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Low-energy effective field theories for lepton universality violation in B decays

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Contents

Introduction

The Standard Model (SM) of particle physics has proven to be highly successful in describing elementary particles and their interactions. The recent observation of a new state compatible with the properties of the Higgs boson at the Large Hadron Collider (LHC) put in place the main missing block for its experimental validation. That notwithstanding, it is widely believed that the SM is the low-energy version of a more fundamental theory. According to the naturalness hypothesis, New Physics (NP) should occur already at the TeV scale, which is being directly tested at the LHC. No NP signal was found during the first run, that achieved a center of mass energy of 7 TeV.
There are however hints for NP in the flavor sector. Recent data in B physics point towards lepton flavor universality (LFU) violation in semi-leptonic B decays, both in charged $b \to c$ transitions as well as in $b \to s$ neutral currents. Additional tensions between SM predictions and experimental measurements in semi-leptonic B decays arise for example in the $B \to K^* \mu^+ \mu^-$ decay, especially in the angular observable P_5'. These deviations from the SM have stimulated an ongoing discussion about possible NP interpretations.
The present work analyses B anomalies and their compatibility with other low-energy observables in a model-independent way. We assume that NP originates at a scale Λ much higher than the electroweak symmetry breaking (EWSB) scale and describe its behaviour below Λ by means of an Effective Field Theory (EFT) approach, as outlined in the first chapter.
In the second chapter we write the effective NP Lagrangian at the scale Λ, the aim being to extend the analysis performed in [1, 2] by enlarging the basis of effective operators in this Lagrangian from two to five well-motivated semi-leptonic operators. Then we build the low-energy NP Lagrangian by computing the quantum effects induced at the GeV scale; a key tool in this sense is provided by renormalization group equations (RGE).
In the third and last chapter we consider the phenomenological implications of the derived NP Lagrangian. After revisiting B anomalies, we study observables receiving NP contribution at loop level, which include Z-pole observables as well as LFU and Lepton Flavor (LF) violating effects in τ decays. Finally we consider a particular scenario motivated by global fits.

Chapter 1

An effective approach to B anomalies

SM predictions have been experimentally verified to a great degree of precision over a wide range of phenomena. However, we also know that the SM is not a complete theory: neutrino oscillations, baryon asymmetry and the evidence for dark matter cannot be explained within the SM. They are indications for the existence of physics beyond the SM.
It is commonly accepted that the SM constitutes a low-energy EFT of a more fundamental theory; even though we still do not know how to extend it, the solution to the hierarchy problem points towards the existence of new degrees of freedom in the TeV range.
Two complementary strategies are currently being used to investigate the TeV scale. Experiments at the high-energy frontier, performed at the LHC at CERN, aim at directly producing and detecting new heavy degrees of freedom, while high-precision experiments investigate virtual effects from NP particles in lower-energy processes. The investigation of B physics fits into this second effort.

1.1 The flavor sector of the Standard Model

The SM of elementary particles is based on the gauge group $G_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$. The matter content consists of fifteen fermion fields and a scalar, the Higgs field [1]

$$L_{iL}(1,2)_{-1/2} \quad e_{iR}(1,1)_{-1} \quad q_{iL}(3,2)_{+1/6} \quad u_{iR}(3,1)_{+2/3} \quad d_{iR}(3,1)_{-1/3} \quad \phi(1,2)_{+1/2}\,, \tag{1.1}$$

where every fermion field appears in three replicas called *flavors*, labelled by the index $i = 1, 2, 3$ and the $SU(2)_L$ doublets L_{iL} and q_{iL} read $L_{iL} = (\nu_{iL}, e_{iL})$ and $q_{iL} = (u_{iL}, d_{iL})$.
The SM Lagrangian is given by the sum of three terms,

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{SM}}^{\text{gauge}} + \mathcal{L}_{\text{SM}}^{\text{Higgs}} + \mathcal{L}_{\text{SM}}^{\text{Yukawa}}\,, \tag{1.2}$$

[1] We denote fermion fields by $\psi(a,b)_Y$, where a, b and Y are the representations under $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, respectively.

where $\mathcal{L}_{\text{SM}}^{\text{gauge}}$, $\mathcal{L}_{\text{SM}}^{\text{Higgs}}$ and $\mathcal{L}_{\text{SM}}^{\text{Yukawa}}$ are the gauge, Higgs and Yukawa Lagrangians, respectively. They are defined as

$$\mathcal{L}_{\text{SM}}^{\text{gauge}} = -\frac{1}{4}G_{\mu\nu}^{A}G^{A} - \frac{1}{4}W_{\mu\nu}^{a}W^{\mu\nu,a} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sum_{\psi'}\bar{\psi}' i\not{D}\,\psi' \,, \tag{1.3}$$

$$\mathcal{L}_{\text{SM}}^{\text{Higgs}} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2(\phi^\dagger\phi) - \lambda(\phi^\dagger\phi)^2 \,, \tag{1.4}$$

$$\mathcal{L}_{\text{SM}}^{\text{Yukawa}} = -\left(\bar{d}_R' Y_d \phi^\dagger q_L' + \bar{u}_R' Y_u \tilde{\phi}^\dagger q_L' + \bar{e}_R' Y_e \phi^\dagger \ell_L' + \text{h.c.}\right) \,, \tag{1.5}$$

where primed fields denote fields in the interaction basis.
$\mathcal{L}_{\text{SM}}^{\text{gauge}}$ has a large global $U(3)^5 = U(3)_\ell^2 \times U(3)_q^3$ flavor symmetry, corresponding to the independent unitary rotations of the fermion fields in flavor space. This symmetry is explicitly broken by the Yukawa Lagrangian, as the couplings $Y_{d,u,e}$ are in general non-diagonal matrices. As a consequence, the residual flavor symmetry group of $\mathcal{L}_{\text{SM}}$ is

$$G_f = U(1)_B \times U(1)_{L_e} \times U(1)_{L_\mu} \times U(1)_{L_\tau} \,, \tag{1.6}$$

where $U(1)_B$ and $U(1)_{L_{e_i}}$ are associated with baryon number and lepton family number conservation, respectively.
In order to diagonalize each Yukawa coupling, two independent unitary matrices are needed

$$Y_u = R_u Y_u^D V_u^\dagger \qquad Y_d = R_d Y_d^D V_d^\dagger \qquad Y_e = R_e Y_e^D V_e^\dagger \,, \tag{1.7}$$

where Y_u^D, Y_d^D and Y_e^D are diagonal and R and V are unitary matrices. For practical purposes it is often convenient to work in the basis where the Yukawas are diagonal, the so-called *mass basis* (denoted by unprimed fields) which is related to the interaction basis by

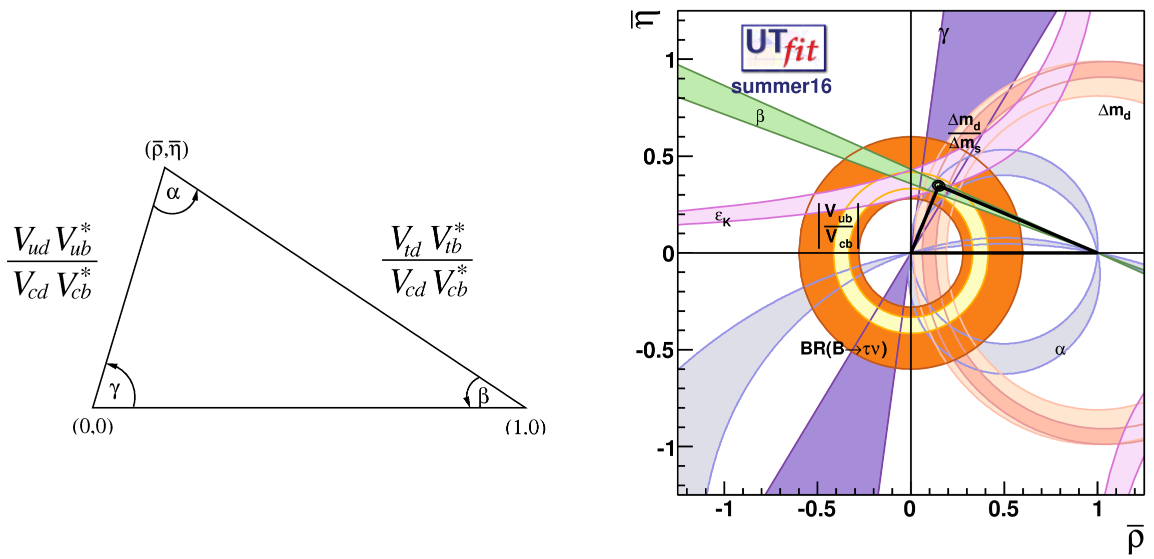
$$\begin{aligned} u_L' &= V_u u_L & d_L' &= V_d d_L & L_L' &= U_e L_L, \\ u_R' &= R_u u_R & d_R' &= R_d d_R & e_R' &= R_e e_R \,. \end{aligned} \tag{1.8}$$

The rotations of the lepton doublet and of the right-handed quarks do not affect $\mathcal{L}_{\text{SM}}^{\text{gauge}}$ and lead to no phenomenological consequences. The same does not hold for left-handed quarks: since in general $Y_u \neq Y_d$, the up and down components of the quark doublet have to be rotated with different matrices in order to diagonalize $\mathcal{L}_{\text{SM}}^{\text{Yukawa}}$. As a result, $\mathcal{L}_{\text{SM}}^{\text{gauge}}$ is not unchanged. In particular, the only term feeling the change of basis is the charged-current interaction involving quarks, which arises from the term $\bar{q}_{iL} i\not{D} q_{iL}$. We have

$$\mathcal{L}_{\text{SM}}^{\text{CC}}|_{\text{quarks}} = -\frac{g_2}{\sqrt{2}}W_\mu^+(\bar{u}_{iL}'\gamma^\mu d_{iL}') + \text{h.c.} = -\frac{g_2}{\sqrt{2}}W_\mu^+(V_{\text{CKM}})_{ij}(\bar{u}_{iL}\gamma^\mu d_{jL}) + \text{h.c.} \,, \tag{1.9}$$

where $V_{\text{CKM}} = V_u^\dagger V_d$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. As we can see, $\mathcal{L}_{\text{SM}}^{\text{CC}}|_{\text{quarks}}$ is flavor-diagonal in the interaction basis; by switching to the mass basis, tree-level Flavor Changing Charged Current (FCCC) transitions arise due to the presence of V_{CKM}.

The CKM matrix is of crucial importance in flavor physics, because it is the only source of flavor-changing transitions in the SM. It can be parametrized in terms of four physical parameters, three angles and a phase. The most used parametrizations are the standard



(a) (b)

Figure 1.1: CKM unitarity triangle and allowed region in the $\bar{\rho}, \bar{\eta}$ plane as obtained by the UTfit collaboration [4].

parametrization, which utilizes three angles θ_{ij} and a complex phase δ, and the Wolfenstein parametrization, where the CKM matrix elements are expanded in powers of the small parameter $\lambda = |V_{us}| \approx 0.22$ [3]. The Wolfenstein parametrization has the advantage of exhibiting the strong hierarchy between the CKM matrix elements in a transparent way. Explicitly it is given by

$$V_{\text{CKM}} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\,, \tag{1.10}$$

where A, ρ, and η are real parameters of order 1. Sometimes the rescaled variables $\bar{\rho}$ and $\bar{\eta}$ are used:

$$\bar{\rho} = \rho\left(1-\frac{\lambda^2}{2}\right) + \mathcal{O}(\lambda^4) \qquad \bar{\eta} = \eta\left(1-\frac{\lambda^2}{2}\right) + \mathcal{O}(\lambda^4)\,. \tag{1.11}$$

Since V_{CKM} is unitary, the following relations hold:

$$\sum_{k=1\dots3} V_{ik}^* V_{ki} = 1 \qquad\qquad \sum_{k=1\dots3} V_{ik}^* V_{kj\neq i}\,. \tag{1.12}$$

These relations are a specific feature of the SM, thus their experimental verification is a powerful consistency check of the model. In particular, the relation

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \qquad \leftrightarrow \qquad [\bar{\rho} + i\bar{\eta}] + [(1-\bar{\rho}) - i\bar{\eta}] + 1 = 0\,, \tag{1.13}$$

is phenomenologically very interesting, because it involves the sum of three terms of the same order in λ. It is usually represented as a triangle in the complex $(\bar{\rho}, \bar{\eta})$ plane (see figure 1.1(a)), the so-called CKM unitarity triangle. The angles and sides of the triangle can be extracted from suitable flavor observables, thus the consistency of eq. (1.13) can be experimentally tested. Since the values of λ and A are determined with good accuracy [4]

$$\lambda = 0.22497 \pm 0.00069 \qquad\qquad A = 0.833 \pm 0.012\,, \tag{1.14}$$

all the the observables sensitive to the CKM matrix elements can be expressed in terms of the remaining parameters $\bar{\rho}$ and $\bar{\eta}$. The resulting constraints are shown in fig. 1.1(b); the graph shows that they are all consistent with a unique value of $\bar{\rho}$ and $\bar{\eta}$ [4]:

$$\bar{\rho} = 0.153 \pm 0.013 \qquad \bar{\eta} = 0.343 \pm 0.011. \tag{1.15}$$

It is therefore clear that the SM provides an extremely good description of flavor physics.

We have seen that FCCC arise already at tree-level in the SM. On the other hand, Flavor Changing Neutral Current (FCNC) processes are highly suppressed in the SM: besides arising at one loop, these processes always involve at least one off-diagonal element of the CKM matrix and are further suppressed by the Glashow-Iliopoulos-Maiani (GIM) mechanism [5]. Therefore, given the high suppression of FCNCs in the SM, FCNC transitions represent golden channels to look for NP effects.
Among the most interesting FCNC processes there are the semi-leptonic B decays, based on the underlying transitions $b \to s\ell^+\ell^-$ and $b \to s\bar{\nu}\nu$, as well as the purely leptonic decays $B_q \to \ell^+\ell^-$ $(q = s, d)$ and the radiative decays $b \to s\gamma$ and $b \to d\gamma$ [2].

1.2 Present status of B anomalies

In the last fifteen years the experimental study of B decays has been carried out at the LHC and at the *B-factories*[3] PEP II and KEKB. The two related experiments, BaBar and Belle, ceased operating in 2008 and 2010 respectively; the upgrade of Belle, Belle II, will start collecting data in early 2018. As to the LHC, three experiments are involved in the study of B physics: ATLAS, CMS and LHCb, where the latter was specially designed for studying the production and the decay of b and c hadrons. An LHCb upgrade is planned for 2019-2020.
Assuming that NP originates at a scale Λ in the TeV range, its effects on weak B decays are suppressed by inverse powers of Λ. It is therefore reasonable to look for NP either in processes which are very suppressed or forbidden within the SM or in observables predicted with high precision in the SM.
An interesting class of B-physics observables falling into the second category is given by ratios of partial widths of semi-leptonic decays with different flavors of leptons in the final state. These ratios, also called R-ratios, are very clean observables, because hadronic uncertainties affecting

[2]For a comprehensive review on rare B decays, see [6].

[3]B factories are asymmetric e^+e^- colliders built with the explicit purpose of producing a huge number of B mesons. They operated at the $\Upsilon(4s)$ resonance, which immediately decays into a B meson-antimeson pair.

the individual branching ratios cancel in the ratio [6]. They test LFU and therefore constitute powerful tests of the SM. In particular we consider the observables $R^{\mu/e}_{K^{(*)}}$ and $R^{\tau/\ell}_{D^{(*)}}$, defined as

$$\begin{aligned} R^{\mu/e}_{K^{(*)}} &= \frac{\mathcal{B}(B \to K^{(*)}\mu^+\mu^-)}{\mathcal{B}(B \to K^{(*)}e^+e^-)}\,, \\ R^{\tau/\ell}_{D^{(*)}} &= \frac{\mathcal{B}(\bar{B} \to D^{(*)}\tau\bar{\nu})_{\rm exp}/\mathcal{B}(\bar{B} \to D^{(*)}\tau\bar{\nu})_{\rm SM}}{\mathcal{B}(\bar{B} \to D^{(*)}\ell\bar{\nu})_{\rm exp}/\mathcal{B}(\bar{B} \to D^{(*)}\ell\bar{\nu})_{\rm SM}} \qquad \ell = e, \mu\,, \end{aligned} \tag{1.16}$$

where $R^{\mu/e}_{K^{(*)}}$ and $R^{\tau/\ell}_{D^{(*)}}$ test μ-e and τ-ℓ ($\ell = e, \mu$) universality, respectively.
In the SM, weak interactions are lepton-flavor universal, because gauge bosons couple in the same way to leptons with different flavors. The only sources of LFU violation (LFUV) are the masses of neutrinos, the masses of the charged leptons and their couplings with the Higgs boson, which have a negligible effect on R-ratios. Hence, $R^{\mu/e}_{K^{(*)}}$ and $R^{\tau/\ell}_{D^{(*)}}$ are expected to be unity. It has been observed that, in principle, radiative QED corrections could induce up to 10% effects in $R^{\mu/e}_{K}$ and $R^{\mu/e}_{K^*}$. However, in practice, the explicit analysis performed in [7] shows that these corrections do not exceed ~ 0.03 in the region $1\ \text{GeV}^2 < q^2 < 6\ \text{GeV}^2$, where q^2 is the di-lepton invariant mass. Therefore, any deviation of $R^{\mu/e}_{K^{(*)}}$ from unity exceeding 1% in this region would constitute a clear signal of NP.
Most notably, recent data point towards LFUV both in charged-current as well as in neutral-current transitions. The statistically most significant data are:

$$\begin{aligned} & R^{\tau/\ell}_{D} = 1.34 \pm 0.17\,, \\ & R^{\tau/\ell}_{D^*} = 1.23 \pm 0.07\,, \\ & R^{\mu/e}_{K}\Big|_{q^2\in[1,6]\text{GeV}^2} = 0.745^{+0.090}_{-0.074} \pm 0.036\,, \\ & R^{\mu/e}_{K^*}\Big|_{q^2\in[1.1,6]\text{GeV}^2} = 0.685^{+0.113}_{-0.069} \pm 0.047\,, \\ & R^{\mu/e}_{K^*}\Big|_{q^2\in[0.045,1.1]\text{GeV}^2} = 0.660^{+0.110}_{-0.070} \pm 0.024\,, \end{aligned} \tag{1.17}$$

where the values for the charged-current anomaly $R^{\tau/\ell}_{D^{(*)}}$ follow from the average [8] of LHCb, Belle and BaBar data [9, 10, 11]. The results for $R^{\mu/e}_{K}$ and $R^{\mu/e}_{K^*}$ come from the LHCb collaboration [12, 13]. In particular, the result for $R^{\mu/e}_{K^*}$ was presented at CERN very recently [13].
B anomalies are not the only tensions in semi-leptonic B decays; recent experimental results have shown other deviations from SM predictions in processes based on the semi-leptonic transition $b \to s\mu^+\mu^-$. The most prominent deviation concerns the angular distribution of $B \to K^*\mu^+\mu^-$, in particular the observable P_5', which exhibits a 3σ deviation from the SM expectation value [14]. Additional tensions arise in the branching ratios $\mathcal{B}(B \to K\mu^+\mu^-)$ and $\mathcal{B}(B \to \phi\mu^+\mu^-)$ [15, 16].

In the present work we investigate B anomalies. To this end we need practical tools to describe NP contributions to observables; rather than building an explicit extension of the SM, we choose an EFT approach.
In the next section, we discuss some general aspects concerning EFTs, and then we focus on their role in describing the SM and NP contribution to semi-leptonic B decays.

1.3 Effective field theories

EFTs are a powerful and versatile tool in quantum field theory, because they provide a general framework to tackle problems involving very different physical scales. A classical example is given by processes whose typical energy E is much smaller than the energy scale of the interaction responsible for the process, which is usually set by the mass M of the corresponding mediator. This is exactly the case for weak B decays, where $E \sim m_b$, which is far below the EW scale as well as the NP scale $\Lambda \sim$ TeV. Therefore, B decays can be analysed by means of an EFT approach both in the SM and in its NP extensions.
Two main tools are used to address this kind of problems: operator product expansion (OPE), and renormalization group (RG). The idea behind OPE is that, if $E \ll M$, for the purpose of computing the amplitude the full theory below M can be replaced by an effective Lagrangian, made up by a series of local operators of dimension $d > 4$

$$\mathcal{L}_{\text{eff}} = \sum_{d>4} \frac{1}{M^{4-d}} \sum_i \mathcal{C}_i^{(d)} Q_i^{(d)} \,, \tag{1.18}$$

where the $Q_i^{(d)}$ are all the possible d-dimensional operators compatible with the symmetry of the theory that can be built using fields lighter than M. In going from the full theory to the OPE series the heavy mediator is removed as a dynamical degree of freedom; its effect is encoded in the dimensionless coefficients $\mathcal{C}_i^{(d)}$, called *Wilson coefficients*.
For our purposes, we can safely truncate the series at $d = 6$, since higher-dimensional operators have a faster decoupling with the scale M and are therefore subdominant. As a result, our effective lagrangian reads

$$\mathcal{L}_{\text{eff}} = \frac{1}{M} \sum_i \mathcal{C}_i^{(5)} Q_i^{(5)} + \frac{1}{M^2} \sum_i \mathcal{C}_i^{(6)} Q_i^{(6)} \,. \tag{1.19}$$

Hereafter we will omit the superscript (6) and Q_i will always denote a six-dimensional operator.
Now, let us suppose that the full theory is known and that we want to find an effective Lagrangian $\mathcal{L}_{\text{eff}}$ reproducing the amplitude computed in the full theory at a given order. Starting from the general effective Lagrangian (1.19), we determine the Wilson cofficients C_i by *matching* $\mathcal{L}_{\text{full}}$ into $\mathcal{L}_{\text{eff}}$, namely by requiring

$$\mathcal{M}_{full} = \mathcal{M}_{eff} = \frac{1}{M^2} \sum_i \mathcal{C}_i(\mu) \left\langle f \right| Q_i(\mu) \left| i \right\rangle \tag{1.20}$$

at the desidered order. Focusing on the one-loop case, we will find something of the form

$$\mathcal{C}_i(\mu) = \mathcal{C}_i(M) + k \frac{\alpha}{4\pi} \ln \frac{M}{\mu} \,, \tag{1.21}$$

where $\mathcal{C}_i(M)$ is the coefficient we would find by doing a tree-level matching, α is the strenght of the interaction originating one-loop corrections and k is some constant. This procedure amounts to computing the Wilson coefficients in ordinary perturbation theory; clearly it has to be repeated for a sufficient number of processes in order to find all the $\mathcal{C}_i$.

The next step is to evaluate the Wilson coefficients at the energy scale of the process we are interested in, in order to compute its physical amplitude. We have to be careful in simply substituting $\mu = E$ in (1.21), because, due to the large difference between E and M, a large logarithm $\alpha \ln \frac{M}{E} \sim 1$ might arise. If this is the case, $\alpha \ln \frac{M}{E}$ is not a good expansion parameter and ordinary perturbation theory breaks down.
The problem is solved by switching from ordinary to RG-improved perturbation theory. To understand this point, we observe that the matrix elements $\langle Q_i \rangle$ in $\mathcal{M}_{\text{eff}}$ have in general two kinds of divergences: a divergence which can be fixed by standard wavefunction or coupling renormalization, and another type of divergence, originating from the fact that we are using operators with dimension higher than four. To fix this second divergence, *operator renormalization* is required.
Defining the matrix Z_Q as

$$Q_i^{(0)}(q) = {Z_Q^{-1}}_{ij}\, Q_j(q)$$

and remembering that $q^{(0)} = \sqrt{Z_\psi}\, q$, the bare Lagrangian can be rewritten as

$$\mathcal{L}_{\text{eff}} = \mathcal{C}_i Q_i^{(0)}(q^{(0)}) = \mathcal{C}_i Z_\psi^2\, {Z_Q^{-1}}_{ij}\, Q_j(q) = \mathcal{C}_i Q_i + \left(Z_\psi^2\, {Z_Q^{-1}}_{ij} - \delta_{ij} \right) \mathcal{C}_i Q_j\,. \tag{1.22}$$

Alternatively, we can renormalize Wilson coefficients, using $\mathcal{C}_i^{(0)} = Z_{\mathcal{C}\,ij}\, \mathcal{C}_j$ with $Z_{\mathcal{C}\,ij} = {Z_Q^{-1}}_{ji}$. Once expressed the bare Lagrangian as the sum of renormalized coefficients and fields plus counterterms, we find $Z_{Q\,ij}$ by requiring the corresponding countertem to cancel the "operator divergences" in $\mathcal{M}_{\text{eff}}$. Note that, as a rule, the one-loop matrix element $\langle Q_i \rangle$ in $\mathcal{M}_{\text{eff}}$ has a divergence proportional not just to Q_i itself, but to different operators Q_j as well; the latter need a counterterm proportional to Q_j to be fixed. For this reason $Z_{Q\,ij}$ is in general a matrix.
At this point we compute the anomalous dimension matrix, defined as

$$\gamma = Z_Q^{-1} \mu \frac{d}{d\mu} Z_Q\,, \tag{1.23}$$

and, by asking the bare Wilson coefficients to be μ-independent, we find that the $\mathcal{C}_i$ obey the following renormalization group equation

$$\mu \frac{d}{d\mu} \mathcal{C}_i(\mu) = \gamma_{ji}\, \mathcal{C}_j(\mu)\,. \tag{1.24}$$

Formally the solution can be written in terms of an evolution matrix $U(\mu, M)$ as $\mathcal{C}_i(\mu) = U(\mu, M)_{ij}\, \mathcal{C}_j(M)$ [17, 18]. $U(\mu, M)$ obeys the same equation as $\vec{\mathcal{C}_i}$ and is given by

$$U(\mu, M)_{ij} = T_g \exp \left[\int_{g(M)}^{g(\mu)} dg\, \frac{\gamma^T(g)}{\beta(g)} \right]_{ij}\,. \tag{1.25}$$

Equation (1.24) is solved using as initial condition $\mathcal{C}_i(M)$, computed by matching the full theory into the effective one. The choice $\mu = M$ enables us to avoid large logarithms, so that ordinary perturbation theory can be used. Then RGE are used to run Wilson coefficients from the high-energy scale M to the energy scale of the process, E; the solution for $C_i(E)$ resums logarithmic corrections $\left(\alpha \ln \frac{M}{E}\right)^n$ to all orders n, yielding the LLA approximation [17]. By re-expanding the result in powers of α, the result in ordinary perturbation theory is recovered. Equation (1.24) shows us another important feature of effective field theories: even if at scale M only a single Wilson coefficient $\mathcal{C}_i$ is non-vanishing, for $\mu < M$ in general new Wilson coefficients will be non zero, hence new operators will arise. This phenomenon is called *operator mixing* and it is due to the fact that generally the anomalous dimension matrix γ_{ij} is not diagonal.

1.3.1 EFT in NP

A common way to describe NP effects on observables is to consider the SM as the renormalizable part of an effective theory, obtained by integrating out heavy degrees of freedom arising at the scale Λ. The main advantage of the EFT approach is that it provides a very general and model-independent description of NP effects by means of a limited number of parameters.
Assuming that NP originates at a scale Λ above the EWSB scale, which we identify with a mass m_{EW}[4], a field theory valid above Λ should be invariant under the SM gauge group $G_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_{\text{em}}$ and contain all SM particles. Since NP degrees of freedom can be integrated out below Λ, in the window below Λ and above m_{EW} the theory can be described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{NP}} \, , \tag{1.26}$$

where $\mathcal{L}_{\text{NP}}$ is the effective Lagrangian describing NP; truncating the OPE series at $d = 6$, we have[5]

$$\mathcal{L}_{\text{NP}} = \frac{1}{\Lambda^2} \sum_i \mathcal{C}_i Q_i \, . \tag{1.27}$$

The operators Q_i are written in terms of SM fields and are invariant under G_{SM}. Reference [19][6] provides the full list of six-dimensional operators compatible with gauge symmetry and conserving baryon number: barring flavor structure and hermitian conjugation, there are 59 of them.

1.3.2 Effective Lagrangians for B anomalies

Effective field theories are widely employed already within the SM. Even if in this case the full theory is known, the EFT approach has the practical advantage of making computations of amplitudes easier and to resum possible large logarithms.
One of the most significant applications of EFT in the SM concerns the description of weak meson decays. As we already observed, these processes are characterized by the presence of different energy scales, since their typical energy lies far below the masses of the W and Z boson.

[4] In our framework we identify $\frac{v}{\sqrt{2}}$, m_t, m_{W} and m_{Z} with a common mass m_{EW}.

[5] There is only one five-dimensional operator, that is the Weinberg operator, which is responsible for the generation of neutrino masses. Since it doesn't play any role in our analysis, we safely neglect it. On the other hand, out of the whole set of six-dimensional operators, we will focus on four-fermion operators.

[6] For a previous analysis, see [20].

