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Introduction

In 1926 G. E. Uhlenbeck and S. Goudsmit were attempting to explain the anomalous Zeeman effect when they conjectured the existence of an additional degree of freedom for the electron: an intrinsic angular momentum $\mathbf{s}$, later called spin by W. E. Pauli, which yields to a magnetic moment $\boldsymbol{\mu}$ able to couple to an external magnetic field. In general, given a particle with charge e and mass m, the relation between $\mathbf{s}$ and $\boldsymbol{\mu}$ involves the so called g-factor through the following equation

$$\boldsymbol{\mu} = \frac{\mathrm{g}e}{2mc}\mathbf{s}\,.$$

The Dirac equation for a spin 1/2 fermion predicts $\mathrm{g} = 2$ exactly but approximately twenty years later, a celebrated paper by J. Schwinger in 1948 showed by means of the renormalization properties of the newborn QED that the actual value is such that $\mathrm{g} \neq 2$. He showed that the so called anomaly $a \equiv \mathrm{g}/2 - 1$, otherwise null, differs from zero by the amount $\alpha/(2\pi)$, with α the fine structure constant. This first result turned out to be independent on the lepton flavor, i.e. it was the same for the muon, the tau and the electron, and it was in perfect agreement with the first measures made in that period on the hyperfine structure of atoms in a constant magnetic field. This discrepancy is naively explained as the effect of vacuum fluctuations: many particles and anti-particles are constantly produced in vacuum and this leads to additional effects which in this particular case modify the interaction law between the particle momentum $\boldsymbol{\mu}$ (or equivalently the spin $\mathbf{s}$) and an external magnetic field. Many other successful tests followed and Quantum Field Theory (QFT) became the general language used to study particle physics at a very precise level. As time passed, the scientific community has been attempting to recover, both experimentally and theoretically, a value for a as precise as possible. But why is it so important? Initially, as we said, it allowed physicists to deep test the QFT approach for the theoretical depiction of particles' properties. On the other hand, during the second half of the XX century many improvements have been done: experiments became increasingly precise and, at the same time, theoretical physicists had to face the effort of heavy computations needed in order to compare them to the results of laboratory tests. As we said, the first order result obtained by Schwinger is independent on the lepton flavor but the higher order contributions gain a dependence on the lepton masses which makes their anomalies interesting from different points of view. For instance, given their small mass, electrons are suitable for measuring the fine structure constant with high precision, because the corrections given by virtual particles (the so called radiative corrections) tend to become smaller and smaller as the perturbation order increases. The importance of the muon measurements is different. Since the muon mass is roughly 207 times bigger than the

electron mass, the higher order contributions remain important. As a consequence, muons are much more sensitive with respect to electrons to any virtual contribution given by particles belonging to some new physics. In particular if Λ is an energy scale beyond which these new particles come into play, then one can show that the QED contribution to the muon anomaly, which is also the leading one, gets a contribution of order m_μ^2/Λ^2. Needless to say, this means that the role of muons in this analysis is remarkable.
These days, at a theoretical level both Quantum Electrodynamics and Weak contributions to a_μ are understood enough to make them comparable with experiments. The real issue is the so called hadronic part, i.e. the one governed by Quantum Chromodynamics (QCD). This is mainly due to the fact that at the energy scale currently employed for the a_μ experiments, the contribution due to quarks and hadrons can not be fully treated in a perturbative way. Non-perturbative approaches involving effective field theories are needed and this makes things much more difficult to handle. In particular, some results rely on experimentally measured quantities, e.g. electron-positron annihilation into hadronic states, which are themselves plagued by consistent uncertainties. Indeed, so far the hadronic contribution to the muon anomaly remains the most hard to treat.
The most precise measurement of the anomalous magnetic moment is the one performed at the Brookhaven National Laboratory (BNL), named E821, where highly relativistic muons are made run into a storage ring and their decay rate is observed in order to infer the anomalous angular velocity. The outcome is given with a relative error of 0.54 ppm. It is worth to stress that another $\mathrm{g}-2$ experiment, named E989, is running at Fermilab for which the same storage ring (the ring inside which muons are trapped after being generated) as in the E821 is used. The first run began in 2018 while a second one started in 2019. This new experiment will deal with a much larger number of muons and it aims at an increase of the precision in the measure of a_μ up to 0.14 ppm. A tiny discrepancy between the theoretical prediction of the muon anomaly a_μ and the value obtained at BNL has been seen and as a consequence there have been many attempts to understand where this discrepancy comes from. Our dissertation places exactly at this point. Among other works, a paper of about two years ago by Morishima *et al.* [16] seemed to blame the mentioned discrepancy to the effect of gravity. In particular they argued that including General Relativistic (GR) effects to the analysis would give a correction for the muon anomaly which fits exactly the current discrepancy between the BNL result and the theoretical prediction. From that moment on, a lot of many other works on the subject have been written, which underlined inconsistencies and misinterpretations inside the paper. Nevertheless nobody actually did the full computation of the gravitational contribution to the muon anomaly a_μ. Our aim is to fully treat the problem of the influence of gravity, at a classical level, on the anomalous magnetic moment of charged particles as leptons, in particular muons. As one could guess from the beginning, the core of the GR correction arises from the acceleration that keeps the observer on Earth's surface preventing him/her to fall towards the center of the planet. Such a precise computation is important in our opinion for three main reasons. First of all it definitively gives a precise estimate on the numerical value of the effect of gravity in this particular case. Secondly, it gives a sort of recipe to treat this kind of problems, which can be useful also for other measurements. Finally, the constant technical improvement makes us think that in the not so distant future experiments could be so precise that we might be able to detect this small discrepancy. Nowadays this effects are completely hidden by experimental uncertainty, as we will show, but maybe one day we will see them.

The material inside this dissertation is organized as follows: in chapter 1 we start recovering the Special Relativistic generalization of the spin evolution law, that is, the so called Bargmann–Michel–Telegdi (BMT) equation. Then we move to the General Relativity framework and we achieve a generalization of the BMT equation in a curved space-time, in particular for a Schwarzschild metric and an observer (the laboratory) standing still on Earth's surface. Finally, an expression for the anomalous angular velocity, which is measured experimentally to obtain a_μ, is given. Comments on other effects such as Earth rotation are commented also thanks to other works on the subject. Then in chapter 2 we make a short review of all of the works written so far after the paper by Morishima et al. [16]. As we are going to show, some of them present interesting clues against the thesis in [16] while others have been less precise but at the same time pointed their attention on important issues in the work. Chapter 3 deals with a brief depiction of the experimental setup and procedure employed at BNL during experiment E821. We are going to analyze the theoretical principles which gave rise to the experimental procedure and then give a look at the procedure itself. At the end of the chapter, the experimental result for the muon anomaly a_μ is given. In chapter 4 we split the anomaly into its contributions as given by the different branches of the Standard Model (SD) of particle physics, i.e. the leading QED contribution, which starts with the result obtained by Schwinger in 1948; then the corrections given by the Weak and Strong (QCD) interactions. The last chapter, chapter 5, finally deals with the explicit computation of the gravitational contribution to the anomalous angular velocity, which directly affects the value of a_μ. This computation, as expected from the beginning, will turn out to be very small. In particular we will see that the whole value of the correction is several orders of magnitude smaller than the current experimental uncertainty.

Chapter 1

Anomalous precession

In this chapter we are going to derive a fully covariant expression for the evolution of a particle's spin in curved space-time moving under the action of an electromagnetic field. Following [1] and [9] we will recover a special-relativistic generalization of the classical result obtained by Uhlenbeck and Goudsmit in 1926. Then we will use the tools given by [6] in order to place the system in a curved space-time.

1.1 BMT equation and Spin precession

In order to explain the Zeeman effect G. E. Uhlenbeck and S. Goudsmit in 1926 conjectured that electrons must be endowed with an intrinsic angular momentum (spin) $\mathbf{s}$ which leads to a magnetic moment $\boldsymbol{\mu}$ given by

$$\boldsymbol{\mu} = \frac{\mathrm{g}e}{2mc}\,\mathbf{s} \tag{1.1}$$

in the particle's rest frame where g is the so called g-factor, e is the particle's charge, m is the particle's mass and c is the speed of light. Thus in presence of an external electromagnetic field the spin interacts with it via magnetic coupling which causes the spin vector itself to perform a precession. If we call $\mathbf{B}'$ the magnetic field felt by the particle, the spin in the particle's rest frame evolves performing a so called *Larmor precession*

$$\left(\frac{d\mathbf{s}}{dt}\right)_{\mathrm{rest}} = \boldsymbol{\mu} \times \mathbf{B}' \,. \tag{1.2}$$

Our aim is to study the corrections induced on this equation by a curved background. Our first step will be the recovering of a special-relativistic version of equation (1.2). As we will see later this is a necessary step to take into account also the quantum corrections given by QFT.

We place ourselves in the Special Relativity framework using the metric signature $(-,+,+,+)$ thus our metric tensor will be $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$, the metric element is $ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu$ and we set $c = 1$. Hence we need to replace all of the vector quantities with their suitable 4-vector versions.

Clearly the 3-velocity $\boldsymbol{\beta}$ will be replaced by the well known 4-velocity $w^\mu = dx^\mu/d\tau = \gamma(1, \boldsymbol{\beta})$ where $\gamma = (1-\boldsymbol{\beta}^2)^{-1/2}$ and τ the particle's proper time defined by $d\tau^2 = -ds^2$. As a consequence of this last definition we have for the 4-velocity $w^2 = w^\mu w_\mu = -1$. The electric and magnetic fields $(\mathbf{E}, \mathbf{B})$ are part of the electromagnetic field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ with $A^\mu = (\phi, \mathbf{A})$ 4-vector potential with ϕ electric scalar potential and $\mathbf{A}$ magnetic vector potential. Now we consider a particle which moves with 4-velocity w^μ with respect to the laboratory frame $\mathcal{K}$ and we look for an expression for the electric and magnetic fields in the particle's frame $\mathcal{K}'$. Given the electromagnetic tensor $F^{\mu\nu}$ in the laboratory frame, we know that the particle experiences an electric and magnetic field given as a function of $F'^{\mu\nu}$ by

$$\begin{aligned} E'^i &= F'^{0i} \,, \\ B'^i &= \frac{1}{2}\epsilon^{ijk} F'_{jk} \,. \end{aligned} \tag{1.3}$$

With ϵ^{ijk} the Levi–Civita symbol. Hence it is natural to define two 4-vector fields E^μ and B^μ which respectively represent the electric and magnetic field as functions of all of the quantities that $\mathcal{K}$ sees as follows

$$\begin{aligned} E^\mu_{(w)} &= F^{\mu\nu} w_\nu \,, \\ B^\mu_{(w)} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} w_\nu F_{\rho\sigma} \,. \end{aligned} \tag{1.4}$$

One can easily verify that in the particle rest frame where $w'^\mu = (1, \mathbf{0})$ equations (1.4) reduce to (1.3). Finally the covariant generalization of the spin vector $\mathbf{s}$ is the spin 4-vector S^μ which in the particle's rest frame $\mathcal{K}'$ reduces to

$$S'^\mu = (0, \mathbf{s}) \,,$$

then the left hand side of (1.2) in a generic frame $\mathcal{K}$ simply becomes

$$\frac{dS^\mu}{d\tau} \,,$$

On the right hand side we must have a linear combination of S^μ and the electromagnetic tensor $F_{\mu\nu}$. Moreover we can also have terms proportional to w^μ and to the 4-acceleration $dw^\mu/d\tau = a^\mu$. We want to avoid terms that are quadratic (or of higher orders) in $F^{\mu\nu}$, thus we can not include the coupling $a^\mu F_{\mu\nu}$ since $a \propto F$ for a particle moving in an electromagnetic field. We must also ignore terms proportional to S^μ because $dS^\mu/d\tau$ is orthogonal to them[1]. Hence the final form of the equation is the following:

$$\frac{dS^\mu}{d\tau} = A_1 F^{\mu\nu} S_\nu + A_2 \left(S_\nu F^{\nu\rho} w_\rho\right) w^\mu + A_3 \left(S_\nu \frac{dw^\nu}{d\tau}\right) w^\mu \,, \tag{1.5}$$

[1]The spin 4-vector modulus does not change in time.

with $\{ A_i \}$ constant. Note that, being the particle's 4-velocity vector w^μ in its rest frame equal to $w'^\mu = (1, \mathbf{0})$, the scalar product $S' \cdot w' = S'^\mu w'_\mu$ goes to zero. Thus in general it holds

$$S \cdot w = 0 \qquad \Rightarrow \qquad S^0 = \boldsymbol{\beta} \cdot \mathbf{S} \,. \tag{1.6}$$

A direct consequence of this last relation is useful for computing the values of the constants A_i. Differentiating the first relation in (1.6) with respect to the particle's proper time we obtain

$$w_\mu \frac{dS^\mu}{d\tau} + S_\mu \frac{dw^\mu}{d\tau} = 0 \,, \tag{1.7}$$

thus inserting equation (1.5) we have

$$(A_1 + A_2) w_\mu F^{\mu\nu} S_\nu + (1 - A_3) S_\nu \frac{dw^\nu}{d\tau} = 0 \,.$$

Since in the most general case we can have also nonelectromagnetic forces it is necessary to impose $A_1 = -A_2$ and $A_3 = 1$. Then from equation (1.2) one gets $A_1 = \mathrm{g}e/2m$, hence the final result is

$$\frac{dS^\mu}{d\tau} = \frac{\mathrm{g}e}{2m} \left[F^{\mu\nu} S_\nu - \left(S_\nu F^{\nu\rho} w_\rho \right) w^\mu \right] + \left(S_\nu \frac{dw^\nu}{d\tau} \right) w^\mu \,. \tag{1.8}$$

This is the covariant generalization of equation (1.2) to a particle in arbitrary motion. If the acceleration is due to the electromagnetic field only, that is

$$\frac{dw^\mu}{d\tau} = \frac{e}{m} F^{\mu\nu} w_\nu \,, \tag{1.9}$$

equation (1.8) becomes

$$\boxed{\frac{dS^\mu}{d\tau} = \frac{e}{m} \left[\frac{\mathrm{g}}{2} F^{\mu\nu} S_\nu - \left(\frac{\mathrm{g}}{2} - 1 \right) \left(S_\nu F^{\nu\rho} w_\rho \right) w^\mu \right]} \,. \tag{1.10}$$

This is the Bargmann–Michel–Telegdi equation (BMT) as it can be found[2] in the original article [1]. It describes the rate of change of the spin 4-vector S^μ in $\mathcal{K}$ with respect to the proper time τ of

[2] The authors used the signature $(+, -, -, -)$ so actually there is a different sign inside the square brackets. The different signature induces a different sign on $F^{\mu\nu}$ by definition. Thus calling F_{BMT} and η_{BMT} the field-strength tensor and the metric in [1] our ones are $F = -F_{\mathrm{BMT}}$ and $\eta = -\eta_{\mathrm{BMT}}$. This clearly brings a different sign whenever there is an odd combination of the terms η and F, e.g.

$$F^{\mu\nu} S_\nu \big|_{\mathrm{BMT}} = F^{\mu\nu} \eta_{\nu\rho} S^\rho |_{\mathrm{BMT}} \quad \longrightarrow \quad F^{\mu\nu} S_\nu \,,$$

$$w_\mu F^{\mu\nu} S_\nu |_{\mathrm{BMT}} = w^\sigma \eta_{\sigma\mu} F^{\mu\nu} \eta_{\nu\rho} S^\rho |_{\mathrm{BMT}} \quad \longrightarrow \quad -w_\mu F^{\mu\nu} S_\nu \,.$$

the particle. Note that by the definitions given in (1.4) the second term in the square brackets of equation (1.10) is proportional to the scalar product $S \cdot E$.

Since our purpose is to compare our results with the experimental ones, we need to fix the laboratory reference frame $\mathcal{K}$ and then recover an equation for the evolution of the spin 3-vector $\mathbf{s}$ in the particle's frame $\mathcal{K}'$ as a function of the physical quantities as seen by $\mathcal{K}$. We start by considering a boost from the laboratory frame $\mathcal{K}$ to the particle frame $\mathcal{K}'$ with boost parameter $\boldsymbol{\beta}$:

$$\mathbf{S}' = \mathbf{s} = \mathbf{S} + \frac{\gamma - 1}{\beta^2}(\mathbf{S} \cdot \boldsymbol{\beta})\boldsymbol{\beta} - \gamma \boldsymbol{\beta} S^0 = \mathbf{S} - \frac{\gamma}{\gamma + 1}(\mathbf{S} \cdot \boldsymbol{\beta})\boldsymbol{\beta}\,, \tag{1.11}$$

where in the last equality we used $\beta^2 = 1 - 1/\gamma^2$ and the result (1.6). Thus in order to obtain an expression for the evolution of $\mathbf{S}' = \mathbf{s}$ we need to derive (1.11) with respect to τ:

$$\frac{d\mathbf{s}}{d\tau} = \frac{d\mathbf{S}}{d\tau} - \frac{d}{d\tau}\left[\frac{\gamma}{\gamma + 1}(\mathbf{S} \cdot \boldsymbol{\beta})\boldsymbol{\beta}\right] . \tag{1.12}$$

Now we need an expression for $d\mathbf{S}/d\tau$. First of all we set

$$F^\mu = \frac{ge}{2m}\left[F^{\mu\nu} S_\nu - \left(S_\nu F^{\nu\rho} w_\rho\right) w^\mu\right] , \tag{1.13}$$

so that $F^\mu = (F^0, \mathbf{F})$; note that from this definition we have that $F^\mu w_\mu = 0$ which implies $F^0 = \boldsymbol{\beta} \cdot \mathbf{F}$. Then from equation (1.8) we have

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F} + \gamma \boldsymbol{\beta}\left(S_\nu \frac{dw^\nu}{d\tau}\right) , \tag{1.14}$$

Using (1.6) we see that

$$S_\nu \frac{dw^\nu}{d\tau} = \gamma \mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}$$

and therefore we can write equation (1.14) as

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F} + \gamma^2 \boldsymbol{\beta}\left(\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right) .$$

We substitute this result into equation (1.12) and write $\mathbf{S}$ in function of $\mathbf{s}$ with a boost from $\mathcal{K}'$ to $\mathcal{K}$

$$\mathbf{S} = \mathbf{s} + \frac{\gamma^2}{\gamma + 1}(\mathbf{s} \cdot \boldsymbol{\beta})\boldsymbol{\beta}\,.$$

After some algebra, equation (1.12) gives

$$\frac{d\mathbf{s}}{d\tau} = \mathbf{F}' + \frac{\gamma^2}{\gamma+1}\left[\mathbf{s}\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{d\tau}\right)\right] . \tag{1.15}$$

Here $F'^{\mu} = (F'^{0}, \mathbf{F}')$ thus using (1.13) we have

$$\mathbf{F}' = \frac{\mathrm{g}e}{2m}\mathbf{s}\times\mathbf{B}' ,$$

which is exactly the magnetic contribution that gives (1.2). The second contribution to the right hand side of equation (1.15) clearly represents an additional precession term due to the relative motion between the particle's rest frame $\mathcal{K}'$ and the laboratory frame $\mathcal{K}$. Remembering that if we call t the laboratory time coordinate we have $d\tau = dt/\gamma$ one can rearrange equation (1.15) as

$$\frac{d\mathbf{s}}{dt} = \frac{\mathbf{F}'}{\gamma} + \boldsymbol{\omega}_{\mathrm{T}}\times\mathbf{s} , \tag{1.16}$$

where we defined

$$\boldsymbol{\omega}_{\mathrm{T}} \equiv \frac{\gamma^2}{\gamma+1}\frac{d\boldsymbol{\beta}}{dt}\times\boldsymbol{\beta} .$$

This term was derived by L. H. Thomas in 1927. We can note that it goes to zero if the 3-acceleration $\mathbf{a} = d\boldsymbol{\beta}/dt$ is parallel to the particle velocity. In the case $\boldsymbol{\beta}\times\mathbf{a} \neq 0$ the Lorentz transformation from $\mathcal{K}$ to $\mathcal{K}'$ actually turns out to be a combination of a boost and a spatial rotation. This explains the presence of the second term in equation (1.16). For a generic Lorentz transformation between $\mathcal{K}$ and $\mathcal{K}'$ the magnetic field $\mathbf{B}$ transforms as

$$\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} ,$$

moreover for a particle moving in an electromagnetic field $(\mathbf{E}, \mathbf{B})$ it can be shown that

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{m\gamma}\left[\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - (\boldsymbol{\beta}\cdot\mathbf{E})\,\boldsymbol{\beta}\right] . \tag{1.17}$$

Putting all these results together we can rewrite equation (1.16) as the so called *Thomas' equation*:

$$\boxed{\frac{d\mathbf{s}}{dt} = \frac{e}{m}\mathbf{s}\times\left[\left(\frac{\mathrm{g}}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{\mathrm{g}}{2} - \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\mathbf{E} - \left(\frac{\mathrm{g}}{2} - 1\right)\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right]} , \tag{1.18}$$

which describes the evolution of the spin 3-vector $\mathbf{s}$ (in $\mathcal{K}'$) with laboratory time as a function of $\mathbf{s}$ and laboratory quantities only. An equation like the last one tells us that $\|\mathbf{s}\| = \mathrm{const}$ so we can extract directly the *spin precession angular velocity* $\boldsymbol{\omega}_{\mathrm{s}}$ directly from (1.18):

$$\boxed{\boldsymbol{\omega}_{\mathrm{s}} = -\frac{e}{m}\left[\left(\frac{\mathrm{g}}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{\mathrm{g}}{2} - \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\mathbf{E} - \left(\frac{\mathrm{g}}{2} - 1\right)\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right]}\,. \tag{1.19}$$

Another important information that we can derive from the previous treatment is the evolution of the projection of the vector $\mathbf{s}$ on the direction of motion $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}/\|\boldsymbol{\beta}\|$. We have

$$\frac{d}{dt}\left(\mathbf{s}\cdot\hat{\boldsymbol{\beta}}\right) = \frac{d}{dt}\left(\mathbf{s}\cdot\frac{\boldsymbol{\beta}}{\beta}\right) = \frac{d\mathbf{s}}{dt}\cdot\hat{\boldsymbol{\beta}} + \frac{1}{\beta}\left[\mathbf{s} - \left(\mathbf{s}\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}}\right]\cdot\frac{d\boldsymbol{\beta}}{dt}\,,$$

with $\beta = \|\boldsymbol{\beta}\|$; which after some manipulations becomes

$$\frac{d}{dt}\left(\mathbf{s}\cdot\hat{\boldsymbol{\beta}}\right) = -\frac{e}{m}\mathbf{s}_\perp\cdot\left[\left(\frac{\mathrm{g}}{2} - 1\right)\hat{\boldsymbol{\beta}}\times\mathbf{B} + \left(\frac{\mathrm{g}\beta}{2} - \frac{1}{\beta}\right)\mathbf{E}\right]\,. \tag{1.20}$$

Here[3] we defined $\mathbf{s}_\perp$ the $\mathbf{s}$ component orthogonal to $\boldsymbol{\beta}$: $\mathbf{s}_\perp = \mathbf{s} - (\mathbf{s}\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}\times\left(\mathbf{s}\times\hat{\boldsymbol{\beta}}\right)$.

