Università degli Studi di Padova

Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”

Dipartimento di Matematica Pura e Applicata,
Università degli Studi di Milano-Bicocca

Corso di Laurea Triennale in Matematica

A formula for computing the exact determining number of Kneser graphs

Relatore:
Prof. Andrea Lucchini

Laureando: Giovanni Mecenero
Matricola: 2020604

Correlatore:
Prof. Pablo Spiga

Anno Accademico 2022/2023

22 Settembre 2023

Contents

Introduction

Your friend wants to guess the graph automorphism you are looking at, which means guessing where each vertex is mapped to. You want to help them by giving some hints, that is telling the image of some vertices. What is the minimum amount of hints you have to give in order for them to be sure to guess correctly? Of course, this depends on the graph we are considering, but your friend knows it, as it is your favourite graph. Can it depend on the automorphism? Are there lucky automorphisms that take fewer hints than others to be guessed? The answer to these last questions is no, in fact, if hinting the image of a set of vertices uniquely determines the automorphism σ_1, then, applying $\sigma_2\sigma_1^{-1}$ to the hints, we get a set of hints that uniquely determines the automorphism σ_2, so σ_2 cannot take more hints than σ_1. This also shows that if hinting the image of a set of vertices allows your friend to guess an automorphism, then the same set of vertices can be used to hint any automorphism. We can conclude that our first question is the same question as "What is the minimal size of a set of vertices of the graph such that if you hint the image of those vertices your friend can always guess the automorphism?".

In this thesis, we will tackle this question, after its formal definition, in the particular case of Kneser graphs. In this case, it turns out that the number we are looking for is the same as the base size $b(n,k)$ of the action of the symmetric group $\mathrm{Sym}(n)$ on the k-subsets of an n-set, at least when $n > 2k$.

The quantity $b(n,k)$ is $\lceil \log_2 n \rceil$ for $n = 2k$, it becomes $\left\lceil \frac{2(n-1)}{k+1} \right\rceil$ when n is sufficiently large, and in general it is the minimum ℓ such that

$$\sum_{\substack{\pi \text{ partition of } n \\ \pi=(1^{c_1},2^{c_2},\dots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} \left(\sum_{\substack{\eta \text{ partition of } k \\ \eta=(1^{b_1},2^{b_2},\dots,k^{b_k})}} \prod_{j=1}^k \binom{c_j}{b_j} \right)^{\ell}$$

is not zero, of course. We just need to prove this last formula now, which is the main goal of this thesis.

Chapter 1

Preliminaries

Given a graph Γ, we let $\mathrm{Aut}(\Gamma)$ be its automorphism group.

Definition 1.1. A set of vertices S is a ***determining set*** of a graph Γ if every automorphism of Γ is uniquely determined by its action on S. This means that, for any two automorphisms $\sigma_1, \sigma_2 \in \mathrm{Aut}(\Gamma)$, if for each $s \in S$ we have $\sigma_1(s) = \sigma_2(s)$, then $\sigma_1 = \sigma_2$.

Definition 1.2. The ***determining number*** of Γ is the minimum cardinality of a determining set of Γ.

Note that every graph has a determining set, and thus a determining number, because the set of all vertices is a determining set. It is also clear that any set containing all but one vertex is a determining set.

For any subset S of vertices of Γ

$$\mathrm{Stab}(S) = \{g \in \mathrm{Aut}(\Gamma) \mid g(v) = v, \forall v \in S\}.$$

Note that $\mathrm{Stab}(S) = \bigcap_{v \in S} \mathrm{Stab}(v)$. This is the pointwise stabilizer of S.

Proposition 1.1. *Let S be a subset of the vertices of the graph Γ. Then S is a determining set for Γ if and only if $\mathrm{Stab}(S) = \{1\}$.*

Proof. If S is a determining set, then whenever $\sigma \in \mathrm{Aut}(\Gamma)$ fixes each $s \in S$, $g = 1$. Thus $\mathrm{Stab}(S) = \{1\}$. Conversely if $\mathrm{Stab}(S) = \{1\}$ and $g, h \in \mathrm{Aut}(\Gamma)$ so that $g(s) = h(s)$ for all $s \in S$ then $h^{-1}g(s) = s$ for all $s \in S \Longrightarrow h^{-1}g = 1 \Longrightarrow g = h$. Therefore, S is a determining set. □

We study the determining number of Kneser graphs.

Definition 1.3. A ***Kneser graph*** $K_{n:k}$ is the graph having vertex set the collection of all k-subsets of $\{1, \ldots, n\}$ where two distinct k-subsets are declared to be adjacent if and only if they are disjoint.

If $n < 2k$, then $K_{n:k}$ is the empty set on $\binom{n}{k}$ vertices, and its determining number is $\binom{n}{k}-1$ since every subset containing all but one vertex is determining, conversely given a subset S of the vertices not containing two vertices, the automorphism that only swaps them would be in $\mathrm{Stab}(S)$, so S is not a determining set.

If $n = 2k$, then $K_{n:k}$ is the disjoint union of $\binom{n}{k}/2$ edges, as every k-subset is only disjoint from its complementary. A determining set S of $K_{n:k}$ must contain at least one vertex from every edge, otherwise the automorphism swapping an edge disjoint from S would fix S. On the other hand, a set containing exactly one vertex per edge is determining, since an automorphism maps edges into edges, thus the determining number for $K_{2k:k}$ is $\binom{2k}{k}/2$.

From now on we only study the case $n > 2k$.
We will prove in Chapter 2 that, in this case, the automorphism group of $K_{n:k}$ is isomorphic to the symmetric group $\mathrm{Sym}(n)$. This will come as a corollary to the Erdős-Ko-Rado theorem.

We now define the base size for the action of a group on a set. In particular we will focus on the action of $\mathrm{Sym}(n)$ on the k-subsets of $\{1, \dots, n\}$.

Definition 1.4. Let G be a permutation group on Ω. For $\Lambda = \{\omega_1, \dots, \omega_k\} \subseteq \Omega$, we write $G_{(\Lambda)}$ for the *pointwise stabilizer* of Λ in G. If $G_{(\Lambda)} = \{1\}$, then we say that Λ is a ***base***.

Definition 1.5. The size of a smallest possible base is known as the ***base size*** of the action of G on Ω.

It is customary to denote the base size by $b(G)$ or (more precisely) by $b_\Omega(G)$.

Let us now focus on the natural action of the symmetric group $\mathrm{Sym}(n)$ on the collection of all the k-subsets of $\{1, \dots, n\}$, that is

$$g \cdot \{a_1, \dots, a_k\} = \{g(a_1), \dots, g(a_k)\}$$

for every $g \in \mathrm{Sym}(n)$ and every $\{a_1, \dots, a_k\}$ k-subset of $\{1, \dots, n\}$. We denote the base size of this action with $b(n,k)$. Note that fixing a k-subset is the same as fixing its complementary. This immediately gives $b(n,k) = b(n, n-k)$.

