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Electric Charges, Magnetic Monopoles and Dyons in ModMax Electrodynamics

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Introduction

Non-linear electrodynamics (NED) theories have been studied since 1933 when Born and Infeld constructed the first non-linear generalization of Maxwell's electrodynamics [1],[2],[3]; it was originally introduced to remove the singularity of the electric field of point-like charges on their space position. Actually many years later Born-Infeld action showed up to be an important element in String Theory; for instance in bosonic open String Theory it arises as an effective low-energy action, while in type II superstring theories it describes an effective worldvolume theory of Dirichlet branes (in 10-dimensional spacetime). Since then many other models have been proposed showing applications in String Theory, gravity, cosmology and condensed matter [4]. For instance, there exist NED models coupled to gravity which give rise to regular cosmological solutions by removing singularities at the initial time of the evolution of the Universe. Other NED models have been used for the description of strongly correlated condensed matter systems via a technique called gravity/CMT correspondence.
Among others, there exists an interesting phenomenon happening to almost all NED theories which is the birefringence of light propagating in a uniform strong electromagnetic background. It consists in a double refraction of ray of light that is split by polarization (with respect to the optical axis of the material) into two rays taking different geodesic paths and the electromagnetic background plays the role of an optical material. The Born-Infeld theory is the only physically consistent NED that does not show this effect. This is an important effect since it can be tested by experiment, but yet to be observed [4].

Source-free Maxwell theory is known to have two important symmetries, namely it is invariant under both duality and conformal transformations. In general in NED theories these symmetries are broken. It has been shown that there exist only two NED theories invariant under both conformal and duality transformations [5]. The first one is the Bialyanicki-Birula theory [6] found in 1983 by passing to the Hamiltonian formulation of Born-Infeld theory and taking the strong field limit (which cannot be taken in Lagrangian formulation). The other one is the so-called ModMax, derived for the first time by Bandos, Lechner, Sorokin and Townsend in 2020 [5]; actually this is the unique maximally symmetric source-free non-linear extension of Maxwell theory [5],[7].

The description of Maxwell Electrodynamics in presence of electric charges has been developed with the use of potentials, which are well defined everywhere if magnetic monopoles do not exist. Actually if only electric charges are present in source-coupled Maxwell Electrodynamics, the duality invariance gets broken. Paul Dirac was one of the firsts to propose a consistent description of Maxwell Electrodynamics with magnetic monopoles. To this end he introduced an auxiliary object, the so-called Dirac's string, to tackle the problem of the consistency of the existence of the electromagnetic vector potential in the presence of the monopoles [8]. Probably the most famous consequence of his theory is that the presence of magnetic monopoles implies the quantization of the electric charges and thus provides an explanation of this empiric fact. Since then many experiments have been conducted for discovering magnetic charges; however they have not yet been observed.

The aim of this work is to study various aspects of ModMax electrodynamics . In particular, since it is the unique maximally symmetric source-free non-linear extension of Maxwell theory, it is interesting to couple to this theory charged sources and see what differences emerge in comparison with Maxwell's theory. We will see that we have the freedom to couple charges with different coupling constants in order to get slightly different descriptions of the electrodynamics and we will discuss physical motivations under these choices. We will present both Lagrangian and Hamiltonian formulations of this source-coupled theory and some solutions of its non-linear field equations, as well as electric-magnetic charge quantization conditions involved. To conclude we will consider ModMax birefringence and its consequences on Compton scattering.

The structure of this work is as follows.
In Chapter 1 we will present ModMax electrodynamics and its symmetries, namely duality and conformal invariance.

In Chapter 2 we will couple to ModMax in three different ways different types of charged particles, namely electric charges, magnetic charges and dyons. Then we will solve ModMax equations sourced by different configurations of dyons. For instance, we will show that the Lienard-Wiechert fields generated by a moving dyon are exact solutions of the ModMax electrodynamics.
In Chapter 3 we will make the Legendre transform from the ModMax Lagrangian to the Hamiltonian formulation with the use of the Dirac constraint formalism and we will then pass to Quantum Mechanics with the aim to find the quantization condition between electric and magnetic charges.
Finally, in Chapter 4 we will study ModMax birefringence and its application to Compton Scattering in the presence of a magnetic background.

1 ModMax Electrodynamics

In 4-dimensional spacetime ModMax electrodynamics is the unique source-free non-linear extension of Maxwell theory invariant under both conformal and duality transformations [5],[7]. In this Chapter we will present its definition and then we will study and prove its symmetries.

1.1 Definition of ModMax

We are in 4-dimensional spacetime with coordinates $x^\mu = (ct, \vec{x})$ and Minkowsky flat metric $\eta_{\mu\nu}$ of signature $(+,-,-,-)$. Given the anti-symmetric field tensor $F^{\mu\nu}$ and its dual ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ with

$$F^{i0} = E^i\,, \qquad F^{ij} = -\epsilon^{ijk}B^k \qquad \text{and} \qquad {}^*F^{i0} = B^i\,, \qquad {}^*F^{ij} = \epsilon^{ijk}E^k \tag{1.1}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively, we define the two Lorentz invariants, the scalar S and the pseudo-scalar P,

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)\,, \qquad P = -\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu} = \vec{E}\cdot\vec{B}\,. \tag{1.2}$$

The Lagrangian density of ModMax is given by

$$\mathcal{L} = \mathcal{L}_\gamma(S,P) = \cosh\gamma\; S + \sinh\gamma\,\sqrt{S^2+P^2} \tag{1.3}$$

where $\gamma \geq 0$ is a dimensionless coupling constant; it is assumed to be non-negative in order to respect the conditions of causality and unitarity. We can see that (1.3) reduces to Maxwell Lagrangian density for $\gamma = 0$, otherwise it describes a non-linear theory.

For further consideration it is convenient to introduce the following anti-symmetric tensor

$$G^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = -\frac{\partial\mathcal{L}}{\partial S}\frac{\partial S}{\partial F_{\mu\nu}} - \frac{\partial\mathcal{L}}{\partial P}\frac{\partial P}{\partial F_{\mu\nu}} = \mathcal{L}_S F^{\mu\nu} + \mathcal{L}_P{}^*F^{\mu\nu} \tag{1.4}$$

$$\text{with} \qquad \mathcal{L}_S = \frac{\partial\mathcal{L}}{\partial S} = \cosh\gamma + \sinh\gamma\frac{S}{\sqrt{P^2+S^2}}\,, \qquad \mathcal{L}_P = \frac{\partial\mathcal{L}}{\partial P} = \sinh\gamma\frac{P}{\sqrt{P^2+S^2}}$$

where when taking the derivative with respect to the components of $F^{\mu\nu}$ we consider $F^{\mu\nu}$ and $F^{\nu\mu}$ as dependent variables.
We can define

$$\vec{D} = \frac{\partial\mathcal{L}}{\partial\vec{E}}\,, \qquad \vec{H} = -\frac{\partial\mathcal{L}}{\partial\vec{B}} \tag{1.5}$$

where $\vec{D}$ and $\vec{H}$ are the electric displacement and magnetic induction fields respectively, and so we have in components

$$G^{i0} = D^i\,, \qquad G^{ij} = -\epsilon^{ijk}H^k \qquad \text{and} \qquad {}^*G^{i0} = H^i\,, \qquad {}^*G^{ij} = \epsilon^{ijk}D^k\,. \tag{1.6}$$

The field equations in a source-free NED theory are written in terms of these four fields as follows

$$\vec{\nabla}\cdot\vec{B} = 0\,, \qquad \partial_t\vec{B} + \vec{\nabla}\times\vec{E} = \vec{0}\,, \qquad \vec{\nabla}\cdot\vec{D} = 0\,, \qquad \partial_t\vec{D} - \vec{\nabla}\times\vec{H} = \vec{0} \tag{1.7}$$

or in a manifestly covariant form

$$\partial_\mu{}^*F^{\mu\nu} = 0\,, \qquad \partial_\mu G^{\mu\nu} = 0\,. \tag{1.8}$$

Thanks to the first group of equations (also known as Bianchi's identities) it follows that $F_{\mu\nu}$ is the curl of a 4-vector potential A_μ, namely

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{1.9}$$

1.2 Duality symmetry

It is known that source-free Maxwell equations are invariant under a duality transformation, which involves a rotation of fields $(F^{\mu\nu},\ {}^*F^{\mu\nu})$. In NED theories this symmetry does not hold anymore in general. However because of the analogy between Maxwell and NED equations in (1.8), one can define a duality transformation involving the fields $(G^{\mu\nu},\ {}^*F^{\mu\nu})$ [6],[9]. In fact this transformation acts as an $SO(2)$ rotation with an angle parameter $\alpha \in [0, 2\pi]$ as follows

$$\begin{cases} G^{'}_{\mu\nu} = G_{\mu\nu}\,\cos\alpha - \ {}^*F_{\mu\nu}\,\sin\alpha \\ {}^*F^{'}_{\mu\nu} = G_{\mu\nu}\,\sin\alpha + \ {}^*F_{\mu\nu}\,\cos\alpha \end{cases} \tag{1.10}$$

Notice that Maxwell duality rotation is recovered when $F^{\mu\nu} = G^{\mu\nu}$, i.e. when the NED equations reduce to the linear case.
This transformation has also an alternative equivalent form if we introduce the complex tensor $G_{\mu\nu} + i\,{}^*F_{\mu\nu}$. Then the duality transformation acts as a change of phase (i.e. as a $U(1)$ transformation)

$$G^{'}_{\mu\nu} + i\,{}^*F^{'}_{\mu\nu} = e^{i\alpha}\left(G_{\mu\nu} + i\,{}^*F_{\mu\nu}\right)\,. \tag{1.11}$$

Since $G_{\mu\nu}$ is not independent but a function of $F_{\mu\nu}$ defined in (1.4) the above transformations are in general not a symmetry of the NED equations of motion. For this to be the case the NED Lagrangian must satisfy a condition which we shall now derive.
With α being small, the infinitesimal transformation takes the form

$$\begin{cases} \delta\,G_{\mu\nu} = -\alpha\,{}^*F_{\mu\nu} \\ \delta\,{}^*F_{\mu\nu} = \alpha\,G_{\mu\nu} \end{cases} \tag{1.12}$$

So in order to have duality invariance the following sequence of relations should hold

$$\alpha\,{}^*F^{\mu\nu} = -\delta\,G^{\mu\nu} = \frac{\partial}{\partial F_{\alpha\beta}}\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\,\delta F_{\alpha\beta} = -\alpha\frac{\partial}{\partial F_{\alpha\beta}}\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\ {}^*G_{\alpha\beta} \qquad \Longleftrightarrow$$

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\frac{\partial^2\mathcal{L}}{\partial F_{\mu\nu}\partial F_{\alpha\beta}}\frac{\partial\mathcal{L}}{\partial F_{\gamma\delta}} = \frac{1}{4}\frac{\partial}{\partial F_{\mu\nu}}\left(\epsilon_{\alpha\beta\gamma\delta}\frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}}\frac{\partial\mathcal{L}}{\partial F_{\gamma\delta}}\right)$$

and if we integrate with respect to $F^{\mu\nu}$ we find (up to a constant that we put to zero)

$$\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu} = \frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}}\frac{\partial\mathcal{L}}{\partial F_{\gamma\delta}}\,.$$

The constant of integration is put to zero because within a conformal invariant theory, if we rescale $F_{\mu\nu}$ the condition must remain the same.
Therefore the condition for the equations of motion to be invariant under these transformations is that the Lagrangian density satisfies

$$F_{\mu\nu}{}^*F^{\mu\nu} = G_{\mu\nu}{}^*G^{\mu\nu} \quad \text{equivalent to} \quad \vec{E}\cdot\vec{B} = \vec{D}\cdot\vec{H}\,. \tag{1.13}$$

Since $^{**} = -1$, for ModMax we find

$$G_{\mu\nu}{}^*G^{\mu\nu} = \left(\mathcal{L}_S F_{\mu\nu} + \mathcal{L}_P{}^*F_{\mu\nu}\right)\left(\mathcal{L}_S{}^*F^{\mu\nu} - \mathcal{L}_P F^{\mu\nu}\right) = F_{\mu\nu}{}^*F^{\mu\nu}\left(\mathcal{L}_S^2 - \mathcal{L}_P^2\right) + F_{\mu\nu}F^{\mu\nu}\left(-2\mathcal{L}_S\mathcal{L}_P\right) =$$

$$= -4P\mathcal{L}_S^2 + 4P\mathcal{L}_P^2 + 8S\mathcal{L}_S\mathcal{L}_P = -4P\left(\cosh^2\gamma - \sinh^2\gamma\right) = F_{\mu\nu}{}^*F^{\mu\nu}\,.$$

Therefore ModMax is invariant under duality transformations.

1.3 Conformal symmetry

Conformal transformations are transformations of coordinates $x \to x'$ for which the metric transforms only with a scale factor $\Lambda(x)$ as

$$\eta_{\mu\nu} \quad \to \quad \eta'_{\mu\nu} = \eta_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = \Lambda(x)\, \eta_{\mu\nu} \,. \tag{1.14}$$

It is possible to prove that a conformal transformation is given by a transformation of coordinates under Poincaré group transformations, scale transformations and special conformal transformations [10].

