Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Master's Degree in Mathematics

Endomorphism algebras of silting complexes

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Introduction

Categories of modules are among the nicest examples of abelian categories. The interesting question from which this thesis starts is to recognise when a certain abelian cocomplete category $\mathcal{C}$ is equivalent to a category of modules $\mathsf{Mod}\, R$. The answer to this question is a classical result obtained by Morita back in 1958. Morita Theory states that such a category $\mathcal{C}$ is equivalent to a category of modules $\mathsf{Mod}\, R$ if and only if $\mathcal{C}$ admits a particular object P, a so-called *progenerator*, i.e. a finitely generated projective generator. Moreover, for such an object, $\mathcal{C}$ is equivalent to $\mathsf{mod}(\mathsf{End}\, P)$. Two non isomorphic progenerators may have different endomorphism ring but equivalent categories of modules. In other words, fixed R ring, the endomorphism rings $S := \mathsf{End}\, P$ for any P progenerator in $\mathsf{mod}\, R$ are exactly the rings for which $\mathsf{mod}\, R$ is equivalent to $\mathsf{mod}\, S$.

In this thesis we consider some objects which extends the notion of a progenerator. We will consider the over a finite-dimensional algebra A and study how the category of modules over their endomorphism ring compare to $\mathsf{mod}\, A$.

The two kinds of objects we will consider are *tilting modules* first and then *silting complexes*. They have weaker assumptions than the classical Morita progenerator, but we still obtain some nice equivalences, which are the main object of interest of this thesis. For tilting modules, we have the Brenner–Butler Theorem, a classical result from the 80s. If A is an algebra and B is the endomorphism ring of a tilting A-module, this theorem yields equivalences between two pair of full subcategories of $\mathsf{mod}\, A$ and $\mathsf{mod}\, B$, respectively, that play a key role in terms of approximation. A similar result holds in the more general case of silting complexes, due to Buan and Zhou.

In the last part we study the properties of the endomorphism algebras of the special objects we are considering. Under some assumptions on A, it turns out that they are characterized by a nice homological property: every indecomposable module is either close to being injective or close to being projective.

Throughout the thesis we will illustrate the results with many examples computed using representations of path algebras of quivers.

In Chapter 1 we establish our notation and we lay the foundations of our work. We present the settings in which we will be working, the categories of finitely generated modules (§1.2) and the derived categories (and more generally triangulated categories(§1.5)). We also briefly introduce the theory of quivers (§1.3), which will be used to construct examples of the results we're stating. Finally, we discuss two important structures that are fundamental in our work: torsion pairs (§1.4) and t-structures (§1.6). In particular we are very interested in the HRS-tilting process which appears many times in connection with the main results.

In Chapter 2 we focus on *tilting* modules (2.1.10). We first show the basics properties of tilting theory and then turn our attention to the Brenner–Butler Theorem. We have that a tilting module induces a *torsion pair* in $\operatorname{mod} A$, $(\mathcal{T}(T), \mathcal{F}(T))$

Taking $B = \operatorname{End}(T)$, we can endow T with a structure of left B-module. The key passage is that ${}_BT$ is a tilting left B-module, which yields a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$. The Theorem of Brenner and Butler (2.2.11) states that there are equivalences of categories between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ and also between $\mathcal{F}(T)$ and $\mathcal{X}(T)$.

In this case by passing to B we don't lose any information on the algebra we started from. Indeed we have that A is exactly the algebra endomorphism of the left B-module T (2.2.5).

In Chapter 3 we enlarge our setting and from the category of modules we pass to the derived category. Here we are interested in the *(2-term) silting complexes* (3.1.2). They can be regarded as a generalization of tilting modules. In the first part of the chapter we present the main results about silting theory.

As with tilting modules, also a 2-term silting complex $\mathbf{P}$ induces a torsion pair in $\operatorname{mod} A$, $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$. Let $B = \operatorname{End}_{D^b(A)}(P)$. In $\operatorname{mod} B$, $\mathbf{P}$ induces a 2-term silting complex $\mathbf{Q}$ that defines a torsion pair $(\mathcal{T}(\mathbf{Q}), \mathcal{F}(\mathbf{Q}))$. It is a recent result of Buan and Zhou [BZ16b] that there exists an equivalence of categories between $\mathcal{T}(P)$ and $\mathcal{F}(Q)$ and there exists an equivalence of categories between $\mathcal{F}(P)$ and $\mathcal{T}(Q)$ (3.2.5). Thus, this theorem of Buan and Zhou generalizes the Brenner–Butler theorem. We study this equivalences in the second section of this chapter.

We then focus on what happens when we repeat the process on $\mathbf{Q}$. In the tilting case we obtain again the starting algebra A. In general this does not happen anymore with silting complexes. Indeed we have that $\operatorname{End}_{\mathcal{D}^b(B)}(\mathbf{Q})$ is an epimorphic image of A and so we lose some information, but not much. Indeed we recover the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$.

Finally, we present another result of Buan and Zhou. Starting from $\operatorname{mod} A$ with A hereditary finite-dimensional algebra (or more generally from an hereditary abelian category) the endomorphism ring of 2-term silting complexes can be characterized by a homological property: being *shod* (3.4.2).

We end the thesis with a chapter dedicated to the study of an interesting example that illustrates the main results of Chapter 3.

Chapter 1

Preliminaries

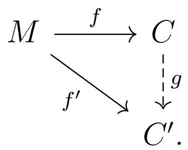
1.1 Approximations

In this section we follow [GT06]. Let $\mathcal{D}$ be a category.

Definition 1.1.1. Let M, N, C be objects of $\mathcal{D}$. We say that $f : M \to C$ is *left minimal* if for any $g : C \to C$ such that $gf = f$ we have that g is an automorphism. Similarly, we say that $h : C \to N$ is *right minimal* if for any $g : C \to C$ such that $hg = h$ we have that g is an automorphism.

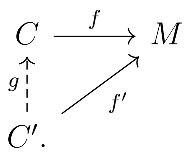
Definition 1.1.2. Let $\mathcal{C} \subseteq \mathcal{D}$ be a class of objects closed under isomorphisms and direct summands. Let $M \in \mathcal{D}$.

We say that a morphism $f \in \mathsf{Hom}_{\mathcal{D}}(M, C)$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-*preenvelope* (or *left approximation*) of M if for any morphism $f' : M \to C'$ with $C' \in \mathcal{C}$ there exists a morphism $g : C \to C'$ such that $f' = gf$:



Equivalently, the map $\mathsf{Hom}_{\mathcal{D}}(f, C') : \mathsf{Hom}_{\mathcal{D}}(C, C') \to \mathsf{Hom}_{\mathcal{D}}(M, C')$ is surjective for each $C' \in \mathcal{C}$. Moreover, we say that f is a $\mathcal{C}$-*envelope* (or *left minimal approximation*) if it is a $\mathcal{C}$-preenvelope and left minimal.

Dually, a morphism $f \in \mathsf{Hom}_{\mathcal{D}}(C, M)$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-*precover* (or *right approximation*) of M if for any morphism $f' : C' \to M$ with $C' \in \mathcal{C}$ there exists a morphism $g : C' \to C$ such that $f' = fg$:



That is, the map $\mathsf{Hom}_{\mathcal{D}}(C', f) : \mathsf{Hom}_{\mathcal{D}}(C', C) \to \mathsf{Hom}_{\mathcal{D}}(C', M)$ is surjective for each $C' \in \mathcal{C}$. Moreover, we say that f is a $\mathcal{C}$-*cover* (or *right minimal approximation*) if it is a $\mathcal{C}$-precover and right minimal.

Definition 1.1.3. A class $\mathcal{C} \subseteq \mathcal{D}$ is called *preenveloping* (or *left functorially finite*) if every object $M \in \mathcal{D}$ admits a $\mathcal{C}$-preenvelope. Moreover, $\mathcal{C}$ is *enveloping* if every object $M \in \mathcal{D}$ admits a $\mathcal{C}$-envelope.

Dually, a class $\mathcal{C} \subseteq \mathcal{D}$ is called *precovering* (or *right functorially finite*) if every object $M \in \mathcal{D}$ admits a $\mathcal{C}$-precover. Moreover, $\mathcal{C}$ is *covering* if every object $M \in \mathcal{D}$ admits a $\mathcal{C}$-cover.

$\mathcal{C}$ is called *functorially finite* if it is both left and right functorially finite.

1.2 Finite-dimensional algebras

Throughout the whole thesis A will always be a finite-dimensional K-algebra (with K field) i.e. an algebra whose underlying vector space has finite-dimension. With $\mathsf{Mod}\,A$ we will denote the category of right A-modules and with $\mathsf{mod}\,A$ its full subcategory of finitely generated modules, that is for any $M \in \mathsf{mod}\,A$ there exist finitely many $m_i \in M$ such that $M = \sum_i m_i A$. Since A is finite-dimensional, we have that all modules in $\mathsf{mod}\,A$ are finite-dimensional. We also denote by $A\,\mathsf{Mod}$ and $A\,\mathsf{mod}$ the respective categories of left A-modules.

In $\mathsf{mod}\,A$ we call $\mathsf{proj}\,A$ and $\mathsf{inj}\,A$ the full subcategories of $\mathsf{mod}\,A$ consisting respectively of projective and injective modules.

We recall that a non-zero module is called *indecomposable* if it cannot be written as direct sum of two non-zero submodules. If $M \in \mathsf{mod}\,A$ is indecomposable then the endomorphism algebra $\mathsf{End}\,M$ is local.

We also recall that, since A is finite-dimensional, then there exists an indecomposable decomposition $A = P_1 \oplus \ldots \oplus P_n$ where P_i are the indecomposable projective right modules. We now state the *Krull–Remak–Schmidt Theorem* that states that under our conditions every module admits a unique decomposition into indecomposable modules. See [ASS06], [AF92].

Theorem 1.2.1 (Krull–Remak–Schmidt)**.** *Let A be a finite-dimensional K-algebra. Then every $M \in \mathsf{mod}\,A$ admits a decomposition*

$$M \cong M_1 \oplus M_2 \oplus \ldots \oplus M_k$$

where M_i are indecomposable modules for all $i = 1, \ldots, k$. The decomposition is unique up to permutation of the indecomposable modules M_i.

We introduce two important functors between categories of modules.

Definition 1.2.2. Let A be a finite-dimensional K-algebra. We define the functor:

$$D := \mathsf{Hom}_K(-, K) : (\mathsf{mod}\,A)^{\mathsf{op}} \longrightarrow A\,\mathsf{mod}\,.$$

For any $M \in \mathsf{mod}\,A$ the left A-module structure of $D(M) = \mathsf{Hom}_K(M, K)$ is given by $(a\varphi)(m) = \varphi(ma)$ for each $a \in A$, $m \in M$ and $\varphi \in \mathsf{Hom}_K(M, K)$. For any $f : M \to N$ in $\mathsf{mod}\,A$ the map $D(f) : D(N) \to D(M)$ is the A-module homomorphism given by $\varphi \mapsto \varphi f$.

D is a duality of categories, called *standard K-duality*. Its quasi-inverse is $D := \mathsf{Hom}_K(-, K) : (A\,\mathsf{mod})^{\mathsf{op}} \to \mathsf{mod}\,A$.

Definition 1.2.3 ([ASS06]). The *Nakayama functor* is defined as:

$$\nu := D\,\mathsf{Hom}_A(-, A) : \mathsf{mod}\,A \to \mathsf{mod}\,A.$$

We have that ν induces an equivalence between $\mathsf{proj}\,A$ and $\mathsf{inj}\,A$.

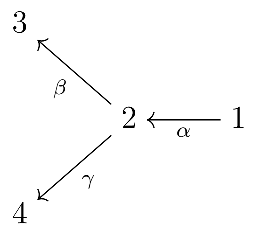
1.3 Quivers

In this section we give a brief introduction to the theory of quivers. We will use them as examples throughout the elaborate. We will only state the main definitions and results. The interested reader may refer to [ASS06], which is our main reference for this section.

Definition 1.3.1. A *quiver* Q is a quadruple (Q_0, Q_1, s, t) where Q_0 and Q_1 are the sets of *points* or *vertices* and *arrows* respectively and s, t are two maps $Q_1 \to Q_0$ that associates to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$ respectively. We will denote an arrow α with source $s(\alpha) = a$ and target $t(\alpha) = b$ simply as $\alpha : a \to b$.

A quiver Q is *finite* if both Q_0 and Q_1 are finite sets. We will only work with finite quivers. A *path* of lenght l from a to b is a composition of arrows $\alpha_1\alpha_2\ldots,\alpha_l$ such that $s(\alpha_1) = a, t(\alpha_l) = b$ and $s(\alpha_{i+1}) = t(\alpha_i)$ for any $i = 1, \ldots, l-1$. For every vertex $a \in Q_0$ we also have the *stationary path* $\varepsilon_a : a \to a$ of length zero. We say that Q is *acyclic* if it contains no cycles, i.e. paths of positive length where the target and source coincides. Q is *connected* if the underlying graph is connected.

Example 1.3.2. The following picture represent an acyclic, connected, finite quiver Q:



where $Q_0 = \{1, 2, 3, 4\}$ and $Q_1 = \{\alpha, \beta, \gamma\}$.

An interesting property of quivers is that every quiver naturally defines an algebra. In the following K will always be an algebraically closed field.

Definition 1.3.3. Let Q be a quiver. The *path algebra* KQ of Q is the K-algebra where the basis of the underlying K-vector space is given by the set of all paths. The product of two path $\alpha_1 \ldots \alpha_l$ and $\beta_1 \ldots \beta_k$ is the path $\alpha_1 \ldots \alpha_l\beta_1 \ldots \beta_k$ if $t(\alpha_l) = s(\beta_1)$ and 0 otherwise.

We have the following properties.

Lemma 1.3.4. *Let Q be a quiver and KQ be its path algebra. Then:*

1) KQ is an associative algebra,

2) KQ has an identity element (which is $\sum_{a \in Q_0} \varepsilon_a$) if and only if Q_0 is finite,

3) KQ is finite-dimensional if and only if Q is finite and acyclic,

4) KQ is hereditary if Q is finite, connected and acyclic.

Example 1.3.5. Consider the quiver of Example 1.3.2. The path algebra KQ has dimension 9 and a base is given by $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \alpha, \beta, \gamma, \alpha\beta, \alpha\gamma\}$. For example we have that $\alpha \circ \varepsilon_2 \circ \beta = \alpha\beta$ and $\varepsilon_1 \gamma = 0$.

Moreover, by the previous lemma, we have that KQ is finite-dimensional and hereditary.

Definition 1.3.6. Let Q be a quiver. We define *relation* in Q an element of KQ of the form:

$$\rho = \sum_{i=1}^{m} \lambda_i \omega_i$$

where $\lambda_i \in K$ are not all zero and ω_i are paths in Q of lenght greater or equal than 2 that have the same source and target.

So relations are just K-linear combinations of paths of length at least two with same source and target. We are only interested in the sets of relations for which the ideal they generate is *admissible*.

Definition 1.3.7. Let Q be a finite quiver. A two-sided ideal $\mathcal{I}$ of KQ is *admissible* if there exists an integer $m \geq 2$ such that:

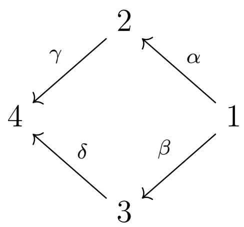
$$R_Q^m \subseteq \mathcal{I} \subseteq R_Q^2$$

where R_Q is the ideal of KQ generated by the arrows of Q.

In this case, we say that Q together with $\mathcal{I}$ is a *bound* quiver. The quotient algebra $KQ/\mathcal{I}$ is called *bound quiver algebra*.

So $\mathcal{I}$ may be induced by a set of relation. In this case, if we have no relation we obviously have $\mathcal{I} = 0$.

Example 1.3.8. An example of a quiver with relation is the following:



with the relation $\gamma\alpha - \delta\beta$. Note that $\mathcal{I} = \langle \gamma\alpha - \delta\beta \rangle$ is an admissible ideal and so this is an example of bound quiver. In the algebra $KQ/\mathcal{I}$ we have $\gamma\alpha = \delta\beta$.

The importance of bound quiver algebras in the theory of finite-dimensional algebras is explained by the following two results.

Proposition 1.3.9. *For a finite quiver Q and an admissible ideal $\mathcal{I}$ we have that the bound quiver algebra $KQ/\mathcal{I}$ is finite-dimensional.*

Recall that an algebra is *connected* if it is not the direct product of two algebras. Moreover, it is called *basic* if, whenever $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents, we have $e_iA \ncong e_jA$ for all $i \neq j$.

Theorem 1.3.10. *Let A be a basic and connected finite-dimensional algebra. Then there exists a quiver Q and an admissible ideal $\mathcal{I}$ of KQ such that $A \cong KQ/\mathcal{I}$.*

We now turn our attention to the modules over the path algebras and we will see how they are connected to quivers.

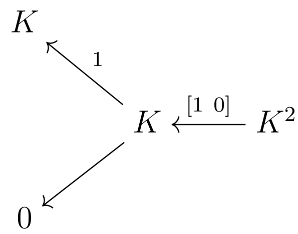
Definition 1.3.11. Let Q be a finite quiver. A *representation* of Q is given by:

1) a K-vector space M_a for each $a \in Q_0$,

2) a K-linear map $\varphi_\alpha : M_a \to M_b$ for each arrow $\alpha : a \to b$ in Q_1.

We will focus on *finite-dimensional representations*, that are representations where all the vector spaces are of finite dimension. They form a category which we denote by $\text{rep}(Q)$.

Example 1.3.12. Consider the quiver of example 1.3.2. An example of representation of it is the following:



By abuse of notation we usually drop the maps and just write the dimension of the vector space associated to every vertex. So we write this representation as:

$$\begin{matrix}1 & & \\ & 1 & 2 \\ 0 & & \end{matrix}$$

Theorem 1.3.13. *Let Q be a finite, connected and acyclic quiver. Then there exists an equivalence of categories $\mathsf{mod}\, KQ \cong \mathsf{rep}(Q)$.*

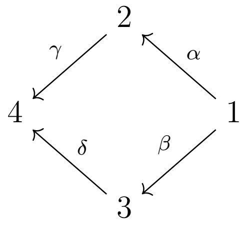
So we can look at the modules over the path algebra of a quiver by looking at the representations of the same quiver. We now give a quick description of the simple, projective and injective modules over $KQ/\mathcal{I}$ by means of representations of the quiver Q bounded by the ideal $\mathcal{I}$. For each vertex $a \in Q_0$ there exists a simple $S(a)$, a projective $P(a)$ and an injective module $I(a)$ associated to it. The simple module is given by $S(a) = (M_b, \varphi_\beta)$ where $\varphi_\beta = 0$ for every $\beta \in Q_1$ and:

$$M_b = \begin{cases} K & \text{if } b = a \\ 0 & \text{if } b \neq a. \end{cases}$$

The projective module $P(a)$ is given by (M_b, φ_b) where M_b is the K-vector space with basis the set of all the $\bar{\omega} = \omega + \mathcal{I}$ with ω path from a to b and, for any $\beta : b \to c$, φ_β is the map given by right multiplication by $\bar{\beta} = \beta + \mathcal{I}$.

Dually the injective module $I(a)$ is given by (M_b, φ_b) where M_b is the K-vector space with basis the set of all the $\bar{\omega} = \omega + \mathcal{I}$ with ω path from b to a and, for any $\beta : b \to c$, φ_β is the map given by left multiplication by $\bar{\beta} = \beta + \mathcal{I}$.

Example 1.3.14. Consider the quiver of Example 1.3.8:



without relations. Then, for example, we have that the simple associated to the vertex 2 is:

$$0\,\begin{smallmatrix}1\\0\end{smallmatrix}\,0$$

whereas $P(1)$ is:

$$2\,\begin{smallmatrix}1\\1\end{smallmatrix}\,1$$

and $I(3)$ is:

$$0\,\begin{smallmatrix}0\\1\end{smallmatrix}\,1$$

Note that if we consider the relation $\gamma\alpha - \delta\beta$, then $P(1)$ becomes:

$$1\,\begin{smallmatrix}1\\1\end{smallmatrix}\,1$$

Furthermore, the ones described before are the only simples, projective and injective modules. Indeed we have a one to one correspondence between Q_0 and each one of the sets of simple modules, of projective modules and of injective modules. Note however that those sets may not be disjoint. For example, in the quiver of Example 1.3.2 we have $S(4) = P(4)$.

We now want to give an outline of *Auslander–Reiten theory*. In particular we are interested in the so called *Auslander–Reiten quiver* that we will largely use in the examples of the next chapters. Firstly we need to define the *Auslander–Reiten translation*.

For any module $M \in \mathsf{mod}\, A$ consider a minimal projective presentation of M:

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

that is, an exact sequence where $p_0 : P_0 \to M$ and $p_1 : P_1 \to \ker p_0$ are projective covers. Applying the A-dual functor

$$(-)^t := \mathsf{Hom}_A(-, A) : \mathsf{mod}\, A \to A\, \mathsf{mod}$$

to it, we obtain the exact sequence:

$$0 \to M^t \to P_0^t \to P_1^t \to \operatorname{coker} p_1^t \to 0$$

We define $\mathrm{Tr}\, M := \operatorname{coker} p_1^t$. Let now $\underline{\mathsf{mod}}A := \mathsf{mod}\, A/\mathcal{P}$, where $\mathcal{P}$ is the ideal of $\mathsf{mod}\, A$ such that $\mathcal{P}(M,N)$ is the subset of all homomorphism between M and N that factors through a projective A-module. Then Tr induces a duality functor $\underline{\mathsf{mod}}A \to A\underline{\mathsf{mod}}$.

Definition 1.3.15. The *Auslander–Reiten translation* is defined as:

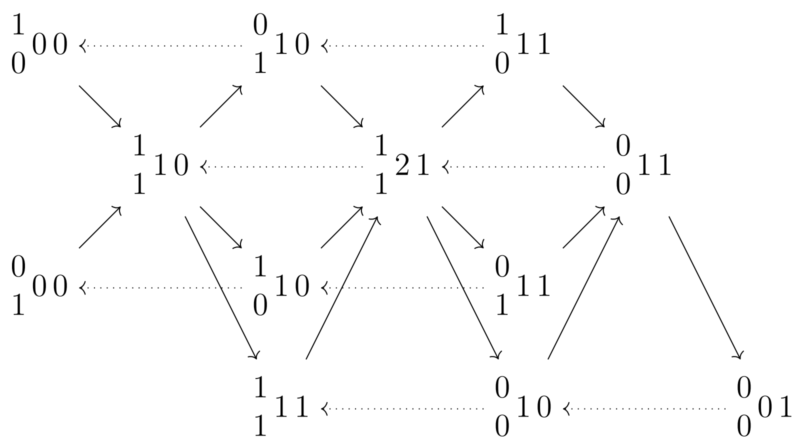
$$\tau := D\mathrm{Tr}$$

with inverse

$$\tau^{-1} = \mathrm{Tr}D.$$

It is interesting to point out that τM is zero if and only if M is projective and dually, $\tau^{-1}N$ is zero if and only if N is injective.

One of the main results of the Auslander–Reiten theory is the Auslander–Reiten quiver. The idea is to describe the category $\mathsf{mod}\, A$ through a quiver. Since according to the Krull–Remak–Schmidt Theorem (1.2.1) every module in $\mathsf{mod}\, A$ can be decomposed into indecomposable modules, it is sufficient to study these modules. Thus, in the Auslander–Reiten quiver, the vertices represent the indecomposable modules and the arrows the so-called *irreducible morphisms*, i.e. morphisms that do not factor through retractions or sections. For example the Auslander–Reiten quiver of the quiver in Example 1.3.2 is:



where the translation τ is indicated by the dotted arrow and goes from right to left. Note that all the injective indecomposable modules are on the right and, as we stated before, they are not images of a translation. Similarly, all the projective modules are on the left.

In each row we have an injective on the right, then all its iterated translations until we reach a projective.

In the quiver we can also see the short exact sequences. Indeed for every indecomposable nonprojective M, there exists an extension between τM and M given by the direct sum of all indecomposable lying in between, with arrows from M and to τM. Note that these short exact sequences never split since the morphisms are irreducible. For example in the quiver above, we have the short exact sequences:

$$0 \to \genfrac{}{}{0pt}{}{1}{1}1\,1 \to \genfrac{}{}{0pt}{}{1}{1}2\,1 \to \genfrac{}{}{0pt}{}{0}{0}1\,0 \to 0$$

$$0 \to \genfrac{}{}{0pt}{}{1}{1}2\,1 \to \genfrac{}{}{0pt}{}{1}{0}1\,1 \oplus \genfrac{}{}{0pt}{}{0}{1}1\,1 \oplus \genfrac{}{}{0pt}{}{0}{0}1\,0 \to \genfrac{}{}{0pt}{}{0}{0}1\,1 \to 0.$$

Not all the short exact sequence are obtained in this way. For example, we also have:

$$0 \to \genfrac{}{}{0pt}{}{1}{0}1\,0 \to \genfrac{}{}{0pt}{}{1}{0}1\,1 \to \genfrac{}{}{0pt}{}{0}{0}0\,1 \to 0.$$

1.4 Torsion Pairs

In this section we will introduce *torsion pairs*, which will be of fundamental importance in the following. Let $\mathcal{A}$ be an abelian category, for example $\mathsf{mod}\, A$ or $\mathsf{Mod}\, A$. The idea is in some sense to generalize in an abelian category what happens when we are dealing with abelian groups, with the notions of torsion subgroups and of torsion-free quotient groups.

Definition 1.4.1. A *torsion pair* is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ such that:

- $\mathsf{Hom}(T, F) = 0$ for each $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
- for each $M \in \mathcal{A}$ there exists a short exact sequence of the form $0 \to T \to M \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

$\mathcal{T}$ is called *torsion class*, whereas $\mathcal{F}$ is the *torsion-free class*. In the following we will refer to the short exact sequence $0 \to T \to M \to F \to 0$ in the definition as the *torsion sequence* of M.

Thus we can see a torsion pair as a pair of subcategories which generate the category $\mathcal{A}$ by extensions and that have no morphisms from one to the other.

We now look at the first properties of torsion pairs. We begin by defining the Hom-orthogonal classes:

$$\mathcal{T}^{\perp} := \{M \in \mathcal{A} \mid \mathsf{Hom}(\mathcal{T}, M) = 0\}$$
$$^{\perp}\mathcal{F} := \{M \in \mathcal{A} \mid \mathsf{Hom}(M, \mathcal{F}) = 0\}$$

Proposition 1.4.2. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Then $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$.*

Proof. We only prove that $\mathcal{F} = \mathcal{T}^{\perp}$, the other equality is dual. By the definition of torsion pair, we already know that $\mathcal{F} \subseteq \mathcal{T}^{\perp}$. To show the other inclusion, let $M \in \mathcal{T}^{\perp}$. By the definition of torsion pair, we have a short exact sequence $0 \to T \to M \to F \to 0$, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Since $M \in \mathcal{T}^{\perp}$ we have no morphisms between $\mathcal{T}$ and M, so the map between T and M in the sequence is the zero map. Thus, by exactness, M is isomorphic to F and so $M \in \mathcal{F}$. □

Proposition 1.4.3. *For each $M \in \mathcal{A}$, the torsion sequence is unique up to (unique) isomorphism.*

Proof. Let us take two torsion sequences ($T_1, T_2 \in \mathcal{T}$ and $F_1, F_2 \in \mathcal{F}$) as in the picture:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1 & \xrightarrow{\alpha_1} & X & \xrightarrow{\beta_1} & F_1 & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & T_2 & \xrightarrow{\alpha_2} & X & \xrightarrow{\beta_2} & F_2 & \longrightarrow & 0 \end{array}$$

Since $\mathsf{Hom}(T_1, F_2) = 0$, we have that the map $\beta_2\alpha_1$ is zero, so that, since T_2 is the kernel of β_2, there exists a unique morphism $u : T_1 \to T_2$ such that $\alpha_2 u = \alpha_1$ by the universal property of the kernel. Similarly, the map $\beta_1\alpha_2$ is zero since it goes from a torsion object to a torsion-free. So, by the universal property of the kernel of β_1 (which is T_1), there exists a unique morphism $v : T_1 \to T_2$ such that $\alpha_2 = \alpha_1 v$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1 & \xrightarrow{\alpha_1} & X & \xrightarrow{\beta_1} & F_1 & \longrightarrow & 0 \\ & & u \downarrow \uparrow v & & \| & & & & \\ 0 & \longrightarrow & T_2 & \xrightarrow{\alpha_2} & X & \xrightarrow{\beta_2} & F_2 & \longrightarrow & 0 \end{array}$$

We now check that u and v are inverse to each other. We have: $\alpha_1 vu = \alpha_2 u = \alpha_1$. Since α_1 is monic, we get $vu = id_{T_1}$. Moreover, $\alpha_2 uv = \alpha_1 v = \alpha_2$, and as before, since α_2 is monic, we get $uv = id_{T_2}$, so that T_1 and T_2 are isomorphic.

