Università degli Studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Laurea Magistrale in Matematica

A generalized Curvature-Dimension inequality in sub-Riemannian geometry

Relatore:
Prof. Davide Barilari

Laureado e Matricola:
Marco Condini
2027568

Anno Accademico 2021/2022

16 Dicembre 2022

To my family and my good friends

Abstract

Riemannian manifolds with a Ricci lower bound is an important researched area of differential geometry, in particular in the context of comparison theorems. One result is a way to characterize Ricci curvature bounds through a condition of the Laplacian: the curvature-dimension inequality

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + \rho\Gamma(f)$$

where n is the dimension of the manifold, Γ_2 and Γ two operators based on Δ, the Laplacian and ρ the bound of the Ricci curvature.
Following this analytic, but equivalent, approach it is possible to find a generalization, which hing only on differential operator, also for non Riemannian cases where the Ricci curvature is not completely defined. In this thesis we study a generalization due to Baudoin and Garofalo for the case of sub-Riemannian manifolds with transverse symmetries.

In Chapter 1 we set down the basic concepts we need throughout the thesis.
Moreover in Chapter 2 we present the Riemannian curvature-dimension inequality and answer the question: what can be said about manifolds with a Ricci lower bound?
Chapter 3 is entirely dedicated to introduce notions of sub-Riemannian manifold.
Finally in Chapter 4 and 5 we introduce the work of Baudoin and Garofalo and apply their theory to two specific cases.

Contents

Mathematical notation

- We are not using Einstein convention for the summation. We will however almost always write $\sum_i$, this notation means implicitly that the sum goes from 1 to the dimension of the manifold.
  In the almost unique cases where one of two edging points is different, it will be denoted in the sum, like $\sum_{i=2}$ or $\sum_i^{n-1}$.

- The set of vector fields of a manifold M in the thesis is denoted as $\mathfrak{X}(M)$. Another possible notation is $\Gamma^\infty(TM)$, where for $\Gamma^\infty(V)$ is meant a smooth section of the tensor bundle V.

Chapter 1

Fundamental concepts

Despite we suppose the reader to be familiar with basic elements of differential geometry, we use this first Chapter for recalling the most important for us, in particular the Ricci curvature and the Laplacian.
The readers, who are not confident with terms like distribution or Levi-Civita connection, can read for example [9].

1.1 The Ricci curvature

We denote by (M, g) a Riemannian manifold, where M is a differential manifold and g is a metric tensor.

Throughout the thesis the letter g will almost always be used to indicate the Riemannian metric. Another notation, which we will sometimes use, are the brackets as in the following way: for $X, Y \in \mathfrak{X}(M)$

$$g(X, Y) = \langle X, Y \rangle$$

We state some symmetry properties of the Riemannian curvature tensor.

Proposition 1.1. *Let ∇ be a connection of a Riemannian manifold (M, g) and R its Riemannian curvature tensor. The following properties hold*

$$g(R(X,Y)Z,K) = -g(R(Y,X)Z,K) \tag{1.1}$$

Moreover, if the connection is the Levi-Civita one, it holds

$$g(R(X,Y)Z,K) = -g(R(X,Y)K,Z) \tag{1.2}$$

and

$$g(R(X,Y)Z,K) = g(R(Z,K)X,Y) \tag{1.3}$$

for $X, Y, Z, K \in \mathfrak{X}(M)$.

Proof. See [9] Proposition 2.5 page 91. □

In order to introduce the Ricci curvature we define the trace of a tensor.

Definition 1.2. Given a $(1,1)$ tensor field $T \in \Gamma^\infty(T^*M \otimes TM)$ we define its trace as

$$trace(T) = \sum_i T_i^i$$

where T_i^i is the representation of the tensor in any frame, meaning for a frame $\{X_1, .., X_n\}$ of vector fields and $\{\alpha^1, .., \alpha^n\}$ of covector fields such that $\alpha^i(X_j) = \delta_i^j$

$$T = \sum_{i,j} T_i^j \alpha^i \otimes X_j$$

Remark 1.3. The trace is independent from the choice of the frame because, since T can be see as a linear map $\tilde{T} \colon \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ given for $Y \in \mathfrak{X}(M)$ by $\tilde{T}(Y) = \sum_{i,j} T_i^j \alpha^i(Y) \otimes X_j = \sum_{i,j} T_i^j Y^i X_j$, then, by linear algebra, an endomorphism has the trace well defined and it coincides with the trace of T.

In the Riemannian case, for an orthonormal frame $\{E_1, .., E_n\}$ it simply becomes

$$trace(T) = \sum_i g(\tilde{T}(E_i), E_i)$$

We are now able to define the Ricci curvature.

Definition 1.4. Let (M, g, ∇) be a Riemannian manifold with a connection and R be the associated curvature tensor.
Let us define for $X, Y \in \mathfrak{X}(M)$ the endomorphism $F_{X,Y} \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ as the map $Z \mapsto R(Z, X)Y$.
Then the Ricci curvature tensor is the $(0,2)$ tensor field $\mathrm{Ric} \in \Gamma^\infty(T^*M^{\otimes 2})$ given by

$$\mathrm{Ric}(X, Y) = trace(F_{X,Y})$$

So given an orthonormal frame $\{E_1, .., E_n\}$

$$\mathrm{Ric}(X, Y) = \sum_i g(R(E_i, X)Y, E_i)$$

Proposition 1.5. *The Ricci curvature tensor associated to the Levi-Civita connection is symmetric, this indicates* $\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X)$

Proof.

$$\mathrm{Ric}(X, Y) = trace(F_{X,Y}) = \sum_i g(F_{X,Y}(E_i), E_i) = \sum_i g(R(E_i, X)Y, E_i)$$

By the symmetries of the Levi-Civita connection it holds

$$\sum_i g(R(E_i, X)Y, E_i) = \sum_i g(R(E_i, Y)X, E_i) = \mathrm{Ric}(Y, X)$$

□

This proposition let us give even a simpler definition in the case of the Levi-Civita connection. In fact, thanks to the polarization identity, the map $\mathrm{Ric}(X,Y)$ is uniquely determined by the value along the pair of same vector fields $\mathrm{Ric}(X,X)$. Since we will work, from now on, mostly with the Riemannian connection we introduce, with a little abuse of notation, a new map

$$\mathrm{Ric}(X) := \mathrm{Ric}(X,X)$$

which would also be called Ricci curvature. The number of arguments will be the discriminant between the two different, but intimately related, maps.

But what is actually the Ricci curvature? As often happens in Mathematics the definition is so general that it's almost impossible to get a clear idea from it. Luckily there is another characterization for manifold of at least two dimensions that hinges on the the concept of sectional curvature.

Definition 1.6. Let $p \in M$ be a point of a Riemannian manifold, and $\sigma_p \subset T_pM$ a two-dimensional subspace of the tangent space at p.
The real number $K(\sigma_p)$ given by

$$K(\sigma_p) = K(X,Y) = \frac{g(R(Y,X)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

where $X, Y \in \sigma_p$ are linear independent, is called sectional curvature.

Proposition 1.7. *The sectional curvature is well-defined, namely it does not depend on the choice of the vectors $X, Y \in \sigma_p$.*

Proof. First we notice that $K(X,Y) = K(Y,X)$ and $K(X,Y) = K(aX,Y)$ with $a \in \mathbb{R}$ by symmetries and linearity of R and g.
Moreover the denominator is the area of the parallelogram determined by the vectors X, Y, i.e. $|X \wedge Y|^2$

$$\begin{aligned}
K(X+aY,Y) &= \frac{g(R(Y,X+aY)X+aY,Y)}{|X+aY \wedge Y|^2} \\
&= \frac{g(R(Y,X)X+aY,Y) + ag(R(Y,Y)X+aY,Y)}{|X \wedge Y|^2} \\
&= \frac{g(R(Y,X)X,Y) + ag(R(Y,X)Y,Y)}{|X \wedge Y|^2} \\
&= \frac{g(R(Y,X)X,Y)}{|X \wedge Y|^2} = K(X,Y)
\end{aligned}$$

by anti-symmetries 1.1 and 1.2 of the curvature .
Since the sectional curvature in invariant under these transformations and with these one can write every other base of σ_p the statement is proved. □

The idea of sectional curvature is nothing more than a generalization of the Gaussian curvature. Roughly speaking it calculates the product of the two principal curvatures at p of the local surface that has σ_p as tangent space, which is the manifold defined locally as the image via exponential map of this tangent space.

Remark 1.8. The Ricci curvature $\mathrm{Ric}|_p(X)$ can be seen, in a certain sense, as the not normalized average of the sectional curvatures containing the fixed vector $X|_p$. Because given a orthonormal basis $\{E_1, .., E_n\}$ it holds at $p \in M$ for $X = \sum_i X^i E_i$

$$\begin{aligned}\mathrm{Ric}|_p(X) = \mathrm{Ric}(X,X) &= \sum_i g(R(E_i,X)X,E_i) = \sum_i K(X,E_i)|X \wedge E_i|^2 \\ &= \sum_i K(X,E_i)|\sum_j X^j E_j \wedge E_i|^2 = \sum_i K(X,E_i)\sum_{j\neq i}(X^j)^2 \\ &= \sum_i K(X,E_i)(\sum_j (X^j)^2 - (X^i)^2) = \|X\|^2 \sum_i K(X,E_i) - \sum_i K(X,E_i)(X^i)^2\end{aligned}$$

Let now $Y \in \mathfrak{X}(M)$ then take $E_1 = \frac{X}{\|X\|}$ and complete it to an orthonormal frame $\{E_1, .., E_n\}$. This means $X^i = \delta^i_1$ so

$$\mathrm{Ric}(X) = \mathrm{Ric}(\|X\|E_1, \|X\|E_1) = \|X\|^2 \mathrm{Ric}(E_1,E_1) = \|X\|^2 \sum_{i=2}^n K(E_1,E_i)$$

Since by anti-symmetry 1.1 $K(E_1,E_1) = g(R(E_1,E_1)E_1,E_1) = -g(R(E_1,E_1)E_1,E_1)$ so it must be zero.

Definition 1.9. We say that a Riemannian manifold M has a b-lower bound, with notation $\mathrm{Ric}(M) \geq b$, if

$$\mathrm{Ric}(X) \geq b\|X\|^2$$

for all $X \in \mathfrak{X}(M)$

We finish the part about Ricci curvature with an example.

Example 1.10. In the case of the sphere S^n embedded in $\mathbb{R}^{n+1}$, it's well known that the Gaussian curvature is constant and equal to 1 everywhere.
This means that for any $p \in M$ the Ricci curvature of a vector $X \in T_p S^n$ is given by

$$\begin{aligned}\mathrm{Ric}(X) &= \|X\|^2 \sum_i K(X,E_i) - \sum_i K(X,E_i)(X^i)^2 \\ &= \|X\|^2 \sum_i 1 - \sum_i (X^i)^2 = \|X\|^2 n - \|X\|^2 = (n-1)\|X\|^2\end{aligned}$$

1.2 The Hessian tensor and the Laplace-Beltrami operator

We move on to another mathematical object involved in the curvature-dimension inequality: the Laplace-Beltrami operator: a second order differential operator which generalized the usual Laplacian of the Euclidean space. Since its definition is based on the Hessian, it is natural to define this concept in a Riemannian manifold first.

Definition 1.11. Let (M, g) be a Riemannian manifold with its associated Levi-Civita connection ∇.
Given a function $f \in C^\infty(M)$ we define its Hessian as the tensor $Hess(f) \in \Gamma^\infty(T^*M^{\otimes 2})$ such that for $X, Y \in \mathfrak{X}(M)$

$$Hess(f)(X,Y) := \langle \nabla_X grad(f), Y \rangle$$

Where $grad(f)$ is the gradient of f.

Proposition 1.12. *The Hessian is a symmetric tensor, namely for* $X, Y \in \mathfrak{X}(M)$

$$Hess(f)(X,Y) = Hess(f)(Y,X)$$

Proof.

$$\begin{aligned} Hess(f)(X,Y) &= \langle \nabla_X grad(f), Y \rangle = \nabla_X \langle grad(f), Y \rangle - \langle grad(f), \nabla_X Y \rangle \\ &= XY(f) - \nabla_X Y(f) \\ &= [X,Y](f) + YX(f) - \nabla_X Y(f) \end{aligned}$$

By torsion-freeness of the connection

$$= YX(f) - \nabla_Y X(f) = Hess(f)(Y,X)$$

□

It exists a very useful equivalent representation of the Hessian of a function based on the covariant derivative of a tensor. Even if it could be at first a bit confusing, it is going to simplify our computation so we present it in the next definition.

Definition 1.13. Given the Levi-Civita connection ∇, the covariant derivative ∇T is defined as the map which takes a vector field X and returns

- If $T = f$ is a smooth function $\nabla f(X) = \nabla_X f = X(f) = df(X)$
- If $T = Y \in \mathfrak{X}(M)$ is a vector field of M, $\nabla T(X) = \nabla_X T \in \mathfrak{X}(M)$
- If $T = df$ is a covector field $(\nabla_X df)(Y) = \nabla_X(df(Y)) - df(\nabla_X Y)$
- If T is a (s,p) tensor field

$$\begin{aligned} \nabla T(Y, X_1, ..X_p, \alpha^1, .., \alpha^s) &:= (\nabla_Y T)(X_1, ..X_p, \alpha^1, .., \alpha^s) \\ &:= Y(T(X_1, ..X_p, \alpha^1, .., \alpha^s)) \\ &- \sum_i^p T(X_1, .., \nabla_Y X_i, ..X_p, \alpha^1, .., \alpha^s) \\ &- \sum_j^s T(X_1, .., X_p, \alpha^1, .., \nabla_Y \alpha^j, .., \alpha^s) \end{aligned}$$

that means for arbitrarily tensor field S, P

$$\nabla_X(S \otimes P) = \nabla_X(S) \otimes P + S \otimes \nabla_X(P)$$

Remark 1.14. We want to stress the fact that the gradient and the covariant derivative of a function look similar, but they are slightly different; they are actually related by the musical isomorphism. In fact, while $\nabla f \in \Gamma^\infty(T^*M)$ the gradient $grad(f) \in \Gamma^\infty(TM)$ and they are related by, for $X \in \mathfrak{X}(M)$,

$$g(grad(f), X) = X(f) = \nabla_X f = \nabla f(X)$$

Given a frame $\{X_1, .., X_n\}$ and a frame of covectors $\{\alpha^1, .., \alpha^n\}$ such that $\alpha^i(X_j) = \delta^i_j$ the difference is even clearer

$$grad(f) = \sum_i \left(\sum_j X_j(f) g^{ij} \right) X_i \qquad \nabla f = \sum_i X_i(f) \alpha^i$$

Given, however, the wide use in the literature of nabla notation for the gradient, we will write the gradient as ∇f, its i-th component as $(\nabla f)^i$ while $\nabla_i f = \nabla_{X_i} f$ will only be used for the covariant derivative.
This distinction will be in our thesis, most of the time, not necessary since in an orthonormal frame it holds $(\nabla f)^i = \nabla_i f$ as we can see from the representation in coordinates.

