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Evaporation of black holes in string theory

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Abstract

This thesis's project studies how black hole evaporation is captured in string theory by so called Quantum Extremal Islands, extremal surfaces that allows to compute the entanglement entropy of Hawking radiation. This project will focus on specific 10d type IIB string theory uplifts of the largely studied 5d Karch-Randall models, doubly-holographic models which allows to study the evaporation of an AdS black hole by coupling it to an external bath via transparent boundary conditions. The main results of recent works on these uplifts will be reproduced and the corresponding Hartman-Maldacena surfaces will be constructed and studied in one of these uplifts, as well as their implications for the Page curve in presence of a non-vanishing dilaton variation.

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Chapter 1

Introduction

Black holes are fascinating objects and, from the theoretical point of view, provide an intriguing arena in which to explore the challenges posed by the reconciliation of general relativity and quantum mechanics.
Beside its defining property of being a region from which no causal observer can escape from, a black hole also obeys the no-hair theorem, which states that all black holes can be completely characterized by their mass, charge and angular momentum. This already brings out some problems, since it implies that the information of whatever falls into the black hole would not be able to be distinguished from outside the event horizon.
In classical general relativity, it can be shown that black holes obey laws that are analogous to the law of thermodynamics. Thus if one considers black holes quantum-mechanically, they behave as thermodynamic objects characterized by a temperature and an entropy. The entropy, also called Bekenstein-Hawking entropy, is given by the area of the event horizon in units of Planck length squared.
Stephen Hawking discovered that black holes emit thermal radiation [**1**]. As a consequence, emitting Hawking radiation, the black hole's mass decreases, the black hole gradually evaporates and eventually disappears. This means that either information is truly lost in black holes, or that information has been preserved by Hawking radiation itself by some mechanism. In addition to the notion of thermodynamic entropy, there is another definition of entropy which is the von Neumann entropy: given a density matrix ρ describing the quantum state of the system, the von Neumann entropy is defined as $S_{vN} = -\mathrm{Tr}[\rho \log \rho]$. It quantifies our ignorance about the precise quantum state of the system, it vanishes for a pure state, it is invariant under time evolution and it is always less than the thermodynamic entropy.
In terms of this entropy we can make a more accurate statement of the information paradox. If one computes the von Neumann entropy of the entanglement of Hawking radiation, one finds that it increases as the number of emitted quanta; but after a black hole evaporates, all that is left is the radiation which is still entangled and so is in a mixed state, in contrast with the fact that the black hole could have been formed from a pure state. But according to unitarity, it is impossible that a pure state evolve into a mixed state, hence the paradox.
The violation of unitarity by the evaporation of the black hole can be explicitly seen by the fact that as the black hole evaporates, the area of the event horizon decreases and the Bekenstein-Hawking entropy decreases as well since it's proportional to the area of the horizon. Due to the entanglement of the Hawking radiation, the von Neumann entropy of the black hole and the one of Hawking radiation itself must always be equal but we run into problem when the increasing von Neumann entropy of Hawking radiation becomes larger than the Bekenstein-Hawking entropy of the black hole (which is always larger than the its von Neumann entropy). Thus, this means that in order to preserve unitarity, at a certain point, called Page time, the entropy of the radiation must begin decreasing, following the so called Page curve [**2**].
Ryu and Takayanagi conjectured a holographic entanglement entropy proposal [**3**], according to which the entanglement entropy of a region of a conformal field theory can be calculated as the area of a codimension two, spacelike extremal surface living in dual AdS theory of gravity.

The proposal has been conjectured firstly in the static case, and then it was derived in time-dependent geometries as well with contributions by Hubeny [**4**]. The static formula, and the time-dependent formula in certain special states, were then derived through the AdS/CFT dictionary by Lewkowycz and Maldacena[**5**]. Subsequently, it was found a quantum generalization of the holographic entanglement entropy prescription[**6**],[**7**],[**8**],[**9**] and this generalization was crucial for a deeper understanding of the calculation of the entropy of Hawking radiation. In particular it was identified an additional contribution to the entanglement entropy of Hawking radiation, namely the islands, regions physically disconnected from the asymptotic radiation region, which turns out to become dominant at late times, leading to a Page curve and thus ensuring the unitary evolution of black hole's evaporation.
Islands have been firstly studied in lower dimensional gravity [**10**], [**11**] and then the first extensions to higher-dimensional setups have appeared in [**12**],[**13**],[**14**],[**15**]. These higher-dimensional setups are essentially based on the Karch-Randall models [**16**],[**17**],[**18**], models of localized gravity which had been firstly constructed and studied between the '90s and the beginning of the new millennium and that have gained new vigor in the last years because they reveal of crucial utility in order to reproduce the Page curve using holography ideas. These models are based on a bulk AdS_5 geometry with a 4d end-of-the-world (ETW) brane extending on an AdS_4 slice, cutting off a part of the AdS_5 bulk geometry and ending on its conformal boundary. The importance of these models in the information paradox context is that they are doubly holographic models: namely they have three different equivalent descriptions which are related to each other by applying the AdS/CFT correspondence twice; in particular in one of these descriptions the model can be viewed as a conformal field theory coupled with a 4-dimensional massive graviton localized on the brane, in turn coupled via transparent boundary conditions to a second conformal field theory living on the uncut-off part of the AdS_5 conformal boundary, which will be called the bath system. Thus if we imagine that an AdS_4 black hole lives on the brane, it will evaporate since the emitted radiation reaching the AdS_4 conformal boundary will be able to escape on the bath system thanks to the transparent boundary conditions. Otherwise an isolated black hole in an asymptotically AdS space would not evaporate because the AdS geometry acts like a box.
The doubly holographic nature of these models ensure also that the quantum extremal surfaces contributing to the entanglement entropy of radiation region, taken as a subsystem of the thermal bath far from the defect, are completely geometrized. In particular the possible quantum extremal surfaces living in the bulk AdS_5 geometry start from the boundary of radiation region and either drop into the horizon or end on the end-of-the-world brane. The first kind of quantum extremal surfaces are called Hartman-Maldacena surface [**19**], and it actually connects the boundary of the radiation region to the boundary of its copy on the second conformal boundary, if we consider the full Penrose diagram of the bulk AdS_5 Schwarzschild black hole. It connects them stretching through an Einstein-Rosen bridge and its area grows linearly in time. On the other side, the second type of extremal surface is the island surface, which stretches outside the horizon and thus its area is constant in time. The competition between these surfaces was proved to lead to a Page curve.
The Karch-Randall models can be seen in one of their three descriptions as boundary CFTs. Holographic string theory configurations realizing interface and defect conformal field theories have been constructed by D'Hoker *et al* [**20**],[**21**] as type IIB string theory uplifts. These type IIB string theory solutions based on $AdS_4 \times S^2 \times S^2 \times \Sigma$, with Σ a Riemann surface, turns out to be completely determined by two harmonic functions on the Riemann surface Σ.
We are going to investigate some examples of these solutions, such as the global $AdS_5 \times S^5$, the supersymmetric Janus solution and the 5-brane doped Janus solution, and most importantly we are going to show how is possible to obtain the 10d uplifts of Karch-Randall models with a suitable choice of the pair of harmonic functions. In particular, the Karch-Randall models with non-gravitating bath, the ones discussed briefly above, can be described by an uplift corresponding to semi-infinite D3-branes terminating on a system of D5- and NS5-branes with suspended D3-branes among the 5-branes. The Karch-Randall models with gravitating bath [**14**] can be

obtained by adding a second end-of-the-world brane intersecting the conformal boundary of the bulk AdS_5 geometry at the same point of the first ETW brane and their corresponding uplift can be engineered with a system of intersecting D5, NS5 and D3-branes without semi-infinite D3-branes since with the additional ETW brane the Karch-Randall geometry becomes compact without any asymptotic AdS_5 region.

Uhlemann **[22]** constructed and studied quantum extremal surfaces on these uplifts, in particular in the uplifts with non-gravitating bath he found that it exists a 10d counterpart of the brane angle of 5d Karch-Randall models, which will be the ratio of suspended and semi-infinite D3-branes, and he numerically found that, as in the Karch-Randall models it exists a critical angle at zero temperature below which islands cease to contribute, it exists a critical value of the ratio within which islands are constrained to exist at zero temperature. Furthermore, he was able to identify 10d versions of the "left/right entanglement entropy" that was found to exhibit a Page curve behaviour in 5d Karch-Randall models with gravitating bath.

Alessandra Gnecchi *et al* studied island surfaces in the same Uhlemann's uplift with non-gravitating bath but allowing a non-vanishing dilaton variation **[23]**, discussing their properties both in relation to the existence of a massive graviton in the same geometric background and to the varying dilaton itself.

After reproducing the main results of both works, the original part of this thesis's project is to construct and study the Hartman-Maldacena surfaces in the same geometric background considered by Gnecchi *et al* with non-vanishing dilaton variation, in particular extending the results obtained by Uhlemann about the area difference between island and Hartman-Maldacena surfaces. Then we try to study the quantum extremal surfaces in a particular solution with gravitating bath and with non-vanishing dilaton variation, even if in the case considered the results will reveal to be trivial and identical to ones of the original solution with vanishing dilaton variation considered by Uhlemann.

This thesis's project is structured as follows. In Chapter 2, we are going to briefly introduce type II superstring theories, focusing at the end on type IIB supergravity since it will be central for our work. In Chapter 3, we describe the other extended objects which exist in type II superstring theories, specifically the D-branes and the NS5-brane, which will be the main ingredients in the brane constructions corresponding to the type IIB geometric backgrounds we are going to work on. In Chapter 4, we review the AdS/CFT correspondence, in particular the motivation of the correspondence between $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions and type IIB superstring theory on $AdS_5 \times S^5$ and the holographic dictionary, finishing the chapter with a discussion on the entanglement entropy and on Ryu and Takayanagi's holographic entanglement entropy proposal. In Chapter 5, we discuss the information paradox and the island prescription, following a brief review of Karch-Randall models with non-gravitating and gravitating bath. In Chapter 6, we discuss the type IIB string theory solutions based on $AdS_4 \times S^2 \times S^2 \times \Sigma$ background geometry found by D'Hoker *et al*, in particular we study the explicit local solution, the global regularity and topology constraints and then many specific examples, like the supersymmetric Janus solution with and without 5-brane sources, the holographic duals of 3d $\mathcal{N} = 4$ SCFTs found by Gaiotto and Witten **[24]**,**[25]** and the uplifts of both Karch-Randall models with non-gravitating and gravitating bath. In Chapter 7, we discuss massive AdS gravity, its embedding in string theory and its range of validity as an effective field theory, but more importantly we are going to show how in the discussed type IIB string theory solutions the lowest-lying spin-2 mode can acquire a tiny mass as a consequence of the no-compactness of the internal manifold **[26]**. Finally the Chapter 8 regards all the work mentioned before about quantum extremal surfaces in the 10d string theory uplifts.

Chapter 2

Superstring theory

The geometric background on which we are going to study the evaporation of black holes and the entanglement entropy of Hawking radiation will be a particular type IIB superstring theory solution. For this reason it's worthwhile to briefly review some basic aspects of superstring theory**[27]**,**[28]**,**[29]**,**[30]**, focusing then on the type IIB superstring theory and on its low energy limit, i.e. the type IIB supergravity action, which describes only the massless modes of the corresponding superstring theory.

The bosonic string theory suffers of two important shortcomings, namely the bosonic string spectrum contains a tachyonic ground state and it exhibits only bosonic excitations. Thus the vacuum instability due to the tachyonic ground state and the lack of fermionic excitations makes the bosonic string quite unrealistic.
Both of these problems are solved in superstring theory. The basic idea is that one adds fermions on the worldsheet by introducing supersymmetry, and their contribution to the zero-point energy on the worldsheet will cancel exactly the negative zero-point energy of the bosonic string, which originally caused the appearance of the tachyonic ground state, solving in this way both the shortcomings mentioned above. The resulting theory will be a tachyonic-free supersymmetric theory in $d = 10$ dimensions consisting of both a bosonic sector, which will be identical to the worldsheet theory of the bosonic string, and a fermionic one.

There exist two major formulations of a superstring theory. Both of them enjoy supersymmetry on the world-sheet and in spacetime, but in the **Ramond-Neveu-Schwarz** (RNS) formulation supersymmetry is manifest only on the world-sheet and in the **Green-Schwarz** (GS) formulation it is manifest only in spacetime. In the following we are going to use the RNS formalism.
The superstring world-sheet action in flat gauge, i.e. the intrinsic world-sheet metric is set to be the flat one $h_{\alpha\beta} = \eta_{\alpha\beta}$, is

$$S = -\frac{1}{4\pi}\int_{\mathcal{M}} d^2\xi(\alpha'^{-1}\partial_\alpha X^\mu\partial^\alpha X_\mu + i\bar{\psi}^\mu_A\gamma^\alpha_{AB}\partial_\alpha\psi_{\mu B}) \tag{2.1}$$

where X^μ are worldsheet scalars, the first piece is the bosonic string worldsheet action in flat gauge and $\psi^\mu_A = (\psi^\mu_+, \psi^\mu_-)^T$ are 2-dimensional worldsheet spinors. α' is related to the string length l_s by $\alpha' = l_s^2$ and to the tension of the fundamental string as $\tau_{F1} = 1/(2\pi\alpha')$. The string world-sheet is the strip $\mathcal{M} = \{(\tau,\sigma)|0 \le \sigma \le \pi, \tau \in \mathbb{R}\}$ and on the two-dimensional world-sheet with flat metric $\eta_{\alpha\beta}$ the Clifford algebra is generated by real two-dimensional γ-matrices with anti-commutation relations

$$\{\gamma^\alpha, \gamma^\beta\}_{AB} = 2\eta^{\alpha\beta}\mathbb{I}_{AB} \tag{2.2}$$

here A, B are spinor indices on the world-sheet and α, β are vector indices on the world-sheet, $\alpha, \beta = 0, 1$.

We assume the worldsheet spinors ψ_A to satisfy the Majorana condition $\psi_A^* = \psi_A$. [1]
In light-cone coordinates $\xi^\pm = \tau \pm \sigma$, (2.1) becomes

$$S = S_B + S_F = \frac{1}{2\pi}\int d^2\xi \frac{2}{\alpha'}\partial_+ X \cdot \partial_- X + \frac{i}{2\pi}\int d^2\xi(\psi_+ \cdot \partial_-\psi_+ + \psi_- \cdot \partial_+\psi_-) \tag{2.4}$$

and this action is invariant under the following supersymmetry transformations

$$\begin{cases} \delta X^\mu = i\sqrt{\frac{\alpha'}{2}}\bar{\varepsilon}\psi^\mu \\ \delta\psi^\mu = \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\gamma^\alpha\partial_\alpha X^\mu \cdot \varepsilon \end{cases} \tag{2.5}$$

where ε is an infinitesimal constant Majorana spinor.
The equations of motion for $X^\mu(\tau,\sigma)$ are given by a relativistic wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = \partial_+\partial_- X^\mu = 0 \tag{2.6}$$

and they must be supplemented by the vanishing of the σ-boundary term[2]

$$\partial_\sigma X^\mu \delta X_\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0 \tag{2.7}$$

Therefore we must impose boundary conditions such that these boundary terms vanish. The consistent boundary conditions are

- Closed string: the boundary terms at $\sigma = 0$ and $\sigma = \pi$ must cancel each other, i.e. this is equivalent to impose periodic boundary conditions defining a closed string

$$X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = \pi) \tag{2.8}$$

- Open string: the boundary terms at $\sigma = 0$ and $\sigma = \pi$ have to vanish separately. Thus for each X^μ, we must impose either Neumann boundary conditions $\partial_\sigma X^\mu = 0$ or Dirichlet boundary conditions $\delta X^\mu = 0$ on one or both string endpoints. Dirichlet boundary conditions for X^μ correspond to string endpoint being fixed in the μ-direction. Thus for each direction the possible combinations of boundary conditions at the endpoints are (NN), (DD), (DN), (ND).

The equation of motion (2.6) can be solved by splitting $X^\mu(\tau,\sigma)$ into left- and right-moving modes depending only on ξ^+ and ξ^- respectively, i.e. $X^\mu(\tau,\sigma) = X_L^\mu(\xi^+) + X_R^\mu(\xi^-)$, which can be decomposed into a Fourier series of the form

$$\begin{aligned} X_R^\mu(\tau - \sigma) &= \frac{x^\mu - \tilde{x}^\mu}{2} + \alpha'\tilde{p}^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)} \\ X_L^\mu(\tau + \sigma) &= \frac{x^\mu + \tilde{x}^\mu}{2} + \alpha' p^\mu(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau+\sigma)} \end{aligned} \tag{2.9}$$

where p^μ and $\tilde{p}^\mu$ are the momenta of the modes and in order to ensure the reality of X^μ the Fourier modes must satisfy $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ and $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$.

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[1] Suppose we choose the following basis of two-dimensional γ-matrices satisfying (2.2)

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

The Majorana condition is that the spinor ψ is equal to its charge conjugate spinor $\psi^c = C\bar{\psi}^T = iC\gamma^{0T}\psi^* = \psi^*$ since the charge conjugation matrix is $C = i\gamma^0$ due to the fact that for a Majorana spinor the Dirac conjugate must satisfy $\bar{\psi} = \psi^\dagger i\gamma^0 = \psi^T C$. Thus in this case the Majorana condition is a reality condition because of the fact that the γ-matrices are real.

[2] The τ-boundary term vanishes by the assumption $\delta X^\mu(\tau = \pm\infty) = 0$

In the case of closed strings, the left- and right moving modes satisfy the periodic boundary conditions if we set $p^\mu = \tilde{p}^\mu$. The $\tilde{x}^\mu$ term cancels in the sum $X_L^\mu(\xi^+) + X_R^\mu(\xi^-)$, while x^μ represents the center of mass position of the string at $\tau = 0$ since

$$\frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu = x^\mu + 2\alpha' p^\mu \tau \tag{2.10}$$

while p^μ represents the total spacetime momentum of the string

$$\int_0^\pi d\sigma\, \Pi^\mu(\tau,\sigma) = p^\mu \tag{2.11}$$

where $\Pi^\mu(\tau,\sigma) = \partial_\tau X^\mu(\tau,\sigma)/(2\pi\alpha')$ is the canonical momentum.
In the case of open strings with Neumann-Neumann (NN) boundary conditions for the target spacetime coordinate X^μ, the solution reads

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma) \tag{2.12}$$

notice that the boundary conditions relate X_L^μ and X_R^μ in such a way the independence of left- and right-moving oscillators drops, namely $\alpha_n^\mu = \tilde{\alpha}_n^\mu$.
In the case of Dirichlet-Dirichlet (DD) boundary conditions on X^μ, $X^\mu(\tau,0) = x_i^\mu$ and $X^\mu(\tau,\pi) = x_f^\mu$, we get

$$X^\mu(\tau,\sigma) = x_i^\mu + \frac{1}{\pi}(x_f^\mu - x_i^\mu)\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma) \tag{2.13}$$

The equation of motion of the world-sheet spinors is the Dirac equation, describing left- and right-moving waves

$$\partial_+\psi_-^\mu = \partial_-\psi_+^\mu = 0 \tag{2.14}$$

provided that the boundary terms of the fermionic action vanish

$$\delta S_F \sim \int d\tau[\psi_+\delta\psi_+ - \psi_-\delta\psi_-]_{\sigma=0}^{\sigma=\pi} \tag{2.15}$$

and, as in the bosonic case, there are two different types of strings satisfying the boundary conditions, namely open and closed strings.

Open superstrings

In the case of open strings, the boundary terms at $\sigma = 0$ and $\sigma = \pi$ must vanish individually and this implies there are two possible Lorentz invariant boundary conditions on world-sheet fermions:

$$\begin{array}{rll} \mathrm{R}: & \psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0) & \psi_+^\mu(\tau,\pi) = \psi_-^\mu(\tau,\pi) \\ \mathrm{NS}: & \psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0) & \psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi) \end{array} \tag{2.16}$$

The Ramond sector (R) corresponds to periodic boundary conditions with integer modes

$$\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau\pm\sigma)} \tag{2.17}$$

whereas the Neveu-Schwartz sector (NS) corresponds to anti-periodic boundary conditions and half integer modes

$$\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau\pm\sigma)} \tag{2.18}$$

where d_n^μ and b_r^μ are Grassmann-valued Fourier modes and the Majorana condition implies that $d_n^\mu = (d_{-n}^\mu)^\dagger$ and $b_r^\mu = (b_{-r}^\mu)^\dagger$.

It can be proved in several ways, for example through the requirement of the absence of negative-norm states in the Fock space of the quantized theory or the requirement of Lorentz invariance of theory in the light-cone gauge, that the critical dimension of a superstring is $d = 10$, unlike the bosonic string whose critical dimension is $d = 26$.
In the NS sector the ground state is a spacetime scalar and is tachyon, whereas the first level excitations form a transverse massless vector $\mathbf{8}_v$ of $SO(8)$.
In the R sector the ground state energy always vanishes because the world-sheet bosons and their superconformal partners have the same moding. The Ramond vacuum is degenerate, since both the ground state itself $|0\rangle_R$ and $d_0^\mu |0\rangle_R$ result to be annihilated by all positive modes. In particular, the Ramond zero-modes satisfy the (ambient spacetime) Clifford algebra

$$\{d_0^\mu, d_0^\nu\} = \frac{1}{2}\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu} \tag{2.19}$$

so the ground state $|0\rangle_R$ is a $Spin(9,1)$ spinor. Being a 10-dimensional spinor, it has $2^{d/2} = 32$ complex components, which actually are real as a consequence of the Majorana condition $d_n^\mu = (d_{-n}^\mu)^\dagger$. In 10 dimensions Majorana spinors can be decomposed further into Weyl spinors, corresponding to the splitting

$$32 = 16 \oplus 16' \tag{2.20}$$

where the prime denotes negative chirality Weyl spinors.
Imposing that the ground state satisfies the Dirac equation, half of the components of $Spin(9,1)$ spinor are eliminated leaving a $Spin(8)$ spinor. The two inequivalent spinor representations of $Spin(8)$, which describe spinors of opposite chirality with eight real components each, are denoted by $\mathbf{8}_s$ and $\mathbf{8}_c$.

The spectrum described above has several problems. First of all, in the NS sector the ground state is a tachyon, a particle with imaginary mass. Secondly, the spectrum is not space-time supersymmetric, since for example there is no fermion in the spectrum with the same mass as the tachyon.
It is possible to turn the RNS string theory into a consistent theory by truncating (or projecting) the spectrum in a such a way it results to be tachyon-free and space-time supersymmetric. This projection is called GSO projection.
First of all, let us define the G-parity operator. In the NS sector it has the following form

$$G = (-1)^{F+1} = (-1)^{\sum_{r=1/2}^{\infty} b_{-r}^i b_r^i + 1} \tag{2.21}$$

where F is world-sheet fermion number. While in the R sector the G-parity operator is defined as

$$G = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d_{-r}^i d_r^i} \tag{2.22}$$

where

$$\Gamma_{11} = \Gamma_0 \Gamma_1 ... \Gamma_9 \tag{2.23}$$

is the chirality operator satisfying

$$(\Gamma_{11})^2 = 1, \qquad \{\Gamma_{11}, \Gamma^\mu\} = 0, \qquad \Gamma_{11}\psi = \pm\psi \tag{2.24}$$

and spinors satisfying the last equation are said to have positive or negative chirality respectively.
The GSO projection consists of keeping only states with a positive G-parity in the NS-sector, whereas in the R sector one can project on states with positive or negative G-parity depending on the chirality of the spinor ground state.
The GSO projection eliminates the open-string tachyon from the spectrum, since it has negative G-parity. The first excited state in the NS sector has positive G-parity, so the massless vector boson becomes the ground state of the projected NS sector, matching nicely with the fact that the ground state in the fermionic sector is a massless spinor. This is a first indication that the spectrum is space-time supersymmetric after the GSO projection, reinforced by noticing that

the NS sector transverse massless vector boson has eight physical polarization states whereas the R ground state is a $\mathbf{8}_s \oplus \mathbf{8}_c$ spinor representation with 16 degrees of freedom, but after the GSO projection only one of the two chiral representation survives with eight spinor degrees of freedom as well.
The ground state for the open string spectrum is the 16-dimensional multiplet $\mathbf{8}_v \oplus \mathbf{8}_c$, if the $\mathbf{8}_s$ chiral spinor representation is suppressed in the GSO projection and viceversa.

Closed superstrings

The closed superstring spectrum is the product of two copies of the open superstring spectrum, with right- and left-moving levels matched.
In the open string spectrum the two choices for the GSO projection were equivalent, but in closed string there are two inequivalent choices, taking the same or opposite projections on the two sides. These give rise to two closed oriented superstring theories, respectively type IIB and type IIA superstring theories.
Thus the massless ground states are given by

$$\begin{aligned} \text{IIA}: &\qquad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \\ \text{IIB}: &\qquad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \end{aligned} \tag{2.25}$$

Both type II theories have the same field content in the NS-NS sector, namely

$$\mathbf{8_v} \oplus \mathbf{8_v} = \Phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \tag{2.26}$$

which are a scalar field called dilaton, an antisymmetric two-form gauge field called Kalb-Ramond field and a symmetric traceless rank-two tensor, the graviton.
In the R-R sector, the IIA and IIB spectra are respectively

$$\begin{aligned} \mathbf{8_s} \oplus \mathbf{8_c} &= [1] + [3] = \mathbf{8_v} \oplus \mathbf{56_t} \\ \mathbf{8_s} \oplus \mathbf{8_s} &= [0] + [2] + [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ \end{aligned} \tag{2.27}$$

here $[n]$ denotes a fully antisymmetric n-tensor, i.e. $[n] \equiv C^{(n)}$ with components $C^{(n)}_{[\mu_1 \cdots \mu_n]}$, with $[4]_+$ denoting the part of the 4-form field having a self-dual field strength with respect to the Hodge $*$-duality. The NS-NS and R-R spectra together form the bosonic components of $d = 10$ type II supergravity theories.
In the NS-R and R-NS sectors the tensor products are

$$\begin{aligned} \mathbf{8_v} \oplus \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c} \\ \mathbf{8_v} \oplus \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s} \end{aligned} \tag{2.28}$$

each sector contains respectively a spin 1/2 dilatino and a spin 3/2 gravitino. In type IIA theory they have opposite chirality, whereas in type IIB the same.

In the ′80s and ′90s, three further consistent superstring theories, known as type I and heterotic superstring theories, with gauge groups $SO(32)$ and $E_8 \times E_8$ have been discovered, and they turn out to be connected with each other and with type II theories by a web of dualities, but from now on we will focus only on type II superstring theories.

Type IIB supergravity

Since we are going to study an uplift of an evaporating black hole model in a type IIB string theory background, let us focus in more detail in this superstring theory and consider in particular the type IIB supergravity, whose field content is given by the massless spectrum of the type IIB superstring. As we have just discussed, the fermionic spectrum consists of two left-handed Majorana-Weyl gravitinos and two right-handed Majorana-Weyl dilatinos, the NS-NS bosons

are given by the metric, the two-form B_2 with field strength H_3 and the dilaton Φ, and finally the R-R bosonic sector consists of the form fields C_0, C_2 and C_4, with the latter must having a self-dual field strength.
Writing down the action of type IIB supergravity presents an issue due to the fact that the term[3]

$$\int d^{10}x|F_5|^2 \tag{2.29}$$

does not incorporate the self-duality condition, thus it describes twice the actual number of propagating degrees of freedom. There are several different ways to deal with this issue, for example one approach consists in not constructing the action but only the field equations and the supersymmetry transformations, whereas another approach, the Pasti-Sorokin-Tonin one, consists in building a manifestly covariant action that reproduces the right number of degrees of freedom by the introduction of an auxiliary scalar field with a compensating gauge symmetry.

Here for simplicity we decide to write an action that gives the correct equations of motion once we impose the self-duality constraint, but which does not satisfy supersymmetry for the reasons we have explained above.
The bosonic part of this action takes the form

$$S = S_{NS} + S_R + S_{CS} \tag{2.30}$$

with

$$\begin{gathered} S_{NS} = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\Big(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\Big) \\ S_R = -\frac{1}{4k^2}\int d^{10}x\sqrt{-g}\Big(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\Big) \\ S_{CS} = -\frac{1}{4k^2}\int C_4\wedge H_3\wedge F_3 \end{gathered} \tag{2.31}$$

where $2k^2 = (2\pi)^7 l_s^8$, $F_{n+1} = dC_n$ as usual and

$$\begin{gathered} \tilde{F}_3 = F_3 - C_0 H_3 \\ \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3 \end{gathered} \tag{2.32}$$

and self-duality is imposed at the level of equation of motions as

$$\tilde{F}_5 = *\tilde{F}_5 \tag{2.33}$$

[3]here we use the convention that $|F_n|^2 = \frac{1}{n!}g^{\mu_1\nu_1}g^{\mu_2\nu_2}...g^{\mu_n\nu_n}F_{\mu_1\mu_2...\mu_n}F_{\nu_1\nu_2...\nu_n}$

Chapter 3

T-duality, D-branes and the NS5-brane

String theory is not only a theory of fundamental one-dimensional strings, but there are also a variety of extended objects, called branes, of different dimensionalities, whose stability depends on the theory considered and its vacuum configuration. One of these are the D-branes[**31**],[**32**], defined as the objects on which open strings end, and one way to motivate their necessity is based on T-duality[**33**], a transformation that relates two different string theories.
Another interesting object is the NS5 brane, which is neither a string or a D-brane, rather it is a solitonic solution to the type IIB supergravity field equations as we will see. The NS5 brane is related to the D5-brane through a weak-strong duality and the reason of prefix "NS" is due to the fact that it couples magnetically to the NS-NS 2-form $B_{\mu\nu}$, as opposed to the D-branes which couple to R-R form fields.

T-duality of closed and open strings

Closed strings

One of the simplest examples to which T-duality applies is the bosonic string in a space-time geometry $\mathbb{R}^{24,1} \times S^1$, so the string target space is a 26-dimensional Minkowski space with one spatial dimension compactified on a circle of radius R. Firstly let us consider the case of a closed bosonic string: we take periodic boundary conditions for the compactified coordinate that we suppose to be X^{25}

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi RW, \quad W \in \mathbb{Z} \tag{3.1}$$

where W is the winding number, which indicates the number of times the string winds around the circle.
The mode expansion of the coordinates X^μ, for $\mu = 0, ..., 24$,

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + +i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau+\sigma)} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)} \tag{3.2}$$

does not change compared to the mode expansion in a flat 26-dimensional Minkowski space, whereas the mode expansion of X^{25} changes to

$$X^{25} = x^{25} + 2\alpha' p^{25}\tau + 2RW\sigma + ... \tag{3.3}$$

with an additional linear term in σ in order to satisfy the boundary condition (3.1). The oscillation mode terms remain untouched under the compactification, thus they will be denoted by dots from now on.
Since X^{25} is periodic, from quantum mechanics we know that the momentum eigenvalue p^{25} must be quantized, in particular it is of the form

$$p^{25} = \frac{K}{R}, \quad K \in \mathbb{Z} \tag{3.4}$$

where the integer K is called the Kaluza-Klein excitation number.
Splitting the expansion into left- and right-movers $X^{25}(\sigma,\tau) = X^{25}_L(\sigma+\tau) + X^{25}_R(\sigma-\tau)$, we obtain

$$\begin{aligned} X^{25}_R &= \frac{1}{2}(x^{25} - \tilde{x}^{25}) + \Big(\alpha'\frac{K}{R} - WR\Big)(\tau-\sigma) + ... \\ X^{25}_L &= \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \Big(\alpha'\frac{K}{R} + WR\Big)(\tau+\sigma) + ... \end{aligned} \tag{3.5}$$

where $\tilde{x}^{25}$ is a constant that cancels in the sum. The usual level-matching condition $N_L = N_R$ and the mass shell relation get modified as follows

$$\begin{aligned} N_R - N_L &= WK \\ \alpha' M^2 &= \alpha'\Big[\Big(\frac{K}{R}\Big)^2 + \Big(\frac{WR}{\alpha'}\Big)^2\Big] + 2N_L + 2N_R - 4 \end{aligned} \tag{3.6}$$

where $N_{L,R}$ are the left- and right-moving excitations number operators.
The above equations are invariant under the interchange $W \leftrightarrow K$, provided that one simultaneously sends $R \to \tilde{R} = \alpha'/R$. This equivalence is called T-duality. In this case, T-duality maps two theories of the same type (one compactified on a circle of radius R and one on a circle of radius $\tilde{R} = \alpha'/R$) into one another and the interchange $W \leftrightarrow K$ means that momentum excitations in one description correspond to winding-mode excitations in the dual description and vice versa.
Notice that this transformation can be expressed equivalently as

$$X^{25}_R \to -X^{25}_R \qquad \text{and} \qquad X^{25}_L \to X^{25}_L \tag{3.7}$$

so the mode expansion of $X^{25}(\sigma,\tau)$ is mapped into

$$\tilde{X}^{25}(\sigma,\tau) = X^{25}_L(\sigma+\tau) - X^{25}_R(\sigma-\tau) = \tilde{x}^{25} + 2\alpha'\frac{K}{R}\sigma + 2RW\tau + ... \tag{3.8}$$

here the coordinate x^{25}, which parametrizes the original circle with periodicity $2\pi R$, has been replaced by the coordinate $\tilde{x}^{25}$, which parametrizes the dual circle with periodicity $2\pi\tilde{R}$, as its conjugate momentum is $\tilde{p}^{25} = RW/\alpha' = W/\tilde{R}$.

Open strings

Let us consider the case of a T-duality transformation applied to a theory containing open strings.
The first thing to notice is that open strings can always be contracted to a point, so they can always be unwound from the periodic dimension and then the winding number is not meaningful in this case. In order to find the T-dual of an open string with Neumann boundary conditions, consider first of all the mode expansion for a space-time coordinate with Neumann boundary conditions

$$X(\tau,\sigma) = x + 2\alpha' p\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}\cos(n\sigma) \tag{3.9}$$

with right- and left-movers mode expansion

$$\begin{aligned} X_R(\tau-\sigma) &= \frac{x-\tilde{x}}{2} + \alpha' p(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau-\sigma)} \\ X_L(\tau+\sigma) &= \frac{x+\tilde{x}}{2} + \alpha' p(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)} \end{aligned} \tag{3.10}$$

Compactifying the corresponding space-time dimension on a circle of radius R and carrying out a T-duality transformation along it gives once again

$$X_R \to -X_R \qquad \text{and} \qquad X_L \to X_L \tag{3.11}$$

and the dual coordinate becomes

$$\tilde{X}(\tau,\sigma) = X_L - X_R = \tilde{x} + 2\alpha' p\sigma + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}\sin(n\sigma) \tag{3.12}$$

from this mode expansion, we can conclude that the dual open string has no momentum in the circular direction, since there is no linear term in τ in the mode expansion of $\tilde{X}$; moreover at $\sigma = 0, \pi$ the position of the string is fixed, since the oscillator term vanish

$$\tilde{X}(\tau,0) = \tilde{x} \quad \text{and} \quad \tilde{X}(\tau,\pi) = \tilde{x} + \frac{2\pi K\alpha'}{R} = \tilde{x} + 2\pi K\tilde{R} \tag{3.13}$$

where it has been used $p = K/R$ and $\tilde{R} = \alpha'/R$ for the dual radius.
Therefore T-duality transforms a bosonic open string with Neumann boundary conditions on a circle of radius R to a bosonic open string with Dirichlet boundary conditions on a circle of radius $\tilde{R} = \alpha'/R$; furthermore the original open string that has momentum and no winding in the circular direction is T-dualized into an open string which has winding but no momentum in the dual circular direction.
Notice that in this case the winding mode is topologically stable because the end-points of the dual open string are fixed by Dirichlet conditions, so the string cannot unwind without breaking.

The hyperplane $\tilde{X} = \tilde{x}$ to which the end points of the dual string are attached is an example of D-brane (where D stands for Dirichlet and brane is an abbreviation of membrane), which is defined as a hypersurface on which an open string can end. More specifically one uses the term Dp-brane, where p denotes the number of spatial dimensions of the brane. In the example we discussed above, the open string with Dirichlet boundary conditions on the dual circle and Neumann boundary conditions in all the remaining directions end on a D24-brane. In general it is possible to compactify n directions on n circles (or on a n-torus) and then perform T-duality transformations in these directions, obtaining dual strings with Dirichlet conditions in the n dual directions and ending on a D$(25-n)$-brane.