Dually, we obtain that F_1 and F_2 are isomorphic using the universal property of the cokernel and the fact that β_1 and β_2 are epic. So the torsion sequence is unique up to isomorphism. □

For any torsion pair $(\mathcal{T}, \mathcal{F})$ we can define a functor

$$t : \mathcal{A} \to \mathcal{T}$$

called *torsion radical*. Indeed, let $X \in \mathcal{A}$, then there exists the torsion sequence:

$$0 \to T \to M \to F \to 0.$$

We define $tM := T$. Note that this is well–defined as a consequence of Proposition 1.4.3. Moreover, for any $f : M \to N$ we have the map $f_T : T_M \to T_N$ given by the universal property of the kernel:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_M & \longrightarrow & X & \longrightarrow & F_M & \longrightarrow & 0 \\ & & f_T \downarrow & & \downarrow f & & & & \\ 0 & \longrightarrow & T_N & \longrightarrow & Y & \longrightarrow & F_N & \longrightarrow & 0. \end{array}$$

We define $tf := f_T$. It is a good definition since f_T is unique as it is given by the universal property of the kernel. It is easy to see that t is indeed a functor. Furthermore it is a subfunctor of the identity functor since the module tM is a submodule of M. Dually we can define the functor $-/t- : \mathcal{A} \to \mathcal{F}$.

So for any $M \in \mathcal{A}$ we can write the torsion sequence as:

$$0 \to tM \to M \to M/tM \to 0$$

We now look at some properties of closure of the torsion and torsion-free classes.

Proposition 1.4.4. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. $\mathcal{T}$ is closed under extensions, quotients, finite sums, direct summands and existing coproducts and colimits. Dually $\mathcal{F}$ is closed under extensions, subobjects, finite sums, direct summands and existing products and limits.*

Proof. We prove the proposition only for $\mathcal{T}$, as it is dual in the case of $\mathcal{F}$.

- Extensions: consider a short exact sequence of the form $0 \to A \to B \to C \to 0$ with $A, C \in \mathcal{T}$. We will show that $B \in \mathcal{T}$. To prove it we use Proposition 1.4.2, so we show that $B \in {}^{\perp}\mathcal{F}$. For each $F \in \mathcal{F}$, apply the contravariant functor $\mathsf{Hom}(-, F)$ to the short exact sequence to obtain:

$$0 \to \mathsf{Hom}(C, F) \to \mathsf{Hom}(B, F) \to \mathsf{Hom}(A, F) \to \mathsf{Ext}^1(C, F) \to \cdots$$

  By hypothesis, $A, C \in \mathcal{T} = {}^{\perp}\mathcal{F}$, so $\mathsf{Hom}(C, F) = \mathsf{Hom}(A, F) = 0$. By exactness we thus have $\mathsf{Hom}(B, F)$, i.e. $B \in {}^{\perp}\mathcal{F} = \mathcal{T}$ as this holds for any $F \in \mathcal{F}$.

- Finite sums: let $A, B \in \mathcal{T}$. We can consider the splitting short exact sequence $0 \to A \to A \oplus B \to B \to 0$. Since $\mathcal{T}$ is closed under extensions, $A \oplus B \in \mathcal{T}$.

- Quotients: let $B \to C \to 0$ be an epimorphism. Applying $\mathsf{Hom}(-, \mathcal{F})$ we get the monomorphism $0 \to \mathsf{Hom}(C, \mathcal{F}) \to \mathsf{Hom}(B, \mathcal{F})$. Since $\mathsf{Hom}(B, \mathcal{F}) = 0$ we have $C \in {}^{\perp}\mathcal{F} = \mathcal{T}$.

- Direct summands: if $A \oplus B \in \mathcal{T}$, then so are both A and B as they are its quotients.

- Coproducts: let $A_i \in \mathcal{T}$ for any i. To show that also $\cup_i A_i \in \mathcal{T}$ we use Proposition 1.4.2. We have the canonical isomorphism:

$$\mathsf{Hom}(\coprod_i A_i, \mathcal{F}) \cong \prod_i \mathsf{Hom}(A_i, \mathcal{F}).$$

  Since $A_i \in \mathcal{T} = {}^{\perp}\mathcal{F}$ for any i, we have $\prod_i \mathsf{Hom}(A_i, \mathcal{F}) = 0$ and so $\mathsf{Hom}(\coprod_i A_i, \mathcal{F}) = 0$. Thus $\coprod_i A_i \in {}^{\perp}\mathcal{F} = \mathcal{T}$.

- Colimits: in an abelian category every colimit is a quotient of a coproduct ([Ste75, IV Proposition 8.4]), so it follows that $\mathcal{T}$ is closed under taking colimits as it is closed under quotients and coproducts.

□

Corollary 1.4.5. *Every simple module is either torsion or torsion-free.*

Proof. Let $0 \to T \to S \to F \to 0$ be the torsion sequence with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. T is then a submodule of S, so either $T = S$ and so $S \in \mathcal{T}$, or $T = 0$ and so $S \in \mathcal{F}$ as it is isomorphic to F. □

It turns out that in $\mathsf{Mod}\, A$ and $\mathsf{mod}\, A$ torsion and torsion-free classes are characterized by the closure properties of Proposition 1.4.4 .

Proposition 1.4.6. *Let $\mathcal{T}$ be a full subcategory of $\mathsf{Mod}\, A$ (or $\mathsf{mod}\, A$) closed under quotients, extensions and existing coproducts. Then $\mathcal{T}$ is a torsion class of some torsion pair. Dually, if $\mathcal{F}$ is closed under subobjects, extensions and existing products, then it is a torsion free class of some torsion pair.*

Proof. We only prove the first statement as the second is dual. For any $M \in \mathsf{Mod}\, A$ (or $\mathsf{mod}\, A$) we define tM to be the trace of $\mathcal{T}$ in M, i.e. :

$$tM := \sum_{\substack{N \subseteq M \\ N \in \mathcal{T}}} N.$$

Note that $tM \in \mathcal{T}$. Indeed, if we are in $\mathsf{Mod}\, A$ this holds since tM is a quotient of $\coprod_{\substack{N\subseteq M \\ N \in \mathcal{T}}} N$ and $\mathcal{T}$ is closed under quotients and coproducts. In the case of $\mathsf{mod}\, A$ put an order on the submodules of M in $\mathcal{T}$ and consider the increasing chain $N_1 \subseteq N_1 + N_2 \subseteq N_1 + N_2 + N_3 \subseteq$ .... It stabilizes to some $\sum_{i=1}^{n} N_i$ since M is Noetherian. Then $tM = \sum_{i=1}^{n} N_i$ is a quotient of $\oplus_{i=1}^{n} N_i$ and so is in $\mathcal{T}$. Thus tM is the largest submodule of M that lies in $\mathcal{T}$. Because of these observations we have that $\mathcal{T} = \{M \mid tM = M\}$. We now show that $t(M/tM) = 0$ for any $M \in \mathsf{Mod}\, A$. Assume that $t(M/tM) = M'/tM$ for some $tM \subseteq M' \subseteq M$. We have the short exact sequence:

$$0 \to tM \to M' \to M'/tM \to 0$$

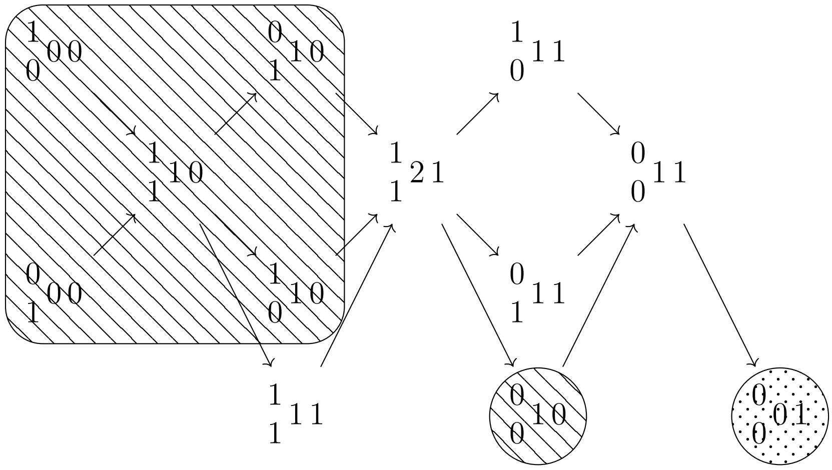
with $tM, M'/tM \in \mathcal{T}$. Since $\mathcal{T}$ is closed under extensions we have: $M' \in \mathcal{T}$ and since it is a submodule of M: $M' \subseteq tM$. Thus $M' = tM$ and so $t(M/tM) = 0$. Define $\mathcal{F} := \{M \mid tM = 0\}$. In particular we have $M/tM \in \mathcal{F}$ for any M. We claim that the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair. For any module M we have the short exact sequence:

$$0 \to tM \to M \to M/tM \to 0$$

where $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. Finally, let $f : T \to F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Since $T \in \mathcal{T}$, $\operatorname{im} f \subseteq F$ belongs to $\mathcal{T}$ as $\mathcal{T}$ is closed under quotients. So $\operatorname{im} f \subseteq tF = 0$ and $\mathsf{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$. Hence we conclude. □

From now on we will be working in $\mathsf{mod}\, A$.

Example 1.4.7. Consider the Auslander–Reiten quiver of the quiver Q of Example 1.3.2. Let $\mathcal{T}$ be the full subcategory of $\mathsf{mod}\, KQ$ given by direct sums of the indecomposable modules in the set with the lines. Similarly let $\mathcal{F}$ be the full subcategory of $\mathsf{mod}\, KQ$ given by direct sums of the indecomposable modules in the dotted set.



Note that $\mathcal{T}$ is closed under quotients and extensions, while $\mathcal{F}$ is closed under subobjects and extension. Then from Proposition 1.4.6 we have that $(\mathcal{T}, \mathcal{F})$ is a torsion pair. We can see for example that the torsion sequence of ${}^{0}_{1}1\,1$ is given by:

$$0 \to {}^{0}_{1}1\,0 \to {}^{0}_{1}1\,1 \to {}^{0}_{0}0\,1 \to 0.$$

Remark 1.4.8. Every torsion class is right functorially finite, while torsion-free classes are left functorially finite. In particular for any module $M \in \operatorname{mod} A$ its right $\mathcal{T}$-approximation is its torsion part while its left $\mathcal{F}$-approximation is its torsion-free part. This comes directly from the universal property of kernel and cokernels.

Definition 1.4.9. A torsion pair $(\mathcal{T}, \mathcal{F})$ is called *splitting* if the torsion sequence splits for any module. It is called *hereditary* if $\mathcal{T}$ is closed under subobjects.

Proposition 1.4.10. *For a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod} A$ the following are equivalent:*

1) $(\mathcal{T}, \mathcal{F})$ is splitting;

2) $\operatorname{Ext}^1_A(F, T) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$;

3) every indecomposable is either in $\mathcal{T}$ or in $\mathcal{F}$.

Proof. The equivalence between (1) and (2) is immediate since $\operatorname{Ext}^1_A(F, T)$ with $F \in \mathcal{F}$ and $T \in \mathcal{T}$ is the group of extensions between $\mathcal{T}$ and $\mathcal{F}$, which is zero if and only if every torsion sequence splits.

To show that (1) implies (3), let M be an indecomposable module. Then, by assumption the torsion sequence $0 \to T_M \to M \to F_M \to 0$ splits, i.e.

$$M = T_M \oplus F_M.$$

Since M is indecomposable, we have either $T_M = M$ and $F_M = 0$ (so M is torsion) or $T_M = 0$ and $F_M = M$ (so M is torsion free).

Conversely, let $M \in \mathsf{mod}\, A$. Decomposing M into indecomposable modules (using Theorem 1.2.1), we can write:

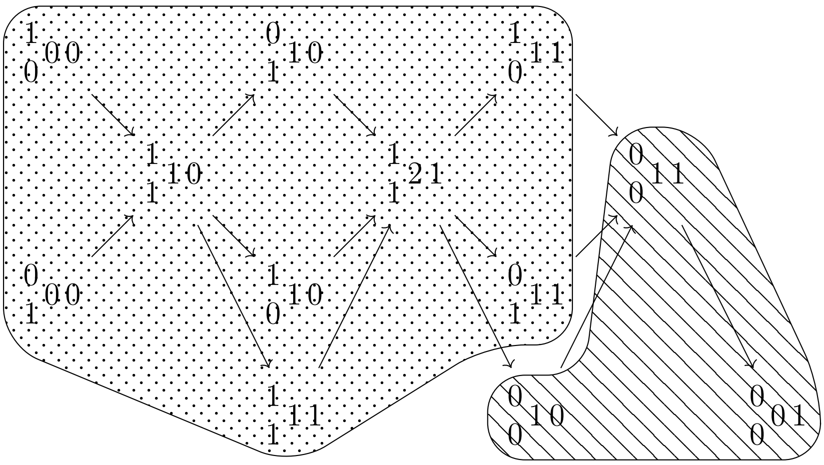
$$M = \bigoplus_{i=1}^{n} M_i = M_T \oplus M_F$$

with M_i indecomposable for each i, $M_T := \bigoplus_{M_i \in \mathcal{T}} M_i$ and $M_F := \bigoplus_{M_i \in \mathcal{F}} M_i$. Note that we have the equality since by assumption every indecomposable is either torsion or torsion free, so each M_i is a direct summand of exactly one between M_T and M_F. Consider now the short exact sequence:

$$0 \to M_T \xrightarrow{\iota_T} M \xrightarrow{\pi_F} M_F \to 0.$$

It is isomorphic to the torsion sequence and it obviously splits. So (3) implies (1) as well. □

Example 1.4.11. The following Auslander–Reiten quiver represents a splitting torsion pair in $\mathsf{mod}\, KQ$ where Q is the quiver of Example 1.3.2. The torsion class is the one denoted with the lines and the torsion-free class is denoted by the dots. Note that we can easily see that it is splitting by the previous proposition, since every indecomposable is either torsion or torsion-free.



Proposition 1.4.12. *Let $\mathcal{C} \subseteq \mathsf{mod}\, A$ a class of modules. Then the pair $({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp})$ is a torsion pair. It is called torsion pair generated by $\mathcal{C}$.*

Proof. Using Proposition 1.4.6, it is easy to see that $\mathcal{C}^{\perp}$ is a torsion-free class. Indeed, consider the monomorphism $X \hookrightarrow Y$ with $Y \in \mathcal{C}^{\perp}$. Since $\mathsf{Hom}_A(\mathcal{C}, -)$ is left exact and $\mathsf{Hom}_A(\mathcal{C}, Y) = 0$ also $\mathsf{Hom}_A(\mathcal{C}, X) = 0$ and so $\mathcal{C}^{\perp}$ is closed under subobjects. Now, consider the short exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in \mathcal{C}^{\perp}$. Applying $\mathsf{Hom}_A(\mathcal{C}, -)$ to it we get $\mathsf{Hom}_A(\mathcal{C}, Y) = 0$ as both $\mathsf{Hom}_A(\mathcal{C}, X)$ and $\mathsf{Hom}_A(\mathcal{C}, Z)$ are zero. So $\mathcal{C}^{\perp}$ is also closed under extensions. Similarly we get that it is closed under direct sums considering the split short exact sequence $0 \to X \to X \oplus Y \to Y \to 0$ with $X, Y \in \mathcal{C}^{\perp}$. By Proposition 1.4.2 we conclude. □

Example 1.4.13. The torsion pair of Example 1.4.7 is the torsion pair generated by $\mathcal{C} = \mathsf{add}\left\{ \begin{smallmatrix}1\\0\end{smallmatrix}0\,0, \begin{smallmatrix}0\\1\end{smallmatrix}0\,0, \begin{smallmatrix}1\\1\end{smallmatrix}1\,0 \right\}$.

Proposition 1.4.14. *If* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair in* $\mathsf{mod}\, A$*, then* $(D\mathcal{F}, D\mathcal{T})$ *is a torsion pair in* $A\,\mathsf{mod}$*.*

Proof. Since D is a duality, we have in particular that it is fully faithful, so that $0 = \mathsf{Hom}(\mathcal{T}, \mathcal{F}) \cong \mathsf{Hom}(D\mathcal{F}, D\mathcal{T})$. Moreover, D is dense, so every $M \in A\,\mathsf{mod}$ is isomorphic to DN for some $N \in \mathsf{mod}\, A$. Consider the torsion sequence of N with respect to $(\mathcal{T}, \mathcal{F})$: $0 \to T \to N \to F \to 0$. Applying D to it we obtain:

$$0 \to DF \to M \to DT \to 0$$

so that $(D\mathcal{F}, D\mathcal{T})$ is a torsion pair. □

1.5 Triangulated Categories

In this section we give a brief presentation of triangulated categories by stating the main results which we will need in the following. In particular, we will focus on the specific examples of homotopic and derived categories which will be the main setting on where we will work. We mainly follow [Har66]. Other useful references are [KS06] and [Nee01].

Definition 1.5.1. Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an autoequivalence. A *triangle* is a sextuple of the form (X, Y, Z, u, v, x) where X, Y, Z are objects of $\mathcal{T}$ and $u : X \to Y, v : Y \to Z, w : Z \to \Sigma X$ are morphisms in $\mathcal{T}$. We will denote triangles in following way:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

A morphism of triangles is a triple of vertical maps that makes all the squares commute.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

We say that $\mathcal{T}$ endowed with a family of triangles is a *triangulated category* if the following axioms are satisfied:

(TR1)
- $X \xrightarrow{\mathsf{id}} X \to 0 \to \Sigma X$ is a triangle.
- Every sextuple isomorphic to a triangle is a triangle.
- Every morphism can be embedded in a triangle. So for any morphism $u : X \to Y$, there exists a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$. Z is called the *cone* of u.

(TR2) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle if and only if $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is a triangle.

(TR3) Given two triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ and two morphism $f : X \to X'$ and $g : Y \to Y'$ there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

(TR4) Consider the triangles $X \xrightarrow{u} Y \xrightarrow{v} U \xrightarrow{w} \Sigma X$, $Y \xrightarrow{a} Z \xrightarrow{b} W \xrightarrow{c} \Sigma Y$ and $X \xrightarrow{au} Z \xrightarrow{d} V \xrightarrow{e} \Sigma X$. There exists $f : U \to V, g : V \to W$ and $h : W \to \Sigma U$ such that $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} \Sigma U$ is a triangle and such that all the squares commutes.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & U & \xrightarrow{w} & \Sigma X \\ \| & & {\scriptstyle a}\downarrow & & {\scriptstyle f}\downarrow & & \| \\ X & \xrightarrow{au} & Z & \xrightarrow{d} & V & \xrightarrow{e} & \Sigma X \\ & & {\scriptstyle b}\downarrow & & {\scriptstyle g}\downarrow & & \\ & & W & = & W & & \\ & & {\scriptstyle c}\downarrow & & {\scriptstyle h}\downarrow & & \\ & & \Sigma Y & \xrightarrow{\Sigma v} & \Sigma U & & \end{array}$$

Note that the morphism h in (TR3) may not be unique. The object Z in (TR1) that completes u to a triangle is called *cone* of u. (TR4) is also called *Octahedral Axiom* and it yields a triangle whenever there exists two morphism that can be composed. The new triangle is given by the cones of the two morphism and their composition.

In the following we will use $X[n]$ to denote $\Sigma^n X$ and call $X[1]$ the shift of X. With $X[-1]$ we will denote the quasi-inverse of $X[1]$.

An important family of functors are the following:

Definition 1.5.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{T} \to \mathcal{A}$ is called *covariant cohomological* if for any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ there exists an exact sequence:

$$\cdots \to H(X[i]) \xrightarrow{H(u[i])} H(Y[i]) \xrightarrow{H(v[i])} H(Z[i]) \xrightarrow{H(w[i])} H(X[i+1]) \to \cdots .$$

If the functor yields the same exact sequence with reversed arrows, it will be called *contravariant cohomological* functor. We will usually denote $H(X[i])$ with $H^i(X)$.

Proposition 1.5.3. *We have the following properties in a triangulated category $\mathcal{T}$:*

1) Two consecutive morphisms in a triangle compose to zero, i.e. for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

we have: $vu = 0$ and $wv = 0$.

2) For any $M \in \mathcal{T}$ the functors:

$$\mathsf{Hom}_{\mathcal{T}}(M,-), \mathsf{Hom}_{\mathcal{T}}(-,M) : \mathcal{T} \to \mathcal{T}$$

are cohomological functors. The former covariant, the latter contravariant.

3) In the axiom (TR3), if f, g are both isomorphism, then also h is an isomorphism.

Proof. 1) Consider the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, we will show that $vu = 0$. By (TR1) there exists the triangle:

$$Z \xrightarrow{\mathsf{id}} Z \to 0 \to Z[1]$$

and by (TR2) we have that:

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is a triangle. Consider the maps $v : Y \to Z$ and $\mathsf{id} : Z \to Z$. By (TR3) there exists a map $X[1] \to 0$ that completes them to a morphism of triangles:

$$\begin{array}{ccccccc} Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \\ \downarrow v & & \downarrow \mathsf{id} & & \downarrow & & \downarrow v[1] \\ Z & \xrightarrow{\mathsf{id}} & Z & \longrightarrow & 0 & \longrightarrow & Z[1]. \end{array}$$

By commutativity we have $0 = v[1]u[1] = (vu)[1] = \Sigma(vu)$. Since Σ is an automorphism we have $vu = 0$. By the same reasoning we can show that $wv = 0$ starting from the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$.

2) Let $M \in \mathcal{T}$. Consider the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$. To prove that $\mathsf{Hom}_{\mathcal{T}}(M,-)$ we have to show that the sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{T}}(M,X) \xrightarrow{\mathsf{Hom}_{\mathcal{T}}(M,u)} \mathsf{Hom}_{\mathcal{T}}(M,Y) \xrightarrow{\mathsf{Hom}_{\mathcal{T}}(M,v)} \mathsf{Hom}_{\mathcal{T}}(M,Z) \to \cdots$$

is exact. It is enough to prove the exactness in $\mathsf{Hom}_{\mathcal{T}}(M,Y)$ as using the other triangles given by (TR2) and the same reasoning we get the exactness in the other degrees.

Since by (1) we have $vu = 0$, for any $f \in \mathsf{Hom}_{\mathcal{T}}(M,X)$, we get:

$$\mathsf{Hom}_{\mathcal{T}}(M,v) \circ \mathsf{Hom}_{\mathcal{T}}(M,u)(f) = vuf = 0$$

so that $\mathrm{im}(\mathsf{Hom}_{\mathcal{T}}(M,u)) \subseteq \ker(\mathsf{Hom}_{\mathcal{T}}(M,v))$.

On the other hand, let $g \in \ker(\mathsf{Hom}_{\mathcal{T}}(M,v)r)$, so that $\mathsf{Hom}_{\mathcal{T}}(M,v)(g) = vg = 0$. For (TR1) there exists the triangle $M \xrightarrow{\mathsf{id}} M \to 0 \to M[1]$. For (TR2) and (TR3) there exists a morphism $f : M \to X$ that completes $g : M \to Y$ and $0 : 0 \to Z$ to a morphism of triangles:

$$\begin{array}{ccccccc} M & \xrightarrow{\mathsf{id}} & M & \longrightarrow & 0 & \longrightarrow & M[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]. \end{array}$$

So by exactness we get: $g = uf = \mathsf{Hom}_{\mathcal{T}}(M, u)(f)$, i.e. $g \in \mathsf{Hom}_{\mathcal{T}}(M, u)$. Hence we have exactness in $\mathsf{Hom}_{\mathcal{T}}(M, Y)$.

3) Consider the morphism of triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow{\scriptstyle \cong} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]. \end{array}$$

We want to show that h is an isomorphism. Apply the covariant homological functor $\mathsf{Hom}_{\mathcal{T}}(Z', -)$ to obtain the commutative diagram with exact columns:

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathsf{Hom}_{\mathcal{T}}(Z', X) & \xrightarrow{f_*} & \mathsf{Hom}_{\mathcal{T}}(Z', X') \\ {\scriptstyle u_*}\downarrow & & \downarrow{\scriptstyle u'_*} \\ \mathsf{Hom}_{\mathcal{T}}(Z', Y) & \xrightarrow{g_*} & \mathsf{Hom}_{\mathcal{T}}(Z', Y') \\ {\scriptstyle v_*}\downarrow & & \downarrow{\scriptstyle v'_*} \\ \mathsf{Hom}_{\mathcal{T}}(Z', Z) & \xrightarrow{h_*} & \mathsf{Hom}_{\mathcal{T}}(Z', Z') \\ {\scriptstyle w_*}\downarrow & & \downarrow{\scriptstyle w'_*} \\ \mathsf{Hom}_{\mathcal{T}}(Z', X[1]) & \xrightarrow{f[1]_*} & \mathsf{Hom}_{\mathcal{T}}(Z', X'[1]) \\ {\scriptstyle u[1]_*}\downarrow & & \downarrow{\scriptstyle u'[1]_*} \\ \mathsf{Hom}_{\mathcal{T}}(Z', Y[1] & \xrightarrow{g[1]_*} & \mathsf{Hom}_{\mathcal{T}}(Z', Y'[1]) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

where f_* denotes $\mathsf{Hom}_{\mathcal{T}}(Z', f)$ and so on. Since f and g are isomorphism, we have that also $f_*, g_*, f[1]_*$ and $g[1]_*$ are isomorphisms. By exactness of the columns and the five lemma we have that h_* is an isomorphism. In particular we have that there exists a morphism $\varphi \in \mathsf{Hom}_{\mathcal{T}}(Z', Z)$ such that $h\phi = h_*(\phi) = \mathsf{id}_{Z'} \in \mathsf{Hom}_{\mathcal{T}}(Z', Z')$.

Similarly, repeating the reasoning using the homological functor $\mathsf{Hom}_{\mathcal{T}}(-, Z)$, we obtain a map $\psi \in \mathsf{Hom}_{\mathcal{T}}(Z', Z)$ such that $\psi h = \mathsf{id}_{Z'}$. So h is an isomorphism and $\phi = \psi$ is its inverse.

$\square$

1.5.1 Homotopy category

For an abelian category $\mathcal{A}$ (e.g. $\mathcal{A} = \mathsf{mod}\, A$) we can define the category of chain complexes $\mathcal{C}(\mathcal{A})$ where the objects are complexes of the form:

$$\mathbf{X} : \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots$$

where $X^i \in \mathcal{A}$ for any $i \in \mathbb{Z}$ and d^i are morphisms in $\mathcal{A}$ such that $d^{i+1}d^i = 0$ for any $i \in \mathbb{Z}$. So in particular we have that $\operatorname{im} d^i \subseteq \ker d^{i+1}$ for each $i \in \mathbb{Z}$. The quotient $\ker d^{i+1} / \operatorname{im} d^i$ is called *cohomology in degree* i and is denoted by $H^i(\mathbf{X})$. A morphism between complexes $f : \mathbf{X} \to \mathbf{Y}$ is a family $\{f_i : X^i \to Y^i\}_{i\in\mathbb{Z}}$ such that every square commutes, i.e. $d^i_{\mathbf{Y}} f^i = f^{i+1} d^i_{\mathbf{X}}$.

We define an equivalence relation on the morphisms of $\mathcal{C}(\mathcal{A})$, the *homotopy equivalence*. We say that $f, g : \mathbf{X} \to \mathbf{Y}$ are homotopic ($f \sim g$) if there exists a family of morphism $h_i : X^i \to Y^{i-1}$ for any $i \in \mathbb{Z}$ such that $f^i - g^i = h^{i+1} d^i_{\mathbf{X}} + d^{i-1}_{\mathbf{Y}} h^i$:

$$\begin{array}{ccccc}
X^{i-1} & \xrightarrow{d^{i-1}_{\mathbf{X}}} & X^i & \xrightarrow{d^i_{\mathbf{X}}} & X^{i+1} \\
 & \swarrow^{h^i} & \downarrow{\scriptstyle f^i - g^i} & \swarrow^{h^{i+1}} & \\
Y^{i-1} & \xrightarrow[d^{i-1}_{\mathbf{Y}}]{} & Y^i & \xrightarrow[d^i_{\mathbf{Y}}]{} & Y^{i+1}.
\end{array}$$

Starting from the category of chain complex we can build the *homotopy category* $\mathcal{K}(\mathcal{A})$. The objects of $K(\mathcal{A})$ are the same of $\mathcal{C}(\mathcal{A})$. For any $\mathbf{X}, \mathbf{Y} \in \mathcal{K}(\mathcal{A})$ the morphisms from $\mathbf{X}$ to $\mathbf{Y}$ are:

$$\mathsf{Hom}_{\mathcal{K}(\mathcal{A})}(\mathbf{X}, \mathbf{Y}) := \mathsf{Hom}_{\mathcal{C}(\mathcal{A})}(\mathbf{X}, \mathbf{Y}) / \sim .$$

So two maps are equal in $\mathcal{K}(\mathcal{A})$ if they are homotopic. We say that a map is *null-homotopic* if it is homotopic to the zero map. We also define the category $\mathcal{K}^+(\mathcal{A})$ as the full subcategory of $\mathcal{K}(\mathcal{A})$ of bounded below complexes, i.e. of complexes $\mathbf{X}$ such that $X^n = 0$ for all $n > \bar{n}$ for some $\bar{n}$. Similarly we define $\mathcal{K}^-(\mathcal{A})$ as the full subcategory of $\mathcal{K}(\mathcal{A})$ of bounded above complexes. Finally the *bounded* homotopy category is $\mathcal{K}^b(\mathcal{A}) := \mathcal{K}^+(\mathcal{A}) \cap \mathcal{K}^-(\mathcal{A})$.