The next proposition expresses how the covariant derivative and the Hessian are related.

Proposition 1.15. *We have the following equality*

$$\nabla^2 f := \nabla \circ \nabla f = Hess(f)$$

Proof. Let $X, Y \in \mathfrak{X}(M)$ then

$$\begin{aligned} Hess(f)(X, Y) &= \langle \nabla_X grad(f), Y \rangle = X(\langle grad(f), Y \rangle) - \langle grad(f), \nabla_X Y \rangle \\ &= X(Y(f)) - \nabla_X Y(f) = X(Y(f)) - \nabla_{\nabla_X Y} f = \nabla_X \nabla_Y (f) = \nabla^2(f)(X, Y) \end{aligned}$$

□

We can now give the definition of the Laplacian in the setting of a Riemannian manifold.

Definition 1.16. Given a Riemannian manifold (M, g), we define the Laplace-Beltrami operator $\Delta \colon C^\infty(M) \to C^\infty(M)$ for $f \in C^\infty(M)$ as

$$\Delta f := \sum_{i,j} Hess(f)(X_i, X_j) \cdot g^{ij}$$

For $\{X_1, .., X_n\}$ a local frame and where g^{ij} are the coefficient in position i, j of the inverse of the metric matrix in the frame, i.e. such that $\sum_i g^{ji} g_{ik} = \delta^j_k$

Remark 1.17. The Laplace operator can be written in coordinates as

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_{x_i} \left(\sqrt{|g|} g^{ij} \partial_{x_j} f \right)$$

Moreover given the frequency of use we are going to refer at this operator only as Laplacian, even if it was actually Beltrami, italian mathematician from the 19th century, the author of this generalization.

The last notion of this section is the norm of a tensor.

Definition 1.18. Given a $(0,p)$ tensor T we define its norm by

$$\|T\| := \sum_{i_1,..,i_p,j_1,..,j_p} g^{i_1 j_1} \cdots g^{i_p j_p} \cdot T_{i_1,..,i_p} \cdot T_{j_1,..,j_p}$$

1.3 The Divergence operator

Definition 1.19. The divergence of a vector field $X \in \mathfrak{X}(M)$ with respect to a smooth volume form ν is defined by

$$\mathrm{div}(X)\nu = \mathcal{L}_X \nu = d \circ \iota_X \nu$$

Where ι is the interior derivative and d is the exterior one. We moreover used in the last equation the famous Cartan magic formula and the fact that the exterior derivative of a volume form is zero.

At the end of this introductory Chapter we prove some properties of the divergence which we will use later.

Theorem 1.20. *Let $X \in \mathfrak{X}(M)$ be a vector field and ν a smooth volume form on M, a differential manifold without boundary.*
Then it holds

$$\int_M \mathrm{div}(X)\nu = 0$$

If either X or ν have compact support.

Proof. It simply follows from Cartan formula and Stokes theorem since

$$\int_M \mathrm{div}(X)\nu = \int_M d \circ \iota_X \nu = \int_{\partial M} \iota_X \nu = 0$$

□

Proposition 1.21. *For $h, f \in C^\infty(M)$, ν a smooth volume form and $X \in \mathfrak{X}(M)$ it holds*

$$\mathrm{div}(fhX) = fX(h) + X(f)h + fh\mathrm{div}(X) \tag{1.4}$$

Proof.

$$\mathrm{div}(fhX)\nu = d \circ \iota_X(fh\nu) = d \circ fh\ \iota_X(\nu) = fh\ d \circ \iota_X(\nu) + d(fh) \wedge \iota_X(\nu)$$

If we analyze the second term

$$d(fh) \wedge \iota_X(\nu) = hd(f) \wedge \iota_X(\nu) + fd(h) \wedge \iota_X(\nu)$$

We pass in coordinates, i.e. let $\{y_1, .., y_n\}$ be coordinates on M such that

$$\nu = dy^1 \wedge .. \wedge dy^n$$

and X^i the component of X on the frame given by this coordinates. Then it turns out

$$hd(f) \wedge \iota_X(\nu) = hd(f) \wedge (\sum_i (-1)^{i-1} dy^i(X) dy^1 \wedge .. \wedge \widehat{dy^i} \wedge .. \wedge dy^n)$$

where hat means omission. So

$$= h \sum_i Y_i(f)((-1)^{i-1})^2 X^i \nu = hX(f)\nu$$

By putting all together we have the result.

$\square$

Chapter 2

Ricci lower bound on a Riemannian manifold

In this Chapter we introduce the Riemannian curvature-dimension inequality. It is a condition which involves the Laplacian and two bilinar forms built from it. After this we prove an important theorem which states the equivalence between a lower bound of the Ricci curvature and the inequality.
This would holds only for a complete manifold.
After that we state some properties of manifold with a Ricci lower bound.

2.1 Curvature-dimension inequality

Before we introduce the two forms, we have to specify as the Laplacian works when it is applied on a product of two functions, its product rule. This will simplify our calculation.

Lemma 2.1. *Let $f, h \in C^\infty(M)$ where (M, g) is a Riemannian manifold then it holds that*

$$\Delta(f \cdot h) = f \cdot \Delta h + \Delta f \cdot h + 2\langle \nabla f, \nabla h\rangle \tag{2.1}$$

Proof. We start by noticing that it holds

$$\nabla(f \cdot h) = \nabla f \cdot h + f \cdot \nabla h \tag{2.2}$$

because for any $X \in \mathfrak{X}(M)$

$$\langle \nabla(f \cdot h), X\rangle = X(f \cdot h) = X(f) \cdot h + f \cdot X(h) = \langle \nabla f, X\rangle \cdot h + \langle \nabla h, X\rangle \cdot f \tag{2.3}$$

And so, passing to the Laplacian given a frame $\{X_1, .., X_n\}$

$$\Delta(f \cdot h) = \sum_{i,j} Hess(f \cdot h)(X_i, X_j) \cdot g^{ij} = \sum_{i,j} g^{ij} \langle \nabla_{X_i}(\nabla(f \cdot h)), X_j\rangle$$

By the compatibility with the metric it follows

$$= \sum_{i,j} g^{ij} \Big(X_i(\langle \nabla(f \cdot h), X_j \rangle) - \langle \nabla(f \cdot h), \nabla_{X_i} X_j \rangle \Big)$$
$$= \sum_{i,j} g^{ij} \Big(X_i(\langle \nabla f \cdot h + f \cdot \nabla h, X_j \rangle) - \langle \nabla f \cdot h + f \cdot \nabla h, \nabla_{X_i} X_j \rangle \Big)$$
$$= \sum_{i,j} g^{ij} \Big(X_i(h \langle \nabla f, X_j \rangle) + X_i(f \langle \nabla h, X_j \rangle) - h \langle \nabla f, \nabla_{X_i} X_j \rangle - f \langle \nabla h, \nabla_{X_i} X_j \rangle \Big)$$

By definition of vector and backwards passage we get

$$= \sum_{i,j} g^{ij} \Big(X_i(f) \langle \nabla h, X_j \rangle + X_i(h) \langle \nabla f, X_j \rangle + h \cdot Hess(f)(X_i, X_j) + f \cdot Hess(h)(X_i, X_j) \Big)$$

Now if we focus only on the first term we have

$$\sum_{i,j} g^{ij} \cdot X_i(f) g(\nabla h, X_j) = \sum_{i,j,k=1}^{n} g^{ij} \cdot g(\nabla f, X_i)(\nabla h)^k g_{kj}$$

Then by symmetry of the Riemannian metric

$$= \sum_{i,j,k} g^{ij} g_{jk} \cdot g(\nabla f, X_i)(\nabla h)^k = \sum_{k,i} \delta^i_k \cdot g(\nabla f, X_i)(\nabla h)^k$$

This means

$$= g(\nabla f, \sum_i (\nabla h)^i X_i) = g(\nabla f, \nabla h)$$

similar for the second term. □

We can now state how the two differential forms are defined and also what they represent thanks to the lemma

Definition 2.2. Let Δ be the Laplacian of a Riemannian manifold (M, g).
We can define two differential bilinear forms $\Gamma, \Gamma_2 \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined for $f, h \in C^\infty(M)$ by

$$\Gamma(f, h) := \frac{1}{2} \Big(\Delta(f \cdot h) - f \cdot \Delta h - \Delta f \cdot h \Big) = \langle \nabla f, \nabla h \rangle$$

and

$$\Gamma_2(f, h) := \frac{1}{2} \Big(\Delta \Gamma(f, h) - \Gamma(f, \Delta h) - \Gamma(\Delta f, h) \Big)$$

With a little abuse of notation we will simply write

1. $\Gamma(f) := \Gamma(f, f) = \|\nabla f\|^2$
2. $\Gamma_2(f) := \Gamma_2(f, f) = \frac{1}{2}\Delta(\|\nabla f\|^2) - \langle \nabla f, \nabla(\Delta f) \rangle = \frac{1}{2}\Delta\Gamma(f) - \Gamma(f, \Delta f)$

The Riemannian curvature-dimension inequality can be written down with respect of this two forms in the following way.

Definition 2.3. We say that a Riemannian manifold (M, g) satisfies the curvature-dimension inequality $CD(\rho, n)$, for $\rho \in \mathbb{R}$ and $n \in \mathbb{N}$ $n > 0$, if for every $f \in C^\infty(M)$ it holds

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + \rho\Gamma(f)$$

Remark 2.4. Some authors call the previous inequality as curvature-dimension condition.

We focus our attention on the equivalence between lower bound of the Ricci curvature and the inequality.
Its proof is intimately connected with the Bochner's formula with gives a relation between the Laplacian, the Hessian and the curvature.

Proposition 2.5 (Bochner's formula)**.** *Let $f \in C^\infty(M)$, then the following equality holds*

$$\Delta(\|\nabla f\|^2) = 2\|\nabla^2 f\|^2 + 2\langle \nabla f, \nabla(\Delta f)\rangle + 2Ric(\nabla f)$$

Proof. The idea is to use a local geodesic frame and commute the derivatives.
Let $p \in M$ and $\{E_1, .., E_n\}$ be an orthonormal frame in a neigborhood of p such that $\nabla_{E_i} E_j = 0$ for all i, j at the point p.
We start by pointing out that in this setting at p holds $\Delta f = \sum_i E_i E_i f$ for $f \in C^\infty(M)$ because

$$\Delta f = \sum_i \nabla_{E_i}\nabla_{E_i} f = \sum_i E_i(\nabla_{E_i} f) - \nabla_{\nabla_{E_i}E_i} f = \sum_i E_i E_i f$$

So by compatibility with the metric and the definition of Hessian

$$\frac{1}{2}\Delta\|\nabla f\|^2 = \frac{1}{2}\sum_i E_i E_i\langle \nabla f, \nabla f\rangle = \sum_i E_i\langle \nabla_{E_i}\nabla f, \nabla f\rangle = \sum_i E_i Hess(f)(E_i, \nabla f)$$

Since the Hessian is symmetric it holds

$$\begin{aligned} &= \sum_i E_i Hess(f)(\nabla f, E_i) = \sum_i E_i\langle \nabla_{\nabla f}\nabla f, E_i\rangle \\ &= \sum_i \langle \nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle + \langle \nabla_{\nabla f}\nabla f, \nabla_{E_i} E_i\rangle \end{aligned}$$

By assumption of a geodesic frame the second term vanishes, leaving

$$\begin{aligned} &= \sum_i \langle \nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle \\ &\qquad = \sum_i [\langle \nabla_{\nabla f}\nabla_{E_i}\nabla f, E_i\rangle + \langle \nabla_{[E_i,\nabla f]}\nabla f, E_i\rangle + \langle R(E_i, \nabla f)\nabla f, E_i\rangle] \end{aligned} \tag{2.4}$$

where we used the definition of Riemannian curvature.
We observe first that $\nabla_{\nabla f} E_i = 0$ because

$$\nabla_{\nabla f} E_i = \nabla_{\sum_k (\nabla f)^k E_k} E_i = \sum_k (\nabla f)^k \nabla_{E_k} E_i = 0$$

Examining the terms separately of equation (2.4)

$$\begin{aligned}\sum_i \langle \nabla_{\nabla f} \nabla_{E_i} \nabla f, E_i \rangle &= \sum_i \left(\nabla f \langle \nabla_{E_i} \nabla f, E_i \rangle - \langle \nabla_{E_i} \nabla f, \nabla_{\nabla f} E_i \rangle \right) \\ &= \nabla f \left(\sum_i Hess(f)(E_i, E_i) \right) \\ &= \nabla f (\Delta f) = \langle \nabla f, \nabla (\Delta f) \rangle\end{aligned}$$

$$\begin{aligned}\sum_i \langle \nabla_{[E_i, \nabla f]} \nabla f, E_i \rangle &= \sum_i Hess(f)([E_i, \nabla f], E_i) \\ &= \sum_i Hess(f)(E_i, \nabla_{E_i} \nabla f - \nabla_{\nabla f} E_i) \\ &= \sum_i Hess(f)(E_i, \nabla_{E_i} \nabla f) \\ &= \sum_i \langle \nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f \rangle\end{aligned}$$

Where in the second passage we use the torsion-freeness of the connection. We recall now that since we have an orthonormal frame it holds $\nabla_{E_i} f = (\nabla f)^i$. If we write ∇f in coordinates and we compute $\nabla_{E_i} \nabla f$.

$$\nabla_{E_i} \sum_j \nabla_{E_j} f \cdot E_j = \sum_j \partial_{E_i} (\nabla_{E_j} f) E_j + \nabla_{E_j} f \nabla_{E_i} E_j = \sum_j \nabla_{E_i} \nabla_{E_j} f \cdot E_j$$

So

$$\sum_i \langle \nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f \rangle = \sum_{j,i} \nabla_{E_i} \nabla_{E_j} f \cdot \nabla_{E_i} \nabla_{E_j} f = \|\nabla^2 f\|^2$$

Summing everything together and substituting in equation (2.4) we get

$$\frac{1}{2} \Delta \|\nabla f\|^2 = \langle \nabla f, \nabla (\Delta f) \rangle + \|\nabla^2 f\|^2 + \sum_i \langle R(E_i, \nabla f) \nabla f, E_i \rangle]$$

By noticing that the last sum is actually the definition of the Ricci curvature, it follows the thesis.