It is therefore reasonable to describe them with a Lagrangian where the massive gauge bosons have been integrated out. The effective Lagrangian we use to compute tree-level amplitudes in substitution to $\mathcal{L}_{\text{SM}}$ is the well-known Fermi Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}}\left(J^{\mu,0}_{\text{SM}} J^0_{\mu,\text{SM}} + J^{\mu,+}_{\text{SM}} J^-_{\mu,\text{SM}}\right) , \tag{1.28}$$

where W and Z are not dynamical degrees of freedom, but their presence is taken into account in the Wilson coefficient, G_{F} in this case.
However, EFTs show their full potential when it comes to taking into account higher-order corrections, for example QCD or electroweak corrections. Following the steps outlined above, we start from a general $U(1)_{em}$ invariant effective Lagrangian. We determine the Wilson coefficients in perturbation theory by matching the full theory into the effective one at the EW scale at one loop. After computing the anomalous dimension matrix, the $\mathcal{C}_i$ are then run down to the low-energy scale with the help of RGE.
Let us consider the B decays on which the neutral-current anomalies $R^{\mu/e}_{K^{(*)}}$ are based, namely $B \to K^{(*)}\ell^+\ell^-$. In the SM, $b \to s$ transitions are described by the following effective Lagrangian, which contains the operators contributing to the semi-leptonic decays $b \to s\ell^+\ell^-$ and $b \to s\bar{\nu}\nu$ and to the radiative decay $b \to s\gamma$ at the quark level [6, 17, 21]

$$\mathcal{L}_{\text{eff}} = -\frac{4G_{\text{F}}}{\sqrt{2}}\left(\lambda^u_{bs}\sum_{i=1}^{2}\mathcal{C}^i O^i_u + \lambda^c_{bs}\sum_{i=1}^{2}\mathcal{C}^i O^i_c - \lambda^t_{bs}\sum_{i=3}^{10}\mathcal{C}^i O^i - \lambda^t_{bs}\mathcal{C}^\nu O^\nu + \text{h.c.}\right) , \tag{1.29}$$

where $\lambda^p_{bs} = V_{pb}V^*_{ps}$. $\mathcal{L}_{\text{eff}}$ contains different types of operators: the current-current operators O^1_p and O^2_p, the QCD penguin operators O^{3-6} (where we sum over $p = u, d, s, c, b$), the electromagnetic and chromomagnetic dipole operators O^7 and O^8 and the semi-leptonic operators O^9, O^{10} and O^ν:

$$\begin{aligned}
O^1_p &= (\bar{s}_L\gamma_\mu T^A p_L)(\bar{p}_L\gamma^\mu T^A b_L) & O^2_p &= (\bar{s}_L\gamma_\mu p_L)(\bar{p}_L\gamma^\mu b_L) \\
O^3 &= (\bar{s}_L\gamma_\mu b_L)\sum_p(\bar{p}\gamma^\mu p) & O^4 &= (\bar{s}_L\gamma_\mu T^A b_L)\sum_p(\bar{p}\gamma^\mu T^A p) \\
O^5 &= (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\rho b_L)\sum_p(\bar{p}\gamma^\mu\gamma^\nu\gamma^\rho p) & O^6 &= (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\rho T^A b_L)\sum_p(\bar{p}\gamma^\mu\gamma^\nu\gamma^\rho T^A p) \\
O^7 &= \frac{e}{16\pi^2}m_b(\bar{s}_L\sigma^{\mu\nu}b_R)F_{\mu\nu} & O^8 &= \frac{g_s}{16\pi^2}m_b(\bar{s}_L\sigma^{\mu\nu}T^A b_R)G^A_{\mu\nu} \\
O^9 &= \frac{e^2}{16\pi^2}(\bar{s}_L\gamma_\mu b_L)(\bar{e}_i\gamma^\mu e_i) & O^{10} &= \frac{e^2}{16\pi^2}(\bar{s}_L\gamma_\mu b_L)(\bar{e}_i\gamma^\mu\gamma_5 e_i) \\
O^\nu &= \frac{e^2}{8\pi^2}(\bar{s}_L\gamma_\mu b_L)(\bar{\nu}_{iL}\gamma^\mu\nu_{iL}) \,.
\end{aligned} \tag{1.30}$$

The sum over repeated flavor indices in the semi-leptonic operators is understood.
In principle we should consider also the chirally-flipped version of the dipole and the semi-leptonic operators; however, they can be neglected, because the correspondent Wilson coefficients are chirally-suppressed due to the $(V - A)(V - A)$ structure of the charged current in the SM.
The operators relevant to the semi-leptonic process $B \to K\ell^+\ell^-$ are O^9 and O^{10}, hence the

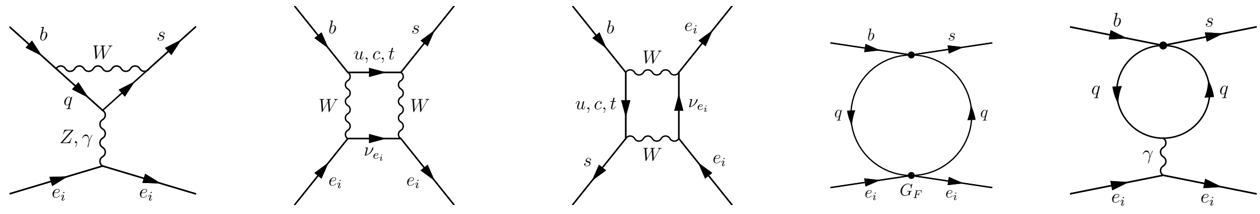


Figure 1.2: Diagrams entering the computation of $\mathcal{C}^9$ and $\mathcal{C}^{10}$ in the full (a)-(c) and in the effective theory (d)-(e).

Lagrangian we need to describe $R_K^{\mu/e}$ in the SM is

$$\mathcal{L}_{\text{eff}}^{\text{NC,SM}} = \frac{4G_F}{\sqrt{2}}\lambda_{bs}^t\left(\mathcal{C}^9 O^9 + \mathcal{C}^{10}O^{10} + \mathcal{C}^\nu O^\nu\right) , \tag{1.31}$$

where for future convenience we have considered also the operator contributing to $B \to K\bar{\nu}\nu$, O^ν. The Feynman diagrams relevant for the computation of $\mathcal{C}^9$ and $\mathcal{C}^{10}$ at the matching scale m_{EW} in the full and in the effective theory are displayed in figure 1.2.
Now the question arises of how new heavy degrees of freedom modify the Lagrangian (1.29) and, as a consequence, (1.31). Essentially NP can affect this low-energy framework in two ways:

- It can modify the operators already present in the SM. Since NP could violate both LFU and LF, in general its presence will determine a contribution with a non-universal and non-diagonal structure in the lepton flavor indices. As a consequence, we need to substitute the implicit structure ii in the semi-leptonic terms with the more generic structure ij.

- It can generate non-negligible contributions to the chirality-flipped versions of O^{7-10} and O^ν - which will be denoted by a primed sign - as well as scalar and tensor operators, defined as

$$O_{ij}^{S(\prime)} = \frac{e}{16\pi^2}(\bar{s}_{L(R)}b_{R(L)})(\bar{e}_i e_j) \qquad O_{ij}^{P(\prime)} = \frac{e}{16\pi^2}(\bar{s}_{L(R)}b_{R(L)})(\bar{e}_i\gamma_5 e_j)$$
$$O_{ij}^{T} = \frac{e}{16\pi^2}(\bar{s}\sigma_{\mu\nu}b)(\bar{e}_i\sigma^{\mu\nu}e_j) \qquad O_{ij}^{T5} = \frac{e}{16\pi^2}(\bar{s}\sigma_{\mu\nu}b)(\bar{e}_i\sigma^{\mu\nu}\gamma_5 e_j)\,.$$

  As observed in [21], the $SU(2)_L \times U(1)_Y$ invariance of the NP Lagrangian above m_{EW} places constrictions on the Wilson coefficients of scalar and tensor operators: tensor operators are excluded ($\mathcal{C}_{ij}^T = \mathcal{C}_{ij}^{T5} = 0$) and scalar operators are not independent, because $\mathcal{C}_{ij}^S = -\mathcal{C}_{ij}^P$ and $\mathcal{C}_{ij}^{S\prime} = \mathcal{C}_{ij}^{P\prime}$.

In our analysis we consider NP effects in the coefficients $C^{9(\prime)}$, $C^{10(\prime)}$ and $C^{\nu(\prime)}$, hence the Lagrangian we use to address $R_K^{\mu/e}$ reads

$$\mathcal{L}_{\text{eff}}^{\text{NC}} = \frac{4G_F}{\sqrt{2}}\lambda_{bs}^t\left(\mathcal{C}_{ij}^9 O_{ij}^9 + \mathcal{C}_{ij}^{9\prime}O_{ij}^{9\prime} + \mathcal{C}_{ij}^{10}O_{ij}^{10} + \mathcal{C}_{ij}^{10\prime}O_{ij}^{10\prime} + \mathcal{C}_{ij}^{\nu}O_{ij}^{\nu} + \mathcal{C}_{ij}^{\nu\prime}O_{ij}^{\nu\prime}\right) , \tag{1.32}$$

where

$$
\begin{aligned}
O^{9}_{ij} &= \frac{e^2}{(4\pi)^2}(\bar{s}_L\gamma^\mu b_L)(\bar{e}_i\gamma_\mu e_j) & O^{9\prime}_{ij} &= \frac{e^2}{(4\pi)^2}(\bar{s}_R\gamma^\mu b_R)(\bar{e}_i\gamma_\mu e_j)\\
O^{10}_{ij} &= \frac{e^2}{(4\pi)^2}(\bar{s}_L\gamma^\mu b_L)(\bar{e}_i\gamma_\mu\gamma_5 e_j) & O^{10\prime}_{ij} &= \frac{e^2}{(4\pi)^2}(\bar{s}_R\gamma^\mu b_R)(\bar{e}_i\gamma_\mu\gamma_5 e_j)\\
O^{\nu}_{ij} &= \frac{e^2}{(4\pi)^2}(\bar{s}_L\gamma^\mu b_L)(\bar{\nu}_i\gamma_\mu(1-\gamma_5)\nu_j) & O^{\nu\prime}_{ij} &= \frac{e^2}{(4\pi)^2}(\bar{s}_R\gamma^\mu b_R)(\bar{\nu}_i\gamma_\mu(1-\gamma_5)\nu_j)\,.
\end{aligned}
\tag{1.33}
$$

In order to describe the decay $B \to K^*\ell^+\ell^-$ we should take also dipole operators into accont. However, they affect $R^{\mu/e}_{K^*}$ only in the low q^2 region [22]; in the central q^2 region, namely $1.1\ \text{GeV}^2 < q^2 < 6\ \text{GeV}^2$, the Lagrangian (1.32) can be used for both $R^{\mu/e}_{K}$ and $R^{\mu/e}_{K^*}$. Note that primed Wilson coefficients enter R_K and R_{K*} with the opposite sign, because K is a pseudoscalar and K^* is a vector.
Summarizing what has been outlined so far, primed operators in $\mathcal{L}^{\text{NC}}_{\text{eff}}$ are negligible in the SM and therefore NP contributions are supposed to dominate. Unprimed operators are already present in the SM, where they arise at one loop, and are modifed by NP in our framework. Since in the SM no lepton-flavor violating operators arise, $(\mathcal{C}^{9}_{\text{SM}})_{ij}$, $(\mathcal{C}^{10}_{\text{SM}})_{ij}$ and $(\mathcal{C}^{\nu}_{\text{SM}})_{ij}$ have the form $\mathcal{C}^{9}_{\text{SM}}\delta_{ij}$, $\mathcal{C}^{10}_{\text{SM}}\delta_{ij}$ and $\mathcal{C}^{\nu}_{\text{SM}}\delta_{ij}$.
Turning to the charged-current anomaly, we observe that in the SM the amplitude of the $b \to c\ell\nu$ transition is dominated by the tree-level exchange of a W boson. We take NP into account by using the effective Lagrangian

$$
\mathcal{L}^{\text{CC}}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}}V_{cb}(\mathcal{C}^{cb}_{L})_{ij}(\bar{c}_L\gamma_\mu b_L)(\bar{e}_{Li}\gamma^\mu e_{Lj})\,, \tag{1.34}
$$

where the SM contribution to $(\mathcal{C}^{cb}_{L})_{ij}$ reads δ_{ij}.

1.4 Global fits and favoured scenarios

SM contributions to the Wilson coefficients in $\mathcal{L}^{\text{NC}}_{\text{eff}}$ are known to next-to-leading order [6, 17]. NP contributions instead are investigated with the help of global fits involving observables sensitive to their presence.
As discussed in the first section, recent experimental results have shown significant tensions between measurements and SM predictions in processes based on the transition $b \to s\mu^+\mu^-$. In the last years several global analyses have investigated these discrepancies, with the common aim of finding out whether they could be softened by NP contributions [23, 24, 25]. Even though they differ in the treatment of theoretical uncertainties and in the choice of the observables included in the fit, all these global analyses point in the same direction: tensions can be relieved by a NP effect in $\mathcal{C}^{9}_{\mu\mu}$ interfering distructively with the SM.
The most recent global analysis of NP in $b \to s\mu^+\mu^-$ is provided by [23][7]. By considering NP contributions in individual Wilson coefficients, the scenario with NP in $\mathcal{C}^{9}_{\mu\mu}$ only exhibits

[7] The observables included in the global fit are angular observables in $B \to K^*\mu^+\mu^-$, the branching ratios $\mathcal{B}(B^{0,\pm} \to K^{0,\pm}_{(*)}\mu^+\mu^-)$, $\mathcal{B}(B_s \to \phi\mu^+\mu^-)$ and $\mathcal{B}(B \to X_s\mu^+\mu^-)$ and angular observables in $B_s \to \phi\mu^+\mu^-$.

Wilson coefficient	best fit point	pull
$\mathcal{C}^9_{\rm NP}$	-1.21	5.2σ
$\mathcal{C}^9_{\rm NP} = -\mathcal{C}^{10}_{\rm NP}$	-0.67	4.8σ
$(\mathcal{C}^9_{\rm NP}, \mathcal{C}^{10}_{\rm NP})$	$(-1.15, +0.26)$	5.0σ
$(\mathcal{C}^9_{\rm NP}, \mathcal{C}^{9\prime}_{\rm NP})$	$(-1.25, +0.59)$	5.3σ
$(\mathcal{C}^9_{\rm NP}, \mathcal{C}^{10\prime}_{\rm NP})$	$(-1.34, -0.39)$	5.4σ

Table 1.1: Best fit points and pulls for scenarios with NP in one or two Wilson coefficients [23]. The subscript $\mu\mu$ is omitted.

the strongest pull; the best fit point is given by $(\mathcal{C}^9_{\rm NP})_{\mu\mu} \approx -1.2$, which is consistent with the previous global fits [24, 25]. Another good scenario is provided by $(\mathcal{C}^9_{\rm NP})_{\mu\mu} = -(\mathcal{C}^{10}_{\rm NP})_{\mu\mu}$. By switching on NP in pairs of Wilson coefficients, the strongest pulls are obtained for NP in the couples $(\mathcal{C}^9_{\mu\mu}, \mathcal{C}^{10}_{\mu\mu})$, $(\mathcal{C}^9_{\mu\mu}, \mathcal{C}^{9\prime}_{\mu\mu})$ and $(\mathcal{C}^9_{\mu\mu}, \mathcal{C}^{10\prime}_{\mu\mu})$, which show a similar behaviour: best fit points give a large shift in $\mathcal{C}^9_{\mu\mu}$ and a small shift in the other operator. These results are summarized in table 1.1.
Reference [23] also points out that although tensions can be coherently explained by a negative $(\mathcal{C}^9_{\rm NP})_{\mu\mu}$, we still cannot rule out that they are caused by an underestimation of hadronic effects. Future measurements of LFU ratios will play a clarifying role in this sense: a confirmation of LFUV would be a clear evidence in favour of NP, because hadronic effects are lepton-flavor universal. Interestingly, experimental results for $R_K^{\mu/e}$ and $R_{K^*}^{\mu/e}$ in the central q^2 region agree with the predictions obtained using the best fit points in $(\mathcal{C}^9_{\mu\mu}, \mathcal{C}^{10}_{\mu\mu})$ and $(\mathcal{C}^9_{\mu\mu}, \mathcal{C}^{9\prime}_{\mu\mu})$ and considering the transition $b \to s e^+ e^-$ to be NP free. The latter hypothesis is motivated by experimental results concerning $b \to s e^+ e^-$ processes [6].
Another recent analysis, [26], bases its fit on LFU observables, specifically the ratios $R_K^{\mu/e}$ and $R_{K^*}^{\mu/e}$ and the LFU differences of $B \to K^* \ell^+ \ell^-$ angular observables $D_{P'_{4,5}}$. By switching on NP in individual Wilson coefficients, it turns out that the fit shows a distinct preference for NP in coefficients involving left-handed quark currents, namely $\mathcal{C}^9_{\mu\mu/ee}$ and $\mathcal{C}^{10}_{\mu\mu/ee}$. On the other hand, an explanation of the measured values of $R_K^{\mu/e}$ and $R_{K^*}^{\mu/e}$ in terms of primed Wilson coefficients only - corresponding to right-handed quark currents - is highly disfavoured. In fact, if primed operators were dominant, we should have opposite anomalies (i.e. if $R_K < 1$, then $R_K* > 1$ and vice versa) because primed Wilson coefficients enter the two ratios with opposite signs.

In section 3.3 we aim at studying a well-motivated scenario inolving NP in a single Wilson coefficient. In this regard, we make two main hypotheses. First, we suppose that the tensions concerning rare $b \to s\mu^+\mu^-$ decays and the LFU ratios $R_{K^{(*)}}^{\mu/e}$ and $R_{D^{(*)}}^{\tau/\ell}$ are to be explained within a common NP framework. Second, we consider the transition $b \to s e^+ e^-$ to be SM like. In view of these assumptions and of the results of [23, 26], two scenarios are singled out, namely $(\mathcal{C}^9_{\rm NP})_{\mu\mu} = -(\mathcal{C}^{10}_{\rm NP})_{\mu\mu}$ and the one with NP in $\mathcal{C}^9_{\mu\mu}$ only. Since the first one has already been investigated in [1, 2], we will address the second one.

Chapter 2

Effective NP Lagrangian at low energy

2.1 Building NP Lagrangian at scale Λ

As stated in the previous chapter, we describe NP contributions above the EWSB scale and below the NP scale Λ - which we imagine being of order TeV - using the effective Lagrangian

$$\mathcal{L}_{\text{NP}} = \frac{1}{\Lambda^2} \sum_i \mathcal{C}_i Q_i \,, \tag{2.1}$$

where Q_i are six-dimensional operators invariant under the SM gauge group $G_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_{\text{em}}$. The complete notation would be $[\mathcal{C}_i]_{prst}[Q_i]_{prst}$, where p, r, s and t are flavor indices; to lighten the notation we will omit them unless they are strictly necessary.
In order to get the most general description of NP contributions, $\mathcal{L}_{\text{NP}}$ at scale Λ, hereafter denoted by $\mathcal{L}^0_{\text{NP}}$, should include the whole set of dimension-six operators compatible with the SM gauge symmetry, with unknown Wilson coefficients $[\mathcal{C}_i(\Lambda)]_{pqrs}$. Such a complete analysis goes far beyond the scope of the present work. Moreover, aside from B anomalies and other tensions in semi-leptonic B decays, there are no other hints of NP at present. It is therefore reasonable to restrict ourselves to operators wich might have an impact on these observables.
That said, there are three main aspects we should keep in mind when building the NP Lagrangian:

- like in [1, 2], we want this NP Lagrangian to simultaneously account for both NC and CC B anomalies;
- we require B anomalies to be explained by a tree-level NP effect within our framework. Our choice should therefore provide a tree-level contribution to the operators involved in the decays of interest ($\mathcal{C}^9_{\mu\mu}$, $\mathcal{C}^{9\prime}_{\mu\mu}$, $\mathcal{C}^{10}_{\mu\mu}$ and $\mathcal{C}^{10\prime}_{\mu\mu}$ for the neutral-current anomaly and $\mathcal{C}^{cb}_L$ for the charged current-anomaly). In particular, as far as the neutral-current process $b \to s\mu^+\mu^-$ is concerned, we are ultimately interested in being able to reproduce the main scenarios favoured by the global fits mentioned in the first chapter [23, 24, 25];
- the SM exhibits a strong hierarchy between the processes on which the anomalies are based: $b \to c\ell\nu$ occurs already at tree level, while $b \to s\ell^+\ell^-$ arises only at one loop. Therefore, we want to naturally extend this hierarchy to NP.

As argued in [21], the complete list of dimension-six gauge invariant operators which could possibly contribute at tree level to $\mathcal{C}^9_{ij}$ and $\mathcal{C}^{10}_{ij}$ is given by three four-fermion semi-leptonic operators - $Q^{(1)}_{\ell q}$,$Q^{(3)}_{\ell q}$ and Q_{qe} - and two operators involving the Higgs, $Q^{(1)}_{Hq}$ and $Q^{(3)}_{Hq}$ (see table 2.1). The operators which could give a tree-level contribution to the primed operators $\mathcal{C}^{9\prime}_{ij}$ and $\mathcal{C}^{10\prime}_{ij}$ are Q_{ed}, Q_{ld} and Q_{Hd}. Analogously, $(\mathcal{C}^{cb}_L)_{ij}$ might receive a tree-level contribution from $Q^{(3)}_{\ell q}$ and $Q^{(3)}_{Hq}$.
It follows that in order to reproduce all possible NP tree-level contributions to the five operators of interest our starting Lagrangian should include all the operators listed above. In the present work we limit our analysis to semi-leptonic operators; we exclude the Higgs operators because their presence at scale Λ would imply a tree-level modification of the Z and W couplings to fermions, which are tightly constrained by LEP measurements. Consequently, the NP Lagrangian at scale Λ will include the following five operators: $Q^{(1)}_{\ell q}$, $Q^{(3)}_{\ell q}$, Q_{ed}, Q_{ld} and Q_{qe}.
We further assume that NP couples only to the third generation in the interaction basis. This choice is supported by the strong constraints on flavor physics involving the first two generations and provides a natural suppression mechanism for the neutral-current anomaly with respect to the charged-current one, because NP couplings to lighter generations are generated by the rotation to the mass basis through small flavor mixing angles.
In conclusion we build $\mathcal{L}^0_{\text{NP}}$ by assuming that NP at scale Λ is dominated by

$$[Q^{(1)}_{lq}]_{3333} = (\bar{\ell}'_{3L}\gamma^\mu \ell'_{3L})(\bar{q}'_{3L}\gamma^\mu q'_{3L}) \tag{2.2}$$
$$[Q^{(3)}_{lq}]_{3333} = (\bar{\ell}'_{3L}\gamma^\mu \tau^a \ell'_{3L})(\bar{q}'_{3L}\gamma^\mu \tau^a q'_{3L}) \tag{2.3}$$
$$[Q_{ld}]_{3333} = (\bar{\ell}'_{3L}\gamma^\mu \ell'_{3L})(\bar{d}'_{3R}\gamma d'_{3R}) \tag{2.4}$$
$$[Q_{ed}]_{3333} = (\bar{e}'_{3R}\gamma^\mu e'_{3R})(\bar{d}'_{3R}\gamma d'_{3R}) \tag{2.5}$$
$$[Q_{qe}]_{3333} = (\bar{q}'_{3L}\gamma q'_{3L})(\bar{e}'_{3R}\gamma^\mu e'_{3R}) \tag{2.6}$$

and write the NP Lagrangian as

$$\mathcal{L}^0_{\text{NP}} = \frac{1}{\Lambda^2}(C_1[Q^{(1)}_{lq}]_{3333} + C_3[Q^{(3)}_{lq}]_{3333} + C_4[Q_{ld}]_{3333} + C_5[Q_{ed}]_{3333} + C_6[Q_{qe}]_{3333}) \,. \tag{2.7}$$

Here $C_1 = [\mathcal{C}^{(1)}_{lq}(\Lambda)]_{3333}$, $C_3 = [\mathcal{C}^{(3)}_{lq}(\Lambda)]_{3333}$ and so on.
With respect to reference [1, 2], where only the purely left-handed operators $Q^{(1)}_{lq}$ and $Q^{(3)}_{lq}$ were considered, we added three operators involving at least one right-handed current. Although the recently announced measurement of $R^{\mu/e}_{K^*}$ seems to discourage operators involving right-handed quark currents [26], they still cannot be excluded.

Operators in 2.7 are written in the interaction basis; later on we will need to move to the mass basis. As we have seen, fields in the two bases are related by the unitary transformations (1.8), while Yukawa matrices are diagonalised as shown in (1.7). For the sake of simplicity, we define the λ and Γ matrices:

$$\begin{aligned} \lambda^u_{ij} &= V^*_{u3i}V_{u3j} & \lambda^d_{ij} &= V^*_{d3i}V_{d3j} & \lambda^e_{ij} &= V^*_{e3i}V_{e3j} & \lambda^{ud}_{ij} &= V^*_{u3i}V_{d3j}\,. \\ \Gamma^d_{ij} &= R^*_{d3i}R_{d3j} & \Gamma^e_{ij} &= R^*_{e3i}R_{e3j} \end{aligned} \tag{2.8}$$

It is straightforward to show that λ and Γ satisfy the following properties:

1. $(\lambda^f)^\dagger = \lambda^f$ and $(\Gamma^f)^\dagger = \Gamma^f \quad (f = u, d, e)$

2. $\lambda^f \lambda^f = \lambda^f$ and $\Gamma^f \Gamma^f = \Gamma^f$

3. $(\lambda^{ud})^\dagger = \lambda^{du}$

4. $\mathrm{Tr}\lambda^f = 1$ and $\mathrm{Tr}\Gamma^f$

5. $\sum_{ij} |\lambda^f_{ij}|^2 = 1$ and $\sum_{ij} |\Gamma^f_{ij}|^2 = 1\,.$ (2.9)

Property 1. follows directly from the definition of λ^f and Γ^f, while the other properties follow from the unitarity of V and R matrices. We observe that λ matrices are redundant, because they are related to the CKM matrix as follows:

5. $\lambda^u = V_{\mathrm{CKM}} \lambda^d V^\dagger_{\mathrm{CKM}}\,,$ (2.10)

6. $\lambda^{ud} = V_{\mathrm{CKM}} \lambda^d\,.$ (2.11)

A general parametrization of λ^f and Γ^f matrices is provided by

$$\lambda^f = \frac{1}{1 + |\alpha_f|^2 + |\beta_f|^2} \begin{pmatrix} |\alpha_f|^2 & \alpha_f \beta_f^* & \alpha_f \\ \alpha_f^* \beta_f & |\beta_f|^2 & \beta_f \\ \alpha_f^* & \beta_f^* & 1 \end{pmatrix} \tag{2.12}$$

$$\Gamma^f = \frac{1}{1 + |\rho_f|^2 + |\sigma_f|^2} \begin{pmatrix} |\rho_f|^2 & \rho_f \sigma_f^* & \rho_f \\ \rho_f^* \sigma_f & |\sigma_f|^2 & \sigma_f \\ \rho_f^* & \sigma_f^* & 1 \end{pmatrix}\,. \tag{2.13}$$

where α_f, β_f, ρ_f and σ_f are complex numbers.
Using relations (1.8) and (2.8) we rewrite the NP Lagrangian at scale Λ in the mass basis, obtaining

$$
\begin{aligned}
\mathcal{L}^0_{\text{NP}} =& \frac{(C_1 - C_3)}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu\lambda_u u_L) + \frac{(C_1 + C_3)}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_L\gamma_\mu\lambda_d d_L) \\
&+ \frac{(C_1 + C_3)}{\Lambda^2}(\bar{\nu}_L\gamma^\mu\lambda_e \nu_L)(\bar{u}_L\gamma_\mu\lambda_d u_L) + \frac{(C_1 - C_3)}{\Lambda^2}(\bar{\nu}_L\gamma^\mu\lambda_e \nu_L)(\bar{d}_L\gamma_\mu\lambda_d d_L) \\
&+ \frac{2C_3}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e \nu_L)(\bar{u}_L\gamma_\mu\lambda_{ud} d_L) + \text{h.c.} + \frac{C_4}{\Lambda^2}(\bar{\nu}_L\gamma^\mu\lambda_e \nu_L)(\bar{d}_R\gamma^\mu\Gamma_d d_R) \\
&+ \frac{C_4}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_R\gamma^\mu\Gamma_d d_R) + \frac{C_5}{\Lambda^2}(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{d}_R\gamma^\mu\Gamma_d d_R) \\
&+ \frac{C_6}{\Lambda^2}(\bar{u}_L\gamma^\mu\lambda_u u_L)(\bar{e}_R\gamma_\mu\Gamma_e e_R) + \frac{C_6}{\Lambda^2}(\bar{d}_L\gamma^\mu\lambda_d d_L)(\bar{e}_R\gamma_\mu\Gamma_e e_R)\,.
\end{aligned} \tag{2.14}
$$

The independent parameters in this expression are the five C_i and the matrices λ^e, λ^d, Γ^e and Γ^d. All information about NP is encoded in the C_i.

