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Corso di Laurea in Fisica

Tesi di Laurea

The Thomas-Bargmann-Michel-Telegdi equation and the Muon $g - 2$ experiment

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Anno Accademico 2020/2021

Abstract

We present a theoretical analysis of the relativistic theory of spin and how it can be used to understand the principles behind the recent Muon $g-2$ (E989) experiment at Fermilab. First, we discuss a covariant generalization of spin and its equation of motion in the presence of electromagnetic fields, the Thomas-Bargmann-Michel-Telegdi equation. We then generalize this equation to account for an electric dipole moment and a general curved spacetime. Finally, after showing why the g-factor is predicted to be $g = 2$ by the Dirac equation, we focus on the E989 experiment at Fermilab and how the anomalous magnetic moment of the muon $a = \frac{g-2}{2}$ is actually measured.

Contents

1 Introduction

For several decades the Standard Model (SM) of particle physics has been one of the pillars of our understanding of the universe, describing three out of the four (known) fundamental forces and all the particles composing ordinary matter. In the hope of unveiling unknown phenomena, physicists have been testing the SM for years with many different experiments, without however obtaining clear evidence of its violation.

Some of the experiments where the SM has shown its predictive power at its best are the measurements of the anomalous magnetic dipole moments of leptons. While in relativistic quantum mechanics these particles are described by the Dirac equation and are thus predicted to have a g-factor $g = 2$, quantum field theory predicts a non-zero value for the anomalous magnetic moment $a = \frac{g-2}{2}$ due to what can be interpreted as the interactions of the leptons with virtual particles. In particular, the g-factor g_e of the electron has been measured with a precision better than a part per trillion [1][1] in what is – as of today – one of the most precise measurement ever made in particle physics.

While measuring a_μ and a_τ experimentally is harder than measuring a_e since muons and taus are unstable, in many SM extensions the contribution of "new physics" of energy scale Λ to the anomalous magnetic moment a_ℓ of a lepton ℓ is expected to scale as m_ℓ^2/Λ^2, where m_ℓ is the mass of the lepton. As a consequence of this trade off, experiments on muons – even if with a larger uncertainty than the ones on electrons – have achieved a high enough precision to be the best candidates for the search of new physics in the anomalous magnetic moments of leptons. In particular, in a series of recent papers released earlier this year [5, 6, 7, 8] the Muon $g-2$ Collaboration, behind the E989 experiment at Fermilab, announced the measurement[2] of the muon a_μ with a precision of 0.46 ppm, confirming a previous result obtained by the E821 experiment at Brookhaven National Laboratory in the early 2000s [9] and increasing the discrepancy with the SM prediction to 4.2 σ [10]. While this result is only relative to Run-1 of E989 and is not yet at the 5 σ confidence level, it strongly hints at new physics beyond the SM. On the other hand, a recent lattice QCD result weakens the discrepancy between the SM prediction and experiments [11].

Even though the theoretical prediction of the muon anomalous magnetic moment requires the use of quantum field theory (for a concise review see [12]), the principles behind the E989 experiment can be understood with a simple relativistic theory of spin. This theory can be constructed by replacing the three-dimensional spin of

[1] There are currently some problems regarding the comparison of the theoretical prediction of g_e with its experimental value, since there are two different measurements of the fine-structure constant α [2, 3] which differ by more than 5 σ and thus yield different theoretical values for g_e [4].

[2] The E989 experiment is still running – aiming to obtain an uncertainty of around 140 ppb – and has only released data from its first run, already reaching a higher precision than Brookhaven E821 and with an uncertainty dominated by the statistical error.

the particle with a four-dimensional vector called four-spin, satisfying a covariant equation of motion called the Thomas-Bargmann-Michel-Telegdi (TBMT) equation.

In section 2 we will therefore briefly introduce the concept of spin and discuss different ways to obtain a relativistic equation for its motion, focusing in particular on deriving the phenomenon of Thomas precession and the TBMT equation, loosely following the approach given by Jackson in [13]. In section 3 we will analyze some of the possible generalizations of the TBMT equation. First we will discuss an alternative formulation of the TBMT equation for the spin tensor $S^{\mu\nu}$ (defined in a way similar to the relativistic angular momentum tensor $L^{\mu\nu}$) and its relation to S^{μ}; we will then modify the regular TBMT equation for the four-spin to account for the presence of an electric dipole moment associated to the spin, another quantity that can be measured in the search for SM violations. We will also obtain a generalized expression of the TBMT equation which is valid in curved spacetime through the introduction of the Fermi-Walker derivative, in an attempt to extend the concepts of Thomas precession and of particle rest frame to general relativity. In section 4 we will discuss the theoretical explanation of why the g-factor is predicted to be $g = 2$ for a spin-$\frac{1}{2}$ particle described by the Dirac equation, briefly introducing the operators associated to angular momentum and the minimal coupling with the electromagnetic field. Finally in section 5 we will discuss the experimental setup of the E989 experiment, how a_μ is actually determined and some of the systematic corrections that must be taken into account to obtain an accurate measurement of a_μ. Our conclusion will be drawn in section 6.

2 Thomas precession and the TBMT equation for the four-spin

2.1 Introduction to spin

The concept of spin was introduced by Uhlenbeck and Goudsmit [14] in 1925 while studying atomic spectra, in order to explain fine-structure intervals and the anomalous Zeeman effect. They hypothesized that an electron possesses an intrinsic angular momentum $\boldsymbol{s}$ called spin, whose value when measured along an axis can only be $\boldsymbol{s} \cdot \hat{\boldsymbol{n}} = \pm\hbar/2$. This quantized behaviour shows that spin is a fundamentally quantum mechanical phenomenon, and in the case of a non-relativistic electron (or other spin-$\frac{1}{2}$ particles) it can be described in terms of Pauli matrices by $\boldsymbol{s} = \frac{\hbar}{2}\boldsymbol{\sigma}$. In order to find a relativistic generalization of spin it is however easier to consider $\boldsymbol{s}$ as a three-dimensional vector rather than an operator.

Since a classical particle with angular momentum $\boldsymbol{\ell}$ has an associated magnetic moment $\boldsymbol{\mu}_{\boldsymbol{\ell}} = \frac{e}{2m}\boldsymbol{\ell}$, it is possible to associate a magnetic moment to the spin with the similar relation

$$\boldsymbol{\mu}_{\boldsymbol{s}} = g\frac{e}{2m}\boldsymbol{s}, \tag{2.1}$$

where the g-factor is a constant that distinguishes the behaviour of the spin magnetic

moment from the angular magnetic moment.
Given that the torque on the angular magnetic moment in a magnetic field $\boldsymbol{B}$ is $\boldsymbol{\mu_\ell} \times \boldsymbol{B}$ and its associated energy is $-\boldsymbol{\mu_\ell} \cdot \boldsymbol{B}$, it is possible to obtain – extending the analogy – the equation of motion for the spin $\boldsymbol{s}$ in the rest frame of the particle and its associated energy

$$\begin{aligned} \frac{d\boldsymbol{s}}{dt} &= \frac{e}{m}\frac{g}{2}\boldsymbol{s} \times \boldsymbol{B} \\ U &= -\frac{e}{m}\frac{g}{2}\boldsymbol{s} \cdot \boldsymbol{B}. \end{aligned} \tag{2.2}$$

In order to extend these equations to a relativistic moving particle and thus express $\frac{d\boldsymbol{s}}{dt}$ in terms of the electric and magnetic field seen by laboratory, we might be tempted to follow a naive approach: we simply replace in the above equation the time t of the rest frame with the proper time τ of the moving particle and $\boldsymbol{B}$ with the magnetic field $\boldsymbol{B'}$ in an instantaneous reference frame co-moving with the particle. This reference frame is obtained via a single boost from the laboratory frame; since the Lorentz transformations for the electromagnetic fields are

$$\begin{aligned} \boldsymbol{E'} &= \gamma\left(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta} \cdot \boldsymbol{E}\right)\boldsymbol{\beta} \\ \boldsymbol{B'} &= \gamma\left(\boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta} \cdot \boldsymbol{B}\right)\boldsymbol{\beta}, \end{aligned} \tag{2.3}$$

the equation of motion for the spin $\boldsymbol{s}$ as a function of the laboratory time t would then become

$$\frac{d\boldsymbol{s}}{dt} = \frac{e}{\gamma m}\frac{g}{2}\boldsymbol{s} \times \boldsymbol{B'}. \tag{2.4}$$

Moreover, if we consider a non-relativistic electron with charge $-q_e$ orbiting around a nucleus with a radial electric field $\boldsymbol{E} = -\hat{\boldsymbol{r}}\frac{dV(r)}{dr}$, we can approximate to first order in $\boldsymbol{\beta}$ the energy and obtain

$$U = \frac{gq_e}{2m}\boldsymbol{s} \cdot \left(\boldsymbol{B} - \frac{\boldsymbol{\ell}}{mr}\frac{dV}{dr}\right), \tag{2.5}$$

with $\boldsymbol{\ell} = m\boldsymbol{r} \times \boldsymbol{\beta}$ the orbital angular momentum. This expression correctly predicts the anomalous Zeeman effect if $g = 2$ but gives a spin-orbit contribution that is twice as large as what is observed experimentally.