Therefore if we consider a set-up of D-branes of various dimensions, in the T-dual formulation these D-branes are getting replaced by D-branes such that:

- if a D-brane wraps a circle that is T-dualized, then the T-dual D-brane doesn't wrap the T-dual circle and vice versa;
- if T-duality transformation is performed on a direction tangent to a Dp-brane, the T-dual brane is a D$(p-1)$-brane, otherwise it is D$(p+1)$-brane if the direction is orthogonal to the original Dp-brane.

T-duality of type II superstrings

Let us examine in the following T-duality transformations for type II superstring theories.
Suppose that the X^9 coordinate of a type II theory is compactified on a circle of radius R and that a T-duality transformation is carried out for this coordinate. The transformation of bosonic coordinates is the same as for the bosonic string, namely

$$X_L^9 \to X_L^9 \qquad \text{and} \qquad X_R^9 \to -X_R^9 \tag{3.14}$$

which interchanges momentum and winding numbers. In the RNS formalism, world-sheet supersymmetry (2.5) requires the world-sheet fermion ψ^9 to transform in the same way as its bosonic superpartner X^9

$$\psi_L^9 \to \psi_L^9 \qquad \text{and} \qquad \psi_R^9 \to -\psi_R^9 \tag{3.15}$$

in particular the zero mode of ψ_R^9 in the Ramond sector transforms $d_0^9 \to -d_0^9$ and, recalling the relation between R-sector zero modes and 10-dimensional Dirac matrices $\Gamma^\mu = \sqrt{2}d_0^\mu$, we conclude that the chirality operator, under a T-duality transformation, behaves as

$$\Gamma_{11} = \Gamma^0\Gamma^1 ... \Gamma^9 \to -\Gamma_{11} \tag{3.16}$$

since, except Γ^9, Dirac matrices stay unchanged under the T-duality transformation.
This implies that after T-duality the chirality of the right-moving Ramond-sector ground state is reversed. The relative chirality of the left-moving and right-moving ground states is what distinguishes the type IIA and type IIB theories. Since only one of these is reversed, it follows that the T-dual of a type IIA theory compactified on a circle of radius R is a type IIB theory compactified on a circle of radius $\tilde{R} = \alpha'/R$ and vice versa.
If several directions are compactified on circles, it is possible to carry out several T-duality transformations along each of these circular directions. In this case an even number of transformations gives back on the dual torus the same type II theory that one started with.
T-duality of type IIA and type IIB superstring theory is a perturbative duality, holding order by order in string perturbation expansion. To prove this, consider the NS-NS part of the low-energy effective action of a type IIA theory, which has the form

$$\frac{1}{g_s^2}\int d^{10}x\mathcal{L}_{NS} \tag{3.17}$$

being the string coupling g_s related to the vacuum expectation value of the dilaton, namely $g_s = e^{\phi_0}$.
The NS-NS part of type IIB supergravity action has the same form with type IIB string coupling $\tilde{g}_s$. Compactifying both type IIA and IIB theory on circles of radii respectively R and $\tilde{R}$, keeping only the zero-mode contributions on the circles, the NS-NS actions become

$$\text{(IIA)}\quad \frac{2\pi R}{g_s^2}\int d^9x\mathcal{L}_{NS} \qquad\qquad \text{(IIB)}\quad \frac{2\pi\tilde{R}}{\tilde{g}_s^2}\int d^9x\mathcal{L}_{NS} \tag{3.18}$$

Taking into account that these two theories are related by the T-duality identification $R\tilde{R} = \alpha'$, one gets the following mapping between string coupling constants

$$\tilde{g}_s = \frac{\sqrt{\alpha'}}{R}g_s \tag{3.19}$$

Since they are proportional to each other, a perturbative expansion in g_s in type IIA corresponds to a perturbative expansion in $\tilde{g}_s$ in type IIB.

D-branes in type II superstring theories

Let us consider now D-branes in type II superstring theories, whose addition to the vacuum configuration gives theories with closed strings in the bulk plus open strings that end on the D-branes themselves.
The presence of the D-branes breaks some of the symmetries of the superstring vacuum. First of all, adding a flat Dp-brane to the Minkowski space vacuum of a type II superstring theory, neglecting its back-reaction on the geometry, breaks the $SO(1,9)$ Lorentz symmetry to $SO(1,p) \times SO(9-p)$. Secondly translational invariance in the transverse directions is spontaneously broken giving rise to the $9-p$ Goldstone scalar fields of the D-brane world-volume theory. Thirdly some or all of the supersymmetries are also broken.
Explicitly regarding the last point, type II superstring theories have $\mathcal{N} = 2$ supersymmetry in the 10-dimensional Minkowski vacuum. Since in ten dimensions each supercharge is a Majorana-Weyl spinor, with 16 real components, it is said there is a total of 32 conserved supercharges. However, vacua containing D-branes can preserve at maximum half of them. The reason is that type II superstring theories have supercharges Q_a constructed purely out of left-moving fields on the worldsheet and $\bar{Q}_a$ constructed out of right-moving ones. The vacuum is invariant under both, but once a D-brane is added to the vacuum, the D-brane imposes boundary conditions relating left- and right-moving fields at the open strings' end points. Thus left- and right-moving fields are no longer independent and as a consequence only a linear combination of the two supercharges Q_a and $\bar{Q}_a$ is a supersymmetry of the full state [**31**].

The D-branes preserving half of supersymmetries are called half-BPS D-branes. Being BPS states, they carry conserved charges that ensure their stability, in particular D-branes couple to R-R fields and thus carry conserved R-R charges.

To determine the conditions for stable p-branes, it is worthwhile to consider the types of conserved charges that they can carry.
The world-volume of a p-brane can couple naturally to a $(p+1)$-form potential

$$\mu_p \int A_{p+1} = \mu_p \frac{1}{(p+1)!} \int A_{\nu_1 \cdots \nu_{p+1}} \frac{\partial x^{\nu_1}}{\partial \sigma_0} \cdots \frac{\partial x^{\nu_{p+1}}}{\partial \sigma_p} d^{p+1}\sigma \tag{3.20}$$

where μ_p is the p-brane charge and this equation generalizes the coupling of a charged particle to a 1-form gauge potential, which has $p=0$ and $\mu_0 = e$

$$e \int A = e \int d\tau A_\mu \frac{\partial x^\mu}{\partial \tau} \tag{3.21}$$

This brane is electrically charged as the electric charge can be obtained through the Gauss's law $\mu_p = \int *F_{p+2}$, where in D dimensions this integral is carried out over a sphere S^{D-p-2}, which is the dimension required to surround a p-brane.
The charge of the magnetic dual branes can be measured by computing the flux $\int_{S^{p+2}} F_{p+2}$ and in D dimensions an S^{p+2} can surround a $(D-p-4)$-brane. Thus, in ten dimensions the magnetic dual of a p-brane is a $(6-p)$-brane.
The Dirac quantization condition for point-like charges can be generalized to charges carried by a dual pairs of p-brane and in ten dimensions one has[1]

$$\mu_p \mu_{6-p} \in 2\pi\mathbb{Z} \tag{3.22}$$

Thus in type II superstring theories a $n-$form gauge field can couple electrically to a $p-$brane with $p = n-1$ and magnetically to a p-brane with $p = 7-n$.
Since the R-R sector of type IIA theory contains gauge fields with $n = 1, 3$, this theory should contain stable Dp-brane with $p = 0, 2, 4, 6$. Whereas the R-R sector of type IIB theory contains n-form gauge field with $n = 0, 2, 4$, so this superstring theory admits stable Dp-branes only for odd values of p. Notice in particular that the 4- form potential C_4 couples both electrically and magnetically to a D3-brane, which is the brane carrying a self-dual charge since the associated field strength is self-dual $F_5 = *F_5$.
The result that type IIA/B theories admit only stable Dp-branes with p even/odd is consistent with the fact that a T-duality transformation maps a type IIA theory with Dp-branes into a type IIB theory with D$(p \pm 1)$-branes and vice versa. Notice that, since R-R fields couple to D-branes, the effect of a T-duality transformation, carried out for the coordinate X^9, on the R-R fields is to add or remove a form 9-index

$$\tilde{C}_9 = C, \quad \tilde{C}_\mu = C_{\mu 9}, \quad \tilde{C}_{\mu\nu 9} = C_{\mu\nu}, \quad \tilde{C}_{\mu\nu\lambda} = C_{\mu\nu\lambda 9} \tag{3.23}$$

so odd-form potentials of type IIA theory are mapped to the even-form potentials of the type IIB theory and vice versa.

D-brane action

Being dynamical objects, it is possible to construct the world-volume action of a D-brane.
The basic idea is that the modes of an open strings which start and end on a given D-brane can be described by fields restricted to the world-volume of the D-brane. In particular, the low-energy dynamics of a Dp-brane will be captured by the $(p+1)$-dimensional effective field

[1]In all superstring theory and M-theory examples it turns out that a single p-brane carries the minimum allowed charge, thus the product of the charges of a single p-brane and its dual $(6-p)$-brane is exactly 2π.

theory of only its world-volume massless fields.
Let us focus on a half-BPS Dp-brane, whose world-volume theory has 16 conserved supercharges. This theory can be constructed in the GS formalism, which maps the Dp-brane world volume into a superspace using

$$X^{\mu}(\sigma) \qquad \text{and} \qquad \theta^{Aa}(\sigma), \quad A = 1, 2 \tag{3.24}$$

in particular $X^{\mu}(\sigma)$ describes the embedding of the Dp-brane in ten-dimensional Minkowski space-time, whereas the pair of Majorana-Weyl spinors extends the embedding to $\mathcal{N} = 2$ superspace. The coordinates σ^{α}, with $\alpha = 0, 1, ...p$, parametrize the Dp-brane world-volume.
The Dp-brane world-volume theory contains also a $U(1)$ gauge field $A_{\alpha}(\sigma)$ since it is part of the spectrum of an open string that starts and ends on the D-brane.
The low-energy dynamics of a Dp-brane in a flat spacetime without background fields is captured by

$$S_{DBI} = -T_{Dp} \int d^{p+1}\sigma \sqrt{-\det(G_{\alpha\beta} + 2\pi\alpha' \mathcal{F}_{\alpha\beta})} \tag{3.25}$$

where T_{Dp} is the tension of the Dp-brane. For type II superstrings in Minkowski space-time, supersymmetry transformations of superspace are given by

$$\begin{cases} \delta\Theta^{Aa} = \varepsilon^{Aa} \\ \delta X^{\mu} = \bar{\varepsilon}^{A}\Gamma^{\mu}\Theta^{A} \end{cases} \tag{3.26}$$

and the invariance is ensured by defining

$$G_{\alpha\beta} = \eta_{\mu\nu}\Pi^{\mu}_{\alpha}\Pi^{\nu}_{\beta}, \qquad \Pi^{\mu}_{\alpha} = \partial_{\alpha}X^{\mu} - \bar{\Theta}^{A}\Gamma^{\mu}\partial_{\alpha}\Theta^{A} \tag{3.27}$$

The remaining tensor is

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + b_{\alpha\beta} \tag{3.28}$$

where $F = dA$ is the usual Maxwell field strength and b is Θ-dependent two-form required to supersymmetrize $\mathcal{F}$. From now on, we will restrict only on the bosonic part of the D-brane action.
The action (3.31) is called Dirac-Born-Infeld action and can be expanded in powers of the field strength

$$S_{DBI} = -T_{Dp} \int d^{p+1}\sigma \sqrt{-\det G}\Big(1 + \frac{(2\pi\alpha')^2}{4} F^{\alpha\beta}F_{\alpha\beta} + \mathcal{O}(F^4)\Big) \tag{3.29}$$

thus the expansion contains a constant energy density term $\int d^{p+1}\sigma\sqrt{-\det G}$ resulting to be the higher-dimensional generalization of the Nambu-Goto action, a quadratic Maxwell-type term and higher-order corrections.
A convenient gauge choice is the static gauge, in which the diffeomorphism symmetry is used to set the first $p+1$ components of X^{μ} equal to the world-volume coordinates σ^{α}, while the other $9-p$ components survive as scalar fields on the world-volume that describe transverse excitations of the brane and, to remark their physical interpretation, let us relabel them as $2\pi\alpha'\Phi^{i}$. In the static gauge, the bosonic part of the DBI action takes the form

$$S_{DBI} = -T_{Dp} \int d^{p+1}\sigma \sqrt{-\det(\eta_{\alpha\beta} + (2\pi\alpha')^2\partial_{\alpha}\Phi^{i}\partial_{\beta}\Phi^{i} + 2\pi\alpha' F_{\alpha\beta})} \tag{3.30}$$

This action can be generalized for the case of a general background with bosonic massless supergravity fields.
The background fields in the NS-NS sector are the space-time metric $g_{\mu\nu}$, the two-form $B_{\mu\nu}$ and the dilaton Φ, which can be pulled back to the world-volume

$$S_{DBI} = -T_{Dp} \int d^{p+1}\sigma e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta} + (2\pi\alpha')^2\partial_{\alpha}\Phi^{i}\partial_{\beta}\Phi^{i} + 2\pi\alpha' F_{\alpha\beta})} \tag{3.31}$$

where we have made an abuse of notation

$$g_{\alpha\beta} + B_{\alpha\beta} = P[g+B]_{\alpha\beta} = (g_{\mu\nu} + B_{\mu\nu})\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu} \tag{3.32}$$

The R-R background fields do not contribute to the DBI action, but with a Chern-Simons type action. As already stated, type II superstring theories involve R-R form fields C_n only with $n = 0, 1, 2, 3, 4$ (in particular odd-rank for type IIA and even-rank for type IIB); hovever for a electric-magnetic symmetric treatment, it is convenient to introduce redundant fields C_n for $n = 5, 6, 7, 8$ such that field strengths satisfy

$$*F_{n+1} = F_{9-n} \tag{3.33}$$

Therefore a Dp-brane action should contain as well a Chern-Simons term which at lowest order takes the form

$$\mu_p \int C_{p+1} \tag{3.34}$$

where μ_p denotes the Dp-brane charge as usual, since a Dp-brane couples electrically to the R-R field C_{p+1}.

It has been specified that this is the Chern-Simons term at lowest order since, in presence of the NS-NS background B field and world-volume gauge fields, the D-brane can couple also to R-R potentials of lower rank. In fact, the complete Chern-Simons term results to be

$$S_{CS} = \mu_p \int \left(C e^{B+2\pi\alpha' F} \right)_{p+1} \qquad \text{with} \quad C = \sum_{n=0}^{8} C_n \tag{3.35}$$

where the subscript $p + 1$ means we have to extract the $(p + 1)$-form piece of the integrand since a p-brane world-volume couples to a $(p + 1)$-form potential. Since the fields B and F are two-forms, it means that a Dp-brane can couple to other R-R fields of lower rank aside from C_{p+1} and subsequently can carry charges associated to D$(p - 2n)$-branes for $n = 0, 1,...,$ even if they are generically smeared over the $(p + 1)$-dimensional world-volume.

In conclusion, let us determine D-brane tensions using T-duality arguments. T-duality exchanges a wrapped Dp-brane in the type IIA theory and an unwrapped D$(p - 1)$-brane in the type IIB theory and vice versa. This fact gives

$$\frac{2\pi R T_{Dp}}{g_s} = \frac{T_{D(p-1)}}{\tilde{g}_s} \tag{3.36}$$

using now the mapping between string coupling constants (3.19), one gets

$$T_{Dp} = \frac{1}{2\pi\sqrt{\alpha'}} T_{D(p-1)} \tag{3.37}$$

and by iteration

$$T_{Dp} = \frac{1}{(2\pi)^p (\alpha')^{p/2}} T_{D0} = \frac{1}{g_s (2\pi)^p (\alpha')^{(p+1)/2}} \tag{3.38}$$

where we set $T_{D0} = (g_s\sqrt{\alpha'})^{-1}$

The tension of a D0-brane can be obtained by exploiting the fact that the M-theory, the 11-dimensional theory containing no strings obtained in the strong coupling limit of both type IIA superstring and $E_8 \times E_8$ heterotic string theories, compactified on a circle of radius R_{11} corresponds to 10-dimensional type IIA superstring theory with string coupling constant $g_s = R_{11}/\sqrt{\alpha'}$. Then the mass of the supergraviton in 11 dimensions is zero

$$M_{11}^2 = -p_M p^M = 0, \qquad M = 0, 1, ..., 9, 11 \tag{3.39}$$

whereas in 10 dimensions

$$M_{10}^2 = -p_\mu p^\mu = p_{11}^2, \qquad \mu = 0, 1, ..., 9 \tag{3.40}$$

since the momentum on the circle in the eleventh direction is quantized, $p_{11} = N/R_{11}$, the spectrum of ten-dimensional masses, which represents a tower of Kaluza-Klein excitations, is

$$M_N^2 = \left(\frac{N}{R_{11}} \right)^2, \qquad N \in \mathbb{Z} \tag{3.41}$$

in particular a type IIA D0-brane can be interpreted, from the point of view of the M-theory, as the first Kaluza-Klein excitation ($N = 1$) of the massless supergravity multiple, following the D0-brane tension formula.

Black-brane solutions and extremal black D-branes

In string theory and M-theory, it has been shown **[34],[35],[36]** that there exist higher-dimensional counterparts of 4-dimensional classical black hole solutions corresponding to p-dimensional extended objects surrounded by an event horizon, i.e. black p-brane, which can be obtained as solutions of the equations of motion associated to the low-energy actions.
We are mainly interested in the construction of black Dp-branes, for which the action required is the type II supergravity action. In this case the NS-NS two form B_2 should not be included since the Dp-brane is not charged under it, thus only the R-R field C_{p+1} needs to be considered in the action, which in the string-frame[2] thus reads

$$S_p = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}\Big[e^{-2\Phi}(R+4(\partial\Phi)^2) - \frac{1}{2}|F_{p+2}|^2\Big] \tag{3.42}$$

where, as usual, p is even for a type IIA action and odd for a type IIB one. In the particular case $p=3$, the self-duality condition $F_5 = *F_5$ needs to be imposed at the level of field equations.
Since the presence of a Dp-brane breaks the $SO(1,9)$ Lorentz symmetry to $SO(1,p)\times SO(9-p)$ as already underlined before, the most general ansatz that we need to impose to find the black Dp-brane solution is

$$\begin{cases} e^{\Phi} = e^{\Phi(r)} \\ ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\delta_{ij}dy^i dy^j \end{cases} \tag{3.43}$$

where x^μ with $\mu = 0,1,...,p$ are the coordinates tangent to the brane, y^i with $i = 1,...,9-p$ are the ones transverse to it and $r = y^i y^j \delta_{ij}$. Substituting the ansatz into the field equations associated to (3.42), we get a system of coupled non-linear differential equations subject to the following boundary conditions

$$\lim_{r\to\infty} g_{MN} = \eta_{MN}, \qquad \lim_{r\to\infty} e^{\Phi(r)} = g_s \tag{3.44}$$

These boundary conditions are chosen such that far away from the brane, i.e. for $r\to\infty$, 10-dimensional flat Minkowski spacetime is recovered and the dilaton approaches a constant value corresponding to the string coupling constant g_s. In addition to the ansatz and to these boundary conditions, we need to impose that the solution must preserve half SUSY, since we are interested in half-BPS black Dp-branes, and this BPS condition turns out to be the higher-dimensional correspondent of extremality condition for 4-dimensional Reissner-Nordström black holes. The BPS requirement leads to the extra constraint

$$(p+1)A(r) + (7-p)B(r) = 0 \tag{3.45}$$

The solution is characterized by an harmonic function $H_p(r)$ and in the string frame is given by

$$\begin{gathered} ds^2 = H_p^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_p^{1/2}\delta_{ij}dy^i dy^j, \qquad H_p(r) = 1 + \Big(\frac{r_p}{r}\Big)^{7-p} \\ F_{p+2} = dH_p^{-1}\wedge dx^0\wedge dx^1\wedge ...\wedge dx^p \\ e^{\Phi} = g_s H_p^{\frac{(3-p)}{4}} \end{gathered} \tag{3.46}$$

An important fact to notice is that for $p=3$ the dilaton is everywhere constant, i.e. $e^{\Phi} = g_s$.
The Hodge-dual field strength results to be

$$*F_{p+2} = (p-7)r_p^{7-p}\omega_{8-p} \tag{3.47}$$

where ω_{8-p} is the volume form for a unit $(8-p)$-sphere. If we integrate $*F_{p+2}$ over S^{8-p} we get the charge of the black Dp-brane, and if we recall that if the Dp-branes are BPS, as it is our case, the charge equals the tension, we can finally find that

$$\Big(\frac{r_p}{l_s}\Big)^{7-p} = (2\sqrt{\pi})^{5-p}\Gamma\Big(\frac{7-p}{2}\Big)g_s N \tag{3.48}$$

[2]the string-frame metric g_{MN} is related to the Einstein-frame metric g^E_{MN}, in terms of which the Ricci scalar appears in the action without the dilaton factor in front, by $g_{MN} = e^{\Phi/2}g^E_{MN}$

where we have used (3.38) and N is the number of coincident Dp-branes. The low-energy field theory is a good and valid description for the solution we have just derived, and subsequently the geometry is smooth on the string scale, only if $g_sQ \propto g_sN \gg 1$, which implies that the characteristic length r_p must be much larger than the string scale l_s.

Type IIB S-duality and the NS5-brane

Type IIB supergravity (2.31) has a non-compact global symmetry $SL(2,\mathbb{R})$. In particular, under a transformation

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in SL(2,\mathbb{R}) \tag{3.49}$$

the two 2-form potentials B_2 and C_2 transform as a doublet

$$\vec{B}_2 \equiv \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \Lambda\vec{B}_2 \tag{3.50}$$

while the complex scalar field τ, defined as

$$\tau = C_0 + ie^{-\Phi} \tag{3.51}$$

transforms non-linearly by the following rule

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.52}$$

The field C_0 is usually referred to as an axion, because of the shift symmetry in the supergravity approximation, and then the complex field τ as an axion-dilaton field.
On the other hand, th $SL(2,\mathbb{R})$ symmetry leaves invariant both the Einstein-frame metric $g^E_{\mu\nu} = e^{-\Phi/2}g_{\mu\nu}$ and the R-R 4-form C_4.
The global symmetry $SL(2,\mathbb{R})$ of type IIB supergravity is not shared by the full type IIB superstring theory since it is broken by a variety of stringy and quantum effects to the infinite discrete subgroup $SL(2,\mathbb{Z})$. One way to show that the residual symmetry group is $SL(2,\mathbb{Z})$ is that under the transformation (3.49) the 2-form coupling of the fundamental string becomes

$$\int B_2' = \int (dB_2 + cC_2) \tag{3.53}$$

so a fundamental string with charge $(1,0)$[3] transforms into a supersymmetric (d,c)-string. Thus the restriction to the $SL(2,\mathbb{Z})$ subgroup is essential to ensure that the charges are integers, as required by the Dirac quantization conditions.
Notice that, if we set the R-R scalar C_0 to zero, the transformation (3.49) with $a = d = 0$ and $c = -b = 1$ corresponds to a S-duality transformation, which is a transformation that relates a string theory with coupling constant g_s to a possibly different theory with coupling constant $1/g_s$; indeed (3.52) becomes $\tau = -1/\tau$ which implies that $\Phi' = -\Phi$ and subsequently $g_s' = 1/g_s$. In this case, the S-duality relates the type IIB superstring theory to itself.
This weak-strong duality takes the F-strings to D-strings into one another since

$$B_2' = C_2, \qquad C_2' = -B_2 \tag{3.54}$$

whereas it leaves the potential C_4 invariant, so it should take the D3-brane to itself.
The D5-brane is a magnetic source for the R-R 2-form charge, so it should be transformed through the weak-strong duality into a magnetic source for the NS-NS 2-form field. This object is neither a string or a D-brane, rather it is a soliton, a localized classical solution to the field

[3]This notation denotes that the F-string has a charge that couples to the NS-NS 2-form B_2 but neither that couples to the R-R 2-form C_2

equations, and it will be called a NS5-brane.
For NS5-brane, the extremal supersymmetric solution reads

$$\begin{aligned} ds^2 &= e^{2\Phi}\delta_{ij}dy^i dy^j + \eta_{\mu\nu}dx^\mu dx^\nu \\ H_{mnp} &= -\varepsilon^l_{ijk}\partial_l\Phi \\ e^{2\Phi} &= e^{2\Phi(\infty)} + \frac{Q}{2\pi^2 r^2}, \qquad r^2 = y^i y^j \delta_{ij} \end{aligned} \tag{3.55}$$

where x^μ with $\mu = 0, 1, ..., 5$ are the coordinates tangent to the brane, y^i with $i = 1, ..., 4$.
The product $T_{D1}T_{D5} = \pi/k_{10}^2$ should be equal to $T_{F1}T_{NS5}$ because of the Dirac quantization condition (which determines the product of the charges) combined with the BPS condition (which encodes the equality between the charges and the tensions), so it follows that the tension of a NS5 brane is

$$T_{NS5} = \frac{2\pi^2\alpha'}{k_{10}^2} = \frac{1}{(2\pi)^5 g_s^2 \alpha'^3} \tag{3.56}$$

The geometry of the metric contains an infinite throat: the point $y^i = 0$ is at infinite distance, and as one approaches it the radius of the angular 3-sphere does not shrink to zero but approaches an asymptotic value $(Q/2\pi^2)^{1/2}$. The dilaton grows in the throat of the NS5-brane, diverging at infinite distance. Therefore the geometry of the throat is $S_3 \times \mathbb{R} \times M_6$, where M_6 is a 6-dimensional Minkowski space.
Moreover, notice that in the Einstein-frame in which $g^E_{\alpha\beta} = e^{-\Phi/2}g_{\alpha\beta}$, we obtain that $ds \propto y^{-3/4}dy$, thus the metric is still singular but the singularity is at finite distance.

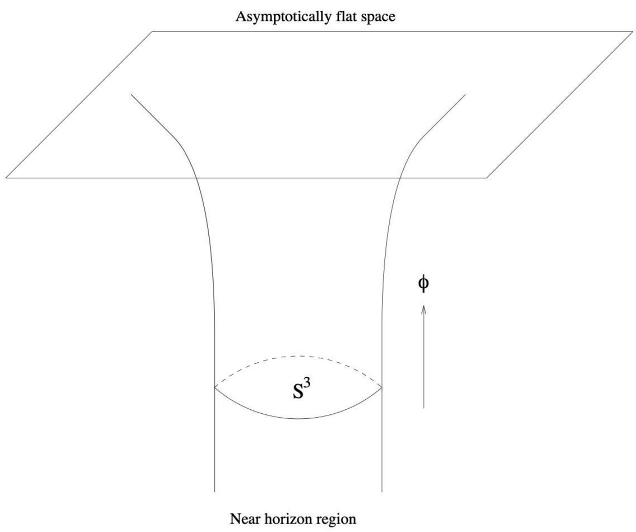


Figure 3.1: Infinite throat of a NS5-brane [**37**]

Chapter 4

AdS/CFT correspondence

One of the most exciting discoveries in theoretical physics in the last decades is the AdS/CFT correspondence[**38**], [**39**],[**40**], a duality, conjectured by Maldacena in 1997, which relates gravity theories on asymptotically Anti-de Sitter spacetimes to conformal field theories.
The most prominent example, which we are going to discuss, is the correspondence between $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions and type IIB superstring theory on $AdS_5 \times S^5$.

More specifically, the strongest form of this AdS/CFT correspondence states that $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ and Yang-Mills coupling constant g_{YM} is dynamically equivalent to type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on $AdS_5 \times S^5$ with curvature radius L and N units of F_5 flux on S^5. The map between the two free parameters g_{YM} and N on the field theory side and the free parameters L/α' and g_s on the gravity side by is given by

$$g_{YM}^2 = 2\pi g_s \qquad \text{and} \qquad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2} \tag{4.1}$$

Thus if the AdS/CFT correspondence holds, the physics of one theory, including operator observables, states, correlation functions and full dynamics, can be mapped onto all the physics of the other. This duality is very peculiar because through it we can map a possible candidate for a theory of quantum gravity, namely type IIB string theory, to a field theory without any gravitational degrees of freedom.
In its strongest form, the correspondence holds for all values of N and all regimes of coupling g_{YM}. However it's more useful and we can obtain more tractable forms of the AdS/CFT correspondence if we take certain limits on both sides in such a way we can obtain new insights into the non-perturbative behaviour, i.e. the strong coupling dynamics, of one theory from the computable weak coupling perturbative behaviour of the other.
For example the t'Hooft limit on the SYM side, in which $\lambda \equiv g_{YM}^2 N$ is fixed as $N \to \infty$, corresponds to $g_s \ll 1$ while keeping $L/\sqrt{\alpha'}$ constant on the AdS side. At leading order in g_s, the AdS side reduces to the classical string theory, since we take into account only tree level diagrams within string perturbation theory, not the entire string loop expansion. In this way classical string theory on $AdS_5 \times S^5$ provides a classical Lagrangian formulation of the large N dynamics of $\mathcal{N} = 4$ SYM theory. This is known as the strong form of the conjecture.
If we take the additional limit $\lambda \to \infty$, we obtain on one side a strongly coupled super Yang-Mills theory and on the other side, since $\sqrt{\alpha'}/L \to 0$, the point-particle limit of type IIB string theory, which is given by type IIB supergravity on $AdS_5 \times S^5$. In this way the correspondence becomes a strong/weak duality and this is referred to as the weak form of the AdS/CFT correspondence.

Motivation of the correspondence

We are going to motivate the weak form of the AdS_5/CFT_4 correspondence and describe the decoupling limit essential for the correspondence itself.
As we have already stressed out many times, superstring theory is not just a theory of closed

strings since it contains also non-perturbative solitonic higher dimensional objects such as the D-branes. The D-branes can be viewed, as we have already seen in the last section, from two different perspectives: the open string perspective and the closed string perspective.

In the open string perspective D-branes can be visualised as higher dimensional objects on which open strings can end. Since we have to treat strings as small perturbations, this perspective is reliable only when the string coupling constant $g_s \ll 1$. In the case of N coincident D-branes, the string coupling becomes $g_s N$, thus the reliability is ensured for $g_s N \ll 1$.
In the closed string perspective, D-branes may be viewed as solitonic solutions of the low-energy limit of superstring theory, namely of supergravity. As we have discussed in the last section, the closed string perspective is reliable only for $g_s N \gg 1$.

If we apply these two perspectives to a stack of N coincident D3-branes in flat spacetime, this can allow us to motivate the correspondence which relates four dimensional $\mathcal{N} = 4$ SYM theory to type IIB superstring theory in $AdS_5 \times S^5$. First of all, let's assume that the stack of N coincident D3-branes extends along the spacetime directions x^μ, with $\mu = 0, 1, 2, 3$, and is transversal to the other six spatial directions x^i, with $i = 4, ..., 9$.

Open string perspective

Consider a type IIB superstring theory in flat 10-dimensional Minkowski spacetime with N coincident D3-branes. The embedding of this stack of D3-brane breaks half of the supersymmetry of type IIB superstring theory in flat spacetime.
Perturbative string theory exhibits two kind of strings in this background: open strings starting and ending on the D3-branes, which can be viewed as excitations of the (3+1)-dimensional brane world-volume, and closed strings, which can be instead seen as excitations of the 10-dimensional flat spacetime.
If we consider geometric configuration at energies $E \ll \alpha'^{-1/2}$, it's equivalent to take only massless excitations into account. In particular, the massless closed string modes can be arranged into a 10-dimensional $\mathcal{N} = 1$ supergravity multiplet and the massless open string ones into a 4-dimensional $\mathcal{N} = 4$ supermultiplet which consists of a gauge field A_μ, corresponding to the bosonic longitudinal excitations, and six real scalar fields ϕ^i, corresponding to the bosonic transverse excitations, as well as fermionic superpartners.
The effective action for all massless string modes can be written as

$$S = S_{bulk} + S_{brane} + S_{int} \tag{4.2}$$

where S_{bulk} describes the closed string modes, S_{brane} the open string modes and S_{int} the interactions between them.
The bulk contribution S_{bulk} is the action of 10-dimensional supergravity

$$\begin{aligned} S_{bulk} &= \frac{1}{2k^2} \int d^{10}x \sqrt{-G} e^{-2\phi} (\mathcal{R} + 4\partial_M \phi \partial^M \phi) + ... \\ &\sim -\frac{1}{2} \int d^{10}x \partial_M h \partial^M h + \mathcal{O}(k) \end{aligned} \tag{4.3}$$

where dots refer to higher derivative terms and the action of fermionic and R-R form fields. The second line represents the lowest order contribution in the metric fluctuations h around the flat metric, obtained by expanding $g = \eta + kh$ with $2k^2 = (2\pi)^7 \alpha'^4 g_s^2$.
On the other side, S_{brane} can be derived from the Dirac-Born-Infeld action 3.31 which for a single D3-brane reads

$$S_{DBI} = -\frac{1}{(2\pi)^3 \alpha'^2 g_s} \int d^4x e^{-\Phi} \sqrt{-\det(\mathcal{P}[g]_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \tag{4.4}$$

where we have set the Kalb-Ramond field $B_{\mu\nu}$ to zero for simplicity. Expanding $e^{-\phi}$ and $g = \eta + kh$, at leading order in α' we find

$$S_{brane} = \frac{1}{2\pi g_s} \int d^4x \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i + \mathcal{O}(\alpha') \Big) \tag{4.5}$$

whereas the interaction action S_{int} can be derived from the Wess-Zumino term, which at leading order in α' reads

$$S_{int} = -\frac{k}{8\pi g_s}\int d^4x\,\Phi F_{\mu\nu}F^{\mu\nu} + \dots \tag{4.6}$$

This holds for a single D3-brane, while for a stack of N coincident D3-branes, the scalars and gauge fields becomes $U(N)$ valued, $\phi^i = \phi^{ia}T_a$ and $A_\mu = A^a_\mu T_a$, we have to trace over the gauge group to ensure gauge invariance, replace the partial with covariant derivatives and add a scalar potential of the form

$$V = \frac{1}{2\pi g_s}\sum_{i,j}\mathrm{Tr}[\phi^i,\phi^j]^2 \tag{4.7}$$

to the action S_{brane} (4.5).
If now one takes the limit $\alpha' \to 0$, S_{brane} becomes just the bosonic part of $\mathcal{N}=4$ SYM action provided we identify $2\pi g_s = g^2_{YM}$, S_{bulk} is just the action of free SUGRA in 10-dimensional Minkowski spacetime and S_{int} vanishes.
Thus the vanishing of S_{int} implies that closed and open strings decouple, with the first which are effectively described by SUGRA in flat 10-dimensional spacetime and the second by $\mathcal{N}=4$ super Yang-Mills theory.