2.2 RGE flow from Λ to m_{EW}

In this section we start from the NP Lagrangian at scale Λ and run it down to m_{EW}. This is achieved by employing the renormalization group equations (RGE) in the limit of exact electroweak symmetry. In particular we will refer to [27], which provides RGE to one-loop accuracy for all 59 six-dimensional operators invariant under G_{SM} listed in [19]. They are written in the format

$$
\dot{\mathcal{C}}_i = 16\pi^2\gamma_{ji}\mathcal{C}_j\,, \tag{2.15}
$$

where $\dot{\mathcal{C}}_i = 16\pi^2\mu\frac{d}{d\mu}\mathcal{C}_i$. We solve them in leading logarithm approximation using $\mathcal{L}^0_{\text{NP}}$ as initial condition; explicitly the leading logarithm solution to these equations reads

$$
\mathcal{C}^i_{prst}(\mu) = \mathcal{C}^i_{prst}(\Lambda) - \frac{1}{16\pi^2}\dot{\mathcal{C}^i}_{prst}(\Lambda)\ln\frac{\Lambda}{\mu}\,. \tag{2.16}
$$

Knowing that the only non-zero $\mathcal{C}_i(\Lambda)$ are the ones in (2.7) and neglecting all Yukawa couplings but that of the top[1] we find out which operators are involved in the RGE flow; they are displayed in table 2.1.
Given the operators Q^i_{prst} involved in the running, the Lagrangian at scale μ will have the form

$$
\mathcal{L}_{\text{NP}} = \frac{1}{\Lambda^2}\sum_i\sum_{prst} C^i_{prst}(\mu)Q^i_{prst}\,; \tag{2.17}
$$

for each Q_i we want to find $\mathcal{C}^i_{prst}(\mu)$ using eq. (2.16). In order to do this we need to determine which flavor structure gives a non-zero $\dot{\mathcal{C}}^i_{prst}(\Lambda)$; then we can compute the sum over the flavor indices in (2.17). This same procedure must be repeated for all the 22 operators appearing in the RGE flow.

[1]This corresponds to writing Y^D_u as $Y^D_u = y_t P_3$, with $P_3 = \text{diag}(0, 0, 1)$.

Leptonic operators	Semi-leptonic operators
$[Q_{\ell\ell}]_{prst} = (\bar\ell'_{pL}\gamma_\mu \ell'_{rL})(\bar\ell'_{sR}\gamma^\mu \ell'_{tR})$	$[Q^{(1)}_{\ell q}]_{prst} = (\bar\ell'_{pL}\gamma_\mu \ell'_{rL})(\bar q'_{sL}\gamma^\mu q'_{tL})$
$[Q_{\ell e}]_{prst} = (\bar\ell'_{pL}\gamma^\mu \ell'_{rL})(\bar e'_{sR}\gamma_\mu e'_{tR})$	$[Q^{(3)}_{\ell q}]_{prst} = (\bar\ell'_{pL}\gamma_\mu \tau^a \ell'_{rL})(\bar q'_{sL}\gamma^\mu \tau^a q'_{tL})$
$[Q_{ee}]_{prst} = (\bar e'_{pR}\gamma_\mu e'_{rR})(\bar e'_{sR}\gamma^\mu e'_{tR})$	$[Q_{\ell u}]_{prst} = (\bar\ell'_{pL}\gamma_\mu \ell'_{rL})(\bar u'_{sR}\gamma^\mu u'_{tR})$
	$[Q_{\ell d}]_{prst} = (\bar\ell'_{pL}\gamma_\mu \ell'_{rL})(\bar d'_{sR}\gamma^\mu d'_{tR})$
	$[Q_{qe}]_{prst} = (\bar q'_{pL}\gamma_\mu q'_{rL})(\bar e'_{sR}\gamma^\mu e'_{tR})$
	$[Q_{eu}]_{prst} = (\bar e'_{pR}\gamma_\mu e'_{rR})(\bar u'_{sR}\gamma^\mu u'_{tR})$
	$[Q_{ed}]_{prst} = (\bar e'_{pR}\gamma_\mu e'_{rR})(\bar d'_{sR}\gamma^\mu d'_{tR})$
Vector operators	**Hadronic operators**
$[Q^{(1)}_{H\ell}]_{pr} = (\phi^\dagger i\overleftrightarrow{D}_\mu\phi)(\bar l'_{pL}\gamma^\mu l'_{rL})$	$[Q^{(1)}_{qq}]_{prst} = (\bar q'_{pL}\gamma_\mu q'_{rL})(\bar q'_{sL}\gamma^\mu q'_{tL})$
$[Q^{(3)}_{H\ell}]_{pR} = (\phi^\dagger i\overleftrightarrow{D}^a_\mu\phi)(\bar l'_{pL}\gamma^\mu \tau^a l'_{rL})$	$[Q^{(3)}_{qq}]_{prst} = (\bar q'_{pL}\gamma_\mu \tau^a q'_{rL})(\bar q'_{sL}\gamma^\mu \tau^a q'_{tL})$
$[Q^{(1)}_{Hq}]_{pR} = (\phi^\dagger i\overleftrightarrow{D}_\mu\phi)(\bar q'_{pL}\gamma^\mu q'_{rL})$	$[Q^{(1)}_{qu}]_{prst} = (\bar q'_{pL}\gamma_\mu q'_{rL})(\bar u'_{sR}\gamma^\mu u'_{tR})$
$[Q^{(3)}_{Hq}]_{pR} = (\phi^\dagger i\overleftrightarrow{D}^a_\mu\phi)(\bar q'_{pL}\gamma^\mu \tau^a q'_{rL})$	$[Q^{(1)}_{qd}]_{prst} = (\bar q'_{pL}\gamma_\mu q'_{rL})(\bar d'_{sR}\gamma^\mu d'_{tR})$
$[Q_{He}]_{pR} = (\phi^\dagger i\overleftrightarrow{D}_\mu\phi)(\bar e'_{pR}\gamma_\mu e'_{rR})$	$[Q_{dd}]_{prst} = (\bar d'_{pR}\gamma_\mu d'_{rR})(\bar d'_{sR}\gamma^\mu d'_{tR})$
$[Q_{Hd}]_{pR} = (\phi^\dagger i\overleftrightarrow{D}_\mu\phi)(\bar d'_{pR}\gamma^\mu d'_{rR})$	$[Q^{(1)}_{ud}]_{prst} = (\bar u'_{pR}\gamma_\mu u'_{rR})(\bar d'_{sR}\gamma^\mu d'_{tR})$

Table 2.1: $SU(2)_L \times U(1)_Y$ invariant operators invoved in the renormalization group evolution of $\mathcal{L}^0_{\rm NP}$ from Λ to the EW scale.

We show how this works for the operator $[Q^{(1)}_{\ell q}]_{prst}$. Only a few terms survive in the equations for $[\mathcal{C}^{(1)}_{\ell q}]_{prst}$ given in [27]:

$$\dot{\mathcal{C}}^{(1)}_{\substack{\ell q\\ prst}}\Big|_\Lambda = \Bigg(\frac{1}{2}[Y_u^\dagger Y_u]_{sv}\mathcal{C}^{(1)}_{\substack{\ell q\\ prvt}} + \frac{1}{2}\mathcal{C}^{(1)}_{\substack{\ell q\\ prsv}}[Y_u^\dagger Y_u]_{vt} + \frac{2}{9}g_1^2\mathcal{C}^{(1)}_{\substack{\ell q\\ prww}}\delta_{st} + \frac{2}{3}g_1^2\mathcal{C}^{(1)}_{\substack{\ell q\\ wwst}}\delta_{pR} - \frac{2}{9}g_1^2\mathcal{C}_{\substack{\ell d\\ prww}}\delta_{st} - g_1^2\mathcal{C}^{(1)}_{\substack{\ell q\\ prst}} + 9g_2^2\mathcal{C}^{(3)}_{\substack{\ell q\\ prsv}}\Bigg)\Bigg|_\Lambda . \tag{2.18}$$

From this expression we see that only the flavor structures $ss33$ and $33st$ give $[\dot{\mathcal{C}}^{(1)}_{\ell q}]_{prst}(\Lambda) \neq 0$. Being careful not to repeat the same term twice, we can write the internal sum in (2.17) for the operator considered as

$$\sum_{prst}\mathcal{C}^{(1)}_{\substack{\ell q\\ prst}}(\mu)\,Q^{(1)}_{\substack{\ell q\\ prst}} = \mathcal{C}^{(1)}_{\substack{\ell q\\ 3333}}(\mu)\,Q^{(1)}_{\substack{\ell q\\ 3333}} + \sum_s\Bigg(\mathcal{C}^{(1)}_{\substack{\ell q\\ 33ss}}(\mu)\,Q^{(1)}_{\substack{\ell q\\ 33ss}} + \mathcal{C}^{(1)}_{\substack{\ell q\\ ss33}}(\mu)\,Q^{(1)}_{\substack{\ell q\\ ss33}}\Bigg) + \sum_{s,t}\mathcal{C}^{(1)}_{\substack{\ell q\\ 33st}}(\mu)\,Q^{(1)}_{\substack{\ell q\\ 33st}}\,, \tag{2.19}$$

with

$$
\begin{cases}
\mathcal{C}^{(1)}_{\underset{3333}{\ell q}}(\mu) = C_1 + \dfrac{L}{16\pi^2}\left(g_1^2 C_1 - 9 g_2^2 C_3\right) \\
\mathcal{C}^{(1)}_{\underset{33ss}{\ell q}}(\mu) = -\dfrac{L}{16\pi^2}\dfrac{2}{9} g_1^2 (C_1 - C_4) \\
\mathcal{C}^{(1)}_{\underset{ss33}{\ell q}}(\mu) = -\dfrac{L}{16\pi^2}\dfrac{2}{3} g_1^2 C_1 \\
\mathcal{C}^{(1)}_{\underset{33st}{\ell q}}(\mu) = -\dfrac{L}{16\pi^2}\dfrac{C_1}{2}\left([Y_u^\dagger Y_u]_{s3}\delta_{3t} + \delta_{s3}[Y_u^\dagger Y_u]_{3t}\right) .
\end{cases}
\tag{2.20}
$$

Here $L = \ln \frac{\Lambda}{\mu}$. After repeating the same procedure for all the operators involved in the running, we find that the effective Lagrangian at the scale $m_{\text{EW}} < \mu < \Lambda$ is given by $\mathcal{L} = \mathcal{L}_{\text{SM}} + \mathcal{L}^0_{\text{NP}} + \mathcal{L}_{\text{eff}}$, where

$$
\mathcal{L}_{\text{eff}} = \delta\mathcal{L}_{\text{SL}} + \delta\mathcal{L}_{\text{L}} + \delta\mathcal{L}_{\text{V}} + \delta\mathcal{L}_{\text{H}} \,. \tag{2.21}
$$

$\mathcal{L}_{\text{eff}}$ describes the contribution induced by renormalization group equations. Explicitly we have

$$
\begin{aligned}
\delta\mathcal{L}_{\text{SL}} = \frac{L}{16\pi^2\Lambda^2}\Bigg[& (g_1^2 C_1 - 9 g_2^2 C_3) Q^{(1)}_{\underset{3333}{lq}} - \frac{2}{9} g_1^2 (C_1 - C_4) Q^{(1)}_{\underset{33ss}{lq}} \\
& - \frac{2}{3} g_1^2 (C_1 + C_6) Q^{(1)}_{\underset{ss33}{lq}} - \frac{1}{2} C_1 \left([Y_u^\dagger Y_u]_{s3}\delta_{3t} + \delta_{s3}[Y_u^\dagger Y_u]_{3t}\right) Q^{(1)}_{\underset{33st}{lq}} \\
& + \left(-3 g_2^2 C_1 + C_3 (6 g_2^2 + g_1^2)\right) Q^{(3)}_{\underset{3333}{lq}} - 2 g_2^2 C_3 Q^{(3)}_{\underset{33ss}{lq}} - \frac{2}{3} g_2^2 C_3 Q^{(3)}_{\underset{ss33}{lq}} \\
& - \frac{1}{2} C_3 \left([Y_u^\dagger Y_u]_{s3}\delta_{3t} + \delta_{s3}[Y_u^\dagger Y_u]_{3t}\right) Q^{(3)}_{\underset{33st}{lq}} - \frac{8}{9} g_1^2 (C_1 - C_4) Q_{\underset{33ss}{lu}} \\
& + 2 [Y_u]_{s3}[Y_u^\dagger]_{3t} C_1 Q_{\underset{33st}{lu}} + 2 g_1^2 C_4 Q_{\underset{3333}{ld}} + \frac{4}{9} g_1^2 (C_1 - C_4) Q_{\underset{33ss}{ld}} \\
& - \frac{2}{3} g_1^2 (C_4 + C_5) Q_{\underset{ss33}{ld}} - 2 g_1^2 C_6 Q_{\underset{3333}{qe}} - \frac{4}{3} g_1^2 (C_1 + C_6) Q_{\underset{33ss}{qe}} \\
& - \frac{1}{2} C_6 \left([Y_u^\dagger Y_u]_{s3}\delta_{3t} + \delta_{s3}[Y_u^\dagger Y_u]_{3t}\right) Q_{\underset{st33}{qe}} + \frac{8}{9} g_1^2 (C_5 - C_6) Q_{\underset{33ss}{eu}} \\
& + 2 [Y_u]_{s3}[Y_u^\dagger]_{3t} C_6 Q_{\underset{33st}{eu}} - 4 g_1^2 C_5 Q_{\underset{3333}{ed}} + \frac{2}{9} g_1^2 (C_5 - C_6) Q_{\underset{ss33}{qe}} \\
& - \frac{4}{9} g_1^2 (C_5 - C_6) Q_{\underset{33ss}{ed}} - \frac{4}{3} g_1^2 (C_4 + C_5) Q_{\underset{ss33}{ed}} \Bigg] ,
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
\delta\mathcal{L}_{\text{L}} = \frac{L}{16\pi^2\Lambda^2}\Bigg[& \left(\frac{2}{3} g_1^2 (C_1 - C_4) + 2 g_2^2 C_3\right) Q_{\underset{33ss}{ll}} - 4 g_2^2 C_3 Q_{\underset{3ss3}{ll}} \\
& + \frac{4}{3} g_1^2 (C_1 - C_4) Q_{\underset{33ss}{le}} - \frac{2}{3} g_1^2 (C_5 - C_6) Q_{\underset{ss33}{le}} \Bigg] ,
\end{aligned}
\tag{2.23}
$$

$$\delta\mathcal{L}_{\mathrm{V}} = \frac{L}{16\pi^2\Lambda^2}\Big[(-6C_1\lambda^u_{33}y_t^2 - \frac{2}{3}g_1^2(C_1-C_4))Q^{(1)}_{\substack{Hl\\33}} + (6C_3\lambda^u_{33}y_t^2 - 2g_1^2(C_1-C_4))Q^{(3)}_{\substack{Hl\\33}} + \frac{2}{3}g_1^2(C_1+C_6)Q^{(1)}_{\substack{Hq\\33}} - \frac{2}{3}g_2^2C_3Q^{(3)}_{\substack{Hq\\33}} + \left(\frac{2}{3}g_1^2(C_5-C_6) - 6C_6\lambda^u_{33}y_t^2\right)Q_{\substack{He\\33}} + \frac{2}{3}g_1^2(C_4+C_5)Q_{\substack{Hd\\33}}\Big]\,, \tag{2.24}$$

$$\delta\mathcal{L}_{\mathrm{H}} = \frac{L}{16\pi^2\Lambda^2}\Big[\frac{2}{9}g_1^2(C_1+C_6)Q^{(1)}_{\substack{qq\\33ss}} + -\frac{2}{3}g_2^2C_3Q^{(3)}_{\substack{qq\\33ss}} + \frac{8}{9}g_1^2(C_1+C_6)Q^{(1)}_{\substack{qu\\33ss}} - \frac{4}{9}g_1^2(C_1+C_6)Q^{(1)}_{\substack{qd\\33ss}} + \frac{2}{9}g_1^2(C_4+C_5)Q^{(1)}_{\substack{qd\\ss33}} + -\frac{4}{9}g_1^2(C_4+C_5)Q_{\substack{dd\\33ss}}\Big]\,. \tag{2.25}$$

The sum over repeated flavor indices is understood.
Note that hereafter we will neglect the hadronic contribution $\delta\mathcal{L}_{\mathrm{H}}$, because fully hadronic operators play no role in our analysis.

2.2.1 Analysis of $\delta\mathcal{L}_V$: modified Z and W couplings

The vector part of the effective Lagrangian encodes the way NP affects Z and W couplings. In particular, we can write $\delta\mathcal{L}_{\mathrm{V}}$ as the sum of three contributions:

$$\delta\mathcal{L}_{\mathrm{V}} = \delta\mathcal{L}^{\mathrm{Z}}_{\mathrm{V}} + \delta\mathcal{L}^{\mathrm{W}}_{\mathrm{V}} + \delta\mathcal{L}^{\mathrm{H}}_{\mathrm{V}}\,. \tag{2.26}$$

$\delta\mathcal{L}^{\mathrm{Z}}_{\mathrm{V}}$ and $\delta\mathcal{L}^{\mathrm{W}}_{\mathrm{V}}$ describe the coupling among fermions and Z and W boson respectively, while $\delta\mathcal{L}^{\mathrm{H}}_{\mathrm{V}}$ involves the Higgs field h. In order to see this, we rewrite the operators in $\delta\mathcal{L}_{\mathrm{V}}$ in the mass basis; taking $\phi = \left(0, \frac{v+h}{\sqrt{2}}\right)$ we get

$$\begin{aligned}
[Q^{(1)}_{H\ell}]_{33} &= \frac{v^2}{2}\frac{g_2}{c_{\mathrm{W}}}Z_\mu[(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L) + (\bar{e}_L\gamma^\mu\lambda_e e_L)] + \dots\\
[Q^{(3)}_{H\ell}]_{33} &= -v^2\frac{g_2}{\sqrt{2}}[W^+_\mu(\bar{\nu}_L\gamma^\mu\lambda_e e_L) + \mathrm{h.c.}] - \frac{v^2}{2}\frac{g_2}{c_{\mathrm{W}}}Z_\mu[(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L) - (\bar{e}_L\gamma^\mu\lambda_e e_L)] + \dots\\
[Q^{(1)}_{Hq}]_{33} &= \frac{v^2}{2}\frac{g_2}{c_{\mathrm{W}}}Z_\mu[(\bar{u}_L\gamma^\mu\lambda_u u_L) + (\bar{d}_L\gamma^\mu\lambda_d d_L)] + \dots\\
[Q^{(3)}_{Hq}]_{pR} &= -v^2\frac{g_2}{\sqrt{2}}[W^+_\mu(\bar{u}_L\gamma^\mu\lambda^{ud}d_L) + \mathrm{h.c.}] - \frac{v^2}{2}\frac{g_2}{c_{\mathrm{W}}}Z_\mu[(\bar{u}_L\gamma^\mu\lambda^u u_L) - (\bar{d}_L\gamma^\mu\lambda^d d_L)] + \dots\\
[Q_{He}]_{33} &= \frac{v^2}{2}\frac{g_2}{c_{\mathrm{W}}}Z_\mu(\bar{e}_R\gamma^\mu\Gamma^e e_R) + \dots\\
[Q_{Hd}]_{33} &= \frac{v^2}{2}\frac{g_2}{c_{\mathrm{W}}}Z_\mu(\bar{d}_R\gamma^\mu\Gamma^e d_R) + \dots\,,
\end{aligned} \tag{2.27}$$

where dots stand for terms involving the Higgs field h. Plugging (2.27) into (2.24), we rewrite

$\delta\mathcal{L}_{\mathrm{V}}^{\mathrm{Z}}$ and $\delta\mathcal{L}_{\mathrm{V}}^{\mathrm{W}}$ compactly as

$$\delta\mathcal{L}_{\mathrm{V}}^{\mathrm{Z}} = -\frac{g_2}{c_{\mathrm{W}}} Z_\mu J^{\mu,0}_{\mathrm{NP}} = -\frac{g_2}{c_{\mathrm{W}}} Z_\mu \sum_f \Big((\Delta g_L^f)_{ij} \bar{f}_{iL}\gamma^\mu f_{jL} + (\Delta g_R^f)_{ij} \bar{f}_{iR}\gamma^\mu f_{jR} \Big) \tag{2.28}$$

$$\delta\mathcal{L}_{\mathrm{V}}^{\mathrm{W}} = -\frac{g_2}{\sqrt{2}} W^+_\mu J^{\mu,-}_{\mathrm{NP}} + \mathrm{h.c.} = -\frac{g_2}{\sqrt{2}} W^+_\mu \Big(\Delta g^\ell_{ij} \bar{\nu}_{iL}\gamma^\mu e_{jL} + \Delta g^q_{ij} \bar{u}_{iL}\gamma^\mu d_{jL} \Big) + \mathrm{h.c.}\,, \tag{2.29}$$

where $c_{\mathrm{W}} = \cos\theta_{\mathrm{W}}$. $\Delta g^f_{L,R}$ and $\Delta g^{q/\ell}$ express the NP modification of the couplings between massive gauge bosons and fermions. The explicit expressions for Z couplings are

$$\begin{cases}
(\Delta g_L^e)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\left[\dfrac{g_1^2}{3}(C_1 - C_4) + g_2^2 C_3 + 3\lambda^u_{33} y_t^2 (C_1 - C_3)\right]\lambda^e_{ij} \\
(\Delta g_L^d)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\dfrac{1}{3}\left[g_2^2 C_3 - g_1^2 (C_1 + C_6)\right]\lambda^d_{ij} \\
(\Delta g_L^u)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\dfrac{1}{3}\left[-g_2^2 C_3 - g_1^2 (C_1 + C_6)\right]\lambda^u_{ij} \\
(\Delta g_L^\nu)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\left[\dfrac{g_1^2}{3}(C_1 - C_4) - g_2^2 C_3 + 3\lambda^u_{33} y_t^2 (C_1 + C_3)\right]\lambda^e_{ij} \\
(\Delta g_R^e)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\left[\dfrac{1}{3}g_1^2 (C_6 - C_5) + 3C_6\lambda^u_{33} y_t^2\right]\Gamma^e_{ij} \\
(\Delta g_R^d)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\left[-\dfrac{1}{3}g_1^2 (C_4 + C_5)\right]\Gamma^d_{ij}\,,
\end{cases} \tag{2.30}$$

while those for W couplings read

$$\begin{cases}
(\Delta g^\ell)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\left[6C_3\lambda^u_{33} y_t^2 - 2g_2^2 C_3\right]\lambda^e_{ij} \\
(\Delta g^q)_{ij} = \dfrac{v^2}{\Lambda^2}\dfrac{L}{16\pi^2}\left[-\dfrac{2}{3}g_2^2 C_3\right]\lambda^{ud}_{ij}\,.
\end{cases} \tag{2.31}$$

Remembering that in the SM the Lagrangians describing the interaction among fermions and massive bosons are given by

$$\mathcal{L}_{\mathrm{Z}}^{\mathrm{SM}} = -\frac{g_2}{c_{\mathrm{W}}} Z_\mu J^{\mu,0}_{\mathrm{SM}} = -\frac{g_2}{c_{\mathrm{W}}} Z_\mu \sum_f \Big((g^f_{L,\mathrm{SM}})_{ij} \bar{f}_{iL}\gamma^\mu f_{jL} + (g^f_{R,\mathrm{SM}})_{ij} \bar{f}_{iR}\gamma^\mu f_{jR} \Big) \tag{2.32}$$

$$\mathcal{L}_{\mathrm{W}}^{\mathrm{SM}} = -\frac{g_2}{\sqrt{2}} W^+_\mu J^{\mu-}_{\mathrm{SM}} + \mathrm{h.c.} = -\frac{g_2}{\sqrt{2}} W^+_\mu \Big((g^\ell_{\mathrm{SM}})_{ij} \bar{\nu}_{iL}\gamma^\mu e_{jL} + (g^q_{\mathrm{SM}})_{ij} \bar{u}_{iL}\gamma^\mu d_{jL} \Big) + \mathrm{h.c.}\,, \tag{2.33}$$

with couplings

$$\begin{cases}
(g^f_{L,\mathrm{SM}})_{ij} = g^f_{L,\mathrm{SM}}\delta_{ij} = (T_3^f - q_f s^2_{\mathrm{W}})\delta_{ij} \\
(g^f_{R,\mathrm{SM}})_{ij} = g^f_{R,\mathrm{SM}}\delta_{ij} = -q_f\delta_{ij}\,, \\
(g^\ell_{\mathrm{SM}})_{ij} = \delta_{ij} \\
(g^q_{\mathrm{SM}}) = (V_{\mathrm{CKM}})_{ij}
\end{cases} \tag{2.34}$$

we can write the full Z and W Lagrangian in a compact notation

$$\mathcal{L}^{\mathrm{tot}}_{\mathrm{Z,W}} = \mathcal{L}^{\mathrm{SM}}_{\mathrm{Z,W}} + \delta\mathcal{L}_{\mathrm{V}}^{\mathrm{Z}} + \delta\mathcal{L}_{\mathrm{V}}^{\mathrm{W}} = -\frac{g_2}{c_{\mathrm{W}}} Z_\mu J^{\mu 0} - \frac{g_2}{\sqrt{2}}\Big(W^+_\mu J^{\mu,-} + \mathrm{h.c.}\Big)\,, \tag{2.35}$$

where $J^{\mu,-} = J^{\mu,-}_{\rm SM} + J^{\mu,-}_{\rm NP}$ and $J^{\mu,0} = J^{\mu,0}_{\rm SM} + J^{\mu,0}_{\rm NP}$.
Let us make some comments about the expressions (2.30) and (2.31). First of all, we immediately notice that the NP contribution to the couplings has a non-diagonal flavor structure. In particular NP introduces a formally tree-level source of lepton flavor violation (LFV) and LFUV, which is absent in the SM. This has important phenomenological implications, which will be discussed in the next chapter.
We also notice that the effective couplings have a dependence on the renormalization scale μ, which must cancel when amplitudes are computed. We check this explicitly for the Z boson decay into a charged lepton pair, $Z \to \bar{e}_i e_j$. The goal is not to compute the full one-loop amplitude for this process, but to check that the μ dependence in the effective couplings Δg^e_L and Δg^e_R cancels properly. The relevant diagrams are shown in figure 2.1(a) and 2.1(b). Specifically 2.1(a) shows the formally tree-level contribution given by $\mathcal{L}^{\rm tot}_{\rm Z,W}$, while 2.1(b) shows the one-loop contribution obtained by inserting the four-fermion interactions contained in $\mathcal{L}^0_{\rm NP}$ in the vertex labelled by C_i. We will denote the two amplitudes by $\mathcal{M}_{\rm tree}$ and $\mathcal{M}_{\rm loop}$ respectively.
$\mathcal{M}_{\rm tree}$ reads

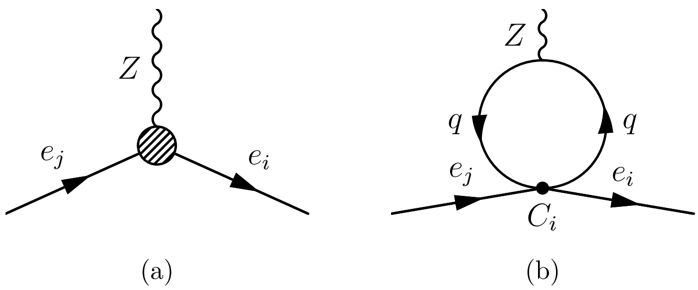


Figure 2.1: Feynman diagrams involved in the process $Z \to \bar{e}_i e_j$.

$$\begin{aligned}\mathcal{M}_{\rm tree} &= -i\frac{g_2}{c_{\rm W}}\bar{u}_i \not{\epsilon} \left[(g^e_{L,\rm SM} + \Delta g^e_L)_{ij} P_L + (g^e_{R,\rm SM} + \Delta g^e_R)_{ij} P_R\right] v_j \\ &\equiv \mathcal{M}_{\rm SM} + \Delta\mathcal{M}_L + \Delta\mathcal{M}_R\,. \end{aligned} \tag{2.36}$$

$\mathcal{M}_{\rm loop}$ receives contribution from six different diagrams, corresponding to the six operators in (2.14) which can be inserted in the vertex:

(a) $\frac{(C_1-C_3)}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu\lambda_u u_L)$

(b) $\frac{(C_1+C_3)}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_L\gamma_\mu\lambda_d d_L)$

(c) $\frac{C_4}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_R\gamma^\mu\Gamma_d d_R)$.