The automorphism group of $K_{n:k}$ is the symmetric group $\mathrm{Sym}(n)$ in its action on k-subsets, as we will show in Corollary 2.2.1, hence, using Proposition 1.1, we deduce that the determining number of $K_{n:k}$ is the base size $b(n,k)$.

In what follows we will study the determining number of $K_{n:k}$ through the study of $b(n,k)$.

There are many partial results on $b(n,k)$, see [1, 2, 3, 5]. For instance, Halasi [5, Theorem 3.2] has proved that

$$b(n,k) = \left\lceil \frac{2(n-1)}{k+1} \right\rceil, \tag{1.1}$$

when $n \geq \lfloor k(k+1)/2 \rfloor + 1$. Strictly speaking, this formula for $b(n,k)$ is proved in [5] when $n \geq k^2$ and has been improved in [2] to $n \geq \lfloor k(k+1)/2 \rfloor + 1$. This result has been improved further in [1, 2, 5], but currently there is no explicit formula for $b(n,k)$, valid for every value of n and k.

Using the principle of inclusion-exclusion, we prove in Chapter 3 an implicit formula for $b(n,k)$ in terms of integer partitions of n, see (3.13).

A result similar to ours was very recently determined independently by Coen del Valle and Colva Roney-Dougal [7], their proof is remarkably different from ours.

Chapter 2

The Erdős-Ko-Rado theorem

Theorem 2.1 (Erdős-Ko-Rado)**.** *Suppose that $\mathcal{A}$ is a family of distinct k-subsets of an n-set X with $n > 2k$ and that each two subsets in $\mathcal{A}$ share at least one element. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

and the equality holds if and only if $\mathcal{A}$ consists of all the k-subsets containing a particular element.

The idea of the following proof is by Katona [6], we will present it as it was taught during a course on Algebraic Combinatorics held by Professor Pablo Spiga at the Galilean School of Higher Education of Padua in 2022.

Proof. We define a cyclic order on X as an ordering on the elements of $X := \{x_1, x_2, \ldots, x_n\}$ with $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1$. Any permutation on X which is an n-cycle gives rise to a cyclic order and conversely, any cyclic order on X gives rise to a permutation which is an n-cycle.

We double count the elements of the family

$$\mathcal{S} := \{(A, C) \mid A \in \mathcal{A},\ C \text{ cyclic order},\ A \text{ interval in } C\},$$

where an interval of a cyclic order C is a subset of X consisting of consecutive elements according to C. Given $A \in \mathcal{A}$, we have $k!(n-k)!$ choices for C with the property that $(A, C) \in \mathcal{S}$. In fact, we may order the elements in A in $k!$ ways and then we can order the elements of $X \setminus A$ in $(n-k)!$ ways. Observe that here we are using $k < n$. Therefore

$$|\mathcal{S}| = |\mathcal{A}|k!(n-k)!.$$

Now, we have $(n-1)!$ cyclic orders on X. We claim that, given a cyclic order C, there are at most k elements of $\mathcal{A}$ that are intervals of C. In fact let C

be the cyclic order $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1$. Suppose $A := \{x_1, \ldots, x_k\} \in \mathcal{A}$ is an interval of C. For every $B \in \mathcal{A}$ which is an interval of C, and $B \neq A$, B must intersect A, so there must be an $i \in \{1, \ldots, k-1\}$ such that either $x_i \notin B$ and $x_{i+1} \in B$, or vice versa. For every $i \in \{1, \ldots, k-1\}$ there is exactly one interval B_1 of C such that $x_i \notin B_1$ and $x_{i+1} \in B_1$, and exactly one interval B_2 such that $x_i \in B_2$ and $x_{i+1} \notin B_2$, and since $2k < n$, $B_1 \cap B_2 = \emptyset$, so at most one of the two is in $\mathcal{A}$. Therefore, apart from A, there can be at most $k-1$ elements of $\mathcal{A}$ that are intervals of C. Therefore

$$|\mathcal{S}| \leq k(n-1)!.$$

We deduce

$$|\mathcal{A}| \leq \frac{k(n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}.$$

We now suppose that $|\mathcal{A}| = \binom{n-1}{k-1}$. We have to show that $\mathcal{A}$ consists of all the k-subsets containing some fixed point.

Suppose that there are two elements x and y such that every k-subset containing x but not y belongs to $\mathcal{A}$. We show that with this additional hypothesis, $\mathcal{A}$ consists of all the k-subsets containing x. Let K be a k-subset not containing x. Since $2k < n$, there is a k-subset L containing x but not y disjoint from K. Thus $K \notin \mathcal{A}$. Therefore, every set in $\mathcal{A}$ contains x, and by considering the cardinality every such set lies in $\mathcal{S}$. So, if $\mathcal{A}$ is a counterexample to the Erdős-Ko-Rado theorem, then for every x and y there is a k-subset not in $\mathcal{A}$ containing x but not y.

We claim that there are two k-subsets K, K' intersecting in $k-1$ elements, such that K is in $\mathcal{A}$ and K' is not. This is true because we can transition from a given k-subset in $\mathcal{A}$ to a given k-subset not in $\mathcal{A}$ changing one element at a time, so there must be a point in the transition where we go from a k-subset K in $\mathcal{A}$ to a k-subset K' not in $\mathcal{A}$ changing only one element.

Let K and K' be as above. Label the points in $K \setminus K'$ and $K' \setminus K$ as 0 and k respectively. Assuming that $\mathcal{A}$ is a counterexample to the Erdős-Ko-Rado theorem, and using what we have shown, choose $K'' \notin \mathcal{A}$ with $0 \in K''$ and $k \notin K''$. Let $K \cap K'' = \{0, \ldots, t-1\}$ with $t < k$. Then number the remaining points of K as $t, \ldots, k-1$, and the remaining points of K'' as $n-k+t, \ldots, n-1$. Number the remaining points with the remaining elements of the integers mod n.
Since we are assuming $|\mathcal{A}| = \binom{n-1}{k-1}$ it must hold true that $|\mathcal{S}| = k(n-1)!$, and in particular for every cyclic order C of X there must be exactly k intervals of C that are elements of $\mathcal{A}$.
As we numbered the elements of X as the integers mod n, consider the natural cyclic order $0 \leq 1 \leq \cdots \leq n-1 \leq 0$. We will denote the intervals of