□

After recalling what is meant be a complete manifold we give the main theorem of this Chapter

Definition 2.6. We say that a Riemannian manifold (M, g) is complete if every geodesic $\gamma \colon I \to M$ can be extend to a maximal geodesic, which has domain the all $\mathbb{R}$.

Theorem 2.7. *On a complete n-dimensional Riemannian manifold (M, g) it holds the following equivalence:*

$$CD(\rho, n) \iff \mathrm{Ric}(M) \geq \rho$$

for $\rho \in \mathbb{R}$

Proof. $\Longleftarrow$). Suppose $\text{Ric}(M) \geq \rho$. We apply the Bochner's formula.

$$\Delta\Gamma(f) := \Delta(\|\nabla f\|) = 2\|\nabla^2 f\|^2 + 2\langle \nabla f, \nabla(\Delta f)\rangle + 2\text{Ric}(\nabla f)$$

$$= 2\|\nabla^2 f\|^2 + 2\Gamma(f, \Delta f) + 2\text{Ric}(\nabla f)$$

By combining this formula with the definition of $\Gamma_2(f)$ we get

$$\Gamma_2(f) = \|\nabla^2 f\|^2 + \text{Ric}(\nabla f) \tag{2.5}$$

Then by Cauchy-Schwarz inequality it follows that

$$\|\nabla^2 f\|^2 \geq \frac{1}{n}(\Delta f)^2 \tag{2.6}$$

In fact let $\tilde{g}(A, B) = tr(A^T B)$ be the trace inner product for the $n \times n$ matrices and $\{E_1, .., E_n\}$ be an orthonormal frame for (M, g) then

$$n\|\nabla^2 f\|^2 = n \sum_{j,k} \nabla_{E_k}\nabla_{E_j} f \cdot \nabla_{E_k}\nabla_{E_j} f$$

If now we define the matrix K such that $K_{kj} := \nabla_{E_k}\nabla_{E_j} f$ we can rewrite it in the following way

$$= tr(Id)tr(K^T \cdot K) = tr(Id^T \cdot Id)tr(K^T \cdot K)$$

$$= \tilde{g}(Id, Id)\tilde{g}(K, K) \geq |\tilde{g}(Id, K)|^2 = |tr(K)|^2 = |\sum_i \nabla_{E_i}\nabla_{E_i} f|^2 = (\Delta f)^2$$

So we get that

$$\Gamma_2(f) = \|\nabla^2 f\|^2 + \text{Ric}(\nabla f) \geq \frac{1}{n}(\Delta f)^2 + \text{Ric}(\nabla f, \nabla f) \geq \frac{1}{n}(\Delta f)^2 + \rho\|\nabla f\|^2$$

Where we used the assumption of a ρ-lower bound for the Ricci curvature.
By using the notation 1. from page 16 we get

$$= \frac{1}{n}(\Delta f)^2 + \rho\Gamma(f)$$

$\Longrightarrow$). Assume $CD(\rho, n)$ holds. We have to prove the lower bound for the Ricci curvature.
Given $p \in M$ and $X \in T_pM$ we can find a small neighborhood of p, $U_p \subset M$ and a function $f \in C^\infty(U_p)$ such that

$$\nabla f|_p = X \qquad \nabla^2 f|_p = 0$$

By a suitable modification of f outside the neighborhood we can assume $f \in C^\infty(M)$, from equation (2.5) above, the Bochner's formula, we get

$$\text{Ric}(X, X) = \text{Ric}(\nabla f|_p, \nabla f|_p) = \Gamma_2(f)|_p - \|\nabla^2 f|_p\|^2$$

We use the assumption of the curvature-dimension inequality,

$$\geq \frac{1}{n}(\Delta f)^2|_p + \rho\Gamma(f)|_p - \|\nabla^2 f|_p\|^2$$

Since $\nabla^2 f|_p = 0$ impies that also $\Delta f|_p = 0$ two terms disappear

$$\geq \rho\Gamma(f)|_p = \rho\|\nabla f|_p\|^2 = \rho\|X\|^2$$

Which is exactly the definition of lower bound since the vector X is arbitrary. $\square$

2.2 Consequence of a Ricci lower bound

We saw how the curvature-dimension inequality captures entirely a lower bounds for the Ricci curvature. We ask ourselves: why is a Ricci lower bound important? The answer is given by the so called Ricci curvature comparison theorems. They are a family of theorems that have as assumption that the Ricci curvature has to be bounded from below.
We are going to present briefly two theorems of this type, without proofs, to give the reader a glimpse of the consequence of such a bound.

Poincarè-Wirtinger inequality The so called Poincarè-Wirtinger inequality is an important inequality in Mathematics. It is one of the fundamental inequalities in the calculus of variations (CoV) together with the Poincarè inequality, which states that in the Sobolev space $W_0^{1,p}$ the Sobolev norm and the L^p norm of the gradient are equivalent.
In the Riemannian setting the equality becomes:

Theorem 2.8. *Assume that (M, g), a complete n-dimensional Riemannian manifold, fulfills for $k \geq -\infty$, $Ric(M) \geq k$ then any $f \in C^\infty(M, [0, \infty])$ satifies*

$$\int_{B(x,R)} \left| f - \left(\frac{1}{vol(B(x,R))} \int_{B(x,R)} f \, dV_g \right) \right| dV_g \leq C \int_{B(x,2R)} |\nabla f| dV_g$$

for $R \leq D \in \mathbb{R}^+$ $C = C(R, n, kD^2)$

Proof. See [15] page 386 corollary 7.1.11 and [7] page 199 theorem 2.11 □

Given such theorem it is a straightforward computation to prove another base fact of calculus of variations: Rellich compactness theorem.
As every person slightly involved in CoV knows, necessary and sufficient conditions to have a minimum of a functional are two:

1. Lower semi continuity of the functional

2. Coercivity of the functional

The last one is a form of compactness, the Rellich theorem states exact conditions to find a converging subsequence needed in CoV.
Having said that, we can present the theorem in a Riemannian environment.

Theorem 2.9. *Assume (M, g) to be a compact n-dimensional Riemannian manifold such that $Ric(M) \geq (n-1)k$ for $k \geq -\infty$ then the following inclusion is compact*

$$W^{1,2}(M) \hookrightarrow L^2(M)$$

which means that $\forall \{x_n\}_n \subset W^{1,2}(M)$ bounded sequence in M exists a subsequence converging to an element of L^2.

Proof. See [15] page 393 theorem 7.1.18 □

Bonnet-Myers The theorem of Bonnet-Myers gives a bound for the diameter with assumption of a lower bound for the Ricci curvature. Before introducing it we start by settling the concept of diameter in a Riemannian setting:

Definition 2.10. Given a Riemannnian manifold (M, g) and a curve $\gamma\colon [a, b] \to M$ we define :

- its length as
$$L_g(\gamma) = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}}\, dt$$
- given two points $p, q \in M$, a notion of distance as
$$d(p, q) = \inf\{L_g(\gamma)|\ \gamma\colon [a, b] \to M \text{ such that } \gamma(a) = p \quad \gamma(b) = q\}$$
- the diameter of a Riemannian manifold M as
$$diam(M) = \sup\{d(p, q)|\ p, q \in M\}$$

Remark 2.11. This notion is slightly different from the usual idea of diameter. We can in fact notice that in the case of a sphere S^2 the "normal" idea of diameter as twice the radius would be 2, but if we think it as embedded in $\mathbb{R}^3$ with the round metric we see that its diameter is actually π, the length of a geodesic between antipodal points.

The next theorem was first proved by Bonnet assuming an inequality on the sectional curvatures of M while Myers weakened it for Ricci curvature in the year 1941, resulting in the following proposition:

Theorem 2.12 (Bonnet-Myers)**.** *Let M be a complete n-dimensional Riemannian manifold with Ricci lower bound* $\mathrm{Ric}(M) \geq \frac{n-1}{r^2}$ *for $r \in \mathbb{R}_{>0}$ then M is compact and with $diam(M) \leq \pi r$*

Proof. See [9] page 200 theorem 3.1 □

As we can notice the diameter bound is not strict, in fact the sphere attains the maximal value. It becomes natural to ask: what happen to manifolds that have the maximal diameter?
Shiu-Yuen Cheng, mathematician from Hong Kong, in 1975 answered with the next theorem, also called maximal diameter rigidity.

Theorem 2.13 (Cheng)**.** *If (M, g) is a complete n-dimensional Riemannian manifold with Ricci lower bound* $\mathrm{Ric}(M) \geq \frac{n-1}{r^2}$ *for $r \in \mathbb{R}_{>0}$ and diam(M)=πr then (M,g) is isometric to $r \cdot S^n$*

Proof. See [15] page 295 theorem 7.2.5 □

Chapter 3

Sub-Riemannian geometry

In this Chapter we briefly introduce basic notions of sub-Riemannian geometry. It is a branch of differential geometry that generalized the usual idea of Riemannian manifold, in fact in sub-Riemannian geometry we can't measure distance in any direction but only in the so called horizontal one, i.e. along a subbundle of the tangent bundle. We will use these new concepts in the next Chapters, when we shall state the generalized curvature-dimension inequality in this new setting.
Let's start by the definition of a bracket generating distribution, a fundamental concept for sub-Riemannian geometry.

Definition 3.1. Given a family of vector fields of M, $\mathcal{F} \subset \mathfrak{X}(M)$, we define $\mathrm{Lie}(\mathcal{F})$ as the smallest subset of $\mathfrak{X}(M)$ such that:

- $\mathcal{F} \subset \mathrm{Lie}\ (\mathcal{F})$
- $X, Y \in \mathrm{Lie}\ (\mathcal{F})$ and $a, b \in \mathbb{R} \implies [X, Y], aX + bY \in \mathrm{Lie}(\mathcal{F})$

We say that a distribution H is bracket generating if it holds $\forall p \in M$

$$\{X|_p \mid X \in \mathrm{Lie}(\Gamma^\infty(H))\} = T_pM$$

Definition 3.2. A Sub-Riemannian manifold is a triple (M, H, g) where:

- M is a differential manifold.
- H is a bracket generating distribution.
- g is a metric on the distribution.

Remark 3.3. The distribution H is also called the set of horizontal directions.

Remark 3.4. As it is easy to see a Riemannian manifold is just a special case of a sub-Riemannian manifold.

Such structure naturally inherits a distance between points. This is based on horizontal curves which are the counterpart of integral curves. Instead of being tangent to a vector field, they must be "tangent" to the distribution:

Definition 3.5. We say that a curve $\gamma\colon [a, b] \to M$ is horizontal for the Sub-Riemannnian manifold (M, H, g) if it is absolutely continuous and $\dot\gamma(t) \in H|_{\gamma(t)}$ for almost every $t \in [a, b]$.

The definition of distance follows in the usual way. In this special setting it is called Carnot–Carathéodory.

Definition 3.6. Given a sub-Riemannian manifold (M, H, g) we define the Carnot-Carathéodory distance between two points $p, q \in M$ as

$$d_{CC}(p, q) = \inf\{L_g(\gamma) \mid \gamma \text{ is a horizontal curve from } p \text{ to } q\}$$

We can't exclude a priori that between two points no horizontal curve exists, so the distance function would not be well-defined.
Thanks to the bracket generating property, though, this event never occurs, as the Chow theorem states.

Theorem 3.7 (Chow)**.** *Every two points of a sub-Riemannian manifold can be connected with a curve tangent to H.*
Moreover the topology induces by the Carnot-Caratheodory distance coincides with the topology of the manifold.

In this setting is possible to define a gradient. Naturally, since the metric is given only on H, it will be a vector field not on all TM.

Definition 3.8. In a sub-Riemannian manifold (M, H, g) we define the horizontal gradient of a function $f \in C^\infty(M)$ as the unique vector field on H, $\nabla f \in \Gamma^\infty(H)$ such that

$$g(\nabla f, X) = X(f)$$

for all $X \in \Gamma^\infty(H)$

Remark 3.9. Sometime, to distinguish it from the normal gradient if the setting is not immediately recognisable, we will write it as $\nabla_H f$

It is possible to state the notion of sub-Laplacian. Even if it is quite different from the one introduce in Chapter 1, we will see later on that in a Riemannian manifold the two objects are the same.

Definition 3.10. A sub-Laplacian, Δ, is the operator on the smooth functions of a sub-Riemannian manifold which depends on a smooth volume form ν, H and g, defined as

$$\Delta(f) = \operatorname{div}(\nabla_H f)$$

Proposition 3.11. *For any sub-Laplacian it holds the following product rule*

$$\Delta(fh) = f\Delta h + h\Delta f + 2g(\nabla_H f, \nabla_H h) \tag{3.1}$$

for $f, h \in C^\infty(M)$

Proof. Let $f, h \in C^\infty(M)$ then

$$\begin{aligned}\Delta(fh) = \operatorname{div}(\nabla_H(fh)) &= \operatorname{div}(\nabla_H(f)h + f\nabla_H(h)) \\ &= \operatorname{div}(\nabla_H(f)h) + \operatorname{div}(f\nabla_H(h))\end{aligned}$$

Where we use equation (2.2) while the linearity of the divergence comes from the linearity of the exterior and interior derivatives.
We look at the terms separately: let ν be the volume form associated to the divergence.

$$\mathrm{div}(h\nabla_H f)\nu = \mathcal{L}_{h\nabla_H f}\nu = d\circ\iota_{h\nabla_H f}\nu = d\circ h\ \iota_{\nabla_H f}\nu = dh\wedge\iota_{\nabla_H f}\nu + h\ d\circ\iota_{\nabla_H f}\nu$$

The last term is just the definition of $h\ \mathrm{div}(\nabla_H f)\nu$ while the other we know from the proof of proposition 1.21 that is equal to $\nabla_H f(h)\nu = dh(\nabla_H f)\nu = g(\nabla_H f, \nabla_H h)\nu$
So $\mathrm{div}(h\nabla_H f) = h\mathrm{div}(\nabla_H f) + g(\nabla_H f, \nabla_H h)$ and putting everything together gives the claim. □

Remark 3.12. We remark that through the previous proof we also find out that the following formula holds true

$$\mathrm{div}(fX) = f\mathrm{div}(X) + g(X, \nabla_H f) = f\mathrm{div}(X) + X(f) \tag{3.2}$$

For any X vector field on H and divergence on a manifold.

We finish this Chapter with a famous example of sub-Riemannian manifold, the Heisenberg group.