To conclude this section we need an expression for the cyclotron angular velocity $\boldsymbol{\omega}_{\mathrm{c}}$, since it is precisely the difference between this quantity and $\boldsymbol{\omega}_{\mathrm{s}}$ that will highlight the anomalous precession of the magnetic moment we are looking for. We start by splitting $\boldsymbol{\beta}$ into its modulus and its unit vector: $\boldsymbol{\beta} = \|\boldsymbol{\beta}\|\,\hat{\boldsymbol{\beta}}$. Now the evolution of the vector is

$$\frac{d\boldsymbol{\beta}}{dt} = \|\boldsymbol{\beta}\|\,\frac{d\hat{\boldsymbol{\beta}}}{dt} + \frac{d\,\|\boldsymbol{\beta}\|}{dt}\hat{\boldsymbol{\beta}}\,.$$

We then scalar multiply both sides by $\hat{\boldsymbol{\beta}}$ and recall that $\hat{\boldsymbol{\beta}}\cdot d\hat{\boldsymbol{\beta}}/dt = 0$ so we have

$$\frac{d\,\|\boldsymbol{\beta}\|}{dt} = \hat{\boldsymbol{\beta}}\cdot\frac{d\boldsymbol{\beta}}{dt}\,,$$

and inserting this in the evolution equation for $\boldsymbol{\beta}$ we obtain

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{1}{\|\boldsymbol{\beta}\|}\left[\frac{d\boldsymbol{\beta}}{dt} - \hat{\boldsymbol{\beta}}\left(\hat{\boldsymbol{\beta}}\cdot\frac{d\boldsymbol{\beta}}{dt}\right)\right] = \frac{1}{\|\boldsymbol{\beta}\|}\,\hat{\boldsymbol{\beta}}\times\left(\frac{d\boldsymbol{\beta}}{dt}\times\hat{\boldsymbol{\beta}}\right)\,. \tag{1.22}$$

This gives the recipe to extract the *cyclotron angular velocity* for the unit vector $\hat{\boldsymbol{\beta}}$:

$$\frac{1}{\|\boldsymbol{\beta}\|}\,\hat{\boldsymbol{\beta}}\times\frac{d\boldsymbol{\beta}}{dt} = \boxed{\boldsymbol{\omega}_{\mathrm{c}} = \frac{e}{m\gamma}\left(\frac{\gamma^2}{\gamma^2-1}\boldsymbol{\beta}\times\mathbf{E} - \mathbf{B}\right)}\,, \tag{1.23}$$

[3]Recall that for three generic 3-vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ it holds

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{1.21}$$

.

where in the last passage we used[4] (1.17).
In the ideal case in which the vectors $\boldsymbol{\omega}_{\mathrm{c}}$ and $\boldsymbol{\omega}_{\mathrm{s}}$ are parallel, we can define the *anomalous spin precession rate* $\boldsymbol{\omega}_{\mathrm{a}}$ at which the spin vector $\mathbf{s}$ turns with respect to the momentum direction $\hat{\boldsymbol{\beta}}$ to be exactly the difference

$$\boxed{\boldsymbol{\omega}_{\mathrm{s}} - \boldsymbol{\omega}_{\mathrm{c}} \;\equiv\; \boldsymbol{\omega}_{\mathrm{a}} = -\frac{e}{m}\left[a\mathbf{B} - \left(a - \frac{1}{\gamma^2 - 1}\right)\boldsymbol{\beta}\times\mathbf{E} - a\left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right]}\,, \tag{1.24}$$

where we have defined $a \equiv \mathrm{g}/2 - 1$, the so called *anomaly*.

1.2 Curved space-time

Now our aim is to generalize the result obtained in the previous section, in order to gain a law for the spin precession that accounts for Earth's gravitational field. In particular we move to the General Relativity framework and put both the particle and the laboratory on Earth and see how the previous laws change when gravity is present.
In order to do that, as before we consider two reference frames: a frame $\mathcal{K}$ with 4-velocity $u = dx/d\tau_u$ (the observer) and a frame $\mathcal{K}'$ with 4-velocity $w = dx/d\tau_w$ (the particle) where we call τ_u and τ_w the former's and the latter's proper time respectively. In a curved space-time the 4-acceleration of a frame which moves with a given velocity, say u, is

$$a^\mu_{(u)} = \left(\frac{Du}{D\tau_u}\right)^\mu = (\nabla_u u)^\mu = \frac{du^\mu}{d\tau_u} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho\,,$$

where $D/D\tau_u = u^\mu\nabla_\mu$ is the covariant derivative and $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols. Since we are parametrizing our observers' world-lines with their proper times, then both their 4-velocities satisfy the condition $u^2 = u^\mu g_{\mu\nu}u^\nu = -1 = w^2$ with $g_{\mu\nu}$ metric tensor of the curved space-time. This clearly implies the well known property

$$u\cdot\frac{Du}{D\tau_u} = 0 = w\cdot\frac{Dw}{D\tau_w}\,. \tag{1.25}$$

The natural way to move from the Special Relativity framework of section 1.1 to the General Relativity one is to make the following substitutions whenever dealing with a covariant equation:

$$\begin{aligned}\frac{d}{dt} &\longrightarrow \frac{D}{D\tau_u}\,,\\ \frac{d}{d\tau} &\longrightarrow \frac{D}{D\tau_w}\,,\\ \eta_{\mu\nu} &\longrightarrow g_{\mu\nu}\,.\end{aligned} \tag{1.26}$$

[4] At first sight it might seem that a term is missing in the final result $\boldsymbol{\omega}_{\mathrm{c}}$ but this term is $\propto \hat{\boldsymbol{\beta}}$ hence it goes to zero when performing the vector product $\boldsymbol{\omega}_{\mathrm{c}}\times\hat{\boldsymbol{\beta}}$.

Since it will be useful we recall that for a curved space-time the volume 4-form is defined as

$$\eta = \frac{1}{4!}\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \sqrt{-g}\,dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\,, \tag{1.27}$$

with $\epsilon_{\mu\nu\rho\sigma}$ Levi–Civita symbol[5] and $g = \det(g_{\mu\nu})$. As we can see, the components of this 4-form are completely anti-symmetric thus we can write

$$\eta = \frac{1}{4!}\,\eta_{\mu\nu\rho\sigma}\,dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\,,$$

so that we can define[6] $\eta_{\mu\nu\rho\sigma} = \sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}$.

1.2.1 Projectors & space-time splitting

As we know u identifies the time direction of frame $\mathcal{K}$ and w identifies the time direction of frame $\mathcal{K}'$. Thus it will be useful to define the projectors $T(u)^\mu_\nu$ and $P(u)^\mu_\nu$, which are respectively the parallel and orthogonal projector with respect to u, as follows:

$$T(u)^\mu_\nu = -u^\mu u_\nu\,, \tag{1.28}$$

$$P(u)^\mu_\nu = \delta^\mu_\nu - T(u)^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu\,. \tag{1.29}$$

These operators allow us to split the observer's space-time $\mathcal{K}$ into a purely spatial part and a purely temporal one. They satisfy the projector's relations

$$T(u)^\mu_\nu u^\nu = u^\mu\,, \tag{1.30}$$

$$P(u)^\mu_\nu u^\nu = 0\,, \tag{1.31}$$

$$P(u)^\mu_\nu + T(u)^\mu_\nu = \delta^\mu_\nu\,, \tag{1.32}$$

$$P(u)^\mu_\alpha P(u)^\alpha_\nu = P(u)^\mu_\nu\,, \tag{1.33}$$

$$T(u)^\mu_\alpha T(u)^\alpha_\nu = T(u)^\mu_\nu\,, \tag{1.34}$$

$$P(u)^\mu_\alpha T(u)^\alpha_\nu = T(u)^\mu_\alpha P(u)^\alpha_\nu = 0\,. \tag{1.35}$$

Given a generic $\binom{p}{q}$ tensor A, in components $A^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}$, its splitting is given by

[5]Note that $\epsilon_{0123} = 1 = \epsilon^{0123}$ because it has to be interpreted as a symbol and not as the component of a tensor.

[6]Now this represents the component of a tensor, hence $\eta^{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ as results from the raising index procedure

$$\eta^{\mu\nu\rho\sigma} = g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\eta_{\alpha\beta\gamma\delta} = \frac{1}{g}\sqrt{-g}\,\epsilon^{\mu\nu\rho\sigma} = -\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\,.$$

The second passage follows from a well known property of the Levi–Civita symbol: $g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\epsilon_{\alpha\beta\gamma\delta} = \det\big(g^{-1}\big)\epsilon^{\mu\nu\rho\sigma}$.

$$A^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} = \big(P(u)+T(u)\big)^{\mu_1}_{\alpha_1}\cdots\big(P(u)+T(u)\big)^{\mu_p}_{\alpha_p}\big(P(u)+T(u)\big)^{\beta_1}_{\nu_1}\cdots\big(P(u)+T(u)\big)^{\beta_q}_{\nu_q}A^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}\,. \quad (1.36)$$

It's worth to stress that the splitting of the metric tensor $g_{\mu\nu}$ is

$$g_{\mu\nu} = -u_\mu u_\nu + P(u)_{\mu\nu}\,, \quad (1.37)$$

with $P(u)_{\mu\nu} = g_{\mu\alpha}P(u)^\alpha_\nu$. This form of the metric tensor highlights the splitting of space and time for the observer u. Given a vector field A such that $A\cdot u = 0$ then A is said to be *spatial* with respect to u, with no surprise, and we will say it belongs to the *local rest space* of u, $A \in \mathrm{LRS}_u = \{\, X \mid X\cdot u = 0\,\}$. From equations (1.25) we can see that $Du/D\tau_u \in \mathrm{LRS}_u$.
By means of $P(u)$ we can define the relative 4-velocity β as the spatial velocity with which the observer u sees the motion of the particle w:

$$\beta = \frac{1}{\gamma}P(u)w\,, \quad (1.38)$$

where the factor $\gamma = -u\cdot w = d\tau_u/d\tau_w$ accounts for the different times of the two frames. From these last definitions and the explicit expression of $P(u)$ (1.29) we get the composition law

$$w = \gamma(u+\beta)\,. \quad (1.39)$$

Notice that by construction $\beta\cdot u = 0$ thus $\beta \in \mathrm{LRS}_u$ and one easily recovers the special-relativistic relation $\beta^2 = 1 - 1/\gamma^2$.
Now given two spatial vector fields A^μ and B^ν we can define the u-scalar product $\cdot_u$ and the u-vector product $\times_u$ as follows

$$A\cdot_u B = P(u)_{\mu\nu}\,A^\mu B^\nu\,, \quad (1.40)$$

$$(A\times_u B)^\mu = \eta(u)^{\mu\nu\rho}A_\nu B_\rho\,. \quad (1.41)$$

where we define $\eta(u)^{\mu\nu\rho} = u_\sigma\eta^{\sigma\mu\nu\rho}$. It can also be shown that if $A, B, C \in \mathrm{LRS}_u$ then a relation formally equal to (1.21) holds:

$$A\times_u(B\times_u C) = (A\cdot_u C)B - (A\cdot_u B)C\,. \quad (1.42)$$

Notice that even if one of the two vector fields only is orthogonal to u, say A, then the scalar product automatically reduces to the u-scalar product, i.e.

$$A\cdot B = g_{\mu\nu}A^\mu B^\nu = \big(-u_\mu u_\nu + P(u)_{\mu\nu}\big)A^\mu B^\nu = A\cdot_u B\,,$$

where we used (1.37). Whenever we will deal with vectors belonging to LRS_u we are going to omit the subscript u in (1.40) and (1.41).

Now given the electromagnetic potential 1-form, $A = A_\mu \, dx^\mu = \phi \, dx^0 + A_i \, dx^i$, taking its exterior derivative we obtain the Faraday 2-form $F = dA = \frac{1}{2!} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$. From this we can define the electric vector field and the magnetic vector field with respect to the observer u as we did before. In this new framework equations (1.4) become

$$\begin{aligned} E^\mu_{(u)} &= F^{\mu\nu} u_\nu \, , \\ B^\mu_{(u)} &= \frac{1}{2} \eta^{\nu\mu\rho\sigma} u_\nu F_{\rho\sigma} = \frac{1}{2} \eta(u)^{\mu\rho\sigma} F_{\rho\sigma} \, . \end{aligned} \tag{1.43}$$

By definition we have $B_{(u)} \cdot u = 0 = E_{(u)} \cdot u$ i.e. $E_{(u)}, B_{(u)} \in \mathrm{LRS}_u$.
Finally the following splitting of the electromagnetic components $F_{\mu\nu}$ will be useful as well:

$$F^{\mu\nu} = u^\mu E^\nu_{(u)} - u^\nu E^\mu_{(u)} + B^\rho_{(u)} \eta(u)_\rho{}^{\mu\nu} \, . \tag{1.44}$$

1.2.2 Cyclotron frequency

Now we want to present the generalization of what we derived in Section 1.1 to curved space-times. In particular we are going to derive an expression for the cyclotron equation of motion, analogous to (1.17), and then another one for the spin precession frequency, analogous to equation (1.18). What we will see is that in both cases a correction due to the observer's 4-acceleration appears.

As for what we did in section 1.1 we are interested in the evolution of all of the quantities with laboratory time, so actually in some cases we might have to switch between the covariant derivative with respect to τ_w and the one with respect to τ_u. This can be easily done as follows

$$\frac{D}{D\tau_w} = \nabla_w = w^\mu \nabla_\mu = \frac{dx^\mu}{d\tau_w} \nabla_\mu = \frac{d\tau_u}{d\tau_w} \frac{dx^\mu}{d\tau_u} \nabla_\mu = \gamma \nabla_u = \gamma \frac{D}{D\tau_u} \, . \tag{1.45}$$

In order to derive the cyclotron equation we start from equation (1.39) and derive both sides with respect to τ_u

$$\frac{Dw}{D\tau_u} = \frac{D}{D\tau_u} \gamma(u + \beta) \quad \Longrightarrow \quad \frac{D\beta}{D\tau_u} = \frac{1}{\gamma} \left(\frac{Dw}{D\tau_u} - (u + \beta) \frac{D\gamma}{D\tau_u} \right) - \frac{Du}{D\tau_u} \, . \tag{1.46}$$

This gives the evolution of the spatial relative velocity β with respect to τ_u. Now we need to project this quantity on LRS_u in order to have the cyclotron equation. Applying $P(u)$ on both sides of equation (1.46) we have

$$P(u) \frac{D\beta}{D\tau_u} = \frac{1}{\gamma} \left(P(u) \frac{Dw}{D\tau_u} - \beta \frac{D\gamma}{D\tau_u} \right) - \frac{Du}{D\tau_u} \, . \tag{1.47}$$

This last equation gives the evolution of β with τ_u as seen by the laboratory frame.
Now, since we assume that our particle w moves under the influence of an electromagnetic field, its 4-acceleration is given by the Lorentz force equation

$$m\left(\frac{Dw}{D\tau_w}\right)^\mu = eF^{\mu\nu}w_\nu\,; \tag{1.48}$$

so we can rearrange the first term on the right hand side of equation (1.47) with the help of equations (1.39), (1.44) and (1.45) to obtain[7]

$$\begin{aligned}\left(\frac{Dw}{D\tau_u}\right)^\mu &= \frac{e}{m\gamma}\left(u^\mu E^\nu - u^\nu E^\mu + B^\rho \eta(u)_\rho{}^{\mu\nu}\right)\gamma\left(u_\nu + \beta_\nu\right)\\ &= \frac{e}{m}\big(E^\mu + (E\cdot\beta)u^\mu + (\beta\times B)^\mu\big)\,,\end{aligned} \tag{1.49}$$

then the projection gives

$$P(u)\frac{Dw}{D\tau_u} = \frac{e}{m}\big(E + \beta\times B\big)\,.$$

The second term on the right hand side becomes

$$\frac{D\gamma}{D\tau_u} = -u\cdot\frac{Dw}{D\tau_u} - w\cdot\frac{Du}{D\tau_u} = \frac{e}{m}E\cdot\beta - \gamma\beta\cdot\frac{Du}{D\tau_u}\,.$$

Thus we find

$$P(u)\frac{D\beta}{D\tau_u} = \frac{e}{m\gamma}\big(E - \beta(E\cdot\beta) + \beta\times B\big) - \left[\frac{Du}{D\tau_u} - \beta\left(\frac{Du}{D\tau_u}\cdot\beta\right)\right]\,, \tag{1.50}$$

that, in a generic coordinate basis, can be put in the more compact form

$$\left(P(u)\frac{D\beta}{D\tau_u}\right)^\mu = \frac{e}{m\gamma}\left[\left(E^\nu - \frac{m\gamma}{e}\frac{Du^\nu}{D\tau_u}\right)(\delta^\mu_\nu - \beta^\mu\beta_\nu) + (\beta\times B)^\mu\right]\,. \tag{1.51}$$

This equation is similar to equation (1.17) where now it appears an effective electric field which shows a dependence on the observer's 4-acceleration. As expected if the observer moves along a geodesic, i.e. it performs a so called free fall, then $Du/D\tau_u = 0$ and equation (1.51) becomes formally identical to equation (1.17).

To recover the cyclotron frequency we need the evolution law of the unit vector $\hat\beta = \beta/\left\|\beta\right\|$. Exactly as we did in section 1.1 for the cyclotron frequency we start by writing $\beta = \left\|\beta\right\|\hat\beta$, then

$$P(u)\frac{D\beta}{D\tau_u} = P(u)\left(\left\|\beta\right\|\frac{D\hat\beta}{D\tau_u} + \hat\beta\,\frac{D\left\|\beta\right\|}{D\tau_u}\right) = \left\|\beta\right\|P(u)\frac{D\hat\beta}{D\tau_u} + \frac{d\left\|\beta\right\|}{d\tau_u}\hat\beta\,.$$

[7] From now on the electric and magnetic vector fields are always referred to the frame u and for simplicity we will omit the subscript (u).

The following result is then easy to show: by scalar multiplying the previous equation by $\hat{\beta}$ we get

$$\hat{\beta} \cdot P(u) \frac{D\beta}{D\tau_u} = \frac{d\,\|\beta\|}{d\tau_u}\,,$$

which allows us to find the evolution of the unit vector as

$$P(u)\frac{D\hat{\beta}}{D\tau_u} = \frac{1}{\|\beta\|}\left[P(u)\frac{D\beta}{D\tau_u} - \hat{\beta}\left(\hat{\beta}\cdot P(u)\frac{D\beta}{D\tau_u}\right)\right] = \frac{1}{\|\beta\|}\,\hat{\beta}\times\left(P(u)\frac{D\beta}{D\tau_u}\times\hat{\beta}\right)\,,$$

where in the last passage we used (1.42). After some algebra we finally obtain

$$P(u)\frac{D\hat{\beta}}{D\tau_u} = \frac{e}{m\gamma}\left[\frac{\gamma^2}{\gamma^2-1}\beta\times\left(E - \frac{m\gamma}{e}\frac{Du}{D\tau_u}\right) - B\right]\times\hat{\beta} \tag{1.52}$$

thus we can define the *cyclotron angular velocity* $\Omega_{\mathrm{c}} \in \mathrm{LRS}_u$

$$\boxed{\Omega_{\mathrm{c}} = \frac{e}{m\gamma}\left[\frac{\gamma^2}{\gamma^2-1}\beta\times\left(E - \frac{m\gamma}{e}\frac{Du}{D\tau_u}\right) - B\right]}\,. \tag{1.53}$$

Again we can see that the 4-acceleration of the observer provides a term that can be seen as a correction to the electric field felt by the particle. As before this correction disappears when the observer is in geodesic motion, $Du/D\tau_u = 0$, and as expected in this case the cyclotron angular velocity (1.53) becomes formally equal to the one we found in the Special Relativity case, (1.23).

