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DUALITY AND SCALING PROPERTIES OF COALESCING RANDOM WALKS AND VOTER MODEL

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0.1 Abstract

Every nice story needs a good setting, Batman wouldn't be who he is if it was not for Gotham or The Lord of the Rings without the Middle Earth. Our story is about a vote and the setting is a multidimensional chess board that enlarges as time passes. It sounds monstrous but it isn't. We will call it Torus $\Lambda(N)$ of $\mathbb{Z}^d$ of side of length N.

Every pivot of the board has a person on it, who starts with an opinion, which can be black or white, Democrat or Republican, Pandoro or Panettone, or, in our case, 0 or 1. We will call them voters. As the time advances every voter can check one of its neighbourhoods and align with its opinion, if it differs from its own.

What we would like to know is if on the enlarging board we will ever have a consensus and, if this happens, how its law can be characterized.

We know from Liggett [11] that if this situation would happen on infinite chess board, i.e. $\mathbb{Z}^d$, we wouldn't have a moment in time where all the voters agree, but, as time advances, every one would take an opinion with a probability that depends on the initial distribution. Instead, if we were to work on a fixed finite board, we would easily find out that the consensus time is finite and it depends on the size of the board and the initial distribution.

What we want to study is a situation in between the two of them. If we were to start on a finite board that enlarges to the infinite one, how does the consensus time increase with respect to the side of the board? What is its asymptotic behaviour?

To be able to study this voter model we read our story, in our strange setting, by turning the book upside down and starting from the end. By doing so we will actually read another story, the story of some people walking around the board, who, when they meet on a pivot, unite into one.

The second model we are referring to is called Coalescing Random Walks. We start with some symmetric random walks on the Torus that, as time advances, move around. The only rule of this model is that, when two of them meet, they coalesce into one and go on together.

As incredible as it may seem, if we draw graphically these two models, we can see that one behaves like the other but with the time inverted. In particular, the event for the voter model "By time t_0 everyone's opinion is the opinion initially

held by person k" is exactly the same as the event for the coalescing random walk process "All particles have coalesced by time t_0, and the walk is at k at time t_0". By noticing this duality, we have an easier way to answer our previous questions. We will just study the second model, which is easier to work on.
We will start by studying the behaviour of just two random walks on the enlarging Torus. We will discover that to have them to meet, we will need two mathematical techniques. The first one is that the two walks have to bound a succession, and the second one is that we need to rescale the time in a way that the result is not trivial and it accelerates the walks.
With these hypotheses we get that, as the board enlarges to infinity, the law of the time of coalescence goes to an exponential law.
The second result is about multiple coalescing random walks and we will need the same hypotheses and scaling procedures to have informations about the time when they will all meet. It will also converge to an exponential law, which will depend on the number of starting walks too. Moreover we will be able to characterize the time when a fixed amount of walks have remained.
As we have characterized the time of coalescence, we can define the stopping times that check the minimum time to have a certain amount of random walks left. What we will learn is that they also converge weakly and in average to a stopping time asymptotically.
Now that we have read our story backwards and learning a lot, we wish to translate this knowledge to the actual story. Luckily, we have a Rosetta stone, a duality identity, that helps us to move the result from one model to the other.
The first result, that we get out of it, is also about a stopping time, in particular it is the consensus time, our first objective. We have that, with the same scaling used for the Coalescing Random Walks, this stopping time converges weakly and on average to a random variable whose law can be explicitly characterized
From this huge result we can also start to ask ourselves other questions. The first one is how does the 'percentage' of voters behave as the torus enlarges? The answer is quite surprising, as it converges to a well known diffusion process, called Wright-Fisher process, with the usual scaling hypotheses. The second one is about the law of the voter model, which we will be able to describe in some particular case.
This story could have many other chapters, like giving the voters more than just two opinions, or with a different graph setting, but we will stop here.
The first chapter will focus on the basic theory of Markov process, with a particular focus on symmetric random walks. The second chapter will be dedicated to the study of the Coalescing Random Walks, with every section focused on just one of the theorems presented previously. The last chapter presents all the results concerning the voter model, the duality and the density process.

Chapter 1

Basic Theory

1.1 Markov process

1.1.1 Discrete-time Markov process

Let S be a set at most countable.

Definition 1.1. A sequence $(X_n)_{n\geq 0}$ of S-valued random variables is a Markov chain if for each $n \geq 1$ and $x_0,\, x_1,\, ,\ldots, x_{n+1} \in S$

$$P(X_{n+1} = x_{n+1} \,|\, X_n = x_n, \ldots,\, X_0 = x_0) = P(X_{n+1} = x_{n+1} \,|\, X_n = x_n). \tag{1.1}$$

If $P(X_{n+1} = x_{n+1} \,|\, X_n = x_n)$ does not depend from n, we say that the chain is time-homogeneous. In this case the transition matrix is defined as

$$P := (p_{yx})_{y,x\in S}, \quad p_{yx} = P(X_{n+1} = x \mid X_n = y). \tag{1.2}$$

The entries are probabilities, and since a transition from any state y must be to some state, it follows that

$$0 \leq p_{yx} \leq 1 \;\forall x \in S, \quad \sum_{x\in S} p_{yx} = 1.$$

The random variable X_0 is called the initial state, and its probability distribution ν,

$$\nu(i) = P(X_0 = i)$$

is the initial distribution. We shall abbreviate $P(A \mid X_0 = i)$ as $P_i(A)$ if we want to fix the initial state, and $P_\nu(\,\cdot\,)$ the probability given that the initial state is distributed according to ν. In particular $P_\nu(A) = \sum_{i\in S} \nu(i) P_i(A)$, with A being an event. From Bayes's rule and in view of the homogeneous Markov property,

$$P(X_k = i_k, \ldots,\, X_1 = i_1,\, X_0 = x_0) = \nu(i_0) p_{i_0 i_1} \cdots p_{i_{k-1} i_k}.$$

In particular, if $\nu_n(i) := P(X_n = i)$,

$$\nu_n^T = \nu_0^T P^n$$

Definition 1.2. A probability distribution π satisfying

$$\pi^T = \pi^T P$$

is called a stationary distribution of the transition matrix P or of its corresponding homogeneous Markov chain.

Definition 1.3. A stationary Markov chain with initial distribution π is reversible if, for all $i, j \in S$,

$$\pi(i)p_{ij} = \pi(j)p_{ji}. \tag{1.3}$$

The last equation is called the detailed balance equation. Clearly, if a distribution π satisfies (1.3), then π is a stationary distribution of P.

Definition 1.4. A stopping time with respect to a stochastic process $(X_n)_{n\geq 0}$ ia random variable τ taking values in $N \cup \{+\infty\}$ and such that, for all integers $m \geq 0$, the event $\{\tau = m\}$ can be expressed in terms of $X_0, X_1, \dots, X_m$.

An example of stopping time is the return time at a state i, defined as

$$\tau_i = \inf\{n > 0 \ : \ X_n = i\}$$

Definition 1.5. A state $i \in S$ is called recurrent if

$$P_i(\tau_i < \infty) = 1,$$

and otherwise it's called transient. A recurrent state is called positive recurrent if $E_i[T_i] < \infty$, else it's called null recurrent.

An important result states that $i \in S$ is recurrent if and only if

$$\sum_{n\geq 0} p_n(i, i) = \infty, \tag{1.4}$$

where $p_n(i, j) = P_i(X_n = j)$.

1.1.2 Continuous-time Markov process

Definition 1.6. A random point process on the positive half-line is a sequence $(T_n)_{n\geq 0}$ of non-negative random variables such that, almost surely,

$$\tau_0 = 0, \quad 0 < \tau_1 < \tau_2 < \cdots, \quad \lim_{n\to+\infty} \tau_n = \infty.$$

For any interval $(a, b] \subset R_+$,

$$N(a, b] := \sum_{n\geq 1} 1_{(a, b]}(\tau_n)$$

is an integer-valued random variable counting the events occurring in the time interval $(a, b]$. For $t > 0$, let $N(t) := N(0, t]$. The family of random variables $N = (N(t))_{t\geq 0}$ is called the counting process of the point process $(\tau_n)_{n\geq 0}$.

Definition 1.7. A counting process N on the positive half-line is called an homogeneous Poisson process (HPP) with intensity $\lambda > 0$ if

- For all times t_i, $i \in \{1, \ldots, k\}$, such that $0 \leq t_1 \leq \cdots \leq t_k$, the random variables $N(t_i, t_{i+1}]$, $i \in \{1, \ldots, k-1\}$, are independent.
- For any interval $(a, b] \subset R_+$, $N(a, b]$ is a Poisson random variable with mean $\lambda(b-a)$.

In particular, for all $k \geq 0$,

$$P(N(a, b] = k) = e^{-\lambda(b-a)} \frac{[\lambda(b-a)]^k}{k!}.$$

In this sense, λ is the the average density points.
The sequence $S_n = T_n - T_{n-1}$ is called the interevent sequence of an HPP and it is i.i.d, with exponential distribution of parameter λ. In particular

$$P(S_n \leq t) = 1 - e^{-\lambda t}, \quad E[S_n] = \lambda^{-1}.$$

The following two results are useful to understand the HPPs.

Theorem 1.1.1. *Let $(N_i)_{i\leq 1}$ be a family of independent HPPs with respective positive intensities $(\lambda_i)_{i\geq 1}$.*

- *Two distinct HPPs of this family have no points in common.*
- *If $\sum_{i\geq 1} \lambda_i = \lambda < \infty$, then*

$$N(t) := \sum_{i\geq 1} N_i(t)$$

defines a counting process of a HPPs with intensity λ.

Theorem 1.1.2. *With the same hypothesis of the previous theorem, in particular that $\sum_{i\geq 1}\lambda_i = \lambda < \infty$, denote by Z the first event time of $N = \sum_{i\geq 1} N_i$ and by J the index of the HPP responsible for it. Then*

$$P(J = i,\, Z \geq a) = P(J = i)P(Z \geq a) = \frac{\lambda_i}{\lambda} e^{-\lambda a}. \tag{1.5}$$

It follows that J and Z are independent, $P(J = i) = \frac{\lambda_i}{\lambda}$, and Z is exponential with mean λ^{-1}.

Definition 1.8. The S-valued process $(X_t)_{t\geq 0}$ is called a continuous-time Markov chain if for all $i,\ j,\ i_1,\dots,\ i_k \in S$, all $t,\ s \geq 0$, and all $s_1,\dots,s_k \geq 0$, with $s_l \leq s$ for all $l \in \{1,\dots,\ k\}$,

$$P(X_{t+s} = j \mid X(s) = i,\, X_{s_1} = i_1,\dots,\, X_{s_k} = i_k) = P(X_{t+s} = j \mid X_s = i), \tag{1.6}$$

whenever both sides are well-defined. This continuous-time Markov chain is called homogeneous if the right-hand side of the equation is independent of s.

An equivalent definition involving the σ-algebra of the random variables is:

Definition 1.9. Let $(X_t)_{t\geq 0}$ be S-valued random variables and $\mathscr{F}_t := \sigma(X_s\,:\, s \leq t)$. Then

$$P(X_t = x \mid \mathscr{F}_s) = P(X_t = x \mid X_s).$$

Let

$$P(t) = (p_{ij}(t))_{i,j\in S}$$

where $p_{ij}(t) := P(X_{t+s} = j \mid X_s = i)$. The family $(P(t))_{t\geq 0}$ is a semigroup, called transition semigroup of the continuous-time HMC. This process has similar properties to the discrete one. That is

$$P(t+s) = P(t)P(s), \quad P(0) = I.$$

The distribution at time t of X_t is the vector $\nu(t) = (\nu_i(t))_{i\in S}$, where $\nu_i(t) = P(X_t = i)$. It is obtained from the initial distribution, as in the discrete form, by the formula

$$\nu(t)^T = \nu(0)^T P(t).$$

For example, let N be an HPP on the positive half-line with intensity $\lambda > 0$. The counting process $(N(t))_{t\geq 0}$ is a continuous-time HMC. For $f\,:\,S \to R$ and $t \geq 0$, we can define $S_t f\,:\,S \to R$ by

$$S_t f(y) := E(f(X_t) \mid X_0 = y) = \sum_{y\in S} f(x)P(X_t = x \mid X_0 = y).$$

In particular $S_t 1 = P(t)$.

Definition 1.10. Let $(X_n)_{n>0}$ be a discrete-time HMC with transition matrix K and let $(\tau_n)_{n\geq 1}$ be an HPP on R_+ with intensity $\lambda > 0$ and associated counting process N. Suppose that $(X_n)_{n\geq 0}$ and N are independent. The S-valued process, defined by

$$X(t) = X_{N(t)}$$

is called a uniform Markov chain. The Poisson process N is called the clock, and the chain $(X_n)_{n\geq 0}$ is called the subordinated chain.

The transition semigroup is

$$P(t) = \sum_{n\geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} K^n$$

Theorem 1.1.3. *Let $(P(t))_{t\geq 0}$ be a continuous transition semigroup on S. For any state i, there exists*

$$q_i = \lim_{h\to 0} \frac{1 - p_{ii}(h)}{h} \in [0,\ +\infty],$$

and for any pair $i,\ j \in S$, there exists

$$q_{ij} = \lim_{h\to 0} \frac{p_{ij}(h)}{h} \in [0,\ +\infty).$$

This theorem allows us to construct the following definition

Definition 1.11. The quantities q_{ij} are called the local characteristics of the semigroup. The matrix

$$L = (q_{ij})_{i,\ j\in S}, \quad q_{ii} := -q_i$$

is called the infinitesimal generator of the semigroup.

Equivalently, if $t \to P(t)$ is continuous,

$$\lim_{t\to 0} \frac{S_t - I}{t} = L$$

exists and $P(t) = e^{tL}$, with L satisfying the algebraic rule $L_{yy} = -\sum_{x\neq y\in S} Lyx$. In particular, letting $\pi_t(x) := P(X_t = x)$, we have

$$\pi_t(x) = \sum_{y\in S} p_{yx}(t)\pi_0(y)$$

therefore

$$\pi_t = \pi_0 P(t) \iff \dot{\pi}_t = \pi_t L.$$

Then a stationary distribution π for the chain satisfies $\pi L = 0$.