Closed string perspective

Let us change point of view and consider N D3-branes in the strongly coupled limit $g_sN \to \infty$. In this limit, the D3-branes can be seen as massive charged objects which act as a source of the several type IIB supergravity fields.
The supergravity solution of N D3-branes, recalling (3.46), is given by

$$\begin{gathered} ds^2 = H_3^{-1/2}dx\cdot dx + H_3^{1/2}dy\cdot dy \\ H_3(r) = 1 + \left(\frac{R}{r}\right)^4 \\ e^\Phi = g_s \\ F_5 = dH_3(r)^{-1}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \end{gathered} \tag{4.8}$$

where $dx\cdot dx$ is the 4-dimensional Lorentz metric along the brane and $dy\cdot dy = dr^2 + r^2 d\Omega_5^2$ is the Euclidean metric in the 6 perpendicular directions.
Once again, using a result we have already found, namely (3.48), the radius L turns out to be

$$R^4 = 4\pi g_s N\alpha'^2 \tag{4.9}$$

The geometric background consists of two different regions for small r and large r respectively. If $r \gg R$, then $H_3(r) \sim 1$ and the metric (4.8) reduces to 10-dimensional flat spacetime; whereas $r \ll R$ corresponds to the near-horizon limit region (the horizon is at $r=0$) and the metric reads

$$ds^2 \simeq \left(\frac{r}{R}\right)^2 dx\cdot dx + \left(\frac{R}{r}\right)^2 dr^2 + L^2 d\Omega_5^2 \tag{4.10}$$

with the change of variable $z = R^2/r$, it takes the form

$$ds^2 \simeq R^2\frac{dx\cdot dx + dz^2}{z^2} + R^2 d\Omega_5^2 \tag{4.11}$$

thus the near horizon geometry is $AdS_5\times S^5$, with both AdS_5 and S^5 having radius R.
Since the temporal component of the metric (4.8) $g_{00} = -H_3(r)^{-1/2}$ is not constant, the energy E_r of an object measured by an observer at a constant position r and the energy E_∞ measured by an observer at infinity are related by

$$E_\infty = \sqrt{-g_{00}}E_r = H_3^{-1/4}E_r \tag{4.12}$$

this implies that as an object approaches the horizon $r = 0$, its energy would appear to decrease for the observer at the infinity.
Therefore if we take the low-energy limit in the background described by (4.8), an observer at infinity would observe two kind of low-energy excitations: massless particles propagating in the bulk region with wavelengths becoming very large and any kind of excitation we bring closer and closer to $r = 0$. These two kind of excitations decouple with each other for the following reasons. The bulk massless particles decouple from the near horizon region because the low-energy absorption cross section goes like $\sigma \sim \omega^3 R^8$ **[41]**, with ω the energy, and this can be interpreted as the fact that the particle's wavelength becomes much bigger than the typical gravitational size of the brane, with the latter being of order R. On the other side, the excitations close to the horizon find more and more difficult to climb the gravitational potential and escape to the asymptotic region.

In conclusion, the low-energy theory consists of the decoupled pieces, the free bulk supergravity and the near horizon region described by fluctuations about the $AdS_5 \times S^5$ solution of type IIB supergravity.

Combination of the two perspectives

In both perspectives, the open and the closed string ones, we have found two decoupled effective theories in the low-energy limit

- Open string perspective: $\mathcal{N} = 4$ SYM theory on flat 4-dimensional spacetime and type IIB supergravity on $\mathbb{R}^{9,1}$;
- Closed string perspective: type IIB supergravity on $AdS_5 \times S^5$ and type IIB supergravity on $\mathbb{R}^{9,1}$.

The two perspective should be equivalent descriptions of the same physics. This implies that, since type IIB supergravity on $\mathbb{R}^{9,1}$ is present in both perspectives, the other two theories, the $\mathcal{N} = 4$ SYM theory in four dimensions and type IIB supergravity on $AdS_5 \times S^5$, should be identified.
A clarification needs to be made: the N coincident D3-branes give rise to a $\mathcal{N} = 4$ gauge multiplet in the adjoint representation of $U(N)$ and this seems in contrast with the statement made at the beginning that the correspondence is between an $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ and a type IIB superstring theory on $AdS_5 \times S^5$.
The fact is that a $U(N)$ gauge theory is essential equivalent to a free $U(1)$ vector multiplet times an $SU(N)$ gauge theory, and the AdS theory is describing the $SU(N)$ part of this gauge theory, since in the closed string perspective the $U(1)$ vector multiplet corresponds to zero modes between the two kinds of low-energy excitations and thus living in the region connecting the near-horizon region with the bulk one. From the AdS point of view, these zero modes live on the boundary, and it looks like we might or might not include them in the AdS theory, having subsequently a correspondence to a $U(N)$ or a $SU(N)$ gauge theory.

An important check of the conjecture is to verify if the two theories have the same symmetries. The 5-dimensional Anti-de-Sitter space has $SO(4, 2)$ as group of isometries which is the same as the conformal group in four dimensions. Moreover the $SO(6)$ symmetry which rotates S^5 can be identified with the $SU(4)_R$ R-symmetry of the conformal field theory. The supergroup corresponding to the bosonic subgroup $SO(4, 2) \times SO(6)$ is the supergroup $SU(2, 2|4)$, and it can be checked also that the two theories have the same number of supercharges: type IIB superstring theory has 32 supercharges as we have already discussed in the last section, whereas 4d $\mathcal{N} = 4$ SYM theory has 16 Poincaré supercharges, and being conformal, has 16 additional superconformal supercharges.
Then, in terms of D-branes' physics, the Yang-Mills coupling is related to the string coupling through

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + C_0 \tag{4.13}$$

where it has been included also the θ angle with is related to the expectation value of the RR scalar C_0. In particular, both the super Yang-Mills theory and the type IIB string theory have an $SL(2,\mathbb{Z})$ self-duality symmetry under which

$$\tau \to \frac{a\tau+b}{c\tau+d} \tag{4.14}$$

In fact, we have already shown that $SL(2,\mathbb{Z})$ is a strong-weak duality symmetry of type IIB string theory in flat space and it should still be a symmetry also in an $AdS_5\times S^5$ background since all the fields which have been turned on, the metric and the 5-form field strength, are invariant under this symmetry as discussed in the last section.

Holographic dictionary

To define more properly the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. A conformal field theory does not have asymptotic states or a S-matrix, thus the natural objects to consider are operators.
First of all, let's choose the following Poincaré patch coordinates (see Appendix A)

$$ds^2 = \frac{R^2}{z^2}(dz^2+dx^2) \tag{4.15}$$

on Euclidean AdS_{d+1} where x_μ are coordinates on $\mathbb{R}^d$ and the boundary is at $z=0$.
The fundamental statement of AdS/CFT correspondence is that

$$Z_{string}\left[\phi(x,z)\Big|_{z=0}=\phi_0(x)\right] = \langle e^{\int d^dx\phi_0(x)\mathcal{O}(x)}\rangle_{CFT} \tag{4.16}$$

where the left-hand side is the full partition function of string theory in an asymptotically AdS_{d+1} space, function of the boundary condition ϕ_0 of the field ϕ on the boundary of AdS, and the right-hand side is the generating function of correlation functions in the d-dimensional conformal field theory, where ϕ_0 is the source of a conformal operator $\mathcal{O}$. Thus each field propagating on AdS space is in a one-to-one correspondence with an operator in the field theory and the boundary value of the bulk field acted as a source for the CFT operator. From this, it follows that there is a relation between the mass of the field and the scaling dimension of the operator in the CFT, as we are going to show in a moment.
For simplicity, let's consider the case in which ϕ is massive scalar field propagating in AdS_{d+1}, then its action is given by

$$S\sim\int d^{d+1}y\sqrt{g}(g^{MN}\partial_M\phi\partial_N\phi+m^2\phi^2) = \int dz\,d^dx\Big(\frac{R}{z}\Big)^{d+1}\Big[\Big(\frac{z}{R}\Big)^2(\partial_z\phi)^2+\Big(\frac{z}{R}\Big)^2(\partial_\mu\phi)^2+m^2\phi^2\Big] \tag{4.17}$$

and the equation of motion is

$$\partial_z\Big(\frac{1}{z^{d-1}}\partial_z\phi\Big)+\partial_\mu\Big(\frac{1}{z^{d-1}}\partial^\mu\phi\Big) = \frac{1}{z^{d+1}}m^2R^2\phi \tag{4.18}$$

Going to Fourier space for the momentum on $\mathbb{R}^d$, $\eta^{\mu\nu}\partial_\mu\partial_\nu=-p^2$, we get

$$z^{d+1}\partial_z\Big(\frac{1}{z^{d-1}}\partial_z\phi\Big)-p^2z^2\phi-m^2R^2\phi=0 \tag{4.19}$$

there are two independent solutions that can be written exactly in terms of Bessel functions, but if we study the asymptotic behavior near the boundary of AdS, i.e. $z\sim 0$, the term with momentum can be neglected and the two independent solutions are power-law

$$\phi\sim\phi_0\,z^{\alpha_-}+\phi_1\,z^{\alpha_+}, \qquad \alpha_\pm=\frac{d}{2}\pm\sqrt{\frac{d^2}{4}+R^2m^2} \tag{4.20}$$

The solution z^{α_-} is the dominant one as $z \to 0$, while the other one always decays, and since the leading contribution approaches a constant only in the case $\alpha_- = 0$, in order to have a consistent prescription one needs to impose the following boundary condition at $z = 0$

$$\phi(x,z)\Big|_{z=\varepsilon} = \varepsilon^{\alpha_-}\phi_0^{ren}(x) \tag{4.21}$$

where, since in AdS several quantities diverge as $z \to 0$, we have introduced a cut-off and imposed the boundary condition at $z = \varepsilon$, which will be sent to zero at the end of computation, and ϕ_0^{ren} is the "renormalized" boundary condition.
If now we perform a rescaling of the coordinates

$$x^\mu \to \lambda x^\mu \qquad z \to \lambda z \tag{4.22}$$

which correspond to an $SO(d,2)$ isometry in AdS_{d+1} and a dilation in the boundary CFT_d, the bulk scalar field ϕ remains invariant, thus $\phi_0^{ren} \to \lambda^{\alpha_-}\phi_0^{ren}(\lambda x)$ and, since we identify the latter as the source of the corresponding boundary conformal operator $\mathcal{O}$, the operator $\mathcal{O}$ must have dimension

$$\Delta = d - \alpha_- = \alpha_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2 m^2} \tag{4.23}$$

So we associate a bulk scalar field of squared mass m^2 to a boundary operator of dimension Δ. Let's make some observations about the identification

$$m^2 R^2 = \Delta(\Delta - d) \tag{4.24}$$

First of all, $m^2 \geq 0$ only for $\Delta \geq d$, so operators with $\Delta \leq d$ correspond to fields with negative squared mass in AdS, but these are not tachyons since their energy is positive as long as the Breitenlohner-Freedman bound **[42]** $m^2R^2 \geq -\frac{d^2}{4}$ is satisfied. The minimal value $m^2R^2 = -\frac{d^2}{4}$ corresponds to $\Delta = \frac{d}{2}$.
The minimal dimension of a conformal operator is given by the unitary bound $\Delta \geq \frac{d-1}{2}$, so the remaining case is the one of operators with $\frac{d-1}{2} \leq \Delta \leq \frac{d}{2}$. To get this operators, we impose the boundary conditions on the other mode $\phi \sim \phi_1 z^{\alpha_+}$, so that we get $\Delta = \alpha_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + R^2m^2}$ and in this case the conformal dimension of the boundary operator can assume values $\frac{d-1}{2} \leq \Delta \leq \frac{d}{2}$.

Similar statement apply to all bulk fields, for which the field $\leftrightarrow$ operator map can be determined explicitly. For example, the boundary value of the bulk metric g_{MN} is the boundary metric $g_{\mu\nu}$, which is the source of the boundary CFT stress tensor $T_{\mu\nu}$, and in this case the mass/dimension relation results to be the same as the one of the massive scalar field

$$m^2 R^2 = \Delta(\Delta - d) \tag{4.25}$$

If our theory of gravity has a gauge field A_M, the dual operator must be a vector J^μ, coupled to the source as

$$\int d^d x A_\mu J^\mu \tag{4.26}$$

Since the bulk theory must be invariant under gauge transformations, $\delta A_M = D_M \lambda$, it follows that

$$0 = \delta \int d^d x A_\mu J^\mu = -\int d^d x \lambda D_\mu J^\mu \tag{4.27}$$

thus the current J_μ should be a conserved current in the boundary conformal field theory. This is an example of a general feature of AdS/CFT correspondence: AdS bulk gauge symmetries correspond to CFT boundary global symmetries.
From the wave equation in AdS, we get that the conformal dimension of the dual operator of a bulk vector field must satisfy

$$m^2 R^2 = (\Delta - 1)(\Delta - d + 1) \tag{4.28}$$

in particular J_μ has dimension $d-1$.
Another important example is provided by the value of the dilaton at infinity, which correspond to the string coupling $g_s = g_{YM}^2$

$$e^{-\phi_\infty} = \frac{1}{g_s} = \frac{2\pi}{g_{YM}^2} \tag{4.29}$$

which acts as the source of the operator $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$; indeed on $AdS_5 \times S^5$ the dilaton is massless and $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ is a marginal operator of dimension 4, consistently with (4.24).

Definition and properties of Entanglement Entropy

One of the main aspects of the study of the evaporation of black holes which will be central in this thesis project is the time evolution of the entanglement entropy between the evaporating black hole and the outgoing Hawking radiation. Before going into detail of how entanglement entropy is encoded and can be computed in holographic systems, let us introduce briefly in the following some basic definitions and properties of entanglement entropy itself.
First of all, consider a quantum mechanical system with many degrees of freedom and let us suppose to divide it into two subsystems A and B, so that the Hilbert space of the total system can be written as $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$. If the density matrix of the total system is ρ, an observer who is only accessible to the subsystem A will feel as if the total system is described by the reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho \tag{4.30}$$

Then we can define the entanglement entropy of the subsystem A as the von Neumann entropy of the reduced density matrix ρ_A

$$S_A = -\mathrm{Tr}_A[\rho_A \log \rho_A] \tag{4.31}$$

This quantity tells us how closely entangled the state of the system ρ is.
This main properties of the entanglement entropy are

- if the density matrix ρ is pure and B is the complement of A, then

$$S_A = S_B \tag{4.32}$$

- for any three subsystems A, B and C that do not intersect each other, the following inequalities, known as strong subadditivity, hold

$$\begin{aligned} S_{A\cup B\cup C} + S_B &\leq S_{A\cup B} + S_{B\cup C} \\ S_A + S_C &\leq S_{A\cup B} + S_{B\cup C} \end{aligned} \tag{4.33}$$

in particular the subadditivity condition is obtained from the first equation by setting B empty

$$S_{A\cup B} \leq S_A + S_B \tag{4.34}$$

Entanglement Entropy in QFT

After having studied entanglement entropy from the quantum mechanics point of view, let's discuss it briefly in a quantum field theory context.
First of all, consider a QFT on a $(d+1)$-dimensional manifold $\mathbb{R} \times \mathcal{M}$, where $\mathbb{R}$ and $\mathcal{M}$ denote the time direction and a d-dimensional space-like manifold respectively, and we consider a d-dimensional submanifold $A \subset \mathcal{M}$ at a fixed time. As before, we call ∂A and B its boundary and its complement with respect to $\mathcal{M}$ respectively.
In this case, with respect to a quantum-mechanical bipartite system, there is an important issue: in a continuum QFT there are UV modes at arbitrarily small scales across the dividing surface ∂A making impossible to split the full Hilbert space. To deal with this problem, we impose a UV cutoff by introducing a "lattice" scale ε_{UV}. With a UV cutoff, the Hilbert space of a finite

region is finite-dimensional and most of the properties seen before apply also in QFT. In the UV, any finite energy is the same as the vacuum state, thus we can restrict to the vacuum $\rho = |0\rangle \langle 0|$ of the full system.
The divergent terms of S_A depend on both the UV physics and on the shape of the submanifold A. In particular, since the entanglement between A and B occurs at the boundary ∂A most strongly, we expect the divergent piece to be a local integral over the entangling surface ∂A

$$S_A^{div} \sim \int_{\partial A} d^{d-1}\sigma \sqrt{h} F[K_{ab}, h_{ab}] \tag{4.35}$$

where K_{ab} and h_{ab} are respectively the extrinsic curvature and the induced metric on ∂A and F is some theory-dependent functional of them.
Now we can expand (4.35) in powers of $K_{ab} \sim 1/L_A$, where L_A quantifies the size of the region A. Recalling that in a pure state $S_A = S_B$, thus also $S_A^{div} = S_B^{div}$ and taking into account that the extrinsic curvature of a surface is $K \sim \nabla n$, with n orthonormal vector to the surface, then this implies that only even powers of K_{ab} are allowed since going from A to its complement B it flips sign

$$S_A^{div} \sim a_1 L_A^{d-1} + a_2 L_A^{d-3} + \cdots \tag{4.36}$$

where the coefficients a_i depend on the theory but not on L_A. Furthermore, if we assume the theory to be scaling invariant, in its vacuum state the only scale is the UV cutoff ε_{UV}, so by dimensional analysis we would have $a_1 \sim \varepsilon_{UV}^{1-d}$, $a_2 \sim \varepsilon_{UV}^{3-d}$, etc.
From (4.36), we can state that the leading divergent contribution is proportional to the area of the boundary ∂A of the region A.
However, the area law (4.36) does not always describe the scaling of entanglement entropy. For example, if we consider a 2-dimensional CFT in the vacuum state of the full system, the entanglement entropy of a region A with length L_A, assuming the full system is infinitely long and by using the replica trick **[43]**, turns out to scale logarithmically with the length

$$S_A = \frac{c}{3} \log \frac{L_A}{\varepsilon_{UV}} \tag{4.37}$$

where c is the central charge of the CFT.

Holographic Entanglement Entropy

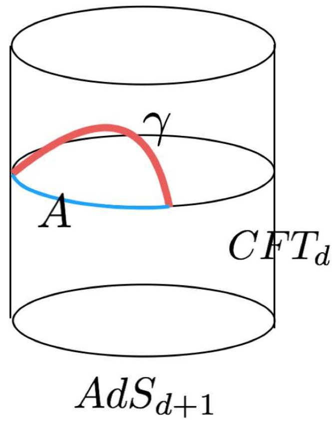
The entanglement entropy has a gravitational interpretation in CFTs with a semiclassical holographic dual. This holographic derivation was conjectured by Ryu and Takayanagi in the static case.
The holographic entanglement entropy S_A in a CFT for a subsystem A is given by

$$S_A = \frac{Area(\gamma_A)}{4G_N} \tag{4.38}$$

where γ_A is a codimension-2, spacelike extremal surface in the dual AdS geometry, anchored to the AdS boundary such that

$$\partial\gamma_A = \partial A \tag{4.39}$$

with the important requirement that the extremal surface γ_A must be homologous, i.e. continuously deformable, to the region A. In the case in which there are multiple extremal surfaces satisfying the above properties, the prescription is to pick the one with the minimal area.



As an example, let's compute the entanglement entropy in AdS_3/CFT_2. Gravitational theories on AdS_3 space of radius R are dual to two dimensional CFTs with central charge $c = 3R/G_N^{(3)}$.
The metric of AdS_3 in the global coordinates (t, ρ, θ) is

$$ds^2 = R^2(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2) \tag{4.40}$$

At the conformal boundary $\rho = \infty$ the metric is divergent, thus we introduce a cutoff ρ_0 which corresponds to the UV cutoff in the dual CFTs. In particular, if L is the total length of the system and a is the UV cutoff in the CFTs, the relation between ρ_0 and a is

$$e^{\rho_0} \sim \frac{L}{a} \tag{4.41}$$

The 2D spacetime for the CFT_2 is identified with the cylinder (t, θ) at the boundary $\rho = \rho_0$ and a subsystem A is identified with a portion of this cylinder given by $0 \leq \theta \leq 2\pi l/L$. Then the extremal surface γ_A is in this case the static geodesic connecting the boundary points $\theta = 0$ and $\theta = 2\pi l/L$ travelling inside AdS_3. The geodesic distance is given by

$$\cosh\left(\frac{L_{\gamma_A}}{R}\right) = 1 + 2\sinh^2 \rho_0 \sin^2\left(\frac{\pi l}{L}\right) \tag{4.42}$$

and assuming a large UV cutoff we obtain the following entanglement entropy

$$S_A \simeq \frac{R}{4G_N^{(3)}} \log\left(e^{2\rho_0} \sin^2 \frac{\pi l}{L}\right) = \frac{c}{3} \log\left[e^{\rho_0} \sin\left(\frac{\pi l}{L}\right)\right] \tag{4.43}$$

which perfectly matches the very well known CFT result. Similarly, if we use Poincaré patch coordinates for AdS_3

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx^2) \tag{4.44}$$

the subsystem A can be identified with the interval $-l/2 \leq x \leq +l/2$ at the boundary $z = 0$. The geodesic γ_A is described by

$$(x, z) = \frac{l}{2}(\cos \lambda, \sin \lambda) \qquad \text{with} \qquad \lambda \in \left(\frac{\varepsilon_{UV}}{l}, \pi - \frac{\varepsilon_{UV}}{l}\right) \tag{4.45}$$

where ε_{UV} is the UV cutoff $\varepsilon_{UV} \sim 2a/l$, thus we obtain the entropy

$$S_A = \frac{c}{3} \log \frac{l}{a} \tag{4.46}$$

which result to be, up to a constant, the small l limit of (4.43), consistent with the CFT result (4.37).

Chapter 5

Information paradox and Karch-Randall models

Information paradox and Page curve

In 1975, Stephen Hawking discovered that black holes emit thermal radiation [**1**]. Using quantum field theory in a classical curved space-time background geometry, he argued that gravitational fields at the horizon are strong enough for quantum mechanical production in the vicinity of the horizon to lead to the emission of thermal radiation. Roughly speaking, the Hawking radiation involves the creation of an entangled pair of particles near the horizon, where one particle falls into the hole, and the other one is emitted as a physical on-shell particle. As a consequence, emitting Hawking radiation, the black hole's mass decreases, the black hole gradually evaporates and eventually disappears. This means that either information is truly lost in black holes, or that information has been preserved by Hawking radiation itself by some mechanism.
In terms of the von Neumann entropy we can make a more accurate statement of the information paradox. If one computes the von Neumann entropy of the emitted Hawking radiation, one finds that it increases as the number of emitted quanta; but after a black hole evaporates, all that is left is the radiation which is still entangled and so is in a mixed state, in contrast with the fact that the black hole could have been formed from a pure state. But according to unitarity, it is impossible that a pure state evolve into a mixed state, hence the paradox.
The violation of unitarity by the evaporation of the black hole can be explicitly seen by the fact that as the black hole evaporates, the area of the event horizon decreases and the Bekenstein-Hawking entropy decreases as well since it's proportional to the area of the horizon. If we assume the black hole has formed from the collapse of a pure state, the von Neumann entropy of the black hole and the one of Hawking radiation itself must always be equal but we run into problem when the increasing von Neumann entropy of Hawking radiation becomes larger than the Bekenstein-Hawking entropy of the black hole, since by definition the von Neumann entropy must be always less than the thermodynamic one. One of the first to propose that unitary must be preserved was Don Page[**2**]. Since we assume to start at a pure state for the black hole, we finally have to arrive at a pure state at the end of the evaporation process to preserve unitarity. This means that the entropy of Hawking radiation must be zero at the end state, thus according to Page this means that it must begin decreasing at some time, called Page time, following the so called Page curve. Arguments for the Page curve rely on fundamental properties of fine-grained entropy. In particular, it is impossible to fix the problem by adding small corrections to the Hawking process and this holds for all orders in perturbation theory. Therefore, if there is a solution for the problem, it should be non-perturbative in gravitational coupling G_N. The major question in the construction of a solution is how to bend the curve down in order to preserve unitarity, a problem sometimes called "information puzzle".

Advances in this direction rely on a deeper understanding of the calculation of the entropy of Hawking radiation, which has been possible thanks to the identification of a new contribution to the entanglement entropy[**7**],[**44**],[**9**]: the islands, regions that are physically disconnected

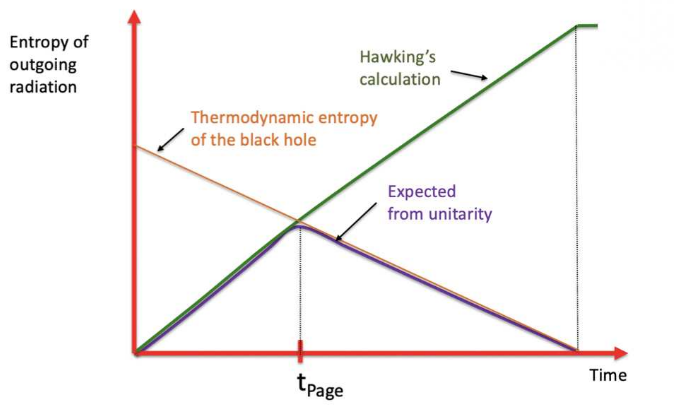


Figure 5.1: Schematic behaviour of the entropy of the outgoing radiation

from the asymptotic radiation region and that dominate the entropy contribution at the Page time, ensuring a unitary evolution of the system.
The starting point is an extension of Ryu-Takayanagi entanglement entropy proposal to a generalised entropy which includes the entropy contribution of quantum external fields **[4]**,**[6]**. Specifically, the gravitational proposal for the von Neumann entropy, i.e. the minimum of the generalised entropy, is

$$S = \min_X \left\{ \mathrm{ext}_X \left[\frac{\mathrm{Area(X)}}{4G_N} + S_{sc}(\Sigma_X) \right] \right\} \tag{5.1}$$

where X is a surface of codimension 2, Σ_X is the region bounded by X and a cutoff surface, and $S_{sc}(\Sigma_X)$ is the von Neumann entropy of quantum fields living on Σ_X in the semiclassical description and the quantity inside the square brackets, including an area term and the contribution of quantum fields, is the generalised entropy which obeys the second law of thermodynamics. The surface X must be chosen so that it minimizes the entropy in the spatial direction and maximises it in the time direction. If there are multiple extremal surfaces, we must take the one corresponding to the global minimum, and this extremal surface will be called the Quantum Extremal Surface (QES). The idea is to start from a cutoff surface outside the black hole and to move it past the horizon into the interior of the black hole to find the minimum.

Entropy of an evaporating black hole

Let's apply the fine-grained entropy formula (5.1) to all stages of the evaporating black hole.
After the black hole forms but before any Hawking radiation has a chance to escape the black hole region, the extremal surface is vanishing, so the first term in (5.1) vanishes, but also the second term vanishes because the collapsing matter that formed the black hole is a pure state, thus at the initial stage of evaporation the entropy vanishes. Once the black hole starts evaporating and emitting Hawking radiation, the von Neumann entropy is not longer zero due to the entanglement with outgoing Hawking radiation and begins increasing due to the pile up of mixed interior modes, whereas the Bekenstein-Hawking entropy, i.e. the thermodynamic entropy, starts decreasing. At the same time, as the Hawking radiation starts escaping the black hole region, a non-vanishing extremal surface appears as well and it turns out to lie very close to the event horizon and the corresponding generalized entropy turns out to be dominated by the area term, thus it can be approximated by the thermodynamic entropy. In particular, since the thermodynamic entropy decreases as black evaporates, this extremal non-vanishing surface yields a decreasing generalized entropy. The final prescription of (5.1) is to take the minimum of generalized entropy among all possible extremal surfaces. Taking into account the time evolution of generalized entropy of vanishing and non-vanishing extremal surfaces, we get a Page curve, with an initial grow due to vanishing extremal surface and a final decrease due to the non-vanishing one.

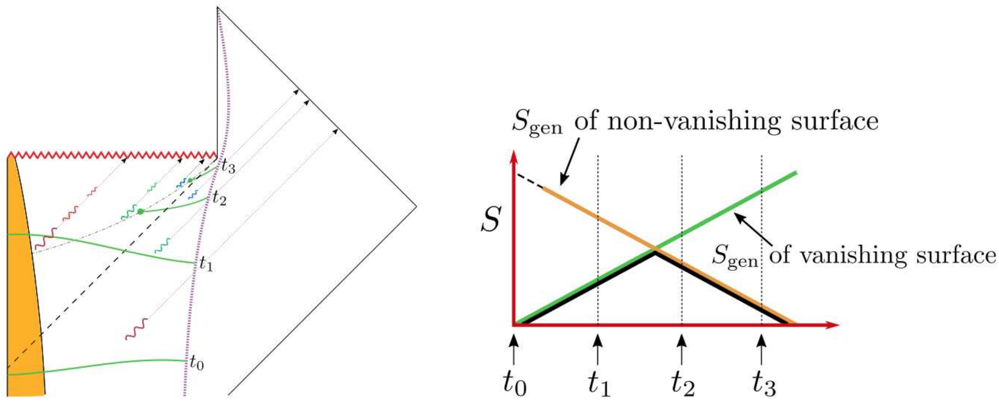


Figure 5.2: Page curve for the fine-grained entropy of the black hole

Thus this seems to indicate a unitary black hole evaporation but it does not address directly the information paradox since the latter concerns the entropy of outgoing Hawking radiation.

Entropy of radiation

So far we have assumed Σ_X to be connected, but in order for the entropy of outgoing Hawking radiation to exhibit a Page curve, we must consider the case in which Σ_X is disconnected. This would increase the area term in (5.1), but in order to ensure the entropy can still be minimised we must jointly decrease the semiclassical entropy contribution. This can happen if we have regions that are far away from any entangled matter; this is precisely our case, since the outgoing radiation is entangled with fields living in the black hole interior. Therefore we can decrease the semiclassical entropy S_{sc} by including a region representing the black hole interior. Disconnected regions like these are called "islands" and the fine-grained entropy of radiation with this island prescription reads

$$S_{rad} = \min_X \Big\{ \text{ext}_X \Big[\frac{\text{Area(X)}}{4G_N} + S_{sc}(\Sigma_{rad} \cup \Sigma_{island}) \Big] \Big\} \tag{5.2}$$

where the area here refers to the area of the boundary of the island, the min/ext procedure is with respect to the location and shape of the island itself, Σ_{rad} is the region from the cutoff surface to infinity accounting for all Hawking radiation that has escaped the black hole region and Σ_{island} refers to any number of islands, disconnected regions contained in the black hole side of the cutoff surface. The necessity of including the interior when calculating the fine-grained entropy in the island prescription was justified through the replica wormholes **[45]**,**[44]**.
As before, we can have the case of vanishing island contribution. In this case the entropy is completely due to the semiclassical entropy contribution of Σ_{rad}, which grows with time as the emitted Hawking radiation quanta. After some time, called scrambling time $t_{scr} = r_s \ln S_{BH}$ **[7]**, a non-vanishing island appears: this island is centered around the origin, its boundary is very near the black hole horizon and moves it up for different times along the cutoff surface. In this case, the semiclassical entropy accounts for both the radiation and the island regions and thus is close to zero since the ingoing Hawking radiation, contained entirely or almost in the island, combines with the outgoing one turning what was before a mixed state into a pure one. This means that the area term dominates and the generalised entropy decreases with time tracking the thermodynamic entropy of the black hole.
As before, the fine-grained entropy is given by the minimum of the generalized entropies of the two cases and this leads to a Page curve, with the rising piece coming from the no-island contribution and the decreasing one from the island contribution.

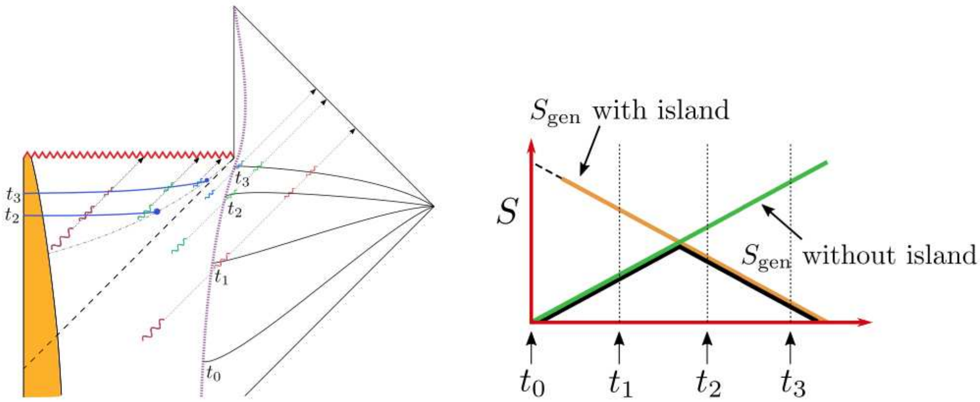


Figure 5.3: Page curve of the fine-grained entropy of emitted radiation

The Page curve of both the evaporating black hole and of the radiation are the same, consistently with the fact that if the black hole has formed from a pure state, the entropies of the black hole and the emitted radiation must be the same as stressed out many times.

Karch-Randall model with non-gravitating bath

One of the most recent and promising developments in the context of quantum gravity have been the calculations of the Page curve in some simplified models using holography ideas. In order to have a theory of gravity under theoretical control, it is convenient to study black holes in asymptotically AdS space so one has a non-perturbative definition of quantum gravity in terms of AdS/CFT correspondence. Since AdS space acts effectively like a box, large black holes in this space do not evaporate.

One way to solve this problem and to have a tractable model of an AdS evaporating black hole consisting in imposing transparent boundary conditions, allowing the radiation of an AdS black hole reaching the boundary of AdS to couple to an external flat space heat bath.

AdS with transparent boundary conditions is thus crucial in the holographic calculations of the Page curves. One of the model that embeds these transparent boundary conditions are is the Karch-Randall model **[16]**,**[17]**,**[18]** describing an end-of-the-world brane (with specific features discussed below) coupled to a non-gravitating bath. Such a system is a doubly-holographic one because it has three descriptions related to each other by applying the AdS/CFT correspondence twice:

1. as an Einstein gravity in an asymptotically AdS_{d+1} space containing a Karch-Randall brane as an end-of-the-world (ETW) brane ;

2. as CFT_d with an intrinsic UV cutoff coupled to d-dimensional graviton on an asymptotically AdS_d space, whose boundary is connected to another d-dimensional CFT on half-Minkowski space via transparent boundary conditions;

3. as non-gravitational CFT_d on half of a Minkowski space with a $(d-1)$-dimensional boundary (i.e. a BCFT_d).

The second holographic interpretation is the one of interest for the black hole information paradox: to pose the paradox, the AdS_d slice is replaced by an AdS_d black hole, obtaining in this way a black hole on the ETW brane coupled to the remaining half of the conformal boundary of the bulk AdS_{d+1}, which serves as a bath.

For simplicity let's assume $d = 4$, thus we consider a $4d$ AdS theory coupled to a CFT_4 living in a flat space bath region, and let's briefly review the simple Karch-Randall geometry with a Planck brane ending on the conformal boundary of the bulk AdS_5 (denoted as $\mathcal{M}$ below).

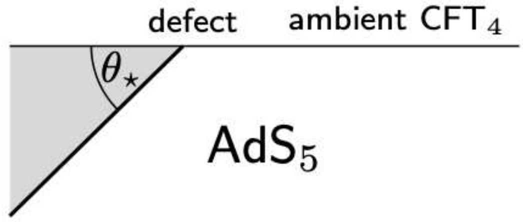


Figure 5.4: Embedding of an AdS$_4$ Karch-Randall brane in a bulk AdS$_5$ space

Therefore we are lead to consider the following action[1]

$$S = \frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5x\sqrt{-g}\Big(R + \frac{12}{L^2}\Big) + \frac{1}{8\pi G_5}\int_{\mathcal{B}} d^4x\sqrt{-h}(K-\lambda) \tag{5.3}$$

where $\mathcal{B}$ denotes the ETW brane with induced metric h_{ab} and should be seen as one of the boundary component of the bulk AdS$_5$ spacetime, L is the AdS$_5$ radius and λ is a quantity proportional to the tension of the brane. $K = h^{ab}K_{ab}$ is the trace of the extrinsic curvature K_{ab} defined by $K_{ab} = \nabla_a n_b$ where n is the unit vector normal to the brane $\mathcal{B}$ with a projection of indices onto $\mathcal{B}$ from $\mathcal{M}$. Moreover we have to remember that the cosmological constant Λ is related to the AdS$_{d+1}$ radius L by $\Lambda = -d(d+1)/2L^2$.
Now let's compute the variation of the above action with respect to the metric

$$\delta S = \frac{1}{16\pi G_5}\int_{\mathcal{B}} \sqrt{-h}\Big[K_{ab}\delta h^{ab} - (K-\lambda)h_{ab}\delta h^{ab}\Big] \tag{5.4}$$

where the terms coming from the variation of the integral on $\mathcal{M}$ vanish because of the Einstein equations on $\mathcal{M}$ and the terms involving the derivative of δh^{ab} vanish thanks to the boundary term. Since the boundary component $\mathcal{B}$ is a brane, let's impose Neumann boundary conditions by setting the coefficients of δh^{ab} to zero

$$K_{ab} - Kh_{ab} + \lambda h_{ab} = 0 \tag{5.5}$$

Now, let's write down the AdS$_5$ line element in Poincaré patch coordinates

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dz^2 + dy^2 + dx_1^2 + dx_2^2) \tag{5.6}$$

here the $y \in \mathbb{R}$ is the horizontal direction in figure 5.4, $z > 0$ is the vertical direction and $\vec{x} = (x_1, x_2)$ represents 2 real transverse directions which are suppressed in the figure together with $t \in \mathbb{R}$.
The brane is the surface $z = -y\tan\theta$, where θ is some angle between 0 and $\pi/2$. We can compute the extrinsic curvature of this surface getting

$$K_{ab} = \frac{\cos\theta}{L}h_{ab} \tag{5.7}$$

and plugging into the boundary condition 5.5 we obtain a relation between the angle θ and the parameter λ representing the tension of the brane

$$\lambda = \frac{3}{L}\cos\theta \tag{5.8}$$

where it has been used $h^{ab}h_{ab} = 4$.
Finally by substituting $y = -z\cot\theta$ in the AdS$_5$ line element and making the change of variable $\tilde{z} = z/\sin\theta$, we can see that the induced metric on the brane is effectively AdS$_4$ with radius

$$l = \frac{L}{\sin\theta} \tag{5.9}$$

[1] The Gibbons-Hawking term of the remaining half of the AdS$_5$ conformal boundary has been omitted for simplicity

It can be shown that when λ becomes equal or larger than the critical tension $\lambda_c = 3/L$, the induced metric on the brane becomes Minkowski or de Sitter and the boundary of the brane is no longer timelike but changes discontinuously to be lightlike or spacelike. From now on, we focus only on subcritical branes with $|\lambda| < \lambda_c$, since they are the only ones relevant for our purposes.
One of the most surprising feature displayed by subcritical Karch-Randall branes is the fact that the d-dimensional graviton localized on the brane picks up a mass. This has been proved by studying the spectrum of linearized gravity fluctuations around the AdS_4 solution in **[18]**, thus from bulk considerations related to description 1), but this can be deduced in a simpler way through description 3) as a direct consequence of the transparent boundary conditions: any gravitational theory on any AdS_d space, not necessarily living in the world-volume of a subcritical Karch-Randall brane, has a description in terms of a CFT_{d-1} with a conserved stress tensor and once we couple this CFT_{d-1} with transparent boundary conditions to another system (in our case the CFT_d living on half of the AdS_{d+1} bulk's conformal boundary) to which it can transfer energy, the stress tensor exhibits a leakage, is not longer conserved and gets an anomalous dimension which translates into a massive graviton in the dual theory according to the AdS/CFT correspondence dictionary.