1.2.3 Spin precession & anomalous precession

In order to achieve an expression for the particle's spin evolution we follow the path of section 1.1. We start by defining a 4-vector S which we call spin and which satisfies the condition

$$S \cdot w = 0\,.$$

We see that $S \in \mathrm{LRS}_w$ thus it is a spatial vector for the particle's frame. Now we want to do the same thing we did in section 1.1, i.e. we are going to achieve an expression for the evolution of the spin vector S with respect to the laboratory's reference frame. Since it will be very useful in the following part, we first split the vector S in order to gain a spatial part of it in the observer's frame. For the splitting we recall (1.36)

$$S = P(u)S + T(u)S = \Sigma - u(S\cdot u) = \Sigma + u(S\cdot\beta)\,,$$

where we defined $\Sigma = P(u)S \in \mathrm{LRS}_u$ which represents the spatial part of the vector S in the laboratory frame. In the last passage we used $u = w/\gamma - \beta$ form the composition relation (1.39); then making the scalar product with β we obtain $S\cdot\beta = \Sigma\cdot\beta$ and then

$$S = \Sigma + u(\Sigma \cdot \beta)\,. \tag{1.54}$$

As we said we need an equation for the evolution of S with respect to the frame carried by the laboratory. Given the laboratory frame $\{ e_\mu \}$, the whole work translates into computing the evolution of the vector field with respect to a frame $\{ E_\mu \}$ which is the boost of $\{ e_\mu \}$ into the particle's frame. Now we use the fact that the boost map is an isometry to note that the evolution of S with respect to $\{ E_\mu \}$ is nothing but the evolution of the boosted spin vector of S into LRS_u, we call it $\mathcal{S}$, with respect to $\{ e_a \}$. This boost can be done following [6], in particular the boost map between local rest spaces $B_{(\mathrm{lrs})}(u, w)\colon \mathrm{LRS}_w \to \mathrm{LRS}_u$ acts on S as follows

$$B_{(\mathrm{lrs})}(u, w)S = \mathcal{S} = \Sigma - \frac{\gamma}{\gamma+1}(\Sigma \cdot \beta)\beta\,. \tag{1.55}$$

Notice that this last equation is formally equal to (1.11). Now we proceed as for the cyclotron equation i.e. we want to end up with an expression for $D\hat{\mathcal{S}}/D\tau_u$ where $\hat{\mathcal{S}}$ is the unit vector of $\mathcal{S}$. However before going on we can notice something that simplifies the work: the curved version of BMT equation (1.10) is simply obtained making the substitutions (1.26)

$$\left(\frac{DS}{D\tau_w}\right)^\mu = \gamma\left(\frac{DS}{D\tau_u}\right)^\mu = \frac{e}{m}\left[\frac{\mathrm{g}}{2}F^{\mu\nu}S_\nu - \left(\frac{\mathrm{g}}{2}-1\right)\left(S_\nu F^{\nu\rho}w_\rho\right)w^\mu\right], \tag{1.56}$$

and again S^2 is constant. Moreover from equation (1.54) one can write $S^2 = \Sigma^2 - (\Sigma \cdot \beta)^2$ and using (1.55) we have $\mathcal{S}^2 = \Sigma^2 - (\Sigma \cdot \beta)^2$ thus we have[8]

$$\|\mathcal{S}\| = \|S\| = \text{const}\,. \tag{1.57}$$

Now we look for an expression that links S to $\mathcal{S}$. From equation (1.54) we can write $\Sigma = S - u(\Sigma \cdot \beta)$ and putting this into equation (1.55) we can write

$$\mathcal{S} = S - u(\Sigma \cdot \beta) - \frac{\gamma}{\gamma+1}(\Sigma \cdot \beta)\beta = S - \frac{\Sigma \cdot \beta}{\gamma+1}(u + w)\,.$$

If we scalar multiply equation (1.55) by β and do some algebra we have $\beta \cdot \Sigma = \gamma\beta \cdot \mathcal{S}$ so we finally get

$$\mathcal{S} = S - \frac{\gamma}{\gamma+1}(\mathcal{S} \cdot \beta)(u + w)\,. \tag{1.58}$$

For the sake of compactness, and also convenience as we will see, we define $f = \gamma(\mathcal{S} \cdot \beta)/(\gamma + 1)$ so that we can write equation (1.58) as

$$\mathcal{S} = S - f(u + w)\,.$$

[8] This comes as no surprise since, as we said before, the boost map is an isometry.

We covariantly derive both sides of this last equation with respect to τ_u:

$$\frac{D\mathcal{S}}{D\tau_u} = \frac{DS}{D\tau_u} - \frac{df}{d\tau_u}(u+w) - f\left(\frac{Du}{D\tau_u} + \frac{Dw}{D\tau_u}\right) . \tag{1.59}$$

Now we use the following trick in order to recover the term $df/d\tau_u$: we scalar multiply both sides of the previous equation by u

$$u \cdot \frac{D\mathcal{S}}{D\tau_u} = -\mathcal{S} \cdot \frac{Du}{D\tau_u} = u \cdot \frac{DS}{D\tau_u} - \frac{df}{d\tau_u}(-1-\gamma) - fu \cdot \frac{Dw}{D\tau_u} ,$$

where the first passage directly follows by deriving the orthogonality condition $\mathcal{S} \cdot u = 0$ with respect to τ_u. In this way we achieve an expression for $df/d\tau_u$

$$\frac{df}{d\tau_u} = \frac{1}{\gamma+1}\left(fu \cdot \frac{Dw}{D\tau_u} - \mathcal{S} \cdot \frac{Du}{D\tau_u} - u \cdot \frac{DS}{D\tau_u}\right) ,$$

now we insert this result into equation (1.59) and project both sides using $P(u)$ to obtain

$$\begin{aligned} P(u)\frac{D\mathcal{S}}{D\tau_u} = P(u)\frac{DS}{D\tau_u} - \frac{1}{\gamma+1}\left(fu \cdot \frac{Dw}{D\tau_u} - \mathcal{S} \cdot \frac{Du}{D\tau_u} - u \cdot \frac{DS}{D\tau_u}\right)\gamma\beta \\ - f\left(\frac{Du}{D\tau_u} + P(u)\frac{Dw}{D\tau_u}\right) . \end{aligned} \tag{1.60}$$

Notice the following: by looking at the terms proportional to $Du/D\tau_u$ and then using (1.42) we have

$$\frac{\gamma}{\gamma+1}\left(\mathcal{S} \cdot \frac{Du}{D\tau_u}\right)\beta - \frac{\gamma}{\gamma+1}(\beta \cdot \mathcal{S})\frac{Du}{D\tau_u} = \frac{\gamma}{\gamma+1}\mathcal{S} \times \left(\beta \times \frac{Du}{D\tau_u}\right) ,$$

which will be the only term depending on the observer's 4-acceleration. Then, as before, we expect it to be the correction term since it is the only one containing the observer's 4-acceleration. Hence (1.60) becomes

$$\begin{aligned} P(u)\frac{D\mathcal{S}}{D\tau_u} = \frac{\gamma}{\gamma+1}\,\mathcal{S} \times \left(\beta \times \frac{Du}{D\tau_u}\right) + P(u)\frac{DS}{D\tau_u} + \frac{1}{\gamma+1}\,u \cdot \frac{DS}{D\tau_u} \\ - \frac{f}{\gamma+1}\,u \cdot \frac{Dw}{D\tau_u}\gamma\beta - fP(u)\frac{Dw}{D\tau_u} . \end{aligned}$$

In order to go on we only need to insert equation (1.56) and then use equations (1.48) and (1.49). After some tedious computation we are left with the following result

$$
\begin{aligned}
P(u)\frac{D\mathcal{S}}{D\tau_u} = \frac{e}{m}\mathcal{S}\times\Bigg[&\left(\frac{\mathrm{g}}{2}-1+\frac{1}{\gamma}\right)B-\left(\frac{\mathrm{g}}{2}-1\right)\frac{\gamma}{\gamma+1}(\beta\cdot B)\beta \\
&-\left(\frac{\mathrm{g}}{2}-\frac{\gamma}{\gamma+1}\right)\beta\times E+\frac{m}{e}\frac{\gamma}{\gamma+1}\beta\times\frac{Du}{D\tau_u}\Bigg],
\end{aligned}
\tag{1.61}
$$

which displays formally the same shape of equation (1.18) apart from the last term inside the square brackets which clearly represents the correction we are looking for, due to the observer's acceleration. From (1.57) we know that $D\mathcal{S}/D\tau_u = \|\mathcal{S}\|\, D\hat{\mathcal{S}}/D\tau_u$ then we can extract the *spin precession angular velocity* $\Omega_{\mathrm{s}} \in \mathrm{LRS}_u$ directly from (1.61) and define

$$
\boxed{
\begin{aligned}
\Omega_{\mathrm{s}} = -\frac{e}{m}\Bigg[&\left(\frac{\mathrm{g}}{2}-1+\frac{1}{\gamma}\right)B-\left(\frac{\mathrm{g}}{2}-1\right)\frac{\gamma}{\gamma+1}(\beta\cdot B)\beta \\
&-\left(\frac{\mathrm{g}}{2}-\frac{\gamma}{\gamma+1}\right)\beta\times E+\frac{m}{e}\frac{\gamma}{\gamma+1}\beta\times\frac{Du}{D\tau_u}\Bigg]
\end{aligned}
}
\tag{1.62}
$$

With no surprise we can note that in the case of a free falling observer this result for the spin angular velocity becomes formally the same as equation (1.19).

Now as we did for the Special Relativity case, we define the rate Ω_{a} at which the spin vector $\mathcal{S}$ turns with respect to the relative momentum direction $\hat{\beta}$ as the difference

$$
\boxed{
\begin{aligned}
\Omega_{\mathrm{s}}-\Omega_{\mathrm{c}} \;\equiv\; \Omega_{\mathrm{a}} = -\frac{e}{m}&\left[aB-\left(a-\frac{1}{\gamma^2-1}\right)\beta\times E-a\left(\frac{\gamma}{\gamma+1}\right)(\beta\cdot B)\beta\right] \\
&-\frac{\gamma}{\gamma+1}\left(\beta\times\frac{Du}{D\tau_u}-\frac{\gamma}{\gamma-1}\frac{Du}{D\tau_u}\right),
\end{aligned}
}
\tag{1.63}
$$

where again we defined $a = \mathrm{g}/2-1$. This definition gives exactly the precession rate we are looking for if the two vectors Ω_{s} and Ω_{c} are parallel. Notice that also $\Omega_{\mathrm{a}} \in \mathrm{LRS}_u$. As expected for a freely falling observer (1.63) reduces to an equation formally identical to (1.24). Thus an observer with non zero 4-acceleration, e.g. the case of an observer (the laboratory) standing still on the surface of Earth, will see additional terms to the anomalous spin precession frequency.

It is worth to say a few words about what we found: we can see the last term in (1.63) as a correction to the magnetic field but, since the terms the terms $Du/D\tau_u$ and $\beta\times Du/D\tau_u$ are not in general parallel to each other neither are they parallel to B, this means that they will correct B both in its direction and in its modulus.

1.3 Adapted frame

All of the results we obtained so far in the curved framework have been given in a coordinate-basis notation and thus they covariantly depend on the coordinate choice. In other words, if we want to

extract any physically significant information from the equations above for the laboratory frame we need to specify a so called adapted frame for the observer u within which we are allowed to write any quantity (vectors and tensors in general) in the way u measures them "with its axes".
Thus a way to proceed is to define a vector basis[9] $\{\, \vec{e}_a(x) \,\}_{a=0}^{3}$ which is coordinate dependent, called *vierbein* ("four legs"; sometimes *tetrad* "group of four"), such that its dual $\{\, e^a(x) \,\}$ allows us to write the metric as follows

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu = \eta_{ab}\, e^a(x) \otimes e^b(x)\,, \tag{1.64}$$

where $(\eta_{ab}) = \mathrm{diag}(-1,1,1,1)$ is the usual flat metric tensor. In other words (for simplicity, we are going to omit the coordinate dependence) the scalar product between vectors of the basis fulfills the orthonormality condition $\vec{e}_a \cdot \vec{e}_b = g(\vec{e}_a, \vec{e}_b) = \eta_{ab}$. Such a frame is obviously referred to as orthonormal frame. If we write these vectors and 1-forms in a coordinate basis we automatically define a transformation matrix $(e^\mu{}_a)$ and its inverse $\left(e_\mu{}^a\right)$ in such a way that $\vec{e}_a = e^\mu{}_a \partial_\mu$ and $e^a = e_\mu{}^a\, dx^\mu$ with the obvious inverse relations $\partial_\mu = e_\mu{}^a \vec{e}_a$ and $dx^\mu = e^\mu{}_a\, e^a$. From the condition (1.64) it follows that

$$g_{\mu\nu}\, e^\mu{}_a\, e^\nu{}_b = \eta_{ab}\,. \tag{1.65}$$

When dealing with Latin indices we are going to put an hat on top of them e.g. for a vector field $(A^a) = (A^{\hat{0}}, A^{\hat{1}}, A^{\hat{2}}, A^{\hat{3}})$. The most convenient way to choose a basis is to define first of all $\vec{e}_{\hat{0}} = u$ with u the 4-velocity of the observer which, as we already know, identifies its time direction and it is a unit time-like vector, $u^2 = -1$. As a consequence we have $u = u^a \vec{e}_a = \vec{e}_{\hat{0}}$ thus $u^a = \delta^a_{\hat{0}}$. The remaining three vectors are then space-like orthonormal vectors i.e. $\vec{e}_{\hat{\imath}} \cdot \vec{e}_{\hat{\jmath}} = \delta_{\hat{\imath}\hat{\jmath}}$ with $\hat{\imath}, \hat{\jmath} = \hat{1}, \hat{2}, \hat{3}$.
Now given a vector field A the quantity A^a, called *flat component* (a is the *flat index*), gives the physical quantity associated to A in the $\vec{e}_a$ direction as seen by the observer which carries the tetrad. If A is a vector field, from the relations above we have

$$A = A^\mu\, \partial_\mu = A^\mu\, e_\mu{}^a\, \vec{e}_a = A^a\, \vec{e}_a\,,$$

while for a generic 1-form B we have

$$B = B_\mu\, dx^\mu = B_\mu\, e^\mu{}_a\, e^a = B_a\, e^a\,,$$

which gives us the way to transform curved indices into flat ones:

$$A^a = e_\mu{}^a A^\mu\,, \tag{1.66}$$

$$B_a = e^\mu{}_a B_\mu\,. \tag{1.67}$$

This rule can be used on generic tensor quantities e.g.

[9] In the following we are going to use the notation present in [5].

$$T^a{}_{bc} = e_\mu{}^a \, e^\nu{}_b \, e^\rho{}_c \, T^\mu{}_{\nu\rho} \, .$$

Notice that with this choice of basis the indices raising-lowering procedure has to be done with the flat metric η_{ab}:

$$A_a = e^\mu{}_a \, A_\mu = e^\mu{}_a \, g_{\mu\nu} \, A^\nu = e^\mu{}_a \, g_{\mu\nu} \, e^\nu{}_b \, A^b = \eta_{ab} \, A^b \, .$$

Moreover it is easy to find the components of A in the u frame:

$$A^{\hat{0}} = -A \cdot \vec{e}_{\hat{0}} \, ,$$
$$A^{\hat{\imath}} = A \cdot \vec{e}_{\hat{\imath}} \, ,$$

where $\hat{\imath} = \hat{1}, \hat{2}, \hat{3}$. From now on we are also going to put an arrow on top of vectors when they are written in the $\{ \, \vec{e}_a \, \}$ basis.
It is important to say that the $\{ \, \vec{e}_a \, \}$ basis is not unique. Indeed, given another basis $\{ \, \vec{e}_{a'} \, \}$, this is linked to the first by a coordinate dependend transformation $\Lambda(x)$ such that

$$\vec{e}_{a'} = \Lambda^a{}_{a'}(x) \, \vec{e}_a \, . \tag{1.68}$$

The orthonormality condition $g(\vec{e}_{a'}, \vec{e}_{b'}) = \eta_{a'b'}$ then translates into the familiar relation

$$\Lambda^a{}_{a'}(x) \, \eta_{ab} \, \Lambda^b{}_{b'}(x) = \eta_{a'b'} \, ,$$

which tells us that at each point x the two basis are linked by a Lorentz transformation.
Another important thing to stress is that the volume 4-form simplifies when we are using this new frame. Indeed we can rewrite the volume 4-form as

$$\eta = \frac{1}{4!} \sqrt{-g} \, \epsilon_{\mu\nu\rho\sigma} \, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \frac{1}{4!} \sqrt{-g} \, \epsilon_{\mu\nu\rho\sigma} \, e^\mu{}_a \, e^\nu{}_b \, e^\rho{}_c \, e^\sigma{}_d \, e^a \wedge e^b \wedge e^c \wedge e^d \, ;$$

now from the properties of the Levi–Civita symbol we have

$$\eta_{abcd} = \sqrt{-g} \, \epsilon_{\mu\nu\rho\sigma} \, e^\mu{}_a \, e^\nu{}_b \, e^\rho{}_c \, e^\sigma{}_d = \sqrt{-g} \, e \, \epsilon_{abcd} = \epsilon_{abcd} \, ,$$

where in the last passage we defined $e = \det{(e^\mu{}_a)}$ and we used the fact that $e = (-g)^{-1/2}$ which can easily be derived from[10] (1.65). Thus

$$\eta_{abcd} = \epsilon_{abcd} \quad \text{and from the flat index raising-lowering procedure} \quad \eta^{abcd} = -\epsilon^{abcd} \, .$$

[10]Actually $e = \pm(-g)^{-1/2}$ but we choose the plus sign in order to preserve the orientation.

Hence with this choice of the basis the vector product defined in (1.41), which is a tensor by definition, becomes the usual vector product

$$(A \times_u B)^a = e^a{}_\mu \, (A \times_u B)^\mu = u_b \, \eta^{bacd} A_c B_d = \epsilon^{\hat{0}acd} A_c B_d \,, \tag{1.69}$$

since $u_b = \eta_{bc} \, \delta^c_{\hat{0}} = \eta_{b\hat{0}} = -\delta_{b\hat{0}}$, which means that the only non vanishing components of the above vector product are the well known

$$(A \times_u B)^{\hat{\imath}} = \epsilon^{\hat{\imath}\hat{\jmath}\hat{\kappa}} A^{\hat{\jmath}} B^{\hat{\kappa}} \,.$$

Thus the anomalous precession frequency (1.63) can be recast as

$$\begin{aligned} \vec{\Omega}_{\mathrm{a}} = -\frac{e}{m} \left[a\vec{B} - \left(a - \frac{1}{\gamma^2 - 1} \right) \vec{\beta} \times \vec{E} - a \left(\frac{\gamma}{\gamma + 1} \right) (\vec{\beta} \cdot \vec{B}) \vec{\beta} \right] \\ - \frac{\gamma}{\gamma + 1} \left(\vec{\beta} \times \frac{D\vec{u}}{D\tau_u} - \frac{\gamma}{\gamma - 1} \frac{D\vec{u}}{D\tau_u} \right) \,, \end{aligned} \tag{1.70}$$

Where $\Omega^{\hat{0}}_{\mathrm{a}} = -\Omega_{\mathrm{a}} \cdot \vec{e}_{\hat{0}} = -\Omega_{\mathrm{a}} \cdot u = 0$ since $\Omega_{\mathrm{a}} \in \mathrm{LRS}_u$ (see previous sections) and

$$\boxed{\begin{aligned} \Omega^{\hat{\imath}}_{\mathrm{a}} = \Omega_{\mathrm{a}} \cdot \vec{e}_{\hat{\imath}} = -\frac{e}{m} \left[aB^{\hat{\imath}} - \left(a - \frac{1}{\gamma^2 - 1} \right) \epsilon^{\hat{\imath}\hat{\jmath}\hat{\kappa}} \beta^{\hat{\jmath}} E^{\hat{\kappa}} - a \left(\frac{\gamma}{\gamma + 1} \right) (\beta^{\hat{\jmath}} B^{\hat{\jmath}}) \beta^{\hat{\imath}} \right] \\ - \frac{\gamma}{\gamma + 1} \left(\epsilon^{\hat{\imath}\hat{\jmath}\hat{\kappa}} \beta^{\hat{\jmath}} \frac{Du^{\hat{\kappa}}}{D\tau_u} - \frac{\gamma}{\gamma - 1} \frac{Du^{\hat{\imath}}}{D\tau_u} \right) \,, \end{aligned}} \tag{1.71}$$

and now all of the vector quantities above are the spatial ones that the observer experiences in its reference frame. The next step is to provide an expression for the laboratory's 4-acceleration in flat components:

$$\left(\frac{Du}{D\tau_u} \right)^{\hat{\imath}} = e_\mu{}^{\hat{\imath}} \left(\frac{Du}{D\tau_u} \right)^\mu \,.$$

Note. Let's consider a vector field $X = X^\mu \partial_\mu = X^a \vec{e}_a$. In general when applying the covariant derivative on X we must be careful because the vectors $\{ \, \vec{e}_a \, \}_{a=0}^3$ may change along the observer's trajectory. That is

$$\frac{DX}{D\tau_u} = \frac{D}{D\tau_u} \left(X^a \vec{e}_a \right) = \frac{dX^a}{d\tau_u} \vec{e}_a + X^a \frac{D\vec{e}_a}{D\tau_u} \,;$$

since each vector $\vec{e}_a$ is normalized, the last term accounts for spatial rotations of the basis. Then if we take the projection over LRS_u we have

$$P(u)\frac{DX}{D\tau_u} = \frac{dX^{\hat{\imath}}}{d\tau_u}\vec{e}_{\hat{\imath}} + X^a P(u)\frac{D\vec{e}_a}{D\tau_u}\,, \tag{1.72}$$

recall that $\vec{e}_{\hat{0}} = u$ so $P(u)\vec{e}_{\hat{0}} = 0$. In particular when we are dealing with vector quantities orthogonal to u (as we did for most of our discussion) we know that $X^{\hat{0}} = 0$ then if $X \in \mathrm{LRS}_u$ (1.72) becomes

$$P(u)\frac{DX}{D\tau_u} = \frac{dX^{\hat{\imath}}}{d\tau_u}\vec{e}_{\hat{\imath}} + X^{\hat{\imath}} P(u)\frac{D\vec{e}_{\hat{\imath}}}{D\tau_u}\,. \tag{1.73}$$

As we will see later on, if a suitable basis is chosen this last term disappears and we can write

$$\begin{aligned} \left(P(u)\frac{DX}{D\tau_u}\right)^{\hat{0}} &= 0\,, \\ \left(P(u)\frac{DX}{D\tau_u}\right)^{\hat{\imath}} &= \frac{dX^{\hat{\imath}}}{d\tau_u}\,. \end{aligned} \tag{1.74}$$

1.4 Schwarzschild geometry

Since we are interested in studying experiments which are performed on the surface of Earth we are going to consider the Schwarzschild solution for a static spherically symmetric matter distribution. With this assumption we are clearly neglecting the non spherical shape of our planet and we are also ignoring its axial rotation. The metric expressed in spherical coordinates[11] (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \tag{1.75}$$

with M the mass of the body generating this geometry. We are using geometrized units $G = 1 = c$ and for the whole analysis it will be $r > 2M = r_S$ with r_S Schwarzshild radius. Thus we can write

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

with inverse

$$(g^{\mu\nu}) = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}.$$

[11] Given an index μ we are going to use both the notation (t, r, θ, ϕ) and $(0, 1, 2, 3)$ with the obvious correspondence; for flat indices, as before we put a hat on them $(\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi})$ or $(\hat{0}, \hat{1}, \hat{2}, \hat{3})$.