Poincaré transformations are defined as

$$x'^\mu = \Omega^\mu{}_\nu\, x^\nu + a^\mu \qquad \text{with} \qquad \Omega^\mu{}_\nu \in SO(1,3) = \{\Omega^\mu{}_\nu \ : \ \Omega\, \eta\, \Omega^T = \eta \ , \ \det \Omega = 1\} \,, \quad a^\mu \in \mathbb{R}^4 \,. \tag{1.15}$$

Scale transformations are defined as $x'^\mu = \alpha x^\mu$ with $\alpha > 0$. Special conformal transformations have their finite form depending on a vector parameter b^μ and are defined as

$$x'^\mu = \beta(x,b)^{-1} \left(x^\mu - x^\nu x_\nu\, b^\mu\right) \qquad \text{with} \qquad \beta(x,b) = 1 - 2\, b_\nu x^\nu + b_\nu b^\nu\, x_\rho x^\rho \,. \tag{1.16}$$

By Noether's theorem we know that given a general tensor field ϕ and a Lagrangian density $\mathcal{L}$, to any invariance of the action $S[\phi] = \int d^4x\, \mathcal{L}$ under a transformation of its variables corresponds a conserved current J^μ defined as

$$\delta S = \int d^4x\, \partial_\mu J^\mu = 0 \qquad \text{with} \qquad J^\mu = \delta x^\mu\, \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \tag{1.17}$$

where the infinitesimal transformations of coordinates and fields are $\delta x^\mu = x'^\mu - x^\mu$, $\delta\phi = \phi'(x) - \phi(x)$ and J^μ is defined up to a total derivative of an antisymmetric tensor $C^{\mu\nu}$, such that $J^\mu \sim J^\mu + \partial_\nu C^{\mu\nu}$. In our case we have (up to the first order)

$$x'^\mu = x^\mu + \epsilon^\mu(x) + O(\epsilon^2) \quad \Longrightarrow \quad \delta x^\mu = \epsilon^\mu(x) \,, \tag{1.18}$$

$$A'_\mu(x') = \frac{\partial x_\mu}{\partial x'_\nu} A_\nu(x) \quad \Longrightarrow \quad A'_\mu(x) + \epsilon^\nu \partial_\nu A_\mu(x) + O(\epsilon^2) = A_\mu(x) - A_\nu(x) \partial_\mu \epsilon^\nu + O(\epsilon^2) \quad \text{thus}$$

$$\delta A_\mu = -\epsilon^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \epsilon^\nu = \epsilon^\nu F_{\mu\nu} - \partial_\mu(\epsilon^\nu A_\nu) \,. \tag{1.19}$$

Therefore we have (using antisymmetry of $F_{\mu\nu}$)

$$J^\mu = \epsilon^\nu \left(\delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} F_{\nu\alpha} \right) - \frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} \partial_\alpha(\epsilon^\nu A_\nu) = -\epsilon^\nu\, T^\mu_{\ \nu} \tag{1.20}$$

where we recognize in the first term $T^{\mu\nu}$ which is called the stress-energy tensor, while the second term is neglected because it can be written in the following way with the use of (1.8)

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} \partial_\alpha(\epsilon^\nu A_\nu) = \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} \epsilon^\nu A_\nu \right) - \epsilon^\nu A_\nu \partial_\alpha \frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} = \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} \epsilon^\nu A_\nu \right) \tag{1.21}$$

which is a total derivative of an antisymmetric tensor.

A sufficient condition for a theory to be invariant under conformal transformations is that the stress-energy tensor is symmetric ($T^{\mu\nu} = T^{\nu\mu}$), traceless ($T^\mu_{\ \mu} = 0$) and conserved ($\partial_\mu T^{\mu\nu} = 0$) on the mass shell, as we will show in the following.

Under Poincaré transformations we have $\epsilon^\mu = \omega^\mu{}_\nu\, x^\nu + a^\mu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, thus

$$\partial_\mu J^\mu = -\omega_{\nu\alpha}\, \partial_\mu(x^\alpha\, T^{\mu\nu}) - a_\nu\, \partial_\mu T^{\mu\nu} = -\omega_{\nu\alpha}\, T^{\alpha\nu} = 0 \,;$$

under scale transformations we have $\epsilon^\mu = (\alpha - 1)\, x^\mu$, thus

$$\partial_\mu J^\mu = (1-\alpha)\, \partial_\mu (x^\nu\, T^\mu_{\ \nu}) = (1-\alpha)\, T^\mu_{\ \mu} = 0\,;$$

under special conformal transformations we have $\epsilon^\mu = 2\, x^\nu b_\nu\, x^\mu - b^\mu\, x^\nu x_\nu$, thus

$$\partial_\mu J^\mu = -2\, b_\mu x_\nu\, T^{\mu\nu} - 2\, x^\alpha b_\alpha\, T^\mu_{\ \mu} + 2\, b_\nu x_\mu\, T^{\mu\nu} = 0\,.$$

For any NED theory the stress-energy tensor takes the form

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial F_{\mu\alpha}} F^\nu_{\ \alpha} = -\eta^{\mu\nu}\mathcal{L} - G^{\mu\alpha} F^\nu_{\ \alpha} = -\eta^{\mu\nu}\mathcal{L} - \mathcal{L}_S\, F^{\mu\alpha} F^\nu_{\ \alpha} - \mathcal{L}_P\, {}^*F^{\mu\alpha} F^\nu_{\ \alpha}\,. \tag{1.22}$$

For ModMax we can see that it is trivially symmetric, traceless

$$T^\mu_{\ \mu} = \eta_{\mu\nu}\, T^{\mu\nu} = -\delta^\mu_{\ \mu}\mathcal{L} - \mathcal{L}_S\, F^{\mu\alpha} F_{\mu\alpha} - \mathcal{L}_P\, {}^*F^{\mu\alpha} F_{\mu\alpha} = -4\mathcal{L} + 4S\mathcal{L}_S + 4P\mathcal{L}_P = 0$$

and conserved. The latter property is proven with the use of (1.8) and noticing that $\partial_\mu \mathcal{L} = -\frac{1}{2} G^{\alpha\beta} \partial_\mu F_{\alpha\beta}$; thus

$$\partial_\mu T^{\mu\nu} = \frac{1}{2} G_{\alpha\beta}\, \partial^\nu F^{\alpha\beta} + \frac{1}{2} G_{\alpha\mu} \left(\partial^\mu F^{\nu\alpha} - \partial^\alpha F^{\nu\mu}\right) = \frac{1}{2} G_{\alpha\beta} \left(\partial^\nu F^{\alpha\beta} + \partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha}\right) = 0\,.$$

Therefore the ModMax action and equations of motion are proven to be invariant under conformal transformations.

2 Coupling Charged Sources to ModMax

As in Maxwell theory, one can couple different types of charges to a NED theory. The most simple ones are the electric charged particles, but one can theoretically assume also the existence of magnetic charges (also called monopoles) as was done by Dirac for Maxwell theory [8]. In this Chapter we intend to couple to ModMax both electric and magnetic charged particles.

2.1 Coupling electric charges

Given the spacetime with coordinates x^μ, we shall describe a point-particle with electric charge e and mass m_e moving along the trajectory $y^\mu(\tau)$ parameterized by a free parameter τ. We also introduce the proper interval s defined by its infinitesimal displacement $ds^2 = \eta_{\mu\nu}\, dy^\mu dy^\nu$. The current of this particle has the form

$$j_e^\mu = e \int d\tau\; \delta^{(4)}\left(x - y(\tau)\right) \frac{dy^\mu}{d\tau}\,. \tag{2.1}$$

In analogy with Maxwell theory, the action which describes the minimal coupling of an electric charged particle to ModMax theory has the following form

$$S\left[A, y\right] = \int d^4x\; \mathcal{L} - \int d^4x\; j_e^\nu A_\nu - m_e \int d\tau \sqrt{\frac{dy^\mu(\tau)}{d\tau}\frac{dy_\mu(\tau)}{d\tau}} \tag{2.2}$$

where $\mathcal{L}$ is the ModMax Lagrangian density (1.3).

Now we derive equations of motion from the variational principle.
Varying the action with respect to A^μ we get [11]

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \qquad \Longrightarrow \qquad \partial_\mu G^{\mu\nu} = j_e^\nu$$

which gives equations of motion of fields together with Bianchi's identities

$$\partial_\mu G^{\mu\nu} = j_e^\nu\,, \qquad \partial_\mu {}^*F^{\mu\nu} = 0\,. \tag{2.3}$$

The variation of the kinetic term of the particle with respect to y^μ gives (choosing $\tau = s$, using integration by parts and assuming compact support)

$$-m_e \int d\tau \frac{1}{\sqrt{\frac{dy^\mu(\tau)}{d\tau}\frac{dy_\mu(\tau)}{d\tau}}} \frac{dy^\mu(\tau)}{d\tau}\frac{d}{d\tau}\left(\delta y_\mu\right) = -m_e \int ds\; \frac{dy^\mu(s)}{ds}\frac{d}{ds}\left(\delta y_\mu\right) = \int ds\; \frac{dp_e^\mu}{ds}\,\delta y_\mu$$

where $p_e^\mu(s) = m_e \dfrac{dy^\mu(s)}{ds}$ is the 4-momentum of the particle. Then the variation of

$$-\int d^4x\; j_e^\nu A_\nu = -e\int d^4x \int d\tau\; \delta^{(4)}\left(x - y(\tau)\right) \frac{dy^\nu}{d\tau} A_\nu(x) = -e\int d\tau\; \frac{dy^\nu}{d\tau} A_\nu(y(\tau))$$

with respect to y^μ gives (using integration by parts, assuming compact support and choosing $\tau = s$)

$$-e\int d\tau\; \frac{d}{d\tau}\left(\delta y^\nu\right) A_\nu(y(\tau)) + \frac{dy^\nu}{d\tau}\partial_\mu A_\nu(y(\tau))\,\delta y^\mu = -e\int ds\; \frac{dy^\nu}{ds}\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right]\delta y^\mu\,.$$

So by extremizing the action with respect to y^μ we find the standard Lorentz force

$$\frac{dp_e^\mu}{ds} = e\, F^{\mu\nu}(y)\, \frac{dy_\nu}{ds}\,. \tag{2.4}$$

2.2 Looking for solutions of ModMax equations of motion

At this point we would like to find solutions of the NED field equations (2.3) in the case of ModMax. The ModMax field equations have the following form

$$\cosh\gamma\,\partial_\mu F^{\mu\nu} + \sinh\gamma\,\partial_\mu\left(\frac{S}{\sqrt{P^2+S^2}}\,F^{\mu\nu} + \frac{P}{\sqrt{P^2+S^2}}\,{}^*F^{\mu\nu}\right) = j_e^\nu\,, \qquad \partial_\mu{}^*F^{\mu\nu} = 0\,. \tag{2.5}$$

These equations are non-linear in general, however for some classes of fields they reduce to Maxwell's equations. For instance, in the case in which electric and magnetic fields are orthogonal i.e. satisfy $P = 0$ (but $S \neq 0$), the equations become linear

$$e^{\eta\gamma}\,\partial_\mu F^{\mu\nu} = j_e^\nu\,, \qquad \partial_\mu{}^*F^{\mu\nu} = 0\,, \tag{2.6}$$

where $\eta = S/|S|$ and takes values $\eta = \pm 1$.
Also, if electric and magnetic fields have the same strength, i.e. $S = 0$ (but $P \neq 0$) the equations linearize

$$\cosh\gamma\,\partial_\mu F^{\mu\nu} = j_e^\nu\,, \qquad \partial_\mu{}^*F^{\mu\nu} = 0. \tag{2.7}$$

More generically we can study the case in which both S and P are nonzero, but proportional to each other, such that $P = \alpha S$ with α being a constant. Then the equations of motion again reduce to linear ones

$$\left(\cosh\gamma + \eta\,\frac{\sinh\gamma}{\sqrt{\alpha^2+1}}\right)\partial_\mu F^{\mu\nu} = j_e^\nu\,, \qquad \partial_\mu{}^*F^{\mu\nu} = 0\,, \tag{2.8}$$

One might have noticed that we have avoided the case in which both S and P are null. This is a very important class of fields called null. For instance source-free plane waves satisfy this condition. The issue is that for these fields the equations (2.5) are not analytic and the limit $S, P \to 0$ is not well defined. One can see that, for instance, taking first $S \to 0$ and then $P \to 0$ produces equations which differ from those obtained by first taking $P \to 0$ and then $S \to 0$. In the case of ModMax, this non-analyticity of the Lagrangian field equations can be resolved by going from the Lagrangian to the Hamiltonian formulation in which the equations are regular for the null fields [4]. We will consider the ModMax Hamiltonian formulation in Chapter 3. Here we just mention that the source-free Hamiltonian equations of ModMax are solvable for null fields. For instance one can find the explicit form of ModMax plane waves using a generalization of Bateman Potentials [12].

Coming back to our simplified equations, we can see that the particular classes of fields mentioned above satisfy equations of motion which are similar to Maxwell ones, but with different coupling constants between the fields and the charges. Therefore the solutions of Maxwell's equations which describe the fields satisfying $P = \alpha S$ with α being a constant are also the solutions of ModMax theory. For instance, in Maxwell's theory the solutions of the equations of motion

$$\partial_\mu F^{\mu\nu} = j_e^\nu\,, \qquad \partial_\mu{}^*F^{\mu\nu} = 0 \tag{2.9}$$

describing the fields generated by a moving electric charged point-particle are called Lienard-Wiechert fields [11]. Their 4-vector potential is

$$A^\mu_{LW} = \frac{e}{4\pi}\frac{v^\mu}{v^\nu L_\nu}\bigg|_{s=s_0} \tag{2.10}$$

where $L^\nu = x^\nu - y^\nu(s)$, $v^\nu(s) = \dfrac{dy^\nu(s)}{ds}$, $\omega^\nu(s) = \dfrac{d^2y^\nu(s)}{ds^2}$ and s_0 is the solution of $L_\alpha L^\alpha = 0$ with condition $x^0 > y^0(s_0)$. The corresponding Lienard-Wiechert field strength has the following form

$$F^{\mu\nu}_{LW} = \frac{e}{4\pi}\frac{1}{(L_\alpha v^\alpha)^3}\left[L^\mu v^\nu + L^\mu L_\beta\left(v^\beta\omega^\nu - \omega^\beta v^\nu\right) - (\mu \leftrightarrow \nu)\right]\bigg|_{s=s_0}\,. \tag{2.11}$$

We can easily see that the Lienard-Wiechert fields satisfy the condition $P = 0$, therefore we can easily adopt them to be a solution of the ModMax equations (2.6) by making the following rescaling

$$F^{\mu\nu}_{MM,LW} = e^{-\gamma}\, F^{\mu\nu}_{LW}\,. \tag{2.12}$$

So the difference of ModMax and Maxwell theory is in the rescaling of the Lienard-Wiechert fields. Let us elaborate on this difference in more detail by considering a charged point-particle fixed at the origin. In Maxwell theory the fields produced by this charge are the Coulomb ones, while in ModMax theory we have almost the same expression, but with the factor which rescales the particle charge. Explicitly we get (in natural units)

$$\vec{E} = e^{-\gamma}\frac{e}{4\pi}\frac{\vec{r}}{r^3}\,, \qquad \vec{B} = \vec{0} \tag{2.13}$$

where $\vec{r} = (x, y, z)$ is the position vector in space. Therefore, from (2.4) it follows that the Coulomb force acted on a test point-particle of charge q at the position $\vec{r}_0$ is given by

$$\vec{F} = e^{-\gamma}\,\frac{eq}{4\pi}\frac{\vec{r}_0}{r_0^3} \tag{2.14}$$

which differs from Maxwell's theory by the factor $e^{-\gamma}$. However the usual Coulomb force is tested in experiments with a very high precision, therefore for ModMax to be a physically consistent theory should have a very small coupling constant γ.
We will see in the following that Coulomb law in the ModMax theory can be made exactly the same as in Maxwell one, or in other words the scale factor e^{γ} can be removed from the field equations (2.6) by rescaling the ModMax Lagrangian.