Moreover, **[18]** shows that the wavefunction of the lightest but massive graviton is normalizable and gives the following relation between the Newton constant on the brane and the brane angle

$$G^{-1} \sim \frac{1}{\theta^{d-2}} \tag{5.10}$$

Moreover it turns out that in the $\theta \to 0$ limit, together with the vanishing of G and the divergence of the curvature radius on the brane l (5.9), the mass of the lightest graviton localized on the brane goes smoothly to zero, so a massless graviton re-emerges.

Extremal surfaces and Page curve

The second holographic interpretation, among the three discussed previously, is the one of interest for the black hole information paradox: to pose the paradox **[10]**,**[12]**,**[13]**,**[14]**, the bulk empty AdS_{d+1} metric is replaced by an AdS_{d+1} black string metric, inducing a planar AdS_d-Schwarzschild black hole on the brane; in this way the black hole on the brane will couple to the CFT_d living on remaining half of the conformal boundary of the bulk AdS_{d+1}, which serves as a bath, and as long as we keep the bath at a finite temperature T matching the Hawking temperature of the black hole on the brane, they will be in stable equilibrium. Even if this setup is not suited to calculate the Page curve of an evaporating black hole, it will be very useful to identify the quantum extremal surfaces contributing to the radiation entanglement entropy, in particular to establish the existence of islands.
In this Karch-Randall setup with a non-gravitating bath, the outgoing radiation can be captured by calculating the entanglement entropy of a region $\mathcal{R}$, which it will be called radiation region, in the ambient space of the BCFT far away from the defect. Naively, $\mathcal{R}$ has nothing to do with the quantum extremal surfaces living on the world-volume of the Karch-Randall brane, but according to the island prescription (5.2), its entanglement entropy must allow contributions from islands located elsewhere, in particular near the brane-world black hole

$$S(\mathcal{R}) = \min_{\mathcal{I}}\{\mathrm{ext}[S_{gen}(\mathcal{R} \cup \mathcal{I})]\} \tag{5.11}$$

where the island $\mathcal{I}$ is a codimension-1 region on the brane disconnected from $\mathcal{R}$, while S_{gen} denotes the generalized entropy functional used in the quantum extremal surface prescription (5.1)

$$S_{gen}(\mathcal{R} \cup \mathcal{I}) = \frac{A(\partial\mathcal{I})}{4G_N} + S_{\mathrm{matter}}(\mathcal{R} \cup \mathcal{I}) \tag{5.12}$$

where $\partial\mathcal{I}$ is the quantum extremal surface, over which we minimize S_{gen} to obtain the von Neumann entropy of radiation region. Since, as proved by Porrati **[46]**,**[47]**, in the context of Karch-Randall braneworlds gravity (hence also graviton mass) can be thought as entirely induced by loops of matter fields on AdS_d, the area term can be included in S_{matter}. Then using holography, the CFT matter in description 2) can be related to its bulk dual in description 1), so that the leading semiclassical order of S_{matter} can be computed using only the geometrical data in the bulk AdS_{d+1}, i.e. using the standard Ryu-Takayanagi minimal area prescription (4.38). Let us say we want to calculate the entanglement entropy of a radiation region located at $y \geq y_0$, for some fixed y_0. As it will be clear in a moment, the full black hole geometry involves a second boundary, thus a second copy of the CFT behind the horizon implying we must include a copy of the $y > y_0$ region in the second CFT as part of the radiation region $\mathcal{R}$ itself.

There are two kinds of extremal surface appear to contribute to the holographic entanglement entropy:

- the Hartman-Maldacena surfaces, stretching between the boundary of the radiation region and the corresponding point in the thermofield double;
- the island minimal surfaces, which start from the bath extending until the gravity system, i.e. the ETW brane.

The Hartman-Maldacena surfaces **[19]** are present independently from the ETW brane and they go straight down through the horizon connecting via an Einstein-Rosen bridge to the second boundary of the fully extended black hole geometry. One very important aspect of these surfaces is that their area grows linearly with time together with the length of the Einstein-Rosen bridge. On the other hand, the presence of the Karch-Randall brane allows the existence of another kind of RT surface, the island surface**[48]**, which does not stretch across the Einstein-Rosen bridge and end on the brane outside the black hole horizon; as a consequence, its area stays constant in time.
By constructing numerically this second kind of extremal surface, **[48]** found that at $t = 0$ this surface has a larger area than the Hartman-Maldacena surface. This means that the entanglement entropy of the radiation region $\mathcal{R}$ is initially due to the Hartman-Maldacena contribution and it grows linearly in time together with the Hartman-Maldacena surface's area. Then at a certain time, the so called Page time, the areas of two kind of extremal surfaces will be equal and after that time the entanglement entropy will be dominated by the island contribution.

A brief digression on Hartman-Maldacena surfaces

It's worthwhile to go into details about Hartman-Maldacena surfaces, in particular how they have been found **[19]** and their physical interpretation.
The metric of an AdS_{d+1} black string has the form

$$ds^2 = \frac{1}{z^2}\left[-(1-z^d)dt^2 + \frac{dz^2}{1-z^d} + dx_{d-1}^2\right] \tag{5.13}$$

and introducing the coordinate ρ, defined as

$$d\rho = \frac{dz}{z\sqrt{(1-z^d)}} \tag{5.14}$$

can be rewritten as

$$ds^2 = -g^2(\rho)dt^2 + h^2(\rho)dx_{d-1}^2 + d\rho^2 \tag{5.15}$$

where the functions h and g are

$$h = \frac{2}{d}\left[\cosh\left(\frac{d}{2}\rho\right)\right]^{2/d}, \qquad g = h\tanh\left(\frac{d}{2}\rho\right) \tag{5.16}$$

This black hole has an horizon at $\rho = 0$ and temperature $T = 1/2\pi$, as can be easily checked.
Since the Hartman-Maldacena surfaces will reveal to be extremal surfaces penetrating the horizon, let's consider the eternal AdS black hole geometry. An eternal black hole is the black hole

with the full, two-sided Penrose diagram and in AdS an eternal black hole has two boundaries, so is dual to two copies of a conformal field theory in a particular pure entangled state.The interior region corresponds to purely imaginary $\rho = ik$ and $t = t_I - i\pi/2$, with t_I real and spacelike in the interior.
In the eternal black hole geometry, temporal symmetry requires that time must run forwards on one copy and backwards on the other copy of the thermofield double, namely in the two conformal field theories living on the two boundaries of the eternal black hole. Since we want to introduce a time dependence in our setup, we choose to make the time run forwards on both copies and we separate each copy in two halves at some time t_b, which will be the same on both sides.

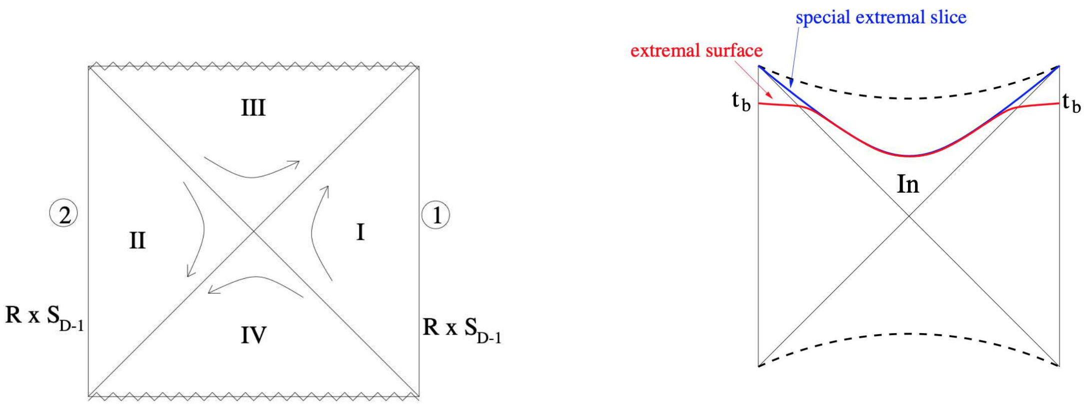


Figure 5.5: Full-extended Penrose diagram of an AdS_{d+1} black hole (left) and extremal surface computing the entanglement entropy (right), sitting very close to a special critical spacelike surface in the interior for $t_b \gg 1$[**19**]

At this point, using the Ryu-Takayanagi formula, we want to calculate the entanglement entropy of the region A, corresponding to the union of the same half of each of the two copies of the thermofield double. Finding the extremal codimension-2 surface that ends on the boundaries of region A amounts to find a function $t(\rho)$ describing how the surface moves in the t, ρ direction. The area, in units of $\vec{x}$ volume, needed to be extremized is

$$\mathcal{A} = V_{d-2} \int dt\, [h(\rho)]^{d-2} \sqrt{-g^2(\rho) + \dot{\rho}^2} \tag{5.17}$$

Since the surface is symmetric under the time reflection $t_I \to -t_I$, we expect that $\dot{\rho} = 0$ at $t_I = 0$. Moreover, since the area integrand $\mathcal{L} = [h(\rho)]^{d-2}\sqrt{-g^2(\rho) + \dot{\rho}^2}$ does not depend explicitly on t, we obtain a quantity conserved along any solution of the equation of motion

$$C = \dot{\rho}\frac{\partial L}{\partial \dot{\rho}} - L = \frac{g^2 h^{d-2}}{\sqrt{-g^2 + \dot{\rho}^2}} = -ig_0 h_0^{d-2} \tag{5.18}$$

where in the last equality we evaluate g and h at the interior point $\rho_0 = ik_0$ at which $\dot{\rho} = 0$.
In particular we can obtain

$$t(\rho) = -i\frac{\pi}{2} - \int_{ik_0}^{\rho} \frac{d\rho'}{g\sqrt{1 - \frac{g^2 h^{2d-4}}{g_0^2 h_0^{2d-4}}}} \tag{5.19}$$

with the integral having a pole at the horizon $\rho = 0$, for this reason the integral must be taken along a contour which avoids the pole.
Then the area is simply given by

$$A = 2V_{d-2} \int_{ik_0}^{\infty} d\rho \frac{h^{d-2}}{\sqrt{1 - \frac{g_0^2 h_0^{2d-4}}{g^2 h^{2d-4}}}} \tag{5.20}$$

where the factor 2 comes from the symmetry of the surface.
This integral is finite as long as $\rho_0 = ik_0$ is not the point where the function $a = (-ig)h^{d-2}$ is extremized, which we will denote with $\rho_{max} = ik_{max}$. Indeed this function a vanishes at the horizon, starts growing in the interior reaching a maximum at ρ_{max} and then it decreases, as can be checked explicitly

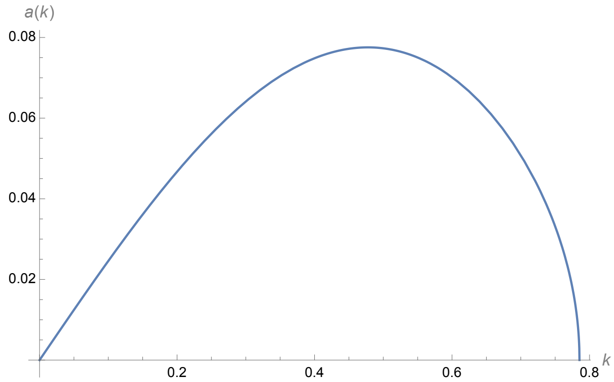


Figure 5.6: Plot of the function $a(k) = (-ig(ik))h^{d-2}(ik)$ with $d = 4$

For finite values of $t_b = t(\rho = \infty)$ we have $k_0 < k_{max}$, but as t_b becomes large we have that $k_0 \to k_{max}$. For large t_b, the minimal extremal surface has two regions: the region where the surface crosses the horizon and goes near the boundary, which gives a constant to (5.20) for $t_b \gg 1$, and the piece of surface in the interior lying at $k \sim k_{max}$ for a long range of the spacelike interior t-direction. The interior piece gives a large contribution linear in t_b to the area, which can be computed by setting $\dot{\rho} = 0$ and $g \to g_{max}$, $h \to h_{max}$ in (5.17). The entanglement entropy for large t_b thus reads

$$S = \frac{A}{4G_N} = \frac{V_{d-2}}{2G_N}(a_{max}t_b + \text{const.}) \tag{5.21}$$

If, instead of a full-extended AdS_{d+1} eternal black hole, we consider an eternal black hole solution with an end of the world brane that cuts it in an half, the holographic computation of entanglement entropy is identical to the one we have just carried out, except for the fact that now the extremal surface ends at the end of the world brane sitting at $t_I = 0$ and the entanglement entropy is the same up to a factor of two, since the end of the world brane cuts off one of the two sides of the full-extended geometry.
The linear growth of entanglement entropy in t_b is due to the fact that the extremal surface is growing along the t_I direction inside the horizon. Indeed the shape of these extremal surfaces resembles the shape of the so called "nice slices" **[49]**, which grow as well since we keep adding space in the interior, making the range of t_I bigger.

Islands and area difference at $t = 0$, $\theta = \pi/2$

A possible way to find the island RT surface that will contribute to the entanglement entropy at late times is to consider the one-parameter family of surfaces ending on the brane and find among these surfaces the one with minimal area.
An alternative approach is to impose that the island RT surface must end orthogonally on the brane: indeed the full Karch-Randall geometry includes two identical copies of the geometry shown in figure 5.4 and to reduce the full Karch-Randall solution to a single-sided copy, we need to impose an orbifold projection across the brane. Thus to be consistent with this orbifold projection, the RT surface must be symmetric in the double-sided geometry with respect to the brane itself.
As an example, we can construct the islands in the simplest Karch-Randall model, the one with $\theta = \pi/2$. In this case the background metric is given by an AdS_5 planar Schwarzschild black

hole

$$ds^2 = \frac{1}{z^2}\Big(-h(z)dt^2 + \frac{dz^2}{h(z)} + dy^2 + d\vec{x}^2\Big), \qquad h(z) = 1 - \Big(\frac{z}{z_h}\Big)^4 \tag{5.22}$$

We can obtain the island extremal surface with $t = 0$ and $y = y(z)$ imposing the boundary conditions $y(0) = y_0$, since the island surface must be anchored at the boundary $y = y_0$ of the radiation region $\mathcal{R}$ located on the conformal boundary $z = 0$, and

$$\frac{1}{y'(z_*)} = 0, \qquad \text{with} \qquad y(z_*) = 0 \tag{5.23}$$

The latter condition is equivalent to impose that at $y = 0$ it must be $z'(0) = 0$, which is the Neumann boundary condition required for consistency with the orbifold projection $y \to -y$ as discussed earlier.
The area per unit volume in $\vec{x}$ space reads

$$A = \int_0^{z_*} dz \frac{1}{z^3}\sqrt{\frac{1}{h} + (y')^2} \tag{5.24}$$

The extremality condition can be easily solved by exploiting the fact that the integrand of the area functional does not depend on y. Once found $y_{ext}(z)$ and fixed its integration constant imposing (5.23), we plug it into the area function which subsequently results to be

$$A = \int_0^{z_*} dz \frac{1}{z^3\sqrt{h}\sqrt{1-(z/z_*)^6}} \tag{5.25}$$

This area has a divergence near $z = 0$, which is the characteristic UV divergence of the entanglement entropy. We can regulate this divergence in two ways, imposing a cutoff $z = \varepsilon$ or computing the difference between the area of the island surface and the area of the Hartman-Maldacena surface, which in this case with $\theta = \pi/2$ it's given by $y = y_0$. In the latter case, the divergence disappear since this is common to both extremal surfaces which are anchored at the point $y = y_0$ on the AdS_5 conformal boundary. The finite area difference is thus given by

$$\Delta A = \int_0^{z_*} dz \frac{1}{z^3\sqrt{h}}\Big(\frac{1}{\sqrt{1-(z/z_*)^6} - 1}\Big) - \int_{z_*}^{z_h} dz \frac{1}{z^3\sqrt{h}} \tag{5.26}$$

where we have used the fact that for the Hartman-Maldacena surface $y = y_0$ the area functional reduces to (5.24) with $y' = 0$ and this surface extends up to the horizon, thus the integration from $z = 0$ to $z = z_h$. In **[13]**, by numerically computing pairs $(y_0, \Delta A)$ for several values of z_*, setting $z_h = 1$, the authors found that the area difference is monotonically increasing and becomes positive for large y_0 and this is the reason why we must take the radiation region $\mathcal{R}$ far away from the defect in order for it to capture the effects of the escaping Hawking radiation.
A natural question now emerges about what happens to extremal surfaces in the smooth massless graviton limit $\theta \to 0$. The Hartman-Maldacena surface is clearly unaffected by this limit, the island surface approach now the AdS_{d+1} conformal boundary on both its endpoints. This means that the island surface will gain an additional divergent area contribution which cannot be cancelled anymore by the area of the Hartman-Maldacena surface. This means that the time t_* at which the area of Hartman-Maldacena surface exceeds the one of the island surface will diverge as $1/\theta^{d-2}$, since the divergence can be seen as arising from the vanishing of the Newton's constant on the brane G (5.10). The divergence of t_* implies that the island contribution will never be important to the entanglement entropy since Hartman-Maldacena surfaces become always dominant.
Actually it was numerically found in **[50]** that at zero temperature (i.e. in empty AdS_{d+1}) the island surface's endpoint on the brane runs to the AdS_{d+1} conformal boundary already if the brane angle drops below a critical angle. At finite temperature, island surfaces still exist below the critical angle but they are anchored beyond a critical anchor point on the brane. This critical anchor point turns out to be an asymptotically decreasing function of brane angle, in particular it decreases from the horizon and it vanishes at the critical angle.

Karch-Randall models with gravitating bath

A gravitating bath can be realized by introducing a second end of the world brane as a bath. The geometry we are going to consider is the one in the figure below.

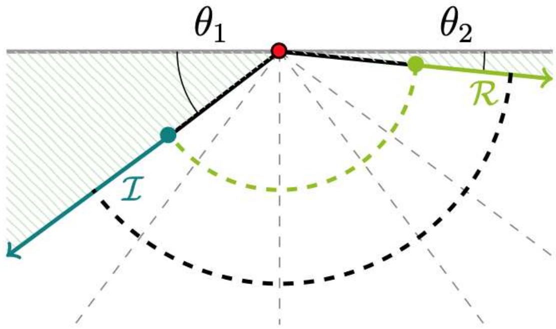


Figure 5.7: Embedding of a Karch-Randall braneworld with two subcritical branes in Anti-de-Sitter space. $\mathcal{R}$ denotes the radiation region, $\mathcal{I}$ the island on the brane, the dashed green line is the candidate RT surface and the dashed black line is the black string horizon **[14]**

We refer to the left brane at an angle θ_1 as the physical brane and the right brane at an angle θ_2 as the bath brane. Both branes, as in the Karch-Randall model with non-gravitating bath, are taken to be subcritical, so the geometry of both the physical and bath brane is asymptotically AdS_d.
Let's start by considering the vacuum solution, i.e. the empty AdS_{d+1} case. There are two useful coordinate systems we can employ. The Poincaré patch coordinate system on AdS_{d+1}, assuming it has unitary curvature radius, reads as usual

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + dy^2 + d\vec{x}^2) \tag{5.27}$$

where, as before, $y \in \mathbb{R}$ is the coordinate in the horizontal direction, $z > 0$ is the coordinate in the vertical direction and $\vec{x}$ corresponds to the coordinate in the $d-2$ real transverse directions suppressed in the figure. The conformal boundary of AdS_{d+1} is at $z = 0$ and the defect, representing the intersection between the conformal boundary and the two branes is at $y = 0$.
However there is a more suitable coordinate system for our subcritical branes' geometry

$$ds^2 = \frac{1}{\sin^2\theta}\Big(\frac{-dt^2 + du^2 + d\vec{x}^2}{u^2} + d\theta^2\Big) \tag{5.28}$$

where $u > 0$ and $\theta \in (0, \pi)$ represents the radial and angular coordinate of the spherical coordinate system centered on the defect. In this coordinate system, the physical and bath branes are located respectively at $\theta = \theta_1$ and $\theta = \pi - \theta_2$. Notice that the metric in each slice at constant θ is an AdS_d geometry, with u playing the role of the associated radial Poincaré patch coordinate. The change of coordinates between two system is given by

$$\begin{cases} z = u\sin\theta \\ y = -u\cos\theta \end{cases} \tag{5.29}$$

In the Poincaré patch coordinate system, the area density per unit $\vec{x}$ volume for an embedding $y(z)$ can be written as

$$\mathcal{A} = \int \frac{dz}{z^{d-1}}\sqrt{1 + y'(z)^2} \tag{5.30}$$

whose extremization gives

$$y'(z) = \pm\frac{z^{d-1}}{\sqrt{z_*^{2(d-1)} - z^{2(d-1)}}} \tag{5.31}$$

where z_* denotes the depth of the turnaround point of the minimal surface and it separates two branches of the solution which goes towards smaller values of z. The only solution which does not have a turnaround point and reaches the Poincaré patch horizon is $y' = 0$, the straight vertical line corresponding to $z_* \to \infty$, while all the other solution turn around and reach either one of the two branes giving rise to an island, the one denoted with $\mathcal{I}$ in the figure.
Using the other coordinate system, we can show that the minimal area surface must end orthogonally to both branes. Indeed, the area functional for the embedding $u(\theta)$ becomes

$$\mathcal{A} = \int_{\theta_1}^{\pi-\theta_2} \frac{d\theta}{(u\sin\theta)^{d-1}} \sqrt{u^2 + u'(\theta)^2} \tag{5.32}$$

In deriving the equation of motion

$$0 = \delta\mathcal{A} = \frac{\delta u}{(u\sin\theta)^{d-1}} \frac{u'}{\sqrt{u^2+u'^2}}\Big|_{\theta_1}^{\pi-\theta_2} - \int_{\theta_1}^{\pi-\theta_2} d\theta(\text{e.o.m.})\delta u \tag{5.33}$$

The first term is the boundary term obtained through the integration by parts, and in order for it to vanish we need to impose Neumann boundary conditions

$$u'(\theta_1) = u'(\pi - \theta_2) = 0 \tag{5.34}$$

from this we get an analytical check of the symmetry arguments based guess of Neumann boundary condition of the island RT surface on the brane we have made before for the single brane configuration ($\theta_2 = 0$).
For the embedding $y(z)$ this orthogonality condition translates, through the use of the change of coordinate (5.29), into

$$y' = -\frac{z}{y} \tag{5.35}$$

consistent once again with (5.23).
If instead of empty bulk AdS_{d+1} we consider the AdS_{d+1} black string, the metric in spherical coordinates (u, θ) reads

$$ds^2 = \frac{1}{u^2\sin^2\theta}\Big[-h(u)dt^2 + \frac{du^2}{h(u)} + d\vec{x}^2 + u^2 d\theta^2\Big], \qquad h(u) = 1 - \frac{u^{d-1}}{u_h^{d-1}} \tag{5.36}$$

The area functional for the embedding function $u(\theta)$ results to be

$$\mathcal{A} = \int_{\theta_1}^{\pi-\theta_2} \frac{d\theta}{(u\sin\theta)^{d-1}} \sqrt{u^2 + \frac{u'(\theta)^2}{h(u)}} \tag{5.37}$$

and the boundary conditions at the branes are still Neumann type as for the empty AdS_{d+1}.
While in the Karch-Randall model with non-gravitating bath one requires a Dirichlet boundary condition on one end of the RT surface, namely we fix y_0 as the endpoint on the BCFT living on the conformal boundary of bulk AdS geometry so that y_0 defines the radiation region $\mathcal{R}$ on the bath, in the case of gravitating bath we cannot define a fixed radiation region and we must minimize over both the endpoints of the RT surface on the branes, in particular we let the boundary of radiation region $\partial\mathcal{R}$ adjust in order to minimize the entropy. This translates into Neumann boundary conditions on both RT surface's endpoints as we have seen.
Taking into account what we have just stated, let's notice that the only island-free RT surface, i.e. the only surface that does not turn around, is the trivial and Hartman-Maldacena like surface $y' = 0$, but this surface can never be orthogonal to the bath brane if $\theta_2 \neq 0$. Therefore, we have a unique always dominating RT surface whose area doesn't grow in time, leading to a flat entropy curve.
Specifically, if one allows the end points of the minimal surface on both ETW branes to be chosen dynamically, at zero temperature (empty AdS_{d+1}) the radiation region $\mathcal{R}$ on the bath would reduce to a point and the entanglement entropy would be zero, while at finite temperature (AdS_{d+1} black string) the minimal surface would settle on the black string horizon. In neither case we get a Page curve.

Left/Right Entanglement Entropy

Although the conventional geometric entanglement entropy of radiation does not lead to a Page curve in the Karch-Randall model with gravitating bath at both zero and finite temperature, the time-dependence can be found in some other form of entanglement entropy for finite temperature setups. The crucial point is that the defect is non-gravitating, thus we should split its degrees of freedom into a "left" and "right" part and call the entanglement entropy between these two parts the left/right entanglement entropy.
In computing the left/right entanglement entropy, we must consider potential RT surfaces that start from the defect where the two branes meet. There are once again two classes of candidate minimal surfaces: the Hartman-Maldacena surface connecting the defect to its thermofield double and the island surface connecting the defect to either brane. The quantum entangling surface is chosen as usual picking the minimal area one between the two candidate surfaces. In this way, choosing to anchor the minimal surface at the remaining point of the conformal boundary of AdS_{d+1}, i.e. at the defect, it would emerge a Page curve as shown in **[14]** and the resulting entanglement entropy can be interpreted as the one between the left and right defect degrees of freedom.

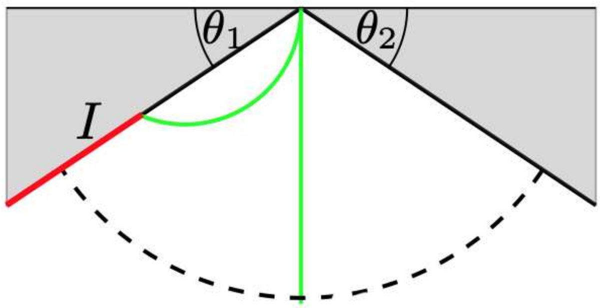


Figure 5.8: Quantum extremal surfaces contributing to left/right entanglement entropy in Karch-Randall models with a gravitating bath. Both kind of quantum extremal surfaces starts from the defect and ends either on one of the two ETW branes or on the horizon **[14]**

Then the authors of **[14]** numerically determined the island surfaces, namely the RT surfaces starting on the defect and ending on one brane with Neumann boundary conditions (5.34) and they found a peculiar behaviour: surfaces, imposed to end orthogonally on either of the two brane and to start from the defect, exist only for branes below a special value for the angle, called the critical angle θ_c. This critical angle depends only on the dimension of the space and the anchor point of the RT surface on the brane results to be a monotonically decreasing function of the brane angle vanishing (i.e. dropping into the defect itself) at the critical angle θ_c.
Above the critical angle $\theta > \theta_c$, it was found that the left/right entanglement entropy is dominated by the so called tiny island surfaces, infinitesimal surfaces obtained as asymptotic limits near the defect, whose area difference with the Hartman-Maldacena surface is $-\infty$, and which are subdominant when the brane is below the critical angle θ_c.
Finally, they found that for much smaller angles than the critical one, the island surface has a larger area than $t = 0$ Hartman-Maldacena surface. This implies as usual that the initial entangling surface is the Hartman-Maldacena one, whose area growth reflects the initial area growth of the entanglement entropy which saturates once the island surface becomes the dominant one. However it was found that at an angle, called Page angle θ_P, which results to be a bit smaller than the critical one, and in the range of angles $\theta \in (\theta_P, \theta_c)$, the island surface is already dominating at $t = 0$ thus leading to a trivial entropy curve.

Chapter 6

Exact half-BPS Type IIB interface solutions

D'Hoker, Estes and Gutperle, in a pair of articles **[20]**,**[21]**, have constructed all type IIB supergravity solutions with 16 supersymmetries, thus half-BPS solutions, on $AdS_4 \times S^2 \times S^2 \times \Sigma$ space-time geometry with $SO(2,3)\times SO(3)\times SO(3)$ symmetry in terms of two harmonic functions on a Riemann surface Σ. These solutions are holographically duals to conformal interface theories with 16 conformal supersymmetries. Indeed the addition of a planar interface to 4-dimensional $\mathcal{N}=4$ super Yang-Mills theory reduces the conformal symmetry group $SO(2,4)$ to $SO(2,3)$, which is precisely the isometry group of AdS_4, and the 32 superconformal symmetries may be reduced to either 0, 4, 8 or 16 with maximal internal symmetry groups $SO(6)$, $SU(3)$, $SU(2)\times U(1)$ and $SO(3)\times SO(3)$ respectively **[51]**.
These geometries are of very interest for our work since the Karch-Randall models are boundary conformal field theories BCFT$_4$, in the case in which the black hole is coupled to a non-gravitating bath, and defect conformal field theories, in the case of gravitating bath. Thus the uplifts in a 10-dimensional string theory background of these defect/interface conformal field theories will be some of this class of geometries corresponding to a suitable choice of the two harmonic functions on Σ.
As stressed several times, the bosonic fields of type IIB supergravity are the metric $g_{\mu\nu}$, the axion-dilaton field τ, the complex 2-form $B_2^c = B_2 + iC_2$ and the real R-R 4-form C_4.
In order to obtain field equations in a simpler form, let us introduce the following composite fields

$$\begin{aligned} B &= \frac{1+i\tau}{1-i\tau} \\ f &= (1-|B|^2)^{-1} \\ P &= f^2 dB \\ Q &= f^2 \mathrm{Im}(B d\bar{B}) \\ G &= f(F_3^c - B\bar{F}_3^c) \\ \tilde{F}_5 &= dC_4 + \frac{i}{4}(B_2^c \wedge \bar{F}_3^c - \bar{B}_2^c \wedge F_3^c) \end{aligned} \tag{6.1}$$

where $F_3^c = dB_2^c$ and the field strength $\tilde{F}_5$ is required to be self-dual as usual

$$\tilde{F}_5 = *\tilde{F}_5 \tag{6.2}$$

In terms of this set of composite fields, the Bianchi identities are given by

$$\begin{aligned} dP - 2iQ \wedge P &= 0 \\ dQ + iP \wedge \bar{P} &= 0 \\ dG - iQ \wedge G + P \wedge \bar{G} &= 0 \\ d\tilde{F}_5 - \frac{i}{2} G \wedge \bar{G} &= 0 \end{aligned} \tag{6.3}$$

while the field equations are given by

$$\begin{aligned}\nabla^\mu P_\mu - 2iQ^\mu P_\mu + \frac{1}{24}G_{\mu\nu\rho}G^{\mu\nu\rho} = 0 \\ \nabla^\rho G_{\mu\nu\rho} - iQ^\rho G_{\mu\nu\rho} - P^\rho \bar{G}_{\mu\nu\rho} + \frac{1}{6}i(\tilde{F}_5)_{\mu\nu\rho\sigma\lambda}G^{\rho\sigma\lambda} = 0 \\ R_{\mu\nu} - P_\mu \bar{P}_\nu - \bar{P}_\mu P_\nu - \frac{1}{96}(\tilde{F}_5^2)_{\mu\nu} - \frac{1}{8}(G_\mu{}^{\rho\sigma}\bar{G}_{\nu\rho\sigma} + \bar{G}_\mu{}^{\rho\sigma}G_{\nu\rho\sigma}) + \frac{1}{48}g_{\mu\nu}G^{\rho\sigma\lambda}G_{\rho\sigma\lambda} = 0\end{aligned} \tag{6.4}$$

The fermionic fields are the dilatino λ and the gravitino ψ_μ, both of which are complex Weyl spinors with opposite chirality. The supersymmetry variations of the fermions are

$$\begin{aligned}\delta\lambda = i(\Gamma\cdot P)\mathcal{C}^{-1}\varepsilon^* - \frac{i}{24}(\Gamma\cdot G)\varepsilon \\ \delta\psi_\mu = D_\mu\varepsilon + \frac{i}{1920}(\Gamma\cdot\tilde{F}_5)\Gamma_\mu\varepsilon - \frac{1}{96}(\Gamma_\mu(\Gamma\cdot G) + 2(\Gamma\cdot G)\Gamma^\mu)\mathcal{C}^{-1}\varepsilon^*\end{aligned} \tag{6.5}$$

where $\mathcal{C}$ is the charge conjugation matrix of the Clifford algebra, which is such that $\mathcal{C}\mathcal{C}^* = \mathbb{I}$ and $\mathcal{C}\Gamma^\mu\mathcal{C}^{-1} = (\Gamma^\mu)^*$. The BPS equations are obtained by setting $\delta\lambda = \delta\psi_\mu = 0$.
The ansatz of type IIB supergravity solution with $SO(2,3)\times SO(3)\times SO(3)$ symmetry for the metric is

$$ds^2 = f_4^2 ds^2_{AdS_4} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + 4\rho^2 dz\bar{z} \tag{6.6}$$

where f_1, f_2, f_4 are functions on Σ. The factor $SO(2,3)$ requires the geometry to contain AdS_4, whereas for the symmetry factor $SO(3)\times SO(3)$ we could have chosen either $S^2\times S^2$ or S^3. But we have to take into account that the Yang-Mills theory with planar interface and maximal supersymmetry has $SO(3)\times SO(3)$ R-symmetry and this reduced R-symmetry split the scalar multiplet into two triplets ϕ^i with $i = 1,3,5$ and $\tilde{\phi}^j$ with $j = 2,4,6$, which transform as **(3,1)** and **(1,3)** respectively. This implies that the natural choice is $S^2\times S^2$, while the two remaining dimensions remain undetermined by the symmetries and the most general space must carry an orientation and a Riemann metric as well, namely it must be a Riemann surface.
Solutions of the form given by the ansatz (6.6) preserving 16 supersymmetries correspond to supergravity fields for which the BPS equations $\delta\lambda = \delta\psi_\mu = 0$ have 16 independent solutions ε. By using the ansatz and symmetry arguments, BPS equations reduce to a integrable system of first order differential equations for two real scalar fields, the dilaton ϕ and the Σ-metric factor ρ, as well as two arbitrary holomorphic functions on Σ. If Bianchi identities and field equations are reduced to $AdS_4\times S^2\times S^2\times\Sigma$ ansatz, they automatically hold whenever ϕ and ρ are solutions to the BPS system of first order equations. By a change of coordinates this integrable system of first order differential equations can be mapped onto a set of linear equations, which is then solved in terms of two holomorphic functions $\mathcal{A}$ and $\mathcal{B}$, or equivalently, two harmonic functions h_1 and h_2.
In writing down the solutions, it is convenient to introduce the dual harmonic functions

$$\begin{aligned}h_1 = -i(\mathcal{A}-\bar{\mathcal{A}}) \qquad &\rightarrow \qquad h_1^D = \mathcal{A}+\bar{\mathcal{A}} \\ h_2 = \mathcal{B}+\bar{\mathcal{B}} \qquad &\rightarrow \qquad h_2^D = i(\mathcal{B}-\bar{\mathcal{B}})\end{aligned} \tag{6.7}$$

which will be defined up to additive real constants, and the following combinations of h_1, h_2 and of their first derivatives (here $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$) :

$$\begin{aligned}W &= \partial h_1\bar{\partial}h_2 + \bar{\partial}h_1\partial h_2 = \partial\bar{\partial}(h_1h_2) \\ N_1 &= 2h_1h_2|\partial h_1|^2 - h_1^2 W \\ N_2 &= 2h_1h_2|\partial h_2|^2 - h_2^2 W\end{aligned} \tag{6.8}$$