We also write the non-vanishing Christoffel symbols, $\Gamma^{\mu}_{\nu\rho}$ for this metric since we are going to need them in the following:

$$
\begin{gathered}
\Gamma^{t}_{rt} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\,, \\
\Gamma^{r}_{tt} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)\,, \quad \Gamma^{r}_{rr} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\,, \\
\Gamma^{r}_{\theta\theta} = 2M-r\,, \quad \Gamma^{r}_{\phi\phi} = (2M-r)\sin^2\theta \\
\Gamma^{\theta}_{r\theta} = \frac{1}{r}\,, \quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta\,, \\
\Gamma^{\phi}_{r\phi} = \frac{1}{r}\,, \quad \Gamma^{\phi}_{\phi\theta} = \cot\theta\,.
\end{gathered}
\tag{1.76}
$$

1.4.1 Static observer

A static observer has in general a 4-velocity given in a coordinate basis by

$$
u^{\mu} = \frac{dx^{\mu}}{d\tau_u} = \frac{\delta^{\mu}_{0}}{\sqrt{-g_{00}}}
\tag{1.77}
$$

which directly follows by the combination of the conditions

- vanishing spatial velocity $(u^{\mu}) = (u^0, 0, 0, 0)$;
- normalization $u^2 = -1$;
- future pointing world-line $u^0 = dt/d\tau_u > 0$.

For a Schwarzschild metric then we have

$$
u^{\mu} = \frac{\delta^{\mu}_{0}}{\sqrt{1-\frac{2M}{r}}}\,.
$$

In this case a tetrad can easily be defined as follows

$$
\begin{aligned}
(\vec{e}_{\hat{0}})^{\mu} &= e^{\mu}{}_{\hat{0}} = \frac{\delta^{\mu}_{0}}{\sqrt{1-\frac{2M}{r}}} = u^{\mu}\,, \\
(\vec{e}_{\hat{1}})^{\mu} &= e^{\mu}{}_{\hat{1}} = \delta^{\mu}_{1}\sqrt{1-\frac{2M}{r}}\,, \\
(\vec{e}_{\hat{2}})^{\mu} &= e^{\mu}{}_{\hat{2}} = \frac{\delta^{\mu}_{2}}{r}\,, \\
(\vec{e}_{\hat{3}})^{\mu} &= e^{\mu}{}_{\hat{3}} = \frac{\delta^{\mu}_{3}}{r\sin\theta}\,.
\end{aligned}
\tag{1.78}
$$

This gives us

$$
\left(e^{\mu}{}_{a}\right) = \begin{pmatrix} \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}
$$

with inverse

$$
\left(e_{\mu}{}^{a}\right) = \begin{pmatrix} \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}
$$

The expression for the 4-acceleration for a static observer is

$$
a^{\mu}_{(u)} = \left(\frac{Du}{D\tau_u}\right)^{\mu} = \frac{du^{\mu}}{d\tau_u} + \Gamma^{\mu}_{\nu\rho}u^{\nu}u^{\rho} = \frac{\Gamma^{\mu}_{00}}{-g_{00}} = \frac{\Gamma^{\mu}_{tt}}{1-\frac{2M}{r}}
$$

thus from (1.76) for the acceleration we have the well known Newton-like form

$$
\left(a^{\mu}_{(u)}\right) = \left(0, \frac{M}{r^2}, 0, 0\right)\,. \tag{1.79}
$$

Now if we want to know which acceleration feels the observer standing still in a laboratory on the surface of Earth we need to rewrite this acceleration in the tetrad basis $\{\,\vec{e}_a\,\}$ (1.78), i.e.

$$
\begin{aligned}
&\left(\frac{Du}{D\tau_u}\right)^{a} = 0 \quad \text{if } a = \hat{t}, \hat{\theta}, \hat{\phi}\,; \\
&\left(\frac{Du}{D\tau_u}\right)^{\hat{r}} = e_{\mu}{}^{\hat{r}}\left(\frac{Du}{D\tau_u}\right)^{\mu} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\,.
\end{aligned}
\tag{1.80}
$$

As expected the only non-vanishing component of this acceleration is the radial one which points outward with respect to the planet's center.

Note (Fermi–Walker derivative). Taking a closer look at what we have obtained so far, it becomes clear that $\vec{e}_{\hat{r}} = a_{(u)}/\left\|a_{(u)}\right\|$ with $a_{(u)} = Du/D\tau_u$. A tetrad built in this way represents what is commonly known as *Fermi–Walker* frame. If an observer is moving along a curve γ with velocity u and acceleration $a_{(u)} \neq 0$ (non-geodesic motion) we can define the Fermi–Walker derivative of a vector field X along γ as

$$\frac{D_{\mathrm{F}}X}{D\tau_u} = \frac{DX}{D\tau_u} + \big(u \cdot X\big)a_{(u)} - \big(a_{(u)} \cdot X\big)u\,. \tag{1.81}$$

Notice that if $X = u = \vec{e}_{\hat{0}}$ the Fermi–Walker derivative vanishes identically so this derivative represents nothing but the natural generalization of the parallel transport of the 4-velocity u along non-geodesic world-lines. In our case it's easy to shows that the same holds for $X = a_{(u)} = \vec{e}_{\hat{1}}$ and $X = \vec{e}_{\hat{2},\hat{3}}$ so we have

$$\frac{D_{\mathrm{F}}\vec{e}_a}{D\tau_u} = 0 \quad \forall a\,,$$

thus the tetrad is said to be Fermi–Walker transported. In particular such a frame turns out to be spatially non-rotating i.e. the vectors $\vec{e}_{\hat{1}}, \vec{e}_{\hat{2}}, \vec{e}_{\hat{3}}$ don't perform any precession when being carried along by the observer.
Now let us look back to equation (1.73) for a moment. We notice that, if $\vec{e}_{\hat{\imath}} \cdot \vec{e}_{\hat{0}} = \vec{e}_{\hat{\imath}} \cdot u = 0\ \forall \hat{\imath} = \hat{1}, \hat{2}, \hat{3}$, then it can easily be shown that

$$\frac{D_{\mathrm{F}}\vec{e}_{\hat{\imath}}}{D\tau_u} = \frac{D\vec{e}_{\hat{\imath}}}{D\tau_u} + u\left(\frac{D\vec{e}_{\hat{\imath}}}{D\tau_u} \cdot u\right) = P(u)\frac{D\vec{e}_{\hat{\imath}}}{D\tau_u}\,. \tag{1.82}$$

Thus, when dealing with the spatial evolution of any vector quantity, that is whenever a $P(u)D/D\tau_u$ term appears, the second term in equation (1.73) naturally disappears if a Fermi–Walker transported basis is chosen.

1.4.2 Freely falling observer

When dealing with a freely falling observer the situations becomes a bit more complicated. In this case by definition $Du/D\tau_u = 0$ so we are actually looking for an expression for the 4-velocity in order to see if we can recover a Special Relativity-like result for the expressions of the anomalous spin frequency.
First of all, since we are interested in a radial free fall, an easy way to recover an expression for the 4-velocity is to start from the line element where we put $d\theta = 0 = d\phi$ and recall that by definition $ds^2 = -d\tau_u^2$. In this way we have $(u^\mu) = (dx^\mu/d\tau_u) = (u^t, u^r, 0, 0)$ and the line element of such a world-line becomes

$$ds^2 = -d\tau_u^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2\,,$$

which can be rearranged to give

$$(u^r)^2 - \left(1 - \frac{2M}{r}\right)^2 \left(u^t\right)^2 + 1 - \frac{2M}{r} = 0\,. \tag{1.83}$$

Now whenever the metric tensor is independent on one of the coordinates, i.e. it exists ρ such that $\partial_\rho g_{\mu\nu} = 0$, then the vector field $\xi_{(\rho)} = \partial_\rho$ satisfies the so called *Killing equation*: $\nabla_\mu \xi_{(\rho)\nu} + \nabla_\nu \xi_{(\rho)\mu} = 0$ and $\xi_{(\rho)}$ is called a *Killing vector*. In such a circumstance one can easily see that along world-lines which are geodesics the quantity $Q_{(\rho)} = g(\xi_{(\rho)}, u)$ remains constant. For the Schwarzschild geometry we have $\partial_t g_{\mu\nu} = 0 = \partial_\phi g_{\mu\nu}$. In particular we are interested in the time variable independence of the metric thus we take the Killing vector $\xi_{(t)} = \partial_t$. This means that the conserved quantity is

$$Q_t = g(\xi_{(t)}, u) = g_{tt} u^t = -\left(1 - \frac{2M}{r}\right) u^t \,\equiv\, -E\,. \tag{1.84}$$

If we put this result inside equation (1.83) we find

$$(u^r)^2 = E^2 - 1 + \frac{2M}{r}\,.$$

From this result adding the conditions

- $dt/d\tau_u > 0$, future pointing world-line;
- $dr/d\tau_u < 0$, the observer is in-falling;
- $u^r(r = R_i) = 0$, the observer starts his/her fall at rest at the initial radius R_i ;

we obtain $E^2 = 1 - 2M/R_i$ and the 4-velocity turns out to be

$$(u^\mu) = \left(\frac{\sqrt{1 - \frac{2M}{R_i}}}{1 - \frac{2M}{r}}, -\sqrt{\frac{2M}{r} - \frac{2M}{R_i}}, 0, 0 \right)\,. \tag{1.85}$$

According to what we said in section 1.3 this will also be our first vector of the tetrad: $\vec{e}_{\hat{0}} = u$, while its easy to see that the same $\vec{e}_{\hat{2}}$ and $\vec{e}_{\hat{3}}$ of the previous section satisfy $\vec{e}_{\hat{2}} \cdot \vec{e}_{\hat{0}} = 0 = \vec{e}_{\hat{3}} \cdot \vec{e}_{\hat{0}}$. Now we only need $\vec{e}_{\hat{1}}$ which has to satisfy $\vec{e}_{\hat{1}} \cdot \vec{e}_{\hat{0}} = \vec{e}_{\hat{1}} \cdot \vec{e}_{\hat{2}} = \vec{e}_{\hat{1}} \cdot \vec{e}_{\hat{3}} = 0$ and $\left\|\vec{e}_{\hat{1}}\right\| = 1$. After some algebra one finds

$$(\vec{e}_{\hat{1}})^\mu = e^\mu{}_{\hat{1}} = \pm \left(\delta^\mu_0 \frac{\sqrt{\frac{2M}{r} - \frac{2M}{R_i}}}{1 - \frac{2M}{r}} - \delta^\mu_1 \sqrt{1 - \frac{2M}{R_i}} \right)\,,$$

then one has to choose the minus sign in order for the three spatial vectors to define a right-handed coordinate frame[12]. This finally gives us

[12]The condition to impose is for example that $(\vec{e}_{\hat{2}} \times_u \vec{e}_{\hat{3}})^\mu = \eta(u)^{\mu\nu\rho} (\vec{e}_{\hat{2}})_\nu (\vec{e}_{\hat{3}})_\rho = u_\alpha \eta^{\alpha\mu\nu\rho} (\vec{e}_{\hat{2}})_\nu (\vec{e}_{\hat{3}})_\rho \stackrel{!}{=} (\vec{e}_{\hat{1}})^\mu$.

$$
\begin{aligned}
\left(e^{\mu}{}_{\hat{0}}\right) &= \left(\frac{\sqrt{1-\frac{2M}{R_i}}}{1-\frac{2M}{r}}, -\sqrt{\frac{2M}{r}-\frac{2M}{R_i}}, 0, 0\right) = \left(u^{\mu}\right) , \\
\left(e^{\mu}{}_{\hat{1}}\right) &= \left(-\frac{\sqrt{\frac{2M}{r}-\frac{2M}{R_i}}}{1-\frac{2M}{r}}, \sqrt{1-\frac{2M}{R_i}}, 0, 0\right) , \\
\left(e^{\mu}{}_{\hat{2}}\right) &= \left(0, 0, \frac{1}{r}, 0\right) , \\
\left(e^{\mu}{}_{\hat{3}}\right) &= \left(0, 0, 0, \frac{1}{r\sin\theta}\right) .
\end{aligned}
\tag{1.86}
$$

While for the transformation matrices we have

$$
\left(e^{\mu}{}_{a}\right) = \begin{pmatrix}
\frac{\sqrt{1-\frac{2M}{R_i}}}{1-\frac{2M}{r}} & -\frac{\sqrt{\frac{2M}{r}-\frac{2M}{R_i}}}{1-\frac{2M}{r}} & 0 & 0 \\
-\sqrt{\frac{2M}{r}-\frac{2M}{R_i}} & \sqrt{1-\frac{2M}{R_i}} & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & \frac{1}{r\sin\theta}
\end{pmatrix}
$$

with inverse

$$
\left(e_{\mu}{}^{a}\right) = \begin{pmatrix}
\sqrt{1-\frac{2M}{R_i}} & \frac{\sqrt{\frac{2M}{r}-\frac{2M}{R_i}}}{1-\frac{2M}{r}} & 0 & 0 \\
\sqrt{\frac{2M}{r}-\frac{2M}{R_i}} & \frac{\sqrt{1-\frac{2M}{R_i}}}{1-\frac{2M}{r}} & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 0 & r\sin\theta
\end{pmatrix}
$$

In this case we can see that the Fermi–Walker derivative is nothing but the standard covariant derivative since $a_{(u)} = 0$. With a little bit of patience one can see that also in this case

$$
\frac{D_{\mathrm{F}}\vec{e}_a}{D\tau_u} = \frac{D\vec{e}_a}{D\tau_u} = 0 \quad \forall a \, .
$$

Chapter 2

Previous works

In this chapter we are going to critically analyze some papers published after the one by Morishima *et al.* [16] which tried to underline inconsistencies and conceptual mistakes that these authors made. In particular we will look at the works [8, 12, 14, 17, 18, 21, 22] and we will highlight what's relevant for us in their treatment.

2.1 Einstein equivalence principle

With his work [22], M. Visser explains that one of the main issues with the paper by Morishima *et al.* is the explicit presence of the Earth's absolute gravitational potential as a correction, $\phi_\oplus = -M_\oplus/R_\oplus$. When dealing with Newtonian gravity, the physically relevant quantity is the difference of absolute potentials. The explicit presence of ϕ implies that our laboratory is somehow able to probe spatial infinity while its finite-size clearly allows only measurements of a finite portion of space only. Another important point directly follows: if we account for the Earth's absolute gravitational potential we get $\phi_\oplus/c^2 \simeq 7 \times 10^{-10}$. Then if we take a look at the absolute gravitational potentials produced by the Sun and by our galaxy on Earth's surface, we find that they are $\phi_\odot \simeq 15\,\phi_\oplus$ and $\phi_{\text{galaxy}} \simeq 2000\,\phi_\oplus$ respectively. This means that, compared to them, the claimed effect of Earth's gravity on particle physics in [16] should be negligible. Moreover the effects of the galaxy should be taken into account as the biggest contributions.

Then it seems that potential differences over the typical length of the laboratory matters rather than the absolute potential. As Visser says, an estimate of this gradient is given by (equation (3.1) in [22]):

$$\Delta\phi \simeq \frac{d\phi}{dh}\Delta h \simeq \phi\,\frac{\text{(size of the laboratory)}}{\text{(radius of the Earth)}}\,,$$

which clearly makes $\Delta\phi$ negligible.

Finally, switching to the General Relativistic treatment, Visser uses the Einstein equivalence principle: one of the implications of this principle is the local flatness of space-time, namely

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\nu\rho\sigma}(x)x^\rho x^\sigma + O(x^3)$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. Again, as the author says, we can "measure" the displacement from flatness of the metric by evaluating the second term over the size of the laboratory. Thus setting $x \sim$ (size of the laboratory) and recalling that $R_{\mu\nu\rho\sigma} \sim \partial^2 g_{\mu\nu} \sim \partial^2\phi \sim \phi/r^2$ we obtain

$$R_{\mu\nu\rho\sigma}\, x^\rho x^\sigma \sim \phi \left[\frac{\text{(size of the laboratory)}}{\text{(radius of the Earth)}}\right]^2$$

which brings a suppression factor even bigger than the previous one.

In conclusion, Visser states that the fact that the results given by Morishima *et al.* have the magnitude they claimed is in contrast with the Einstein equivalence principle and thus can not be correct.
This is in perfect agreement with what we obtained in the previous chapter. The gravitational corrections we obtained for the anomalous precession angular velocity are additive terms that depend on $Du^{\hat{r}}/D\tau_u \propto M/r^2 = -d\phi/dr$ which indeed is the gradient of the potential.

2.2 Coordinate vs. physical quantities

H. Nikolić in [17] follows what Visser said and shows the place of error: the motion of a particle in an electromagnetic field and in a flat space-time is governed by the equation (1.17) which, considering the electric field $\mathbf{E}$ null and the velocity $\boldsymbol{\beta}$ orthogonal to the magnetic field $\mathbf{B}$, after the manipulation by Morishima *et al.* becomes

$$\frac{d\boldsymbol{\beta}}{dt} \simeq (1 + 3\epsilon^2\phi)\frac{e}{m}\boldsymbol{\beta} \times \mathbf{B}\,, \tag{2.1}$$

with $\epsilon = c^{-1}$ in which appears ϕ as a claimed gravitational correction. The problem is now the meaning of that t as time variable. Morishima *et al.* treat that t as the physical time variable while it is only a coordinate time variable. This can be explained by showing that for a particle performing an horizontal orbit on Earth's surface ($dr = 0$), defining a new time variable $dt' = (1 + 2\epsilon^2\phi)^{1/2}dt$, the Schwarzschild line element loses its dependence on ϕ:

$$ds^2 = -dt'^2 + r^2 d\Omega^2\,.$$

Moreover t' is the actual physical time that an observer at fixed radius r would measure with its clock. This is nothing but what we did in the previous chapter when we chose a basis of orthonormal vectors (the vielbein) and projected all of the quantities on this basis.

Another work, by D. Venhoek [21], explores this problem with a little more detail: he starts underlining that Morishima *et al.* made a mistake while computing $d\boldsymbol{\beta}/dt$ which actually differs from (2.1):

$$\frac{d\boldsymbol{\beta}}{dt} \simeq (1 + \epsilon^2\phi)\frac{e}{m}\boldsymbol{\beta} \times \mathbf{B}\,.$$

Then he does essentially what we did in the previous chapter but in a less general fashion: he asks himself whether the equation motion of the particle written in such a way represents what we need i.e.

the variation of the spatial (physical) position of the particle over a variation of the (physical) time both measured in the laboratory. The answer is no. In fact if we follow Venhoek's paper, at some point he defines $\boldsymbol{\beta}$ in the same way we did[1] in (1.38): the physical time t' the observer experiences is related to the coordinate time t by the previous relation $dt' \simeq (1+\epsilon^2\phi)dt$ (at first order in ϵ^2) while the physical spatial line element is $d\mathbf{x}' \simeq (1-\epsilon^2\phi)d\mathbf{x}$ (at first order in ϵ^2)[2]. Thus the physical spatial velocity of the particle as seen by the laboratory is (we use the same notation as [21])

$$\boldsymbol{\beta}_{\mathrm{ph}} \simeq \frac{1-\epsilon^2\phi}{1+\epsilon^2\phi}\boldsymbol{\beta}\,,$$

where with $\boldsymbol{\beta} = d\mathbf{x}/dt$ we keep referring to the relative velocity in [16]. The problem is that Morishima *et al.* treated $\boldsymbol{\beta}$ as the velocity with which the particle moves with respect to the observer while this is clearly not the case. Then Venhoek writes the rate of change of $\boldsymbol{\beta}_{\mathrm{ph}}$ with laboratory time for a particle in an electromagnetic field as

$$\frac{d\boldsymbol{\beta}_{\mathrm{ph}}}{dt'} \simeq (1-\epsilon^2\phi)\frac{d\boldsymbol{\beta}_{\mathrm{ph}}}{dt}\,,$$

which gives him the following equation

$$(1-\epsilon^2\phi)\frac{d\boldsymbol{\beta}_{\mathrm{ph}}}{dt} = \frac{e}{m\gamma_{\mathrm{ph}}}\big(\mathbf{E} + \boldsymbol{\beta}_{\mathrm{ph}}\times\mathbf{B} - (\boldsymbol{\beta}_{\mathrm{ph}}\cdot\mathbf{E})\boldsymbol{\beta}_{\mathrm{ph}}\big) \tag{2.2}$$

with $\gamma_{\mathrm{ph}} = (1-\boldsymbol{\beta}_{\mathrm{ph}}^2)^{-1/2}$. This perfectly matches the flat evolution of the relative velocity (1.17). This comes as no surprise because we are doing nothing but choosing a local inertial frame for the laboratory and more importantly we are ignoring gravity: the rate at which the relative velocity of the particle evolves in the laboratory frame is obviously $P(u)D\beta/D\tau_u$, with β defined as (1.38) (see equation (1.50) in section 1.2.2).
In conclusion the first mistake in all of the treatment is the identification (using our notation)

$$\frac{dw^i}{dt} = \frac{d\boldsymbol{\beta}^i}{dt}$$

[1] One can see this by rewriting equation (1.38) as follows

$$\beta = \frac{1}{\gamma}P(u)w = P(u)\frac{d\tau_w}{d\tau_u}\frac{dx}{d\tau_w} = \frac{P(u)dx}{d\tau_u} = \frac{d\mathbf{x}'}{d\tau_u}$$

where $d\mathbf{x}'$ is the physical spatial displacement of the particle and $d\tau_u$ is the physical time interval within which the displacement takes place, both of them as seen by the laboratory.

[2] These two results come from the (post-Newtonian) expansion of the *isotropic* Schwarzschild metric:

$$ds^2 = -\frac{\left(1+\frac{\phi}{2}\right)^2}{\left(1-\frac{\phi}{2}\right)^2}dt^2 + \left(1-\frac{\phi}{2}\right)^4\left(dx^2+dy^2+dz^2\right)\,,$$

which gives

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)d\mathbf{x}^2 + O(\epsilon^4) = -(dt')^2 + (d\mathbf{x}')^2 + O(\epsilon^4)\,,$$

that is, the metric assumes the usual flat Minkowski form up to order ϵ^4.

by Morishima *et al.*, that is, they don't project the 4-velocity w on the space of the observer and so this can not represent the spatial velocity the laboratory sees. Moreover they deal with a non-covariant equation of motion for the evolution of $\boldsymbol{\beta}$. Last but not least, they treat as physical quantities that are coordinate.