2.2 The particle rest frame

As shown by Thomas [15], the partially incorrect conclusions of this naive approach (equations (2.4) and (2.5)) are not due to the fact that we have not used operators to describe the spin, but are instead caused by a relativistic effect, nowadays called the Thomas precession.

In particular the problem in our previous argument lies in the definition of the rest frame of the particle, in which (2.2) holds true. Let us define the rest frame of the particle as the succession of reference frames where the particle is instantaneously at rest, with each frame obtained from the previous one via a pure (infinitesimal) boost.

To be more concrete, let us consider a particle moving with velocity $\boldsymbol{\beta}$ at time t. A generic Lorentz transformation can be written as $\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$, with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ the six parameters determining the transformation, and $J^{\mu\nu} = -J^{\nu\mu}$ the representatives of the basis of $\mathfrak{so}(1,3)$ on the generic vector space V in which the Lorentz transformations acts, satisfying the commutation relation [16][3]

$$\left[J^{\mu\nu}, J^{\alpha\beta}\right] = 2i\left(J^{\alpha[\mu}g^{\nu]\beta} - J^{\beta[\mu}g^{\nu]\alpha}\right). \tag{2.6}$$

In three-dimensional notation, it is possible to identify $\eta^i = \omega^{0i}$ as the rapidity of the boost, $\theta^i = -\frac{1}{2}\epsilon^{ijk}\omega^{jk}$ as the angle of rotation, $K^i = J^{0i}$ and $R^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ the generators of boosts and rotations respectively, satisfying

$$\left[R^i, R^j\right] = i\epsilon^{ijk}R^k,\ \left[R^i, K^j\right] = i\epsilon^{ijk}K^k,\ \left[K^i, K^j\right] = -i\epsilon^{ijk}R^k. \tag{2.7}$$

The Lorentz transformation from the lab frame to the rest frame of the particle at time t is therefore $\Lambda(t) = e^{i\boldsymbol{\eta}\cdot\boldsymbol{K}}$, where $\boldsymbol{\eta} = \left(\tanh^{-1}\beta\right)\hat{\boldsymbol{\beta}}$ is the particle rapidity. After a time δt – in the lab frame – the particle gains a velocity $\boldsymbol{\delta\beta'}$ (and a rapidity $\boldsymbol{\delta\eta'}$) in its rest frame and the Lorentz transformation from the lab to its rest frame at time $t+\delta t$ is thus, according to our definition of rest frame, $\Lambda(t+\delta t) = e^{i\boldsymbol{\delta\eta'}\cdot\boldsymbol{K}}e^{i\boldsymbol{\eta}\cdot\boldsymbol{K}}$, with $e^{i\boldsymbol{\delta\eta'}\cdot\boldsymbol{K}} \approx \mathbb{1} + i\boldsymbol{\delta\eta'}\cdot\boldsymbol{K}$.

The key point is that, since Lorentz transformations do not commute, the transformation $\Lambda(t+\delta t)$ is not a pure boost from the lab frame, but is a combination of a boost and a rotation, contrary to what was assumed when obtaining (2.4); if the transformation were a pure boost, it would simply be $\Lambda'(t+\delta t) = e^{i(\boldsymbol{\eta}+\boldsymbol{\delta\eta})\cdot\boldsymbol{K}}$, with $\boldsymbol{\eta}+\boldsymbol{\delta\eta}$ the particle rapidity in the lab frame at time $t+\delta t$. Both $\Lambda(t+\delta t)$ and $\Lambda'(t+\delta t)$ are transformations from the lab frame to a frame moving with velocity $\boldsymbol{\beta}+\boldsymbol{\delta\beta}$ (with respect to the lab), therefore they can only differ by a rotation which can be computed by determining $\Delta = \Lambda(t+\delta t)\Lambda'^{-1}(t+\delta t)$.

2.3 Thomas precession

In order to evaluate this relative transformation, it is convenient to rewrite Λ'^{-1} as $\Lambda'^{-1} = e^{-i\boldsymbol{\eta}\cdot\boldsymbol{K}}A$ for some matrix A, so that $\Delta = \left(\mathbb{1} + i\boldsymbol{\delta\eta'}\cdot\boldsymbol{K}\right)A$. Expanding the matrix exponential as a power series and keeping only the terms up to first order in

[3] We are following the sign convention on the exponential and thus on the commutators used in [16], together with the metric signature $(+,-,-,-)$. For a more in depth yet not too technical analysis on $SO(1,3)$ see [17].

$\boldsymbol{\delta\eta}$ yields

$$e^{-i(G+\delta G)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[G^n + G^{n-1}(\delta G) + G^{n-2}(\delta G)G + \right. \\ \left. \cdots + G(\delta G)G^{n-2} + (\delta G)G^{n-1} \right], \tag{2.8}$$

where $G = \boldsymbol{\eta}\cdot\boldsymbol{K}$ and $\delta G = \boldsymbol{\delta\eta}\cdot\boldsymbol{K}$ have been introduced for convenience. It is possible to prove – by repeatedly exchanging δG and G with the help of their commutator – that the following relation holds for the n-th term in the square brackets:

$$\begin{aligned}[\dots]_n =& \binom{n}{0} G^n + \binom{n}{1} G^{n-1}(\delta G) + \binom{n}{2} G^{n-2}[\delta G, G] + \\ & \binom{n}{3} G^{n-3}[[\delta G, G], G] + \binom{n}{4} G^{n-4}[[[\delta G, G], G], G] \\ & + \dots + \binom{n}{n} [[\dots[\delta G, G], \dots G], G].\end{aligned} \tag{2.9}$$

Finally, isolating the terms with a different number of (nested) commutators and recomposing the matrix exponentials (by changing the summation index) yields

$$e^{-i(G+\delta G)} = e^{-iG} \left(\mathbb{1} - i\delta G + \frac{(-i)^2}{2!}[\delta G, G] + \frac{(-i)^3}{3!}[[\delta G, G], G] + \dots \right). \tag{2.10}$$

The commutators can be easily computed by recursion using (2.7):

$$\begin{aligned}&[\delta G, G] = -i\,(\boldsymbol{\delta\eta}) \cdot (\boldsymbol{\eta} \times \boldsymbol{R}) \\ &[[\delta G, G], G] = (\boldsymbol{\delta\eta} \cdot \boldsymbol{\eta})\, G - \eta^2\,(\delta G) \\ &[[[\delta G, G], G], G] = -\eta^2 [\delta G, G].\end{aligned} \tag{2.11}$$

By recognizing the Taylor series at $\eta = 0$ of $\cosh\eta$ and $\sinh\eta$ when plugging (2.11) into (2.10), we can express the matrix A as

$$A = \mathbb{1} - i \left(\frac{\sinh\eta}{\eta} \right) \boldsymbol{\delta\eta} \cdot \boldsymbol{K} + i \frac{\sinh\eta - \eta}{\eta^3} (\boldsymbol{\delta\eta} \cdot \boldsymbol{\eta})\, \boldsymbol{\eta} \cdot \boldsymbol{K} + i \frac{\cosh\eta - 1}{\eta^2} \boldsymbol{\delta\eta} \cdot (\boldsymbol{\eta} \times \boldsymbol{R}). \tag{2.12}$$

The last step in order to obtain Δ is to express $\boldsymbol{\eta}$, $\boldsymbol{\delta\eta}$ and $\boldsymbol{\delta\eta'}$ in terms of $\boldsymbol{\beta}$ and $\boldsymbol{\delta\beta}$. Considering only terms up to first order – as done previously – $\boldsymbol{\delta\eta'} = \boldsymbol{\delta\beta'}$; $\boldsymbol{\delta\beta'}$ can be obtained from the velocity $\boldsymbol{\beta} + \boldsymbol{\delta\beta}$ thanks to the formula for the relativistic addition of velocities, and therefore $\boldsymbol{\delta\beta'} = \gamma \left(\boldsymbol{\delta\beta}_\perp + \gamma \boldsymbol{\delta\beta}_{||} \right)$. By variation of the definition of rapidity, $\boldsymbol{\delta\eta} = \frac{\eta}{\beta} \boldsymbol{\delta\beta}_\perp + \gamma^2 \boldsymbol{\delta\beta}_{||}$. Finally combining all these relations together yields, after some algebra,

$$\Delta = \mathbb{1} + i \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\delta\beta}_\perp \times \boldsymbol{\beta}) \cdot \boldsymbol{R} = \mathbb{1} + i \boldsymbol{\delta\Omega} \cdot \boldsymbol{R}. \tag{2.13}$$