The dilaton solution[1] is given by

$$e^{4\phi} = \frac{N_2}{N_1} \tag{6.9}$$

[1]in this case we use $\phi \equiv \Phi/2$ for the dilaton

while metric factors, in the case in which $W \leq 0$, explicitly read

$$\begin{aligned} f_4^8 &= 16\frac{N_1 N_2}{W^2} & \rho^8 &= \frac{N_1 N_2 W^2}{h_1^4 h_2^4} \\ f_1^8 &= 16 h_1^8 \frac{N_2 W^2}{N_1^3} & f_2^8 &= 16 h_2^8 \frac{N_1 W^2}{N_2^3} \end{aligned} \tag{6.10}$$

Type IIB supergravity is invariant under the strong-weak duality, under which we have the following transformation rules

$$h_1 \leftrightarrow h_2, \qquad \phi \leftrightarrow -\phi \tag{6.11}$$

Moreover, let's underline the following combinations of metric factors, which will result to be much useful later on

$$\begin{aligned} f_1^2 f_4^2 &= 4e^{2\phi} h_1^2 \\ f_2^2 f_4^2 &= 4e^{-2\phi} h_2^2 \\ \rho^4 f_1^2 f_2^2 &= 4W^2 \end{aligned} \tag{6.12}$$

The complex 3-form field strength $F_3^c = H_3 + iF_3$ decomposes into the real NS-NS form H_3 and the real R-R form F_3, which in conformal coordinates z on Σ take the form

$$H_3 = \omega^{45} \wedge db_1 \qquad F_3 = \omega^{67} \wedge db_2 \tag{6.13}$$

where ω^{45} and ω^{67} are the volume forms of unit-radius spheres S_1^2 and S_2^2, and

$$\begin{aligned} b_1 &= 2ih_1 \frac{h_1 h_2(\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2)}{N_1} + 2h_2^D \\ b_2 &= 2ih_2 \frac{h_1 h_2(\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2)}{N_2} - 2h_1^D \end{aligned} \tag{6.14}$$

The first contributions to b_1 and b_2 are well-defined single-valued functions on Σ, since both h_1 and h_2 are single-valued, as well as their derivatives, but the dual harmonics $h_{1,2}^D$ are not single-valued in general. As a consequence, if we consider a 3-cycle M_3 decomposed as $M_3 = C_1 \times (S_1^2)^{n_1} \cup C_2 \times (S_2^2)^{n_2}$, with $C_{1,2}$ two closed curves in Σ and n_1 and n_2 the integers representing the number of times M_3 wraps the spheres S_1^2 and S_2^2, the flux through the 3-cycle is given by

$$\oint_{M_3} F_3^c = 8\pi n_1 \oint_{C_1} dh_2^D - 8\pi n_2 i \oint_{C_2} dh_1^D \tag{6.15}$$

where the two contour integrals can be solved by applying the residue theorem.
Finally the 5-form reads

$$F_5 = -4f_4^4 \omega^{0123} \wedge \mathcal{F} + 4f_1^2 f_2^2 \omega^{45} \wedge \omega^{67} \wedge (*_2 \mathcal{F}) \tag{6.16}$$

where ω^{0123} is the volume form of the unit-radius AdS_4, $\mathcal{F}$ is a 1-form on Σ with the property that $f_4^4 \mathcal{F}$ is closed and $*_2$ denotes Poincaré duality with respect to the Σ metric. The explicit expression for $\mathcal{F}$ is given by

$$f_4^4 \mathcal{F} = dj_1 \qquad \text{with} \qquad j_1 = 3\mathcal{C} + 3\bar{\mathcal{C}} - 3\mathcal{D} + i\frac{h_1 h_2}{W}(\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2) \tag{6.17}$$

where $\mathcal{C}$ and $\mathcal{D}$ are defined respectively as $\mathcal{C} = \mathcal{A}\partial\mathcal{B} - \mathcal{B}\partial\mathcal{A}$ and $\mathcal{D} = \bar{\mathcal{A}}\mathcal{B} + \mathcal{A}\bar{\mathcal{B}}$.
Once again, in evaluating closed contour integral of dj_1, the last term does not contribute being single-valued

$$\oint_C dj_1 = 3\oint_C d(\mathcal{C} + \bar{\mathcal{C}} - \mathcal{D}) \tag{6.18}$$

The above expressions give the local form of the general solution for the ansatz (6.6).

Global regularity and topology constraints

Although any pair of harmonic functions gives a local solution of the supergravity field equations, global consistency requires regularity and topology conditions[**20**].
First of all, the dilaton and the metric functions must be non-singular in the interior of Σ and on the boundary $\partial\Sigma$, except possibly at isolated points on the latter. For example, as we are going to see, an asymptotic $AdS_5 \times S^5$ region, which may be appear in the limit $\mathrm{Re}(z) \to \infty$, is an example of isolated point in $\partial\Sigma$ where f_4 diverges, as well as singularities of probe D5 and/or NS5 branes, for which the metric factors f_1, f_2 and f_4 diverges.
Excluding the isolated points on $\partial\Sigma$, the remaining part of the boundary consists of open segments and we require they correspond to regular interior points, which can be achieved by demanding that throughout each such segment one of the two S^2, but not both, shrink to zero size, implying that the product $f_1 f_2$ must vanish along $\partial\Sigma$ except on the isolated points.
Exploiting this fact, the third combination of metric factors (6.12) and the non-singularity of ρ^2 on $\partial\Sigma$, it follows the vanishing of W on $\partial\Sigma$. Furthermore, by requiring that $\partial\Sigma$ has only a single connected component, it follows that $f_1 f_2$ and W cannot vanish in the interior of Σ, except at isolated points, and by continuity their sign must remain constant throughout the interior of Σ.
A final constraint is that the AdS_4-radius f_4 cannot vanish, since its vanishing would imply the vanishing of both the metric factors f_1 and f_2 originating an unphysical singularity in the type IIB geometry. Combining the non-vanishing of f_4 and the fact that ϕ is non-singular (except possibly on isolated points) with (6.12), we get that $f_1 = 0$ if and only if $h_1 = 0$ and the same conclusion for f_2 and h_2. In particular if on an open segment of $\partial\Sigma$ $f_1 = h_1 = 0$, namely h_1 satisfies a vanishing Dirichlet boundary condition, it can be proved that h_2 must obey a Neumann boundary condition on the same segment and vice versa, so

$$h_1 = \partial_n h_2 = 0 \qquad \text{on } \partial\Sigma_+ \qquad\qquad h_2 = \partial_n h_1 = 0 \qquad \text{on } \partial\Sigma_- \tag{6.19}$$

where ∂_n denotes the derivative normal to the boundary $\partial\Sigma$, while $\partial\Sigma_+$ and $\partial\Sigma_-$ form a partition of $\partial\Sigma$ and they are such that their closures intersect at isolated point.
In the explicit examples of solutions we are going to investigate, we assume that the Riemann surface is given by the infinite strip

$$\Sigma = \left\{ z = x + iy \in \mathbb{Z} \;\middle|\; 0 \leq y \leq \frac{\pi}{2} \right\} \tag{6.20}$$

Global $AdS_5 \times S^5$ solution

The simplest solution is the global $AdS_5 \times S^5$ which is obtained by the following choice of the pair of harmonic functions $h_{1/2}$

$$h_1 = -i\alpha \sinh z + c.c. \qquad \text{and} \qquad h_2 = \hat{\alpha} \cosh z + c.c. \tag{6.21}$$

and they gives using (6.10)

$$\begin{aligned} \rho^4 &= |\alpha\hat{\alpha}| & f_4^2 &= 4\rho^2 \cosh^2\Big(\frac{z+\bar{z}}{2}\Big) \\ f_1^2 &= 4\rho^2 \sin^2\Big(\frac{z-\bar{z}}{2i}\Big) & f_2^2 &= 4\rho^2 \cos^2\Big(\frac{z-\bar{z}}{2i}\Big) \end{aligned} \tag{6.22}$$

writing $z = x + iy$ one recognizes immediately the $AdS_5 \times S^5$ metric

$$ds^2 = L^2[dx^2 + \cosh^2(x) ds^2_{AdS_4} + dy^2 + \sin^2(y) ds^2_{S_1^2} + \cos^2(y) ds^2_{S_2^2}] \tag{6.23}$$

with radius $L^4 = 16|\alpha\hat{\alpha}|$ and constant dilaton $e^{2\phi} = |\hat{\alpha}/\alpha|$.

Supersymmetric Janus solution

A one-parameter deformation of $AdS_5 \times S^5$ solution, whose original non-supersymmetric version was found in [**52**], is given by the supersymmetric Janus configuration given by the real harmonic functions

$$h_1 = -i\alpha \sinh\left(z - \frac{\delta\phi}{2}\right) + c.c. \qquad \text{and} \qquad h_2 = \hat{\alpha}\cosh\left(z + \frac{\delta\phi}{2}\right) + c.c. \tag{6.24}$$

where $\delta\phi$ is a real parameter, which corresponds to the difference of the values that the dilaton assumes in the two asymptotic regions, as we are going to show now.
In the asymptotic regions $x \to \pm\infty$ the Janus geometry approaches an $AdS_5 \times S^5$ one. In particular, for the case $x \to -\infty$, by defining $\mu = e^x$ and by expanding around $\mu = 0$ one can find

$$\begin{aligned} f_4^4 &\approx \frac{|\alpha\hat{\alpha}|}{\cosh(\delta\phi)}\frac{1}{\mu^4} \qquad & \rho^4 &\approx |\alpha\hat{\alpha}|\cosh(\delta\phi) \\ f_1^2 &\approx 4\rho^2 \sin^2 y \qquad & f_2^2 &\approx 4\rho^2\cos^2 y \end{aligned} \tag{6.25}$$

so the asymptotic metric takes the typical form of an $AdS_5 \times S^5$ metric with radius and dilaton

$$L_{5-}^4 = 16|\alpha\hat{\alpha}|\cosh(\delta\phi) \qquad e^{2\phi_-} = \left|\frac{\hat{\alpha}}{\alpha}\right| e^{-\delta\phi} \tag{6.26}$$

A similar result can be found in the limit $x \to +\infty$, where the radius of the $AdS_5 \times S^5$ region is the same and the dilaton is the same up a flip in the sign

$$L_{5+}^4 = 16|\alpha\hat{\alpha}|\cosh(\delta\phi) \qquad e^{2\phi_+} = \left|\frac{\hat{\alpha}}{\alpha}\right| e^{+\delta\phi} \tag{6.27}$$

Thus it results clear that the dilaton difference $\phi_+ - \phi_-$ corresponds to the parameter $\delta\phi$ as anticipated at the beginning.
After having studied the asymptotic limits of the solution, we want to check its regularity.
The harmonic functions, after some simplification, take the form

$$\begin{aligned} h_1 &= 2\alpha\cosh\left(x - \frac{\delta\phi}{2}\right)\sin y \\ h_2 &= 2\hat{\alpha}\cosh\left(x + \frac{\delta\phi}{2}\right)\cos y \end{aligned} \tag{6.28}$$

and we obtain that

$$\begin{aligned} W &= -\alpha\hat{\alpha}\cosh\delta\phi\sin 2y \\ N_1 &= 2\alpha^2\hat{\alpha}\cos y\, D(x,y) \\ N_2 &= 2\alpha\hat{\alpha}^2\sin y\, N(x,y) \end{aligned} \tag{6.29}$$

where the functions N and D are given by

$$\begin{aligned} N(x,y) &= \cosh\left(x - \frac{\delta\phi}{2}\right) + \cosh\left(3x + \frac{\delta\phi}{2}\right) + 2\Big[\cosh\left(x + \frac{3\delta\phi}{2}\right) + \cos 2y \sinh\left(x + \frac{\delta\phi}{2}\right)\sinh\delta\phi\Big] \\ D(x,y) &= \Big[\cos 2y + \cosh\big(2x - \delta\big)\Big]\cosh\left(x + \frac{\delta\phi}{2}\right) + 2\cosh\left(x - \frac{\delta\phi}{2}\right)\cosh\delta\phi\sin y^2 \end{aligned}$$

Thus the dilaton will be given by

$$e^{4\phi} = \frac{1}{2}\left(\frac{\hat{\alpha}}{\alpha}\right)^2 \frac{\cosh\left(x + \frac{\delta\phi}{2}\right)}{\cosh\left(x - \frac{\delta\phi}{2}\right)}\frac{N(x,y)}{D(x,y)} \tag{6.30}$$

The absence of singularities is ensured by the fact that both $N(x,y)$ and $D(x,y)$ are free of

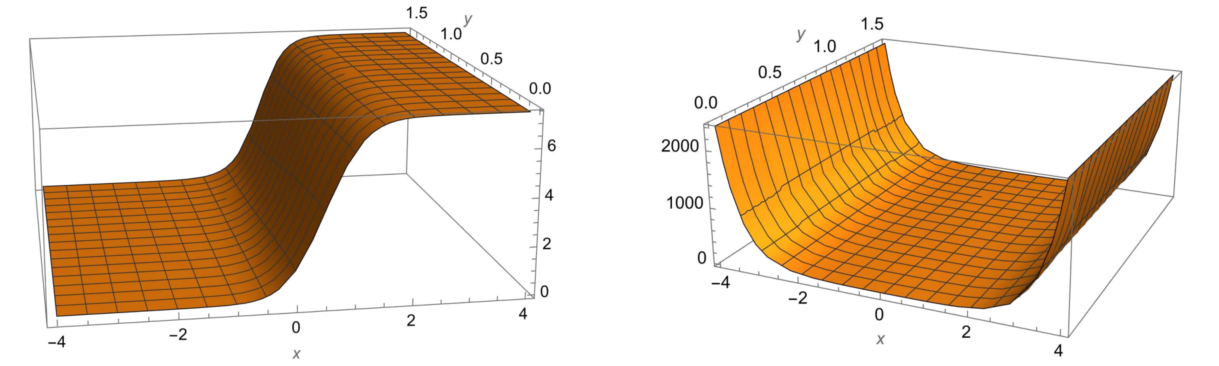


Figure 6.1: Three dimensional plot of the dilaton $e^{4\phi}$ (left) and of the metric factor f_4^2 (right) for the supersymmetric Janus solution as a function of x and y, for $\delta\phi = 1$

zeros being strictly positive over their entire domain.

Then we can find that the the S^2-metric factors $f_{1,2}$ are never singular as a result of the positivity of $N(x, y)$ and $D(x, y)$ as well (the same holds for the AdS_4 metric factor f_4 and for the Σ metric factor ρ) and they goes like $f_1 \sim \sin y$ and $f_2 \sim \cos y$, thus they shrink to zero size only on the boundaries of Σ, i.e. at $y = 0$ and $y = \pi/2$, and in particular they don't vanish simultaneously on $\partial\Sigma$, consistently with regularity constraints.

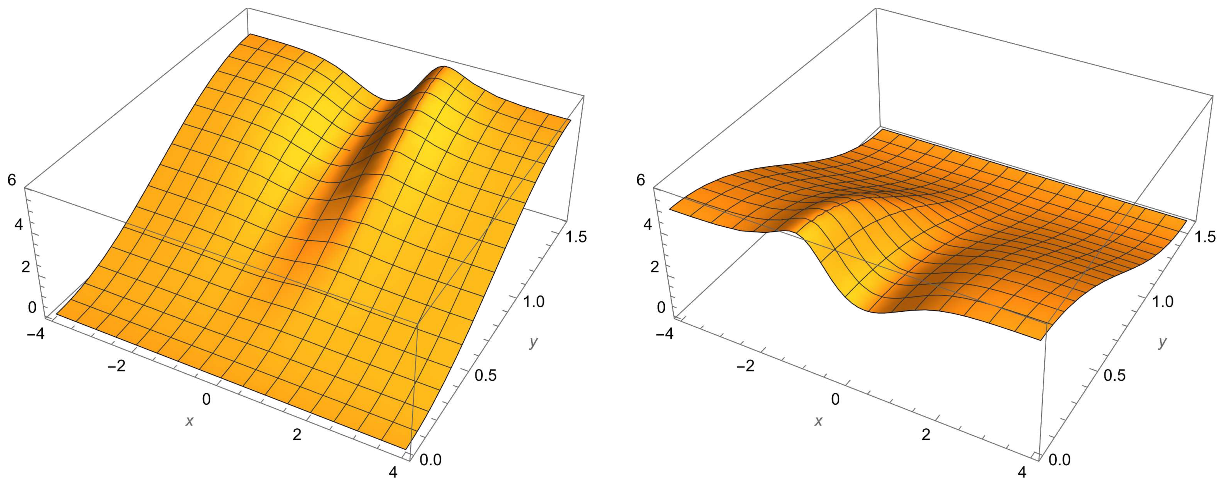


Figure 6.2: Three dimensional plot of the metric factors f_1^2 (left) and f_2^2 (right) for the supersymmetric Janus solution as a function of x and y, for $\delta\phi = 1$

The holographic interpretation of the Janus solution is an interface conformal field theory: the dual 4-dimensional field theory lives on two 4-dimensional half spaces glued together through a three-dimensional interface. Let's see this in more detail.

The boundary of the bulk geometry corresponds to the limit $x \to \pm\infty$ at which the AdS_4 metric blows up as can be seen from (6.25). However there is an additional component of the boundary of the bulk geometry; indeed, employing the Poincaré patch metric for AdS_4

$$ds^2_{AdS_4} = \frac{1}{z^2}(-dt^2 + dw_1^2 + dw_2^2 + dz^2) \tag{6.31}$$

we don't have to forget the boundary component at $z \to 0$. The 10-dimensional asymptotic metric, in the limit $x \to \pm\infty$ and $z \to 0$, recalling the asymptotic behaviour of the metric factors (6.25), is given by

$$ds^2 \sim \frac{1}{z^2\mu^2}\left(z^2 d\mu^2 + \frac{dw_1^2 + dw_2^2 - dt^2 + dz^2}{4\cosh\delta\phi} + z^2\mu^2(dy^2 + \sin^2 y ds^2_{S_1^2} + \cos^2 y ds^2_{S_2^2})\right) + \mathcal{O}(\mu^2)$$

where $\mu = e^{\mp x}$, so that the limit $x \to \pm\infty$ corresponds to the limit $\mu \to 0$.
Thus the boundary is made of three components, two asymptotic 4-dimensional half spaces which are glued together at a 3-dimensional interface at $z \to 0$. Two 4-dimensional SYM theory live in the two halves, with different values of the coupling constant g_{YM}: recall that, according to the AdS/CFT correspondence, the coupling constant of the 4-dimensional $\mathcal{N} = 4$ SYM is identified with the asymptotic value of the dilaton, in particular in this case we get

$$x \to \pm\infty : \frac{g_{YM}^2}{2\pi} = e^{\phi_\pm} \tag{6.32}$$

Janus solutions doped with 5-branes

Now let us consider the solution corresponding to the following choice of real harmonic functions

$$\begin{aligned} h_1 &= \Big[-i\alpha \sinh(z-\beta) - \gamma \ln\Big(\tanh\Big(\frac{i\pi}{4} - \frac{z-\delta}{2}\Big)\Big)\Big] + \text{c.c.} \\ h_2 &= \Big[\hat{\alpha} \cosh\Big(z-\hat{\beta}\Big) - \hat{\gamma} \ln\Big(\tanh\Big(\frac{z-\hat{\delta}}{2}\Big)\Big)\Big] + \text{c.c.} \end{aligned} \tag{6.33}$$

where the parameters $(\alpha, \beta, \gamma, \delta)$ and $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ are all real and $\alpha\gamma$, $\hat{\alpha}\hat{\gamma}$ must be non-negative in order for the solution to be regular inside Σ. Indeed, if $\alpha\gamma$ or $\hat{\alpha}\hat{\gamma}$ were negative, the solution would exhibit curvature singularities supported on a 1-dimensional curve in the interior of Σ [**20**].
This solution describes the near-horizon geometry of stacks of intersecting D3-branes, NS5-branes and D5-branes and preserves the same super-Poincaré and R-symmetries as the brane constructions of linear quiver gauge theories discussed in Appendix B.
Notice that, setting $\gamma = \hat{\gamma} = 0$, we recover the Janus geometry (6.24), and setting furthermore $\beta = \hat{\beta} = 0$ the solution reduces to the global $AdS_5 \times S^5$ (6.21).
The background geometry defined by the harmonic functions (6.33) approaches asymptotically two $AdS_5 \times S^5$ regions at $x \to \pm\infty$, with the values of the radius and the dilaton given by

$$L_{5\pm}^4 = 16|\alpha^\pm \hat{\alpha}^\pm| \cosh\Big(\beta^\pm - \hat{\beta}^\pm\Big) \qquad \text{and} \qquad e^{2\phi_\pm} = \Big|\frac{\hat{\alpha}^\pm}{\alpha^\pm}\Big| e^{\pm(\beta^\pm - \hat{\beta}^\pm)} \tag{6.34}$$

where

$$\alpha^\pm = \alpha\sqrt{1 + \frac{4\gamma}{\alpha} e^{\pm(\delta-\beta)}} \qquad \text{and} \qquad e^{\beta^\pm} = e^\beta \Big(1 + \frac{4\gamma}{\alpha} e^{\pm(\delta-\beta)}\Big)^{\pm 1/2} \tag{6.35}$$

with similar expressions holding for $\hat{\alpha}^\pm$, $\hat{\beta}^\pm$.
The other important feature of this solution is the presence of singularities on the boundary of the strip at $z = \delta + i\pi/2$ and $z = \hat{\delta}$, which describe the presence of stacks of respectively D5 and NS5 branes.
One way to see that a stack of NS5 branes is located at $z = \hat{\delta}$ is to zoom in on the singular region near $z = \hat{\delta}$ by choosing the coordinate $z = re^{i\psi} + \hat{\delta}$ and expanding in small r. This gives at leading order

$$h_1 = 2\frac{\gamma}{\cosh(\delta - \hat{\delta})} r \sin\psi \qquad h_2 = -2\hat{\gamma} \ln\Big(\frac{r}{2}\Big) \tag{6.36}$$

For the dilaton we obtain at leading order

$$e^{4\phi} = \Big(\frac{\hat{\gamma}\cosh(\delta - \hat{\delta})}{\gamma}\Big)^2 \frac{\Big|\ln\Big(\frac{r}{2}\Big)\Big|}{r^2} \tag{6.37}$$

which diverges as $r \to 0$. The metric factors at leading order read

$$\begin{aligned} f_4^2 = f_2^2 &= 4\sqrt{\frac{\gamma\hat{\gamma}}{\cosh(\delta-\hat{\delta})}}\sqrt{r}\left|\ln\left(\frac{r}{2}\right)\right|^{\frac{3}{4}} \\ \rho^2 &= \sqrt{\gamma\hat{\gamma}}\frac{1}{r^{\frac{3}{2}}\left|\ln\left(\frac{r}{2}\right)\right|^{\frac{3}{4}}} \\ f_1^2 &= 4\sqrt{\gamma\hat{\gamma}}\frac{r^{\frac{1}{2}}}{\left|\ln\left(\frac{r}{2}\right)\right|^{\frac{1}{4}}}\sin^2\psi \end{aligned} \tag{6.38}$$

Thus in the vicinity of $r=0$, we find the metric of the warped space $(AdS_4 \times S_2^2) \times_w S^3$

$$ds^2 = f_4^2(r)\Big[ds^2_{AdS_4} + ds^2_{S_2^2}\Big] + 4r^2\rho^2(r)\Big[d\psi^2 + \sin^2\psi ds^2_{S_1^2} + \frac{1}{r^2}dr^2\Big] \tag{6.39}$$

Together with the behaviour of the dilaton, this suggests a supergravity solution corresponding to the NS5 brane geometry (3.55), whose world-volume consists of $AdS_4 \times S^2$ at $z = \hat{\delta}$. The presence of a D5-brane stack at $z = \delta + i\pi/2$ can be showed analogously or more directly by carring out a S-duality transformation which exchanges the two harmonic functions and flips the sign of the dilaton, and transforming the complex coordinate $z \to i\pi/2 - z$.

Another way to show that the singularity $z = \hat{\delta}$ signals the presence of NS5 brane source consists in identifying first of all a non-contractile 3-circle that can support the NS-NS 3-form H_3 flux. This 3-cycle is given by the fibration of the sphere S_1^2 over an open curve $\mathcal{I}$ in Σ, which starts and ends on the lower boundary of the infinite strip enclosing the the point $z = \hat{\delta}$; since the sphere S_1^2 shrinks to zero in the lower boundary, this 3-cycle is topologically a 3-sphere. Through this cycle there is no RR 3-form flux since the RR 3-form F_3 is proportional to the volume of the sphere S_2^2 (6.13). Thus the flux of the NS-NS 3-form H_3 through $\mathcal{C}_3^{\hat{\delta}} = S_1^2 \times \mathcal{I}$ or equivalently the NS5 charge reads

$$Q_{NS5}^{\hat{\delta}} = \int_{\mathcal{C}_3^{\hat{\delta}}} H_3 = \int_{\mathcal{I}} db_1 \int_{S_1^2} \omega^{45} = 4\pi \int_{\mathcal{I}} db_1 = 4\pi b_1|_{\partial\mathcal{I}} \tag{6.40}$$

exploiting the fact that h_1 vanishes on the real z-axis and recalling the expressions of b_1 (6.14) and h_2^D (6.7), we get

$$Q_{NS5}^{\hat{\delta}} = 8\pi h_2^D|_{\partial\mathcal{I}} \qquad \text{with} \qquad h_2^D = \hat{\zeta} + \Big[i\hat{\alpha}\cosh\Big(z-\hat{\beta}\Big) - i\hat{\gamma}\ln\Big(\tanh\Big(\frac{z-\hat{\delta}}{2}\Big)\Big)\Big] + \text{c.c.}$$

in total accordance to (6.15) with $n_1 = 1$. It results that the 5-brane charge contribution is completely due to the imaginary part of the logarithmic function, which in turn depends on the choice of the branch cut. The most natural choice is to put the logarithmic cut outside the infinite strip Σ; with this choice h_1^D has a discontinuity of $2\pi\gamma$ at the singularity on the upper boundary of the strip and h_2^D has a discontinuity of $-2\pi\hat{\gamma}$ at the singularity on the lower boundary.

Finally we can write

$$Q_{NS5}^{\hat{\delta}} = -16\pi^2\hat{\gamma} \tag{6.41}$$

and repeating an analogue reasoning for the singularity $z = \delta + i\pi/2$, we find a non-vanishing flux of the R-R 3-form F_3 through $\mathcal{C}_3^{\delta+i\pi/2} = S_2^2 \times \mathcal{I}$, where now $\mathcal{I}$ starts and ends on the upper boundary of the infinite strip enclosing the the point $z = \delta + i\pi/2$, and the D5 charge results to be

$$Q_{D5}^{\delta+i\pi/2} = 16\pi^2\gamma \tag{6.42}$$

These charges are local, gauge invariant and conserved. They can be expressed in terms of the number 5-branes exploiting the fact that they are quantized in units of $2k^2T_5$, where

$2k^2 = (2\pi)^7(\alpha')^4$ is the gravitational coupling constant, and T_5 is the tension of a D5-brane, which can be derived from (3.38). The tension of a NS5-brane is equal to the one of a D5-brane up to a factor $1/g_s$ (3.56), but since we have kept the dilaton arbitrary so far, we can set it so that $g_s = 1$ and the tension of NS5-branes and D5-branes is thus the same. Then we get

$$\begin{aligned} Q^{\hat{\delta}}_{NS5} &= -16\pi^2\hat{\gamma} = -4\pi^2\alpha' N^{\hat{\delta}}_{NS5} \\ Q^{\delta+i\pi/2}_{D5} &= 16\pi^2\gamma = 4\pi^2\alpha' N^{\delta+i\pi/2}_{D5} \end{aligned} \tag{6.43}$$

The D3-brane charge is more subtle to deal with. Indeed, from the second equation of (2.32), taking the differential we obtain

$$d\tilde{F}_5 = *j_{D3} = dF_5 - H_3 \wedge F_3 \tag{6.44}$$

and taking the differential once again, taking into account that $dH_3 = *j_{NS5}$ and $dF_3 = *j_{D5}$, we get

$$d(*j_{D3}) = -(*j_{NS5}) \wedge F_3 + H_3 \wedge (*j_{D5}) \tag{6.45}$$

then the D3-source current is not conserved, implying the associated charge is neither conserved nor local.
However, it's possible to obtain a charge that turns out to be local, conserved and quantized at cost of gauge invariance, by defining

$$*j^{Page}_{D3} = *j_{D3} + (*j_{NS5}) \wedge C_2 - B_2 \wedge (*j_{D5}) \tag{6.46}$$

whose associated charge will be called the Page charge.
In fact, previously the non-conservation of the charge was due to the fact that the field strength $\tilde{F}_5$ satisfies an anomalous Bianchi identity, whereas the Page charge definition amounts to redefine the R-R 5-form field strength as

$$\begin{aligned} F_5 &\to F'_5 = F_5 + C_2 \wedge H_3 \\ F_5 &\to F''_5 = F_5 - B_2 \wedge F_3 \end{aligned} \tag{6.47}$$

which ensures the non-anomalous Bianchi identity, thus the conservation of the current, but now the resulting 5-form field strength associated to j^{Page}_{D3} is no longer gauge invariant.
Which of the two choices (6.47) is the suitable one depends on the case under consideration, in particular it depends on which of the two form, B_2 or C_2, can be defined globally on the 5-cycle over which one wishes to integrate.
For example, if we consider the 5-cycle $\mathcal{C}^{\hat{\delta}}_5 = S^2_2 \ltimes \mathcal{C}^{\hat{\delta}}_3$, we know that the integral of $F_3 = dC_2$ on $\mathcal{C}^{\hat{\delta}}_3$ vanishes as said before, thus C_2 can be defined globally on $\mathcal{C}^{\hat{\delta}}_5$ and the D3-brane Page charge (in the follow whenever we talk about D3-brane charge, we will implicitly refer to the Page one) will be

$$Q^{\hat{\delta}}_{D3} = \int_{\mathcal{C}^{\hat{\delta}}_5} F_5 + C_2 \wedge H_3 = 4\pi b_2|_{z=\hat{\delta}} Q^{\hat{\delta}}_{NS5} \tag{6.48}$$

similarly B_2 can be defined globally on $\mathcal{C}^{\delta+i\pi/2}_5 = S^2_1 \ltimes \mathcal{C}^{\delta+i\pi/2}_3$ and the D3-brane Page charge reads

$$Q^{\delta+i\pi/2}_{D3} = \int_{\mathcal{C}^{\hat{\delta}}_5} F_5 - B_2 \wedge F_3 = -4\pi b_1|_{z=\delta+i\pi/2} Q^{\delta+i\pi/2}_{D5} \tag{6.49}$$

The integrals have been computed assuming to take the cycles to lie very close to the 5-brane singularities, so that the gauge potentials are constant, implying in particular the vanishing of F_5 on the integration cycles, and taking into account that the integrals over the 3-form fluxes coincide with the 5-brane charges as computed before (6.41),(6.42).
Computing b_1 and b_2 with the choice of harmonic functions (6.33) and choosing the "canonical gauge" $\zeta = \hat{\zeta} = 0$, we get

$$\begin{aligned} Q^{\delta+i\pi/2}_{D3} &= 2^8\pi^3\Big(\hat{\alpha}\gamma\sinh{(\delta-\hat{\beta})} - 2\gamma\hat{\gamma}\arctan{(e^{\hat{\delta}-\delta})}\Big) \\ Q^{\hat{\delta}}_{D3} &= 2^8\pi^3\Big(\alpha\hat{\gamma}\sinh{(\hat{\delta}-\beta)} + 2\hat{\gamma}\gamma\arctan{(e^{\hat{\delta}-\delta})}\Big) \end{aligned} \tag{6.50}$$

Closing the $AdS_5 \times S^5$ asymptotic regions

As already pointed out, the limit $x \to \pm\infty$ corresponds to $AdS_5 \times S^5$ regions with radii (6.34). An interesting fact emerges in the limit $\alpha = \hat{\alpha} = 0$, in which the radii vanish and the asymptotic $AdS_5 \times S^5$ are smoothly capped off and get replaced by regions homeomorphic to $AdS_4 \times B_6$, where B_6 is the 6-dimensional ball.
To prove this, let's consider the harmonic functions (6.33) with $\alpha = \hat{\alpha} = 0$

$$\begin{aligned} h_1 &= -\gamma \ln\Big(\tanh\Big(\frac{i\pi}{4} - \frac{z-\delta}{2}\Big)\Big)\Big] + \text{c.c.} \\ h_2 &= -\hat{\gamma} \ln\Big(\tanh\Big(\frac{z-\hat{\delta}}{2}\Big)\Big)\Big] + \text{c.c.} \end{aligned} \tag{6.51}$$

introducing now the coordinate

$$r^2 = 2(e^{2\delta} + e^{2\hat{\delta}})e^{\mp 2x} \tag{6.52}$$

and proceeding in computing the metric factors in an expansion around $r = 0$, one can get in the limit $x \to \pm\infty$:

$$ds^2 \simeq L_4^2[ds^2_{AdS_4} + dr^2 + r^2(\sin^2(y)ds^2_{S_1^2} + \cos^2(y)ds^2_{S_2^2} + dy^2)] \tag{6.53}$$

with

$$L_4^4 = \frac{16\gamma\hat{\gamma}}{\cosh(\delta - \hat{\delta})} \tag{6.54}$$

This metric locally describes an $AdS_4 \times \mathbb{R}^6$ geometry, thus in this case the asymptotic limits correspond to regular interior regions of the solution.
Thus the metric corresponding to the choice of harmonic functions (6.51) describes a warped product $AdS_4 \times_w M_6$, where M_6 is a compact 6-dimensional manifold, with singularities at the location of the 5-branes. In particular, notice that the AdS_4 radius is proportional to $(\gamma\hat{\gamma})^{1/4}$, so the presence of both D5 and NS5 branes is required for the regular solution.
Another interesting fact to underline is that, since the asymptotic $AdS_5 \times S^5$ have been smoothly closed, there are no semi-infinite D3-branes or D3-branes stretched between the two asymptotic regions as were present in the 5-branes doped Janus solution, and ,being present only one stack of D5-branes and one of NS5-branes, we get that the net number of D3-branes ending on the NS5-brane stack is equal to the net number of D3-branes ending on the D5-brane stack. This can be checked through (6.50) by setting $\alpha = \hat{\alpha} = 0$ and recalling (6.43), from which we get

$$N_{D3}^{\delta+i\pi/2} = -N_{D3}^{\hat{\delta}} = -N_{D5}^{\delta+i\pi/2} N_{NS5}^{\hat{\delta}} \frac{2}{\pi} \arctan(e^{\hat{\delta}-\delta}) \tag{6.55}$$

The background described above is an example of a gravity dual of the $\mathcal{N} = 4$ SCFTs labeled by the pair of partitions $(\rho, \hat{\rho})$. The simplest possible partitions in this example are

$$\rho = \hat{\rho}: \qquad N = 1 + 1 + ... + 1 \tag{6.56}$$

corresponding to having the same number of each kind of brane, which can be obtained in the special case $\gamma = \hat{\gamma}$ and $\hat{\delta} - \delta = \ln\tan\frac{\pi}{2N}$.