(d) $\frac{C_5}{\Lambda^2}(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{d}_R\gamma^\mu\Gamma_d d_R)$

(e) $\frac{C_6}{\Lambda^2}(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{u}_L\gamma^\mu\lambda_u u_L)$

(f) $\frac{C_6}{\Lambda^2}(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{d}_L\gamma^\mu\lambda_d d_L)$

Up- and down-type quarks run in the loop. The diagrams corresponding to (a)-(c) cancel the μ dependence in Δg^e_L, the ones with (d)-(f) cancel the μ dependence in Δg^e_R. Denoting these two amplitudes as $\mathcal{M}_L$ and $\mathcal{M}_R$ respectively, so that $\mathcal{M}_{\rm loop} = \mathcal{M}_L + \mathcal{M}_R$, we should find that $\mathcal{M}_L + \Delta\mathcal{M}_L$ and $\mathcal{M}_R + \Delta\mathcal{M}_R$ do not depend on the renormalization scale.
We show the explicit computation in the case of $\mathcal{M}_L$, starting from the loop diagram given by

the insertion of (a). Let k and p denote the loop and the Z momentum respectively; Z is taken to be on shell, so that $p^2 = m_Z^2$. In dimensional regularization we find

$$\mathcal{M}_a = 3\,\frac{C_1 - C_3}{\Lambda^2}\,\frac{g_2}{c_W}\,\varepsilon^\nu\,\lambda^e_{ij}\,(\bar{u}_i\gamma^\mu P_L v_j)\sum_l \lambda^u_{ll}\,\mathcal{I}_l\,, \tag{2.37}$$

where the integral $\mathcal{I}_l$ is given by

$$\mathcal{I}_l = \mu^{4-D}\int \frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}\left[(\not{k} + m_{u_l})\gamma_\mu P_L(\not{k}+\not{p}+m_{u_l})\gamma_\nu(g^u_L P_L + g^u_R P_R)\right]}{(k^2 - m^2_{u_l})((k+p)^2 - m^2_{u_l})}\,. \tag{2.38}$$

Hereafter for simplicity we write g^f_L, g^f_R instead of $g^f_{L,\mathrm{SM}}$, $g^f_{R,\mathrm{SM}}$. Using (A.2) we compute the trace in the numerator, obtaining

$$\begin{aligned}\mathrm{Tr}\,[\dots] &= g^u_L\,\mathrm{Tr}\left[\not{k}\gamma_\mu(\not{k}+\not{p})\gamma_\nu P_L\right] + g^u_R\,m^2_{u_l}\,\mathrm{Tr}\,[\gamma_\mu P_L\gamma_\nu P_R]\\ &= 2g^u_L\left[k_\mu(k+p)_\nu + k_\nu(k+p)_\mu - k\cdot(k+p)\eta_{\mu\nu} + 2ik^\alpha p^\beta\epsilon_{\alpha\mu\beta\nu}\right] + 2g^u_R m^2_{u_l}\eta_{\mu\nu}\,.\end{aligned} \tag{2.39}$$

We then recast the denominator using the Feynman parametrization (A.3) with $\mathrm{A} = (k+p)^2 - m^2_{u_l}$ and $\mathrm{B} = k^2 - m^2_{u_l}$. Noting that $Ax + B(1-x) = (k+px)^2 - (m^2_{u_l} - p^2x(1-x)) = (k+px)^2 - \Delta$ we get

$$\mathcal{I}_l = \mu^{4-D}\int_0^1 dx \int \frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}[\dots]}{((k+px)^2 - \Delta)^2}\,. \tag{2.40}$$

By performing the shift $k \to k - px$, we obtain

$$\mathrm{Tr}[\dots]\ \to 2\eta_{\mu\nu}\left[g^u_L\left(\left(\frac{2}{D} - 1\right)k^2 + p^2x(1-x)\right) + g^u_R m^2_{u_l}\right]\,, \tag{2.41}$$

where we have kept only even powers of k and used the substitution $k_\mu k_\nu \to \frac{k^2}{D}\eta_{\mu\nu}$[2]. We also used the fact that terms proportional to $p_\mu p_\nu$ give a zero contribution in the massless limit thanks to the Dirac equation. The integral becomes

$$\begin{aligned}\mathcal{I}_l &= 2\eta_{\mu\nu}\int dx\left[-\frac{g^u_L}{2}\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\frac{k^2}{(k^2-\Delta)^2} + \left(g^u_R m^2_{u_l} + g^u_L p^2 x(1-x)\right)\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-\Delta)^2}\right]\\ &\simeq \frac{2i\eta_{\mu\nu}}{(4\pi)^2}\int dx\left(2m^2_{u_l}(g^u_L - g^u_R) - 2g^u_L m^2_Z x(1-x)\right)\ln\left(\frac{\mu^2}{m^2_{u_l} - m^2_Z x(1-x)}\right)\,.\end{aligned} \tag{2.42}$$

Here we used the dimensional regularization integrals (A.5) and (A.4) and worked in the $\overline{\mathrm{MS}}$ scheme, keeping only leading logarithms. Plugging this into (2.37) and remembering that the only non-zero quark mass is m_t^2, we find

$$\begin{aligned}\mathcal{M}_a = &-\frac{6i}{(4\pi)^2}\frac{C_1 - C_3}{\Lambda^2}\frac{g_2}{c_W}(\bar{u}\not{\varepsilon}\lambda^e P_L v)\left[2g^u_A\lambda^u_{33}m_t^2\ln\mu^2 - 2g^u_L m^2_Z\int dx\ x\,(1-x)\ln\mu^2\right.\\ &-\lambda^u_{33}\int dx\left(2m_t^2 g^u_A - 2g^u_L m^2_Z x(1-x)\right)\ln(m_t^2 - m^2_Z x(1-x))\\ &\left.+\,2g^u_L m^2_Z(1-\lambda^u_{33})\int dx\ x\,(1-x)\ln(-m^2_Z x\,(1-x))\right]\,.\end{aligned} \tag{2.43}$$

[2]We can do this because $\int d^Dk\ k_\mu\ k_\nu\ f(k^2) = \frac{\eta_{\mu\nu}}{D}\int d^Dk\ k^2\ f(k^2)$

The computation for the (b) operator is analogous and gives

$$\begin{aligned}\mathcal{M}_b = &-\frac{6i}{(4\pi)^2}\frac{C_1+C_3}{\Lambda^2}\frac{g_2}{c_{\text{W}}}(\bar{u}\not{\epsilon}\lambda^e P_L v)\left[-2g_L^d m_{\text{Z}}^2\int dx\; x\,(1-x)\ln\mu^2\right.\\ &\left.+2m_{\text{Z}}^2 g_L^d\int dx\; x\,(1-x)\ln\left(-m_{\text{Z}}^2 x\,(1-x)\right)\right]\,,\end{aligned} \tag{2.44}$$

where we considered all down-type quarks to be massless. $\mathcal{M}_c$ can be easily obtained by substituting $C_1+C_3\to C_4$ and $g_L^d\to g_R^d$ in the expression for $\mathcal{M}_b$.
In conclusion, summing the amplitudes (a)-(c) we obtain

$$\begin{aligned}\mathcal{M}_L = &-\frac{i}{(4\pi)^2}\frac{g_2}{c_{\text{W}}}(\bar{u}\not{\epsilon}\lambda^e P_L v)\frac{v^2}{\Lambda^2}\left[\frac{3}{2}\lambda_{33}^u y_t^2(C_1-C_3)\int dx\ln\frac{m_t^2-m_{\text{Z}}^2\,x\,(1-x)}{\mu^2}\right.\\ &-6\left(\frac{g_1^2}{6}(C_1-C_4)+\frac{g_2^2}{2}C_3\right)\int dx\; x\,(1-x)\ln\frac{-m_{\text{Z}}^2\,x\,(1-x)}{\mu^2}\\ &\left.+3\lambda_{33}^u\left(\frac{1}{2}-\frac{2}{3}s_{\text{W}}^2\right)\frac{g_2^2}{c_{\text{W}}^2}\int dx\; x\,(1-x)\ln\frac{m_t^2-m_{\text{Z}}^2\,x\,(1-x)}{-m_{\text{Z}}^2 x\,(1-x)}\right]\,.\end{aligned} \tag{2.45}$$

By comparing this expression with $\Delta\mathcal{M}_L$ we see that the μ-dependence cancels in the sum, which can be written in a compact way as

$$\mathcal{M}_L+\Delta\mathcal{M}_L = -i\frac{g_2}{c_{\text{W}}}(\bar{u}_i\not{\epsilon}\lambda^e v_j)(\delta g_L^e)_{ij}\,, \tag{2.46}$$

where

$$(\delta g_L^e)_{ij} = \frac{v^2}{\Lambda^2}\left[\frac{3}{2}\lambda_{33}^u y_t^2(C_1-C_3)\mathcal{I}_1-6\left(\frac{g_2^2}{2}C_3+\frac{g_1^2}{6}(C_1-C_4)\right)\mathcal{I}_2+3\lambda_{33}^u\left(\frac{1}{2}-\frac{2}{3}s_{\text{W}}^2\right)\frac{g_2^2}{c_{\text{W}}^2}\mathcal{I}_3\right]\frac{\lambda_{ij}^e}{16\pi^2}\,, \tag{2.47}$$

and the integrals $\mathcal{I}_{1-3}$ are defined as

$$\begin{aligned}\mathcal{I}_1 &= \int_0^1 dx\ln\frac{m_t^2-m_{\text{Z}}^2\,x\,(1-x)}{\Lambda^2}\\ \mathcal{I}_2 &= \int_0^1 dx\; x\,(1-x)\ln\frac{-m_{\text{Z}}^2\,x\,(1-x)}{\Lambda^2}\\ \mathcal{I}_3 &= \int_0^1 dx\; x\,(1-x)\ln\frac{m_t^2-m_{\text{Z}}^2\,x\,(1-x)}{-m_{\text{Z}}^2\,x\,(1-x)}\,.\end{aligned} \tag{2.48}$$

In order to find δg_R^e from δg_L^e we just need to take $C_3=0$ and substitute C_1, C_4, λ_{ij}^e with C_6, C_5, Γ_{ij}^e respectively. We obtain

$$(\delta g_R^e)_{ij} = \frac{v^2}{\Lambda^2}\left[\frac{3}{2}\lambda_{33}^u y_t^2 C_6\mathcal{I}_1-g_1^2(C_6-C_5)\mathcal{I}_2+3\lambda_{33}^u\left(\frac{1}{2}-\frac{2}{3}s_{\text{W}}^2\right)\frac{g_2^2}{c_{\text{W}}^2}\mathcal{I}_3\right]\frac{\Gamma_{ij}^e}{16\pi^2}\,. \tag{2.49}$$

In leading logarithm approximation, that is keeping only terms of the form $\ln\frac{\Lambda}{m_{t,Z}}$ and neglecting finite terms, the effective Z couplings to charged leptons read

$$\begin{aligned}(\delta g_L^e)_{ij} &= \frac{v^2}{\Lambda^2}\left[3\lambda_{33}^u y_t^2(C_1-C_3)\ln\frac{\Lambda}{m_t}+g_2^2C_3\ln\frac{\Lambda}{m_{\text{Z}}}+\frac{g_1^2}{3}(C_1-C_4)\ln\frac{\Lambda}{m_{\text{Z}}}\right]\frac{\lambda_{ij}^e}{16\pi^2}\\ (\delta g_R^e)_{ij} &= \frac{v^2}{\Lambda^2}\left[3\lambda_{33}^u y_t^2 C_6\ln\frac{\Lambda}{m_t}+\frac{g_1^2}{3}(C_6-C_5)\ln\frac{\Lambda}{m_{\text{Z}}}\right]\frac{\Gamma_{ij}^e}{16\pi^2}\,.\end{aligned} \tag{2.50}$$

Note that here we resctricted our analysis to Z couplings to leptons; obviously a similar procedure could be followed for the effective Z couplings to quarks by considering the decay $Z \to \bar{q}q$ and for the effective W couplings by considering the processes $W \to \bar{\nu}e$ and $W \to \bar{u}d$.

2.2.2 Integrating out W and Z at tree level

Heavy degrees of freedom, namely Z, W and the top quark, have to be integrated out when going below the EW scale; we start by integrating out the massive bosons at tree level.
Given the complete Lagrangian (2.35), we integrate out Z and W at tree level exactly like in the Fermi theory. This gives

$$\begin{aligned}\mathcal{L}^{\text{tot}}_{\text{Z,W}} &= -\frac{2}{v^2}\left(J^{\mu,0}J^0_\mu + J^{\mu,+}J^-_\mu\right)\\ &\approx -\frac{2}{v^2}\left(J^{\mu,0}_{\text{SM}}J^0_{\mu,\text{SM}} + J^{\mu,+}_{\text{SM}}J^-_{\mu,\text{SM}}\right) - \frac{2}{v^2}\left(2J^0_{\mu,\text{SM}}J^{\mu,0}_{\text{NP}} + J^{\mu,+}_{\text{NP}}J^-_{\mu,\text{SM}} + \text{h.c.}\right)\,. \end{aligned} \tag{2.51}$$

We keep only terms linear in the NP contribution, i.e. linear in Δg. The first term is the Fermi Lagrangian and belongs to $\mathcal{L}_{\text{SM}}$, while the second one enters the effective Lagrangian, and we denote it by $\delta\mathcal{L}^*_{\text{V}}$. At this point our effective Lagrangian reads

$$\mathcal{L}_{\text{eff}} = \delta\mathcal{L}_{\text{L}} + \delta\mathcal{L}_{\text{SL}} + \delta\mathcal{L}^*_{\text{V}}\,. \tag{2.52}$$

The next step is to rewrite $\mathcal{L}_{\text{eff}}$ in the mass basis using (1.8). Operators with a 3333, $33ss$, $ss33$ and $3ss3$ flavor structure appear in the Lagrangian with coefficients which do not depend on flavor indices. They can all be rewritten using the following two simple relations

$$\bar{f}'_{3L}\gamma_\mu f'_{3L} = \bar{f}_L\gamma_\mu\lambda^f f_L \tag{2.53}$$
$$\sum_s \bar{f}'_s\gamma_\mu f'_s = \bar{f}\gamma_\mu f\,. \tag{2.54}$$

The first one follows directly from (1.8) and (2.8); the same relation holds for right-handed fields once we substitute λ^f with Γ^f. The second relation holds for both left- and right-handed fields and can be easily obtained by remembering that rotation matrices are unitary.
Particular attention is needed when considering operators with structure $33st$ in $\delta\mathcal{L}_{\text{SL}}$, because their coefficients depend on the flavor indices s,t. From $Y_u = R_u Y_u^D V_u$ and $Y_u^D = P_3\, y^t$ we have

$$\begin{aligned}[][Y_u^\dagger Y_u]_{33} &= y_t^2\ \lambda^u_{33}\\ [Y_u^\dagger Y_u]_{s3} &= y_t^2\ V_{us3}V^*_{s33}\\ [Y_u^\dagger Y_u]_{3t} &= y_t^2\ V_{u33}V^*_{ut3}\\ [Y_u]_{s3}[Y_u^\dagger]_{3t} &= y_t^2\ \lambda^u_{33}\ R_{us3}R^*_{ut3}\,. \end{aligned} \tag{2.55}$$

Using these relations and the unitarity of rotation matrices we can show that the following two relations hold

$$\sum_{st}\left([Y_u^\dagger Y_u]_{s3}\delta_{3t} + \delta_{s3}[Y_u^\dagger Y_u]_{3t}\right)V_{uti}V^*_{usj} = y_t^2(P_3\lambda^u + \lambda^u P_3)_{ij} \tag{2.56}$$
$$\sum_{st}[Y_u]_{s3}[Y_u^\dagger]_{3t}R^*_{usj}R_{uti} = y_t^2\ \lambda^u_{33}\ \delta_{3j}\delta_{i3}\,. \tag{2.57}$$

These can be used to rewrite the terms containing the Yukawas in $\mathcal{L}_{\text{SL}}$. For example, in

$$\sum_{st}\left([Y_u^\dagger Y_u]_{s3}\delta_{3t}+\delta_{s3}[Y_u^\dagger Y_u]_{3t}\right)Q^{(3)}_{\substack{lq\\33st}} \tag{2.58}$$

we can write $Q^{(3)}_{\substack{lq\\33st}}$ as

$$\begin{aligned}Q^{(3)}_{\substack{lq\\33st}} &= (\bar{\ell}'_{3L}\gamma_\mu\ell'_{3L})(\bar{q}'_{sL}\gamma^\mu q'_{tL})\\ &= \sum_{ij}(\bar{\ell}_L\gamma_\mu\lambda^e\ell_L)(V_{uti}V^*_{usj}\bar{u}_{jL}\gamma^\mu u_{iL}+V_{dti}V^*_{dsj}\bar{d}_{jL}\gamma^\mu d_{iL})\\ &= \sum_{ij}(\bar{\ell}_L\gamma_\mu\lambda^e\ell_L)V_{uti}V^*_{usj}(\bar{u}_{jL}\gamma^\mu u_{iL}+\bar{d}^{\text{CKM}}_{jL}\gamma^\mu d^{\text{CKM}}_{iL})\,,\end{aligned} \tag{2.59}$$

where $d_L^{\text{CKM}}=V_{\text{CKM}}d_L$ and we used the property $\lambda^u=V_{\text{CKM}}\lambda^d V^\dagger_{\text{CKM}}$. Plugging (2.59) into (2.58) we get

$$\begin{aligned}&\sum_{ij}(\bar{\ell}_L\gamma_\mu\lambda^e\ell_L)\sum_{st}\left([Y_u^\dagger Y_u]_{s3}\delta_{3t}+\delta_{s3}[Y_u^\dagger Y_u]_{3t}\right)V_{uti}V^*_{usj}(\bar{u}_{jL}\gamma^\mu u_{iL}+\bar{d}^{\text{CKM}}_{jL}\gamma^\mu d^{\text{CKM}}_{iL})\\ &=(\bar{\ell}_L\gamma_\mu\lambda^e\ell_L)\left(\bar{u}_L(\lambda_uP_3+P_3\lambda_u)u_L+\bar{d}^{\text{CKM}}_L(\lambda_uP_3+P_3\lambda_u)d^{\text{CKM}}_L\right)\,,\end{aligned} \tag{2.60}$$

where we used (2.56).
By carefully writing $\mathcal{L}_{\text{eff}}$ in the mass basis using these properties we get the effective Lagrangian at $\mu=m_{\text{EW}}$ with Z and W integrated out at tree level:

$$\mathcal{L}_{\text{eff}}=\frac{1}{\Lambda^2}\sum_i C_i(m_{\text{EW}})Q_i=\frac{1}{16\pi^2\Lambda^2}\ln\frac{\Lambda}{m_{\text{EW}}}\sum_i\xi_iQ_i\,. \tag{2.61}$$

The operators Q_i and their coefficients ξ_i are listed in the tables below.

Q_i	ξ_i
$(\bar{\nu}_L\gamma_\mu\nu_L)(\bar{\nu}_L\gamma_\mu\lambda_e\nu_L)$	$-6y_t^2\lambda^u_{33}(C_1+C_3)$
$(\bar{\nu}_L\gamma_\mu\nu_L)(\bar{e}_L\gamma_\mu\lambda_e e_L)$	$-6y_t^2\lambda^u_{33}(C_1+C_3)$
$(\bar{\nu}_L\gamma_\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu\nu_L)$	$-12y_t^2\lambda^u_{33}C_3$
$(\bar{e}_L\gamma_\mu\lambda_e\nu_L)(\bar{\nu}_L\gamma_\mu e_L)$	$-12y_t^2\lambda^u_{33}C_3$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{e}_L\gamma_\mu e_L)$	$\frac{4}{3}e^2\left[(C_1-C_4)+3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1+C_3\right]\left(-\frac{1}{2}+s^2_{\text{W}}\right)$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{e}_R\gamma_\mu e_R)$	$\frac{4}{3}e^2\left[(C_1-C_4)+3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1+C_3)\right)s^2_{\text{W}}$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu e_L)$	$\frac{4}{3}e^2\left[(C_1-C_4)-3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1-C_3\right)\left(-\frac{1}{2}+s^2_{\text{W}}\right)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_R\gamma_\mu e_R)$	$\frac{4}{3}e^2\left[(C_1-C_4)-3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1-C_3)\right)s^2_{\text{W}}$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{e}_L\gamma_\mu e_L)$	$-\frac{4}{3}e^2(C_5-C_6)-12C_6\lambda^u_{33}y_t^2(-\frac{1}{2}+s^2_{\text{W}})$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{e}_R\gamma_\mu e_R)$	$-\frac{4}{3}e^2(C_5-C_6)-12C_6\lambda^u_{33}y_t^2s^2_{\text{W}}$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{\nu}_L\gamma^\mu\nu_L)$	$-6C_6\lambda^u_{33}y_t^2$

Table 2.2: Leptonic operators: running from Λ to m_{EW}.

Q_i	ξ_i
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu\lambda_u u_L)$	$(g_1^2+3g_2^2)C_1-(g_1^2+15g_2^2)C_3$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{d}_L^{\rm CKM}\gamma^\mu\lambda_u d_L^{\rm CKM})$	$(g_1^2+3g_2^2)C_1-(g_1^2+15g_2^2)C_3$
$(\bar{e}_L\gamma^\mu e_L)(\bar{u}_L\gamma_\mu\lambda_u u_L)$	$-\frac{4}{3}e^2(C_1+C_6-C_3)$
$(\bar{e}_R\gamma^\mu e_R)(\bar{u}_L\gamma_\mu\lambda_u u_L)$	$-\frac{4}{3}e^2(C_1+C_6-C_3)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_L^{\rm CKM}\gamma_\mu\lambda_u d_L^{\rm CKM})$	$(g_1^2-3g_2^2)(C_1+C_3)$
$(\bar{e}_L\gamma^\mu e_L)(\bar{d}_L^{\rm CKM}\gamma_\mu\lambda_u d_L^{\rm CKM})$	$-\frac{4}{3}e^2(C_1+C_6+C_3)$
$(\bar{e}_R\gamma^\mu e_R)(\bar{d}_L^{\rm CKM}\gamma_\mu\lambda_u d_L^{\rm CKM})$	$-\frac{4}{3}e^2(C_1+C_6+C_3)$
$(\bar{e}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\lambda_\mu\lambda_u d_L^{\rm CKM})$	$-6g_2^2C_1+2(6g_2^2+g_1^2)C_3$
$(\bar{e}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\gamma_\mu d_L^{rmCKM})$	$-12y_t^2C_3$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\gamma_\mu u_L)$	$-\frac{8}{9}e^2\left[(C_1-C_4)+3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1+C_3\right)\left(\frac{1}{2}-\frac{2}{3}s_W^2\right)$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_R\gamma_\mu u_R)$	$-\frac{8}{9}e^2\left[(C_1-C_4)+3C_3\right]+8y_t^2\lambda^u_{33}\left(C_1+C_3\right)\left(-\frac{1}{2}+s_{\rm W}^2\right)$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{d}_L\gamma_\mu d_L)$	$+\frac{4}{9}e^2\left[(C_1-C_4)+3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1+C_3\right)\left(-\frac{1}{2}+\frac{1}{3}s_{\rm W}^2\right)$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{d}_R\gamma_\mu d_R)$	$+\frac{4}{9}e^2\left[(C_1-C_4)+3C_3\right]-4y_t^2\lambda^u_{33}\left(C_1+C_3\right)s_{\rm W}^2$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu u_L)$	$-\frac{8}{9}e^2\left[(C_1-C_4)-3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1-C_3\right)\left(\frac{1}{2}-\frac{2}{3}s_{\rm W}^2\right)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_R\gamma_\mu u_R)$	$-\frac{8}{9}e^2\left[(C_1-C_4)-3C_3\right]+8y_t^2\lambda^u_{33}\left(C_1-C_3\right)\left(-\frac{1}{2}+s_{\rm W}^2\right)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_L\gamma_\mu d_L)$	$+\frac{4}{9}e^2\left[(C_1-C_4)-3C_3\right]-12y_t^2\lambda^u_{33}\left(C_1-C_3\right)\left(-\frac{1}{2}+\frac{1}{3}s_{\rm W}^2\right)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_R\gamma_\mu d_R)$	$+\frac{4}{9}e^2\left[(C_1-C_4)-3C_3\right]-4y_t^2\lambda^u_{33}\left(C_1-C_3\right)s_{\rm W}^2$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}_L\gamma_\mu e_L)$	$-\frac{4}{3}e^2(C_4+C_5)$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}_R\gamma_\mu e_R)$	$-\frac{4}{3}e^2(C_4+C_5)$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{u}_L\gamma_\mu u_L)$	$\frac{8}{9}e^2(C_5-C_6)-12y_t^2\lambda^u_{33}C_6(\frac{1}{2}-\frac{2}{3}s_{\rm W}^2)$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{u}_R\gamma_\mu u_R)$	$\frac{8}{9}e^2(C_5-C_6)+8y_t^2\lambda^u_{33}C_6s_{\rm W}^2$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{d}_L\gamma_\mu d_L)$	$-\frac{4}{9}e^2(C_5-C_6)-12y_t^2\lambda^u_{33}C_6(-\frac{1}{2}+\frac{1}{3}s_{\rm W}^2)$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{d}_R\gamma_\mu d_R)$	$-\frac{4}{9}e^2(C_5-C_6)-4y_t^2\lambda^u_{33}C_6s_{\rm W}^2$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}_L\gamma_\mu\lambda_e e_L)$	$2g_1^2C_4$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{\nu}_L\gamma_\mu\lambda_e\nu_L)$	$2g_1^2C_4$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}_R\gamma_\mu\Gamma_e e_R)$	$-4g_1^2C_5$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{u}_L\gamma^\mu\lambda_u u_L)$	$-2g_1^2C_6$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{d}_L\gamma^\mu\lambda_d d_L)$	$-2g_1^2C_6$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\gamma_\mu\lambda_u u_L)$	$(g_1^2-3g_2^2)(C_1+C_3)$

Table 2.3: Semi-leptonic operators: running from Λ to $m_{\rm EW}$.

Q_i	ξ_i
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\gamma_\mu(\lambda_u P_3 + P_3\lambda_u)u_L)$	$-\frac{1}{2}y_t^2(C_1+C_3)$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{d}_L^{\text{CKM}}\gamma^\mu(\lambda_u P_3 + P_3\lambda_u)d_L^{\text{CKM}})$	$-\frac{1}{2}y_t^2(C_1-C_3)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu(\lambda_u P_3 + P_3\lambda_u)u_L)$	$-\frac{1}{2}y_t^2(C_1-C_3)$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_L^{\text{CKM}}\gamma_\mu(\lambda_u P_3 + P_3\lambda_u)d_L^{\text{CKM}})$	$-\frac{1}{2}y_t^2(C_1+C_3)$
$(\bar{e}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\lambda_\mu(\lambda_u P_3 + P_3\lambda_u)d_L^{\text{CKM}})$	$-y_t^2 C_3$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{u}_L\gamma_\mu(\lambda_u P_3 + P_3\lambda_u)u_L)$	$-\frac{1}{2}C_6 y_t^2$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{d}_L^{\text{CKM}}\gamma_\mu(\lambda_u P_3 + P_3\lambda_u)d_L^{\text{CKM}})$	$-\frac{1}{2}C_6 y_t^2$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{u}_{3R}\gamma_\mu u_{3R})$	$2y_t^2\lambda_{33}^u C_6$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_{3R}\gamma_\mu u_{3R})$	$2y_t^2\lambda_{33}^u C_1$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_{3R}\gamma_\mu u_{3R})$	$2y_t^2\lambda_{33}^u C_1$

Table 2.4: Semi-leptonic operators with projectors or u_3: running from Λ to m_{EW}.

2.3 Consistency checks

Operators in tables 2.2, 2.3 and 2.4 are generated by one-loop electroweak corrections to the operators in $\mathcal{L}_{\text{NP}}^0$. In this regard two types of diagrams need to be considered:

- Electroweak penguins
- Current-current diagrams

These corrections involve the exchange of Z, W and γ, as well as the exchange of the Higgs and of pseudo-Goldstone bosons. The latter arise from the fact that we choose to work in the R_ξ ($\xi = 1$) gauge. We observe that in our framework Higgs and pseudo-Goldstone bosons interact only with the top quark, because we neglected all other Yukawa couplings; thus corrections with exchange of Higgs and pseudo-Goldstone bosons are responsible for the generation of operators containing the projectors P_3 and the currents with the top $(\bar{u}_{3R}\gamma^\mu u_{3R})$ (see table 2.4). We do not compute all these corrections by hand to find the corresponding anomalous dimension matrix; as stated at the beginning of this chapter, we rely on the renormalization group equations given by [27]. However, we do some explicit consistency checks. These enable us to further check the correctness of RGE and to learn how to compute the relevant types of one-loop corrections, which will be useful for the second part of the running - from m_{EW} down to 1 GeV.

In general in order to compute the full one-loop amplitude of a given process in the effective theory described by Lagrangian $\mathcal{L} = \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{NP}}^0 + \mathcal{L}_{\text{eff}}$ we need to consider three types of contributions: a genuinely tree-level contribution, indipendent from the renormalization scale μ, a formally tree-level contribution induced by RGE (i.e. deriving from $\mathcal{L}_{\text{eff}}$, which we shall refer to as "RGE term") and a one-loop contribution, arising from the interplay of one of the

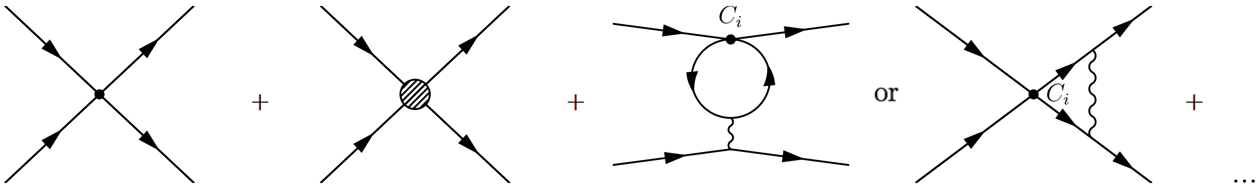


Figure 2.2: General form for the one-loop amplitude of a given process in the effective field theory described by $\mathcal{L}$. From left to right we have a purely tree-level contribution, a RGE contribution - denoted by a shaded circle - and one-loop diagrams (either penguins or current-current diagrams).

four-fermions operators in $\mathcal{L}^0_{\rm NP}$ and SM vertices. For consistency, the μ dependence in the one-loop term should be canceled by the RGE term, provided the wavefunction renormalization has been taken into account.