Example 3.13 (Heisenberg group)**.** Consider the space $M = \mathbb{R}^3$, the distribution H given as the span of the following two vector fields

$$X(x,y,z) := \partial_x - \frac{y}{2}\partial_z \qquad Y(x,y,z) := \partial_y + \frac{x}{2}\partial_z \tag{3.3}$$

and the metric g on the distribution given by, for $v_1 = a_1X + b_1Y$ and $v_2 = a_2X + b_2Y$ as

$$g(v_1, v_2) = a_1\cdot a_2 + b_1\cdot b_2$$

We can prove that the triple (M, H, g) is indeed a sub-Riemannian manifold, in fact we only have to prove that H is bracket generating.
If we calculate the Lie bracket of X and Y we get

$$[X,Y] = \partial_x(\frac{x}{2})\partial_z - \partial_y(-\frac{y}{2})\partial_z = \partial_z$$

And the span of $X, Y, [X,Y]$ is the all tangent space of M at every point, i.e. all $\mathbb{R}^3$, as the three vectors fields are always linearly independent

$$det\left(\begin{bmatrix}1 & 0 & -\frac{y}{2}\\ 0 & 1 & +\frac{x}{2}\\ 0 & 0 & +1\end{bmatrix}\right) = 1$$

This property characterizes the Lie algebra of the Heisenberg group, also called Heisenberg algebra. Actually it defines the Heisenberg group itself.

Definition 3.14. A 3-dimensional Heisenberg group is the simply connected Lie group whose Lie algebra is spanned by 3 linearly independent vector fields X, Y, Z such that

$$[X,Y] = Z \quad [X,Z] = 0 \quad [Y,Z] = 0$$

The Heisenberg group in this way is well defined because we know by the Lie's third theorem (see for example [16]) a Lie algebra uniquely, up to isomorphism, determines a simply connected group. It makes sense to speak about the Heisenberg group, which is unique up to isomorphism.
For example, two isomorphic groups are

$$G_1 := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} |x, y, z \in \mathbb{R} \right\}$$

and

$$G_2 := \left\{ (\mathbb{R}^3, \cdot) | (x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')) \right\}$$

We can see that they are both Heisenberg groups by computing their Lie algebra, we do this only for the the group G_2.
We start by determining the left multiplication

$$L_{(s,t,u)}(x, y, z) = (s, t, u) \cdot (x, y, z) = (x + s, y + t, z + u + \frac{1}{2}(sy - tx))$$

and its differential

$$dL_{(s,t,u)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-t}{2} & \frac{s}{2} & 1 \end{bmatrix}$$

We can check that the two vectors X, Y from equation (3.3) and $Z := [X, Y]$ are left invariant and ultimately, in the Lie algebra

$$dL_{(s,t,u)}(X) = \partial_{x'} - (\frac{t}{2} + \frac{y}{2})\partial_{z'} \quad dL_{(s,t,u)}(Y) = \partial_{y'} + (\frac{t}{2} + \frac{x}{2})\partial_{z'} \quad dL_{(s,t,u)}(Z) = \partial_{z'}$$

while

$$X|_{L_{(s,t,u)}((x,y,z))} = \partial_{x'} - \frac{y'}{2}\partial_{z'} \quad Y|_{L_{(s,t,u)}((x,y,z))} = \partial_{y'} + \frac{x'}{2}\partial_{z'} \quad Z|_{L_{(s,t,u)}((x,y,z))} = \partial_{z'}$$

Since $(x', y', z') = L_{(s,t,u)}((x, y, z)) = (x+s, y+t, u+z+\frac{1}{2}(sy+tx))$, the three vector fields are indeed left invariant and they span the Heisenberg algebra.
Naturally this is the group product that gives the Heisenberg group, defined at the beginning of this example, its structure.

Chapter 4

Generalized curvature-dimension inequality

This generalization of the curvature-dimension inequality is introduced for the first time by the authors F. Baudoin and N. Garofalo in this paper [4] from where the main parts of this Chapter is taken.
The idea of the two mathematicians is to define a reasonable extension of the inequality to any differential operators even when the premise of a Riemannian setting is missing or incomplete.
Hence we can not fully rely on the metric tensor but on the underlining differential manifold. To do this they follow an axiomatic formulation: they request 3 additional hypotheses on the manifold which must be fulfilled in order to say that the new and expanded inequality holds.
As we will see the ambient, in which the authors thought the inequality, is mostly analytic, so the big work we are going to do in the section 4.1 will be to translate these notions of analysis in our geometrical way of thinking.

The first definition is the idea of smooth measure in a manifold, that is nothing else then the measure associated to a smooth volume form.

Definition 4.1. Given a manifold M, μ is a smooth measure if it exists a smooth volume form ν such that

$$\mu(A) = \int_A \nu$$

for A Borel set of M

With integration with respect to μ we will mean

$$\int_M f \, d\mu = \int_M f\nu$$

for $f \in C_0^\infty(M)$

Remark *4.2.* In the definition above we have to ask the functions to be compactly supported, otherwise the integral could not be well-defined and for some choices of f it could get the value $+\infty$.
So anytime in the future a integration is involved, the right set of definition shall be $C_0^\infty(M)$.

Another notion we must state since is required for the setting of the generalized curvature-dimension inequality is the idea of locally sub-ellipticity for a differential operator:

Definition 4.3. We say that an operator L is locally subelliptic if for any $A \Subset M$ there exist C, ϵ such that the following estimate holds for any $f \in C_0^\infty(A)$

$$\|f\|^2_{H^{2\epsilon}} \leq C\left[\|Lf\|^2_{L^2} + \|f\|^2_{L^2}\right]$$

where $\|f\|_{H^{2\epsilon}}$ is the Sobolev norm.

We are now able to present the setting for the new inequality with a clear idea of what everything means.

Let M be a differential manifold endowed with a smooth measure μ and $L\colon C^\infty(M) \to C^\infty(M)$ be a locally sub-elliptic operator such that $L1 = 0$, where 1 is the constantly 1 function.
Moreover we assume that the operator is symmetric with respect to μ and non positive, by this is meant

$$\int_M fL(h)\,d\mu = \int_M L(f)h\,d\mu \qquad \int_M fL(f)\,d\mu \leq 0$$

for all $f, h \in C_0^\infty(M)$.
A possible form for L in coordinates is

$$L = -\sum_i X_i^* X_i$$

where X_i^* is the adjoint of X_i with respect to the L^2 scalar product.
In fact if we write $\langle f, h\rangle := \int_M fh\,d\mu$ the inner product, we can see that

$$\begin{aligned}\int_M fL(h)\,d\mu = \langle f, L(h)\rangle = \langle f, -\sum_i X_i^* X_i h\rangle \\ = -\sum_i \langle X_i f, X_i h\rangle = \langle -\sum_i X_i^* X_i f, h\rangle = \int_M L(f)h\,d\mu\end{aligned}$$

Moreover

$$\int_M fL(f)\,d\mu = \langle f, Lf\rangle = \langle f, -\sum_i X_i^* X_i f\rangle = -\sum_i \langle X_i f, X_i f\rangle = -\sum_i \|X_i f\|^2 \leq 0$$

Now the question becomes obviously how the adjoint looks like.
We recall, we proved in the first Chapter, that the integral of a divergence associated to a smooth volume form is zero if something was compactly supported, in particular

$$\int_M \operatorname{div}(Y)\,d\mu = 0$$

for any Y vector field compactly supported in M.
Using now equation (1.4) also from the first Chapter and this property we get

$$0 = \int_M \operatorname{div}(fhX)\,d\mu = \int_M fX(h) + X(f)h + fh\operatorname{div}(X)\,d\mu$$

for $f, h \in C_0^\infty(M)$; that we can written as

$$\int_M X(h)f = \int_M -X(f)h - \operatorname{div}(X)fh\, d\mu$$

We have shown that the adjoint of X_i with respect to the L^2 product is

$$X_i^* = -X_i - \operatorname{div}(X_i)$$

So if an operator in coordinates is

$$L = \sum_i X_i X_i + \operatorname{div}(X_i)X_i$$

then it is suitable for the generalized curvature-dimension inequality.
Now that we discuss how the operator can look like we keep forward with the preparation for the new curvature-dimension inequality, following the procedure as in the "old" one we introduce a symmetric form based on the operator L.

Definition 4.4. Given the operator L we define a symmetric, first-order differential bilinear form, $\Gamma\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$

$$\Gamma(f,h) = \frac{1}{2}\left(L(fh) - fL(h) - hL(f)\right)$$

and, by a little abuse of notation, the so called carré du champ operator

$$\Gamma(f) := \Gamma(f,f) = \frac{1}{2}\left[L(f^2) - 2fL(f)\right]$$

Moreover we define a bilinear second order form Γ_2 on $C^\infty(M)$ defined by Γ and L in the following way

$$\Gamma_2(f,h) = \frac{1}{2}\left(L\Gamma(f,h) - \Gamma(f, Lh) - \Gamma(Lf, h)\right)$$

and we will write as before

$$\Gamma_2(f) = \Gamma_2(f,f)$$

Until this point we are not too far from what we had written for the curvature-dimension inequality in the Riemannian case; also there, given the Laplace operator, we used two billinear forms build on it. The peculiarity of this new inequality is the object we are going to define below

Definition 4.5 (Extrinsic carré du champ)**.** Assume on M another symmetric, first-order differential bilinear form $\Gamma^Z\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

$$\Gamma^Z(fk, h) = f\Gamma^Z(k,h) + k\Gamma^Z(f,h)$$

We assume moreover that if, $\Gamma^Z(f) := \Gamma^Z(f,f)$, it holds

$$\Gamma^Z(f) \geq 0$$

The extrinsic carré du champ, as we will see in the setting of sub-Riemannian geometry with transverse symmetries, can be thought as the counterpart of Γ when the metric is defined only on a subbundle of the tangent bundle and it will not be determined by the manifold.

Remark 4.6. By the close relation between $\Gamma^Z(f)$ and $\Gamma^Z(f,f)$, with extrinsic carré du champ we will denote both operators.

Also in this case we have to define another second-order differential form, by keeping in mind that this operator can be thought as a extrinsic Γ we are not surprise that the definition is identical to the one for Γ_2

Definition 4.7. Given the operator L and the extrinsic carré du champ Γ^Z we define a second order differential form Γ_2^Z as

$$\Gamma_2^Z(f,h) = \frac{1}{2}\left(L\Gamma^Z(f,h) - \Gamma^Z(f,Lh) - \Gamma^Z(Lf,h)\right)$$

We now have all the objects to define the generalized curvature-dimension inequality. As we had written, the authors follow an axiomatic way, actually there are three hypotheses, together with the inequality, that must be fulfill so that a manifold satisfies CD.

Hypothesis 4.8. *The semigroup $P_t := e^{tL}$, that are the operators such that $P_t f(p) = u(t,p)$ solves the system*

$$\begin{cases} \partial_t u = Lu \\ u(0,p) = f(p) \end{cases}$$

must be stochastically complete that is, for $t \geq 0$, $P_t 1 = 1$ and for every $f \in C_0^\infty(M)$ and $T \geq 0$, one has

$$\sup_{t\in[0,T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty$$

Hypothesis 4.9. *There exists an increasing sequence $\{h_k\}_{k\in\mathbb{N}} \subset C_0^\infty(M)$ such that $h_k \nearrow 1$ on M and*

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to 0 \qquad \text{as } k \to \infty$$

Hypothesis 4.10. *For any $f \in C^\infty(M)$ one has*

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f))$$

This last hypothesis is a relation between the two carré du champ operators, a sort of commutativity.
We are ready to present the main part of this Chapter and thesis: the inequality.

Definition 4.11. We shall say that M satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, k, d)$ with respect to L and Γ^Z if the three hypotheses above are satisfied and there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $k \geq 0$ and $0 < d \leq \infty$ such that

$$\Gamma_2(f) + r\Gamma_2^Z(f) \geq \frac{1}{d}(L(f))^2 + \left(\rho_1 - \frac{k}{r}\right)\Gamma(f) + \rho_2\Gamma^Z(f) \tag{4.1}$$

for every $f \in C^\infty(M)$ and every $r > 0$.

4.1 Sub-Riemannian manifolds with transverse symmetries

In the section above we defined the new inequality in the general and analytic way presented by the authors, now we want to restrict our attention to a specific family of manifolds: sub-Riemannian manifolds with transverse symmetries.
We know already what a sub-Riemannian manifold is but we have no clues what are transverse symmetries, what this means we see in the next lines.

Definition 4.12. Given a sub-Riemannian manifold (M, H, g), the Lie algebra of sub-Riemannian Killing vector fields V is the Lie algebra of vectors fields whose elements are Killing for (M, g), this means that their flow is a local isometry with respect to the sub-Riemannian metric.
More precisely $Z \in V$ if it fulfills these two properties:

- For $X, Y \in H$ it holds

$$\mathcal{L}_Z g(X,Y) := Z(g(X,Y)) - g([Z,X],Y) - g(X,[Z,Y]) = 0 \tag{4.2}$$

- If $X \in H$ then $[Z, X] \in H$

Definition 4.13. Given a sub-Riemannian manifold (M, H, g) we say that it has transverse symmetries if it exists a Lie algebra of sub-Riemannian Killing vector fields $(V, [,])$ such that for all $p \in M$ it holds

$$T_pM = H|_p \oplus V|_p$$

Remark 4.14. The subbundle V is called the set of vertical directions in contraposition to the set of horizontal directions H.
The objects on which all the next theory will be base, are only M, H, V and of course the metric g that we recall is only defined on the distribution H.

In this setting as operator L can be chosen the sub-Laplacian.