What we have defined as the rest frame of the particle is therefore rotating with respect to the laboratory frame with angular velocity

$$\boldsymbol{\omega_T} = \frac{\boldsymbol{\delta\Omega}}{\delta t} = \frac{\gamma^2}{\gamma+1}\frac{d\boldsymbol{\beta}}{dt} \times \boldsymbol{\beta}, \tag{2.14}$$

a phenomenon now called Thomas precession. This means that the result (2.4) obtained earlier is valid for a reference frame which is stationary with respect to the lab, but which is also rotating. The correct equation of motion for the spin is therefore

$$\frac{d\boldsymbol{s}}{dt} = \left(\frac{d\boldsymbol{s}}{dt}\right)_{rot} + \boldsymbol{\omega_T} \times \boldsymbol{s} = \boldsymbol{s} \times \left(\frac{e}{\gamma m}\frac{g}{2}\boldsymbol{B'} - \boldsymbol{\omega_T}\right). \tag{2.15}$$

If we now consider a particle moving in an electromagnetic field with no other external forces and ignore the contribution to the force due to a magnetic dipole moment in a non-uniform field, its equation of motion $\frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}\right)$ can be expressed as

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma m}\left[\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B} - \boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \boldsymbol{E}\right)\right]. \tag{2.16}$$

Plugging this result into (2.15) yields

$$\frac{d\boldsymbol{s}}{dt} = \frac{e}{m}\boldsymbol{s} \times \left[\left(a + \frac{1}{\gamma}\right)\boldsymbol{B} - a\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta} \cdot \boldsymbol{B}\right)\boldsymbol{\beta} - \left(a + \frac{1}{\gamma+1}\right)\boldsymbol{\beta} \times \boldsymbol{E}\right], \tag{2.17}$$

where

$$a = \frac{g}{2} - 1 \tag{2.18}$$

is called the anomalous magnetic moment. If we now consider a non-relativistic electron, the correct version of equation (2.5) becomes

$$U = \frac{q_e}{m}\boldsymbol{s} \cdot \left(\frac{g}{2}\boldsymbol{B} - \frac{g-1}{2}\frac{\boldsymbol{\ell}}{mr}\frac{dV}{dr}\right). \tag{2.19}$$

Note that this equation correctly predicts both the anomalous Zeeman effect and fine-structure intervals due to spin-orbit interactions (with $g = 2$). Note also that this equation can be also interpreted as the hamiltonian for a non-relativistic quantum mechanical system, by simply replacing $\boldsymbol{\ell}$ and $\boldsymbol{s}$ with their corresponding operators; this approach can be used in perturbation theory to study the fine structure of the hydrogen atom.

2.4 Four-spin

While (2.15) is the correct equation of motion for the spin of a relativistic particle, the equation itself is not manifestly covariant and it only describes the spin $\boldsymbol{s}$ in the particle reference frame in terms of other quantities measured in the laboratory. In order to find a more elegant expression – that clearly holds in every reference frame

– it is necessary to generalize the three-dimensional vector $\boldsymbol{s}$ to a covariant tensor. There are two main possible approaches: one consists in generalizing the spin in analogy to what is done with the non-relativistic angular momentum, defining an antisymmetric spin tensor $S^{\mu\nu}$; the other instead considers a four-vector S^μ called four-spin. Both $S^{\mu\nu}$ and S^μ must reduce to the ordinary spin $\boldsymbol{s}$ for a particle at rest. While these two approaches are both valid, we will now concentrate on the latter, which is by far the most popular (we will discuss the former in section 3.1).

In order for a four-vector S^μ to describe the three-dimensional quantity $\boldsymbol{s}$, the only requirement that must be fulfilled is that S^μ reduces to $\boldsymbol{s}$ if the particle is stationary; this is accomplished if $S^0 = 0$ and $S^i = s^i$ when $\boldsymbol{\beta} = 0$ (and the four-velocity $U^\mu = \big(1,\, 0,\, 0,\, 0\big)$). This constraint can thus be expressed covariantly as

$$U_\mu S^\mu = 0. \tag{2.20}$$

It is also expected, since (2.15) shows that the spin precesses but does not change in magnitude, that the following relation holds:

$$S^\mu S_\mu = -\boldsymbol{s}\cdot\boldsymbol{s} = \text{const.} \tag{2.21}$$

It is possible to obtain the component of the four-spin explicitly as a function of $\boldsymbol{s}$ using a Lorentz transformation:

$$S^\mu = \Big(\gamma\boldsymbol{\beta}\cdot\boldsymbol{s}, \quad \boldsymbol{s} + \tfrac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{s}\right)\boldsymbol{\beta}\Big). \tag{2.22}$$

2.5 The TBMT equation

Let us now find a relativistic equation of motion for S^μ. As suggested by (2.15), we expect that the equation is of the form

$$\frac{dS^\mu}{d\tau} = A_1 F^{\mu\nu} S_\nu + A_2 U^\mu \Big(F^{\alpha\beta} S_\alpha U_\beta\Big) + A_3 U^\mu \left(W_\nu S^\nu\right), \tag{2.23}$$

where the first term is suggested by $\boldsymbol{s}\times\boldsymbol{B}$, the second term by $\boldsymbol{s}\times(\boldsymbol{\beta}\times\boldsymbol{E})$ and the third by $\boldsymbol{s}\times\Big(\frac{d\boldsymbol{\beta}}{dt}\times\boldsymbol{\beta}\Big)$; note also that $W^\mu = \frac{dU^\mu}{d\tau}$ and A_1, A_2 and A_3 are constants to be determined[4].
Applying the total derivative to condition (2.20), however, implies that $\frac{dS^\mu}{d\tau}U_\mu + S^\mu W_\mu = 0$; if we insert (2.23) in this last relation we obtain

$$\left(A_1 + A_2\right) F^{\mu\nu} U_\mu S_\nu + \left(A_3 + 1\right) S^\mu U_\nu = 0. \tag{2.24}$$

This implies that for the equation of motion to satisfy (2.20), $A_2 = -A_1$ and $A_3 = -1$. Finally, A_1 can be obtained by comparing (2.23) with the equation (2.2) for $\boldsymbol{s}$

[4] Possible terms like $W^\mu\left(U_\nu S^\nu\right)$ have been discarded as equal to 0 due to the constraint (2.20).

when the particle is at rest; the correct equation of motion for the four-spin thus becomes

$$\frac{dS^\mu}{d\tau} = \frac{e}{m}\frac{g}{2}\left[F^{\mu\nu}S_\nu + U^\mu\left(F^{\alpha\beta}S_\alpha U_\beta\right)\right] - U^\mu\left(W_\nu S^\nu\right). \tag{2.25}$$

If we neglect the force exerted on the magnetic moment in a non-uniform field, we can take the equation for a particle in an electromagnetic field

$$\frac{dU^\mu}{d\tau} = \frac{e}{m}F^{\mu\nu}U_\nu, \tag{2.26}$$

and combine it with (2.25), obtaining the Thomas-Bargmann–Michel–Telegdi equation [18]

$$\frac{dS^\mu}{d\tau} = \frac{e}{m}\left[\frac{g}{2}F^{\mu\nu}S_\nu + aU^\mu\left(F^{\alpha\beta}S_\alpha U_\beta\right)\right]. \tag{2.27}$$

Note that (2.25) satisfies both the constraints (2.20) and (2.21). It is also possible to show, albeit with quite a bit of algebra and with the help of (2.16) and (2.22), that the TBMT equation reduces to the relativistic equation of motion (2.17) for the spin $\boldsymbol{s}$ obtained heuristically in section 2.3.

3 Generalizations of the TBMT equation

3.1 The spin tensor

As suggested previously, it is also possible to generalize $\boldsymbol{s}$ with an antisymmetric spin tensor $S^{\mu\nu}$, instead of the four-spin S^μ, that mimics more closely the behaviour of the angular momentum tensor $L^{\mu\nu}$. We can build a theory for $S^{\mu\nu}$ following the exact logic of sections 2.4 and 2.5.