We have the following important result ([KS06, Theorem 11.2.6]):

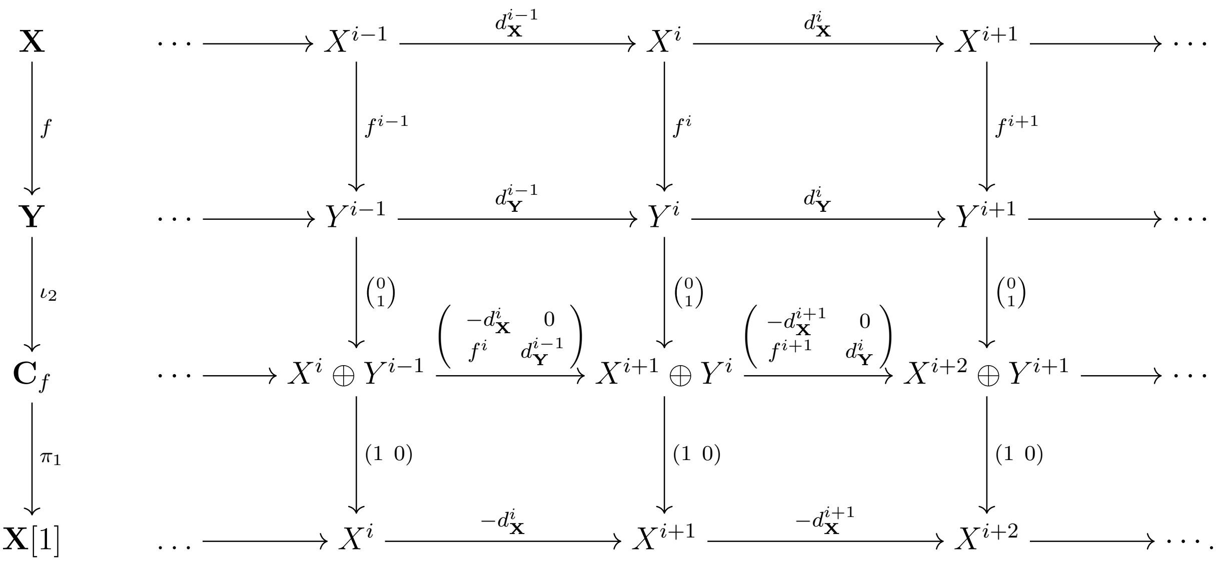
Proposition 1.5.4. *$\mathcal{K}(\mathcal{A})$ is a triangulated category.*

Here we only show how we define the translation and the cone of a morphism. First of all we need to define the shift functor. For any complex $\mathbf{X}$ we define the complex $\Sigma\mathbf{X} = \mathbf{X}[1]$ as $(\Sigma\mathbf{X})^i := \mathbf{X}^{i+1}$ and $d^i_{\Sigma\mathbf{X}} := -d^{i+1}_{\mathbf{X}}$:

$$\mathbf{X} : \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots$$

$$\mathbf{X}[1] : \cdots \to X^0 \xrightarrow{-d^0} X^1 \xrightarrow{-d^1} X^2 \to \cdots .$$

Given a morphism $f : \mathbf{X} \to \mathbf{Y}$ we define the cone $\mathbf{C}_f := \mathbf{X}[1] \oplus \mathbf{Y}$ with differential $\begin{pmatrix} -d_{\mathbf{X}} & 0 \\ f & d_{\mathbf{Y}} \end{pmatrix}$:



It is easy to check that the diagram is commutative.

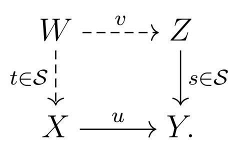
1.5.2 Derived category

Starting from the homotopy category, we now construct the so called *derived category* that will be the main setting on which we will work in Chapter 3. It is constructed using the process of *localization of categories*. As a reference for derived categories, see [GM03].

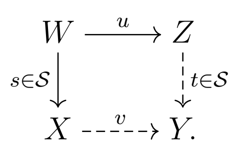
Definition 1.5.5. Let $\mathcal{C}$ be a category and $\mathcal{S}$ a class of morphisms in $\mathcal{C}$. We say that $\mathcal{S}$ is a *localizing class* if the following holds:

(LC1) $\mathcal{S}$ is closed under composition and $\mathsf{id}_X \in \mathcal{S}$ for any object X of $\mathcal{C}$.

(LC2) For any morphisms $u : X \to Y$, $s : Z \to Y$ with $s \in \mathcal{S}$, there exist morphisms $v : W \to Z$, $t : W \to X$ with $t \in \mathcal{S}$ such that the following diagram commutes:



Dually, for any morphisms $u : W \to Z$, $s : W \to X$ with $s \in \mathcal{S}$, there exist morphisms $v : X \to Y$, $t : Z \to Y$ with $t \in \mathcal{S}$ such that the following diagram commutes:



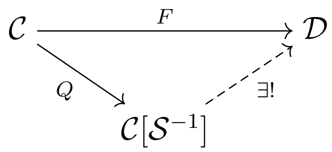
(LC3) For any morphisms $f, g : X \to Y$, the following are equivalent:

a) there exists $s : Y \to Y'$ such that $sf = sg$

b) there exists $t : X' \to X$ such that $ft = gt$

Definition 1.5.6. Let $\mathcal{C}$ be a category and $\mathcal{S}$ a localizing class. The *localization of* $\mathcal{C}$ *at* $\mathcal{S}$ is a category $\mathcal{C}[\mathcal{S}^{-1}]$ together with a functor $Q : \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ such that:

- $Q(s)$ is an isomorphism for all $s \in \mathcal{S}$;
- every functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$ factors uniquely through Q:



Remark 1.5.7. The construction in [GM03] may not give a category with Hom-sets as opposed to Hom-classes. In our case, this will never happen (Remark 1.5.12).

So the localized category $\mathcal{C}[\mathcal{S}^{-1}]$ is the smallest category where all the morphisms in $\mathcal{S}$ are invertible. Note that generally there is no need for $\mathcal{S}$ to be a localizing class, this assumption is made to have a nice description of the localized category.

The idea behind the derived category is that we are only interested in the cohomologies of our complexes, so to build the derived category of the abelian category $\mathcal{A}$ we localize the homotopy category at *quasi-isomorphism*, making them invertible.

Definition 1.5.8. A morphism $f \in \mathsf{Hom}_{\mathcal{K}(\mathcal{A})}(\mathbf{X}, \mathbf{Y})$ is called *quasi-isomorphism* if the induced map in cohomology $H^n(f) : H^n(\mathbf{X}) \to H^n(\mathbf{Y})$ is an isomorphism. We denote with $\mathfrak{quiso}$ the class of all quasi-isomorphisms.

It turns out that $\mathfrak{quiso}$ forms a localizing class ([GM03, Theorem III.4.4]), so it makes sense to consider the localization at $\mathfrak{quiso}$.

Definition 1.5.9. We define the *derived category* of $\mathcal{A}$ as the category:

$$\mathcal{D}(\mathcal{A}) := \mathcal{K}(\mathcal{A})[\mathfrak{quiso}^{-1}].$$

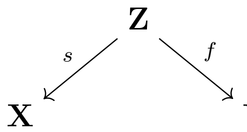
Similarly we define $\mathcal{D}^*(\mathcal{A}) := \mathcal{K}^*(\mathcal{A})[\mathfrak{quiso}^{-1}]$ for $* = +, -, b$.

Example 1.5.10. Projective and injective resolutions of a module are quasi isomorphic to the module. So if $\mathbf{P} : P^{-n} \to \cdots \to P^0$ is a projective resolution of M then the two complexes:

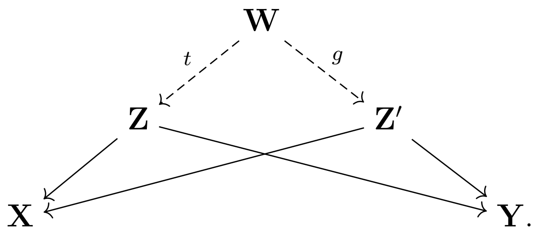
$$\cdots \to 0 \to P^{-n} \to \cdots \to P^0 \to 0 \to \cdots$$
$$\cdots \to 0 \to M \to 0 \to \cdots$$

are isomorphic in $\mathcal{D}(\mathsf{mod}\, A)$.

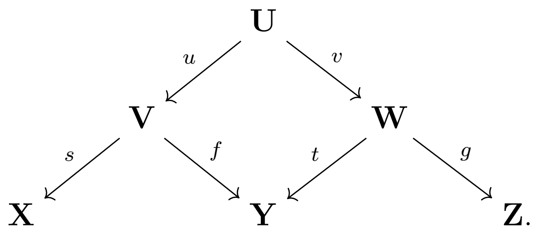
So the objects of $\mathcal{D}(\mathcal{A})$ are the same of $\mathcal{C}(\mathcal{A})$. Given two objects $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{D}(\mathcal{A})$ a morphism between them in the derived category is an equivalence class of the so-called *roofs*, that are diagrams of the form:



where $s \in \mathfrak{quiso}$ and f is a morphism in $\mathcal{K}(\mathcal{A})$. Two such roofs are equivalent if there exists a quasi-isomorphism t and a morphism g such that the two triangles in the picture commutes:



Given two roofs $\mathbf{X} \overset{s}{\leftarrow} \mathbf{V} \overset{f}{\rightarrow} \mathbf{Y}$ and $\mathbf{Y} \overset{t}{\leftarrow} \mathbf{W} \overset{g}{\rightarrow} \mathbf{Z}$, we define their composition as the big roof $\mathbf{X} \overset{su}{\longleftarrow} \mathbf{U} \overset{gv}{\longrightarrow} \mathbf{Z}$ where v and the quasi-isomorphism u exists by (LC2):



With this definition of composition it is easy to verify that it is associative and that the identity is the roof $\mathbf{X} \overset{1}{\leftarrow} \mathbf{X} \overset{1}{\rightarrow} \mathbf{X}$.

Moreover we have that $\mathcal{D}(\mathcal{A})$ is an additive category and that it is triangulated, with the same shift functor of $\mathcal{K}(\mathcal{A})$.

For the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ we have the following fundamental results that will be very useful later on.

Proposition 1.5.11 ([Har66, Proposition 4.7])**.** *Let $\mathcal{P}$ be the full subcategory of projective objects in $\mathcal{A}$ and $\mathcal{I}$ the full subcategory of injective objects in $\mathcal{A}$. Then the natural functors:*

$$p : \mathcal{K}^{-}(\mathcal{P}) \longrightarrow \mathcal{D}^{-}(\mathcal{A})$$

$$i : \mathcal{K}^{+}(\mathcal{I}) \longrightarrow \mathcal{D}^{+}(\mathcal{A})$$

are fully faithful.

Moreover, if $\mathcal{A}$ has enough projectives, then p is an equivalence of categories and dually, if $\mathcal{A}$ has enough injectives, then i is an equivalence of categories.

Remark 1.5.12. The previous proposition shows that $\mathcal{D}^{-}$ and $\mathcal{D}^{+}$ are categories with Hom-sets.

We will largely use the previous proposition when we will work in $\mathsf{mod}\, A$. In particular we will use that all the maps in $\mathcal{D}^b(\mathsf{mod}\, A)$ are simply maps in $K^b(\mathsf{proj}\, A)$. We will usually denote by $\mathcal{D}(A)$ the derived category of $\mathsf{mod}\, A$.

Proposition 1.5.13. *Consider an abelian category $\mathcal{A}$ and its derived category $\mathcal{D}(\mathcal{A})$. Then:*

- *The functor $Q : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ that sends every object X to the complex concentrated in zero with X in degree zero is fully faithful. Moreover, Q is an equivalence of categories between $\mathcal{A}$ and the full subcategory of $\mathcal{D}(\mathcal{A})$ that has non-zero cohomology only in degree zero.*
- *For any $X, Y \in \mathcal{A}$. We have the isomorphism:*
$$\mathsf{Ext}^n_{\mathcal{A}}(X, Y) \cong \mathsf{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$$
for any $n \in \mathbb{Z}$.

1.6 t-Structures

As we said, torsion pairs can be defined in abelian categories. They can also be generalized to triangulated categories as in [IY08]. Here we are interested in a special class of torsion pairs, the so called *t-structures*. We will follow [BBD82].

Definition 1.6.1. Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is a couple of full subcategories $(\mathcal{U}, \mathcal{V})$ such that:

1) $\mathcal{U}[1] \subseteq \mathcal{U}$ and $\mathcal{V}[-1] \subseteq \mathcal{V}$,
2) $\mathsf{Hom}(\mathcal{U}, \mathcal{V}[-1]) = 0$,
3) $\mathcal{U} * \mathcal{V}[-1] := \{X \in \mathcal{D} \mid \text{there exists a triangle } U \to X \to V[-1] \to U[1] \text{ with } U \in \mathcal{U}, V \in \mathcal{V}\} = \mathcal{D}$.

$\mathcal{U}$ is called *aisle* of the t-structure, whereas $\mathcal{V}$ is the *coaisle*.

We can see that 2) and 3) are similar to the condition for a torsion pair. In this sense we can look at t-structures as the torsion pairs with the torsion class closed under positive shifts and the torsion free class closed under negative shifts. To know more about this refer to [IY08]. Similarly to the case of torsion pairs we have that if $(\mathcal{U}, \mathcal{V})$ is a t-structure, then we have that $\mathcal{U} = {}^{\perp}\mathcal{V}$ and $\mathcal{U} = \mathcal{V}^{\perp}$.

A fundamental example of a t-structure is the following.

Example 1.6.2. Let $\mathcal{D}(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. Define:
$$\mathcal{D}^{\leq 0} := \{\mathbf{X} \in \mathcal{D}(\mathcal{A}) : H^i(\mathbf{X}) = 0 \ \forall i > 0\}$$
$$\mathcal{D}^{\geq 0} := \{\mathbf{X} \in \mathcal{D}(\mathcal{A}) : H^i(\mathbf{X}) = 0 \ \forall i < 0\}.$$
The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure in $\mathcal{D}(\mathcal{A})$ and it is called *standard t-structure* in $\mathcal{D}(\mathcal{A})$. For example, to verify (1), observe that $\mathcal{D}^{\leq 0}[1] = \{X[1] \in \mathcal{D}(\mathcal{A}) : H^i(X[1]) = 0 \ \forall i > 0\} = \{X[1] \in \mathcal{D}(\mathcal{A}) : H^{i+1}(X) = 0 \ \forall i > 0\} = \{X[1] \in \mathcal{D}(\mathcal{A}) : H^i(X) = 0 \ \forall i > -1\} \subseteq \mathcal{D}^{\leq 0}$.

Similarly to what we did in the example we can define:

$$\mathcal{D}^{\leq n} := \{X \in \mathcal{D}(\mathcal{A}) : H^i(X) = 0 \ \forall i > n\}$$

$$\mathcal{D}^{\geq n} := \{X \in \mathcal{D}(\mathcal{A}) : H^i(X) = 0 \ \forall i < n\}.$$

Using the same reasoning of the previous example, we can actually see these are just the shifts of the standard t-structure:

$$\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[n] \qquad \mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[n]$$

We have that $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is a t-structure. This follows from a more general statement: *shifts of t-structures are t-structures.* To show this, let us take $(\mathcal{U}, \mathcal{V})$ t-structure in $\mathcal{D}$ triangulated category. We will show that $(\mathcal{U}[i], \mathcal{V}[i])$ is a t-structure in $\mathcal{D}$ for any $i \in \mathbb{Z}$. By definition of t-structure we have $\mathcal{U}[1] \subseteq \mathcal{U}$, by applying the shift i times to both sides we get $\mathcal{U}[i+1] \subseteq \mathcal{U}[i]$ and similarly for $\mathcal{V}$. Next, consider $f \in \mathsf{Hom}(\mathcal{U}[i], \mathcal{V}[i-1])$ (for any $i \in \mathbb{Z}$), then $f[-i] \in \mathsf{Hom}(\mathcal{U}, \mathcal{V}[-1]) = 0$, so that $f = 0$ since the shift is an autoequivalence. Finally, to get the canonical triangle of any $X \in \mathcal{D}$, consider $X[-i]$. Since $(\mathcal{U}, \mathcal{V})$ is a t-structure, there exists the canonical triangle $U \to X[-i] \to V[-1] \to U[1]$ which is just the triangle $U[i] \to X \to V[i-1] \to U[i+1]$.

Proposition 1.6.3. *For any $X \in \mathcal{D}$ the triangle given by the t-structure is unique up to isomorphism.*

Proof. Let us consider two triangles as in the picture:

$$\begin{array}{ccccccc}
U & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & V[-1] & \longrightarrow & U[1] \\
 & & \| & & & & \\
U' & \xrightarrow{\alpha'} & X & \xrightarrow{\beta'} & V'[-1] & \longrightarrow & U'[1]
\end{array}$$

with $U, U' \in \mathcal{U}$ and $V, V' \in \mathcal{V}$.

Applying $\mathsf{Hom}(U, -)$ to the second triangle yields the exact sequence:

$$0 = \mathsf{Hom}(U, V'[-2]) \to \mathsf{Hom}(U, U') \xrightarrow{\mathsf{Hom}(U,\alpha')} \mathsf{Hom}(U, X) \to \mathsf{Hom}(U, V'[-1]) = 0$$

since $V'[-2], V'[-1] \in \mathcal{V}[-1]$. Thus $\mathsf{Hom}(U, \alpha')$ is an isomorphism, so there exists a unique morphism $f \in \mathsf{Hom}(U, U')$ such that $\mathsf{Hom}(U, \alpha')(f) = a$, i.e. $\alpha' f = \alpha$. Similarly, applying $\mathsf{Hom}(U', -)$ to the first triangle we get the isomorphism $\mathsf{Hom}(U', U) \xrightarrow{\mathsf{Hom}(U',\alpha)} \mathsf{Hom}(U', X)$, so that there exists a unique morphism $g \in \mathsf{Hom}(U', U)$ such that $\alpha g = \alpha'$.

$$\begin{array}{ccccccc}
U & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & V[-1] & \longrightarrow & U[1] \\
{\scriptstyle f}\downarrow\uparrow{\scriptstyle g} & & \| & & & & \\
U' & \xrightarrow{\alpha'} & X & \xrightarrow{\beta'} & V'[-1] & \longrightarrow & U'[1]
\end{array}$$

We now check that f and g are inverse to each other. For the same reason as before, applying $\mathsf{Hom}(U, -)$ to the first triangle yields the isomorphism $\mathsf{Hom}(U, U) \xrightarrow{\mathsf{Hom}(U,\alpha)}$

$\mathsf{Hom}(U, X)$. We have $\mathsf{Hom}(U, \alpha)(gf) = \alpha g f = \alpha' f \alpha = \mathsf{Hom}(U, \alpha)(1_U)$, so since $\mathsf{Hom}(U, \alpha)$ is iso: $gf = 1_U$. Similarly, applying $\mathsf{Hom}(U', -)$ to the second triangle yields $fg = 1'_U$, so U and U' are isomorphic.

Finally, V and V' are isomorphic since the first two terms of the triangles are.

□

Given a t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathcal{D}$ triangulated category, we can define the *truncation* functors:

$$\tau^{\leq 0} : \mathcal{D} \to \mathcal{U}$$
$$\tau^{\geq 0} : \mathcal{D} \to \mathcal{V}$$

in the following way. For any $X \in \mathcal{D}$ we have the triangle given by the t-structure:

$$U \to X \to V[-1] \to U[1]$$

and we define $\tau^{\leq 0}(X) := U$ and $\tau^{\geq 0}(X) := V$. Then, for any $f : X \to Y$, using the same reasoning as in the previous proof, we obtain the commutative diagram:

$$\begin{array}{ccccccc} U_X & \longrightarrow & X & \longrightarrow & V_X[-1] & \longrightarrow & U_X[1] \\ {\scriptstyle f_U}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f_V}\downarrow & & \\ U_Y & \longrightarrow & Y & \longrightarrow & V_Y[-1] & \longrightarrow & U_Y[1] \end{array}$$

with $U_X, U_Y \in \mathcal{U}$ and $V_X, V_Y \in \mathcal{V}$. We define $\tau^{\leq 0}(f) := f_U$ and $\tau^{\geq 0}(f) := f_V[1]$. It is easy to see that with this definition $\tau^{\leq 0}$ and $\tau^{\geq 0}$ are indeed functors. Note in particular that they are well defined by Proposition 1.6.3.

In the same way we also define:

$$\tau^{\leq i} : \mathcal{D} \to \mathcal{U}[i]$$
$$\tau^{\geq i} : \mathcal{D} \to \mathcal{V}[i]$$

for any $i \in \mathbb{Z}$. Note that for any $X \in \mathcal{D}$ and $i \in \mathbb{Z}$ we have the triangle:

$$\tau^{\leq i} X \to X \to \tau^{\geq i+1} X \to \tau^{\leq i} X[1].$$

Example 1.6.4. Let $\mathcal{D} = \mathcal{D}(\mathcal{A})$ for some abelian category $\mathcal{A}$ and consider the standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. For any complex $\mathbf{X} \in \mathcal{D}(\mathcal{A})$:

$$\mathbf{X} : \cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots$$

we have:

$$\tau^{\leq 0}(\mathbf{X}) : \cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} \ker d^o \to 0 \to \cdots$$

and

$$\tau^{\geq 0}(\mathbf{X}) : \cdots \to 0 \to \frac{X^{-1}}{\ker d^{-1}} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots .$$

In particular we have the short exact sequence:

$$0 \to \tau^{\leq 0}(\mathbf{X}) \to \mathbf{X} \to \tau^{\geq 1}(\mathbf{X}) \to 0$$

which yields the triangle:

$$\tau^{\leq 0}(\mathbf{X}) \to \mathbf{X} \to \tau^{\geq 1}(\mathbf{X}) \to \tau^{\leq 0}(\mathbf{X})[1].$$

Proposition 1.6.5. *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathcal{D}$ triangulated category. We have the adjoint pairs:*

$$\iota_{\mathcal{U}} : \mathcal{U} \leftrightarrows \mathcal{D} : \tau^{\leq 0}$$
$$\tau^{\geq 0} : \mathcal{D} \leftrightarrows \mathcal{V} : \iota_{\mathcal{V}}$$

where $\iota_{\mathcal{U}}$ and $\iota_{\mathcal{V}}$ are the inclusions of $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{D}$ respectively.

Proof. Refer to [BBD82, Proposition 1.3.3] and [KV88, Proposition 1.1] □

Definition 1.6.6. Given a t-structure $(\mathcal{U}, \mathcal{V})$, we define the *heart* of the t-structure as

$$\mathcal{H} := \mathcal{U} \cap \mathcal{V}.$$

We have the following fundamental result about the heart of a t-structures from [BBD82]:

Theorem 1.6.7. *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure on a triangulated category $\mathcal{D}$ and let $\mathcal{H}$ be its heart. Then $\mathcal{H}$ is an abelian category. Moreover short exact sequences in $\mathcal{H}$ are triangles in $\mathcal{D}$ with all the objects in $\mathcal{H}$.*

Definition 1.6.8. We define the *n-th cohomology* with respect to the t-structure as the functor:

$$H^n : \mathcal{D} \longrightarrow \mathcal{H}$$
$$X \mapsto \tau^{\leq n}\tau^{\geq n}(X).$$

Note that we have $\tau^{\leq n}\tau^{\geq n} = \tau^{\geq n}\tau^{\leq n}$ up to natural equivalence, so we can equivalently define:

$$H^n(X) := \tau^{\geq n}\tau^{\leq n}(X).$$

Note also that we have $H^n(X) = H^0(X[n])$.

Example 1.6.9. In the case of $\mathcal{D} = \mathcal{D}(\mathcal{A})$ with $\mathcal{A}$ abelian category and the standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ we have that the cohomology functors with respect to the t-structure coincide with the standard notion of cohomology. Indeed we have that:

$$H^n(\mathbf{X}) = \cdots \to 0 \to \frac{X^{n-1}}{\ker d^{n-1}} \xrightarrow{d^{-1}} X^n \to 0 \to \cdots$$

which has cohomology only in degree n and equal to $\ker d^n / \operatorname{im} d^{n-1}$. Thus $H^n(\mathbf{X})$ is isomorphic to the complex with the classical cohomology concentrated in degree n.

1.6.1 HRS-tilting

In this section the HRS-tilting process, introduced by Happel, Reiten and Smalø in [HRS96]. It is a process to construct a new t-structure starting from an existing t-structure and a torsion pair in its heart.

In [HRS96, Proposition 2.1, Corollary 2.2] this construction was made for the derived category of an abelian category with the standard t-structure. Here we present a generalized version with an arbitrary triangulated category and t-structure as in [Pol07] and [FMT16, Proposition 1.8]. The proof follows [Pav22].

Proposition 1.6.10. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{D} = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t-structure on $\mathcal{C}$ and denote the shifts $\mathcal{D}^{\leq 0}[n]$ and $\mathcal{D}^{\geq 0}[n]$ respectively by $\mathcal{D}^{\leq -n}$ and $\mathcal{D}^{\geq -n}$. Let $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ be its heart and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{H}$. Then there exists a t-structure $(\mathcal{D}_{\mathbf{t}}^{\leq 0}, \mathcal{D}_{\mathbf{t}}^{\geq 0})$ given by:*

$$\mathcal{D}_{\mathbf{t}}^{\leq 0} = \left\{ X \in \mathcal{D}^{\leq 0} : H_{\mathcal{D}}^{0}(X) \in \mathcal{T} \right\} = \mathcal{D}^{\leq 0}[1] * \mathcal{T}$$

$$\mathcal{D}_{\mathbf{t}}^{\geq 0} = \left\{ X \in \mathcal{D}^{\geq -1} : H_{\mathcal{D}}^{-1}(X) \in \mathcal{F} \right\} = \mathcal{F}[1] * \mathcal{D}^{\geq 0}$$

where $H_{\mathcal{D}}^{i}$ denote the i-th cohomology with respect to the t-structure $\mathcal{D}$. This t-structure is called HRS-tilt of $\mathcal{D}$ with respect to $\mathbf{t}$. The heart of this t-structure is given by:

$$\mathcal{H}_{\mathbf{t}} = \left\{ X \in \mathcal{H} : H_{\mathcal{D}}^{0}(X) \in \mathcal{T},\ H_{\mathcal{D}}^{-1}(X) \in \mathcal{F} \right\}.$$

Moreover, we have that $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{H}_{\mathbf{t}}$.

Proof. We obviously have $\mathcal{D}_{\mathbf{t}}^{\leq 0}[1] \subseteq \mathcal{D}_{\mathbf{t}}^{\leq 0}$ and $\mathcal{D}_{\mathbf{t}}^{\geq 0}[-1] \subseteq \mathcal{D}_{\mathbf{t}}^{\geq 0}$. Consider now $\mathsf{Hom}_{\mathcal{D}}(X, Y)$ with $X \in \mathcal{D}_{\mathbf{t}}^{\leq 0}$ and $Y \in \mathcal{D}_{\mathbf{t}}^{\geq 0}[-1]$. Using Proposition 1.6.5 we have:

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(X, Y) &= \mathsf{Hom}_{\mathcal{D}}(iX, Y) = \mathsf{Hom}_{\mathcal{D}}(X, \tau^{\leq 0} Y) = \\ &= \mathsf{Hom}_{\mathcal{D}}(X, i\tau^{\leq 0} Y) = \mathsf{Hom}_{\mathcal{H}}(H_{\mathcal{D}}^{0}(X), H_{\mathcal{D}}^{0}(Y)) = 0 \end{aligned}$$

since $H_{\mathcal{D}}^{0}(X) \in \mathcal{T}$ and $H_{\mathcal{D}}^{0}(Y) = H_{\mathcal{D}}^{-1}(Y[1]) \in \mathcal{F}$ as $Y[1] \in \mathcal{D}_{\mathbf{t}}^{\geq 0}$.