Claim 4.15. *The sub-Laplacian Δ is a symmetric and non positive operator.*

Proof. It is sufficient to show that it is possible to write it in a frame as

$$L = \sum_i E_i E_i + \text{div}(E_i) E_i$$

as our theory says.
So let $\{E_1, .., E_d\}$ be an orthonormal frame for the distribution H, then the horizontal gradient can be written as $\nabla_H f = \sum_i E_i(f)E_i$ since the frame is orthonormal then $(\nabla_H f)^i = \nabla_i(f) = E_i(f)$.
If we apply equation (3.2) we get that the sub-Laplacian is

$$\Delta(f) = \text{div}(\nabla_H f) = \text{div}(\sum_i (\nabla_H f)^i E_i) = \sum_i \text{div}(E_i(f)E_i)$$

$$= \sum_i E_i(E_i(f)) + E_i(f)\text{div}(E_i)$$

and it must be symmetric and non positive. $\square$

We would like to see if the sub-Laplacian is also a locally sub-elliptic operator. This follows from the bracket generating property of H, in fact it holds the following theorem

Theorem 4.16. *Given a bracket generating distribution $D = span\{X_1, .., X_d\}$ and $A \Subset M$ there exists $\epsilon \in (0, 1)$, $C > 0$ such that for every $f \in C_0^\infty(A)$ it holds*

$$\|f\|_{H^\epsilon} \leq C\|f\|_{W_X^{1,2}}$$

Where $\|f\|_{H^\epsilon}$ is the Sobolev norm and the second norm is

$$\|f\|^2_{W_X^{1,2}} = \sum_i \|X_i f\|^2_{L^2} + \|f\|^2_{L^2}$$

Proof. See [6] page 6 proposition 1.1 and [11] □

Expanding the norm of the derivative one gets the sub-ellipticity estimate, in fact

$$\sum_i \|X_i f\|^2_{L^2} = \sum_i \langle X_i f, X_i f\rangle = \sum_i \langle X_i^* X_i f, f\rangle = \langle -\Delta f, f\rangle \leq |\langle -\Delta f, f\rangle|$$

By Cauchy–Schwarz inequality and the property $\|-\Delta f\|_{L^2} = \|\Delta f\|_{L^2}$.

$$\leq \|-\Delta f\|_{L^2}\|f\|_{L^2} = \|\Delta f\|_{L^2}\|f\|_{L^2}$$

Moreover using the fact $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$ so $2ab \leq a^2 + b^2$ with $b = \|f\|_{L^2}$ and $a = \|\Delta f\|_{L^2}$ we get

$$\sum_i \|X_i f\|^2_{L^2} \leq \frac{1}{2}\left(\|\Delta f\|^2_{L^2} + \|f\|^2_{L^2}\right)$$

then adding to both side $\|f\|^2_{L^2}$ it becomes

$$\|f\|^2_{W_X^{1,2}} \leq \frac{3}{2}\left(\|\Delta f\|^2_{L^2} + \|f\|^2_{L^2}\right)$$

We can combine the theorem and the inequality above to find

$$\|f\|^2_{H^\epsilon} \leq \tilde{C}\left(\|\Delta f\|^2_{L^2} + \|f\|^2_{L^2}\right)$$

that is exactly the definition of sub-ellipticity for Δ.

After we proved that Δ is a suitable operator for the generalized curvature-dimension inequality we move our attention to the problem of finding the bilinear form associated to Δ.
We apply the formula from the previous section: for $f, k \in C^\infty(M)$

$$\Gamma(f, k) = \frac{1}{2}\left(\Delta(fk) - f\Delta(k) - k\Delta(f)\right)$$

then by proposition 3.1 it gets

$$\Gamma(f, k) = g(\nabla_H k, \nabla_H f)$$

and the carrè du champ operator is simply the squared norm of the horizontal gradient

$$\Gamma(f) = g(\nabla_H f, \nabla_H f)$$

What can a good form for the extrinsic carré du champ operator be?.
We said it is not canonically arising from the manifold but chosen in a way that it satisfies some assumptions. The main idea however is that it's the counterpart of Γ. So it seems reasonable to chose for Γ^Z a shape which recalls the original operator.

Definition 4.17. Given a sub-Riemannian manifold with transverse symmetries and a metric g_V on V, we call vertical gradient $\nabla_V f$ of a smooth function f on M the unique vector field on V such that

$$g_V(\nabla_V f, X) = X(f)$$

for $X \in \Gamma^\infty(V)$

We set as extrinsic carré du champ the bilinear form Γ^Z given by

$$\Gamma^Z(f,k) = g_V(\nabla_V f, \nabla_V k)$$

for $f, k \in C^\infty(M)$

Remark 4.18. Hereafter g_V will denote the metric on the vertical direction and g_H the one on the horizontal directions.

We still have to check if the extrinsic carré du champ satisfies the two properties of its definition: to be positive semidefinite and to act like a derivation in its arguments.
It is positive semidefinite since g_V is a metric.
The derivation property also comes easily from the definition of gradient, in fact using equation (2.2) we get

$$\Gamma^Z(hf,k) = g_V(\nabla_V hf, \nabla_V k) = g_V(f\nabla_V h + h\nabla_V f, \nabla_V k) = f\Gamma^Z(h,k) + h\Gamma^Z(f,k)$$

We proved that Γ^Z is indeed a good choice for the extrinsic carré du champ operator. We state the two bilinear second order forms Γ_2 and Γ_2^Z, they are

$$\Gamma_2(f,k) := \frac{1}{2}\left(\Delta(g_H(\nabla_H f, \nabla_H k)) - g_H(\nabla_H f, \nabla_H \Delta k) - g_H(\nabla_H \Delta f, \nabla_H k)\right)$$

With

$$\begin{aligned}\Gamma_2(f) = \Gamma_2(f,f) &= \frac{1}{2}\Delta(g_H(\nabla_H f, \nabla_H f)) - g_H(\nabla_H f, \nabla_H \Delta f)\\ &= \frac{1}{2}\Delta(g_H(\nabla_H f, \nabla_H f)) - \Gamma(f, \Delta f)\end{aligned}$$

and since Γ_2^Z is defined in the same way

$$\Gamma_2^Z(f,k) := \frac{1}{2}\left(\Delta(g_V(\nabla_V f, \nabla_V k)) - g_V(\nabla_V f, \nabla_V \Delta k) - g_V(\nabla_V \Delta f, \nabla_V k)\right)$$

$$\begin{aligned}\Gamma_2^Z(f) &= \frac{1}{2}\Delta(g_V(\nabla_V f, \nabla_V f)) - g_V(\nabla_V f, \nabla_V \Delta f) \\ &= \frac{1}{2}\Delta(g_V(\nabla_V f, \nabla_V f)) - \Gamma^Z(f, \Delta f)\end{aligned}$$

The natural continuation of the thesis would be, at this point, to find conditions so that the 3 hypotheses for the generalized curvature-dimension inequality are satisfied in this setting.
However, since some new concepts are involved in this process, it is reasonable to introduce them first and postpone the hypotheses for later.
We are going to define adapted frames for the sub-Riemannian manifold, that will help us to write the objects in coordinates.
We must take care that an adapted frame can not be a random choice of vector fields, but it must respect the nature of the manifold, i.e. it must preserve the transverse symmetries.
Since there is no hope in find a global frame, the definition has to be only local. Let's fixed the notation for the dimensions of the distribution H and the Lie algebra V, respectively s and t.

Definition 4.19. For $p \in M$ a sub-Riemannian manifold with transverse symmetries, we call a set of vector fields $\{X_1, .., X_s, Z_1, .., Z_t\}$, defined in a sufficiently small neighborhood of p, an adapted frame if

1. $Z_1, .., Z_t \in V$
2. $\{X_1|_p, .., X_s|_p\}$ is an g_H-orthonormal base for $H|_p$
3. $\{Z_1|_p, .., Z_t|_p\}$ is an g_V-orthonormal base for $V|_p$
4. the following relations hold

$$[X_i, X_j] = \sum_n^s \omega_{ij}^n X_n + \sum_m^t \gamma_{ij}^m Z_m$$

$$[X_i, Z_j] = \sum_n^s \delta_{ij}^n X_n$$

where ω_{ij}^n, γ_{ij}^m are smooth functions antisymmetric with respect to the lower indeces by the antisymmetry of the Lie bracket, i.e. for $r = 1, .., s$ $l = 1, .., t$

$$\omega_{ij}^r = -\omega_{ji}^r \qquad \gamma_{ij}^l = -\gamma_{ji}^l$$

Moreover δ_{ij}^n must be smooth functions such that Z_k are Killing vector fields, hence they must satisfy property (4.2). To find out what this is we start with the definition

$$\begin{aligned}Z_k(g_H(X_i, X_j)) &= g_H([Z_k, X_i], X_j) + g_H(X_i, [Z_k, X_j]) \\ &= \sum_n^s \delta_{ik}^n g_H(X_n, X_j) + \sum_p^s \delta_{jk}^p g_H(X_i, X_p)\end{aligned}$$

then by g_H-orthonormality of $\{X_1, .., X_d\}$

$$0 = \delta^j_{ik} + \delta^i_{jk}$$

So the functions must be such that

$$\delta^j_{ik} = -\delta^i_{jk}$$

We want now to define a privileged connection on the manifold. Given the geometry of the sub-Riemannian manifold it is not possible to define the Levi-Civita connection; the metric is defined only on a part of the tangent bundle and the uniqueness based on the compatibility with the metric would be missed.
We can't either use the Levi-Civita connection of $(M, g = g_h \oplus g_V)$ since it would depend on the choice of g_V while the objects arising from the manifold, we said, must depend only on (M, H, g_H, V).
Therefore the right connection that preserves the geometry, must be the following:

Proposition 4.20. *There exists a unique affine connection ∇ on (M, H, g_H) with the following properties:*

1. $\nabla g_H = 0$
2. *If X and Y are horizontal vector fields then $\nabla_X Y$ remains horizontal*
3. *If Z is vertical then $\nabla Z = 0$*
4. *If X and Y are horizontal vector fields and Z is vertical, then the torsion vector field $T(X, Y)$ is vertical while $T(X, Z) = 0$*

Proof. Before starting with the proof we recall briefly what is meant by torsion:

Definition 4.21. Given a connection ∇ on M, its torsion is the tensor field $T\colon \mathfrak{X}(M)\times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Let $\tilde{g}_V$ be any metric on the vertical set. We set $g_R(X, Y) = g_H(X_1, Y_1) + \tilde{g}_V(X_2, Y_2)$ where $X, Y \in \mathfrak{X}(M)$ and X_1, Y_1 are their horizontal parts and X_1, Y_1 are their vertical parts.
We turn M into a Riemannian manifold (M, g_R), which has unique Levi-Civita connection ∇^R.
We can define now the connection ∇ on M that acts in the following way

- If X, Y are horizontal vector fields $\nabla_X Y = \pi_H(\nabla^R_X Y)$ where $\pi_H\colon \Gamma^\infty(TM) \to \Gamma^\infty(H)$ is the projection
- If X is horizontal and Z is vertical $\nabla_Z X = [Z, X]$
- If Z is vertical vector field $\nabla Z = 0$

We just have to check if the last prescription of the proposition is fulfilled.
Let X, Y, Z as point 4. of the above proposition.

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \pi_H(\nabla^R_X Y) - \pi_H(\nabla^R_Y X) - [X,Y]$$

then by linearity of the projection and torsion-freeness of the Levi-Civita connection

$$= \pi_H(\nabla^R_X Y - \nabla^R_Y X) - [X,Y] = \pi_H([X,Y]) - [X,Y] = -\pi_V([X,Y])$$

and

$$T(X,Z) = \nabla_X Z - \nabla_Z X - [X,Z] = 0 - [Z,X] - [X,Z] = 0$$

Uniqueness follows by noticing that the connection is defined on every possible direction in a fixed way, then if 2 such connections existed their difference would be zero everywhere.
Furthermore we also notice that since g_R makes the decomposition between H and V orthogonal, the horizontal part of the Levi-Civita connection does not depend on the choice of g_V, namely ∇ for horizontal vector fields is the unique torsionfree and compatible with g_H connection on (H, g_H). □

The connection we just introduced will be the privileged connection of this setting. It can be thought as the reciprocal of the Levi-Civita in the Riemannian case, in fact that ambient can be seen as a degenerate case of a sub-Riemannian manifold where the distribution H is the all tangent space.
In particular the connection above in that case would be the Levi-Civita, since the projection π_H is the identity if the distribution is the all TM.

Remark 4.22. From now on with the notation ∇ we will indicate the connection from proposition 4.20.

We can finally have a closer look on the 3 hypotheses of the inequality. It turns out that in this setting they are implied by 2 geometrical assumptions on the manifold: completeness and the manifold of Yang-Mills type.
The proof of the fact that this two properties are enough for the fulfillment of the 3 hypotheses required long computation, which we will not display. We will just briefly give an hint in the next paragraphs.

Hypothesis 4.8 Hypothesis 4.8 is the stochastic completeness of the semigroup of Δ; as it seems, it is a purely analytic request.
Another possible geometrical assumption one can state is to ask that the manifold is of Yang-Mills type.

Definition 4.23. A sub-Riemannian manifold of transverse symmetries is said to be of Yang-Mills type if for every horizontal vector field Y, it holds

$$\sum_i (\nabla_{X_i} T)(X_i, Y) = 0$$

where T is the torsion tensor with respect to ∇ and $\{X_1, .., X_s, Z_1, .., Z_t\}$ is an adapted frame.

Proposition 4.24. *Let M be a sub-Riemannian manifold with transverse symmetries of Yang-Mills type.*
Suppose in addition that (M, d_{CC}) is a complete metric space, where d_{CC} is the Carnot–Carathéodory distance, then if the sub-Laplacian satisfy equation (4.1) then hypothesis 4.8 is fulfilled.

Proof. See [4] page 194 theorem 4.3 □

Hypothesis 4.9 We want to show that the following theorem implies that Hypothesis 4.9 in our setting is fulfilled.

Theorem 4.25. *Let M be a complete Riemannian manifold. Then it exits $\{h_k\}_{k\in\mathbb{N}} \subset C_0^\infty(M)$ such that $h_k \nearrow 1$ on M and*

$$\|\nabla h_k\|_\infty \to 0$$

Proof. See [2] page 230 Lemma 10.7 and [10] □

First of all we recall that by theorem of Hopf-Rinow completeness of a Riemannian manifold from definition 2.6 is equivalent to the completeness for the metric space whose distance comes from the Riemannian metric.
Secondly we remember that we can turn the sub-Riemannian manifold in a Riemannian one with help of the vertical metric as we did in the proposition of the connection.
So let (M, g) be the Riemannian manifold, where $g = g_H \oplus g_V$ and we denote d the distance function from the Riemannian metric while d_{CC} the Carnot Carathéodory distance.