Since the connection between non-relativistic and relativistic angular momentum is given by $\ell^i = \frac{1}{2}\epsilon^{ijk}L^{ij}$, we expect $s^i = \frac{1}{2}\epsilon^{ijk}S^{jk}$ to hold as well in the rest frame of the particle, along with the other components satisfying $S^{0i} = 0$. This last constraint can be expressed as

$$S^{\mu\nu}U_\nu = 0, \tag{3.1}$$

a condition similar to (2.20). Once again we also expect

$$S^{\mu\nu}S_{\mu\nu} = 2\boldsymbol{s}\cdot\boldsymbol{s} = \text{const}. \tag{3.2}$$

Finally, we can guess the terms of the equation of motion for the spin tensor taking inspiration from (2.15) or (2.25); taking into account the fact that the spin tensor is antisymmetric, we expect the equation of motion to be of the form

$$\frac{dS^{\mu\nu}}{d\tau} = A_1' F^{\alpha[\mu}S^{\nu]}{}_\alpha + A_2' U^{[\mu}S_\alpha{}^{\nu]}F^{\alpha\beta}U_\beta + A_3' U^{[\mu}S_\alpha{}^{\nu]}W^\alpha. \tag{3.3}$$

Imposing the condition $\frac{dS^{\mu\nu}}{d\tau}U_\nu + S^{\mu\nu}W_\nu = 0$ and obtaining the remaining constants – as done previously – by comparing (3.3) with the rest frame equation (2.2), we get the equivalent of (2.25) for the spin tensor:

$$\frac{dS^{\mu\nu}}{d\tau} = g\frac{e}{m}\left(F^{\alpha[\mu}S^{\nu]}{}_\alpha + U^{[\mu}S_\alpha{}^{\nu]}F^{\alpha\beta}U_\beta\right) - 2U^{[\mu}S_\alpha{}^{\nu]}W^\alpha. \tag{3.4}$$

Combining this equation with (2.26) it is finally possible to obtain an alternative TBMT equation for the spin tensor:

$$\frac{dS^{\mu\nu}}{d\tau} = \frac{2e}{m}\left(\frac{g}{2}F^{\alpha[\mu}S^{\nu]}{}_\alpha + aU^{[\mu}S_\alpha{}^{\nu]}F^{\alpha\beta}U_\beta\right). \tag{3.5}$$

Comparing this equation with the proper TBMT equation for the four-spin allows us to determine that it is possible to convert from the two different generalizations of $\boldsymbol{s}$ via the following duality transformations:

$$\begin{aligned} S^{\mu\nu} &= \epsilon^{\mu\nu\alpha\beta}U_\alpha S_\beta \\ S^\mu &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}U_\nu S_{\alpha\beta}. \end{aligned} \tag{3.6}$$

While the four-spin S^μ is used in most applications, the spin tensor $S^{\mu\nu}$ can be useful when discussing symmetry transformations. Since $L^{\mu\nu}$ is a tensor under parity reversal and a pseudo-tensor under time reversal, the spin tensor is expected to behave in the same way; this also implies that, thanks to the above duality transformations, S^μ is a pseudo-vector under both time and parity reversals. Note that the behaviour of S^μ can also be inferred from (2.22) given that $\boldsymbol{s}$, like $\boldsymbol{\ell}$, is a pseudo-vector under parity and time reversals.

Finally, the spin tensor is also useful in expressing the energy (2.2) associated to the spin in the particle rest frame as

$$U = \frac{e}{m}\frac{g}{4}S^{\mu\nu}F_{\mu\nu}. \tag{3.7}$$

Note that $S^{\mu\nu}F_{\mu\nu}$ is a scalar under parity and time reversals and thus preserves both parity and time symmetry.

3.2 Electric dipole moment

One obvious generalization of the TBMT equation regards the possible presence of an electric dipole moment (EDM) associated with the spin $\boldsymbol{s}$. While in classical mechanics there is no EDM associated with the orbital angular momentum $\boldsymbol{\ell}$, it is quite easy to extend our theory to include the possibility of an EDM associated with the spin $\boldsymbol{s}$, following the same approach given in section 2.

Let us start from the rest frame of the particle and define the spin electric dipole moment by

$$\boldsymbol{d_s} = \eta\frac{e}{2m}\boldsymbol{s}. \tag{3.8}$$

The torque on an electric dipole $\boldsymbol{d}$ is $\boldsymbol{d} \times \boldsymbol{E}$, with an associated energy of $-\boldsymbol{d} \cdot \boldsymbol{E}$; we can therefore write the equation of motion for the spin in the rest frame of the particle as

$$\frac{d\boldsymbol{s}}{dt} = \frac{e}{m}\boldsymbol{s} \times \left(\frac{g}{2}\boldsymbol{B} + \frac{\eta}{2}\boldsymbol{E}\right), \tag{3.9}$$

in analogy to how we defined the magnetic dipole moment in (2.1). It is now possible to directly write the correct equation of motion for the spin for a relativistic moving particle, given the Lorentz transformations for the electromagnetic field (2.3) and taking into account the Thomas precession; we thus obtain

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{s} \times \left(\frac{e}{\gamma m}\frac{g}{2}\boldsymbol{B'} + \frac{\eta}{2}\frac{e}{\gamma m}\boldsymbol{E'} - \boldsymbol{\omega_T}\right). \tag{3.10}$$

If the only forces acting on the particle are electromagnetic and, once again, we neglect terms due to field gradients, the equation of motion can be written as

$$\begin{aligned}\frac{d\boldsymbol{s}}{dt} =& \frac{e}{m}\boldsymbol{s} \times \left[\left(a + \frac{1}{\gamma}\right)\boldsymbol{B} - a\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{B}\right)\boldsymbol{\beta} - \left(a + \frac{1}{\gamma+1}\right)\boldsymbol{\beta}\times\boldsymbol{E}\right] + \\ & \frac{e}{m}\boldsymbol{s} \times \left[b\boldsymbol{E} - b\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}\right)\boldsymbol{\beta} - b\boldsymbol{\beta}\times\boldsymbol{B}\right],\end{aligned} \tag{3.11}$$

where the quantity $b = \frac{\eta}{2}$ is defined in analogy to the anomalous magnetic moment $a = \frac{g}{2} - 1$.

Let us now try to find a generalized TBMT equation which also accounts for the presence of an EDM. We can once again start by guessing the form of the equation of motion as

$$\begin{aligned}\frac{dS^\mu}{d\tau} =& A_1 F^{\mu\nu} S_\nu + A_2 U^\mu \left(F^{\alpha\beta} S_\alpha U_\beta\right) + A_3 U^\mu \left(W_\nu S^\nu\right) + \\ & A_4 G^{\mu\nu} S_\nu + A_5 U^\mu \left(G^{\alpha\beta} S_\alpha U_\beta\right).\end{aligned} \tag{3.12}$$

Note that the additional terms we expect due to the EDM are almost identical to the one due to the magnetic moment, with the electromagnetic tensor $F^{\mu\nu}$ replaced by its dual $G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. Imposing the constraint (2.20) and determining the remaining coefficient by comparing equation (3.12) with the one in the rest frame, we obtain

$$\begin{aligned}\frac{dS^\mu}{d\tau} =& \frac{e}{m}\left[\left(\frac{g}{2}F^{\mu\nu} - \frac{\eta}{2}G^{\mu\nu}\right)S_\nu + U^\mu\left(\frac{g}{2}F^{\alpha\beta} - \frac{\eta}{2}G^{\alpha\beta}\right)S_\alpha U_\beta\right] \\ & - U^\mu \left(W_\nu S^\nu\right).\end{aligned} \tag{3.13}$$

Expressing S^μ in terms of $\boldsymbol{s}$ allows us to deduce correctly (3.10) from this equation for the four-spin. Using the equation of motion for the particle trajectory we can obtain the generalized TBMT equation which accounts for an EDM:

$$\frac{dS^\mu}{d\tau} = \frac{e}{m}\left[\left(\frac{g}{2}F^{\mu\nu} - \frac{\eta}{2}G^{\mu\nu}\right)S_\nu + U^\mu\left(aF^{\alpha\beta} - bG^{\alpha\beta}\right)S_\alpha U_\beta\right]. \tag{3.14}$$

Finally, using the spin tensor, the energy associated to the spin $\boldsymbol{s}$ in the rest frame of the particle can be written once again in a quasi-covariant form as

$$U = \frac{e}{2m} S^{\mu\nu} \left(\frac{g}{2} F_{\mu\nu} - \frac{\eta}{2} G_{\mu\nu} \right). \tag{3.15}$$

Note that, while $S^{\mu\nu} F_{\mu\nu}$ is a scalar, $S^{\mu\nu} G_{\mu\nu}$ is a pseudo-scalar under time and parity reversals, therefore the presence of an EDM directly breaks both time and parity symmetry of electromagnetism. Note also that the interaction of S^μ with the electromagnetic field for a particle with an EDM, as seen in both equation (3.13) and equation (3.15), is the same as the one from a regular particle with $\eta = 0$ in the presence of the field $F^{\mu\nu} - \frac{\eta}{g} G^{\mu\nu}$ (which corresponds to replacing $\boldsymbol{B}$ with $\boldsymbol{B} + \frac{\eta}{g}\boldsymbol{E}$ and $\boldsymbol{E}$ with $\boldsymbol{E} - \frac{\eta}{g}\boldsymbol{B}$). This sort of symmetry is however broken by the fact that the equation of motion (2.26) of the particle is the same both with and without an EDM, so the TBMT equation (3.14) cannot simply be obtained from (2.27) by replacing $F^{\mu\nu}$ with $F^{\mu\nu} - \frac{\eta}{g} G^{\mu\nu}$.