1.2 Random Walks

1.2.1 Symmetric Random Walks

We now study some simple cases of random walks and we try to explore the concepts of recurrent and transient states. We will often use the identity (1.4).
We will study firstly the 1-dimensional case. Let X_0 be a random variable with values in $\mathbb{Z}$. Let $\{Z_n\}_{n\geq 1}$ be a sequence of i.i.d random variables, independent of X_0, taking the values +1 or -1, and with the probability distribution $P(Z_n = +1) = p$, where $p \in (0,1)$. The process $(X_n)_{n\geq 1}$ defined by $X_{n+1} = X_n + Z_{n+1}$ is an homogeneous Markov chain, and it's called the random walk on $\mathbb{Z}$.
The nonzero terms of its transition matrix are

$$p_{i,\,i+1} = p, \quad p_{i,\,i-1} = 1-p.$$

We can easily notice that all the states are connected, so if one state is recurrent, then all of them are, and vice-versa. We study the origin. The first observation we need to take is that $p_{00}(2n+1) = 0$, because, if the walk starts in the origin, it needs an even amount of steps to come back. Then

$$p_{00}(2n) = \frac{(2n)!}{n!n!} p^n (1-p)^n$$

because for every step in one direction, the walk has to take one in the other way to come back.
By Sterling's equivalence formula $n! \sim \frac{n}{e}\sqrt{2\pi n}$ we get

$$p_{00}(2n) \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

The nature of the series $\sum_{n=0}^{\infty} p_{00}(n)$ depends on p. If $p = \frac{1}{2}$, then the sum diverges and the origin is a recurrent state. If $p \neq \frac{1}{2}$, in which case $4p(1-p) < 1$, the sum converges and then the origin is a transient state.
Similarly the 2-dimensional case has a recurrent the origin. We just consider the case in which the jumps in a neighbourhood state happen with probability $p = \frac{1}{4}$, called symmetric random walk. We use the identity (1.4). Again $p_{00}(2n+1) = 0$, but

$$p_{00}(2n) = \frac{1}{4^{2n}} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}$$

and by using Sterling's identity $p_{00}(2n) \sim \frac{1}{\pi n}$. Then the origin is a recurrent state for the symmetric random walks.

The 3-dimensional case differs from the previous ones. The not null-terms of the transition matrix are given by

$$p_{x,\,x\pm e_i} = \frac{1}{6}.$$

Clearly, $p_{00}(2n+1) = 0$ for all $n \geq 0$, and

$$p_{00}(2n) = \sum_{0 \leq i+j \leq n} \frac{(2n)!}{(i!j!(n-i-j)!)^2} \left(\frac{1}{6}\right)^{2n}.$$

This can be rewritten as

$$p_{00}(2n) = \sum_{0 \leq i+j \leq n} \frac{1}{4^n} \binom{2n}{n} \left(\frac{n!}{i!j!(n-i-j)!}\right)^2 \left(\frac{1}{3}\right)^{2n}.$$

Using the trinomial formula

$$\sum_{0 \leq i+j \leq n} \frac{n!}{i!j!(n-i-j)!} \left(\frac{1}{3}\right)^n = 1,$$

we obtain the bound

$$p_{00}(2n) \leq K_n \frac{1}{4^n} \binom{2n}{n} \left(\frac{1}{3}\right)^n$$

where $K_n = \max_{0 \leq i+j \leq n} \frac{n!}{i!j!(n-i-j)!}$.
By maximizing K_n the values we can get

$$p_{00}(2n) \sim \frac{n!}{(n/3)!(n/3)!2^{2n}e^n} \binom{2n}{n}.$$

By Stirling's equivalence formula, then

$$p_{00}(2n) \sim \frac{3\sqrt{3}}{2(\pi n)^{3/2}}$$

so it's the general term of a divergent series. Then state 0 is therefore transient. What we can conclude then it's that for $d = 1,\,2$ all the states are recurrent for the symmetric random walks, i.e. the walk will pass through every state an infinite amount of times, while for $d \geq 3$ all the states are transient, i.e. the walk will be in each state just few times.

1.2.2 Brownian Motion

We now want to show how to construct a Brownian motion by using the symmetric random walks. We seek this result just to see the potential of the symmetric random walks.
We consider a family of i.i.d random variables $\{X_k\}_{k\geq 0}$ such that $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$ and let $S_n = \sum_{k=1}^{n} X_k$, $\forall n \in \mathbb{N}$, with $S_0 = 0$. Clearly

$$S_n = \begin{cases} 0, & \text{elsewhere} \\ \binom{n}{k} 2^{-n}, & \text{for } |k| \leq n. \end{cases}$$

We consider, for each $n \in \mathbb{N}$, the process

$$Y_t^n = \frac{S_{[nt]}}{\sqrt{n}} + \frac{nt - [nt]}{\sqrt{n}} X_{[nt]+1}$$

for $t \in \mathbb{R}^+$, where $[\cdot]$ is the integer part of a real number.
The succession $\{Y_t^n\}_n$ is a linear interpolation of points of S_n with a rescaling of time n and space $\sqrt{n}$.
We know that

$$\frac{S_n}{\sqrt{n}} \to^d N(0,\, 1),$$

by the central limit Theorem, then

$$\frac{S_{[nt]}}{\sqrt{n}} = \frac{S_{[nt]}}{\sqrt{nt}} \cdot \sqrt{t} \to^d \sqrt{t} \cdot N(0,\, 1) =^d N(0,\, t),$$

with the last equivalence being on distribution. Moreover

$$\frac{nt - [nt]}{\sqrt{n}} X_{[nt]+1} \to^p 0,$$

as $n \to \infty$, since the scalar term 'wins' over the random variables.
We state a useful lemma

Lemma 1.2.1. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be successions of random variables with values in the same probability space. If $X_n \to^d X$ and $Y_n \to^p 0$ as $n \to \infty$, then $X_n + Y_n \to^d X$.*

Proof. See [3]. □

By this lemma we then have that

$$Y_t^n \to^d N(0,\, t).$$

We recall the definition of a Brownian motion B_t, that is a stochastic Gaussian process with average $\mu_t = 0$, co-variance $k(s,\, t) = \min\{s,\, t\}$ and continuous trajectories a.e..

Theorem 1.2.2. *For every $t_1, \dots, t_k \in \mathbb{R}^k$, $k \in \mathbb{N}$*

$$(Y^n_{t_1}, \dots, Y^n_{t_k}) \to^d (B_{t_1}, \dots, B_{t_k})$$

as $n \to \infty$.

Proof. See [3]. □

This results states that, as we see the random variables as processes with respect to the time, we have convergence of the finite dimensional law.
To check the convergence of processes we just need the tightness, as it follows from Prokorov's Theorem. The final result is

Theorem 1.2.3. *The law of $(Y^n_t)_{t\geq 0}$ converges to the law of $(B_t)_{t\geq 0}$.*

Proof. See [3]. □

We now can understand the potentiality and the complexity of the object we are about to use, this little introduction will helps us to have a better image of the symmetric random walk.

1.3 Voter model on $\mathbf{Z}^d$

We state a result in the case of the voter model on $\mathbb{Z}^d$. We write η^μ_t the model at the time t with initial distribution μ, with $L(\eta^\mu_t)$ being the law of η^μ_t.
We denote F as the set of invariant probability measures for η_t. F is a convex set with respect to the sum of measures and multiplication with scalars, and we denote F_e as the set of extreme points, i.e., the set of elements of F which are not a linear combination of other two elements.
For $0 \leq \theta \leq 1$ let μ_θ be the product measure with density θ, i.e., $\mu_\theta(\{\eta(x) = 1\}) = \theta$ for all $x \in \mathbb{Z}^d$. We have

Theorem 1.3.1. *If $d \leq 2$, then $F_e = \{\mu_0, \mu_1\}$ and $L(\eta^{\mu_\theta}_t) \rightharpoonup (1-\theta)\mu_0 + \theta\mu_1$, as $t \to \infty$. If $d \geq 3$, then there are probability measures ν_θ, such that $F_e = \{\nu_\theta, 0 \leq \theta \leq 1\}$ and $L(\eta^{\mu_\theta}_t) \rightharpoonup \nu_\theta$ as $t \to \infty$.*

Proof. See Theorem V.1.8 in [11]. □

This Theorem shows that, on the whole $\mathbb{Z}^d$, we don't have a moment in time where all the voters agree almost surely, but instead every one of them would have an opinion with a probability based on the measure ν_θ, which depends solely on $\theta \in [0.\ 1]$. As we will see later on, working on the torus of $\mathbb{Z}^d$, not only we will have consensus in a finite amount of time, but also we will be able to find a law.

Chapter 2

Coalescing random walks

In this chapter we will work on the set $S := \mathbb{Z}^d$ with $d \geq 2$. We will consider a sequence of finite systems by taking $\Lambda(N) = \mathbb{Z}^d \cap [-N/2, N/2)^d$, $N = 2, 4, \ldots$. The set $\Lambda(N)$ is regarded as a torus and we write $p^N(x,y)$ for the transition function of simple symmetric random walk on $\Lambda(N)$. That is, given two states,

$$p^N(x,y) = \sum_{z \in \mathbb{Z}^d} p(x,z) 1_{(y \equiv z \bmod(N))}.$$

We need to define

$$s_N = \begin{cases} N^2, & d = 1, \\ N^2 \log N, & d = 2, \\ N^d, & d \geq 3, \end{cases} \tag{2.1}$$

a rescalating system that will appear often throughout the proves.

2.1 Case of two Coalescing Random Walks

2.1.1 Definitions and useful quantities

The coalescing random walk system ξ_t has as state space the sets of parts of $\mathbb{Z}^d$ and $\xi_t(A)$ is the set of occupied sites at time t when the initial state is $A \subset \mathbb{Z}^d$.
Each particle independently executes simple symmetric rate 1 continuous random walk on $\mathbb{Z}^d$, that means the walk has equal probability to move to a neighbourhood state with the jump distributed as $\exp(1)$.
When a particle lands on a site already occupied they coalesce into one. We define ξ_t^N the process restricted to $\Lambda(N)$. The behavior of this object on a finite system obviously differs from a infinity one; for example, if $d \geq 3$, in the first case all the walks are eventually bound to meet, but it doesn't happen in the second case,

essentially because the random walks on $\mathbb{Z}^d$ are transient, as we have seen in the previous chapter.
We define the constant

$$G = \begin{cases} \frac{1}{6}, & d = 1, \\ \frac{2}{\pi}, & d = 2, \\ \sum_{n \geq 0} p_n(x,\, y), & d \geq 3. \end{cases} \tag{2.2}$$

Where $p_n(x,\, y)$ is the n-th iterate of $p(x,\, y)$. It's interesting to know the nature of this constant for $d \geq 3$, it is generated by a particular case of the Green's function. Let X_n be a random walk on $\mathbb{Z}^d$, and $x,\, y \in \mathbb{Z}^d$. We define the Green's generating function to be the power series in $\xi \in \mathbb{C}$

$$G(x,\, y;\, \xi) := \sum_{n=0}^{\infty} \xi^n P_x(X_n = y) = \sum_{n=0}^{\infty} \xi^n p_n(x,\, y)$$

that is absolutely convergent for $|\xi| < 1$. There is a particular interpretation of the sum for $\xi \leq 1$. Suppose T is a random variable independent from X_n with a geometric distribution,

$$P(T = j) = \xi^{j-1}(1-\xi), \quad j = 1,\, ,\ldots$$

We think of T as a 'killing time' for the walk, where $1-\xi$ is the killing rate. At each time j, if the walker has not already been killed, the process is killed with that probability, independently from the position of the walker. If the random walk starts at the origin, then the expected number of visits to x before being killed is given by

$$\begin{aligned} E\left[\sum_{j<T} 1_{(X_j=x)}\right] &= E\left[\sum_{j=0}^{\infty} 1_{(X_j=x,\, T>j)}\right] \\ &= \sum_{j=0}^{\infty} P_0(X_j = x,\, T > j) \\ &= \sum_{j=0}^{\infty} p_j(x,0)\xi^j = G(x,\, 0;\, \xi). \end{aligned}$$

Clearly a random walk is transient if and only if $G(0,\, 0;\, 1) < \infty$, from (1.4), in which case the escape probability is $G(0,\, 0;\, 1)^{-1}$. For a transient random walk, we then define the Green's function to be

$$G(x,\, y) := G(x,\, y;\, 1) = \sum_{n=0}^{\infty} p_n(x,\, y).$$

As we are working with a symmetric random walk, $G(x,\, y) = G(y,\, x)$, and also $G(x,\, x) = G(0,\, 0)$ for any $x,\, y \in \mathbb{Z}^d$. In particular, G is independent from the states $x,\, y \in \mathbb{Z}^d$; indeed, thanks to the previous observations, we just need to see $G(0,\, 0) = G(0,\, e_1)$. So

$$G(0,\, 0) = \sum_{n=0}^{\infty} p_n(0,\, 0) = \sum_{n=1}^{\infty} p_n(0,\, 0) = \sum_{n=0}^{\infty} p_n(0,\, e_1) = G(0,\, e_1)$$

since the probability of coming back to the origin in the first step is null, and

$$p_{n+1}(0,\, x) = \sum_{y \in Z^d} p(0,\, y) p_n(0,\, x-y)$$

where we choose $x = 0$. In fact

$$p_n(0,\, 0) = \sum_{y=\pm e_i,\, i \in \{1,\dots,d\}} \frac{1}{2d} p_n(0,\, y) = p_n(0, e_1)$$

because $p_n(0,\, e_i) = p_n(0,\, e_j)$ for $i,\, j \in \{1, \dots, d\}$.

2.1.2 Convergence in law of the coalescing time

We start by simplifying the object of the study. Let X_t^N, $t \geq 0$, be a simple symmetric rate 1 continuous time random walk on the torus $\Lambda(N)$ and let H^N be the hitting time of the origin. In particular $H^N = \inf\{t > 0 \mid X_t^N = 0\}$.

Theorem 2.1.1. *Assume $a_N \to \infty$ and $a_N = o(N)$ as $N \to \infty$. For $d = 2$ assume in addition that $a_N \sqrt{\log N}/N \to \infty$. Then, uniformly in $t \geq 0$ and $x \in \Lambda(N)$ with $|x| \geq a_N$, then*

$$P_x(H^N/s_N > t) \to \exp(-t/G).$$

For $x_1,\, x_2 \in \Lambda(N)$, we may see $\xi_t^N(x_1) - \xi_t^N(x_2)$ as a rate 2 random walk on $\Lambda(N)$ until the time that the random walks meet. The theorem implies that

$$P(|\xi_{ts_N}^N(\{x_1,\, x_2\})| = 1) = P_{x_1-x_2}(H^N/s_N > 2t) \to 1 - e^{-\frac{2t}{G}}, \tag{2.3}$$

as $N \to \infty$, provided that $|x_1 - x_2| \geq a_N$. Indeed, if we have two random walks, and their difference, as a process, hits 0, this means they have met and, as consequence, coalesced.