We now prove that $\mathcal{D} = \mathcal{D}_{\mathbf{t}}^{\leq 0} * \mathcal{D}_{\mathbf{t}}^{\geq 0}$. Let $X \in \mathcal{D}$. We have the canonical triangle of $\tau_{\mathcal{D}}^{\leq 1} X$:

$$\tau_{\mathcal{D}}^{\leq -1} X \to \tau_{\mathcal{D}}^{\leq 0} X \to H_{\mathcal{D}}^{0}(X) \to \tau_{\mathcal{D}}^{\leq -1} X[1].$$

Moreover, $H_{\mathcal{D}}^{0}(X) \in \mathcal{H}$, so the torsion sequence given by $(\mathcal{T}, \mathcal{F})$ yields the triangle:

$$tH_{\mathcal{D}}^{0}(X) \to H_{\mathcal{D}}^{0}(X) \to fH_{\mathcal{D}}^{0}(X) \to (tH_{\mathcal{D}}^{0}(X))[1].$$

Considering these two triangles, by the octahedral axiom we have that there exists U and the triangle given by the third column:

$$\begin{array}{ccccccc}
tH_{\mathcal{D}}^{0}(X) & \longrightarrow & H_{\mathcal{D}}^{0}(X) & \longrightarrow & fH_{\mathcal{D}}^{0}(X) & \longrightarrow & (tH_{\mathcal{D}}^{0}(X))[1] \\
\| & & \downarrow & & \downarrow & & \| \\
tH_{\mathcal{D}}^{0}(X) & \longrightarrow & (\tau_{\mathcal{D}}^{\leq -1} X)[1] & \longrightarrow & U[1] & \longrightarrow & (tH_{\mathcal{D}}^{0}(X))[1] \\
 & & \downarrow & & \downarrow & & \\
 & & (\tau_{\mathcal{D}}^{\leq 0} X)[1] & = & (\tau_{\mathcal{D}}^{\leq 0} X)[1] & & \\
 & & \downarrow & & \downarrow & & \\
 & & (H_{\mathcal{D}}^{0}(X))[1] & \longrightarrow & (fH_{\mathcal{D}}^{0}(X))[1] & &
\end{array}$$

Since $tH_{\mathcal{D}}^{0}(X) \in \mathcal{T}$ and $\tau_{\mathcal{D}}^{\leq -1} X \in \mathcal{D}^{\leq 0}[1]$, from the second row we get that $U \in \mathcal{D}_{\mathbf{t}}^{\leq 0}$.

Now, consider the diagram given by the octahedral axiom, where the first row is the shift of the triangle in the third column of the previous diagram and the second column is the canonical triangle of X given by the t-structure $\mathcal{D}$:

$$
\begin{array}{ccccccc}
U & \longrightarrow & \tau_{\mathcal{D}}^{\leq 0} X & \longrightarrow & f H^0_{\mathcal{D}}(X) & \longrightarrow & U[1] \\
\| & & \downarrow & & \downarrow & & \| \\
U & \longrightarrow & X & \longrightarrow & V & \longrightarrow & U[1] \\
 & & \downarrow & & \downarrow & & \\
 & & \tau_{\mathcal{D}}^{\geq 1} X & = & \tau_{\mathcal{D}}^{\geq 1} X & & \\
 & & \downarrow & & \downarrow & & \\
 & & (\tau_{\mathcal{D}}^{\leq 0} X)[1] & \longrightarrow & (f H^0_{\mathcal{D}}(X))[1] & &
\end{array}
$$

As before, the third column shows that $V \in \mathcal{D}_{\mathbf{t}}^{\geq 0}$. So the second row of this last diagram shows that $\mathcal{D} = \mathcal{D}_{\mathbf{t}}^{\leq 0} * \mathcal{D}_{\mathbf{t}}^{\geq 0}$. Thus $(\mathcal{D}_{\mathbf{t}}^{\leq 0}, \mathcal{D}_{\mathbf{t}}^{\geq 0})$ is a t-structure.

The description of $\mathcal{H}$ follows directly from the definition of heart. Moreover, we have:

$$\mathcal{H}_{\mathbf{t}} = \mathcal{D}_{\mathbf{t}}^{\leq 0} \cap \mathcal{D}_{\mathbf{t}}^{\geq 0} = (\mathcal{D}^{\leq 0}[1] * \mathcal{T}) \cap (\mathcal{F}[1] * \mathcal{D}^{\geq 0}) = \mathcal{F}[1] * \mathcal{T}.$$

Finally we have $\mathsf{Hom}_{\mathcal{H}}(\mathcal{F}[1], \mathcal{T}) = 0$ since $\mathcal{F}[1] \subseteq \mathcal{D}^{\leq 0}[1]$ and $\mathcal{T} \subseteq \mathcal{D}^{\geq 0}$. □

Chapter 2

Tilting Theory

2.1 First properties

Let A be a finite-dimensional K-algebra. We will work on $\mathsf{mod}\,A$, the category of finite-dimensional modules over A.

We begin by defining two class of modules which we will use later and by studying some of their closure properties.

Definition 2.1.1. We define the Ext^1-orthogonal of $T \in \mathsf{mod}\,A$ as

$$T^{\perp_1} := \{M \in \mathsf{mod}\,A : \mathsf{Ext}^1(T, M) = 0\}.$$

Lemma 2.1.2. *$T^{\perp_1}$ is closed under extensions.*

Proof. Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A, C \in T^{\perp_1}$. Applying the covariant functor $\mathsf{Hom}(T, \_)$ we obtain the long exact sequence

$$\begin{aligned} 0 \to \mathsf{Hom}(T, A) \to \mathsf{Hom}(T, B) \to \mathsf{Hom}(T, C) \to \\ \to \mathsf{Ext}^1(T, A) \to \mathsf{Ext}^1(T, B) \to \mathsf{Ext}^1(T, C) \to \cdots . \end{aligned}$$

Since both A and C are Ext^1-orthogonal to T we get $\mathsf{Ext}^1(T, A) = 0$ and $\mathsf{Ext}^1(T, C) = 0$, so by exactness $\mathsf{Ext}^1(T, B) = 0$, i.e. $B \in T^{\perp_1}$. □

Lemma 2.1.3. *If $\mathsf{pd}\,T \leq 1$ then $T^{\perp_1}$ is closed under quotients.*

Proof. Let $0 \to A \to B \to C \to 0$ with $B \in T^{\perp_1}$. Applying $\mathsf{Hom}(T, -)$ as before we get:

$$\cdots \to \mathsf{Ext}^1(T, B) \to \mathsf{Ext}^1(T, C) \to \mathsf{Ext}^2(T, A) \to \cdots .$$

We have $\mathsf{Ext}^1(T, B) = 0$ since $B \in T^{\perp_1}$ and $\mathsf{Ext}^2(T, A) = 0$ since $\mathsf{pd}\,T \leq 1$, so by exactness $\mathsf{Ext}^1(T, C) = 0$, i.e. $C \in T^{\perp_1}$. □

Definition 2.1.4. For any module $T \in \mathsf{mod}\,A$ we define $\mathsf{gen}\,T$ as the family of quotients of copies of T:

$$\mathsf{gen}\,T := \{M \in \mathsf{mod}\,A \mid \exists\, T^d \twoheadrightarrow M \text{ for some } d \geq 0\}$$

Lemma 2.1.5. $\mathsf{gen}\,T$ *is closed under quotients.*

Proof. It is immediate, since if $M \in \mathsf{gen}\,T$ and N is a quotient of M, then the composition of maps $T^d \twoheadrightarrow M$ (coming from the fact that $M \in \mathsf{gen}\,T$) and $M \twoheadrightarrow N$ (coming from the fact that N is a quotient of M) makes N an element of $\mathsf{gen}\,T$. ☐

Lemma 2.1.6. *If* $\mathsf{gen}\,T \subseteq T^{\perp_1}$*, then* $\mathsf{gen}\,T$ *is closed under extensions.*

Proof. Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A, C \in \mathsf{gen}\,T$. Then there exists $n \geq 0$ and an epimorphism $T^n \twoheadrightarrow C$. Let now P be the pullback of the two maps $B \to C$ and $T^n \twoheadrightarrow C$. By the properties of the pullback we have the following diagram with horizontal exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & T^n & \longrightarrow & 0 \\ & & \| & & \downarrow\!\!\!\downarrow & \lrcorner & \downarrow\!\!\!\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

In particular, note that the map between P and B is epic since it is the pullback of an epimorphism.

Now, by hypothesis, $A \in \mathsf{gen}\,T \subseteq T^{\perp_1}$, so $\mathsf{Ext}^1(T^n, A) = \mathsf{Ext}^1(T, A)^n = 0$. This means that $P \cong A \oplus T^n$. Thus, since $A \in \mathsf{gen}\,T$, there exists $m \geq 0$ and an epimorphism $T^{m+n} = T^m \oplus T^n \twoheadrightarrow A \oplus T^n = P$, which composed with $P \twoheadrightarrow B$ gives that $B \in \mathsf{gen}\,T$. ☐

We can now define the modules that will be the main interest of this chapter, the *tilting modules*. We start by requiring some weaker conditions.

Definition 2.1.7. Let $T \in \mathsf{mod}\,A$. We say that T is a *partial tilting module* if it satisfies the following conditions:

(T1) $\mathsf{pd}\,T \leq 1$,

(T2) $\mathsf{Ext}^1_A(T, T) = 0$.

Note that any projective module is partial tilting. It is interesting to note also that the direct sum of two partial tilting modules may not be partial tilting. In fact if T_1, T_2 are partial tilting then $\mathsf{Ext}^1_A(T_1, T_1) = \mathsf{Ext}^1_A(T_2, T_2) = 0$ but $\mathsf{Ext}^1_A(T_1, T_2)$ and $\mathsf{Ext}^1_A(T_2, T_1)$ may not be zero.

Remark 2.1.8. If T is partial tilting, then $\mathsf{gen}\,T \subseteq T^{\perp_1}$. In fact by (T2) we have that $T \in T^{\perp_1}$ and, since by (T1) $\mathsf{pd}\,T \leq 1$, $T^{\perp_1}$ is closed by quotients. Since $T^{\perp_1}$ is trivially closed under direct sums as well, we get the claim. This means that, if T is partial tilting, then both $\mathsf{gen}\,T$ and $T^{\perp_1}$ are closed under extension and quotients, so they are torsion classes.

Proposition 2.1.9. *If* $T \in \mathsf{mod}\,A$ *is a partial tilting module, then both* $\mathsf{gen}\,T$ *and* $T^{\perp_1}$ *are torsion classes. The corresponding torsion free classes are*

$$T^{\perp} := \{M \in \mathsf{mod}\,A \mid \mathsf{Hom}_A(T, M) = 0\}$$

and

$$\mathsf{cogen}(\tau T) := \{M \in \mathsf{mod}\,A \mid \exists\, M \hookrightarrow T^d \text{ for some } d \geq 0\}.$$

Proof. Omitted, refer to [ASS06, Lemma 2.3]. □

We saw that starting from a partial tilting module we get two torsion pairs. They become interesting if we add more conditions on the module we're considering.

Definition 2.1.10. Let $T \in \operatorname{mod} A$. We say that T is a *tilting module* if T is partial tilting and moreover it satisfies the following condition:

(T3) There exists a short exact sequence of the form:

$$0 \to A \to T_0 \to T_1 \to 0$$

where $T_0, T_1 \in \operatorname{add} T$.

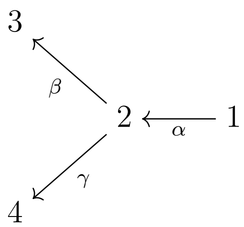
Notice that the condition (T3) can be changed with the following: for any indecomposable projective A-module P there exists a short exact sequence of the form:

$$0 \to P \to T_0 \to T_1 \to 0$$

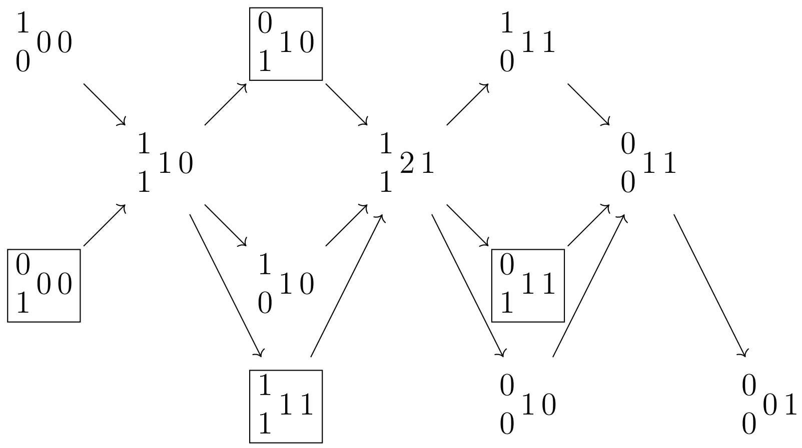
with $T_0, T_1 \in \operatorname{add} T$, since A is the sum of all the indecomposable projectives.

Remark 2.1.11. Every tilting module is faithful since it cogenerates A.

Example 2.1.12. Let Q be the quiver of Example 1.3.2:



The KQ-module $T = {}^{0}_{1}11 \oplus {}^{1}_{1}11 \oplus {}^{0}_{1}10 \oplus {}^{0}_{1}00$ is tilting. Indeed, since the quiver is acyclic, finite and connected, KQ is hereditary and so clearly $\operatorname{pd} T \leq 1$. From the Auslander–Reiten quiver it is easy to see that there are no extensions between any two direct summands of T, so $\operatorname{Ext}^1_A(T, T) = 0$.



Finally, for any projective P we have a short exact sequence $0 \to P \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathsf{add}\, T$, namely:

$$0 \to {}^{1}_{0}0\,0 \to {}^{1}_{1}1\,1 \to {}^{0}_{1}1\,1 \to 0$$

$$0 \to {}^{0}_{1}0\,0 \to {}^{0}_{1}0\,0 \to 0 \to 0$$

$$0 \to {}^{1}_{1}1\,0 \to {}^{1}_{1}1\,1 \oplus {}^{0}_{1}1\,0 \to {}^{0}_{1}1\,1 \to 0$$

$$0 \to {}^{1}_{1}1\,1 \to {}^{1}_{1}1\,1 \to 0 \to 0.$$

The next lemma is known as *Bongartz's Lemma* and it states that we can always obtain a tilting module starting from a partial tilting. It is proven firstly in [Bon81], here we follow [ASS06] in the proof.

Lemma 2.1.13. *Let $T \in \mathsf{mod}\, A$ be a partial tilting module. Then there exists $E \in \mathsf{mod}\, A$ such that $T \oplus E$ is a tilting module.*

Proof. $\mathsf{Ext}^1_A(T, A)$ is a finite-dimensional vector space. Take $e_1, \ldots, e_n$ as a basis. Then each e_i is an extension from T to A, so it can be represented by a short exact sequence $0 \to A \xrightarrow{f_i} E_i \xrightarrow{g_i} T \to 0$. Consider now the maps $f : A^n \to \oplus_{i=1}^n E_i$ and $g : \oplus_{i=1}^n E_i \to T^n$ given by:

$$f = \begin{bmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{bmatrix} \qquad\qquad g = \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_n \end{bmatrix}.$$

They yields the short exact sequence:

$$0 \to A^n \xrightarrow{f} \bigoplus_{i=1}^n E_i \xrightarrow{g} T^n \to 0.$$

Consider the map $k : A^n \to A$ that sums all the components, i.e. $k = [1, 1, \ldots, 1]$. Let E be the pushout of f and k. By the properties of the pushout there exists a commutative diagram with exact rows of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^n & \xrightarrow{\ f\ } & \bigoplus_{i=1}^n E_i & \xrightarrow{\ g\ } & T^n & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle u} & & \Big\| & & \\ 0 & \longrightarrow & A & \xrightarrow{\ v\ } & E & \xrightarrow{\ w\ } & T^n & \longrightarrow & 0. \end{array}$$

We claim that E is the module we were looking for, i.e. that $T \oplus E$ is tilting. The short exact sequence $0 \to A \xrightarrow{v} E \xrightarrow{w} T^n \to 0$ is enough to show (T1) and (T3). In fact it is exactly the sequence that we ask for in (T3) since $E, T^n \in \mathsf{add}(T \oplus E)$. Moreover, since $\mathsf{pd}\, T \leq 1$ and A is projective, we get that $\mathsf{pd}\, E \leq 1$ from the horseshoe lemma and so $\mathsf{pd}(T \oplus E) \leq 1$ and we get (T1).

It is left to show that $\mathsf{Ext}^1_A(T \oplus E, T \oplus E) = 0$. To do this, call $t_i : T \to T^n$ the i-th inclusion. We can then consider the map:

$$\mathsf{Ext}^1_A(t_i, A) : \mathsf{Ext}^1_A(T^n, A) \to \mathsf{Ext}^1_A(T, A)$$

which sends every short exact sequence of the form $0 \to A \to X \to T^n \to 0$ to the class of short exact sequences $0 \to A \to Y \to T \to 0$, where Y is the pullback of the maps $X \to T^n$ and t_i. Denote by e the representative in $\mathsf{Ext}^1_A(T^n, A)$ which corresponds to the short exact sequence $0 \to A \xrightarrow{v} E \xrightarrow{w} T^n \to 0$. We want to show that $\mathsf{Ext}^1_A(t_i, A)(e) = e_i$ for each $i = 1, \ldots, n$. Let $a_i : A \to A^n$ and $u_i : E_i \to \oplus_{i=1}^n E_i$ for each $i = 1, \ldots, n$ be the other inclusions to the i-th component. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_i} & & \downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle t_i} & & \\
0 & \longrightarrow & A^n & \xrightarrow{f} & \bigoplus_{i=1}^{n} E_i & \xrightarrow{g} & T^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle u} & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{v} & E & \xrightarrow{w} & T^n & \longrightarrow & 0.
\end{array}$$

Now, ka_i is just the identity on A, so composing the vertical maps we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & T & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle uu_i} & & \downarrow{\scriptstyle t_i} & & \\
0 & \longrightarrow & A & \xrightarrow{v} & E & \xrightarrow{w} & T^n & \longrightarrow & 0.
\end{array}$$

Since the first vertical arrow is the identity, by the characterization of the pullback, we have that E_i is the pullback of w and t_i, so we get $\mathsf{Ext}^1_A(t_i, A)(e) = e_i$ as we wanted. Consider again the short exact sequence $0 \to A \xrightarrow{v} E \xrightarrow{w} T^n \to 0$:

- applying $\mathsf{Hom}_A(T, -)$ to it, we obtain the long exact sequence:

  $$\cdots \to \mathsf{Hom}_A(T, T^n) \xrightarrow{\delta} \mathsf{Ext}^1_A(T, A) \to \mathsf{Ext}^1_A(T, E) \to \mathsf{Ext}^1_A(T, T^n) \to \cdots$$

  where $\mathsf{Ext}^1_A(T, T^n) = 0$ since T is partial tilting. By what we prove before we have $e_i = \mathsf{Ext}^1_A(t_i, A)(e) = \delta(t_i)$, so every element of the basis of $\mathsf{Ext}^1_A(T, A)$ is in the image of δ, thus δ is surjective. Then by exactness $\mathsf{Ext}^1_A(T, E) = 0$;

- applying $\mathsf{Hom}_A(-, E)$ to it, we obtain the long exact sequence:

  $$\cdots \to \mathsf{Hom}_A(A, E) \to \mathsf{Ext}^1_A(T^n, E) \to \mathsf{Ext}^1_A(E, E) \to \mathsf{Ext}^1_A(A, E) \to \cdots$$

where $\mathsf{Ext}^1_A(A, E) = 0$ by the projectivity of A and $\mathsf{Ext}^1_A(T^n, E) = \mathsf{Ext}^1_A(T, E)^n = 0$ by additivity of Ext and from the previous point;

- applying $\mathsf{Hom}_A(-, T)$ to it, we obtain the long exact sequence:

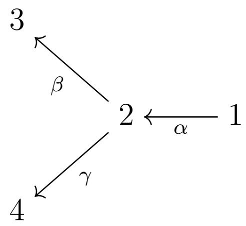
$$\cdots \to \mathsf{Hom}_A(A, T) \to \mathsf{Ext}^1_A(T^n, T) \to \mathsf{Ext}^1_A(E, T) \to \mathsf{Ext}^1_A(A, T) \to \cdots$$

  where $\mathsf{Ext}^1_A(A, T) = 0$ by the projectivity of A and $\mathsf{Ext}^1_A(T^n, T) = 0$ since T is partial tilting. Then by exactness $\mathsf{Ext}^1_A(E, T) = 0$.

So, since also $\mathsf{Ext}^1_A(T, T)$ is zero, we get $\mathsf{Ext}^1_A(T \oplus E, T \oplus E) = 0$ and so $T \oplus E$ is tilting. □

The short exact sequence $0 \to A \to E \to T^n \to 0$ built in the proof is called *Bongartz's exact sequence*.

Remark 2.1.14. Note that the completion to a tilting module is not unique. Indeed for example consider the quiver Q:



and the partial tilting module $T = {}^{0}_{1}1\,0$ on KQ. Then both

$$T_1 = {}^{0}_{1}1\,0 \oplus {}^{1}_{1}1\,0 \oplus {}^{1}_{1}1\,1 \oplus {}^{0}_{1}0\,0$$

and

$$T_2 = {}^{0}_{1}1\,0 \oplus {}^{0}_{1}1\,1 \oplus {}^{1}_{1}1\,1 \oplus {}^{0}_{1}0\,0$$

are completion of T to a tilting module.

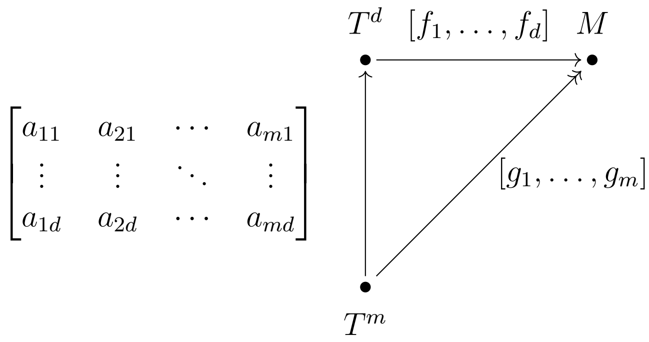
We now want to give a characterization of tilting module according to the two classes we previously built and studied. We start with a lemma.

Lemma 2.1.15. *Let $T \in \mathsf{mod}\, A$ such that $\mathsf{gen}\, T = T^{\perp_1}$. Then for any $M \in \mathsf{gen}\, T$ there exists a short exact sequence*

$$0 \to K \to T^d \to M \to 0$$

with $K \in T^{\perp_1}$.

Proof. Let $M \in \mathsf{gen}\, T$. $\mathsf{Hom}_A(T, M)$ is a finite-dimensional vector space, so take $f_1, \dots, f_d$ to be a base. Consider the map $f = (f_1, \dots, f_d) : T^d \to M$. This is an epimorphism. In fact, since $M \in \mathsf{gen}\, T$, there exists an epimorphism $g = (g_1, ..., g_m) : T^m \twoheadrightarrow M$ for some $m \geq 0$. Since every g_i is a map from T to M, we can write $g_i = \sum_{j=1}^{d} a_{ij} f_j$ for suitable $a_{ij} \in A$. If we consider the map $a : T^m \to T^d$ given by the matrix $(a_{ij})^t$ with $i = 1, \dots, m$ and $j = 1, \dots, d$, we have that $g = fa$, so f is epic since g is.



Now, let K be the kernel of f to obtain the short exact sequence:

$$0 \to K \to T^d \to M \to 0.$$

It is left to show that $K \in \mathsf{gen}\, T$. To prove it apply $\mathsf{Hom}_A(T, -)$ to the short exact sequence, obtaining:

$$\cdots \to \mathsf{Hom}_A(T, T^d) \xrightarrow{\mathsf{Hom}_A(T,f)} \mathsf{Hom}_A(T, M) \to \mathsf{Ext}^1_A(T, K) \to \mathsf{Ext}^1_A(T, T^d) \to \cdots .$$

Notice that $\mathsf{Ext}^1_A(T, T^d) = 0$ since obviously $T \in \mathsf{gen}\, T = T^{\perp_1}$. From the same reasoning as before (taking $m = 1$) we get that $\mathsf{Hom}_A(T, f)$ is epic since every morphism from T to M factors through f. So by exactness we have $\mathsf{Ext}^1_A(T, K) = 0$, i.e. $K \in T^{\perp_1}$. □

Proposition 2.1.16. *Let T be partial tilting. Then T is tilting if and only if* $\mathsf{gen}\, T = T^{\perp_1}$.

Proof. $\Leftarrow$ Since T is partial tilting, by Bongartz's Lemma there exists a module E such that $T \oplus E$ is tilting. In particular, this means that $\mathsf{Ext}^1_A(E, T) = 0$ and also $\mathsf{Ext}^1_A(T, E) = 0$ since $\mathsf{Ext}^1_A(T \oplus E, T \oplus E)$ is zero. The latter equivalence yields $E \in T^{\perp_1}$ so, by hypothesis, $E \in \mathsf{gen}\, T$. By the previous lemma there exists a short exact sequence of the form $0 \to K \to T^d \to E \to 0$ with $K \in T^{\perp_1}$. Now, since $T \oplus E$ is tilting, we have $\mathsf{pd}\, E \leq 1$, so that by Lemma 2.1.3, $E^{\perp_1}$ is closed under quotient. As we mentioned before, $\mathsf{Ext}^1_A(E, T) = 0$, so $T \in E^{\perp_1}$ and also $\mathsf{gen}\, T \subseteq E^{\perp_1}$. So $K \in E^{\perp_1}$. This means that $\mathsf{Ext}^1_A(E, K) = 0$ so the short exact sequence $0 \to K \to T^d \to E \to 0$ splits and E is a direct summand of T^d. This means that $E \in \mathsf{add}\, T$. So T was already a tilting module.

$\Rightarrow$ Since T is partial tilting we already know that $\mathsf{gen}\, T \subseteq T^{\perp_1}$. To complete the proof we want to show that taken $M \in T^{\perp_1}$ there exists $d \geq 0$ and an epimorphism $T^d \twoheadrightarrow M$. A is projective, so there exists a map $A^d \twoheadrightarrow M$. Since T is tilting by hypothesis, there exists a short exact sequence $0 \to A \to T_0 \to T_1 \to 0$. We can then consider the short exact sequence $0 \to A^d \to T_0^d \to T_1^d \to 0$. Let P be the pushout of the two morphisms $A^d \to T_0^d$ and $A^d \twoheadrightarrow M$. By the properties of the pushout we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^d & \longrightarrow & T_0^d & \longrightarrow & T_1^d & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & T_1^d & \longrightarrow & 0. \end{array}$$

Since $M \in T^{\perp_1}$ we have $\mathsf{Ext}^1_A(T, M) = 0$, but also $\mathsf{Ext}^1_A(\mathsf{add}\, T, M) = 0$. So in particular: $\mathsf{Ext}^1_A(T_1^d, M) = 0$. This means that the short exact sequence $0 \to M \to P \to T_1^d \to 0$ splits, thus there exists an epimorphism $P \twoheadrightarrow M$. We then get the claim considering the composition of epimorphisms $T_1^d \twoheadrightarrow P \twoheadrightarrow M$.

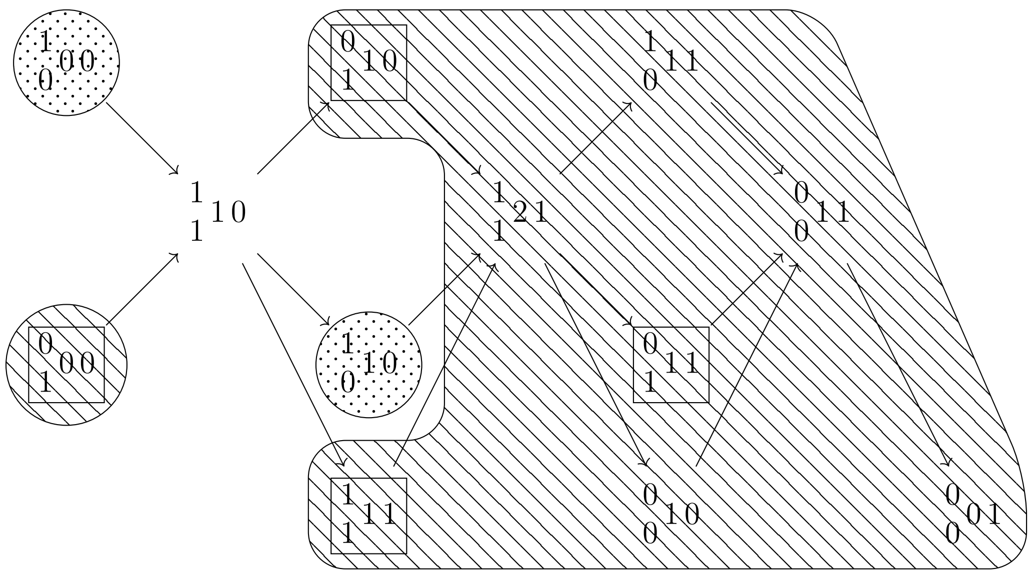
□

We saw that when starting from a partial tilting module T we obtain two (in general different) torsion pairs $(T^{\perp_1}, \mathsf{cogen}(\tau T))$ and $(\mathsf{gen}\, T, T^{\perp})$. From the last proposition we get that in the case T is tilting the two torsion pair are the same and so we can talk about *the* torsion pair induced by T. We will denote it with $(\mathcal{T}(T), \mathcal{F}(T))$.

