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Analysis of extremes in the branching Brownian motion

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Introduction

Records and extremes are fascinating us in all areas of life and they are of a big importance. We are constantly interested in knowing how big, how small, how rainy, how hot things may possibly be. In many cases, these questions relate to very variable and highly unpredictable phenomena. A look at historical data can help: given the past observations, what can we say about what to expect in the future? Of course, a look at the data will reveal no obvious rules, hence we have to model them as a stochastic process and hence our predictions on the future will be statistical: we have to make assertions on the probability of certain events. These events, however, are rather particular. In fact, they will be rare events and related to the worst things that may happen, in other words, to extremes. This is why a big branch of statistics, extreme value statistics, has been developed and studied throughout the years.

In this thesis we will focus on the analysis of extremes in the branching Brownian motion (BBM). We are interested in a model in which particles are independent, move in space according to some Markovian process and branch. Branching Brownian motion in $\mathbb{R}$ can be thought as a particle that starts at a point $x \in \mathbb{R}$ and moves according to a Brownian motion. After a random time distributed as an exponential random variable, the particle splits into daughter particles, which start to move as independent Brownian motions, that in turn split after independent exponential lifetime, and so on.
BBM has been introduced for the first time in the second half of XX century and then studied over the last 50 years. It has been very important in the study of the *Kolmogorov* equation, or *Fisher-Kolmogorov, Petrovsky, Piskunov* (*F-KPP*) equation, the basic prototype of parabolic differential equations. It appears as a model in ecology, population genetics, epidemiology. The *F-KPP* equation has had a central place in the study of wave-like phenomena and has aroused great interest in the physicists, since it is one of the simplest example of a semilinear parabolic equation

that admits traveling waves. The equation was first considered in 1937 by Fisher in *The advance of advantageous genes* and by Kolmogorov, Petrovsky and Piskunov in *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique.* Over the years, this equation has been studied by many authors, such as Kolmogorov, Skorokhod, McKean, Bramson, both analytically and probabilistically. In particular, McKean in [14] provided a representation of solutions of the *F-KPP* equation in terms of BBM and started studying the asymptotics of position of the rightmost particle, whose centering term was found by Bramson in [6]. In the following years, many authors refined Bramson's results on the convergence in distribution of the maximum in different ways. For example, Harris and Hardy in *A new formulation of the spine approach to branching diffusions* in 2006 and Harris and Roberts in *The many-to-few lemma and multiple spines* in 2011 used spine decompositions and their relation to probability tilting to study the convergence in distribution of the maximum, while Roberts in *A simple path to asymptotics for the frontier of a branching Brownian motion* in 2011 approached differently, trying to simplify the study by counting only particles that stay below a certain line. The final result always concerned the law of the maximum of the BBM, which has been proven to be a random shift of the Gumbel distribution.

Recently, the field has experienced a revival with many remarkable contributions and repercussions into other areas, such as Gaussian free fields, like Zeitouni dealt in [15]. In particular, the study of BBM has started focusing not only on the position of the rightmost particle, but on the whole point process formed by the rightmost particles, namely the extremal process of the branching Brownian motion. A full convergence result has been obtained independently by Arguin, Bovier and Kistler in [1], [2], [3] on the one hand, and by Aidekon, Berestycki, Brunet and Shi in *Branching Brownian motion seen from its tip* on the other. Many of these results have been proven for the branching random walk as well, see for example Madaule in *Convergence in law for the branching random walk seen from its tip.*

In the last years, mathematicians have started studying modified branching random walks and branching Brownian motions: they introduced the *selection* mechanism. By selection we mean the process of killing particles, which can be interpreted as the effect of natural selection on a population, or viewed in the framework of fronts under the effect of noise. Indeed, real life systems are well approximated by introducing a noise, that causes a tremendous change in the results obtained in the classic versions of branching random walks and branching Brownian motions.

In the first chapter we will state some basic definitions and facts on extreme value theory. In particular we will introduce Poisson point processes and Laplace functionals, that will be central in the study of two basic models of disordered systems: the REM and the GREM. The REM, or *random energy model*, was introduced by Derrida in 1980 as a basic model for particle systems. Indeed, the REM is defined by an Hamiltonian in which there are 2^N independent variables, distributed as standard normal variables. If the random variables we are considering are correlated, we have to deal with the GREM, or *generalized random energy model.* As we will underline, the study of the partition function of these models and the description of the asymptotic behaviour are very similar to the study of the extremes of our BBM.

In the second chapter we will give the basic definitions of graph theory, since this will be the structure on which we will construct branching random walks and the branching Brownian motion. After that, we will define branching random walks, the discrete counterpart of BBM, and state the main results on the maximal displacement and the expectation. In particular, we will prove the convergence in distribution of the maximum and obtain from that a law of large numbers. We will focus on the proof of these results, providing upper and lower bounds for $\frac{M_n}{n}$, where M_n denotes the maximum over n variables.

In the third chapter we will rigorously define the branching Brownian motion and state and prove all the results on the maximal displacement. In particular we will prove that the probability that the maximum stays under a certain value is a solution of the *F-KPP* equation. After that we will study the convergence in distribution of $M_t - m_t$, where the centering term m_t is of the form $\sqrt{2}t - \frac{3}{2\sqrt{2}} + C + o(1)$. We will then introduce the main object of this thesis: the extremal process of branching Brownian motion. The aim of this section will be to prove that the extremal process converges weakly to a cluster Poisson point process. In order to do this, we will define a point process $\mathcal{E}_t$ that encodes the statistics of the extremal particles of the BBM. We will first prove the existence of its limiting process and then we will introduce an auxiliary process Π_t, proving that it is a Poisson point process and that it is equal in law to the process $\mathcal{E}_t$.

In the last chapter, we will characterize the paths of the extremal particles throughout three theorems and a corollary, that will enable us to visualize the extremal process through a quite precise graph. We will prove that extremal particles cannot fluctuate too wildly in the upward direction: they stay below the so-called *upper envelope*, that we will define. After that we will prove that the paths lie well below the interpolating line $s \mapsto \frac{s}{t} m_t$. In this way, also unconstrained paths does not hit the upper envelope. In fact, they stay with high probability under the *entropic* *envelope*. Finally, we will introduce the *lower envelope* and prove that extremal paths lie above it. As a result, we will find a tube, a region in which extremal particles spend most of their time with overwhelming probability. Localization of paths leads us to a final theorem on genealogy of extremal particles: in the large t-limit, ancestors of such particles split either within a distance of order 1 from time 0 or within a distance of order 1 from time t.

Chapter 1

Extreme value theory

In this chapter, we are going to introduce some basic ideas on extreme value theory. We will study extremal processes in the iid case and we will try to generalize towards more complicated models. We will introduce a toy model in order to better understand how to study the extremal processes of branching random walks and branching Brownian motion.

1.1 Extreme value distribution of iid sequences

We start working with a family of real valued, independent and identically distributed random variables $X_i, i \in \mathbb{N}$, with the following distribution function: $F(x) = \mathbb{P}[X_i \leq x]$. We are interested in the extremal distribution of this collection of random observations from the same distribution. Setting $M_n = \max_{i=1}^n X_i$, it is easy to observe that

$$\mathbb{P}[M_n \leq x] = (F(x))^n.$$

Taking the limit,

$$\lim_{n\to\infty} (F(x))^n = \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

This means that any value that the variables X_i can exceed with positive probability will be eventually exceeded after sufficiently many independent trials.

To get something more interesting, we must rescale the random variable M_n. We would like to find two sequences a_n and b_n and a distribution function $G(x)$, such that

$$\lim_{n\to\infty} \mathbb{P}[a_n(M_n - b_n)] = G(x).$$

The distributions we are looking for arise as limits of the form $F^n(a_n x + b_n) \to G(x)$, where F^n is the n-th convolution of F. We want such limits to have particular properties, for example we will require max-stability. Let us start by being as general as possible with regard to the allowed distributions F. We are going to classify all max-stable distributions modulo the equivalence below and to determine their domains of attraction. We need some basic definitions.

Definition 1.1. Two distributions F and G are said to be ***equivalent***(or of the same type) if $\exists a > 0, b \in \mathbb{R}$ such that

$$F(ax + b) = G(x). \tag{1.1}$$

Definition 1.2. A distribution function G is ***max-stable*** if, for all $n \in \mathbb{N}$, there exist $a_n > 0$ and $b_n \in \mathbb{R}$, such that, for all $x \in \mathbb{R}$,

$$G^n(a_n^{-1}x + b_n) = G(x), \tag{1.2}$$

where G^n is the n-th convolution of G.

Definition 1.3. A sequence $F_n, n \in \mathbb{N}$, of probability distribution functions ***converges weakly*** to a probability distribution function F, namely $F_n \xrightarrow{w} F$, if

$$\lim_{n\to\infty} F_n(x) = F(x) \tag{1.3}$$

for all points x where F is continuous.

We can now observe that different choices of the scaling sequences a_n and b_n can lead only to equivalent distributions $G_*(x) = G(ax + b)$, with $a > 0, b \in \mathbb{R}$. This is known as Khintchine's theorem:

Theorem 1.4. *Let F_n, $n \in \mathbb{N}$, be distribution functions and let G be a non degenerate distribution function. Let $a_n > 0$ and $b_n \in \mathbb{R}$ be sequences such that*

$$F_n(a_n x + b_n) \xrightarrow{w} G(x). \tag{1.4}$$

Then there are constants $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$, and a non-degenerate distribution function G_*, such *that*

$$F_n(\alpha_n x + \beta_n) \xrightarrow{w} G_*(x) \tag{1.5}$$

if and only if

$$a_n^{-1}\alpha_n \to a, \qquad \frac{\beta_n - b_n}{a_n} \to b \tag{1.6}$$

and

$$G_*(x) = G(ax + b), \tag{1.7}$$

where $a > 0, b \in \mathbb{R}$.

We are now going to state a proposition that shows us that the only distributions that can occur as extremal distribution are max-stable distributions and that, viceversa, any max-stable distribution is the limit of a suitable extremal distribution.

Proposition 1.5. (i) *A probability distribution* G *is max-stable if and only if there exist probability distribution functions* F_n *and constants* $a_n > 0$ *and* $b_n \in \mathbb{R}$*, such that, for all* $k \in \mathbb{N}$*,*

$$F_n(a_{nk}^{-1} + b_{nk}) \xrightarrow{w} G^{\frac{1}{k}}. \tag{1.8}$$

(ii) G *is max-stable if and only if there exist a probability distribution function* F *and constants* $a_n > 0$ *and* $b_n \in \mathbb{R}$ *such that*

$$F^n(a_n^{-1}x + b_n) \xrightarrow{w} G(x). \tag{1.9}$$

We have seen that the only distributions that can occur as extremal distributions are max-stable distributions. We will now classify these distributions:

Theorem 1.6. *Any max-stable distribution is of the same type as one of the following:*

(i) *The Gumbel distribution,*

$$G(x) = e^{-e^{-x}} \tag{1.10}$$

(ii) *The Fréchet distribution with parameter* $\alpha > 0$*,*

$$G(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-x^{-\alpha}} & \text{if } x > 0 \end{cases} \tag{1.11}$$

(iii) *The Weibull distribution with parameter* $\alpha > 0$*,*

$$G(x) = \begin{cases} e^{-(-x)^\alpha} & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \tag{1.12}$$

Corollary 1.7. *Let* X_i*,* $i \in \mathbb{N}$*, be a sequence of iid random variables. Let us assume that there exist sequences* $a_n > 0$*,* $b_n \in \mathbb{R}$*, and a non-degenerate probability distribution function* G*, such that*

$$\mathbb{P}[a_n(M_n - b_n) \leq x] \xrightarrow{w} G(x). \tag{1.13}$$

Then $G(x)$ *is of the same type as one of the three max-stable distributions.*

Now we would like to find some criteria to decide for a given distribution F to which distribution the maximum of iid variables with this distribution corresponds.

Definition 1.8. Let X_i, $i \in \mathbb{N}$ be a sequence of iid random variables, distributed according to F. If 1.13 holds with G extremal distribution, we say that F belongs to the ***domain of attraction*** of G.

Theorem 1.9. *Let* $x_F = \sup\{x : F(x) < 1\}$*. The following conditions are necessary and sufficient for a distribution function* F *to belong to the domain of attraction of the three extremal types:*

(i) *Fréchet:*

$$x_F = \infty, \tag{1.14}$$

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \tag{1.15}$$

for every $x > 0$*,* $\alpha > 0$*.*

(ii) *Weibull:*

$$x_F < \infty, \tag{1.16}$$

$$\lim_{h\to 0} \frac{1 - F(x_F - xh)}{1 - F(x_F - h)} = x^{\alpha}, \tag{1.17}$$

for every $x > 0$*,* $\alpha > 0$*.*

(iii) *Gumbel:*

$$\exists g(t) > 0, \tag{1.18}$$

$$\lim_{t\to x_F} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x}, \tag{1.19}$$

for every x.

Once we know how the maximum behaves, we are allowed to ask for more. What is the joint distribution of the maximum, the second largest, the third largest...?
We will state basic results on the variable M_n^k, that represents the value of the k-th largest of the first n variables X_i.

Definition 1.10. Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be real numbers. We denote by $M_n^1, \ldots, M_n^n$ its ***order statistic***, that is, for some permutation π of n numbers, $M_n^k = X_{\pi(k)}$ and

$$M_n^n \leq M_n^{n-1} \leq \ldots \leq M_n^2 \leq M_n^1 = M_n. \tag{1.20}$$

We also denote by $S_n(u) = \#\{i \leq n : X_i > u\}$ the number of exceedances of the level u. Observe that according to this notation we have

$$\mathbb{P}[M_n^k \leq u] = \mathbb{P}[S_n(u) < k]. \tag{1.21}$$

Theorem 1.11. *Let X_i, $i \in \mathbb{N}$, be a sequence of iid random variables with distribution F. If there exists a sequence u_n such that*

$$n(1 - F(u_n)) \longrightarrow \tau, \tag{1.22}$$

with $0 < \tau < \infty$, then

$$\mathbb{P}[M_n^k \leq u_n] = \mathbb{P}[S_n(u_n) < k] \longrightarrow e^{-\tau} \sum_{s=0}^{k-1} \frac{\tau^s}{s!}, \tag{1.23}$$

hence the k-th largest of the first n variables X_i is distributed as a Poisson random variable as $n \to \infty$.

Theorem 1.12. *Let $u_n^1 > u_n^2 > \ldots > u_n^r$ be such that, for all $\ell \in \mathbb{N}$, $n(1 - F(u_n^\ell)) \to \tau_\ell$, with $0 < \tau_1 < \tau_2 < \ldots < \tau_r < \infty$.*
Then, under the assumptions of Theorem 1.11, with $S_n^i = S_n(u_n^i)$,

$$\begin{aligned} \mathbb{P}[S_n^1 = k_1, S_n^2 - S_n^1 = k_2, \quad \ldots \quad , S_n^r - S_n^{r-1} = k_r] \to \\ \frac{\tau_1^{k_1}}{k_1!} \frac{(\tau_2 - \tau_1)^{k_2}}{k_2!} \quad \ldots \quad \frac{(\tau_r - \tau_{r-1})^{k_r}}{k_r!} e^{-\tau_r}, \end{aligned} \tag{1.24}$$

namely the joint distribution of the variables S_n is the product of Poisson distribution functions.

1.2 Extremal processes

We are now going to develope the theory of extremal values studying point processes, that are useful to describe the probabilistic structure of point sets in some metric space. We will work in the space $\mathbb{R}^d$. A convenient way to represent a collection of points x_i in $\mathbb{R}^d$ is by associating to them a point measure. We start by recalling some facts about point processes.

Definition 1.13. A ***point measure*** is a measure μ on $\mathbb{R}^d$ such that there exists a countable collection of points, $\left\{x_i \in \mathbb{R}^d, i \in \mathbb{N}\right\}$, such that,

$$\mu = \sum_{i=1}^{\infty} \delta_{x_i}, \tag{1.25}$$

where

$$\delta_{x_i}(A) = \begin{cases} 1 & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A \end{cases} \tag{1.26}$$

is the Dirac measure, $\forall A \in \mathscr{B}$, where $\mathscr{B}$ is the Borel sigma$-$algebra.

Definition 1.14. The ***support*** of μ is the set

$$S_\mu = \left\{x \in \mathbb{R}^d : \mu(x) \neq 0\right\}. \tag{1.27}$$

Definition 1.15. A ***simple*** point measure is a point measure μ, such that for all $x \in \mathbb{R}^d$, we have $\mu(x) \leq 1$.

Let $M_p(\mathbb{R}^d)$ be the set of all point measures on $\mathbb{R}^d$ and $\mathscr{M}_p(\mathbb{R}^d)$ the smallest sigma-algebra that contains all subsets of $M_p(\mathbb{R}^d)$ of the form

$$\left\{\mu \in M_p(\mathbb{R}^d) : \mu(F) \in B, \text{with } F \in \mathscr{B}(\mathbb{R}^d), B \in \mathscr{B}\left([0,\infty)\right)\right\}.$$

Definition 1.16. Let $\mathscr{F}$ be a sigma-algebra and let $N : (\Omega, \mathscr{F}, \mathbb{P}) \longrightarrow (M_p(\mathbb{R}^d), \mathscr{M}_p(\mathbb{R}^d)$ be a measurable map from a probability space to the space of point measures, i.e. a random variable taking values in $M_p(\mathbb{R}^d)$. Then N is called a ***point process***.

Definition 1.17. The ***intensity measure*** λ of a point process is the following quantity

$$\lambda(A) = \mathbb{E}[N(A)], \tag{1.28}$$

for $A \in \mathscr{B}$.

Definition 1.18. Let $\mathbb{Q}$ be a probability measure on $(M_p, \mathscr{M}_p)$. The ***Laplace transform*** of $\mathbb{Q}$ is a map ψ from non-negative Borel functions on $\mathbb{R}^d$ to $\mathbb{R}_+$, defined as

$$\psi(f) = \int_{M_p} \exp\Big(-\int_{\mathbb{R}^d} f(x)\mu(dx)\Big)\mathbb{Q}(d\mu). \tag{1.29}$$

Definition 1.19. Denote by P_N the law of $N : (\Omega, \mathscr{F}, \mathbb{P}) \longrightarrow (M_p(\mathbb{R}^d), \mathscr{M}_p(\mathbb{R}^d)$, point process, and let $\omega \in \Omega$. The ***Laplace functional*** of N is:

$$\begin{aligned} \psi_N(f) &= \mathbb{E}[e^{-N(f)}] = \int e^{-N(\omega,f)}\mathbb{P}(d\omega) = \\ &= \int_{M_p} \exp\Big(-\int_{\mathbb{R}^d} f(x)\mu(dx)\Big)P_N(d\mu) \end{aligned} \tag{1.30}$$

We will now give the definition of Poisson point processes and a characterization in terms of Laplace functionals.

Definition 1.20. Let λ be a σ-finite and positive measure on $\mathbb{R}^d$. A ***Poisson point process*** with intensity measure λ is a point process N such that

(i) For any $A \in \mathscr{B}(\mathbb{R}^d)$ and $k \in \mathbb{N}$,

$$\mathbb{P}[N(A) = k] = \begin{cases} e^{-\lambda(A)}\frac{(\lambda(A))^k}{k!} & \text{if } \lambda(A) < \infty \\ 0 & \text{if } \lambda(A) = \infty \end{cases} \tag{1.31}$$

(ii) If $A, B \in \mathscr{B}$ are disjoint sets, then $N(A)$ and $N(B)$ are independent random variables.

A Poisson point process with intensity measure λ is usually denoted by $\mathscr{P}_\lambda$.

Lemma 1.21. *A point process N on $\mathbb{R}^d$ is a Poisson point process with intensity measure λ if and only if*

$$\psi_N(\phi) = \exp\Big(-\int (e^{-\phi(x)} - 1)\lambda(dx)\Big), \tag{1.32}$$

for all continuous functions ϕ with compact support.

Remark 1.22. The *clustering* operation is performed when each point of some point process is replaced by another point process. If the original process is a Poisson point process, the resulting one is called ***Poisson cluster point process***.

Remark 1.23. Instead of working in $\mathbb{R}^d$ we can handle with a more general setting. Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}$. Elements of $\mathcal{M}$ are in correspondence with the positive linear functionals on $C_c(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support. The space $\mathcal{M}$ is endowed with the vague topology: $\mu_n \to \mu$ in $\mathcal{M}$ if for any $\phi \in C_c(\mathbb{R})$, $\int \Phi d\mu_n \to \int \phi d\mu$.
In this setting, we define a point process as a random measure that is integer-valued almost surely. A sufficient condition for their convergence is the convergence of Laplace functionals.

We now formulate the first theorem on Poisson convergence in terms of triangular arrays, since this will be convenient for the application to the REM. As the REM is an iid model, we need results on iid variables.

Theorem 1.24. *Let X_i^n be a triangular array of random variables with support in $\mathbb{R}_+$. Assume that for any $n \in \mathbb{N}$, all the X_i^n, $i \in \mathbb{N}$ are iid. Let ν be a σ-finite measure on $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$, such that $\nu([x_0, \infty)) < \infty$, for some $x_0 \in [0, \infty)$. Assume that, for a given sequence $a_n \to \infty$, we have the following convergence in probability at all continuity points of ν:*

$$a_n \mathbb{P}[X_1^n > x] \xrightarrow{p} \nu((x, \infty)). \tag{1.33}$$

Then the following hold:

$$\sum_{i=1}^{[a_n]} \delta_{X_i^n} \to \mathscr{P}_\nu, \tag{1.34}$$

and

$$\sum_{i \in \mathbb{N}} \delta_{(\frac{i}{a_n}, X_i^n)} \to \mathscr{P}_{dx \times \nu}. \tag{1.35}$$

Proof. Let ϕ be a continuous non-negative function with compact support. Then

$$\begin{aligned} \psi_n(\phi) &= \mathbb{E}\left[\exp\left\{-\sum_{i=1}^{\infty} \phi\left(\frac{i}{a_n}, X_i^n\right)\right\}\right] \qquad (1.36) \\ &= \prod_{i=1}^{\infty} \mathbb{E}\left[\exp\left\{-\phi\left(\frac{i}{a_n}, X_i^n\right)\right\}\right] \\ &= \prod_{i=1}^{\infty} \left(1 + \mathbb{E}\left[\exp\left\{-\phi\left(\frac{i}{a_n}, X_1^n\right)\right\} - 1\right]\right). \end{aligned}$$

Since ϕ has compact support, all sums and products are finite. Notice that

$$\begin{aligned} & \lim_{n\to\infty} \prod_{i=1}^{\infty} \left(\mathbb{E}\left[\exp\left\{ -\phi\left(\frac{i}{a_n}, X_1^n \right) \right\} - 1 \right] \right) \\ = \quad & \exp\left\{ \lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{\infty} a_n \mathbb{E}\left[e^{-\phi\left(\frac{i}{a_n}, X_1^n\right)} - 1 \right] \right\}. \end{aligned} \tag{1.37}$$

By hypothesis, we have

$$a_n \mathbb{P}[X_1^n > x] \xrightarrow{p} \nu((x, \infty)). \tag{1.38}$$

Using partial integration and the previous convergence in probability, we obtain, for a differentiable ϕ

$$\lim_{n\to\infty} a_n \mathbb{E}\left[e^{-\phi(y, X_1^n)} - 1 \right] = \int \left(e^{-\phi(y,x)} - 1 \right) d\nu(x) \tag{1.39}$$

and since the sum over i turns into the Lebesgue integral, we conclude that

$$\lim_{n\to\infty} \psi_n(\phi) = \exp\left\{ \int \left(e^{-\phi(y,x)} - 1 \right) d\nu(x) dy \right\}, \tag{1.40}$$

which is the Laplace functional of the Poisson point process. Notice that we can extend the proof to continuous ϕ by a standard approximation argument. □

Notice that in order to prove the previous theorem, we used Laplace functionals. This will be the same method we will apply to branching Brownian motion.
What happens if we introduce sums of independent random variables? Let us state two classical results before studying the particular case of triangular arrays.

Theorem 1.25. *Let X_i be iid random variables with support in $\mathbb{R}_+$ and assume that there exists $\alpha \in (0,1)$ such that*

$$n\mathbb{P}[X_1 > n^{\frac{1}{\alpha}} x] \to c x^{-\alpha} \tag{1.41}$$

with $c > 0$.
Then

$$S_n(t) = n^{-\frac{1}{\alpha}} \sum_{i=1}^{[t_n]} X_i \to V_{\alpha,c}(t), \tag{1.42}$$

where $V_{\alpha,c}(0, 0, \nu_{\alpha,c})$ is a stable Lèvy subordinator, with

$$\nu_{\alpha,c}(dx) = c\alpha x^{-\alpha-1} \mathbb{I}_{x>0} dx \tag{1.43}$$

and convergence is in law with respect to the Skorokhod (J_1)-topology.

Theorem 1.26. *Let X_i be iid random variables and assume that*

(i) $\mathbb{E}[X_1] = \mu$ *exists and it is finite,*

(ii) *there exists $\alpha \in (1,2)$ such that*

$$n\mathbb{P}[X_1 > n^{\frac{1}{\alpha}}x] \to c_+ x^{-\alpha} \tag{1.44}$$

and

$$n\mathbb{P}[X_1 < -n^{\frac{1}{\alpha}}x] \to c_- x^{-\alpha} \tag{1.45}$$

with $c_+, c_- > 0$,

then

$$n^{-\frac{1}{\alpha}} \sum_{i=1}^{[t_n]} (X_i - \mathbb{E}[X_i \mathbb{I}_{|X_i| \le n^{\frac{1}{\alpha}}}]) \to V_{\alpha,c_+,c_-}(t), \tag{1.46}$$

where $V_{\alpha,c_+,c_-}(0,0,\nu_{\alpha,c_+,c_-})$ is a stable Lèvy process, with

$$\nu_{\alpha,c_+,c_-}(dx) = c_+\alpha x^{-\alpha-1}\mathbb{I}_{x>0}dx + c_-\alpha(-x)^{-\alpha-1}\mathbb{I}_{x<0}dx \tag{1.47}$$

and convergence is in law with respect to the Skorokhod (J_1)-topology.

The special cases $\alpha = 2$ and $\alpha = 1$ require some extra care.

Theorem 1.27. *Assume that the hypothesis of theorem 1.26 are satisfied and $\alpha = 2$. Then*

$$\frac{1}{\sqrt{\frac{c_+ + c_-}{2} n \ln n}} \sum_{i=1}^{[nt]} (X_i - \mu) \to B_t, \tag{1.48}$$

where B_t is a standard Brownian motion.

Theorem 1.28. *Assume that the hypothesis of Theorem 1.25 are satisfied and $\alpha = 1$. Then*

$$\frac{1}{cn \ln n} \sum_{i=1}^{[nt]} X_i \to t. \tag{1.49}$$

As we noticed before, we will need to formulate the results in terms of triangular arrays.