Proposition 4.26. *With the assumptions above it exist $s \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for closed enough pair of points $p, q \in M$*

$$d(p,q) \leq d_{CC}(p,q) \leq C d(p,q)^s$$

Proof. We only prove the first inequality, for the second one can see for example: [5] page 118 section 1.2

$$\begin{aligned} d(p,q) &= \inf\{L_g(\gamma) |\ \gamma \text{ is a curve from } p \text{ to } q\} \\ &\leq \inf\{L_g(\gamma) |\ \gamma \text{ is a } H\text{-horizontal curve from } p \text{ to } q\} = d_{CC}(p,q) \end{aligned}$$

□

So if we suppose (M, d_{CC}) to be a complete metric space then (M, d) is complete as well.
Then by theorem above $\{h_k\}_{k\in\mathbb{N}} \subset C_0^\infty(M)$ such that $h_k \nearrow 1$ on M and

$$\|\nabla h_k\|_\infty \to 0$$

Claim 4.27.

$$\|\nabla h_k\|_\infty \to 0 \implies \|\Gamma(h_k)\|_\infty \to 0$$

Proof. Let $\{X_1, .., X_s, Z_1, .., Z_t\}$ be an adapted frame. Then by definition of the metric

$$\nabla h_k = \nabla_H h_k + \nabla_V h_k$$

so

$$\|\nabla h_k\|_\infty \to 0 \implies \|\nabla_H h_k\|_\infty \to 0$$

and in particular $\forall p \in M$

$$g_H(\nabla_H h_k, \nabla_H h_k)|_p = \sum_i (g_H)_{i,j}(p)(\nabla_H h_k)^i(p)(\nabla_H h_k)^j(p) \leq \tilde{C}(p)\|\nabla_H h_k\|_\infty^2 \to 0$$

Since the metric is a tensor field , the map $p \mapsto g_H(\nabla_H h_k, \nabla_H h_k)|_p$ is smooth and at every points it goes to zero hence its supremum norm also goes to zero.
The carrè du champ operator becomes

$$\|\Gamma(h_k)\|_\infty = \|g_H(\nabla_H h_k, \nabla_H h_k)\|_\infty \to 0 \qquad \text{as } k \to \infty$$

□

We sum everything together with the next proposition

Proposition 4.28. *If (M, d_{CC}) is a complete metric space then hypothesis 4.9 holds for the sub-Laplacian.*

Hypothesis 4.10 In a sub-Riemannian manifold with transverse symmetries with our choice for Δ and Γ^Z hypothesis 4.10 is satisfied. In order to prove this we use an adapted frame and we write both objects in coordinates

Proposition 4.29. *In the setting of a sub-Riemannian manifold with transverse symmetries with $L = \Delta$ and Γ^Z as before, hypothesis 4.10 is fulfilled*

Proof.

$$\Gamma^Z(f, \Gamma(f)) = g_V(\nabla_V f, \nabla_V \Gamma(f)) = g_V(\nabla_V f, \nabla_V g_H(\nabla_H f, \nabla_H f))$$

Since in an adopted frame $\nabla_H f = \sum_i X_i(f)X_i$ and $\nabla_V(f) = \sum_j Z_j(f)Z_j$ we get

$$\begin{aligned}
&= g_V(\nabla_V f, \nabla_V \sum_j X_j(f)^2) \\
&= \sum_j g_V(\sum_i Z_i(f)Z_i, \sum_k Z_k((X_j(f))^2)Z_k) \\
&= \sum_j g_V(\sum_i Z_i(f)Z_i, 2\sum_k Z_k(X_j(f))X_j(f)Z_k) \\
&= 2\sum_j \sum_k Z_k(f)X_j(f)Z_kX_j(f)
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&= 2\sum_j \sum_k Z_k(f)X_j(f)\left([Z_k, X_j] + X_j Z_k\right)(f) \\
&= 2\sum_j \sum_k Z_k(f)X_j(f)\left(X_j Z_k - [X_j, Z_k]\right)(f) \\
&= 2\sum_k Z_k(f)X_j(f)\left(\sum_j X_j Z_k - \sum_j \sum_m \delta_{jk}^m X_m\right)(f) \\
&= 2\sum_k \sum_j Z_k(f)X_j(f)X_j Z_k(f)
\end{aligned}
$$

where the delta functions vanish by their antisymmetry property.
We obverse that if we calculate $\Gamma(f, \Gamma^Z(f))$ we just do the same passages as (4.3) but getting $X_j Z_k$ instead of $Z_k X_j$. So we proved

$$
\Gamma^Z(f, \Gamma(f)) = \Gamma(f, \Gamma^Z(f))
$$

that was the claim. □

So in a complete sub-Riemannian manifold with transverse symmetries of Yang-Mills type the 3 hypotheses are naturally satisfied and we can care only about the inequality with respect to Δ and Γ^Z.
Actually thanks to this special setting, we can do more; since we have a privileged connection that preserved the geometry of the manifold we are able to construct the Ricci curvature and the Riemann curvature tensor.
It is moreover possible to imitate what we did in Chapter 2: find Bochner's formulas for the vertical and the horizontal directions.
Ultimately we can find an equivalence between bounds of objects related to the Ricci and torsion tensor and the curvature-dimension inequality.
We state here only the last part, starting from introducing two operator, for an in-depth study one can see [4].

Remark 4.30. From now on μ will not be any measure but the one associated with the Riemannian volume of $(M, g = g_H \oplus g_V)$.

Definition 4.31. Let ∇ be the connection and Ric its associated Ricci curvature tensor.
Let moreover T be the torsion tensor with respect to ∇, then for $f \in C^\infty(M)$ we define

$$
\mathcal{R}(f) = \mathrm{Ric}(\nabla_H f, \nabla_H f) + \sum_{lk}\left(-(\nabla_{X_l}T)(X_l, X_k))f)(X_k f) + \frac{1}{4}(T(X_l, X_k)f)^2\right)
$$

$$
\mathcal{T}(f) = \sum_i g_V(T(X_i, \nabla_H f), T(X_i, \nabla_H f)) = \sum_i \|T(X_i, \nabla_H f)\|_V^2
$$

where $\{X_1, .., X_s, Z_1, .., Z_t\}$ is an adapted frame for M.

We finish this Chapter with the equivalence theorem for a sub-Riemannian manifold with transverse symmetries.

Theorem 4.32 (Equivalence theorem)**.** *For a sub-Riemannian manifold (M, H, g_H) with transverse symmetries complete and of Yang-Mills type, it holds the following equivalence:*

$$CD(\rho_1, \rho_2, k, s) \iff \begin{cases} \mathcal{R}(f) \geq \rho_1 \Gamma(f) + \rho_2 \Gamma^Z(f) \\ \mathcal{T}(f) \leq k\Gamma(f) \end{cases} \qquad \forall f \in C^\infty(M)$$

for s the dimension of H and constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$ and $k \geq 0$.

Sketch of the proof. The idea as we said is to follow the procedure of the ordinary curvature-dimension inequality. We introduce the two Bochner's formulas, one for the vertical direction and one for the horizontal.

Proposition 4.33. *For every $f \in C^\infty(M)$ it holds*

$$\Gamma_2(f) = \|\nabla_H^2 f\|^2 + \mathcal{R}(f) + \mathcal{S}(f) \qquad \textit{Horizontal Bochner's formula}$$

$$\Gamma_2^Z(f) = \|\nabla_H \nabla_V f\|^2 \qquad \textit{Vertical Bochner's formula}$$

where for $\{X_1, .., X_s, Z_1, .., Z_t\}$ adapted frame

$$\mathcal{S}(f) = -2\sum_i g(\nabla_{X_i} \nabla_V f, T(X_i, \nabla_H f))$$

The proof of this two equalities is based on writing the two bilinear forms in an adapted frame of the manifold and after many computations recognizing that the two sides are equal.
With this proposition is possible to prove the equivalence theorem: let's start with the direction $\Longleftarrow$.
With an adapted frame thanks to the Cauchy-Schwarz inequality is possible to prove

$$\frac{1}{s}(\Delta f)^2 \leq \|\nabla_H^2 f\|^2$$

Appling now the horizontal Bochner formula and the assumption one get

$$\frac{1}{s}(\Delta f)^2 \leq \Gamma_2(f) - \rho_1\Gamma(f) - \rho_2\Gamma^Z(f) + \mathcal{S}(f)$$

Writing the objects in the frame is possible to find a relation between $\mathcal{S}$ and $\mathcal{T}$ that turns the expression above in

$$\frac{1}{s}(\Delta f)^2 \leq \Gamma_2(f) + \nu\Gamma_2^Z(f) + \frac{1}{\nu}\mathcal{T}(f) - \rho_1\Gamma(f) - \rho_2\Gamma^Z(f)$$

Then we can use the second bound of the assumption, getting the curvature-dimension inequality.
$\Longrightarrow$). Given $p \in M$, $U \in H|_p$ and $K \in V|_p$, in an adapted frame $\{X_1, .., X_s, Z_1, , ., Z_t\}$ it is possible to find $f \in C^\infty(M)$ such that

- $\nabla_H f|_p = U$

- $\nabla_V f|_p = K$
- $\nabla^2_H f|_p = 0$
- $X_j Z_m f|_p = \frac{1}{\nu}\sum_i \gamma^m_{ij} K^i$

for $\nu > 0$ and $[X_i, X_j] = \sum_n^s \omega^n_{ij} X_n + \sum_m^t \gamma^m_{ij} Z_m$.
Using this special function in the curvature-dimension inequality gives

$$\Gamma_2(f)|_p + \nu\Gamma_2^Z(f)|_p \geq \left(\rho_1 - \frac{k}{\nu}\right) g_H(U,U) + \rho_2 g_V(K,K)$$

Moreover from the horizontal Bochner's formula, by the properties of the function we get

$$\Gamma_2(f)|_p = \mathcal{R}(f)|_p + \mathcal{S}(f)|_p$$

and the relation between $\mathcal{S}$ and $\mathcal{T}$ of this special function is

$$\mathcal{S}(f)|_p = -\frac{2}{\nu}\mathcal{T}(f)|_p$$

The vertical Bochner's formula of the function f is

$$\Gamma_2^Z(f)|_p = \|\nabla_H \nabla_V f|_p\|^2 = \frac{1}{\nu}\mathcal{T}(f)|_p$$

Hence the inequality becomes

$$\mathcal{R}(f)|_p - \frac{1}{\nu}\mathcal{T}(f)|_p \geq \left(\rho_1 - \frac{k}{\nu}\right) g_H(U,U) + \rho_2 g_V(K,K)$$

It can be proved by using the adapted frame that the left-hand side only depends on U and K, the horizontal and vertical gradient of f, $\mathcal{R}(f)|_p =: \mathcal{R}(U,K)$ and $\mathcal{T}(f)|_p =: \mathcal{T}(U)$.
It is just proved that $\forall p \in M$, $U \in H|_p$ and $K \in V|_p$ for $\nu > 0$

$$\mathcal{R}(U,K) - \frac{1}{\nu}\mathcal{T}(U) \geq \left(\rho_1 - \frac{k}{\nu}\right) g_H(U,U) + \rho_2 g_V(K,K)$$

So for all $h \in C^\infty(M)$ taking for $p \in M$ $\nabla_H h|_p = U$ and $\nabla_V h|_p = K$ holds

$$\mathcal{R}(h) - \frac{1}{\nu}\mathcal{T}(h) \geq \left(\rho_1 - \frac{k}{\nu}\right)\Gamma(h) + \rho_2\Gamma^Z(h) \tag{4.4}$$

By letting $\nu \to \infty$ first we get

$$\mathcal{R}(h) \geq \rho_1\Gamma(h) + \rho_2\Gamma^Z(h) \tag{4.5}$$

then subtracting (4.5) to (4.4) we have the condition:

$$-\frac{1}{\nu}\mathcal{T}(h) \geq -\frac{k}{\nu}\Gamma(h)$$

by multiplying for $-\nu$ both sides we get the conclusion. □

Chapter 5

Riemannian cases: with and without magnetic field

In this Chapter we are going to apply the theory of the generalized curvature-dimension inequality in two sub-Riemannian manifolds with transverse symmetries: a Riemannian manifold and a sub-Riemannian manifold from a magnetic field.
The purposes are to find how all the objects look like and how the equivalence theorem 4.32 is reformulated in these two different settings.

5.1 Riemannian case

We want to see how the generalized curvature-dimension inequality becomes on a Riemannian manifold. This will also give us the chance to check if it is indeed a generalization, namely we should find the same result of Chapter 2, for at least some choices of the values involved: the inequality should look like definition 2.3 and the equivalence theorem 4.32 as 2.7

Let (M, g) be a Riemannian manifold. We can easily see it as a sub-Riemannian one (M, H, g_H) where for all $p \in M$ it holds

$$T_pM = H|_p$$

and the metric $g_H = g$.
We assume the same hypotheses of Chapter 2: let dV_g be the Riemmanian volume form as the one associated to our measure and Δ as the operator.
Given the geometry of the distribution, the vertical direction set can only be empty so the request of being Killing vector fields is trivially satisfied.
We are so in a degenerate case of sub-Riemannian manifold with transverse symmetries.
Hence to use the theory we developed in second part of Chapter 4 we must just check that the two definitions of Laplacian, one based on the Hessian and one on the divergence, coincide.

Proposition 5.1. *In a Riemannian manifold, the Laplacian defined as definition 1.16 and as the sub-Laplacian for $H = TM$ are the same.*

Proof. Let $p \in M$ and $X \in T_pM$. We can always choose an orthonormal frame $\{E_1, .., E_n\}$ such that in the point $p \in M$ it holds $\nabla_{E_i} E_j|_p = 0$ for all i, j .
Moreover let $\{\alpha^1, .., \alpha^n\}$ covectors which are the dual frame of $\{E_1, .., E_n\}$ i.e.

$$\alpha^i(E_j) = \delta^i_j$$

Then dV_g, the Riemannian volume, is written as

$$dV_g = \alpha^1 \wedge ... \wedge \alpha^n$$

Moreover we know, since the frame is orthonormal, it must hold the formula

$$\alpha^i(X) = \langle X, E_i \rangle$$

In order to compute the Lie derivative of dV_g, we start by computing the interior derivative

$$\begin{aligned} \iota_X(dV_g) = \iota_X(\alpha^1 \wedge .., \wedge \alpha^n) &= \sum_i (-1)^{i-1} \alpha^1 \wedge .. \wedge \iota_X(\alpha^i) \wedge .. \wedge \alpha^n \\ &= \sum_i (-1)^{i-1} \alpha^i(X) \alpha^1 \wedge .. \wedge \widehat{\alpha}^i \wedge .. \wedge \alpha^n \end{aligned}$$

where the hat means that that term is omitted.
Now we pass to the exterior derivative

$$\begin{aligned} d(\iota_X(dV_g)) &= \sum_j \sum_i (-1)^{i-1} E_j(\alpha^i(X)) \alpha^j \wedge \alpha^1 \wedge .. \wedge \widehat{\alpha}^i \wedge .. \wedge \alpha^n \\ &= \sum_i E_i(\alpha^i(X)) \alpha^1 \wedge .. \wedge \alpha^i \wedge .. \wedge \alpha^n = \sum_i E_i(\alpha^i(X)) dV_g \end{aligned}$$

then

$$\operatorname{div}(X) = \sum_i E_i(\alpha^i(X))$$

We recall that ∇ is Levi-Civita and we have an orthonormal frame of geodesics

$$= \sum_i E_i(\alpha^i(X)) = \sum_i E_i \langle X, E_i \rangle = \sum_i \langle \nabla_{E_i} X, E_i \rangle + \langle X, \nabla_{E_i} E_i \rangle = \sum_i \langle \nabla_{E_i} X, E_i \rangle$$

By choosing $X = \nabla f$ we get

$$\operatorname{div}(\nabla f) = \sum_i \langle \nabla_{E_i} \nabla f, E_i \rangle = \sum_i Hess(f)(E_i, E_i)$$

$\square$

It follows that Δ is a locally sub-elliptic operator and since we don't have vertical directions, the carré du champ operator and Γ_2 coincide, by definition, with the one of Chapter 2 while the extrinsic and Γ_2^Z are always zero.
An adapted frame is any orthonormal frame of (M, g) and the connection of definition 4.20, as we observed before, is the Levi-Civita. This means that its torsion tensor always vanishes and the Ricci curvature is the usual one on M.
By torsion-freeness of the connection the manifold is of Yang-Mills type thus all 3 hypotheses are satisfied once we assume M to be a complete manifold.