2.3 Magnitude of the correction

Even assuming the procedure performed by Morishima *et al.* for the relativistic correction right, in his paper [8] P. Guzowski tries to convince the reader that the magnitude of this correction is way too small than the one claimed in [16] and thus it is absorbed in the systematic uncertainties of the experiment. As we will see the aim of the experiment is to set the magnitude of β in such a way that the coefficient of the $\beta \times E$ term goes to zero, $a - 1/(\gamma_{\mathrm{m}}^2 - 1) = 0$, and also to set the direction of β in such a way that it is orthogonal to the magnetic field, $\beta \cdot B = 0$. The quantity γ_{m} is called *magic gamma* while the associated momentum $p_{\mathrm{m}} = m\sqrt{\gamma_{\mathrm{m}}^2 - 1}$ is called *magic momentum*. Now if we consider the gravitational correction given by Morishima *et al.* the condition for the coefficient of the first term to be zero becomes

$$\gamma_\phi^2 = \frac{(1 + a_\mu)(1 - \epsilon^2\phi)}{a_\mu(1 - \epsilon^2\phi) - 4\epsilon^2\phi}$$

which causes a deviation from the magic momentum of $\sim 3\,\mathrm{keV}$. This translates into a displacement of the orbit from the circular one of $\sim 10\,\mu\mathrm{m}$ which has to be compared to the $500\,\mu\mathrm{m}$ uncertainty on the position of the electric quadrupoles. Thus the effect of gravity as obtained in [16] is negligible.

Another comment on the weight of the correction proposed by Morishima *et al.* comes from the work by Miller and Lee Roberts [14]. Again the authors show that the effect claimed in [16] contributes with an electric field correction which gives a relative displacement on a_μ of $\Delta a_\mu / a_\mu = 1.4 \times 10^{-10}$. This value is completely negligible when compared to the 0.54 ppm uncertainty of BNL (Brookhaven National Laboratory) E821 and to the expected 0.140 ppm at Fermilab.

2.4 GR effects

With their work [12] A. Lázló and Z. Zimborás tried to estimate exactly the effect of gravity on the muon spin precession by using the theory of General Relativity. They start by writing the orbit of the particle in the Schwarzschild space-time as a closed circular orbit before they mention the physical process which causes this kind of motion. The strategy is to write down an expression for the particle's world-line and try to compute the Thomas' and Larmor's precessions for the spin vector. The former naturally arises from simply knowing the shape of the orbit while the latter comes from the BMT equation. The authors work with an explicit coordinate basis notation and moreover they use the time coordinate in place of the proper time; this can lead to a misinterpretation between physical quantities and mere coordinate objects. The final result is the anomalous angular velocity, as we did at the end of section 1.2.3, and it is Taylor expanded at first order about $r_S = 0$ in order to reproduce the Special Relativity limit plus a gravitational correction.

The authors treat the problem in the context of the g − 2 experiments and the so called electric dipole moment (EDM) search experiments. Then they make the assumption that there is no electric field (which is equivalent to say that the term $\vec{\beta}\times\vec{E}$ in equation (1.70) is set to zero) and that the magnetic field is orthogonal to the particle's world-line (i.e. the term $\vec{\beta}\cdot\vec{B}$ is set to zero). At first they compute the General Relativistic correction to the cyclotron frequency $\vec{\Omega}_{\rm c}$ observed by the laboratory. They claim that this deviation is negligible and then they proceed with the correction to the anomalous precession frequency. As expected this coincides with the special-relativistic case analyzed in chapter 1 in the limit $r_S \to 0$:

$$\|\Omega_{\rm a}\|\big|_{r_S\,=\,0} = |a\boldsymbol{\omega}_{\rm c}|\gamma\,,$$

with $a = {\rm g}/2 - 1$ as usual and $|\boldsymbol{\omega}_{\rm c}| = eB/m\gamma$. For the first order correction they obtain

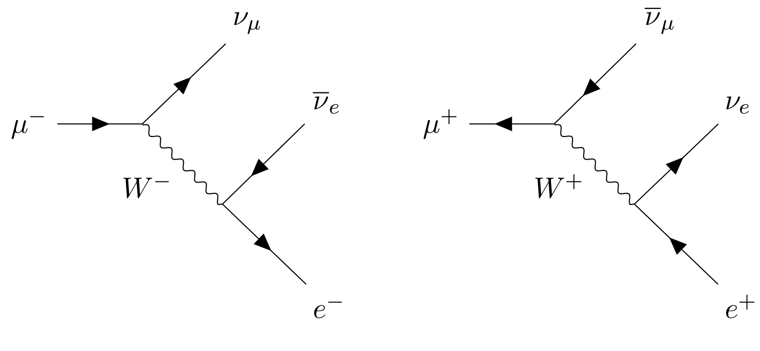
$$r_S\left(\frac{d}{dr_S}\|\Omega_{\rm a}\|\bigg|_{r_S\,=\,0}\right) = -|a\boldsymbol{\omega}_{\rm c}|\gamma\,\frac{r_S}{R}\frac{L^2}{R^2}$$

with L radius of the storage ring. Thus the result is

$$\|\Omega_{\rm a}\| \simeq |a\boldsymbol{\omega}_{\rm c}|\gamma\left(1-\frac{r_S}{R}\frac{L^2}{R^2}\right)$$

at first order in r_S, where the second term in brackets provides a correction of $\sim -2\times10^{-21}$ which clearly goes beyond the experimental sensitivity.
One problem we can highlight in the work by Lázló and Zimborás is that the final estimate of the correction shows a dependence on the laboratory size through the ratio L/R. In particular the issue is where this term comes from. In [12] the authors seem to interpret the gravitational corrections given by the metric in a way similar to the one we find in [16] by Morishima *et al.*, that is, they use the metric as a correction while, using an adapted frame (see chapter 1), one can easily see that all these terms are reabsorbed into the purely geometrical quantities to give the physical ones.

Another work that we want to analyze, and which aims to give a formal GR treatment of the problem as well, is the paper by A. Notari and D. Bertacca [18]. The authors ask themselves in which way gravity impacts on the experimental measurements. In particular, as we will comment later, the goal of the experimental setup is to detect the electron produced by the $\mu^\pm$ decays



and whose energy is above a certain threshold. These data are then used to infer the anomalous frequency. Notari and Bertacca then focus their attention on the contribution of space-time curvature on the energy of these detected electrons. In particular it can be shown that in a flat space-time, in absence of electric field and if the motion of the muon is perpendicular to the magnetic field the electron energy in the laboratory frame shows an oscillatory behavior. This means that if we call $\mathcal{E}$ the electron energy in the laboratory frame then the frequency is

$$\omega^2 = -\frac{1}{\mathcal{E}}\frac{d^2\mathcal{E}}{d\tau_u^2} = \omega_{\mathrm{a}}^2 = \left(\frac{e}{m}aB\right)^2 .$$

Then they analyze corrections due to deviations on the previously mentioned ideal situation to the measured frequency i.e. the presence of an electric field, the fact that the muon velocity will never be perfectly orthogonal to the magnetic field and the influence of gravity, that is, the non-flatness of space-time. It is interesting to note that they treat the problem of gravity by looking at different shapes of the metric tensor: separately a rotating metric and a Schwarzschild metric[3] in order to reproduce motion and shape of the Earth. With no surprise in the former case curvature contributes with a relative magnitude of $\Delta\omega/\omega_{\mathrm{a}} \approx 1.2 \times 10^{-12}$ which is due to a Coriolis-like term arising from rotation; in the latter case the relative corrections are even smaller with $\Delta\omega/\omega_{\mathrm{a}} \approx 8 \times 10^{-16}$ coming from the leading order Schwarzschild correction and which is proportional to the gravitational acceleration on the surface of the planet $g_T = 9.8\,\mathrm{m\,s^{-2}}$.

We would like to underline that all of the results mentioned above are actually corrections to the measured frequency recorded by the laboratory. Our aim is a little bit different: we want to gain an expression for the correction that General Relativity introduces to the QFT prediction of the muon anomalous magnetic moment.

Note. We would like to underline an issue in the work by Notari and Bertacca: in section 5.2.1 of [18] they write the metric for a flat uniform rotating reference frame as

$$ds^2 = -dt^2 + dr^2 + r^2\big(d(\varphi + \omega t)\big)^2 + dz^2 ,$$

that is a flat Minkowski metric in cylindrical coordinates (r, ϕ, z) where we made the substitution $\phi \to \phi' = \varphi + \omega t$. We stress that, as we are going to see later, any Coriolis term inside the evolution equation of the particle velocity $\boldsymbol{\beta}$, arising from a rotating metric, is an object proportional to $\boldsymbol{\omega} \times \boldsymbol{\beta}$ with $\boldsymbol{\omega}$ the angular velocity vector which gives rise to the rotation of the frame. Since the particle is moving along a closed ring and since, as we are going to highlight in the following, only the contribution of the average along this path of all the physical quantities will matter in the end, it is natural to state that even if the magnitude of such a Coriolis correction was different from zero it would be totally ineffective and would not produce any modification to the anomalous magnetic moment as far as the analysis of chapter 1 is concerned.
Moreover, the metric of a rotating object must be asymptotically flat as we move to the spatial infinity

[3] Actually they also analyze a gravitational waves metric but it is presented more like an exercise.

$r \to \infty$ in spherical coordinates (r, θ, ϕ). This means that the leading metric term given by the axial body rotation should be

$$\propto \frac{xdy - ydx}{r^3}\, dt = \frac{r^2 d\phi}{r^3} dt\,.$$

Hence any contribution given by a flat rotating term would gain a suppression factor of order $R_\oplus^{-3}$ and it would become totally negligible.

Chapter 3

Experimental setup

In this chapter we are going to briefly analyze the experimental setup used in the E821 experiment at Brookhaven National Laboratory (BNL). We will enter in some details on how the the final value of a_μ is determined and we will provide a brief discussion on systematic perturbations due to two important corrections. We are going to closely follow the works by G. W. Bennet *et al.* [3] and J. M. Paley [19].

3.1 Principles

The aim of the experiment is to measure the evolution of the angle, we call it ϑ, between the muon spin vector $\mathbf{s}$ in the muon rest frame and its momentum $\mathbf{p}$ in the laboratory frame. Before the E821 experiment at BNL there were actually three main experiments which aimed to measure the muon anomalous magnetic moment precession and thus a_μ. Each of these experiments were performed by the Conseil Européen pour la Recherche Nucléaire (CERN). The first of them took place in 1961, the second started in 1966 while the third began in 1969 and its results were published in 1979. As time passed the precision of the measurements has been gradually increased from the $\sim 4300\,\mathrm{ppm}$ of CERN I, through the $270\,\mathrm{ppm}$ of CERN II and finally the $\sim 7.3\,\mathrm{ppm}$ of CERN III, which translates into a reduction of the uncertainty of ~ 600 times in less than 20 years.
Technically speaking, while in the first two experiments a non-homogeneous magnetic field has been used in order to vertical-focus the muons inside the storage ring, from the third one on (CERN III) an electric quadrupole field has been introduced and there was no need for a magnetic gradient anymore. This constituted an important improvement since the employment of an homogeneous magnetic field made the determination of its magnitude much easier. Furthermore, the muons γ-parameter was raised from 12 to 29.3 and this had two important consequences: first of all a larger Lorentz factor made it possible to see much more oscillations (since the muon life-time in the laboratory frame is dilated); on the other hand the precise value $\gamma = 29.3 \doteq \gamma_{\mathrm{m}}$, called *magic gamma*, makes any dependence of the anomalous precession frequency on the electric field disappear.

We already know from section 1.1 that the evolution law of ϑ in presence of an electromagnetic field is governed by equation (1.24). This equation can be simplified by a suitable choice of the direction $\boldsymbol{\beta}$ and its modulus (i.e. the magnitude of the particle's velocity in the laboratory frame): if the motion is orthogonal to the magnetic field, i.e. $\boldsymbol{\beta} \cdot \mathbf{B} = 0$, and γ is set in such a way that $a_\mu - 1/(\gamma_{\mathrm{m}}^2 - 1) = 0$

(to be precise $\gamma_{\mathrm{m}} = 29.3$ for the muon at E821) then the evolution of ϑ with respect to the laboratory time t is simply given by

$$\frac{d\vartheta}{dt} = \omega_{\mathrm{a}_\mu} = a_\mu \frac{|e|}{m} \langle B \rangle \,, \tag{3.1}$$

where as usual $a_\mu = \mathrm{g}_\mu/2 - 1$ while now $\langle B \rangle$ refers to the magnetic field averaged over the muon ring trajectories. Now we can easily see that, if we are able to measure the magnetic field strength and the anomalous frequency ω_{a_μ}, then we obtain a_μ.
Obviously the previous simplifications are affected by a statistical uncertainty which will be discussed later on.

Muon production starts by the collision of a proton beam with a Ni target, which among other particles produces pions $\pi^\pm$. Pions weakly decay producing muons: $\pi^+ \to \mu^+ \, \nu_\mu$ and $\pi^- \to \mu^- \, \overline{\nu}_\mu$. Due to the parity violation of this process and to the pion's zero spin, negative (positive) muons are produced in such a way that their spin direction is (anti-)aligned with their momentum, hence a simple momentum-selection of muons is equivalent to a spin polarization.
Now the method to measure the evolution of ϑ comes from the well known parity-violating nature of the muon decay chain $\mu \to \; e \, \overline{\nu} \, \nu$ (with this notation we refer both to $\mu^\pm$ decay chain with $e^\pm$ product). In particular we move to the muon rest frame (all the quantities showing a star * are taken in the muon frame) and we define $y = E^*/E^*_{\max} \in [0,1]$, with E^* the e particle energy and[1] $E^*_{\max} = 52.8\,\mathrm{MeV}$, and ϑ^*_{s} the angle between the electron velocity and the muon spin both measured in the muon rest frame. The differential probability of the electron decay turns out to be

$$\frac{d^2\mathcal{P}}{dy \, d(\cos\vartheta^*_{\mathrm{s}})} = n^*(y) \left[1 + \frac{e}{|e|} A^*(y) \cos\vartheta^*_{\mathrm{s}} \right] \,, \tag{3.2}$$

where now e is the particle charge. The functions n^* and A^* explicitly are

$$n^*(y) = y^2(3 - 2y) \,, \qquad A^*(y) = \frac{2y-1}{3-2y} \,,$$

thus $n^*(y)$ is always positive while $A^*(y) \lesseqgtr 0$ if $y \lesseqgtr 0.5$. This means that for $y > 0.5$ the differential probability (3.2) reaches its maximum at $\vartheta^*_{\mathrm{s}} = \pi$ for the e^- (at $\vartheta^*_{\mathrm{s}} = 0$ for the e^+), hence electrons (positrons) are mainly produced anti-parallel (parallel) to the muon spin vector. This also implies that the angular distribution of the e-particles in the muon rest frame rotates in the same way as the spin does i.e. with a frequency of ω_{a_μ}. Moreover integrating over all the possible energies we obtain

$$\frac{d\mathcal{P}}{d(\cos\vartheta^*_{\mathrm{s}})} = \frac{1}{2} \left(1 + \frac{e}{|e|} \frac{1}{3} \cos\vartheta^*_{\mathrm{s}} \right) \,,$$

[1] The maximum energy with which the e particle can be produced in the muon rest frame is

$$E^*_{\max} = \frac{m_\mu^2 + m_e^2 - (m_{\nu_\mu} + m_{\overline{\nu}_e})^2}{2m_\mu} \simeq \frac{m_\mu}{2} = 52.8\,\mathrm{MeV} \,.$$

Recall that $m_\mu/m_e \simeq 207$ and that $\sum_{i=e,\mu,\tau} m_{\nu_i} \lesssim 0.15\,\mathrm{eV}$ (upper limit on the sum of neutrino masses).

which again shows that the distribution of decayed electrons (positrons) is peaked backwards (forward) with respect to the muon spin direction. Applying a boost we can obtain an expression for the electron energy in the laboratory frame:

$$E_{\text{lab}} = \gamma(E^* + \beta p^* \cos\vartheta^*) \simeq \gamma E^*(1+\cos\vartheta^*)\,, \tag{3.3}$$

where we used the fact that both the muon and the e-particle produced are highly relativistic thus the energy $E^* \simeq |\mathbf{p}^*| = p^*$ and $\beta \simeq 1$, while ϑ^* is the angle between the muon flight direction $\boldsymbol{\beta}$ and the e momentum $\mathbf{p}^*$ in the muon frame. Naturally also the quantities n^* and A^* change when moving into the laboratory frame and we will call them simply n and A. Then a threshold energy E_{th} is chosen, above which e-particles in the laboratory frame will be detected. Thus from (3.3) we see that this translates into a selection of an angles range in the muon rest frame as follows:

$$E \geq E_{\text{th}} \quad \Longrightarrow \quad \theta^* \leq \arccos\left(\frac{E_{\text{th}}}{\gamma E^*} - 1\right)\,.$$

In this way the number of electron (positrons) $N(t)$ above E_{th} will display a damped oscillatory behavior with frequency ω_{a_μ} as follows

$$N(t) = N_0 \,\exp\left(-\frac{t}{\gamma\tau_\mu}\right)\left[1 + \frac{e}{|e|}A\cos\bigl(\omega_{\text{a}_\mu}t + \phi\bigr)\right]\,,$$

where τ_μ is the muon lifetime while N_0, A (the so called *asymmetry*) and ϕ are parameters which implicitly depend on E_{th}. A simple interpolation process will then give ω_{a_μ}.
In order to recover an expression for a_μ we need a measure of the mean magnetic field $\langle B\rangle$ felt by the muons. This can be achieved by Nuclear Magnetic Resonance (NMR) i.e. the measure of protons' Larmor spin precession frequency in water:

$$\omega_{\text{L}_p} = \text{g}_p \frac{e_p}{2m_p}\langle B\rangle\,,$$

this, with the help of equation (3.1), allows us to rewrite a_μ as

$$a_\mu = \frac{\omega_{\text{a}_\mu}}{\omega_{\text{L}_p}}\frac{m_\mu}{m_p}\frac{\text{g}_p}{2} = \frac{\omega_{\text{a}_\mu}}{\omega_{\text{L}_p}}\frac{m_\mu}{m_p}\frac{\text{g}_p}{\text{g}_\mu}(1+a_\mu)\,,$$

where we used the relation $1/2 = \text{g}_\mu^{-1}(1+a_\mu)$. Now if we define $\mathcal{R} = \omega_{\text{a}_\mu}/\omega_{\text{L}_p}$ and $\lambda = \text{g}_\mu m_p/(\text{g}_p m_\mu)$ we obtain

$$a_\mu = \frac{\mathcal{R}}{\lambda - \mathcal{R}}\,. \tag{3.4}$$

This combination of constant was proposed at BNL E821 since the muon-proton magnetic moment ratio $\lambda = |\mu_{\mu^-}|/\mu_p$ is known from previous experiments on studies on the muonium hyperfine structure (see [15] and equation (1.1)) while $\mathcal{R}$ is measured with this experiment.

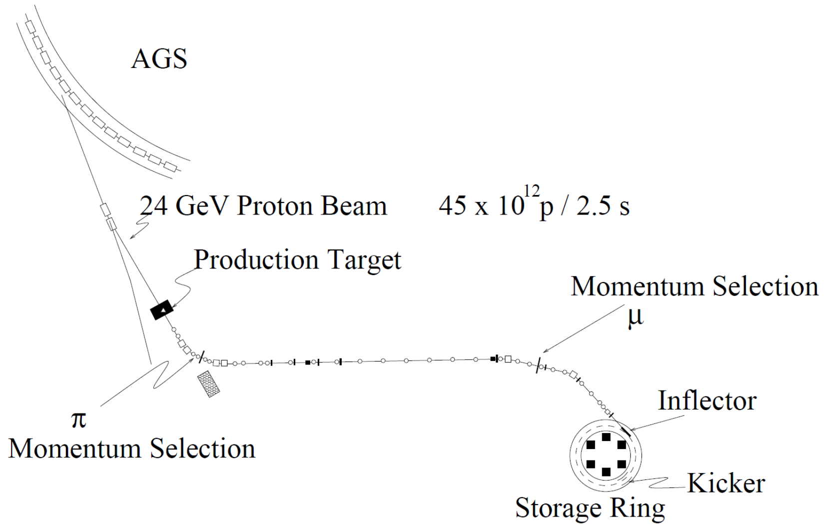


Figure 3.1: Top-view of the beam-line (figure taken from [19]).

Note. As we said before when stating that $\boldsymbol{\beta} \cdot \mathbf{B} = 0$ and $a_\mu - 1/(\gamma_m^2 - 1) = 0$ we are making an idealization. Actually among all the particles of the beam there are particles whose momentum differs from the magic momentum and there are also particles which do not perform a path exactly orthogonal to the magnetic field. Hence corrections need to be taken into account and typically one relies on averages over the muon distribution ensemble in order to see if these corrections are above or below the sensitivity of the experimental setup. These two corrections are called *radial E field correction*, the oscillation of γ about $\gamma_{\rm m}$, and *pitch correction*, the non-orthogonality of the particle's velocity to $\mathbf{B}$.

3.2 Beam-line

At first a linear accelerator (LINAC) speeds up protons until they reach a momentum of 200 MeV/c then a Booster accelerates them to a momentum of 1.6 GeV/c. At this point they enter the so called Accelerating Gradient Synchrotron (AGS) which raises the momentum to 24 GeV/c. From the AGS protons are then thrown towards the Ni target and the impact produces many particles among which also low-energy charged pions $\pi^\pm$. After the production there is a first momentum-selection of pions ($p_\pi = 1.017\,p_{\rm m}$, with $p_{\rm m} = 3.094$ GeV/c magic momentum; see previous section) that enter a straight path along which roughly one third of them decays into muons. At the end of the straight path there is the second momentum-selection ($p_\mu = p_{\rm m}$) before the final injection of muons into the g − 2 storage ring. A top view of the beam path between the AGS and the injection point is shown in figure 3.1. It is worth to stress that all of these momentum-selections (which are actually also charge-selections) also guarantee the elimination of particles which otherwise could be detected as noise.

As we said in the previous section, when muons are momentum selected they are also automatically spin-polarized as a consequence of parity-violation of pions decay. In particular the forward going selection of muons results into a $\sim 95\%$ of longitudinal polarization.