Theorem 1.29. *Let X_i^n be a triangular array of random variables with support in $\mathbb{R}_+$. Assume that for any $n \in \mathbb{N}$, all the X_i^n, $i \in \mathbb{N}$ are iid. Assume that there exist $\alpha \in (0,1)$ and sequences $a_n \to \infty$ such that*

$$a_n \mathbb{P}[X_1^n > x] \to x^{-\alpha}. \tag{1.50}$$

Assume that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} a_n \mathbb{E}[\mathbb{I}_{X_1^n \leq \epsilon} X_1^n] = 0, \tag{1.51}$$

then

$$S_n(t) = \sum_{i=1}^{[ta_n]} X_i^n \to V_\alpha(t), \tag{1.52}$$

where $V_\alpha(0, 0, \nu_\alpha)$ is a stable Lèvy subordinator, with

$$\nu_\alpha(dx) = \alpha x^{-\alpha-1} \mathbb{I}_{x>0} dx \tag{1.53}$$

and convergence is in law with respect to the Skorokhod (J_1)-topology.

Proof. We decompose $S_n(t)$ into the central and extreme parts:

$$S_n(t) = \sum_{i=1}^{[a_n t]} X_i^n \mathbb{I}_{X_i^n \leq \epsilon} + \sum_{i=1}^{[a_n t]} X_i^n \mathbb{I}_{X_i^n > \epsilon} = S_n(t)^{\leq} + S_n(t)^{>}. \tag{1.54}$$

From Theorem 1.24, we get that

$$S_n(t)^{>} \to \int_\epsilon^\infty \int_0^t x \mathscr{P}(dx, ds), \tag{1.55}$$

where $\mathscr{P}$ is the Poisson point process with intensity $\alpha x^{-\alpha-1} dx ds$. Therefore

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} S_n(t)^{>} = V_\alpha(t). \tag{1.56}$$

The control of the other term $S_n(t)^{\leq}$ is given by the assumption 1.51 □

Theorem 1.30. *Let X_i^n be a triangular array of random variables with support in $\mathbb{R}_+$. Assume that for any $n \in \mathbb{N}$, all the X_i^n, $i \in \mathbb{N}$ are iid. Assume that there exists $\alpha \in (1,2)$ and sequences*

a_n such that

$$\begin{aligned} a_n\mathbb{P}[X_1^n] &\to c_+x^{-\alpha} && (1.57)\\ a_n\mathbb{P}[X_1^n < -x] &\to c_-x^{-\alpha}, && (1.58)\end{aligned}$$

where $c_+, c_- > 0$.

Assume that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} a_n\mathbb{E}[\mathbb{I}_{|X_1^n-\mathbb{E}[X_1^n]|\leq\epsilon}(X_1^n - \mathbb{E}[X_1^n])^2] = 0. \tag{1.59}$$

Then

$$S_n(t) = \sum_{i=1}^{[ta_n]} (X_i^n - \mathbb{E}[X_i^n\mathbb{I}_{|X_i^n|\leq 1}]) \to V_{\alpha,c_=,c_-}(t), \tag{1.60}$$

where $V_{\alpha,c_=,c_-}(0,0,\nu_\alpha)$ is a stable Lèvy process, with

$$\nu_\alpha(dx) = c_+\alpha x^{-\alpha-1}\mathbb{I}_{x>0}dx + c_-\alpha(-x)^{-\alpha-1}\mathbb{I}_{x<0}dx \tag{1.61}$$

and convergence is in law with respect to the Skorokhod (J_1)-topology.

Proof. Fix $\epsilon > 0$.

$$\begin{aligned}\sum_{i=1}^{[a_nt]}\left(X_i^n - \mathbb{E}\left[X_i^n\mathbb{I}_{X_i^n\leq\epsilon}\right]\right) &= \sum_{i=1}^{[a_nt]}\left(X_i^n\mathbb{I}_{|X_i^n|\leq\epsilon} - \mathbb{E}\left[X_i^n\mathbb{I}_{|X_i^n|\leq\epsilon}\right]\right)\\ &+ \sum_{i=1}^{[a_nt]}\left(X_i^n\mathbb{I}_{X_i^n>\epsilon} - \mathbb{E}\left[X_i^n\mathbb{I}_{1\geq X_i^n>\epsilon}\right]\right)\\ &+ \sum_{i=1}^{[a_nt]}\left(X_i^n\mathbb{I}_{X_i^n<-\epsilon} - \mathbb{E}\left[X_i^n\mathbb{I}_{-1\leq X_i^n<-\epsilon}\right]\right)\\ &= S_n(t)^{\leq} + \left(S_n(t)^+ - \mathbb{E}\left[S_n(t)^+\right]\right)\\ &+ \left(S_n(t)^- - \mathbb{E}\left[S_n(t)^-\right]\right). \end{aligned} \tag{1.62}$$

From condition 1.59, we can observe that $S_n(t)^{\leq} \to 0$ in probability, as $n\to\infty$ and $\epsilon\to 0$. The other two terms reconstruct a pure jump Lévy process:

$$\left(S_n(t)^+ - \mathbb{E}\left[S_n(t)^+\right]\right) \to \int_\epsilon^\infty\int_0^t x\,(\mathscr{P}(dx,ds) - \mathbb{I}_{x\leq 1}\nu(dx)dt), \tag{1.63}$$

where $\nu(dx) = c_+\alpha x^{-\alpha-1}dx$. The same holds for $S_n(t)^-$. Putting all together, we get the thesis.

□

Theorem 1.31. *Let X_i^n be a triangular array of random variables with support in $\mathbb{R}_+$. Assume that for any $n \in \mathbb{N}$, all the X_i^n, $i \in \mathbb{N}$ are iid. Assume that there exists a sequence a_n such that*

$$a_n \mathbb{P}[X_1^n > x] \to c_+ x^{-2}, \tag{1.64}$$

$$a_n \mathbb{P}[X_1^n < -x] \to c_- x^{-2}. \tag{1.65}$$

Then

$$\frac{1}{\sqrt{(c_+ + c_-)\frac{(\ln a_n)}{2}}} \sum_{i=1}^{[ta_n]} (X_i^n - \mathbb{E}[X_i^n]) \to B_t, \tag{1.66}$$

where B_t is a standard Brownian motion.
If $X_i^n \geq 0$ and there exists a sequence a_n, such that

$$a_n \mathbb{P}[X_1^n > x] \to x^{-1}, \tag{1.67}$$

then

$$\frac{1}{\ln a_n} \sum_{i=1}^{[ta_n]} X_i^n \to t. \tag{1.68}$$

Proof. In both cases, normalizing with the extra logarithm kills the large terms in the sum. For the rest of the proof we need to compute moments. The first and the second ones diverge, while the truncated moments produce the factors $\ln a_n$. All higher moments do not have extra logarithms, hence we end up with the moments of the constants and of the Gaussian random variables. □

1.3 Disordered systems and the REM

Extreme value theory is fundamental in the analysis of the REM, the *random energy model*, that is one the most simple model of a disordered system.
Disordered systems are studied in statistical mechanics and they have to do with probability. In fact, one of the basic axioms is that the properties of a system can be described by defining a probability measure on the space of configurations, namely $\{-1, +1\}^{\mathbb{Z}^d}$. The proper measure to choose is the so called Gibbs measure, which is defined through an Hamiltonian H, that represents the energy of the system. Let $\sigma = \{\sigma_x\}_{x \in \mathbb{Z}^d}$ be a configuration, Z_β a normalizing constant, ρ the

uniform measure on the configuration space. We define the **Hamiltonian H**

$$H(\sigma) = -\sum_{x,y\in\mathbb{Z}^d, ||x-y||_1=1} \sigma_x\sigma_y - h\sum_{x\in\mathbb{Z}^d} \sigma_x, \tag{1.69}$$

and the ***Gibbs measure***

$$\mu(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} \rho(\sigma). \tag{1.70}$$

To give sense to these two expressions we first need to start from a finite set Λ instead of $\mathbb{Z}^d$. The measure ρ on the finite space $\{-1,,+1\}^\Lambda$ in finite volumes is the product Bernoulli measure:

$$\rho_\Lambda(\sigma_\Lambda = s_\Lambda) = \prod_{x\in\Lambda} \rho_x(\sigma_x = s_x), \tag{1.71}$$

with $\rho_x(\sigma_x = +1) = \rho_x(\sigma_x = -1) = \frac{1}{2}$.
We can extend this to infinite volumes. We give $\mathscr{S} = \{-1,+1\}^{\mathbb{Z}^d}$ the structure of a measure space, with the product topology of the discrete topology on $\{-1,+1\}$. The sigma-algebra $\mathscr{F}$ on $\mathscr{S}$ is just the product sigma-algebra.
Let now $\mathscr{A}_\lambda$ be a cylinder event. Then we observe that

$$\rho(\mathscr{A}_\lambda) = \rho_\lambda(\mathscr{A}_\lambda). \tag{1.72}$$

We then have an a-priori probability space, $(\mathscr{S}, \mathscr{F}, \rho)$.
What about the Hamiltonian? It is natural to define it as the energy of an infinite-volume configuration within a finite volume Λ:

$$H_\Lambda(\sigma) = -\sum_{x\vee y\in\Lambda, ||x-y||_1=1} \sigma_x\sigma_y - h\sum_{x\in\Lambda} \sigma_x. \tag{1.73}$$

Notice that this expression, in contrast to 1.69, contains the energy corresponding to the interactions between spins in Λ with those outside Λ.
There are classical results that ensure that there exists probability measures μ_β on $(\mathscr{S}, \mathscr{F})$ and they are Gibbs measures for the Hamiltonian H and inverse temperature β.
To study disordered systems we would like to have simple models with an easy to study Hamiltonian. The REM is a model in which the corresponding Gaussian process whose Hamiltonian is an iid field. We will work on the state space $\mathscr{S}_N = \{-1,+1\}^N$. The Hamiltonian of this model is

given by

$$H_N(\sigma) = -\sqrt{N} X_\sigma, \tag{1.74}$$

where X_σ, with $\sigma \in \mathscr{S}_N$, are 2^N iid standard normal random variables.

We are now ready to compute the partition function of the REM. We need this more general result first:

Theorem 1.32. *Let Z_i be iid random variables in the domain of attraction of the Gumbel distribution. Then, for a given sequence $a_n \to \infty$, there exist sequences b_n and c_n such that*

$$a_n \mathbb{P}[Z_1 > b_n^{-1}(\ln c_n + z)] \to e^{-z}. \tag{1.75}$$

Assume that, for $s > 1$,

$$\int_{-\infty}^{0} e^{sz} a_n \mathbb{P}[Z_1 > b_n^{-1}(\ln c_n + z)] dz \to \int_{-\infty}^{0} e^{(s-a)x} dx. \tag{1.76}$$

Set $X_i^n = c_n^{-\frac{1}{\alpha}} e^{\alpha^{-1} b_n Z_i}$. Then:

(i) *For $\alpha \in (1, 2)$,*

$$\sum_{i=1}^{[ta_n]} (X_i^n - \mathbb{E}[X_i^n \mathbb{I}_{X_i^n \leq 1}]) \to V_\alpha, \tag{1.77}$$

where $V_\alpha(0, 0, \nu_\alpha)$ is the α-stable Lèvy process, with

$$\nu_\alpha(dx) = \alpha x^{-1-\alpha} dx \mathbb{I}_{x>0}. \tag{1.78}$$

(ii) *For $\alpha \in (0, 1)$,*

$$\sum_{i=1}^{[ta_n]} X_i^n \to V_\alpha, \tag{1.79}$$

where $V_\alpha(0, 0, \nu_\alpha)$ is the α-stable Lèvy subordinator, with

$$\nu_\alpha(dx) = \alpha x^{-1-\alpha} dx \mathbb{I}_{x>0}. \tag{1.80}$$

(iii) *For $\alpha = 2$,*

$$\frac{1}{\sqrt{\frac{1}{2} \ln a_n}} \sum_{i=1}^{[ta_n]} (X_i^n - \mathbb{E}[X_i^n]) \to B_t, \tag{1.81}$$

where B_t is the standard Brownian motion.

(iv) *For $\alpha = 1$,*

$$\frac{1}{\ln a_n} \sum_{i=1}^{[ta_n]} X_i^n \to t. \tag{1.82}$$

Proof. We only have to verify conditions 1.50 and 1.57.

$$\mathbb{P}\left[c_n^{-\frac{1}{\alpha}} \exp\left\{\alpha^{-1} b_n Z_1\right\} > x\right] = \mathbb{P}\left[Z_1 > b_n^{-1}\left(\ln c_n + \ln x^{\alpha}\right)\right]. \tag{1.83}$$

Set $\alpha = 1$ and $\ln x = z$. Since Z_1 is in the domain of attraction of the Gumbel distribution by hypothesis, there exist sequences b_n, c_n such that

$$a_n \mathbb{P}\left[Z_1 > b_n^{-1}\left(\ln c_n + z\right)\right] \to e^{-z}. \tag{1.84}$$

Then

$$\begin{aligned} a_n \mathbb{P}\left[Z_1 > b_n^{-1}\left(\ln c_n^{\frac{1}{\alpha}} + z\right)\right] &= n\mathbb{P}\left[Z_1 > b_n^{-1}\left(\ln c_n + \alpha z\right)\right] \\ &\longrightarrow e^{-\alpha z} = x^{-\alpha}. \end{aligned} \tag{1.85}$$

By condition 1.76, all computations and limits can be passed through integrals and consequently we can control the terms corresponding to the $Z_n^{\leq}$ parts of the sums. □

The particular case when the random variables are Gaussian corresponds to the computation of the partition function of the random energy model(REM):

Theorem 1.33. *Denote by $\mathscr{P}$ the Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}dx$. The partition function of the REM has the following fluctuations:*

(i) *If $\beta < \sqrt{\frac{\ln 2}{2}}$, then*

$$e^{\frac{N}{2}(\ln 2 - \beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}[Z_{\beta,N}]} \xrightarrow{d} \mathscr{N}(0,1). \tag{1.86}$$

(ii) *If $\beta = \sqrt{\frac{\ln 2}{2}}$, then*

$$e^{\frac{N}{2}(\ln 2 - \beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}[Z_{\beta,N}]} \xrightarrow{d} \mathscr{N}(0,\frac{1}{2}). \tag{1.87}$$

(iii) *If* $\sqrt{\frac{\ln 2}{2}} < \beta < \sqrt{2\ln 2}$, *then*

$$\exp \frac{N}{2}(\sqrt{2\ln 2} - \beta)^2 + \frac{1}{2\alpha}[\ln(N\ln 2) + \ln 4\pi] \ln \frac{Z_{\beta,N}}{\mathbb{E}[Z_{\beta,N}]}$$
$$\xrightarrow{d} \int_{-\infty}^{\infty} e^{\alpha^{-1}z}(\mathscr{P}(dz) - e^{-z}dz). \tag{1.88}$$

(iv) *If* $\beta = \sqrt{2\ln 2}$, *then*

$$\exp \frac{1}{2}[\ln N \ln 2 + \ln 4\pi](\frac{Z_{\beta,N}}{\mathbb{E}[Z_{\beta,N}]} - \frac{1}{2} + \frac{\ln N \ln 2 + ln 4\pi}{4\sqrt{\pi N \ln 2}})$$
$$\xrightarrow{d} \int_{-\infty}^{0} e^{z}(\mathscr{P}(dz) - e^{-z}dz) + \int_{0}^{\infty} e^{z}\mathscr{P}dz. \tag{1.89}$$

(v) *If* $\beta > \sqrt{2\ln 2}$, *then*

$$\exp -N[\beta\sqrt{2\ln 2} - \ln 2] + \frac{1}{2\alpha}[\ln(N\ln 2) + \ln 4\pi] Z_{\beta,N}$$
$$\xrightarrow{d} \int_{-\infty}^{\infty} e^{\alpha^{-1}z}\mathscr{P}(dz). \tag{1.90}$$

Once we get these results on the partition function, we can try to describe the asymptotic behaviour of the Gibbs measure. Since we are looking for a result on the convergence in distribution of random measures, we need a topology on the spin configuration space, in order to make it uniformly compact. We map the hypercube $\mathscr{S}_N$ into the interval $(0,1]$, through the map f:

$$\begin{aligned} f : \mathscr{S}_N &\rightarrow (0,1] \\ \sigma &\mapsto r_N(\sigma) = 1 - \sum_{i=1}^{N}(1-\sigma_i)2^{-i-1} \end{aligned} \tag{1.91}$$

and define the point measure $\tilde{\mu}_{\beta,N}$on $(0,1]$:

$$\tilde{\mu}_{\beta,N} = \sum_{\sigma\in\mathscr{S}_N} \delta_{r_N(\sigma)}\mu_{\beta,N}(\sigma). \tag{1.92}$$

The results we are going to state hold with respect to this topology. They are interesting for physicists since they describe the behaviour at the high and at the low temperature.
At the high temperature phase, namely when $\beta \leq \sqrt{2\ln 2}$, the limit is the same as for $\beta = 0$, whereas at the low temperature we need to introduce a Poisson point process $\mathscr{R}$ on the strip

$(0,1] \times \mathbb{R}$, with intensity measure $\frac{1}{2}dy \times e^{-x}dx$. Denote the atoms of this process by (Y_k, X_k). For $\alpha > 1$, define a new point process $\mathscr{M}_\alpha$ on $(0,1] \times (0,1]$, whose atoms are (Y_k, w_k), where

$$w_k = \frac{e^{\alpha^{-1}} X_k}{\int \mathscr{R}(dy, dx) e^{\alpha x}}. \tag{1.93}$$

The followings hold:

Theorem 1.34. *Let λ be the Lebesgue measure on $[0,1]$. If $\beta \leq \sqrt{2 \ln 2}$, then*

$$\tilde{\mu}_{\beta,N} \longrightarrow \lambda, a.s.. \tag{1.94}$$

Theorem 1.35. *If $\beta > \sqrt{2 \ln 2}$ and $\alpha = \frac{\sqrt{2 \ln 2}}{\beta}$, then*

$$\tilde{\mu}_{\beta,N} \xrightarrow{d} \tilde{\mu}_\beta = \int_{(0,1] \times (0,1]} \mathscr{M}_\alpha(dy, dw) \delta_y w. \tag{1.95}$$

1.4 The GREM

If we try to generalize the case in which the random variables we are considering are iid, the next step is to study the correlated case. This gives rise to the GREM (Generalized Random Energy model).

Assume that A is the distribution function of a measure, whose support is a finite number n of points $x_1, \dots, x_n \in [0,1]$. Let a_i be the mass of the atoms x_i and set $\ln \alpha_i = (x_i - x_{i-1}) \ln 2$, for $i = 1, \dots, n$, with $x_0 = 0$, such that $\sum_{i=1}^n a_i = 1$ and $\prod_{i=1}^n \alpha_i = 2$.

Let X_σ be a Gaussian process. We can find an explicit representation of it.

Set $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$, where $\sigma_i \in \mathscr{S}_{N \frac{\ln \alpha_i}{\ln 2}}$ and assume that $x_1 > 0$, $x_n = 1$ and $a_i > 0$ for all i. Let $X_{\sigma_1}, X_{\sigma_1 \sigma_2}, \dots, X_{\sigma_1 \cdots \sigma_n}$ be independent standard Gaussian variables, where $\sigma_i \in \mathscr{S}_{N \frac{\ln \alpha_i}{\ln 2}}$. Then the Gaussian process can be constructed in the following way:

$$X_\sigma = \sqrt{a_1} X_{\sigma_1} + \sqrt{a_2} X_{\sigma_1 \sigma_2} + \dots + \sqrt{a_n} X_{\sigma_1 \sigma_2 \cdots \sigma_n}. \tag{1.96}$$

Following the construction we did in the case of the REM, we can start studying the case in which $n = 2$ and then we can generalize. We obtain the following

Theorem 1.36. *Let $0 < a_i < 1$, $\alpha_i > 1$, $i = 1, 2, \dots, n$, such that $\sum_{i=1}^n a_i = 1$. Set $\bar{\alpha} = \prod_{i=1}^n \alpha_i$.*

Then the following point process

$$\sum_{\sigma} \delta_{u^{-1}_{\bar{\alpha},N}(\sqrt{a_1}X_{\sigma_1}+\sqrt{a_2}X_{\sigma_1\sigma_2}+\ldots+\sqrt{a_n}X_{\sigma_1\sigma_2\cdots\sigma_n})} \tag{1.97}$$

converges weakly to the Poisson point process $\mathscr{P}$ *on* $\mathbb{R}$ *with intensity measure* $Ke^{-x}dx$, $K \in \mathbb{R}$ *if and only if*

$$a_i + a_{i+1} + \ldots + a_n \geq \frac{\ln(\alpha_i \alpha_{i+1} \ldots \alpha_n)}{\ln \bar{\alpha}} \tag{1.98}$$

for all $i = 2, 3, \ldots, n$, *or equivalently if* $A(x) \leq x$, *for all* $x \in (0, 1)$.
Moreover, if all inequalities in 1.98 are strict, then $K = 1$. *If some of them are equalities, then* $0 < K < 1$.

Theorem 1.37. *Let* $\alpha_i \geq 1$ *and set* $\bar{\alpha} = \prod_{i=1}^{k} \alpha_i$. *Let* $Y_{\sigma_1}, Y_{\sigma_1\sigma_2}, \ldots, Y_{\sigma_1\cdots\sigma_k}$ *be identically distributed random variables, such that the following vectors are independent:*

$$\left(Y_{\sigma_1}\right)_{\sigma_1 \in \{-1,1\}^{N\frac{\ln \alpha_1}{\ln \bar{\alpha}}}}, \ldots, \left(Y_{\sigma_1\sigma_2\cdots\sigma_k}\right)_{\sigma_k \in \{-1,1\}^{N\frac{\ln \alpha_k}{\ln \bar{\alpha}}}}. \tag{1.99}$$

Define $v_{N,1}(x), \ldots, v_{N,k}(x)$, *functions on* $\mathbb{R}$, *such that we have these weak convergences:*

$$\sum_{\sigma_1} \delta_{v_{N,1}(Y_{\sigma_1})} \rightarrow \mathscr{P}_1 \tag{1.100}$$

$$\sum_{\sigma_2} \delta_{v_{N,2}(Y_{\sigma_1\sigma_2})} \rightarrow \mathscr{P}_2 \ \forall \sigma_1 \tag{1.101}$$

$$\cdots$$

$$\sum_{\sigma_k} \delta_{v_{N,k}(Y_{\sigma_1\sigma_2\cdots\sigma_k})} \rightarrow \mathscr{P}_k \ \forall \sigma_1, \cdots, \sigma_{k-1}, \tag{1.102}$$

where $\mathscr{P}_1, \ldots, \mathscr{P}_k$ *are Poisson point processes on* $\mathbb{R}$ *with intensity measures* $K_1 e^{-x}dx, \ldots, K_k e^{-x}dx$, *for some constants* $K_1, \ldots, K_k$. *Then the point process on* $\mathbb{R}^k$

$$\mathscr{P}^{(k)}_N = \sum_{\sigma_1, \cdots, \sigma_k} \delta_{(v_{N,1}(Y_{\sigma_1}), v_{N,2}(Y_{\sigma_1\sigma_2}), \cdots, v_{N,k}(Y_{\sigma_1\sigma_2\cdots\sigma_k}))} \tag{1.103}$$

converges weakly to a point process $\mathscr{P}^{(k)}$ *on* $\mathbb{R}^k$.

Definition 1.38. The point process $\mathscr{P}^{(k)}$ is called ***Poisson cascades*** with k levels.

Combining these two previous theorems we get:

Theorem 1.39. *Let* $\alpha_i \geq 1$, $0 < a_i < 1$, *such that* $\prod_{i=1}^{n} \alpha_i = 2$ *and* $\sum_{i=1}^{n} a_i = 1$. *Let* $J_1, J_2, \ldots, J_m \in \mathbb{N}$ *be indexes such that* $0 = J_0 < J_1 < J_2 < \ldots < J_m = n$.
Denote by $\bar{a}_l = \sum_{i=J_{l-1}+1}^{J_l} a_i$ *and by* $\bar{\alpha}_l = \prod_{i=J_{l-1}+1}^{J-l} \alpha_i$, *for* $l = 1, 2, \ldots, m$. *Set*

$$\bar{X}^{\sigma_1 \cdots \sigma_{J_{l-1}}}_{\sigma_{J_{l-1}+1}\sigma_{J_{l-1}+2}\sigma_{J_l}} = \frac{1}{\sqrt{\bar{a}_l}} \sum_{i=1}^{J_l - J_{l-1}} \sqrt{a_{J_{l-1}+i}} X_{\sigma_{J_1} \cdots \sigma_{J_{l-1}+i}}. \tag{1.104}$$

Assume that for all $l = 1, 2, \ldots, m$ *and all* k *such that* $J_{l-1} + 2 \leq k \leq J_l$

$$\frac{(a_k + a_{k+1} + \ldots + a_{J_{l-1}} + a_{J_l})}{\bar{a}_l} \geq \frac{\ln(\alpha_k \alpha_{k+1} \ldots \alpha_{J_{l-1}} \alpha_{J_l})}{\ln(\bar{\alpha}_l)}. \tag{1.105}$$

Then the point process

$$\mathscr{P}_N^{(m)} = \sum_{\sigma} = \delta_{\left(u^{-1}_{\bar{\alpha}_1,N}(\bar{X}_{\sigma_1 \cdots \sigma_{J_1}}), \cdots, u^{-1}_{\bar{\alpha}_m,N}\left(\bar{X}^{\sigma_1 \cdots \sigma_{J_{m-1}}}_{\sigma_{J_{m-1}+1} \cdots \sigma_{J_m}}\right)\right)} \tag{1.106}$$

converges weakly to the process $\mathscr{P}^{(m)}$ *on* $\mathbb{R}^m$ *defined in Theorem 1.37, with constants* $K_1, \ldots, K_m$. *If all* $J_l - J_{l-1} - 1$ *inequalities in 1.105, for* $k = J_{l-1} + 2, \ldots, J_l$ *are strict, then* $K_l = 1$. $0 < K_l < 1$ *otherwise.*

We have constructed all possible point processes. We now find the extremal process by choosing the one that yields the largest values.
Set $\gamma_l = \frac{\sqrt{\bar{a}_l}}{\sqrt{2 \ln \bar{\alpha}_l}}$, for $l = 1, 2, \ldots, m$. Observe that $\gamma_1 > \gamma_2 > \ldots > \gamma_m$. Define the function $U_{J,N}$:

$$U_{J,N}(x) = \sum_{l=1}^{m} \left(\sqrt{2N\bar{a}_l \ln \bar{\alpha}_l} - N^{-\frac{1}{2}} \gamma_l \frac{(\ln(N(\ln \bar{\alpha}_l)) + \ln 4\pi)}{2} \right) + N^{-\frac{1}{2}} x \tag{1.107}$$

and the point process

$$\mathscr{E}_n = \sum_{\sigma \in \{-1,1\}^N} \delta_{U^{-1}_{J,N}}(X_\sigma). \tag{1.108}$$

Theorem 1.40. (i) *As* $N \to \infty$, *the point process* $\mathscr{E}_n$ *converges weakly to the point process*

$$\mathscr{E} = \int_{\mathbb{R}^m} \mathscr{P}^{(m)}(dx_1, \ldots, dx_m) \delta_{\sum_{l=1}^{m} \gamma_l x_l}, \tag{1.109}$$

where $\mathscr{P}^{(m)}$ *is the Poisson cascade of Theorem 1.39.*

(ii) $\mathscr{E}$ *exists since* $\gamma_1 > \ldots > \gamma_m$. *It is the cluster point process on* $\mathbb{R}$ *containing an almost surely* *finite number of points in any interval of the form* $[b, \infty)$*,* $b \in \mathbb{R}$*. The probability that there exists at least one point of* $\mathscr{E}$ *in the interval* $[b, \infty)$ *is decreasing exponentially as* $b \to \infty$*.*

As we did in the REM, we could now try to compute the asymptotics of the partition function and the asymptotic behaviour of the Gibbs measure, but we're not going to state any theorem since the proofs, the computations and the notation are heavy and technical.