this cyclic order as $[a,b] = \{a, a+1, \dots, b\}$, where the numbers are intended mod n. The interval $K = [0, k-1]$ is in $\mathcal{A}$, so for every $i \in \{0, 1, \dots, k-2\}$ exactly one between $[i-k+1, i]$ and $[i+1, i+k]$ must be in $\mathcal{A}$. We know that $K' = [1, k]$ is not in in $\mathcal{A}$ so $[1-k, 0]$ must be in $\mathcal{A}$. Now by induction $[i-k+1, i] \in \mathcal{A}$ for every $i \in \{0, \dots, k-2\}$, in fact if $[i-k+1, i] \in \mathcal{A}$ for a given $i \in \{0, \dots, k-3\}$, then $[i-k+2, i+1] \in \mathcal{A}$, as one between $[i-k+2, i+1]$ and $[i+2, i+k+1]$ must be in $\mathcal{A}$ but the latter is disjoint from $[i-k+1, i]$. This leads to a contradiction because for $i = t-1 \leq k-2$ we know that $K'' = [t-k, t-1] \notin \mathcal{A}$. □

The first part of the theorem, i.e. $\mathcal{A} \leq \binom{n-1}{k-1}$, holds even in the case $n = 2k$. We can see this by partitioning the k-subsets in couples of complementary subsets: it is clear that it is necessary and sufficient to take at most one element from every couple to create a family $\mathcal{A}$ such that each two subsets in $\mathcal{A}$ intersect, and there are exactly $\binom{2k-1}{k-1} = \binom{2k}{k}/2$ couples. From this, we also see that in this case we cannot conclude that if $\mathcal{A} = \binom{n-1}{k-1}$ then $\mathcal{A}$ consists of all the k-subsets containing a particular element. For example, for $X = \{1, 2, 3, 4\}$ and $k = 2$, each two of the sets $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$ intersect, and $\binom{4-1}{2-1} = 3$, but there is no element shared by all three.

An independent set on a graph is a subset of the vertices such that no two vertices in the subset are adjacent. In a Kneser graph $K_{n:k}$, an independent set is a subset $\mathcal{A}$ of its vertices such that every two vertices of $\mathcal{A}$ must intersect, thus it satisfies the Erdős-Ko-Rado hypothesis. So we can rewrite the Erdős-Ko-Rado theorem in the following form:

Theorem 2.2 (Erdős-Ko-Rado on Kneser graphs)**.** *The maximum size for an independent set on a Kneser graph $K_{n:k}$ with $n > 2k$ is $\binom{n-1}{k-1}$. Furthermore, an independent set of size $\binom{n-1}{k-1}$ consists of the k-subsets of $\{1, 2, \dots, n\}$ containing a particular point.*

Corollary 2.2.1. *If $n > 2k$, then the automorphism group of $K_{n:k}$ is isomorphic to the symmetric group* $\mathrm{Sym}(n)$*.*

Proof. Let X denote $K_{n:k}$ and $X(i)$ denote the set of all the k-subsets containing the point i, for all $i = 1, \dots, n$. Let σ be an automorphism of $K_{n:k}$. Observe that $X(i)$ is an independent set since any two of its elements intersect in i. Now, σ maps independent sets in independent sets, so $\sigma(X(i))$ is an independent set of X of size $\binom{n-1}{k-1}$, thus by Theorem 2.2 it is of the form $X(j)$ for some $j = 1, \dots, n$. This induces a permutation g_σ on $1, \dots, n$, in fact since $X(i_1) \neq X(i_2)$ for $i_1 \neq i_2$, then $\sigma(X(i_1)) \neq \sigma(X(i_2))$, or σ would not be an automorphism. If σ is not the identity then g_σ is not the identity, in fact if $A = \{a_1, a_2, \dots, a_k\} \subseteq \{1, \dots, n\}$ is not fixed by σ, say $a_1 \notin \sigma(A)$, then $X(a_1) \neq \sigma(X(a_1))$, so g_σ is not $1_{\mathrm{Sym}(n)}$. Now we just need to check that every $g \in \mathrm{Sym}(n)$ is g_σ for some σ, but this is easy since if σ_g is the automorphism of $K_{n:k}$ induced by g, then $\sigma_g(X(i)) = X(g(i))$ for every $i = 1, \ldots, n$, so $g = g_{\sigma_g}$. □

Remark. We constructed an isomorphism between $\mathrm{Sym}(n)$ and $\mathrm{Aut}(K_{n:k})$, which maps $g \in \mathrm{Sym}(n)$ into σ_g, which is the action of g on the vertices of $K_{n:k}$, as we saw that $g_{\sigma_g} = g$. We can thus conclude that the automorphism group of $K_{n:k}$ is the symmetric group $\mathrm{Sym}(n)$ in its action of k-subsets.

Chapter 3

The Formula

Let n,k and ℓ be positive integers with $1 \leq k \leq n/2$. We let

$$F = \binom{\{1, \ldots, n\}}{k}$$

be the collection of all k-subsets of $\{1, \ldots, n\}$ and we let F^ℓ be the collection of all ℓ-tuples of k-subsets of $\{1, \ldots, n\}$. In particular,

$$|F^\ell| = \binom{n}{k}^\ell.$$

In what follows, when we refer to subsets of $\{1, \ldots, n\}$ being fixed by a permutation we mean setwise fixed, instead when we refer to a tuple being fixed by a permutation we mean pointwise. In particular a tuple of subsets of $\{1, \ldots, n\}$ is fixed by a permutation if each subset in the tuple is setwise fixed.

For each $g \in \mathrm{Sym}(n)$, we let

$$F_g = \{\alpha \in F \mid g \cdot \alpha = \alpha\}$$

be the collection of all k-subsets of $\{1, \ldots, n\}$ fixed by g. Therefore, the cartesian product F_g^ℓ is the collection of all ℓ-tuples of k-subsets of $\{1, \ldots, n\}$ which are fixed by g, that is the ℓ-tuples of k-subsets such that each k-subset in the ℓ-tuple is fixed by g. We can also see this as an action of $\mathrm{Sym}(n)$ on F^ℓ, this way $F_g^\ell = \{\beta \in F^\ell \mid g \cdot \beta = \beta\}$. For instance, for each $1 \leq i < j \leq n$, $F_{(i\,j)}$ is the collection of all k-subsets of $\{1, \ldots, n\}$ fixed by the transposition swapping i and j. Moreover, we let

$$H_\ell = \{\beta \in F^\ell \mid \text{ if } g \cdot \beta = \beta, \text{ then } g = 1\}$$

denote the collection of all ℓ-tuples of k-subsets of $\{1, \ldots, n\}$ which are only fixed by the identity, and $h_\ell := |H_\ell|$.
Since each element of F^ℓ is either fixed by some non-identity element of $\mathrm{Sym}(n)$ or is only fixed by the identity, exclusively, we have

$$F^\ell \setminus H_\ell = \bigcup_{g \in \mathrm{Sym}(n) \setminus \{1\}} F^\ell_g.$$

Observe that

$$h_\ell = 0 \text{ if and only if } \ell < b(n,k). \tag{3.1}$$

In fact, by definition of $b(n,k)$, there exists an ℓ-tuple of k-subsets which is only fixed by the identity if and only if $\ell \geq b(n,k)$.