2.3 Different scaling of the ModMax Lagrangian

Let us rescale the ModMax Lagrangian density as follows

$$\mathcal{L} = e^{-\gamma}\,\left(\cosh\gamma\, S + \sinh\gamma\,\sqrt{S^2 + P^2}\right)\,. \tag{2.15}$$

Then

$$G^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = e^{-\gamma}\left(\cosh\gamma + \sinh\gamma\frac{S}{\sqrt{P^2+S^2}}\right)F^{\mu\nu} + e^{-\gamma}\left(\sinh\gamma\frac{P}{\sqrt{P^2+S^2}}\right)\,{}^*F^{\mu\nu} \tag{2.16}$$

and for $P = 0$ the field equations now become the same as in (2.9). These have (for $S > 0$) the standard Lienard-Wiechert fields as solutions, while the Lorentz force remains the same as in (2.4). Therefore now the Coulomb law takes exactly the same form as in Maxwell's theory and does not allow to distinguish these two theories.

Let us note that the rescaling of ModMax Lagrangian has an important consequence: duality-invariance condition (1.13) gets modified as follows

$$G_{\mu\nu}{}^*G^{\mu\nu} = e^{-2\gamma}F_{\mu\nu}{}^*F^{\mu\nu} \tag{2.17}$$

and so the duality transformation involves the rotation of the fields ($G^{\mu\nu}$, $e^{-\gamma}\,{}^*F^{\mu\nu}$) in contrast to that in (1.10).

2.4 Coupling magnetic charges

In standard Maxwell theory it is possible (at least theoretically) to assume the existence of a magnetic current j^{μ}_g and generalize accordingly Maxwell's equations. However such a generalization brings the issue of the existence of the electromagnetic 4-vector potential (at least defined in the whole space). Dirac managed to formulate a consistent electromagnetic theory with magnetic charges (called monopoles) and a potential defined everywhere except for a line going from the monopole to infinity (the so-called "Dirac's string") [8]. We would like to carry out a similar construction for ModMax.

We suppose that in addition to an electric charged point-particle we have a point-like monopole of charge g and mass m_g moving along a trajectory $z(\tau)$ parameterized by τ. We have the magnetic current

$$j_g^\mu = g \int d\tau\, \delta^{(4)}\left(x - z(\tau)\right) \frac{dz^\mu}{d\tau}\,. \tag{2.18}$$

In analogy with Maxwell theory, we modify Bianchi's identities in (2.3) by introducing the magnetic current as follows

$$\partial_\mu {}^*F^{\mu\nu} = j_g^\nu\,. \tag{2.19}$$

As happens in Maxwell theory, we can assume that $F^{\mu\nu}$ can be written in terms of a vector potential defined everywhere except at least in one point on every closed surface enveloping the magnetic monopole; in the best case we can assume that this fails only on a continuous line going from the monopole to infinity (the Dirac's string). We now parameterize the string worldsheet with a time parameter τ and a space parameter $\sigma \in [0;\infty)$ and since it starts from the monopole we can write its embedding in spacetime as follows

$$w^\mu(\tau,\sigma) = z^\mu(\tau) + u^\mu(\tau,\sigma) \qquad \text{with the condition} \qquad u^\mu(\tau,0) = 0\,. \tag{2.20}$$

We now generalize the electromagnetic field strength as follows

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - {}^*C_{\mu\nu} \quad \Longrightarrow \quad \partial_\mu {}^*F^{\mu\nu} = j_g^\nu = \partial_\mu C^{\mu\nu} \tag{2.21}$$

where $C^{\mu\nu}$ is a tensor that has its support on the string worldsheet. It has the following form

$$C_{\mu\nu}(x) = -g \iint d\tau d\sigma \left(\frac{\partial w_\mu}{\partial \tau}\frac{\partial w_\nu}{\partial \sigma} - \frac{\partial w_\nu}{\partial \tau}\frac{\partial w_\mu}{\partial \sigma}\right) \delta^{(4)}\left(x - w(\tau,\sigma)\right)\,. \tag{2.22}$$

To check that this field satisfies the equations (2.21) we need to use Stokes' Theorem, which states that given two functions U, V defined on a surface S of variables x, y with boundary ∂S, the following equality holds

$$\iint_S dxdy \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right) = \int_{\partial S} U \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right)\,. \tag{2.23}$$

Here we have a surface with parameters (τ,σ) and the boundary (not at infinity) is given only by the trajectory of the monopole at $\sigma = 0$, therefore

$$\begin{gathered}\partial_\mu C^{\mu\nu} = +g \iint d\tau d\sigma \left(\frac{\partial w^\mu}{\partial \tau}\frac{\partial w^\nu}{\partial \sigma} - \frac{\partial w^\nu}{\partial \tau}\frac{\partial w^\mu}{\partial \sigma}\right) \frac{\partial \delta^{(4)}\left(x - w\right)}{\partial w^\mu} = \\ g \iint d\tau d\sigma \left(\frac{\partial \delta^{(4)}\left(x - w\right)}{\partial \tau}\frac{\partial w^\nu}{\partial \sigma} - \frac{\partial w^\nu}{\partial \tau}\frac{\partial \delta^{(4)}\left(x - w\right)}{\partial \sigma}\right) = \\ g \int d\tau\, \delta^{(4)}\left(x - w(\tau,0)\right) \frac{\partial w^\nu(\tau,0)}{\partial \tau} = g \int d\tau\, \delta^{(4)}\left(x - z(\tau)\right) \frac{dz^\mu}{d\tau} = j_g^\nu\,.\end{gathered}$$

We define the action in analogy with the one in (2.2) as

$$S\left[A,y,z,u\right] = \int d^4x\, \mathcal{L} - \int d^4x\, j_e^\nu A_\nu - m_e \int d\tau \sqrt{\frac{dy^\mu(\tau)}{d\tau}\frac{dy_\mu(\tau)}{d\tau}} - m_g \int d\tau \sqrt{\frac{dz^\mu(\tau)}{d\tau}\frac{dz_\mu(\tau)}{d\tau}} \tag{2.24}$$

where $\mathcal{L}$ is as in (1.3) but with $F_{\mu\nu}$ defined in (2.21).
We can see that the action depends on the vector potential, the trajectories of the charges and the Dirac's string. The latter must however be unphysical, i.e. the physical effects should not depend on

the string position in space. A necessary condition for this to happen is that the string equations of motion are not independent but follow from the other ones.

Let us see which equations of motion we get. Varying the action with respect to A^μ we get again the electromagnetic field equations

$$\partial_\mu G^{\mu\nu} = j_e^\nu \tag{2.25}$$

accompanied by the Bianchi's identities sourced by the magnetic current (2.19).
Varying the action with respect to y^μ we get

$$\frac{dp_e^\mu}{ds} = e\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right]\Big|_y \frac{dy_\nu}{ds} \tag{2.26}$$

and this is equal to the standard Lorentz force induced by the field strength (2.21) if $C^{\mu\nu}(y) = 0$ which is true if the electric particle never passes through the string, so we impose the condition (known as Dirac's veto) $y(\tau) \neq w(\tau, \sigma)$. This implies that on the string the electric current is zero and equations (2.25) reduce to

$$j_e^\mu(w) = 0\,, \qquad \partial_\mu G^{\mu\nu}(w) = 0\,. \tag{2.27}$$

Varying the action with respect to u^μ we get

$$\int d^4x\, \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \delta(-{}^*C_{\alpha\beta}) = \int d^4x\, {}^*G^{\alpha\beta}\delta C_{\alpha\beta} = -g\int d^4x\, {}^*G^{\alpha\beta}\delta \iint d\tau d\sigma \frac{\partial w_\alpha}{\partial \tau}\frac{\partial w_\beta}{\partial \sigma}\delta^{(4)}\left(x - w\right) =$$

$$-g\iint d\tau d\sigma\, {}^*G^{\alpha\beta}(w)\left(\frac{\partial \delta w_\alpha}{\partial \tau}\frac{\partial w_\beta}{\partial \sigma} - \frac{\partial \delta w_\alpha}{\partial \sigma}\frac{\partial w_\beta}{\partial \tau}\right) + \frac{\partial^* G^{\alpha\beta}(w)}{\partial w^\mu}\frac{\partial w_\alpha}{\partial \tau}\frac{\partial w_\beta}{\partial \sigma}\delta w^\mu =$$

$$-g\iint d\tau d\sigma\left(\frac{\partial({}^*G^{\alpha\beta}\delta w_\alpha)}{\partial \tau}\frac{\partial w_\beta}{\partial \sigma} - \frac{\partial({}^*G^{\alpha\beta}\delta w_\alpha)}{\partial \sigma}\frac{\partial w_\beta}{\partial \tau}\right) +$$

$$-g\iint d\tau d\sigma \frac{\partial^* G_{\alpha\beta}(w)}{\partial w^\mu}\left(\frac{\partial w^\alpha}{\partial \tau}\frac{\partial w^\beta}{\partial \sigma}\delta w^\mu - \frac{\partial w^\mu}{\partial \tau}\frac{\partial w^\beta}{\partial \sigma}\delta w^\alpha + \frac{\partial w^\beta}{\partial \tau}\frac{\partial w^\mu}{\partial \sigma}\delta w^\alpha\right)\,.$$

We are left with two integrals. The first one turns out to be null by Stokes' Theorem in (2.23) because on the boundary $\delta u^\mu(\tau, 0) = 0$, while the second one, upon rearranging indices and noticing that $\delta w^\mu = \delta u^\mu$, takes the form

$$g\iint d\tau d\sigma\,\left(\frac{\partial^* G_{\alpha\beta}}{\partial w^\mu} + \frac{\partial^* G_{\beta\mu}}{\partial w^\alpha} + \frac{\partial^* G_{\mu\alpha}}{\partial w^\beta}\right)\frac{\partial w^\mu}{\partial \tau}\frac{\partial w^\beta}{\partial \sigma}\delta u^\alpha\,.$$

Therefore the variation of the action with respect to u^μ produces the Dirac string equations of motion

$$\left(\frac{\partial^* G_{\alpha\beta}}{\partial w^\mu} + \frac{\partial^* G_{\beta\mu}}{\partial w^\alpha} + \frac{\partial^* G_{\mu\alpha}}{\partial w^\beta}\right)\frac{\partial w^\mu}{\partial \tau}\frac{\partial w^\beta}{\partial \sigma} = 0 \tag{2.28}$$

which are identically satisfied because of Dirac's veto (2.27). This indicates that the string does not have independent dynamics and is hence unphysical.
Varying the action with respect to z^μ we get

$$-g\int ds\, {}^*G_{\alpha\beta}(z)\delta z^\alpha \frac{dz^\beta}{ds} + m_g\int ds\, \frac{d^2 z^\mu}{ds^2}\delta z_\mu\,.$$

The first contribution is obtained by varying the purely electromagnetic part Lagrangian of the action in the same way as we did for the string, applying Stokes' Theorem in (2.23) and using condition (2.28), while the second contribution is given by the variation of the monopole kinetic part of the action. Denoting the monopole 4-momentum by $p_g^\mu(s) = m_g\dfrac{dz^\mu(s)}{ds}$, we get the Lorentz force acting on the magnetic charged particle

$$\frac{dp_g^\mu}{ds} = g\, {}^*G^{\mu\nu}\frac{dz_\nu}{ds}\,. \tag{2.29}$$

Let us conclude this Section by noticing that there exists a formulation for describing monopoles developed by Wu and Yang [8] which allows to avoid Dirac's veto (and so which makes the string unobservable). Actually this formulation turns out to be valid not only for magnetic charges in Maxwell's theory, but for all NEDs. We will discuss it in the Appendix A.1.

2.5 Different choices for coupling charges

We shall start this Section from making an observation about equations of motion of fields produced by the action (2.24). The form of equations (2.19) and (2.25) allow us to extend duality rotation to currents in order that they remain form-invariant under duality (note that this is not a real symmetry, since it mixes electric and magnetic constant charges). To do so we just need to apply the duality rotation in (1.10) to the 2-vector (j_e^μ, j_g^μ).
Let us now consider the case of the rescaled ModMax Lagrangian (2.15). In this case if we want to preserve duality symmetry with the condition (2.17), equations of motion would assume the form

$$\partial_\mu G^{\mu\nu} = j_e^\nu\,, \qquad e^{-\gamma}\,\partial_\mu {}^*F^{\mu\nu} = j_g^\nu \tag{2.30}$$

which means that we effectively rescale the value of the magnetic charge in comparison to (2.19). This brings a new definition of $C^{\mu\nu}$ in (2.22) which becomes

$$C_{\mu\nu}(x) = -e^{\gamma}\,g \iint d\tau d\sigma \left(\frac{\partial w_\mu}{\partial \tau}\frac{\partial w_\nu}{\partial \sigma} - \frac{\partial w_\nu}{\partial \tau}\frac{\partial w_\mu}{\partial \sigma}\right)\delta^{(4)}\left(x - w(\tau,\sigma)\right)\,. \tag{2.31}$$

Thus varying the action (2.24) (with $\mathcal{L}$ the one in (2.15)) with respect to its variables we get the equations of motion of fields in (2.30); Lorentz force for electric particles (2.26) and condition of Dirac's veto (2.28) remain the same while Lorentz force for magnetic particles becomes

$$\frac{dp_g^\mu}{ds} = e^{\gamma}\,g\,{}^*G^{\mu\nu}\frac{dz_\nu}{ds}\,. \tag{2.32}$$

If we want to get rid of the factor $e^{-\gamma}$ in (2.30) we could change the definition of magnetic current, so that it becomes dependent on γ as follows

$$j_g^\mu = e^{-\gamma}\,g \int d\tau\,\delta^{(4)}\left(x - z(\tau)\right)\frac{dz^\mu}{d\tau}\,. \tag{2.33}$$

With this new definition we restore the equation of motion of fields (2.25) and (2.19), $C^{\mu\nu}$ recovers its former form (2.22) as well as the Lorentz force for magnetic particles (2.29).