Chapter 2

Branching random walks

2.1 Setting and definitions

Branching random walks (BRWs) and their continuous time version, branching Brownian motions (BBMs), are models that try to describe the evolution of a population of particles where spatial motion is present. To introduce these models we need a spatial structure that will help us to better understand them: trees. We will now give some basic definitions and results on graph theory.

Definition 2.1. An ***oriented graph*** G is a pair (V, E), where V is a finite set and $E \subseteq V \times V$. A ***non-oriented graph*** G is a pair (V, E), where V is a finite set and E is a set of non ordered pairs (v_1, v_2), such that $v_1, v_2 \in V$.
The elements of V are called ***vertices*** and the elements of E are said to be the ***edges***.

Definition 2.2. Let $G = (V, E)$ be an oriented graph. A ***direct path*** is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ such that $(v_i, v_{i+1}) \in E$, $\forall i = 0, 1, \ldots, k-1$.
k is the ***length*** of the path.
A path is said to be ***simple*** if no vertex, apart from v_0 and v_k, is repeated.
A path is said to be ***closed*** if $v_0 = v_k$.
A path is a ***cycle*** if it is simple and closed.

Definition 2.3. Let $G = (V, E)$ be an oriented graph. A ***non-direct path*** is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ such that $(v_i, v_{i+1}) \in E$, or $(v_{i+1}, v_i) \in E$, $\forall i = 0, 1, \ldots, k-1$.

Definition 2.4. An oriented graph is said to be ***connected*** if for any pair of vertices v and $w \in V$, there exists a non-direct path connecting v and w.

Definition 2.5. Let $G = (V, E)$ be an oriented graph. The ***degree*** of a vertex v is the number of edges that touch v.
The ***degree*** of the graph is defined as

$$\deg(G) = \max \{\deg(v)|v \in V\} \tag{2.1}$$

Definition 2.6. Let τ be an oriented graph. τ is said to be an ***oriented tree*** if

(i) τ is connected,

(ii) τ has no non-oriented cyles.

Remark 2.7. Vertices of a tree are also called ***nodes***.

Remark 2.8. Observe that Definition 2.6 implies that in a tree, given any two vertices v and $w \in V$, there is a unique path connecting v and w.

Definition 2.9. A ***rooted tree*** is a tree in which one vertex has been designated to be the root. We will denote the root $\emptyset$.

Definition 2.10. A **k**$-$***ary tree*** is a rooted tree in which each vertex has no more than k children.

Definition 2.11. The ***distance*** $|v|$ of a vertex v from the root is the length of the shortest path from v to the root, which is unique, by Remark 2.8
We write $\emptyset \leftrightarrow v$ for the collection of vertices, or edges, on this path.
If $v, w \in V$, we denote by $\rho(v, w)$ the length of the shortest path connecting v and w and we write $v \leftrightarrow w$ for the collection of vertices on this path.

We now have the tools to introduce our first model, the branching random walk.
Let τ be an oriented tree rooted at a vertex $\emptyset$, with vertex set V and edge set E.
The collection $D_n := \{v \in V : |v| = n\}$ represents the n-th generation, while for $v \in D_m$ and $n > m$, we can define the collection of descendants of v in D_n: $D_n^v = \{w \in D_n : \rho(w, v) = n - m\}$.

Definition 2.12. Let $\{X_e\}_{e \in E}$ be a family of real valued independent real valued random variables associated to the edges of the tree τ with the following assumptions:

(i) The variables X_e are independent and identically distributed with common law μ,

(ii)

$$\mathbb{E}_\mu[e^{\lambda X_e}] =: e^{\Lambda(\lambda)} < \infty, \lambda \in \mathbb{R}, \tag{2.2}$$

(*iii*) the tree τ is a k-ary tree.

Set $S_v = \sum_{e\in\emptyset\leftrightarrow v} X_e$, for $v \in V$. The ***branching random walk*** **(BRW)** is the collection of random variables $\{S_v\}_{v\in V}$.

We also introduce the large deviation rate function, associated with Λ:

$$I(x) = \sup_{\lambda\in\mathbb{R}}(\lambda x - \Lambda(\lambda)), \tag{2.3}$$

where $\Lambda(\lambda) = \log \mathbb{E}[e^{\lambda X_e}]$.

Remark 2.13. Let x^* be the unique point such that $x^* > \mathbb{E}_\mu[X_e]$ and $I(x^*) = \log k$. Then $I(x^*) = \lambda^* x^* - \Lambda(\lambda^*)$, with $x^* = \Lambda'(\lambda^*)$ and $I'(x^*) = \lambda^*$.

2.2 Main results on the maximal displacement

In this section we will focus on some results on the maximal displacement of the BRW:

$$M_n = \max_{v\in D_n} S_v. \tag{2.4}$$

In order to do this we need to first study the independent case. This means we have to consider the following collection of i.i.d. random variables, $\{\tilde{S}_v\}_{v\in D_n}$, where $\tilde{S}_v \sim S_v$. $\tilde{M}_n$ will be the maximum over these variables, namely $\tilde{M}_n = \max_{v\in D_n} \tilde{S}_v$.
As a first step, we are going to prove the convergence in distribution of $\tilde{M}_n - \tilde{m}_n$, where

$$\tilde{m}_n = nx^* - \frac{1}{2I'(x^*)}\log n. \tag{2.5}$$

Theorem 2.14. *There exists a constant C such that*

$$\mathbb{P}[\tilde{M}_n \leq \tilde{m}_n + x] \overset{n\to\infty}{\to} \exp\{-Ce^{-I'(x^*)x}\}. \tag{2.6}$$

Proof. Let $a_n = o(\sqrt{n})$. For a large deviation estimate we have

$$\mathbb{P}[\tilde{S}_v > nx^* - a_n] \sim \frac{C}{\sqrt{n}} \exp\left\{-nI\left(x^* - \frac{a_n}{n}\right)\right\}, \tag{2.7}$$

where with $\sim$ we denote the same asymptotic behaviour. I is smooth at x^*, by definition, since

x^* is in the interior of the domain of I, hence, using Taylor expression at the second order,

$$nI\left(x^* - \frac{a_n}{n}\right) = nI(x^*) - I'(x^*)a_n + o(1). \tag{2.8}$$

We recall that $I(x^*) = \ln k$. Substituting this and 2.8 in 2.7, we obtain

$$\begin{aligned}
\mathbb{P}[\tilde{M}_n \leq nx^* - a_n] &= \left(1 - \mathbb{P}[\tilde{S}_v > nx^* - a_n]\right)^{k^n} && (2.9)\\
&\sim \left(1 - \frac{C}{k^n\sqrt{n}} e^{I'(x^*)a_n + o(1)}\right)^{k^n} && (2.10)\\
&\sim \exp\left\{\frac{-Ce^{I'(x^*)a_n + o(1)}}{\sqrt{n}}\right\} && (2.11)
\end{aligned}$$

We can now choose $a_n = \ln \frac{n}{2I'(x^*)} - x$ and obtain

$$\mathbb{P}[\tilde{M}_n \leq \tilde{m}_n + x] \sim \exp\left\{-Ce^{-I'(x^*)x} + o(1)\right\} \tag{2.12}$$

□

Remark 2.15. We can deduce from 2.14 that

$$\frac{\tilde{M}_n}{n} \underset{n\to\infty}{\longrightarrow} x^*, \tag{2.13}$$

almost surely.

Our current aim is to show that the same result holds also for M_n. We are going to prove that

Theorem 2.16.

$$\frac{M_n}{n} \underset{n\to\infty}{\longrightarrow} x^*, \tag{2.14}$$

almost surely.

Proof. Let c be a constant. It is easy to check that $\frac{M_n}{n} \underset{n\to\infty}{\longrightarrow} c$ almost surely. In fact, associate each vertex in D_n with a word $a_1, \ldots, a_n$, where $a_i \in \{1, \ldots, k\}$. Introduce an arbitrary order on the vertices of D_n and define

$$v_m^* = \min\{v \in D_m : S_v \geq \max_{w \in D_m} S_w\}. \tag{2.15}$$

For $n > m$, we denote by

$$D_n^{v_m^*} = \{w \in D_n : \rho(w, v_m^*) = n - m\} \tag{2.16}$$

the collection of descendants of v_m^*. Let also

$$M_n^m = \max_{w \in D_n^{v_m^*}} S_w - S_{v_m^*}. \tag{2.17}$$

From the definitions we gave, we have that $M_n \geq M_m + M_n^m$. Observe also that M_n has all moments. This means that we can apply the subadditive ergodic theorem, obtaining

$$\frac{M_n}{n} \to c,$$

almost surely, for some constant c. We now have to identify c.

We would like to find an upper and a lower bound for $\frac{M_n}{n}$. We start by defining $\bar{Z}_n = \sum_{v \in D_n} \mathbb{I}_{\{S_v > n(1+\epsilon)x^*\}}$. Applying the definition of expectation in the first inequality and Chebychev's inequality in the last one, we obtain

$$\mathbb{E}[\bar{Z}_n] \leq k^n \mathbb{P}[S_v > n(1+\epsilon)x^*] \leq k^n e^{-nI((1+\epsilon)x^*)}. \tag{2.18}$$

Since I is strictly monotone at x^*, we get $\mathbb{E}[\bar{Z}_n] \leq e^{-nc(\epsilon)}$, for some $c(\epsilon) > 0$. Then,

$$\mathbb{P}[M_n > n(1+\epsilon)x^*] \leq \mathbb{E}[\bar{Z}_n] \leq e^{-c(\epsilon)n}. \tag{2.19}$$

It follows that

$$\limsup_{n \to \infty} \frac{M_n}{n} \leq x^*, \text{ almost surely.} \tag{2.20}$$

A natural way to proceed to obtain a lower bound would have been to define

$$\underline{Z}_n = \sum_{v \in D_n} \mathbb{I}_{\{S_v > n(1-\epsilon)x^*\}} \tag{2.21}$$

and to show that $\underline{Z}_n \geq 1$ with high probability. However, the correlation between the events $\{S_v > n(1-\epsilon)x^*\}$ and $\{S_w > n(1-\epsilon)x^*\}$, when $v \neq w$, is too large and so it is useful to find other events whose probability is similar, but whose correlation is much smaller and easy to handle.

Recall that, for $v \in D_n$ and $t \in \{0, \ldots, n\}$, the ancestor of v at t is $v_t = \{w \in D_t : \rho(v, w) = n - t\}$.

Set $S_v(t) = S_{v_t}$, where $S_v = S_v(n)$ for $v \in D_n$. Define the event

$$B_v^\epsilon = \{|S_v(t) - x^* t| \le \epsilon n, t = 1, \ldots, n\}, \tag{2.22}$$

and

$$Z_n = \sum_{v \in D_n} \mathbb{I}_{B_v^\epsilon}. \tag{2.23}$$

We recall a basic large deviation result:

Lemma 2.17. *Under the assumptions 2.2,*

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln \mathbb{P}[B_v^\epsilon] = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \ln \mathbb{P}[B_v^\epsilon] = -I(x^*). \tag{2.24}$$

By Theorem 2.17 and the strict monotonicity of I at x^* we obtain a lower bound,

$$\mathbb{E}[Z_n] \ge e^{-c(\epsilon)n}. \tag{2.25}$$

Fix now a pair of vertices $v, w \in D_n$ with $\rho(v, w) = 2r$. The number of such ordered pairs is $k^{n+r-1}(k-1)$.

$$\begin{aligned}
\mathbb{P}[B_v^\epsilon \cap B_w^\epsilon] &= \mathbb{P}[|S_v(t) - x^* t| \le \epsilon n, t = 1, \ldots, n-r] \\
&\quad \cdot \ \mathbb{E}[\mathbb{P}[|S_v(t) - x^* t| \le \epsilon n, t = n-r+1, \ldots, n | S_v(n-r)]]^2 \\
&\le \mathbb{P}[|S_v(t) - x^* t| \le \epsilon n, t = 1, \ldots, n-r] \\
&\quad \cdot \ \mathbb{P}[|S_v(t) - x^* t| \le 2\epsilon n, t = 1, \ldots, r]^2,
\end{aligned}$$

where we used independence in the first equality and homogeneity in the second inequality. Applying Theorem 2.17 on the two factors, we obtain

$$\mathbb{P}[B_v^\epsilon \cup B_w^\epsilon] \le e^{-(n-r)I(x^*) - 2rI(x^*) + c(\epsilon)n}, \tag{2.26}$$

where $c(\epsilon) \overset{\epsilon \to 0}{\longrightarrow} 0$. It also holds that

$$\mathbb{E}[Z_n^2] \le \sum_{r=0}^{n} k^{n+r} e^{-(n+r)I(x^*) + c(\epsilon)n} = e^{c(\epsilon)n}. \tag{2.27}$$

Recall that, from Cauchy-Schwartz, for any non-negative, integer valued random variable Z,

$$\mathbb{P}[Z \geq 1] \geq \frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}. \tag{2.28}$$

From equations 2.25 and 2.28, we have that, for any $\delta > 0$,

$$\mathbb{P}[\exists v \in D_n : S_v \geq n(1-\delta)x^*] \geq e^{-o(n)}. \tag{2.29}$$

Fix now $\epsilon > 0$ and pick a value x such that $\mathbb{P}[X_e > x] > \frac{1}{k}$, and consider the tree τ_ϵ of depth ϵn. This tree corresponds to independent bond percolation on τ in levels, $1, \dots, \epsilon n$, where we kept only those edges e such that $X_e > x$. Since $k\mathbb{P}[X_e > x] > 1$, the percolation is supercritical and there exists a constant C, independent of ϵ, such that

$$\mathbb{P}[\{|\tau_\epsilon \cup D_{n\epsilon}| > e^{\epsilon C n}\}\}] > C_x > 0,$$

with $C_x \overset{x \to -\infty}{\longrightarrow} 1$. By independence, we can conclude that

$$\mathbb{P}[M_n \geq n(1-\epsilon)x^* + n\epsilon x] \geq C_x(1-(1-e^{-o(n)})^{e^{\epsilon C_n}}) \overset{n\to\infty}{\longrightarrow} C_x. \tag{2.30}$$

We can now take $n \to \infty$, $\epsilon \to 0$ and $x \to -\infty$ and obtain

$$\liminf_{n\to\infty} \frac{M_n}{n} \geq x^*, \text{ almost surely.} \tag{2.31}$$

We have

$$\limsup_{n\to\infty} \frac{M_n}{n} \leq x^* \leq \liminf_{n\to\infty} \frac{M_n}{n} \tag{2.32}$$

and thus we can conclude obtaining 2.14. □

2.2.1 Expectation

We have just provided a sort of law of large numbers for M_n. It could be interesting to know what is the behaviour of the expectation of M_n.

Theorem 2.18.

$$\mathbb{E}[M_n] = nx^* - \frac{3}{2I'(x^*)} \log n + O(1) \tag{2.33}$$

We are not proving the theorem we just stated since the proof is long and technical. A complete proof can be found in [15], which is based on well-known Bramson's results([5]). The idea is to find a lower and upper bounds on the right tail of M_n, using a second moment method. This allows us to find an upper bound for the expectation. It is also possible to find a lower bound for $\mathbb{E}[M_n]$, through a first moment method. Upper and lower bounds together give the result we are looking for.

Chapter 3

Branching Brownian motion

3.1 Setting and definitions

In this chapter, we will describe a model, whose aim is to study a system of particles that are independent (they don't interact), move in space according to some process and branch. A branching Brownian motion can be described as follows: a particle starts at a certain point $x \in \mathbb{R}$ and moves according to a Brownian motion. After a random time, represented by an exponential random variable with parameter β, the particle splits and it is replaced by daughter particles, which also start to move according to independent Brownian motions and split after independent exponential random times and so on.

As we did in the study of branching random walks, we are going to use the structure of a tree to formally define branching Brownian motion.

Let τ be a rooted oriented k-ary tree. We set

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n \text{ and } \bar{\mathcal{U}} = \mathcal{U} \cup \mathbb{N}^{\infty},$$

with the convention that $\mathbb{N}^0 = \{\emptyset\}$, and think of an element $u \in \mathcal{U}$, that is a sequence $u = (u_1, u_2, \ldots, u_n)$ of integers, as the label of a node of our tree τ at the $n-$th branch. For example $u = (1, 2, 3)$ represents the third child of the second child of the first child of the root $\emptyset$.

We also define a map

$$\begin{aligned} p : \mathcal{U} \backslash \emptyset &\rightarrow \mathcal{U} \\ (u_1, u_2, \ldots, u_n) &\rightarrow (u_1, u_2, \ldots, u_{n-1}), \end{aligned} \tag{3.1}$$

that defines the parent of u.

We also introduce a reproduction mechanism: when a particle splits, it splits into $k \geq 1$ particles with probabilities μ_k, where the following hold:

(i) $\sum_{k=1}^{\infty} \mu_k = 1$,

(ii) $\sum_{k=1}^{\infty} k\mu_k = 2$,

(iii) $K = \sum_{k=1}^{\infty} k(k-1)\mu_k < \infty$.

We now need to enrich our tree τ associating to each $u \in \tau$ a lifetime $\sigma_u \geq 0$, where the σ_u's are exponential random variables. We then define the birth-time of u:

$$b_u = \sum_{v<u} \sigma_v, \tag{3.2}$$

where $v < u$ means that v is an ancestor of u, that is $p(u) = v$. The death-time of u is:

$$d_u = b_u + \sigma_u. \tag{3.3}$$

Furthermore, we require that each particle u performs a Brownian motion

$$Y_u : \mathbb{R}^+ \to E,$$

where E is the space in which the particles are living. Without loss of generality, we can think of E as a subset of $\mathbb{R}$, $E \subseteq \mathbb{R}$.

The tree we are now working on is defined as the triplet

$$\mathfrak{t} = (\tau, \sigma, Y) = (\tau, \{\sigma_u, (Y_u(s), s \geq 0), u \in \tau\}). \tag{3.4}$$

Let now $\mathcal{N}_t = \{u \in \tau : b_u \leq t \leq d_u\} \subseteq \mathcal{U}$ be the set of particles that are alive at time t and let $\omega = (\tau, (\sigma_u, Y_u)_{u\in\tau})$. We can define inductively the position in E of the particle u at time t:

$$X_u(t) = X_u(t, \omega) := Y_u(t - b_u) + X_{p(u)}(b_u^-). \tag{3.5}$$

Definition 3.1. We define

$$X(t) = (X_u(t), u \in \mathcal{N}_t) \tag{3.6}$$

to be the ***branching Brownian motion process***, whose natural filtration is

$$\mathscr{F}_t = \sigma\{X(s), s \leq t\}. \tag{3.7}$$

Definition 3.2. The *standard (or dyadic) branching Brownian motion* is obtained with the following choices:

(i) τ is a regular binary tree,

(ii) the σ_u are independent and identically distributed:

$$\sigma_u \sim \exp(1),$$

(iii) the Y_u are standard Brownian motions.

In other words we can think of branching Brownian motion as a cloud of particles, growing in size and shape, that starts from a single particle u which performs a Brownian motion $X_u(t)$, such that $X_u(0) = 0$, that continues for an exponential holding time T with parameter β independent of X_u, with $\mathbb{P}[T > t] = e^{-\beta t}$. At time T, the particle splits independently of X_u and T into k offsprings with probabilities μ_k.

Remark 3.3. In the dyadic case, we have $\mu_2 = 1$.

Each of these particles continue performing independent Brownian motions starting at $X_u(T)$ and they are subject to the same splitting rule. After a time t, the resulting tree has a random number of particles, $n(t)$, such that $\mathbb{E}[n(t)] = e^t$.
Notice that, at time t, each particle u is at position $X_u(t)$, which is a centered Gaussian variable with variance t, exactly like a Brownian motion.

Remark 3.4. The $X_u(t)$'s are not independent, since they are correlated by their genealogical history.

3.2 Main results on the maximal displacement

Let X be a branching Brownian motion with reproduction law $(\mu_k)_{k\geq 0}$ and branching rate β. We start the study of our model by looking at the position of the rightmost particle, namely

$$M_t = \max_{u \in \mathcal{N}_t} X_u(t). \tag{3.8}$$

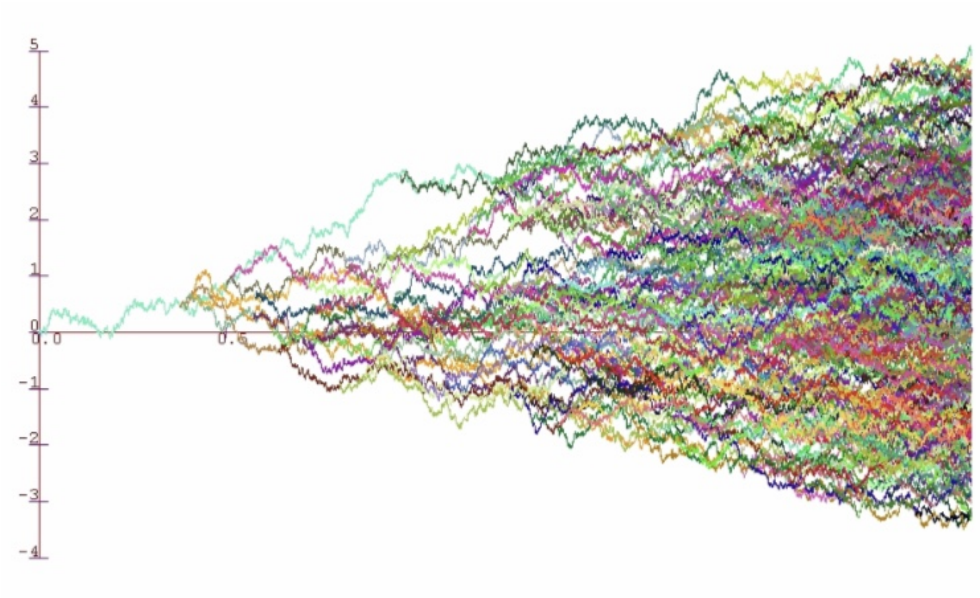


Figure 3.1: A realization of a dyadic Brownian motion

For simplicity, from now on, we are going to deal with standard Brownian motions ($\beta = 1$ and $\mu_2 = 1$). The position of the rightmost particle can be studied through the analysis of the *F-KPP* equation, that we will briefly introduce.

3.2.1 *F-KPP* equation

We introduce the following partial differential equation, the *Kolmogorov* equation or *F-KPP* (Fisher or Kolmogorov-Petrovsky-Piscounov) equation :

$$u_t = \frac{1}{2} u_{xx} + g(u), \tag{3.9}$$

where $u = u(t,x) : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$ and $g \in C^1[0,1]$, such that $g(0) = g(1) = 0$, $g(u) > 0$ for $u \in (0,1)$ and $g'(0) = \beta > 0$, $g'(u) \leq \beta$ for $u \in (0,1]$. In our study, we will always consider the case $g(u) = \beta u(1-u)$, that is equal to $u - u^2$ in the dyadic case, with Heavyside initial condition

$$u(0,x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{3.10}$$

F-KPP equation has been well studied through the years and it appears in several models related to reaction-diffusion phenomena and front propagation, since it is one of the simplest example of a semilinear parabolic equation which admits traveling wave solutions. It will be useful also in our model: let us state some basic results on this equation. First, notice that *F-KPP* is sufficiently well behaved so that we can establish existence and uniqueness of the solution under measurable data. What about the asymptotic behaviour of solutions?

Theorem 3.5. *Let u be a solution of 3.9 such that $0 \leq u(0,x) \leq 1$. Then*

$$u(t, m_t + x) \to w(x), \text{ uniformly in } x \text{ as } t \to \infty, \tag{3.11}$$

where the centering term $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + C + o(1)$, and w is the unique solution of the ordinary differential equation

$$\frac{1}{2}w'' + \sqrt{2}w' - w^2 + w = 0. \tag{3.12}$$

if and only if

(i) *for some $h > 0$, $\limsup_{t\to\infty} \frac{1}{t} \ln \int_t^{t(1+h)} u(0,y) \leq -\sqrt{2}$,*

(ii) *for some $\nu > 0$, $M > 0$, $N > 0$, $\int_x^{x+N} u(o,y)dy > \nu$ for all $x \leq -M$.*

Remark 3.6. Any function w that solves 3.12 is called a traveling wave solution of 3.9 with speed $\sqrt{2}$. Indeed it is easily checked that $u(t,x) = w(x - \sqrt{2}t)$ is a solution of 3.9.
More in general, if w_λ solves

$$\frac{1}{2}w'' + \lambda w - w^2 + w = 0, \tag{3.13}$$

then $u(t,x) = w(x - \lambda t)$ is also a solution.

3.2.2 Maximum of the BBM

Here we state the main results on the maximal displacement of branching Brownian motion.

Theorem 3.7. *Let $u(t,x) = \mathbb{P}_0[M_t \leq x]$. Then u satisfies 3.9 with Heavyside initial condition.*

Proof. Recall that we are in the dyadic case, namely with $\beta = 1$ and $p_2 = 1$.
We need to compute $u(t+dt,x) - u(t,x)$ up to terms of order $o(dt)$, where dt is small. We decompose the interval of time $[0,dt]$, according to what happens to the branching Brownian motion.
With probability $(1 - dt) + o(dt)$, the branching Brownian motion doesn't branch. Conditionally to this event

$$\mathbb{P}[M_{t+dt} \leq x] = \mathbb{P}[M_t \leq x - B_{dt}] = u(t, x - B_{dt}), \tag{3.14}$$

where B is a branching Brownian motion.