We want to show that

$$F^\ell \setminus H_\ell = \bigcup_{1 \leq i < j \leq n} F^\ell_{(i\,j)}. \tag{3.2}$$

For this, we need to prove that if an element of F^ℓ is fixed by some non-identity permutation g, then it is also fixed by a transposition. We can use the following.

Lemma 3.1. *Given a non-identity permutation $g \in \mathrm{Sym}(n)$ there exists a transposition τ such that for every $S \subseteq \{1, \ldots, n\}$ fixed by g, S is also fixed by τ.*

Proof. Let $(a_1\, a_2\, \ldots\, a_i)$ be one of the cycles of the decomposition of g in disjoint cycles of order at least two. We can assume $i \geq 2$ because g is not the identity. Then $S \cap \{a_1, \ldots, a_i\}$ is either $\emptyset$ or $\{a_1, \ldots, a_i\}$. In fact, let's suppose that $a_j \in S$. Then since S is fixed by g and $g(a_j) = a_{j+1}$ then a_{j+1} must be in S too (we intend the indices $\bmod i$) and so $\{a_1, \ldots, a_i\} \subseteq S$. Similarly if $a_j \notin S$ then $\{a_1, \ldots, a_i\} \cap S = \emptyset$ because S fixed by g implies $\overline{S}$ fixed by g, where $\overline{S} := \{1, \ldots, n\} \setminus S$.
Then a_1 and a_2 are either both in S or both not in S, so the transposition $(a_1\, a_2)$ fixes S, since it only swaps two elements in S or outside S. □

So if each k-subset of an ℓ-tuple is fixed by g then there exists a transposition τ that also fixes each of them, so τ fixes the ℓ-tuple, and thus we have (3.2).

Now, let $\binom{\{1,\ldots,n\}}{2}$ denote the set of all 2-subsets of $\{1, \ldots, n\}$. From (3.2), using inclusion-exclusion, we obtain

$$\binom{n}{k}^\ell = |F^\ell| = h_\ell + \sum_{\emptyset \neq \Gamma \subseteq \binom{\{1,\ldots,n\}}{2}} (-1)^{|\Gamma|-1} \left| \bigcap_{\{i,j\} \in \Gamma} F^\ell_{(i\,j)} \right|. \tag{3.3}$$

Now, given a subset $\emptyset \neq \Gamma \subseteq \binom{\{1,\dots,n\}}{2}$, we write

$$F^\ell_\Gamma := \bigcap_{\{i,j\}\in\Gamma} F^\ell_{(i\,j)}$$

and, with a slight abuse of terminology, we let $F^\ell_\emptyset := F^\ell$. With this notation, from (3.3), we get

$$h_\ell = \sum_{\Gamma\subseteq\binom{\{1,\dots,n\}}{2}} (-1)^{|\Gamma|}|F^\ell_\Gamma|. \tag{3.4}$$

In what follows, we identify $\Gamma \subseteq \binom{\{1,\dots,n\}}{2}$ with a graph on $\{1,\dots,n\}$ having edge set Γ. In particular, we borrow some notation from graph theory.

Given $\Gamma \subseteq \binom{\{1,\dots,n\}}{2}$, we let $\pi(\Gamma)$ be the partition of n where the parts are the cardinalities of the connected components of Γ. In other words, let $X_1, X_2, \dots, X_t$ be the connected components of Γ ordered with $|X_1| \geq |X_2| \geq \cdots \geq |X_t|$. Then

$$\pi(\Gamma) := (|X_1|, |X_2|, \dots, |X_t|).$$

We now make two important remarks. First:

Lemma 3.2. *Let Γ be a graph having vertex set $\{1,\dots,n\}$ and having connected components $X_1, X_2, \dots, X_t$. Then*

$$\langle (i\,j) \mid \{i,j\} \in \Gamma\rangle = \mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2) \times \cdots \times \mathrm{Sym}(X_t).$$

Proof. For each $i \in \{1,\dots,t\}$, let Γ_i be the restriction of Γ to X_i, that is the graph on X_i with edge set $\Gamma \cap \binom{X_i}{2}$.
As $X_i \cap X_j = \emptyset$ for $i \neq j$ and as every edge in Γ is in Γ_i for some i, we deduce

$$\langle (i\,j) \mid \{i,j\} \in \Gamma\rangle = \langle (i\,j) \mid \{i,j\} \in \Gamma_1\rangle \times \cdots \times \langle (i\,j) \mid \{i,j\} \in \Gamma_t\rangle. \tag{3.5}$$

Now we just need to prove that if Γ is connected then

$$\langle (i\,j) \mid \{i,j\} \in \Gamma\rangle = \mathrm{Sym}(n),$$

indeed applying this in (3.5) to every connected component we get the thesis. We prove this by induction on n.

Base case: $n = 1$. The group generated by the empty set is just $1 = \mathrm{Sym}(1)$.

Inductive case: $n \geq 2$. We take a leaf of a spanning tree of Γ, suppose without loss of generality that it is the vertex n and let $\tilde{\Gamma}$ be the restriction of Γ to $\{1,\dots,n-1\}$. By inductive hypothesis, we know that

$$\langle (i\,j) \mid \{i,j\} \in \tilde{\Gamma}\rangle = \mathrm{Sym}(n-1)$$

since $\tilde{\Gamma}$ is connected. As Γ is connected, there is an edge $\{i, n\}$ for some $i \in \{1, \dots, n-1\}$, so

$$\mathrm{Sym}(n) \geq \langle (i\,j) \mid \{i,j\} \in \Gamma \rangle \geq \langle \mathrm{Sym}(n-1), (i,n) \rangle = \mathrm{Sym}(n). \qquad \square$$

In other words, the group generated by the transpositions corresponding to elements in Γ is a direct product of symmetric groups.