Solution for non-gravitating baths

The Karch-Randall models describe an AdS black hole coupled to an external thermal bath that can be either gravitating or non-gravitating as we have already seen in the last chapter. They admit three different descriptions and in one of them they can be seen as boundary conformal field theories (BCFTs) or as 3-dimensional defect conformal field theories, depending on which the bath is non-gravitating or gravitating respectively.
The uplifts of these models on the type IIB supergravity background studied so far can be carried out by choosing suitable representative solutions dual to BCFTs and 3d SCFTs.

The solution for non-gravitating baths can be obtained with the following choice of the harmonic functions

$$\begin{aligned} h_1 &= -i\frac{\pi\alpha'}{4}Ke^z - \frac{\alpha'}{4}N_5 \ln\left[\tanh\left(\frac{i\pi}{4} - \frac{z}{2}\right)\right] + \text{c.c.} \\ h_2 &= \frac{\pi\alpha'}{4}\hat{K}e^z - \frac{\alpha'}{4}N_5 \ln\left[\tanh\left(\frac{z}{2}\right)\right] + \text{c.c.} \end{aligned} \tag{6.57}$$

In this case, the limit $x \to -\infty$ leads to a regular point in the internal space, whereas an $AdS_5 \times S^5$ geometry emerges in the limit $x \to +\infty$, and from this follows that the solution is dual to $\mathcal{N} = 4$ SYM on a half space coupled to a 3d $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory on the boundary. Setting $r = e^{-2x}$ and expanding around $r = 0$, it is possible to find at leading order

$$\begin{aligned} f_4^4 &\approx \frac{\alpha'^2\pi^3(K\hat{K})^2}{4(K+\hat{K})N_5}\frac{1}{r^2} & \rho^4 &\approx \frac{\pi\alpha'^2}{4}(K+\hat{K})N_5 \\ f_1^2 &\approx 4\rho^2\sin^2 y & f_2^2 &\approx 4\rho^2\cos^2 y \end{aligned} \tag{6.58}$$

and the value of the radius of AdS_5 and S^5 and dilaton are given by

$$L_5^4 = 4\pi\alpha'^2(K+\hat{K})N_5 \qquad e^{2\phi} = \frac{\hat{K}}{K} \tag{6.59}$$

Notice in particular that we have a vanishing dilaton variation in the AdS_5 throat if $K = \hat{K}$. As the Janus solution doped with 5-branes, there is a stack of D5-branes at $z = i\pi/2$ and a stack of NS5-branes at $z = 0$, which can be verified through a flux calculation perfectly identical to the doped Janus since it only depends on the logarithmic terms of the harmonic functions that result to be the same in the two cases. In particular, from (6.43), it results that the parameter N_5 corresponds effectively to the number of 5-branes in each stack.
The D3-branes flux computations give in this case

$$\begin{aligned} Q^0_{D3} &= -16\pi^4\alpha'^2\left[\frac{N_5^2}{2} + N_5K\right] \\ Q^{i\pi/2}_{D3} &= -16\pi^4\alpha'^2\left[\frac{N_5^2}{2} + N_5\hat{K}\right] \\ Q^{semi-\infty}_{D3} &= 4\pi^3L_5^4 = 16\pi^4\alpha'^2(K+\hat{K})N_5 \end{aligned} \tag{6.60}$$

where we exploit the results (6.48) and (6.49) taking into account that

$$b_1(0,y) = \frac{1}{2}\pi\alpha'(N_5 + 2\hat{K}\sin y) \qquad \text{and} \qquad b_2(0,y) = \frac{1}{2}\pi\alpha'(N_5 + 2K\cos y) \tag{6.61}$$

while the semi-infinite D3-branes' charge as been calculated recalling that

$$Q^{semi-\infty}_{D3} = \int_{S^5} F_5 = 4\text{Vol}(S^5) = 4\pi^3 L_5^4 \tag{6.62}$$

finally, remembering the D3-brane charge is measured in units of $2k^2T_{D3} = 16\pi^4\alpha'^2$, we get that the brane configuration consists of $(K+\hat{K})N_5$ semi-infinite D3-branes ending on the system of N_5 D5-branes and N_5 NS5-branes, $N_5\hat{K}$ D3-branes ending on the D5-branes, N_5K D3-branes ending on the NS5-branes and $N_5^2/2$ D3-branes suspended between the D5- and NS5-branes.

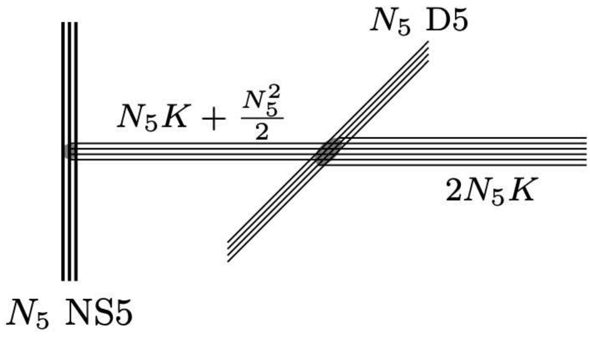


Figure 6.3: Brane configuration for the representative non-gravitating bath solution with $K = \hat{K}$ **[22]**

As we are going to show next, an important feature in this solution is the presence of a lowest-lying massive graviton as a consequence of the non-compactness of the internal manifold. A crucial parameter both for graviton's mass and for the contribution of quantum island surfaces to radiation's entanglement entropy at zero temperature will be

$$\alpha = \frac{N_5}{K} \tag{6.63}$$

which is, for $K = \hat{K}$ or equivalently for $\delta\phi = 0$, the ratio of number of suspended D3-branes between 5-branes and the number of semi-infinite D3-branes; thus it quantifies the number of "boundary" over "bulk" degrees of freedom. In the case of non-vanishing dilaton variation the corresponding quantity is

$$\alpha = \sqrt{\frac{N_5^2}{K\hat{K}}} \tag{6.64}$$

which can expressed as (6.63) for generic $\delta\phi$ through the redefinitions $K \to Ke^{\delta\phi}$ and $\hat{K} \to Ke^{-\delta\phi}$.

Solution for gravitating baths

The representative solution for gravitating baths, holographic dual of a 3d $T^{\rho}_{\hat{\rho}}[SU(N)]$ SCFT, corresponds the following choice of the harmonic functions

$$\begin{aligned} h_1 &= -\frac{\alpha'}{4}\frac{N_5}{2}\Big[\ln\Big(\tanh\Big(\frac{i\pi}{4} - \frac{z-\delta}{2}\Big)\Big) + \ln\Big(\tanh\Big(\frac{i\pi}{4} - \frac{z+\delta}{2}\Big)\Big)\Big] + \text{c.c.} \\ h_2 &= -\frac{\alpha'}{4}\frac{N_5}{2}\Big[\ln\Big(\tanh\Big(\frac{z-\delta}{2}\Big)\Big) + \ln\Big(\tanh\Big(\frac{z+\delta}{2}\Big)\Big)\Big] + \text{c.c.} \end{aligned} \tag{6.65}$$

The resulting geometric can be considered an extension of (6.51) to two stacks of D5-branes and two stacks of NS5-branes. In particular there are two $N_5/2$ D5-brane groups located at $z = \pm\delta + i\pi/2$ and two $N_5/2$ NS5-brane groups located at $z = \pm\delta$.
In particular, in presence of more than one singularity for each kind of 5-brane, (6.50) with $\alpha = \hat{\alpha} = 0$, since the asymptotic $AdS_5 \times S^5$ regions are capped off, can be generalized, in term of the number of D3-branes, as

$$\begin{aligned} N^a_{D3} &= -N^a_{D5}\sum_{b=1}^{\hat{q}} \hat{N}^b_{NS5}\frac{2}{\pi}\arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \\ \hat{N}^b_{D3} &= -\hat{N}^b_{NS5}\sum_{a=1}^{q} N^a_{D5}\frac{2}{\pi}\arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \end{aligned} \tag{6.66}$$

where we assume D5-brane singularities $\delta_1, \delta_2, ..., \delta_q$ and NS5-brane singularities $\hat{\delta}_1, \hat{\delta}_2, ..., \hat{\delta}_{\hat{q}}$. In our case, it results to be present $\frac{N_5^2}{4}\Delta$ suspended D3-branes between both the two stacks of D5-branes and the two stacks of NS5-branes, with $\Delta = \frac{1}{2} + \frac{2}{\pi}\arctan e^{2\delta}$, and $N_5^2/2$ suspended D3-branes between both the groups of D5-branes and the groups of NS5-branes. Thus the parameter δ determines how the D3-branes terminate on 5-branes, and if $\delta = 0$ we return to the case (6.51) in which the number of D3-branes terminating on each group of 5-branes are equal.

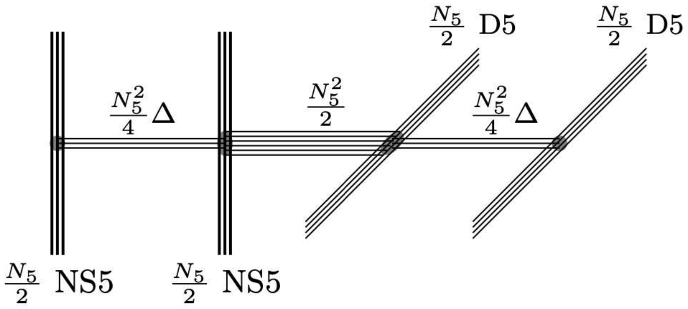


Figure 6.4: Brane configuration for the representative gravitating bath solution **[22]**

Introducing now the coordinate

$$r^2 = 4e^{2\delta}e^{2x} \tag{6.67}$$

and proceeding in computing the harmonic functions in an expansion around $r = 0$, one can get in the limit $x \to -\infty$

$$N_1 = N_2 = \frac{1}{512}(1 + e^{2\delta})^4 N_5^4 \cos y \sin y \; r^4 + \mathcal{O}(r^5) \tag{6.68}$$

thus $e^{4\phi} = N_2/N_1 = 1$ and the same result holds also in the limit $x \to +\infty$, following the vanishing of the dilaton at infinity. A non-vanishing dilaton can be obtained for example by changing the number of NS5 branes in each stack to be $\hat{N}_5/2$, i.e. by changing the parameter of the harmonic function h_2 from N_5 to $\hat{N}_5$, obtaining in this way at infinity

$$e^{2\phi} = \frac{\hat{N}_5}{N_5} \tag{6.69}$$

Chapter 7

Massive AdS gravity

Massive gravity have a long story, but the AdS massive gravity is known to be the "easier case" with respect to the Minkowski or de Sitter gravity, since the van Dam-Veltman-Zakharov discontinuity is not present in Anti de Sitter space[**53**], where the massless graviton limit is smooth whenever the square of the graviton mass vanishes faster than the cosmological constant. However questions remain concerning both the consistency of massive AdS classical theories and the range of validity if viewed as effective field theories around a given classical background. The answers of these questions are hoped to be found by embedding massive AdS gravity in a UV-complete theory like string theory.
In the solutions of type IIB string theory derived by D'Hoker *et al* and discussed so far, the lowest-lying spin-2 mode can acquire a tiny mass. To show this, first of all let us rewrite (6.6) in the following form

$$ds^2 = L_4^2(y)\bar{g}_{\mu\nu}dx^\mu dx^\nu + g_{ij}dy^i dy^j \tag{7.1}$$

where the Greek indices run over $\{0,1,2,3\}$, $\bar{g}_{\mu\nu}$ is the metric of the unit-radius AdS_4 and Latin indices labels the remaining coordinates, y^i, of the internal manifold M_6, which is the warped product of the two 2-spheres over the Riemann surface as usual.
In [**25**] it was derived the equation for the spectrum of the spin-2 Kaluza-Klein excitations around any solution with maximal symmetry in four dimensions. In particular, restricting to our supergravity solution with background metric (7.1) and to perturbations of the four-dimensional metric only

$$ds^2 = L_4^2(y)(\bar{g}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + g_{ij}dy^i dy^j \tag{7.2}$$

we can look for factorized solutions of the linearized field equations of the following form

$$h_{\mu\nu}(x,y) = h^{[tt]}_{\mu\nu}(x|\lambda)\psi(y|\lambda) \tag{7.3}$$

where $h^{[tt]}_{\mu\nu}$ solves the Pauli-Fierz equations for a massive spin-2 particle in AdS_4

$$(\bar{\Box}^{(2)}_x - \lambda)h^{[tt]}_{\mu\nu} = 0 \qquad \text{and} \qquad \bar{\nabla}^\mu h^{[tt]}_{\mu\nu} = \bar{g}^{\mu\nu}h^{[tt]}_{\mu\nu} = 0 \tag{7.4}$$

where the four-dimensional Pauli-Fierz mass is related to λ as

$$\lambda + 2 = m^2(y)L_4^2(y) \tag{7.5}$$

The two equations (7.4) force $h^{[tt]}_{\mu\nu}$ to be transverse and traceless. The linearized Einstein equations with the above ansatz (7.3) reduce to a single-order equation for the internal-space wave-function $\psi(y)$

$$\mathcal{M}^2\psi := -\frac{L_4^{-2}}{\sqrt{g}}(\partial_i L_4^4\sqrt{g}g^{ij}\partial_j)\psi = (\lambda+2)\psi \tag{7.6}$$

An interesting fact is that the mass operator $\mathcal{M}^2$ only depends on the background metric, and on no other fields: this is consequence of the fact that all other background fields enter the linearized equations for spin-2 modes via the total energy-momentum tensor, which can be re-expressed in terms of the background curvature.

The eigenmode equation must be supplemented with a space of admissable $\psi(y)$, which in turn requires the introduction of a norm. With canonically-normalized fields in 10 dimensions, the Kaluza-Klein reduction of the inner product reads

$$\langle\psi_1|\psi_2\rangle = \int_{M_6} d^6y\sqrt{g}L_4^2\psi_1^*\psi_2 \tag{7.7}$$

We conclude therefore that the normalizable excitations are those for which their norm is finite. One immediate corollary is that the mass eigenvalue m^2 is non-negative, and this follows from

$$m^2 \propto \langle\psi|\mathcal{M}^2|\psi\rangle = -\int d^6y\psi^*(\partial_i L_4^4\sqrt{g}g^{ij}\partial_j)\psi = \int d^6y\sqrt{g}L_4^4|\partial\psi|^2 \geq 0 \tag{7.8}$$

with the mild assumption that, upon integrating by parts, one picks no contributions either from singularities or from asymptotic regions.
Another simple consequence, with the same mild assumption as above, is that from (7.8) a massless graviton must necessarily have $\psi(y)$ =constant. Thus, from normalizability condition, it follows that

$$\text{the existence of a massless graviton} \qquad \Longleftrightarrow \qquad \int d^6y\sqrt{g}L_4^2 < \infty \tag{7.9}$$

This condition is automatically satisfied for compact internal manifolds with smooth $L_4^2(y)$. Thus the AdS_4 vacuum, where the internal manifold M_6 is compact, includes a massless low-lying graviton and a Kaluza-Klein tower of massive modes. Conversely, the background allowing for a massive graviton must include non-compact internal manifold and in order to examine quantitatively how it can acquire a small mass, we consider specifically a compact region in the vicinity of the 5-brane singularities, called bag, attached in a matching region to a semi-infinite Janus throat. In particular, as we are going to see, the tiny mass can be obtained if the radius of the Janus throat L_5 is much smaller than the AdS_4 radius L_4, which is tied to the bag-size L_{bag} as a consequence of the scale non-separation problem **[54]**,**[55]**.
However, this can be already understood holographically**[26]**. If one considers a 3d CFT in its own, dual to the above AdS_4 solution, its stress energy tensor is conserved, and as a consequence it has canonical conformal dimension $\Delta = 3$, translating, through the AdS/CFT correspondence holographic dictionary (4.28), in the presence of an AdS_4 massless graviton

$$m_g^2L_4^2 = \Delta(\Delta - 3) \tag{7.10}$$

Once the 3d CFT is coupled to a 4d $\mathcal{N} = 4$ SYM, corresponding holographically to the attachment of a semi-infinite throat to the bag solution, the three-dimensional stress energy tensor is no longer conserved since it leaks in the extra bulk dimension, thus its canonical dimension acquires an anomalous dimension $\Delta = 3 + \varepsilon$ which corresponds holographically to a correction to the AdS_4 graviton mass $m_g^2L_4^2 \sim \varepsilon$.
Therefore requiring a tiny graviton mass amounts to require a weak stress energy tensor leakage in the extra bulk dimension, which in turn amounts to require that the number of boundary degrees of freedom is much greater than the bulk ones or equivalently, in terms of the dual solutions, that the number of semi-infinite D3-branes is much smaller than the number of D3-branes suspended between the 5-branes in the bag region.

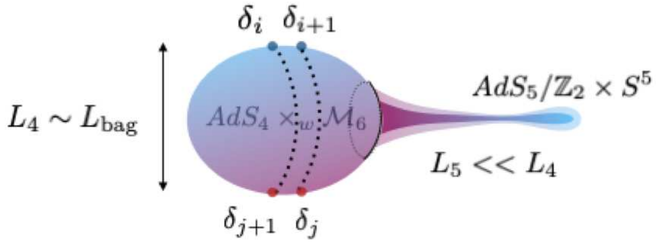


Figure 7.1: The geometric background of interest allowing a small mass low-lying graviton **[23]**

We are not interested in solving the entire spectrum problem, but only in finding the smallest eigenvalue of the mass operator $\mathcal{M}^2$

$$\lambda_0 + 2 = \mathbf{min}_\psi \Big[\int_{M_6} d^6y\sqrt{g}L_4^4(g^{ij}\partial_i\psi^*\partial_j\psi)\Big] \qquad \text{with} \qquad \int_{M_6} d^6y\sqrt{g}L_4^2|\psi|^2 = 1 \tag{7.11}$$

If M_6 is reduced to the compact $\bar{M}_6$ by truncating the Janus throat as it has been done in subsection 6, the minimum would correspond to the massless graviton with constant wavefunction

$$\psi_0(y) = \Big(\int_{\bar{M}_6} d^6y\sqrt{g}L_4^2\Big)^{-1/2} \tag{7.12}$$

and we identify it with ψ_{bag}.
As can be verified starting from the supersymmetric Janus solution, $\sqrt{g}L_4^2$ reaches a minimum value L_5^8 inside the semi-infinite throat and then diverges as $L_5^8\cosh^8 x$ at infinity where the 10-dimensional geometry asymptotes to $AdS_5 \times S^5$. This implies that normalizable wavefunctions should therefore go to zero inside the semi-infinite throat.
Therefore, at leading order in L_5/L_{bag}, the contribution to the norm comes exclusively from the bag, while the contribution to the graviton mass comes only from the bottom of the throat where $\sqrt{g}L_4^2$ is minimal. The matching region contributes to neither and can be shrunk to a point for our purposes. The minimal eigenvalue problem can be reformulated thus as a variational problem in the Janus geometry:

$$\lambda_0+2 \simeq \mathbf{min}_\psi \Big[\int_{\text{throat}} \sqrt{g}L_4^4(g^{ij}\partial_i\psi^*\partial_j\psi)\Big] \qquad \text{with} \qquad \psi \to \begin{cases}\psi_{bag} & \text{in the matching region}\\ 0 & \text{at } \infty\end{cases} \tag{7.13}$$

Taking into account that in the Janus geometry

$$\frac{1}{4}\sqrt{g}L_4^4\rho^{-2} = L_4^4f_1^2f_2^2 = 16h_1^2h_2^2 = \frac{L_5^8}{16}\sin^2(2\tau)\mathcal{G}(x) \tag{7.14}$$

where we used the metric factors' combinations (6.12), the expression of the harmonic functions of the supersymmetric Janus (6.28) and τ instead of the usual y as imaginary coordinate of the Riemann surface, and the function $\mathcal{G}(x)$ reads

$$\mathcal{G}(x) = \Big(\frac{\cosh 2x + \cosh\delta\phi}{\cosh\delta\phi}\Big)^2 \tag{7.15}$$

finally knowing that the lightest spin-2 eigenfunction is only function of x **[26]**, we get

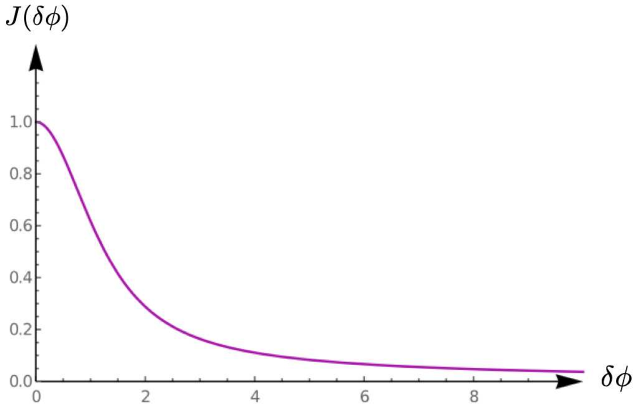
$$\lambda_0 + 2 = \mathbf{min}_\psi \Big[\frac{\pi^3}{4}L_5^8\int_{x_c}^{\infty} dx\, \mathcal{G}(x)(\partial_x\psi)^2\Big] \qquad \text{with} \qquad \psi \to \begin{cases}\psi_{bag} & \text{at } x = x_c\\ 0 & \text{at } x = \infty\end{cases} \tag{7.16}$$

where we have cutoff the integral at some large negative value x_c at the boundary of matching region, since ψ would have been a non-normalizable mode in the complete Janus geometry.
The solution of the variational problem , defining $a = \cosh\delta\phi$, gives

$$\lambda_0 + 2 = \frac{3\pi^3}{4}L_5^8\psi_{bag}^2 J(a) \tag{7.17}$$

where the Janus correction factor $J(a)$ reads

$$J(a)^{-1} = \frac{3a^3}{(a^2-1)^{3/2}}\log\Big[a + \sqrt{a^2-1}\Big] - \frac{3a^2}{(a^2-1)} \tag{7.18}$$



As $\delta\phi$ goes from 0 to ∞, $J(a)$ decreases monotonically from 1 to 0, thus for $AdS_5 \times S_5$ throats for which the dilaton is constant $J(a)$ is trivially the identity, whereas in general it has the effect of reducing the graviton mass. Also the vanishing of the graviton mass at large dilaton variation can be understood holographically: $\delta\phi$ is the difference between the value of the dilaton at the entry of the throat and its value at infinity, with the latter determining the coupling constant g_{YM} of the dual 4d $\mathcal{N} = 4$ super Yang-Mills theory. If one takes g_{YM} to zero, meaning that the dilaton variation $\delta\phi$ is taken to diverge, the bulk CFT decouples from the 3d CFT, dual to the bag solution, restoring the conservation of the stress energy tensor and sending the graviton mass to zero. However this limit is singular, since reaching this point in the dilaton moduli space would bring down an infinite tower of modes which invalidate the effective field theory itself, beside the fact the supergravity approximation would break eventually down in this limit.
Finally let's express ψ_{bag}, defined in (7.12), as

$$\psi_{bag}^{-2} = \int_{\bar{M}_6} d^6y\sqrt{g}L_4^2 = V_6 \left\langle L_4^2 \right\rangle_{bag} \simeq V_6 L_{bag}^2 \tag{7.19}$$

where V_6 is the volume of the bag, and $\left\langle L_4^2 \right\rangle_{bag}$ is the average AdS_4 squared radius which is of the same order as the Kaluza-Klein scale, due once again to the scale non-separation problem. Therefore the contribution to the graviton mass (7.5) reads

$$\bar{m}_g^2 \equiv m_g^2 L_4^2 = \frac{3\pi^3}{4V_6}\left(\frac{L_5}{L_{bag}}\right)^8 J(\cosh\delta\phi) \tag{7.20}$$

at the leading order in the radii ratio.
In the case of the solution for non-gravitating bath (6.57), the radii are given by $L_5^4 \sim N_5 K$ and $L_{bag}^4 \sim N_5^2$ and the graviton mass thus takes the form

$$\bar{m}_g^2 \equiv m_g^2 L_4^2 = \frac{3\pi^3}{4V_6}\left(\frac{K}{N_5}\right)^2 J(\cosh\delta\phi) = \frac{3\pi^3}{4V_6}\frac{1}{\alpha^2} J(\cosh\delta\phi) \tag{7.21}$$

thus it is proportional to the inverse of microscopic ratio α (6.63). The dependence of the graviton mass on the ratio α was expected because of the reasoning we have made before, namely the stress energy tensor leakage (and subsequently the dual graviton mass) decreases as the ratio of boundary over bulk degrees of freedom increases.
As we are going to discuss in the last chapter, the ratio α represents the 10d counterpart of the inverse of the brane angle θ of 5d Karch-Randall models. Thus, as in Karch-Randall setups islands are not expected to contribute for $\theta < \theta_{crit}$ at zero temperature, in the 10d string theory realizations of these models islands are not expected to contribute for $\alpha > \alpha_{crit}$. This means that graviton mass must be large than a critical value in order for islands' contribution

$$\bar{m}_g > \bar{m}_{g,crit} \sim \frac{1}{\alpha_{crit}} \tag{7.22}$$

Furthermore, the graviton mass formula (7.20) holds only if Janus throat radius L_5 is much smaller than the bag size $L_{bag} \sim L_4$, equivalently only if $N_5 \gg K$, meaning that the critical value α_{crit} must be sufficiently large to ensure this regime.
Finally let's make a comment on the breakdown of AdS massive gravity theory in regime of small graviton masses ($\bar{m}_g \ll 1$, i.e. $K \ll N_5$). The effective field theory breaks down at the scale **[56]**

$$\Lambda_* = \frac{m_g^{1/3} M_P^{1/3}}{L_4^{1/3}} \tag{7.23}$$

in an AdS background with fixed radius L_4, i.e. with fixed number of 5-branes N_5. M_P is the 4d Planck mass, which we will set to unit, and it is already clear, as it must be, that $m_g < \Lambda_*$. The critical value of the graviton mass can be either outside the regime of EFT validity or inside it but at an energy scale below or above the graviton mass. In order for the critical mass to be at an energy scale below the EFT breakdown, it turns out that we must require

$$K_{crit} < K^{1/3} N_5^{5/6} \ll N_5^{7/6} \tag{7.24}$$

where we have used the breakdown energy scale formula (7.23), the graviton mass formula (7.20) and exploited that, due to the no-scale separation problem, $\left\langle L_4^{1/2} \right\rangle_{bag} \sim L_{bag}^{1/2} \sim N_5^{1/4}$.
More stringent is thus the requirement that the graviton mass is above the critical mass value which yields

$$K_{crit} < K \tag{7.25}$$

Chapter 8

Islands and Page curve in type IIB string theory

In the following we are going to construct and study the quantum extremal surfaces in the type IIB string theory uplifts of the Karch-Randall models. In particular we briefly motivate how the type IIB solutions corresponding to the choices of harmonic functions (6.57) and (6.65) represents the 10d uplift of 5d Karch-Randall models with respectively non-gravitating and gravitating bath. Then we will find the extremality conditions determining the island and Hartman-Maldacena surfaces, we will describe the numerical relaxation method to solve them and to construct the extremal surfaces themselves. We proceed to study and analyse these surfaces, in particular we are going to reproduce the main results of Uhlemann **[22]** in the case of vanishing dilaton variation both for the uplifts of non-gravitating and gravitating bath solutions and the results regarding island surfaces for solutions with non-gravitating bath in presence of non-vanishing dilaton variation found in **[23]**. The original work will consist in constructing the Hartman-Maldacena surfaces for solutions with non-gravitating bath in presence of non-vanishing dilaton variation and extending the results found by Uhlemann regarding the area difference between the extremal surfaces at $t = 0$ for non-vanishing dilaton variation. The target of extending the results of extremal surfaces for a particular solution with gravitating bath with non vanishing dilaton variation reveals however trivial results with respect to the vanishing dilaton variation case.

Construction of quantum extremal surfaces in type IIB string theory uplifts

Karch-Randall models can be uplift in type IIB string theory background and 4d black holes coupled to non-gravitating and gravitating baths are realized through 10d black hole solutions based on the $AdS_4 \times S^2 \times S^2 \times \Sigma$ type IIB solutions discussed extensively in section 6.
In particular the type IIB string theory realization of the Karch-Randall model with a non-gravitating bath, as previously anticipated in 6, can be obtained with the choice of the harmonic functions h_1, h_2 (6.57), which we recall here (setting for simplicity $\alpha' = 1$ from now on)

$$h_1 = -i\frac{\pi}{4}Ke^z - \frac{N_5}{4}\ln\left[\tanh\left(i\frac{\pi}{4} - \frac{z}{2}\right)\right] + \text{c.c.} = \frac{\pi}{2}Ke^x \sin y - \frac{N_5}{4}\ln\left(\frac{\cosh x - \sin y}{\cosh x + \sin y}\right)$$
$$h_2 = \frac{\pi}{4}\hat{K}e^z - \frac{N_5}{4}\ln\left[\tanh\left(\frac{z}{2}\right)\right] + \text{c.c.} = \frac{\pi}{2}\hat{K}e^x \cos y - \frac{N_5}{4}\ln\left(\frac{\cosh x - \cos y}{\cosh x + \cos y}\right)$$

the reason of the uplift is that the asymptotic region $AdS_5 \times S^5$ at $x \to +\infty$ corresponds to the AdS_5 part of the 5d Karch-Randall model, whereas the region with the 5-brane sources is the string theory version of the end-of-the-world brane itself. Moreover, we recall that the dilaton variation in the asymptotic throat amounts to

$$e^{2\delta\phi} = \frac{\hat{K}}{K} \tag{8.1}$$

The solution for the Karch-Randall models with gravitating bath can be obtained instead with the choice of harmonic functions (6.65)

$$
\begin{aligned}
h_1 &= -\frac{N_5}{8}\Big[\ln\Big(\frac{\cosh(x-\delta)-\sin y}{\cosh(x-\delta)+\sin y}\Big)+\ln\Big(\frac{\cosh(x+\delta)-\sin y}{\cosh(x+\delta)+\sin y}\Big)\Big] \\
h_2 &= -\frac{N_5}{8}\Big[\ln\Big(\frac{\cosh(x-\delta)-\cos y}{\cosh(x-\delta)+\cos y}\Big)+\ln\Big(\frac{\cosh(x+\delta)-\cos y}{\cosh(x+\delta)+\cos y}\Big)\Big]
\end{aligned}
\tag{8.2}
$$

which, as discussed in 6, describes a 5-branes doped Janus solution in which the asymptotic $AdS_5 \times S^5$ regions are closed off. The capping off of the asymptotic region at $x \to +\infty$ corresponds to the introduction of the second end-of-the-world brane in the Karch-Randall model with gravitating bath. The entire 10d solution corresponds to the remaining wedge of AdS_5.

To introduce the temperature, we need to replace the AdS_4 by a finite temperature AdS_4 black hole and in doing this replacement we still obtain a solution to the type IIB SUGRA field equations. The corresponding metric is

$$
ds_4^2 = \frac{dr^2}{b(r)} + e^{2r}(-b(r)dt^2 + ds^2_{\mathbb{R}^2}), \qquad b(r) = 1 - e^{3(r_h - r)} \tag{8.3}
$$

where $r = r_h$ is the horizon radius, as can be verified by imposing $b(r) = 0$, and the conformal boundary is at $r \to \infty$.
The horizon temperature of an AdS planar black hole can be derived by expanding the associated Euclidean metric in the near horizon limit

$$
ds^2 \simeq 3e^{2r_h}(r - r_h)d\tau^2 + \frac{1}{r - r_h}dr^2 + e^{2r_h}ds^2_{\mathbb{R}_2} \tag{8.4}
$$

where $\tau = -it$ is the Euclidean imaginary time. Defining $R^2 := 4(r - r_h)/3$ and neglecting the angular part of the metric, the near horizon metric takes the form of a Rindler one

$$
ds^2 \simeq \frac{9e^{2r_h}}{4}R^2 d\tau^2 + dR^2 \tag{8.5}
$$

Finally imposing the periodicity of the Euclidean time $\tau \sim \tau + \beta$ so to avoid conical singularity, one finds that the horizon temperature is given by

$$
T = \frac{1}{\beta} = \frac{3e^{r_h}}{4\pi} \tag{8.6}
$$

Notice that the temperature of the AdS planar black hole (8.3) vanishes in the limit $r_h \to -\infty$, or equivalently if $b(r) = 1$.
In certain cases, it is useful to use a tortoise radial coordinate

$$
du = \frac{dr}{\sqrt{b(r)}} \qquad \Rightarrow \qquad u = \frac{2}{3}\cosh^{-1}\big(e^{\frac{3}{2}(r - r_h)}\big) \tag{8.7}
$$

using this radial coordinate, the exterior region of the black hole is described by $u > 0$, the horizon is at $u = 0$ and the metric becomes

$$
ds_4^2 = du^2 + e^{2r_h}\cosh^{4/3}\Big(\frac{3u}{2}\Big)\Big[-\tanh^2\Big(\frac{3u}{2}\Big)dt^2 + ds^2_{\mathbb{R}^2}\Big] \tag{8.8}
$$

Islands surfaces

The extremal surfaces are 8-dimensional surfaces in the 10d geometry wrapping both S^2 and partially the Riemann surface Σ and the AdS_4 black hole geometry. Thus their embeddings can be described in terms of the AdS_4 radial coordinate $r(x, y)$ for any given point of Σ and the induced metric reads

$$
ds_\gamma^2 = e^{2r}f_4^2 ds^2_{\mathbb{R}^2} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + 4\rho^2(dx^2 + dy^2) + \frac{f_4^2}{b(r)}(dx\partial_x r + dy\partial_y r)^2 \tag{8.9}
$$

where the metric (8.3) has been used for the AdS_4 black hole geometry.
From the induced metric, the area of the surface in units of the volume of $\mathbb{R}^2 \times S_1^2 \times S_2^2$ results to be given by

$$\mathcal{A} = 32 \int dxdy\, e^{2r} f \sqrt{1 + g(\nabla r)^2} \tag{8.10}$$

where we exploited the combination of metric factors (6.12) and defined the functions $f(x, y)$ and $g(x, y)$ as

$$f = |h_1 h_2 W|, \qquad g = \frac{1}{2b(r)} \left| \frac{h_1 h_2}{W} \right| \tag{8.11}$$

and ∇ is the covariant derivative with respect to the metric on Σ.
The extremality condition is found by solving the Euler-Lagrange equation of the integrand $\mathcal{L}$ of the area functional

$$0 = \frac{1}{\mathcal{L}} \Big(\frac{\delta \mathcal{L}}{\delta r} - \nabla \frac{\delta \mathcal{L}}{\delta(\nabla r)} \Big) = \frac{1}{1 + g(\nabla r)^2} \Big[2 - \nabla(g \nabla r) + \frac{1}{2} g \nabla r \cdot \nabla \ln \Big(\frac{1 + g(\nabla r)^2}{b(r) f^2} \Big) \Big] \tag{8.12}$$

Let's discuss now the boundary conditions we need to impose to surfaces extending along Σ. The induced metric near the lower component of the strip $y = 0$, recalling that $f_1 \sim 4y^2\rho^2$ (6.38), reads

$$ds_\gamma^2 \simeq e^{2r} f_4^2 ds_{\mathbb{R}^2}^2 + f_2^2 ds_{S_2^2}^2 + 4\rho^2(dx^2 + dy^2 + y^2 ds_{S_1^2}^2) + \frac{f_4^2}{b(r)} (dx \partial_x r + dy \partial_y r)^2 \tag{8.13}$$

the contribution $(\partial_y r)^2 dy^2$ may introduce a conical singularity in the (y, S_1^2) part of the induced metric on the surface. To obtain a smooth metric, we must impose a Neumann boundary condition of the embedding function $r(x, y)$ on the lower boundary of the strip, and analogously it can be checked that the same must hold on the upper boundary

$$\partial_y r(x, y)|_{y=0, \frac{\pi}{2}} = 0 \tag{8.14}$$

At $x \to -\infty$, the background geometry closes off smoothly, whereas for the surface to be smooth the limit $\lim_{x \to -\infty} r(x, y)$ should be independent of y. Recalling the asymptotic behaviour of metric factor at $x \to -\infty$ (6.54), the induced metric on the minimal surface in this limit becomes

$$ds_\gamma^2 \simeq L_4^2 \Big[e^{2r} ds_{\mathbb{R}^2}^2 + dv^2 + v^2 \Big(dy^2 + \sin^2 y ds_{S_1^2}^2 + \cos^2 y ds_{S_2^2}^2 \Big) + (dx \partial_x r)^2 \frac{dv^2}{v^2} \Big] \tag{8.15}$$

where in this case we have used as expansion coordinate $v = 2e^x$. The metric in the round bracket is the line element of S^5 and a smooth $\mathbb{R}^8$ with no conical singularity is obtained imposing

$$\lim_{x \to -\infty} e^{-x} \partial_x r(x, y) = 0 \tag{8.16}$$

which together with (8.14) forms the 10-dimensional analog of the Neumann boundary conditions imposed at the ETW brane in the 5d Karch-Randall models.
The boundary condition at $x \to +\infty$ is different depending on which we are studying the solution with a non-gravitating or a gravitating bath. In the former case at $x \to +\infty$ there is an $AdS_5 \times S^5$ region and a Dirichlet boundary condition anchoring the surface is imposed, while for the latter case the limits $x \to \pm\infty$ both lead to regular boundary points and the boundary condition at $x \to +\infty$ is identical as the one at $x \to -\infty$ with $x \to -x$. Thus in summary

$$\begin{aligned} \lim_{x \to +\infty} r(x, y) = r_R(y) = r_R \qquad & \text{for non-gravitating bath} \\ \lim_{x \to +\infty} e^{+x} \partial_x r(x, y) = 0 \qquad & \text{for gravitating bath} \end{aligned} \tag{8.17}$$

the reason the Dirichlet boundary condition at $x \to +\infty$ for the non-gravitating bath solution is independent of y lies on the fact that the form of $r_R(y)$ can be determined considering the global $AdS_5 \times S^5$ solution (6.21) for which $|h_1 h_2 / W| \propto 2 \cosh^2 x$, which is independent of y.