To summarize, we have three different choices for coupling charges to ModMax assuming the canonical minimal coupling to electric currents:

1. Non-rescaled ModMax (1.3) and standard definition of magnetic current (2.18).
2. Rescaled ModMax (2.15) and standard definition of magnetic current (2.18): this preserves duality symmetry with the modified condition (2.17) and makes the standard Lienard-Wiechert fields (2.11) to be a solution of ModMax, but the sourced Bianchi's identities acquire the explicit dependence on γ.
3. Rescaled ModMax (2.15) and new definition of magnetic current (2.33) containing the dependence on γ: this removes the explicit dependence on γ in the equations of motion (2.30), but modifies the formal duality transformation of electric and magnetic currents so that now we have to apply (1.10) to the 2-vector $(j_e^\mu,\, e^{\gamma}\,j_g^\mu)$.

From now on within this paper we shall refer to one of these different choices with their relative number. In Chapter 3 we will see that the Dirac quantization condition for the electric and magnetic charges changes from case to case.

2.6 Coupling dyons

Upon having assumed the existence of magnetic charges, one can also assume the existence of particles carrying both electric and magnetic charges; such kind of particles are called dyons. We would like to couple dyons to ModMax. We start from considering a dyon with mass m forming the electric and magnetic currents along $z^\mu(\tau)$

$$j_e^\mu = e \int d\tau \; \delta^{(4)} \left(x - z(\tau)\right) \frac{dz^\mu}{d\tau} \,, \qquad j_g^\mu = g \int d\tau \; \delta^{(4)} \left(x - z(\tau)\right) \frac{dz^\mu}{d\tau} \,. \tag{2.34}$$

The Dirac's string has the same form (and assume same notation) of the previous one in (2.20), as well as the field strength (2.21).
In analogy of what have been done previously, we write the action of this configuration as

$$S\left[A, z, u\right] = \int d^4x \; \mathcal{L} - \int d^4x \; j_e^\nu A_\nu - m \int d\tau \sqrt{\frac{dz^\mu(\tau)}{d\tau}\frac{dz_\mu(\tau)}{d\tau}} \tag{2.35}$$

where $\mathcal{L}$ is the one in (1.3).

Like above, varying the action with respect to A^μ we get equations of fields

$$\partial_\mu G^{\mu\nu} = j_e^\nu \tag{2.36}$$

together with Bianchi's identities

$$\partial_\mu {}^*F^{\mu\nu} = j_g^\nu \,. \tag{2.37}$$

Varying the action with respect to z^μ we find the Lorentz force for dyons

$$\frac{dp^\mu}{ds} = \left(e\, F^{\mu\nu} + g\, {}^*G^{\mu\nu}\right) \frac{dz_\nu}{ds} \tag{2.38}$$

while varying with respect to u^μ we find same condition (2.28), which is identically satisfied if we assume that the dyon trajectory cannot cross the string due to Dirac's veto.

If we assume other coupling choices, such as number 2, the action becomes as (2.35) with $\mathcal{L}$ in (2.15) and we get equations of fields and Lorentz force for dyons

$$\partial_\mu G^{\mu\nu} = j_e^\nu \,, \qquad e^{-\gamma}\, \partial_\mu {}^*F^{\mu\nu} = j_g^\nu \,, \qquad \frac{dp^\mu}{ds} = \left(e\, F^{\mu\nu} + e^\gamma g\, {}^*G^{\mu\nu}\right) \frac{dz_\nu}{ds} \tag{2.39}$$

while condition (2.28) stays the same.

If we make coupling choice number 3 we have the action (2.35) with $\mathcal{L}$ in (2.15) and we restore equations (2.36), (2.37) and (2.38).

2.7 Generalization of Lienard-Wiechert fields

Now we would like to generalize Lienard-Wiechert fields in presence of dyons; in other words we would like to find the fields produced by a moving point-dyon with a given trajectory that is not influenced by its own generated fields. We shall see that we can use our knowledge of standard Lienard-Wiechert fields (2.11) to solve the problem.
We want to solve equations (2.36), (2.37) simultaneously. We start from finding solution to equations

$$\partial_\mu K^{\mu\nu} = \int d\tau \; \delta^{(4)} \left(x - z(\tau)\right) \frac{dz^\mu}{d\tau} \,, \qquad \partial_\mu {}^*K^{\mu\nu} = 0 \tag{2.40}$$

which, in analogy with (2.11), is given by

$$K^{\mu\nu} = \frac{1}{4\pi} \frac{1}{(L_\alpha u^\alpha)^3} \left[L^\mu v^\nu + L^\mu L_\beta \left(v^\beta \omega^\nu - \omega^\beta v^\nu \right) - (\mu \leftrightarrow \nu) \right] \Big|_{s=s_0} \tag{2.41}$$

where $L^\nu = x^\nu - z^\nu(s)$, $v^\nu(s) = \dfrac{dz^\nu(s)}{ds}$, $\omega^\nu(s) = \dfrac{d^2 z^\nu(s)}{ds^2}$ and s_0 the solution of $L_\alpha L^\alpha = 0$ with condition $x^0 > z^0(s_0)$.

We therefore make the ansatz for the field produced by the dyon as follows

$$F^{\mu\nu} = e^{-\gamma}\, e\, K^{\mu\nu} - g\, {}^*K^{\mu\nu}\,. \tag{2.42}$$

It is clear that equations (2.37) are satisfied.
We have the identity ${}^*K^{\mu\nu}K_{\mu\nu} = 0$. We also introduce the notation $K^2 = K^{\mu\nu}K_{\mu\nu}$; explicitly it becomes

$$K^{\mu\nu}K_{\mu\nu} = \frac{2}{(4\pi)^2\,(L_\alpha u^\alpha)^6}\left[L_\beta L^\beta(1 - 2L_\delta\omega^\delta) - (L_\delta u^\delta)^2\right] \tag{2.43}$$

and imposing condition $L_\alpha L^\alpha = 0$ we always have $K^2 < 0$.
So now we evaluate the two Lorentz scalars S, P resulting to be proportional to each other

$$-4S = F^{\mu\nu}F_{\mu\nu} = (e^{-2\gamma}\, e^2 - g^2)\, K^2\,, \quad -4P = {}^*F^{\mu\nu}F_{\mu\nu} = 2e^{-\gamma}\, eg\, K^2 \quad \Longrightarrow \quad P = \frac{2e^{-\gamma}\, eg}{e^{-2\gamma}\, e^2 - g^2} S\,.$$

We then evaluate $G^{\mu\nu}$ from (1.4) remembering the proportionality between S, P and that $-K^2 > 0$

$$G^{\mu\nu} = \left(\cosh\gamma + \sinh\gamma\, \frac{e^{-2\gamma}\, e^2 - g^2}{e^{-2\gamma}\, e^2 + g^2}\, \frac{-K^2}{|-K^2|}\right) F^{\mu\nu} + \sinh\gamma\, \frac{2e^{-\gamma}\, eg}{e^{-2\gamma}\, e^2 + g^2}\, \frac{-K^2}{|-K^2|}\, {}^*F^{\mu\nu} \quad \Longrightarrow$$

$$G^{\mu\nu} = e\, K^{\mu\nu} - e^{-\gamma} g\, {}^*K^{\mu\nu} \tag{2.44}$$

and this solves equations (2.36). Therefore our ansatz (2.42) is correct and it is the generalized Lienard-Wiechert field for ModMax (with coupling choice number 1).

We can proceed in analogue way to find Lienard-Wiechert fields for different coupling choices. For coupling choice number 2 we get

$$F^{\mu\nu} = e\, K^{\mu\nu} - e^{\gamma}\, g\, {}^*K^{\mu\nu}\,, \qquad G^{\mu\nu} = e\, K^{\mu\nu} - e^{-\gamma}\, g\, {}^*K^{\mu\nu} \tag{2.45}$$

and satisfy equations (2.39). For coupling choice number 3

$$F^{\mu\nu} = e\, K^{\mu\nu} - g\, {}^*K^{\mu\nu}\,, \qquad G^{\mu\nu} = e\, K^{\mu\nu} - e^{-2\gamma}\, g\, {}^*K^{\mu\nu} \tag{2.46}$$

and satisfy equations (2.36), (2.37).

Now given the fields, one can see how a dyon interacts with them in the assumption that its charges are small enough to not change perceptibly the fields. To do so we need to consider the Lorentz force caused by the fields produced by the dyon described above acting on a test-dyon of electric and magnetic charges q and p in a point of the space x^μ and having a 4-velocity $\overline{v}^\mu$. We denote with the index $J = 1, 2, 3$ the different coupling choices as before. We get (doing some trivial calculation)

$$\frac{dp_1^\mu}{ds} = \left[e^{-\gamma}(qe + pg)\, K^{\mu\nu} + (pe - qg)\, {}^*K^{\mu\nu}\right]\overline{v}_\nu\,, \tag{2.47}$$

$$\frac{dp_2^\mu}{ds} = \left[(qe + pg)\, K^{\mu\nu} + e^{\gamma}(pe - qg)\, {}^*K^{\mu\nu}\right]\overline{v}_\nu\,, \tag{2.48}$$

$$\frac{dp_3^\mu}{ds} = \left[(qe + e^{-2\gamma} pg)\, K^{\mu\nu} + (pe - qg)\, {}^*K^{\mu\nu}\right]\overline{v}_\nu\,. \tag{2.49}$$

2.8 Looking for solutions of ModMax equations of motion with dyons

We have found the solution for the fields produced by a moving dyon in ModMax electrodynamics. In this Section we would like to find solutions for other non-trivial configurations of dyons. Explicitly and in a non-covariant form equations of motion (2.36), (2.37) read (with $j^\mu = (\rho, \vec{j})$)

$$\vec{\nabla} \cdot \vec{B} = \rho_g \,, \qquad \partial_t \vec{B} + \vec{\nabla} \times \vec{E} = -\vec{j}_g \,, \qquad \vec{\nabla} \cdot \vec{D} = \rho_e \,, \qquad \partial_t \vec{D} - \vec{\nabla} \times \vec{H} = -\vec{j}_e \,. \tag{2.50}$$

Since equations of motion are non-linear, we cannot simply sum up solutions to get a new one, so we cannot predict easily for instance the fields produced by two moving dyons. The idea is to try to find configurations with a nice relation between the two Lorentz scalars S, P (1.2) in order to simplify our equations as done in Section 2.2: thus we shall proceed making ansatzes and evaluating their Lorentz scalars.

We work in the 4-dimensional spacetime with coordinates $(t, x, y, z) = (t, \vec{r})$ and we start from the static case. Unfortunately for two fixed dyons of arbitrary charges we could not find a solution easily; however if we require that their charges are equal, the problem becomes very simple; actually we can find a bigger class of solutions imposing a condition on their charge densities. We ask that the charge density of our configuration satisfies

$$\rho_e(\vec{r}) = e f(\vec{r}) \,, \qquad \rho_g(\vec{r}) = g f(\vec{r}) \tag{2.51}$$

where e, g are two electric and magnetic charges and $f(\vec{r})$ is a function with dimension of length at power -3. Our ansatz for fields produced by this configuration is

$$\vec{E}(\vec{r}) = e^{-\gamma} \frac{e}{4\pi} \int d^3r' \, f(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \,, \qquad \vec{B}(\vec{r}) = \frac{g}{4\pi} \int d^3r' \, f(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \,. \tag{2.52}$$

In this case we find that

$$P/S = \frac{2e^{-\gamma} \, eg}{e^{-2\gamma} e^2 - g^2} \tag{2.53}$$

which is constant, so the two Lorentz scalars are proportional to each other and we see easily that the electric displacement and the magnetic induction fields are (remembering the definition of $G^{\mu\nu}$ (1.4))

$$\vec{D}(\vec{r}) = \frac{e}{4\pi} \int d^3r' \, f(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \,, \qquad \vec{H}(\vec{r}) = e^{-\gamma} \frac{g}{4\pi} \int d^3r' \, f(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \,. \tag{2.54}$$

Knowing that

$$\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = -\vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \,, \qquad \vec{\nabla}^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \, \delta^{(3)}(|\vec{r} - \vec{r}'|)$$

one can easily check that these fields are a solution (and so the solution) for our charge densities. This is a quite big class of configurations; for instance two fixed dyons of same charges, every uniformly fixed charged object (a plane, a sphere etc), but also many more non uniform configurations.
Finding classes of solutions in the dynamic case is far more complicated and nothing interesting has been found.

To conclude this Section we want to specify that we solved equations of motion related to coupling choice number 1. If one is interested to see which form the solutions assume in the other choices, he has to do the following steps. To deal with the coupling choice number 2 one has to send the fields $(\vec{E}, \vec{B}) \to (e^\gamma \vec{E}, e^\gamma \vec{B})$; for the coupling choice number 3 one has also to send the magnetic charge $g \to e^{-\gamma} g$.

3 Dirac's Charge Quantization for ModMax

While studying Maxwell theory with the presence of electric and magnetic charges, Paul Dirac ended up to a quantization condition that must hold between the two kinds of charges; this seems to explain why electric charge gets quantized, however existence of magnetic charges is required, and these have not been observed yet. In this Chapter we will consider an analogue problem for ModMax with the aim to figure out whether the electric-magnetic charge quantization condition gets modified. One way to derive Dirac's quantization is to pass to the Hamiltonian formulation as Quantum Mechanics requires [8]. To do so we need to perform the Legendre transform from the Lagrangian to the Hamiltonian.