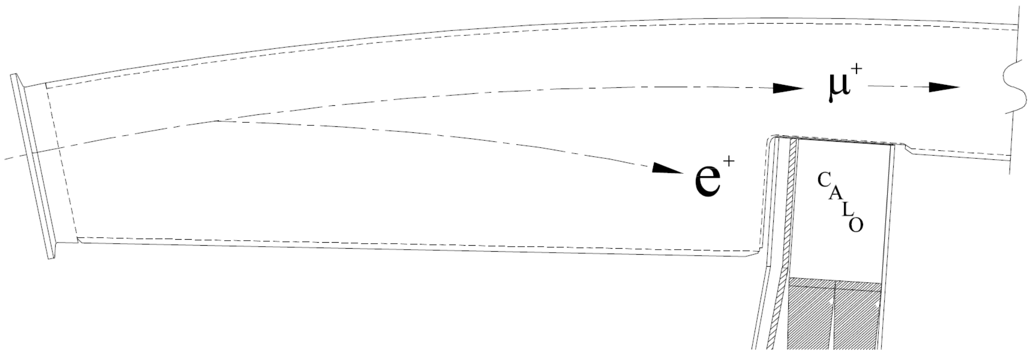


Figure 3.2: Top-view of the position of an electromagnetic calorimeters (CALO) (figure taken from [19]).

3.3 Storage Ring & Outcome

When the beam enters the storage ring, the muons begin their circular motion provided by the vertical magnetic field giving the so called horizontal focusing of the particles. The vertical focusing of the beam is given by an electric quadrupole. The magnetic field is set in order to reach the value of 1.4513 T at the equilibrium orbit i.e. at a distance of 7.112 m from the center of the ring.
One of the main improvement developed for the current BNL E821 experiment with respect to the last one performed at CERN (CERN III) is that in the E821 muons are directly injected inside the storage ring while in CERN III pions were injected instead. This leads to a much higher number of stored muons that end up in the stable orbit and also reduces the previously high background which represented a problem for the detectors.
While muons travel inside the storage ring they decay and, as we said before, the products of this decay, $e^{\pm}$, are recorded since their energy will oscillate exactly at the anomalous frequency ω_{a_μ}. Many electromagnetic calorimeters (CALO) made of lead, scintillating fiber and epoxy are placed all around the ring. These detectors are placed in the inner part of the ring, very close to the ring wall as shown in figure 3.2.

The outcomes of the experiment are the anomalous frequency ω_{a} and the proton Larmor frequency ω_{L_p} given by NMR (see section 3.1) from which we write the ratio $\mathcal{R} = \omega_{\mathrm{a}}/\omega_{\mathrm{L}_p}$ in order to use equation (3.4). Multiple runs are made using either positive and negative muons thus two results are obtained for the ratio $\mathcal{R}$, $\mathcal{R}_{\mu^+}$ and $\mathcal{R}_{\mu^-}$. Then an average over these two values is taken and it gives [3]

$$\mathcal{R}_{\mathrm{avg}}^{\mathrm{E821}} = 0.003\,707\,206\,4(20)\,.$$

As we said earlier, this has to be combined with the ratio $\lambda = |\mu_\mu|/\mu_p$ which is given by [7]

$$\lambda = \frac{|\mu_\mu|}{\mu_p} = \frac{\mathrm{g}_\mu}{\mathrm{g}_p}\frac{m_p}{m_\mu} = 3.183\,345\,39(10)\,.$$

This, using equation (3.4), gives the following result for a_μ [3]:

$$a_\mu^{\mathrm{E821}} = \frac{\mathcal{R}_{\mathrm{avg}}^{\mathrm{E821}}}{\lambda - \mathcal{R}_{\mathrm{avg}}^{\mathrm{E821}}} = 116\,592\,080(63) \times 10^{-11}\,, \tag{3.5}$$

where the uncertainty, 0.54 ppm, is the quadratic sum of a statistical contribution of 54×10^{-11} (0.46 ppm) and a systematic contribution of 33×10^{-11} (0.28 ppm).

Note. We want to stress that at a theoretical level the only anomalous angular velocity component we are going to consider is the vertical one. Where by "vertical" we mean the direction orthogonal to the ideal orbital plane of muons. This is consistent with equation (3.1) where in place of B it appears the average magnetic field taken over the particle path $\langle B \rangle$. When performing the average of a vector which rotates, whenever a component in the orbital plane is non-zero its mean over the closed path automatically goes to zero too.

3.4 Corrections

This section is dedicated to the study of the two main effects which can affect the measurement of a_μ i.e. the radial E-field correction and the pitch correction. Again we will follow [14] and [19].

3.4.1 Radial E-field correction

The radial E-field correction is the one arising from the fact that not all of the muons are traveling exactly at the magic gamma. Thus for a muon with momentum $p = p_\mathrm{m} + \delta p$ the second term on the right hand side of equation (1.24) does not vanish and contributes to ω_a with a value[2]

$$\boldsymbol{\omega}'_\mathrm{a} = -\frac{e}{m}\left[a_\mu \mathbf{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\boldsymbol{\beta} \times \mathbf{E}\right] .$$

Focusing on the vertical component of the angular velocity, the only component of the vector field that contributes is the one laying on the orbital planet. Thus the vector product can be written as $\boldsymbol{\beta} \times \mathbf{E} = \beta_\phi E_r \hat{z} = \beta E_r \hat{z}$ where we are using cylindrical coordinates (r, ϕ, z) with the origin at the center of the storage ring. Then the relative displacement becomes

$$\frac{\delta\omega_\mathrm{a}}{\omega_\mathrm{a}} = \frac{\omega'_\mathrm{a} - \omega_\mathrm{a}}{\omega_\mathrm{a}} = -\frac{\beta E_r}{a_\mu B_z}\left(a_\mu - \frac{1}{\gamma^2 - 1}\right) = -\frac{\beta E_r}{B_z}\left(1 - \frac{\gamma_\mathrm{m}^2 - 1}{\gamma^2 - 1}\right) \simeq -2\,\frac{\beta E_r}{B_z}\frac{\delta p}{p_\mathrm{m}}\,,$$

where we used $a_\mu = 1/(\gamma_\mathrm{m}^2 - 1)$ and $m^2(\gamma^2 - 1) = p^2$ then we kept only first order terms in δp. All the above quantities can be linked to experimental parameters. It is important to stress that if R_0 is the equilibrium radius of the orbit of a muon with γ_m the other muons with $\gamma \neq \gamma_\mathrm{m}$ will have a different equilibrium radius when put in the same magnetic field. We call this radius $R_e = R_0 + r_e$ where now r_e represents the displacement from the R_0 orbit.
First of all by using a simple dynamical model for the orbiting muons we can write

[2] We work in the hypothesis that $\boldsymbol{\beta} \cdot \mathbf{B} = 0$.

$$\frac{\delta p}{p_{\mathrm{m}}} = (1-n)\frac{r_e}{R_0}$$

where $n = \kappa R_0/(\beta B_z)$ with κ the quadrupole gradient strength[3]. Since, as we said before, we are interested in the mean value of the the quantities felt by the muons along their path now we average the radial electric field. We find

$$\langle E_r \rangle = \kappa r = \beta B_z n \frac{r}{R_0} \, ,$$

which leads to the expression

$$\frac{\delta \omega_{\mathrm{a}}}{\omega_{\mathrm{a}}} = -2n(1-n)\beta^2 \frac{r r_e}{R_0^2} \, .$$

Taking the average of this contribution and recalling that over an oscillation period $\langle r \rangle = r_e$, we get the final radial electric field correction C_E

$$C_E = \left\langle \frac{\delta \omega_{\mathrm{a}}}{\omega_{\mathrm{a}}} \right\rangle = -2n(1-n)\beta^2 \frac{r_e^2}{R_0^2} \, . \tag{3.6}$$

Experiment E821 data in the 2001 run have been collected at two different n values and gave the following E-field corrections $C_E^{(1)} = 0.470(54)\,\mathrm{ppm}$ and $C_E^{(2)} = 0.500(54)\,\mathrm{ppm}$ with the first at a lower n value than the second one.

3.4.2 Pitch correction

The pitch correction has to be taken into account because not all of the muons enter the storage ring with a velocity $\boldsymbol{\beta}$ orthogonal to the magnetic field $\mathbf{B}$. Thus in this case the $\boldsymbol{\beta}\cdot\mathbf{B}$ term in equation (1.24) does not vanish and the new anomalous angular velocity becomes[4]

$$\boldsymbol{\omega}'_{\mathrm{a}} = -\frac{e}{m}\left[a_\mu \mathbf{B} - a_\mu \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right] \, .$$

By moving into the rotating frame of the particle and developing a simple model that accounts for the existence of vertical $\boldsymbol{\beta}$ components, after some algebra it can be shown that the correction to the vertical component of $\boldsymbol{\omega}_{\mathrm{a}}$ is simply

$$\omega'_{\mathrm{a}} = -\frac{e}{m} a_\mu B_z \left(1 - \frac{\psi^2}{2}\right) \, ,$$

[3]If E_r is the radial component of the electric field we have

$$\kappa = \frac{\partial E_r}{\partial r} \, .$$

[4]Here we are supposing $\gamma = \gamma_{\mathrm{m}}$ thus the term $\boldsymbol{\beta} \times \mathbf{E}$ goes to zero.

where ψ is the angle the vector $\boldsymbol{\beta}$ forms with the horizontal plane. Clearly the angle ψ is not constant and oscillates because of the presence of the electric quadrupole that keeps the beam vertically focused. In particular we have $\psi = \psi_0 \cos(\omega_y t)$ with $\omega_y = \omega_{\mathrm{c}}\sqrt{n}$ the so called vertical *betatron oscillation*, n field index (see previous section) and ω_{c} cyclotron frequency (see section 1.1). Thus the relative correction is

$$\frac{\delta\omega_{\mathrm{a}}}{\omega_{\mathrm{a}}} = \frac{\omega'_{\mathrm{a}} - \omega_{\mathrm{a}}}{\omega_{\mathrm{a}}} = -\frac{\psi^2}{2}\,,$$

from which we can compute the pitch correction C_B as the time average of this quantity over a period i.e.

$$C_B = \left\langle \frac{\delta\omega_{\mathrm{a}}}{\omega_{\mathrm{a}}} \right\rangle = -\frac{\langle\psi^2\rangle}{2}\,. \tag{3.7}$$

Again in 2001 this correction has been computed for two different values of n and gave the results $C_B^{(1)} = 0.270(36)\,\mathrm{ppm}$ and $C_B^{(2)} = 0.320(36)\,\mathrm{ppm}$ and again the first is taken at a lower n value than the second one.

3.4.3 Total pitch & E-field correction

Now the only thing left is to sum these two corrections and compute the total correction due to the pitch correction and radial electric field correction. We call it C_T and it is

$$C_T^{(1)} = 0.740(65)\,\mathrm{ppm}\,, \qquad C_T^{(2)} = 0.820(65)\,\mathrm{ppm}\,. \tag{3.8}$$

These are the uncertainties on the value of ω_{a} given by the radial E-field correction and the pitch correction. Thus every GR effect which could influence any of the quantities involved into the computation of ω_{a} has to display a magnitude above the uncertainties (3.8) in order for the experimental procedure to see it.

Chapter 4

Quantum Field Theory

In this chapter we are going to discuss how Quantum Field Theory (QFT) deals with the problem of a_μ determination. In particular we aim at a review on the different contributions that the Standard Model predicts for the muon anomalous magnetic moment.
At the end of the chapter we will recover what we did in chapter 1 and we will use equation (1.71) to give a final value of the correction we found within the General Relativity (GR) framework. For the introductory part we will closely follow the works [11, 13, 20].

4.1 Introduction: QED

As we briefly said in chapter 1, Uhlenbeck and Goudsmit supposed that electrons are endowed with a magnetic momentum which is directly linked to the so called intrinsic spin angular momentum. To see how the QFT describes the coupling phenomenon, we need to recover the Dirac theory for spin 1/2 fermions. We start by writing the Lagrangian of a spin 1/2 particle interacting with an external electromagnetic field $(A_e^\mu) = (\varphi_e, \mathbf{A}_e)$ in the context of Quantum Electrodynamics:

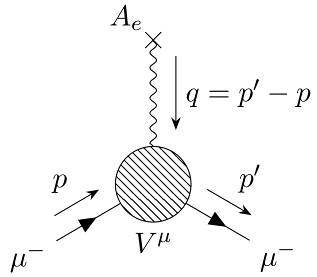
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - j^\mu (A_\mu + A_{e\mu}) \,,$$

where $\psi(x)$ is the spinor field, $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, $A^\mu(x)$ is the radiation field with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ as before, $j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x)$ is the current term and e and m are the particle's charge and mass[1] respectively. The term proportional to ξ^{-1} is a gauge fixing term needed to recover a consistent quantization of the electromagnetic field. The *slashed notation* means $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ where $\{ \gamma^\mu \}_{\mu=0,1,2,3}$ are the so called Dirac gamma matrices which fulfill the relations[2]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \,.$$

[1] Unless explicitly specified, along this chapter we will use m referring to the muon mass, that is $m = m_\mu$.
[2] Now we change the metric signature in order to be consistent with the standard QFT literature: from now on $(\eta_{\mu\nu}) = (\eta_{\mu\nu})^{-1} = (\eta^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$.

At first order in the external field, the Feynman graphic representing the interaction between the muon and the external electromagnetic field has the following form



whose amplitude is given by

$$\mathcal{M} = -ie\bar{u}(\mathbf{p'})V^\mu u(\mathbf{p})A_{e\mu}(\mathbf{q})\,. \tag{4.1}$$

Now the vertex function V^μ in the most general case, i.e. accounting for Lorentz invariance and Lorenz gauge condition $\partial_\mu A_e^\mu(x) = 0$, becomes

$$\bar{u}' V^\mu u = \bar{u}' \left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}}{2m}q_\nu \right] u \tag{4.2}$$

where $q^\nu = (p' - p)^\nu$, $\bar{u}' = \bar{u}(\mathbf{p'})$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. The function $F_1(q^2)$ is also known as *Dirac form factor* while $F_2(q^2)$ is called *Pauli form factor*.

Now we need a link between the above amplitude and the muon anomalous magnetic moment. Hence we write down the non-relativistic quantum Hamiltonian for a particle with magnetic moment $\boldsymbol{\mu}$ in an external electromagnetic field[3] $(\mathbf{E}, \mathbf{B})$

$$\mathcal{H} = \frac{(\mathbf{p} - e\mathbf{A}_e)^2}{2m} - \boldsymbol{\mu} \cdot \mathbf{B} + e\varphi_e$$

where $\mathbf{B} = \nabla \times \mathbf{A}_e$ while[4] $\mathbf{E} = -\nabla\varphi_e - \dot{\mathbf{A}}_e$. This is useful since in the non-relativistic limit we can write both an expression for the scattering amplitude following from the non-relativistic Hamiltonian above and a relation between the latter and $\mathcal{M}$. In the Born approximation the scattering amplitude is given by

$$f = -\frac{m}{2\pi}\int \Psi'^\dagger(\mathbf{x}, \mathbf{p'})V(\mathbf{x})\Psi(\mathbf{x}, \mathbf{p})\, d^3\mathbf{x}\,, \tag{4.3}$$

[3] We deal with static conditions, hence none of the quantities we are considering is time dependent.

[4]
$$\dot{\mathbf{A}}_e = \frac{\partial \mathbf{A}_e}{\partial t}\,.$$

where $\Psi(\mathbf{x},\mathbf{p}) = \chi e^{i\mathbf{x}\cdot\mathbf{p}}$ and $\Psi'(\mathbf{x},\mathbf{p}') = \chi' e^{i\mathbf{x}\cdot\mathbf{p}'}$, with χ and χ' accounting for the spin degrees of freedom[5], and V has the following form

$$V = -\frac{e}{2m}(\mathbf{p}\cdot\mathbf{A}_e + \mathbf{A}_e\cdot\mathbf{p}) - \boldsymbol{\mu}\cdot\mathbf{B} + e\varphi_e\,. \tag{4.4}$$

Putting the expression (4.4) into equation (4.3), by means of a Fourier transform we obtain

$$f = -\frac{m}{2\pi}\chi'^{\dagger}\left[-\frac{e}{2m}(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}_e(\mathbf{q}) + e\varphi_e(\mathbf{q}) - i\boldsymbol{\mu}\cdot\big(\mathbf{q}\times\mathbf{A}_e(\mathbf{q})\big)\right]\chi\,, \tag{4.5}$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ and $\big(A_e^{\mu}(\mathbf{q})\big) = \big(\varphi_e(\mathbf{q}), \mathbf{A}_e(\mathbf{q})\big)$ is the Fourier transform of $A_e^{\mu}(\mathbf{x})$:

$$A_e^{\mu}(\mathbf{q}) = \int A_e^{\mu}(\mathbf{x})\, e^{-i\mathbf{q}\cdot\mathbf{x}}\, d^3\mathbf{x}\,.$$

As previously mentioned, we need a relation between f and the non-relativistic limit of $\mathcal{M}$ and this is given by the following relation

$$\lim_{|\mathbf{p}|\ll m} \mathcal{M} = 4\pi i f\,, \tag{4.6}$$

which is valid if the spinor fields are normalized to $2m$ i.e. $\bar{u}u = u^{\dagger}\gamma^0 u = 2m = \bar{u}'u'$, where we used a shorthand notation: $|\mathbf{p}| \ll m$ means both $|\mathbf{p}'|, |\mathbf{p}| \ll m$. Now we need to compute $\lim_{|\mathbf{p}|\ll m}\mathcal{M}$. To do this we recall the Dirac–Pauli representation of the gamma matrices, i.e.

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\mathbb{I}_2 \end{pmatrix}\,, \quad \gamma^k = \begin{pmatrix} \mathbb{O}_2 & \sigma_k \\ -\sigma_k & \mathbb{O}_2 \end{pmatrix} \quad k = 1,2,3\,,$$

where $\mathbb{I}_2$ and $\mathbb{O}_2$ are the identity and null 2×2 matrices respectively while $\{\,\sigma_k\,\}_{k=1,2,3}$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

while the spinor field $u(\mathbf{p})$ is given by

$$u(\mathbf{p}) = \sqrt{E+m}\begin{pmatrix} \chi \\ \dfrac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix} \xrightarrow[|\mathbf{p}|\ll m]{} \sqrt{2m}\begin{pmatrix} \chi \\ 0 \end{pmatrix}\,, \tag{4.7}$$

[5]They are the well known Pauli spinors. In principle $\chi = \chi_r$ with $r = 1,2$ which labels the spin state with

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,.$$

with $E = \sqrt{\mathbf{p}^2 + m^2}$. If now we put equation (4.7) into the amplitude (4.1), by means of equation (4.2), as $q \to 0$ and keeping only linear terms in $\mathbf{p}$ and $\mathbf{p}'$, we obtain

$$\mathcal{M} = 2m(-ie)\chi'^{\dagger}\left[F_1(0)\left(\varphi_e(\mathbf{q}) - \frac{1}{2m}\mathbf{A}_e(\mathbf{q})\cdot(\mathbf{p}'+\mathbf{p})\right) - \frac{i}{2m}\big(F_1(0)+F_2(0)\big)\boldsymbol{\sigma}\cdot\big(\mathbf{q}\times\mathbf{A}_e(\mathbf{q})\big)\right]\chi\,.$$

Now the relation (4.6) allows us to write

$$F_1(0) = 1\,, \qquad \boldsymbol{\mu} = \frac{e}{2m}\big(1+F_2(0)\big)\boldsymbol{\sigma} = \frac{e}{2m}\,2\big(1+F_2(0)\big)\mathbf{s}\,,$$

where we used the well known relation $\mathbf{s} = \frac{\boldsymbol{\sigma}}{2}$. If we recall equation (1.1) we can relate our g-factor with $F_2(0)$:

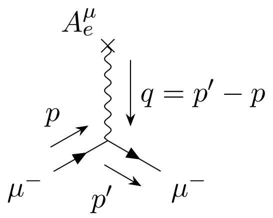
$$2\big(1+F_2(0)\big) = \mathrm{g}_\mu \quad \Rightarrow \quad a_\mu = \frac{\mathrm{g}_\mu - 2}{2} = F_2(0)\,, \tag{4.8}$$

thus $\mathrm{g}_\mu \neq 2$ only if $F_2(0) \neq 0$.
Now the perturbative approach in QED allows us to write the so called anomaly a_μ as a series

$$a_\mu = \sum_{n=0}^{\infty} a_\mu^{(n)} = \sum_{n=0}^{\infty} c_n \left(\frac{\alpha}{\pi}\right)^n\,, \qquad \alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137.04}\,, \tag{4.9}$$

where α is the well known *fine structure constant*.