This theorem explains also the nature of s_N. On the torus, the average time, as N grows, of first passing through the origin converges to Gs_N, because the sequence of random variables converges in distribution to an exponential. Then s_N is the only rescaling that gives a non-trivial limit. We could see it as an acceleration on time that the system needs to not converge to a trivial situation.

2.1.3 Useful results

Lemma 2.1.2. *If $t_N \to \infty$, then*

$$\lim_{N\to\infty} \sup_{u\geq t_N N^2} \sup_{x\in\Lambda(N)} N^d |p_u^N(x,\,0) - N^{-d}| = 0. \tag{2.4}$$

For $d = 2$, if $a_N \to \infty$ and $a_N = o(N)$ as $N \to \infty$, then there is a finite constant K such that

$$\limsup_{N\to\infty} \sup_{u\geq 1} \sup_{|x|\geq a_N,\, x\in\Lambda(N)} a_N^2 p_u^N(x,\,0) \leq K. \tag{2.5}$$

Proof. We first prove (2.4).
It is not reductive to consider t_N a sequence of integers, then it suffices to prove

$$\lim_{N\to\infty} \sup_{x\in\Lambda(N)} N^d |p_{t_N N^2}^N(x,\,0) - N^{-d}| = 0.$$

For if $u \leq t_N N^2$ and $x \in \Lambda(N)$,

$$\begin{aligned} N^d |p_u^N(x,\,0) - N^{-d}| &= N^d \left| \sum_y p_{u-t_N N^2}^N(x,\,y) \left(p_{t_N N^2}^N(y,\,0) - N^{-d} \right) \right| \\ &\leq \sup_y N^d |p_{t_N N^2}^N(y,\,0) - N^{-d}| \to 0 \end{aligned}$$

since we are working with stochastic transitions.
By a Corollary 2.2.3 of [2], applied around $p_1(x,\,y)$, we have for $t = 1,\,2,\ldots,$

$$p_t(x,\,0) = \left(\frac{d}{2\pi t}\right)^{d/2} \exp\left(-\frac{d|x|^2}{2t}\right) \left[1 + \sum_{r=1}^{d} t^{-r/2} B_r\left(\frac{x}{\sqrt{t}}\right)\right] + e(x,\,t) \tag{2.6}$$

where each B_r is a polynomial, depending on d, of degree at most r and

$$t^{d/2} \sum_{x\in Z^d} |e(x,\,t)| \to 0 \quad \text{as } t \to \infty.$$

We are essentially approximating the behaviour of the distribution starting in x and passing in 0 with a Gaussian distribution, which depends on some polynomial terms that decrease to null as the time goes to infinity.
Since $p_u^N(x,\,0) = \sum_{z\in\mathbb{Z}^d} p_u(x,\,Nz)$ because we are working on a torus and by using

the previous expression, it follows that

$$\begin{aligned}N^d p_u^N(x,\,0) = N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{z\in Z^d}\exp\left(-\frac{d|x-Nz|^2}{2u}\right)\\ +N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{z\in Z^d}\exp\left(-\frac{d|x-Nz|^2}{2u}\right)\sum_{r=1}^{d}u^{-r/2}B_r\left(\frac{x-Nz}{\sqrt{u}}\right)\\ +N^d\sum_{z\in Z^d}e(x-Nz,\,u).\end{aligned} \tag{2.7}$$

We consider $u = t_N N^2$ and we analyze the three parts of this expansion.
For the first term, fix $R > 0$, let $I = [-\frac{1}{2},\ \frac{1}{2}]^d$ and $x' = x/N$. Then we expand again the expression

$$\begin{aligned}N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{z\in Z^d}\exp\left(-\frac{d|x-Nz|^2}{2u}\right) = N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{|z|\le R}\exp\left(-\frac{dN^2|x'-z|^2}{2u}\right)\\ +N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{|z|>R}\int_{I+z}\exp\left(-\frac{dN^2|x'-z|^2}{2u}\right)dy,\end{aligned}$$

from the basic $\int_{I+z} dy = (z+\frac{1}{2}-z+\frac{1}{2})^d = 1^d = 1$. Then for some constant C, the first sum in the right-hand side above is bounded from above by

$$\frac{CN^dR^d}{u^{d/2}} = \frac{CR^2}{t_N^{d/2}} \to 0$$

as $N \to \infty$ for a fixed R, moreover the series is surely finite because it has a finite amount of addends.
For the second sum, observe that $x' \in I$, with N big enough, and we can choose κ_R such that $0 < \kappa_R < 1$, $\kappa_R \to 1$ as $R \to \infty$ and

$$\kappa_R \le \frac{|y|}{|x'-z|} \le \frac{1}{\kappa_R}$$

with $|z| \ge R$ and $y \in I+z$. Then

$$\begin{aligned}N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{|z|>R}\int_{I+z}\exp\left(-\frac{dN^2|x'-z|^2}{2u}\right)dy\\ \le N^d\left(\frac{d}{2\pi u}\right)^{d/2}\sum_{|z|\ge R-1}\int_{I+z}\exp\left(-\frac{dN^2\kappa_R^2|y|^2}{2u}\right)dy\\ \le N^d\left(\frac{d}{2\pi u}\right)^{d/2}\left(\int_{-\infty}^{+\infty}\exp\left(-\frac{dN^2\kappa_R^2 r^2}{2u}\right)dr\right)^d = \frac{1}{\kappa_R^d} \to 1 \quad \text{as } R\to\infty.\end{aligned}$$

by first summing on the whole Z^d and using Fubini-Tonelli, and then substituting the variable in the Gaussian integral.
On the other hand, using the other side of the previous inequality,

$$\begin{aligned} N^d \left(\frac{d}{2\pi u}\right)^{d/2} \sum_{|z|>R} \int_{I+z} \exp\left(-\frac{dN^2|x'-z|^2}{2u}\right) dy \\ \geq N^d \left(\frac{d}{2\pi u}\right)^{d/2} \int_{|y|>R+1} \exp\left(-\frac{dN^2|y|^2}{2u\kappa_R^2}\right) dy \\ = \kappa_R^d (2\pi)^{-d/2} \int_{|y|>c_N} \exp(-|y|^2/2)\, dy, \end{aligned}$$

where, from the substitution of variable,

$$C_N = (R+1)\sqrt{\frac{dN^2}{\kappa_R^2 u}} = (R+1)\sqrt{\frac{d}{\kappa_R^2 t_N}} \to 0$$

as $N \to \infty$ for a fixed R. Hence

$$\begin{aligned} \liminf_{n\to\infty} N^d \left(\frac{d}{2\pi u}\right)^{d/2} \sum_{|z|>R} \int_{I+z} \exp\left(-\frac{dN^2|x'-z|^2}{2u}\right) dy \\ = \kappa_R^d (2\pi)^{-d/2} \int_{|y|>c_N} \exp(-|y|^2/2)\, dy \\ = \kappa_R^d. \end{aligned}$$

Since $\kappa_R \to 1$ as $R \to \infty$, we have proved

$$N^d \left(\frac{d}{2\pi t_N N^2}\right)^{d/2} \sum_{z\in Z^d} \exp\left(-\frac{d|x-Nz|^2}{2t_N N^2}\right) \to 1.$$

Since each B_r is a polynomial and considering always $u = t_N N^2$, the upper limit of the second term of (2.7) is finite, as the exponential takes over. It implies

$$\lim_{N\to\infty} N^d \left(\frac{d}{2\pi t_N N^2}\right)^{d/2} \sum_{z\in Z^d} \exp\left(-\frac{d|x-Nz|^2}{2t_N N^2}\right) d(t_N N^2)^{-r/2} B_r\left(\frac{x-Nz}{\sqrt{t_N N^2}}\right) = 0$$

For the third term of (2.7), by the definition of $e(x,\, t)$,

$$N^d \sum_{z\in Z^d} e(x-Nz,\, t_N N^2) \to 0$$

So, by choosing wisely the steps of the time, we have proven that the first limit (2.4) does converge to 0.
For the second inequality (2.5) it suffices to prove that

$$\limsup_{N\to\infty} \sup_{k\geq 1} \sup_{|x|\geq a_N,\, x\in\Lambda(N)} a_N^2 p_k^N(x,\, 0) \leq \infty$$

where k is a positive integer. Indeed, if $k \leq t \leq k+1$, then

$$\begin{aligned} p_{k+1}^N(x,\, 0) &\geq p_t^N(x,\, 0) p_{k+1-t}^N(0,\, 0) \\ &\geq p_t^N(x,\, 0) p_{k+1-t}(0,\, 0) \\ &\geq c_0 p_t^N(x,\, 0), \end{aligned}$$

where $c_0 = \inf\{p_s(0,\, 0) \mid 0 \leq s \leq 1\} > 0$.
Let $x' = x/N$ as before. We expand $a_N^2 p_k^N(x,\, 0)$ as in (2.7) and the main contribution is

$$\frac{a_N^2}{\pi k} \sum_{z\in Z^d} \exp\left(-\frac{|x-Nz|^2}{k}\right) = \frac{a_N^2}{\pi k}\left[\exp\left(-\frac{|x|^2}{k}\right) + \sum_{z\neq 0} \exp\left(-\frac{N^2|x'-z|^2}{k}\right)\right].$$

The first term above is bounded, and for some finite constant C it holds

$$\frac{a_N^2}{\pi k} \exp\left(-\frac{|x|^2}{k}\right) \leq \frac{a_N^2}{\pi k} \exp\left(-\frac{a_N^2}{k}\right) \leq C$$

since we consider $|x| \geq a_N$, for all k and N.
For the second term we use the same trick of the integral of the previous inequality, in particular

$$\begin{aligned} \frac{a_N^2}{\pi k} \sum_{z\neq 0} \exp\left(-\frac{N^2|x'-z|^2}{k}\right) &= \frac{a_N^2}{\pi k} \sum_{z\neq 0} \int_{z+I} \exp\left(-\frac{N^2|x'-z|^2}{k}\right) dy \\ &\leq \frac{a_N^2}{\pi k} \sum_{z\neq 0} \int_{z+I} \exp\left(-\frac{N^2\kappa_1^2|y|^2}{k}\right) dy \\ &\leq \frac{a_N^2}{\pi k} \int_{R^2} \exp\left(-\frac{N^2\kappa_1^2|y|^2}{k}\right) \\ &= C\frac{a_N^2\kappa_1^2}{N^2} \to 0 \quad \textbf{as } N\to\infty. \end{aligned}$$

This proves that

$$\limsup_{N\to\infty} \sup_{k\geq 1} \sup_{|x|\geq a_N,\, x\in\Lambda(N)} \frac{a_N^2}{\pi k} \sum_{z\in Z^d} \exp\left(-\frac{|x-Nz|^2}{k}\right) < \infty$$

The other two errors terms behave as in the previous inequality so their analysis is omitted. □

The goal now is to prove Theorem (2.1.1). Let $p_t^N(x,\, y) = P_x(X_t^N = y)$, and let F_N and G_N be the Laplace transforms

$$F_N(x,\, \lambda) = \int_0^\infty e^{-\lambda t} P_x(H^N \in dt), \tag{2.8}$$

$$G_N(x,\, \lambda) = \int_0^\infty e^{-\lambda t} p_t^N(x,\, 0)\, dt.$$

defined for $\lambda > 0$ and $x \in \Lambda(N)$. The following identity emerges

$$G_N(x,\, \lambda) = \int_0^\infty e^{-t\lambda} \int_0^t P_x(H^N \in du) p_{t-u}^N(0,\, 0)\, dt$$

that is the Laplace transform of the probability that the random walk hits two times the origin starting from the state x, and from this one may derive the fundamental relation

$$F_N(x,\, \lambda) = \frac{G_N(x,\, \lambda)}{G_N(0,\, \lambda)}.$$

The characteristic function of discrete time random walk on Z^d is

$$\begin{aligned} \phi(\theta) &:= \sum_{x\in Z^d} e^{ix\cdot\theta} p(0,\, x) \\ &= \frac{1}{2d} \sum_{j=1}^d e^{i\theta_j} + e^{-i\theta_j} \\ &= \frac{1}{d} \sum_{j=1}^d \cos(\theta_j), \quad \theta \in R^d \end{aligned} \tag{2.9}$$

since the jumps can reach only to the neighbourhood states of the center.
In this model the relation between the characteristic function and the Laplace transform previously defined is

$$G_N(x,\, \lambda) = \frac{1}{N^d} \sum_{y\in\Lambda(N)} \frac{e^{i2\pi x\cdot y/N}}{1+\lambda-\phi(2\pi y/N)}, \tag{2.10}$$

see [12] for the proof. This identity is used to find some interesting results.
We want to study

$$E_x[e^{-\lambda H^N/s_N}] = F_N(x,\, \lambda/s_N) = \frac{G_N(x,\, \lambda/s_N)}{G_N(0,\, \lambda/s_N)}.$$

For $d = 2$, $s_N = N^2 \log N$,

$$\begin{aligned} \frac{G_N(0,\, \lambda/N^2\log N)}{\log N} &= \lambda^{-1} + \frac{1}{N^2\log N} \sum_{0\neq y\in\Lambda(N)} \frac{1}{1+\lambda/N^2\log N - \phi(2\pi y/N)} \\ &\sim \lambda^{-1} + \frac{1}{N^2\log N} \sum_{0\neq y\in\Lambda(N)} \frac{1}{1-\phi(2\pi y/N)}, \end{aligned}$$

where the λ^{-1} appears from the origin component of the sum and the symbol $\sim$ means that, given $f(N)$ and $g(N)$, $F(N)/G(N) \to 1$ as $N \to \infty$. We wish to study the convergence of the series as $N \to \infty$. We define

$$\varphi_N := \frac{1}{N^2} \sum_{0 \neq y \in \Lambda(N)} \frac{1}{1-\phi(2\pi y/N)} = \frac{4}{N^2} \sum_{y_1=1,\, y_2=1}^{[(N-1)/2]^2} \frac{1}{1-\frac{1}{4}(\cos(2\pi y_1/N)+\cos(2\pi y_2/N))}$$

where we transpose the sum just on the first positive quadrant, and we always consider $N = 0,\ 2,\ 4, \ldots$. As $N \to \infty$ the sum alone approaches a divergent integral, since $1-\phi(2\pi y/N)$ becomes small rapidly as $(2\pi y/N) \to 0$. One can approximate ϕ by

$$\phi(2\pi x/N) \sim 1-(2\pi^2/N^2)(x_1^2+x_2^2)+\ldots$$

See [13] and [12]. The range of the summation is then divided into two parts; the first part containing those $(x_1,\ x_2)$ such that $(x_1^2+x_2^2)^{1/2} < \alpha N$, where α small enough so that the approximation is good for all those points, and the second containing the remaining. The contribution of the second set to φ remains bounded as $N \to \infty$, as the first set grows with N, so we don't consider it. Then

$$\varphi_N = \frac{4}{N^2} \sum_{1 \leq (y_1^2+y_2^2)^{1/2} < \alpha N} \frac{N^2}{2\pi^2(y_1^2+y_2^2)}$$

as $N \to \infty$, the sum is well approximated by the following integral, express in polar coordinates

$$\sum_{1 \leq (y_1^2+y_2^2)^{1/2} < \alpha N} \frac{1}{(y_1^2+y_2^2)} \sim \int_1^{\alpha N} \frac{2\pi r}{r^2}\, dr = 2\pi \log \alpha N.$$