3.3 Curved spacetime

Another possible generalization of the TBMT equation regards the presence of gravity and thus of a curved spacetime. It is possible to obtain the correct generalization of the TBMT equation directly from (2.27) heuristically, by simply replacing the two total derivatives $\frac{d}{d\tau}$ with total covariant derivatives $\frac{D}{D\tau}$. If we want to show why this procedure gives the correct result, however, we need to generalize the concept of the Thomas precession for a particle moving in curved spacetime; in particular we want to work directly with a four-vector, instead of deriving the Thomas precession for a three-dimensional vector and then guessing the corresponding terms in the covariant equation of motion.

Let us start from a particle in a general spacetime and in the absence of electromagnetic fields. We expect that, in the rest frame of the particle, the spin interacts only with electromagnetic fields and not with gravity, since the rest frame equation (2.2) for $\boldsymbol{s}$ must hold; $\boldsymbol{s}$ will therefore be stationary in this frame. In order to describe the rest frame of the particle, let us consider a frame field along the particle trajectory with the time axis parallel to U^μ and that is not spatially rotating. Note that, thus far, we are just restating in a more formal manner the definition of the rest frame of the particle – in which (2.2) holds – given in section 2.2.

If the frame field is described by $e_a = e_a{}^\mu \partial_\mu$, with $e_0{}^\mu = U^\mu$, the four-spin can be described in the non-coordinate basis by the relation

$$S^\mu \partial_\mu = S^a e_a{}^\mu \partial_\mu = S^a e_a. \tag{3.16}$$

where the vierbeins $e_a{}^\mu \in \mathrm{GL}(4,\mathbb{R})$ satisfy $g(e_a, e_b) = \eta_{ab}$, with $g(\cdot,\cdot)$ the scalar product and η_{ab} the Minkowski metric. Since the four-spin does not change in the particle frame, its components in the non-coordinate basis must be constant and thus $\frac{dS^a}{d\tau} = 0$. Taking the total covariant derivative of both sides of (3.16) and imposing this last condition yields[5]

$$\frac{dS^\mu}{d\tau} + \Gamma^\mu{}_{\nu\alpha} U^\nu S^\alpha - S^a \frac{De_a{}^\mu}{D\tau} = 0. \tag{3.17}$$

The vierbeins satisfy by definition $e_{a\mu} e_b{}^\mu = \eta_{ab}$; taking the total covariant derivative of this relation implies $\frac{De_{a\mu}}{D\tau} e_b{}^\mu + e_{a\mu} \frac{De_b{}^\mu}{D\tau} = 0$. To fulfill this condition we must have

$$\frac{De_a{}^\mu}{D\tau} = \Omega^{\mu\nu} e_{a\nu}, \tag{3.18}$$

with $\Omega^{\mu\nu}$ antisymmetric and thus $i\Omega^\mu{}_\nu \in \mathfrak{so}(1,3)$. Note that the same type of relation holds when we have three rigid perpendicular versors moving in euclidean three-dimensional space, since $\frac{d\hat{\boldsymbol{n}}_{\boldsymbol{a}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{n}}_{\boldsymbol{a}}$ can be written as $\frac{dn_a{}^i}{dt} = \epsilon^{ijk} \omega^j n_a{}^k$ and $i\epsilon^{ijk}\omega^j \in \mathfrak{so}(3)$.

Let us find an expression for $\Omega^{\mu\nu}$ that also satisfies the condition that the Lorentz transformation generated instantaneously by $\Omega^{\mu\nu}$ is a boost, as seen in the rest frame of the particle. Since $e_0{}^\mu = U^\mu$, $\frac{De_0{}^\mu}{D\tau} = W^\mu = \Omega^{\mu\nu} U_\nu$, this condition is satisfied if we take

$$\Omega^{\mu\nu} = W^\mu U^\nu - U^\mu W^\nu + C^{\mu\nu}. \tag{3.19}$$

with $C^{\mu\nu}$ an antisymmetric tensor such that $C^{\mu\nu} U_\nu = 0$. To find the correct $C^{\mu\nu}$ that gives a non-rotating transformation, let us consider a generic space-like vector B^μ carried along the rest frame – which thus satisfies $B^\mu U_\mu = 0$ – that is also perpendicular to the four-acceleration ($B^\mu W_\mu = 0$). If the frame is not rotating, the total covariant derivative of this vector must be null and therefore

$$\frac{DB^\mu}{D\tau} = \Omega^{\mu\nu} B_\nu = C^{\mu\nu} B_\nu = 0. \tag{3.20}$$

To satisfy this condition we can therefore take – since B^μ is quite generic – $C^{\mu\nu} = 0$[6].

We can now take $\Omega^{\mu\nu} = W^\mu U^\nu - U^\mu W^\nu$ and (3.18) and plug them into equation (3.17) to obtain

$$\frac{D_F S^\mu}{D\tau} = \frac{DS^\mu}{D\tau} + \left(U^\mu W_\nu - W^\mu U_\nu \right) S^\nu = 0. \tag{3.21}$$

The differential operator $\frac{D_F}{D\tau}$ that we have defined is called the Fermi-Walker derivative and it coincides with the regular covariant derivative when $W^\mu = 0$ and the particle is following a geodesic. With the Fermi-Walker derivative we can also extend the concept of parallel transport to curves that are not geodesics by imposing $\frac{D_F S^\mu}{D\tau} = 0$, obtaining what is called the Fermi-Walker transport.

[5] We are following the notation of [19] with the convention that $D_\nu e_\mu = e_\alpha \Gamma^\alpha{}_{\nu\mu}$.

[6] It is possible to show – see [20] – that if the tetrad is rotating, $C^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} U_\alpha \omega_\beta$, with ω^μ a vector that satisfies $\omega^\mu U_\mu = 0$ and describes the angular velocity of the rotation.

We have thus shown that the four-spin of a particle moving in a general spacetime and in the absence of electromagnetic fields is simply Fermi-Walker transported along the particle trajectory. With the help of the Fermi-Walker derivative we can now finally write equation (2.25), originally valid only in a flat spacetime, in the generalized form

$$\frac{D_F S^\mu}{D\tau} = \frac{e}{m}\frac{g}{2}\left[F^{\mu\nu}S_\nu + U^\mu\left(F^{\alpha\beta}S_\alpha U_\beta\right)\right], \tag{3.22}$$

valid for any spacetime. Note that since condition (2.20) is still valid, the term $-W^\mu U_\nu S^\nu$ in the Fermi-Walker derivative disappears. If we now consider gravity and electromagnetism as the only forces acting on the particle and insert the equation of motion $W^\mu = \frac{e}{m}F^{\mu\nu}U_\nu$ into (3.22), we get the generalized TBMT equation in curved spacetime

$$\frac{DS^\mu}{D\tau} = \frac{e}{m}\left[\frac{g}{2}F^{\mu\nu}S_\nu + aU^\mu\left(F^{\alpha\beta}S_\alpha U_\beta\right)\right], \tag{3.23}$$

which is simply the regular TBMT equation with the total derivative replaced by the total covariant derivative. Note that in the presence of an EDM, as seen in section 3.2, we can obtain the curved spacetime extension of equation (3.13) by simply replacing the total time derivative $\frac{d}{d\tau}$ in (3.13) with $\frac{D}{D\tau}$, or by replacing $F^{\mu\nu}$ with $F^{\mu\nu} - \frac{\eta}{g}G^{\mu\nu}$ in (3.22).

Equation (3.23) is not only useful when dealing with spin in a gravitational field but can also be used to analytically solve problems in the case of particular electromagnetic field configurations. For example, when dealing with fields with cylindrical or spherical symmetry, we can use cylindrical or spherical coordinates (or even a rotating frame of reference) together with equation (3.23) instead of trying to directly solve the regular TBMT equation.