With probability $dt + o(dt)$, there exists exactly one branching event. Conditionally to this event

$$\begin{aligned} \mathbb{P}[M_{t+dt} \leq x] &= (\mathbb{P}[M_t \leq x - B_{dt}])(\mathbb{P}[M_t \leq x - B'_{dt}]) && (3.15) \\ &= (\mathbb{P}[M_t \leq x])^2 + o(1), && (3.16) \end{aligned}$$

where B and B' are correlated Brownian motions. We have

$$\begin{aligned} \mathbb{P}[M_{t+dt} \leq x] - \mathbb{P}[M_t \leq x] &= (1-dt)\mathbb{P}[M_t \leq x - B_{dt}] \\ &+ dtu^2(t,x) + o(dt) - u(t,x) \\ &= \mathbb{E}[u(t, x - B_{dt})] - u(t,x) \\ &+ dt[u^2(t,x) - u(t,x)] + o(dt) \\ &+ dt(u(t, x - B_{dt}) + u(t,x)) \\ &= \mathbb{E}[u(t, x - B_{dt})] - u(t,x) \\ &+ dt[u^2(t,x) - u(t,x)] + o(dt). && (3.17) \end{aligned}$$

Recall that if g is a smooth enough function, $v(t,x) = \mathbb{E}[g(B_{dt})]$ solves the heat equation $v_t = \frac{1}{2}v_{xx}$ and so, if we write $g(z) = u(t,z)$, then $\mathbb{E}[u(t, x - B_{dt})] = \mathbb{E}_x[g(B_{dt})]$ and

$$\lim_{dt \to 0} \frac{\mathbb{E}[u(t, x - B_{dt})] - u(t,x)}{dt} = \frac{1}{2}\frac{\delta^2}{\delta x^2}u(t,x). \tag{3.18}$$

Hence we obtain

$$\lim_{dt \to 0} \frac{u(t+dt, x) - u(t,x)}{dt} = \frac{1}{2}\frac{\delta^2}{\delta x^2}u(t,x) + [u^2(t,x) - u(t,x)], \tag{3.19}$$

which concludes the proof. □

From the previous theorem we can characterize the behaviour of the maximum in the limit of large times.

Theorem 3.8. *Almost surely,*

$$\lim_{t \to \infty} \frac{M_t}{t} = \sqrt{2} \tag{3.20}$$

and

$$\lim_{t\to\infty} M_t - \sqrt{2}t = -\infty. \tag{3.21}$$

Proof. In order to prove the previous theorem, we need to change probability measure, using a new one, easier to handle and to introduce dyadic Brownian motion with spine.

Definition 3.9. A spatial tree with spine is a pair $(\mathfrak{t}, \xi)$, where $\mathfrak{t} = (\tau, \sigma, Y)$ is a spatial tree and $\xi \subset \mathcal{U}$ such that

(i) $|\{\xi \cap \mathcal{N}_t\}| \leq 1, \forall t \geq 0,$

(ii) $u \in \xi \Rightarrow v \in \xi$, for each $v < u$.

In other words, a spine is a distinguished line of descent in a tree. For $v \in \xi \cap \mathcal{N}_t$, denote by $\xi_t = v$ the label of the spine, and $\Xi(t) = X_{\xi_t}(t)$ for the position of the spine particle. We can define a law $\tilde{\mathbb{P}}$ on the space of marked trees, such that if the pair $(\mathfrak{t}, \xi)$ has law $\tilde{P}$ then $\mathfrak{t}$ is a standard Brownian motion. We also introduce

$$\mathcal{G}_t = \sigma(\Xi(s), s \leq t), t \geq 0, \tag{3.22}$$

the natural filtration of Ξ.
Given a functional of continuous paths $F : C_{[0,t)} \to \mathbb{R}$, we define the following quantities:

$$\begin{aligned} \zeta(t) &= F(\Xi(s), s \leq t), && (3.23)\\ \zeta_u(t) &= F(X_u(s), s \leq t) \text{ for } u \in \mathcal{N}_t, && (3.24)\\ Z(t) &= e^{-t} \sum_{u \in \mathcal{N}_t} \zeta_u(t). && (3.25) \end{aligned}$$

Suppose we choose a path functional F such that $Z(t)$ is positive and with mean one. We can apply Girsanov theorem and define a new probability measure $\mathbb{Q}$ on τ by the relation

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z(t). \tag{3.26}$$

Observe that if $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, then almost sure events under $\mathbb{Q}$ are also almost sure events under $\mathbb{P}$.
Let now $\lambda \in \mathbb{R}$, $c_\lambda = \frac{\lambda}{2} + \frac{m\beta}{2}$. If we use $\zeta(t) = e^{-\lambda\Xi(t) - \lambda^2 \frac{t}{2}}$, $Z(t)$ turns out to be the so-called

additive exponential martingale

$$W_\lambda(t) = \sum_{u \in \mathcal{N}_t} e^{-\lambda(X_u(t)+c_\lambda t)}. \tag{3.27}$$

Chauvin and Rouault in [10] proved that, under $\mathbb{Q}$, the process $\Xi(t)$ is a Brownian motion with drift $-\lambda$ and so, almost surely, $\liminf \frac{M_t}{t} \geq \liminf \frac{\Xi(t)}{t} = \lambda$. Define now $\lambda^* = \sqrt{2\beta m}$ that it is equal to $\lambda^* = \sqrt{2}$ in the dyadic case. Observe that the function $\lambda \mapsto c_\lambda$ has its maximum on $(-\infty, 0)$ at $-\lambda^*$ and its minimum on $(0, \infty)$ at λ^*. We need also the following theorem :

Theorem 3.10. *The limit $W_\lambda = \lim_{t\to\infty} W_\lambda(t)$ exists $\mathbb{P}$-almost surely and*

(i) *If $|\lambda| \geq \lambda^*$, then $W(\lambda) = 0$ $\mathbb{P}$-almost surely,*

(ii) *if $|\lambda| \in [0, \lambda^*)$, then $W(\lambda)$ is a $L^1(\mathbb{P})$ limit and $\mathbb{P}[W_\lambda > 0] = 1$.*

For the proof of this theorem, see [4].

Denote now $c^* = c_{\lambda^*} = \lambda^*$. By this theorem, we can say that all the martingales defined as in 3.27 converge. Furthermore $W_\lambda(t) \to 0$, $\mathbb{P}$-almost surely as soon as $|\lambda| \geq \lambda^*$. Notice also that $e^{\lambda^*(M_t + c_{-\lambda^*}t)} \leq W_{-\lambda^*}(t) \to 0$ as soon as $|\lambda| \geq \lambda^*$. Observe that $c_{-\lambda^*} = -c^*$. Thus

$$M_t + c_{-\lambda^*}t = M_t - c^*t \to -\infty \tag{3.28}$$

that is our thesis (note that $c^* = \sqrt{2}$ in the dyadic case).

Furthermore $\limsup \frac{M_t}{t} \leq c^*$. We now need to prove the converse bound. $\mathbb{Q}_\lambda$ and $\mathbb{P}$ are equivalent, when $\lambda \in (-\lambda^*, 0]$. We know, from previous observations, that under $\mathbb{Q}_\lambda$ the process $\Xi(t)$ is a Brownian motion with drift $-\lambda$ and so $\liminf \frac{M_t}{t} \geq |\lambda|$, $\mathbb{Q}_\lambda$-almost surely and so $\mathbb{P}$-almost surely as well. Since λ is arbitrary in $(-\lambda^*, 0]$, we obtain $\liminf \frac{M_t}{t} \geq \lambda^*$, $\mathbb{P}$-almost surely. Hence, in the standard case,

$$\limsup \frac{M_t}{t} \leq \sqrt{2} \leq \liminf \frac{M_t}{t} \Rightarrow \lim \frac{M_t}{t} = \sqrt{2}. \tag{3.29}$$

□

Theorem 3.11. *Let $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + o(1)$. Then $M_t - m_t$ converges in distribution and there exists a random variable Z such that*

$$\mathbb{P}[M_t - m_t \leq y] \to \mathbb{E}[\exp -cZe^{\sqrt{2}y}], \text{ as } t \to \infty \tag{3.30}$$

where c is a constant, $c > 0$.

Proof. Let $\hat{M}_t = M_t - m_t$. Observe that, for each $u \in \mathcal{N}_t$, $M_t = M_{t-s} + X_u(s)$ and $m_t = m_{t-s} + 0(1)$. We have

$$\begin{aligned}
\mathbb{P}[\hat{M}_t \leq y] &= \mathbb{E}[\mathbb{P}[\hat{M}_t \leq y|\mathcal{F}_s]] \\
&= \mathbb{E}[\prod_{u \in \mathcal{N}_s} \mathbb{P}[M_{t-s} \leq m_t + y - X_u(s)|X_u(s)]] \\
&= \mathbb{E}[\prod_{u \in \mathcal{N}_s} (1 - \mathbb{P}[M_{t-s} > m_{t-s} + y - (X_u(s) - \sqrt{2}s) + o(1)|X_u(s)])].
\end{aligned}$$

From 3.21, we know that $M_s - \sqrt{2}s \to -\infty$ as $s \to \infty$. This means that we should have $y - (X(s) - \sqrt{2}s) \gg 1$, for all $u \in \mathcal{N}_s$. Hence we can use the following lemma, that we are not going to prove:

Lemma 3.12. *There exists a constant c such that*

$$\mathbb{P}[M_t > m_t + y] \sim cye^{-\sqrt{2}y}, \text{ as } y \to \infty. \tag{3.31}$$

Let $Z_s = \sum_{u \in \mathcal{N}_s} (\sqrt{2}s - X_u(s))e^{-\sqrt{2}(\sqrt{2}s - X_u(s))}$.

$$\begin{aligned}
\mathbb{P}[\hat{M}_t \leq y] &\sim_t \mathbb{E}[\prod_{u \in \mathcal{N}_s} (1 - c(y + \sqrt{2}s - X_u(s))e^{-\sqrt{2}(y + \sqrt{2}s - X_u(s))})] \\
&\sim_s \mathbb{E}[\exp\{-cZ_s e^{-\sqrt{2}y}\}].
\end{aligned}$$

Hence we proved that

$$\lim_{t \to \infty} \mathbb{P}[M_t - m_t \leq y] \sim \mathbb{E}[\exp\{-cZ_s e^{-\sqrt{2}y}\}] \text{ as } s \to \infty. \tag{3.32}$$

Notice that the left-hand side does not depends on s. This means that the right-hand side must have a limit for $s \to \infty$. Hence $Z_s \to Z$ in distribution. So we obtain 3.30 □

3.2.3 Refinements

From Theorem 3.11 we can say that the law of the maximum of branching Brownian motion is a random shift of the Gumbel distribution. As we observed before, Lalley and Sellke in [12] provided a characterization of the limiting law of the maximal displacement. Let us define the so-called derivative martingale:

$$Z(t) = \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_u(t)) \exp\left(-\sqrt{2}(\sqrt{2}t - X_u(t))\right). \tag{3.33}$$

They proved that $Z(t)$ converges almost surely to a strictly positive random variable Z, that is the one of Theorem 3.11. If we admit this we get

$$w(x) = \mathbb{E}[\exp\left(-cZe^{-\sqrt{2}x}\right)]. \tag{3.34}$$

We can obtain a result also on the expectation of M_t:

Theorem 3.13.

$$\mathbb{E}[M_t] = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + o(1) \text{ as } t \to \infty. \tag{3.35}$$

Proof. We only sketch the proof. Details can be found in [15]. The aim is to find an upper bound and a lower bound for the expectation. From the estimate 2.7 we obtain the upper bound $\mathbb{E}[M_t] \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + o(1)$. The corresponding lower bound can be obtained once we show

$$\lim_{z \to -\infty} \limsup_{t \to \infty} \int_{-\infty}^{z} \mathbb{P}[M_t \leq m_t + y] dy. \tag{3.36}$$

□

3.3 The extremal process of the branching Brownian motion

We have precise information on the maximum of branching Brownian motion, while less is known on the full statistics of the particles close to the maximal one.

Let $X(t)$ be the branching Brownian motion process and call $N(t)$ the number of particles alive at time t, namely $N(t) = |\mathcal{N}_t|$. Let $X_1(t) \leq X_2(t) \leq \ldots \leq X_{N(t)}(t)$ be their positions enumerated from left to right. The statistics of such particles are completely encoded in the following extremal process:

$$\mathcal{E}_t = \sum_{k \leq N(t)} \delta_{X_k(t) - m_t}. \tag{3.37}$$

We are going to study the extremal process in the limit of large times. In Chapter 1, we have studied classical results for extremal processes of correlated random variables, like the GREM. However these models have a rather simple hierarchical structure that involves only a finite number of hierarchies, while our BBM has infinite levels of branching. Fortunately, branching Brownian motion is just at the borderline where correlations start to effect the extremes, so we can go beyond the simple Poisson structures and open the way towards the rigorous study of complex extremal structures.

Our aim is to prove that the extremal process of branching Brownian motion, in the limit of large times, converges weakly to a cluster point process: the limiting process is a randomly shifted Poisson cluster point process, where the position of the clusters is a Poisson process with intensity measure with exponential density.

Theorem 3.14. *Let*

$$\mathcal{P}_Z = \sum_{i\in\mathbb{N}} \delta_{p_i} \overset{law}{=} PPP(cZ\sqrt{2}e^{-\sqrt{2}x}dx), \tag{3.38}$$

and let $\{\mathcal{D}^{(i)}, i \in \mathbb{N}\}$ *be a family of independent copies of the following gap-process on* $(-\infty, 0]$*:*

$$\mathcal{D} = \sum_j \delta_{\Delta_j}, \tag{3.39}$$

where $\Delta_j = \xi_j - \max_i \xi_i$ *and the* ξ_i*'s are atoms of the limiting point process of* $\bar{\mathcal{E}}_t = \sum_{k\le N(t)} \delta_{X_k(t)-\sqrt{2}t}$. *Then the point process* $\mathcal{E}_t = \sum_{k\le N(t)} \delta_{X_k(t)-m_t}$ *converges in law as* $t \to \infty$ *to a Poisson cluster point process* $\mathcal{E}$*:*

$$\mathcal{E} = \lim_{t\to\infty} \mathcal{E}_t \overset{law}{=} \sum_{i,j} \delta_{p_i+\Delta_j^{(i)}}. \tag{3.40}$$

In order to prove this theorem, we need some intermediate steps. We first prove the existence of the limiting process of $\mathcal{E}_t$ and then we introduce an auxiliary point process Π_t (see 3.53), proving that this is a Poisson point process. Finally we prove that our original process $\mathcal{E}_t$ is equal in law to the process Π_t.

3.3.1 Existence of the limit

Theorem 3.15. *The point process* $\mathcal{E}_t = \sum_{k\le N(t)} \delta_{X_k(t)-m_t}$ *converges in law to a point process* $\mathcal{E}$.

Proof. We need to show that the Laplace functional of the extremal process $\mathcal{E}_t$ of our branching Brownian motion converges.

Let $\phi \in \mathcal{C}_{\mathcal{C}}(\mathbb{R})$. We define such functional:

$$\psi_t(\phi) = \mathbb{E}\left[\exp\left\{-\int \phi(y)\mathcal{E}_t(dy)\right\}\right]. \tag{3.41}$$

In [1] it has been proved that, for any bounded measurable set $B \subset \mathbb{R}$,

$$\lim_{N\to\infty} \lim_{t\to\infty} \mathbb{P}[\mathcal{E}_t(B) > N] = 0. \tag{3.42}$$

This means that the limiting process must be locally finite.

Denote now by $\max \mathcal{E}_t = \max_{k \le N(t)} X_k(t) - m_t$ and let $\delta > 0$. Define

$$v(t, \delta + m_t) = \mathbb{E}\left[\prod_{k=1}^{N(t)} \mathbb{I}_{\{X_k(t) - m_t \le \delta\}}\right] = \mathbb{P}[\max \mathcal{E}_t \le \delta]. \tag{3.43}$$

Applying Theorem 3.5 to this function v, it holds that

$$\lim_{\delta\to\infty} \lim_{t\to\infty} 1 - v(t, \delta + m_t) = \lim_{\delta\to\infty} w(\delta) = 0. \tag{3.44}$$

We write

$$\begin{aligned}\mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\right] &= \mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\mathbb{I}_{\{\max \mathcal{E}_t \le \delta\}}\right] \\ &+ \mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\mathbb{I}_{\{\max \mathcal{E}_t > \delta\}}\right]. \end{aligned} \tag{3.45}$$

In order to prove convergence, we want to show that this quantity, in the limit, is strictly smaller than 1.

By 3.44, the second term of the right hand side goes to 0. In fact:

$$\begin{aligned}&\limsup_{\delta\to\infty} \limsup_{t\to\infty} \mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\mathbb{I}_{\{\max \mathcal{E}_t > \delta\}}\right] \\ &\qquad \le \limsup_{\delta\to\infty} \limsup_{t\to\infty} \mathbb{P}[\max \mathcal{E}_t > \delta] = 0. \end{aligned} \tag{3.46}$$

Denote now by $\psi_t^\delta(\phi)$ the first term of the right hand side:

$$\psi_t^\delta(\phi) = \mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\mathbb{I}_{\{\max \mathcal{E}_t \le \delta\}}\right]. \tag{3.47}$$

Our aim is to show that the limit $\lim_{\delta\to\infty} \lim_{t\to\infty} \psi_t^\delta(\phi) = \psi(\phi)$ exists and it is strictly smaller than 1. Set $g_\delta(x) = e^{-\phi(x)}\mathbb{I}_{\{x \le \delta\}}$ and

$$u_\delta(t, x) = \mathbb{E}\left[\prod_{k \le N(t)} g_\delta(-x + X_k(t))\right]. \tag{3.48}$$

By Lemma 5.6, u_δ turns out to be solution of the *F-KPP* equation 3.9 with $u(0,x) = g_\delta(-x)$. Moreover, $g_\delta(-x) = 1$ for x large enough in the positive, $g_\delta(-x) = 0$ for x large enough in the negative. Hence conditions of Theorem 3.5 are satisfied. By Theorem 3.5,

$$u_\delta(t, x + m_t) = \mathbb{E}\left[\prod_{k=1}^{N(t)} g_\delta(-x + X_k(t) - m_t)\right] \tag{3.49}$$

converges as $t \to \infty$ uniformly in x. We have

$$\begin{aligned} \psi_t^\delta(\phi) &= \mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\mathbb{I}_{\{\max \mathcal{E}_t \leq \delta\}}\right] \\ &= \mathbb{E}\left[\prod_{k\leq N(t)} \exp\left\{-\phi(X_k(t) - m_t)\right\}\mathbb{I}_{\{X_k(t)-m_t\leq\delta\}}\right] \\ &= \mathbb{E}\left[\prod_{k\leq N(t)} g_\delta(X_k(t) - m_t)\right] = u_\delta(t, 0 + m_t). \end{aligned}$$

Hence $\lim_{t\to\infty} \psi_t^\delta(\phi) = \lim_{t\to\infty} u_\delta(t, 0 + m_t) = \psi^\delta(\phi)$ exists.
Moreover, the function $\delta \mapsto \psi^\delta(\phi)$ is increasing and bounded by construction and therefore $\lim_{\delta\to\infty} \psi^\delta(\phi)$ exists. Since the maximum is an atom of $\mathcal{E}_t$ and ϕ is non-negative, there is also the following bound:

$$\mathbb{E}\left[\exp\left\{-\int \phi d\mathcal{E}_t\right\}\mathbb{I}_{\{\max \mathcal{E}_t\leq\delta\}}\right] \leq \mathbb{E}\left[\exp\left\{-\phi(\max \mathcal{E}_t)\right\}\mathbb{I}_{\{\max \mathcal{E}_t\leq\delta\}}\right]. \tag{3.50}$$

The limit for $t \to \infty$ and $\delta \to \infty$ of the right-hand side exists and it is strictly smaller than 1, since the re-centered maximum converges in law to $w(x)$, by Theorem 3.5. Hence

$$\psi(\phi) = \lim_{\delta\to\infty} \lim_{t\to\infty} \psi_t^\delta(\phi) < 1. \tag{3.51}$$

□

3.3.2 The auxiliary process

At this point, we need to define an auxiliary process. Introduce a new probability space $(\Omega', \mathcal{F}', P)$ and denote by E the expectation with respect to P. Let $Z : \Omega' \to \mathbb{R}_+$ be a random variable with distribution as that of the limit of the derivative martingale 3.33. Recall that the law of the standard BBM is denoted by $\mathbb{P}$.

Let η be a Poisson point process on $(-\infty, 0)$, shifted by $\frac{1}{\sqrt{2}} \ln Z$, with intensity measure

$$\sqrt{\frac{2}{\pi}}(-x)e^{-\sqrt{2}x}dx, \tag{3.52}$$

and $(\eta_i, i \in \mathbb{N})$ its atoms. For each $i \in \mathbb{N}$, we consider independent BBMs on $(\Omega', \mathcal{F}', P)$ with drift $-\sqrt{2}$, namely $\{X_k^{(i)}(t) - \sqrt{2}t; k \leq N^{(i)}(t)\}$, where $N^{(i)}(t)$ is the number of particles of the BBM i at time t.

The auxiliary point process that we need is the superposition of these iid BBMs with drift and shifted by $\eta_i + \frac{1}{\sqrt{2}} \ln Z$:

$$\Pi_t = \sum_{i,k} \delta_{\frac{1}{\sqrt{2}} \ln Z + \eta_i + X_k^{(i)}(t) - \sqrt{2}t}. \tag{3.53}$$

Proposition 3.16. *Let Π_t^{ext} be the point process obtained by retaining from Π_t the maximal particles of the BBMs, namely*

$$\Pi_t^{ext} = \sum_{i} \delta_{\frac{1}{\sqrt{2}} \ln Z + \eta_i + \max_k X_k^{(i)}(t) - \sqrt{2}t}. \tag{3.54}$$

Then

$$\lim_{t\to\infty} \Pi_t^{ext} \overset{law}{=} PPP(cZ\sqrt{2}e^{-\sqrt{2}x}dx) \tag{3.55}$$

as a point process on $\mathbb{R}$. In particular, the maximum of the cluster-extrema has the same law of the limit law of the maximum of BBM.

Proof. Let $\eta = (\eta_i)$ be a Poisson process on $(-\infty, 0)$ with intensity measure $\sqrt{\frac{2}{\pi}} e^{-\sqrt{2}y}dy$ and let $M^{(i)}(t) = \max_k X_k^{(i)}(t)$. We want to show that

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}\left[\exp\left\{-\sum_i \phi(\eta_i + M^{(i)}(t) - \sqrt{2}t)\right\}\right] \\ &\quad = \exp\left\{-C\int_{\mathbb{R}} (1 - e^{-\phi(a)})\sqrt{2}e^{-\sqrt{2}a}da\right\}. \end{aligned} \tag{3.56}$$

Since η is a Poisson process and the $M^{(i)}$'s are iid, we obtain

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{-\sum_i \phi(\eta_i + M^{(i)}(t) - \sqrt{2}t)\right\}\right] \\ &= \exp\left\{-\int_{-\infty}^0 \mathbb{E}\left[1 - e^{-\phi(y+M_t-\sqrt{2}t)}\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy\right\}, \end{aligned} \tag{3.57}$$

where M_t is distributed as $M^{(i)}(t)$. If we set $h(x) = 1 - e^{-\phi(x)}$ and take the limit, we obtain the thesis applying Lemma 5.2. □

Theorem 3.17. *Let $\mathcal{E}_t$ be the extremal process defined as in 3.37. Then*

$$\lim_{t\to\infty} \mathcal{E}_t \overset{law}{=} \lim_{t\to\infty} \Pi_t. \tag{3.58}$$

Proof. From Proposition 3.16, we know that Π_t is a Poisson point process. Thus, we can use the form of the Laplace functional of a Poisson process:

$$\begin{aligned} & \mathbb{E} \left[\exp\{-\int \phi(x)\Pi_t(dx)\}\right] \\ = & \mathbb{E} \left[\exp\{ -\int_{-\infty}^{0} \left(1 - \mathbb{E}\left[\exp\{-\sum_{k\le N(t)} \phi(x + X_k(t) - \sqrt{2}t + \frac{1}{\sqrt{2}}\ln Z)\}\right]\right) \right. \\ & \left. \cdot \sqrt{\frac{2}{\pi}(-x)e^{-\sqrt{2}x}dx} \ \}\right] \end{aligned} \tag{3.59}$$

Setting

$$u(t,x) = 1 - \mathbb{E}\left[\exp\left\{-\sum_{k\le N(t)} \phi(-x + X_K(t))\right\}\right], \tag{3.60}$$

we can write 3.59 as

$$\mathbb{E}\left[\exp\left\{-\sqrt{\frac{2}{\pi}}\int_0^\infty u(t, x + \sqrt{2}t - \frac{1}{\sqrt{2}}\ln Z)xe^{\sqrt{2}x}dx\right\}\right].$$

By Lemma 5.8,

$$\begin{aligned} \lim_{t\to\infty} & \sqrt{\frac{2}{\pi}}\int_0^\infty u(t, x + \sqrt{2}t - \frac{1}{\sqrt{2}}\ln Z)xe^{\sqrt{2}x}dx \\ = & Z\sqrt{\frac{2}{\pi}}\lim_{t\to\infty}\int_0^\infty u(t, x + \sqrt{2}t)xe^{\sqrt{2}x}dx, \end{aligned}$$

and the limit exists and it is strictly positive by Lemma 5.9. This implies that the Laplace functionals of $\lim_{t\to\infty}\Pi_t$ and of the extremal process of the BBM are equal. □

3.3.3 Proof of the main theorem

Proof. We have to show that, given $\phi : \mathbb{R} \to \mathbb{R}_+$ continuous and with compact support, the Laplace functional $\psi_t(\phi)$ of the extremal process $\mathcal{E}_t$ of the BBM satisfies the following equivalence:

$$\lim_{t\to\infty} \psi_t(\phi) = \mathbb{E}\left[\exp\left\{-CZ\int_{\mathbb{R}} \mathbb{E}\left[1 - e^{-\int \phi(y+z)\mathcal{D}(dz)}\right]\sqrt{2}e^{-\sqrt{2}y}dy\right\}\right], \tag{3.61}$$

where $\mathcal{D}$ is the point process of Corollary 5.4.

By Theorem 3.17, it holds

$$\lim_{t\to\infty} \psi_t(\phi) = \lim_{t\to\infty} \mathbb{E}\left[\exp\left\{-\sum_{i,k} \phi(\eta_i + \frac{1}{\sqrt{2}}\ln Z + X_k^{(i)}(t) - \sqrt{2}t)\right\}\right]. \tag{3.62}$$

Since η is a Poisson point process, we can write

$$\begin{aligned} \lim_{t\to\infty} \quad & \mathbb{E}\left[\exp\left\{-\sum_{i,k} \phi(\eta_i + \frac{1}{\sqrt{2}}\ln Z + X_k^{(i)}(t) - \sqrt{2}t)\right\}\right] \\ = \quad & \mathbb{E}\,[\,\exp\{\,-Z\lim_{t\to\infty}\int_{-\infty}^{0} \mathbb{E}\left[1 - \exp\left\{-\int \phi(x+y)\bar{\mathcal{E}}_t(dx)\right\}\right] \cdot \\ & \sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy\ \}\ ]\,. \end{aligned} \tag{3.63}$$

Let now $\mathcal{D}_t$ be the gap process defined in 5.5. It holds

$$\begin{aligned} \lim_{t\to\infty} \quad & \int_{-\infty}^{0} \mathbb{E}\left[1 - \exp\{-\int \phi(x+y)\bar{\mathcal{E}}_t(dx)\}\right] \\ = \lim_{t\to\infty} \quad & \int_{-\infty}^{0} \mathbb{E}\left[f\left(\int \{T_{y+\max\bar{\mathcal{E}}_t}\phi(z)\}\mathcal{D}_t(dz)\right)\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy, \end{aligned}$$

where $T_x\phi(y) = \phi(y+x)$ and f is a function that is continuous, bounded on $[0,\infty)$ and such that $f(x) = 1 - e^{-x}$. Notice that $f(0) = 0$.