Second: given $\emptyset \neq \Gamma \subseteq \binom{\{1,\dots,n\}}{2}$, an ℓ-tuple of k-subsets of $\{1, \dots, n\}$ is fixed by every transposition corresponding to elements in Γ if and only if it is fixed by every element of $\langle (i\,j) \mid \{i,j\} \in \Gamma \rangle$, in other words

$$F^\ell_\Gamma := \bigcap_{\{i,j\} \in \Gamma} F^\ell_{(i\,j)} = \bigcap_{g \in \langle (i\,j) \mid \{i,j\} \in \Gamma \rangle} F^\ell_g. \tag{3.6}$$

In fact if g and h fix an ℓ-tuple then so do g^{-1} and gh.
By Lemma 3.2 we know that $\langle (i\,j) \mid \{i,j\} \in \Gamma \rangle$ only depends on $\pi(\Gamma)$ up to isomorphisms, so we can show the following.

Lemma 3.3. *For every* $\Gamma_1, \Gamma_2 \subseteq \binom{\{1,\dots,n\}}{2}$ *with* $\pi(\Gamma_1) = \pi(\Gamma_2)$ *we have* $|F^\ell_{\Gamma_1}| = |F^\ell_{\Gamma_2}|$.

Proof. Let $X_1, X_2, \dots, X_t$ be the connected components of Γ_1 and similarly let $Y_1, Y_2, \dots, Y_t$ be the connected components of Γ_2, both ordered by cardinality. Let $g \in \mathrm{Sym}(n)$ be a permutation such that $g(X_i) = Y_i$ for all $i = 1, \dots, t$, it exists because $\pi(\Gamma_1) = \pi(\Gamma_2)$. Given $\beta \in F^\ell_{\Gamma_1}$, we know that $h \cdot \beta = \beta$ for all h in $\langle (i\,j) \mid \{i,j\} \in \Gamma_1 \rangle$ by (3.6). Moreover we know that $g\mathrm{Sym}(X_i)g^{-1} = \mathrm{Sym}(Y_i)$, and so by Lemma 3.2 we get

$$g \langle (i\,j) \mid \{i,j\} \in \Gamma_1 \rangle g^{-1} = \langle (i\,j) \mid \{i,j\} \in \Gamma_2 \rangle.$$

Now for every $h \in \langle (i\,j) \mid \{i,j\} \in \Gamma_1 \rangle$ we have

$$ghg^{-1} \cdot (g \cdot \beta) = ghg^{-1}g \cdot \beta = g \cdot (h \cdot \beta) = g \cdot \beta,$$

that means $g \cdot \beta \in F^\ell_{\Gamma_2}$ by (3.6).
Similarly we can show that $g^{-1} \cdot \gamma \in F^\ell_{\Gamma_1}$ for every $\gamma \in F^\ell_{\Gamma_2}$, and this shows that there is a bijection between $F^\ell_{\Gamma_1}$ and $F^\ell_{\Gamma_2}$, and in particular $|F^\ell_{\Gamma_1}| = |F^\ell_{\Gamma_2}|$. $\square$

Since the cardinality of these sets depends only on a partition, for each partition π of n, we let f^ℓ_π be the cardinality of $|F^\ell_\Gamma|$, where Γ is an arbitrary graph with $\pi = \pi(\Gamma)$.

Fix π a partition of n. We write π in "exponential" notation, that is, $\pi = (1^{c_1}, 2^{c_2}, \dots, n^{c_n})$ where c_i denotes the number of parts in π equal to i. We now want to show the following

Lemma 3.4. *Given a partition $\pi = (1^{c_1}, 2^{c_2}, \dots, n^{c_n})$ of n it holds*

$$\sum_{\substack{\Gamma \subseteq \binom{\{1,\dots,n\}}{2} \\ \pi(\Gamma)=\pi}} (-1)^{|\Gamma|} = (-1)^{n-\sum_{i=1}^{n} c_i} \frac{n}{\prod_{i=1}^{n} i^{c_i} c_i!}. \tag{3.7}$$

Proof. First, we show (3.7) in the special case $\pi = (1^0, 2^0, \dots, n^1)$, that is, the trivial partition consisting of one part of size n, and then we will deduce the general case from that. In particular, we want to show

$$\sum_{\substack{\Gamma \subseteq \binom{\{1,\dots,n\}}{2} \\ \pi(\Gamma)=(n)}} (-1)^{|\Gamma|} = (-1)^{n-1}(n-1)!.$$

This equality has a combinatorial interpretation: Among all connected graphs on n labelled vertices, the difference between the number of those with an even number of edges and those with an odd number of edges is $(-1)^{n-1}(n-1)!$.

We will show this by induction on n, with base case $n = 1$: the only graph on 1 vertex has an even number of edges, and $(-1)^0 0! = 1$. Assume now the thesis to be true for $n-1$.

Let p_n and d_n be the number of connected graphs on n labelled vertices with an even number of edges and with an odd number of edges, respectively.

Let P_n and D_n be the number of graphs on n labelled vertices with an even number of edges and with an odd number of edges, respectively. In fact

$$P_n = \binom{\binom{n}{2}}{0} + \binom{\binom{n}{2}}{2} + \dots$$

$$D_n = \binom{\binom{n}{2}}{1} + \binom{\binom{n}{2}}{3} + \dots$$

We can count the number of disconnected graphs with an even (odd) number of edges on n labelled vertices in two ways: on one hand it is $P_n - p_n$ (respectively $D_n - d_n$), on the other hand, we can count the number of rooted disconnected graphs with an even (odd) number of edges on n labelled vertices (rooted means with an highlighted vertex, the root) and then divide this number by n, as a graph can be rooted in n different ways.

To count the number of rooted disconnected graphs, we first choose the connected component containing the root: for every possible cardinality $i = 1, \dots, n-1$ we can choose the connected component in $\binom{n}{i}$ ways, inside of which we have i ways to choose the root.

If we want the graph to have an even number of edges then either both the connected component of the root and the rest have an even number of edges, or they have both an odd number of edges.

If we want the graph to have an odd number of edges then either the connected component of the root has an even number of edges and the rest have an odd number of edges, or vice versa. Therefore, we have

$$P_n - p_n = \frac{1}{n}\sum_{i=1}^{n-1} i\binom{n}{i}(p_i P_{n-i} + d_i D_{n-i}) \tag{3.8}$$

$$D_n - d_n = \frac{1}{n}\sum_{i=1}^{n-1} i\binom{n}{i}(p_i D_{n-i} + d_i P_{n-i}). \tag{3.9}$$

Taking (3.9)-(3.8) we get

$$p_n - d_n + D_n - P_n = \frac{1}{n}\sum_{i=1}^{n-1} i\binom{n}{i}(D_{n-i} - P_{n-i})(p_i - d_i). \tag{3.10}$$

From the Binomial theorem, we have

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k} = 0 \quad \forall m > 0.$$

This implies $D_n - P_n = 0$ whenever $\binom{n}{2} > 0$, that is, for every $n \geq 2$. For $n = 1$, $D_1 - P_1 = -1$, since the only graph on 1 vertex has an even number of edges. So (3.10) becomes

$$p_n - d_n = \frac{1}{n}(n-1)n(-1)(p_{n-1} - d_{n-1}) = (-1)^{n-1}(n-1)!$$

where the last equality follows by our inductive hypothesis.