Example 2.1.17. Consider the tilting module

$$T = {}^{0}_{1}11 \oplus {}^{1}_{1}11 \oplus {}^{0}_{1}10 \oplus {}^{0}_{1}00$$

from Example 2.1.12. Since in a torsion pair both torsion class and torsion-free class are closed under direct sum, we can see the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ induced by T looking at the Auslander–Reiten quiver:



The direct summands of T are denoted with a rectangle, whereas $\mathcal{F}(T)$ is denoted with the dots and $\mathcal{T}(T)$ with the lines. Note that we have that $\mathcal{T}(T)$ is exactly $\mathsf{gen}\, T$ and $T^{\perp_1}$, as from the previous proposition, and dually $\mathcal{F}(T)$ is $T^{\perp}$ and $\mathsf{cogen}(\tau T)$.

Remark 2.1.18. Note that every injective module is torsion in the torsion pair induced by T tilting. Indeed, take E injective, then $\mathsf{Ext}^1(T, E) = 0$ and so $E \in T^{\perp_1} = \mathcal{T}(T)$.

On the other hand, if the projective which are torsion are in $\mathsf{add}\, T$. Indeed, take $P \in \mathcal{T} = \mathsf{gen}\, T$ projective, then there exists an epimorphism $T^m \twoheadrightarrow P$ for some $m \geq 0$. By the projectivity of P, this epimorphism splits, so we get that P is a direct summand of T^m, thus $P \in \mathsf{add}\, T$.

As an immediate consequence of 2.1.15 we get the following result:

Corollary 2.1.19. *Let $T \in \mathsf{mod}\, A$ be tilting and $M \in T^{\perp_1}$. There exists an exact sequence of the form:*

$$\cdots \to T_2 \to T_1 \to T_0 \to M \to 0$$

with $T_i \in \mathsf{add}\, T$ for any $i \geq 0$.

Proof. By 2.1.15, since $M \in T^{\perp_1}$, there exists a short exact sequence:

$$0 \to L_0 \to T_0 \to M \to 0$$

with $L_0 \in T^{\perp_1}$ and $T_0 \in \mathsf{add}\, T$. If we continue to apply 2.1.15 to L_i with $i \geq 1$ we obtain the short exact sequences:

$$0 \to L_{i+1} \to T_{i+1} \to L_i \to 0$$

with $L_{i+1} \in T^{\perp_1}$ and $T_{i+1} \in \mathsf{add}\, T$. Glueing these short exact sequences together we obtain the long exact sequence of the claim. □

2.2 Brenner–Butler Theorem

In this section we reach the main result of the chapter: the *Brenner and Butler Theorem*. It states that whenever we have a tilting module, there exists an equivalence of full subcategories of $\mathsf{mod}\, A$ and the categories of modules over the endomorphism algebra of the tilting module. The main result was first proven in [BB80]. We will follow [ASS06].

Let A be a finite-dimensional algebra over a field. Take $T \in \mathsf{mod}\, A$ and let $B = \mathsf{End}\, T$. We can give T a natural left B-module structure, where the product on the left is given by: $b \cdot t = b(t)$ with $t \in T$ and $b \in B = \mathsf{End}\, T$. Since any $b \in B$ is a right A-module homomorphism, we get that the left B-module structure is compatible with the right A-module structure of T, in particular ${}_BT_A$ is a bimodule. So for any $M \in \mathsf{mod}\, A$, $\mathsf{Hom}_A(T, M)$ has a natural structure of right B-module given by the left B-module structure of T: $f \cdot b(t) = f(bt)$ for any $f \in \mathsf{Hom}_A(T, M)$, $t \in T$, $b \in B$.

So we obtain that the functor $\mathsf{Hom}_A(T, -)$ maps $\mathsf{mod}\, A$ to $\mathsf{mod}\, B$. We begin by giving some first results about this functor.

Lemma 2.2.1. *Let A be an algebra, $T \in \mathsf{mod}\, A$ and $B = \mathsf{End}\, T$. The functor $\mathsf{Hom}_A(T, -)$ induces an equivalence of categories between $\mathsf{add}\, T$ and $\mathsf{proj}\, B$.*

Proof. It is clear that $\mathsf{Hom}_A(T,-)$ maps $\mathsf{add}\,T$ into $\mathsf{proj}\,B$, indeed:

$$\mathsf{Hom}_A(T,\mathsf{add}\,T) = \mathsf{add}\,\mathsf{Hom}(T,T) = \mathsf{add}\,B = \mathsf{proj}\,B$$

by additivity of Hom. Moreover, let $P \in \mathsf{mod}\,B$ be indecomposable and projective. Then it is a summand of $B = \mathsf{Hom}_A(T,T)$. So $P \cong \mathsf{Hom}_A(T,T_0)$ for some $T_0 \in \mathsf{add}\,T$. Thus in particular the functor is dense. To show that it is an equivalence of categories we are left to show that it is full and faithful. To do this, consider the following chain of isomorphisms with $M \in \mathsf{mod}\,A$:

$$\mathsf{Hom}_B(\mathsf{Hom}_A(T,T),\mathsf{Hom}_A(T,M)) \cong \mathsf{Hom}_B(B,\mathsf{Hom}_A(T,M)) \cong \mathsf{Hom}_A(T,M).$$

By the additivity of Hom we then get that for any $T_0 \in \mathsf{add}\,T$:

$$\mathsf{Hom}_B(\mathsf{Hom}_A(T,T_0),\mathsf{Hom}_A(T,M)) \cong \mathsf{Hom}_A(T_0,M).$$

This shows that $\mathsf{Hom}_A(T,-)$ is full and faithful on $\mathsf{add}\,T$, so it is an equivalence of categories between $\mathsf{add}\,T$ and $\mathsf{proj}\,B$. □

Note that for now we didn't ask anything on T. If we ask for T to be tilting we can say more. In particular we have the torsion class $\mathcal{T}(T)$ induced by T and this next result shows that $\mathsf{Hom}_A(T,\mathcal{T}(T))$ is a full subcategory of $\mathsf{mod}\,B$ closed under extensions.

Proposition 2.2.2. *Let $T \in \mathsf{mod}\,A$ be tilting and $M,N \in \mathcal{T}(T)$. We have the functorial isomorphisms:*

1) $\mathsf{Hom}_A(M,N) \cong \mathsf{Hom}_B(\mathsf{Hom}_A(T,M),\mathsf{Hom}_A(T,N))$*;*

2) $\mathsf{Ext}^1_A(M,N) \cong \mathsf{Ext}^1_B(\mathsf{Hom}_A(T,M),\mathsf{Hom}_A(T,N))$*.*

Proof. 1) By Corollary 2.1.19, stopping at the second iteration, we get the exact sequence:

$$0 \to L_1 \to T_1 \to T_0 \to M \to 0$$

with $L_1 \in \mathcal{T}(T)$ and $T_1, T_0 \in \mathsf{add}\,T$. Applying to it firstly $\mathsf{Hom}_A(T,-)$ and then $\mathsf{Hom}_B(-,\mathsf{Hom}_A(T,N))$ we obtain the exact sequence:

$$0 \to \mathsf{Hom}_B(\mathsf{Hom}_A(T,M),\mathsf{Hom}_A(T,N)) \to \mathsf{Hom}_B(\mathsf{Hom}_A(T,T_0),\mathsf{Hom}_A(T,N)) \to$$
$$\to \mathsf{Hom}_B(\mathsf{Hom}_A(T,T_1),\mathsf{Hom}_A(T,N))$$

since $\mathsf{Ext}^1_A(T,L_1) = 0$. On the other hand, applying to the first exact sequence $\mathsf{Hom}_A(-,N)$ we get the exact sequence:

$$0 \to \mathsf{Hom}_A(M,N) \to \mathsf{Hom}_A(T_0,N) \to \mathsf{Hom}_A(T_1,N).$$

So we have a commutative diagram of this form, with the isomorphisms that follow from Lemma 2.2.1:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathsf{Hom}_A(M,N) & \dashrightarrow & \mathsf{Hom}_B(\mathsf{Hom}_A(T,M),\mathsf{Hom}_A(T,N)) \\ \downarrow & & \downarrow \\ \mathsf{Hom}_A(T_0,N) & \xrightarrow{\cong} & \mathsf{Hom}_B(\mathsf{Hom}_A(T,T_0),\mathsf{Hom}_A(T,N)) \\ \downarrow & & \downarrow \\ \mathsf{Hom}_A(T_1,N) & \xrightarrow{\cong} & \mathsf{Hom}_B(\mathsf{Hom}_A(T,T_1),\mathsf{Hom}_A(T,N)). \end{array}$$

The dotted arrow exists by the universal property of the kernel and it is and isomorphism since the other two horizontal maps are.

2) Let T^* be the exact sequence given by Corollary 2.1.19:

$$T^* : \cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M \to 0$$

with $T_i \in \mathsf{add}\, T$ for any $i \geq 0$. Consider the complex $\mathsf{Hom}_A(T, T^*)$. It is exact. Indeed, we can decompose T^* into the short exact sequences $0 \to \ker d_i \to T_i \to \operatorname{im} d_i \to 0$ with $\ker d_i, \operatorname{im} d_i \in \mathcal{T}(T)$. The functor $\mathsf{Hom}_A(T, -)$ is exact on each of these short exact sequences since $\mathsf{Ext}^1(T, \ker d_i) = 0$ (from $\ker d_i \in \mathcal{T}(T)$). So we can compose the short exact sequences:

$$0 \to \mathsf{Hom}_A(T, \ker d_i) \to \mathsf{Hom}_A(T, T_i) \to \mathsf{Hom}_A(T, \operatorname{im} d_i) \to 0$$

obtaining a long exact sequence, that is exactly the complex $\mathsf{Hom}_A(T, T^*)$. Now, since every $T_i \in \mathsf{add}\, T$, by Lemma 2.2.1 we have that $\mathsf{Hom}_A(T, T_i) \in \mathsf{proj}\, B$, so $\mathsf{Hom}_A(T, T^*)$ represents a projective resolution of $\mathsf{Hom}_A(T, M)$ in $\mathsf{mod}\, B$. Thus, by definition of Ext^1_B as derived functor, we get that $\mathsf{Ext}^1_B(\mathsf{Hom}_A(T, M), \mathsf{Hom}_A(T, N))$ is the first cohomology group of the complex $\mathsf{Hom}_B(\mathsf{Hom}_A(T, T^*), \mathsf{Hom}_A(T, N))$. By part (1) this last complex is isomorphic to $\mathsf{Hom}_A(T^*, N)$, so we want to show that $\mathsf{Ext}^1_A(M, N)$ is isomorphic to the first cohomology group of $\mathsf{Hom}_A(T^*, N)$. To show this, consider the exact sequence:

$$\begin{aligned} 0 \to \mathsf{Hom}_A(M, N) \to \mathsf{Hom}_A(T_0, N) \xrightarrow{\mathsf{Hom}_A(j,N)} \mathsf{Hom}_A(L, N) \to \\ \to \mathsf{Ext}^1_A(M, N) \to \mathsf{Ext}^1_A(T_0, N) \to \cdots \end{aligned}$$

obtained by applying $\mathsf{Hom}_A(-, N)$ to the short exact sequence $0 \to \ker d_0 \xrightarrow{j} T_0 \xrightarrow{d_0} M \to 0$. Since $T_0 \in \mathsf{add}\, T$ and $N \in \mathcal{T}(T) = T^{\perp_1}$, we have $\mathsf{Ext}^1_A(T_0, N) = 0$. So we get that $\mathsf{Ext}^1_A(M, N) \cong \operatorname{coker} \mathsf{Hom}_A(j, N)$ is isomorphic to the first cohomology group of the complex $\mathsf{Hom}_A(T^*, N)$. This concludes the proof.

□

We have seen that T has a structure of B-A-bimodule. In the case that T_A is a tilting module, we obtain that also ${}_BT$ is tilting. To show it we will need the following remark.

Remark 2.2.3. If T_A is tilting, then we have:

$$D({}_BT) \cong \mathsf{Hom}_A(T, DA).$$

Indeed we have the following chain of isomorphisms:

$$D({}_BT) \cong D({}_BT_A \otimes_A A) = \mathsf{Hom}_K({}_BT_A \otimes_A A, K) \cong \mathsf{Hom}_A(T, \mathsf{Hom}_K(A, K)) = \mathsf{Hom}_A(T, DA)$$

where the third isomorphism comes from the adjunction $- \otimes_A A \dashv \mathsf{Hom}_K(A, -)$.

Proposition 2.2.4. *Let T_A be a tilting module. Then ${}_BT$ is tilting, where $B = \mathsf{End}_A T$.*

Proof. We prove the tilting axioms for ${}_BT$:

(T1) we want to show that $\mathsf{pd}\, {}_BT \leq 1$. From (T3) of T_A there exists a short exact sequence $0 \to A \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathsf{add}\, T$. Applying $\mathsf{Hom}_A(-, {}_BT_A)$ to it we obtain the exact sequence:

$$0 \to \mathsf{Hom}_A(T_1, T) \to \mathsf{Hom}_A(T_0, T) \to \mathsf{Hom}_A(A, T) \to \mathsf{Ext}^1_A(T_1, T) \to \cdots .$$

Since $T_1 \in \mathsf{add}\, T$ and T_A is tilting, we have $\mathsf{Ext}^1_A(T_1, T) = 0$. Moreover $\mathsf{Hom}_A(A, T) = \mathsf{Hom}_A(A, {}_BT_A) \cong {}_BT$ and $\mathsf{Hom}_A(T_1, T), \mathsf{Hom}_A(T_0, T)$ are projective B-modules since $T_1, T_0 \in \mathsf{add}\, T$. So the last sequence represents a projective resolution of ${}_BT$ and thus $\mathsf{pd}\, {}_BT \leq 1$.

(T2) we check that $\mathsf{Ext}^1_B(T, T) = 0$. Recall that DA is injective in $\mathsf{mod}\, A$ since A is projective. In particular $DA \in \mathcal{T}(T)$. So, using the second point of Proposition 2.2.2 and Remark 2.2.3, we get:

$$\mathsf{Ext}^1_B(DT, DT) \cong \mathsf{Ext}^1_B(\mathsf{Hom}_A(T, DA), \mathsf{Hom}_A(T, DA)) \cong \mathsf{Ext}^1_B(DA, DA) = 0$$

from which we get $\mathsf{Ext}^1_B(T, T) = 0$.

(T3) to build the short exact sequence $0 \to B \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathsf{add}\, {}_BT$, consider the projective resolution of T_A: $0 \to P_1 \to P_0 \to T_A \to 0$ and apply $\mathsf{Hom}_A(-, {}_BT_A)$ to it to obtain the exact sequence:

$$0 \to \mathsf{Hom}_A(T_A, {}_BT_A) \to \mathsf{Hom}_A(P_0, {}_BT_A) \to \mathsf{Hom}_A(P_1, {}_BT_A) \to \mathsf{Ext}^1_A(T, T) \to \cdots .$$

Since T_A is tilting $\mathsf{Ext}^1_A(T, T) = 0$. Moreover, $\mathsf{Hom}_A(P_0, {}_BT_A), \mathsf{Hom}_A(P_1, {}_BT_A)$ are in $\mathsf{add}\, {}_BT$ and $\mathsf{Hom}_A(T_A, {}_BT_A) \cong {}_BB$, so we get the short exact sequence we were looking for.

□

From ${}_BT$ we can also recover the initial algebra A as the opposite of its endomorphism algebra:

Lemma 2.2.5. *The canonical homomorphism*

$$A \to \mathsf{End}({}_BT)^{\mathsf{op}}$$

$$a \mapsto (t \mapsto ta)$$

is an isomorphism.

Proof. First note that A and $\mathsf{End}({}_BT)^{\mathsf{op}}$ are isomorphic as K vector spaces, since:

$$A \cong \mathsf{End}\, DA \cong \mathsf{End}\, \mathsf{Hom}_A(T, DA) \cong \mathsf{End}\, DT$$

where the second isomorphism follows from the fact that $DA \in \mathcal{T}(T)$ and Proposition 2.2.2 (1), whereas the last isomorphism follows from Remark 2.2.3. So in particular we get $\dim_K A = \dim_K \mathsf{End}({}_BT)$. To prove that the given homomorphism is in fact an isomorphism we just need to check that it is monic. Let $a \in A$ be in the kernel. We thus get that $Ta = 0$, which yields $a = 0$ since T is faithful being tilting. □

So we have seen that if T is tilting as a right A-module, then it is tilting also as a left B-module. This means that ${}_BT$ induces a torsion pair in $B\,\mathsf{mod}$, $(\mathcal{T}({}_BT), \mathcal{F}({}_BT))$, given by:

$$\mathcal{T}({}_BT) = {}_BT^{\perp_1} = \{{}_BM \mid \mathsf{Ext}^1_B(T, M) = 0\} = \mathsf{gen}\, {}_BT$$

$$\mathcal{F}({}_BT) = {}_BT^{\perp} = \{{}_BM \mid \mathsf{Hom}_B(T, M) = 0\} = \mathsf{cogen}\, \tau({}_BT).$$

We are interested in a torsion pair in $\mathsf{mod}\, B$ and not in $B\,\mathsf{mod}$. So the idea is to use Proposition 1.4.14 and consider the torsion pair:

$$(\mathcal{X}(T), \mathcal{Y}(T)) := (D\mathcal{F}({}_BT), D\mathcal{T}({}_BT)).$$

To have a better description of the new torsion pair we built in $\mathsf{mod}\, B$ we use the following lemma:

Lemma 2.2.6. *Let B be a finite-dimensional K-algebra. There exist functorial isomorphisms:*

$$\mathsf{Hom}_B(X, DY) \cong D(X \otimes_B Y)$$

and

$$\mathsf{Ext}^1_B(X, DY) \cong D\, \mathsf{Tor}^B_1(X, Y)$$

for any $X, Y \in \mathsf{mod}\, B$.

Proof. The first isomorphism follows from the adjunction $- \otimes_B T \dashv \mathsf{Hom}_K(T, -)$. For the second one refer to [ASS06, Proposition A.4.11]. □

Proposition 2.2.7. *Any tilting module $T \in \mathsf{mod}\, A$ induces a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\mathsf{mod}\, B$ given by:*

$$\mathcal{X}(T) = \{X_B \mid \mathsf{Hom}_B(X, DT) = 0\} = \{X_B \mid X \otimes_B T = 0\}$$

$$\mathcal{Y}(T) = \{Y_B \mid \mathsf{Ext}^1_B(Y, DT) = 0\} = \{Y_B \mid \mathsf{Tor}^B_1(Y, T) = 0\}$$

Proof. The first two equivalences follows directly from how we defined the two classes and using the fact that D is an equivalence of categories. The other equivalences follows from the isomorphisms $\mathsf{Hom}_B(X, DT) \cong D(X \otimes_B T)$ and $\mathsf{Ext}^1_B(Y, DT) \cong D\,\mathsf{Tor}^B_1(Y, T)$ of Lemma 2.2.6. □

Remark 2.2.8. Note that $\mathcal{Y}(T)$ contains all the projective right B-modules as $\mathsf{Ext}^1_B(P, DT)$ is obviously zero for any P projective.

We now introduce two lemmas that will be useful in the proof of the Brenner and Butler Theorem. They state that if we restrict to $\mathcal{T}(T)$ and $\mathcal{Y}(T)$, then the functors $\mathsf{Hom}_A(T, -)$ and $- \otimes_B T$ becomes quasi-inverse.

Lemma 2.2.9. *Let T_A be a tilting module. For any $Y_B \in \mathcal{Y}(T)$ we have an isomorphism:*

$$\delta_Y : Y_B \to \mathsf{Hom}_A(T, Y \otimes_B T)$$

$$y \mapsto (t \mapsto y \otimes t).$$

Proof. We first consider $\delta_{DT} : (DT)_B \to \mathsf{Hom}_A(T, DT \otimes_B T)$. We have:

$$\mathsf{Hom}_B(T, T) \cong \mathsf{Hom}_B(DT, DT) \cong D(DT \otimes_B T)$$

where the first isomorphism follows from the fully faithfulness of D, while the second is simply $\mathsf{Hom}_B(X, DT) \cong D(X \otimes_B T)$ with $X = DT$. In particular we get $DT \otimes_B T \cong D\,\mathsf{Hom}_B(DT, DT) \cong DA$ from Lemma 2.2.5. So $\mathsf{Hom}_A(T, DT \otimes_B T) \cong \mathsf{Hom}_A(T, DA) \cong DT$ from 2.2.3 and thus δ_{DT} is an isomorphism. By additivity oh Hom, we get that also δ_{T^*} is an isomorphism for any $T^* \in \mathsf{add}\, DT$.

Now we want to show the claim for an arbitrary $Y_B \in \mathcal{Y}(T)$. There exists a short exact sequence $0 \to Y \to T^* \to Z \to 0$ with $T^* \in \mathsf{add}\, DT$ and $Z \in \mathcal{Y}(T)$. Indeed, since ${}_BT$ is tilting and $DY \in \mathcal{T}(T)$, there exists the short exact sequence in $B\,\mathsf{mod}$ $0 \to Y' \to T' \to DY \to 0$ with $Y' \in \mathcal{T}(T)$ and $T' \in \mathsf{add}\, T$ by Lemma 2.1.15. Applying D (which is exact since $\mathsf{mod}\, K$ is semisimple) to it we obtain $0 \to Y \to DT' \to DY' \to 0$ with $T^* := DT' \in \mathsf{add}\, T$ and $Z := DY' \in \mathcal{T}(T)$.

So from $Y \in \mathcal{Y}(T)$ we have the short exact sequence

$$0 \to Y \to T^*_0 \to Y_0 \to 0$$

and since $Y_0 \in \mathcal{Y}(T)$ we also have:

$$0 \to Y_0 \to T^*_1 \to Y_1 \to 0$$

which combines into the exact sequence:

$$0 \to Y \to T^*_0 \to T^*_1 \to Y_1 \to 0 \tag{2.1}$$

with $Y_1 \in \mathcal{T}(T)$ and $T^*_0, T^*_1 \in \mathsf{add}\, DT$. In particular, since $Y_0, Y_1 \in \mathcal{Y}(T)$, we have $\mathsf{Tor}^B_1(Y_0, T) = 0$ and $\mathsf{Tor}^B_1(Y_1, T) = 0$. So applying $- \otimes_B T$ to the two short exact sequence we obtain the short exact sequences:

$$0 \to Y \otimes_B T \to T^*_0 \otimes_B T \to Y_0 \otimes_B T \to 0$$

and

$$0 \to Y_0 \otimes_B T \to T_1^* \otimes_B T \to Y_1 \otimes_B T \to 0$$

which combines into the exact sequence:

$$0 \to Y \otimes_B T \to T_0^* \otimes_B T \to T_1^* \otimes_B T \to Y_1 \otimes_B T \to 0.$$

Applying $\mathsf{Hom}_A(T,-)$ to it we obtain:

$$0 \to \mathsf{Hom}_A(T, Y \otimes_B T) \to \mathsf{Hom}_A(T, T_0^* \otimes_B T) \to \mathsf{Hom}_A(T, T_1^* \otimes_B T) \to \mathsf{Hom}_A(T, Y_1 \otimes_B T).$$

Considering this last sequence and 2.1 we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathsf{Hom}_A(T, Y \otimes_B T) & \longrightarrow & \mathsf{Hom}_A(T, T_0^* \otimes_B T) & \longrightarrow & \mathsf{Hom}_A(T, T_1^* \otimes_B T) \\
 & & \delta_Y \big\uparrow & & \delta_{T_0^*} \big\uparrow \cong & & \delta_{T_1^*} \big\uparrow \cong \\
0 & \longrightarrow & Y & \longrightarrow & T_0^* & \longrightarrow & T_1^*
\end{array}$$

Since both $\delta_{T_0^*}$ and $\delta_{T_1^*}$ are isomorphism, then δ_Y is an isomorphism by the five lemma. □

In this lemma we also have a characterisation of the modules in the torsion class of a tilting module in terms of the canonical homomorphism $\mathsf{Hom}_A(T,M) \otimes_B T \to M$.

Lemma 2.2.10. *Let $M, T \in \mathsf{mod}\, A$ and $B = \mathsf{End}\, T$. Consider the A-module homomorphism ε_M given by:*

$$\mathsf{Hom}_A(T,M) \otimes_B T \to M$$
$$f \otimes t \mapsto f(t).$$

We have:

1) $M \in \mathsf{gen}\, T$ if and only if ε_M is an epimorphism.

2) If T is tilting, then $M \in \mathcal{T}(T)$ if and only if ε_M is an isomorphism.

Proof. 1) Let $M \in \mathsf{gen}\, T$. Consider the short exact sequence:

$$0 \to K \to T^d \xrightarrow{f} M \to 0$$

given by Lemma 2.1.15, where $f = (f_1, \dots, f_d)$ with $f_1, \dots, f_d$ basis of $\mathsf{Hom}_A(T,M)$. By construction of f, $\mathsf{Hom}_A(T,f)$ is an epimorphism, so applying $\mathsf{Hom}_A(T,-)$ to the short exact sequence yields:

$$0 \to \mathsf{Hom}_A(T,K) \to \mathsf{Hom}_A(T,T^d) \to \mathsf{Hom}_A(T,M) \to 0.$$

Now, applying $- \otimes_B T$ to it and considering the respective ε_- we get the commutative diagram:

$$\begin{array}{ccccccc}
\mathsf{Hom}_A(T,K) \otimes_B T & \longrightarrow & \mathsf{Hom}_A(T,T^d) \otimes_B T & \longrightarrow & \mathsf{Hom}_A(T,M) \otimes_B T & \longrightarrow & 0 \\
\varepsilon_K \big\downarrow & & \varepsilon_{T^d} \big\downarrow & & \varepsilon_M \big\downarrow & & \\
K & \longrightarrow & T^d & \xrightarrow{\ f\ } & M & \longrightarrow & 0
\end{array}$$

Note that ε_{T^d} is an isomorphism. Indeed, by additivity of Hom and by the properties of tensor product we have:

$$\mathsf{Hom}_A(T, T^d) \otimes_B T \cong \mathsf{Hom}_A(T, T)^d \otimes_B T \cong B^d \otimes_B T \cong T^d.$$

Since $f\,\varepsilon_{T_d}$ is epic (since both are), by commutativity of the diagram we have ε_M epic.

Conversely, let ε_M be surjective. $\mathsf{Hom}_A(T, M)$ is a finitely generated B-module, so there exists an epimorphism $g : B^m \to \mathsf{Hom}_A(T, M)$. So we have $\varepsilon_M \circ g \otimes T : B^m \twoheadrightarrow M$. By the isomorphism $T^m \cong B^m \otimes_B T$ we get that $M \in \mathsf{gen}\, T$.

2) If ε_M is an isomorphism, then by 1) $M \in \mathsf{gen}\, T = \mathcal{T}(T)$ since T is tilting.