In our calculations we check that this cancellation actually takes place for some significant cases. In particular we consider electroweak penguins and current-current corrections with Z, W and γ exchange; we do not check the consistency for operators generated by the exchange of the Higgs or pseudo-Goldstone bosons. For simplicity we choose four-fermion processes involving particles with definite chiralities and mediated by a single Q_i in the RGE term.

Note that all external momenta can be taken to be zero, because the cancellation must take place independently from the particles momenta. In fact, the ξ_i do not depend on momenta, so the one-loop term they have to cancel can't depend on momenta either. Clearly if we were interested in the physical amplitude of the process we should take into account physical, on shell external momenta.
Hereafter - unless otherwise specified - dots stand for terms independent from the renormalization scale.

2.3.1 Consistency check for $(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu e_L)$

We start by considering the neutral-current lepton-flavor-violating process $\mu^-_L e^+_R \to e^-_L e^+_R$. The relevant diagrams are displayed in figure 2.3.
The only tree-level contribution comes from $\mathcal{L}_{\rm eff}$. In particular, given the chiralities of the particles involved, the only relevant operator in $\mathcal{L}_{\rm eff}$ is $(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu e_L)$, whose ξ is given in table (2.3). Therefore the tree-level amplitude reads

$$\mathcal{M}_{\rm tree} = \frac{i}{16\pi^2\Lambda^2}\ln\frac{\Lambda^2}{\mu^2}\left[\frac{2}{3}(C_1 - C_4 - 3C_3) - 6y_t^2\lambda^u_{33}(C_1 - C_3)\left(-\frac{1}{2} + s^2_{\rm W}\right)\right]$$
$$\lambda^e_{12}(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L v)\,. \tag{2.62}$$

The diagrams contributing to the process at one loop are the penguins in figure 2.3(b) and 2.3(c), which involve a quark loop and the exchange of a photon and of a Z boson respectively. Starting from the electromagnetic penguin, we observe that the operators we can insert in the vertex labelled with C_i are the ones involving a charged lepton current and an up/down-type

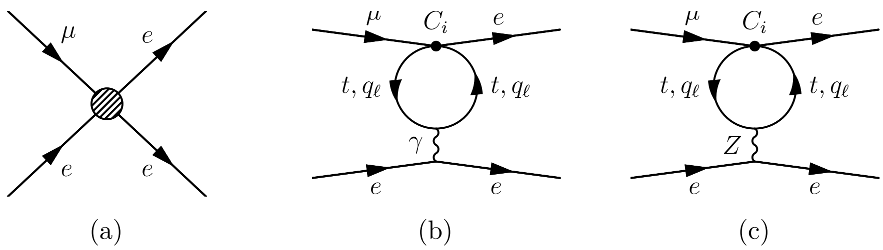


Figure 2.3: Diagrams contributing to the process $\mu_L^- e_R^+ \to e_L^- e_R^+$ in the effective theory above $m_{\rm EW}$. From left to right: RGE contribution, electromagnetic penguin and Z penguin.

quark current:

$$\begin{aligned}\mathcal{L}^0_{\rm NP} =& \frac{C_1 - C_3}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu\lambda_u u_L) + \frac{C_1 + C_3}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_L\gamma_\mu\lambda_d d_L) \\ &+ \frac{C_4}{\Lambda^2}(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}_R\gamma_\mu\Gamma_d d_R) + \dots\,, \end{aligned} \tag{2.63}$$

where dots stand for operators irrelevant to the process. Taking all three relevant operators into account, we obtain the following expression for the amplitude of the electromagnetic penguin

$$\begin{aligned}\mathcal{M}_\gamma = \frac{i}{16\pi^2\Lambda^2}\Bigg[&\frac{4}{3}e^2(C_1 - C_3)\left(\lambda^u_{11}\ln\frac{\mu^2}{m_u^2} + \lambda^u_{22}\ln\frac{\mu^2}{m_c^2} + \lambda^u_{33}\ln\frac{\mu^2}{m_t^2}\right) \\ &- \frac{2}{3}e^2(C_1 + C_3)\left(\lambda^d_{11}\ln\frac{\mu^2}{m_d^2} + \lambda^d_{22}\ln\frac{\mu^2}{m_s^2} + \lambda^d_{33}\ln\frac{\mu^2}{m_b^2}\right) \\ &- \frac{2}{3}e^2 C_4\left(\Gamma^d_{11}\ln\frac{\mu^2}{m_d^2} + \Gamma^d_{22}\ln\frac{\mu^2}{m_s^2} + \Gamma^d_{33}\ln\frac{\mu^2}{m_b^2}\right)\Bigg]\lambda^e_{12}(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L u)\,. \end{aligned} \tag{2.64}$$

The detailed computation can be found in the appendix section A.2.1.1.
As to the Z penguin, in theory we should consider exactly the same operator insertions we considered for the electromagnetic one. However, its amplitude is proportional to the Yukawa coupling of the quark running in the loop[3], so it is sufficient to consider the diagram with the virtual top. The final result reads

$$\mathcal{M}_{\rm Z} = \frac{i}{16\pi^2\Lambda^2}\left[-6y_t^2\lambda^u_{33}(C_1 - C_3)\left(-\frac{1}{2} + s^2_{\rm W}\right)\ln\frac{\mu^2}{m_t^2}\right]\lambda^e_{12}(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L u)\,. \tag{2.65}$$

The complete computation is given in the appendix (see section A.2.1.2).
Summing all contributions we get

$$\begin{aligned}\mathcal{M}_{\rm tot} &= \mathcal{M}_{\rm tree} + \mathcal{M}_\gamma + \mathcal{M}_{\rm Z} \\ &= \frac{i}{16\pi^2\Lambda^2}\lambda^e_{12}(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L u)\Bigg[\frac{2}{3}e^2(C_1 - C_4 - 3C_3)\ln\frac{\Lambda^2}{M^2} \\ &\quad -6y_t^2\lambda^u_{33}(C_1 - C_3)\left(-\frac{1}{2} + s^2_{\rm W}\right)\ln\frac{\Lambda^2}{m_t^2}\Bigg]\,, \end{aligned} \tag{2.66}$$

[3]This can be easily understood by observing that m_t is the only mass scale besides Λ.

where M^2 arises from the μ-independent part in $\mathcal{M}_\gamma$ and is defined as

$$M^2 = \left[\frac{\left(m_u^{2\lambda_{11}^u}\, m_c^{2\lambda_{22}^u}\, m_t^{2\lambda_{33}^u}\right)^{2(C_1-C_3)}}{\left(m_d^{2\lambda_{11}^d}\, m_s^{2\lambda_{22}^d}\, m_b^{2\lambda_{33}^d}\right)^{C_1+C_3}\left(m_d^{2\Gamma_{11}^d}\, m_s^{2\Gamma_{22}^d}\, m_b^{2\Gamma_{33}^d}\right)^{C_4}}\right]^{\frac{1}{C_1-3C_3-C_4}} . \tag{2.67}$$

As expected, the renormalization scale dependence cancels in the sum.

2.3.2 Consistency check for $(\bar{e}_R\gamma^\mu\Gamma^e e_R)(\bar{d}_R\gamma_\mu\Gamma^d d_R)$

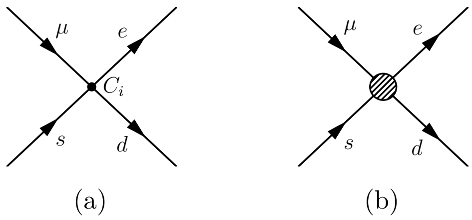


Figure 2.4: Tree-level contribution to the process $\mu_R^- e_L^+ \to d_R\bar{s}_L$. On the left, the purely tree-level contribution from $\mathcal{L}_{\rm NP}^0$; on the right, the RGE contribution from $\mathcal{L}_{\rm eff}$.

Now we are interested in doing a consistency check involving the computation of curren-current corrections. For this purpose we consider the process $\mu_R^- e_L^+ \to d_R\bar{s}_L$; diagrams contributing to the process at tree level and at one loop are shown in figures 2.4 and 2.5.
At tree level the process receives a contribution from both $\mathcal{L}_{\rm NP}^0$ and $\mathcal{L}_{\rm eff}$ through the operator $(\bar{e}_R\gamma^\mu\Gamma^e e_R)(\bar{d}_R\gamma_\mu\Gamma^d d_R)$:

$$\mathcal{M}_{\rm tree} = \frac{iC_5}{\Lambda^2}\left(1 - \frac{2g_1^2}{16\pi^2}\ln\frac{\Lambda^2}{\mu^2}\right)\Gamma_{12}^e\,\Gamma_{12}^d\,(\bar{v}\gamma^\mu P_R u)(\bar{u}\gamma_\mu P_R v)\,. \tag{2.68}$$

One-loop contributions involve the exchange of γ and Z. Diagrams 2.5(c)-2.5(h) concern the vertex correction and wavefunction renormalization. They give no contribution; in fact, as we expect from the QED Ward identity, we find

$$\mathcal{M}_c^\gamma + \mathcal{M}_d^\gamma + \mathcal{M}_e^\gamma = 0 \qquad\qquad \mathcal{M}_f^\gamma + \mathcal{M}_g^\gamma + \mathcal{M}_h^\gamma = 0\,. \tag{2.69}$$

Given the fact that leptons are massless in our framework, the computation for the Z diagrams proceeds in the same way, hence we get

$$\mathcal{M}_c^z + \mathcal{M}_d^z + \mathcal{M}_e^z = 0 \qquad\qquad \mathcal{M}_f^z + \mathcal{M}_g^z + \mathcal{M}_h^z = 0\,. \tag{2.70}$$

Regarding the remaining diagrams, we compute only $\mathcal{M}_i^{\gamma,z}$ and $\mathcal{M}_k^{\gamma,z}$, because $\mathcal{M}_i^{\gamma,z} = \mathcal{M}_j^{\gamma,z}$ and $\mathcal{M}_k^{\gamma,z} = \mathcal{M}_l^{\gamma,z}$. Explicit calculation (see A.2.2) gives

$$\mathcal{M}_i^Z + \mathcal{M}_k^Z = \frac{i}{16\pi^2\Lambda^2}\left(-3\frac{g_2^2}{c_{\rm W}^2}\,g_R^e\,g_R^d\,C_5\right)\left[\ln\frac{\mu^2}{m_{\rm Z}^2} + 1\right]\Gamma_{12}^e\Gamma_{12}^d\,(\bar{v}\gamma_\mu P_R u)\,(\bar{u}\gamma^\mu P_R v)\,. \tag{2.71}$$

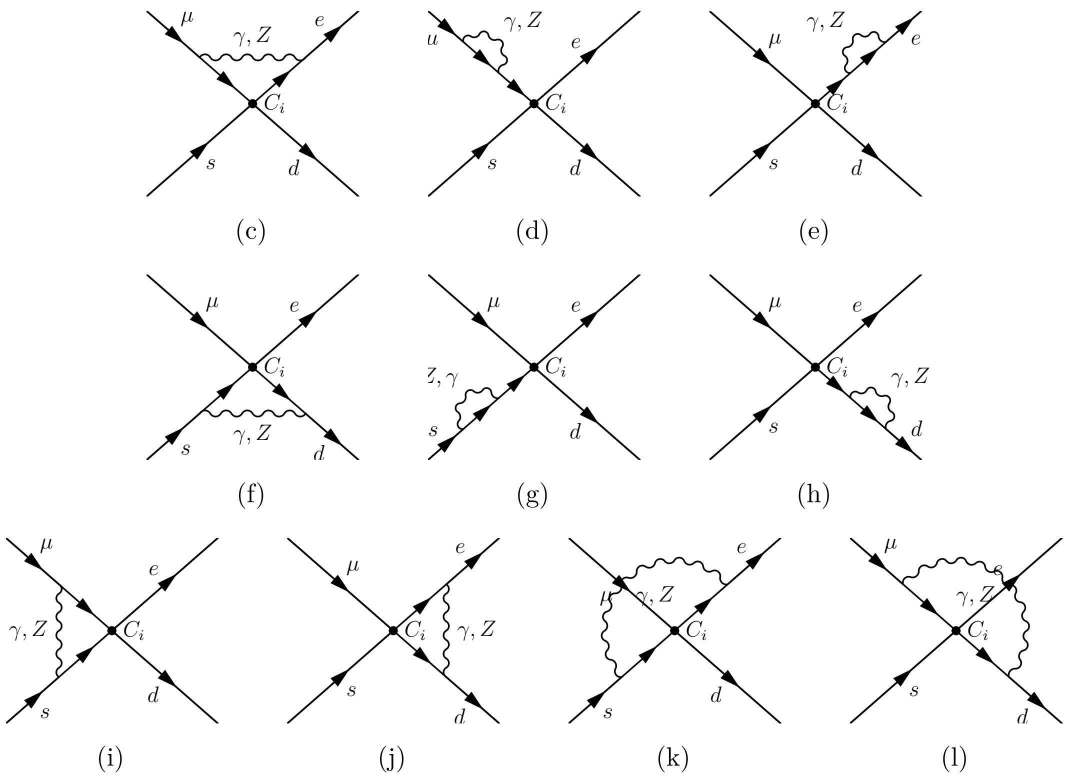


Figure 2.5: One-loop contribution to the process $\mu_R^- e_L^+ \to d_R \bar{s}_L$. Diagrams 2.5(c)-2.5(h) represent the vertex correction and wavefunction renormalization for the two currents in the operator $(\bar{e}_R \gamma^\mu \Gamma^e e_R)(\bar{d}_R \gamma_\mu \Gamma^d d_R)$. The remaining current-current corrections to the operator considered are given by 2.5(i)-2.5(l).

$\mathcal{M}_i^\gamma + \mathcal{M}_k^\gamma$ can be obtained by substituting $m_{\text{Z}}^2 \to m_\gamma^2$ and $\frac{g_2}{c_{\text{W}}}\, g_R^f \to e\, q_f$ in this expression[4]

$$\mathcal{M}_i^\gamma + \mathcal{M}_k^\gamma = \frac{i}{16\pi^2\Lambda^2}\left(-3e^2 q_e q_d C_5\right)\left[\ln\frac{\mu^2}{m_\gamma^2} + 1\right]\Gamma_{12}^e \Gamma_{12}^d \left(\bar{v}\gamma_\mu P_R u\right)\left(\bar{u}\gamma^\mu P_R v\right) , \tag{2.72}$$

hence the total amplitude at one loop reads

$$\begin{aligned}\mathcal{M}_{\text{loop}} &= \mathcal{M}_{\text{tot}}^Z + \mathcal{M}_{\text{tot}}^\gamma = 2\left(\mathcal{M}_i^Z + \mathcal{M}_k^Z\right) + 2\left(\mathcal{M}_i^\gamma + \mathcal{M}_k^\gamma\right) \\ &= -\frac{6i}{16\pi^2\Lambda^2}\left(\frac{g_2^2}{c_{\text{W}}^2}\, g_R^e\, g_R^d\, C_5 + e^2 q_e q_d C_5\right)\ln\mu^2\, \Gamma_{12}^e\, \Gamma_{12}^d\, \left(\bar{v}\gamma_\mu P_R u\right)\left(\bar{u}\gamma^\mu P_R v\right) + \dots .\end{aligned} \tag{2.73}$$

Remembering that $q_e = -1$, $q_d = -\frac{1}{3}$, $g_R^e = -s_{\text{W}}^2$, $g_R^d = -\frac{1}{3}s_{\text{W}}^2$ and $g_1 c_{\text{W}} = e = g_2 s_{\text{W}}$ we find

$$\mathcal{M}_{\text{loop}} = \frac{i}{16\pi^2\Lambda^2}\left(-2g_1^2 C_5\right)\ln\mu^2\, \Gamma_{12}^e\, \Gamma_{12}^d\, \left(\bar{v}\gamma_\mu P_R u\right)\left(\bar{u}\gamma^\mu P_R v\right) + \dots , \tag{2.74}$$

which cancels exactly the renormalization scale dependence in eq. (2.68).

[4]We use m_γ as an IR regulator.

2.3.3 Consistency check for $(\bar{e}_L\gamma^\mu\lambda^e e_L)(\bar{u}_L\gamma_\mu\lambda^u u_L)$

Another interesting case is the one where current-current corrections with W boson exchange appear; as an example we consider the process $\mu^-_L e^+_R \to u_L\bar{c}_R$. Again, we have a tree-level contribution from both $\mathcal{L}^0_{\rm NP}$ and $\mathcal{L}_{\rm eff}$, where the relevant operator is $(\bar{e}_L\gamma^\mu\lambda^e e_L)(\bar{u}_L\gamma_\mu\lambda^u u_L)$:

$$\mathcal{M}_{\rm tree} = \frac{i}{\Lambda^2}\left[(C_1 - C_3) + \frac{1}{32\pi^2}\Big(C_1(g_1^2+3g_2^2) - C_3(g_1^2+15g_2^2)\Big)\ln\frac{\Lambda^2}{\mu^2}\right]\lambda^e_{12}\lambda^u_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}\gamma_\mu P_L v)\,.$$

At one loop we need to compute current-current corrections involving Z, γ and W exchange. Current-current Z and γ corrections are based on the insertion of a single $\mathcal{L}^0_{\rm NP}$ operator, $(\bar{e}_L\gamma^\mu\lambda^e e_L)(\bar{u}_L\gamma_\mu\lambda^u u_L)$. Using the result obtained in the previous subsection with the substitutions $q_d \to q_u$, $g^e_R\, g^d_R \to g^e_L\, g^u_L$ snd $C_5 \to (C_1 - C_3)$, we find

$$\mathcal{M}^\gamma_{tot} + \mathcal{M}^Z_{tot} = \frac{i}{16\pi^2\Lambda^2}\left(\frac{g_1^2}{2}+\frac{3g_2^2}{2}\right)(C_1 - C_3)\ln\mu^2\;\lambda^e_{12}\;\lambda^u_{12}\;(\bar{v}\gamma_\mu P_R u)\,(\bar{u}\gamma^\mu P_R v) + \dots \tag{2.75}$$

Current-current W corrections instead are based on the insertion of four different $\mathcal{L}^0_{\rm NP}$ operators:

- $(\bar{e}_L\gamma^\mu\lambda^e e_L)(\bar{u}_L\gamma_\mu\lambda^u u_L)$ for 2.6(d), 2.6(e), 2.6(g) and 2.6(h)
- $(\bar{\nu}_L\gamma^\mu\lambda^e \nu_L)(\bar{u}_L\gamma_\mu\lambda^u u_L)$ for 2.6(c)
- $(\bar{e}_L\gamma^\mu\lambda^e e_L)(\bar{d}_L\gamma_\mu\lambda^d d_L)$ for 2.6(f)
- $(\bar{e}_L\gamma^\mu\lambda_e \nu_L)(\bar{u}_L\gamma_\mu\lambda_u d_L^{\rm CKM})$ for 2.6(i) and 2.6(j)

Differently from the case with Z and γ, here diagrams describing wave function renormalization and vertex corrections give a non-zero contribution. The results read

$$\begin{aligned}\mathcal{M}_c &= \frac{i}{16\pi^2\Lambda^2}\left(\frac{g_2^2}{2}(C_1+C_3)\right)\ln\mu^2\;\lambda^e_{12}\;\lambda^u_{12}\;(\bar{v}\gamma_\mu P_L u)\,(\bar{u}\gamma^\mu P_L v) + \dots\\ \mathcal{M}_d &= -\frac{i}{16\pi^2\Lambda^2}\left(\frac{g_2^2}{4}(C_1-C_3)\right)\ln\mu^2\;\lambda^e_{12}\;\lambda^u_{12}\;(\bar{v}\gamma_\mu P_L u)\,(\bar{u}\gamma^\mu P_L v) + \dots\end{aligned} \tag{2.76}$$

For the last two diagrams, 2.6(i) and 2.6(j), we need to sum over all possible down quarks in the loop. For the amplitude $\mathcal{M}_i$ we find

$$\mathcal{M}_i = \frac{i}{16\pi^2\Lambda^2}(-4g_2^2C_3)\left[\ln\frac{\mu^2}{m^2_{\rm W}}+1\right]\;\lambda^e_{12}\;\lambda^u_{12}\,(\bar{v}\gamma_\mu P_L u)\,(\bar{u}\gamma^\mu P_L v)\;, \tag{2.77}$$

where the explicit computation can be found in A.2.3.1. Since $\mathcal{M}_c = \mathcal{M}_f$, $\mathcal{M}_d = \mathcal{M}_e = \mathcal{M}_g = \mathcal{M}_h$ and $\mathcal{M}_i = \mathcal{M}_j$, the total contribution given by one-loop W corrections is

$$\mathcal{M}^{W}_{\rm tot} = 2\mathcal{M}_c + 4\mathcal{M}_d + 2\mathcal{M}_g = \frac{i}{16\pi^2\Lambda^2}(-6g_2^2C_3)\ln\mu^2\;\lambda^e_{12}\;\lambda^u_{12}\,(\bar{v}\gamma_\mu P_L u)\,(\bar{u}\gamma^\mu P_L v) + \dots\,, \tag{2.78}$$

therefore the full one-loop contribution reads

$$\begin{aligned}\mathcal{M}_{\rm loop} &= \mathcal{M}^\gamma_{\rm tot} + \mathcal{M}^{Z}_{\rm tot} + \mathcal{M}^{W}_{\rm tot}\\ &= \frac{i}{32\pi^2\Lambda^2}\Big(C_1(g_1^2+3g_2^2) - C_3(g_1^2+15g_2^2)\Big)\ln\mu^2\;\lambda^e_{12}\;\lambda^u_{12}\;(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L v) + \dots\,,\end{aligned} \tag{2.79}$$

which cancels the μ dependence in the formally tree-level RGE contribution.

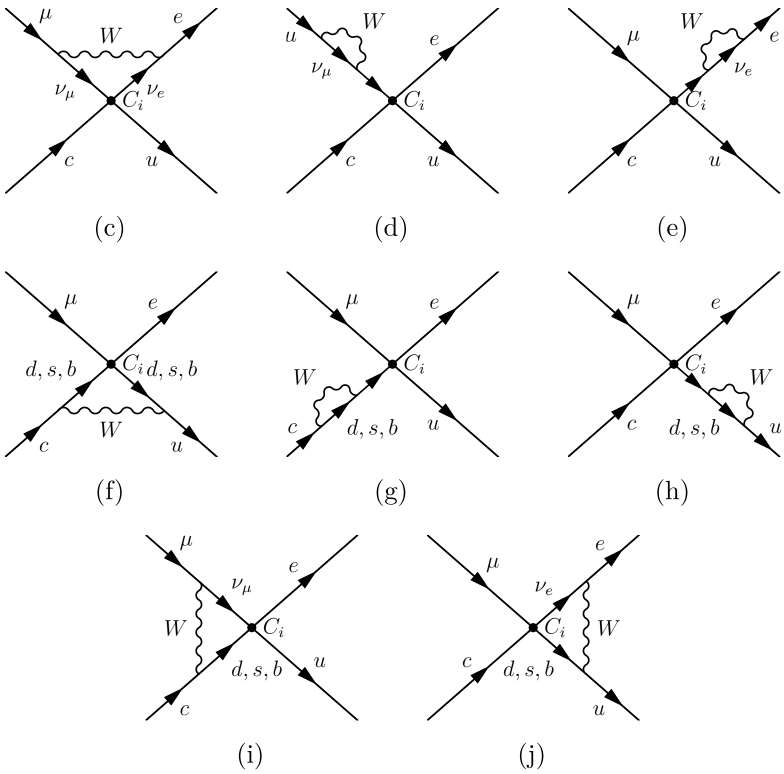


Figure 2.6: One-loop contribution to the process $\mu_L^- e_R^+ \to u_L \bar{c}_R$. Note that in this case the analogous of diagrams 2.5(k) and 2.5(l) cannot be built.

2.3.4 Consistency check for $(\bar{\nu}_L \gamma^\mu \lambda^e e_L)(\bar{e}_L \gamma_\mu \nu_L)$

As a final example we consider a charged-current process, the tau decay $\tau \to \nu_\tau e \bar{\nu}_e$. The corresponding diagrams are shown in figure 2.7. At tree level it receives both a SM and a RGE

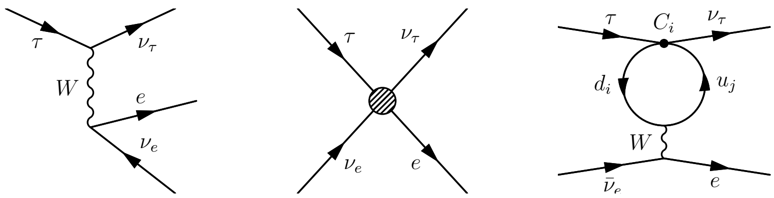


Figure 2.7: Diagrams contributing to the process $\tau \to \nu_\tau e \bar{\nu}_e$. From left to right, SM tree-level contribution, RGE contribution and penguin with W exchange.

contribution from the purely leptonic operator $(\bar{\nu}_L \gamma^\mu \lambda^e e_L)(\bar{e}_L \gamma_\mu \nu_L)$ in $\mathcal{L}_{\text{eff}}$

$$\mathcal{M}_{\text{tree}} = \left[-\frac{i}{m_{\text{W}}^2} + \frac{i}{16\pi^2 \Lambda^2} (-6 y_t^2 \lambda_{33}^u C_3) \ln \frac{\Lambda^2}{\mu^2} \, \lambda_{33}^e \right] (\bar{u} \gamma^\mu P_L u)(\bar{u} \gamma^\mu P_L v) \,. \tag{2.80}$$

The only diagram contributing at one loop is the penguin with exchange of a W boson and insertion of the $\mathcal{L}_{\text{NP}}^0$ operator $(\bar{d}_L \lambda^{du} \gamma^\mu u_L)(\bar{\nu}_L \gamma_\mu \lambda^e e_L)$. In general we should sum over all possible combinations of up and down quarks in the loop, but given our choice for Yukawa couplings the only relevant couples are those involving the top and the down quarks d, s, b. Following the same steps we saw fot the Z penguin, we find the following final result

$$\mathcal{M}_{\text{loop}} = \frac{i}{16\pi^2\Lambda^2}(-6y_t^2\lambda_{33}^u C_3)\ln\frac{\mu^2}{m_t^2}\ \lambda_{33}^e\ (\bar{u}\gamma^\mu P_L u)(\bar{u}\gamma^\mu P_L v) \tag{2.81}$$

As expected, the μ-dependence cancels in the sum of the two amplitudes.
Like this operator, all purely leptonic charged-current operators are not renormalized by QED interactions. In fact, using the Fierz identity, the operator $(\bar{\nu}_{\tau L}\gamma_\mu\tau_L)(\bar{e}_L\gamma^\mu\nu_{eL})$ can be rewritten as $(\bar{\nu}_{\tau L}\gamma_\mu\nu_{eL})(\bar{e}_L\gamma^\mu\tau_L)$, and the charged lepton current $(\bar{e}_L\gamma^\mu\tau_L)$ is not affected by renormalization due to the QED Ward identity.

2.4 Explicit matching at the EW threshold

In section 2.2 we obtained the effective Lagrangian at the scale $m_{\text{EW}} < \mu < \Lambda$, eq. (2.21). This Lagrangian contains all SM fields as dynamical degrees of freedom: heavy fields - W, Z, top quark - and light fields, denoted by ϕ_ℓ. Below the EW threshold heavy fields can be integrated out; the theory is described by a new effective Lagrangian where only light fields appear as dynamical degrees of freedom. As a rule, this Lagrangian will be the most general Lagrangian we can write using the allowed d.o.f. and compatible with the residual $U(1)_{\text{em}}$ symmetry

$$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda^2}\sum_i \bar{\mathcal{C}}_i(\mu)Q_i(\phi_\ell)\,, \tag{2.82}$$

where Q_i are $U(1)_{\text{em}}$ invariant operators containing only light SM fields and $\bar{\mathcal{C}}_i(\mu)$ are their Wilson coefficients. The $\bar{\mathcal{C}}_i(\mu)$ are a priori unknown; they are determined by matching the high-energy effective Lagrangian into the low-energy one.
As discussed in the first chapter, the idea behind the matching procedure is to compute the amplitude of a certain physical process within the high and the low-energy effective theory; these amplitudes are then required to be equal at the matching scale at a given order in the parameters. As an example we consider once more the process $\mu_L^- e_R^+ \to e_L^- e_R^+$. Figures 2.3 and 2.8 show the diagrams contributing to the process in the effective theories described by (2.52)[5] and (2.82). The penguins with light quarks in the loop cancel in the matching and we are left with the equation in figure 2.9. Since the two penguins give a contribution $\propto \ln\frac{m_{\text{EW}}^2}{m_t^2}$, which is zero in the approximation $m_t \approx m_{\text{EW}}$, we find

$$\mathcal{C}(m_{\text{EW}}) = \bar{\mathcal{C}}(m_{\text{EW}})\,, \tag{2.83}$$

where $\mathcal{C}(m_{\text{EW}})$ and $\bar{\mathcal{C}}(m_{\text{EW}})$ denote the coefficients of the operator $(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu e_L)$ in the Lagrangian above and below the EW threshold.