We can also use (3.23) together with a frame field along the particle trajectory in order to obtain a solution to the equation of motion of S^μ. In particular if we take vierbeins that satisfy

$$\frac{De_a{}^\mu}{D\tau} = \frac{e}{m}F^{\mu\nu}e_{a\nu}, \tag{3.24}$$

the four-velocity U^a is a constant and $U^b\Gamma^a{}_{bc} = \frac{e}{m}F^a{}_c$. We can therefore cast the generalized TBMT equation (3.23) as

$$\frac{dS^a}{d\tau} = a\frac{e}{m}\left(F^a{}_b + U^aU^cF_{bc}\right)S^b. \tag{3.25}$$

This equation admits the perturbative solution

$$S^a(\tau) = \left(\mathcal{T}e^{\int_0^\tau \Omega^a{}_b(s)ds}\right)S_\mathrm{i}{}^b, \tag{3.26}$$

given that we can solve the equation of motion of the particle and thus know $F^{ab} = F^{ab}\left(x^\mu(\tau)\right)$. Note that $\Omega^a{}_b = a\frac{e}{m}\left(F^a{}_b + U^aU^cF_{bc}\right)$, $S_\mathrm{i}{}^b$ is the initial four-spin of the particle and $\mathcal{T}$ indicates time-ordering, which is necessary when F^{ab} is not constant along the particle trajectory.

Finally, this approach actually gives a fully analytical solution for the four-spin motion when applied to the case of a flat spacetime and with a constant electromagnetic field. Equation (3.24) is solved by $e_a{}^\mu(\tau) = e^{\tau \frac{e}{m} F^\mu{}_\nu} \eta^\nu{}_a$, choosing the initial vierbeins aligned with the laboratory axes; we also have $F^{ab} = F^{\alpha\beta} \eta_\alpha{}^a \eta_\beta{}^b$ – the electric and magnetic fields are identical in the laboratory and in the particle frame – since $\left[F^{\mu\nu}, e^{\tau \frac{e}{m} F^\mu{}_\nu}\right] = 0$. Finally the four-spin is given by (3.26), without the time-ordering operator; putting back everything together yields

$$S^\mu(\tau) = e^{\tau \frac{e}{m} F^\mu{}_\nu} e^{\tau a \frac{e}{m} \left(F^\nu{}_\alpha + U_{\rm i}{}^\nu U_{\rm i}{}^\beta F_{\alpha\beta}\right)} S^\alpha_{\rm i}, \tag{3.27}$$

where $U_{\rm i}{}^\nu$ is the initial four-velocity in the lab frame.

4 Dirac equation and $g = 2$

4.1 Orbital and spin angular momentum operators

The wave equation describing the motion of a relativistic spin-$\frac{1}{2}$ particle – discovered by Dirac in an attempt to create a first order wave equation in place of the Klein-Gordon equation – is the Dirac equation [21]:

$$i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) = 0. \tag{4.1}$$

The wave function ψ is a field that at each point in spacetime has value $\psi(x) \in \mathbb{C}^4$ and the gamma matrices γ^μ are four different 4×4 matrices that satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{4.2}$$

associated to the scalar product of the Minkowski metric. The first two components of the wave function describe the spin up and spin down components of a spin-$\frac{1}{2}$ particle as done by the Pauli equation, while the last two components describe the spin up and down components of the corresponding antiparticle, a particle with the same physical properties like spin and mass but with opposite electric charge.

In order to find the operators associated to orbital and spin angular momentum – following an approach similar to the one used in non-relativistic quantum mechanics – we can use the fact that these operators are the generators of Lorentz transformations on the spatial and the spin part of the wave function respectively. In particular, since Lorentz transformations are described by six generators $J^{\mu\nu}$, we need to find a set of operators $\mathcal{L}^{\mu\nu}$ and $\mathcal{S}^{\mu\nu}$ that represent the Lie algebra $\mathfrak{so}(1,3)$ and thus satisfy the commutation relations (2.6). These operators will therefore generate projective unitary representations of $SO^+(1,3)$, which are isomorphic to unitary representations of its double cover $Spin(1,3) \simeq SL(2,\mathbb{C})$. We also require that $\frac{1}{2}\epsilon^{ijk}\mathcal{L}^{jk} = \ell^k$

and that $\frac{1}{2}\epsilon^{ijk}\mathcal{S}^{jk}$ acts on each couple of components of $\psi(x)$ as $\frac{\sigma^i}{2}$, given that we want to describe a spin-$\frac{1}{2}$ particle.

A set of operators that follows all these requirements is given by

$$\begin{aligned} \mathcal{L}^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \\ \mathcal{S}^{\mu\nu} &= \frac{\sigma^{\mu\nu}}{2} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \end{aligned} \tag{4.3}$$

where $\mathcal{L}^{\mu\nu}$ generalizes the relativistic angular momentum tensor[7] and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ extends the Pauli matrices.

Note that the total angular momentum $\mathcal{J}^{\mu\nu} = \mathcal{L}^{\mu\nu} + \mathcal{S}^{\mu\nu}$ still satisfies the commutation relations of $\mathfrak{so}(1,3)$ and thus generates Lorentz transformations of the whole wave function.

We can now define the operator corresponding to the four-spin S^μ in special relativity; this operator is called the Pauli-Lubański pseudo-vector and is defined by

$$\mathcal{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu \mathcal{J}_{\alpha\beta}. \tag{4.4}$$

Note that we have defined this operator in terms of the four-momentum and not the four-velocity because in quantum mechanics we are usually using the momentum instead of the velocity and because this expression is also valid for massless particles. Note also that this implies that $S^\mu = \frac{1}{m}\left\langle \mathcal{W}^\mu \right\rangle_\psi$ for a massive particle, and that $\mathcal{W}^\mu p_\mu = 0$ as expected from the classical constraint (2.20).

Finally, the operator $\mathcal{W}^\mu \mathcal{W}_\mu$ commutes with p^μ (the generators of translations) and $\mathcal{J}^{\mu\nu}$ (the generators of rotations), and is therefore a Casimir operator of the Pointcaré group. The eigenvalues of $\mathcal{W}^\mu \mathcal{W}_\mu$ and $p^\mu p_\mu$ (the other Casimir operator) can be used to label the irreducible representations of the Pointcaré group; this is important because physical states of particles transform under these representations [17]. In particular, given the properties of the Pointcaré algebra, the eigenvalues are $p^\mu p_\mu = M^2$ and $\mathcal{W}^\mu \mathcal{W}_\mu = -M^2 S(S+1)$, with M the mass of the particle and S (a positive integer or half-integer) the spin of the particle [22]. Using the Dirac equation (4.1), it is possible to show that $\mathcal{W}^\mu \mathcal{W}_\mu = -\frac{3}{4}m^2$, and thus (4.1) correctly describes a spin-$\frac{1}{2}$ particle.

4.2 Interaction with the electromagnetic field

In order to introduce an electromagnetic interaction into the Dirac equation we can once again follow the same approach used in non-relativistic quantum mechanics and impose that the wave function $\psi(x)$ is locally $U(1)$ invariant.

[7] Note that while the orbital angular tensor momentum generates unitary transformations, this is not true for $\mathcal{S}^{\mu\nu}$ which instead only generates projective unitary transformations, like in the non-relativistic case.

More formally let us consider a $U(1)$ gauge transformation of the wave function $\psi'(x) = e^{-ie\Lambda(x)}\psi(x)$, with e the electric charge acting as a coupling constant; since $\partial_\mu\psi'(x) = e^{-ie\Lambda(x)}\left[\partial_\mu\psi(x) - ie\partial_\mu\Lambda(x)\psi(x)\right] \neq e^{-ie\Lambda(x)}\partial_\mu\psi(x)$, if we want our theory to be gauge invariant we must replace the partial derivative ∂_μ with a new gauge covariant derivative D_μ that satisfies

$$D'_\mu\psi'(x) = e^{-ie\Lambda(x)}D_\mu\psi(x). \tag{4.5}$$

It is possible to define such a derivative by introducing a connection[8] one-form expressed locally as $\mathcal{A}(x) = eA_\mu(x)dx^\mu$, with $A_\mu(x) \in \mathfrak{u}(1) = \mathbb{R}$. Choosing four basis vector e_a for $\mathbb{C}^4$ – so that $\psi(x) = \psi^a(x)e_a(x)$ – it is possible to define the covariant derivative of e_a as $D_\mu(e_a) = ieA_\mu{}^b{}_a e_b$, where $A_\mu{}^b{}_a$ is a representation of A_μ acting on the left on $\mathbb{C}^4$; from now on we will assume that $\mathcal{A}$ acts naturally on $\mathbb{C}^4$. Imposing the condition (4.5) yields the behaviour of the connection under gauge transformation $A'_\mu = A_\mu + \partial_\mu\Lambda$. Note also that the curvature tensor associated with the connection, also called Yang-Mills field strength, is defined by $[D_\mu, D_\nu] = ieF_{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic tensor. The gauge covariant derivative is therefore

$$D_\mu\psi(x) = \left(\partial_\mu + ieA_\mu\right)\psi(x). \tag{4.6}$$

Finally if we want to include the electromagnetic interaction into the Dirac equation, we can simply replace the partial derivative with the covariant derivative and obtain

$$i\gamma^\mu D_\mu\psi(x) - m\psi(x) = 0. \tag{4.7}$$

This way of coupling the Dirac field and the electromagnetic field is called minimal coupling.