3.1 Legendre transform

Given a general Lagrangian $L(q, \dot{q})$ dependent on generalized coordinates q and velocities $\dot{q}$ (here q and $\dot{q}$ can be interpreted as arrays of variables), we define Legendre transform as follows

$$(q, \dot{q}) \quad \to \quad (q, p) = \left(q, \frac{\partial L}{\partial \dot{q}}\right) \tag{3.1}$$

where p are the generalized momenta associated to coordinates q.
We now shall write Legendre transform for ModMax. We start from considering the action minimally coupled with only pure electric and magnetic charges (we do not consider dyons). We have written its Lagrangian formulation with the action (2.24) which is linked to the Lagrangian through the relation

$$S = \int dt\, L\,. \tag{3.2}$$

So we need to specify a parametrization for our free parameter τ, and a proper choice is to set $\tau = t$; then for our trajectories we set

$$y^0 = z^0 = t\,, \qquad w^0 = t \qquad \Longrightarrow \qquad u^0(\tau = t, \sigma) = 0\,. \tag{3.3}$$

We have as generalized coordinates

$$\Big(A^\mu(\vec{x}), \vec{y}, \vec{z}, \vec{u}(\sigma)\Big) \qquad \text{with} \qquad \vec{x} \in \mathbb{R}^3\,, \quad \sigma \in (0, \infty)\,; \tag{3.4}$$

notice that we have the continuous parameters $\vec{x}$, σ defining continuous sets of coordinates $A^\mu(\vec{x})$, $\vec{u}(\sigma)$ and we did not include $\sigma = 0$ since the string (2.20) brings condition $\vec{u}(t, 0) = 0$.
We define the associated generalized velocities as follows

$$\dot{A}^\mu(\vec{x}) = \frac{dA^\mu(\vec{x})}{dt} = \partial_0 A^\mu(\vec{x})\,, \qquad \dot{\vec{y}} = \frac{d\vec{y}}{dt} = \vec{v}_e\,, \qquad \dot{\vec{z}} = \frac{d\vec{z}}{dt} = \vec{v}_g\,, \qquad \dot{\vec{u}}(\sigma) = \frac{d\vec{u}(\sigma)}{dt}\,. \tag{3.5}$$

The Lagrangian is given by

$$L = \int d^3x\, \mathcal{L} - e\, A^0(\vec{y}) + e\, \vec{v}_e \cdot \vec{A}(\vec{y}) - m_e\sqrt{1 - \vec{v}_e^{\,2}} - m_g\sqrt{1 - \vec{v}_g^{\,2}} \tag{3.6}$$

where $\mathcal{L}$ is the ModMax in (1.3).

Legendre transform defines the following conjugated momenta.
For the fields $A^\mu(\vec{x})$ we have

$$p_A^\mu(\vec{x}) = \frac{\partial L}{\partial(\partial_0 A_\mu(\vec{x}))} = \frac{\partial \mathcal{L}}{\partial F_{0\mu}}(\vec{x}) = -G^{0\mu}(\vec{x}) \quad \Longleftrightarrow \quad \vec{p}_A(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \vec{E}} = \vec{D}(\vec{x})\,, \qquad p_A^0(\vec{x}) = 0\,. \tag{3.7}$$

For y^i we have

$$p_y^i = \frac{\partial L}{\partial \dot{y}^i} = e\ A_i(\vec{y}) + m_e \frac{v_e^i}{\sqrt{1-\vec{v}_e^2}}\,. \tag{3.8}$$

For z^i it is useful to use the following relations

$$\frac{\partial w^0}{\partial t} = 1\,, \qquad \frac{\partial w^i}{\partial t} = v_g^i + \dot{u}^i\,, \qquad \frac{\partial w^0}{\partial \sigma} = 0\,, \qquad \frac{\partial w^i}{\partial \sigma} = \frac{\partial u^i}{\partial \sigma}$$

so using antisymmetry of $G^{\mu\nu}$ and properties of $\delta^{(4)}(x-w)$ we get

$$\int d^3x\ \frac{\partial}{\partial \dot{z}^i}\mathcal{L} = \int d^3x\ \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}}\frac{\partial(-\,^*C^{\mu\nu})}{\partial v_g^i} = g\int d^3x\ ^*G_{\mu\nu}(\vec{x})\frac{\partial}{\partial v_g^i}\iint dtd\sigma\ \frac{\partial w^\mu}{\partial t}\frac{\partial w^\nu}{\partial \sigma}\delta^{(4)}\left(x - w(t,\sigma)\right) =$$

$$g\int d\sigma\ ^*G_{jm}(\vec{w})\ \delta_i^{\ j}\ \frac{\partial u^m}{\partial \sigma} = g\int d\sigma\ \epsilon_{imn}D^n\frac{\partial u^m}{\partial \sigma} = -g\int d\sigma\left[\vec{D}\times\frac{\partial \vec{u}}{\partial \sigma}\right]^i\,;$$

overall we have that the conjugated momenta to z^i are

$$p_z^i = \frac{\partial L}{\partial \dot{z}^i} = -g\int d\sigma\left[\vec{D}(\vec{w}(\sigma))\times\frac{\partial \vec{u}}{\partial \sigma}\right]^i + m_g\frac{v_g^i}{\sqrt{1-\vec{v}_g^2}}\,. \tag{3.9}$$

For the string $u^i(\sigma)$ we get (doing similar calculations as above) for $\sigma > 0$

$$\int d^3x\ \frac{\partial}{\partial \dot{u}^i(\sigma)}\mathcal{L} = g\int d\sigma'\ ^*G_{\mu\nu}(\vec{w})\frac{\partial}{\partial \dot{u}^i(\sigma)}\left(\frac{\partial w^\mu}{\partial t}\frac{\partial w^\nu}{\partial \sigma'}\right) = g\int d\sigma'\ ^*G_{jm}(\vec{w})\ \delta_i^{\ j}\ \delta(\sigma-\sigma')\ \frac{\partial u^m}{\partial \sigma'} \qquad \Longrightarrow$$

$$p_u^i(\sigma) = \frac{\partial L}{\partial \dot{u}^i(\sigma)} = -g\left[\vec{D}(\vec{w}(\sigma))\times\frac{\partial \vec{u}}{\partial \sigma}\right]^i\,, \qquad \sigma > 0\,. \tag{3.10}$$

3.2 The ModMax Hamiltonian

To derive the Hamiltonian $H(q,p)$ associated to $L(q,\dot{q})$ we need to invert Legendre transform (3.1) with respect to p in order to find $\dot{q} = \dot{q}(q,p)$, then H is given by

$$H(q,p) = \left[\dot{q}\cdot\frac{\partial L}{\partial \dot{q}} - L(q,\dot{q})\right]\Bigg|_{\dot{q}=\dot{q}(q,p)}\,. \tag{3.11}$$

Notice that Legendre transform is required to be invertible, otherwise we could not write $\dot{q}$ in terms of (q,p). However we will see that in our specific case is not possible (at least for all momenta). This complication showed up also in Maxwell theory, and Dirac developed a formalism that revealed to be very powerful to deal with this problem but not only; it is known as the constrained Hamiltonian formalism.
Within this formalism we look to singular parts of Legendre transform as primary constraints, such as constraints between coordinates and momenta that hold without using equations of motion. We do the inversion of Legendre transform when possible writing the so-called Canonical Hamiltonian, and then add the constraints in a proper way to get the so-called Total Hamiltonian. We will describe the details step by step.

In our case we have that from equations (3.7) and (3.10) derive

$$\varphi^0(\vec{x}) = p_A^0(\vec{x}) \approx 0 \qquad \text{and} \qquad \varphi^i(\sigma) = p_u^i(\sigma) + g\left[\vec{D}(\vec{w}(\sigma))\times\frac{\partial \vec{u}}{\partial \sigma}\right]^i \approx 0 \tag{3.12}$$

where $\approx$ stands for "weak equality", such as an equality that does not necessary hold anymore if we consider a variation of the quantities involved (so $X \approx 0$ means $X = 0$ but does not imply $\delta X = 0$).

To make a consistent description of the dynamics of the system we need that the constraints remain null throughout their time evolution, which means that their total time derivative must be null (with weak equality); these requirements are called consistency conditions.

We start from writing the Canonical Hamiltonian H_C; it is given with same procedure as in the non-pathological case, but omitting contributions of momenta involved in the singular parts of Legendre transform. In our specific case we do not consider $p_A^0(\vec{x})$ and $p_u^i(\sigma)$. The details of calculations are contained in Section A.2 in the Appendix and we find that the Canonical Hamiltonian is given by

$$H_C = \int d^3x \left[\mathcal{H} + e\,\delta^{(3)}(\vec{x}-\vec{y})\,A^0(\vec{x}) - A^0(\vec{x})\,\vec{\nabla}\cdot\vec{D}\right] + \sqrt{\vec{\pi}_e^2 + m_e^2} + \sqrt{\vec{\pi}_g^2 + m_g^2} \tag{3.13}$$

where we recognize the two effective momenta

$$\vec{\pi}_e = \vec{p}_y - e\,\vec{A}(\vec{y})\,, \qquad \vec{\pi}_g = \vec{p}_z + g\int d\sigma\;\vec{D}(\vec{w}(\sigma)) \times \frac{\partial \vec{u}}{\partial \sigma} \tag{3.14}$$

and the source-free Hamiltonian density of ModMax

$$\mathcal{H} = \mathcal{H}_\gamma(\vec{D},\vec{B}) = \frac{1}{2}\cosh\gamma\left(\vec{D}^2 + \vec{B}^2\right) - \frac{1}{2}\sinh\gamma\sqrt{\left(\vec{D}^2 + \vec{B}^2\right)^2 - 4\left(\vec{D}\times\vec{B}\right)^2}\,. \tag{3.15}$$

We specify that here $\vec{B}$ depends on the vector potential and the string through the relation

$$\vec{B}(t,\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) - g\int d\sigma\,\frac{\partial\vec{u}(t,\sigma)}{\partial\sigma}\,\delta^{(3)}\left(\vec{x} - \vec{z}(t) - \vec{u}(t,\sigma)\right)\,. \tag{3.16}$$

We also incidentally specify that (3.15) is the Hamiltonian we were talking about in Section 2.2 which admits source-free null field solutions and in this formulation we have as independent variables $\vec{B}$, $\vec{D}$.

To get the Total Hamiltonian H_T we have to add to the Canonical Hamiltonian all the constraints (related to the momenta omitted before) weighted with some arbitrary multipliers, usually called Lagrange's multipliers.
In our case we define four arbitrary multipliers $\lambda^0(\vec{x})$, $\vec{\lambda}(\sigma)$ (actually they are a continuous set of multipliers dependent on the continuous parameters $\vec{x}\in\mathbb{R}^3$ and $\sigma\in(0,\infty)$) which allows us to write

$$H_T = H_C + \int d^3x\;\lambda^0(\vec{x})\varphi^0(\vec{x}) + \int d\sigma\;\vec{\lambda}(\sigma)\cdot\vec{\varphi}(\sigma)\,. \tag{3.17}$$

This is the Hamiltonian that we wanted. This construction guarantees that we restore the same dynamics of the Lagrangian formalism under some conditions that will be discussed in the next Section.

3.3 Conservation of constraints

Firstly we need to verify that consistency conditions hold (four in total in our case). The best case is that constraints are conserved during time evolution of the system just by satisfying equations of motion (otherwise we need to define other so-called secondary constraints and verify further consistency conditions). Fortunately this is the case we will find.
To evaluate the total time derivative of a function in Hamiltonian formulation we need to use Poisson Brackets (PB). Given the Hamiltonian $H(q,p)$, we denote the PB between two functions $f(q,p;t)$ and $g(q,p;t)$ (can be also explicitly time dependent) as follows

$$\{f,g\} = \frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q}\,. \tag{3.18}$$

We then have that total derivative of a function $f(q, p; t)$ is given by

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,. \tag{3.19}$$

Going to our specific case, the total derivative of a general function f dependent on our generalized coordinates, momenta and time assumes the form

$$\begin{aligned}\frac{df}{dt} = &\int d^3x\, \frac{\partial f}{\partial A_\mu(\vec{x})}\frac{\partial H_T}{\partial p^\mu_A(\vec{x})} + \int d\sigma\, \frac{\partial f}{\partial u^i(\sigma)}\frac{\partial H_T}{\partial p^i_u(\sigma)} + \frac{\partial f}{\partial y^i}\frac{\partial H_T}{\partial p^i_y} + \frac{\partial f}{\partial z^i}\frac{\partial H_T}{\partial p^i_z} + \\ &- \int d^3x\, \frac{\partial f}{\partial p^\mu_A(\vec{x})}\frac{\partial H_T}{\partial A_\mu(\vec{x})} - \int d\sigma\, \frac{\partial f}{\partial p^i_u(\sigma)}\frac{\partial H_T}{\partial u^i(\sigma)} - \frac{\partial f}{\partial p^i_y}\frac{\partial H_T}{\partial y^i} - \frac{\partial f}{\partial p^i_z}\frac{\partial H_T}{\partial z^i} + \frac{\partial f}{\partial t}\,.\end{aligned} \tag{3.20}$$

Therefore first consistency condition reads

$$\frac{d\varphi^0(\vec{x})}{dt} = \frac{dp^0_A(\vec{x})}{dt} = \vec{\nabla}\cdot\vec{D}(\vec{x}) - e\,\delta^{(3)}(\vec{x}-\vec{y}) \approx 0 \tag{3.21}$$

which is weakly null thanks to equations of motion of fields.
For the other three consistency conditions the details about the calculations are contained in Section A.3 in the Appendix and in the end we get

$$\frac{d\varphi_i}{dt}(\sigma) = g\,\epsilon_{irm}\,\frac{\partial u^r}{\partial\sigma}\left[\left(\lambda^m(\sigma) + \frac{\partial H_T}{\partial \pi^m_g}\right)\partial_n D^n(\vec{w}(\sigma)) - e\,\frac{\partial H_T}{\partial \pi^m_e}\,\delta^{(3)}(\vec{y}-\vec{w}(\sigma))\right] \approx 0 \tag{3.22}$$

which are weakly null thanks to Dirac's veto (for $\sigma > 0$).