To solve the partial differential equation (8.12) numerically, we introduce an auxiliary external time parameter τ in such a way the embedding is described by a τ-dependent function $r(x, y, \tau)$ and its τ-evolution is given by

$$\partial_\tau r(x,y,\tau) = -\frac{1}{1+g(\nabla r)^2}\Big[2 - \nabla(g\nabla r) + \frac{1}{2} g\nabla r \cdot \nabla \ln\Big(\frac{1+g(\nabla r)^2}{b(r)f^2}\Big)\Big] \tag{8.18}$$

For a generic τ we get a trial surface satisfying the boundary conditions, and by waiting for a sufficiently long relaxation time $\tau_{max} \gg 1$ in order for the extremality condition to stabilise, the trial surface will dynamically settle on the minimal area configuration.
To numerically implement the relaxation, first of all we perform the change of variable $\xi = \tanh x$ in order to get a finite domain, then this domain is discretized into a rectangular lattice with equidistant points, the derivatives are discretized using second-order finite differences and the boundary conditions are implemented such that they are compatible with the second-order accuracy of the finite differences. In this way, the partial differential equation turns into a set of coupled algebraic equations and then these equations are then solved with the relaxation method (8.18), through which they are replaced by a set of first order differential equations.
The resulting extremality condition after the coordinate transformation $\xi = \tanh x$ can be obtained by performing the change of variable either directly in the extremality condition (8.12) or from the beginning in the area functional (8.10); in both cases now the extremality condition reads

$$\begin{aligned} 0 = \frac{1}{1+g(\tilde{\nabla} r)^2}\Big\{2 - \Big[\partial_y g \partial_y r + (-1+\xi^2)^2 \partial_\xi g \partial_\xi r + g\Big(\partial_y^2 r + (-1+\xi^2)(2\xi\partial_\xi r + (-1+\xi^2)^2\partial_\xi^2 r)\Big)\Big]\Big\} \\ + \frac{\frac{1}{2}g}{1+g(\tilde{\nabla} r)^2}\Big\{-2\Big[\frac{\partial_y f}{f}\partial_y r + (-1+\xi^2)^2\frac{\partial_x f}{f}\partial_x r\Big] \\ + \frac{1}{1+g(\tilde{\nabla} r)^2}\Big[\Big(\partial_y g\partial_y r + (-1+\xi^2)^2\partial_\xi g \partial_\xi r\Big)(\tilde{\nabla} r)^2 + 4g(-1+\xi^2)^2\partial_y r \partial_\xi r \partial_{\xi y}^2 r + \\ 4g\xi(-1+\xi^2)^3(\partial_\xi r)^3 + 2g(-1+\xi^2)^4(\partial_\xi r)^2\partial_\xi^2 r + 2g(\partial_y r)^2\partial_y^2 r\Big] - 3\frac{e^{3(r_h - r)}(\tilde{\nabla} r)^2}{1-e^{3(r_h-r)}}\Big\} \end{aligned}$$

where r, g, f are now functions of (ξ, y) and $(\tilde{\nabla} r)^2 = (J\nabla r)^2 = (\partial_y r)^2 + (-1+\xi^2)^2(\partial_\xi r)^2$, with J the Jacobian of the coordinate transformation.
We typically use a 30×30 grid and the residuals have been decreased to $\mathcal{O}(10^{-12})$ or less.

Hartman-Maldacena surfaces for non-gravitating bath solutions

For the non-gravitating bath solutions, let's use the tortoise radial coordinate u and parametrize the embedding of $t = 0$ Hartman-Maldacena surfaces using $x(u, y)$ instead of $r(x, y)$, for reasons of convenience which will be clear in a moment.
These extremal surfaces extends from $u_R = u(r_R)$, imposed by the Dirichlet boundary condition (8.17), through the horizon $u = 0$ into the thermofield double; in particular we are going to consider surfaces anchor at the same point r_R in the thermofield double, thus symmetric with respect to reflection across $u = 0$ at $t = 0$.
Exploiting the reflection symmetry, we can restrict to $u \geq 0$ and subsequently find the curve $x_h(y)$ along which the extremal surfaces end on the horizon.
The induced metric on the surface results to be

$$\begin{aligned} ds^2 = e^{2r_h}\cosh^{4/3}\Big(\frac{3u}{2}\Big) f_4^2 ds^2_{\mathbb{R}^2} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + \\ +[f_4^2 + 4\rho^2(\partial_u x)^2]du^2 + 4\rho^2[dy^2(1+(\partial_y x)^2) + 2(\partial_u x)(\partial_y x)du dy] \end{aligned} \tag{8.19}$$

and the area in units of $S_1^2 \times S_2^2 \times \mathbb{R}^2$ volume is

$$\mathcal{A} = 32\int du dy\, e^{2r_h}\cosh^{4/3}\Big(\frac{3u}{2}\Big) f\sqrt{g\Big((1+(\partial_y x)^2\Big) + (\partial_u x)^2} \tag{8.20}$$

where here the functions f and g are now defined as

$$f = |h_1 h_2 W|, \qquad g = \frac{1}{2}\left|\frac{h_1 h_2}{W}\right| \tag{8.21}$$

The extremality condition is given by

$$\begin{aligned}0 = \frac{1}{g(1+(\partial_y x)^2)+(\partial_u x)^2}\Big\{ & \frac{\partial_x f}{f}[g(1+(\partial_y x)^2)+(\partial_u x)^2] + \frac{1}{2}\partial_x g(1+(\partial_y x)^2) - 2\tanh\Big(\frac{3}{2}u\Big)\partial_u x \\ & -\frac{\partial_y f}{f} g\partial_y x - \frac{\partial_u f}{f}\partial_u x - \frac{2g^2\partial_y^2 x + 2g(\partial_u x)^2(\partial_y^2 x) + 2\partial_y g(\partial_y x)(\partial_u x)^2}{2[g(1+(\partial_y x)^2)+(\partial_u x)^2]} + \\ & -\frac{(\partial_y g \partial_y x + 2\partial_u^2 x) g(1+(\partial_y x)^2) - 4g\partial_u x \partial_y x \partial_{uy}^2 x}{2[g(1+(\partial_y x)^2)+(\partial_u x)^2]}\Big\}\end{aligned} \tag{8.22}$$

with boundary conditions

$$\partial_y x(u,y)|_{y=0,\frac{\pi}{2}} = 0, \qquad \lim_{u\to u(r_R)} x(u,y) = +\infty \tag{8.23}$$

the Neumann boundary conditions on the upper and lower boundary of the strip can be found by analogous arguments to the ones which leads to (8.14) and the Dirichlet boundary condition is the corresponding of (8.17) in the $x(u,y)$ parametrization.
In the one-sided perspective, as said previously, the surface should end at the horizon and the symmetry under reflection across $u = 0$ translates into the following Neumann boundary condition

$$\partial_u x(u,y)|_{u=0} = 0 \tag{8.24}$$

which ensures the vanishing of the boundary terms coming out of the variation of the area functional.
In this case the domain of the embedding function $x(u,y)$ is already finite, but we perform the change of variable $\xi = \tanh x$ anyway since it helps to implement numerically the Dirichlet boundary conditions at $u \to u(r_R)$ as

$$\lim_{u\to u(r_R)} \xi(u,y) = 1 \tag{8.25}$$

while the partial derivative equation (8.22) with this change of variable reads

$$\begin{aligned}0 = & \frac{(1-\xi^2)\partial_\xi f}{f} + \frac{[(-1+\xi^2)^2+(\partial_y\xi)^2](1-\xi^2)\partial_\xi g}{2g[(-1+\xi^2)^2+(\partial_y\xi)^2]+2(\partial_u\xi)^2} + \\ & \frac{-1+\xi^2}{g[(-1+\xi^2)^2+(\partial_y\xi)^2]+(\partial_u\xi)^2}\Big[2\tanh\Big(\frac{3}{2}u\Big)\partial_u\xi + g\frac{\partial_y f}{f}\partial_y\xi + \frac{\partial_u f}{f}\partial_u\xi + \partial_y\xi\partial_y g\Big] + \\ & + \frac{1-\xi^2}{\{g[(-1+\xi^2)^2+(\partial_y\xi)^2]+(\partial_u\xi)^2\}^2}\Big\{[(-1+\xi^2)^2+(\partial_y\xi)^2]\Big(\frac{1}{2}\partial_u\xi\partial_u g + \frac{1}{2}g\partial_y\xi\partial_y g - g\partial_u^2\xi\Big) + \\ & +(\partial_u\xi)^2 g(-2\xi+2\xi^3-\partial_y^2\xi) - g^2(-1+\xi^2)[(-1+\xi^2)\partial_y^2\xi - 2\xi(\partial_y\xi)^2] + 2g\partial_u\xi\partial_y\xi\partial_{uy}^2\xi\Big\}\end{aligned}$$

as for the island surfaces, we typically use a 30×30 grid and the residuals have been decreased to $\mathcal{O}(10^{-12})$ or less.

Analysis of extremal surfaces for vanishing dilaton variation

First of all, let's construct and analyse extremal surfaces in 10d solutions with non-gravitating bath and vanishing dilaton variation in the asymptotic throat, i.e. with $K = \hat{K}$.

The parameters that control the solution and subsequently the shape of the extremal surfaces are the anchor point $r_R = \lim_{\xi\to 1} r(\xi, y)$ in the asymptotic $AdS_5 \times S^5$ region, the horizon

radius r_h and N_5/K, namely the ratio of the number of suspended D3-branes between the 5-branes and the number of semi-infinite D3-branes. For simplicity in the following we set $r_h = 0$, except when we will consider surfaces at zero temperature which are obtained with $b(r) = 1$ or equivalently $r_h \to -\infty$.
Examples of island surfaces obtained with fixed $\alpha = N_5/K = 2.5$ and varying anchor points are

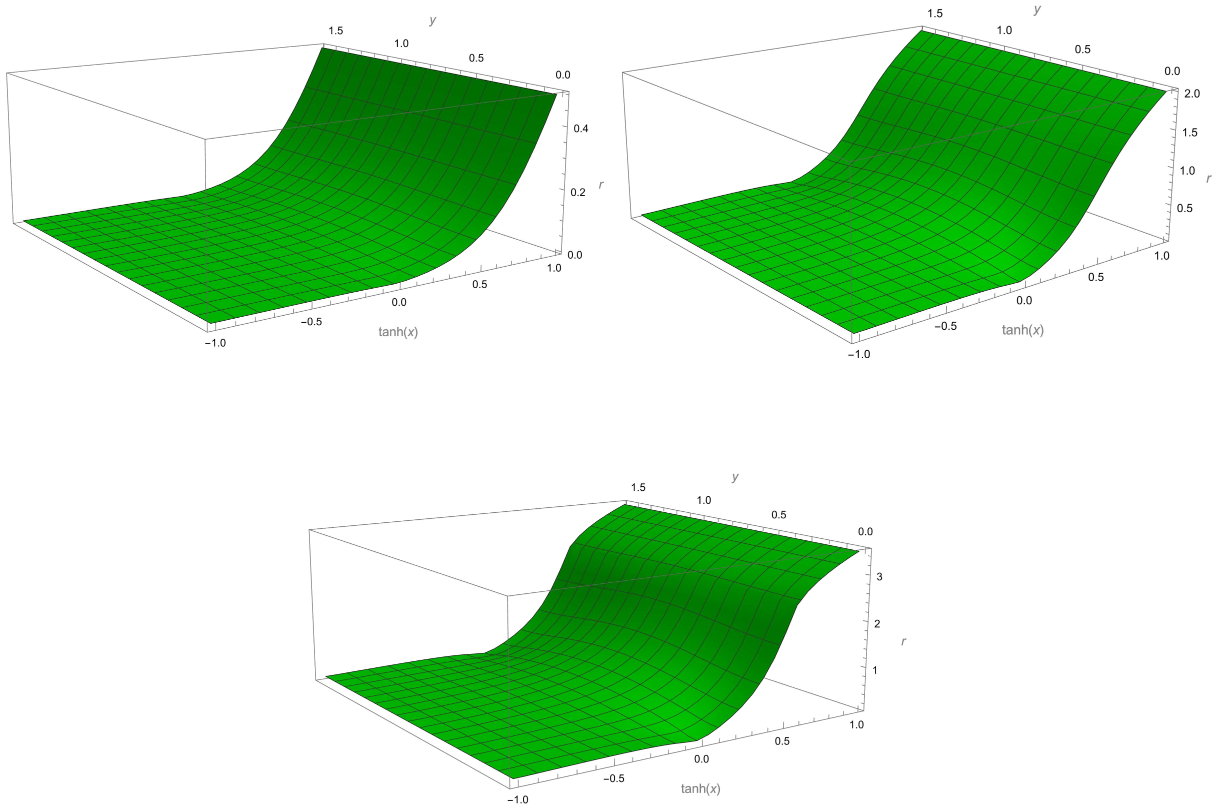
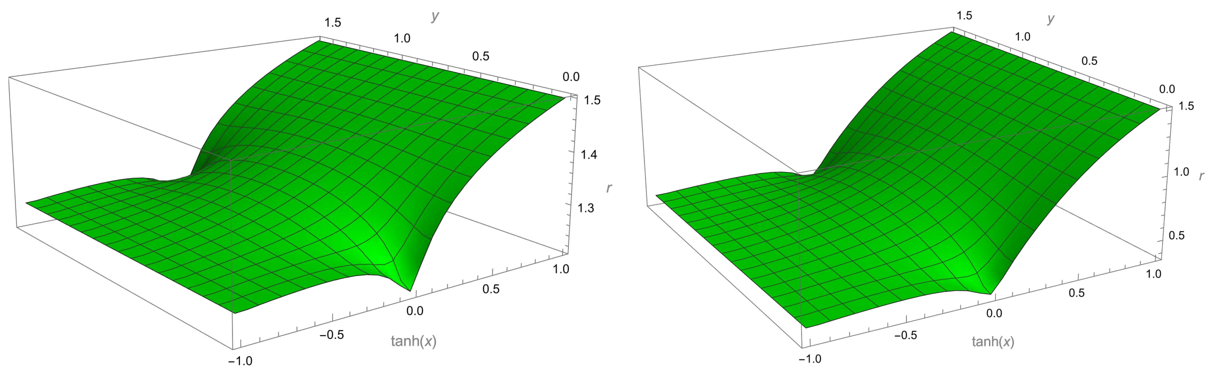


Figure 8.1: Island surfaces obtained with $\alpha = 2.5$, $r_h = 0$ and with the anchor point $r_R = \{0.5; 2; 3.5\}$ from top left to bottom

As the anchor point r_R is decreased towards the horizon, the entire surface moves towards it. Then notice that due to the symmetry of D5/NS5 branes in the supergravity solution (8) under S-duality combined with $z \to i\pi/2 - z$, the Einstein-frame metric is invariant under $y \to \pi/2 - y$ and, since their boundary conditions respect this symmetry, the same holds for the minimal surfaces.

In the following we report examples of island surfaces with fixed anchor point $r_R = 1.5$ and varying $\alpha = N_5/K$



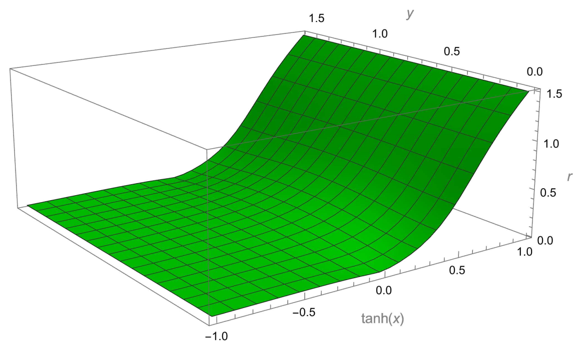


Figure 8.2: Island surfaces obtained with $r_R = 1.5$, $r_h = 0$ and with ratio $\alpha = \{0.2; 1; 2.5\}$ from top left to bottom

We can notice that as α decreases at fixed r_R, the effect of the poles in correspondence of the 5-brane sources is more pronounced. This is due to the fact that, as we are going to see later, increasing α the point r_L where the island surfaces close off at $x \to -\infty$ moves towards the Poincaré horizon, thus for small α the effect of 5-brane singularities at which the surface must drop logarithmically into the Poincaré horizon itself **[22]** is more accentuated.

Some examples of $t = 0$ Hartman-Maldacena surfaces are displayed below.

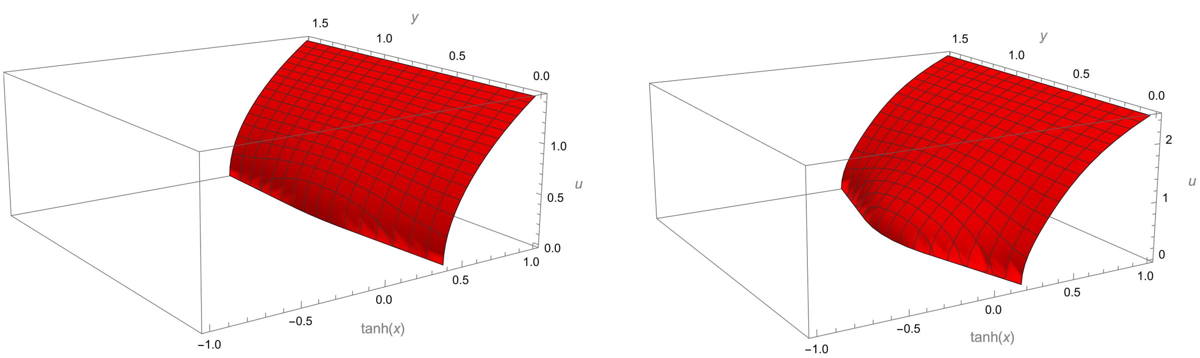


Figure 8.3: Hartman-Maldacena surfaces with $N_5/K = 2$ and $r_R = \{1, 2\}$ from left to right

These surfaces drop into the horizon along a curve $\xi_h(y)$ which is located before the 5-branes at $\xi = 0$. More precisely, as one gradually increases the anchor point r_R, the surfaces reach smaller values of x and the corresponding curve $\xi_h(y)$ starts to shift towards negative values in the interior of Σ, whereas its boundary points $\xi_h(0)$ and $\xi_h(\pi/2)$ remain at positive values without reaching the position of the 5-branes at $\xi = 0$.

Once we have solved the PDEs numerically and we have constructed the extremal surfaces, we can compare the areas of the island and $t = 0$ Hartman-Maldacena surfaces in order to study the evolution of the entanglement entropy. Their areas are divergent as any entanglement entropy in four dimensions, but instead of regulating the two areas separately, we directly compute the area difference between the two kinds of surfaces, anchored at the same point r_R, which turns out to be finite as usual.

For numerical stability, we take the difference between the areas at the level of the integrands, at least in the region of large x (in our computations we have performed the integral of the difference from $\xi = 0.95$). Since the Hartman-Maldacena surfaces have been obtained with a different parametrization with respect the island surfaces, specifically their embedding is described by $\xi(r, y)$, the interpolation function $\xi_{HM}(r, y)$ obtained from the numerical solution has been inverted with respect to the first argument and the derivatives of $r_{HM}(\xi, y)$ can be easily obtained in terms of the derivatives of $\xi_{HM}(r, y)$; then $r_{HM}(\xi, y)$ and its derivatives have been used in the area functional (8.10). The area difference has been computed as the integral of the difference between the integrands in the range $\{(\xi, y)|\xi \in [0.95, 0.999], y \in [0, \pi/2]\}$, then this has been summed up with the island area computed in the range $\{(\xi, y)|\xi \in [-1, 0.95], y \in [0, \pi/2]\}$ and finally it has been subtracted the Hartman-Maldacena area computed in the range $\{(\xi, y)|\xi \in$

$[\xi_{min}, 0.95], y \in [0, \pi/2]\}$, where ξ_{min} is the minimum of the curve $\xi_h(y)$. The integration of the three contributions has been carried out numerically as well and the dependence on the choice of the cut-off $\xi_{max} = 0.999$ has been verified to be very mild.

This area difference has been studied as a function of the anchor point r_R in the asymptotic $AdS_5 \times S^5$ region with different choice of the ratio N_5/K and the results are shown in the following.

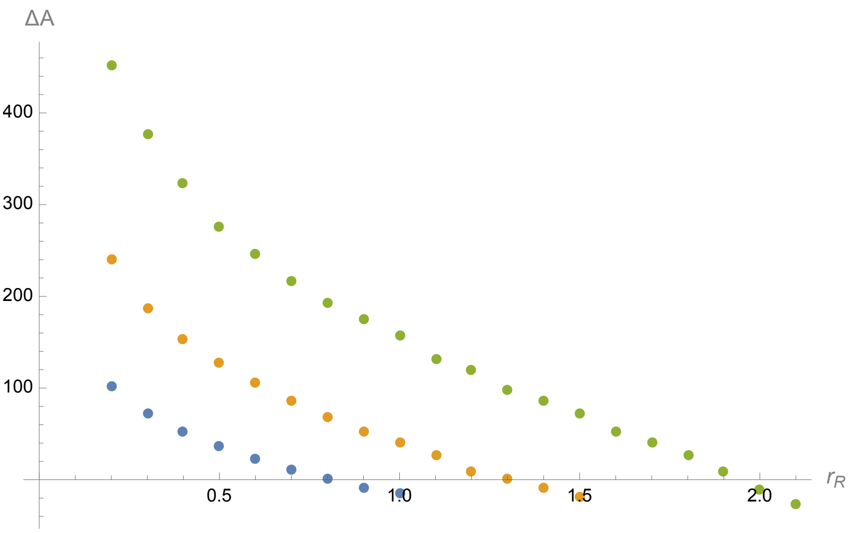


Figure 8.4: Area difference $\Delta\mathcal{A} = \mathcal{A}_{island} - \mathcal{A}_{HM}$ curves as a function of the anchor point r_R, from bottom to top, for $N_5/K = \{1.2; 1.6; 2\}$ with $K = 1$

We can see that for each value of $\alpha = N_5/K$, the area of $t = 0$ Hartman-Maldacena surface is smaller than the area of the island surface if the anchor point r_R is not too far from the horizon $r_h = 0$. In these cases, in other words for radiation regions far in the bath system, the Hartman-Maldacena surface initially dominates over the island surface, contributing to the entropy of radiation region. The former will increase in time until the island surface will become the minimal surface, consequently bounding the entropy and leading to a Page curve.
Each curve in **fig. 8.4** shows a transition point r_R^{trans} at with the area of the two surfaces are equal at $t = 0$, suggesting constant entropies for $r_R > r_R^{trans}$. Moreover, beyond values r_R^* near the end point of each curve, the relaxation extends the Hartman-Maldacena trial surface all the way to $\xi \to -1$ and a equilibrium configuration is not reached. Since our numerical relaxation method does not capture extremal surfaces for which the area functional does not take a local minimum, for these anchor point values the Hartman-Maldacena surface can become a saddle point implying the relaxation's transition towards the island surface. Another possibility is that these surfaces extending over negative ξ values also on the boundary of Σ may become relevant for some reason. The value of r_R^* starts small for small α, increases with α diverging towards the critical value $\alpha_{crit} \approx 4$ (it will be clear later why we call it "critical") and for larger α Hartman-Maldacena surfaces can be found without any bound on r_R.
Finally another important thing to notice is that the area difference $\Delta\mathcal{A} = \mathcal{A}_{island} - \mathcal{A}_{HM}$ is larger for larger α. In terms of the 5d Karch-Randall models this can be interpreted in the following point of view: larger N_5/K corresponds to a boundary conformal field theory (third doubly holographic description at the beginning of section 5) with more 3d defect degrees of freedom than 4d bulk degrees of freedom, which amounts to a larger tension of the ETW brane; a larger tension implies a smaller angle θ between the brane and the bulk AdS_5 conformal boundary (5.8). Thus we expect the island surface to have larger area with respect to $t = 0$ Hartman-Maldacena surface for smaller θ, which is reasonable since the ETW brane is further from the bath.

Analysis of extremal surfaces for non-vanishing dilaton variation

In the case of non-vanishing dilaton variation in the $AdS_5 \times S^5$ throat, the parameters that control the shape of the extremal surfaces are once again the anchor point $r_R = \lim_{\xi \to 1} r(\xi, y)$ in the asymptotic $AdS_5 \times S^5$ region, the horizon radius r_h and the ratio α which now is given by

$$\alpha = \sqrt{\frac{N_5^2}{K\hat{K}}} \tag{8.26}$$

As before, in the following we set $r_h = 0$, except when we will consider surfaces at zero temperature which are obtained with $b(r) = 1$ or equivalently $r_h \to -\infty$. The first thing to notice is that while in the $\delta\phi = 0$ case the extremal surfaces are symmetric under $y \to \frac{\pi}{2} - y$ and in particular the boundary points of the Hartman-Maldacena surface's horizon curve $\xi_h(y)$, namely $\xi_h(0)$ and $\xi_h(\pi/2)$, are equal due to the latter symmetry, in the $\delta\phi \neq 0$ case the $y \to \frac{\pi}{2} - y$ symmetry is spoiled by the fact that $K \neq \hat{K}$ and this implies that the boundary points of the Hartman-Maldacena surface's horizon curve are different, in particular from numerical solutions $\xi_h(0) < \xi_h(\pi/2)$.
Then if we fix α and the anchor point r_R, as $\delta\phi$ increases it has been found that the curve $\xi_h(y)$ moves towards larger positive values as can be observed in the plots below.

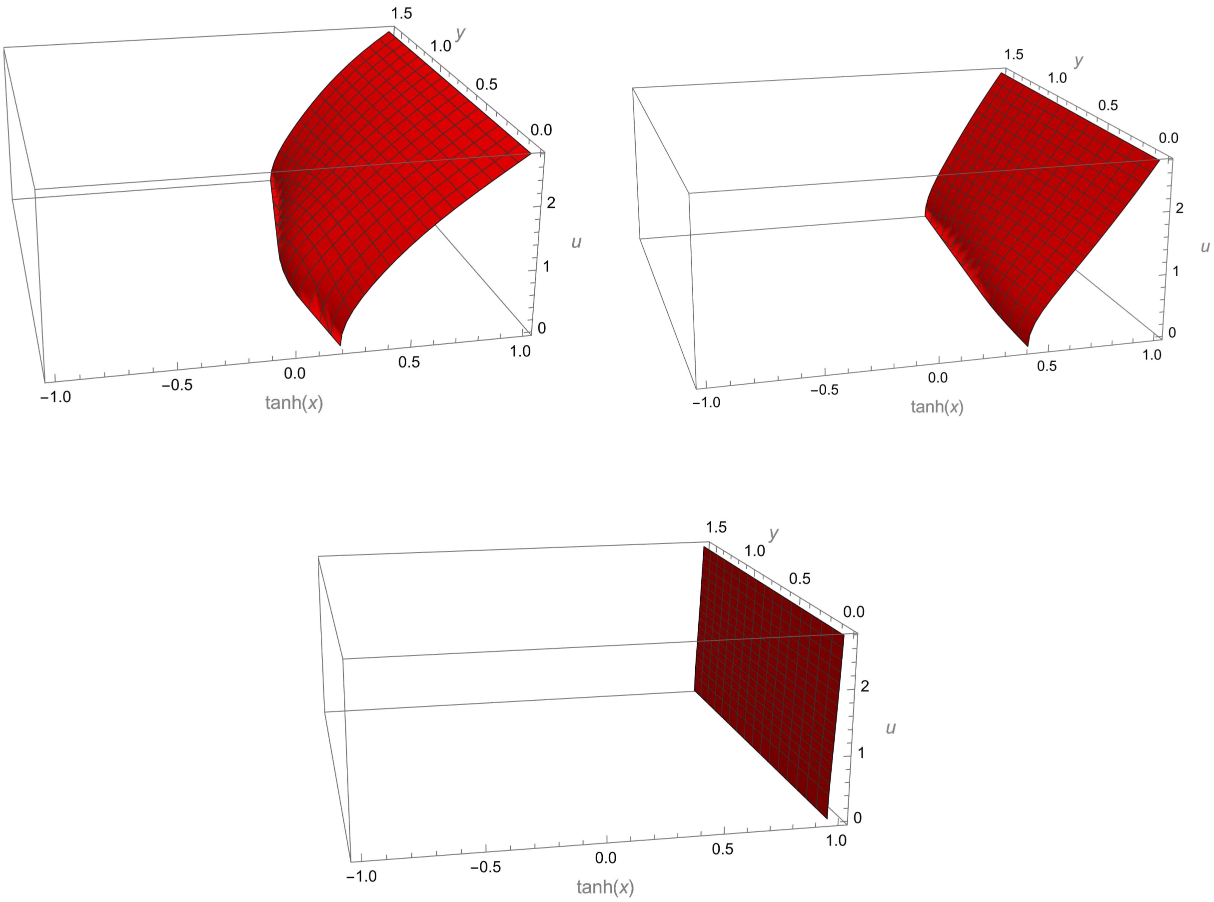


Figure 8.5: Hartman-Maldacena surfaces with fixed $\alpha = 2$ and $r_R = 2.3$, and with $\delta\phi = \{1.2; 2; 5\}$ from top left to bottom

Actually increasing $\delta\phi$, the entire surface asymptotes $\xi = 1$ and for high enough $\delta\phi$ we do not find an extremal Hartman-Maldacena-like surface. The reason is that tuning $\delta\phi$ to high values, we get in a regime of small graviton masses (7.20) and the background becomes closer to the one with a gravitating bath, for which the graviton mass vanishes because of the compactness of the internal geometry. In particular, we have seen that in a Karch-Randall model with gravitating

bath the Hartman-Maldacena surface is not a quantum extremal surface, i.e. does not contribute the the conventional geometric entanglement entropy of outgoing radiation. Thus this behavior is well captured by the 10d uplift under consideration.
Next we can repeat the same analysis concerning the area difference we have previously done for the case of vanishing dilaton and study how an area difference $\Delta\mathcal{A} = \mathcal{A}_{island} - \mathcal{A}_{HM}$ curve at fixed α modifies as $\delta\phi$ changes.
As an example, we consider the cases $\alpha = 1.2$ and $\alpha = 2$ and we study the behaviour of the area difference curve for $\delta\phi = \{0.8; 1.2; 1.6\}$ and $\delta\phi = \{0.8; 1.2\}$ respectively.

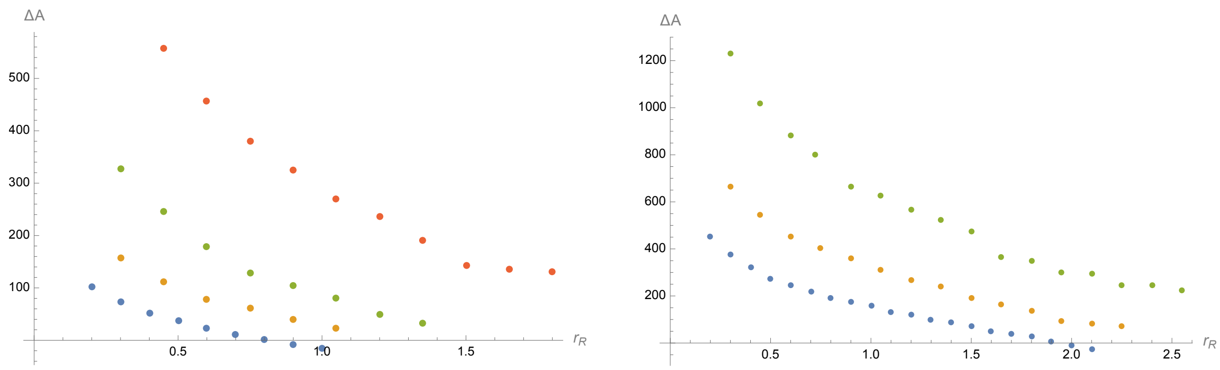


Figure 8.6: Area difference $\Delta\mathcal{A} = \mathcal{A}_{island} - \mathcal{A}_{HM}$ curves as a function of the anchor point r_R with $\alpha = 1.2$ (left) and $\alpha = 2$ (right), with $\delta\phi = \{0; 0.8; 1.2; 1.6\}$ from bottom to top

As we can see from the above figure, the non-vanishing dilaton variation produces several effects on the area difference curve.
First of all, the values of the curve become larger as $\delta\phi$ increases. A second important behaviour, which is related to the first, is that in general there is not a transition point r_R^{trans} at which the two quantum extremal surfaces' areas are equal at $t = 0$. As in the $\delta\phi = 0$ case, the area difference curves have an end point, since increasing gradually the anchor point r_R the two boundary points of the Hartman-Maldacena surface's horizon curve $\xi_h(y)$ shifts gradually towards $\xi = 0$, and once the point $\xi_h(0)$ goes over the NS5-branes singularity, the trial Hartman-Maldacena surface does not settle anymore on equilibrium.

Island surfaces at zero temperature and critical brane setups

In the Karch-Randall model it has been showed the existence of a critical angle θ_c defined as the angle at which at zero temperature the end-point on the ETW brane of a surface anchored at a fixed point in the bath system diverges towards the Poincaré horizon and below which island surfaces disappear. It exists a 10d counterpart of this critical angle in the type IIB string theory uplift under consideration as we are going to prove now.
Let's suppose to work with vanishing dilaton variation firstly. If we fix the anchor point $r_R = \lim_{x\to+\infty} r(x, y)$ of zero temperature island surface in the asymptotic $AdS_5 \times S^5$, the other anchor point $r_L = \lim_{x\to-\infty} r(x, y)$ at which the surface closes off at $x \to -\infty$ moves towards the Poincaré horizon as $\alpha = N_5/K$ increases, in particular the difference $r_R - r_L$ increases linearly for small α and then it diverges in correspondence of a critical value $\alpha_{crit} \approx 4$.
The critical value α_{crit} has been found by gradually increasing α and verifying if the difference $\Delta r = r_R - r_L$ changes as the relaxation time increases and jointly if residuals decrease or remain finite. Once we reaches a value of α for which the residuals are irreducible keeping driving the anchor points further apart with increasing relaxation time, it means we have found the critical value α_{crit}.
Taking into account the role of θ_c in the Karch-Randall model with non-gravitating bath at zero temperature and the behaviour of island surfaces for brane angles close to it, these results are consistent with the identification we have discussed before between N_5/K and the inverse of the brane angle $1/\theta$ of 5d Karch-Randall models. For black hole solutions with finite temperature,

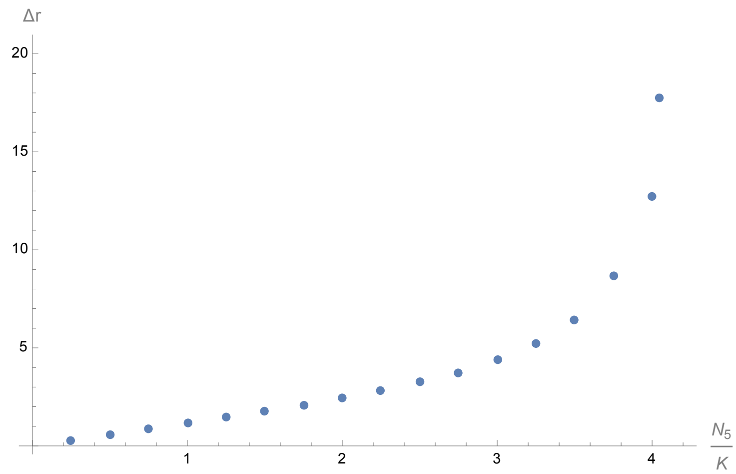


Figure 8.7: $\Delta r = r_R - r_L$ as a function of $\alpha = N_5/K$ for $\delta\phi = 0$

this behavior is regulated and island surfaces can be found also beyond the critical ratio since numerically the endpoint r_L appears stuck below a critical value, similarly to what happens in the 5d Karch-Randall model with non-gravitating bath at finite temperature.
For $\delta\phi \neq 0$, it still exists a critical value α_{crit}, which is pushed to higher values for increasing $\delta\phi$ as can be seen below.