[5] The matching can be performed starting directly from (2.52), where Z and W have already been integrated out at tree level.

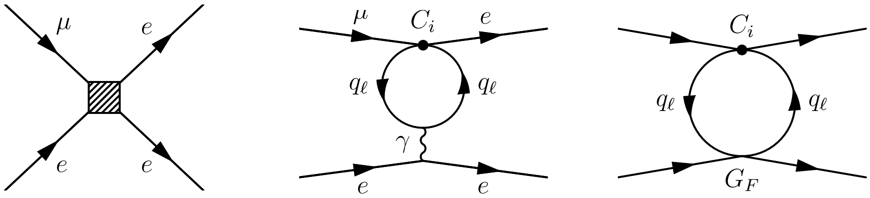


Figure 2.8: Diagrams contributing to the process in the effective theory below $m_{\rm EW}$. Here the RGE contribution is denoted by a shaded square.

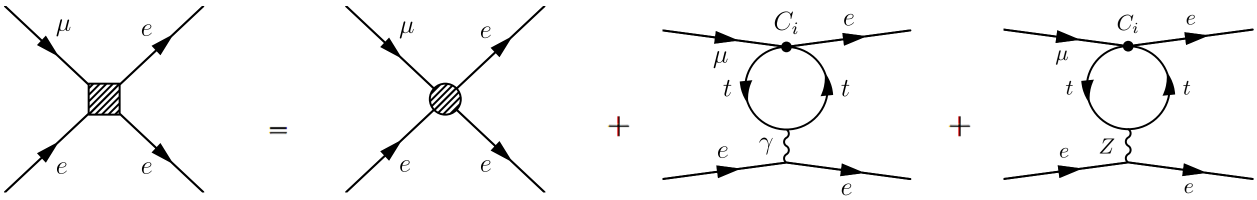


Figure 2.9: Explicit matching for the operator $(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu e_L)$ at the EW scale. Matching between EFT below $m_{\rm EW}$ (on the left) and EFT above $m_{\rm EW}$ (on the right).

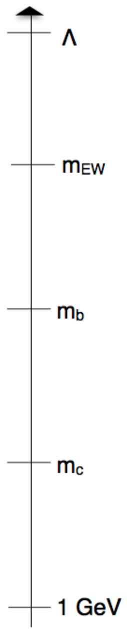
It can be easily seen that this is a general result, valid for all operators listed in tables 2.2-2.4. As a consequence in order to write the Lagrangian (2.82) we just need to take equation (2.52) and to remove t_L and t_R from the final result.

2.5 RGE flow from $m_{\rm EW}$ to the GeV scale

In the previous section we derived the effective Lagrangian at the EW scale after integrating out the massive bosons W and Z and the top quark

$$\mathcal{L}_{\rm eff} = \frac{1}{\Lambda^2}\sum_i \bar{\mathcal{C}}_i(m_{\rm EW})Q_i = \frac{1}{16\pi^2\Lambda^2}\ln\frac{\Lambda}{m_{\rm EW}}\sum_i \xi_i Q_i\,. \tag{2.84}$$

At this point our aim is to find the effective Lagrangian at the GeV scale; in order to do this we need to run the Wilson coefficients from the EW scale down to $\mu \sim 1$ GeV. We observe that in our setup the running below the EW scale does not generate any new operators.
In going from the EW scale to the GeV scale two energy thresholds must be crossed, corresponding to the masses of the bottom and the charm quark, m_b and m_c. When crossing a threshold we integrate out the corresponding matter field, thus reducing the number of dynamical degrees of freedom in the theory. In this sense in every energy range we have a different effective theory:



effective Lagrangian	degrees of freedom
$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda^2}\sum_i \mathcal{C}_i(\mu) Q_i$	$W,\ Z\ , u, c, t, d, s, b, e, \mu, \tau$
$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda^2}\sum_i \bar{\mathcal{C}}_i(\mu) Q_i$	$u, c, d, s, b, e, \mu, \tau$
$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda^2}\sum_i \tilde{\mathcal{C}}_i(\mu) Q_i$	u, c, d, s, e, μ, τ
$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda^2}\sum_i \mathcal{C}^*_i(\mu) Q_i$	u, d, s, e, μ, τ

Confining effective theories need to be matched at the corresponding threshold. Like in the case of the electroweak threshold, matching at m_c and m_b gives $\bar{\mathcal{C}}_i(\mu_b) = \tilde{\mathcal{C}}_i(\mu_b)$, $\tilde{\mathcal{C}}_i(m_c) = \mathcal{C}^*_i(m_c)$ for each Q_i.
Working in leading logarithm approximation, in different ranges the Wilson coefficient for a given Q_i has the form

$$\begin{cases} \bar{\mathcal{C}}_i(\mu) = \bar{\mathcal{C}}_i(m_{\text{EW}}) - \dfrac{b_i}{16\pi^2}\ln\dfrac{m_{\text{EW}}}{\mu} & m_b < \mu < m_{\text{EW}} \quad (2.85)\\ \tilde{\mathcal{C}}_i(\mu) = \tilde{\mathcal{C}}_i(m_b) - \dfrac{c_i}{16\pi^2}\ln\dfrac{m_b}{\mu} & m_c < \mu < m_b \quad (2.86)\\ \mathcal{C}^*_i(\mu) = \mathcal{C}^*_i(m_c) - \dfrac{d_i}{16\pi^2}\ln\dfrac{m_c}{\mu} & 1\text{GeV} < \mu < m_c\,, \quad (2.87)\end{cases}$$

where the coefficients b_i, c_i and d_i have to be determined. Using the matching conditions at m_b and m_c we find the following expression for $\mathcal{C}^*_i(\mu)$

$$\begin{aligned}\mathcal{C}^*_i(\mu) &= \mathcal{C}_i(m_{\text{EW}}) - \frac{1}{16\pi^2}\left(b_i\ \ln\frac{m_{\text{EW}}}{m_b} + c_i\ \ln\frac{m_b}{m_c} + d_i\ \ln\frac{m_c}{\mu}\right)\\ &\equiv \mathcal{C}_i(m_{\text{EW}}) + \delta\mathcal{C}_i(\mu)\,, \end{aligned} \tag{2.88}$$

where $\delta\mathcal{C}_i(\mu)$ encodes the running from m_{EW} to $\mu \sim 1$ GeV. Rewriting it as

$$\delta\mathcal{C}_i(\mu) = \frac{1}{16\pi^2}\ln\frac{m_{\text{EW}}}{\mu}\ \delta\xi_i \tag{2.89}$$

with

$$\delta\xi_i = -\frac{1}{\ln\frac{\mu}{m_{\text{EW}}}}\left(b_i\ \ln\frac{m_{\text{EW}}}{m_b} + c_i\ \ln\frac{m_b}{m_c} + d_i\ \ln\frac{m_c}{\mu}\right)\,, \tag{2.90}$$

the effective Lagrangian at scale $1\ \text{GeV} < \mu < m_c$ reads

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{1}{\Lambda^2}\sum_i \mathcal{C}_i(m_{\text{EW}})Q_i + \frac{1}{\Lambda^2}\sum_i \delta\mathcal{C}_i(\mu)Q_i \\ &= \frac{1}{16\pi^2\Lambda^2}\ln\frac{\Lambda}{m_{\text{EW}}}\sum_i \xi_i Q_i + \frac{1}{16\pi^2\Lambda^2}\ln\frac{m_{\text{EW}}}{\mu}\sum_i \delta\xi_i Q_i\,. \end{aligned} \tag{2.91}$$

It is completely determined once we fix b_i, c_i, d_i for every Q_i. The procedure we used to find these coefficients is the following:

1. Choose a four-fermion process which receives RGE contribution only from Q_i.
2. Within each energy range write the diagrams contributing to the process at the desired order, one loop in our case. As stated previously, the process will in general receive a truly tree-level contribution, a RGE contribution and a one-loop contribution with insertion of $\mathcal{L}^0_{\text{NP}}$ operators. Being under the EWSB scale, the only one-loop diagrams we can build are those involving the exchange of a virtual photon and/or of the virtual fermions which are still dynamical degrees of freedom in the energy range considered. As a consequence there are just two kinds of diagrams involved in this computation: electromagnetic current-current corrections and electromagnetic penguins, both of wich have been analysed in the previous section. Clearly Wilson coefficients whose operator cannot be generated by electromagnetic current-current or penguin corrections to the operators in $\mathcal{L}^0_{\text{NP}}$ are not renormalized by QED and have $\delta\xi = 0$.
3. Range by range require the RGE term to cancel the renormalization scale dependence emerging from the loop diagram. The three resulting conditions enable us to determine b_i, c_i and d_i.

Note that this procedure is completely equivalent to employing renormalization group equations for dimension-six $U(1)_{\text{em}}$-invariant operators; finding b_i, c_i and d_i in fact basically amounts to computing the anomalous dimension matrix for the Q_i. As a matter of principle, one could attempt to recover these equations starting from the RGE for G_{SM}-invariant operators [27] by switching off the $SU(2)_L$ gauge coupling and by appropriately substituting hypercharges with electric charges.

Operators whose Wilson coefficients keep running below the electroweak threshold can be divided in two main categories:

- current-current operators, given by the product of two $V \pm A$ currents. They have the form $Q = (\bar{f}\gamma_\mu M^f P f)(\bar{g}\gamma_\mu M^g P g)$, where f and g are fermions, P is a projector and $M^{f/g}$ is either $\lambda^{f/g}$ or $\Gamma^{f/g}$;
- penguin operators, appearing as product of a $V \pm A$ current with a vector current: $Q = (\bar{f}\gamma_\mu M^f P f)(\bar{g}\gamma_\mu g)$. For our purpose it is convenient to analyse separately penguins with a $V \pm A$ quark current and penguins with a $V \pm A$ lepton current.

In the next section we show how to apply procedure 1- 3 to these categories.

2.5.1 Current-current operators

In order to find $\delta\xi$ for an operator of the form $Q = (\bar{f}\gamma_\mu M^f P f)(\bar{g}\gamma_\mu M^g P g)$ we consider the process $\bar{f}_i f_j \to g_k \bar{g}_\ell$. The one-loop contribution below the EW threshold is given by electromagnetic current-current diagrams with insertion of Q itself. Independently from the energy range, the one-loop amplitude reads

$$\mathcal{M}_{\text{loop}} = \frac{i}{16\pi^2\Lambda^2}\Big(\pm 6\, e^2\, q_f\, q_g\, C\Big) \ln\mu^2\, M^1_{ij}\, M^2_{k\ell}\, (\bar{v}\gamma_\mu P u)(\bar{u}\gamma_\mu P v) + \dots\,, \tag{2.92}$$

where C is the coefficient of Q in $\mathcal{L}^0_{\text{NP}}$ and the computation is performed along the lines of eq. (2.72). We choose the minus sign if Q is of the kind LL or RR, the plus sign if Q is of the kind LR or RL [6].
The RGE term must cancel the renormalization scale dependence in $\mathcal{M}_{\text{loop}}$ within each range. In the interval $m_b < \mu < m_{\text{EW}}$ eq.(2.85) gives

$$\mathcal{M}_{\text{RGE}} = \frac{i}{16\pi^2\Lambda^2}\frac{b}{2}\ln\mu^2 \dots\,. \tag{2.93}$$

For the μ-dependence to cancel in the sum, $b = \mp 12\, e^2\, q_f\, q_g\, C$.
The same procedure has to be repeated sistematically; since $\mathcal{M}_{\text{loop}}$ is unchanged all the way from m_{EW} downwards, we find $b = c = d$. This implies

$$\delta\xi = -b = \mp 12\, e^2\, q_f\, q_g\, C\,. \tag{2.94}$$

2.5.2 Penguin operators

In the case of penguins we analyse a process with two fermions of the same family in final state, $\bar{f}_i f_j \to g_k \bar{g}_k$.

Hadronic $V \pm A$ current

If the $V \pm A$ current contains quarks, the one-loop contribution to the process $\bar{f}_i f_j \to g_k \bar{g}_k$ is given by electromagnetic penguins with charged leptons in the loop. The inserted $\mathcal{L}^0_{\text{NP}}$ operators have the form $C_L(\bar{f}\gamma_\mu M^f P f)(\bar{e}\gamma_\mu \lambda^e P_L e)$ and $C_R(\bar{f}\gamma_\mu M^f P f)(\bar{e}\gamma_\mu \Gamma^e P_R e)$.
Remembering (2.64)[7], the general formula for the amplitude reads

$$\mathcal{M}_{\text{loop}} = \frac{i}{16\pi^2\Lambda^2}\left[-\frac{2e^2}{3} q_g q_e \sum_i (C_L \lambda^e_{ii} + C_R \Gamma^e_{ii}) \ln\frac{\mu^2}{m^2_{ei}}\right] M^f_{ij} (\bar{v}\gamma_\mu P u)(\bar{u}\gamma_\mu v) + \dots\,, \tag{2.95}$$

where the index i runs over the charged leptons in the loop. Since we assume $m_\tau \sim 1$ GeV, in the whole range from the EW scale down to 1 GeV all charged leptons are dynamical degrees of freedom and we need to sum over $i = 1, 2, 3$. Substituting $q_e = -1$ and using the property $\text{Tr}\lambda^e_{ii} = \text{Tr}\Gamma^e = 1$ we obtain

$$\mathcal{M}_{\text{loop}} = \frac{i}{16\pi^2\Lambda^2}\left[\frac{2e^2}{3} q_g (C_L + C_R)\right] \ln\mu^2 M^f_{ij} (\bar{v}\gamma_\mu P u)(\bar{u}\gamma_\mu v) + \dots\,. \tag{2.96}$$

[6] The origin of this difference lies in the spinor current contractions (A.6)-(A.9)
[7] Note that this expressione must be divided by three because leptons have no color.

Applying the same reasoning followed for current-current operators, we find $b = c = d$. Therefore $\delta\xi$ is

$$\delta\xi = -b = \frac{4e^2}{3} q_g (C_L + C_R)\,. \tag{2.97}$$

Leptonic $V \pm A$ current

In the case of a leptonic $V \pm A$ current, the relevant diagrams at one loop are electromagnetic penguins with a quark loop. For a given f, three operators can be inserted in the penguin: $(\bar{f}\gamma_\mu M^e P f)(\bar{u}\gamma_\mu \lambda^u P_L u)$, $(\bar{f}\gamma_\mu M^e P f)(\bar{d}\gamma_\mu \lambda^d P_L d)$ and $(\bar{f}\gamma_\mu M^e P f)(\bar{d}\gamma_\mu \Gamma^d P_R d)$. Denoting by C_L^u, C_L^d and C_R^d their coefficients in $\mathcal{L}^0_{\rm NP}$, the total amplitude is given by

$$\mathcal{M}_{\rm loop} = \frac{-2ie^2 q_g}{16\pi^2\Lambda^2}\left[q_u C_L^u \sum_i \lambda^u_{ii} \ln\frac{\mu^2}{m^2_{ui}} + q_d \sum_j \left(C_L^d \lambda^d_{jj} + C_R^d \Gamma^d_{jj}\right)\ln\frac{\mu^2}{m^2_{di}}\right] M^e_{ij}(\bar{v}\gamma_\mu P u)(\bar{u}\gamma_\mu v) + \dots\,, \tag{2.98}$$

where the indices i, j run over the up and down quarks in the loop. In this case $b \neq c \neq d$, because the number of "active" quarks diminishes when crossing the m_b and m_c threshold. For $m_b < \mu < m_{\rm EW}$ all quarks but the top are active, so we sum over $i = 1, 2$ and $j = 1, 2, 3$. From (2.85) we get

$$b = 4e^2\, q_g[q_u C_L^u(1 - \lambda^u_{33}) + q_d(C_L^d + C_R^d)]\,. \tag{2.99}$$

We proceed analogously for the ranges $m_c < \mu < m_b$ and $1\ \text{GeV} < \mu < m_c$. Keeping (2.86) and (2.87) in mind we find the following expressions for the coefficients c and d

$$c = 4e^2 q_g\left[q_u C_L^u(1-\lambda^u_{33}) + q_d C_L^d(1-\lambda^d_{33}) + q_d C_R^d(1-\Gamma^d_{33})\right] h \tag{2.100}$$

$$d = 4e^2\, q_g\left[q_u C_L^u(1-\lambda^u_{33} - \lambda^u_{22}) + q_d C_L^d(1-\lambda^d_{33}) + q_d C_R^d(1-\Gamma^d_{33})\right]\,, \tag{2.101}$$

thus the final expression for $\delta\xi$ reads

$$\delta\xi = -\frac{4e^2 q_g}{3}\left[2C_L^u - C_L^d - C_R^d + \left(C_L^d\hat{\lambda}^d_{33} + C_R^d\hat{\Gamma}^d_{33}\right)\ln\frac{m_b}{\mu} - 2C_L^u\left(\lambda^u_{33} + \hat{\lambda}^u_{22}\ln\frac{m_c}{\mu}\right)\right]\,, \tag{2.102}$$

where $\hat{\lambda} = \frac{\lambda}{\ln\frac{\mu}{m_{\rm EW}}}$.

Equations (2.94), (2.97) and (2.102) allow us to compute $\delta\xi_i$ for every operator Q_i; results are shown in tables 2.6 and 2.5. The low-energy effective Lagrangian is now completely determined.

Q_i	$\delta\xi_i$
$(\bar{u}_L\gamma_\mu\lambda_u u_L)(\bar{e}\gamma^\mu e)$	$-\frac{4}{3}e^2(C_1+C_6-C_3)$
$(\bar{d}_L^{\text{CKM}}\gamma_\mu\lambda_u d_L^{\text{CKM}})(\bar{e}\gamma^\mu e)$	$-\frac{4}{3}e^2(C_1+C_6+C_3)$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}\gamma_\mu e)$	$-\frac{4}{3}e^2(C_4+C_5)$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}\gamma_\mu u)$	$-\frac{8}{9}e^2\left\{C_1-C_4+3C_3-2(C_1+C_3)(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
-	$\left\{+\left[(C_1-C_3)\hat{\lambda}^d_{33}+C_4\hat{\Gamma}^d_{33}\right]\ln\frac{m_b}{\mu}\right\}$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{d}\gamma_\mu d)$	$+\frac{4}{9}e^2\left\{C_1-C_4+3C_3-2(C_1+C_3)(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
-	$\left\{+\left[(C_1-C_3)\hat{\lambda}^d_{33}+C_4\hat{\Gamma}^d_{33}\right]\ln\frac{m_b}{\mu}\right\}$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}\gamma_\mu u)$	$-\frac{8}{9}e^2\left\{C_1-C_4-3C_3-2(C_1-C_3)(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
-	$\left\{+\left[(C_1+C_3)\hat{\lambda}^d_{33}+C_4\hat{\Gamma}^d_{33}\right]\ln\frac{m_b}{\mu}\right\}$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{d}\gamma_\mu d)$	$+\frac{4}{9}e^2\left\{C_1-C_4-3C_3-2(C_1-C_3)(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
-	$\left\{+\left[(C_1+C_3)\hat{\lambda}^d_{33}+C_4\hat{\Gamma}^d_{33}\right]\ln\frac{m_b}{\mu}\right\}$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{u}\gamma_\mu u)$	$+\frac{8}{9}e^2\left\{C_5-C_6-(C_5\hat{\Gamma}^d_{33}+C_6\hat{\lambda}^d_{33})\ln\frac{m_b}{\mu}+2C_6(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{d}\gamma_\mu d)$	$-\frac{4}{9}e^2\left\{C_5-C_6-(C_5\hat{\Gamma}^d_{33}+C_6\hat{\lambda}^d_{33})\ln\frac{m_b}{\mu}+2C_6(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}_L\gamma_\mu\lambda_e e_L)$	$4e^2C_4$
$(\bar{d}_R\gamma^\mu\Gamma_d d_R)(\bar{e}_R\gamma_\mu\Gamma_e e_R)$	$-4e^2C_5$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{u}_L\gamma^\mu\lambda_u u_L)$	$-8e^2C_6$
$(\bar{e}_R\gamma_\mu\Gamma_e e_R)(\bar{d}_L\gamma^\mu\lambda_d d_L)$	$4e^2C_6$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{u}_L\gamma_\mu\lambda_u u_L)$	$+8e^2(C_1-C_3)$
$(\bar{e}_L\gamma^\mu\lambda_e\nu_L)(\bar{u}_L\gamma_\mu\lambda_u d_L^{\text{CKM}})$	$-8e^2C_3$

Table 2.5: Semi-leptonic operators: running from m_{EW} to $\mu\sim 1$ GeV. Operators with $\delta\xi_i=0$ are omitted.

Q_i	$\delta\xi_i$
$(\bar{\nu}_L\gamma^\mu\lambda_e\nu_L)(\bar{e}\gamma_\mu e)$	$\frac{4}{3}e^2\left\{C_1-C_4+3C_3-2(C_1+C_3)(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
-	$\left\{+\left[(C_1-C_3)\hat{\lambda}^d_{33}+C_4\hat{\Gamma}^d_{33}\right]\ln\frac{m_b}{\mu}\right\}$
$(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}\gamma_\mu e)$	$\frac{4}{3}e^2\left\{C_1-C_4-3C_3-2(C_1-C_3)(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$
-	$\left\{+\left[(C_1+C_3)\hat{\lambda}^d_{33}+C_4\hat{\Gamma}^d_{33}\right]\ln\frac{m_b}{\mu}\right\}$
$(\bar{e}_R\gamma^\mu\Gamma_e e_R)(\bar{e}\gamma_\mu e)$	$-\frac{4}{3}e^2\left\{C_5-C_6-(C_5\hat{\Gamma}^d_{33}+C_6\hat{\lambda}^d_{33})\ln\frac{m_b}{\mu}+2C_6(\lambda^u_{33}+\hat{\lambda}^u_{22}\ln\frac{m_c}{\mu})\right\}$

Table 2.6: Leptonic operators: running from m_{EW} to $\mu\sim 1$ GeV. Operators with $\delta\xi_i=0$ are omitted.

Chapter 3

Phenomenological implications

This chapter addresses the phenomenological consequences of Lagrangian (2.7) and of its RGE-improved evolution at low energy. Our main goal is to understand the limits imposed by the experimental bounds on selected observables on the values of $R^{\tau/\ell}_{D^{(*)}}$ and $R^{\mu/e}_{K}$. Keeping this in mind, we start by analysing B anomalies and other B-decay channels receiving NP contribution at tree level. We choose not to investigate the observable $R^{\mu/e}_{K^*}$, because the discrepancy with respect to the SM concerns not only the central q^2 region, but also the region at low q^2. As shown in [22], the NP contribution to $R^{\mu/e}_{K^*}$ in the low bin is given by the dipole Wilson coefficient $\mathcal{C}^{7}_{\text{NP}}$ (which is tightly constrained by $b \to s\gamma$ transitions [25, 28]) and by the coefficients $(\mathcal{C}^{9(\prime)}_{\text{NP}})_{\mu\mu}$ and $(\mathcal{C}^{10(\prime)}_{\text{NP}})_{\mu\mu}$, whose impact in this region is limited and cannot account for the measured anomaly.
Subsequently, we take one-loop phenomenology into consideration, focusing on observables which are so tightly constrained experimentally that the NP contribution plays a major role despite showing up only at one loop. In particular we consider Z-pole observables and τ LFUV and LFV decays. Finally, we thoroughly analyse a phenomenologically relevant scenario, where NP affects dominantly the Wilson coefficient $\mathcal{C}^9$.
The NP contribution to the observables is parametrized in terms of the free parameters of $\mathcal{L}^{0}_{\text{NP}}$, namely the five C_i and the matrices λ^e, λ^d, Γ^e and Γ^d. Before starting the phenomenological analysis we significantly simplify our setup by assuming $\lambda \approx \Gamma$. Given the absence of LFUV and LFV in processes involving the first two generations, we also assume $\lambda^{e/d}_{1i} = \Gamma^{e/d}_{1i} = 0$ for $i = 1, 2, 3$ and $\lambda^{e/d}_{22} \approx \left|\lambda^{e/d}_{23}\right|^2 \ll \lambda^{e/d}_{33}$[1]. Moreover, λ^{e}_{23} and λ^{d}_{23} are taken to be real. The parameters involved in our analysis are therefore C_1, C_3, C_4, C_5, C_6 and $\lambda^{e/d}_{23}$.

[1]As a consequence, $\lambda^{e/d}_{33} \approx 1$.

3.1 Tree-level phenomenology

3.1.1 B anomalies

In our framework B anomalies receive NP contribution at tree level. In order to compute this contribution explicitly, we need to match the low-energy Lagrangians $\mathcal{L}_{\text{eff}}^{\text{NC}}$ and $\mathcal{L}_{\text{eff}}^{\text{CC}}$ with the NP Lagrangian $\mathcal{L}_{\text{NP}}^{0}$. Strictly speaking $\mathcal{L}_{\text{eff}}^{\text{NC}}$ and $\mathcal{L}_{\text{eff}}^{\text{CC}}$ should be matched to the Lagrangian obtained by running the Wilson coefficients down to $\mu = m_B$, but RGE induced terms are generally negligible with respect to tree-level ones[2]. From equations (1.32), (1.34) and (2.14) we find

$$\begin{cases} (\mathcal{C}_{\text{NP}}^{9})_{ij} = \dfrac{4\pi^2}{e^2\lambda_{bs}^t}\dfrac{v^2}{\Lambda^2}\lambda_{23}^d\lambda_{ij}^e[C_1 + C_3 + C_6] + \dots \\ (\mathcal{C}_{\text{NP}}^{10})_{ij} = \dfrac{4\pi^2}{e^2\lambda_{bs}^t}\dfrac{v^2}{\Lambda^2}\lambda_{23}^d\lambda_{ij}^e[-C_1 - C_3 + C_6] + \dots \\ (\mathcal{C}_{\text{NP}}^{\prime 9})_{ij} = \dfrac{4\pi^2}{e^2\lambda_{bs}^t}\dfrac{v^2}{\Lambda^2}\Gamma_{23}^d\lambda_{ij}^e[C_4 + C_5] + \dots \\ (\mathcal{C}_{\text{NP}}^{\prime 10})_{ij} = \dfrac{4\pi^2}{e^2\lambda_{bs}^t}\dfrac{v^2}{\Lambda^2}\Gamma_{23}^d\lambda_{ij}^e[-C_4 + C_5] + \dots \\ (\mathcal{C}_{\text{NP}}^{\nu})_{ij} = \dfrac{4\pi^2}{e^2\lambda_{bs}^t}\dfrac{v^2}{\Lambda^2}\lambda_{23}^d\lambda_{ij}^e[C_1 - C_3] + \dots \\ (\mathcal{C}_{\text{NP}}^{\prime\nu})_{ij} = \dfrac{4\pi^2}{e^2\lambda_{bs}^t}\dfrac{v^2}{\Lambda^2}\Gamma_{23}^d\lambda_{ij}^e C_4 + \dots \\ (\mathcal{C}_{L,\text{NP}}^{cb})_{ij} = -\dfrac{v^2}{\Lambda^2}\dfrac{\lambda_{23}^{ud}}{V_{cb}}C_3\lambda_{ij}^e + \dots , \end{cases} \tag{3.1}$$

where dots stand for subleading RGE terms.
We focus on the ratios $R_K^{\mu/e}$ and $R_{D^{(*)}}^{\tau/\ell}$, defined in (1.16) and subject to the experimental bounds (1.17). In our framework $R_K^{\mu/e}$ can be written as

$$\begin{aligned} R_K^{\mu/e} &\approx \frac{\left|\mathcal{C}_{\mu\mu}^{9} + \mathcal{C}_{\mu\mu}^{9\prime}\right|^2 + \left|\mathcal{C}_{\mu\mu}^{10} + \mathcal{C}_{\mu\mu}^{10\prime}\right|^2}{\left|\mathcal{C}_{ee}^{9} + \mathcal{C}_{ee}^{9\prime}\right|^2 + \left|\mathcal{C}_{ee}^{10} + \mathcal{C}_{ee}^{10\prime}\right|^2} \\ &\approx \frac{|\mathcal{C}_{\text{SM}}^{9} + (\mathcal{C}_{\text{NP}}^{9})_{\mu\mu} + (\mathcal{C}_{\text{NP}}^{9\prime})_{\mu\mu}|^2 + |\mathcal{C}_{\text{SM}}^{10} + (\mathcal{C}_{\text{NP}}^{10})_{\mu\mu} + (\mathcal{C}_{\text{NP}}^{10\prime})_{\mu\mu}|^2}{|\mathcal{C}_{\text{SM}}^{9} + (\mathcal{C}_{\text{NP}}^{9})_{ee} + (\mathcal{C}_{\text{NP}}^{9\prime})_{ee}|^2 + |\mathcal{C}_{\text{SM}}^{10} + (\mathcal{C}_{\text{NP}}^{10})_{ee} + (\mathcal{C}_{\text{NP}}^{10\prime})_{ee}|^2} . \end{aligned} \tag{3.2}$$

$(\mathcal{C}_{\text{NP}}^{9})_{ee}$, $(\mathcal{C}_{\text{NP}}^{9\prime})_{ee}$, $(\mathcal{C}_{\text{NP}}^{10})_{ee}$ and $(\mathcal{C}_{\text{NP}}^{10\prime})_{ee}$ can be neglected because $\lambda_{11}^e = 0$. Working linearly in NP contributions and substituting $\mathcal{C}_{\text{SM}}^{9} \approx -\mathcal{C}_{\text{SM}}^{10} \approx 4.2$ [6], we find

$$R_K^{\mu/e} \approx 1 + \frac{2}{\mathcal{C}_{\text{SM}}^{9}}\frac{\pi}{\alpha\lambda_{bs}^t}\frac{v^2}{\Lambda^2}\lambda_{23}^d\lambda_{22}^e\left(C_1 + C_3 + C_4\right) , \tag{3.3}$$

which results in the numerical expression

$$R_K^{\mu/e} \approx 1 - \frac{0.28}{\Lambda^2(\text{TeV}^2)}\frac{\lambda_{22}^e\lambda_{23}^d}{10^{-3}}\left(C_1 + C_3 + C_4\right) . \tag{3.4}$$

[2]This is true unless particular cancellations among parameters take place; we suppose that this is not the case.