By Proposition 5.1, there exist A_1 and A_2 such that

$$\begin{aligned} \int_{-\infty}^{0} \quad & \mathbb{E}\left[f\left(\int \left\{T_{y+\max\bar{\mathcal{E}}_t}\phi(z)\right\}\right)\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy \\ = \quad & \Omega_t(A_1,A_2) + \int_{-A_2\sqrt{t}}^{-A_1\sqrt{t}} \mathbb{E}\left[f\left(\int \left\{T_{y+\max\bar{\mathcal{E}}_t}\phi(z)\right\}\right)\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy, \end{aligned} \tag{3.64}$$

where $\Omega_t(A_1, A_2)$ is the error term that satisfies

$$\lim_{A_1 \to 0, A_2 \to \infty} \sup_{t \geq t_0} \Omega_t(A_1, A_2) = 0. \tag{3.65}$$

Let now m_ϕ be the minimum of the support of ϕ. Notice that

$$f\left(\int \left\{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\right\}\right) = 0 \text{ when } y + \max \bar{\mathcal{E}}_t < m_\phi. \tag{3.66}$$

Moreover $\mathbb{P}\left[\left\{y + \max \bar{\mathcal{E}}_t = m_\phi\right\}\right] = 0$. Hence

$$\begin{aligned}
& \mathbb{E}\left[f\left(\int \left\{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\right\} \mathcal{D}_t(dz)\right)\right] \\
= \quad & \mathbb{E}\left[f\left(\int \left\{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\right\} \mathcal{D}_t(dz)\right) \mathbb{I}_{\left\{y+\max \bar{\mathcal{E}}_t > m_\phi\right\}}\right] \\
& \left[f\left(\int \left\{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\right\} \mathcal{D}_t(dz)\right) \Big| y + \max \bar{\mathcal{E}}_t > m_\phi\right] \mathbb{P}\left[y + \max \bar{\mathcal{E}}_t > m_\phi\right].
\end{aligned} \tag{3.67}$$

By Corollary 5.4, the conditional law of $\left\{(\mathcal{D}_t, y + \max \bar{\mathcal{E}}_t) \Big| y + \max \bar{\mathcal{E}}_t > m_\phi\right\}$ exists in the limit and the convergence is uniformly in $y \in [-A_1\sqrt{t}, -A_2\sqrt{t}]$. The convergence applies to the random variable $\int \left\{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\right\} \mathcal{D}_t(dz)$, by Lemma 5.5. Therefore

$$\begin{aligned}
& \lim_{t \to \infty} \mathbb{E}\left[f\left(\int \left\{T_{y+\max \bar{\mathcal{E}}_t} \phi(z)\right\} \mathcal{D}_z(dz)\right) \Big| y + \max \bar{\mathcal{E}}_t > m_\phi\right] \\
& = \int_{m_\phi}^{\infty} \mathbb{E}\left[f\left(\int (T_y \phi(z)) \mathcal{D}(dz)\right)\right] \frac{\sqrt{2} e^{-\sqrt{2}y} dy}{e^{-\sqrt{2}m_\phi}}.
\end{aligned} \tag{3.68}$$

By Proposition 5.1 and Lemma 5.2, it holds that

$$\int_{-A_2\sqrt{t}}^{-A_1\sqrt{t}} \mathbb{P}\left[y + \max \bar{\mathcal{E}}_t > m_\phi\right] \sqrt{\frac{2}{\pi}} (-y) e^{-\sqrt{2}y} dy = C e^{-\sqrt{2}m_\phi} + \Omega_t(A_1, A_2). \tag{3.69}$$

Combining equations 3.67, 3.68, 3.69, we obtain that 3.63 converges to

$$\mathbb{E}\left[\exp\left(-CZ \int_{\mathbb{R}} \mathbb{E}\left[1 - e^{-\int \phi(y+z)\mathcal{D}(dz)}\right] \sqrt{2} e^{-\sqrt{2}y} dy\right)\right], \tag{3.70}$$

that is the limiting Laplace transform of the extremal process of the BBM by 3.61.

□

3.4 Related models and applications

Starting from the branching random walks and the branching Brownian motions we defined, it is possible to study models that are more similar to the systems we find in real life. In a population, we do not have an exponential growth of the individuals: by *selection* we mean the process of killing particles, which can be interpreted as the effect of natural selection. We will now briefly introduce two selection mechanisms: the BBM with absorption and the N-BBM.

3.4.1 Branching Brownian motion with absorption

Let $(X(t), t \geq 0)$ be a usual branching Brownian motion with drift μ started from $x > 0$ and let $\mathcal{N}(t)$ be the set of particles alive at time t for this full BBM. We want to kill particles when they hit the origin. Define

$$\mathcal{N}_{\text{abs}}(t) = \left\{ u \in \mathcal{N}_t : \inf_{s \leq t} X_u(s) > 0 \right\}. \tag{3.71}$$

Then $(\{X_u(t), u \in \mathcal{N}_{\text{abs}}(t)\}, t \geq 0)$ is the branching Brownian motion with absorption: we just kept all the particles whose path has not touched 0. Obviously we could also kill particles when they hit a generic line.

The first question we may ask about is whether this process survives or not. Kesten in 1978 in *Branching Brownian motion with absorption* proved the following result:

Theorem 3.18. *Let $\xi = \inf\{t \geq 0 : \mathcal{N}_{abs}(t) = \emptyset\}$ be the extinction time of the BBM with absorption. Then $\mathbb{P}[\xi < \infty] = 1$ if and only if $\mu \leq -\sqrt{2}$.*

From this first result, mathematicians started studying the process of absorption, finding a criterion for almost sure extinction also for branching random walks. This process has recently aroused interest because of David Aldous' conjecture. He conjectured that $\mathbb{E}[Z \ln Z] = \infty$, where Z is a random variable that denotes the number of particles with critical drift that cross the origin for the first time. Aldous and Pemantle provided an incomplete proof, while Aidekon and Maillard tried to refine the result.

3.4.2 N-BBM

We will first introduce the N-BRW. We have a population of size N with asexual reproduction. Each individual $i \leq N$ is completely characterized by a number $x_i \in \mathbb{R}$ which represents its selection advantage, that we can interpret as his fitness. At a given time the population is a collection of N points on the real line. Time is discrete and at each generation the whole population is entirely renewed according to this mechanism:

(i) Reproduction-mutation: each individual has k offsprings (after this step we momentarily have kN particles) and the relative positions with respect to their parents are given by iid copies of a certain displacement law ρ.

(ii) Selection: we just keep the N rightmost particles among the kN just created.

Brunet and Derrida found an interesting conjecture on the speed of the system. Let us denote $X_1(t) \leq X_2(t) \leq \ldots \leq X_N(t)$ the positions of the particles at time t. We have that

$$\lim_{t\to\infty} \frac{X_1(t)}{t} = \lim_{t\to\infty} \frac{X_N(t)}{t} = v_N \tag{3.72}$$

for a certain velocity v_N, which depends on N.

Conjecture 3.19. *It holds that*

$$v_N \to v_\infty < \infty \tag{3.73}$$

as $N \to \infty$ *and*

$$v_\infty - v_N = \frac{c}{2(\ln N)^2} - c\frac{3\ln\ln N}{(\ln N)^3} + \cdots \tag{3.74}$$

where c *is a constant which is explicitly determined in terms of the displacement law.*

The first order of this correction has been proved by Bérard and Gouéré in *Brunet-Derrida behaviour of branching-selection particle systems on the line.*
It should be possible to extend this result also to the case of the N-BBM.

In the continuous case, we still have a cloud of N particles. The evolution of the population is following the same steps as the discrete case:

(i) Reproduction-mutation: each particle moves in $\mathbb{R}$ according to an independent Brownian motion and branches at rate β into k new particles, with probabilities p_k which then start to follow the same behaviour and so on. At this stage we have the general branching Brownian motion.

(ii) Selection: at each branching event, we kill the leftmost particles to keep the population size constant.

We are going to state the Brunet and Derrida's other conjecture, but first we have to briefly introduce Λ-coalescents.
Suppose we have a countable population of individuals $i = 1, 2, \ldots$ for which you can follow their ancestral lines of descendance. At the time of the most recent common ancestor of two individuals, i and j for instance, their lineage coalesce. So if we look at the k first individuals, their ancestral lineages trace a tree with k leaves. The coalescent is a Markov process which describes how the lineages merge when we go backward in time.

Definition 3.20. In a ***Λ-coalescent***, there is a point process (t_i, p_i) in $\mathbb{R}_+ \times [0, 1]$ with intensity $dt \otimes p^{-2}\Lambda(dp)$, where Λ is a finite measure on $[0, 1]$. At time t_i of an atom, we select by independent coin flipping a proportion p of the active lineages and merge them in a single one.

In particular, a ***Bolthausen-Snitzman coalescent*** is the Λ-coalescent we obtain when $\Lambda(dp) = dp\mathbb{I}_{\{p\in[0,1]\}}$.

Conjecture 3.21. *Let T_p be the time of the most recent common ancestor of p individuals, the statistics $\frac{\mathbb{E}[T_p]}{\mathbb{E}[T_2]}$ converge to the values which are the same as those obtained for the Bolthausen-Snitzman coalescent. The sequence $\frac{\mathbb{E}[T_p]}{\mathbb{E}[T_2]}$ characterizes the distribution Λ.*

This conjecture can be reformulated by saying that on a timescale of order $(\log N)^3$, the genealogy of a population converges to a Bolthausen-Snitzman coalescent.

Despite the simplicity of the N-BBM, it is very difficult to analyze it rigorously, because of the strong interaction between the particles, the impossibility to describe it exactly through differential equations and the fact that the shifts in the position of the system do not occur instantaneously but gradually over the timescale $\log^2 N$. This is the reason why several results have been obtained by an approximation of the N-BRW and the N-BBM with absorption at a linear barrier.

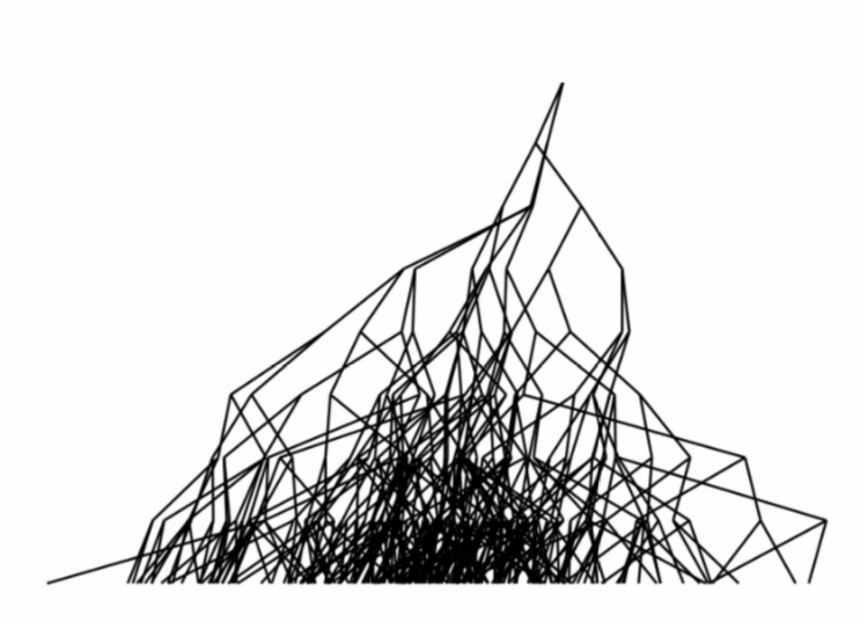


Figure 3.2: A branching Brownian motion

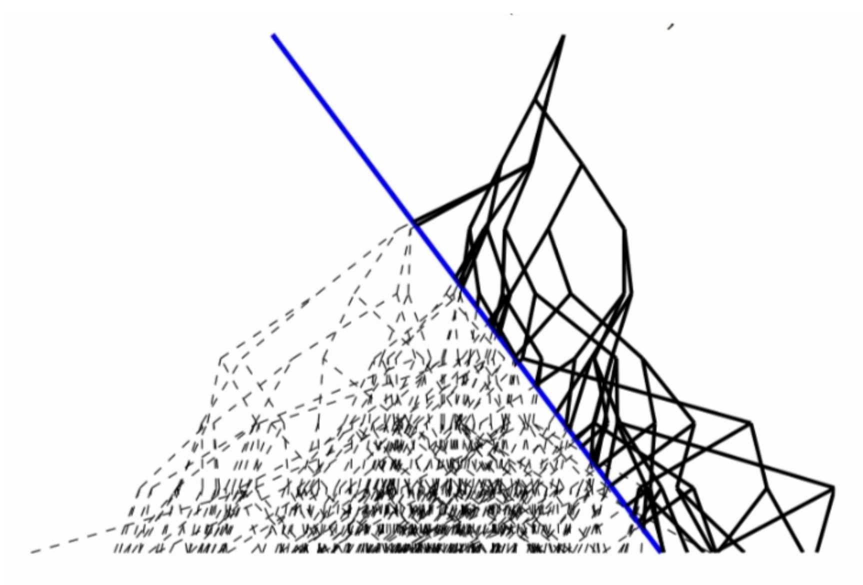


Figure 3.3: A branching Brownian motion with absorption

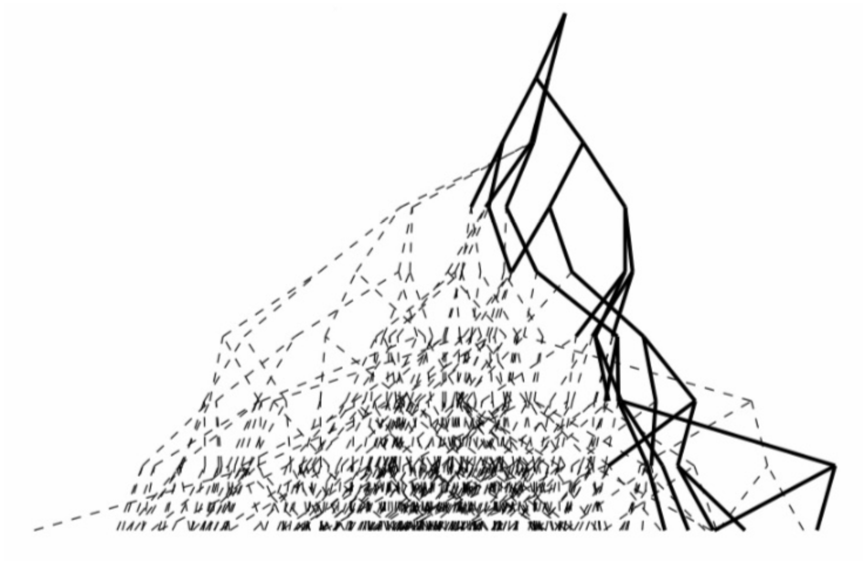


Figure 3.4: A N-BBM, where $N = 6$

Chapter 4

Localization of paths

We can say something more about extremal particles and their genealogical distance. Visualizing the paths that the particles perform during their lifetime would be helpful and that is why we are going to try to draw a picture of such paths.

4.1 Upper envelope

As a first step, we prove that extremal particles cannot fluctuate too wildly in the upward direction. Let $\gamma > 0$ and set

$$f_{t,\gamma}(s) = \begin{cases} s^{\gamma} & \text{if } 0 \leq s \leq \frac{t}{2} \\ (t-s)^{\gamma} & \text{if } \frac{t}{2} \leq s \leq t \end{cases} \tag{4.1}$$

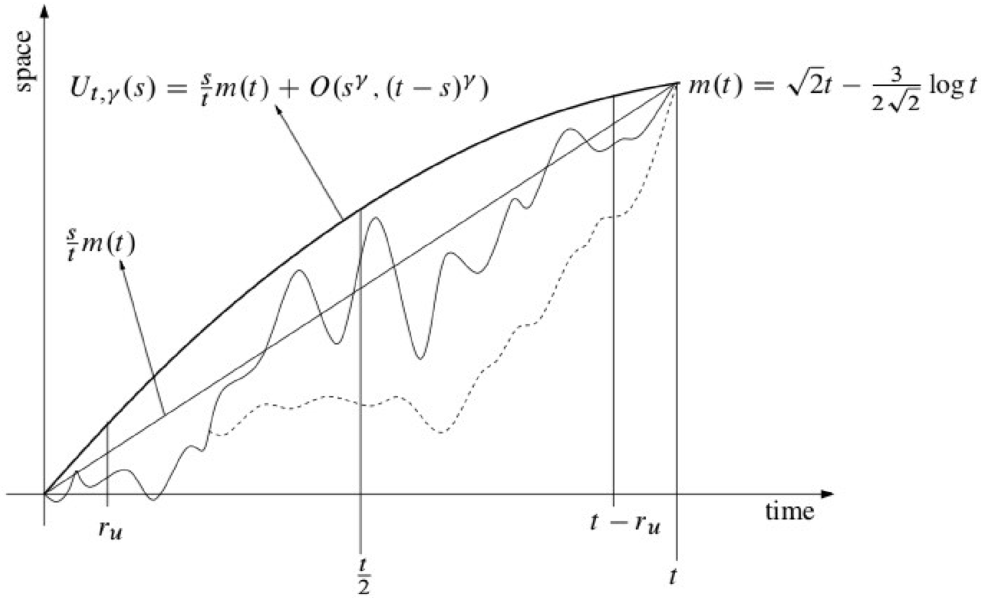
Define the *upper envelope* $U_{t,\gamma}$ as

$$U_{t,\gamma}(s) = \frac{s}{t} m_t + f_{t,\gamma}(s), \tag{4.2}$$

where we recall that $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + o(1)$.

Theorem 4.1. *Let $0 < \gamma < \frac{1}{2}$. Let $y \in \mathbb{R}$ and $\epsilon > 0$ be fixed. There exists $r_u = r_u(\gamma, y, \epsilon)$ such that for $r \geq r_u$ and for any $t > 3r$,*

$$\mathbb{P}[\exists k \leq N(t) : X_k(s) > y + U_{t,\gamma}(s) \text{ for some } s \in [r, t-r]] < \epsilon. \tag{4.3}$$



Proof. To prove this theorem, we are going to find an upper bound to the probability

$$\mathbb{P}[\exists k \leq N(t) : X_k(s) > y + m_s + f_{t,\gamma}(s) \text{ for some } s \in [r, t-r]]. \tag{4.4}$$

An upper bound for the probability we just introduced implies an upper bound for the probability in 4.3: in fact, for $s > e$, we have that $\frac{\ln s}{s} > \frac{\ln t}{t}$ and the function $\frac{\ln x}{x}$ is decreasing for $x > e$. Hence $\frac{s}{t}m_t > m_s$ and so

$$U_{t,\gamma}(s) = \frac{s}{t}m_t + f_{t,\gamma}(s) > m_s + f_{t,\gamma}(s) \tag{4.5}$$

which implies

$$\begin{aligned} & \mathbb{P}[\exists k \leq N(t) : X_k(s) > y + m_s + f_{t,\gamma}(s) \text{ for some } s \in [r, t-r]] \\ \leq \quad & \mathbb{P}[\exists k \leq N(t) : X_k(s) > y + m_s + f_{t,\gamma}(s) \text{ for some } s \in [r, t-r]]. \end{aligned} \tag{4.6}$$

We first prove that the maximum of the process at integer times doesn't cross the upper envelope and then we will extend the result to all times.

Denote by $\lceil s \rceil$ the smallest integer greater or equal to s. Our current aim is to prove the following result:

Lemma 4.2. *Let $0 < \gamma < \frac{1}{2}$. For any $y \in \mathbb{R}$ and $\epsilon > 0$, there exists $r'(\gamma, y, \epsilon)$ such that for*

$r > r'(\gamma, y, \epsilon)$ and $t > 3r$, it holds that

$$\mathbb{P}[\exists s \in [r, t-r] : \max_{k \leq N(s)} X_k(\lceil s \rceil) \geq y + m(\lceil s \rceil) + f_{t,\gamma}(\lceil s \rceil)] < \epsilon. \tag{4.7}$$

We can write

$$\begin{aligned} & \left\{ \exists s \in [r, t-r] : \max_{k \leq N(s)} X_k(\lceil s \rceil) \geq y + m(\lceil s \rceil) + f_{t,\gamma}(\lceil s \rceil) \right\} \qquad (4.8) \\ = \quad & \bigcup_j \left\{ \max_{k \leq N(j)} X_k(j) \geq y + m(j) + f_{t,\gamma}(j) \right\}, \end{aligned}$$

where $j = \lceil r \rceil, \lceil r \rceil + 1, \lceil t - r \rceil$. Choose r large enough so that $y + f_{t,\gamma}(r) > 0$. We now use the following estimate on the right tail of the maximal displacement, obtained by Bramson in [5]:

$$\mathbb{P}[\max_{k \leq N(t)} X_k(t) \geq m_t + Y] \leq \kappa (1 + Y)^2 e^{-\sqrt{2} Y}, \tag{4.9}$$

with $0 < Y < \sqrt{t}$ and $\kappa > 0$ a numerical constant.
We apply the previous estimate on the event 4.8, choosing $t = j$ and $Y = y + f_{t,\gamma}(j)$ and using the symmetry of the curve $f_{t,\gamma}$. We get

$$\begin{aligned} & \mathbb{P}[\exists s \in [r, t-r] : \max_{k \leq N(s)} X_k(\lceil s \rceil) \geq y + m(\lceil s \rceil) + f_{t,\gamma}(\lceil s \rceil)] \\ & \qquad \leq 2\kappa \sum_{j=\lceil r \rceil}^{\lceil \frac{t}{2} \rceil + 1} (1 + (j^\gamma + y))^2 \exp\left\{ -\sqrt{2}(j^\gamma + y) \right\} \quad . \end{aligned} \tag{4.10}$$

The right-hand side is summable, hence the probability can be made arbitrarily small by taking the limits $t \to \infty$ and then $r \to \infty$ and we obtain the claim.
We now want to extend the result to all $s \in [r, t-r]$. Observe that we have this dichotomy:

$$\max_{k \leq N(\lceil s \rceil)} X_k(\lceil s \rceil) < m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}(\lceil s \rceil)}, \tag{4.11}$$

or

$$\max_{k \leq N(\lceil s \rceil)} X_k(\lceil s \rceil) \geq m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}(\lceil s \rceil)}. \tag{4.12}$$

It follows that

$$\begin{aligned}
&\mathbb{P}\left[\exists s \in [r, t-r] : \max_{k \le N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s)\right] \\
\le\ &\mathbb{P}\left[\exists s \in [r, t-r] : \max_{k \le N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s)\ ,\right. \\
&\left. \max_{k \le N(\lceil s \rceil)} X_k(\lceil s \rceil) \ge m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}(\lceil s \rceil)}\right] \\
+&\mathbb{P}\left[\exists s \in [r, t-r] : \max_{k \le N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s)\ ,\right. \\
&\left. \max_{k \le N(\lceil s \rceil)} X_k(\lceil s \rceil) < m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s \rceil)\right].
\end{aligned} \tag{4.13}$$

We have that

$$\begin{aligned}
&\mathbb{P}\left[\exists s \in [r, t-r] : \max_{k \le N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s)\ ,\right. \\
&\left. \max_{k \le N(\lceil s \rceil)} X_k(\lceil s \rceil) \ge m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}(\lceil s \rceil)}\right] \\
\le\ &\mathbb{P}[\exists s \in [r, t-r] : \max_{k \le N(\lceil s \rceil)} X_k(\lceil s \rceil) \ge m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}(\lceil s \rceil)}] \qquad (4.14) \\
&\le \frac{\epsilon}{2},
\end{aligned}$$

where the last inequality holds from 4.2, by choosing $r > r'(y, \frac{\gamma}{2}, \frac{\epsilon}{2})$. So it remains to bound

$$\begin{aligned}
&\mathbb{P}\left[\exists s \in [r, t-r] : \max_{k \le N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s)\ ,\right. \\
&\left. \max_{k \le N(\lceil s \rceil)} X_k(\lceil s \rceil) < m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s \rceil)\right].
\end{aligned} \tag{4.15}$$

We define the stopping time $\mathcal{S}$, where we choose $r = 1$ to avoid technicalities:

$$\mathcal{S} = \inf\left\{s \in [1, t-1] : \max_{k \le N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s)\right\}. \tag{4.16}$$

We can rewrite the probability 4.15, conditioning on $\mathcal{S}$:

$$\mathbb{P}\Bigg[\exists s \in [r, t-r] : \max_{k \leq N(s)} X_k(s) > y + m_s + f_{t,\gamma}(s) \ , \\ \max_{k \leq N(\lceil s \rceil)} X_k(\lceil s \rceil) < m(\lceil s \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s \rceil)\Bigg] \tag{4.17}$$
$$= \int_r^{t-r} \mathbb{P}[\max_{k \leq N(\lceil s' \rceil)} X_k(\lceil s' \rceil) < y + m(\lceil s' \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) \mid \mathcal{S} = s'] \mathbb{P}[\mathcal{S} \in ds'].$$

Without loss of generality, we can assume $r > 2$. We have that $r > \lceil r \rceil - 1$ and $t - r < \lceil t \rceil - \lceil r \rceil + 1$. Hence

$$\int_r^{t-r} \mathbb{P}[\max_{k \leq N(\lceil s' \rceil)} X_k(\lceil s' \rceil) < y + m(\lceil s' \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) \mid \mathcal{S} = s'] \mathbb{P}[\mathcal{S} \in ds'] \\ < \sum_{j=\lceil r \rceil}^{\lceil t \rceil - \lceil r \rceil + 1} \int_j^{j+1} \mathbb{P}\left[\max_{k \leq N(\lceil s' \rceil)} X_k(\lceil s' \rceil) < y + m(\lceil s' \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) \mid \mathcal{S} = s'\right] \\ \times \mathbb{P}[\mathcal{S} \in ds']. \tag{4.18}$$

This means that it remains to show that

$$\mathbb{P}[\max_{k \leq N(\lceil s' \rceil)} X_k(\lceil s' \rceil) < y + m(\lceil s' \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) \mid \mathcal{S} = s'] \to 0, \tag{4.19}$$

uniformly in s', as $r \to \infty$. By the definition of $\mathcal{S}$, this probability is bounded by the probability that the offspring at time $\lceil s' \rceil$ of the maximum at time s', make a jump smaller than

$$m(\lceil s' \rceil) - m(s') + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) - f_{t,\gamma}(s'). \tag{4.20}$$

By the Markov property of the BBM, this is exactly

$$\mathbb{P}[\max_{k \leq N(\lceil s' \rceil - s')} X_k(\lceil s' \rceil - s') < m(\lceil s' \rceil) - m(s') + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) - f_{t,\gamma}(s')]. \tag{4.21}$$