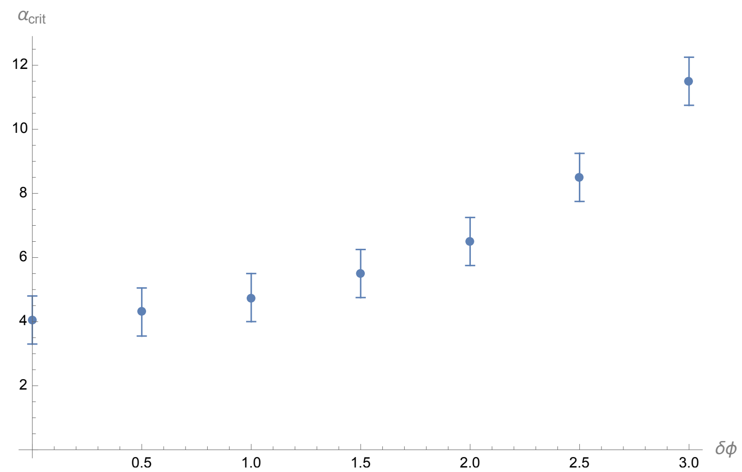


Figure 8.8: Monotonic increase of α_{crit} with dilaton variation $\delta\phi$ across the geometry

For values $\delta\phi > 3$, divergences at small α appear as well, even though island surfaces can be found for larger α, and these signals code's loss of predictivity; thus we present numerical results only for $\delta\phi \leq 3$.
Recalling the fact that the graviton mass is proportional to the ratio of bulk over boundary degrees of freedom, namely $m_g L_4 \propto \alpha^{-1}$, in the case $\delta\phi = 0$ the critical value $\alpha_{crit} \approx 4$ turns out to be not large enough to put the islands in the regime of validity of AdS massive gravity, which in string theory is under theoretical control only for small graviton mass $m_g^2 L_4^2 \ll 1$. Thus the crucial important consequence of non-vanishing dilaton variation is to increase the value of the critical ratio α_{crit} so that it is big enough to ensure the possibility to find at zero temperature islands in the regime of small graviton masses.

Extremal surfaces in a gravitating bath

As in the Karch-Randall models with gravitating bath, if we consider their 10d string theory realizations corresponding to the type IIB string theory solution (6.65), there is no an asymptotic

$AdS_5 \times S^5$ region, which was the natural place to geometrically define the radiation region at $x = +\infty$ and to anchor minimal surfaces in the 10d solutions with non-gravitating bath. As a consequence, minimal surfaces extending from $x = -\infty$ to $x = +\infty$ satisfy Neumann boundary conditions on both ends and are found to settle onto the horizon leading to flat entropy curve, similarly to what happens in the corresponding 5d Karch-Randall model.
A Page curve can arise considering surfaces that divide the internal space and which we expect to compute non-geometric entanglement entropies, in perfect analogy to the surfaces starting from the defect in the 5d KR models with gravitating bath. In particular, due to the reflection symmetry of the solution (6.65) under $x \to -x$, the natural 10d analog of the defect is the locus $x(y) = 0$, which will serve as an anchor point for the 8d minimal surfaces wrapping the spatial part of AdS_4, both the two 2-spheres S^2 and a curve in Σ depending on the tortoise radial coordinate u.
The Hartman-Maldacena surface at $t = 0$ can be found by imposing the trial surface to be anchored at the locus $x(y) = 0$ both for $u \to \infty$ and in the thermofield double and letting the relaxation method to settle on a equilibrium configuration. This procedure selects $x(u, y) = 0$ as the Hartman-Maldacena surface at $t = 0$.
For the island surfaces, we must impose they are anchored as well along $x(y) = 0$ for $u \to \infty$ and their embedding $u(x, y)$ must satisfy Neumann boundary conditions both at $x \to \pm\infty$. Exploiting the invariance of the supergravity solution under $x \to -x$, we can construct the island surfaces extending from $x = 0$ to $x \to -\infty$.
Since in this case we want the embedding function to be $u(x, y)$, we start by rewriting the induced metric (8.9) using the tortoise radial coordinate u

$$ds^2_\gamma = e^{2r_h} \cosh^{4/3} \frac{3u}{2} f_4^2 ds^2_{\mathbb{R}^2} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + 4\rho^2(dx^2 + dy^2) + f_4^2(dx\partial_x u + dy\partial_y u)^2 \tag{8.27}$$

then the area density functional reads

$$\mathcal{A} = 32 \int dxdy\, e^{2r_h} \cosh^{4/3}\Big(\frac{3u}{2}\Big) f \sqrt{1 + g(\nabla u)^2} \tag{8.28}$$

where ∇ is the covariant derivative with respect to the metric on Σ and here the functions f and g are defined as

$$f = |h_1 h_2 W|, \qquad g = \frac{1}{2}\Big|\frac{h_1 h_2}{W}\Big| \tag{8.29}$$

We set as usual $r_h = 0$ and we can notice that the area functional written in the tortoise radial coordinate u is very similar formally to the one written in the radial coordinate r, since the difference is only the non-derivative factor in the area integrand $\mathcal{L}$ and a factor $1/b(r)$ in the definition of $g(x, y)$.
The extremality condition is found by solving the Euler-Lagrange equation of the integrand $\mathcal{L}$ of the area functional

$$0 = \frac{1}{\mathcal{L}}\Big(\frac{\delta\mathcal{L}}{\delta u} - \nabla\frac{\delta\mathcal{L}}{\delta(\nabla u)}\Big) = \frac{1}{1 + g(\nabla u)^2}\Big[2\tanh\Big(\frac{3u}{2}\Big) - \nabla(g\nabla u) + \frac{1}{2}g\nabla u \cdot \nabla \ln\Big(\frac{1 + g(\nabla u)^2}{f^2}\Big)\Big]$$

with Neumann boundary conditions on both upper and lower boundary of the strip Σ, and at $x \to -\infty$ (8.16)

$$\partial_y u(x, y)|_{y=0,\frac{\pi}{2}} = 0, \qquad \lim_{x\to-\infty} e^{-x}\partial_x u(x, y) = 0 \tag{8.30}$$

whereas at $x \to 0$ we must impose that the embedding function $u(x, y)$ diverges

$$\lim_{x\to 0} u(x, y) = +\infty \tag{8.31}$$

Then we perform the change of coordinates $\xi = \tanh x$ and $\zeta(\xi, y) = \tanh[u(\xi, y)]$ respectively in order to make the domain of the embedding function finite and to implement numerically the

Dirichlet boundary condition at $\xi \to 0$. The resulting extremality condition in these coordinates becomes

$$\begin{aligned}&\frac{(-1+\zeta^2)^2}{2[(1-\zeta^2)^2+g((\partial_y\zeta)^2+(-1+\xi^2)^2(\partial_\xi\zeta)^2]}\Big[\frac{g}{-1+\zeta^2}\Big(-2\frac{\partial_y f}{f}\partial_y\zeta-2(-1+\xi^2)^2\frac{\partial_\xi f}{f}\partial_\xi\zeta+\\&+\frac{\partial_y\zeta\partial_y h+(-1+\xi^2)^2\partial_\xi\zeta\partial_\xi h}{h}\Big)-4\tanh\Big(\frac{3}{2}\operatorname{arctanh}\zeta\Big)-2\frac{\partial_y g\partial_y\zeta+(-1+\xi^2)^2\partial_\xi g\partial_\xi\zeta}{-1+\zeta^2}+\\&+2\frac{g}{(-1+\zeta^2)^2}\Big(2\zeta(\partial_y\zeta)^2-(-1+\zeta^2)\partial_y^2\zeta-(1-\xi^2)(-2\xi(-1+\zeta^2)\partial_\xi\zeta+\\&+2(-1+\xi^2)\zeta(\partial_\xi\zeta)^2-(-1+\xi^2)(-1+\zeta^2)\partial_\xi^2\zeta)\Big)\Big]=0\end{aligned} \tag{8.32}$$

where the function $h(\xi, y)$ is defined as

$$h(\xi, y) = 1 + \frac{g(\xi, y)[(\partial_y\zeta)^2 + (-1+\xi^2)^2(\partial_\xi\zeta)^2]}{(-1+\zeta^2)^2} \tag{8.33}$$

Examples of island surfaces for different values of the 5-brane source location δ on Σ are shown below.

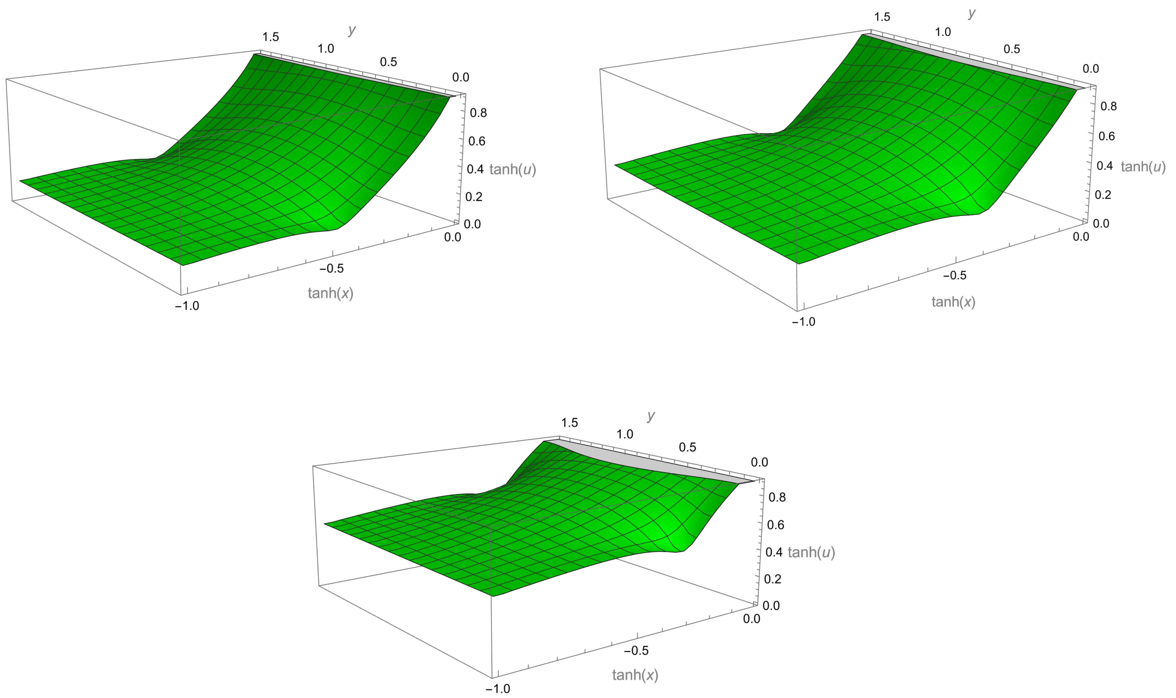


Figure 8.9: Island surfaces in gravitating bath solutions with $\delta = \{0.5; 0.4; 0.3\}$ from top left to bottom

As we can see from the plots above and quantitatively from the figure below, the cap-off point $\tanh u_L$ exponentially falls off $\tanh u_L = 1.17e^{-4.28\delta}$ towards the black hole horizon $u = 0$ as δ increases, however towards small values of δ the cap-off point grows rapidly diverging for a critical value $\delta_c \approx 2.8$. Below this critical value, the relaxation method does not settle the trial island surface on an equilibrium configuration. Notice moreover how the island surfaces seem to bulge towards the horizon in correspondence of the location of the 5-branes, even if this local behaviour is not well captured by our numerical solution method.

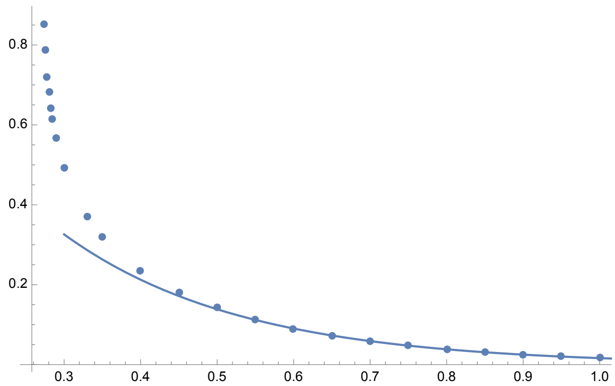


Figure 8.10: Cap-off point $\tanh u_L$ as a function of 5-branes location δ

Uhlemann [**22**] has also calculated the area difference between the island and $t = 0$ Hartman-Maldacena surfaces and he found out that for larger δ the area of $t = 0$ Hartman-Maldacena surface is smaller than the one of island surface, ensuring a Page curve as time grows. In particular the area difference at $t = 0$ displays a value of δ, called the Page value δ_P, such that the area difference vanishes and from numerical data it results to be $\delta_P \approx 0.29$, thus close but slightly larger than δ_c. In particular for $\delta_c < \delta < \delta_P$, the island surface is already dominant with respect to $t = 0$ Hartman-Maldacena surface.
Furthermore, in the range $\delta < \delta_c$ we have found numerically trial island surfaces approaching the AdS_4 conformal boundary almost everywhere on Σ, except near the 5-branes locations where they reach the horizon. These can be interpreted as the 10d version of tiny island surfaces, giving a dominant contribution in 5d Karch-Randall models with gravitating bath in the regime above the critical ETW brane angle.
Thus these 10d results are perfectly consistent with the results found out with the 5d Karch-Randall model with gravitating bath if we identify the inverse of the brane angle of the 5d KR description with the brane stack position δ of the 10d string theory uplift.

As anticipated in 6, a 10d uplift of the Karch-Randall model with gravitating bath and with non-vanishing dilaton can be obtained with the following choice of the harmonic functions

$$\begin{aligned} h_1 &= -\frac{N_5}{8}\left[\ln\left(\frac{\cosh(x-\delta)-\sin y}{\cosh(x-\delta)+\sin y}\right)+\ln\left(\frac{\cosh(x+\delta)-\sin y}{\cosh(x+\delta)+\sin y}\right)\right] \\ h_2 &= -\frac{\hat{N}_5}{8}\left[\ln\left(\frac{\cosh(x-\delta)-\cos y}{\cosh(x-\delta)+\cos y}\right)+\ln\left(\frac{\cosh(x+\delta)-\cos y}{\cosh(x+\delta)+\cos y}\right)\right] \end{aligned} \tag{8.34}$$

for which the dilaton variation reads

$$e^{2\delta\phi} = \frac{\hat{N}_5}{N_5} \tag{8.35}$$

unfortunately with this choice of the harmonic functions, the function g results the same as the one with the previous choice, since in the ratio $h_1 h_2/W$ the factors in front of h_1 and h_2 drops out, whereas the factor in front of $f = |h_1 h_2 W|$ changes but it drops out in the extremality condition since the function f always appears through the ratio $(\partial f/f)$ as can be seen from (8.32).
Another possibility to obtain a non-vanishing dilaton variation could be with the choice

$$\begin{aligned} h_1 &= -\frac{N_5}{8}\ln\left(\frac{\cosh(x-\delta)-\sin y}{\cosh(x-\delta)+\sin y}\right)-\frac{\hat{N}_5}{8}\ln\left(\frac{\cosh(x+\delta)-\sin y}{\cosh(x+\delta)+\sin y}\right) \\ h_2 &= -\frac{\hat{N}_5}{8}\ln\left(\frac{\cosh(x-\delta)-\cos y}{\cosh(x-\delta)+\cos y}\right)-\frac{N_5}{8}\ln\left(\frac{\cosh(x+\delta)-\cos y}{\cosh(x+\delta)+\cos y}\right) \end{aligned} \tag{8.36}$$

but this choice cannot yield a valid type IIB solution since the harmonic functions do not satisfy the required S-duality transformation.

Chapter 9

Conclusions

We have presented UV-complete string theory settings exhibiting the emergence of entanglement islands and Page curves for black holes in 4d theories of gravity. These gravity theories certainly differ from the one we experience in nature but they have dynamical gravitons, whose mass can be controlled, and they show versions of information paradox whose resolution can be analyzed by applying AdS/CFT dualities.

In particular, starting from the bottom-up 5d Karch-Randall models, we have studied specific examples of d'Hoker's type IIB string theory solutions based on $AdS_4 \times S^2 \times S^2 \times \Sigma$ background geometry, checking their consistency with global constraints, studying the asymptotic regions and the possibly presence of 5-brane singularities, together with the fluxes and the brane configuration. Among these examples, a supersymmetric 5-branes doped Janus solution with one asymptotic $AdS_5 \times S^5$ region closed off has been identified as the 10d string theory realization of Karch-Randall model with non-gravitating bath and similarly a solution, resulting to be an holographic dual of 3d SCFTs, has been selected as a representative 10d uplift of Karch-Randall model with gravitating bath. On this 10d uplifts, following Uhlemann's and Gnecchi *et al* works **[22]**,**[23]**, we have constructed numerically the quantum extremal surfaces, i.e. the island and Hartman-Maldacena surfaces, contributing to the entanglement entropy of outgoing radiation and whose competition leads to a Page curve. Most of the work has been focused on the uplifts with non-gravitating bath, on which we have reproduced the Uhlemann's results in the case of constant dilaton field, more specifically we have reproduced the $t = 0$ area difference curves as a function of the surfaces' anchor point in the asymptotic $AdS_5 \times S^5$, studying how they behave as the microscopic parameter α changes and noticing it always exist an anchor point value at which the area difference vanishes, which suggests to take the radiation region far in the bath system if one wants to obtain a Page curve, in line with what has been found in the Karch-Randall models **[13], [12]**. Then always about Uhlemann's results, we have verified the presence of a critical value of the microscopic parameter α above which at zero temperature the islands cease to contribute to the entanglement entropy, similarly to what happens in the Karch-Randall models, and we have verified, in line with Gnecchi *et al* work, how island surfaces change in the same geometric background allowing a non-vanishing dilaton variation, and how α_{crit} gets modified as $\delta\phi$ varies. In particular α_{crit} increases as $\delta\phi$ increases and this ensures we can tune dilaton variation in such a way to obtain a critical value of α big enough to the island surfaces to contribute in the regime of validity of massive AdS gravity as effective field theory, namely of small graviton masses.

In this thesis's project we have constructed and studied the Hartman-Maldacena surfaces in the uplifts with non-gravitating bath and in presence of a varying dilaton field. In particular we studied the area difference curves with fixed α and varying the dilaton variation $\delta\phi$ finding that the area difference increases as $\delta\phi$ increases and noticing there is no more an anchor point value in the asymptotic $AdS_5 \times S^5$ such that the area difference vanishes. It follows that for all the anchor point values for which the area difference curve is defined, i.e. for all the values for which we can numerically find an Hartman-Maldacena surface settling on an equilibrium configuration, it is possible to get a Page curve for the time evolution of radiation entanglement entropy.

Concerning the 10d realizations of Karch-Randall models with gravitating bath, we have checked the existence of a critical value of 5-brane location δ, which in this solution plays the role of the brane angle θ of the Karch-Randall model, below which island surfaces cease to exist. The brane configuration has been slightly modified in order to allow a non-vanishing dilaton variation but unfortunately the extremality condition determining the island surfaces remain unaltered.
It would be interesting to study more complex brane configurations corresponding to more general 10d solutions for both non-gravitating and gravitating bath, in which it may be expected a more complicated phase structure. Furthermore it would be desirable to investigate how the critical parameters in supergravity solutions translate on the CFT side and study their implications, for example through CFT correlation functions.

Appendix A

Anti de Sitter space

Anti de Sitter space is a maximally symmetric Lorentzian space with constant negative curvature and it is a solution of Einstein equations with negative cosmological constant.
The geometrical properties of a d-dimensional Anti de Sitter spacetime can be derived by looking at this space as an hypersurface embedded in a spacetime with $d+1$ dimensions. The embedding equations can be written as

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = l^2 \tag{A.1}$$

in a flat $(d+1)$-dimensional space with signature

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2 \tag{A.2}$$

where l is a constant of dimension of a length that parametrizes the radius of curvature of the AdS hypersurface. By construction, this space has the isometry $SO(2, d-1)$.

The d-dimensional metric can be obtained by choosing a proper parametrization of the $d+1$ embedding coordinates.
A global parametrization is constructed as follows. Given the form of the embedding equation (A.1) we consider two spheres $X_0^2 + X_d^2 = r_1^2$, $\sum_{i=1}^{d-1} X_i^2 = r_2^2$ of radii r_1, r_2, such that

$$r_1^2 - r_2^2 = l^2 \tag{A.3}$$

This equation is solved by setting $r_1 = l\cosh\rho$, $r_2 = l\sinh\rho$ where $\rho \in \mathbb{R}^+$. Taking the parametrization in spherical coordinates and replacing in the line element (A.2) one gets

$$\begin{aligned} ds^2 &= -(dr_1^2 + r_1^2 d\psi^2) + dr_2^2 + r_2^2 d\Omega_{d-2}^2 \\ &= l^2(-\cosh^2\rho\, d\psi^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-2}^2) \end{aligned} \tag{A.4}$$

This is the global parametrization of AdS since all points of hyperboloid are taken into account exactly once.
Notice that the time-like coordinate ψ is an angular coordinate $\psi \in [-\pi, \pi[$ and, as a consequence, this global parametrization of AdS gives closed time-like curves. This can be avoided if one passes over the universal covering surface[1] by taking a new time-like coordinate $t \in \mathbb{R}$.
The change of coordinate $\tan\theta = \sinh\rho$ gives the metric

$$ds^2 = \frac{l^2}{\cos\theta^2}(-dt^2 + d\theta^2 + \sin\theta^2 d\Omega_{d-2}^2) \tag{A.5}$$

where $0 \le \theta < \pi/2$ for any $d > 2$.
If $d = 2$, the coordinate $\rho \in \mathbb{R}$ since r_2 can assume any real value due to the degeneracy of $d-2$

[1] this procedure of unwrapping the circle of time coordinate is legitimate because the space is not simply connected, so the time circle cannot be topologically reduced to a point

sphere, thus the range of the coordinate θ is $-\pi/2 \leq \theta \leq \pi/2$. This implies that the Penrose diagram of the global AdS_2, obtained from (A.5) by dropping the overall conformal factor, is an infinite strip between $\theta = -\pi/2$ and $\theta = +\pi/2$, whereas the Penrose diagram of AdS_d is the ball $B_{d-1} \times \mathbb{R}$ (actually it is one of the hemisphere times $\mathbb{R}$ since $0 \leq \theta < \pi/2$). Moreover, it follows that the boundary of conformally compactified AdS_d is $\mathbb{R} \times S^{d-2}$, which is equal to the conformal compactification of the flat Minkowski space $\mathbb{R}^{d-2,1}$.

Another useful parametrization is provided by Poincaré coordinate system (t, r, x_i) defined by the following relations

$$\begin{aligned} X_0 &= \frac{1}{2r}(1 + r^2(l^2 + \vec{x}^2 - t^2)) \\ X_i &= lrx_i, \quad (i = 1, ..., d-2) \\ X_{d-1} &= \frac{1}{2r}(1 - r^2(l^2 - \vec{x}^2 + t^2)) \\ X_d &= lrt \end{aligned} \tag{A.6}$$

Inserting into (A.2) we get the desired line element

$$ds^2 = l^2\Big(\frac{dr^2}{r^2} + r^2(-dt^2 + d\vec{x}^2)\Big) \tag{A.7}$$

where $t, x_i \in \mathbb{R}$. Notice from (A.6) that

$$r = \frac{X_0 - X_{d-1}}{l^2} \tag{A.8}$$

we conclude we have two different Poincaré charts. The first chart is the region $r > 0$, that means $X_0 > X_{d-1}$ and corresponds to one half of the hyperboloid; the other half $X_0 < X_{d-1}$ corresponds to the second Poincaré chart, i.e. the region $r < 0$. The Poincaré AdS_d space is the portion of AdS_d corresponding to one of these two charts (the $r > 0$ is usually chosen). Due to the fact that we cannot cover the entire AdS_d with an unique Poincaré patch, we have a degenerate Killing horizon at $r = 0$. The conformal boundary is attained at $r \to \infty$ and the line element becomes:

$$ds^2 = l^2r^2(-dt^2 + d\vec{x}^2) \tag{A.9}$$

so the Poincaré AdS_d space conformal boundary is topologically a $(d-1)$-dimensional Minkowski space. In terms of the coordinate $z = 1/r$ the line element (A.7) becomes

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + dz^2 + d\vec{x}^2) \tag{A.10}$$

up to a conformal factor, this is just like the d-dimensional Minkowski space, thus its Penrose diagram is the same. The latter is a triangular region, thus it does not cover the whole spacetime region of the Penrose diagram of the global AdS_d space as expected. Let us briefly construct the Penrose diagram of a Minkowski space

$$ds^2 = -dt^2 + dz^2 + d\vec{x}^2 \tag{A.11}$$

then we introduce the light-cone coordinates

$$u_\pm = t \pm z \quad \Rightarrow \quad ds^2 = -du_+du_- + d\vec{x}^2 \tag{A.12}$$

notice that $z = \frac{(u_+ - u_-)}{2} \geq 0$ implies $u_+ \geq u_-$, so these are infinite range coordinates. Let us proceed defining new finite range coordinates

$$u_\pm = \tan \tilde{u}_\pm \tag{A.13}$$

such that $\tilde{u}_+, \tilde{u}_- \in (-\pi/2, +\pi/2)$ and $\tilde{u}_+ \geq \tilde{u}_-$. Finally defining

$$\tilde{u}_\pm = \tau \pm \theta \tag{A.14}$$

the metric, by dropping an overall conformal factor and reducing to two dimension in the coordinate (τ, θ), becomes

$$ds^2 = -d\tau^2 + d\theta^2 \tag{A.15}$$

which describes a flat compact two dimensional space since $|\tilde{u}_{\pm}| \leq 0$, thus $|\tau \pm \theta| \leq \pi/2$, and $\theta \geq 0$, so the Carter-Penrose diagram is a triangular region as claimed.

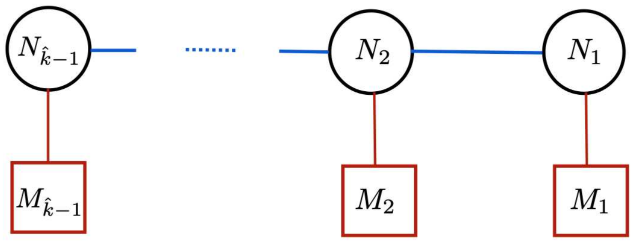
Appendix B

$D = 3$, $\mathcal{N} = 4$ SCFTs

The theories $T^{\rho}_{\hat{\rho}}(SU(N))$, introduced by Gaiotto and Witten in **[24]** and argued to flow in the IR to 3d $\mathcal{N} = 4$ SCFTs under certain constraints we are going to discuss, are labeled by a pair of partitions ρ and $\hat{\rho}$ of N, which uniquely determine a three dimensional $\mathcal{N} = 4$ supersymmetric gauge theory in the UV limit.
The gauge group of $T^{\rho}_{\hat{\rho}}(SU(N))$ is

$$G = U(N_1) \times U(N_2) \times ... \times U(N_{\hat{k}-1}) \tag{B.1}$$

then the theory has one hypermultiplet in the bifundamental representation of each pair of neighboring factors $U(N_i) \times U(N_{i+1})$, as well as M_i hypermultiplets in the fundamental representation of each $U(N_i)$ factor of the gauge group.
This SUSY gauge theory is summarized by the linear quiver diagram shown in the following**[25]**,



where the circles denote the gauge group factors and the squares and blue lines stand for fundamental and bi-fundamental hypermultiplets respectively.
Any two partitions ρ and $\hat{\rho}$ of N determine completely the gauge theory data $\{N_j, M_j\}$. Two useful parametrizations of each of the two partitions are given by

$$\begin{aligned} \rho: \qquad & N = l_1 + ... + l_k = \sum_l l M_l \\ \hat{\rho}: \qquad & N = \hat{l}_1 + ... + \hat{l}_{\hat{k}} = \sum_{\hat{l}} \hat{l} \hat{M}_{\hat{l}} \end{aligned} \tag{B.2}$$

where the l_i are positive non increasing integers, $l_1 \geq l_2 \geq ... \geq l_k > 0$, while M_l is the number of times the integer l occurs in the partition, and the same hold for the parametrization of $\hat{\rho}$. One can associate to ρ and $\hat{\rho}$ a Young tableau whose rows have length respectively l_1,...,l_k and $\hat{l}_1$,...,$\hat{l}_{\hat{k}}$. With this choice of parametrizations, M_j is precisely the number of hypermultiplets in the fundamental representation of $U(N_j)$ gauge group factor, with

$$N_1 = k - \hat{l}_1, \qquad N_j = N_{j-1} + m_j - \hat{l}_j, \quad j = 2, ..., \hat{k} - 1 \tag{B.3}$$

where $m_l = m_{l-1} - M_l$ counts the number of terms that are equal or bigger than l in the partition ρ. The set m_l is a non-increasing sequence of positive integers defining the partition ρ^T, whose

Young tableau is the transpose of the one associated to partition ρ.
The condition $\rho^T > \hat{\rho}$, implying $\hat{k} > l_1$ and thus $N_i \in \mathbf{N}$, $M_l = 0$ for $l \geq \hat{k}$, is necessary in order for the linear quiver associated to the triplet $(\rho, \hat{\rho}, N)$ to make sense. The conjecture of Gaiotto and Witten [**24**] is that all these quiver gauge theories $T^{\rho}_{\hat{\rho}}(SU(N))$ flow to a non trivial infrared fixed point. Since $\rho^T > \hat{\rho}$ implies $\hat{\rho}^T > \rho$, also $T^{\hat{\rho}}_{\rho}(SU(N))$ is expected to flow to an infrared superconformal field theory and it's believed to coincide with the one of $T^{\rho}_{\hat{\rho}}(SU(N))$.
The linear quiver gauge theories $T^{\rho}_{\hat{\rho}}(SU(N))$ can be realized as low-energy limits of certain type IIB brane configurations, as we are going to see in the follow.

The basic brane configuration consist of:
- a set of k D5-branes spanning the dimensions (012456);
- a set of $\hat{k}$ NS5-branes spanning the dimensions (012789);
- a set of D3-branes stretched among the five-branes along (0123).

	0	1	2	3	4	5	6	7	8	9
D5	X	X	X		X	X	X			
NS5	X	X	X					X	X	X
D3	X	X	X	X						

Such a configuration preserve 1/4 of the supersymmetries of type IIB string theory, namely 8, which correspond to the three dimensional $\mathcal{N} = 4$ Poincaré supersymmetries of $T^{\rho}_{\hat{\rho}}(SU(N))$[1].
The R-symmetry of $T^{\rho}_{\hat{\rho}}(SU(N))$, which coincides with the R-symmetry of the infrared SCFT, is the $SO(4) \simeq SU(2)_1 \times SU(2)_2$ rotation symmetry which is manifest in the brane configuration in the $(456) \times (789)$ dimensions.
The vector multiplets live on the D3-branes, and since these branes have finite extension along the x^3 direction, the corresponding four-dimensional world-volume theory at large distances becomes a three-dimensional gauge theory. On the other hand, hypermultiplets arise from open strings stretching between the D3-branes and the D5-branes, or between two stacks of D3-branes ending on the same NS5-brane from the left and right. The only relevant data of this brane configuration in the infrared limit are the total numbers k and $\hat{k}$ of D5- and NS5-branes respectively (but not their ordering along the x^3 segment, since rearrangements of the five-branes change the phase of the gauge theory but not its superconformal limit) and their linking numbers defined as

$$\begin{aligned} l_i &= -n_i + R_i^{NS5} \qquad\qquad i = 1, ..., k \\ \hat{l}_j &= \hat{n}_j + L_j^{D5} \qquad\qquad j = 1, ..., \hat{k} \end{aligned} \tag{B.4}$$

where n_i is the number of D3-branes ending on the ith D5-brane from right minus the of the number of D3-branes ending on it from the left, $\hat{n}_j$ is the same difference referred to the jth NS5-brane, R_i^{NS5} is the number of NS5-branes lying to the right of the ith D5-brane and L_j^{D5} is the number of D5-branes lying to the left of the jth NS5-brane. The linking numbers are invariant under five-brane movements because when a D5 brane moves past a NS5 brane in the direction from left to right, a D3-brane stretching between the two is created.

A brane configuration realizing a linear quiver gauge theory $T^{\rho}_{\hat{\rho}}(SU(N))$ can be depicted as N D3-branes in the middle ending on the left on a collection of NS5-branes and on the right on a collection of D5-branes. Then, from (B.4), it follows that the net number of D3-branes that terminate on each five-brane is precisely its linking number and in addition

$$N = l_1 + ... + l_k = \hat{l}_1 + ... + \hat{l}_{\hat{k}} \tag{B.5}$$

These are the two partitions ρ and $\hat{\rho}$ of N that label the theory $T^{\rho}_{\hat{\rho}}(SU(N))$: the partition $\hat{\rho}$ encodes the linking numbers of NS5-branes, while ρ encodes the linking numbers of D5-branes.

[1]The Poincaré superalgebra supercharges are Weyl spinors with $2^{(D-1)/2} = 2$ real components each, for a total of 8 SUSY d.o.f.

Since the relative order of five-brane of the same kind is irrelevant in the infrared, it is possible to arrange them so that $l_1 \geq l_2 \geq ... \geq l_k$ and $\hat{l}_1 \geq l_2 \geq ... \geq \hat{l}_k$, where $i = 1$ and $j = 1$ are the innermost five-branes, while $i = k$ and $j = \hat{k}$ are the outermost ones. Notice that S-duality of type IIB string theory exchanges the two type of five-branes, and thus two partitions realizing the mirror gauge theory $T^{\hat{\rho}}_{\rho}(SU(N))$.

The condition $\rho^T > \hat{\rho}$, discussed earlier as necessary for the linear quiver associated to the gauge theory $T^{\rho}_{\hat{\rho}}(SU(N))$ to make sense, can be recovered in the brane configuration in the weaker form $\rho^T \geq \hat{\rho}$ as a requirement for unbroken supersymmetry; however, when the inequality is saturated it can be shown that the corresponding quiver gauge theory breaks down to pieces that flow to non-interacting SCFTs in the infrared.

When the strong form of the constraint is satisfied, the brane configuration, consists of a connected linear chain of $\hat{k}$ NS5-branes, attached in pairs by N_j D3-branes. The D5-branes intersect the D3-branes being detached from them at the same time. The gauge theory data that can be derived from this configuration agrees precisely with the gauge content of $T^{\rho}_{\hat{\rho}}(SU(N))$: there is one $U(N_j)$ gauge group factor for every set of N_j stretched D3-branes, one hypermultiplet in the bifundamental representation for each adjacent pair of NS5-branes and one hypermultiplet in the fundamental representation of the corresponding gauge group for each D5-brane.

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