The ratio describing the charged-current anomaly, $R_{D^{(*)}}^{\tau/\ell}$, is given by

$$R_{D^{(*)}}^{\tau/\ell} = \frac{\sum_j |(\mathcal{C}_L^{cb})_{3j}|^2}{\sum_j |(\mathcal{C}_L^{cb})_{\ell j}|^2} \,. \tag{3.5}$$

Keeping only linear NP contributions and neglecting λ_{11}^e and λ_{22}^e with respect to λ_{33}^e, we find

$$R_{D^{(*)}}^{\tau/\ell} \approx 1 - 2\frac{v^2}{\Lambda^2}\frac{\lambda_{23}^{ud}}{V_{cb}} C_3 \lambda_{33}^e \tag{3.6}$$

Using the relation $\lambda^{ud} = V_{\rm CKM}\lambda^d$, λ_{23}^{ud} can be expressed as $\lambda_{23}^{ud} = V_{cs}\lambda_{23}^d + V_{cb}\lambda_{33}^d$. Substituting numerical quantities, we end up with the following expression

$$R_{D^{(*)}}^{\tau/\ell} \approx 1 - 0.12\frac{C_3}{\Lambda^2(\text{TeV}^2)}\lambda_{33}^e\left(\frac{V_{cs}}{V_{cb}}\lambda_{23}^d + \lambda_{33}^d\right) \,. \tag{3.7}$$

3.1.2 Further tree-level phenomenology in B decays

Besides B anomalies, our framework predicts tree-level NP effects in other B-decay channels:

- An important example is given by the decay $B \to K\bar{\nu}\nu$, which is strictly related to the neutral-current anomaly. In fact, NP enters this process through the coefficients $\mathcal{C}_{\rm NP}^{\nu}$ and $\mathcal{C}_{\rm NP}^{\nu\prime}$; as equation (3.1) shows, these are closely linked to the coefficients describing NP in $b \to s\ell^+\ell^-$, $\mathcal{C}_{\rm NP}^{9(\prime)}$ and $\mathcal{C}_{\rm NP}^{10(\prime)}$. This happens because they ultimately originate from the same $SU(2)_L \times U(1)_Y$ invariant operators in $\mathcal{L}_{\rm NP}^0$.
  We consider the observable $R_K^{\nu\nu}$, defined as

  $$R_K^{\nu\nu} = \frac{\mathcal{B}(B \to K\bar{\nu}\nu)}{\mathcal{B}(B \to K\bar{\nu}\nu)_{\rm SM}} \,, \tag{3.8}$$

  which is subject to the experimental constraint $R_K^{\nu\nu} < 4.3$. In our framework $R_K^{\nu\nu}$ can be expressed as

  $$R_K^{\nu\nu} = \frac{\sum_{ij}\left|\mathcal{C}_{ij}^{\nu} + \mathcal{C}_{ij}^{\nu\prime}\right|^2}{3\,|\mathcal{C}_{\rm SM}^{\nu}|^2} = \frac{\sum_{ij}|\mathcal{C}_{\rm SM}^{\nu}\delta_{ij} + (\mathcal{C}_{\rm NP}^{\nu})_{ij} + (\mathcal{C}_{\rm NP}^{\nu\prime})_{ij}|^2}{3\,|\mathcal{C}_{\rm SM}^{\nu}|^2} \,. \tag{3.9}$$

  By expanding the numerator and using the property $\sum_{ij}|\lambda_{ij}^e|^2 = 1$ and $\text{Tr}\lambda^e = 1$, we find

  $$R_K^{\nu\nu} \approx 1 + \frac{2}{3}\frac{\lambda_{23}^d}{|\mathcal{C}_{\rm SM}^{\nu}|}\frac{\pi}{\alpha\lambda_{bs}^t}\frac{v^2}{\Lambda^2}(C_1 - C_3 + C_4) + \frac{1}{3}\left(\frac{\lambda_{23}^d}{|\mathcal{C}_{\rm SM}^{\nu}|}\frac{\pi}{\alpha\lambda_{bs}^t}\frac{v^2}{\Lambda^2}(C_1 - C_3 + C_4)\right)^2 \,. \tag{3.10}$$

  Since $\mathcal{C}_{\rm SM}^{\nu} \approx -6.4$, we get the numerical result

  $$R_K^{\nu\nu} \approx 1 + 0.6\left(\frac{\lambda_{23}^d}{0.01}\frac{C_1 - C_3 + C_4}{\Lambda^2(\text{TeV}^2)}\right) + 0.3\left(\frac{\lambda_{23}^d}{0.01}\frac{C_1 - C_3 + C_4}{\Lambda^2(\text{TeV}^2)}\right)^2 \,. \tag{3.11}$$

- The presence of a non-zero λ^e_{23} implies non-vanishing branching ratios for lepton-flavor-violating decays involving third generation leptons, for example $B_s \to \tau\mu$ and $B \to K\tau\mu$. The branching ratio for this second process reads [29]

$$\mathcal{B}(B \to K\tau\mu) \approx 2 \cdot 10^{-9} \left(9.6\,|(\mathcal{C}^9_{\mathrm{NP}})_{\mu\tau} + (\mathcal{C}'^9_{\mathrm{NP}})_{\mu\tau}|^2 + 10\,|(\mathcal{C}^{10}_{\mathrm{NP}})_{\mu\tau} + (\mathcal{C}'^{10}_{\mathrm{NP}})_{\mu\tau}|^2\right) . \tag{3.12}$$

Employing the relation $\frac{(\mathcal{C}^9_{\mathrm{NP}})_{\mu\tau}}{(\mathcal{C}^9_{\mathrm{NP}})_{\mu\mu}} = \lambda^e_{23}/\lambda^e_{22}$, we find

$$\mathcal{B}(B \to K\tau\mu) \approx 2 \cdot 10^{-9} \left(\frac{\lambda^e_{23}}{\lambda^e_{22}}\right)^2 \left(9.6\,|(\mathcal{C}^9_{\mathrm{NP}})_{\mu\mu} + (\mathcal{C}'^9_{\mathrm{NP}})_{\mu\mu}|^2 + 10\,|(\mathcal{C}^{10}_{\mathrm{NP}})_{\mu\mu} + (\mathcal{C}'^{10}_{\mathrm{NP}})_{\mu\mu}|^2\right) . \tag{3.13}$$

Given the order of magnitude we expect for $\mathcal{C}^9$, $\mathcal{C}^{9\prime}$,$\mathcal{C}^{10}$, $\mathcal{C}^{10\prime}$ to account for B anomalies, this is well below the current experimental bound $\mathcal{B}(B \to K\tau\mu) \leq 4.8 \times 10^{-5}$ [30].

- Our framework predicts a tree-level NP contribution also to the purely leptonic decay $B_s \to \bar{\mu}\mu$, which is sensitive to $\mathcal{C}^{10(\prime)}_{\mathrm{NP}}$.

- LFUV in $B \to D\ell\nu$ is related to LFUV in $B \to \ell\nu$, described by the observable $R^{\tau/\ell}_{B\tau\nu}$ [2]. The experiment Belle II, which should start data taking in early 2018, is expected to measure this quantity to a 5% precision.

3.2 One-loop phenomenology

Renormalization group evolution induces two main effects. First, as discussed in section 2.2.1, Z and W couplings to fermions are modified with respect to the SM. Second, as we can see from equation (2.91) and the related tables, at low energies a purely leptonic Lagrangian is induced. As a consequence, we expect LFV and LFUV effects in Z, W and τ observables. In this section we examine some of them.

3.2.1 Z-pole observables

The NP modification to Z couplings (2.30) explicitly breaks both LF and LFU in weak interactions. The consequent deviations of Z-pole observables from SM expectation values are tightly constrained by LEP measurements of the Z decay widths, left-right and forward-backward asymmetry. Remembering the definition of the axial and vector couplings

$$v_\ell = (g^\ell_L)_{\ell\ell} + (g^\ell_R)_{\ell\ell} \qquad\qquad a_\ell = (g^\ell_L)_{\ell\ell} - (g^\ell_R)_{\ell\ell}\,, \tag{3.14}$$

we consider the observables $\frac{v_\tau}{v_e}$ and $\frac{a_\tau}{a_e}$, which quantify the universality of Z couplings to charged leptons. Experimental bounds on these couplings are derived from the measured values of asymmetries and are given by [31]

$$\frac{v_\tau}{v_e} = 0.959\;(29) \qquad\qquad \frac{a_\tau}{a_e} = 1.0019\;(15)\,. \tag{3.15}$$

In our framework they read

$$\frac{v_\tau}{v_e} \approx 1 - \frac{2}{1-4s_{\mathrm{W}}^2}\left[(\delta g_{L}^{e})_{33} - (\delta g_{L}^{e})_{11} + (\delta g_{R}^{e})_{33} - (\delta g_{R}^{e})_{11}\right] \tag{3.16}$$

$$\frac{a_\tau}{a_e} \approx 1 - 2\left[(\delta g_{L}^{e})_{33} - (\delta g_{L}^{e})_{11} - (\delta g_{R}^{e})_{33} + (\delta g_{R}^{e})_{11}\right] . \tag{3.17}$$

Plugging (2.50) into these expressions we obtain

$$\begin{aligned}\frac{v_\tau}{v_e} &\approx 1 - \frac{2\lambda^e_{33}}{1-4s_{\mathrm{W}}^2}\left[3\lambda_u^{33} y_t^2 (C_1 - C_3 + C_6)\ln\frac{\Lambda}{m_t} + g_2^2 C_3 \ln\frac{\Lambda}{m_t} + \frac{g_1^2}{3}(C_1 - C_4 + C_6 - C_5)\ln\frac{\Lambda}{m_{\mathrm{Z}}}\right] \\ \frac{a_\tau}{a_e} &\approx 1 - 2\lambda^e_{33}\left[3\lambda_u^{33} y_t^2 (C_1 - C_3 - C_6)\ln\frac{\Lambda}{m_t} + g_2^2 C_3 \ln\frac{\Lambda}{m_t} + \frac{g_1^2}{3}(C_1 - C_4 - C_6 + C_5)\ln\frac{\Lambda}{m_{\mathrm{Z}}}\right] ,\end{aligned} \tag{3.18}$$

and substituting numerical values we get the following estimate

$$\begin{aligned}\frac{v_\tau}{v_e} &\approx 1 - \frac{0.05}{\Lambda^2(\mathrm{TeV}^2)}\left[(C_1 - C_3 + C_6) + 0.2C_3 + 0.02\,(C_1 - C_4 + C_6 - C_5)\right] \\ \frac{a_\tau}{a_e} &\approx 1 - \frac{0.004}{\Lambda^2(\mathrm{TeV}^2)}\left[(C_1 - C_3 - C_6) + 0.2C_3 + 0.02\,(C_1 - C_4 - C_6 + C_5)\right] .\end{aligned} \tag{3.19}$$

Another important observable is the number of neutrinos N_ν, which is extracted from the invisible Z width. Taking the NP modification of Z couplings to neutrinos into account, N_ν can be approximated by

$$N_\nu \approx 3 + 4(\delta g_L^\nu)_{33} \approx 3 + \frac{0.008}{\Lambda^2}\left[(C_1 + C_3) - 0.2C_3 + 0.02\,(C_1 - C_4)\right] . \tag{3.20}$$

The experimental bound reads $N_\nu = 2.9840 \pm 0.0082$ [31] .

3.2.2 Purely leptonic effective Lagrangian

The effective low-energy Lagrangian (2.91) contains a purely leptonic Lagrangian $\mathcal{L}^\ell_{\mathrm{eff}}$. Taking into account the explicit values of the ξ_i and $\delta\xi_i$ for leptonic operators, we can write it as

$$\begin{aligned}\mathcal{L}^\ell_{\mathrm{eff}} = -\frac{4G_F}{\sqrt{2}}&\Big[(\bar{e}_L\gamma_\mu\lambda^e e_L)\sum_f(\bar{f}\gamma^\mu f)(2g^f_{\mathrm{SM}}c^e_t - Q_\psi c^e_\gamma) + (\bar{e}_R\gamma_\mu\lambda^e e_R)\sum_f(\bar{f}\gamma^\mu f)(2g^f_{\mathrm{SM}}c^{e\,\prime}_t - Q_f c^{e\,\prime}_\gamma) \\ &+ c^{cc}_t(\bar{e}_L\gamma_\mu\lambda^e\nu_L)(\bar{\nu}_L\gamma_\mu e_L + \bar{u}_L\gamma_\mu V_{\mathrm{CKM}} d_L) + \mathrm{h.c.}\Big] ,\end{aligned} \tag{3.21}$$

where $f = \{\nu_L, e_L, e_R, u_L, u_R, d_L, d_R\}$ and $g^f_{\rm SM}$ is the Z coupling to the f field in the SM. The coefficients c^e_t, c^e_γ, $c^{e\,\prime}_t$, $c^{e\,\prime}_\gamma$ are defined as

$$\begin{cases} c^e_t = \dfrac{3v^2}{32\pi^2\Lambda^2} y_t^2 (C_1 - C_3)\lambda^u_{33} \ln \dfrac{\Lambda^2}{m^2_{\rm EW}} \\ c^{e\,\prime}_t = \dfrac{3v^2}{32\pi^2\Lambda^2} y_t^2 C_6 \lambda^u_{33} \ln \dfrac{\Lambda^2}{m^2_{\rm EW}} \\ c^e_\gamma = \dfrac{v^2}{48\pi^2\Lambda^2} e^2 \left[(3C_3 - C_1 + C_4) \ln \dfrac{\Lambda^2}{\mu^2} + 2(C_1 - C_3)\left(\lambda^u_{33} \ln \dfrac{m^2_{\rm EW}}{\mu^2} + \lambda^u_{22} \ln \dfrac{m^2_c}{\mu^2} \right) \right. \\ \qquad\qquad\qquad \left. -\lambda^d_{33}\,(C_1 + C_3 + C_4) \ln \dfrac{m_b^2}{\mu^2} \right] \\ c^{e\,\prime}_\gamma = \dfrac{v^2}{48\pi^2\Lambda^2} e^2 \left[(C_6 - C_5) \ln \dfrac{\Lambda^2}{\mu^2} + 2C_6 \left(\lambda^u_{33} \ln \dfrac{m^2_{\rm EW}}{\mu^2} + \lambda^u_{22} \ln \dfrac{m^2_c}{\mu^2} \right) - \lambda^d_{33}\,(C_6 + C_5) \ln \dfrac{m_b^2}{\mu^2} \right] \\ c^{cc}_t = \dfrac{3v^2}{16\pi^2\Lambda^2} y_t^2 C_3 \lambda^u_{33} \ln \dfrac{\Lambda^2}{m_t^2}\ . \end{cases}$$

Leptonic processes triggered by this Lagrangian get a NP contribution from a combination of c^{cc}_t, c^e_γ, c^e_t and/or from their primed counterparts. In general this contribution will depend on the renormalization scale μ; this dependence must cancel when computing the matrix element. The scale dependence is encoded in the coefficients c^e_γ, $c^{e\,\prime}_\gamma$ and c^{cc}_t, because they include the electromagnetic contribution, which runs from Λ to the low-energy scale μ. The contribution coming from the loop diagram with the top instead runs only from the higher scale Λ to $m_{\rm EW}$, because for lower values of μ the top is integrated out. It is worth noting that the coefficient c^{cc}_t has no electromagnetic NP contribution; this happens because fully leptonic charged-current operators, as already observed, aren't renormalized by QED effects.
Lagrangian (3.21) manifestly generates both LFV and LFUV processes. Given the hierarchy in λ^e, NP effects are maximized in transitions involving the third generation. As a consequence, we focus on τ decays. In the SM leptonic τ decays $\tau \to \ell\bar\nu_\ell\nu_\tau$ ($\ell = \mu, e$) proceed trough the tree-level exchange of a W boson, with the universal coupling associated with the charged-current interactions. Due to the W coupling universality, all these decay modes have equal amplitudes in the SM, provided that final fermion masses are neglected. The NP contribution instead violates LF and LFU; we consider $\tau \to \ell\bar\nu\nu$ as an example of LFUV and $\tau \to 3\mu$ as an example of LFV.

3.2.3 $\tau \to \ell\bar\nu\nu$

LFU breaking effects in $\tau \to \ell\bar\nu\nu$ (with $\ell_{1,2} = e, \mu$) are described by the observables

$$R_\tau^{\tau/\ell_{1,2}} = \frac{\mathcal{B}(\tau \to \ell_{2,1}\nu\bar\nu)_{\rm exp}/\mathcal{B}(\tau \to \ell_{2,1}\nu\bar\nu)_{\rm SM}}{\mathcal{B}(\mu \to e\nu\bar\nu)_{\rm exp}/\mathcal{B}(\mu \to e\nu\bar\nu)_{\rm SM}}\,, \tag{3.22}$$

which are subject to the strong experimental constraints [32]

$$R_\tau^{\tau/\mu} = 1.0022 \pm 0.0030 \qquad\qquad R_\tau^{\tau/e} = 1.0060 \pm 0.0030\,. \tag{3.23}$$

The process $\tau \to \ell \bar{\nu}_j \nu_i$ receives a contribution both from the SM Lagrangian and from $\mathcal{L}^\ell_{\text{eff}}$. The SM contribution reads

$$\mathcal{L}_{\text{SM}} = -\frac{4G_F}{\sqrt{2}} \delta_{\ell j}\delta_{i3} (\bar{\ell}_L \gamma_\mu \nu_{jL})(\bar{\nu}_{iL}\gamma^\mu \tau_L) = -\frac{4G_F}{\sqrt{2}} \delta_{\ell j}\delta_{i3} (\bar{\ell}_L \gamma_\mu \tau_L)(\bar{\nu}_{iL}\gamma^\mu \nu_{jL}) \,, \tag{3.24}$$

where we used the Fierz identity $(\bar{\nu}_{iL}\gamma_\mu \tau_L)(\bar{\ell}_L \gamma^\mu \nu_{jL}) = (\bar{\nu}_{iL}\gamma_\mu \nu_{jL})(\bar{\ell}_L\gamma^\mu \tau_L)$. The part of $\mathcal{L}^\ell_{\text{eff}}$ we are interested in is given by

$$\begin{aligned}\mathcal{L}^\ell_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} \Big\{ &\Big[c_t^e \delta_{ij} \lambda^e_{\ell 3} + c_t^{cc}(\lambda^e_{\ell j}\delta_{i3} + \lambda^e_{i3}\delta_{\ell j})\Big] (\bar{\ell}_L \gamma_\mu \tau_L)(\bar{\nu}_{iL}\gamma^\mu \nu_{jL}) \\ &+ c_t^{e\prime} \delta_{ij}\lambda^e_{\ell 3} (\bar{\ell}_R \gamma_\mu \tau_R)(\bar{\nu}_{iL}\gamma^\mu \nu_{jL}) \Big\} \,. \end{aligned} \tag{3.25}$$

Therefore the full Lagrangian contributing to $\tau \to \ell \bar{\nu}\nu$ is

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} \Big\{ (C_L^{\tau\ell})_{ij} (\bar{\ell}_L \gamma_\mu \tau_L)(\bar{\nu}_{iL}\gamma^\mu \nu_{jL}) + (C_R^{\tau\ell})_{ij} (\bar{\ell}_R \gamma_\mu \tau_R)(\bar{\nu}_{iL}\gamma^\mu \nu_{jL}) \Big\} \,, \tag{3.26}$$

where $(C_L^{\tau\ell})_{ij} = \delta_{\ell j}\delta_{i3} + c_t^e \delta_{ij}\lambda^e_{\ell 3} + c_t^{cc}(\lambda^e_{\ell j}\delta_{i3} + \lambda^e_{i3}\delta_{\ell j})$ and $(C_R^{\tau\ell})_{ij} = c_t^{e\prime}\lambda^e_{\ell 3}\delta_{kj}$. The ratio $R_\tau^{\tau/\ell}$ can be expressed in terms of these coefficients as follows

$$R_\tau^{\tau/\ell_{1,2}} = \frac{\sum_{ij} |(C_L^{\tau\ \ell_{2,1}})_{ij}|^2 + |(C_R^{\tau\ \ell_{2,1}})_{ij}|^2}{\sum_{ij} |(C_L^{\ell_{2,1}\ e})_{ij}|^2 + |(C_R^{\ell_{2,1}\ e})_{ij}|^2} \,. \tag{3.27}$$

Working linearly in NP contribution, only the interference between the SM term and the charged-current term survives, giving

$$R_\tau^{\tau/\ell} \simeq 1 + 2\, c_t^{cc} \lambda^e_{33} \approx 1 + 0.008\, \lambda^e_{33} \frac{C_3}{\Lambda^2(\text{TeV})} \,. \tag{3.28}$$

3.2.4 $\tau \to 3\mu$

One of the most studied LFV processes generated by $\mathcal{L}^\ell_{\text{eff}}$ is the decay $\tau \to 3\mu$, which is forbidden in the SM[3]. The only contribution is given by $\mathcal{L}^\ell_{\text{eff}}$

$$\begin{aligned}\mathcal{L}^\ell_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} \lambda^e_{23} \big[&(c_{LR} - c_t^e)(\mu_L \gamma_\mu \tau_L)(\bar{\mu}_L \gamma^\mu \mu_L) + c_{LR}(\mu_L \gamma_\mu \tau_L)(\bar{\mu}_R \gamma^\mu \mu_R) \\ &+ (c'_{LR} - c_t^{e\prime})(\mu_R \gamma_\mu \tau_R)(\bar{\mu}_L \gamma^\mu \mu_L) + c'_{LR}(\mu_R \gamma_\mu \tau_R)(\bar{\mu}_R \gamma^\mu \mu_R)\big] + \ldots \,, \end{aligned} \tag{3.29}$$

where $c^{(\prime)}_{LR} = 2 s^2_{\text{W}} c_t^{e\,(\prime)} + c_\gamma^{e\,(\prime)}$. Adapting the formula given in ref. [33] we find

$$\Gamma(\tau \to 3\mu) = \frac{G_F^2 m_\tau^5}{192\pi^3} |\lambda^e_{23}|^2 \Big[2(c_{LR} - c_t^e)^2 + c_{LR}^2 + 2c'^2_{LR} + (c'_{LR} - c_t^{e\prime})^2 \Big] \,. \tag{3.30}$$

[3]This is rigorously true if we consider neutrinos to be massless; taking neutrino mixing into account $\tau \to 3\mu$ receives a one-loop SM contribution. However the corresponding branching ratio is tiny $\mathcal{B} \lesssim 10^{-40}$, far below the current experimental limit. An observation of $\tau \to 3\mu$ in the near future would therefore be an unambiguous signal of NP, beyond that needed to account for neutrino masses.

If at least one among $C_1 - C_3$ and C_6 is non-zero the term proportional to the Yukawa coupling gives the leading contribution; in this case we neglect the electromagnetic contribution given by c^e_γ, obtaining

$$\Gamma(\tau \to 3\mu) = \frac{G_F^2 m_\tau^5}{192\pi^3} |\lambda^e_{23}|^2 \left[c_t^{e\,2}(12 s_{\rm W}^4 - 8 s_{\rm W}^2 + 2) + c_t^{e\,\prime\,2}(12 s_{\rm W}^4 - 4 s_{\rm W}^2 + 1) \right] . \tag{3.31}$$

Knowing that the τ lifetime is $\tau_\tau = (290.17 \pm 0.53 \pm 0.33) \cdot 10^{-15} s$ [34] we find the following numerical expression for the branching ratio

$$\mathcal{B}(\tau \to 3\mu) \approx \left(\frac{\lambda^e_{23}}{0.3} \right)^2 \left[5.0 \frac{(C_1 - C_3)^2}{\Lambda^4(\text{TeV}^4)} + 4.5 \frac{C_6^2}{\Lambda^4(\text{TeV}^4)} \right] \cdot 10^{-8} . \tag{3.32}$$

The current experimental bound reads $\mathcal{B}(\tau \to 3\mu) \leqslant 2.1 \cdot 10^{-8}$ [30].

3.3 Study of a motivated scenario (only C_9)

So far we obtained the general analytic expressions for B anomalies and for other significant observables; no specific hypothesis on the coefficients C_i in $\mathcal{L}^0_{\rm NP}$ has been made yet. In this section we specify our analysis to a phenomenologically relevant case. In view of the considerations in section 1.4, we examine the scenario where only $(\mathcal{C}^9_{\rm NP})_{\mu\mu}$ is non-vanishing. Operators containing a right-handed quark current are absent in this scenario; the only difference with respect to [1, 2] is the presence of a non-zero C_6. By imposing $(\mathcal{C}^{10}_{\rm NP})_{\mu\mu} = (\mathcal{C}^{9\prime}_{\rm NP})_{\mu\mu} = (\mathcal{C}^{10\prime}_{\rm NP})_{\mu\mu} = 0$ we obtain the following conditions on the C_i

$$C_1 + C_3 = C_6 \qquad\qquad C_4 = C_5 = 0 . \tag{3.33}$$

Taking the NP scale to be $\Lambda \approx 1$ TeV, the free parameters in this setup are C_1, C_3, λ^d_{23} and λ^e_{23}. Our initial assumptions about the structure of λ^e and λ^d straightforwardly imply $|\lambda^{e,d}_{23}| \leq 0.5$; we can further restrict the bounds on λ^e_{23}, because $|\lambda^e_{22}| \leq 0.1$ [35] and $\lambda^e_{22} \approx |\lambda^e_{23}|^2$ imply $|\lambda^e_{23}| \lesssim 0.3$. As to $C_{1,3}$, we can safely assume $|C_{1,3}| \leq 3$.
Given (3.33), B anomalies read

$$\begin{aligned} R_D^{\tau/\ell} &= 1 - 0.12\, \frac{C_3}{\Lambda^2} \lambda^e_{33} \left(\frac{V_{cs}}{V_{cb}} \lambda^d_{23} + \lambda^d_{33} \right) \\ R_K^{\mu/e} &= 1 - \frac{0.28}{\Lambda^2} \frac{\lambda^e_{22}\lambda^d_{23}}{10^{-3}} \left(C_1 + C_3 \right) . \end{aligned} \tag{3.34}$$

On the other hand, the expressions for the relevant observables - which we shall refer to as *constraint observables* - simplify to

$$\begin{aligned} R_K^{\nu\nu} &= 1 + 0.6 \left(\frac{\lambda^d_{23}}{0.01} \frac{C_1 - C_3}{\Lambda^2} \right) + 0.3 \left(\frac{\lambda^d_{23}}{0.01} \frac{C_1 - C_3}{\Lambda^2} \right)^2 \\ \frac{v_\tau}{v_e} &= 1 - \frac{0.05\, \lambda^e_{33}}{\Lambda^2} (2\, C_1 + 0.2\, C_3 + 0.02\, (2\, C_1 + C_3)) \\ \frac{a_\tau}{a_e} &= 1 + 0.007\, \lambda^e_{33} \frac{C_3}{\Lambda^2} \end{aligned}$$

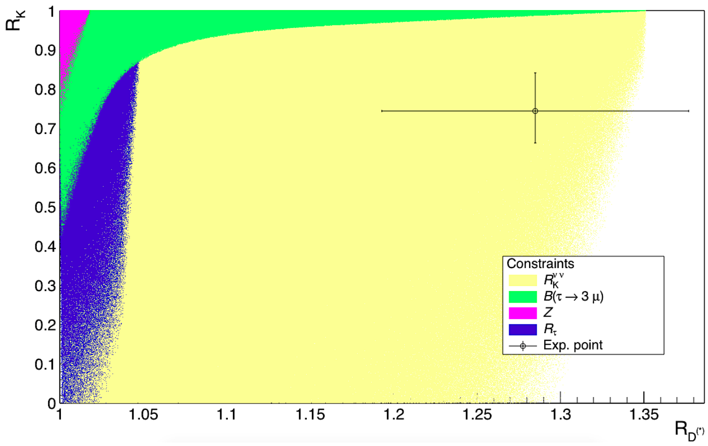


Figure 3.1: Impact of one-loop-induced constraints on the values of $R_{D^{(*)}}^{\tau/\ell}$ and $R_K^{\mu/e}$ for $C_1 \in \{-3, 3\}$, $C_3 \in \{-3, 0\}$, $\lambda^e_{23} \in \{-0.3, 0.3\}$ and $\lambda^d_{23} \in \{-0.04, -0.001\}$. The experimental point shows the value of $R_K^{\mu/e}$ and the combined value of $R_D^{\tau/\ell}$ and $R_{D^*}^{\tau/\ell}$.

$$
\begin{aligned}
&N_\nu = 3 + \frac{0.008\lambda^e_{33}}{\Lambda^2}(C_1 + C_3 - 0.2\, C_3 + 0.02\, C_1) \\
&R_\tau^{\tau/\ell_{1,2}} = 1 + 0.008\, \lambda^e_{33} \frac{C_3}{\Lambda^2} \\
&\mathcal{B}(\tau \to 3\mu) = \left(\frac{\lambda^e_{23}}{0.3}\right)^2 \left[5.0 \frac{(C_1 - C_3)^2}{\Lambda^4} + 4.5 \frac{(C_1 + C_3)^2}{\Lambda^4}\right] \cdot 10^{-8}\,.
\end{aligned} \tag{3.35}
$$

It is interesting to observe that although two neutral-current operators are present in $\mathcal{L}^0_{\text{NP}}$, the ratio $\frac{a_\tau}{a_e}$ turns out to depend exclusively on the Wilson coefficient of the charged-current operator $Q^{(3)}_{\ell q}$ at scale Λ, C_3. Choosing $|\lambda^d_{23}| \lesssim V_{cb}$ in order to avoid too much fine tuning when reproducing the CKM matrix, there is a strong relation between the allowed departure from 1 of $R_{D^{(*)}}^{\tau/\ell}$, $\frac{a_\tau}{a_e}$ and $R_\tau^{\tau/\ell_{1,2}}$. Since the latter are tightly constrained by experimental bounds, $\delta R_{D^{(*)}}^{\tau/\ell} = R_{D^{(*)}}^{\tau/\ell} - 1$ will be tightly constrained as well.