Hence

$$\lim_{N \to \infty} \frac{\varphi_N}{\log N} = \frac{2}{\pi} = G, \quad \text{with } d = 2,$$

for a fixed α.
For $d \geq 3$, $s_N = N^{-d}$, and $I = [-1/2,\ 1/2]^d$ instead

$$\begin{aligned} G_N(0,\ \lambda N^{-d}) &= \lambda^{-1} + \frac{1}{N^d} \sum_{0 \neq y \in \Lambda(N)} \frac{1}{1+\lambda/N^d-\phi(2\pi y/N)} \\ &\sim \lambda^{-1} + \frac{1}{N^d} \sum_{0 \neq y \in \Lambda(N)} \frac{1}{1-\phi(2\pi y/N)} \\ &\to \lambda^{-1} + \int_I \frac{1}{1-\phi(2\pi\theta)}\, d\theta \end{aligned}$$

As we consider the limit on N on every component of $\frac{1}{N^d}$ and the sum converging to the integral on the enlarging torus, that then shrinks to I by substituting the variable.
If $X = (X_1, \dots, X^d)$ is a Z^d-valued random variable with characteristic function ϕ, then for every $x \in Z^d$,

$$P(X = x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi(\theta) e^{-ix\cdot\theta} \, d\theta.$$

Indeed

$$\phi(\theta) = E[e^{iX\cdot\theta}] = \sum_{y \in Z^d} e^{iX\cdot\theta} P(X = y),$$

and then

$$\int_{[-\pi, \pi]^d} \phi(\theta) e^{-ix\cdot\theta} \, d\theta = \sum_{y \in Z^d} P(X = y) \int_{[-\pi, \pi]^d} e^{i(y-x)\cdot\theta} \, d\theta.$$

By the dominated convergence we can interchange the sum and the integral, and if $x, \ y \in Z^d$

$$\int_{[-\pi, \pi]^d} e^{i(y-x)\cdot\theta} \, d\theta = \begin{cases} (2\pi)^d, & y = x, \\ 0, & y \neq x. \end{cases}$$

More, if a Z^d-valued random process is i.i.d with characteristic function ϕ, and let $S_n = \sum_{i=1}^n X_i$ then, for all $x \in Z^d$

$$P(S_n = x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi^n(\theta) e^{-ix\cdot\theta} \, d\theta.$$

Now

$$\begin{aligned} G(0, \ x, \ \xi) = \sum_{n=0}^{\infty} \xi^n p_n(0, \ x) &= \sum_{n=0}^{\infty} \xi^n \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi^n(\theta) e^{-ix\cdot\theta} \, d\theta \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \sum_{n=0}^{\infty} (\xi \phi(\theta))^n e^{-ix\cdot\theta} \, d\theta \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - \xi \phi(\theta)} e^{-ix\cdot\theta} \, d\theta. \end{aligned}$$

The interchange is again possible thanks to the dominated convergence. For $\xi = 1$, finally,

$$\lim_{N \to \infty} G_N(0, \ \lambda N^{-d}) = \frac{1}{\lambda} + G \tag{2.11}$$

for $d \geq 3$.

2.1.4 Proof of the main Theorem

Now everything we need to prove the Theorem is ready 2.1.1.

Proof Theorem 2.1.1. It suffices to show that for any fixed $\lambda > 0$, uniformly in $|x| \geq a_N$,

$$E_x\left[\exp\left(-\frac{\lambda H^N}{s_N}\right)\right] \to \frac{1}{1+\lambda G}$$

as $N \to \infty$. Since for each N the left side is a monotone function of λ and the right side is continuous in λ, it follows that the convergence must hols uniformly in $\lambda \geq 0$. The uniformity and continuity imply that for any bounded continuous function f on $[0, \infty)$, uniformly in $|x| \geq a_N$,

$$E_x[f(H^N/s_N)] \to G^{-1}\int_0^\infty f(v)e^{-v/G}\,dv$$

as $N \to \infty$. By using an approximation of the identity function that, for any fixed $t > 0$, uniformly in $|x| \geq a_N$, the monotonicity implies that

$$P_x(H^N/s_N > t) \to e^{-t/G}, \quad N \to \infty$$

must hold uniformly in $t \geq 0$.
We recall that

$$E_x[e^{-\lambda H^N/s_N}] = \frac{G_N(x,\, \lambda/s_N)}{G_N(0,\, \lambda/s_N)}$$

and that

$$G_N(0,\, \lambda N^2 \log N) \to \frac{1}{\lambda} + G, \quad d = 2$$
$$G_N(0,\, \lambda N^{-d}) \to \frac{1}{\lambda} + G, \quad d \geq 3.$$

Thus it remains to prove that, uniformly in $|x| \geq a_N$,

$$\frac{1}{\log N}\int_0^\infty e^{-\lambda t/N^2 \log N} p_t^N(x,\, 0)\,dt \to \lambda^{-1}, \quad d = 2$$
$$\int_0^\infty e^{-\lambda t/N^d} p_t^N(x,\, 0)\,dt \to \lambda^{-1}, \quad d \geq 3.$$

***Case d* = 2**. We assume $a_N = o(N)$ and $a_N\sqrt{\log N}/N \to \infty$ as $N \to \infty$. Let $t_N \leq a_N\sqrt{\log N}/N$ such that $t_N \to \infty$ and $t_N = o(\log N)$ and break the integral in two parts. The first is

$$\frac{1}{\log N}\int_0^{t_N N^2} e^{-\lambda t/N^2\log N} p_t^N(x,\,0)\,dt \le \frac{1}{\log N}\left(1+\int_1^{t_N N^2}\frac{K}{a_N^2}e^{-\lambda t/N^2\log N}\,dt\right)$$
$$\le \frac{1}{\log N}\left(1+\frac{KN^2\log N}{\lambda a_N^2}(1-e^{-\lambda t_N/\log N})\right)$$
$$\le \frac{1}{\log N}\left(1+\frac{KN^2 t_N}{a_N^2}\right)\to 0$$

as $N\to\infty$, by using (2.5) and the hypothesis on t_N. The second part is

$$\frac{1}{\log N}\int_{t_N N^2}^{\infty} e^{-\lambda t/N^2\log N} p_t^N(x,\,0)\,dt \le \frac{1+o(1)}{N^2\log N}\left(\int_{t_N N^2}^{\infty} e^{-\lambda t/N^2\log N}\,dt\right)$$
$$= \frac{1+o(1)}{\lambda}\exp\left(-\frac{\lambda t_N}{\log N}\right)\to \lambda^{-1}$$

as $N\to\infty$, where we used (2.4).
Case d $\geq$ **3**. We need another estimate to continue

$$P(|X_t|\ge t^{1/2}\log t)\le C/t^2,\quad t\ge 0.$$

This comes from Markov's inequality $P(X\ge a)\le\frac{E[X]}{a}$, in the exponential form, with X positive random variable. In particular

$$P(|X_t|\ge a)=P(e^{\lambda|X_t|}\ge e^{\lambda a})\le\frac{E[e^{\lambda|X_t|}]}{e^{\lambda a}}.$$

By substituting $a=t^{1/2}\log t$, $\lambda=2/t^{1/2}$, and $C=E[e^{\lambda|X_t|}]$, we get the estimate, noticing that the mean of the exponential is finite. For any finite set $\Gamma\subset Z^d$ we have

$$p_t^N(x,\,0)=\sum_{z\in Z^d}p_t(x,\,Nz)$$
$$\le|\Gamma\cap(x+NZ^d)|p_t(0,\,0)+\sum_{z\notin\Gamma}p_t(0,\,x+Nz).$$

Since $p_t(0,\,0)\le C/t^{d/2}$, the choice $\Gamma=[-t^{1/2}\log t,\,t^{1/2}\log t]^d$ gives

$$p_t^N(x,\,0)\le\frac{C}{t^{d/2}}\left(\frac{t^{1/2}\log t}{N}\right)^d+P(|X_t|\ge t^{1/2}\log t)$$
$$\le C\left(\left(\frac{\log t}{N}\right)^d+\frac{1}{t^2}\right).$$

Now we break the integral in two parts.

$$\begin{aligned}
0 &\leq \int_0^{N^2 \log N} e^{-\lambda t/N^d} p_t^N(x,\, 0)\, dt \\
&\leq \int_0^{N^2 \log N} p_t^N(x,\, 0)\, dt \\
&\leq T \sup_{|x|\geq a_N} \sup_{0\leq t\leq T} p_t^N(x,\, 0) + C \int_T^{N^2 \log N} \left(\frac{(\log t)^d}{N^d} + \frac{1}{t^2} \right) dt \\
&= T \sup_{|x|\geq a_N} \sup_{0\leq t\leq T} p_t^N(x,\, 0) + C \left[\frac{t}{N^d} \sum_{k=0}^{d} (-1)^{d-k} \frac{d!}{k!} (\log t)^k - \frac{1}{t} \right] \Big|_T^{N^2 \log N} \\
&\to \frac{C}{T}
\end{aligned}$$

as $N \to \infty$ for a fixed T, since $p_t^N(x,\, 0) \to 0$ for an escaping state x in a finite time. And then for $T \to \infty$

$$\lim_{N\to\infty} \int_0^{N^2 \log N} e^{-\lambda t/N^d} p_t^N(x,\, 0)\, dt = 0.$$

Now, using (2.4) the second part is

$$\int_{N^2 \log N}^{\infty} e^{-\lambda t/N^d} p_t^N(x,\, 0)\, dt = \frac{(1+o(1))}{N^d} \int_{N^2 \log N}^{\infty} e^{-\lambda t/N^d}\, dt \to \lambda^{-1}$$

as $N \to \infty$, by substituting the variable. This completes the proof. $\square$

2.2 Multiple Coalescing Random Walks

As we have studied the case of two random walks and we have estimated their time of coalescence, it's time to consider n starting random walks and their behaviour. We define the functions:

$$q_{n,\, k}(t) = \sum_{j=k}^{n} \frac{(-1)^{j+k}(2j-1)(j+k-2)!\binom{n}{j}}{k!(k-1)!(j-k)!\binom{n+j-1}{j}} \exp\left(-t \binom{j}{2} \right). \tag{2.12}$$

$$q_{\infty,\, k}(t) = \sum_{j=k}^{\infty} \frac{(-1)^{j+k}(2j-1)(j+k-2)!}{k!(k-1)!(j-k)!} \exp\left(-t \binom{j}{2} \right). \tag{2.13}$$

This two quantities are tied to the distributions of a Markov chain D_t defined on the positive integers, fixing the initial state. In particular, a state n goes in the state $n-1$ at (exponential) rate $\binom{n}{2}$. Then $P_n(D_t = k) = q_{n,\, k}(t)$.
Now we can state:

Theorem 2.2.1. *Assume $d \geq 2$, T>0 and $n \geq 2$, and a_N satisfies the assumptions of Theorem 2.1.1. Then, uniformly in $0 \leq t \leq T$ and $A^N = \{x_1, \dots, x_n\} \subset \Lambda(N)$ with $|x_i - x_j| \geq a_N$ for $i \neq j$,*

$$P(|\xi^N_{ts_N}(A^N)| = k) \to q_{n,\,k}(2t/G) \tag{2.14}$$

for $1 \leq k \leq n$.

Proof. We first define some object that will help during this proof

$$\tau^N(i,\ j) := \inf\{t \geq 0\ :\ |\xi^N_t(\{x_i,\ x_j\})| = 1\},$$
$$\hat{\tau}^N := \min_{i \neq j} \tau^N(i,\ j),$$
$$H^N_t(i,\ j) := \{\tau^N(i,\ j) \leq ts_N\},$$
$$F^N_t(i,\ j) := \{\hat{\tau}^N = \tau^N(i,\ j) \leq ts_N\},$$

Then the relation between these objects can be expressed as

$$\begin{aligned} P(H^N_t(i,\ j)) = P(F^N_t(i,\ j)) \\ + \sum_{\{k,\,l\} \neq \{i,\,j\}} \sum_{y_\alpha,\,y_\beta} \int_0^{ts_N} P(\hat{\tau}^N = \tau^N(k,\ l) \in du, \xi^N_u(x_i) = y_\alpha,\ \xi^N_u(x_j) = y_\beta) \cdot \\ \cdot P(|\xi^N_{ts_N - u}(\{y_\alpha,\ y_\beta\})|) \end{aligned} \tag{2.15}$$