Now we need to evaluate PB between the constraints; we would like that they all commute and so are all first-class; this tells us that we restored degrees of freedom of the Lagrangian formalism and the complete dynamic description of the system. In our case the only non-trivial PB to evaluate are

$$\{\varphi^i(\sigma), \varphi^j(\sigma')\} = \int d\sigma''\left(\frac{\partial\varphi^i(\sigma)}{\partial u^m(\sigma'')}\frac{\partial\varphi^j(\sigma')}{\partial p^m_u(\sigma'')} - \frac{\partial\varphi^j(\sigma')}{\partial u^m(\sigma'')}\frac{\partial\varphi^i(\sigma)}{\partial p^m_u(\sigma'')}\right)\,. \tag{3.23}$$

Actually the result is already contained in (3.22), and explicitly we have

$$\{\varphi^i(\sigma), \varphi^j(\sigma')\} = g\,\epsilon_{ijk}\,\frac{\partial u^k}{\partial\sigma}\,\delta(\sigma-\sigma')\,\partial_n D^n(\vec{w}(\sigma))\,. \tag{3.24}$$

We can see that also these PB are null because of Dirac's veto. Therefore all constraints are first-class.

3.4 Dirac's quantization condition

At this point we are ready to make the transition to Quantum Mechanics. In particular we have that first-class constraints become conditions that the state vector ψ has to satisfy. In our case constraints (3.12) read

$$\varphi^j(\sigma)\psi = 0 \qquad \Longleftrightarrow \qquad \left[-i\frac{\partial}{\partial u^j(\sigma)} + g\,\epsilon_{jlm}D^l(\vec{w}(\sigma))\frac{\partial u^m}{\partial\sigma}\right]\psi(\vec{u}) = 0\,. \tag{3.25}$$

The solution to this equation is given by

$$\psi(\vec{u}) = \exp\left\{-ig\int_S d\vec{S}\cdot\vec{D}\right\}\psi(\vec{u}_0) \qquad \text{with} \qquad d\vec{S} = \frac{\partial\vec{u}'}{\partial\sigma}d\sigma\times d\vec{u}'(\sigma) \tag{3.26}$$

where $\vec{u}_0$ corresponds to some fixed position of the string and S is a surface going from $\vec{w}_0 = \vec{z} + \vec{u}_0$ to $\vec{w}$. It is easy to check that this is a solution since (the other calculations are trivial)

$$\frac{\partial}{\partial u^i(\sigma)} \int \epsilon_{mjk} \frac{\partial u'^j}{\partial \sigma'} d\sigma' \, du'^k \, D^m(\vec{x}) = \epsilon_{mjk} \delta^{ik} \int d\sigma' \, \delta(\sigma - \sigma') \frac{\partial u^j}{\partial \sigma'} D^m(\vec{w}(\sigma')) = \epsilon_{imj} \frac{\partial u^j}{\partial \sigma} D^m(\vec{w}(\sigma)) \, .$$

Now, since the argument of the exponent in (3.26) is purely imaginary, if we make a whole rotation of 2π of the string, the following relation must hold

$$g \int_S d\vec{S} \cdot \vec{D} = 2\pi N \tag{3.27}$$

with $N \in \mathbb{Z}$ and S being a closed surface. Therefore, indicating with V the volume enclosed by S, using Stokes' theorem and using equations of motion we finally end up to the quantization condition, such as

$$g \int_S d\vec{S} \cdot \vec{D} = g \int_V d^3x \, \vec{\nabla} \cdot \vec{D} = g \int_V d^3x \, j_e^0(x) = ge = 2\pi N \qquad \Longrightarrow \qquad \frac{eg}{2\pi} \in \mathbb{Z} \, . \tag{3.28}$$

We can see that this result is independent on γ and is the same that Dirac found for Maxwell theory.

If we now change our Lagrangian so that we make coupling choice number 2, we find analogue Legendre transform, however since $C^{\mu\nu}$ gained a factor e^{γ} (2.31) we get instead of (3.9) and (3.10)

$$p_z^i = \frac{\partial L}{\partial \dot{z}^i} = -e^{\gamma} g \int d\sigma \left[\vec{D}(\vec{w}(\sigma)) \times \frac{\partial \vec{u}}{\partial \sigma} \right]^i - m_g \frac{v_g^i}{\sqrt{1 - \vec{v}_g^2}} \, , \tag{3.29}$$

$$p_u^i(\sigma) = \frac{\partial L}{\partial \dot{u}^i(\sigma)} = -e^{\gamma} g \left[\vec{D}(\vec{w}(\sigma)) \times \frac{\partial \vec{u}}{\partial \sigma} \right]^i \, . \tag{3.30}$$

The Hamiltonian stays formally the same as in (3.17), but $\mathcal{H}$ changes: it is not only rescaled by the overall factor $e^{-\gamma}$, but also the field $\vec{D}$ is rescaled as $\vec{D} \to e^{\gamma} \vec{D}$; thus we have

$$\mathcal{H} = \frac{1}{2} e^{-\gamma} \cosh \gamma \left(e^{2\gamma} \, \vec{D}^2 + \vec{B}^2 \right) - \frac{1}{2} e^{-\gamma} \sinh \gamma \sqrt{ \left(e^{2\gamma} \, \vec{D}^2 + \vec{B}^2 \right)^2 - 4 \, e^{2\gamma} \left(\vec{D} \times \vec{B} \right)^2 } \, . \tag{3.31}$$

Also the following quantities become

$$\vec{\pi}_g = \vec{p}_z + e^{\gamma} g \int d\sigma \, \vec{D}(\vec{w}(\sigma)) \times \frac{\partial \vec{u}}{\partial \sigma} \, , \qquad \varphi^i(\sigma) = p_u^i(\sigma) + e^{\gamma} g \left[\vec{D}(\vec{w}(\sigma)) \times \frac{\partial \vec{u}}{\partial \sigma} \right]^i \, . \tag{3.32}$$

Consistency conditions remain valid (they just gain a factor e^{γ} in front of g as shown in (3.32)) and our state vector ψ and quantization condition become

$$\psi(\vec{u}) = \exp \left\{ -i e^{\gamma} g \int_S d\vec{S} \cdot \vec{D} \right\} \psi(\vec{u}_0) \qquad \Longrightarrow \qquad e^{\gamma} \frac{eg}{2\pi} \in \mathbb{Z} \, . \tag{3.33}$$

This time the quantization condition is manifestly dependent on γ.

If we make coupling choice number 3, we restore the former Legendre transform, the vector state ψ and the quantization condition (3.28); the Hamiltonian stays the same as in (3.17) but with $\mathcal{H}$ as in (3.31).

We finish this Section just mentioning that the same procedure will not work if we consider a single dyon coupled to ModMax since the integral done in (3.28) turns out to be not well defined.

4 Compton Effect in ModMax

Compton effect is a well known scattering process which involves a photon interacting with an electron which results in the increase of the wavelength of the scattered photon. This effect was described for the first time by Compton in 1923 assuming the radiation interacting with the electron to show particle-like behavior, which means considering light made of particles called photons.
In [13] Compton effect has been studied in a magnetic background for general NED theories whose Lagrangians satisfy some constraints; in particular the mixed derivative with respect to S, P of the Lagrangian density $\mathcal{L}(S, P)$ was assumed to be null, which is not the case for ModMax. In this Chapter our aim is to study Compton effect in a magnetic background for ModMax, and we will do it for two different orientations of $\vec{B}$ and then try to interpret the results.

4.1 Plane waves in uniform magnetic background field

To start we need to find ModMax plane wave solutions in a magnetic background. As we have already mentioned, within source-free ModMax Lagrangian formulation it is not possible to find plane wave solutions to equations of motion in the vacuum. However it is possible to find weak electromagnetic waves in presence of a constant strong background field as shown in [5], leading to the birefringence effect, which consists in a double refraction of ray of light that is split by polarization (with respect to the optical axis of the material) into two rays taking different geodesic paths and the electromagnetic background plays the role of an optical material. In particular, for a generic non-linear Lagrangian density in the form $\mathcal{L}(S, P)$ with uniform constant background fields $\vec{E}$, $\vec{B}$, weak plane wave solutions are described by the wave 4-vector $k^\mu = (\omega, \vec{k})$ satisfying [6]

$$k^2 = n_\pm (k^\rho F_{\rho\mu})(k_\nu F^{\nu\mu}) \tag{4.1}$$

where $F^{\mu\nu}$ is the background field and $n_\pm$ are the so-called "birefringence indices" since they determine two forms of the dispersion law. For ModMax the birefringence indices are [5]

$$n_- = 0\,, \qquad n_+ = \frac{\tanh\gamma}{\sqrt{S^2+P^2} - \tanh\gamma\, S}\,. \tag{4.2}$$

In 3-vector notation we find that (4.1) reads

$$\omega^2 - \vec{k}^2 = n_\pm \left[(\vec{k}\cdot\vec{E})^2 - (\omega\vec{E} + \vec{k}\times\vec{B})^2\right]\,. \tag{4.3}$$

For $n_- = 0$ we get usual plane waves with $\omega = |\vec{k}|$ travelling with the speed of light. For n_+, if we only take a uniform magnetic field $\vec{B}$ as the background, we see that the equation reduces to

$$\omega^2 - \vec{k}^2 = \frac{2\tanh\gamma}{\vec{B}^2(1+\tanh\gamma)}\left[(\vec{k}\cdot\vec{B})^2 - \vec{k}^2\vec{B}^2\right] = \vec{k}^2\frac{2\tanh\gamma}{1+\tanh\gamma}\left(\cos^2\varphi - 1\right) \qquad \Longrightarrow$$

$$\omega^2 = \vec{k}^2\left(\cos^2\varphi + e^{-2\gamma}\sin^2\varphi\right) \qquad \text{with} \qquad \cos\varphi = \frac{\vec{k}\cdot\vec{B}}{|\vec{k}||\vec{B}|} \tag{4.4}$$

so we get a plane wave solution moving slower than light since $\gamma > 0$ (actually this result motivates the choice of non-negative γ).
Indicating with $(\mathcal{E}, \vec{p})$ the 4-momentum of the photons of the plane wave (which is equal to the wave 4-vector $(\omega, \vec{k})$ since in natural units $\hbar = 1$), the energy-momentum relation for the subluminal propagating wave follows trivially from (4.4) to be

$$\mathcal{E}^2 = \vec{p}^2\left(\cos^2\varphi + e^{-2\gamma}\sin^2\varphi\right) \qquad \text{with} \qquad \cos\varphi = \frac{\vec{p}\cdot\vec{B}}{|\vec{p}||\vec{B}|}\,. \tag{4.5}$$

We can see that for different angles between $\vec{p}$ and $\vec{B}$ we get different relations.

We conclude this Section by mentioning that the dispersion relation (4.4) will remain the same also if we consider a rescaled ModMax Lagrangian. This is because the birefringence indices are ratios of combinations of derivatives of the Lagrangian with respect to S, P and then are independent on the scale factor of the Lagrangian.

4.2 Compton effect with orthogonal magnetic field

As in [13] we first consider a magnetic field orthogonal to the direction of the incoming photon momentum and assume that the electron is at rest before scattering. Hence, the initial 4-momentum of the electron is $(m_e, \vec{0})$, where $m_e = \lambda_e^{-1}$ are the electron mass and the Compton wavelength respectively; upon scattering with the photon the electron acquires the 4-momentum $(\mathcal{E}_e, \vec{p}_e)$. The 4-momentum of the incoming photon is $(\mathcal{E}, \vec{p})$ and of the outgoing one is $(\mathcal{E}', \vec{p}')$. Because of our choice of the orientation of $\vec{B}$ we have $\vec{p} \cdot \vec{B} = 0$ and so from (4.5) we have the following energy-momentum relations

$$\mathcal{E}^2 = e^{-2\gamma}\,\vec{p}^2\,, \qquad \mathcal{E}'^2 = \vec{p}'^2 f^2(\gamma,\varphi) \quad \text{where} \quad f^2(\gamma,\varphi) = \cos^2\varphi + e^{-2\gamma}\sin^2\varphi\,, \tag{4.6}$$

and φ is the angle between $\vec{p}'$ and $\vec{B}$.

Conservation of the 4-momentum implies that

$$\mathcal{E}_e^2 = (\mathcal{E} + m - \mathcal{E}')^2\,, \qquad \vec{p}_e^2 = (\vec{p} - \vec{p}')^2 = \vec{p}^2 + \vec{p}'^2 - 2\,\vec{p}\cdot\vec{p}'\,. \tag{4.7}$$

Remembering that $\mathcal{E}_e^2 - \vec{p}_e^2 = m^2$ and using (4.6) we find

$$p^2(e^{-2\gamma} - 1) + p'^2(f^2 - 1) - 2pp'(e^{-\gamma} f - \cos\theta) + 2m\,(e^{-\gamma}p - p'\,f) = 0 \tag{4.8}$$

where $p = |\vec{p}|$, $p' = |\vec{p}'|$ and θ is the angle between $\vec{p}$ and $\vec{p}'$. Written in terms of the wavelengths of the photons $\lambda = 1/p$ and $\lambda' = 1/p'$ the above equation becomes

$$\lambda'^2(e^{-2\gamma} - 1) + \lambda^2(f^2 - 1) - 2\lambda\lambda'(e^{-\gamma} f - \cos\theta) + 2\frac{\lambda\lambda'}{\lambda_e}\,(e^{-\gamma}\lambda' - \lambda\,f) = 0\,. \tag{4.9}$$

Solving for λ' we find (considering only the positive root)

$$\lambda' = \frac{1}{2 - 2\lambda_e/\lambda\,\sinh\gamma}\Big\{ f\lambda + \lambda_e(f - \cos\theta) + \tag{4.10}$$

$$+\sqrt{[e^{\gamma} f\lambda + \lambda_e(f - e^{\gamma}\cos\theta)]^2 + 2e^{\gamma}\lambda\lambda_e(1 - f^2)(1 - \lambda_e/\lambda\,\sinh\gamma)}\Big\}\,.$$

One can see that for $\gamma = 0$ we have $f = 1$ and the usual Compton scattering formula is restored as one should expect.

The above expression is very complicated; however we expect γ to be very small, in particular a consequence of PVLAS experiment (which was thought to test vacuum birefringence in QED) allows us to estimate a bound on the value of γ to be $\gamma \leq 3 \cdot 10^{-22}$ [4]. Hence it is reasonable to perform a Taylor expansion of (4.10) up to the first order in γ which results in the following simpler expression

$$\lambda' = \lambda + \lambda_e(1 - \cos\theta) + \gamma\lambda\left[1 - \frac{\sin^2\varphi}{1 + \lambda_e/\lambda\,(1 - \cos\theta)}\right]\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right] + O(\gamma^2)\,. \tag{4.11}$$

We see that in ModMax the difference of the wavelengths of the incoming and outgoing photons is a bit greater than that in Maxwell's electrodynamics. Therefore, with a sufficiently precise measure one might be able to estimate (at least an upper bound on) the value of γ from the Compton effect.