Conversely, note that for any $T' \in \mathsf{add}\, T$ we have that $\varepsilon_{T'}$ is an isomorphism from the fact that $\mathsf{Hom}_A(T, T) \otimes_B T \cong B \otimes_B T \cong T$ and by additivity of Hom. Now, let $M \in \mathcal{T}(T)$. Applying Lemma 2.1.15 to M and subsequently to L_0 we obtain the short exact sequences:

$$0 \to L_0 \to T_0 \to M \to 0$$

$$0 \to L_1 \to T_1 \to L_0 \to 0$$

with $T_0, T_1 \in \mathsf{add}\, T$ and $L_0, L_1 \in \mathcal{T}(T)$, which combines into:

$$0 \to L_1 \to T_1 \to T_0 \to M \to 0.$$

Applying $\mathsf{Hom}_A(T, -)$ to the two short exact sequences and combining them we obtain the exact sequence:

$$0 \to \mathsf{Hom}_A(T, L_1) \to \mathsf{Hom}_A(T, T_1) \to \mathsf{Hom}_A(T, T_0) \to \mathsf{Hom}_A(T, M) \to 0$$

since $\mathsf{Ext}^1_A(T, L_0) = \mathsf{Ext}^1_A(T, L_1) = 0$ as $L_0, L_1 \in \mathcal{T}(T)$. Applying $- \otimes_B T$ to this sequence and considering the respective ε_- we get the commutative diagram:

$$\begin{array}{ccccccc}
\mathsf{Hom}_A(T, T_1) \otimes_B T & \longrightarrow & \mathsf{Hom}_A(T, T_0) \otimes_B T & \longrightarrow & \mathsf{Hom}_A(T, M) \otimes_B T & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varepsilon_{T_1}} & & \downarrow{\scriptstyle \varepsilon_{T_0}} & & \downarrow{\scriptstyle \varepsilon_M} & & \\
T_1 & \longrightarrow & T_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Since both ε_{T_0} and ε_{T_1} are isomorphisms, then so is ε_M by the five lemma.

□

We can now prove the main theorem of this section: the Brenner–Butler Theorem.

Theorem 2.2.11. *Let A be a finite-dimensional K-algebra. Let $T \in \mathsf{mod}\, A$ be a tilting module and $B = \mathsf{End}\, T$. Consider the torsion pairs induced by T in $\mathsf{mod}\, A$ and $\mathsf{mod}\, B$: $(\mathcal{T}(T), \mathcal{F}(T))$ and $(\mathcal{X}(T), \mathcal{Y}(T))$. Then:*

1) $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ are equivalent categories via the quasi-inverse functors $\mathsf{Hom}_A(T, -)$ and $- \otimes_B T$:

$$\mathsf{Hom}_A(T, -) : \mathcal{T}(T) \leftrightarrows \mathcal{Y}(T) : - \otimes_B T.$$

2) $\mathcal{F}(T)$ *and* $\mathcal{X}(T)$ *are equivalent categories via the quasi-inverse functors* $\mathsf{Ext}^1_A(T,-)$ *and* $\mathsf{Tor}^B_1(-,T)$*:*

$$\mathsf{Ext}^1_A(T,-) : \mathcal{F}(T) \leftrightarrows \mathcal{X}(T) : \mathsf{Tor}^B_1(-,T).$$

Proof. 1) Consider $M \in \mathcal{T}(T)$. Using the isomorphism of Lemma 2.2.6 we have:

$$D\,\mathsf{Hom}_A(T,M) \cong D\,\mathsf{Hom}_A(T,DDM) \cong D(D(T \otimes_A DM)) \cong T \otimes_A DM.$$

Since $M \in \mathsf{gen}\,T_A$ we have $DM \in \mathsf{gen}\,{}_AT$, so that $T \otimes_A DM \in \mathsf{gen}\,{}_BT = \mathcal{T}({}_BT)$. So, by the isomorphism, $D\,\mathsf{Hom}_A(T,M) \in \mathcal{T}({}_BT)$ and thus $\mathsf{Hom}_A(T,M) \in \mathcal{Y}(T) = D\mathcal{T}({}_BT)$. So $\mathsf{Hom}_A(T,-)$ maps $\mathcal{T}(T)$ to $\mathcal{Y}(T)$. Moreover, since T is tilting, by Lemma 2.2.10 we have that $M \cong \mathsf{Hom}_A(T,M) \otimes_B T$, hence $- \otimes_B T \circ \mathsf{Hom}_A(T,-)$ is naturally isomorphic to the identity functor in $\mathcal{T}(T)$

Conversely, let $Y \in \mathcal{Y}(T)$. Then $Y \otimes_B T \in \mathsf{gen}\,T = \mathcal{T}(T)$. Moreover, by Lemma 2.2.9: $Y \cong \mathsf{Hom}_A(T, Y \otimes_B T)$. Hence also $\mathsf{Hom}_A(T,-) \circ - \otimes_B T$ is naturally isomorphic to the identity functor in $\mathcal{Y}(T)$ and so the two functors are quasi-inverse between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.

2) Consider $N \in \mathcal{F}(T)$. Let $E \in \mathsf{mod}\,A$ injective such that $N \hookrightarrow E$. Taking L as the cokernel of this map we obtain the short exact sequence:

$$0 \to N \to E \to L \to 0.$$

Since E is injective it belongs to $\mathcal{T}(T)$ and thus also $L \in \mathcal{T}(T)$ since $\mathcal{T}(T)$ is closed under quotients. Applying $\mathsf{Hom}_A(T,-)$ to the short exact sequence we get the exact sequence:

$$0 \to \mathsf{Hom}_A(T,N) \to \mathsf{Hom}_A(T,E) \to \mathsf{Hom}_A(T,L) \to \mathsf{Ext}^1_A(T,N) \to \mathsf{Ext}^1_A(T,E) \to \cdots .$$

Now, $\mathsf{Hom}_A(T,N) = 0$ since $N \in \mathcal{F}(T)$ and $\mathsf{Ext}^1_A(T,E) = 0$ since E is injective. So the sequence just becomes the short exact sequence:

$$0 \to \mathsf{Hom}_A(T,E) \to \mathsf{Hom}_A(T,L) \to \mathsf{Ext}^1_A(T,N) \to 0.$$

Apply $- \otimes_B T$ to get the exact sequence:

$$0 \to \mathsf{Tor}^B_1(\mathsf{Ext}^1_A(T,N),T) \to \mathsf{Hom}_A(T,E) \otimes_B T \to \\ \to \mathsf{Hom}_A(T,L) \otimes_B T \to \mathsf{Ext}^1_A(T,N) \otimes_B T \to 0$$

since $\mathsf{Tor}^B_1(\mathsf{Hom}_A(T,L),T) = 0$ by the fact that $L \in \mathcal{T}(T)$ and so by 1) $\mathsf{Hom}_A(T,L) \in \mathcal{Y}(T)$. Consider now the following commutative diagram, where ε_E and ε_L are isomorphisms from Lemma 2.2.10:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathsf{Tor}_1^B(\mathsf{Ext}_A^1(T,N),T) & \overset{\alpha}{\dashrightarrow} & N \\
\downarrow & & \downarrow \\
\mathsf{Hom}_A(T,E)\otimes_B T & \xrightarrow[\varepsilon_E]{\cong} & E \\
\downarrow & & \downarrow \\
\mathsf{Hom}_A(T,L)\otimes_B T & \xrightarrow[\varepsilon_L]{\cong} & L \\
\downarrow & & \downarrow \\
\mathsf{Ext}_A^1(T,N)\otimes_B T & \dashrightarrow & 0 \\
\downarrow & & \\
0. & &
\end{array}
$$

The map α is induced by the others passing to the kernel. Since ε_E and ε_L are isomorphism, then also the dotted maps are. In particular we obtain:

$$\mathsf{Ext}_A^1(T,N)\otimes_B T = 0$$

and so $\mathsf{Ext}_A^1(T,N)\in\mathcal{X}(T)$ and:

$$\mathsf{Tor}_1^B(\mathsf{Ext}_A^1(T,N),T)\cong N$$

and thus $\mathsf{Tor}_1^B(-,T)\circ\mathsf{Ext}_A^1(T,-)\simeq\mathsf{id}_{\mathcal{F}(T)}$.

Conversely, take $X_B\in\mathcal{X}(T)$. Dually as before, take $P\in\mathsf{mod}\,A$ projective such that $P\twoheadrightarrow X$. Taking Y to be the kernel we have the short exact sequence $0\to Y\to P\to X\to 0$. Since P is projective, $P\in\mathcal{Y}(T)$ and so also $Y\in\mathcal{Y}(T)$ as $\mathcal{Y}(T)$ is closed under subojects. Applying $-\otimes_B T$ to the short exact sequence we get the short exact sequence:

$$0\to\mathsf{Tor}_1^B(X,T)\to Y\otimes_B T\to P\otimes_B T\to 0$$

since $\mathsf{Tor}_1^B(P,T)=0$ as $P\in\mathcal{Y}(T)$ and $X\otimes_B T=0$ as $X\in\mathcal{X}(T)$. Applying $\mathsf{Hom}_A(T,-)$ to it we get:

$$\mathsf{Hom}_A(T,\mathsf{Tor}_1^B(X,T))\to\mathsf{Hom}_A(T,Y\otimes_B T)\to\mathsf{Hom}_A(T,P\otimes_B T)\to\mathsf{Ext}_A^1(T,\mathsf{Tor}_1^B(X,T))\to 0$$

because $\mathsf{Ext}_A^1(T,Y\otimes_B T)=0$ since $Y\in\mathcal{Y}(T)$ and so, by (1), $Y\otimes_B T\in\mathcal{T}(T)$. In particular we have the following commutative diagram, where δ_Y and δ_P are isomorphism from Lemma 2.2.9:

$$
\begin{array}{ccc}
0 & \dashrightarrow & \mathsf{Hom}_A(T, \mathsf{Tor}_1^B(X,T)) \\
\downarrow & & \downarrow \\
Y & \xrightarrow[\delta_Y]{\cong} & \mathsf{Hom}_A(T, Y \otimes_B T) \\
\downarrow & & \downarrow \\
P & \xrightarrow[\delta_P]{\cong} & \mathsf{Hom}_A(T, P \otimes_B T) \\
\downarrow & & \downarrow \\
X & \overset{}{\underset{\beta}{\dashrightarrow}} & \mathsf{Ext}_A^1(T, \mathsf{Tor}_1^B(X,T)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

where β is induced by the other maps passing to the cokernel. Since δ_Y and δ_P are isomorphisms, also the dotted maps are. So we have:

$$\mathsf{Hom}_A(T, \mathsf{Tor}_1^B(X,T)) = 0$$

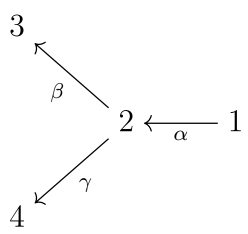
and thus $\mathsf{Tor}_1^B(X,T) \in \mathcal{F}(T)$ and:

$$\mathsf{Ext}_A^1(T, \mathsf{Tor}_1^B(X,T)) \cong X.$$

So $\mathsf{Ext}_A^1(T,-) \circ \mathsf{Tor}_1^B(-,T) \simeq \mathsf{id}_{\mathcal{X}(T)}$ and thus $\mathsf{Ext}_A^1(T,-)$ and $\mathsf{Tor}_1^B(-,T)$ are quasi-inverse between $\mathcal{F}(T)$ and $\mathcal{X}(T)$. This concludes the proof.

□

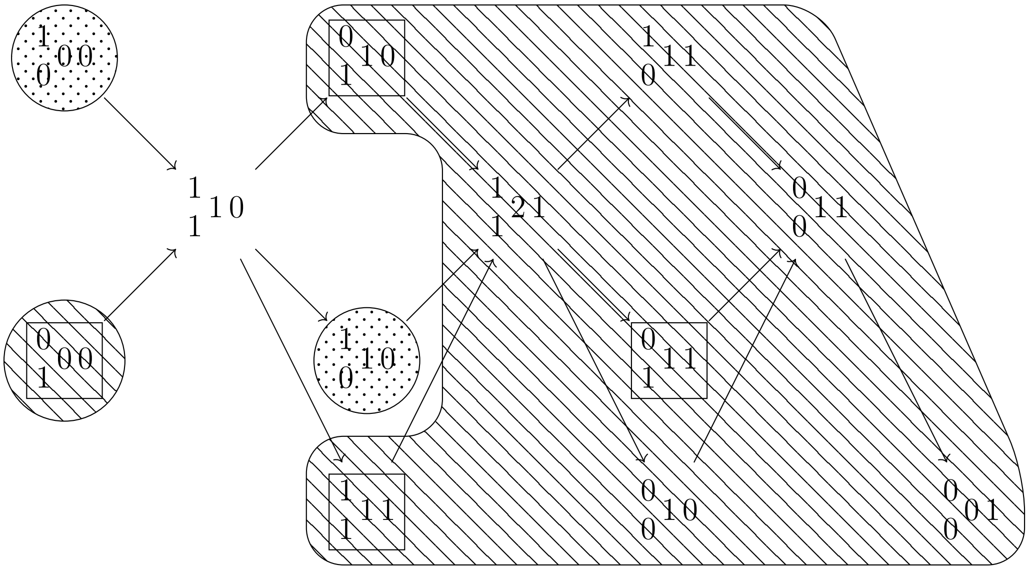
Example 2.2.12. We show an example of the equivalence given by the Brenner–Butler Theorem. Consider again the quiver:



and the tilting module:

$$T = {}^{0}_{1}1\,1 \oplus {}^{1}_{1}1\,1 \oplus {}^{0}_{1}1\,0 \oplus {}^{0}_{1}0\,0.$$

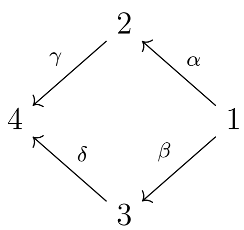
Recall that the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ is:



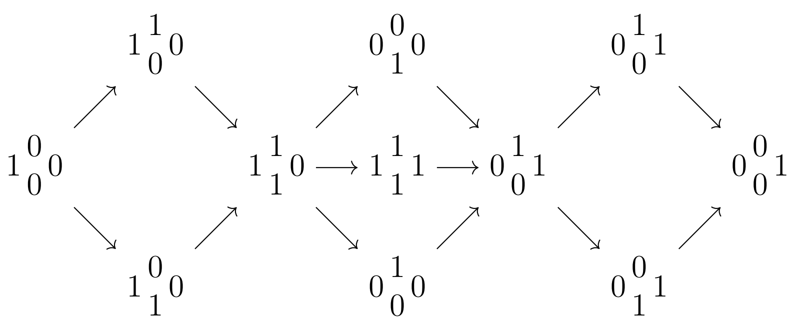
We have that the endomorphism algebra of T_A is:

$$B = \mathsf{End}_A(T) = \begin{pmatrix} K & K & K & K \\ 0 & K & 0 & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}$$

which is represented by the quiver:



with the commutative relation $\gamma\alpha - \delta\beta$. The Auslander–Reiten quiver of B is:



We want to find the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ and see the equivalences given by the theorem. From Proposition 2.2.7 we know that $\mathcal{X}(T) = \{X_B \mid \mathsf{Hom}_B(X, DT) = 0\}$ and

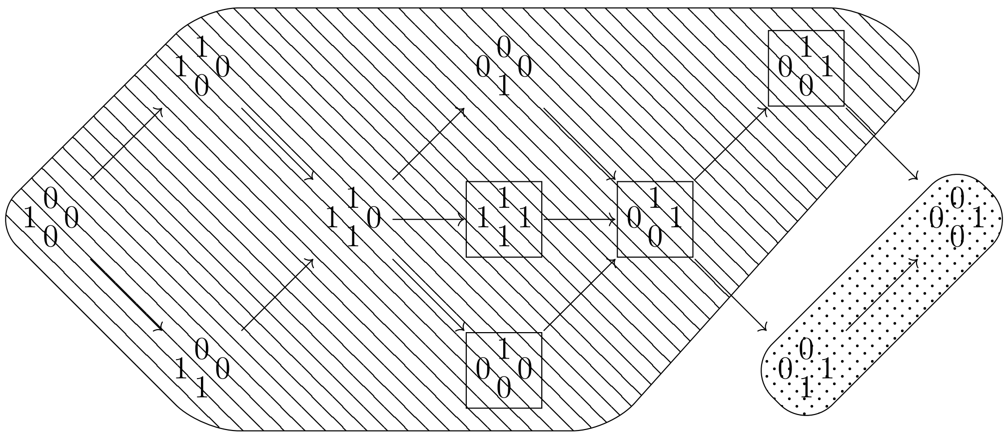
$\mathcal{Y}(T) = \{Y_B \mid \mathsf{Ext}^1_B(Y, DT) = 0\}$. By Remark 2.2.3 we know that $D(_BT) = \mathsf{Hom}_A(T, DA)$ where DA is the direct sum of all the injectives, so:

$$DA = {}^{1}_{0}1\,1 \oplus {}^{0}_{0}1\,1 \oplus {}^{0}_{1}1\,1 \oplus {}^{0}_{0}0\,1.$$

Thus we get:

$$D(_BT) = 1\,{}^{1}_{1}1 \oplus 0\,{}^{1}_{0}0 \oplus 0\,{}^{1}_{1}1 \oplus 0\,{}^{1}_{0}1$$

and the torsion pair (on the indecomposable) is given by:



where $\mathcal{X}(T)$ is denoted by the dots and $\mathcal{Y}(T)$ is denoted by the lines. Note that we have the equivalences stated by the Brenner–Butler Theorem: every module in $\mathcal{T}(T)$ corresponds via $\mathsf{Hom}_A(T, -)$ to a module in $\mathcal{Y}(T)$ and every module in $\mathcal{F}(T)$ corresponds to a module in $\mathcal{X}(T)$ via $\mathsf{Ext}^1_A(T, -)$. For example we have that:

$$\mathsf{Hom}_A(T, {}^{1}_{1}2\,1) = 1\,{}^{1}_{1}0 \in \mathcal{Y}(T)$$

and

$$\mathsf{Ext}^1_A(T, {}^{1}_{0}0\,0) = 0\,{}^{0}_{1}1 \in \mathcal{X}(T).$$

Remark 2.2.13. We can look at the equivalence given by the Brenner–Butler Theorem from another point of view, using the HRS-tilting. In fact, $\mathsf{mod}\, A$ is equivalent to the heart of the standard t-structure on $\mathcal{D}(A)$. In $\mathsf{mod}\, A$, given a tilting module T we have the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ induced by T. The HRS-tilting process gives us a new t-structure and in particular an abelian category $\mathcal{H}$, which is its heart. Thanks to a theorem of Happel, Reiten and Smalø ([HRS96, Theorem 4.3]) we have that $\mathcal{H}$ is exactly the category of modules over the opposite of the endomorphism algebra of T, so $\mathcal{H} \cong \mathsf{mod}\, B$. Finally, the HRS-tilting states that $(\mathcal{F}(T)[1], \mathcal{T}(T))$ is a torsion pair in $\mathcal{H}$. Since obviously $\mathcal{F}(T)[1]$ is equivalent to $\mathcal{F}(T)$, we get the result of the Brenner–Butler Theorem: we have found a torsion pair in $\mathsf{mod}\, B$ where the torsion class is equivalent to the torsion-free class of the torsion pair in $\mathsf{mod}\, A$ and vice versa. The Brenner–Butler Theorem also gives us the quasi-inverse functors.

Chapter 3

Silting Theory

3.1 First properties

In this section we take a first look at silting complexes.

Definition 3.1.1. Let $\mathcal{D}$ be a triangulated category. A full subcategory $\mathcal{T} \subseteq \mathcal{D}$ is said to be:

- triangulated if it is closed under shifts and cones;
- thick if it is triangulated and closed under direct summands.

If $\mathcal{X}$ is a family of objects in $\mathcal{D}$, we denote by $\mathsf{thick}(\mathcal{X})$ the smallest thick subcategory of $\mathcal{D}$ which contain $\mathcal{X}$.

We recall that the category $\mathcal{K}^b(\mathsf{proj}\, A)$ is the bounded homotopy category with finitely generated projective right A-modules in each degree.

Definition 3.1.2. A complex $\mathbf{X} \in \mathcal{K}^b(\mathsf{proj}\, A)$ is called *silting* if

1. $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{X}, \mathbf{X}[n]) = 0$ for each $n > 0$;
2. $\mathsf{thick}(\mathbf{X}) = \mathcal{K}^b(\mathsf{proj}\, A)$.

If a complex satisfies only the first condition we say that it is *presilting*.

We will look at them embedded in the bounded derived category, thanks to the monomorphism: $\mathcal{K}^b(\mathsf{proj}\, A) \hookrightarrow \mathcal{D}^b(A)$.

Remark 3.1.3. From the isomorphism $\mathcal{K}^b(\mathsf{proj}\, A) \simeq \mathcal{D}^b(A)$ (by Proposition 1.5.11), we have that in the first point of the definition we can look at the morphisms between $\mathbf{X}$ and its shifts in the homotopy category and not in the derived category.

In the following we will be mainly interested in 2-term silting objects, that are silting complexes with non-zero modules only in degree 0 and -1. So they are complexes of the form:

$$... \to 0 \to 0 \to P_{-1} \xrightarrow{d} P_0 \to 0 \to 0 \to ...$$

with P_0 and P_{-1} projective. Note that in this case the first condition in the definition of silting complex becomes $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{X}, \mathbf{X}[1]) = 0$. In fact we have only two terms and, by the previous remark, the morphisms are in the homotopy category, so we automatically get that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{X}, \mathbf{X}[n]) = 0$ for $n > 1$.

Example 3.1.4. The complex $A = A[0]$ with the algebra A in degree 0 and zero everywhere is silting. Firstly, it is obviously in $\mathcal{K}^b(\mathsf{proj}\, A)$ since A is projective. Now, since A is a projective module, $\mathsf{Hom}_{\mathcal{D}(A)}(A, A[n]) \simeq \mathsf{Ext}^n(A, A) = 0$. Finally, using the triangulated structure of $\mathcal{K}(A)$, up to shifting, we can get sums of A in each degree, and then any projective is a direct summand of it. Thus we get that we can generate all the category $\mathcal{K}^b(\mathsf{proj}\, A)$ from A via cones, shifts and direct summands, i.e. $\mathsf{thick}(A) = \mathcal{K}^b(\mathsf{proj}\, A)$.

Example 3.1.5. Every tilting module is a 2-term silting complex. In fact, let T be a tilting module. Take $\mathbf{P} = (P_{-1} \to P_0)$ to be its minimal projective resolution. Then $\mathbf{P} \cong T$ in $\mathcal{D}^b(A)$ and so $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}[1]) = \mathsf{Hom}_{\mathcal{D}^b(A)}(T, T[1]) = \mathsf{Ext}^1(T, T) = 0$ since T is tilting. So $\mathbf{P}$ is presilting. Moreover, consider the short exact sequence:

$$0 \to A \to T_0 \to T_1 \to 0$$

with $T_0, T_1 \in \mathsf{add}\, T$. In particular $T_0, T_1 \in \mathsf{thick}(\mathbf{P})$. The short exact sequence corresponds to a triangle $A \to T_0 \to T_1 \to A[1]$ and since $\mathsf{thick}(\mathbf{P})$ is closed under cocones, we have that $A \in \mathsf{thick}(\mathbf{P})$. So $\mathsf{thick}(\mathbf{P}) = \mathcal{K}^b(\mathsf{proj}\, A)$ and $\mathbf{P} \cong T$ is silting.

Conversely, we have that a 2-term silting $\mathbf{P}$ is a module if and only if it has no cohomology in degree -1, and it this case $\mathbf{P} \cong H^0(\mathbf{P})$ is tilting. In particular this holds if and only if $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$.

For any 2-term complex we have this interesting remark:

Remark 3.1.6. Let $\mathbf{X} \in \mathcal{D}^b(A)$ be a 2-term complex. There exists a triangle:

$$H^{-1}(\mathbf{X})[1] \to \mathbf{X} \to H^0(\mathbf{X}) \to H^{-1}(\mathbf{X})[2].$$

Indeed consider the standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and the canonical sequence of $\mathbf{X}$:

$$\tau^{\leq -1}\mathbf{X} \to \mathbf{X} \to \tau^{\geq 0}\mathbf{X} \to \tau^{\leq -1}\mathbf{X}[1].$$

But, $\tau^{\geq 0}\mathbf{X}$ is the complex $\cdots \to 0 \to \operatorname{im} d \to X_0 \to 0 \to \cdots$ which is isomorphic to $H^0(\mathbf{X})$ in $\mathcal{D}^b(A)$. Also, $\tau^{\leq -1}\mathbf{X}$ is the complex $\cdots \to 0 \to \ker d \to 0 \to 0 \to \cdots$ with $\ker d$ in degree -1. This means that $\tau^{\leq -1}\mathbf{X} \cong H^{-1}(\mathbf{X})[1]$. So the canonical sequence of $\mathbf{X}$ becomes the triangle we were looking for.

Lemma 3.1.7. *Let $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ be a 2-term complex. Then for any $\boldsymbol{X} \in \mathcal{D}^b(A)$ and any $i \in \mathbb{Z}$ we have the short exact sequence:*

$$0 \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, H^{i-1}(\boldsymbol{X})[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, \boldsymbol{X}[i]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, H^i(\boldsymbol{X})) \to 0.$$

Proof. Recall that we have $H^{i-1}(\mathbf{X}) = H^{-1}(\mathbf{X}[i])$ and $H^i(\mathbf{X}) = H^0(\mathbf{X}[i])$. So we want to show the existence of the short exact sequence:

$$0 \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^{-1}(\mathbf{X}[i])[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}[i]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^0(\mathbf{X}[i])) \to 0.$$

Without losing any generality we can consider $\mathbf{X}$ instead of $\mathbf{X}[i]$. Consider the canonical triangle given by the truncations on the standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$:

$$\tau^{\leq 0}\mathbf{X} \to \mathbf{X} \to \tau^{\geq 1}\mathbf{X} \to \tau^{\leq 0}\mathbf{X}[1]$$

with $\tau^{\leq 0}\mathbf{X} \in \mathcal{D}^{\leq 0}$ and $\tau^{\geq 1}\mathbf{X} \in \mathcal{D}^{\geq 1} = \mathcal{D}^{\geq 0}[-1]$. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ to the triangle we obtain the exact sequence:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq 1}\mathbf{X}[-1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq 0}\mathbf{X}) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq 1}\mathbf{X})$$

Now, both $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq 1}\mathbf{X}[-1])$ and $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq 1}\mathbf{X})$ are zero since $\tau^{\geq 1}\mathbf{X}[-1]$ and $\tau^{\geq 1}\mathbf{X}$ are zero in degrees 0 and -1. By exactness, we thus get that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq 0}\mathbf{X}) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X})$.

Consider now $\tau^{\leq 0}\mathbf{X}$ and its truncation triangle given by the t-structure $(\mathcal{D}^{\leq -2}, \mathcal{D}^{\geq -2})$:

$$\tau^{\leq -2}\mathbf{X} \to \tau^{\leq 0}\mathbf{X} \to \tau^{\geq -1}\tau^{\leq 0}\mathbf{X} \to \tau^{\leq -2}\mathbf{X}[1]$$

where $\tau^{\leq -2}\mathbf{X} \in \mathcal{D}^{\leq -2}$ and $\tau^{\geq -1}\tau^{\leq 0}\mathbf{X} \in \mathcal{D}^{\geq -1}$. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ to the triangle we obtain the exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq -2}\mathbf{X}) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq 0}\mathbf{X}) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq -1}\tau^{\leq 0}\mathbf{X}) \to$$
$$\to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq -2\mathbf{X}}[1]) \to \cdots .$$

Again, $\tau^{\leq -2}\mathbf{X}$ and $\tau^{\leq -2\mathbf{X}}[1]$ are zero in degrees 0 and -1, so $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq -2}\mathbf{X}) = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq -2}\mathbf{X}[1]) = 0$. Thus $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\leq 0}\mathbf{X}) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq -1}\tau^{\leq 0}\mathbf{X})$ by exactness and so:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \tau^{\geq -1}\tau^{\leq 0}\mathbf{X}).$$

Notice that we have restricted X to the terms in degree -1 and 0 and so we can consider $\mathbf{X}$ to be a 2-term complex. Then using Remark 3.1.6 we get a triangle:

$$H^{-1}(\mathbf{X})[1] \to \mathbf{X} \to H^0(\mathbf{X}) \to H^{-1}(\mathbf{X})[2].$$

Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ we get the exact sequence:

$$0 = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^0(\mathbf{X})[-1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^{-1}(\mathbf{X})[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}) \to$$
$$\to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^0(\mathbf{X})) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^{-1}(\mathbf{X})[2]) = 0.$$

This concludes the proof. □

The previous lemma states that to know the morphisms between a 2-term complex and a shifted bounded complex, we can look at the morphism between the 2-term complex and the cohomologies of the complex in the right degrees.