Hence

$$\mathbb{P}[\max_{k \leq N(\lceil s' \rceil)} X_k(\lceil s' \rceil) < y + m(\lceil s' \rceil) + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) \mid \mathcal{S} = s'] \tag{4.22}$$
$$\leq \mathbb{P}[\max_{k \leq N(\lceil s' \rceil - s')} X_k(\lceil s' \rceil - s') < m(\lceil s' \rceil) - m(s') + f_{t,\frac{\gamma}{2}}(\lceil s' \rceil) - f_{t,\gamma}(s')] \quad .$$

Recall that the expected number of offsprings at time t is e^t and let X be a standard Brownian motion. By Markov's inequality we have that

$$\mathbb{P}\left[\max_{k\leq N(\lceil s'\rceil - s')} X_k(\lceil s'\rceil - s') < m(\lceil s'\rceil) - m(s') + f_{t,\frac{\gamma}{2}}(\lceil s'\rceil) - f_{t,\gamma}(s')\right]$$
$$\leq e^{\lceil s'\rceil - s'}\mathbb{P}[X(\lceil s'\rceil - s') < m(\lceil s'\rceil) - m(s') + f_{t,\frac{\gamma}{2}}(\lceil s'\rceil) - f_{t,\gamma}(s')]. \tag{4.23}$$

Observe that, since $\lceil s'\rceil - s' < 1$,

$$e^{\lceil s'\rceil - s'} < e \tag{4.24}$$

and, by the definition of m_t,

$$m(\lceil s'\rceil) - m(s') < \sqrt{2}. \tag{4.25}$$

Moreover, from the definition of $f_{t,\gamma}$ 4.1, we get that, for $r \leq s' \leq \frac{t}{2}$,

$$f_{t,\frac{\gamma}{2}}(\lceil s'\rceil) - f_{t,\gamma}(s') = -s'^{\gamma}\left(1 - \frac{\lceil s'\rceil^{\frac{\gamma}{2}}}{s'^{\gamma}}\right) \leq -\frac{1}{2}r^{\gamma}, \tag{4.26}$$

where we choose r large enough to get the factor $\frac{1}{2}$. On the other hand, for $\frac{t}{2} \leq s' \leq t - r$, it holds

$$f_{t,\frac{\gamma}{2}}(\lceil s'\rceil) - f_{t,\gamma}(s') = -(t - s')^{\gamma}\left(1 - \frac{(t - \lceil s'\rceil)^{\frac{\gamma}{2}}}{(t - s')^{\gamma}}\right) \leq -\frac{1}{2}r^{\gamma}. \tag{4.27}$$

Hence, by 4.24, 4.25, 4.26, 4.27, we obtain

$$e^{\lceil s'\rceil - s'}\mathbb{P}[X(\lceil s'\rceil - s') < m(\lceil s'\rceil) - m(s') + f_{t,\frac{\gamma}{2}}(\lceil s'\rceil) - f_{t,\gamma}(s')]$$
$$\leq e\mathbb{P}\left[X(\lceil s'\rceil - s') < \sqrt{2} - \frac{1}{2}r^{\gamma}\right] \quad . \tag{4.28}$$

This probability tends to 0 as $r \to \infty$,uniformly in s', since $X(\lceil s'\rceil - s')$ is by definition a Gaussian variable of variance $\lceil s'\rceil - s' \leq 1$.
Thus, by 4.22, we obtain 4.19. □

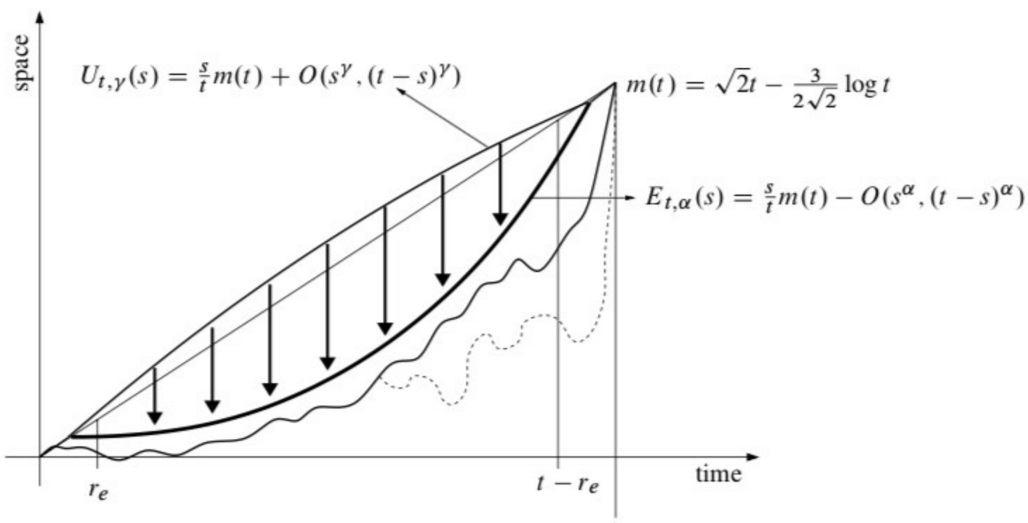
This theorem tells us that almost all the paths of the extremal particles never cross the upper envelope, but due to the strong fluctuations of the unconstrained paths, particles that at some point are close to the line $s \mapsto \frac{s}{t}m_t$ have plenty of chances to hit the upper envelope in the remaining time. We are going to state a theorem that ensures that the paths lie well below the interpolating line for most of the time: the upper envelope can be replaced by a lower entropic envelope, under which paths of extremal particles lie with high probability.

Let f be as in 4.1 and $\alpha > 0$. Define the *entropic envelope* $E_{t,\alpha}(s)$ as

$$E_{t,\alpha}(s) = \frac{s}{t} m_t - f_{t,\alpha}(s). \tag{4.29}$$

Theorem 4.3. *Let $D \subset \mathbb{R}$ a compact set and $0 < \alpha < \frac{1}{2}$. Set $\bar{D} = \sup\{x \in D\}$. For any $\epsilon > 0$, there exists $r_e = r_e(\alpha, D, \epsilon)$ such that for $r \geq r_e$ and $t > 3r$,*

$$\mathbb{P}[\exists k \leq N(t) : X_k(t) \in m_t + D, \text{ but } \exists s \in [r, t-r] : X_k(s) \geq \bar{D} + E_{t,\alpha}(s)] < \epsilon. \tag{4.30}$$



Proof. In order to prove this theorem, we start by choosing $0 < \gamma < \alpha < \frac{1}{2}$. Denote by $\underline{D} = \inf\{x \in D\}$. Taking $y = \underline{D}$ in Theorem 4.1, we know that paths of extremal particles must remain below the upper envelope $U_{t,\gamma}$ for most of the time. This means that it suffices to show that

$$\begin{aligned} \mathbb{P}\big[\exists k \leq N(t) : X_k(t) \in m_t + D, X_k(s) \leq \underline{D} + U_{t,\gamma}(s) \forall s \in [r, t-r], \\ \text{but } \exists s \in [r, t-r] : X_k(s) \geq \bar{D} + E_{t,\alpha}(s)\big] \to 0, \end{aligned} \tag{4.31}$$

as $r \to \infty$, uniformly in $t > 3r$, where we recall that $\bar{D} = \sup\{x \in D\}$. Applying Markov's inequality and observing that $\mathbb{E}[N(t)] = e^t$, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\exists k \le N(t) : X_k(t) \in m_t + D, X_k(s) \le \underline{D} + U_{t,\gamma}(s) \forall s \in [r, t-r],\right. \\
&\qquad\qquad \left.\text{but } \exists s \in [r, t-r] : X_k(s) \ge \bar{D} + E_{t,\alpha}(s)\right] \\
&\le e^t \mathbb{P}\left[X(t) \in m_t + D, X(s) \le \underline{D} + U_{t,\gamma}(s) \forall s \in [r, t-r],\right. \\
&\qquad\qquad \left.\text{but} \exists s \in [r, t-r] : X(s) \ge \bar{D} + E_{t,\alpha}(s)\right] \\
&= e^t \mathbb{P}\left[X(t) \in m_t + D, X(s) \le \underline{D} + \frac{s}{t} m_t + f_{t,\gamma}(s) \forall s \in [r, t-r],\right. \\
&\qquad\qquad \left.\text{but } \exists s \in [r, t-r] : X(s) \ge \bar{D} + \frac{s}{t} m_t - f_{t,\alpha}(s)\right], \qquad (4.32)
\end{aligned}$$

where we used the definitions of $U_{t,\gamma}$ and $E_{t,\alpha}$.

Consider now the event $\{X(t) \in m_t + D\}$. We can rewrite it as $\{X(t) \in m_t + D\} = \{X(t) - \bar{D} \le m_t \le X(t) - \underline{D}\}$. With this observation we can replace the condition on the paths in the above probability:

$$X(s) \le \underline{D} + \frac{s}{t} m_t + f_{t,\gamma}(s) \forall s \in [r, t-r] \qquad (4.33)$$

can be written as

$$\begin{aligned}
X(s) &\le \underline{D} + \frac{s}{t}(X(t) - \underline{D}) + f_{t,\gamma}(s) \\
&= \bar{D}\frac{t-s}{t} + \frac{s}{t} X(t) + f_{t,\gamma}(s) \forall s \in [r, t-r] \qquad (4.34)
\end{aligned}$$

and

$$\exists s \in [r, t-r] : X(s) \ge \bar{D} + \frac{s}{t} X(t) - f_{t,\alpha}(s) \qquad (4.35)$$

is the same as saying that

$$\begin{aligned}
\exists s \in [r, t-r] : X(s) &\ge \bar{D} + \frac{s}{t}(X(t) - \bar{D}) - f_{t,\alpha}(s) \\
&= \bar{D}\frac{t-s}{t} + \frac{s}{t} X(t) - f_{t,\alpha}(s). \qquad (4.36)
\end{aligned}$$

Combining 4.34 and 4.36 and recalling that $X(t)$ is independent of the Brownian bridge $\xi_t(s), 0 \le$

$s \leq t$, we get

$$
\begin{aligned}
e^t \mathbb{P}\Big[X(t) \in m_t + D, X(s) \leq \underline{D} + \frac{s}{t} m_t + f_{t,\gamma}(s) \forall s \in [r, t-r],\\
\text{but } \exists s \in [r, t-r] : X(s) \geq \bar{D} + \frac{s}{t} m_t - f_{t,\alpha}(s)\Big],\\
\leq e^t \mathbb{P}\Big[X(t) \in m_t + D, X(s) - \frac{s}{t} X(t) \leq \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s) \forall s \in [r, t-r]\\
\text{but } \exists s \in [r, t-r] : X(s) - \frac{s}{t} X(t) \geq \bar{D}\frac{t-s}{t} - f_{t,\alpha}(s)\Big]\\
= e^t \mathbb{P}\left[X(t) \in m_t + D\right]\\
\times \mathbb{P}\Big[\xi_t(s) \leq \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s) \forall s \in [r, t-r]\\
\text{but } \exists s \in [r, t-r] : \xi_t(s) \geq \bar{D}\frac{t-s}{t} - f_{t,\alpha}\Big].
\end{aligned}
\tag{4.37}
$$

Using the same tail estimation we applied in the proof of 4.1 ([5]), we observe that

$$
e^t \mathbb{P}\left[X(t) \in D + m_t\right] \leq \kappa t \int_D \exp\left\{\sqrt{2} x\right\} dx \tag{4.38}
$$

for some $\kappa > 0$ and $t \geq 2$. Hence, by the bounds 4.32, 4.37, 4.38, the claim of the Theorem will follow if we show that

$$
\begin{aligned}
t\mathbb{P}\Big[\xi_t(s) \leq \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s) \forall s \in [r, t-r],\\
\text{but } \exists s \in [r, t-r] : \xi_t(s) \geq \bar{D}\frac{t-s}{t} - f_{t,\alpha}(s)\Big] \to 0
\end{aligned}
\tag{4.39}
$$

as $r \to \infty$ uniformly in $t > 3r$, that is what we are going to prove now. Observe that, by definition of complement of a set,

$$
\begin{aligned}
\left\{\exists s \in [r, t-r] : \xi_t(s) \geq \bar{D}\frac{t-s}{t} - f_{t,\alpha}(s)\right\}^C\\
\subseteq \left\{\xi_t(s) \leq \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s) \forall s \in [r, t-r]\right\}.
\end{aligned}
\tag{4.40}
$$

Hence

$$\begin{aligned} t\mathbb{P}\Big[\xi_t(s) \le \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s)\forall s \in [r, t-r], \\ \text{but } \exists s \in [r, t-r] : \xi_t(s) \ge \bar{D}\frac{t-s}{t} - f_{t,\alpha}(s)\Big] \\ = t\Big(\mathbb{P}\Big[\xi_t(s) \le \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s)\forall s \in [r, t-r]\Big] \\ -\mathbb{P}\Big[\xi_t(s) \le \bar{D}\frac{t-s}{t} - f_{t,\alpha}(s)\forall s \in [r, t-r]\Big]\Big). \end{aligned} \tag{4.41}$$

We now need to define the following functions:

$$f(s) = \bar{D}\frac{t-s}{t}, \tag{4.42}$$

$$F(s) = f(s) + f_{t,\gamma}(s), \tag{4.43}$$

$$\bar{F}(s) = f(s) + f_{t,\alpha}(s), \tag{4.44}$$

$$\underline{F}(s) = f(s) - f_{t,\alpha}(s). \tag{4.45}$$

Using these definition and the same notation as in Subsection 5.1, we rewrite 4.41:

$$\begin{aligned} t\Big(\mathbb{P}\Big[\xi_t(s) \le \bar{D}\frac{t-s}{t} + f_{t,\gamma}(s)\forall s \in [r, t-r]\Big] \\ -\mathbb{P}\Big[\xi_t(s) \le \bar{D}\frac{t-s}{t} - f_{t,\alpha}(s)\forall s \in [r, t-r]\Big]\Big) \\ = t\left(P^0\left[B^F[r, t-r]\right] - P^0\left[B^{\underline{F}}[r, t-r]\right]\right) \\ = tP^0\left[B^0[r, t-r]\right]\frac{P^0\left[B^F[r, t-r]\right]}{P^0\left[B^0[r, t-r]\right]}\left(1 - \frac{P^0\left[B^{\underline{F}}[r, t-r]\right]}{P^0\left[B^F[r, t-r]\right]}\right). \end{aligned} \tag{4.46}$$

By the definitions 4.43, 4.44, 4.45, it is clear that $\underline{F} \le F \le \bar{F}$. Moreover, we can choose r large enough to get $\underline{F} \le 0 \le F$ on the interval $[r, t-r]$. By Lemma 5.12, we obtain

$$\begin{aligned} P^0\left[B^{\underline{F}}[r, t-r]\right] \le P^0\left[B^0[r, t-r]\right] &\le P^0\left[B^F[r, t-r]\right] \\ &\le P^0\left[B^{\bar{F}}[r, t-r]\right]. \end{aligned} \tag{4.47}$$

Hence,

$$\frac{P^0\left[B^{\underline{F}}[r,t-r]\right]}{P^0\left[B^{\bar{F}}[r,t-r]\right]} \leq \frac{P^0\left[B^{\underline{F}}[r,t-r]\right]}{P^0\left[B^{F}[r,t-r]\right]} \leq 1 \quad \leq \quad \frac{P^0\left[B^{F}[r,t-r]\right]}{P^0\left[B^{0}[r,t-r]\right]}$$
$$\leq \quad \frac{P^0\left[B^{\bar{F}}[r,t-r]\right]}{P^0\left[B^{\underline{F}}[r,t-r]\right]}. \tag{4.48}$$

Thus, by the bounds we just showed,

$$tP^0\left[B^0[r,t-r]\right]\frac{P^0\left[B^{F}[r,t-r]\right]}{P^0\left[B^{0}[r,t-r]\right]}\left(1-\frac{P^0\left[B^{\underline{F}}[r,t-r]\right]}{P^0\left[B^{F}[r,t-r]\right]}\right)$$
$$\leq tP^0\left[B^0[r,t-r]\right]\frac{P^0\left[B^{\bar{F}}[r,t-r]\right]}{P^0\left[B^{\underline{F}}[r,t-r]\right]}\left(1-\frac{P^0\left[B^{\underline{F}}[r,t-r]\right]}{P^0\left[B^{\bar{F}}[r,t-r]\right]}\right). \tag{4.49}$$

Moreover, the Brownian bridge is symmetric around the x axis, and so we have

$$\frac{P^0\left[B^{\underline{F}}[r,t-r]\right]}{P^0\left[B^{\bar{F}}[r,t-r]\right]} = \frac{P^0\left[B_{-\underline{F}}[r,t-r]\right]}{P^0\left[B_{-\bar{F}}[r,t-r]\right]}$$
$$= \frac{\mathbb{P}\left[\xi_t(s) > -f(s)+f_{t,\alpha}(s), r \leq s \leq t-r\right]}{\mathbb{P}\left[\xi_t(s) > -f(s)-f_{t,\alpha}(s), r \leq s \leq t-r\right]}, \tag{4.50}$$

where in the second row we just used the definition of $P^0\left[B\right]$ as in Subsection 5.1. Recall that our aim is to show that 4.39 converges to 0 as $r \to \infty$ uniformly in t. By the previous observations this will follow if we show that 4.49 converges to 0 as $r \to \infty$ uniformly in t.

Observe that Lemma 5.14 implies

$$\lim_{r\to\infty}\lim_{t\to\infty} tP^0\left[B^0[r,t-r]\right] = +\infty. \tag{4.51}$$

Hence we need to find uniform bounds on 4.50 to compensate 4.51, otherwise we can not obtain the convergence we are looking for.

We introduce the function

$$\beta_{r,t}(s) = \begin{cases} 2r^{\alpha-1}s, & 0 \leq s \leq r, \\ 2s^{\alpha}, & r \leq s \leq \frac{t}{2} \\ 2(t-s)^{\alpha}, & \frac{t}{2} \leq s \leq t-r, \\ 2r^{\alpha-1}(t-s), & t-r \leq s \leq t, \end{cases} \tag{4.52}$$

assuming, without loss of generality, that $t > 3r$. Hence we can rewrite the numerator of equation 4.50 using the function just introduced:

$$\begin{aligned} &\mathbb{P}\left[\xi_t(s) > -f(s) + f_{t,\alpha}(s), r \leq s \leq t-r\right] \\ &= \mathbb{P}\left[\xi_t(s) - \beta_{r,t}(s) > -f(s) - f_{t,\alpha}(s), r \leq s \leq t-r\right]. \end{aligned} \tag{4.53}$$

We now need to use a change of measure.
Recall that if P_a is the law of the Brownian bridge on $[0,t]$ with drift $a(s) \in L^2[0,t]$ and $A \subset C[0,t]$ is a Borel set, Girsanov's formula says that

$$P_a[A] = \mathbb{E}\left[\exp\left\{\int_0^t a(s)d\xi_t(s) - \frac{1}{2}\int_0^t a^2(s)ds + \frac{1}{2t}\left(\int_0^t a(s)ds\right)^2\right\}; A\right]. \tag{4.54}$$

The process $\xi_t(s) - \beta_{r,t}(s)$ is a diffusion with drift $a(s) = -\beta'_{r,t}(s)$. Set

$$A_{r,t} = \{\xi : \xi_t(s) > -f(s) - f_{t,\alpha}(s), s \in [r, t-r]\}. \tag{4.55}$$

We can apply Girsanov's formula:

$$\begin{aligned} &\mathbb{P}\left[\xi_t(s) - \beta_{r,t}(s) > -f(s) - f_{t,\alpha}(s), r \leq s \leq t-r\right] \\ &= \mathbb{E}\left[\exp\left\{-\int_0^t \beta'_{r,t}(s)d\xi_t - \frac{1}{2}\int_0^t \beta'_{r,t}(s)^2 ds - \frac{1}{2t}\left(\int_0^t \beta'_{r,t}(s)ds\right)^2\right\} : A_{r,t}\right] \\ &= \mathbb{E}\left[\exp\left\{-\int_0^t \beta'_{r,t}(s)d\xi_t - \frac{1}{2}\int_0^t \beta'_{r,t}(s)^2 ds\right\}; A_{r,t}\right], \end{aligned} \tag{4.56}$$

since $\int_0^t \beta'_{r,t}(s)ds = 0$.
Consider

$$A^1_{r,t} = A_{r,t} \cup \{\xi : \xi_t(s) > \Lambda_t(s), r \leq s \leq t-r\} \subset A_{r,t}, \tag{4.57}$$

where

$$\Lambda(s) = \begin{cases} 2s^\theta, & 0 \leq s \leq \frac{t}{2}, \\ 2(t-s)^\theta, & \frac{t}{2} \leq s \leq t, \end{cases} \tag{4.58}$$

with $\frac{1}{2} < \theta < 1 - \alpha$. We would like to control the behaviour of our Brownian bridge on $A^1_{r,t}$.

We start by estimating $\int_0^t \beta'_{r,t}(s)d\xi_t(s)$. Observe that

$$\beta'_{r,t}(s) = \begin{cases} 2r^{\alpha-1}, & 0 \le s \le r, \\ 2\alpha s^{\alpha-1}, & r \le s \le \frac{t}{2} \\ -2\alpha(t-s)^{\alpha-1}, & \frac{t}{2} \le s \le t-r, \\ -2r^{\alpha-1}(t-s), & t-r \le s \le t, \end{cases} \tag{4.59}$$

Moreover, $-\beta'' \ge 0$ and $\xi_t(s) \le \Lambda_t(s)$. Hence, using integration by parts, we obtain

$$\int_0^t \beta'_{r,t}(s)d\xi_t(s) = -\int_r^{t-r} \beta''_{r,t}(s)\xi_t(s)ds \le -\int_r^{t-r} \beta''_{r,t}(s)\Lambda_t(s)ds. \tag{4.60}$$

Define now

$$l_{r,t}(s) = \begin{cases} 2r^{\theta-1}s, & 0 \le s \le r, \\ 2s^{\theta}, & r \le s \le \frac{t}{2} \\ 2(t-s)^{\theta}, & \frac{t}{2} \le s \le t-r, \\ 2r^{\theta-1}(t-s), & t-r \le s \le t, \end{cases} \tag{4.61}$$

Using again integration by parts, we have

$$\int_0^t \beta'_{r,t}(s)dl_{r,t}(s) = -\int_r^{t-r} \beta''_{r,t}(s)\Lambda_t(s)ds. \tag{4.62}$$

Thus, combining 4.60 and 4.62,

$$-\int_0^t \beta'_{r,t}(s)d\xi_t(s) \ge -\int_0^t \beta'_{r,t}(s)dl_{r,t}(s). \tag{4.63}$$

It can be checked that

$$-\int_0^t \beta'_{r,t}(s)dl_{r,t}(s) \ge -\kappa_1 r^{\alpha+\theta-1}, \tag{4.64}$$

for some $\kappa_1 > 0$ and

$$-\frac{1}{2}\int_0^t \beta'_{r,t}(s)^2 ds \ge -\kappa_2 r^{2\alpha-1}, \tag{4.65}$$

for some $\kappa_2 > 0$. Therefore 4.56 is bounded from below:

$$\mathbb{E}\left[\exp\left\{-\int_0^t \beta'_{r,t}(s)d\xi_t - \frac{1}{2}\int_0^t \beta'_{r,t}(s)^2 ds\right\}; A_{r,t}\right] \ge \exp\left\{-\kappa_3 r^{\alpha+\theta-1}\right\}\mathbb{P}\left[A^1_{r,t}\right], \tag{4.66}$$

for some constant $\kappa_3 > 0$. Notice that $\exp\left\{-\kappa_3 r^{\alpha+\theta-1}\right\} \to 1$ for r large, since for our choice

$\alpha + \theta - 1 < 0$.

Recall that we were looking for uniform bounds on 4.50. We reduced this problem to showing that

$$\frac{\mathbb{P}\left[A^1_{r,t}\right]}{\mathbb{P}\left[A_{r,t}\right]} \to 1 \tag{4.67}$$

as $r \to \infty$, uniformly in t. Firstly, observe that

$$\frac{\mathbb{P}\left[A^1_{r,t}\right]}{\mathbb{P}\left[A_{r,t}\right]} = \mathbb{P}\left[\xi_t(s) < \Lambda_t(s), r \leq\leq t - r \mid A_{r,t}\right]. \tag{4.68}$$

Since $f(s) = \frac{t-s}{t}\bar{D}$, there exists a $r_0 = r_0(\bar{D}, C, \alpha)$ such that

$$- f(s) - f_{t,\alpha}(s) < 0 \text{ for } r \leq s \leq t - r, \tag{4.69}$$

for $r \geq r_0$.

Therefore we can apply Lemma 5.12: for all $r \geq r_0$

$$\begin{aligned} &\mathbb{P}\left[\xi_t(s) < \Lambda_t(s), r \leq\leq t - r \mid A_{r,t}\right] \\ &\geq \mathbb{P}\left[\xi_t(s) < \Lambda_t(s), r \leq s \leq t - r \mid \xi_t(s) > 0, r \leq s \leq t - r\right] \\ &= P^0\left[B^{\Lambda_t}[r, t-r] \mid B_0[r, t-r]\right] \geq 1 - \kappa_4 \sum_{k=r}^{\infty} k \exp\left\{-Ck^{2\theta-1}\right\} \\ &= 1 - \kappa_4 \sum_{k=r}^{\infty} k \exp\left\{-Ck^{1-2\alpha}\right\}, \end{aligned} \tag{4.70}$$

for some $\kappa_4 > 0$, where the second inequality comes from Lemma 5.13. We obtained a lower bound on 4.50. Therefore 4.49 is, up to numerical constants, smaller than

$$\begin{aligned} & tP^0\left[B^0[r, t-r]\right] \frac{\sum_{k=r}^{\infty} k \exp\left\{-Ck^{1-2\alpha}\right\}}{1 - \sum_{k=r}^{\infty} k \exp\left\{-Ck^{1-2\alpha}\right\}} \\ \leq\ & tP^0\left[B^0[r, t-r]\right] \frac{\int_r^{\infty} x \exp\left\{-Cx^{1-2\alpha}\right\} dx}{1 - \int_r^{\infty} x \exp\left\{-Cx^{1-2\alpha}\right\} dx} \\ \leq\ & t\frac{2r}{t-2r} \frac{\int_r^{\infty} x \exp\left\{-Cx^{1-2\alpha}\right\} dx}{1 - \int_r^{\infty} x \exp\left\{-Cx^{1-2\alpha}\right\} dx}, \end{aligned} \tag{4.71}$$

where the last inequality holds for Lemma 5.14. The integral in the last term converges to 0 as $r \to \infty$, since for our choice $\alpha < \frac{1}{2}$. Hence the whole expression converges to 0 as $r \to \infty$, uniformly in t, which was our claim. □

4.2 Lower envelope and the tube

We need another piece of information about paths of extremal particles : they cannot lie too low. The following theorem ensures this result:

Theorem 4.4. *Let $D \subset \mathbb{R}$ be a compact set and let $\frac{1}{2} < \beta < 1$. Set $\bar{D} = \sup\{x \in D\}$. For any $\epsilon > 0$ there exists $r_l = r_l(\beta, D, \epsilon)$ such that for $r \geq r_l$ and $t > 3r$,*

$$\mathbb{P}[\exists k \leq N(t) : X_k(t) \in m_t + D, \text{ but } \exists s \in [r, t-r] : X_k(s) \leq \bar{D} + E_{t,\beta}(s)] < \epsilon. \tag{4.72}$$

Proof. Let α be such that $0 < \alpha < \frac{1}{2} < \beta$ and set $\underline{D} = \inf\{x : x \in D\}$. By Theorem 4.3, we know that particles have to stay below the entropic envelope. Hence, to prove the theorem, it suffices to show that

$$\begin{aligned}\mathbb{P}\,[\exists k \leq N(t) : X_k(t) \in m_t + D, X_k(s) \leq \underline{D} + E_{t,\alpha}(s) \forall s \in [r, t-r] \\ \text{but } \exists s \in [r, t-r] : X_k(s) \leq \bar{D} + E_{t,\beta}(s)\ ] \to 0\end{aligned} \tag{4.73}$$

as $r \to \infty$, uniformly in $t > 3r$.