4.3 Compton effect with parallel magnetic field

Let us orient the uniform magnetic field $\vec{B}$ along the momentum $\vec{p}$ of the incoming photon. With this configuration, indicating as before with θ the angle between $\vec{p}$ and $\vec{p}'$ (which is the same between $\vec{B}$ and $\vec{p}'$ by construction), we see that the energy-momentum relation (4.5) takes the following form

$$\mathcal{E}^2 = \vec{p}^2 \,, \qquad \mathcal{E}'^2 = \vec{p}'^2 f^2(\gamma, \theta) \quad \text{where} \quad f^2(\gamma, \theta) = \cos^2\theta + e^{-2\gamma}\sin^2\theta \,. \tag{4.12}$$

Imposing the conservation of the 4-momentum and proceeding as above we find the relation

$$\lambda' - f\lambda - \frac{\lambda\lambda_e}{2\lambda'}(1 - f^2) - \lambda_e f + \lambda_e \cos\theta = 0 \,. \tag{4.13}$$

Solving for λ' we find (considering only the positive root)

$$\lambda' = \frac{1}{2} f\lambda + \frac{1}{2}\lambda_e(f - \cos\theta) + \frac{1}{2}\sqrt{\lambda_e^2(f - \cos\theta)^2 + \lambda^2 f^2 + 2\lambda\lambda_e(1 - f\cos\theta)} \,. \tag{4.14}$$

One can see that for $\gamma = 0$ we have $f = 1$ and the usual Compton scattering formula is restored, while if we perform the Taylor expansion up to the first order in γ we get

$$\lambda' = \lambda + \lambda_e(1 - \cos\theta) - \gamma\,\frac{\lambda\,\sin^2\theta}{1 + \lambda_e/\lambda\,(1 - \cos\theta)}\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right] + O(\gamma^2) \,. \tag{4.15}$$

In contrast with the previous case we found that the expected wavelength variation is a bit less than the one we would measure in Maxwell electrodynamics.

4.4 Interpretation of Compton effect in ModMax

Instead of Maxwell electrodynamics, ModMax scattering process involves both the non-linearity of the electrodynamics and the possible change of velocity of the scattered photon due to a different polarization with respect to the magnetic field. Hence we could think that the difference with respect to Maxwell electrodynamics of the variation of the photon energy due to Compton effect is the sum of two contributions: one is due to the variation of photon dispersion relation (and so its velocity) in the magnetic background while the other is intrinsic to ModMax. Our aim is to guess the explicit form of the intrinsic contribution given by ModMax in the final expression of variation of photon wavelength in Compton scattering. This will hopefully give us a physical interpretation of ModMax Compton scattering behaviour.

To show this we have to search for Compton scattering with particular configurations in which the scattered photon has the same velocity as the incident one. This is quite hard computationally so we restrict to the scattering that happens on the xy-plane. We have (with same notation as before) an incoming photon with 4-momentum $(\mathcal{E}, \vec{p})$ and an outgoing one with $(\mathcal{E}', \vec{p}')$; the background magnetic field forms an angle $-\varphi$ with $\vec{p}$ and has no z-component, while the angle between $\vec{p}$ and $\vec{p}'$ is θ. We have then the following energy-momentum relations

$$E^2 = \vec{p}^2 f^2 \qquad \text{where} \qquad f^2 = f^2(\gamma, \varphi, \theta) = \cos^2\varphi + e^{-2\gamma}\sin^2\varphi \,, \tag{4.16}$$

$$E'^2 = \vec{p}'^2 f'^2 \qquad \text{where} \qquad f'^2 = f'^2(\gamma, \varphi, \theta) = \cos^2(\varphi + \theta) + e^{-2\gamma}\sin^2(\varphi + \theta) \,. \tag{4.17}$$

Proceeding as in the previous Section we have to solve the equation

$$\lambda'^2(f^2 - 1) + \lambda^2(f'^2 - 1) - 2\lambda\lambda'(ff' - \cos\theta) + \frac{2\lambda\lambda'}{\lambda_e}(f\lambda' - f'\lambda) = 0 \,. \tag{4.18}$$

Solving for λ' we find

$$\lambda' = \frac{f'}{2f - \lambda_e/\lambda(1-f^2)}\Big\{\lambda + \lambda_e(f - \cos\theta) + \tag{4.19}$$

$$+\frac{1}{f'}\sqrt{2\lambda\lambda_e(f - f'\cos\theta) + f'^2\lambda^2 + \lambda_e^2[f'^2 + f^2 - 2ff'\cos\theta - 1 - \cos^2\theta]}\Big\}.$$

For the considered configuration we see that the incident and scattered photons have equal velocities when $\theta = 0,\ -2\varphi,\ \pi - 2\varphi,\ \pi$. When $\theta = 0$ the photon does not scatter and $\lambda' = \lambda$. For the other cases, introducing

$$\Delta\lambda(\theta) = \lambda'(\theta) - \lambda - \lambda_e(1 - \cos\theta)\,, \tag{4.20}$$

we get the following expressions up to the first order in γ

$$\Delta\lambda(-2\varphi) = \frac{\gamma\,\lambda_e \sin^2\varphi\,(1-\cos 2\varphi)}{1 + \lambda_e/\lambda\,(1 - \cos 2\varphi)}\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos 2\varphi)\right], \tag{4.21}$$

$$\Delta\lambda(\pi - 2\varphi) = \frac{\gamma\,\lambda_e \sin^2\varphi\,(1-\cos(\pi - 2\varphi))}{1 + \lambda_e/\lambda\,(1 - \cos(\pi - 2\varphi))}\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos(\pi - 2\varphi))\right], \tag{4.22}$$

$$\Delta\lambda(\pi) = \frac{\gamma\,\lambda_e \sin^2\varphi\,(1-\cos\pi)}{1 + \lambda_e/\lambda\,(1 - \cos\pi)}\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\pi)\right]. \tag{4.23}$$

These expressions have a common pattern. Thus our claim is that the intrinsic contribution of ModMax in λ' (proportional to γ) is given by

$$\Delta\lambda = \gamma\frac{\lambda_e \sin^2\varphi\,(1-\cos\theta)}{1 + \lambda_e/\lambda\,(1 - \cos\theta)}\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right] \tag{4.24}$$

where φ is the angle between the magnetic field and the incoming photon momentum, while θ is the angle between the incoming and outgoing photon momenta. The other contribution proportional to γ is believed to be given by the change of velocity of the photon induced by the magnetic background, and is expected to become null if velocities of incoming and outgoing photons are equal.

Actually this claim is confirmed by the form of (4.11). We can rewrite $\Delta\lambda$ in the following way

$$\Delta\lambda = \gamma\left[\frac{\lambda(1-\sin^2\varphi)}{1 + \lambda_e/\lambda\,(1-\cos\theta)} + \frac{\lambda_e(1-\cos\theta)}{1 + \lambda_e/\lambda\,(1-\cos\theta)}\right]\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right]; \tag{4.25}$$

in this form if the velocities of the incoming and outgoing photons are equal (which means $\varphi = \pm\pi/2$) we are left only with the intrinsic term of ModMax. In (4.15) this contribution does not appear since $\varphi = 0$.

Hence we could interpret the difference between Maxwell and ModMax Compton scattering in the following way. We have an intrinsic contribution of ModMax which always gives a greater wavelength and lower energy of the scattered photon than the ones expected in Maxwell's theory.
Furthermore, we have a contribution related to change of the velocity of the outgoing photons in comparison of the incoming ones (due to the birefringence) which gives a greater or lower wavelength of the outgoing photon than expected from Maxwell case if the velocity of the outgoing photon increases or decreases respectively (and there is no change if the velocities remain equal) as one can see by looking at (4.25) and (4.15) respectively.

Conclusion

In this work we have studied ModMax electrodynamics coupled to charged sources. We have seen that we produce different dynamics with different couplings to the ModMax Lagrangian, showing some similarities and differences from Maxwell electrodynamics. In particular the Lorentz force acting on electric particles can be made the same as in Maxwell's theory by choosing a suitable rescaling of the ModMax Lagrangian; however, in the presence of magnetic charges the Lorentz force in ModMax always differs from that in Maxwell's theory. We also found that Lienard-Wiechert fields produced by a moving dyon are exact solutions of ModMax equations of motion.
We then studied ModMax birefringence in a constant magnetic background and Compton scattering, which exhibited differences from Maxwell's electrodynamics independently by ModMax coupling choices.

It is still to be understood whether ModMax can be regarded as a fundamental extension of Maxwell electrodynamics or it is just an effective field theory for the description of certain models, for instance of specific materials in condensed matter theory. These are interesting challenges for future research.

A Appendix

In this Appendix we shall derive explicitly some equations that have been given without a proof and we shall motivate some brief comments made during this paper.

A.1 Avoiding Dirac's veto

On the Dirac's string the potential A^μ is not well defined (so it shows a singularity); however the electromagnetic fields are well defined. It is reasonable to think that this singularity has no physical implications. Wu and Yang made a construction in which this singularity is avoided [8], and is suitable for any general Lagrangian density $\mathcal{L}$ dependent on the field strength $F^{\mu\nu}$.

We start from considering the trajectory of the magnetic charged particle $z(\tau)$; $\forall\tau$ we consider two overlapping open sets of the 3-dimensional space $R_a(\tau)$ and $R_b(\tau)$ whose union gives the whole space. In other words, indicating the whole space with $\mathbb{R}^3$ of coordinates $\{x^1, x^2, x^3\}$, we can define those sets as all $\mathbb{R}^3$ minus two non overlapping lines going from the magnetic particle to infinity. For instance we could take R_a as the whole space minus a vertical straight line going from the monopole to infinity along the negative x^3-axis and R_b as the whole space minus a vertical straight line going from the monopole to infinity along the positive x^3-axis. In formulas $\forall\tau$

$$R_a(\tau) := \mathbb{R}^3 \setminus \{x^3 \leq z^3(\tau),\ x^1 = z^1(\tau),\ x^2 = z^2(\tau)\}\,, \tag{A.1}$$

$$R_b(\tau) := \mathbb{R}^3 \setminus \{x^3 \geq z^3(\tau),\ x^1 = z^1(\tau),\ x^2 = z^2(\tau)\}\,. \tag{A.2}$$

At every moment in the regions are well defined the vector potentials $A_\mu^{(a)}$ and $A_\mu^{(b)}$ and the field strength is defined to be

$$F_{\mu\nu} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} = \partial_\mu A_\nu^{(b)} - \partial_\nu A_\mu^{(b)} \tag{A.3}$$

thus it is independent on a, b. Therefore in the overlapping region $R_{ab} := R_a \cap R_b$ the potentials can differ only by a gauge transformation, which means

$$A_\mu^{(a)} - A_\mu^{(b)} = \alpha_\mu \qquad \Longrightarrow \qquad \partial_\mu\alpha_\nu - \partial_\nu\alpha_\mu = 0 \quad \text{in} \quad R_{ab}\,. \tag{A.4}$$

We also have another condition that we see considering a closed surface enveloping the monopole, and doing a line integral on a closed loop Γ on the surface and in R_{ab}

$$\oint_\Gamma \alpha_\mu d\xi^\mu = g \tag{A.5}$$

since thanks to Stokes' theorem we have that the integral is equal to the outward magnetic flux on the closed surface. This condition has to be taken as a consistency condition of α_μ. We also have an equivalence between the equation

$$\partial_\mu F^{\mu\nu} = j_g^\nu \tag{A.6}$$

and the definition of the potentials thanks to condition (A.5).

Now we build up the new action as follows

$$S\left[A, y, z\right] = \int d^4x\, \mathcal{L} - e\int^* d\tau\, \frac{dy^\nu}{d\tau} A_\nu(y(\tau)) - m_e \int d\tau \sqrt{\frac{dy^\mu(\tau)}{d\tau}\frac{dy_\mu(\tau)}{d\tau}} - m_g \int d\tau \sqrt{\frac{dz^\mu(\tau)}{d\tau}\frac{dz_\mu(\tau)}{d\tau}} \tag{A.7}$$

where

$$\int^* dy^\mu A_\mu(y) = \int_{-\infty}^Q dy^\mu A_\mu^{(b)}(y) + \int_Q^\infty dy^\mu A_\mu^{(a)}(y) + \beta(Q) \quad \text{with} \quad \beta(Q) = \int^Q d\xi^\mu \left(A_\mu^{(a)} - A_\mu^{(b)}\right) \tag{A.8}$$

where Q is a point on $y(\tau) \cap R_{ab}$ and the integral $\beta(Q)$ is taken along a path in R_{ab} from ∞ to Q. $\beta(Q)$ is important to make the action independent on Q as it is clear to see. Also if we move Q on a path defined as Γ in (A.5) we can see that $\beta(Q|\phi = 2\pi) - \beta(Q|\phi = 0) = g$ because of (A.5), so we must take the quantity $e\beta(Q) \mod eg$ if we want the action to be continuous with respect to the distortion of the world lines.