4.3 g-factor of a spin-$\frac{1}{2}$ particle

Let us now determine the g-factor of a particle satisfying the Dirac equation (4.7). If we apply the operator $-\left(i\gamma^\mu D_\mu + m\right)$ to equation (4.7) we get, after some algebra,

$$D^\mu D_\mu\psi(x) + m^2\psi(x) + e\mathcal{S}^{\mu\nu}F_{\mu\nu}\psi(x) = 0. \tag{4.8}$$

This equation is identical to the Klein-Gordon equation describing a spin 0 particle with minimal coupling with the electromagnetic field, but with the additional term $e\mathcal{S}^{\mu\nu}F_{\mu\nu}$ due to the presence of the spin.

If we consider a constant weak magnetic field with an associated vector potential $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{x}$, equation (4.8) – excluding second order terms in $\boldsymbol{A}$ – becomes

$$\partial^\mu\partial_\mu\psi(x) + m^2\psi(x) - e\boldsymbol{B}\cdot\left(\boldsymbol{\ell} + 2\boldsymbol{\mathcal{S}}\right)\psi(x) = 0, \tag{4.9}$$

[8] Even more formally we can first construct a principal $U(1)$ bundle $P(\mathbb{R}^4, U(1))$ and its associated vector bundle $P \times_\rho \mathbb{C}^4$; $\mathcal{A}$ can thus be seen as a local Lie-algebra valued one-form obtained from an Ehresmann connection one-form $\omega \in \mathfrak{u}(1) \otimes T^*P$ and a local section σ from the relation $\mathcal{A} = \sigma^*\omega$, while ψ can be seen as a section of $P \times_\rho \mathbb{C}^4$, see [19].

where $\mathcal{S}^i = \frac{1}{2}\epsilon^{ijk}\mathcal{S}^{jk}$ is the operator for spin angular momentum. To obtain the Schrödinger equation from (4.9) let us consider a wave function of the form $\psi(x) = e^{-imt}\left(\phi(x)\ ,\ 0\right)$ with $\phi(x)$ a two component spinor that oscillates slowly with respect to e^{-imt}. Therefore by placing this wave function in (4.9) and neglecting second derivatives in time of the spinor we obtain

$$i\frac{\partial\phi(x)}{\partial t} = -\frac{\nabla^2\phi(x)}{2m} - \frac{e\boldsymbol{B}}{2m}\cdot(\boldsymbol{\ell} + 2\boldsymbol{s})\,\phi(x) \tag{4.10}$$

with $\boldsymbol{s} = \frac{\boldsymbol{\sigma}}{2}$, and therefore we can conclude that the g-factor of a particle described by the Dirac equation is $g = 2$.

5 Fermilab E989 and the measurement of a_μ

5.1 Principles behind the experiment

The idea behind the E989 experiment, and in general of all past and present experiments aiming to determine the muon $g-2$, is to directly measure the rate of precession of the muon spin when placed in a constant magnetic field. Since the muon decays with a mean lifetime of $\tau = 2.2\ \mu$s, it is not suitable to be contained in a Penning trap, as done with electrons, and thus a_μ must be measured from a beam of moving muons. Since a beam of moving charged particles in a constant magnetic field experiences the Lorentz force, the muons are set to move along a circular trajectory inside what is called a storage ring, where strong (usually superconducting) magnets are used to generate a constant vertical magnetic field[9] $\boldsymbol{B} = B\hat{\boldsymbol{y}}$.

It is the muon decay itself that allows us to determine the direction of the muon spin. The main decay mode of the positive muon is $\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu$; the weak force does not conserve parity and the high energy positron of the decay is mainly emitted along the direction of the spin $\boldsymbol{s}$. It is therefore possible to use trackers and electromagnetic calorimeters placed around the storage ring to track the positron trajectory and thus deduce the muon spin orientation at the time of the positron emission.

The connection between the spin precession frequency $\boldsymbol{\omega_s}$ and the anomalous magnetic moment in a generic electromagnetic field is given by equation (2.17):

$$\boldsymbol{\omega_s} = -\frac{e}{m}\left[\left(a + \frac{1}{\gamma}\right)\boldsymbol{B} - a\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta} - \left(a + \frac{1}{\gamma+1}\right)\boldsymbol{\beta}\times\boldsymbol{E}\right]. \tag{5.1}$$

Note however that the particle is also rotating in the storage ring – as described by equation (2.16) – with an angular velocity

$$\boldsymbol{\omega_C} = -\frac{e}{m}\left[\frac{1}{\gamma}\boldsymbol{B} - \frac{\gamma}{\gamma^2-1}\boldsymbol{\beta}\times\boldsymbol{E}\right], \tag{5.2}$$

[9] In this section we use cylindrical coordinates, with x as the radial component and y as the vertical component, following the notation used in many of the papers by the Muon $g-2$ collaboration such as [5].

where we neglected the electric field parallel to the velocity of the particle, since we expect $\boldsymbol{E}$ to be close or equal to zero and that the particles move in a circular trajectory.

What we are actually measuring in the experiments is therefore the relative angular velocity

$$\boldsymbol{\omega_a} = \boldsymbol{\omega_s} - \boldsymbol{\omega_C} = -\frac{e}{m}\left[a\boldsymbol{B} - a\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{B}\right)\boldsymbol{\beta} - \left(a - \frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\boldsymbol{E}\right], \tag{5.3}$$

that simplifies to $\boldsymbol{\omega_a} = -\frac{e}{m}a\boldsymbol{B}$ if there are no electric fields and the particle trajectories are exactly perpendicular to the magnetic field. If a particle enters the storage ring polarized with its spin parallel to its velocity, the spin will rotate relative to the velocity in the same plane of the cyclotron motion and the anomalous magnetic moment can be obtained from the relative angular velocity as $a_\mu = m\omega_a/eB$.

5.2 Detection of an electric dipole moment

Since the Dirac equation does not predict an electric dipole moment for spin-$\frac{1}{2}$ particles and the SM predicts [23] $\eta_\mu \sim 1.38 \times 10^{-38}$ e cm, the search for a muon EDM can be an alternative way to probe for physics beyond the SM. In particular due to equation (3.11) the precession frequency of the muon spin changes; therefore equation (5.3) is no longer valid and needs a replacement.

The particle trajectories do not depend on the presence of a muon EDM and therefore we only need to modify (5.1), which becomes

$$\begin{aligned}\boldsymbol{\omega_s'} = &-\frac{e}{m}\left[\left(a+\frac{1}{\gamma}\right)\boldsymbol{B} - a\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{B}\right)\boldsymbol{\beta} - \left(a+\frac{1}{\gamma+1}\right)\boldsymbol{\beta}\times\boldsymbol{E}\right]\\ &-\frac{e}{m}\left[b\boldsymbol{E} - b\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}\right)\boldsymbol{\beta} - b\boldsymbol{\beta}\times\boldsymbol{B}\right],\end{aligned} \tag{5.4}$$

and thus the relative angular velocity becomes

$$\begin{aligned}\boldsymbol{\omega_a'} = &-\frac{e}{m}\left[a\boldsymbol{B} - a\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{B}\right)\boldsymbol{\beta} - \left(a-\frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\boldsymbol{E}\right]\\ &-\frac{e}{m}\left[b\boldsymbol{E} - b\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}\right)\boldsymbol{\beta} - b\boldsymbol{\beta}\times\boldsymbol{B}\right].\end{aligned} \tag{5.5}$$

The principal consequence of this equation is that if the particle moves in a circular trajectory, perpendicular to a constant magnetic field $\boldsymbol{B}$, the spin precession plane is tilted radially by an angle $\theta = \arctan(\beta b/a)$ and the measured angular velocity is $\omega_a' = \omega_a/\cos\theta$. This means that it is not possible to obtain a from ω_a' alone, but we can only get a certain combination of a and b from the experimental measurement. Luckily, the fact that the precession plane is tilted allows us to determine (or set an upper bound on) b by measuring the asymmetry in the number of positron emitted above and below the trajectory plane, and how this value changes in time [24].