This is the probability that the random walks starting from the states $i,\ j$ are actually the first to coalesce plus the probability that all the other random walks do not do the same, considering the adjustment on the time.
We want to assume that $|y_\alpha - y_\beta| \geq a_N$, if not then the integral converges to 0 as $N \to \infty$. In particular

$$\varepsilon_N := \int_0^{Ts_N} P(\hat{\tau}^N = \tau^N(k,\ l) \in du, |\xi^N_u(x_i) - \xi^N_u(x_j)| \leq a_N) \to 0$$

as $N \to \infty$ and with $t_N N^2 \leq Ts_N$.
There are two case to consider; let $X^N_u(x),\ x \in \Lambda(N)$, be independent random walks, with $X^N_0(x) = x$. We must show, as $N \to \infty$,

$$\int_0^{Ts_N} P(\hat{\tau}^N = \tau^N(1,\ 2) \in du,\ |\xi^N_u(x_3) - \xi^N_u(x_4)| \leq a_N) \to 0$$
$$\int_0^{Ts_N} P(\hat{\tau}^N = \tau^N(1,\ 2) \in du,\ |\xi^N_u(x_3) - \xi^N_u(x_1)| \leq a_N) \to 0$$

where, without losing generality, we choose to fix the two random walks that meet first.
The only difference between the two integrals is the walks it considers, in the first case the two walks are the ones that don't coalesce, in the second case one does. We need to study both for generality. Luckily, we just need to prove the second limit as the other one behaves similarly; from the independence of random walks $X_u^N(x)$, as x varies, we have

$$\begin{aligned}
&\int_0^{Ts_N} P(\hat{\tau}^N = \tau^N(1,\,2) \in du,\, |\xi_u^N(x_3) - \xi_u^N(x_1)| \leq a_N) \\
&\leq P(\tau^N(1,\,2) \leq t_N N^2) + \int_{t_N N^2}^{Ts_N} P(\tau^N(1,\,2) \in du,\, |X_u^N(x_3) - X_u^N(x_1)| \leq a_N) \\
&= P(\tau^N(1,\,2) \leq t_N N^2) + \sum_{y \in \Lambda(N)} \int_{t_N N^2}^{Ts_N} P(\tau^N(1,\,2) \in du,\, X_u^N(x_1) = y) P(|X_u^N(x_3) - y| \leq a_N),
\end{aligned}$$

where we have denoted by $\tau^N(1,\,2)$ the coalescing time of $X_u^N(x_1)$ and $X_u^N(x_2)$. Without loss of generality choose t_N such that $t_N \to \infty$ and $t_N/\log N \to 0$. Then the right-hand side of the previous equation is bounded from above by

$$\begin{aligned}
&P_{x_1-x_2}(H^N \leq 2t_N N^2) + C\frac{a_N^d}{N^d} P(\tau^N(1,\,2) \in [t_N,\, Ts_N]) \\
&\qquad \leq P_{x_1-x_2}\left(\frac{H^N}{s_N} \leq \frac{2t_N N^2}{s_N}\right) + C\frac{a_N^d}{N^d} \to 0
\end{aligned}$$

where we use Theorem 2.1.1 and (2.4), and the hypothesis $t_N N^2/s_N \to 0$.
So, from now on, $|y_\alpha - y_\beta| \geq a_N$, and $y := y_\alpha - y_\beta$. In this case

$$\begin{aligned}
P(|\xi_{ts_N-u}^N(\{y_\alpha,\, \alpha_\beta\})| = 1) &= P_y(H^N \leq 2(ts_N - u)) \\
&= 1 - \exp\left[-2\left(t - \frac{u}{s_N}\right)/G\right] + \varepsilon_N
\end{aligned}$$

by Theorem 2.1.1.

$$\begin{aligned}
&\sum_{y_\alpha,\, y_\beta} \int_0^{ts_N} P(\hat{\tau}^N = \tau^N(k,\, l) \in du, \xi_u^N(x_i) = y_\alpha,\, \xi_u^N(x_j) = y_\beta) \cdot P(|\xi_{ts_N-u}^N(\{y_\alpha,\, y_\beta\})|) \\
&= \int_0^{ts_N} P(\hat{\tau}^N = \tau^N(k,\, l) \in du) \left(1 - \exp\left[-2\left(t - \frac{u}{s_N}\right)/G\right]\right) + \varepsilon_N \\
&= \left[P(F_u^N(k,\, l)) \left(1 - \exp\left[-2\left(t - \frac{u}{s_N}\right)/G\right]\right)\right]\Big|_0^{t/s_N} + \frac{2}{G}\int_0^t P(F_{u'}^N(k,\, l)) e^{-2(t-u')/G} du' + \varepsilon_N \\
&= \frac{2}{G}\int_0^t P(F_{u'}^N(k,\, l)) e^{-2(t-u')/G} du' + \varepsilon_N
\end{aligned}$$

where we integrate by parts and notice that the distribution and the exponential are 0 in 0 and t/s_N respectively, and then we substitute the variable $u' = u/s_N$. Since

$$P(H_t^N(i,\ j)) \to 1 - e^{-2t/G}$$

and by using (2.15)

$$1 - e^{-2t/G} = P(F_t^N(i,\ j)) + \frac{2}{G} \sum_{\{i,j\} \neq \{l,k\}} \int_0^t P(F_u^N(k,\ l)) e^{-2(t-u')/G} du + \varepsilon_N.$$

A summation over $i,\ j$ gives

$$\binom{n}{2}(1 - e^{-2t/G}) = q^N(t) + \frac{2}{G}\left(\binom{n}{2} - 1\right) e^{-2t/G} \int_0^t q^N(u) e^{2u/G} du + \varepsilon_N \quad (2.16)$$

considering

$$q^N(t) = P(\hat{\tau}^N \le ts_N).$$

The solution of the equation (2.16) converges to the solution of

$$\binom{n}{2}(1 - e^{-2t/G}) = q(t) + \frac{2}{G}\left(\binom{n}{2} - 1\right) e^{-2t/G} \int_0^t q(u) e^{2u/G} du \qquad (2.17)$$

as $N \to \infty$, uniformly in A^N and t, we call such solution $q_N(t)$ and $q(t)$. So, we define $\alpha_N(t) = q_N(t) - q(t)$ and it satisfies

$$|\alpha_N(t)| \le \frac{2}{G}\left(\binom{n}{2} - 1\right) e^{-2t/G} \int_0^t |\alpha_N(u)| e^{2u/G} du + \varepsilon_N \le C \int_0^t |\alpha_N(u)| e^{2u/G} du + \varepsilon_N$$

by subtracting one of the previous equation to the other. We can now use the Gronwall's Lemma and we get

$$|\alpha_N(t)| \le C' \varepsilon_N \to 0$$

as $N \to \infty$.
The solution of (2.17) is

$$q(t) := 1 - \exp\left(-2t\binom{n}{2}/G\right),$$

it can be seen by just substituting the function in the equation.
We conclude the proof by induction. The induction hypothesis is that for a_N satisfying the assumptions, uniformly for $[0,\ T]$ and $A^N = \{x_1, \dots,\ x_n\} \subset \Lambda(N)$ such that $|x_i - x_j| \ge a_N$ for $i \ne j$,

$$P(|\xi_{ts_N}^N(A^N)| = k) \to q_{n,\,k}(t), \quad 1 \le k \le n.$$

We have already studied the special cases $k = 1, n$. The induction step is to prove that uniformly for $t \in [0, T]$ and $B^N = \{y_1, \dots, y_{n+1}\} \subset \Lambda(N)$ such that $|y_i - y_j| \geq a_N$ for $i \neq j$,

$$
\begin{aligned}
P(|\xi^N_{ts_N}(B^N)| = k) &\to \binom{n+1}{2} \frac{2}{G} \int_0^t \exp\left(-2u\binom{n+1}{2}/G\right) q_{n,k}(t-u)\,du \\
&= q_{n+1,k}(t), \quad 1 \leq k \leq n.
\end{aligned}
$$

To prove this let $\sigma^N = \inf\{t \geq 0 \,:\, |\xi^N_t(B^N)| = n\}$ and fix $k \leq n$. Then

$$
P(|\xi^N_{ts_N}(B^N)| = k) = \sum_{C^N} \int_0^{ts_N} P(\sigma^N \in du,\, \xi^N_u(B^N) = C^N) P(|\xi^N_{ts_N - u}(C^N)| = K),
$$

where $C^N = \{z_1, \dots, z_n\} \subset \Lambda(N)$, as we have already seen

$$
\int_0^{ts_N} P(\sigma^N \in du,\, \xi^N_u(B^N) = \{z_1, \dots, z_n\},\, |z_i - z_j| \leq a_N \text{ for some } i \neq j) \to 0,
$$

and so, by using the induction hypothesis, we obtain

$$
\begin{aligned}
&P(|\xi t s_N{}^N(B^N)| = k) \\
&\quad = \sum_{C^N} \int_0^{ts_N} P(\sigma^N \in du,\, \xi^N_u(B^N) = C^N) \left(q_{n,k}(t - u/s_N) + \varepsilon_N\right) + \varepsilon_N \\
&\quad = \int_0^{ts_N} P(\sigma^N \in du) q_{n,k}(t - u/s_N) + \varepsilon_N \\
&\quad \to \binom{n+1}{2} \frac{2}{G} \int_0^t \exp\left(-2u\binom{n+1}{2}/G\right) q_{n,k}(t - u/s_N)\,du
\end{aligned}
$$

as required. This completes the proof. □

2.3 Mean of the coalescing times

We state the main Theorem

Theorem 2.3.1. *Let $\xi^N_0 = \Lambda(N)$ and let $\sigma^N_j = \inf\{t \geq 0 \,:\, \xi^N_t = j\}$. There are random variables σ_j such that for $j = 1, 2, \dots$ as $N \to \infty$,*

$$
\sigma^N_j / s_N \rightharpoonup \sigma_j \quad \text{and} \quad E[\sigma^N_j / s_N] \to E[\sigma_j] \tag{2.18}
$$

If $d \geq 2$, then

$$
P(\sigma_j \leq s) = \sum_{k=1}^{j} q_{\infty,k}(2s/G), \quad s \geq 0, \tag{2.19}
$$

and $E[\sigma_1] = G$.

2.3.1 Useful Lemmas

To tackle the proof of this theorem we need a series of lemmas. First we define

$$g_N(t) = \begin{cases} N/\sqrt{t}, & d = 1, \\ N^2 \log(1+t)/t, & d = 2, \\ N^d/t, & d \geq 3, \end{cases}$$

that will help in the estimates.

Lemma 2.3.2. *If $B \subset A \subset \Lambda(N)$ and $h_s(A) = \min_{x,\, y \in \Lambda(N)} P_{x-y}(H^N \leq s)$, then*

$$E[|\xi_s(N)(B)|] \leq |B| - (|B| - 1)h_s(A). \tag{2.20}$$

Proof. We can assume that $|B| \geq 1$, so fix $x_0 \in B$ and define

$$Z_s = \sum_{x \in B - \{x_0\}} 1_{(\xi_s^N(x) = \xi_s^N(x_0))},$$

that is the number of walks that coalesce with the walk starting at x_0. For sure

$$|\xi_s^N(B)| \leq |B| - Z_s$$

and

$$E[Z_s] \geq (|B| - 1)h_s(A)$$

then we just take expectation and we get the final result. □

Lemma 2.3.3. *If $t \leq r \leq r+s \leq 2t$, $\frac{1}{2}E[|\xi_{2t}^N|] \geq 4^{-d}E[|\xi_t^N|] \geq 2^d$ and A_t is a cube in $\Lambda(N)$ of side $[8N/E[|\xi_t^N|]^{1/d}]$, where $[t]$ denotes the greatest integer less or equal to t, then*

$$E[|\xi_{r+s}^N|] \leq E[|\xi_r^N|](1 - \frac{1}{2}h_s(A_t)) \leq E[|\xi_r^N|]\exp(-\frac{1}{2}h_s(A_t)). \tag{2.21}$$

Proof. Let B_i, $1 \leq i \leq n(t)$, be disjoint cubes covering $\Lambda(N)$, each B_i no larger than A_t, with

$$n(t) \leq \left(\frac{NE[|\xi_t^N|]^{1/d}}{8N} + 1 \right)^d \leq \left(\frac{E[|\xi_t^N|]^{1/d}}{4} \right)^d = \frac{E[|\xi_t^N|]}{4^d} \leq \frac{1}{2}E[|\xi_{2t}^N|] \leq \frac{1}{2}E[|\xi_r^N|].$$

If we ignore the coalescence of particles starting in different B_i's, then the Markov property implies

$$E[|\xi_{r+s}^N|] \leq \sum_{C_i \subset B_i,\ 1 \leq i \leq n(t)} P(\xi_r^N \cap B_i = C_i) \sum_j E[|\xi_s^N(C_j)|].$$

Using the previous lemma and writing h_s for $h_s(A_t)$ we have

$$\begin{aligned}\sum_i E[|\xi_s^N(C_i)|] &\le \sum_i [|C_i| - (|C_i|-1)h_s] \\ &= (1-h_s)\sum_i |C_i| + h_s n(t) \\ &\le (1-h_s)\sum_i |C_i| + \frac{1}{2}h_s E[|\xi_r^N|].\end{aligned}$$

And combing the estimates we obtain the result

$$\begin{aligned}E[|\xi_{r+s}^N|] &\le \sum_{C_i\subset B_i,\ 1\le i\le n(t)} P(\xi_r^N \cap B_i = C_i)\left((1-h_s)\sum_j |C_j| + \frac{1}{2}h_s E[|\xi_r^N|]\right) \\ &= (1-\frac{1}{2}h_s(A_t))E[|\xi_r^N|].\end{aligned}$$

As for the exponential form we use the simple consideration $(1+x) \le e^x$ for $x \in R$. □

Lemma 2.3.4. *If $f_N(t) = E[|\xi_t^N|]/g_N(t)$, then there exist finite constants M_d such that*

$$\begin{aligned}f_N(t) &\le M_d, \quad 0 \le t \le 4,\ N = 2,\ 4, \dots, \\ f_N(2t) &\le \max\{M_d,\ f_N(t)\}, \quad t \ge 0,\ N = 2,\ 4, \dots.\end{aligned}$$

Proof. If $t \le 4$, then since $|\xi_t^N| \le N^d$, the first inequality holds with $M_d = 4$. Now if cannot apply (2.21), then either

$$E[|\xi_t^N|] \le 8^d$$

or

$$E[|\xi_{2t}^N|] \le 2\cdot 4^{-d} E[|\xi_t^N|]$$

and in either case, for all d the second inequality holds with $M_d = 8^d$. So we may assume that the result of the previous lemma is true, in which case iteration of (2.21) gives

$$E[|\xi_{2t}^N|] \le E[|\xi_t^N|]\exp\left(-\frac{1}{2}\left[\frac{t}{s}\right]h_s(A_t)\right).$$

If B is a square of side $b \ge 8$, then there are positive constants α_d such that

$$h_{b^2}(B) \ge \begin{cases}\alpha_1, & d=1,\\ \alpha_2/\log b, & d=2,\\ \alpha_3/b^{d-2}, & d\ge 3.\end{cases}$$

by Lemma 5 of [4]. We can see that the nature of this object is similar to the one of s_N, rescaled.
We now let s depend on t by setting s to be the square of the side A_t, so $s = s_t := (8N/E[|\xi_t^N|]^{1/d})^2$. We may assume that $s_t \leq t/2$, else

$$f_N(t) = \frac{E[|\xi_t^N|]}{g_N(t)} < \frac{128^{d/2}N^d}{t^{d/2}N^d k_d(t)} = 128^{d/2}k'_d(t) \leq 128^{d/2}$$

because $t/2 < (8N/E[|\xi_t^N|]^{1/d})^2$ and where $k_d(t)$ is the temporal part of $g_N(t)$ and one can see easily that $0 < k'_d(t) := 1/t^{d/2}k_d(t) < 1$ for $t \geq 2$. With this choice we have

$$\left[\frac{t}{s_t}\right] \geq \frac{t}{2s_t} \geq \frac{tE[|\xi_t^N|]^{2/d}}{128N^2}$$

and

$$h_{b^2}(B) \geq \begin{cases} \alpha_1, & d = 1, \\ \alpha_2/\log\left(\frac{8N}{E[|\xi_t^n|]^{1/2}}\right), & d = 2, \\ \alpha_3/\left(\frac{8N}{E[|\xi_t^n|]^{1/2d}}\right)^{d-2}, & d \geq 3. \end{cases}$$