We can now use this idea to avoid Dirac's veto. We start from considering the part of the action containing the electric current; this integral is not well defined if the trajectory of the charged particle Γ_e intersects the string. Assume that this happens in a point Q of the space, then let be R and S two other points on $\Gamma_e \cap R_{ab}$ very closed to Q. So now

$$I = e\int_{\Gamma_e}^{*} dy^\mu A_\mu(y) = e\int_{-\infty}^{S} dy^\mu A_\mu^{(b)}(y) + e\int_{R}^{\infty} dy^\mu A_\mu^{(a)}(y) + e\int_{S}^{R} dy^\mu A_\mu^{(a)}(y) + e\beta(S) - e\beta(R)\,. \quad \text{(A.9)}$$

If we take the limit for $R, S \to Q$ the third term of I vanishes and the difference of $e\beta$ becomes the line integral of $A_\mu^{(b)} - A_\mu^{(a)}$ on a path Γ' not intersecting the string going from S to R and I stayed defined $\mod eg$. For an infinitesimal path Γ' we simply have

$$I = e\int_{\Gamma_e} dy^\mu A_\mu^{(b)}(y) = e\int_{\Gamma_e'} dy^\mu A_\mu^{(a)}(y) \quad \text{(A.10)}$$

where Γ_e' is the same as Γ_e except for a small contour around the string. Taking this as a definition, we easily find usual Lorentz force for the electric charged particle varying the total action with respect to y^μ since for this new part we have (here J stands for a or b)

$$\delta I = e\int_{\Gamma_e'} d\tau \frac{dy^\nu}{d\tau}\left[\partial_\mu A_\nu^{(J)} - \partial_\nu A_\mu^{(J)}\right]\delta y^\mu = e\int_{\Gamma_e} d\tau \frac{dy^\nu}{d\tau} F_{\mu\nu}\delta y^\mu \qquad \mod eg\,.$$

This is because doing a small contour around the string makes the string unobservable (at the cost to define the action $\mod eg$ in order to be continuous as a functional of trajectories).

A.2 Calculation of the Hamiltonian

In this Section we shall evaluate the Canonical Hamiltonian (3.13).
We see that the variables (3.14) will help us doing the inversions of (3.8) and (3.9); for instance for $\dot{\vec{y}}$, squaring both sides of equation (3.8), we can find

$$\vec{v}_e^{\,2} = \frac{\vec{\pi}_e^{\,2}}{\vec{\pi}_e^{\,2} + m_e^2} \qquad \Longrightarrow \qquad \sqrt{1 - \vec{v}_e^{\,2}} = \frac{m_e}{\sqrt{\vec{\pi}_e^{\,2} + m_e^2}}$$

so with easy calculation we get (doing the same for $\vec{v}_g$)

$$\vec{v}_e = \frac{\vec{\pi}_e}{\sqrt{\vec{\pi}_e^{\,2} + m_e^2}}\,, \qquad \vec{v}_g = \frac{\vec{\pi}_g}{\sqrt{\vec{\pi}_g^{\,2} + m_g^2}}\,. \quad \text{(A.11)}$$

The Canonical Hamiltonian using (3.11) is given by

$$H_C = \int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}\partial_0 A_\mu - \mathcal{L}\right] + e\, A^0(\vec{y}) + \vec{p}_y \cdot \vec{v}_e + m_e\sqrt{1 - \vec{v}_e^{\,2}} - e\, \vec{v}_e \cdot \vec{A}(\vec{y}) + \vec{p}_z \cdot \vec{v}_g + m_g\sqrt{1 - \vec{v}_g^{\,2}}\,.$$

We need to write it in a better way. Firstly we shall introduce the following Hamiltonian density

$$\mathcal{H} = \vec{D} \cdot \vec{E} - \mathcal{L} \quad \text{(A.12)}$$

which stands for the Hamiltonian density conjugated to the Lagrangian density $\mathcal{L}$ in a source-free system. For ModMax Lagrangian density (1.3) we explicitly get (3.15) as shown in [5].
Also we can see that since

$$B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk} \qquad \text{and} \qquad \frac{1}{2}\epsilon^{mnl}\,{}^*C^{nl} = \frac{1}{2}\epsilon^{mnl}\epsilon^{nl0k}C^{0k} = \frac{1}{2}\epsilon^{mnl}\epsilon^{nlk}C^{0k} = -C^{0m}$$

we have explicitly

$$B^i(x) = -\frac{1}{2}\epsilon^{ijk}F^{jk}(x) = \epsilon^{ijk}\partial_j A^k(x) + C^{0i}(x) = \epsilon^{ijk}\partial_j A^k(x) - g\iint d\sigma dt' \frac{\partial u^i}{\partial\sigma}(t',\sigma)\,\delta^{(4)}(x-w)$$

which gives the explicit expression for $\vec{B}$ in (3.16).

Now we shall do the following calculation that will be useful later

$$\int d^3x\, {}^*C_{i0}G^{i0} = \frac{1}{2}\int d^3x\; \epsilon_{i0jm}C^{jm}G^{i0} = g\;\epsilon_{ijm}\int d\sigma\; D^i(\vec{w}(\sigma))(v_g^j + \dot{u}^j)\frac{\partial u^m}{\partial\sigma}\,.$$

Thus using the above result we can see that

$$\int d^3x\left[\frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)}\partial_0 A_\mu - \mathcal{L}\right] + e\;A^0(\vec{y}) = \int d^3x\;\left[\vec{D}\cdot\vec{E} - \mathcal{L}\right] + e\;\delta^{(3)}(\vec{x}-\vec{y})A^0(\vec{x}) + G^{\mu0}\partial_\mu A_0 - {}^*C_{\mu0}G^{\mu0} =$$

$$\int d^3x\;\left[\mathcal{H} + e\;\delta^{(3)}(\vec{x}-\vec{y})A^0(\vec{x}) + \partial_i(A_0G^{i0}) - A_0\partial_i G^{i0} - {}^*C_{i0}G^{i0}\right] =$$

$$\int d^3x\;\left[\mathcal{H} + e\;\delta^{(3)}(\vec{x}-\vec{y})A^0(\vec{x}) - A^0\vec{\nabla}\cdot\vec{D}\right] + \vec{v}_g\cdot\left(g\int d\sigma\;\vec{D}(\vec{w}(\sigma))\times\frac{\partial\vec{u}}{\partial\sigma}\right)$$

because the total derivative vanishes (assuming compact support) and we ignore the part with $\dot{\vec{u}}$. We also can write in a more elegant way the following expressions

$$\vec{p}_y\cdot\vec{v}_e + m_e\sqrt{1-\vec{v}_e^2} - e\;\vec{v}_e\cdot\vec{A}(\vec{y}) = \frac{\vec{p}_e\cdot\vec{\pi}_e}{\sqrt{\vec{\pi}_e^2+m_e^2}} + \frac{m_e^2}{\sqrt{\vec{\pi}_e^2+m_e^2}} - \frac{e\;\vec{\pi}_e\cdot\vec{A}(\vec{y})}{\sqrt{\vec{\pi}_e^2+m_e^2}} = \sqrt{\vec{\pi}_e^2+m_e^2}\,;$$

$$\vec{p}_z\cdot\vec{v}_g + \vec{v}_g\cdot\left(g\int d\sigma\;\vec{D}(\vec{w}(\sigma))\times\frac{\partial\vec{u}}{\partial\sigma}\right) + m_g\sqrt{1-\vec{v}_g^2} = \frac{\vec{\pi}_g\cdot\vec{\pi}_g}{\sqrt{\vec{\pi}_g^2+m_g^2}} + \frac{m_g^2}{\sqrt{\vec{\pi}_g^2+m_g^2}} = \sqrt{\vec{\pi}_g^2+m_g^2}\,.$$

Grouping all together we end up with the Canonical Hamiltonian (3.13).

A.3 Calculation of the total derivative of constraints

In this Section we shall evaluate equations (3.22). The full derivative of the constraints is given by

$$\frac{d\varphi_i}{dt}(\sigma) = \frac{dp_u^i(\sigma)}{dt} + g\;\epsilon_{ijk}\left[\frac{dD^j(\vec{x})}{dt}\frac{\partial u^k}{\partial\sigma}\delta^{(3)}(\vec{x}-\vec{w}) + D^j(\vec{x})\frac{\partial u^k}{\partial\sigma}\frac{d}{dt}\delta^{(3)}(\vec{x}-\vec{w}) + D^j(\vec{w})\frac{\partial}{\partial\sigma}\frac{du^k}{dt}\right] \tag{A.13}$$

We then have to evaluate further total derivatives. We start from evaluating the following partial derivatives which will be useful later.

$$\frac{\partial\pi_g^m}{\partial u^i(\sigma)} = g\;\epsilon_{mnl}\,\partial_i D^n(w(\sigma))\,\frac{\partial u^l(\sigma)}{\partial\sigma} - g\;\epsilon_{mni}\,\partial_r D^n(w(\sigma))\,\frac{\partial u^r(\sigma)}{\partial\sigma}\,; \tag{A.14}$$

$$\frac{\partial\varphi^m(\sigma')}{\partial u^i(\sigma)} = g\;\epsilon_{mnl}\,\frac{\partial u^l}{\partial\sigma'}\,\partial_i D^n\;\delta(\sigma-\sigma') + g\;\epsilon_{mni}\;D^n\,\frac{\partial}{\partial\sigma'}\,\delta(\sigma-\sigma')\,; \tag{A.15}$$

$$\frac{\partial\mathcal{H}}{\partial B^l(\vec{x})} = D^m\frac{\partial E^m}{\partial B^l} - \frac{\partial\mathcal{L}}{\partial E^r}\frac{\partial E^r}{\partial B^l} - \frac{\partial\mathcal{L}}{\partial B^l} = -\frac{\partial\mathcal{L}}{\partial B^l} = H^l(\vec{x})\,; \tag{A.16}$$

$$\frac{\partial B^l(\vec{x})}{\partial u^i(\sigma)} = g\,\frac{\partial u^l}{\partial\sigma}\,\partial_i\delta^{(3)}(\vec{x}-\vec{w}(\sigma)) - g\,\delta_i{}^l\int d\sigma'\;\delta^{(3)}(\vec{x}-\vec{w}(\sigma'))\,\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma')\,; \tag{A.17}$$

$$\frac{\partial B^l(\vec{x}')}{\partial A^j(\vec{x})} = \epsilon_{lmj}\,\frac{\partial}{\partial x'^m}\delta^{(3)}(\vec{x}-\vec{x}')\,. \tag{A.18}$$

Using (A.15),(A.16) and (A.17) we find that the total derivative of the string momentum is

$$\frac{dp_u^i(\sigma)}{dt} = -\frac{\partial H_T}{\partial u^i(\sigma)} = -\frac{\partial H_T}{\partial \pi_g^m}\frac{\partial \pi_g^m}{\partial u^i(\sigma)} - \int d\sigma'\, \lambda^m(\sigma')\,\frac{\partial \varphi^m(\sigma')}{\partial u^i(\sigma)} - \int d^3x\, \frac{\partial \mathcal{H}}{\partial u^i(\sigma)} \qquad \Longrightarrow$$

$$\frac{dp_u^i(\sigma)}{dt} = -\frac{\partial H_T}{\partial \pi_g^m}\frac{\partial \pi_g^m}{\partial u^i(\sigma)} - g\,\epsilon_{mnl}\,\lambda^m(\sigma)\,\frac{\partial u^l}{\partial \sigma}\,\partial_i D^n(\vec{w}(\sigma)) + \tag{A.19}$$

$$+g\,\epsilon_{mni}\,\lambda^m(\sigma)\,\partial_r D^n(\vec{w}(\sigma))\,\frac{\partial u^r}{\partial \sigma} - g\,\frac{\partial u^l}{\partial \sigma}\,(\partial_l H^i(\vec{w}(\sigma) - \partial_i H^l(\vec{w}(\sigma))\,.$$

The total derivative of $\delta^{(3)}(\vec{x}-\vec{w}(\sigma))$ is given by

$$\frac{d}{dt}\delta^{(3)}(\vec{x}-\vec{w}(\sigma)) = \frac{\partial \delta^{(3)}(\vec{x}-\vec{w}(\sigma))}{\partial z^j}\frac{\partial H_T}{\partial p_z^j} + \int d\sigma' \frac{\partial \delta^{(3)}(\vec{x}-\vec{w}(\sigma))}{\partial u^j(\sigma')}\frac{\partial H_T}{\partial p_u^j(\sigma')} \qquad \Longrightarrow$$

$$\frac{d}{dt}\delta^{(3)}(\vec{x}-\vec{w}(\sigma)) = -\left(\lambda^j(\sigma) + \frac{\partial H_T}{\partial \pi_g^j}\right)\partial_j \delta^{(3)}(\vec{x}-\vec{w}(\sigma))\,. \tag{A.20}$$

Using (A.16) and (A.18) we find the total derivative of the field momentum $\vec{p}_A(\vec{x}) = \vec{D}(\vec{x})$

$$\frac{dD^j(\vec{x})}{dt} = -\frac{\partial H_T}{\partial A^j(\vec{x})} = e\,\frac{\partial H_T}{\partial \pi_e^m}\frac{\partial A^m(\vec{y})}{\partial A^j(\vec{x})} - \int d^3x'\,\frac{\partial \mathcal{H}}{\partial B^l(\vec{x}')}\frac{\partial B^i(\vec{x}')}{\partial A^j(\vec{x})} \qquad \Longrightarrow$$

$$\frac{dD^j(\vec{x})}{dt} = e\,\frac{\partial H_T}{\partial \pi_e^j}\,\delta^{(3)}(\vec{x}-\vec{y}) - \epsilon_{jml}\,\partial_m H^l(\vec{x})\,. \tag{A.21}$$

Lastly we need the total derivative of the string which is given by

$$\frac{du^k(\sigma)}{dt} = \frac{\partial H_T}{\partial p_u^k(\sigma)} = \lambda^k(\sigma)\,. \tag{A.22}$$

Now by substituting (A.19), (A.20), (A.21) and (A.22) in (A.13) and doing straightforward calculations we end up with

$$\frac{d\varphi_i}{dt}(\sigma) = eg\,\epsilon_{ijk}\delta^{(3)}(\vec{y}-\vec{w}(\sigma))\frac{\partial H_T}{\partial \pi_e^j}\frac{\partial u^k}{\partial \sigma} - g\,\frac{\partial u^r}{\partial \sigma}\left(\lambda^m(\sigma) + \frac{\partial H_T}{\partial \pi_g^m}\right)\Big[\epsilon_{mni}\partial_r D^n + \epsilon_{rnm}\partial_i D^n + \epsilon_{inr}\partial_m D^n\Big]\,.$$

Using the fact that $\epsilon_{irm}\epsilon^{irm} = 6$ and

$$\epsilon^{irm}\left(\epsilon_{mni}\partial_r D^n + \epsilon_{rnm}\partial_i D^n + \epsilon_{inr}\partial_m D^n\right) = 3\epsilon^{irm}\epsilon_{mni}\partial_r D^n = -6\delta^r{}_n\partial_r D^n = -6\partial_n D^n$$

we find the expression in (3.22).

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