5.3 Experimental setup of Fermilab E989

Let us now focus our attention on the experimental setup of the E989 experiment. The experiment utilizes the same storage ring with a central orbit radius of $R_0 = 7.112$ m and the same 1.45 T superconducting magnets as Brookhaven E821 but with many improvements, including in particular a 2.5 times improved magnetic field intrinsic uniformity [25]. The muon beam – provided by the Fermilab Muon Campus – consists in ~ 120 ns long bunches of 3.1 GeV/c muons with an average longitudinal polarization of approximately 95 %; sixteen individual bunches of muons are injected every 1.4 s in the storage ring by an inflector and are then placed into their correct trajectories by a fast pulsed-kicker magnet, for a total of ~ 5000 stored muons per fill.

The muons are then kept from spiraling vertically out of the constant magnetic field region by means of four sections of electrostatic quadrupole plates that provide weak vertical focusing. Usually the presence of a non-zero electric field modifies the relative precession frequency, as seen by equation (5.3). The particular value of the muon beam energy, however, has been chosen so that

$$a - \frac{1}{\gamma^2 - 1} = 0; \tag{5.6}$$

therefore the term proportional to $\boldsymbol{\beta} \times \boldsymbol{E}$ disappears from (5.3) and the weak electric fields used for focusing the beam do not influence $\boldsymbol{\omega_a}$. The particular value of $\gamma = 29.304$ that satisfies the condition (5.6) is often called the "magic γ" [26].

While the focusing of both E821 and E989 is provided by quadrupolar electric fields and thus the muons must move at the magic γ, there are different possible ways to focus the beam that do not require electric fields, thus allowing more freedom in the choice of the particle energy and the storage ring radius. As an example, let us consider the experiment proposed in [27] for the measurements of the muon anomalous magnetic moment and EDM at the J-PARC muon facility. In this proposal very weak magnetic focusing is used, allowing for muons with $\gamma = 3$ moving on a circle with a radius of only 333 mm. The highly uniform 3 T magnetic field is provided by MRI-type superconducting solenoid magnet and the injection system is also different from the one in E989, with the muons injected vertically rather than horizontally.

Focusing back on the E989 experiment, the positrons emitted by the decaying muons are then detected by a series of twenty-four calorimeters, each containing a 9×6 array of PbF_2 crystals, placed around the storage ring; light emitted by PbF_2 is then converted to an electric signal by silicon photomultipliers. Reconstruction of the waveforms allows to determine ω_a. Finally, the magnetic field is determined via pulsed proton NMR, with 378 fixed NMR probes placed above and below the storage volume that measure the Larmor precession frequency $\tilde{\omega}'_p(T_r)$ of protons shielded in a spherical water sample at a fixed reference temperature T_r.

5.4 Measurement of a_μ and important corrections

It is now possible, putting everything back together, to express the muon anomalous magnetic moment as a function of only experimental quantities[10]:

$$a_\mu = \frac{\omega_a}{\tilde{\omega}'_p(T_r)} \frac{\mu'_p(T_r)}{\mu_e(H)} \frac{\mu_e(H)}{\mu_e} \frac{m_\mu}{m_e} \frac{g_e}{2}. \tag{5.7}$$

Note that the only quantity measured by Fermilab E989 is $\mathcal{R}'_\mu = \omega_a/\tilde{\omega}'_p(T_r)$, while all other quantities are known from other experiments. These quantities are in order: $\mu'_p(T_r)/\mu_e(H)$ the ratio between the proton magnetic moment in a spherical sample of H_2O at $T_r = 34.7°C$ and the electron magnetic moment in hydrogen; $\mu_e(H)/\mu_e$ the ratio between the electron magnetic moments of an electron in hydrogen and a free electron[11]; m_μ/m_e is the ratio of the masses of the muon and the electron; g_e the electron g-factor.

Let us briefly discuss some of the corrective terms that must be taken into account to eliminate systematic biases in the measured value of $\mathcal{R}'_\mu$. First of all, the biggest corrective term is due to the fact that the muons in the beam do not all have the same exact energy – corresponding to the magic γ – but they have a certain energy distribution that is instead peaked at slightly higher γ value, with a non-negligible width; this means that the muons move on average on a larger radius $x_e + R_0$ and therefore experience an outward radial electric field, due to the focusing electric quadrupoles. This implies, coupled with the fact the muons do not exactly move at the magic γ, a corrective factor of $C_e = \Delta\omega_a/\omega_a = 2n(1-n)\beta^2 \left\langle x_e^2 \right\rangle / R_0^2$, where $n = \kappa R_0/\beta c B$ is called the field index (and κ the quadrupole field strength $\kappa = \partial_y E_y$).

Another important correction we need to make is due to the fact that the muons do not move perfectly perpendicular to the magnetic field and thus the term $\boldsymbol{\beta} \cdot \boldsymbol{B} \neq 0$ in equation (5.3). In particular the muons oscillate vertically with a frequency $\omega_z = \omega_C \sqrt{n}$, where ω_C is the cyclotron frequency; since this frequency is much faster than ω_a, the pitching can be averaged out [26] and it is possible to obtain the correction term $C_p = \Delta\omega_a/\omega_a = n \left\langle A_y^2 \right\rangle / 4R_0^2$, where A_y is the maximum amplitude of the betatron vertical oscillation.

Other smaller corrections that have been taken into account are linked to beam dynamics – with a term due to the average phase of muon losses and a phase-acceptance term – and to the fast magnetic transients synchronized with the injections that influence the average field seen by the beam, due to charging of the electric quadrupoles and the firing of the kicker magnets.

Putting all the corrections together yields $a_\mu^{\text{FNAL}} = 116592040(54) \cdot 10^{-11}$, with the corrective terms accounting for 544 ppb of $\mathcal{R}'_\mu$ and the total uncertainty – domi-

[10] Once again we are following the notation of [5].

[11] Note that this factor is not measured experimentally but is instead computed from QED with negligible uncertainty.

nated by the statistical uncertainty – accounting for 462 ppb of $\mathcal{R}'_\mu$ [5]. This result agrees with the one from E821 and the two can thus be averaged to obtain $a_\mu^{\mathrm{EXP}} = 116592061(41) \cdot 10^{-11}$, with a precision of 0.35 ppm. The difference between the experimental value and the SM prediction of [10] is $a_\mu^{\mathrm{EXP}} - a_\mu^{\mathrm{SM}} = 251 \pm 59 \cdot 10^{-11}$, with a significance of 4.2 σ. However, a recent lattice QCD result weakens this discrepancy [11].

Finally note that equation (5.7) is based on the assumption that the muon EDM $d_\mu = 0$. This assumption is justified by the current experimental limit on the muon EDM of $|d_\mu| < 3.2 \cdot 10^{-19}$ e cm set by Brookhaven E821, a value with an influence on a_μ too small with respect to systematic and statistical uncertainties of E989's Run-1. In the future the expected precision of 140 ppb on a_μ at the end of all the E989's runs is not going to be limited by the current limit on the muon EDM since E989 is also expected to achieve an EDM sensitivity of $\sim 10^{-21}$ e cm [24].

6 Conclusions

In this work we built from the ground up a simple relativistic theory of spin and used it to explain the principles behind the Fermilab E989 experiment. In section 2 – starting from the behaviour of the three-dimensional spin $\boldsymbol{s}$ of a particle at rest – we first obtained heuristically an equation of motion for $\boldsymbol{s}$ in the case of a relativistic moving particle, by finding an appropriate definition of the particle rest frame and then taking into account the effect of the Thomas precession. We then confirmed the validity of this equation by constructing a covariant generalization of the spin – the four-spin S^μ – and deriving its equation of motion, the TBMT equation.

In section 3 we discussed possible generalizations of the TBMT equation: after providing an alternative formulation that utilizes the spin tensor $S^{\mu\nu}$ instead of S^μ, we generalized the TBMT equation to the case of an electric dipole moment associated to the spin and the presence of a curved spacetime. In particular, we constructed a new operator, the Fermi-Walker derivative, which naturally incorporates the effect of the Thomas precession in a covariant theory and discussed how it can be used as an efficient way to solve the TBMT equation analytically in particular field configurations.

In section 4 we briefly introduced the Dirac equation describing a relativistic spin-$\frac{1}{2}$ particle and the operators associated to the orbital and spin angular momentum. We added the interaction with the electromagnetic field through minimal coupling by defining a connection one-form $\mathcal{A}$ and using it to obtain a $U(1)$ gauge covariant derivative. In the approximation of a weak magnetic field and a non-relativistic moving particle we then showed that the Dirac equation reduces to the Schrödinger equation with $g = 2$.

Finally, in section 5 we discussed the theoretical principles behind the E989 experiment at Fermilab, also taking into account the possible presence of a muon electric dipole moment. We described the experimental setup of this experiment, focusing in particular on the so-called “magic γ” condition. We discussed how a_μ is actually determined and which are the most important corrections that need to be applied to make an accurate measurement.

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