Now that we have concluded the special case we need to deduce the general case. Given $\pi = (1^{c_1}, \ldots, n^{c_n})$, the number of ways that the set $\{1, \ldots, n\}$ can be partitioned into π is

$$\frac{n!}{\prod_{i=1}^{n} c_i!(i!)^{c_i}}.$$

Now, given a specific partition $\mathcal{P}$ of $\{1, \ldots, n\}$ realizing π, the sum among all graphs Γ with connected components $\mathcal{P}$ is

$$\sum_{\substack{\Gamma\subseteq\binom{\{1,\ldots,n\}}{2} \\ \text{with conn. comp.s} \\ \mathcal{P}}} (-1)^{|\Gamma|} = \prod_{\mathcal{S}\in\mathcal{P}} \sum_{\substack{\gamma\subseteq\binom{\mathcal{S}}{2} \\ \pi(\gamma)=\mathcal{S}}} (-1)^{|\gamma|} = \prod_{\mathcal{S}\in\mathcal{P}} (-1)^{|\mathcal{S}|-1}(|\mathcal{S}|-1)! =$$

$$= \prod_{i=1}^{n} (-1)^{(i-1)c_i} (i-1)!^{c_i} = (-1)^{n-\sum_{i=1}^n c_i} \prod_{i=1}^{n} (i-1)!^{c_i}.$$

So in conclusion

$$\sum_{\substack{\Gamma\subseteq \binom{\{1,\ldots,n\}}{2} \\ \pi(\Gamma)=\pi}} (-1)^{|\Gamma|} = \frac{n!}{\prod_{i=1}^n c_i!(i!)^{c_i}} (-1)^{n-\sum_{i=1}^n c_i} \prod_{i=1}^{n} (i-1)!^{c_i} =$$

$$= (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}. \qquad \square$$

Using Lemma 3.4 in (3.4), we get

$$h_\ell = \sum_{\substack{\pi \text{ partition of } n \\ \pi=(1^{c_1},2^{c_2},\ldots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} f_\pi^\ell. \tag{3.11}$$

Lemma 3.5. *Given an integer partition $\pi = (1^{c_1}, \ldots, n^{c_n})$, we have*

$$f_\pi^\ell = \left(\sum_{\substack{\eta \text{ partition of } k \\ \eta=(1^{b_1},2^{b_2},\ldots,k^{b_k})}} \prod_{j=1}^{k} \binom{c_j}{b_j} \right)^{\ell}. \tag{3.12}$$

Proof. Given $\Gamma \subseteq \binom{\{1,\ldots,n\}}{2}$ satisfying $\pi(\Gamma) = \pi$ and with connected components $X_1, \ldots, X_t$, we want to calculate how many elements of F^ℓ are fixed by $\langle (i\,j) \mid \{i,j\} \in \Gamma\rangle$. This is the number of k-subsets of $\{1, \ldots, n\}$ that are fixed by $\langle (i\,j) \mid \{i,j\} \in \Gamma\rangle$, raised to the ℓ, since in order for an element of F^ℓ to be fixed, each of its k-subsets has to be fixed.

To calculate this number, we first notice that in order for a k-subset S to be fixed, either $X_i \subseteq S$ or $X_i \cap S = \emptyset$. For $|X_i| = 1$ this is trivial. For $|X_i| \geq 2$ we already showed in Lemma 3.1 that if S is fixed by a cycle of order at least 2, then either all the elements of the cycle are in S or none of them are. If $X_i = \{a_1, \ldots, a_s\}$ then

$$(a_1, \ldots, a_s) \in \mathrm{Sym}(X_i) \leq \mathrm{Sym}(X_1) \times \cdots \times \mathrm{Sym}(X_t) = \langle (i\,j) \mid \{i,j\} \in \Gamma\rangle.$$

So $X_i \subseteq S$ or $X_i \subseteq \overline{S}$ for every $i = 1\ldots,t$. This condition is also sufficient for a k-subset to be fixed by $\langle (i\,j) \mid \{i,j\} \in \Gamma\rangle$, since this is a direct product of the symmetric groups of the X_is, so its action on S is only to internally shuffle the X_is which are completely inside of S.

Thus the number we are looking for is the number of ways we can create a set of order k combining different X_is, so, using the exponential notation

$$f_\pi^\ell = |F_\Gamma^\ell| = \left(\sum_{\substack{\eta \text{ partition of } k \\ \eta=(1^{b_1},2^{b_2},\dots,k^{b_k})}} \prod_{j=1}^{k} \binom{c_j}{b_j} \right)^\ell . \qquad \square$$

Finally, from (3.11) and (3.12), we find the beautiful equality

$$h_\ell = \sum_{\substack{\pi \text{ partition of } n \\ \pi=(1^{c_1},2^{c_2},\dots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} \left(\sum_{\substack{\eta \text{ partition of } k \\ \eta=(1^{b_1},2^{b_2},\dots,k^{b_k})}} \prod_{j=1}^{k} \binom{c_j}{b_j} \right)^\ell . \tag{3.13}$$

From (3.1) it follows that to find $b(n,k)$ it is sufficient to calculate h_ℓ using this formula for increasing values of ℓ until we get a non-zero value: $b(n,k)$ is the smallest positive integer ℓ such that $h_\ell \neq 0$.

Chapter 4

Results

We have implemented this formula in a computer and we are reporting in Table 4.1 the values of $b(n,k)$, for every $k \leq 14$. We hope that these values can be of some help to shed some light on $b(n,k)$ when k is large.

Observe that in Table 4.1, for a given $k \leq 14$, we are reporting only the values of $b(n,k)$ when $n \leq \lfloor k(k+1)/2 \rfloor$, because when $n \geq \lfloor k(k+1)/2 \rfloor + 1$ we may simply use (1.1) to compute $b(n,k)$. In particular, when $k = 1$, we have $\lfloor k(k+1)/2 \rfloor = 1 \leq 2k$ and hence $b(n,k) = \lceil 2(n-1)/2 \rceil = n-1$, for every $n \geq 2$. When $k = 2$, we have $\lfloor k(k+1)/2 \rfloor = 3 \leq 2k$ and hence $b(n,k) = \lceil 2(n-1)/3 \rceil$, for every $n \geq 4$. For this reason, in Table 4.1, we are only including the values of $k \geq 3$.