Lemma 3.1.8. *Let $X \in \mathsf{mod}\, A$. There exists a functorial isomorphism:*

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, X) \cong \mathsf{Hom}_A(H^0(\boldsymbol{P}), X)$$

and a monomorphism:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\boldsymbol{P}), X[1]) \hookrightarrow \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, X[1]).$$

Proof. By Remark 3.1.6, there exists the triangle:

$$H^{-1}(\mathbf{P})[1] \to \mathbf{P} \to H^0(\mathbf{P}) \to H^{-1}(\mathbf{P})[2].$$

Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(-, X)$ to the triangle we obtain the long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(H^{-1}(\mathbf{P})[2], X) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\mathbf{P}), X) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X) \to$$
$$\to \mathsf{Hom}_{\mathcal{D}^b(A)}(H^{-1}(\mathbf{P})[1], X) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\mathbf{P})[-1], X) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}[-1], X) \to \cdots .$$

We have $\mathsf{Hom}_{\mathcal{D}^b(A)}(H^{-1}(\mathbf{P})[2], X) = 0$ and $\mathsf{Hom}_{\mathcal{D}^b(A)}(H^{-1}(\mathbf{P})[1], X) = 0$ since X is concentrated in degree 0, while the cohomologies are in degree -2 and -1 respectively. So by exactness we get the isomorphism. On the other hand, we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\mathbf{P})[-1], X) = \mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\mathbf{P}), X[1])$ and $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}[-1], X) = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[1])$. Again, by exactness, we get the monomorphism. □

We have seen that to a tilting module we can associate a torsion pair given by the Ext^1-orthogonal and the Hom-orthogonal. We want to generalize it to 2-term silting. So, define:

$$\mathcal{T}(\mathbf{P}) := \{X \in \mathsf{mod}\, A | \, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[1]) = 0\}$$
$$\mathcal{F}(\mathbf{P}) := \{Y \in \mathsf{mod}\, A | \, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y) = 0\}.$$

Note that, if P is a module, these definitions coincide with the ones given in the tilting case.

We have another description of the pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ that will be useful later. Recalling that $\mathsf{gen}\, X$ denotes the full subcategory of the objects generated by X and that dually $\mathsf{cogen}\, X$ denotes the full subcategory of the objects cogenerated by X, we have the following result in [HKM02]:

Proposition 3.1.9. *Let $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ be a 2-term silting complex. Then we have:*

$$(\mathcal{T}(\boldsymbol{P}), \mathcal{F}(\boldsymbol{P})) = (\mathsf{gen}\, H^0(\boldsymbol{P}), \mathsf{cogen}\, H^{-1}(\nu \boldsymbol{P})).$$

Definition 3.1.10. Let $\mathcal{C}$ be a full subcategory of $\mathsf{mod}\, A$. We say that $M \in \mathcal{C}$ is Ext-*projective* in $\mathcal{C}$ if $\mathsf{Ext}^1_A(M, \mathcal{C}) = 0$. We say that $M \in \mathcal{C}$ is Ext-*injective* in $\mathcal{C}$ if $\mathsf{Ext}^1_A(\mathcal{C}, M) = 0$.

The definition of Ext-projective and Ext-injective generalise the one of projective or injective to a full subcategory. Indeed, being Ext-projective in $\mathcal{C}$ means exactly to behave like a projective in the subcategory, and dually with the injectives. For example any module T is Ext-projective in $\mathcal{T}(T)$.

Proposition 3.1.11. *Let $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ be a 2-term silting complex and $(\mathcal{T}(\boldsymbol{P}), \mathcal{F}(\boldsymbol{P}))$ be its associated torsion pair. Then:*

1) *for any $X \in \mathsf{mod}\, A$, we have $X \in \mathsf{add}\, H^0(\boldsymbol{P})$ if and only if X is Ext-projective in $\mathcal{T}(\boldsymbol{P})$;*

2) *for any $X \in \mathcal{T}(\boldsymbol{P})$, there exists a short exact sequence*
$$0 \to L \to T_0 \to X \to 0$$
with $T_0 \in \mathsf{add}\, H^0(\boldsymbol{P})$ and $L \in \mathcal{T}(\boldsymbol{P})$;

3) *for any $X \in \mathsf{mod}\, A$, we have $X \in \mathsf{add}\, t\nu A$ if and only if X is Ext-injective in $\mathcal{T}(\boldsymbol{P})$;*

4) *for any $X \in \mathcal{T}(\boldsymbol{P})$, there exists a short exact sequence*
$$0 \to X \to T_0 \to L \to 0$$
with $T_0 \in \mathsf{add}\, t\nu A$ and $L \in \mathcal{T}(\boldsymbol{P})$;

5) *for any $X \in \mathsf{mod}\, A$, we have $X \in \mathsf{add}\, H^{-1}(\nu\boldsymbol{P})$ if and only if X is Ext-injective in $\mathcal{F}(\boldsymbol{P})$;*

6) *for any $X \in \mathcal{F}(\boldsymbol{P})$, there exists a short exact sequence*
$$0 \to X \to F_0 \to L \to 0$$
with $F_0 \in \mathsf{add}\, H^{-1}(\nu\boldsymbol{P})$ and $L \in \mathcal{F}(\boldsymbol{P})$;

7) *for any $X \in \mathsf{mod}\, A$, we have $X \in \mathsf{add}\, A/tA$ if and only if X is Ext-projective in $\mathcal{F}(\boldsymbol{P})$;*

8) *for any $X \in \mathcal{F}(\boldsymbol{P})$, there exists a short exact sequence*
$$0 \to L \to F_0 \to X \to 0$$
with $F_0 \in \mathsf{add}\, A/tA$ and $L \in \mathcal{F}(\boldsymbol{P})$.

Proof. We prove only the first four statements as the others are dual.

1) $\Rightarrow$ Let $Y \in \mathcal{T}(\mathbf{P})$, so we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1]) = 0$. From the monomorphism of Lemma 3.1.8 we get also $\mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\mathbf{P}), Y[1]) = 0$. By the isomorphism $\mathsf{Hom}_{\mathcal{D}^b(A)}(H^0(\mathbf{P}), Y[1]) \cong \mathsf{Ext}^1_A(H^0(\mathbf{P}), Y)$ we get that $H^0(\mathbf{P})$ (and by additivity $\mathsf{add}\, H^0(\mathbf{P})$) is Ext-projective.

   $\Leftarrow$ Let M be Ext-projective in $\mathcal{T}(\mathbf{P})$. Consider $T_0 \xrightarrow{\alpha} M$ an $\mathsf{add}\, H^0(\mathbf{P})$-precover of M. Since $M \in \mathcal{T}(\mathbf{P}) = \mathsf{gen}\, H^0(\mathbf{P})$, we can consider α epic. Taking L as the kernel of α we get the short exact sequence:
$$0 \to L \to T_0 \to M \to 0.$$

Since α is an $\mathsf{add}\, H^0(\mathbf{P})$-precover, we have that $\mathsf{Hom}_A(H^0(\mathbf{P}), \alpha)$ is an epimorphism and, by the isomorphism of Lemma 3.1.8, also $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \alpha)$ is epic. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ to the short exact sequence we obtain the exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0) \twoheadrightarrow \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, M) \to \\ \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0[1]) \to \cdots .$$

Since $T_0 \in \mathsf{add}\, H^0(\mathbf{P})$ we have that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0[1]) = 0$. Indeed, shifting the triangle from Remark 3.1.6 for $\mathbf{P}$ and then applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$, we get the exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^0(\mathbf{P})[1]) \to \\ \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^{-1}(\mathbf{P})[3]) \to \cdots .$$

Since both $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}[1])$ and $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^{-1}(\mathbf{P})[3])$ are zero (by the fact that $\mathbf{P}$ is a 2-term silting), we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^0(\mathbf{P})[1]) = 0$ and so also $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathsf{add}\, H^0(\mathbf{P})[1]) = 0$ by additivity of Hom. In particular $T_0 \in \mathcal{T}(\mathbf{P})$. Since $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0[1]) = 0$, we get by exactness that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L[1]) = 0$, so $L \in \mathcal{T}(\mathbf{P})$. By assumption M is Ext-projective in $\mathcal{T}(\mathbf{P})$, and since both T_0 and L are in $\mathcal{T}(\mathbf{P})$, the short exact sequence splits. So $M \in \mathsf{add}\, H^0(\mathbf{P})$.

2) This follows from the previous proof by replacing M with an arbitrary $X \in \mathcal{T}(\mathbf{P})$.

3) Refer to [ASS06, Proposition VI.1.11]. Recall that $X \in \mathsf{add}\, t\nu A$ means that X is the torsion part of an injective module.

4) Let $X \in \mathcal{T}(\mathbf{P})$. Consider an injective envelope $\alpha : X \hookrightarrow I$. I is injective, i.e. $I \in \mathsf{add}\, \nu A$. Consider the torsion sequence of I with respect to the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$:

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & & \big\downarrow{\scriptstyle\alpha} & & & & \\ 0 & \longrightarrow & tI & \xrightarrow{\beta} & I & \xrightarrow{\gamma} & I/tI & \longrightarrow & 0. \end{array}$$

Since $X \in \mathcal{T}(\mathbf{P})$ and $I/tI \in \mathcal{F}(\mathbf{P})$, $\gamma\alpha = 0$. By the universal property of the kernel, there exists $\alpha' : X \to tI$ such that $\beta\alpha' = \alpha$. α' is monic since α is. Let L be the cokernel of α' to get the short exact sequence:

$$0 \to X \xrightarrow{\alpha'} tI \to L \to 0.$$

This is the sequence we were looking for. Indeed, $tI \in \mathsf{add}\, t\nu A$ and, by (3), it is Ext-injective in $\mathcal{T}(\mathbf{P})$, in particular $tI \in \mathcal{T}(\mathbf{P})$. So, $L \in \mathcal{T}(\mathbf{P})$ since $\mathcal{T}(\mathbf{P})$ is closed under quotients being a torsion class.

□

Now, for any $m \in \mathbb{Z}$, define the two full subcategories of $\mathcal{D}^b(A)$:

$$\mathcal{D}^{\leq m}(\mathbf{P}) := \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}[i]) = 0,\ i > m\right\}$$

$$\mathcal{D}^{\geq m}(\mathbf{P}) := \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}[i]) = 0,\ i < m\right\}.$$

Proposition 3.1.12. *Let $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ be a 2-term silting complex. Then*

1) $(\mathcal{T}(\boldsymbol{P}), \mathcal{F}(\boldsymbol{P}))$ is a torsion pair in $\mathsf{mod}\, A$;

2) $(\mathcal{D}^{\leq m}(\boldsymbol{P}), \mathcal{D}^{\geq m}(\boldsymbol{P}))$ is a t-structure in $\mathcal{D}^b(A)$ for each $m \in \mathbb{Z}$.

Proof. The proof of (1) can be found in [HKM02]. Here we only show (2). Consider the standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of the triangulated category $\mathcal{D}^b(A)$. The heart of this t-structure is isomorphic to $\mathsf{mod}\, A$. By (1) we know that $\mathbf{t} = (\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ is a torsion pair in $\mathsf{mod}\, A$. So we have that there exists a t-structure $(\mathcal{D}^{\leq 0}_{\mathbf{t}}, \mathcal{D}^{\geq 0}_{\mathbf{t}})$ in $\mathcal{D}^b(A)$ given by:

$$\mathcal{D}^{\leq 0}_{\mathbf{t}} = \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid H^0(\mathbf{X}) \in \mathcal{T}(\mathbf{P}), H^k(\mathbf{X}) = 0\ \forall k > 0\right\}$$

$$\mathcal{D}^{\geq 0}_{\mathbf{t}} = \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid H^{-1}(\mathbf{X}) \in \mathcal{F}(\mathbf{P}), H^k(\mathbf{X}) = 0\ \forall k < 0\right\}$$

that is the HRS-tilt of $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with respect to $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$. We have that $(\mathcal{D}^{\leq 0}_{\mathbf{t}}, \mathcal{D}^{\geq 0}_{\mathbf{t}})$ is exactly $(\mathcal{D}^{\leq 0}(\mathbf{P}), \mathcal{D}^{\geq 0}(\mathbf{P}))$. Indeed we have:

$$\begin{aligned}
\mathcal{D}^{\leq 0}(\mathbf{P}) &= \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}[i]) = 0\ \forall i > 0\right\} \\
&= \{\mathbf{X} \in \mathcal{D}^b(A) \mid \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^{i-1}(\mathbf{X})[1]) = 0\ \forall i > 0 \\
&\qquad\qquad \text{and } \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^i(\mathbf{X})) = 0\ \forall i > 0\} \\
&= \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid H^i(\mathbf{X}) = 0\ \forall i > 0 \text{ and } \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, H^0(\mathbf{X})[1]) = 0\right\} \\
&= \left\{\mathbf{X} \in \mathcal{D}^b(A) \mid H^i(\mathbf{X}) = 0\ \forall i > 0 \text{ and } H^0(\mathbf{X}) \in \mathcal{T}(\mathbf{P})\right\} = \mathcal{D}^{\leq 0}_{\mathbf{t}}
\end{aligned}$$

where the second equivalence follows from 3.1.7 and the third one follows from the fact that for a module M we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, M) = 0 = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, M[1])$ if and only if $M = 0$. Dually, we have $\mathcal{D}^{\geq 0}_{\mathbf{t}} = \mathcal{D}^{\geq 0}(\mathbf{P})$. Finally, for an arbitrary $m \in \mathbb{Z}$ it suffices to start with the t-structure $(\mathcal{D}^{\leq m}, \mathcal{D}^{\geq m})$ and apply the same reasoning. □

Remark 3.1.13. Note that $\mathcal{T}(\mathbf{P}) = \mathcal{D}^{\leq 0}(\mathbf{P}) \cap \mathsf{mod}\, A$. In fact, we obviously have that $\mathcal{D}^{\leq 0}(\mathbf{P}) \cap \mathsf{mod}\, A \subseteq \mathcal{T}(\mathbf{P})$. On the other hand, let $X \in \mathcal{T}(\mathbf{P})$. By definition it is in $\mathsf{mod}\, A$. Moreover, we have that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[i]) = 0$ for any $i > 2$, since $\mathbf{P}$ is a 2-term silting. The fact that $X \in \mathcal{T}$ precisely tells us that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[1]) = 0$, so, combining the two, we have that $X \in \mathcal{D}^{\leq 0}(\mathbf{P})$. Dually, we have that $\mathcal{F} = \mathcal{D}^{\geq 1}(\mathbf{P}) \cap \mathsf{mod}\, A$.

Consider now the t-structure $(\mathcal{D}^{\leq 0}(\mathbf{P}), \mathcal{D}^{\geq 0}(\mathbf{P}))$ and call $\mathcal{C}(\mathbf{P}) := \mathcal{D}^{\leq 0}(\mathbf{P}) \cap \mathcal{D}^{\geq 0}(\mathbf{P})$ its heart. We recall that $\mathcal{C}(\mathbf{P})$ is an abelian category and that the short exact sequences in $\mathcal{C}(\mathbf{P})$ are the triangles in $\mathcal{D}^b(A)$ with terms in $\mathcal{C}(\mathbf{P})$.

We now study the main properties of $\mathcal{C}(\mathbf{P})$ that will be useful later.

Theorem 3.1.14. *Let $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ be a 2-term silting complex. Then:*

1) $(\mathcal{F}(\boldsymbol{P})[1], \mathcal{T}(\boldsymbol{P}))$ *is a torsion pair in* $\mathcal{C}(\boldsymbol{P})$*.*

2) $\boldsymbol{X} \in \mathcal{D}^b(A)$ *is in* $\mathcal{C}(\boldsymbol{P})$ *if and only if* $H^0(\boldsymbol{X}) \in \mathcal{T}(\boldsymbol{P})$*,* $H^{-1} \in \mathcal{F}(\boldsymbol{P})$ *and* $H^i(\boldsymbol{X}) = 0$ *for any* $i \neq 0, 1$*.*

3) $\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, -) : \mathcal{C}(\boldsymbol{P}) \to \mathsf{mod}\, B$ *is an equivalence of categories.*

Proof. (1) and (2) follows directly from the fact that $(\mathcal{D}^{\leq 0}(\mathbf{P}), \mathcal{D}^{\geq 0}(\mathbf{P}))$ is the HRS-tilt of the standard t-structure with respect to the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$. For a proof of (3) refer to [HKM02]. □

Define:

$$\mathcal{X}(\mathbf{P}) := \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{F}(\mathbf{P})[1]),$$
$$\mathcal{Y}(\mathbf{P}) := \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{T}(\mathbf{P})).$$

They are full subcategories of $\mathsf{mod}\, B$ by Theorem 3.1.14. From the same theorem we get these two immediate corollaries.

Corollary 3.1.15. *Let* $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ *be a 2-term silting complex. Then* $(\mathcal{X}(\boldsymbol{P}), \mathcal{Y}(\boldsymbol{P}))$ *is a torsion pair in* $\mathsf{mod}\, B$ *and*

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, -) : \mathcal{T}(\boldsymbol{P}) \to \mathcal{Y}(\boldsymbol{P})$$
$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, -[1]) : \mathcal{F}(\boldsymbol{P}) \to \mathcal{X}(\boldsymbol{P})$$

are equivalences of subcategories which sends short exact sequences with terms in $\mathcal{T}(\boldsymbol{P})$ *(or* $\mathcal{F}(\boldsymbol{P})$*) to short exact sequences in* $\mathsf{mod}\, B$*.*

Proof. This comes directly from Theorem 3.1.14 (3), from the definition of $\mathcal{X}(\mathbf{P})$ and $\mathcal{Y}(\mathbf{P})$ and from the fact that $\mathcal{T}(\mathbf{P})$ and $\mathcal{F}(\mathbf{P})[1]$ are in $\mathcal{C}(\mathbf{P})$. □

Corollary 3.1.16. *Let* $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ *be a 2-term silting complex and* $M \in \mathcal{T}(\boldsymbol{P}), N \in \mathcal{F}(\boldsymbol{P})$*. We have the functorial isomorphisms:*

$$\mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, M), \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, N[1])) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(M, N[1]) \cong \mathsf{Ext}^1_A(M, N)$$
$$\mathsf{Ext}^1_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, M), \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, N[1])) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(M, N[2]) \cong \mathsf{Ext}^2_A(M, N).$$

Proof. These isomorphism derives directly from Theorem 3.1.14 (3) and from the isomorphism $\mathsf{Hom}_{\mathcal{D}^b(A)}(M, N[i]) \cong \mathsf{Ext}^i_A(M, N)$. □

This next theorem is the silting version of the Bongartz Lemma for tilting modules. In particular it states that any presilting complex is a direct summand of a silting complex and thus we can always complete a presilting to a silting.

Theorem 3.1.17. [BZ16b] *Let* $\boldsymbol{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ *be a 2-term presilting complex. Then there exists a 2-term complex* $\boldsymbol{E} \in \mathcal{K}^b(\mathsf{proj}\, A)$ *such that* $\boldsymbol{P} \oplus \boldsymbol{E}$ *is a 2-term silting complex. Moreover there exists a triangle:*

$$A \to \boldsymbol{E} \to \boldsymbol{P}' \to A[1]$$

with $\boldsymbol{P}' \in \mathsf{add}\, \boldsymbol{P}$*.*

Proof. We build $\mathbf{E}$ starting from the triangle. We start by considering $A[1]$ and we take $f : \mathbf{P}' \to A[1]$ to be an $\mathsf{add}\,\mathbf{P}$-precover of it. We then take $\mathbf{E}$ to be the cocone of f, thus obtaining the triangle:

$$A \to \mathbf{E} \to \mathbf{P}' \xrightarrow{f} A[1].$$

We now check that $\mathbf{P} \oplus \mathbf{E}$ is a silting complex. First, obviously, by additivity of Hom we have:

$$\mathsf{Hom}(\mathbf{P} \oplus \mathbf{E}, \mathbf{P} \oplus \mathbf{E}[1]) = \mathsf{Hom}(\mathbf{P}, \mathbf{P}[1]) \oplus \mathsf{Hom}(\mathbf{P}, \mathbf{E}[1]) \oplus \mathsf{Hom}(\mathbf{E}, \mathbf{P}[1]) \oplus \mathsf{Hom}(\mathbf{E}, \mathbf{E}[1]).$$

We show that they are all zero.

1) $\mathsf{Hom}(\mathbf{P}, \mathbf{P}[1]) = 0$ since $\mathbf{P}$ is silting.

2) $\mathsf{Hom}(\mathbf{P}, \mathbf{E}[1])$: apply $\mathsf{Hom}(\mathbf{P}, -)$ to the triangle to get the exact sequence:

$$\mathsf{Hom}(\mathbf{P}, \mathbf{P}') \xrightarrow{\mathsf{Hom}(\mathbf{P}, f)} \mathsf{Hom}(\mathbf{P}, A[1]) \to \mathsf{Hom}(\mathbf{P}, \mathbf{E}[1]) \to \mathsf{Hom}(\mathbf{P}, \mathbf{P}'[1])$$

Now, $\mathsf{Hom}(\mathbf{P}, \mathbf{P}'[1]) = 0$ since $\mathbf{P}$ is silting and $\mathbf{P}' \in \mathsf{add}\,\mathbf{P}$. Moreover $\mathsf{Hom}(\mathbf{P}, f)$ is epic since f is an $\mathsf{add}\,\mathbf{P}$-precover (and $\mathbf{P} \in \mathsf{add}\,\mathbf{P}$), so the map

$$\mathsf{Hom}(\mathbf{P}, A[1]) \to \mathsf{Hom}(\mathbf{P}, \mathbf{E}[1])$$

is the zero map. Thus, by exactness, we get $\mathsf{Hom}(\mathbf{P}, \mathbf{E}[1]) = 0$.

3) $\mathsf{Hom}(\mathbf{E}, \mathbf{P}[1])$: apply $\mathsf{Hom}(-, \mathbf{P})$ to the triangle to get the exact sequence:

$$\cdots \to \mathsf{Hom}(\mathbf{P}'[-1], \mathbf{P}) \to \mathsf{Hom}(\mathbf{E}[-1], \mathbf{P}) \to \mathsf{Hom}(A[-1], \mathbf{P}) \to \cdots$$

which is just:

$$\cdots \to \mathsf{Hom}(\mathbf{P}', \mathbf{P}[1]) \to \mathsf{Hom}(\mathbf{E}, \mathbf{P}[1]) \to \mathsf{Hom}(A, \mathbf{P}[1]) \to \cdots.$$

We have that $\mathsf{Hom}(\mathbf{P}', \mathbf{P}[1]) = 0$ since $\mathbf{P}' \in \mathsf{add}\,\mathbf{P}$ and also $\mathsf{Hom}(A, \mathbf{P}[1]) = 0$ since A is concentrated in degree zero and $\mathbf{P}[1]$ has no term in degree zero. So, by exactness, $\mathsf{Hom}(\mathbf{E}, \mathbf{P}[1]) = 0$.

4) $\mathsf{Hom}(\mathbf{E}, \mathbf{E}[1])$: apply $\mathsf{Hom}(-, \mathbf{E})$ to the triangle to get the exact sequence:

$$\cdots \to \mathsf{Hom}(\mathbf{P}'[-1], \mathbf{E}) \to \mathsf{Hom}(\mathbf{E}[-1], \mathbf{E}) \to \mathsf{Hom}(A[-1], \mathbf{E}) \to \cdots$$

which is just:

$$\cdots \to \mathsf{Hom}(\mathbf{P}', \mathbf{E}[1]) \to \mathsf{Hom}(\mathbf{E}, \mathbf{E}[1]) \to \mathsf{Hom}(A, \mathbf{E}[1]) \to \cdots.$$

As before, we have that $\mathsf{Hom}(\mathbf{P}', \mathbf{E}[1]) = 0$ by (2), since $\mathbf{P}' \in \mathsf{add}\,\mathbf{P}$, and also $\mathsf{Hom}(A, \mathbf{P}[1]) = 0$, since A is concentrated in degree zero and $\mathbf{E}[1]$ has no term in degree zero. So, by exactness, $\mathsf{Hom}(\mathbf{E}, \mathbf{E}[1]) = 0$.

Finally, from to the triangle, we have that $A \in \mathsf{thick}(\mathbf{P} \oplus \mathbf{E})$, so that $\mathsf{thick}(\mathbf{P} \oplus \mathbf{E}) = \mathcal{K}^b(\mathsf{proj}\,A)$. This concludes the proof. □

In particular, from this theorem, we get that if $\mathbf{P} \in K^b(\mathsf{proj}\,A)$ is a silting complex we have a triangle:

$$\Delta_{\mathbf{P}} : A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1].$$

3.2 A silting theorem

In this section we present the generalization of the theorem of Brenner and Butler to the case of silting complexes. It is due to Buan and Zhou in the paper [BZ16b].

We start with some results that will be necessary later. We recall that with $|X|$ we denote the number of indecomposable direct summand of X up to isomorphism.

Lemma 3.2.1. *Let $\boldsymbol{P} \in K^b(\mathsf{proj}\, A)$ be a presilting complex. Then, $\boldsymbol{P}$ is silting if and only $|\boldsymbol{P}| = |A|$.*

Proof. Omitted. Refer to [AIR14, Proposition 3.3]. □

Lemma 3.2.2. *Let $\boldsymbol{P} \in K^b(\mathsf{proj}\, A)$. Then, for any $\boldsymbol{P'} \in \mathsf{add}\, \boldsymbol{P}$ and $\boldsymbol{X} \in \mathcal{D}^b(A)$, there is a functorial isomorphism:*

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P'}, \boldsymbol{X}) \cong \mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, \boldsymbol{P'}), \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, \boldsymbol{X})).$$

Proof. If $\mathbf{P}' = \mathbf{P}$ we have:

$$\mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}), \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X})) = \mathsf{Hom}_B(B, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X})) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{X}).$$

The result then follows from the additivity of Hom. □

Lemma 3.2.3. *Let $0 \to t_X \to X \to f_X \to 0$ be the torsion sequence of $X \in \mathsf{mod}\, A$ associated to the torsion pair $(\mathcal{T}(\boldsymbol{P}), \mathcal{F}(\boldsymbol{P}))$. Then we have the isomorphisms:*

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, X) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, t_X),$$

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, X[1]) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, f_X[1]).$$

Proof. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ to the torsion sequence we obtain the long exact sequence of derived functor. Since $X \in \mathsf{mod}\, A$, using that $\mathsf{Ext}^1(\mathbf{P}, -) = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -[1])$, it becomes:

$$\begin{aligned} 0 \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, t_X) &\to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f_X) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, t_X[1]) \to \\ &\to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f_X[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, t_X[2]) = 0. \end{aligned}$$

Since $t_X \in \mathcal{T}(\mathbf{P})$ and $f_X \in \mathcal{F}(\mathbf{P})$ the sequence becomes:

$$\begin{aligned} 0 \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, t_X) \to &\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X) \to 0 \to \\ &\to 0 \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f_X[1]) \to 0 \end{aligned}$$

so, by exactness, we get the claim.

□

Lemma 3.2.4. *Let $\boldsymbol{X} \in \mathcal{K}^b(\mathsf{proj}\, A)$. If $H^0(\boldsymbol{X}) \cong H^{-1}(\nu \boldsymbol{X}) \cong 0$ then $\boldsymbol{X} \cong 0$.*

Proof. Let $\mathbf{X} = (X_{-1} \xrightarrow{x} X_0)$. From $H^0(\mathbf{X}) = X_0/\operatorname{im} x = 0$ we get that x is an epimorphism. Since X_0 is projective, x splits. Similarly, from $H^{-1}(\nu\mathbf{X}) = \ker \nu x = 0$, we get that νx is a monomorphism. Since ν is an equivalence between $\mathsf{proj}\, A$ and $\mathsf{inj}\, A$, we have that X_{-1} is injective and so νx splits. So νx is a split monomorphism in $\mathsf{inj}\, A$, thus x is a split monomorphism in $\mathsf{proj}\, A$. As it was already a split epimorphism, we have that x is an isomorphism. The complex is then exact and so isomorphic to zero in $\mathcal{D}^b(A)$. □

Obviously, also the converse hold, so we actually have an if and only if.

We are now ready to state and prove the first important result. We show that a 2-term silting complex in $\mathsf{mod}\, A$ induces a 2-term silting complex in $\mathsf{mod}\, B$. Moreover we obtain a result similar to the Theorem of Brenner and Butler: we have the equivalence between the classes of two torsion pairs in $\mathsf{mod}\, A$ and $\mathsf{mod}\, B$.

Let $\mathbf{P} \in \mathcal{K}^b(\mathsf{proj}\, A)$ be a 2-term silting complex. We have seen that there exists a triangle of the form $A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1]$ with $\mathbf{P}', \mathbf{P}'' \in \mathsf{add}\, \mathbf{P}$ which is unique up to homotopy equivalence. From it, we can build the 2-term complex $\mathbf{Q}$:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'').$$

We have that $\mathbf{Q}$ is unique up to isomorphism and moreover $\mathbf{Q} \in \mathcal{K}^b(\mathsf{proj}\, B)$. In fact we have that $\mathbf{P}, \mathbf{P}', \mathbf{P}'' \in \mathsf{add}\, \mathbf{P}$, so that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}')$ and $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'')$ are both in $\mathsf{End}(\mathsf{add}\, \mathbf{P}) = \mathsf{add}\, \mathsf{End}(\mathbf{P}) = \mathsf{add}\, B$ which is just $\mathsf{proj}\, B$.