At order zero in α, the anomaly has to be exactly $a_\mu^{(0)} = 0$ i.e. $\mathrm{g}_\mu = 2$ and this can be seen by writing the tree level vertex amplitude



$$= -ie\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p})A_{e\mu}(\mathbf{q})\,. \tag{4.10}$$

We can decompose the current term $\bar{u}'\gamma^\mu u$ by means of the so called *Gordon identity*, that is

$$\bar{u}'\gamma^\mu u = \bar{u}'\left(\frac{1}{2m}(p'+p)^\mu + \sigma^{\mu\nu}q_\nu\frac{i}{2m}\right)u\,,$$

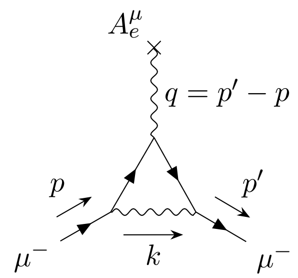
with $q^\nu = (p'-p)^\nu$. These two terms are sometimes called *convection current* (the first term) and *spin current* (the second one) and the latter clearly reflects what we said before i.e. that at the tree level $\mathrm{g}_\mu = 2$ exactly. Thus taking a static magnetic potential $(A_e^\mu) = (0, \mathbf{A}_e)$ we have

$$\bar{u}' \not{A}_e u = \bar{u}' \left(-\frac{1}{2m} (\mathbf{p}' + \mathbf{p}) \cdot \mathbf{A}_e + \sigma^{ij} \mathbf{A}_e^i \mathbf{q}^j \frac{i}{2m} \right) u \,,$$

which in the non-relativistic limit, by means of (4.6), (4.7) and again keeping only linear terms in $\mathbf{p}$ and $\mathbf{p}'$, gives $\mathrm{g}_\mu = 2$. Thus the series (4.9) becomes

$$a_\mu = \sum_{n=1}^{\infty} a_\mu^{(n)} = \sum_{n=1}^{\infty} c_n \left(\frac{\alpha}{\pi}\right)^n = c_1 \left(\frac{\alpha}{\pi}\right) + c_2 \left(\frac{\alpha}{\pi}\right)^2 + c_3 \left(\frac{\alpha}{\pi}\right)^3 + \cdots \,. \tag{4.11}$$

At first order in α the contribution to the anomaly is given by the following vertex modification



$$= -ie\bar{u}(\mathbf{p}')\Gamma^\mu(p', p) u(\mathbf{p}) A_{e\mu}(\mathbf{q}) \,. \tag{4.12}$$

Where now the vertex $\Gamma^\mu(p', p)$ is given by

$$\begin{aligned} \Gamma^\mu(p', p) &= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\alpha \frac{i(\not{p}' - \not{k} + m)}{(p' - k)^2 - m^2} \gamma^\mu \frac{i(\not{p} - \not{k} + m)}{(p - k)^2 - m^2} \gamma^\beta \frac{-i\eta_{\alpha\beta}}{k^2} \\ &= -\frac{ie^2}{(2\pi)^4} \int \frac{d^4k}{k^2} \gamma^\alpha \frac{\not{p}' - \not{k} + m}{(p' - k)^2 - m^2} \gamma^\mu \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2} \gamma_\alpha \,. \end{aligned} \tag{4.13}$$

As we can easily see this integral is both infrared (IR, $k \to 0$) and ultraviolet (UV, $k \to \infty$) divergent:

$$\Gamma^\mu(p, p') \xrightarrow[k \to 0]{\text{IR}} -\frac{ie^2}{(2\pi)^4} \gamma^\mu \int \frac{d^4k}{k^2} \frac{p' \cdot p}{(p' \cdot k)(p \cdot k)} \,,$$

$$\Gamma^\mu(p, p') \xrightarrow[k \to \infty]{\text{UV}} -\frac{ie^2}{(2\pi)^4} \gamma^\mu \int \frac{d^4k}{k^4} \,.$$

However this is not a problem for our purpose, since both the previous divergences will affect only the Dirac form factor F_1 while in order to gain information about the muon's anomaly we look only at the Pauli form factor F_2. Thus working on the integral (4.13) we can use the Feynman parametrization for the denominator, then perform the integration over the virtual momentum k and neglect all the terms containing γ^μ only, for the reason we mentioned above. The final result reads

$$\Gamma^\mu(p', p)\big|_{a_\mu} = \frac{\alpha}{2\pi} \frac{i\sigma^{\mu\nu}}{2m} q_\nu \,,$$

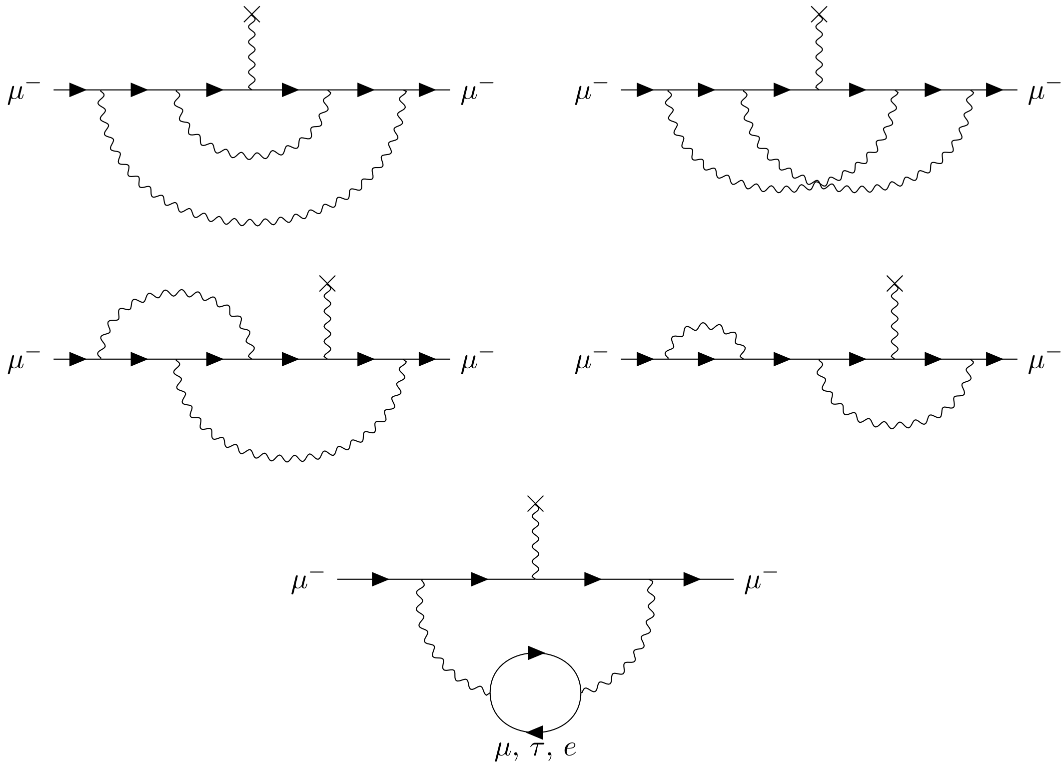


Figure 4.1: Diagrams involved in the computation of $a_\mu^{(2)} = \mathcal{O}(\alpha^2)$.

where with this notation we refer to the part of $\Gamma^\mu(p', p)$ contributing to the Dirac form factor i.e. to the anomaly a_μ. A comparison with (4.2) leads to the identification

$$F_2(0) = a_\mu^{(1)} + \mathcal{O}(\alpha^2) = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) \quad \Rightarrow \quad c_1 = \frac{1}{2},$$

which is the well known result obtained by Schwinger in 1948; hence at first order in α

$$\mathrm{g}_\mu = 2\left(1 + \frac{\alpha}{2\pi}\right).$$

At second order in α the diagrams in figure 4.1 are involved. Notice that also the mirrored versions of both the third and fourth diagrams have to be taken into account. Moreover the last one accounts for all the three vacuum polarization loops given by μ, τ or e. Splitting the latter contribution we can write the order α^2 term of the anomaly series as

$$a_\mu^{(2)} = \left(\frac{\alpha}{\pi}\right)^2 \left(w^{(2)}_{\mathrm{vp},e} + w^{(2)}_{\mathrm{vp},\tau} + w^{(2)}\right),$$

where $w^{(2)}$ refers to diagrams in which only muons appear, while $w^{(2)}_{\mathrm{vp},e(\tau)}$ represents the contribution given by the electron (tau) loop. Given the different lepton masses $m_e \ll m_\mu \ll m_\tau$ the vacuum polarization terms involving e and τ need to be treated in different ways. After some lengthy calculation one finds [13]

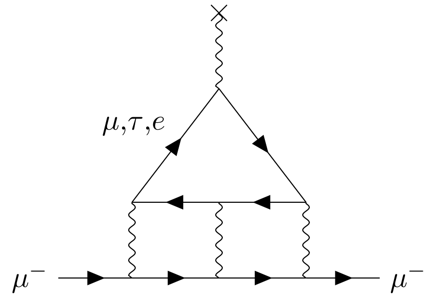
$$w^{(2)}_{\mathrm{vp},e} = \frac{1}{3}\ln\left(\frac{m}{m_e}\right) - \frac{25}{36} + \frac{\pi^2}{4}\frac{m_e}{m} + \mathcal{O}\left(\frac{m_e^2}{m^2}\right),$$

$$w^{(2)}_{\mathrm{vp},\tau} = \frac{1}{45}\left(\frac{m}{m_\tau}\right)^2 ,$$

$$w^{(2)} = \frac{197}{144} + \frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln 2 + \frac{\pi^2}{12},$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function. It is worth to stress that due to the condition $m_\tau \gg m$ the contribution given by $w^{(2)}_{\mathrm{vp},\tau}$ is heavily suppressed.

At third order in α things become very challenging since the so called light-by-light diagrams come in:

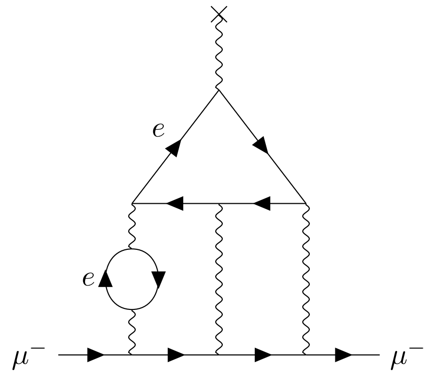


Again we can split the different contributions in order to write the third order term of the anomaly series (4.11) as

$$a^{(3)}_\mu = \left(\frac{\alpha}{\pi}\right)^3 \left(w^{(3)}_{\mathrm{lbl},e} + w^{(3)}_{\mathrm{lbl},\tau} + w^{(3)}_{\mathrm{vp}} + w^{(3)}\right) ,$$

where $w^{(3)}_{\mathrm{lbl},e(\tau)}$ gives the light-by-light loop containing e (τ), $w^{(3)}_{\mathrm{vp}}$ again accounts for the vacuum polarization contributions and $w^{(3)}$ contains all the diagrams in which only muons appear. Notice that $w^{(3)}_{\mathrm{vp}}$ accounts for multiple vacuum polarization loops of the same fermion and of different fermions as well $w^{(3)}_{\mathrm{vp}} = w^{(3)}_{\mathrm{vp},e} + w^{(3)}_{\mathrm{vp},\tau} + w^{(3)}_{\mathrm{vp},e,\tau}$ where the first and the second terms account for diagrams in which at least one vacuum polarization diagram, containing electrons and τ respectively, appears; while the last term contains both electron and τ vacuum polarization diagrams.
Among all of the vacuum polarization contributions, for which analytical expressions are known, the two terms $w^{(3)}_{\mathrm{vp},e,\tau}, w^{(3)}_{\mathrm{vp},\tau} \sim 10^{-11}$ give a contributions smaller than the current experimental sensitivity; the only significant vacuum polarization contribution is given by the electron loops $w^{(3)}_{\mathrm{vp},e}$. Considering the light-by-light scattering diagrams, the τ contribution goes below the experimental precision as well with $w^{(3)}_{\mathrm{lbl},\tau} \sim 10^{-11}$ while the greater result is given by $w^{(3)}_{\mathrm{lbl},e}$. The explicit expressions for these contributions are quite lengthy and we omit them.

At fourth order in α the only results available nowadays are given numerically. The importance of the $\mathcal{O}(\alpha^4)$ corrections to a_μ stems from the fact that at this order we can compare the theory with the experimental results whose precision has been greatly enhanced with time. The main $\mathcal{O}(\alpha^4)$ contribution to the anomaly a_μ is given by the following graph



obviously accounting for all the three permutations in the position of the e vacuum polarization; this is nothing but an electron light-by-light scattering with an electronic vacuum polarization which as in the previous case give rise to much of the magnitude of this correction. An important feature of this term, let's call it $w^{(4)}_{\mathrm{lbl,vp},e}$, is that nowadays it is clear that most of its value is provided by a few, very well understood Feynman diagrams. This means that any discrepancy found between theoretical and experimental value of a_μ at order $\sim 100 \times 10^{-11}$ is not given by a lack of knowledge of QED processes. Moreover, despite only two research groups were able to give results arising from this term so far, the outcomes are consistent.

Finally for the fifth order correction in α we can follow the previous reasoning and find that the main contribution is given by a graph which displays a light-by-light electron scattering with two electron vacuum polarization loops inside the photon propagator. Nevertheless the full calculation of the five loops QED contribution to a_μ is still in progress.

The final result for the QED contribution up to order $\mathcal{O}(\alpha^5)$ is [10]

$$a_\mu^{\mathrm{QED}} = 116\,584\,718.851(360) \times 10^{-11} \tag{4.14}$$

We underline that the QED part of a_μ is the dominant one.

Note. Since all the existing particles have to be taken into account when computing a lepton's anomaly a, it could be interesting to know how a particle's mass, say m_p, affects its value. In the two cases $m_p \geq m$ and $m_p \ll m$ for a lepton of mass m the anomaly variation $\delta a_{\mu,p}$ is given by

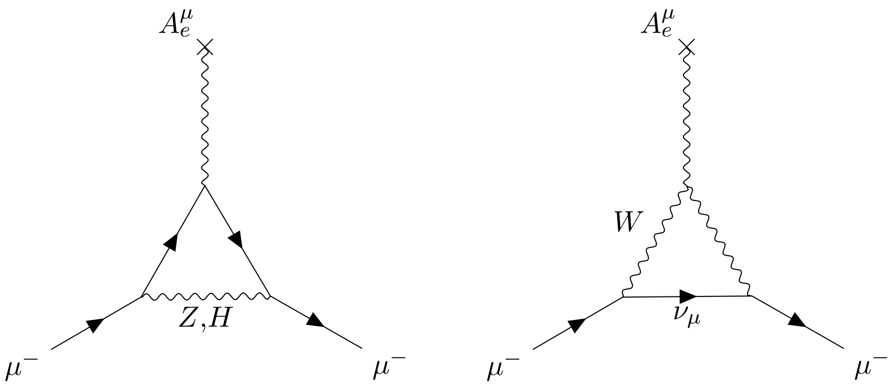
$$\delta a_{\mu,p} \sim \begin{cases} \left(\dfrac{\alpha}{\pi}\right)^{n_p} \left(\dfrac{m}{m_p}\right)^2 \ln^{k_p} \dfrac{m_p}{m} & m_p \geq m \\ \left(\dfrac{\alpha}{\pi}\right)^{n_p} \ln^{k_p} \dfrac{m}{m_p}\,, \quad \text{and } k_p < n_p & m_p \ll m\,, \end{cases} \tag{4.15}$$

where n_p and k_p depend on the order of perturbation that gives rise to these contributions. Now we look at corrections on the muon and electron anomalies given by other particles. The electron is the lightest charged particle, thus from (4.15) it follows that all the contribution to a_e are suppressed by a factor m_e/m_p which is at least $m_e/m_\mu \sim 10^{-4}$ hence corrections to a_e are governed by powers of

α/π. On the other hand corrections to the anomalous magnetic moment of the muon a_μ are governed by powers of $(\alpha/\pi)\ln(m/m_p)$ thus contributions given either by electrons (for which $\ln(m_\mu/m_e) \sim 5$) and hadrons (typically ρ mesons for which m_ρ is roughly of the same order of m_μ). This is useful because on one hand, since the effect of hadron and new physics does not affect the electron anomaly, measurements of a_e allows a precise determination of the fine structure constant α; on the other hand the muon anomaly a_μ is much more sensitive to physics beyond the Standard Model (BSM): given an energy scale Λ at which some new physics becomes relevant modifying QED, the following new contribution should be of order $(m_\mu/\Lambda)^2$.

4.2 Electroweak corrections

The one loop electroweak corrections to the value of a_μ are given by exchange of W and Z bosons but also include contribution by the Higgs boson H. The corresponding Feynman diagrams give the higher order corrections to the muon anomaly and are the following:



We recall that for each boson involved it holds $m_{W,Z,H} \ll m_\mu = m$ and this provides a very simple form for this contribution. The one loop result reads [20]

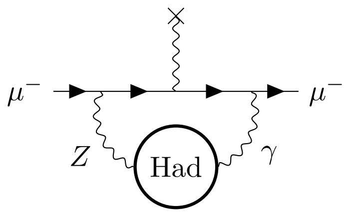
$$a_\mu^{11} = \frac{G_F}{\sqrt{2}}\frac{m^2}{24\pi^2}\left[\underbrace{10}_{W} + \underbrace{(1-4\sin^2\theta_W)^2 - 5}_{Z} + \mathcal{O}\left(\frac{m^2}{m_{W,H}^2}\right)\right] = 194.8\times 10^{-11}\,, \quad \text{for } m_H = 150\,\text{GeV}\,.$$

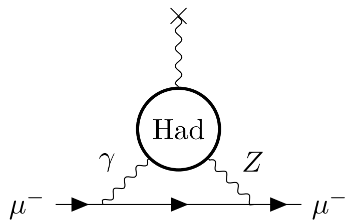
Where G_F is the Fermi coupling constant defined by the relation $G_F/\sqrt{2} = (g_W/m_W)^2$ with g_W the weak coupling constant. The angle θ_W is the Weinberg (or weak) mixing angle given by the relation $\cos\theta_W = m_W/m_Z$, where m_Z is the experimental value of the Z mass while m_W is the SM prediction of the W mass as a function of the Higgs mass m_H. Since [13] $114\,\text{GeV} \leq m_H \leq 250\,\text{GeV}$ the value above is given by the choice [20] $m_H = 150\,\text{GeV}$ as written.
The Higgs boson contribution turns out to be $< 3\times 10^{-14}$ (given by the lower bound on Higgs boson's mass) and thus can be neglected.

Higher order contributions are given by diagrams containing two loops. The complete calculation in the 't Hooft–Feynman gauge involves ~ 1700 diagrams and thus the technical issue is non-negligible. Moreover a lot of particles enter the computation, thus the result carry dependence on all their masses, that is, the uncertainties grow. Initially the two loops corrections were thought to be irrelevant with respect to the one loop ones but former calculations lead some authors in the early '90s to a result that appears as follows [10]

$$a_\mu^{2l} \sim -10 \left(\frac{\alpha}{\pi}\right) a_\mu^{1l} \left(\ln \frac{m_Z}{m} + 1\right) \sim -40 \times 10^{-11}$$

and it actually brings a sensitive reduction of $a_{\mu,1l}$. It is important to stress that, among all these two loops diagrams, there are some which involve light-quark exchanges[6]. In particular there appear contributions from the following two graphs

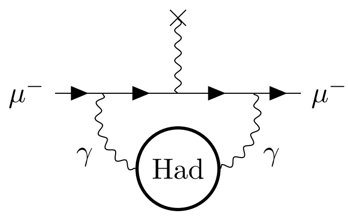




that is, a γ–Z hadronic mixing and the so called quark triangle with two virtual γ and Z bosons and one external γ. The reason why they are relevant is that for these diagrams the perturbative approach of Quantum Chromodynamics (QCD) fails. In the next section we will briefly see how to deal with such hadronic contributions.

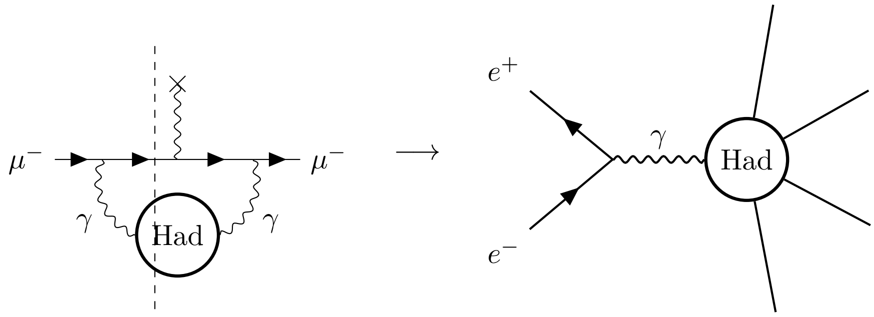
4.3 Hadronic corrections

The leading order hadronic contribution to a_μ^{QED} is $\mathcal{O}(\alpha^2)$ and it is given by the hadronic vacuum polarization in the photon propagator of the modified vertex function



Unfortunately for this kind of process perturbation theory can not be employed since it involves long distance QCD. Nevertheless it can be computed via dispersion relations, following by analyticity and unitarity, relying on data from the electron–positron annihilation. In particular the following cut in the diagram is performed

[6]For particles heavy enough, the associated energy scale is such that the strong coupling constant is small and this allows for a perturbative QCD (pQCD) treatment. One example is the top quark with $m_t \sim 170\,\text{GeV}$.



(4.16)

and after the cut the substitution $\mu \to e$ has been done. This substitution is necessary because we want to rely on the available experimental data for the electron-positron annihilation process[7]. The previous diagram means that all the final hadronic states have to be considered. This allows to write the following dispersion integral for the hadronic leading order contribution a_μ^{hlo} to the muon anomaly as a function of observable quantities

$$a_\mu^{\text{hlo}} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} K(s)\sigma^{(0)}(s)\, ds = \frac{\alpha^2}{3\pi^2} \int_{4m_\pi^2}^{\infty} K(s)R(s)\, \frac{ds}{s}\,,$$

where

$$R(s) = \frac{\sigma^{(0)}(s)}{\frac{4\pi\alpha^2}{3s}}\,, \qquad K(s) = \int_0^1 \frac{x^2(1-x)}{x^2 + (1-x)\frac{s}{m_\mu^2}}\, dx\,.$$

Here $\sigma^{(0)}(s)$ is the total experimental cross section of the e^+e^- annihilation into hadrons while $\sigma^{(0)}/R(s)$ is the high energy limit of the Born cross section for muon pair production. This hadronic contribution turns out to be $a_\mu^{\text{hlo}} \sim 7000 \times 10^{-11}$.

Moving on, the order α^3 hadronic correction is mainly given by three loops diagrams containing at least one hadronic loop. Some examples are given in figure 4.2. For the first two hadronic vacuum polarization contributions, the same procedure used in the previous paragraph and $e^+e^- \to \big(\text{hadrons}\big)$ data as well are employed. The interesting part is the hadronic light-by-light graph, the third in figure 4.2, for which no link with experiments exists and the calculation of which must rely on theoretical tools only although, we highlight, such a term can not be computed from first principles. The contribution of this hadronic light-by-light term to the Pauli form factor $F_2(0)$, which we recall is identified with the anomalous magnetic moment a_μ itself (equation (4.8)), involves integration of a particular tensor

$$\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3) = \int d^4x_1\, d^4x_2\, d^4x_3\, e^{i(q_1\cdot x_1 + q_2\cdot x_2 + q_3\cdot x_3)} \left\langle 0 | T\{j_\mu(q_1) j_\nu(q_2) j_\rho(q_3) j_\sigma(0)\} | 0 \right\rangle\,,$$

with j^μ the light quark part of the electromagnetic current and q_i is the momentum of the i^{th} outgoing quark[8]:

[7] This procedure follows the important paper [2].

[8] Note that the u now stands for the quark up spinor field $u(x) = \psi_u(x)$.