This can be clearly seen in the graph displayed in figure 3.1, which shows the allowed regions for $R_K^{\mu/e}$ and $R_{D^{(*)}}^{\tau/\ell}$ when bounds on constraint observables[4] are imposed. Each constraint observable is evaluated for a given quadruple (C_1, C_3, λ^e_{23}, λ^d_{23}) in parameter space; if the result is compatible with the experimental value within 2σ, the values of $R_K^{\mu/e}$ and $R_{D^{(*)}}^{\tau/\ell}$ for that particular quadruple are plotted with the color associated to the observable considered. The same procedure is repeated for quadruples randomly distributed in parameter space. Note that the bound on the observable $R_K^{\nu\nu}$ is imposed a priori on all quadruples.

Altough all observables receiving NP contribution at one loop impose strong bounds on B anomalies, Z-pole observables set the strictest limits, forcing $\delta R_{D^{(*)}}^{\tau/\ell}$ to be $\lesssim 0.02$. Like in [1, 2], we conclude that current data on constraint observables challenge a simultaneous explanation of the present values of $R_K^{\mu/e}$ and $R_{D^{(*)}}^{\tau/\ell}$.

[4]Note that Z-pole observables are considered as a single constraint.

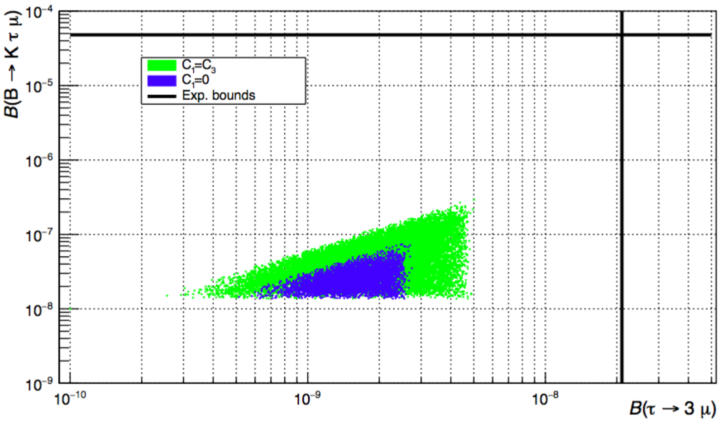


Figure 3.2: $\mathcal{B}(\tau \to 3\mu)$ vs. $\mathcal{B}(B \to K\tau\mu)$ within our model for two different configurations of C_1 and C_3, imposing all constraints but $R^{\tau/\ell}_{D^{(*)}}$. We let parameters vary in the ranges $C_3 \in \{-3, 3\}$, $\lambda^e_{23} \in \{-0.3, 0.3\}$, $\lambda^d_{23} \in \{-0.04, 0.04\}$ and $\Lambda \in \{1, 2\}$ TeV.

In the plot of figure 3.2 we analyse the correlation between the predictions for two LFV decays in our model, namely the semi-leptonic process $B \to K\tau\mu$ and the purely leptonic decay $\tau \to 3\mu$. The graph shows that the loop-induced process $\tau \to 3\mu$ is a much more sensitive probe of the considered scenario than the tree-level observable $B \to K\tau\mu$, due to the current and expected future experimental resolutions.

Conclusions

In the last few years the experimental collaborations of LHCb, Belle and Babar reported indications of LFUV in the processes $B \to K^{(*)}\ell\ell$ and $B \to D^{(*)}\ell\bar{\nu}$. Since the universality of weak interactions is one of the key predictions of the SM, these results have triggered a large interest about possible NP interpretations.
In the present work we have analysed $R^{\mu/e}_{K^{(*)}}$ and $R^{\tau/\ell}_{D^{(*)}}$ in a model-independent way, assuming that NP originates at a scale $\Lambda \sim 1$ TeV. We have started by building the NP Lagrangian at the scale Λ in terms of five six-dimensional semi-leptonic operators. Then we have derived the low-energy effective Lagrangian extensively: we have addressed running effects from Λ to the EW scale by employing one-loop RGE in the limit of exact electroweak symmetry, and, after integrating out heavy degrees of freedom, we have described the evolution down to 1 GeV using RGE dominated by the electromagnetic interaction.
In the last part of our work, we have studied the most relevant phenomenological consequences of the derived Lagrangian. After considering B anomalies and other B decays receiving NP contribution at tree level, we have investigated the phenomenology arising at one loop. Since the most important effects of the running are the modification of the Z and W couplings and the generation of a purely leptonic effective lagragian, we have focused on LFUV and LFV in Z-pole observables and in τ decays. Motivated by the global fits regarding NP in $b \to s\mu^+\mu^-$ transitions, we have finally considered the scenario where only the Wilson coefficient C^9 receives NP contributions, with the aim of quantifying the impact of one-loop-induced constraints on $R^{\tau/\ell}_{D^{(*)}}$ and $R^{\mu/e}_{K}$. We have found that in this scenario Z-pole observables set the strictest bound on the allowed values for B anomalies, because they imply $R^{\tau/\ell}_{D^{(*)}} \lesssim 1.02$. This result conveys the same message as [1, 2]: electroweak radiative corrections challenge a simultaneous explanation of $R^{\mu/e}_{K}$ and $R^{\tau/\ell}_{D^{(*)}}$ and cannot be ignored when addressing B anomalies.
Finally, we have identified the leptonic decay $\tau \to 3\mu$ as the most promising channel to test LFV effects in our framework.

Appendix A

A.1 Formulas

γ relations

$$
\begin{aligned}
&\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\\
&\gamma^\mu \gamma \nu \gamma_\mu = -2\gamma^\nu\\
&\gamma^\mu \gamma \nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho}\\
&\gamma^\mu \gamma \nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu\\
&\gamma_\mu \gamma_\alpha \gamma_\nu = \eta_{\mu\alpha}\gamma_\nu + \eta_{\alpha\nu}\gamma_\mu - \eta_{\mu\nu}\gamma_\alpha + i\epsilon_{\mu\alpha\nu\rho}\gamma^\rho\gamma^5
\end{aligned}
\tag{A.1}
$$

Trace formulas

$$
\begin{aligned}
&\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu}\\
&\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})\\
&\mathrm{Tr}[\gamma^5] = \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0\\
&\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma}\\
&\mathrm{Tr}[\underbrace{\gamma^\mu \gamma^\nu ...}_{\text{odd}}] = 0\\
&\mathrm{Tr}[\gamma^5 \underbrace{\gamma^\mu \gamma^\nu ...}_{\text{odd}}] = 0
\end{aligned}
\tag{A.2}
$$

Feynman parameters

$$
\frac{1}{\mathrm{AB}} = \int_0^1 dx \frac{1}{[\mathrm{A}x + \mathrm{B}(1-x)]^2}
\tag{A.3}
$$

Dimensional regularization

$$
(\mu^2)^{2-\frac{D}{2}} \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^2} = -\frac{2i\Delta}{(4\pi)^2}\left(-\frac{2}{\epsilon} - \ln 4\pi - \frac{\mu^2}{\Delta} + \gamma - 1 + o(\epsilon)\right)
\tag{A.4}
$$

$$
(\mu^2)^{2-\frac{D}{2}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^2}\left(\frac{2}{\epsilon} + \ln 4\pi + \ln \frac{\mu^2}{\Delta} - \gamma + o(\epsilon)\right)
\tag{A.5}
$$

Spinor currents contractions

$$(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_{R(L)}u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_{R(L)}v)\,\eta^{\alpha\beta} = 16\,(\bar{v}\gamma_\mu P_{R(L)}u)\,(\bar{u}\gamma^\mu P_{R(L)}v) \tag{A.6}$$

$$(\bar{v}\gamma^\nu\gamma^\alpha\gamma^\mu P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_{R(L)}v\right)\eta_{\alpha\beta} = 4\,(\bar{v}\gamma_\mu P_{R(L)}u)\,(\bar{u}\gamma^\mu P_{R(L)}v) \tag{A.7}$$

$$\left(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_{L(R)}u\right)(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_{R(L)}v)\,\eta^{\alpha\beta} = 4\,(\bar{v}\gamma_\mu P_{R(L)}u)\,(\bar{u}\gamma^\mu P_{L(R)}v) \tag{A.8}$$

$$(\bar{v}\gamma^\nu\gamma^\alpha\gamma^\mu P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_{L(R)}v\right)\eta_{\alpha\beta} = 16\,(\bar{v}\gamma_\mu P_{R(L)}u)\left(\bar{u}\gamma^\mu P_{L(R)}v\right) \tag{A.9}$$

Proof: Using the last relation in (A.1) we show that

$$\begin{aligned}
(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_{R(L)}u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_{R(L)}v) =& (\bar{v}\gamma_\nu P_{R(L)}u)\,(\bar{u}\gamma_\alpha\gamma_\beta\gamma^\nu P_{R(L)}v)\\
&+ (\bar{v}\gamma_\mu P_{R(L)}u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma_\alpha P_{R(L)}v)\\
&+ 2\,(\bar{v}\gamma_\alpha P_{R(L)}u)\,(\bar{u}\gamma_\beta P_{R(L)}v)\\
&+ i\epsilon_{\mu\alpha\nu\rho}\,(\bar{v}\gamma^\rho P_{R(L)}u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_{R(L)})
\end{aligned}$$

$$\begin{aligned}
(\bar{v}\gamma^\nu\gamma^\alpha\gamma^\mu P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_{R(L)}v\right) =& (\bar{v}\gamma^\mu P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma^\alpha P_{R(L)}v\right)\\
&+ (\bar{v}\gamma^\nu P_{R(L)}u)\,(\bar{u}\gamma^\alpha\gamma_\beta\gamma_\nu P_{R(L)}v)\\
&+ 2\,(\bar{v}\gamma^\alpha P_{R(L)}u)\left(\bar{u}\gamma^\beta P_{R(L)}v\right)\\
&+ i\epsilon^{\nu\alpha\mu\rho}\,(\bar{v}\gamma_\rho P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_{R(L)}\right)
\end{aligned}$$

$$\begin{aligned}
(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_{R(L)}u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_{R(L)}v) =& (\bar{v}\gamma_\nu P_{R(L)}u)\,(\bar{u}\gamma_\alpha\gamma_\beta\gamma^\nu P_{R(L)}v)\\
&+ (\bar{v}\gamma_\mu P_{R(L)}u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma_\alpha P_{R(L)}v)\\
&+ 2\,(\bar{v}\gamma_\alpha P_{R(L)}u)\,(\bar{u}\gamma_\beta P_{R(L)}v)\\
&- i\epsilon_{\mu\alpha\nu\rho}\,(\bar{v}\gamma^\rho P_{R(L)}u)\left(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_{L(R)}\right)
\end{aligned}$$

$$\begin{aligned}
(\bar{v}\gamma^\nu\gamma^\alpha\gamma^\mu P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_{L(R)}v\right) =& (\bar{v}\gamma^\mu P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma^\alpha P_{L(R)}v\right)\\
&+ (\bar{v}\gamma^\nu P_{R(L)}u)\left(\bar{u}\gamma^\alpha\gamma_\beta\gamma_\nu P_{L(R)}v\right)\\
&+ 2\,(\bar{v}\gamma^\alpha P_{R(L)}u)\left(\bar{u}\gamma^\beta P_{L(R)}v\right)\\
&- i\epsilon^{\nu\alpha\mu\rho}\,(\bar{v}\gamma_\rho P_{R(L)}u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_{L(R)}\right)
\end{aligned}$$

where we used $\gamma^5 P_R = P_R$ and $\gamma^5 P_L = -P_L$. These imply (A.6)-(A.9).

A.2 Detailed computations for consistency checks

A.2.1 Consistency check for $(\bar{e}_L\gamma^\mu\lambda_e e_L)(\bar{e}_L\gamma_\mu e_L)$

A.2.1.1 Electromagnetic penguin

$\mathcal{M}_\gamma$ receives contribution from the insertion of the three operators in (2.63). We start by inserting the operator with coefficient $(C_1 - C_3)$. Letting ℓ be the momentum carried by the photon and k the momentum circulating in the loop, the amplitude reads

$$\mathcal{M} = 3\cdot\frac{2}{3}e^2\frac{C_1-C_3}{\Lambda^2}\sum_i\lambda^u_{ii}\int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}[(\not\ell+\not k+m_{u_i})\gamma_\mu P_L(\not k+m_{u_i})\gamma_\nu]}{\ell^2((\ell+k)^2-m^2_{u_i})(k^2-m^2_{u_i})}\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}\gamma^\nu P_L v)\,,$$

where we multiplied by 3 in order to account for the three quark colors. Note that we cannot choose immediately $\ell = 0$; we need to check that the pole cancels first. Using trace formulas in (A.2) we find

$$\begin{aligned}\mathrm{Tr}[(\not\ell+\not k+m_{u_i})\gamma_\mu P_L(\not k+m_{u_i})\gamma_\nu] =& 2\left((\ell+k)_\mu k_\nu+(\ell+k)_\nu k_\mu-\eta_{\mu\nu}k\cdot(\ell+k)\right.\\ &\left.+m^2_{u_i}+i(\ell+k)^\alpha k^\beta\epsilon_{\alpha\beta\mu\nu}\right)\,.\end{aligned}$$

We then recast the denominator using the Feynman parametrization (A.3) with A $=(\ell+k)^2-m^2_{u_i}$ and B $=k^2-m^2_{u_i}$. Noting that

$$Ax+B(1-x)=(k+\ell x)^2-m^2_{u_i}-\ell^2x(1-x)=(k+\ell x)^2-\Delta\,,$$

where $\Delta = m^2_{u_i}+\ell^2x(1-x)$, we get

$$\int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}[(\not\ell+\not k+m_{u_i})\gamma_\mu P_L(\not k+m_{u_i})\gamma_\nu]}{((\ell+k)^2-m^2_{u_i})(k^2-m^2_{u_i})}=\int_0^1dx\int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}[(\not\ell+\not k+m_{u_i})\gamma_\mu P_L(\not k+m_{u_i})\gamma_\nu]}{((k+\ell x)^2-\Delta)^2}\,.$$

Performing the shift $k\to k-\ell x$ the trace becomes

$$\mathrm{Tr}[\dots]\;\to 2\left[\left(\frac{2}{D}-1\right)k^2+\left(m^2_{u_i}+\ell^2x(1-x)\right)\right]\eta_{\mu\nu}-2\ell_\mu\ell_\nu x(1-x)\,,$$

where we have kept only even powers of k and used the substitution $k_\mu k_\nu\to\frac{k^2}{D}\eta_{\mu\nu}$[1]. Note that the chiral stucture of the current giving the loop content plays no role, since the γ_5 dependence in the trace cancels after the shift.
The term $\propto\ell_\mu\ell_\nu$ does not contribute because, thanks to Dirac equation, it yields zero when contracted to the fermionic currents; then

$$\begin{aligned}\mathcal{M} =&\frac{4e^2(C_1-C_3)}{\ell^2\Lambda^2}\sum_i\lambda^u_{ii}\int_0^1dx\left[\left(\frac{2}{D}-1\right)\int\frac{d^Dk}{(2\pi)^D}\frac{k^2}{(k^2-\Delta)^2}\right.\\ &\left.+\left(m^2_{u_i}+\ell^2x(1-x)\right)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-\Delta)^2)}\right]\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}\gamma^\mu P_L v)\,.\end{aligned}$$

[1]This can be done because $\int d^Dkk_\mu k_\nu f(k^2)=\frac{\eta_{\mu\nu}}{D}\int d^Dkk^2f(f^2)$

Using dimensional regularization integrals (A.5) and (A.4) and eliminating divergences in the $\overline{\text{MS}}$ scheme, we find

$$\mathcal{M} = \frac{i}{16\pi^2}\frac{8e^2}{\ell^2}\frac{C_1 - C_3}{\Lambda^2}\sum_i \lambda^u_{ii}\int_0^1 dx\,\left(\ell^2 x(1-x)\right)\ln\frac{\mu^2}{\Delta}\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}\gamma^\mu P_L v)\,.$$

The pole in $\ell = 0$ cancels, so ℓ can be safely taken to be 0. The final result yields

$$\mathcal{M} = \frac{i}{16\pi^2\Lambda^2}\left[\frac{4}{3}e^2(C_1 - C_3)\sum_i \lambda^u_{ii}\ln\frac{\mu^2}{m^2_{u_i}}\right]\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}\gamma^\mu P_L v)\,.$$

In order to compute the contribution deriving from the other two operators in (2.63) we just need to change the charge of the particle running in the loop ($\frac{4}{3}e^2 \to -\frac{2}{3}e^2$) and to substitute λ^u with λ^d and Γ^d.

In conclusion, the final result reads

$$\begin{aligned}\mathcal{M}_\gamma = \frac{i}{16\pi^2\Lambda^2}\Bigg[&\frac{4}{3}e^2(C_1 - C_3)\left(\lambda^u_{11}\ln\frac{\mu^2}{m_u^2} + \lambda^u_{22}\ln\frac{\mu^2}{m_c^2} + \lambda^u_{33}\ln\frac{\mu^2}{m_t^2}\right)\\ &-\frac{2}{3}e^2(C_1 + C_3)\left(\lambda^d_{11}\ln\frac{\mu^2}{m_d^2} + \lambda^d_{22}\ln\frac{\mu^2}{m_s^2} + \lambda^d_{33}\ln\frac{\mu^2}{m_b^2}\right)\\ &-\frac{2}{3}e^2 C_4\left(\Gamma^d_{11}\ln\frac{\mu^2}{m_d^2} + \Gamma^d_{22}\ln\frac{\mu^2}{m_s^2} + \Gamma^d_{33}\ln\frac{\mu^2}{m_b^2}\right)\Bigg]\lambda^e_{12}(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L u)\,.\end{aligned}$$

A.2.1.2 *Z* penguin

The amplitude for the Z penguin with the top quark running in the loop reads

$$\begin{aligned}\mathcal{M}_Z = &-3\lambda^u_{33}\frac{C_1 - C_3}{\Lambda^2}\frac{g_2^2}{c^2_{\mathrm{W}}}\frac{\eta^{\nu\rho}}{\ell^2 - m^2_{\mathrm{Z}}}\int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}\left[(\not{\ell} + \not{k} + m_t)\gamma_\mu P_L(\not{k} + m_t)(g^t_L\gamma_v P_L + g^t_R\gamma_v P_R)\right]}{((\ell + k)^2 - m_t^2)(k^2 - m_t^2)}\\ &\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}(g^e_L\gamma_\rho P_L + g^e_R\gamma_\rho P_R)v)\,.\end{aligned}$$

Since the propagator is massive we can safely take $\ell = 0$, hence

$$\begin{aligned}\mathcal{M}_Z = &3\lambda^u_{33}\frac{C_1 - C_3}{\Lambda^2}\frac{g_2^2}{c^2_{\mathrm{W}}m^2_{\mathrm{Z}}}\int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}\left[(\not{k} + m_t)\gamma_\mu P_L(\not{k} + m_t)(g^t_L\gamma_v P_L + g^t_R\gamma_v P_R)\right]}{(k^2 - m_t^2)^2}\\ &\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}(g^e_L\gamma^\nu P_L + g^e_R\gamma^\nu P_R)v)\,.\end{aligned}$$

We then compute the trace using (A.2)

$$\mathrm{Tr}\left[\ldots\right] = \frac{g^t_L}{2}\mathrm{Tr}\left[\not{k}\gamma_\mu\not{k}\gamma_\nu\right] + \frac{g^t_L}{2}m_t^2\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] = 2\eta_{\mu\nu}\left[g^t_L\left(\frac{2}{D} - 1\right)k^2 + g^t_R m_t^2\right]\,,$$

and find the following expression for the amplitude

$$\begin{aligned}\mathcal{M}_Z = &6\lambda^u_{33}\frac{C_1 - C_3}{\Lambda^2}\frac{g_2^2}{c^2_{\mathrm{W}}m^2_{\mathrm{Z}}}\left[g^t_L\left(\frac{2}{D} - 1\right)\int\frac{d^Dk}{(2\pi)^D}\frac{k^2}{(k^2 - m_t^2)^2} + g^t_R m_t^2\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 - m_t^2)^2}\right]\\ &\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}(g^e_L\gamma_\mu P_L + g^e_R\gamma_\mu P_R)v)\\ = &6\lambda^u_{33}\frac{C_1 - C_3}{\Lambda^2}\frac{g_2^2}{c^2_{\mathrm{W}}m^2_{\mathrm{Z}}}(-2g^t_A)\ln\frac{\mu^2}{m_t^2}\lambda^e_{12}(\bar{v}\gamma^\mu P_L u)(\bar{u}(g^e_L\gamma_\mu P_L + g^e_R\gamma_\mu P_R)v)\,.\end{aligned}$$

In the last step we used the dimensional regularization integrals (A.5) and (A.4). Remembering that $m_t^2 = \frac{2m_Z^2 c_W^2}{g_2^2} y_t^2$ the final result is

$$\mathcal{M}_Z = \frac{i}{16\pi^2\Lambda^2}\lambda^e_{12}(\bar{v}\gamma_\mu P_L u)(\bar{u}\gamma^\mu P_L u)\left[-6y_t^2\lambda^u_{33}(C_1 - C_3)\left(-\frac{1}{2}+s_W^2\right)\ln\frac{\mu^2}{m_t^2}\right] .$$

A.2.2 Consistency check for $(\bar{e}_R\gamma^\mu\Gamma^e e_R)(\bar{d}_R\gamma_\mu\Gamma^d d_R)$

A.2.2.1 Computation of $\mathcal{M}_i^Z$

Taking all external particles to have momentum p, the amplitude of the diagram in figure 2.5(i) is given by

$$\mathcal{M}_i^Z = -\frac{g_2^2}{c_W^2}\, g_R^e\, g_R^d\, \frac{C_5}{\Lambda^2}\,\Gamma^e_{12}\,\Gamma^d_{12}\int\frac{d^Dk}{(2\pi)^D}\frac{(k-p)^\alpha(k-p)^\beta\,(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_R u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_R v)}{(k^2-m_Z^2)(k-p)^4 *} .$$

We take $p=0$ and rewrite the numerator using $k^\alpha k^\beta = \frac{k^2}{D}\eta^{\alpha\beta}$ and the spinor contraction (A.6), obtaining

$$\begin{aligned}
\mathcal{M}_i^Z &= -\frac{g_2^2}{c_W^2}\, g_R^e\, g_R^d\, \frac{C_5}{\Lambda^2}\,\Gamma^e_{12}\,\Gamma^d_{12}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\alpha k^\beta\,(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_R u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_R v)}{(k^2-m_Z)^2k^4} \\
&= -4\frac{g_2^2}{c_W^2}\, g_R^e\, g_R^d\,\frac{C_5}{\Lambda^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k^2-m_Z^2)}\Gamma^e_{12}\,\Gamma^d_{12}\;(\bar{v}\gamma_\mu P_R u)\,(\bar{u}\gamma^\mu P_R v) \\
&= -4\frac{g_2^2}{c_W^2}g_R^e g_R^d\frac{C_5}{\Lambda^2}\int_0^1 dx\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-m_Z^2x)^2}\Gamma^e_{12}\Gamma^d_{12}\,(\bar{v}\gamma_\mu P_R u)\,(\bar{u}\gamma^\mu P_R v) ,
\end{aligned}$$

where in the last step we rewrote the denominator using (A.3). Next we compute the integrals over the loop momentum using (A.5); the result in the $\overline{\text{MS}}$ subtraction scheme reads

$$\mathcal{M}_i^Z = \frac{i}{16\pi^2\Lambda^2}\left(-4\frac{g_2^2}{c_W^2}\, g_R^e\, g_R^d\, C_5\right)\left[\ln\frac{\mu^2}{m_Z^2}+1\right]\Gamma^e_{12}\Gamma^d_{12}\,(\bar{v}\gamma_\mu P_R u)\,(\bar{u}\gamma^\mu P_R v) .$$

A.2.2.2 Computation of $\mathcal{M}_k^Z$

The computation is similar to the $\mathcal{M}_k^Z$ one, except for the γ-structure appearing in the numerator. Using the spinor current contraction (A.7) we find

$$\begin{aligned}
\mathcal{M}_k^Z &= \frac{g_2^2}{c_W^2}g_R^e g_R^d\frac{C_5}{\Lambda^2}\Gamma^e_{12}\Gamma^d_{12}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\alpha k^\beta\,(\bar{v}\gamma^\nu\gamma^\alpha\gamma^\mu P_R u)\left(\bar{u}\gamma_\mu\gamma^\beta\gamma_\nu P_R v\right)}{(k^2-m_Z)^2k^4} \\
&= \frac{i}{16\pi^2\Lambda^2}\left(\frac{g_2^2}{c_W^2}g_R^e g_R^d C_5\right)\left[\ln\frac{\mu^2}{m_Z^2}+1\right]\Gamma^e_{12}\Gamma^d_{12}\,(\bar{v}\gamma_\mu P_R u)\,(\bar{u}\gamma^\mu P_R v) . \qquad (A.10)
\end{aligned}$$

A.2.3 Consistency check for $(\bar{e}_L\gamma^\mu\lambda^e e_L)(\bar{u}_L\gamma_\mu\lambda^u u_L)$

A.2.3.1 Computation of $\mathcal{M}_i$

Let $p=0$ be the momentum of the external particles and k the loop momentum. W have

$$\mathcal{M}_i = -\frac{g_2^2}{2}\frac{2C_3}{\Lambda^2}\lambda^e_{12}\sum_i[\lambda^u V]_{1i}[V^\dagger]_{i2}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\alpha k^\beta\,(\bar{v}\gamma_\mu\gamma_\alpha\gamma_\nu P_L u)\,(\bar{u}\gamma^\mu\gamma_\beta\gamma^\nu P_L v)}{k^4(k^2-m_W^2)} .$$

We observe that $\sum_i [\lambda^u V_{\text{CKM}}]_{1i} [V^\dagger_{\text{CKM}}]_{i2} = \lambda^u_{12}$ due to λ properties.
Using $k^\alpha k^\beta = \frac{k^2}{D} \eta^{\alpha\beta}$ and the spinor current contraction (A.6) we find

$$\mathcal{M}_i = -\frac{4g_2^2 C_3}{\Lambda^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k^2 - m_{\text{W}}^2)} \lambda^e_{12} \lambda^u_{12} \left(\bar{v}\gamma_\mu P_L u\right) \left(\bar{u}\gamma^\mu P_L v\right) \, .$$

The Feynman parametrization is the same we used for $\mathcal{M}_c$; the final result in the $\overline{\text{MS}}$ scheme reads

$$\mathcal{M}_i = \frac{i}{16\pi^2 \Lambda^2} (-4 g_2^2 C_3) \left[\ln \frac{\mu^2}{m_{\text{W}}^2} + 1 \right] \lambda^e_{12} \lambda^u_{12} \left(\bar{v}\gamma_\mu P_L u\right) \left(\bar{u}\gamma^\mu P_L v\right) \, .$$

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