Let now $\xi_t(s) = X(t) - \frac{s}{t}X(s), 0 \leq s \leq t$ be a Brownian bridge independent of $X(t)$. Recall that on the event $\{X(t) \in m_t + D\}$, we have that $X(t) - \bar{D} \leq m_t \leq X(t) - \underline{D}$. Using this fact and by Markov's inequality, we get

$$\begin{aligned}\mathbb{P}\,[\exists k \leq N(t) : X_k(t) \in m_t + D, X_k(s) \leq \underline{D} + E_{t,\alpha}(s) \forall s \in [r, t-r] \\ \text{but } \exists s \in [r, t-r] : X_k(s) \leq \bar{D} + E_{t,\beta}(s)\ ] \\ \leq e^t \mathbb{P}\,[X(t) \in m_t + D] \\ \times \mathbb{P}\left[\xi_t(s) \leq \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\alpha}(s) \forall s \in [r, t-r]\right. \\ \left.\text{but } \exists s \in [r, t-r] : \xi_t(s) \leq \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\beta}(s)\right].\end{aligned} \tag{4.74}$$

Since $X(t)$ is Gaussian, we have that

$$e^t \mathbb{P}\,[X(t) \in D] \leq \kappa_1 t \int_D e^{-\sqrt{2}x} dx \tag{4.75}$$

for some $\kappa_1 > 0$. Moreover, if we set $\mathrm{diam}(D) = |\bar{D}| + |\underline{D}|$, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\alpha}(s) \forall s \in [r, t-r]\right. \\
&\left.\text{but } \exists s \in [r, t-r] : \xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\beta}(s)\right] \\
&\le \mathbb{P}\left[\xi_t(s) \le \mathrm{diam}(D) - f_{t,\alpha}(s) \forall s \in [r, t-r]\right. \\
&\left.\text{but } \exists s \in [r, t-r] : \xi_t(s) \le \mathrm{diam}(D) - f_{t,\beta}(s)\right]. \qquad (4.76)
\end{aligned}$$

By the previous inequalities, the claim of the theorem will follow if we prove that

$$\begin{aligned}
&\kappa_1 t \int_D e^{-\sqrt{2}x} dx \mathbb{P}\left[\xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\alpha}(s) \forall s \in [r, t-r]\right. \\
&\left.\text{but } \exists s \in [r, t-r] : \xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\beta}(s)\right] \to 0
\end{aligned}$$

as $r \to \infty$, uniformly in $t > 3r$, that is the same as asking

$$\begin{aligned}
&t\mathbb{P}\left[\xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\alpha}(s) \forall s \in [r, t-r]\right. \\
&\left.\text{but } \exists s \in [r, t-r] : \xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\beta}(s)\right] \to 0
\end{aligned}$$

as $r \to \infty$, uniformly in $t > 3r$.

Without loss of generality, we can take $0 < a < 1$ such that $2a\beta - 1 > 0$. Let $D \subset \mathbb{R}$ be a given compact. We can find $\tilde{r} = \tilde{r}(\alpha, \beta, D, a)$ such that for $r \ge \tilde{r}$we have

$$\mathrm{diam}(D) - f_{t,\alpha}(s) \le 0 \qquad (4.77)$$

and

$$\mathrm{diam}(d) - f_{t,\beta}(s) \le -f_{t,a\beta}(s) \qquad (4.78)$$

for all $s \in [r, t-r]$. Thus, recalling also that the Brownian bridge is symmetric around the x-axis,

$$\begin{aligned}
&t\mathbb{P}\left[\xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\alpha}(s) \forall s \in [r, t-r]\right. \\
&\left.\text{but } \exists s \in [r, t-r] : \xi_t(s) \le \left\{\bar{D} - \frac{s}{t}\underline{D}\right\} - f_{t,\beta}(s)\right] \\
&\le t\mathbb{P}\left[\xi_t(s) \le 0 \forall s \in [r, t-r] \text{ but } \exists s \in [r, t-r] : \xi_t(s) \le -f_{t,a\beta}(s)\right] \\
&= t\mathbb{P}\left[\xi_t(s) \ge 0 \forall s \in [r, t-r] \text{ but } \exists s \in [r, t-r] : \xi_t(s) \ge f_{t,a\beta}(s)\right]. \qquad (4.79)
\end{aligned}$$

Moreover

$$
\begin{aligned}
t\mathbb{P}\left[\xi_t(s) \geq 0 \forall s \in [r, t-r] \text{ but } \exists s \in [r, t-r] : \xi_t(s) \geq f_{t,a\beta}(s)\right]& \\
= t\left(\mathbb{P}\left[\xi_t(s) \geq 0 \forall s \in [r, t-r]\right] - \mathbb{P}\left[0 \leq \xi_t(s) \leq f_{t,a\beta}(s) \forall s \in [r, t-r]\right]\right)& \\
= tP^0\left[B_0[r, t-r]\right]\left(1 - P^0\left[B^{f_{t,a\beta}}[r, t-r] \mid B_0\right]\right)& \\
\leq \kappa t P^0\left[B_0[r, t-r]\right] \int_r^\infty x e^{-x^\delta} dx,& \qquad (4.80)
\end{aligned}
$$

where the last inequality holds by Lemma 5.13 and $\kappa > 0$, and $\delta = 2a\beta - 1$. By Lemma 5.14, we have

$$
tP^0\left[B_0[r, t-r]\right] \leq \kappa \frac{rt}{t-2r} \qquad (4.81)
$$

and

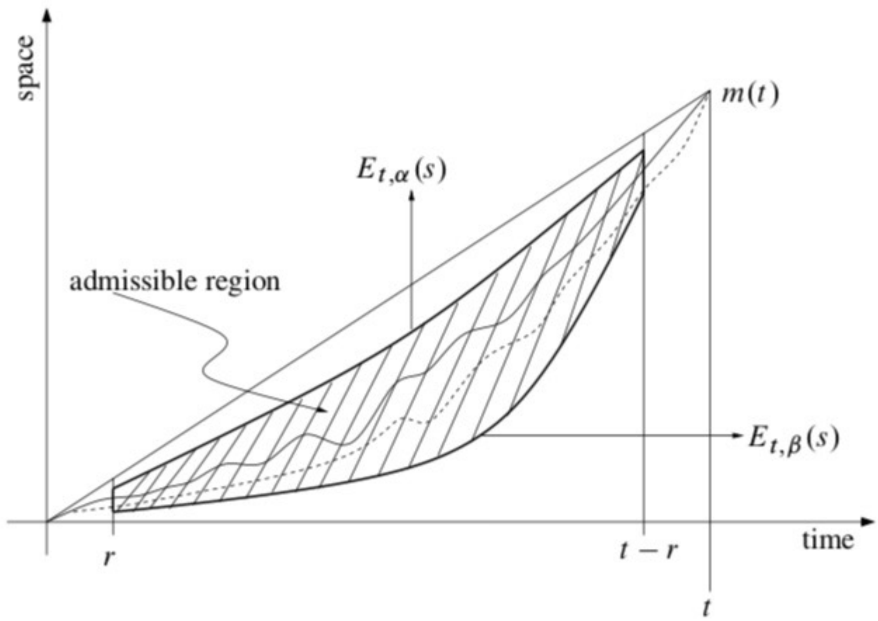
$$
\frac{rt}{t-2r} < \kappa r \qquad (4.82)
$$

if $t > 3r$. Moreover $\int_r^\infty x e^{-x^\delta}$ tends to 0 faster than any power of r, as $r \to \infty$. Hence we obtain that 4.80 tends to 0 as $r \to \infty$, uniformly on $t > 3r$, which proves the theorem. □

Theorems 4.3 and 4.4 provide an explicit characterized tube, a space-time region, where paths of extremal particles spend most of their time with overwhelming probability:

Corollary 4.5. *Let $D \subset \mathbb{R}$ be a compact set. Let $0 < \alpha < \frac{1}{2} < \beta < 1$. For any $\epsilon > 0$ there exists $r_1 = r_1(\alpha, \beta, D, \epsilon)$ such that for $r \geq r_1$ and $t > 3r$,*

$$
\begin{aligned}
\mathbb{P}[\forall k \leq N(t) : \quad & X_k(t) \in m_t + D, \\
& \bar{D} + E_{t,\beta}(s) \leq X_k(s) \quad \leq \bar{D} + E_{t,\alpha}(s) \forall s \in [r, t-r]] \geq 1 - \epsilon.
\end{aligned}
$$



4.3 Genealogy of extremal particles

The localization of paths is very useful also for proving the following strong result on genealogy of extremal particles: we prove that in the large t-limit, such particles descend with overwhelming probability from ancestors having split either within a distance of order 1 from time 0, or within a distance of order 1 from time t.

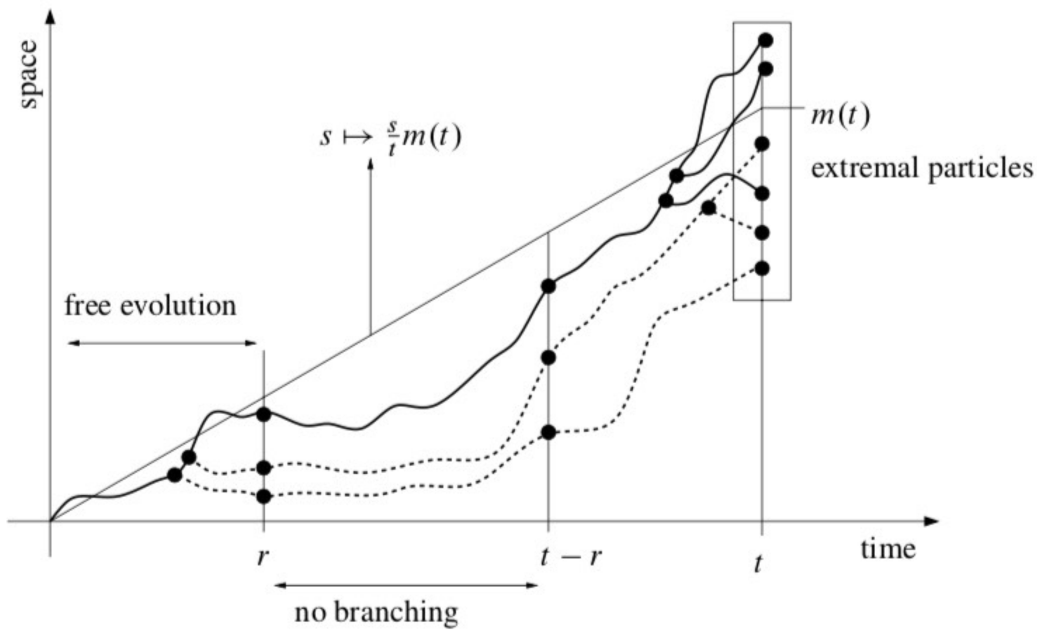
Let $u, v \in \mathcal{N}_t$. Recall that, conditionally upon the branching mechanism, it holds

$$\mathbb{E}[X_u(t)X_v(t)] = Q_t(u,v), \tag{4.83}$$

where $Q_t(u,v) = \sup\{s \leq t : X_u(s) = X_v(s)\}$ is the time of first branching.

Theorem 4.6. *For any compact set $D \subset \mathbb{R}$,*

$$\lim_{r\to\infty} \sup_{t>3r} \mathbb{P}[\exists u, v \in \mathcal{N}_t(D) : Q_t(u,v) \in (r, t-r)] = 0 \tag{4.84}$$



Proof. Let $D \subset \mathbb{R}$ be a compact set. By definition of compactness, there exists $\underline{D} \leq \bar{D} \in \mathbb{R}$ such that $D \subset [\underline{D}, \bar{D}]$. In order to prove the theorem, we need to find $r_0(D,\epsilon)$ and $t_0 = t_0(D,\epsilon)$ such that for $r \geq r_0$ and $t > \max\{t_0, 3r\}$ we have

$$\mathbb{P}\left[\exists i,j \leq N(t) : X_i(t), X_j(t) \in m_t + D \text{ and } Q_t(i,j) \in [r, t-r]\right] < \epsilon. \tag{4.85}$$

Corollary 4.5 tells us that there exists $r_{=}r_1(D,\epsilon)$ such that the extremal particles that reach D at time t satisfy, for $t > 3r_1$,

$$\begin{aligned} \bar{D} + \frac{s'}{t}m_t - f_{t,\beta}(s') &\leq X_i(s) \\ &\leq \bar{D} + \frac{s'}{t}m_t - f_{t,\alpha}(s') \forall s' \in [r_1, t-r_1] \end{aligned} \tag{4.86}$$

with probability at least $1-\epsilon$. Denote by $\Xi_{D,t}$ the set of paths $X(s')$ satisfying 4.86 and $X(t) \in m_t + D$. Moreover, denote by $\Xi_{D,t}^{[s,t-r_1]}$ the set of paths that satisfy 4.86 for all $s' \in [s, t-r_1]$, for $s \geq r_1$ and $X(t) \in m_t + D$.

Let now $K = \sum_k p_k k(k-1)$, where $\{p_k\}$ is the offspring distribution. We need the following lemma ([5]):

Lemma 4.7. *Let μ_s be the Gaussian measure of variance s. It holds*

$$\begin{aligned} &\mathbb{E}[|\{(i,j) : X_i, X_j \in \Xi_{D,t} i \neq j\}|] \quad (4.87) \\ &= K e^t \int_0^t e^{t-s} ds \int_{-\infty}^{\infty} d\mu_s(y) \mathbb{P}[X \in \Xi_{D,t} | X(s) = y] \mathbb{P}[X \in \Xi_{D,t}^{[s,t-r_1]} | X(s) = y]. \end{aligned}$$

In our case, the event considered includes a condition on $Q_t(i,j)$ and so the statement changes:

$$\begin{aligned} &\mathbb{E}[|\{(i,j) i \neq j : X_i, X_j \in \Xi_{D,t}, Q_t(i,j) \in [r, t-r]\}|] \quad (4.88) \\ &= K e^t \int_r^{t-r} e^{t-s} ds \int_{-\infty}^{\infty} d\mu_s(y) \mathbb{P}[X \in \Xi_{D,t} | X(s) = y] \mathbb{P}[X \in \Xi_{D,t}^{[s,t-r_1]} | X(s) = y]. \end{aligned}$$

Our aim is to prove that there exists $r_0 = r_0(D, \epsilon)$ and $t_0 = t_0(D, \epsilon)$, such that, for $r > r_0$ and $t > \max\{t_0, 3r\}$, 4.88 is smaller than ϵ. By Markov's inequality and Corollary 4.5, with $r_0 > r_1$, will imply 4.85. We want to bound the term $\mathbb{P}\left[x \in \Xi_{D,t}^{[s,t-r_1]|X(s)=y}\right]$, uniformly in y.
Observe that

$$s' \mapsto \bar{D} + \frac{s'}{t} m_t - f_{t,\alpha}(s') \quad (4.89)$$

is a convex function and equals $m_t + \bar{D}$ at time t:

$$\bar{D} + \frac{t}{t} m_t - f_{t,\alpha}(t) = m_t + \bar{D}. \quad (4.90)$$

From the picture below we can see that

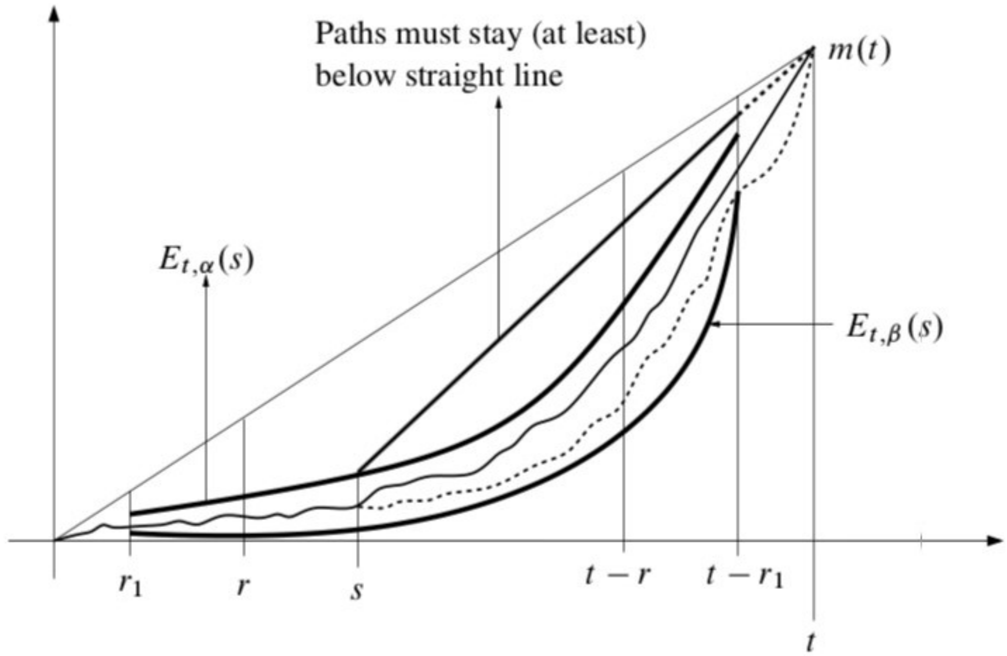
$$\begin{aligned} &\left\{ X(s') \leq \bar{D} + \frac{s'}{t} m_t - f_{t,\alpha}(s') \forall s' \in [s, t-r_1] \right\} \quad (4.91) \\ &\subseteq \left\{ X(s') \leq \bar{D} + \frac{\left(1 - \frac{s}{t}\right) m_t + f_{t,\alpha}(s')}{t-s} (s'-s) + \right. \\ &\left. \bar{D} + \frac{s}{t} m_t + f_{t,\alpha}(s) - y \forall s' \in [s, t-r_1] \right\}. \quad (4.92) \end{aligned}$$

Set

$$a = \bar{D} + \frac{s}{t} m_t - f_{t,\alpha}(s) - y \quad (4.93)$$

and

$$b = \frac{\left(1 - \frac{s}{t}\right) m_t + f_{t,\alpha}(s)}{t-s}. \quad (4.94)$$



We now subtract $X(s)$ and $\frac{s'-s}{t-s}X(t)$ from $X(s')$ and shift the time s' by s, obtaining

$$\begin{aligned} &\mathbb{P}\left[X \in \Xi^{[}_{D,t} s, t-r_1] | X(s) = y\right] \\ \leq \mathbb{P}\Bigg[X(s') - \frac{s'}{t-s}X(t-s) \leq a + bs' - \frac{s'}{t-s}X(t-s), \forall s' \in [0, t-s-r_1], \\ &X(t-s) \in m_t - y + D\Big]. \end{aligned} \tag{4.95}$$

Recall that $D \subseteq [\underline{D}, \bar{D}]$. Set $Z_1 = \bar{D} + \frac{s}{t}m_t - f_{t,\alpha}(s) - y$ and $Z_2 = \bar{D} - \underline{D}$. We find the following upper bound

$$\begin{aligned} &\mathbb{P}\Bigg[X(s') - \frac{s'}{t-s}X(t-s) \leq a + bs' - \frac{s'}{t-s}X(t-s), \forall s' \in [0, t-s-r_1], \\ &\qquad X(t-s) \in m_t - y + D\Big] \\ &\leq \mathbb{P}\Bigg[X(s') - \frac{s'}{t-s}X(t-s) \leq \left(1 - \frac{s'}{t-s}\right)Z_1 + \frac{s'}{t-s}Z_2, \forall s' \in [0, t-s-r_1], \\ &\qquad X(t-s) \geq m_t - y + \underline{D}\Big]. \end{aligned} \tag{4.96}$$

By definition, $X(s') - \frac{s'}{t-s}X(t-s)$ is a Brownian bridge $\xi_{t-s}(s')$. Hence, by the independence property of it,

$$\mathbb{P}\left[X(s') - \frac{s'}{t-s}X(t-s) \leq \left(1 - \frac{s'}{t-s}\right) Z_1 + \frac{s'}{t-s} Z_2, \forall s' \in [0, t-s-r_1], \right.$$
$$\left. X(t-s) \geq m_t - y + \underline{D}\right]$$
$$= \mathbb{P}\left[\xi_{t-s}(s') \leq \left(1 - \frac{s'}{t-s}\right) Z_1 + \frac{s'}{t-s} Z_2 \forall s' \in [0, t-s-r_1]\right]$$
$$\times \mathbb{P}\left[X(t-s) \geq m_t - y + \underline{D}\right]. \tag{4.97}$$

Moreover, by Lemma 5.14, it holds

$$\mathbb{P}\left[\xi_{t-s}(s') \leq \left(1 - \frac{s'}{t-s}\right) Z_1 + \frac{s'}{t-s} Z_2 \forall s' \in [0, t-s-r_1]\right]$$
$$\leq \frac{2Z_1}{t-s-r_1}\left(\frac{r_1}{t-s} Z_1 + \left(1 - \frac{r_1}{t-s}\right) Z_2 + \sqrt{r_1}\right). \tag{4.98}$$

Recall now the $X(s) = y$ lies between the entropic and the lower envelope, by Corollary 4.5. Consequently

$$\bar{D} + \frac{s}{t} m_t - f_{t,\beta}(s) - f_{t,\alpha}(s) \leq X(s) \leq \bar{D} + \frac{s}{t} m_t - f_{t,\alpha}(s) \tag{4.99}$$

and thus

$$0 \leq Z_1 \leq f_{t,\beta}(s) - f_{t,\alpha}(s) \leq \kappa f_{t,\beta}(s), \tag{4.100}$$

for some $\kappa > 0$, independent of t and r.

Recall that we are looking for a $r_0 = r_0(D, \epsilon)$ and $t_0 = t_0(D, \epsilon)$ such that for $r \geq r_0$ and $t > \max\{t_0, 3r\}$, the thesis of the theorem holds. We would like to specify the choice of r_0 to make 4.98 small. Observe that in the integral 4.88, $r \leq t - s$. Hence, if we choose $r > 2r_1$, we obtain $\frac{1}{t-s-r_1} \leq \frac{2}{t-s}$. So, we require $r_0 > 2r_1$. Moreover, by 4.98,

$$\frac{r_1 Z_1}{t-s} \leq \frac{r_1 \kappa f_{t,\beta}(s)}{t-s} \leq \kappa r_1 (t-s)^{\beta - 1}, \tag{4.101}$$

where, since $r \leq t - s$, we can choose $r_0^{1-\beta} > r_1$, so that $\frac{r_1 Z_1}{t-s} < 1$. Without loss of generality, we can also require $\sqrt{r_0} \geq \max\{Z_2, 1\}$. Thus we obtain

$$\frac{2Z_1}{t-s-r_1}\left(\frac{r_1}{t-s} Z_1 + \left(1 - \frac{r_1}{t-s}\right) Z_2 + \sqrt{r_1}\right)$$
$$\leq \frac{\kappa\sqrt{r} Z_1}{t-s} \leq \frac{\kappa\sqrt{r} f_{t,\beta}(s)}{t-s}. \tag{4.102}$$

We now want to bound $\mathbb{P}[X(t-s) \geq m_t - y + \underline{D}]$. Since $y \leq \frac{s}{t}m_t - f_{t,\alpha}$ and X is Gaussian, we can use a well-known estimate of the Gaussian density, obtaining

$$\mathbb{P}[X(t-s) \geq m_t - y + \underline{D}] \leq \kappa \frac{e^{-(t-s)} e^{\frac{3}{2}\frac{t-s}{t}\ln t} e^{-\sqrt{2} f_{t,\alpha}(s)}}{(t-s)^{\frac{1}{2}}}. \tag{4.103}$$

Finally, putting together 4.102 and4.103, we find what we were looking for:

$$\mathbb{P}\left[X \in \Xi_{D,t}^{[s,t-r_1]} | X(s) = y\right] \leq \kappa\sqrt{r} \frac{e^{-(t-s)} e^{\frac{3}{2}\frac{t-s}{t}\ln t} e^{-\sqrt{2} f_{t,\alpha}(s)}}{(t-s)^{\frac{3}{2}}}. \tag{4.104}$$

We can now bound the whole expression in 4.88:

$$\begin{aligned} &Ke^t \int_r^{t-r} e^{t-s} ds \int_{-\infty}^{\infty} d\mu_s(y) \mathbb{P}[X \in \Xi_{D,t} | X(s) = y] \mathbb{P}[X \in \Xi_{D,t}^{[s,t-r_1]} | X(s) = y] \\ &\qquad \leq \kappa e^t \mathbb{P}[X \in \Xi_{D,t}] \sqrt{r} \int_r^{t-r} \frac{e^{-(t-s)} e^{\frac{3}{2}\frac{t-s}{t}\ln t} e^{-\sqrt{2} f_{t,\alpha}(s)}}{(t-s)^{\frac{3}{2}}} ds. \end{aligned} \tag{4.105}$$

We have to show that this can be made arbitrarily small by taking r_0 large.
Let $t > 3r$, which ensures that $t \geq 3r_1$. Using the definition of $\Xi_{D,t}$ and the bounds on D, we obtain

$$\begin{aligned} \mathbb{P}[X \in \Xi_{D,t}] &\leq \mathbb{P}\left[X(s) \leq \frac{s}{t}m_t + \bar{D} \forall s \in [r_1, t-r_1], X(t) \in m_t + D\right] \\ &\leq \mathbb{P}\left[\xi_t(s) \leq \bar{D} \forall s \in [r_1, t-r_1]\right] \mathbb{P}\left[X(t) \geq m_t + \underline{D}\right]. \end{aligned} \tag{4.106}$$

By Lemma 5.14 and the fact that $t \geq 3r_1$, we have

$$\xi_{\approx}(\sim) \leq \bar{\mathbb{D}} \forall \sim \in [\searrow_{\nVdash}, \approx - \searrow_{\nVdash}] \leq \kappa \frac{r_1}{t - 2r_1} \leq \kappa \frac{r_1}{t}. \tag{4.107}$$

Moreover,

$$\mathbb{P}\left[X(t) \geq m_t + \underline{D}\right] \leq \kappa t e^{-t}. \tag{4.108}$$

Hence the term $e^t \mathbb{P}\left[X \in \Xi_{D,t}\right] \sim r_1$, uniformly for $t \geq 3r_1$. We now concentrate on the integral in 4.105. We split the domain of integration into the intervals $\left[r, \frac{t}{2}\right]$ and $\left[\frac{t}{2}, t-r\right]$. In the first interval we have:

$$\begin{aligned}&\kappa\sqrt{r}\int_r^{\frac{t}{2}}\frac{e^{-(t-s)}e^{\frac{3}{2}\frac{t-s}{t}\ln t}e^{-\sqrt{2}f_{t,\alpha}(s)}}{(t-s)^{\frac{3}{2}}}ds\\&\leq\kappa\sqrt{r}\int_r^{\infty}s^{\beta}e^{-(\sqrt{2})s^{\alpha}}ds\leq\kappa\kappa' r^{\frac{3}{2}}e^{-r^{\alpha}},\end{aligned}\tag{4.109}$$

for some $\kappa'=\kappa'(\alpha)$.

$$\kappa\kappa' r^{\frac{3}{2}}e^{-r^{\alpha}}\to 0,\tag{4.110}$$

as $r\to\infty$. On the second interval, we change variable $s\mapsto t-s$:

$$\kappa\sqrt{r}\int_{\frac{t}{2}}^{t}\frac{e^{-(t-s)}e^{\frac{3}{2}\frac{t-s}{t}\ln t}e^{-\sqrt{2}f_{t,\alpha}(s)}}{(t-s)^{\frac{3}{2}}}ds=\kappa\sqrt{r}\int_r^{\frac{t}{2}}\frac{e^{\frac{3}{2}\frac{s}{t}\ln t}s^{\beta}e^{-\sqrt{2}s^{\alpha}}}{s^{\frac{3}{2}}}ds.\tag{4.111}$$

Now we can split again the integration domain into $[r,t^{\delta}]$ and $[t^{\delta},\frac{t}{2}]$, with $0<\delta<1$. For t large enough we have

$$\kappa\sqrt{r}\int_r^{t^{\delta}}\frac{e^{\frac{3}{2}\frac{s}{t}\ln t}s^{\beta}e^{-\sqrt{2}s^{\alpha}}}{s^{\frac{3}{2}}}ds\leq\kappa\sqrt{r}\int_r^{\infty}\frac{s^{\beta}e^{-\sqrt{2}s^{\alpha}}}{s^{\frac{3}{2}}}ds\leq\kappa\sqrt{r}e^{-r^{\alpha}}.\tag{4.112}$$

This goes to 0 as $r\to\infty$. On the second interval

$$\begin{aligned}&\kappa\sqrt{r}\int_{t^{\delta}}^{\frac{t}{2}}\frac{e^{\frac{3}{2}\frac{s}{t}\ln t}s^{\beta}e^{-\sqrt{2}s^{\alpha}}}{s^{\frac{3}{2}}}ds\\&\leq\kappa\sqrt{r}t^{\frac{3}{4}}\int_{t^{\delta}}^{\frac{t}{2}}\frac{s^{\beta}e^{-\sqrt{2}s^{\alpha}}}{s^{\frac{3}{2}}}ds\leq\kappa\sqrt{r}t^{\frac{7}{4}}e^{-t^{\alpha\delta}}\leq\kappa t^{\frac{13}{4}}e^{-t^{\alpha\delta}},\end{aligned}\tag{4.113}$$

which can be made arbitrarily small for t_0 large enough, since $t>\max\{t_0,3r\}$. The claim follows. □

Chapter 5

Appendix

Let $X^{(i)}(t)$ be a family of iid branching Brownian motion processes, defined as in Chapter 3 and let $Z = \lim_{t\to\infty} Z(t)$, where $Z(t)$ is the derivative martingale defined as in 3.33. Let η be a Poisson process, defined on $(-\infty, 0]$ and shifted by $\frac{1}{\sqrt{2}} \ln Z$, with intensity

$$\sqrt{\frac{2}{\pi}}(-x)e^{-\sqrt{2}x}dx,$$

and let $(\eta_i, i \in \mathbb{N})$ be its atoms.