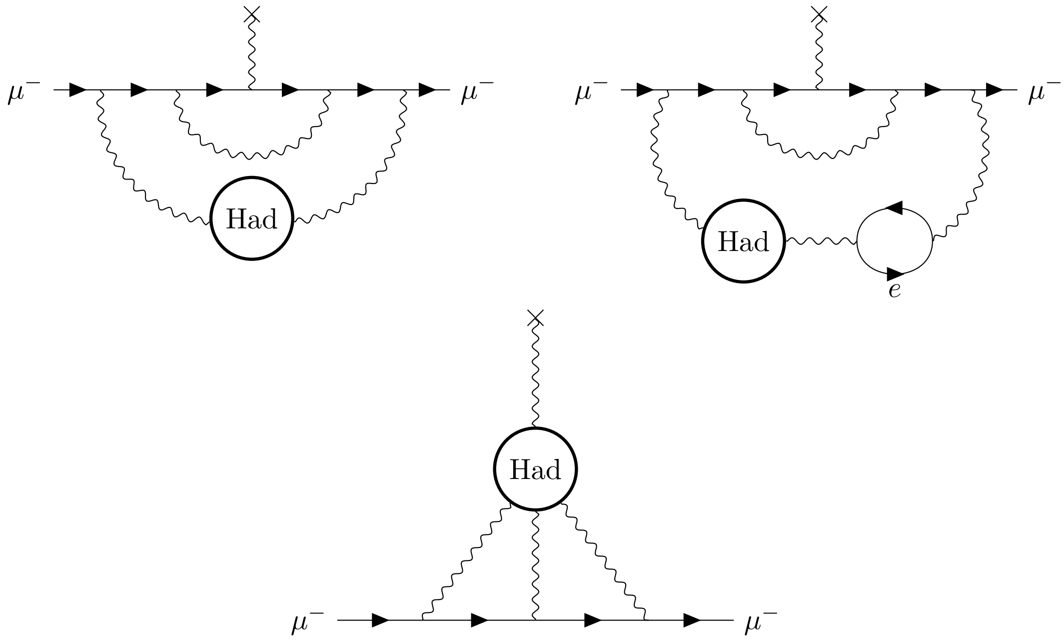


Figure 4.2: Some $\mathcal{O}(\alpha^3)$ hadronic loop contributions.

$$j^\mu(x) = \frac{2}{3}(\bar{u}\gamma^\mu u) - \frac{1}{3}(\bar{d}\gamma^\mu d)(x) - \frac{1}{3}(\bar{s}\gamma^\mu s)(x)\,.$$

The main problem is that the structure of $\Pi_{\mu\nu\rho\sigma}$ has no leptonic counterpart. Nevertheless this tensor is shown to be dominated by the exchange of pseudo-scalar mesons such as π^0, η, η' and these exchange are well described by the effective Wess–Zumino–Witten Lagrangian which reads [11]

$$\mathcal{L}_{\text{WZW}} = \frac{\alpha}{\pi}\frac{N_\text{c}}{12F_\pi}\left(\pi^0 + \frac{1}{\sqrt{3}}\eta_8 + 2\sqrt{\frac{2}{3}}\eta_0\right)\tilde{F}_{\mu\nu}F^{\mu\nu}$$

with N_c number of colors, F_π the pion decay constant, π^0 is the neutral pion field, the couple of pseudo-scalar fields η_0 and η_8 mix to give the physical states η and η' and $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}/2$. Another detail is that in the effective field theory photons couple to hadrons via vector mesons as ρ^0. In particular a way to insert vector mesons (other examples are ϕ and ω vector-mesons) into the theory is through the so called Resonance Lagrangian Approach (RLA) including the Vector-Meson Dominance (VMD) which provides an extension of the low-energy (up to $\sim 1\,\text{GeV}$) QCD effective field theory given by Chiral Perturbation Theory, thus reflecting the chiral symmetry properties of QCD. Hence what is done is a splitting of the QCD contribution into two parts. The first is the short distance tail which involves quark loops and can be studied with the perturbation theory (pQCD). The second is the so called long distance tail for which perturbative expansion breaks down and we must make use of the effective field theory. Since there is no deep knowledge on what is actually going on in the latter case, every attempt to give an estimate of the QCD influence on a_μ is plagued by its own uncertainty. This uncertainties arise from the fact that effective field theories for the treatment of non-perturbative QCD differ from one another by the model employed and thus by the particles they have to account for.

The final results for the hadronic light-by-light contribution to the anomalous magnetic moment a_μ^{hlbl} is given by [10]

$$a_\mu^{\mathrm{hlbl}} = 103.4(288) \times 10^{-11} \, .$$

4.4 Theoretical anomaly value

We present the current value of a_μ as it is given in the Standard Model (SD) given the contributions discussed in the above sections of this chapter. Accounting for the QED contribution a_μ^{QED}, the Weak contribution a_μ^{W} and the hadron contribution a_μ^{QCD} we have [10]

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{QCD}} = 116\,591\,776(44) \times 10^{-11} \, . \tag{4.17}$$

Which has to be compared with the experimental value obtained by E821 experiment at BNL, see equation (3.5).

Chapter 5

GR anomaly correction

In this last chapter we are going to give an estimate of the gravitational correction to the muon anomalous magnetic moment measurement. For this purpose we start from the conclusions obtained in chapter 1 and compute the final result.

5.1 Computation

We rewrite equation (1.71) because it will be our starting point for the whole analysis[1]:

$$\boldsymbol{\Omega}_{\mathrm{a}} = -\frac{e}{m}\left[a\mathbf{B} - \left(a - \frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\mathbf{E} - a\left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right] - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\times\mathbf{a} - \frac{\gamma}{\gamma-1}\,\mathbf{a}\right), \quad (5.1)$$

where we wrote

$$\mathbf{a}^{\hat{\imath}} \equiv \left(\frac{Du}{D\tau_u}\right)^{\hat{\imath}}.$$

As we said before, since we are making use of a local coordinate basis, all of the 3-vector quantities whose components appear in equation (5.1) are the physical ones the observer (the laboratory in our case) sees. Before we start it is important to stress that we are going to look only at the vertical component of the anomalous angular velocity, because every physical vector quantity which lays in the particle's plane of motion, when averaged over the ring path, will give no contribution.
A suitable 3-dimensional unit vector pointing vertically, i.e. orthogonal to the particle's orbit plane and pointing outwards with respect to the Earth center, is given by

$$\mathbf{n} = \cos\theta\,\vec{e}_{\hat{r}} - \sin\theta\,\vec{e}_{\hat{\theta}}\,.$$

[1]Since in the adapted frame all the vector quantities are lacking the $\hat{0}$ component, we use the boldface notation to refer to vectors belonging to the observer's space.

The observer is the laboratory on Earth surface hence its 4-velocity in the adapted frame is $u^a = \delta^a_{\hat{0}}$ and its 4-acceleration is given by equations (1.80):

$$\mathbf{a} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \vec{e}_{\hat{r}}\,.$$

The particle is performing a horizontal path at constant velocity thus $\boldsymbol{\beta}$ has only a constant ϕ-component, that is

$$\boldsymbol{\beta} = \beta\,\vec{e}_{\hat{\phi}}\,.$$

where β is the particle velocity in the laboratory frame. The electric field is irrelevant because at the magic momentum p_{m} (see chapter 2 and chapter 3) its contribution is null. The magnetic field is vertical with respect to the particle's orbit thus it is proportional to $\mathbf{n}$ with proportionality constant given by the experimentally measured value of the magnetic field:

$$\mathbf{B} = B\,\mathbf{n}\,.$$

Now we are able to write the spatial components of the anomalous precession angular velocity as

$$\boldsymbol{\Omega}_{\mathrm{a}} = \left(-\frac{e}{m}aB\right)\mathbf{n} - \frac{\gamma}{\gamma+1}\left[\frac{M\beta}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\vec{e}_{\hat{\theta}} - \frac{\gamma}{\gamma-1}\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\vec{e}_{\hat{r}}\right]. \tag{5.2}$$

As we already said, in order to extract information from this equation we need to project the angular velocity $\boldsymbol{\Omega}_{\mathrm{a}}$ along the vertical unit vector $\mathbf{n}$. If we do so we obtain

$$\omega'_{\mathrm{a}} = \boldsymbol{\Omega}_{\mathrm{a}}\cdot\mathbf{n} = -\frac{e}{m}aB + \underbrace{\frac{\gamma}{\gamma+1}\frac{GM}{r^2c}\left(1-\frac{2GM}{rc^2}\right)^{-\frac{1}{2}}\left(\frac{\gamma}{\gamma-1}\cos\theta + \beta\sin\theta\right)}_{\delta g} = \omega_{\mathrm{a}} + \delta g\,, \tag{5.3}$$

where we restored the c and G factors. The first term is exactly the flat value for the anomalous angular velocity in the Special Relativistic framework we found in section 1.1, which it is nothing but equation (1.24) once one imposes orthogonality between $\boldsymbol{\beta}$ and $\mathbf{B}$ and puts $\gamma = \gamma_{\mathrm{m}}$. The second term represents the General Relativistic correction to the anomalous angular velocity. We are going to call this term simply δg.

Now we have to put some numbers inside equation (5.3). First of all, the observer i.e. the laboratory is on the Earth's surface so we set $r = R_\oplus$ and $m = M_\oplus$; we are dealing with a muon beam so that $e = Q_{e^-} = -1.602\,176\,620\,8(98)\times 10^{-19}\,\mathrm{C}$ and $m = m_\mu = 105.658\,374\,5(24)\,\mathrm{MeV/c^2}$. The nominal value of the magnetic field is [3] $B = 1.4513\,\mathrm{T}$ and we take the experimental result [3] $a = a_\mu^{\mathrm{E821}} = 116\,592\,080(63)\times 10^{-11}$. For the leading term ω_{a} we find the following value

$$\omega_{\mathrm{a}} = -\frac{Q_{e^-}}{m_\mu} a_\mu^{\mathrm{E821}} B = 1.439\,341\,1(8)\,\mathrm{MHz}\,, \tag{5.4}$$

where the uncertainty is given by quadratic sum of the uncertainty of each quantity and it corresponds to 0.56 ppm which reflects the noticeable fact that the anomaly has the biggest uncertainty.
Now, our z axis passes through the center of the particle's orbit ring and it is orthogonal to it, pointing outward with respect to the Planet center. Hence the angle $\theta = \Theta$ is such that $R_\oplus \sin\Theta = L$ where we named L the equilibrium orbit radius from the center of the torus. Following [3] we are going to take $\gamma = \gamma_{\mathrm{m}} = 29.3(1)$ (the maximum γ range of the stored muons is given by the experimental momentum acceptance which is [19] $\gamma_{\mathrm{m}} \pm 0.5\%$). In order to evaluate δg we split it into two parts:

$$\delta g = \underbrace{\frac{\gamma_{\mathrm{m}}}{\gamma_{\mathrm{m}}+1}\frac{GM_\oplus}{R_\oplus^2 c}\left(1-\frac{2GM_\oplus}{R_\oplus c^2}\right)^{-\frac{1}{2}}}_{1} \underbrace{\left(\frac{\gamma_{\mathrm{m}}}{\gamma_{\mathrm{m}}-1}\sqrt{1-\frac{L^2}{R_\oplus^2}}+\beta\frac{L}{R_\oplus}\right)}_{2}\,.$$

Let's begin analyzing the second contribution: we proceed by Taylor expanding in $L/R_\oplus \sim 10^{-6}$ around zero, given that $L = 7.112\,\mathrm{m}$ (nominal value, [3]) and $R_\oplus = 6.3781 \times 10^6\,\mathrm{m}$ (nominal value), which gives

$$2 = \frac{\gamma_{\mathrm{m}}}{\gamma_{\mathrm{m}}-1} + \beta\frac{L}{R_\oplus} + \mathcal{O}\left(\frac{L^2}{R_\oplus^2}\right)\,.$$

We then see that we need to keep only the constant term $\gamma_{\mathrm{m}}/(\gamma_{\mathrm{m}}-1)$. This happens because 2 has to be multiplied by[2] 1 $\sim g_T/c \sim 3\times10^{-8}\,\mathrm{Hz}$, with $g_T = GM_\oplus/R_\oplus^2$ the surface acceleration. This means that even keeping the first order term $\beta L/R_\oplus \sim 10^{-6}$ ($\beta \sim 1$) would give a contribution of 1 $\times 10^{-6} \sim 10^{-14}\,\mathrm{Hz}$ which is well below the uncertainty on the leading term (5.4). We can already foresee that the contribution δg will turn out to be very small.
We are left with the quantity

$$\delta g = \frac{\gamma_{\mathrm{m}}^2}{\gamma_{\mathrm{m}}^2-1}\frac{GM_\oplus}{R_\oplus^2 c}\left(1-\frac{2GM_\oplus}{R_\oplus c^2}\right)^{-\frac{1}{2}}\,,$$

which, recalling the definition of Earth's Schwarzschild radius $r_S = 2GM_\oplus/c^2$, can be rewritten as

$$\delta g = \frac{\gamma_{\mathrm{m}}^2}{\gamma_{\mathrm{m}}^2-1}\frac{c r_S}{2R_\oplus^2}\left(1-\frac{r_S}{R_\oplus}\right)^{-\frac{1}{2}}\,.$$

Taking for the Schwarzschild radius the value $r_S = 8.870\,056\,580(18)\,\mathrm{mm}$ we have

$$\delta g = 3.272\,20(4)\times10^{-8}\,\mathrm{Hz}\,. \tag{5.5}$$

[2]This rough estimate is given considering $\gamma_{\mathrm{m}}/(\gamma_{\mathrm{m}}\pm1) \sim 1$ for $\gamma_{\mathrm{m}} = 29.3$ and $2GM_\oplus/(R_\oplus c^2) = r_S/R_\oplus \sim 10^{-9} \ll 1$.

As we said earlier, the gravitational correction to ω_a is so small that goes beyond the uncertainty on the leading term (5.4).

Now we want to see how this result modifies the value of the anomaly a_μ by using the tools of chapter 3. In particular we recall equation (3.4)

$$a_\mu = \frac{\mathcal{R}'}{\lambda - \mathcal{R}'}\,,$$

where $\mathcal{R}' = \omega'_\mathrm{a}/\omega_{\mathrm{L}_p}$. Thus we have

$$a_\mu = \frac{\frac{\omega'_\mathrm{a}}{\omega_{\mathrm{L}p}}}{\lambda - \frac{\omega'_\mathrm{a}}{\omega_{\mathrm{L}p}}} = \frac{\frac{\omega_\mathrm{a}+\delta g}{\omega_{\mathrm{L}p}}}{\lambda - \frac{\omega_\mathrm{a}+\delta g}{\omega_{\mathrm{L}p}}} = \frac{\mathcal{R}\left(1+\frac{\delta g}{\omega_\mathrm{a}}\right)}{\lambda - \mathcal{R}\left(1+\frac{\delta g}{\omega_\mathrm{a}}\right)} = \frac{1+\frac{\delta g}{\omega_\mathrm{a}}}{\left(\frac{\lambda}{\mathcal{R}}-1\right) - \frac{\delta g}{\omega_\mathrm{a}}}\,.$$

where we have $\mathcal{R} = \omega_\mathrm{a}/\omega_{\mathrm{L}_p}$. Thus from the experimental values of λ and $\mathcal{R}$ we know that the ratio $\lambda/\mathcal{R} \sim 8\times 10^2$ and the term $(-1+\lambda/\mathcal{R}) \gg \delta g/\omega_\mathrm{a} \sim 2\times 10^{-14}$. This allows us to write

$$a_\mu \simeq \frac{\mathcal{R}}{\lambda-\mathcal{R}}\left(1+\frac{\delta g}{\omega_\mathrm{a}}\right)\left[1+\frac{\frac{\delta g}{\omega_\mathrm{a}}}{\left(\frac{\lambda}{\mathcal{R}}-1\right)}\right] \simeq \frac{\mathcal{R}}{\lambda-\mathcal{R}}\left(1+\frac{\delta g}{\omega_\mathrm{a}}\right)\,.$$

From what we said earlier we can compute the ratio which gives the contribution

$$\frac{\delta g}{\omega_\mathrm{a}} = 2.273\,40(3)\times 10^{-14}\,. \tag{5.6}$$

Considering the current precision in the measurement of a_μ is $\delta a_\mu/a_\mu = 0.54\,\mathrm{ppm}$, and noticing that the needed sensitivity in order to detect the effect of GR corrections should be at least the product between the value of the anomaly and the ratio (5.6), i.e. $\sim 1.16\times 10^{-3}\times 2\times 10^{-14} \sim 10^{-17} = 10^{-11}$ ppm, we can conclude that this correction is absolutely invisible nowadays.

Notes. We want to stress three important things.

(I) At the beginning of this section we said that the magnetic field is proportional to the vector $\mathbf{n}$ which is orthogonal to the orbit plane of the beam or, in other words, which is parallel to the z axis. In particular we do not exactly know how the vertical setup has been achieved and this opens a problem: maybe we should account for a magnetic field that is proportional to the radial unit vector $\vec{e}_{\hat{r}}$ rather than $\mathbf{n}$. This could be the case if, for instance, the experimental setup for the vertical direction of $\mathbf{B}$ has been set by means of a "plumb line". In that case the direction of $\mathbf{B}$ is not the one of $\mathbf{n}$ anymore but it becomes the one of $\vec{e}_{\hat{r}}$. In that case we mus again impose that the proportionality is the magnetic field module B, then $\mathbf{B} = B\,\vec{e}_{\hat{r}}$. Then, in that case when performing the scalar product $\boldsymbol{\Omega}_\mathrm{a}\cdot\mathbf{n}$ in equation (5.3) the leading term turns out to be proportional to $\mathbf{B}\cdot\mathbf{n} = B\cos\theta$, but we recall that $\theta = \Theta = \arcsin(L/R_\oplus)$ then

$$\mathbf{B}\cdot\mathbf{n} = B\sqrt{1-\frac{L^2}{R_\oplus^2}} = B\left[1+\mathcal{O}\left(\frac{L^2}{R_\oplus^2}\right)\right] ,$$

which, as we already said, would produce an additive term whose magnitude is well below the experimental sensitivity. Hence, even not knowing what procedure has been used, we can safely say that our computation works well in both cases as far as the current experimental sensitivity is concerned.

II During the whole computation above, we assumed that

i. $\mathbf{B}$ is always orthogonal to the muon velocity: $\mathbf{B}\cdot\boldsymbol{\beta} = 0$;

ii. the muons move exactly at the magic momentum: $\gamma = \gamma_{\mathrm{m}}$.

Clearly there are muons whose velocity is not perfectly orthogonal to **B** and also muons which are not moving precisely at the magic momentum γ_{m}. However the effects of these systematic discrepancies are analyzed at the end of chapter 3 (see section 3.4 and [14, 19] for details).

III There is no gravitational contribution affecting the ω_{L_p} term. When performing NMR the probes are standing still, thus no motion with respect to the laboratory (no relative spatial velocity β) is present for them. Since there is no motion the spin angular velocity for the probes is the Larmor angular velocity only. Following the analysis of chapter 1, a first gravitational correction stems from the expression for the cyclotron angular velocity Ω_{c}, equation (1.53), which is clearly null because the particles in the probe are not moving with respect to the laboratory. This also leads us to the identification $\Omega_{\mathrm{a}} \equiv \Omega_{\mathrm{s}} - \Omega_{\mathrm{c}} = \Omega_{\mathrm{s}}$. Moreover the gravitational correction arising from the spin precession equation (1.62) is again proportional to the spatial velocity of the particles β which is null. Again one may ask himself if the magnetic field orientation could influence the result obtained from NMR for ω_{L_p}. The same analysis we have done could be applied also in this case and the answer would be the same: given the current experimental sensitivity, such corrections are invisible.

5.2 Final remark

From the General Relativistic work of chapter 1 we were able to give a precise estimate of the gravitational correction to the anomalous angular velocity which reflects on a modification of the anomaly as it is measured in current experiments such as the E821 at BNL. What we did was to account for the observer's acceleration i.e. the one arising from his/her standing still on Earth's surface. The main result we obtained is the correction term δg to the anomalous angular velocity and the ratio between this quantity and the leading term ω_{a} which read respectively

$$\delta g = 3.272\,20(4)\times 10^{-8}\,\mathrm{Hz}\,,$$

$$\frac{\delta g}{\omega_{\mathrm{a}}} = 2.273\,40(3)\times 10^{-14}\,.$$

As expected, from a theoretical point of view the computational outcome is so tiny that the effect is overwhelmed by the experimental uncertainty. Hence we can safely say that now we are very far from being able to detect such a correction.

Conclusion

From the QFT point of view the Standard Model (SM) of particle physics predicts that, given the contributions of QED, Weak and Strong interactions, the value of the anomaly for the muon is the sum of these three contributions: $a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{QCD}}$. The most recent and precise data available right now are the one obtained at BNL, experiment E821, and gave an outcome of a_μ^{E821} with a precision of 0.54 ppm. The numerical results are the following [3, 10]

$$a_\mu^{\mathrm{SM}} = 116\,591\,776(44) \times 10^{-11}\,,$$

$$a_\mu^{\mathrm{E821}} = 116\,592\,080(63) \times 10^{-11}\,.$$

Here we gave a ful General Relativistic generalization of the Bargman–Michel–Telegdi equation which rules the evolution of a particle's spin 4-vector when it is placed in a region with an electromagnetic field. This allowed us to study the influence of the gravitational field on the anomalous magnetic moment of the muon for which the quantity a_μ, the anomaly, is measured experimentally. To be more precise, we have obtained a correction to the anomalous angular velocity. We called this quantity δg. This quantity is a purely classical result and arises from the fact that the observer is standing still on Earth's surface and thus it is an accelerated observer. Calling ω_{a} the flat Special Relativistic anomalous angular velocity, i.e the one which arises in absence of curvature, we obtain a relative correction given by

$$\frac{\delta g}{\omega_{\mathrm{a}}} = 2.273\,40(3) \times 10^{-14}\,.$$

As it can be seen in chapter 5, in order to experimentally see the GR correction to the anomaly, we call it a_μ^{GR}, the accuracy should be at least of 10^{-11} ppm. This clearly means that nowadays these effects are invisible.

So far the GR corrections to the anomaly have been neglected and with this work we can precisely say why this is fully justified, given the current experimental accuracy in determining the value of a_μ. However, what can be said at this point is that since the constant and unstoppable increasing of the experimental precision we have seen in the last decades has no reason to stop, this trend could make us able to detect such corrections in a reasonable amount of time. By that time maybe it will be useful to have a full treatment of such a problem in a formal and precise way, as we tried to achieve in the previous chapters of this dissertation. We also think that this analysis could be useful when dealing with different measurements.

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