Furthermore, for $n = 2k$, $b(n,k)$ is not the determining number of the Kneser Graph $K_{n:k}$, as we discussed in Chapter 1, but we can still study its value with our formula, and in this case Z. Halasi in [5] showed that $b(n, n/2) = \lceil \log_2 n \rceil$.

$n \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14
6	3											
7	-											
8	-	3										
9	-	4										
10	-	4	4									
11	-	-	4									
12	-	-	4	4								
13	-	-	5	4								
14	-	-	5	5	4							
15	-	-	5	5	4							
16	-	-	-	5	5	4						
17	-	-	-	5	5	5						

18	-	-	-	6	5	5	5					
19	-	-	-	6	5	5	5					
20	-	-	-	6	6	5	5	5				
21	-	-	-	6	6	5	5	5				
22	-	-	-	-	6	6	5	5	5			
23	-	-	-	-	6	6	6	5	5			
24	-	-	-	-	6	6	6	5	5	5		
25	-	-	-	-	7	6	6	6	5	5		
26	-	-	-	-	7	6	6	6	5	5	5	
27	-	-	-	-	7	7	6	6	6	5	5	
28	-	-	-	-	7	7	6	6	6	6	5	5
29	-	-	-	-	-	7	7	6	6	6	6	5
30	-	-	-	-	-	7	7	6	6	6	6	6
31	-	-	-	-	-	7	7	7	6	6	6	6
32	-	-	-	-	-	8	7	7	6	6	6	6
33	-	-	-	-	-	8	7	7	7	6	6	6
34	-	-	-	-	-	8	8	7	7	6	6	6
35	-	-	-	-	-	8	8	7	7	7	6	6
36	-	-	-	-	-	8	8	7	7	7	6	6
37	-	-	-	-	-	-	8	8	7	7	7	6
38	-	-	-	-	-	-	8	8	7	7	7	6
39	-	-	-	-	-	-	8	8	8	7	7	7
40	-	-	-	-	-	-	9	8	8	7	7	7
41	-	-	-	-	-	-	9	8	8	8	7	7
42	-	-	-	-	-	-	9	8	8	8	7	7
43	-	-	-	-	-	-	9	9	8	8	7	7
44	-	-	-	-	-	-	9	9	8	8	8	7
45	-	-	-	-	-	-	9	9	8	8	8	7
46	-	-	-	-	-	-	-	9	9	8	8	8
47	-	-	-	-	-	-	-	9	9	8	8	8
48	-	-	-	-	-	-	-	9	9	9	8	8
49	-	-	-	-	-	-	-	9	9	9	8	8
50	-	-	-	-	-	-	-	10	9	9	8	8
51	-	-	-	-	-	-	-	10	9	9	9	8
52	-	-	-	-	-	-	-	10	9	9	9	8
53	-	-	-	-	-	-	-	10	10	9	9	8
54	-	-	-	-	-	-	-	10	10	9	9	9
55	-	-	-	-	-	-	-	10	10	9	9	9
56	-	-	-	-	-	-	-	-	10	10	9	9
57	-	-	-	-	-	-	-	-	10	10	9	9
58	-	-	-	-	-	-	-	-	10	10	9	9

59	-	-	-	-	-	-	-	-	10	10	10	9
60	-	-	-	-	-	-	-	-	11	10	10	9
61	-	-	-	-	-	-	-	-	11	10	10	9
62	-	-	-	-	-	-	-	-	11	10	10	10
63	-	-	-	-	-	-	-	-	11	11	10	10
64	-	-	-	-	-	-	-	-	11	11	10	10
65	-	-	-	-	-	-	-	-	11	11	10	10
66	-	-	-	-	-	-	-	-	11	11	10	10
67	-	-	-	-	-	-	-	-	-	11	11	10
68	-	-	-	-	-	-	-	-	-	11	11	10
69	-	-	-	-	-	-	-	-	-	11	11	10
70	-	-	-	-	-	-	-	-	-	11	11	11
71	-	-	-	-	-	-	-	-	-	12	11	11
72	-	-	-	-	-	-	-	-	-	12	11	11
73	-	-	-	-	-	-	-	-	-	12	11	11
74	-	-	-	-	-	-	-	-	-	12	11	11
75	-	-	-	-	-	-	-	-	-	12	12	11
76	-	-	-	-	-	-	-	-	-	12	12	11
77	-	-	-	-	-	-	-	-	-	12	12	11
78	-	-	-	-	-	-	-	-	-	12	12	11
79	-	-	-	-	-	-	-	-	-	-	12	12
80	-	-	-	-	-	-	-	-	-	-	12	12
81	-	-	-	-	-	-	-	-	-	-	12	12
82	-	-	-	-	-	-	-	-	-	-	12	12
83	-	-	-	-	-	-	-	-	-	-	12	12
84	-	-	-	-	-	-	-	-	-	-	13	12
85	-	-	-	-	-	-	-	-	-	-	13	12
86	-	-	-	-	-	-	-	-	-	-	13	12
87	-	-	-	-	-	-	-	-	-	-	13	12
88	-	-	-	-	-	-	-	-	-	-	13	13
89	-	-	-	-	-	-	-	-	-	-	13	13
90	-	-	-	-	-	-	-	-	-	-	13	13
91	-	-	-	-	-	-	-	-	-	-	13	13
92	-	-	-	-	-	-	-	-	-	-	-	13
93	-	-	-	-	-	-	-	-	-	-	-	13
94	-	-	-	-	-	-	-	-	-	-	-	13
95	-	-	-	-	-	-	-	-	-	-	-	13
96	-	-	-	-	-	-	-	-	-	-	-	13
97	-	-	-	-	-	-	-	-	-	-	-	14
98	-	-	-	-	-	-	-	-	-	-	-	14
99	-	-	-	-	-	-	-	-	-	-	-	14

100	-	-	-	-	-	-	-	-	-	-	-	14
101	-	-	-	-	-	-	-	-	-	-	-	14
102	-	-	-	-	-	-	-	-	-	-	-	14
103	-	-	-	-	-	-	-	-	-	-	-	14
104	-	-	-	-	-	-	-	-	-	-	-	14
105	-	-	-	-	-	-	-	-	-	-	-	14

Table 4.1: Some values for $b(n,k)$

It appears, as intuition would suggest, that for every k and $n_1 > n_2 \geq 2k$ it holds $b(n_1,k) \geq b(n_2,k)$, and also that for every n and $k_1 < k_2 \leq n/2$ it holds $b(n,k_1) \geq b(n,k-2)$. These properties are in fact true and are proved in [5].