Remark 5.2. This is the first and, as we will see, the only difference from theory of Chapter 2. While there the assumption of completeness was necessary only for theorem 2.7, here it is requested also for the inequality. Naturally this comes from the variety of settings for the new CD and from the assiomatic approach which the authors used.

After the assumption made above we can state that on M holds the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, k, n)$ if there exist constant $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $k \geq 0$ and $0 < n \leq \infty$ such that

$$\Gamma_2(f) \geq \frac{1}{n}\Delta(f)^2 + \left(\rho_1 - \frac{k}{l}\right)\Gamma(f)$$

for all $f \in C^\infty(M)$ and $l > 0$.
So to have the same CD of Chapter 2 we just set $k = 0$, $CD(\rho_1, \rho_2, 0, n)$ for an arbitrarily ρ_2.

Remark 5.3. The constant $\rho_2 > 0$ does not appear in the inequality so the curvature-dimension condition depends only on n and ρ_1 $CD(\rho_1, \rho_2, 0, n) = CD(\rho_1, n)$ as Chapter 2.

It remains to rephrase theorem 4.32 in this setting. For this aim we see what become the two operator $\mathcal{R}$ and $\mathcal{T}$:

$$\mathcal{R}(f) = Ric(\nabla f, \nabla f) + 0$$

$$\mathcal{T}(f) = 0$$

both by the torsion-freeness of the connection.
Hence the equivalence theorem becomes

Theorem 5.4. *Suppose (M, g) to be a complete n-dimensional Riemannian manifold and $\rho_1 \in \mathbb{R}$.*
Then it holds the following equivalence:

$$CD(\rho_1, n) \iff Ric(\nabla f) \geq \rho_1 g(\nabla f, \nabla f) \quad \forall f \in C^\infty(M)$$

This is exactly theorem 2.7. The right-hand side is indeed a ρ_1-lower bound since the Ricci curvature $\mathrm{Ric}(X)$ at $p \in M$ depends only on $X|_p$ and, following the idea of the proof of theorem 2.7, locally it is always possible to find $f \in C^\infty$ such that

$$\nabla f|_p = X|_p$$

5.2 Sub-Riemannian manifold from a magnetic field

Let $(\tilde{M}, \tilde{g})$ be a n-dimensional Riemannian manifold and A be a differential 1-form. Consider the following constrained problem:

Problem For $c \in \mathbb{R}$ and $p, q \in \tilde{M}$ fixed, find

$$\min \left\{ L(\gamma) \mid \gamma \text{ is a curve from } p \text{ to } q \text{ and } \int_\gamma A = c \right\}$$

What we have in mind is to rephrase the above problem into a sub-Riemannian manifold where the horizontal curves σ will be in a 1:1 correspondence with the curves γ in $\tilde{M}$ which satisfy the constrain: this means that the constrain must be incorporated in the new curve.
We want to construct the relation $\gamma \overset{\Phi}{\mapsto} \sigma$. Since we have an integral constrain and the only way to relate it with a curve is $\int_\gamma A$, we will always have two informations $(\gamma, \int_\gamma A)$. So it makes sense to enlarge the manifold

$$M = \tilde{M} \times \mathbb{R}$$

where the last coordinate is the integral.
Let's see what happen once we have a parametrization of the original curve $\gamma\colon [a, b] \to \tilde{M}$.
The integral becomes

$$\int_\gamma A = \int_a^b A(\dot{\gamma}(t))\, dt$$

thus the new curve $\sigma\colon [a, b] \to M$ $\sigma(t) = \Phi(\gamma(t))$ must be

$$t \mapsto \left(\gamma(t), \int_a^t A(\dot{\gamma}(t)) \right) dt$$

As we said σ is set to be horizontal, so we have to compute the distribution H such that

$$\dot{\sigma}(t) = (\dot{\gamma}(t), A(\dot{\gamma}(t))) \in H|_{\gamma(t)}$$

To do so we introduce a frame in M $\{X_1, ..., X_n, Z\}$ such that the first n vector fields are a frame for $\tilde{M}$, then $\dot{\gamma}(t)$ can be decompose as

$$\dot{\gamma}(t) = \sum_i \alpha^i(t) X_i|_{\gamma(t)}$$

for α^i smooth functions and

$$A(\dot{\gamma}(t)) = \sum_i \alpha^i(t) A(X_i)|_{\gamma(t)}$$

Hence

$$\dot{\sigma}(t) = \sum_i \alpha^i(t) X_i|_{\gamma(t)} + \alpha^i(t) A(X_i)|_{\gamma(t)} Z|_{\gamma(t)} = \sum_i \alpha^i(t) \left(X_i|_{\gamma(t)} + A(X_i)|_{\gamma(t)} Z|_{\gamma(t)} \right)$$

So we found that the curve is always tangent to H, the span of the n vector fields $\{Y_i\}$ defined by

$$Y_i := X_i + A(X_i) Z \tag{5.1}$$

Remark 5.5. Since in the frame the first n components are related to $\tilde{M}$ it follows that Z must be a multiple of the vector field associated to the last coordinate, so we can always take $Z = \partial_z$ where z for a curve $\sigma(t) = (\gamma(t), z(\gamma(t)))$ in M is

$$z(\gamma(t)) = \int_\gamma A(\dot{\gamma}(t))dt$$

and by differentiating

$$\dot{z}(\gamma(t))\dot{\gamma}(t) = A(\dot{\gamma}(t))$$

that can be written independently from γ as

$$dz = A$$

That is why, from now on, we will always choose canonically $Z = \partial_z$.
Moreover, since the last component was add artificially, the vector fields X_i, so also Y_i, will only depend on the coordinates of $\tilde{M}$, hence not on z .

To give a possible sub-Riemannian structure to (M, H) we introduce a metric g_H on the distribution H defined for $v, w \in H$ as

$$g(v, w) = \tilde{g}(\pi_{\tilde{M}}(v), \pi_{\tilde{M}}(w))$$

where $\pi_{\tilde{M}}$ is the projection onto $\tilde{M}$.
Of course the distribution will not always be bracket generating, it will depend on the form of A. We can though calculate under which conditions of the first order of Lie bracket, the manifold (M, H, g_H) is sub-Riemannian.

$$\begin{aligned}
[Y_i, Y_j] &= [X_i + A(X_i)\partial_z, X_j + A(X_j)\partial_z] \\
&= [X_i, X_j] + [A(X_i)\partial_z, X_j] - [A(X_j)\partial_z, X_i] + [A(X_i)\partial_z, A(X_j)\partial_z] \\
&= [X_i, X_j] + [A(X_i)\partial_z, X_j] - [A(X_j)\partial_z, X_i]
\end{aligned}$$

because from the remark the functions $A(X_i)$ never depend on z.
Focusing only on the last two terms

$$[A(X_j)\partial_z, X_i] = A(X_j)\partial_z X_i - X_i A(X_j)\partial_z = -X_i A(X_j)\partial_z$$

So it becomes

$$[Y_i, Y_j] = [X_i, X_j] + X_i(A(X_j))\partial_z - X_j(A(X_i))\partial_z$$

By applying the famous formula for the exterior derivative of a 1-form ω, i.e $d\omega(K, L) = K(\omega(L)) - L(\omega(K)) - \omega([K, L])$ everything becomes

$$[Y_i, Y_j] = [X_i, X_j] + dA(X_i, X_j)\partial_z + A([X_i, X_j])\partial_z$$

If we suppose that $[X_i, X_j] = \sum_k c_{ij}^k X_k$ for c_{ij}^k smooth functions then

$$= \sum_k c_{ij}^k X_k + \sum_k c_{ij}^k A(X_k)\partial_z + dA(X_i, X_j)\partial_z$$

$$= \sum_k c_{ij}^k Y_k + dA(X_i, X_j)\partial_z$$

Given the relevance we call $B = dA$ and we see that $\{Y_l, [Y_i, Y_j]\}$ for i, j fixed and $l \in \{1, .., n\}$ generate the all tangent space, using $\{Y_1, .., Y_n, \partial_z\}$ as frame since ∂_z is independent from $\{Y_i\}$, if

$$det \left(\begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ & & .... & \\ c_{ij}^1 & c_{ij}^2 & ... & B_{ij} \end{bmatrix} \right) \neq 0$$

The distribution is therefore surely bracket generating if for some indeces i, j it holds for all $p \in M$

$$B_{ij}(p) \neq 0$$

But this is not the only condition, for example if the function for j fixed

$$\sum_i |B_{ij}|$$

never vanishes or even conditions involving the second-order Lie brackets.
For a sub-Riemannian manifold from a magnetic vector field we will means the triple (M, H, g_H) when the distribution H is bracket generating.

Remark 5.6. B is called magnetic field. This comes from the case where $\tilde{M} = \mathbb{R}^3$ and we are in presence of a magnetic vector field $\tilde{B}$. Given the particularity of the setting, it is possible to relate a vector field with a differential 2-form, exactly B. Moreover it's possible to relate the magnetic potential $\tilde{A}$ also with a 1-form, exactly A.
So if the curve $\gamma\colon [a, b] \to \mathbb{R}^3$ is a closed loop we can give a meaning to the value $z(\gamma(b))$. We can thing it is as the magnet flux through any surfaces S which have gamma as boundary.
The solution of the problem above instead can be physically interpreted as the trajectory of a particle going from p to q subject to a constained by c magnetic field $\tilde{B}$ which has $\tilde{A}$ as potential. For a deep explanation of this fact one can see for example [14]

Example 5.7 (Heisenberg group)**.** Let $M = \mathbb{R}^3$ with the usual Euclidean metric g_E and $B_{12} = 1$ constant. Naturally if $\{x_i\}$ are the canonical coordinates it holds

$$[\partial_{x_i}, \partial_{x_j}] = 0 \quad \forall i, j$$

which means that all the functions c_{ij}^k vanish.
So the Lie brackets of $\{Y_1, Y_2, \partial_z\}$ are

$$[Y_1, Y_2] = \partial_z \quad [Y_1, \partial_z] = 0 = [Y_2, \partial_z]$$

We immediately recognize the Heisenberg algebra.
So the sub-Riemannian manifold from a magnetic field $(\mathbb{R}^3, H, g_H)$ with H=span$\{Y_1, Y_2\}$ could be the Heisenberg group from Chapter 2. We only have to check if g_H defined as before and $\widehat{g}$ as Chapter 2 coincide.
Let $v = a_1Y_1 + a_2Y_2$ and $u = b_1Y_1 + b_2Y_2$ then

$$g_H(v, u) = g_E(a_1\partial_{x_1} + a_2\partial_{x_2}, b_1\partial_{x_1} + b_2\partial_{x_2}) = a_1b_1 + a_2b_2 = \widehat{g}(v, u)$$

Now that we illustrated our setting we can pass to study what happen to the generalized curvature-dimension inequality.

Let's suppose from now on that the distribution is bracket generating. We want to check if (M, H, g_H) is a sub-Riemannian manifold with transverse symmetries: to the set of horizontal direction H we oppose V=span$\{\partial_z\}$.
Since ∂_z is not linearly dependent from $\{Y_i\}$ and the distribution has one dimension less that the manifold, V hast to be the Lie algebra of vertical directions.
Moreover $Z = a\partial_z \in V$, for $a \in \mathbb{R}$, are Killing vector since we observed that Y_i do not depend on z, so

$$Zg_H(Y_i, Y_j) = a\partial_z(\tilde{g}_{ij}) = 0$$

and for the same reason

$$[Z, Y_i] = 0$$

therefore both properties are naturally fulfilled and V is a Lie algebra of Killing vector field.
As before the sub-Laplacian Δ will be our operator and we fix the measure μ already: it is the one associated to the Riemannian volume form of $(M, g = g_H \oplus g_V)$ for an arbitrary choice of g_V, metric on V.
We look for an adapted frame to be able to write everything in coordinates and find the connection:
When are $\{Y_i\}$ orthonormal vector with respect to the metric g_H?

$$g_H(Y_i, Y_j) = g_H(X_i + A(X_i)\partial_z, X_j + A(X_j)\partial_z) = \tilde{g}(X_i, X_j)$$

So $\{Y_i\}$ are g_H-orthonormal if and only if $\{X_i\}$ are $\tilde{g}$-orthonormal, hence we assume so.
By the arbitrariness of g_V, vertical metric, an orthonormal frame on V just translates into a choice of $Z \in V$. The set $\{Y_i, Z\}$ is an adapted frame with

$$\omega_{ij}^k = c_{ij}^k \quad \gamma_{ij} = B_{ij} \quad \delta_{ij}^k = 0$$

since we already calculated the Lie brackets before.
In this frame the sub-Laplacian Δ is naturally sub-elliptic and the two first order bilinear forms Γ, Γ^Z are

$$\Gamma(f, k) = g_H(\nabla_H f, \nabla_H k)$$
$$\Gamma^Z(f, k) = g_V(\nabla_V f, \nabla_V k)$$

for $f, k \in C^\infty(M)$. In the adapted frame the two gradients are written as

$$\nabla_H f = \sum_i Y_i(f)Y_i \qquad \nabla_V f = Z(f)Z$$

Hence the two carré du champ operators in the frame become

$$\Gamma(f) = \sum_i (Y_i(f))^2$$

$$\Gamma^Z(f) = (Z(f))^2$$

So we can try to write down also the Laplacian:

Proposition 5.8. *With the assumptions above is holds*

$$\Delta f = \sum_i Y_i Y_i f$$

Proof. By formula (3.2)

$$\Delta f = \operatorname{div}(\nabla_H f) = \operatorname{div}(\sum_i Y_i(f)Y_i) = \sum_i Y_i Y_i f + \operatorname{div}(Y_i)Y_i(f)$$

then let dY_i, dZ be the the coverctors such that

$$dY_i(Y_j) = \delta_i^j \quad dZ(Z) = 1 \quad dZ(Y_j) = 0 \quad \forall j$$

Then a volume form $\tilde{\nu}$ can be written as

$$\tilde{\nu} = \nu dY_1 \wedge .. \wedge dY_n \wedge dZ$$

for $\nu \in C^\infty(M)$. We start by the definition of divergence:

$$\operatorname{div}(Y_i)\tilde{\nu} = d \circ \iota_{Y_i}\tilde{\nu} = \sum_i (-1)^{i-1} d\nu \wedge dY_1 \wedge .. \wedge \widehat{dY}_i \wedge .. \wedge dZ$$

where the hat means omission.

$$= \sum_i Y_i(\nu) dY_1 \wedge .. \wedge dY_n \wedge dZ = \sum_i \frac{Y_i(\nu)}{\nu}\tilde{\nu}$$

But the volume form $\tilde{\nu}$ is exactly the Riemannian volume of $g_H \oplus g_V$ then $\nu = 1$ and the claim is proved. □

In order to affirm if the sub-Riemannian manifold satisfies the generalized curvature-dimension inequality, in addition to assume (M, d_{CC}) complete, we suppose that the manifold is of Yang-Mills type.
The CD inequality is fulfilled if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $k \geq 0$ and $0 < n \leq \infty$ such that

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{n}(\Delta(f))^2 + \left(\rho_1 - \frac{k}{l}\right)\Gamma(f) + \rho_2 Z(f)$$

for every $f \in C^\infty(M)$ and every $l > 0$.