We now consider the cases $d = 1$, $d = 2$ and $d \geq 3$ separately.
$(d = 1)$: Utilizing what we have done till now

$$\begin{aligned} f_N(2t) &= \frac{\sqrt{2t}E[|\xi_{2t}^N|]}{N} \\ &\leq \frac{\sqrt{2t}E[|\xi_t^N|]}{N}\exp\left[-\frac{1}{2}\frac{tE[|\xi_t^N|]^2}{128N^2}\alpha_1\right] \\ &= f_N(t)\exp\left(\log\sqrt{2} - \frac{\alpha_1}{256}f_N^2(t)\right) \end{aligned}$$

then the result holds and $M_1 = \max\{\sqrt{128},\ \sqrt{128\log 2/\alpha_1}\}$.
$(d = 2)$: As the previous case we have

$$\begin{aligned} f_N(2t) &= \frac{2tE[|\xi_{2t}^N|]}{N^2\log 2t} \\ &\leq \frac{2tE[|\xi_t^N|]}{N^2\log 2t}\exp\left[-\frac{1}{2}\frac{tE[|\xi_t^N|]^2}{128N^2}\frac{\alpha_2}{\log(8N/(E[|\xi_t^N|]^{1/2}))}\right] \\ &= f_N(t)\left(\frac{\log t}{\log 2t}\right)^2\exp\left(\log 2 - \frac{\alpha_2}{256}f_N(t)\frac{\log t}{\log(8\sqrt{t}/\sqrt{\log t}) - \frac{1}{2}\log f_N(t)}\right) \\ &\leq f_N(t)\exp\left(\log 2 - \frac{\alpha_2}{256}f_N(t)\right) \end{aligned}$$

unless

$$\frac{\log t}{\log(8\sqrt{t}/\sqrt{\log t}) - \frac{1}{2}\log f_N(t)} \leq 1.$$

Since the denominator is positive $(f_N(t) \leq t/\log t$ this can happen only if

$$\log f_N(t) \leq 4\left(\log\frac{8\sqrt{t}}{\sqrt{\log t}} - \log t\right) \leq 4\log 8,$$

and putting all of the pieces together we have the inequality with $M_2 = \{128,\ 256\log 2/\alpha_2\}$.
$(d \geq 3)$: As above,

$$\begin{aligned} f_N(2t) &= \frac{2tE[|\xi_{2t}^N|]}{N^d} \\ &\leq \frac{2tE[|\xi_t^N|]}{N^d}\exp\left[-\frac{1}{2}\frac{tE[|\xi_t^N|]^{2/d}}{128N^2}\alpha_d\left(\frac{E[|\xi_t^N|]^{1/d}}{8N}\right)^{d-2}\right] \\ &= f_N(t)\exp\left(\log 2 - \frac{\alpha_d}{4\cdot 8^d}f_N(t)\right) \end{aligned}$$

and so the inequality for $d \geq 3$ holds with $M_d = \max\{128^{d/2},\ 8^d\log 2/\alpha_d\}$. □

Lemma 2.3.5. *There are finite constants c_d such that*

$$E[|\xi_t^N|] \leq c_d\max\{1,\ g_N(t)\} \tag{2.22}$$

for $t > 0$ and $N = 2,\ 4,\ldots$.

Proof. We first fix $t > 0$ and $N = 2,\ 4,\ldots$. Because of the previous lemma

$$f_N(2t) \leq \max\{M_d,\ f_N(t)\},\quad t \geq 0,\ N = 2,\ 4,\ldots.$$

where $f_N(t) = E[|\xi_t^N|]/g_N(t)$. If $f_N(t) \leq M_d$ then

$$E[|\xi_{2t}^N|] \leq M_d g_N(2t)$$

and we just need to rescale the time. Else

$$E[|\xi_{2t}^N|] \leq \frac{g_N(2t)}{g_N(t)}E[|\xi_t^N|] \leq E[|\xi_t^N|]$$

with $\frac{g_N(2t)}{g_N(t)} \leq 1$ for $t > 0$, $N = 2,\ 4$. Since the t is arbitrary we have, in this case, that $E[|\xi_t^N|]$ decreases with respect to time, as we could expect. Then $0 \leq E[|\xi_t^N|] \leq C$. We take the maximum of the two and we get the result. □

This theorem gives a scale of the maximal value of $E[|\xi_t^N|]$ with respect to time and to the increasing of the torus.

2.3.2 Proof of the main Theorem

Proof Theorem 2.3.1. We first prove the first convergence. Fix $t > 0$, $j \geq 1$ and a_N as in Theorem 2.1.1. Now fix $n \geq 2$ and select $A^N = \{x_1, \dots, x_n\} \subset \Lambda(N)$, $|x_\alpha - x_\beta| \geq a_N$ for $\alpha \neq \beta$. Then since $\xi_t^N(A^N) \subset \xi_t^N(\Lambda(N))$,

$$P(|\xi_{ts_N}^N(\Lambda(N))| \leq j) \leq P(|\xi_{ts_N}^N(A^N)| \leq j) \to \sum_{k=1}^{j} q_{n,\,k}(e^{-2t/G})$$

as $N \to \infty$ by Theorem 2.2.1. Letting $n \to \infty$ we obtain

$$\limsup_{N\to\infty} P(|\xi_{ts_N}^N(\Lambda(N))|) \leq \sum_{k=1}^{j} q_{\infty,\,k}(e^{-2t/G}).$$

For the reverse inequality fix M and $\delta_1,\ \delta_2 > 0$. By Markov's inequality and by lemma 2.3.5

$$P(|\xi_{\delta_1 s_N}^N(\Lambda(N))| \geq M) \leq \frac{E[|\xi_{\delta_1 s_N}^N|]}{M} \leq \frac{c_d}{\delta_1 M}.$$

As in the proof of the Theorem 2.2.1 and the usual construction with random independent random walks that, uniformly in $k \leq M$ and $\{y_1,\ \dots,\ y_k\} \subset \Lambda(N)$,

$$P(\xi_{\delta_1 s_N}^N(\Lambda(N)) = \{y_1, \dots, y_k\},\ \exists\, z_1 \neq z_2 \in \xi_{\delta_2 s_N}^N(\Lambda(N)),\ |z_1 - z_2| \leq a_N) = \varepsilon_N \to 0.$$

Combining these remarks with the Theorem 2.2.1 applied to $\xi_{(t-\delta_2)s_N}^N(\xi_{\delta_2 s_N}^N(\Lambda(N)))$ and letting

$$A_l(z_1, \dots, z_l) := \{\xi_{\delta_2 s_N}^N(\Lambda(N)) = \{z_1, \dots, z_l\},\ |z_\alpha - z_\beta| \geq a_N \text{ for } \alpha \neq \beta\},$$

we have

$$\begin{aligned}
P(|\xi_{ts_N}^N(\Lambda(N))| \leq j) &\geq \sum_{l=1}^{M} P(|\xi_{(r-\delta_2)s_N}^N(\{z_1, \dots, z_l\})| \leq j \mid A_(z_1, \dots, z_l)) \\
&\quad \times P(A_l(z_1, \dots, z_l) \mid |\xi_{\delta_1 s_N}^N(\Lambda(N))| \leq M) \times P(|\xi_{\delta_1 s_N}^N(\Lambda(N)) \leq M) \\
&\geq \left(1 - \frac{c_d}{\delta_1 M}\right) \sum_{l=1}^{M} \left(\sum_{k=1}^{j} q_{l,\,k}(e^{-2(t-\delta_2)/G}) + \varepsilon_N \right) \\
&\quad \times P(A_l(z_1, \dots, z_l) \mid |\xi_{\delta_1 s_N}^N(\Lambda(N))| \leq M) \\
&\geq \left(1 - \frac{c_d}{\delta_1 M}\right) \left(\sum_{k=1}^{j} q_{\infty,\,k}(e^{-2(t-\delta_2)/G}) \right) + \varepsilon_N.
\end{aligned}$$

If we first let $N \to \infty$, then $M \to \infty$ and then $\delta_1,\ \delta_2 \to 0$, we obtain

$$\liminf_{N\to\infty} P(|\xi^N_{ts_N}(\Lambda(N))| \le j) \ge \sum_{k=1}^{j} q_{\infty,\,k}(e^{-2t/G}).$$

Then

$$\lim_{N\to\infty} P(|\xi^N_{ts_N}(\Lambda(N))| = j) = q_{\infty,\,j}(e^{-2t/G}).$$

which is enough to prove the weak convergence.

The moment convergence follows from the weak convergence provided that the sequence σ^N_j/s_N is uniformly integrable, from Vitali's theorem. Since

$$P(\sigma^N_j \le 1) = P(|\xi^N_{ts_N}(\Lambda(N))| \le j) \to \sum_{k=1}^{j} q_{\infty,\,k}(e^{-2/G)}$$

as $N \to \infty$, there exists δ_j such that for all $N = 2,\ 4,\dots$ we have

$$P(\sigma^N_j/s_N \le 1) \ge \delta_j.$$

Now for any $A \subset \Lambda(N)$, since $\xi^N_t(A) \subset \xi^N_t(\Lambda(N))$, we must also have that for all $N = 2,\ 4,\dots,$

$$P(\sigma^N_j/s_N \le 1 \mid \xi^N_0 = A) \ge \delta_j.$$

Then, the Markov property and iteration lead to

$$P(\sigma^N_j/s_N \ge n) \le (1-\delta_j)^n,$$

and since this limit vanishes as $N \to \infty$, we have the uniform integrability. $\square$

Chapter 3

Voter Model

3.1 Duality

3.1.1 Voter model and duality

We imagine that at the initial moment every point on the torus is a person with either the opinion 0 or 1. As the time advances every person can check one of its neighbourhoods and align with its opinion, if it differs from its own. So the state space is $\{0, 1\}^{\Lambda(N)}$, where we indicate $\eta_t^N(x)$ is the state of the component at site x at time t on the torus $\Lambda(N)$. The transitions are

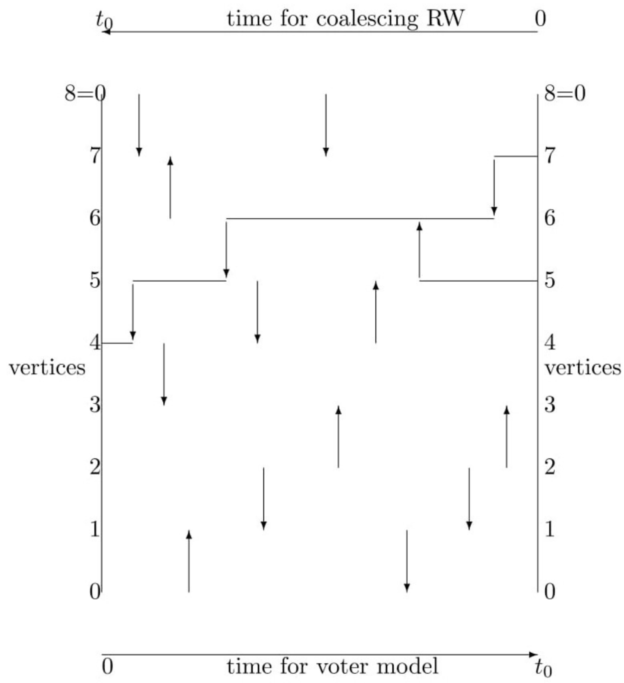
$$\eta_t^N \to 1 - \eta_t^N(x), \quad \text{at rate} \sum_{y \in \Lambda(N)} p^N(x, y) 1_{(\{\eta_t^N(x) \neq \eta_t^N(y)\})}.$$

In this case, each voter waits a random time which is exponential with parameter 1, then selects a neighbourhood voter according to p^N and adopts its opinion.
We write $A \subset \eta^N$ for $A \subset \Lambda(N)$ and $\eta^N \in \{0, 1\}^{\Lambda(N)}$ to mean $A \subset \{x : \eta^N(x) = 1\}$. The first duality equation we need is

$$P_{\eta^N}(B \subset \eta_t^N) = P(\xi_t^N(B) \subset \eta^N) \tag{3.1}$$

where P_{η^N} indicates that the voter model η_t^N starts with $\eta_0^N = \eta^N$.
We try to understand the duality of the two models just by 'picturing' it, and actually we don't need much more. The explanation is that the two processes can be seen by looking to the same situation in two different ways.
In the voter model, we interpret time as increasing left-to-right from 0 to t_0, and we see an arrow $j \to i$ as time t as meaning that person j adopts i's opinion at time t. In the coalescing random walk model, we interpret time as increasing right-to-left from 0 to t_0, and we see an arrow $j \to i$ at time t as meaning that the walk at state j at time t jumps to state i, and coalesces with the cluster at i, if any it's there.

For fixed t_0, we can regard both processes as constructed from the same Poissan process of arrows. For any vertices i, j, k the event for the voter model "The opinions of the people i and j at time t_0 are both the opinion initially held by k" is exactly the same as the event for the coalescing random walk process "The walks starting at i and at j have coalesced before time t_0 and their cluster is at vertex k at time t_0".



In the particular case of the previous picture, the horizontal lines indicate part of the trajectories. In terms of the coalescing random of the coalescing random walks, the particles starting at 5 and 7 coalesce, and the walk is at 4 at time t_0. We can see that exactly three of the initial opinions survive, i.e that the random walks coalesce into three walks.
In particular, the event for the voter model "By time t_0 everyone's opinion is the opinion initially held by person k" is exactly the same as the event for the coalescing random walk process "All particles have coalesced by time t_0, and the walk is at k at time t_0".

3.1.2 Consensus time convergence

As we start η_t^N in product measure with density θ, the previous identity leads to the second duality equation

$$P(B \subset \eta_t^N) = E\left[\theta^{|\xi_t^N(B)|}\right]. \tag{3.2}$$

Indeed

$$\begin{aligned} P(B \subset \eta_t^N) &= \sum_{\eta^N \subset \Lambda(N)} P_{\eta^N}(B \subset \eta_t^N) \cdot P(\eta^N) \\ &= \sum_{\eta^N \subset \Lambda(N)} P(\xi_t^N(B) \subset \eta^N) \cdot P(\eta^N) = E\left[\theta^{|\xi_t^N(B)|}\right]. \end{aligned}$$

We define also

$$\tau^N = \inf\{t \geq 0 \mid \eta_t^N \equiv 0 \text{ or } 1 \text{ on } \Lambda(N)\}, \tag{3.3}$$

which is clearly a stopping time on the torus.