Proposition 5.1. *Let $y \in \mathbb{R}$ and $\epsilon > 0$ be given. There exist two constants A_1 and A_2 with $0 < A_1 < A_2 < \infty$ and t_0, depending only on y and ϵ, such that*

$$\sup_{t\geq 0} \mathbb{P}[\exists i, k : \eta_i + X_k^{(i)}(t) - \sqrt{2}t \geq y, \eta_i \notin [-A_1\sqrt{t}, -A_2\sqrt{t}] < \epsilon. \tag{5.1}$$

Lemma 5.2. *Let $b \in \mathbb{R}$ and $h(x) = \mathbb{I}_{[b,\infty)}(x)$, $x \in \mathbb{R}$. Let C be a constant in $\mathbb{R}$. Then*

$$\lim_{t\to\infty} \int_{-\infty}^{0} \left\{ \int_{\mathbb{R}} h(x)\mathbb{P}[y + \max_i X_i(t) - \sqrt{2}t \in dx] \right\} \sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy = Ce^{\sqrt{2}b}. \tag{5.2}$$

Moreover, the convergence also holds for a continuous function $h(x)$ that is bounded and is zero when x is small enough:

$$\lim_{t\to\infty} \int_{-\infty}^{0} \left\{ \int_{\mathbb{R}} h(x)\mathbb{P}[y + \max_i X_i(t) - \sqrt{2}t \in dx] \right\} \sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy$$
$$= \int_{\mathbb{R}} h(z)\sqrt{2}Ce^{-\sqrt{2}z}dz. \tag{5.3}$$

Theorem 5.3. *Let $x = a\sqrt{2}t + b$ for some $a < 0$, $b \in \mathbb{R}$. The point process*

$$\sum_{k\leq N(t)} \delta_{x+X_k(t)-\sqrt{2}t} \text{ conditioned on the event } \{x + \max_k X_k(t) - \sqrt{2}t > 0\} \tag{5.4}$$

converges in law as $t \to \infty$ to a well-defined point process $\bar{\mathcal{E}}$. The limit does not depend on a and b, and the maximum of $\bar{\mathcal{E}}$ has the law of an exponential random variable.

Corollary 5.4. *Let $\mathcal{D}$ as in 3.39 and $\bar{\mathcal{E}}$ the point process obtained in Theorem 5.3. Let $x = a\sqrt{t}$, $a < 0$. In the limit $t \to \infty$, the random variables*

$$\mathcal{D}_t = \sum_i \delta_{X_i(t)-\max_j X_j(t)} \tag{5.5}$$

and $x + \max \bar{\mathcal{E}}$ are conditionally independent on the event $\{x + \max \bar{\mathcal{E}} > b\}$ for any $b \in \mathbb{R}$. More precisely, for any bounded continuous function f, h and $\phi \in \mathcal{C}_{\mathcal{C}}(\mathbb{R})$, it holds

$$\lim_{t\to\infty} \mathbb{E}\left[f\left(\int \phi(z)\mathcal{D}_t(dz)\right) h(x + \max \bar{\mathcal{E}}_t) \mid x + \max \bar{\mathcal{E}}_t > b\right] \tag{5.6}$$

$$= \quad \mathbb{E}\left[f\left(\int \phi(z)\mathcal{D}(dz)\right)\right] \int_b^\infty h(y)\frac{\sqrt{2}e^{-\sqrt{2}y}dy}{e^{-\sqrt{2}b}}. \tag{5.7}$$

Moreover, convergence is uniform in $x = a\sqrt{t}$ for a belonging to a compact set.

Lemma 5.5. *Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}$. Let (μ_t, X_t) be a sequence of random variables on $(\mathcal{M}\times\mathbb{R}, \mathbb{P})$ that converges to (μ, X), that is: for any bounded continuous function f, h on $\mathbb{R}$ and any $\phi \in \mathcal{C}_{\mathcal{C}}(\mathbb{R})$ it holds*

$$\mathbb{E}\left[f\left(\int \phi(y)\mu_t(dy)\right) h(X_t)\right] \to \mathbb{E}\left[f\left(\int \phi(y)\mu(dy)\right) h(X)\right]. \tag{5.8}$$

Then, for any $\phi \in \mathcal{C}_{\mathcal{C}}(\mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}$ bounded and continuous,

$$\mathbb{E}\left[g\left(\int \phi(y + X_t)\mu_t(dy)\right)\right] \to \mathbb{E}\left[g\left(\int \phi(y + X)\mu(dy)\right)\right]. \tag{5.9}$$

Lemma 5.6. *Let $f : \mathbb{R} \to [0, 1]$. The function*

$$v(t, x) = \mathbb{E}\left[\prod_{k=1}^{N(t)} f(x + X_k(t))\right] \tag{5.10}$$

is a solution of the F-KPP *equation 3.9 with initial condition* $v(0,x) = f(x)$*.*

Proposition 5.7. *Let* u *be a solution of the* F-KPP *equation 3.9 with initial condition satisfying the assumption of Theorem 3.5 and*

$$y_0 = \sup\{y : u(0,y) > 0\} < \infty. \tag{5.11}$$

Define

$$\psi(r,t,X+\sqrt{2}t) = \frac{e^{-\sqrt{2}X}}{\sqrt{2\pi(t-r)}} \int_0^\infty u(r,y'+\sqrt{2}r)e^{y'\sqrt{2}}e^{-\frac{(y'-X)^2}{2(t-r)}} \tag{5.12}$$

$$\left(1 - e^{-2y'\frac{X+\frac{3}{2\sqrt{2}\ln t}}{t-r}}\right) dy'. \tag{5.13}$$

Then

$$\lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{\frac{3}{2}}}{\frac{3}{2\sqrt{2}\ln t}} \psi(r,t,x+\sqrt{2}t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(r,y+\sqrt{2}r)ye^{y\sqrt{2}}dy. \tag{5.14}$$

Lemma 5.8. *Let* u *be a solution of the* F-KPP *equation 3.9 with initial condition satisfying the assumption of Theorem 3.5 and*

$$y_0 = \sup\{y : u(0,y) > 0\} < \infty. \tag{5.15}$$

Let

$$C = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty ye^{y\sqrt{2}}u(r,y+\sqrt{2}r)dy, \tag{5.16}$$

whose existence is ensured by Proposition 5.7. Then, for any $x \in \mathbb{R}$ *it holds*

$$\lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty ye^{y\sqrt{2}}u(r,x+y+\sqrt{2}r)dy = Ce^{-\sqrt{2}x}. \tag{5.17}$$

Lemma 5.9. *Let* $\phi : \mathbb{R} \to [0,\infty)$ *be a non-negative continuous function with compact support. Let* $u(t,x)$ *and* $u_\delta(t,x)$ *be solutions of the* F-KPP *equation 3.9, with initial condition* $u(0,x) = 1 - e^{-\phi(-x)}$ *and* $u_\delta(0,x) = 1 - e^{-\phi(-x)}\mathbb{I}_{\{-x\le\delta\}}$ *respectively. Set*

$$C(\delta,\phi) = \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(t,y+\sqrt{2}t)ye^{\sqrt{2}y}dy \tag{5.18}$$

Then $\lim_{\delta\to\infty} C(\delta,\phi)$ *exists and*

$$C(\phi) = \lim_{\delta\to\infty} C(\delta,\phi) = \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(t, y+\sqrt{2}t) y e^{\sqrt{2}y} dy. \tag{5.19}$$

Moreover

$$\lim_{t\to\infty} u(t, x+m_t) = 1 - \mathbb{E}\left[\exp\left\{-C(\phi) Z e^{-\sqrt{2}x}\right\}\right], \tag{5.20}$$

where we recall that $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + o(1)$.

5.1 Brownian bridges and their properties

In this subsection we set some notations and facts on Brownian bridges.

Denote by $\{X(s), s \geq 0\}$ a standard Brownian motion.

Definition 5.10. A ***Brownian bridge*** is defined as

$$\xi_t(s) = X(s) - \frac{s}{t} X(t),\ 0 \leq s \leq t, \tag{5.21}$$

that is a new Gaussian process starting and ending at time t in 0.

We assume that both X and ξ_t are defined on $\mathcal{C}([0,t],\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra. Denote by P^0 the corresponding measure of ξ_t on $\mathcal{C}([0,t],\mathcal{B})$.

Let $l : [s_1, s_2] \to \mathbb{R}$ be a curve. We denote by $B_l[s_1,s_2]$, or just B_l if there is no risk of confusion, the set of paths lying strictly above l on the interval $[s_1,s_2]$ and by $B^l[s_1,s_2]$, or simply B^l, the set of paths lying strictly below l.

In this setting, we can now state some useful facts concerning Brownian bridges.

Lemma 5.11. *Let* $\xi_t(s)$ *be the Brownian bridge defined in 5.10. Then*

(i) $\xi_t(s)$ *is a strong Markov process;*

(ii) $\xi_t(s)$ *is independent of* $X(t)$, *for* $0 \leq s \leq t$.

Lemma 5.12. *Let* l_1, l_2 *and* Λ *be curves such that* $l_1(s) \leq l_2(s) \leq \Lambda(s)$ *for all* $s \in [0,t]$ *and* $P^0[B_{l_2}[0,t]] > 0$. *Then*

$$P^0[B^\Lambda \mid B_{l_1}] \geq P^0[B^\Lambda \mid B_{l_2}] \tag{5.22}$$

and

$$P^0[B_\Lambda \mid B_{l_1}] \leq P^0[B_\Lambda \mid B_{l_2}]. \tag{5.23}$$

The following theorem gives us uniform bounds on conditional probabilities of a Brownian bridge to stay below a certain curves and allows us to compare probabilities that the Brownian bridge hits curves that are close to one another.

Lemma 5.13. *Set*

$$\Lambda_t(s) = \begin{cases} Cs^\epsilon & \text{for } 0 \le s \le \frac{t}{2} \\ C(t-s)^\epsilon & \text{for } \frac{t}{2} \le s \le t \end{cases}$$

with $\epsilon > \frac{1}{2}$ *and* $C > 0$*. Then*

$$P^0[B^{\Lambda_t}[r, t-r] \mid B_0[r, t-r]] \to 1, \tag{5.24}$$

uniformly in $t > 3r$ *as* $r \to \infty$*. More precisely, if* a *is a fixed constant* $a > 0$*, and* $\delta = 2\epsilon - 1 > 0$*, it holds*

$$P^0[B^{\Lambda_t}[r, t-r] \mid B_0[r, t-r]] \ge 1 - 2aC \sum_{k=r}^{\infty} k e^{-Ck^\delta}. \tag{5.25}$$

Lemma 5.14. *Let* $Z_1, Z_2 \ge 0$ *and* $r_1, r_2 \ge 0$*. Set* $Z(r_1) = \left(1 - \frac{r_1}{t}\right) Z_1 + \frac{r_1}{t} Z_2$ *and* $Z(r_2) = \frac{r_2}{t} Z_1 + \left(1 - \frac{r_2}{t}\right) Z_2$*. Then, for* $t > r_1 + r_2$*,*

$$\begin{aligned} \mathbb{P}\left[\xi_t(s) \le \left(1 - \frac{s}{t}\right) Z_1 + \frac{s}{t} Z_2, r_1 \le s \le t \le t - r_2\right] \\ \le \frac{2}{t - r_1 - r_2} \prod_{i=1,2} \left(Z(r_i) + \sqrt{r_i}\right) \quad . \end{aligned} \tag{5.26}$$

5.2 Simulations

In this section we will present a Matlab code, that represents the BBM. We start with a certain number of particles, which we choose distributed according to a Poisson process in the plane with the scaled Lebesgue mean measure. Each particle performs a Brownian motion in the plane and has a lifetime that is distributed as an exponential random variable. When the lifetime ends, the particles either divides into two daughter particles with probability 1−van_pr or dies with a probability van_pr. If we set van_pr= 0, we are choosing a BBM without selection: particles never die and we can see that the resulting graph is a cloud of particles that keep growing in size and shape (fig.5.1). Otherwise we can add a noise, asking that particles can die with a certain probability. The resulting graph (fig.5.2) shows a significant difference in the number of alive particles.

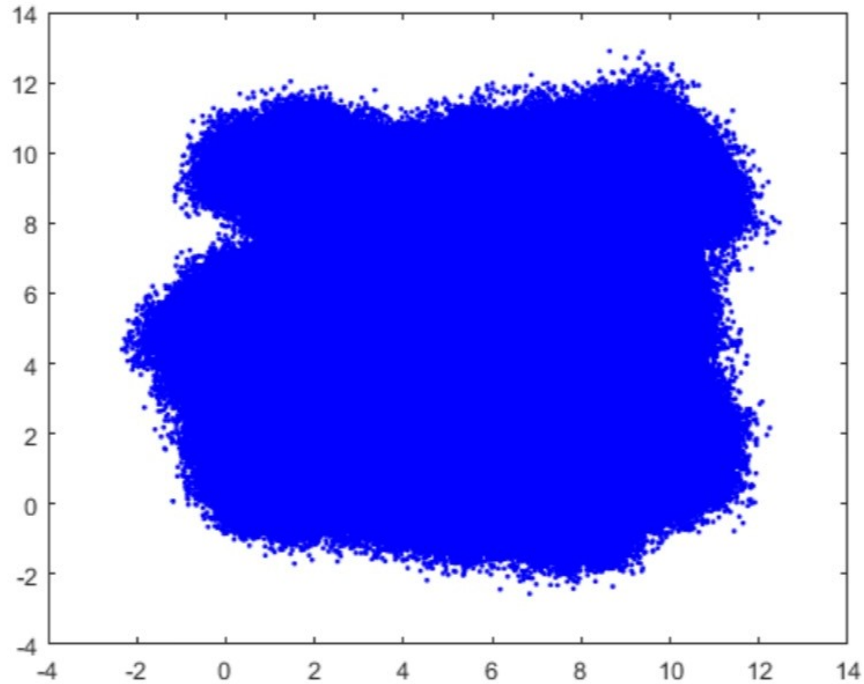


Figure 5.1: A BBM is a cloud of particles growing in size and shape

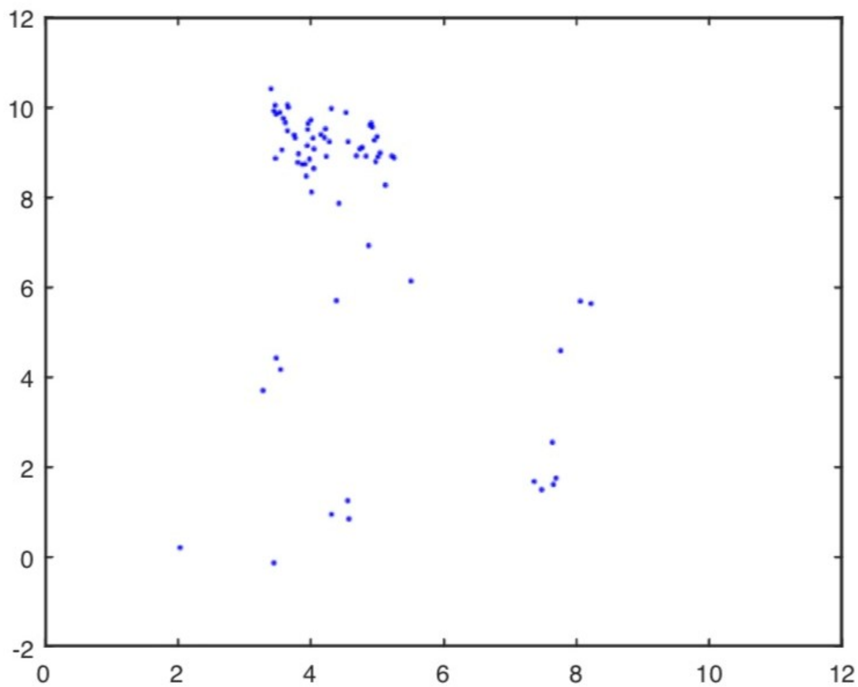


Figure 5.2: A BBM with selection

```
function [pt_conf, fmat]=multibbm(nu, lambda, van_pr, maxt, dt, ...
                 domain)
% MULTIBBM Simulate a branching Brownian motion in the plain and make
%   an animation.
%
%   Initially particles are distributed according to a Poisson
%   process in the plain with the scaled Lebesgue mean measure. The
%   particle follows a Brownian motion in the plane and has
%   exp(lambda) distributed lifetime. When the lifetime ends, the
%   particle either divides into two particles with probability 1-van_pr
%   or dies with probability van_pr.
%
% [pt_conf]=multibbm(nu, lambda, van_pr, maxt
%                   [, dt, domain])
%
% Inputs:
%   nu - intensity of the Poisson process for the initial
%     configuration
%   lambda - parameter of the exponential distribution
%     of the lifetime (note that expectation=1/lambda)
%   van_pr - probability for a particle to vanish
%   maxt - time interval
%   dt - time discretisation step. Optional, default dt=0.01.
%   domain - bounds for the region. A 4-dimensional vector in
%     the form [x_min x_max y_min y_max]. Optional, default value
%     [0 10 0 10].
%
% Outputs:
%   pt_conf - "configuration of the particles". A cell array
%     describing the system dynamics. An element k is a N_k x 2
%     matrix with the coordinates of the particles alive after time
%     k*dt.
%

% Authors: R.Gaigalas, I.Kaj
% v1.8 Created 07-Nov-01
%      Modified 24-Nov-05 Changed variable names and comments
%      Modified 10-Jan-06 Corrected the bug with maxt and
%        parenthesis in l114

 if (nargin<1) % default parameter values
   nu = 0.7; % intensity of the Poisson process
   lambda = 20.0; % parameter of the lifetime distribution
   van_pr = 0.5;  % probability to vanish
   maxt = 0.7;   % time interval
 end

 if (nargin<5) % default parameter values
   dt = 0.01;
   domain = [0 10 0 10];
 end

 disp('Generating BBM');
```

```
xmin = domain(1); % bounds of the initial domain
xmax = domain(2);
ymin = domain(3);
ymax = domain(4);
clear domain;

% initial number of particles Poisson distributed
% with intensity proportional to the area
area = (xmax-xmin)*(ymax-ymin);
ini_part = poissrnd(nu*area)

%given the number of particles, coordinates are uniformly
% distributed in the plain
pt_coor = rand(ini_part, 2);
pt_coor(:, 1) = pt_coor(:, 1)*(xmax-xmin)+xmin;
pt_coor(:, 2) = pt_coor(:, 2)*(ymax-ymin)+ymin;

% remaining lifetime - exp(lambda) distributed
pt_life = -1/lambda*log(rand(ini_part, 1));
% create the first frame for the movie
pt_conf = cell(1, 1);
pt_conf{1} = pt_coor;

extinct = 0; % flag for the whole process
nsteps = round(maxt/dt)+1;

c_step = ceil(nsteps/100);

for t=2:nsteps

  % shorten the remaining lifetime by dt
  pt_life = pt_life-dt;

  % generate new coordinates for the particles alive:
  %     add the increment of BM
  i_alive = find(pt_life>0);
  pt_coor(i_alive, :) = pt_coor(i_alive, :) ...
                        +randn(length(i_alive), 2)*dt^0.5;

  % find dying particles and decide if they will die
  % or will divide
  i_dying = find(pt_life<=0);
  runi = rand(1, length(i_dying));
  % set the vanished particles to NaN
  pt_life(i_dying(find(runi<=van_pr))) = NaN;

  % replicate the particles that need that
  i_rep = i_dying(find(runi>van_pr));
  nrep = length(i_rep);
  % generate new lifetimes for the parents
  pt_life(i_rep) = -1/lambda*log(rand(nrep, 1));
  % generate lifetimes for the children and add to the array
  pt_life = [pt_life; -1/lambda*log(rand(nrep, 1))];

  % add the coordinates of one of the newborn particles to the
  % array
  pt_coor = [pt_coor; pt_coor(i_rep, :)];

  % create a new frame
  pt_conf = [pt_conf cell(1, 1)];
  % add the particles alive and the "second child" particles
  pt_conf{t} = [pt_coor(i_alive, :); pt_coor(i_rep, :)];

```

```
  % test if there are any particles alive
  if all(isnan(pt_life))
    extinct = 1;
    break;
  end

  % display the progress
  if (rem(t, c_step)==0)
     fprintf('\r %i%% done', t/c_step);
  end
end

fprintf(1, '\r 100%% done\n');

if (extinct)
  fprintf('\nExtinct after t=%f\n',(t-1)*dt);
else
  fprintf('\n%d particles alive after t=%f\n', ...
          size(pt_conf{nsteps}, 1), maxt);
end

% animate
figure(1)
fmat = bbmplot(pt_conf, 1);
```

Bibliography

[1] L.P. Arguin, A. Bovier, N.Kistler, *Genealogy of extremal particles in branching Brownian motion*, Comm. Pure Appl. Math. **64**, 2011.

[2] L.P. Arguin, A.Bovier, N.Kistler, *Poissonian statistics in the extremal process of branching Brownian motion*, Annals Appl. Probab., 22, 1693-1711, 2012.

[3] L.P. Arguin, A.Bovier, N.Kistler, *The extremal process of branching Brownian motion*, Probab. Theory Related Fields, 157, 535-574, 2013.

[4] J.Berestycki, *Topics on branching Brownian motion*, University of Oxford, EBP XVIII, August 7, 2014.

[5] M.Bramson, *Maximal displacement of branching Brownian motion*, Comm. Pure Appl. Math. **31**, 31, 531-581, 1978.

[6] M.Bramson, *Convergence of solutions of the Kolmogorov equation to traveling waves*, Mem. Amer. Math. Soc., 44, 285, 1983.

[7] A.Bovier, *Extreme values of random processes*, Insitute for Applied Mathematics, Friedrich-Wilhelms-Univeritat Bonn.

[8] A.Bovier, *From spin glasses to branching Brownian motion-and back?*, Insitute for Applied Mathematics, Friedrich-Wilhelms-Univeritat Bonn, 2015.

[9] A.Bovier, *Statistical mechanics, extreme values and disordered systems*, Insitute for Applied Mathematics, Rheinische Friedrich-Wilhelms-Univeritat Bonn, 2017.

[10] B. Chauvin and A. Rouault, F-KPP *equation and supercritical branching Brownian motion in the supercritical speed area. Application to spatial trees.*, Probab. Theory Related Fields,80, 299-314, 1988.

[11] A.Karr, *Point processes and their statistical inference*, 1991.

[12] S.P. Lalley and T.Sellke, *A conditional limit theorem for frontier of a branching Brownian motion*, Ann. Prob., 15, 1052-1061, 1987.

[13] P.Maillard, *Branching Brownian motion with selection*, Université Pierre et Marie Curie, Paris VI, 2012.

[14] H.P. McKean, *Application of Brownian motion to the equation of Kolmogorv-Petrovskii-Piskunov*, Comm. Pure Appl. Math, 28, 323-331, 1975.

[15] O.Zeitouni, *Branching random walks and Gaussian fields. Notes for Lectures*, Department of Mathematics, Weizmann Institute, Rehovot, Israel and University of Minnesota, Minneapolis, USA, 2012.