The purpose of this last part is to turn the equivalence theorem in this setting, we hence have to compute $\mathcal{R}$ and $\mathcal{T}$.
To do so we pass in determining the connection: in this case it can be compute from the proof of proposition 4.20, so if ∇^R is the Levi-Civita connection of $(M, g = g_H \oplus g_V)$ then ∇ is for our adapted frame

- $\nabla_{Y_i} Y_k = \pi_H(\nabla^R_{Y_i} Y_k)$ for $\pi_H \colon \Gamma^\infty(TM) \to \Gamma^\infty(H)$ the projection.
- $\nabla_Z Y_j = [Z, Y_j] = 0$

- $\nabla Z = 0$

Still following the proof we get the formula for the torsion of the connection

$$T(Y_i, Y_j) = -\pi_V([Y_i, Y_j]) = -\pi_V(\sum_k c_{ij}^k Y_k + B_{ij} Z) = -B_{ij} Z$$

Hence the torsion of ∇ is zero if any of the argument is vertical otherwise is equal to the opposite of the magnetic field along Z.

Finally now by definition

$$\mathcal{T}(f) = \sum_i \|T(Y_i, \nabla_H f)\|_V^2 = \sum_i \|T(Y_i, \sum_j Y_j(f) Y_j)\|^2 = \sum_i \|\sum_j Y_j(f) T(Y_i, Y_j)\|^2$$

so

$$\mathcal{T}(f) = \sum_i \left(-\sum_j Y_j(f) B_{ij} \right)^2 = \sum_i \left(\sum_j Y_j(f) B_{ij} \right)^2$$

Also by definition $\mathcal{R}$ is

$$\mathcal{R}(f) = \mathrm{Ric}(\nabla_H f, \nabla_H f) + \sum_{lk} \left(-(\nabla_{Y_l}(T(Y_l, Y_k))f)(Y_k f) + \frac{1}{4}(T(Y_l, Y_k)f)^2 \right)$$

where each term is

- $\mathrm{Ric}(\nabla_H f, \nabla_H f)$ is the Ricci curvature associated to the connection ∇.
- $\sum_k (\sum_l \nabla_{Y_l}(T(Y_l, Y_k))f)) \cdot (Y_k f) = 0$ because by Yang-Mills condition the first factor is zero.
- $\frac{1}{4} \sum_{lk} (T(Y_l, Y_k)f)^2 = \frac{1}{4} \sum_{lk} (-B_{lk} Z(f))^2 = \frac{1}{4} \sum_{lk} (B_{lk} Z(f))^2$

Ultimately $\mathcal{R}$ becomes

$$\mathcal{R}(f) = \mathrm{Ric}(\nabla_H f, \nabla_H f) + \frac{1}{4} \sum_{lk} (B_{lk} Z(f))^2 \tag{5.2}$$

The equivalence theorem can be state down as

Theorem 5.9. *In a complete with respect to the Carnot-Carathéodory distance sub-Riemannian manifold from a magnetic field that satisfies the Yang-Mills condition, it holds the following equivalence:*

$$CD(\rho_1, \rho_2, k, n) \iff \begin{cases} \mathcal{R}(f) \geq \rho_1 g_H(\nabla_H f, \nabla_H f) + \rho_2 (Z(f))^2 \\ \mathcal{T}(f) \leq k g_H(\nabla_H f, \nabla_H f) \end{cases} \qquad \forall f \in C^\infty(M)$$

for constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $k \geq 0$ and n the dimension of the distribution

Example 5.10. Suppose $\tilde{M} = \mathbb{R}^2$ with the Euclidean metric and A a 1-form, then the sub-Riemannian manifold is $M = \mathbb{R}^3$ while the magnetic field $B = dA$ is identified by a smooth function $h \in C^\infty(\tilde{M})$ since

$$B(X_1, X_2) = -B(X_2, X_1) =: h(x_1, x_2)$$

for $\{x_1, x_2\}$ the standard coordinates of $\mathbb{R}^2$ and $\{X_1, X_2\}$ its corresponding frame. With the setting an adapted frame is $\{Y_1, Y_2, Z\}$ where Z is the normal vector with respect to the vertical metric g_V and Y_i is defined by (5.1).
Since the vector fields $\{X_2, X_2\}$ comes from the Euclidean coordinates their Lie bracket vanishes and

$$0 = [X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 \implies c_{12}^1 = c_{12}^2 = 0$$

that in terms of adapted frame means for $ij \in \{1, 2\}$

$$\omega_{ij}^k = 0 \qquad \gamma_{12} = B_{12} = h = -\gamma_{12} \qquad \delta_{ij}^k = 0$$

We suppose that the distribution is bracket generating and we postpone the proof of under which condition it is of Yang-Mills type for later.
The inequality, Γ, Γ^Z don't change from section above, we can investigate how $\mathcal{T}$ and $\mathcal{R}$ turn in this setting.
We can calculate how the operator $\mathcal{T}$ looks like

$$\begin{aligned}
\mathcal{T}(f) &= \sum_i \left(\sum_j Y_j(f) B_{ij} \right)^2 \\
&= (Y_2(f)B_{12})^2 + (Y_1(f)B_{21})^2 = (Y_2(f)h)^2 + (-Y_1(f)h)^2 \\
&= h^2(Y_1(f)^2 + Y_2(f)^2) = h^2 g_H(\nabla_H f, \nabla_H f)
\end{aligned}$$

We compute every term of formula (5.2) separately

$$\frac{1}{4}\sum_{lk}(B_{lk}Z(f))^2 = \frac{1}{4}(B_{12}Z(f))^2 + (B_{21}Z(f))^2 = \frac{1}{4}(hZ(f))^2 + (-hZ(f))^2 = \frac{1}{2}h^2 Z(f)^2$$

Claim 5.11. *The Ricci curvature associated to ∇ is zero*

Proof. Let ∇^R be the Levi-Civita connection of the manifold $(\mathbb{R}^3, g = g_H \oplus g_V)$ and $\{Y_1, Y_2, Z\}$ the adapted frame defined above, which, we recall is g-orthonormal.
Applying Koszul formula for the orthonormal frame, $i, j, k \in \{1, 2\}$

$$2g(\nabla^R_{Y_i} Y_j, Y_k) = g([Y_i, Y_j], Y_k) - g([Y_j, Y_k], Y_i) + g([Y_k, Y_i], Y_j)$$

and substituting $[Y_i, Y_j] = \sum_k c_{ij}^k Y_k + dA(X_i, X_j)Z = B_{ij}Z$ one get for $i, j, k \in \{1, 2\}$

$$2g(\nabla^R_{Y_i} Y_j, Y_k) = 0$$

because Y_k and Z are g-orthogonal.
So $\nabla_{Y_i} Y_j = \pi_H(\nabla^R_{Y_i} Y_j) = 0$.

The Ricci curvature can be compute starting by the Riemannian curvature that is for $P, K, L \in \mathfrak{X}(\mathbb{R}^3)$

$$R(P,K)L = [\nabla_X, \nabla_K]L - \nabla_{[X,K]}L$$

Naturally if any of the vector fields is vertical the curvature vanishes. Focusing only in the case of horizontal vector fields we get for $i, j, k \in \{1, 2\}$

$$R(Y_i, Y_j)Y_k = [\nabla_{Y_i}, \nabla_{Y_j}]Y_k - \nabla_{[Y_i,Y_j]}Y_k = \nabla_{B_{ij}Z}Y_k = B_{ij}\nabla_Z Y_k = 0$$

The curvature is the zero tensor and its trace, i.e. the Ricci curvature, is of course zero.

□

We can try to compute under which condition the manifold is of Yang-Mills type, but before we need to compute the covariant derivative of the torsion. Let $\tilde{T} \in \Gamma^\infty(T^*M^{\otimes 2} \otimes TM)$ be the torsion tensor field such that

$$\tilde{T}(X, Y, df) = df(T(X, Y)) = T(X, Y)(f)$$

then it holds

$$\tilde{T}(Y_i, Y_j, df) = -B_{ij}Z(f) = -B \otimes Z(Y_i, Y_j, df)$$

that means

$$\tilde{T} = -B \otimes Z$$

then by the definition of covariant derivative

$$\nabla_{Y_i}\tilde{T} = -\nabla_{Y_i}B \otimes Z - B \otimes \nabla_{Y_i}Z = -\nabla_{Y_i}B \otimes Z$$

this means that we only care about the covariant derivative of the magnetic field. So let K be an horizontal vector field

$$\begin{aligned}
\sum_i (\nabla_{Y_i}B)(Y_i, K) &= \sum_i Y_i(B(Y_i, K)) - B(\nabla_{Y_i}Y_i, K) - B(Y_i, \nabla_{Y_i}K) \\
&= \sum_{ij} Y_i(K^j B(Y_i, Y_j)) - B(Y_i, Y_i(K^j)Y_j + K^j \nabla_{Y_i}Y_j) \\
&= \sum_{ij} Y_i(K^j)B(Y_i, Y_j) + K^j Y_i(B(Y_i, Y_j)) - Y_i(K^j)B(Y_i, Y_j) \\
&= \sum_{ij} K^j Y_i(B_{ij})
\end{aligned}$$

In order to the initial sum be zero for any choice of horizontal vector field K, it must holds for any $j \in \{1, 2\}$

$$\sum_i Y_i(B_{ij}) = 0$$

that in our setting becomes

$$Y_1(h) = 0 \qquad Y_2(h) = 0$$

so the manifold is of Yang-Mills type if the magnetic field h is a constant different from zero, otherwise the distribution is not bracket generating.
The condition $\mathcal{T}(f) \leq k g_H(\nabla_H f, \nabla_H f)$ get simplified in

$$h^2 g_H(\nabla_H f, \nabla_H f) \leq k g_H(\nabla_H f, \nabla_H f)$$

Finally the equivalence theorem becomes

Theorem 5.12. *For $(\mathbb{R}^3, H, g_H)$ sub-Riemannian manifold from the magnetic field $h \in \mathbb{R} \setminus \{0\}$ that is complete with respect to the Carnot-Carathéodory distance it holds the following equivalence:*

$$CD(\rho_1, \rho_2, k, 2) \iff (\frac{1}{2}h^2 - \rho_2)Z(f)^2 \geq \rho_1 g_H(\nabla_H f, \nabla_H f) \qquad \forall f \in C^\infty(\mathbb{R}^3)$$

for constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $k \geq h^2$.

So for example we deduce that the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, k, 2)$ for this manifold holds when $\rho_1 = 0$, $0 \leq \rho_2 \leq \frac{1}{2}h^2$ and $k \geq h^2$.

Bibliography

[1] Andrei A Agrachev, Davide Barilari, and Ugo Boscain. Introduction to Riemannian and sub-Riemannian geometry. 2014.

[2] Davide Barilari, Ugo Boscain, and Mario Sigalotti. *Geometry, analysis and dynamics on sub-Riemannian manifolds*, volume 1. EMS, 2016.

[3] Davide Barilari, Ugo Boscain, and Mario Sigalotti. *Geometry, analysis and dynamics on sub-Riemannian manifolds*, volume 2. EMS, 2016.

[4] Fabrice Baudoin and Nicola Garofalo. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *Journal of the European Mathematical Society*, 19(1):151–219, 2016.

[5] A. Bellaïche, A. Bellaiche, J.J. Risler, and J.J. Risler. *Sub-Riemannian Geometry*. Progress in Mathematics. Springer Basel AG, 1996.

[6] Marco Bramanti. On the proof of Hörmander's hypoellipticity theorem. *Le Matematiche*, 75(1):3–26, 2020.

[7] Jeff Cheeger and Tobias H Colding. Lower bounds on ricci curvature and the almost rigidity of warped products. *Annals of mathematics*, 144(1):189–237, 1996.

[8] Xianzhe Dai and Guofang Wei. Comparison geometry for Ricci curvature. *preprint*, 2012.

[9] Manfredo Perdigao Do Carmo and J Flaherty Francis. *Riemannian geometry*, volume 6. Springer, 1992.

[10] Robert Everist Greene and Hung-Hsi Wu. *Function theory on manifolds which possess a pole*, volume 699. Springer, 2006.

[11] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119:147–171, 1967.

[12] Enrico Le Donne. Lecture notes on sub-Riemannian geometry from the Lie group viewpoint, 2021. cvgmt preprint.

[13] John M Lee. *Riemannian manifolds: an introduction to curvature*, volume 176. Springer Science & Business Media, 2006.

[14] Alessandro Minuzzo. On magnetic fields and sub-Riemannian geodesics. Master's thesis, University of Padua, 2016.

[15] Peter Petersen. *Riemannian geometry*, volume 171. Springer, 2016.

[16] Jean-Pierre Serre. *Lie algebras and Lie groups: 1964 lectures given at Harvard University*. Springer, 2009.