Theorem 3.1.1. *There are random variables τ depending on the dimension d such that as $N \to \infty$,*

$$\tau^N/s_N \rightharpoonup \tau \text{ and } E[\tau^N/s_N] \to E[\tau].$$

If $d \geq 2$, then

$$P(\tau \leq s) = \sum_{k=1}^{\infty} [\theta^k + (1-\theta)^k] q_{\infty,k}(2s/G), \quad s \geq 0,$$

and $E[\tau] = G[\theta \log \theta + (1-\theta)\log(1-\theta) + 1]$.

Proof. Fix $t \geq 0$. Then

$$\begin{aligned} P(\tau^N \leq ts_N) &= P(\eta_{ts_N}^N \equiv 1 \text{ or } 0) \\ &= E\left[\theta^{|\xi_{ts_N}^N(\Lambda(N))|}\right] + E\left[(1-\theta)^{|\xi_{ts_N}^N(\Lambda(N)))|}\right] \\ &\to \sum_{k=1}^{\infty} [\theta^k + (1-\theta)^k] q_{\infty,k}(-2t/G) \end{aligned}$$

by Theorem (2.3.1). To obtain the moment convergence we note that

$$\begin{aligned} P(\tau^N > t) &= 1 - E\left[\theta^{|\xi_{ts_N}^N(\Lambda(N)))|}\right] + E\left[(1-\theta)^{|\xi_{ts_N}^N(\Lambda(N))|}\right] \\ &= \sum_{k=2}^{\infty} [\theta^k + (1-\theta)^k] P(|\xi_t^N(\Lambda(N))| = k) \\ &\leq P(|\eta_t^N(\Lambda(N))| \geq 2) = P(\sigma^N > t), \end{aligned}$$

which means that τ^N is stochastically smaller than σ_1^N and since σ_1^N/s_N is uniformly integrable, so is τ^N/s_N. This is enough to have the convergence of expectations. For the explicit calculation the computation is

$$\begin{aligned}
\int_0^\infty P(\tau > t)\,dt &= \sum_{k=2}^{\infty}[\theta^k + (1-\theta)^k]\int_0^\infty P_\infty(D_{2t/G} = k)\,dt \\
&= \sum_{k=2}^{\infty}[\theta^k + (1-\theta)^k]\frac{G}{2}E[\text{holding time in state } k] \\
&= \sum_{k=2}^{\infty}[\theta^k + (1-\theta)^k]\frac{G}{2}\binom{k}{2}^{-1} \\
&= G\sum_{k=2}^{\infty}\frac{[\theta^k + (1-\theta)^k]}{k(k-1)} \\
&= G\sum_{k=2}^{\infty}\left(\theta\frac{\theta^{k-1}}{k-1} + (1-\theta)\frac{(1-\theta)^{k-1}}{k-1} - \frac{\theta^k}{k} - \frac{(1-\theta)^k}{k}\right) \\
&= G\left(-\theta\log(1-\theta) - (1-\theta)\log\theta - \sum_{k=2}^{\infty}\left(\frac{\theta^k}{k} + \frac{(1-\theta)^k}{k}\right)\right) \\
&= G\left(-\theta\log(1-\theta) - (1-\theta)\log\theta - \sum_{k=1}^{\infty}\left(\frac{\theta^k}{k} + \frac{(1-\theta)^k}{k}\right) + \theta + (1-\theta)\right) \\
&= G(-\theta\log(1-\theta) - (1-\theta)\log\theta + \log\theta + \log(1-\theta) + 1) \\
&= G(\theta\log\theta + (1-\theta)\log(1-\theta) + 1).
\end{aligned}$$

using the following Taylor expansion around 0

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

for $x \in (0,1)$. □

3.2 Density process

3.2.1 Coalescing Random Walk with different starting time

Before going ahead we need to establish another coalescing random walk result, where we consider random walks that start moving at different times. For $t_1 < \cdots < t_k$ and $A_i \subset \Lambda(N)$ let $\eta_t^N(A_1,\ t_1;\ldots;\ A_k,\ t_k)$ denote a coalescing random walk system in which random walks start each point of A_i at time t_i (they are frozen until this time) and then execute the coalescing random walk motion. For $t > t_k$ and positive integers m, n_i define

$$q_{n_1,\,m}(t_1,\,t) = q_{n_1,\,m}(t-t_1),$$

$$\begin{aligned} q_{n_1,\dots,\,n_k,\,m}(t_1,\dots,\,t_k,\,t) = &\sum_{l_1}\cdots\sum_{l_{k-1}} q_{n_1,\,l_1}(t_2-t_1)q_{n_2+l_1,\,l_2}(t_3-t_2) \\ &\times\cdots\times q_{n_{k-1}+l_{k-2},\,l_{k-1}}(t_k-t_{k-1})q_{n_k+l_{l-1},\,m}(t-t_k). \end{aligned}$$

Hence

$$\begin{aligned} &q_{n_1,\dots,\,n_k,\,m}(t_1,\dots,\,t_k,\,t) \\ &\quad = \sum_l q_{n_1,\dots,\,n_{k-1},\,l}(t_1,\dots,\,t_{k-1},\,t_k)q_{n_k+l,\,m}(t-t_k) \\ &\quad = \sum_l q_{n_1,\,l}(t_2-t_1)q_{l+n_2,\,n_3,\dots,\,n_k,\,m}(t_2,\dots,\,t_k,\,t). \end{aligned}$$

Theorem 3.2.1. *Assume $d \geq 2$, fix $T > 0$, $k > 0$, $n_1,\dots,\,n_k$, let $a_N \to \infty$, $a_n = o(N)$ as $N \to \infty$. Then for fixed $0 \leq t_1 < \cdots < t_k < t$, uniformly for $A_i = \{x_i^j\,,\ j = 1,\dots,\,n_i\} \subset \Lambda(N)$ such that $|x_i^\alpha - x_i^\beta| \geq a_N$ for all i and $\alpha \neq \beta$,*

$$P(|\xi^N_{ts_N}(A_1,\,t_1s_N,\dots,\,A_k,\,t_ks_N)| = m) \to q_{n_1,\dots,\,n_k,\,m}(2t_1/G,\dots,\,2t_k/G,\,2t/G).$$

Proof. The $k = 1$ result follows from Theorem 2.2.1, so we assume now that $k \geq 2$ and proceed by induction. We will run the system until time t_2s_N and look at $\eta^N_{ts_N}(A_1,\,t_1s_N) = \{y_1,\dots,\,y_l\}$. By constructing independent random walks we can see

$$P(\exists\, y_\alpha,\,y_\beta \in \xi^N_{t_1s_N}(A_1,\,t_1s_N),\ y_\alpha \neq y_\beta \text{ and } |y_\alpha - y_\beta| \leq a_N) \leq Ca_N^d/N^d$$

$$P(\exists\, y_\alpha \in \xi^N_{t_1s_N}(A_1,\,t_1s_N),\ x_\beta \in A_2 \text{ with } |y_\alpha - x_\beta| \leq a_N) \leq Ca_N^d/N^d$$

as in the proof of Theorem 2.2.1. Thus

$$\begin{aligned} P(|\xi^N_{ts_N}| = m) &= \varepsilon_N + \sum_l \sum_{y_1,\dots,\,y_l} P(\xi^N_{ts_N}(A_1,\,ts_N) = \{y_1,\dots,\,y_l\}) \\ &\times P(|\xi^N_{ts_N-t_2s_N}(A_2\cup\{y_1,\dots,\,y_l\},\,0,\,A_3,\,t_3s_N-t_2s_N,\dots,\,A_k,\,t_ks_N-t_2s_N)| = m) \\ &\to \sum_l q_{n_1,\,l}(2(t_2-t_1)/G)\times q_{n_2+l,\,n_3,\dots,\,n_k,\,m}(0,\,2(t_3-t_2)/G,\dots,\,2(t_k-t_2)/G,\,(t-t_2)/G) \\ &= q_{n_1,\,,n_2,\dots,\,n_k,\,m}(2t_1/G,\dots,\,2t_k/G,\,2t/G). \end{aligned}$$

where the sum on the y_i is over $|y_\alpha - y_\beta| \geq a_N$ and $|y_\alpha - x_\beta| \geq a_N$, with $x_\beta \in A_2$. We have the result by using Theorem 2.2.1, the induction hypothesis and the previous observations. □

By using this theorem and the duality equation (3.1), we get

$$P_\eta(B_i \subset \eta_{t_i},\ 1 \le i \le k) = P(\xi_{t_k}(B_k,\ 0,\ B_{k-1},\ t_k - t_{k-1}, \dots,\ B_1,\ t_k - t_1) \subset \eta)$$

for $0 \le t_1 \le \cdots \le t_k$. In particular, using the equation (3.2), we see

$$P_\eta(B_i \subset \eta_{t_i},\ 1 \le i \le k) = E\left[\theta^{|\xi_{t_k}(B_k,\ 0,\ B_{k-1},\ t_k - t_{k-1}, \dots,\ B_1,\ t_k - t_1)|}\right]. \tag{3.4}$$

3.2.2 The Wright-Fisher diffusion process

Now we introduce the Wright-Fisher diffusion Y_t. We know study the discrete case that will bring us to the continuous one, which will appear in a further result. The Wright-Fisher model assumes that the total population remains at a constant level N and focuses on the changes in the relative proportions of the different types.
Fluctuations of the total population, provided that they do not become too small, result in time-varying rates in the Wright-Fisher model but do not change the main qualitative features of the conclusions.
The classical model is a discrete time model of a population with constant size N and types $E = \{1,\ 2\}$. Let Z_n be the number of type 1 individuals at time n. Then Z_n is a Markov chain with state space $\{0, \dots,\ N\}$ and transition probabilities:

$$P(Z_{n+1} = j \mid Z_n = i) = \binom{N}{j}\left(\frac{i}{N}\right)^j\left(1 - \frac{i}{N}\right)^{N-j}, \quad j = 0, \dots, N.$$

We interpret that at generation $n+1$ this involves binomial sampling with probability $p = \frac{X_n}{N}$, that is, the current empirical probability of type 1. It's easy to find the moment generating function, the mean and the variation

$$M(\theta) = \left(pe^\theta\right)^N, \quad E(X) = Np, \quad Var(X) = Np(1-p).$$

Now we will the passage from the discrete time to the continuous time process.

Theorem 3.2.2. *Assume that $N^{-1}X_0^N \to p_0$ as $N \to \infty$.Then*

$$(p_N(t))_{t \ge 0} \equiv (N^{-1}X^N_{[Nt]})_{t \ge 0} \to (p(t))_{t \ge 0}$$

where $(p(t))_t$ is a Markov diffusion process with state space $[0,\ 1]$ and with generator

$$Gf(p) = \frac{1}{2}p(1-p)\frac{d^2}{dp^2}f(p) \tag{3.5}$$

if $f \in C^2([0,\ 1])$. This is equivalent to the pathwise unique solution of the SDE

$$dp(t) = \sqrt{p(t)(1-p(t))}dB(t), \quad p(0) = p_0.$$

Proof. See Theorem 5.2 on [8]. □

We define the Wright-Fisher diffusion process Y_t as the one generated by the equation (3.5) in the previous Theorem.
There are two very important equations which connect the Wright-Fisher diffusion Y_t and the death process D_t. They are

$$E_\theta[Y_t^m] = E_m[\theta^{D_t}] = \sum_{j=1}^{m} \theta^j q_{m,\,j}(t) \tag{3.6}$$

and

$$E_\theta\left[\prod_{i=1}^{k} Y_{t_i}^{m_i}\right] = \sum_{j\geq 1} \theta^j q_{n_k,\dots,\,n_1,\,j}(0,\, t_k - t_{k-1}, \dots,\, t_k - t_1,\, t_k). \tag{3.7}$$

The first equation can be studied in [15] at chapter 5.
The equation (3.7) is an iteration of (3.6). Indeed, for $k \geq 2$, by using the Markov property of our process

$$\begin{aligned}
E_\theta\left[\prod_{i=1}^{k} Y_{t_i}^{m_i}\right] &= E_\theta\left[E_{Y_{t_{k-1}}}[Y_{t_k}^{m_k}] Y_{t_1}^{m_1} \cdots Y_{t_{k-1}}^{m_{k-1}}\right] \\
&= \sum_{j\geq 1} q_{n_k,\, n_{k-1},\, j}(0,\, t_k - t_{k-1},\, t_k) E_\theta\left[Y_{t_1}^{m_1} \cdots Y_{t_{k-1}}^{m_{k-1}+j}\right] \\
&\;\vdots \\
&= \sum_{j\geq 1} \theta^j q_{n_k,\dots,\,n_1,\,j}(0,\, t_k - t_{k-1}, \dots,\, t_k - t_1,\, t_k)
\end{aligned}$$

where $0 \leq t_1 \leq \cdots \leq t_k$.

3.2.3 The density process

We want to study the density process, that is

$$\Delta_t^N = \frac{1}{N^d} \sum_{x \in \Lambda(N)} \eta_t^N(x).$$

It represents the 'percentage' of voters with still opinion 1 on the torus $\Lambda(N)$ at the time t. The following result shows that the particle density on $\Lambda(N)$ fluctuates like the Wright-Fisher diffusion with the time scale s_N.

Theorem 3.2.3. *If $d \geq 2$, then as $N \to \infty$,*

$$\Delta_{ts_N}^N \rightharpoonup Y_{2t/G} \tag{3.8}$$

as processes.

As we may see $\Delta^N_{ts_N}$ a sequence of random variables with values in $C([0,\,\infty))$ with the relative distributions, to converge weakly as processes means that the sequence of laws converges to the law of a random variable.
As consequence of Prohorov's Theorem, it is equivalent to have the weak convergence of finite-dimensional distributions, and tightness. In this particular case we will study the first property and, by noticing that the density process is actually a martingale, conclude with the help of a Preposition in [1].