Another proof that $b(n,1) = n-1$

Our formula can be used to find again the known result $b(n,1) = n-1$, as follows.
First of all, note that with $k=1$ and $\ell \geq 1$ the formula (3.13) becomes

$$h_\ell = \sum_{\substack{\pi \text{ partition of } n \\ \pi=(1^{c_1},2^{c_2},\ldots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} c_1^\ell := h_\ell(n), \tag{4.1}$$

as the only partition of 1 is $(1^{b_1},\ldots,k^{b_k}) = (1^1)$. The notation $h_\ell(n)$ with n specified and $\ell \geq 1$ will be useful later.
We will prove $b(n,1) = n-1$ by induction on n. First, we prove the following Lemma.

Lemma 4.1. *For $n \geq 2$ it holds*

$$h_0(n) := \sum_{\substack{\pi \textit{ partition of } n \\ \pi=(1^{c_1},2^{c_2},\ldots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} = 0.$$

(Note that this is a definition for $h_0(n)$, since $h_\ell(n)$ from above was defined for $\ell \geq 1$)

Proof. By Lemma 3.4, we get

$$h_0(n) = \sum_{\pi \text{ part. of } n} \sum_{\substack{\Gamma \subseteq \binom{\{1,\ldots,n\}}{2} \\ \pi(\Gamma)=\pi}} (-1)^{|\Gamma|} = \sum_{\Gamma \subseteq \binom{\{1,\ldots,n\}}{2}} (-1)^{|\Gamma|}.$$

We conclude by the binomial theorem, as $\binom{n}{2} \geq 1$ and

$$\sum_{\Gamma \subseteq \binom{\{1,\ldots,n\}}{2}} (-1)^{|\Gamma|} = \sum_{k=0}^{\binom{n}{2}} (-1)^k \binom{\binom{n}{2}}{k} = 0. \quad \square$$

We now proceed with the proof by induction that $h_\ell(n) = 0$ for $1 \leq \ell < n-1$ and $h_{n-1}(n) \neq 0$.

Base case: $n = 2$. The partitions of 2 in exponential notation are $(2,0)$ are $(0,1)$, so (4.1) becomes

$$h_\ell(2) = (-1)^{2-2} \frac{2!}{1^2 \cdot 2! \cdot 2^0 \cdot 0!} 2^\ell + (-1)^{2-1} \frac{2!}{1^0 \cdot 0! \cdot 2^1 \cdot 1!} 0^\ell = 2^\ell,$$

which is non-zero for $\ell = 1$.

Inductive case: let now assume that $h_\ell(n-1) = 0$ for $1 \leq \ell < n-2$ and $h_{n-2}(n-1) \neq 0$. We want to show $h_\ell(n) = 0$ for $1 \leq \ell < n-1$ and $h_{n-1}(n) \neq 0$. The induction is based on the fact that we can obtain the partitions of n from the partitions of $n-1$ simply adding a component of cardinality 1 (increasing b_1 by 1, where $\mu = (1^{b_1}, \ldots, (n-1)^{b_{n-1}})$ is a generic partition of $n-1$), in this way we obtain every partition of n with $c_1 \neq 0$, but partitions of n with $c_1 = 0$ don't count in $h_\ell(n)$, since $c_1^\ell = 0$. With this idea, we calculate $h_\ell(n)$.

$$\begin{aligned}
h_\ell(n) &= \sum_{\substack{\pi \text{ partition of } n \\ \pi = (1^{c_1}, 2^{c_2}, \ldots, n^{c_n})}} (-1)^{n - \sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} c_1^\ell \\
&= \sum_{\substack{\pi \text{ partition of } n \\ \pi = (1^{c_1}, 2^{c_2}, \ldots, n^{c_n}) \\ c_1 \neq 0}} (-1)^{n - \sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} c_1^\ell \\
&= \sum_{\substack{\mu \text{ partition of } n-1 \\ \mu = (1^{b_1}, 2^{b_2}, \ldots, (n-1)^{b_{n-1}})}} (-1)^{n - (b_1+1) - \sum_{i=2}^{n-1} b_i} \frac{n(n-1)!}{(b_1+1) \prod_{i=1}^{n-1} i^{b_i} b_i!} (b_1+1)^\ell \\
&= n \cdot \sum_{\substack{\mu \text{ partition of } n-1 \\ \mu = (1^{b_1}, 2^{b_2}, \ldots, (n-1)^{b_{n-1}})}} (-1)^{n-1-\sum_{i=1}^{n-1} b_i} \frac{(n-1)!}{\prod_{i=1}^{n-1} i^{b_i} b_i!} (b_1+1)^{\ell-1},
\end{aligned}$$

but

$$(b_1+1)^{\ell-1} = 1 + \sum_{i=1}^{\ell-1} \binom{\ell-1}{i} b_1^i,$$

so

$$h_\ell(n) = n \cdot \left(h_0(n-1) + \sum_{i=1}^{\ell-1} \binom{\ell-1}{i} h_i(n-1) \right). \tag{4.2}$$

Now, $h_0(n-1) = 0$ by Lemma 4.1, and $h_i(n-1) = 0$ for $1 \leq i < n-2$, so $h_\ell(n) = 0$ for $\ell - 1 < n - 2$, but $h_{n-1}(n) \neq 0$ since $h_{n-2}(n-1) \neq 0$, concluding.

We also showed that $h_{n-1}(n) = n!$, as $h_1(2) = 2^1 = 2!$ and for every $n \geq 3$ equality (4.2) yields $h_{n-1}(n) = n \cdot h_{n-2}(n-1)$. This makes sense, as the number of $(n-1)$-tuples of 1-subsets of $\{1, \dots, n\}$ which are only fixed by the identity are exactly the $(n-1)$-tuples of distinct 1-subsets, which, considering the different orderings, there are $n!$ of.

Bibliography

[1] D. Boutin, Identifying graph automorphisms using determining sets. *Electronic J. Combin.* **13** (2006), #R78.

[2] J. Cáceres, D. Garijo, A. González, A. Márquez, M. L. Puertas, The determining number of Kneser graphs, *Discrete Math. Theor. Comput. Sci.* **15** (2013), 1–14.

[3] A. Das, H. K. Dey, Determining number of Kneser graphs: exact values and improved bounds, *Discrete Math. Theor. Comput. Sci.* **24** (2022), no. 1, Paper No. 10, 9 pp.

[4] C. Godsil, G. Royle, *Algebraic graph theory*, Graduate texts in mathematics **207**, Springer, New York, 2001.

[5] Z. Halasi, On the base size for the symmetric group acting on subsets, *Studia Sci. Math. Hungar.* **49** (2012), 492–500.

[6] G. O. H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, *J. Combinatorial Theory Ser. B* **13** (1972), 183–184.

[7] C. del Valle, C. M. Roney-Dougal, The base size of the symmetric group acting on subset, `https://arxiv.org/abs/2308.04360`.