Proof. We first prove that, as we fix $t \geq 0$ and $m \geq 1$, we have

$$\Delta^N_{ts_N} \rightharpoonup Y_{2t/G}$$

by showing that

$$E\left[\left(\Delta^N_{ts_N}\right)^m\right] \to E_\theta[Y^m_{2t/G}]$$

as $N \to \infty$.
We define a_N as in Theorem 2.1.1 and

$$\begin{aligned}
E\left[\left(\Delta^N_{ts_N}\right)^m\right] &= N^{-md} \sum_{x_1,\dots,\,x_m \in \Lambda(N)} P(\eta^N_{ts_N}(x_i) = 1,\ 1 \leq i \leq m) \\
&= N^{-md} \sum_{\substack{x_1,\dots,\,x_m \in \Lambda(N) \\ |x_\alpha - x_\beta| \geq a_N,\ \alpha \neq \beta}} P(\eta^N_{ts_N}(x_i) = 1,\ 1 \leq i \leq m) + \varepsilon_N \\
&= N^{-md} \sum_{\substack{x_1,\dots,\,x_m \in \Lambda(N) \\ |x_\alpha - x_\beta| \geq a_N,\ \alpha \neq \beta}} E\left[\theta^{|\xi^N_{ts_N}(\{x_1,\dots,\,x_m\})|}\right] + \varepsilon_N \\
&= N^{-md} \sum_{\substack{x_1,\dots,\,x_m \in \Lambda(N) \\ |x_\alpha - x_\beta| \geq a_N,\ \alpha \neq \beta}} \sum_{j=1}^m \theta^j q_{m,\,j}(e^{-2t/G}) + \varepsilon_N \\
&\to \sum_{j=1}^m \theta^j q_{m,\,j}(e^{-2t/G}) = E_\theta[Y^m_{2t/G}]
\end{aligned}$$

by using (3.6).
Now we can prove the weak convergence of finite-dimensional distributions. Fix $k \geq 2$, $m_i \geq 1$ and $0 \leq t_1 < \cdots < t_k$. We need to see

$$E\left[((\Delta^N_{t_1 s_N})^{m_1} \cdots (\Delta^N_{t_k s_N})^{m_k}\right] \to E\left[Y^{m_1}_{2t_1/G} \cdots Y^{m_k}_{2t_k/G}\right]$$

which is enough to prove

$$\left(\Delta^N_{t_1 s_N}, \dots,\, \Delta^N_{t_k s_N}\right) \rightharpoonup \left(Y_{2t_1/G}, \dots,\, Y_{2t_k/G}\right).$$

Using the previous computation

$$
\begin{aligned}
&E\left[\left(\Delta^N_{t_1 s_N}\right)^{m_1}\cdots\left(\Delta^N_{t_k s_N}\right)^{m_k}\right]\\
&= N^{-d(m_1+\cdots+m_k)} \sum_{\substack{x^1,\dots,x^k\\ x^i=(x^i_1,\dots,x^i_m),\, x^i_j\in\Lambda(N)}} P(\eta^N_{t_i s_N}(x^i_j)=1,\ 1\le i\le k,\ 1\le j\le m_i)\\
&= \Sigma_1+\Sigma_2
\end{aligned}
$$

where Σ_1 contains all the terms $x^1,\dots,\ x^k$ such that $|x^i_\alpha - x^i_\beta| \geq a_N$, $\alpha\neq\beta$, and Σ_2 contains all the other terms. As in the proof of Theorem 2.2.1, $\Sigma_2=\varepsilon_N\to 0$ as $N\to\infty$ and by the equation (3.4), with $B^i=\{x^i_j,\ 1\le j\le m_i\}$, a term of Σ_1 is

$$
\begin{aligned}
&P(\eta^N_{t_i s_N}(x^i_j)=1,\ 1\le i\le k,\ 1\le j\le m_i)\\
&\quad = E\left[\theta^{|\xi^N_{t_k s_N}(B_k,\,0,\,B_{k-1},\,t_k-t_{k-1},\dots,\,B_1,\,t_k-t_1)|}\right]\\
&\quad \to \sum_m \theta^m q_{n_k,\dots,\,n_1,\,m}(0,\ 2(t_k-t_{k-1})/G,\dots,\ 2(t_k-t_1)/G,\ t_k)\\
&\quad = E\left[Y^{m_1}_{2t_1/G}\cdots Y^{m_k}_{2t_k/G}\right]
\end{aligned}
$$

by (3.7).

Proposition 1.2 of [1] states that if a process is a martingale, which converges weakly for every finite-dimensional distribution, the limit is continuous and for each $t\geq 0$ the process is uniformly integrable, then it converges weakly as process. We notice that $M^N(t):=\Delta^N_{ts_N}$ is clearly bounded and measurable, so uniformly integrable. Moreover Y_t is continuous in t and we have already proved that the finite-dimensional distributions converge weakly. We then just need to prove that $M^N(t)$ is a martingale. For a fixed N and for $\eta^N_t=\eta$, we have that

$$
r(\eta)=\sum_{x,\,y\in\Lambda(N)} \eta(x)(1-\eta(y))p^N(x,\,y)
$$

then $|\eta^N_t|\to|\eta^N_t|+1$ at rate $r(\eta)$ and $|\eta^N_t|\to|\eta^N_t|-1$ also at rate $r(\eta)$. We study the easier case whit discrete steps. As the rate of having a voter moving from opinion 1 to 0 and viceversa is the same, we get that

$$
E\left[\Delta^N_{(n+1)s_N}-\Delta^N_{ns_N}\mid \Delta^N_{ns_N}\right]=\frac{1}{N^d}r(\eta)(1-1)=0,
$$

and we conclude. □

3.2.4 Approximation of the Law of the Voter Model

Theorem 3.2.4. *Assume* $d \geq 3$*,* $A \subset Z^d$ *is finite and* ζ *is fixed. If* $t_N \to \infty$ *and* $t_N/N^d \to t \in [0, \infty]$ *as* $N \to \infty$*, then*

$$P(\eta^N_{t_N}(x) = \zeta(x),\, x \in A) \to \int_{[0,1]} P(Y_{2t/G} \in d\theta')\nu_{\theta'}(\eta(x) = \zeta(x),\, x \in A).$$

Proof. We are going to show first that

$$P(\eta^N_{t_N}(x) = 1,\, x \in A) \to \int_{[0,1]} P(Y_{2t/G} \in d\theta')\nu_{\theta'}(\eta(x) = 1,\, x \in A), \tag{3.9}$$

and we assume firstly that $t_N/N^d \to t \in (0, +\infty)$. To prove (3.9) ....
Let $\eta_\infty(A) := \lim_{t\to\infty} |\eta_t(A)|$ and let $p_n(A) = P(\eta_\infty(A) = n)$. By Theorem 1.3.1 and the duality equation (3.1),

$$\begin{aligned}
\nu_\theta(\eta(x) = 1,\, x \in A) &= \lim_{t\to\infty} P(\eta_t(A) = 1) \\
&= \lim_{t\to\infty} E\left[\theta^{|\xi_t(A)|}\right] \\
&= \sum_{n=1}^{|A|} p_n(A)\theta^n \\
&= \sum_{n=1}^{|A|} p_n(A)E_n\left[\theta^{D(2t/G)}\right].
\end{aligned}$$

by using the duality equation (3.6) on the last passage. This and an application of duality to the left-hand side of (3.9) show that it suffices to prove

$$E\left[\theta^{|\xi^N_{t_N}(A)|}\right] \to \sum_{n=1}^{|A|} p_n(A)E_n\left[\theta^{D(2t/G)}\right].$$

We introduce a collection of independent random walks on Z^d, $\{X_t(x),\, x \in Z^d\}$, where $X_0(x) = x$. If $T_N \to \infty$, $T_N = o(N^d)$ and $a_N \to \infty$, $a_N = o(N)$ as $N \to \infty$, we have for all $x \in A$, $\varepsilon > 0$

$$P(|X_t(x)| \leq N^{1-\varepsilon},\, 0 \leq t \leq T_N) \to 1,$$

which it's equal to show

$$P(\sup_{t\in[0,\, T_N]} |X_t(x)| > N^{1-\varepsilon}) \to 0.$$

By squaring and, thanks to it, using the Doob's inequality on the first term, we get

$$P(\sup_{t\in[0,\,T_N]}|X_t(x)|>N^{1-\varepsilon})\le\frac{E[X_t^2(x)]}{N^{2-2\varepsilon}}.$$

But $E[X_{T_N}^2(x)] = dE[(X_{T_N}^1(x))^2] = d\mathrm{Var}(X_{T_N}^1(x)) = T_N$, where we consider $X_{T_N}^1$ as the first component of the random walk X_t and since the symmetric random walk is actually a martingale.
We now take a T_N smaller than $N^{2-2\varepsilon}$, like $N^{2-3\varepsilon}$, and then

$$P(\sup_{t\in[0,\,T_N]}|X_t(x)|>N^{1-\varepsilon})\le\frac{E[X_t^2(x)]}{N^{2-2\varepsilon}}\le\frac{N^{2-3\varepsilon}}{N^{2-2\varepsilon}}\to 0$$

as $N\to\infty$. We also need to see

$$P(|X_{T_N}(x)|\le a_N)\to 0.$$

We use a Gaussian approximation

$$P(|X_{T_N}|\le a_N)\sim P(|N(0,1)|\le\frac{a_N}{\sqrt{T_N}}\le\varepsilon,$$

where we have to use the same T_N as before but we can choose a a_N that suits our needs, like $a_N = N^{1-2\varepsilon}$.
Now we define random walks on $\Lambda(N)$ as $X_t^N(x) = (X_t(x)\ \mathrm{mod} N) - N/2$. Using the $X_t(x)$ and $X_t^N(x)$ it's clear that we can construct the processes $\xi_t(A)$ and $\xi_t^N(A)$ such that $\xi_t(A) = \xi_t^N(A)$ for all $t\le\gamma_N$, where $\gamma_N := \inf\{t\ge 0\ :\ |\xi_t(x)|\ge N^{1-\varepsilon}$ for some $x\in A\}$.
Let $B := \{x_1,\ldots,\,x_n\}\subset A$, we have

$$\begin{aligned}
E\left[\theta^{|\xi_{t_N}^N(A)|}\right] &= \sum_{n=1}^{|A|}\sum_{|B|=n}P(\xi_{T_N}^N(A)=B)E\left[\theta^{|\xi_{t_N-T_N}^N(B)|}\right]\\
&= \sum_{n=1}^{|A|}\sum_{|B|=n}P(\xi_{T_N}(A)=B)E\left[\theta^{|\xi_{t_N-T_N}^N(B)|}\right]+\varepsilon_N\\
&= \sum_{n=1}^{|A|}\sum_{\substack{|B|=n,\,x_j,\,x_i\in B,\\ |x_i-x_j|\ge a_N\text{ for }i\ne j}}P(\xi_{T_N}(A)=B)E\left[\theta^{|\xi_{t_N-T_N}^N(B)|}\right]+\varepsilon_N\\
&= \sum_{n=1}^{|A|}P(\xi_{T_N}(A)=n)E_n\left[\theta^{D(2t_N/GN^d-2T_N/GN^d)}\right]+\varepsilon_N\\
&\to \sum_{n=1}^{|A|}p_N(A)E_n\left[\theta^{D(2t/G)}\right]
\end{aligned}$$

as required. □

3.3 Multiple voter model

In the view of modelizing an election situation, considering just two opinions could be seen as restricting. We consider $d \geq 2$. Given $\Lambda \subset Z^d$ and $\kappa < \infty$, we define the κ-type voter model η_t on Λ with transition matrix p^Λ with the state space $\{0,\ 1, \dots,\ \kappa - 1\}^\Lambda$ and transitions

$$\eta_t^\Lambda(x) \to i \text{ at rate } \sum_{y \in \Lambda} p^\Lambda(x,\ y) 1_{(\{\eta_t^\Lambda(y) = i\})}$$

for $i \neq \eta_t^\Lambda(x)$. As before, for $\theta = (\theta_1, \dots,\ \theta_{\kappa-1})$ let μ_θ denote product measure on $\{0,\ 1, \dots,\ \kappa - 1\}^\Lambda$, $\mu_\theta(\eta(x) = i) = \theta_i$.
As we go through with the proves and definitions of Theorems 2.3.1 and 3.2.3, we can notice they don't relay heavily on the algebraic properties of our first voter model's opinion, so 0 and 1. Then, we could expect that the extension to more opinions should give similar results. And indeed it is so.
Let τ_j^N be the time it takes the process to reach exactly j types, i.e.

$$\tau_j^N := \inf\{t \geq 0 \ : \ \exists A \subset \{0,\ 1, \dots,\ \kappa - 1\},\ |A| = j,\ \eta_t^N(x) \in A \text{ for all } x \in \Lambda(N)\}$$

and let Δ_t^N be the κ-vector $(\Delta_t^N(0), \dots,\ \Delta_t^N(\kappa - 1))$, where, as before,

$$\Delta_t^N(i) := \frac{1}{N^d} \sum_{x \in \Lambda(N)} 1_{(\{\eta_t^N(x) = i\})}$$

Let Y_t be the κ-type Wright-Fisher diffusion, which has generator

$$Gf(p) = \frac{1}{2} \sum_{i,\ j=0}^{\kappa-1} p_i(\delta_{ij} - p_j) \frac{\partial^2}{\partial p_i \partial p_j} f(p)$$

and lives on the state space $\{p = (p_0, \dots,\ p_{\kappa-1}) \ : \ p_i \geq 0,\ \sum_i p_i = 1\}$. The generator is constructed in a way such that, if $\kappa = 2$, we have the previous case. We remember how we used the discrete time case to construct the Theorem 3.2.2 with the state space of just two elements. Here, similarly, we start with a state space of κ elements and we define the neutral κ-allele Wright-Fisher model, which is a Markov chain Z_n with state space $E = \{e_1, \dots,\ e_\kappa\}$ and transition probabilities

$$P(Z_{n+1} = (\beta_1, \dots,\ \beta_\kappa) \mid Z_n = (\alpha_1, \dots, \alpha_\kappa)) = \frac{N!}{\beta_1! \cdots \beta_\kappa!} \left(\frac{\alpha_1}{N}\right)^{\beta_1} \cdots \left(\frac{\alpha_\kappa}{N}\right)^{\beta_\kappa}.$$

By this model we get the generator previously defined as in Theorem 3.2.2.
If η_t^N has initial distribution μ_θ, then there are random variables τ_j such that

$$\frac{\tau_j}{s_N} \rightharpoonup \tau_j, \quad E\left[\tau_j^N / s_N\right] \to E[\tau_j]$$

and

$$\Delta^N_{ts_N} \rightharpoonup Y_{2t/G}, \quad Y_0 = (\theta_0, \dots, \theta_{\kappa-1})$$

as processes.
The proof of these results can be find on [6], it relies on the fact that one needs to show that for any $A \subset \{0, \dots, \kappa-1\}$

$$P(\forall x \in \Lambda(N), \ \eta^N_{ts_N} \notin A) \to \sum_{j=1}^{\infty} \left[\sum_{i \notin A} \theta_i \right]^j q_{\infty,\, j}(2s/G),$$

and then, by using the inclusion-exclusion, get an explicit representation for the distribution of τ_j. One also needs to extend the duality equation to the κ-allele Wright-Fisher process situation.
The number of opinions could be extended to infinity and, using some precautions, the results would be similar. If one is interested, check [6].

[7][5][14][9] [10]

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