Proposition 3.2.5. *The complex $\mathbf{Q}$ is a 2-term silting complex in $\mathcal{K}^b(\mathsf{proj}\, B)$. Moreover we have $\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P})$ and $\mathcal{F}(\mathbf{Q}) = \mathcal{Y}(\mathbf{P})$.*

Proof. We begin the proof by showing that $\mathbf{Q}$ is presilting, i.e. that $\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathbf{Q}[1]) = \mathsf{Hom}_{\mathcal{K}^b(\mathsf{proj}\, B)}(\mathbf{Q}, \mathbf{Q}[1]) = 0$. Let $\alpha \in \mathsf{Hom}_{\mathcal{K}^b(\mathsf{proj}\, B)}(\mathbf{Q}, \mathbf{Q}[1])$, we will show it is null-homotopic. Then α is of the form:

$$\begin{array}{ccccccc}
\mathbf{Q}: & & 0 & \longrightarrow & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)} & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'') \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle\alpha_{-1}} & & \downarrow{\scriptstyle 0} \\
\mathbf{Q}[1]: & & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)} & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'') & \longrightarrow & 0.
\end{array}$$

So $\alpha_{-1} \in \mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'), \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}''))$. Then by Lemma 3.2.2 there exists a unique $h \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', \mathbf{P}'')$ such that $\alpha_{-1} = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, h)$. Consider the diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\
 & & \downarrow{\scriptstyle h} & & & & \\
\mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] & \xrightarrow{-e[1]} & \mathbf{P}'[1].
\end{array}$$

Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(A, -)$ to the second triangle we get that

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(A, f) : \mathsf{Hom}_{\mathcal{D}^b(A)}(A, \mathbf{P}') \to \mathsf{Hom}_{\mathcal{D}^b(A)}(A, \mathbf{P}'')$$

is epic since $\mathsf{Hom}_{\mathcal{D}^b(A)}(A, A[1]) = 0$. So there exists $h_1 : A \to \mathbf{P}'$ such that $fh_1 = he$. Similarly, applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(-, A[1])$ to the first triangle, we get that $\mathsf{Hom}_{\mathcal{D}^b(A)}(f, A[1])$ is an epimorphism, so that there exists $h_2 : \mathbf{P}'' \to A[1]$ such that $h_2 f = gh$. We thus obtain a morphism of triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_1[1]}\downarrow \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] & \xrightarrow{-e[1]} & \mathbf{P}'[1]. \end{array}$$

We now show that h is null-homotopic. Consider the map $h_2 : \mathbf{P}'' \to A[1]$. Since $g : \mathbf{P}'' \to A[1]$ is an $\mathsf{add}\,\mathbf{P}$-precover, h_2 must factor through g. So there exists $h_3 : \mathbf{P}'' \to \mathbf{P}''$ such that $h_2 = gh_3$.

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h}\downarrow & {\scriptstyle h_3}\swarrow & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_1[1]}\downarrow \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] & \xrightarrow{-e[1]} & \mathbf{P}'[1]. \end{array}$$

We have $g(h - h_3 f) = gh - gh_3 f = gh - h_2 f = 0$ by commutativity. Then, by the triangles axioms (TR1) and (TR3), there exists $h_4 : \mathbf{P}' \to \mathbf{P}'$ such that $fh_4 = h - h_3 f$:

$$\begin{array}{ccccccc} \mathbf{P}' & \xrightarrow{1_{\mathbf{P}'}} & \mathbf{P}' & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbf{P}'[1] \\ {\scriptstyle h_4}\downarrow & & {\scriptstyle h-h_3 f}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle h_1[1]}\downarrow \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] & \xrightarrow{-e[1]} & \mathbf{P}'[1] \end{array}$$

We thus have $h = fh_4 + h_3 f$. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$, by additivity, we get:

$$\alpha = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, h) = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)\,\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, h_4) + \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, h_3)\,\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f).$$

Hence α is null-homotopic, so that it is zero in $\mathsf{Hom}_{\mathcal{K}^b(\mathsf{proj}\,B)}(\mathbf{Q}, \mathbf{Q}[1])$ and thus $\mathbf{Q}$ is presilting.

To show that it is silting we use Lemma 3.2.1 and show that $|\mathbf{Q}| = |A|$. Obviously we have $|\mathbf{Q}| \leq |A|$ since in A we have all indecomposable projectives as direct summands. So we show $|\mathbf{Q}| \geq |A|$. Take $A = \bigoplus_{i=1}^{n} P_i$ so that $P_1, P_2, ..., P_n$ are all the indecomposable, pairwise non-isomorphic, projective A-modules. For each $i = 1, ..., n$ consider an $\mathsf{add}\,\mathbf{P}$-envelope $P_i \xrightarrow{e_i} \mathbf{P}'_i$. Using that e and e_i are preenvelopes we obtain maps $a : \mathbf{P}'_i \to \mathbf{P}'$ and $b : \mathbf{P}' \to \mathbf{P}'_i$ such that $ae_i = e\iota_i$ and $e_i \pi_i = be$. Since e_i is an envelope, ba is an automorphism in P_i. So, up to precomposing with the inverse of ba, we obtain that $\mathbf{P}'_i$ is a direct summand of $\mathbf{P}'$. We can then take the cone $\mathbf{P}''_i$ of e_i and complete the morphisms we have to a morphism of triangle by (TR3), so there exists $c : \mathbf{P}''_i \to \mathbf{P}''$ and $d : \mathbf{P}'' \to \mathbf{P}''_i$. Since both 1_{P_i} and ba are isomorphisms, also dc is. So, as before, $\mathbf{P}''_i$ is a direct summand of $\mathbf{P}''$.

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ {\scriptstyle \pi_i}\downarrow\uparrow{\scriptstyle \iota_i} & & {\scriptstyle b}\downarrow\uparrow{\scriptstyle a} & & {\scriptstyle d}\downarrow\uparrow{\scriptstyle c} & & \\ P_i & \xrightarrow{e_i} & \mathbf{P}'_i & \xrightarrow{f_i} & \mathbf{P}''_i & \xrightarrow{g_i} & P_i[1] \end{array}$$

We thus get the triangles:

$$P_i \xrightarrow{e_i} \mathbf{P}'_i \xrightarrow{f_i} \mathbf{P}''_i \xrightarrow{g_i} P_i[1]$$

for each $i = 1, ..., n$. By construction, their direct sum is a direct summand of $\Delta_{\mathbf{P}}$. So, denoting by $\mathbf{Q}_i$ the 2-term complex

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'_i) \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f_i)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}''_i)$$

in $K^b(\mathsf{proj}\, B)$, we get that $\bigoplus_{i=1}^n \mathbf{Q}_i$ is isomorphic to a direct summand of $\mathbf{Q}$. To prove our statement we show that each $\mathbf{Q}_i$ is nonzero and any two of them have no direct summand in common.

If $\mathbf{Q}_i \cong 0$, then we obviously have $H^0(\mathbf{Q}_i) \cong 0$ and $H^{-1}(\nu\mathbf{Q}_i) \cong 0$. We get:

$$H^0(\mathbf{Q}_i) = \operatorname{coker} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f_i) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, P_i[1]) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, P_i/tP_i[1]),$$

where the first isomorphism follows from the fact that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'_i[1]) = 0$, while the second isomorphism follows from Lemma 3.2.3. On the other hand, we get:

$$\begin{aligned} H^{-1}(\nu\mathbf{Q}_i) &= \ker \nu\, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f_i) \cong \ker \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \nu f_i) \\ &\cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \nu P_i) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, t\nu P_i). \end{aligned}$$

Here the first isomorphism comes from the fact that:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \nu\mathbf{P}) \cong D\, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}) \cong DB \cong \nu B \cong \nu\, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P})$$

and by additivity of $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$, whereas the second isomorphism follows from the fact that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \nu\mathbf{P}''_i[1]) = 0$ and the last one again from Lemma 3.2.3. So, if $\mathbf{Q}_i \cong 0$, we have that $t\nu P_i \in \mathcal{F}(P)$ and $P_i/tP_i \in \mathcal{T}(P)$ so that they are both zero. This means that P_i is torsion, while νP_i is torsion free. P_i is projective, in particular is Ext-projective in $\mathcal{T}(\mathbf{P})$, so by Proposition 3.1.11 we have $P_i \in \mathsf{add}\, H^0(\mathbf{P})$. Again using the projectivity of P_i, there exists a morphism $P_i \to P^0$, so $P_i \in \mathsf{add}\,\mathbf{P}$. Similarly, using Proposition 3.1.11 and the fact that νP_i is injective, we obtain $P_i[1] \in \mathsf{add}\,\mathbf{P}$. But this is a contradiction since $1_{P_i} \in \mathsf{Hom}_{\mathcal{D}^b(A)}(P_i, P_i[1]) = 0$. So $\mathbf{Q}_i \not\cong 0$ for every $i = 1, ..., n$.

We need now to check that $\mathbf{Q}_i$ and $\mathbf{Q}_j$ have no common direct summands for $i \neq j$. If, for $i \neq j$, $H^0(\mathbf{Q}_i)$ and $H^0(\mathbf{Q}_j)$ have a common direct summand, then by the fact that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ is an equivalence, also P_i/tP_i and P_j/tP_j should have a common direct summand. But this is not possible, since we have that (if $P_i/tP_i \neq 0$) P_i is a projective cover of P_i/tP_i and P_i is indecomposable and not isomorphic to P_j. So $H^0(\mathbf{Q}_i)$ and $H^0(\mathbf{Q}_j)$ have no common direct summands. Similarly, also $H^{-1}(\nu\mathbf{Q}_i)$ and $H^{-1}(\nu\mathbf{Q}_j)$ have no common direct summands, since νP_i is an injective envelope of $t\nu P_i$ (if $t\nu P_i \neq 0$). So, if $\mathbf{Q}_i$ and $\mathbf{Q}_j$ have a common direct summand $\mathbf{X}$, then $H^0(\mathbf{X}) \cong H^{-1}(\nu\mathbf{X}) \cong 0$ and, by Lemma 3.2.4, we get $\mathbf{X} \cong 0$. Thus $|\mathbf{Q}| \geq |A|$ and so $\mathbf{Q}$ is silting.

It is left to prove that $\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P})$ and $\mathcal{F}(\mathbf{Q}) = \mathcal{Y}(\mathbf{P})$. We prove only the former as the proof of the latter is similar. From Proposition 3.1.9 we have a characterization of $\mathcal{T}(\mathbf{Q})$ as $\mathsf{gen}\, H^0(\mathbf{Q})$. We just showed that $H^0(\mathbf{Q}) \cong \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, A/tA[1])$, so we'll prove that $\mathcal{X}(\mathbf{P}) = \mathsf{gen}\,\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, A/tA[1])$. Recall that we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -[1]) : \mathcal{F}(\mathbf{P}) \to \mathcal{X}(\mathbf{P})$ which is an equivalence. Since by definition $A/tA[1] \in \mathcal{F}(\mathbf{P})$, we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, A/tA[1]) \in \mathcal{X}(\mathbf{P})$. Being a torsion class, $\mathcal{X}(\mathbf{P})$ is closed under quotients, so we get $\mathsf{gen}(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, A/tA[1])) \subseteq \mathcal{X}(\mathbf{P})$.

On the other hand, take $X \in \mathcal{X}(\mathbf{P})$. From the same equivalence we have the existence of $X' \in \mathcal{F}(\mathbf{P})$ such that $X = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X'[1])$. From Proposition 3.1.11 there exists a short exact sequence of the form

$$0 \to L \to F_0 \to X' \to 0$$

with $L \in \mathcal{F}(\mathbf{P})$ and $F_0 \in \mathsf{add}(A/tA)$. It induces the triangle

$$L \to F_0 \to X' \to L[1]$$

in $\mathcal{D}^b(A)$. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -[1])$ we obtain the long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, F_0[1]) \to$$
$$\to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X'[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L[2]) = 0 \to \cdots .$$

So we have:

$$X = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X'[1]) \in \mathsf{gen}(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, F_0[1])) \subseteq \mathsf{gen}(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, A/tA[1]))$$

since $F_0 \in \mathsf{add}(A/tA)$. This concludes the proof. □

3.3 Silting twice

In the tilting case we started from a finite-dimensional algebra A and a tilting module T over it. We then took B to be the endomorphism algebra of T and a tilting module over it, which happened to be T as a left B-module. We showed in Lemma 2.2.5 that taking the endomorphism algebra of ${}_BT$ we can recover the initial algebra A.

In the silting case we proceeded in the same way. We began by considering a finite-dimensional algebra A and a silting complex $\mathbf{P} \in K^b(\mathsf{proj}\,A)$. Again, we took B to be the endomorphism algebra of $\mathbf{P}$ and, in $K^b(\mathsf{proj}\,B)$, we built a complex $\mathbf{Q}$ which is silting by Proposition 3.2.5. Our aim now is to relate the endomorphism algebra of $\mathbf{Q}$, $\mathsf{End}_{\mathcal{D}^b(B)}(\mathbf{Q}) =: \bar{A}$, to the starting algebra A, generalizing what we had in the tilting case.

We thus want to define an algebra homomorphism $\phi_{\mathbf{P}} : A \to \bar{A}$. We will see that it is an epimorphism and that it is an isomorphism if $\mathbf{P}$ is tilting. To build $\phi_{\mathbf{P}}$, consider the triangle $\Delta_{\mathbf{P}}$. In particular, consider the map $\mathbf{P}' \xrightarrow{f} \mathbf{P}''$ and define $\hat{\mathbf{Q}} \in \mathsf{add}\,\mathbf{P}$ to be the complex given by $\mathbf{P}' \xrightarrow{f} \mathbf{P}''$ in degree -1 and 0, and zero in all the other degrees. Now, as we saw before, $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -) : \mathsf{add}\,\mathbf{P} \to \mathsf{proj}\,B$ is an equivalence of categories, so the functor $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ induces an equivalence of triangulated categories $K^b(\mathsf{add}\,\mathbf{P})$ and $K^b(\mathsf{proj}\,B)$, which sends $\hat{\mathbf{Q}}$ to $\mathbf{Q}$. So we obtain the isomorphisms:

$$\mathsf{End}_{K^b(\mathsf{add}\,\mathbf{P})}(\hat{\mathbf{Q}}) \cong \mathsf{End}_{K^b(\mathsf{proj}\,B)}(\mathbf{Q}) \cong \mathsf{End}_{\mathcal{D}^b(B)}(\mathbf{Q}) = \bar{A}.$$

In order to construct $\phi_{\mathbf{P}} : A \to \bar{A}$, we build an algebra homomorphism $\mathsf{End}_A(A) \to \mathsf{End}_{K^b(\mathsf{add}\,\mathbf{P})}(\hat{\mathbf{Q}})$. In the following, we will represent $\mathbf{P}'$ as:

$$\mathbf{P}' : P^{-1} \xrightarrow{p} P^0$$

and $\mathbf{P}''$ as the cone of $A \xrightarrow{e} \mathbf{P}'$:

$$\mathbf{P}'' : P^{-1} \oplus A \xrightarrow{(-p\ \ e)} P^0.$$

Moreover, in the triangle $\Delta_{\mathbf{P}}$:

$$A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1]$$

the maps e, f, g are given by:

$$\begin{array}{ccc} 0 & \xrightarrow{0} & A \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle e} \\ P^{-1} & \xrightarrow{p} & P^0 \end{array} \qquad \begin{array}{ccc} P^{-1} & \xrightarrow{p} & P^0 \\ {\scriptstyle \binom{-1}{0}}\downarrow & & \downarrow{\scriptstyle 1} \\ P^{-1} \oplus A & \xrightarrow{(-p\ \ e)} & P^0 \end{array} \qquad \begin{array}{ccc} P^{-1} \oplus A & \xrightarrow{(-p\ \ e)} & P^0 \\ {\scriptstyle (0\ \ 1)}\downarrow & & \downarrow{\scriptstyle 0} \\ A & \xrightarrow{0} & 0 \end{array}$$

Let $a \in \mathsf{End}_A(A)$. We want to find $b : \mathbf{P}' \to \mathbf{P}'$ and $c : \mathbf{P}'' \to \mathbf{P}''$ such that the following diagram commutes:

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle a[1]} \\ A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1]. \end{array}$$

To build b consider the diagram:

$$\begin{array}{ccccccc} \mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \\ & & {\scriptstyle a}\downarrow & & & & \\ \mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''. \end{array}$$

Since $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}''[-1], \mathbf{P}') = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', \mathbf{P}'[1]) = 0$ as $\mathbf{P}', \mathbf{P}'' \in \mathsf{add}\,\mathbf{P}$, there exists a map $b : \mathbf{P}' \to \mathbf{P}'$ such that $be = ea$:

$$\begin{array}{ccccccc} \mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \\ \mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''. \end{array}$$

Then $b = (b_1, b_2)$ is a chain map:

$$\begin{array}{ccc} P^{-1} & \xrightarrow{p} & P^0 \\ {\scriptstyle b_1}\downarrow & & \downarrow{\scriptstyle b_2} \\ P^{-1} & \xrightarrow{p} & P^0 \end{array}$$

and so $b_2 p = p b_1$.

The central square in the diagram with the two triangles is commutative in $K^b(\operatorname{proj} A)$, so the map $ea - be = (0, ea - b_2 e) : A \to \mathbf{P}'$ is zero, i.e. is null-homotopic. Thus, there exists a map $t : A \to P^{-1}$ such that $pt = ea - b_2 e$:

$$\begin{array}{ccc} 0 & \xrightarrow{\ 0\ } & A \\ {\scriptstyle 0}\downarrow & \overset{t}{\swarrow} & \downarrow{\scriptstyle ea-b_2e} \\ P^{-1} & \xrightarrow{\ p\ } & P^0 \end{array}$$

Consider now the map $c \in \operatorname{End}_{\mathcal{D}^b(A)}(\mathbf{P}')$ given by:

$$\begin{array}{ccc} P^{-1} \oplus A & \xrightarrow{(-p\ \ e)} & P^0 \\ {\scriptstyle \begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle b_2} \\ P^{-1} \oplus A & \xrightarrow{(-p\ \ e)} & P^0. \end{array}$$

It is a chain map (i.e. the previous diagram commutes). Indeed, using the definition of t and the fact that b is a chain map, we have:

$$(-p\ \ e)\begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix} = (-pb_1 \quad -pt + ea) = (-b_2 p\ \ b_2 e) = b_2(-p\ \ e).$$

Moreover we have that $cf = fb$ and $a[1]g = gc$. Indeed, for the former, both cf and fb are equal to b_2 in degree 0, whereas: $(cf)_{-1} = \begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix}\binom{-1}{0} = \binom{-b_1}{0} = \binom{-1}{0} b_1 = (fb)_{-1}$. Similarly, both gc and $a[1]g$ are zero in degree 0 and $(gc)_{-1} = (0\ \ 1)\begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix} = (0\ \ a) = a(0\ \ 1) = (a[1]g)_{-1}$.

Thus, (a, b, c) is a morphism of triangles and we obtain the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\ e\ } & \mathbf{P}' & \xrightarrow{\ f\ } & \mathbf{P}'' & \xrightarrow{\ g\ } & A[1] \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle a[1]} \\ A & \xrightarrow{\ e\ } & \mathbf{P}' & \xrightarrow{\ f\ } & \mathbf{P}'' & \xrightarrow{\ g\ } & A[1]. \end{array}$$

So for any $a \in \operatorname{End}_A(A)$ we build $(b, c) \in \operatorname{End}_{K^b(\operatorname{add} \mathbf{P})}(\hat{\mathbf{Q}})$. In the following proposition we show that this assignment is a surjective algebra homomorphism.

Proposition 3.3.1. *Let $\phi_{\boldsymbol{P}}$ be the map defined by:*

$$\phi_{\boldsymbol{P}} : \operatorname{End}_A(A) \to \operatorname{End}_{K^b(\operatorname{proj} A)}(\hat{\boldsymbol{Q}})$$
$$a \mapsto (b, c)$$

where $b = (b_1, b_2)$ and $c = \left(\begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix}, b_2\right)$. Then $\phi_{\boldsymbol{P}}$ is a well-defined algebra homomorphism. It is surjective and we have:

$$\ker \phi_{\boldsymbol{P}} = \left\{ \begin{matrix} v\alpha u \mid u \in \mathsf{Hom}_{\mathcal{D}^b(A)}(A, \boldsymbol{P}_1), \alpha \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}_1, \boldsymbol{P}_2[-1]), \\ v \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}_2[-1], A) \textit{ where } \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathsf{add}\, \boldsymbol{P} \end{matrix} \right\}.$$

Moreover, $\ker \phi_{\boldsymbol{P}} = 0$ *if and only if* $\mathsf{Hom}_{\mathcal{D}^b(A)}(\boldsymbol{P}, \boldsymbol{P}[-1]) = 0$.

Proof. We begin by showing that $\phi_{\mathbf{P}}$ is well-defined. Recall that to define $\phi_{\mathbf{P}}$ we made a choice on b and consequently on c. We need to show that two different choices of b give homotopic maps, so that they are equal in $K^b(\mathsf{add}\,\mathbf{P})$. So, for any $a \in \mathsf{End}_A(A)$, let (b^1, c^1) and (b^2, c^2) be two maps in $\mathsf{End}_{K^b(\mathsf{proj}\,A)}$ that make the following diagram commute:

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \\ {\scriptstyle b^i}\downarrow & & \downarrow{\scriptstyle c^i} \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \end{array}$$

for $i = 1, 2$. Recall that we have $b^i = (b_1^i, b_2^i)$ and $c^i = \left(\begin{pmatrix} b_1^i & t^i \\ 0 & a \end{pmatrix}, b_2^i \right)$ for $i = 1, 2$. To show that (b^1, c^1) and (b^2, c^2) are homotopic, we show that their difference:

$$(b^0, c^0) := (b^1, c^1) - (b^2, c^2) = \left((b_1^1 - b_1^2, b_2^1 - b_2^2), \left(\begin{pmatrix} b_1^1 - b_1^2 & t^1 - t^2 \\ 0 & 0 \end{pmatrix}, b_2^1 - b_2^2 \right) \right) =$$
$$= \left((b_1^0, b_2^0), \left(\begin{pmatrix} b_1^0 & t^0 \\ 0 & 0 \end{pmatrix}, b_2^0 \right) \right)$$

is null-homotopic. Consider the map $\mu : \mathbf{P}'' \to \mathbf{P}'$ given by:

$$\begin{array}{cccc} \mathbf{P}'' & P^{-1} \oplus A & \xrightarrow{(-p\ \ e)} & P^0 \\ {\scriptstyle \mu}\downarrow & \downarrow{\scriptstyle (-b_1^0\ \ -t^0)} & & \downarrow{\scriptstyle b_2^0} \\ \mathbf{P}' & P^{-1} & \xrightarrow[p]{} & P^0. \end{array}$$

First, note that μ is a chain map. Indeed:

$$p(-b_1^0 \quad -t^0) = (-pb_1^1 + pb_1^2 \quad -pt^1 + pt^2) = (-b_2^1 p + b_2^2 p \ \ b_2^1 e - b_2^2 e) = b_2^0(-p \ \ e)$$

by definition of t^1, t^2 and since b^1, b^2 are chain maps. We now check that μ is the desired homotopy, i.e. it is such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \\ {\scriptstyle (b_1^0,\, b_2^0)}\downarrow & \swarrow{\scriptstyle \mu} & \downarrow{\scriptstyle \left(\left(\begin{smallmatrix} b_1^0 & t^0 \\ 0 & 0 \end{smallmatrix}\right),\, b_2^0\right)} \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \end{array}$$

Indeed:

$$\mu f = ((-b_1^0 \quad -t^0), b_2^0) \circ \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, 1 \right) = \left((-b_1^0 \quad -t^0) \begin{pmatrix} -1 \\ 0 \end{pmatrix}, b_2^0 \right) = (b_1^0, b_2^0)$$

and

$$f\mu = \left(\begin{pmatrix}-1\\0\end{pmatrix}, 1\right) \circ ((-b_1^0 \quad -t^0), b_2^0) = \left(\begin{pmatrix}-1\\0\end{pmatrix}(-b_1^0 \quad -t^0), b_2^0\right) = \left(\begin{pmatrix}b_1^0 & t^0\\0 & 0\end{pmatrix}, b_2^0\right).$$

So (b^1, c^1) and (b^2, c^2) are homotopic and thus $\phi_{\mathbf{P}}$ is well-defined.

Now, $\phi_{\mathbf{P}}$ is an algebra homomorphism. Indeed, let $a, \bar{a} \in \mathsf{End}_A(A)$. Then

$$\phi_{\mathbf{P}}(a\bar{a}) = \left(\tilde{b}, \left(\begin{pmatrix}\tilde{b_1} & \tilde{t}\\0 & a\bar{a}\end{pmatrix}, \tilde{b_2}\right)\right)$$

with $\tilde{b}$ such that $\tilde{b}e = ea\bar{a}$ and $\tilde{t}$ such that $p\tilde{t} = ea\bar{a} - \tilde{b_2}e$. On the other hand we have:

$$\phi_{\mathbf{P}}(a)\phi_{\mathbf{P}}(\bar{a}) = \left(b\bar{b}, \left(\begin{pmatrix}b_1 & t\\0 & a\end{pmatrix}\begin{pmatrix}\bar{b_1} & \bar{t}\\0 & \bar{a}\end{pmatrix}, b_2\bar{b_2}\right)\right) = \left((b_1\bar{b_1}, b_2\bar{b_2}), \left(\begin{pmatrix}b_1\bar{b_1} & b_1\bar{t} + t\bar{a}\\0 & a\bar{a}\end{pmatrix}, b_2\bar{b_2}\right)\right)$$

with $be = ea$, $\bar{b}e = e\bar{a}$ and $pt = ea - b_2e$, $p\bar{t} = e\bar{a} + \bar{b_2}e$. To show that they are equal it suffices to show that $b\bar{b}$ and $b_1\bar{t} + t\bar{a}$ satisfy the condition on $\tilde{b}$ and $\tilde{t}$ respectively. We have:

$$(b\bar{b})e = be\bar{a} = ea\bar{a}$$

using the properties of $\bar{b}$ and b. In particular, $b_2\bar{b_2} = \tilde{b_2}$. Moreover:

$$\begin{aligned} p(b_1\bar{t} + t\bar{a}) &= pb_1\bar{t} + pt\bar{a} = b_2p\bar{t} + ea\bar{a} - b_2e\bar{a} = b_2(e\bar{a} - \bar{b_2}e) + ea\bar{a} - b_2e\bar{a} \\ &= -b_2\bar{b_2}e + ea\bar{a} = e(a\bar{a}) - \tilde{b_2}e \end{aligned}$$

using the properties of $t, \bar{t}$ and the fact that b is a chain map. So we have $\phi_{\mathbf{P}}(a\bar{a}) = \phi_{\mathbf{P}}(a)\phi_{\mathbf{P}}(\bar{a})$ and so $\phi_{\mathbf{P}}$ is an algebra homomorphism.

We now check that $\phi_{\mathbf{P}}$ is surjective. Let $(b, c) \in \mathsf{End}_{K^b(\mathsf{add}\,\mathbf{P})}(\hat{\mathbf{Q}})$ defined by:

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' \\ {\scriptstyle (b_1, b_2)}\downarrow & & \downarrow{\scriptstyle \left(\begin{pmatrix}c_1 & c_2\\c_3 & c_4\end{pmatrix}, c_0\right)} \\ \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' \end{array}$$

We will show that it is the image of some $a \in \mathsf{End}_A(A)$ via $\phi_{\mathbf{P}}$. In particular, since we are in $K^b(\mathsf{add}\,\mathbf{P})$, we will show that (b, c) is homotopy equivalent to a map of the form:

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' \\ {\scriptstyle (b_1,\, b_2)}\downarrow & & \downarrow{\scriptstyle \left(\begin{pmatrix}b_1 & t\\0 & a\end{pmatrix}, b_2\right)} \\ \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' \end{array}$$

for some $a \in \mathsf{End}_A(A)$ and some $t : A \to P^{-1}$ such that $pt = ea - b_2e$ (as, by construction of $\phi_{\mathbf{P}}$, t has to be an homotopy between $ea - b_2e$ and zero). Since (b, c) is a chain map we have $cf = fb$ in $K^b(\mathsf{proj}\,A)$, i.e. they are homotopic. Now:

$$cf = \left(\begin{pmatrix}c_1 & c_2\\c_3 & c_4\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}, c_0\right) = \left(\begin{pmatrix}-c_1\\-c_3\end{pmatrix}, c_0\right)$$

and

$$fb = \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} b_1, b_2 \right) = \left(\begin{pmatrix} -b_1 \\ 0 \end{pmatrix}, b_2 \right).$$

So there exists a homotopy $\binom{x}{y} : P^0 \to P^{-1} \oplus A$ such that the following diagram commutes:

$$\begin{tikzcd} P^{-1} \arrow[r, "p"] \arrow[d, "\binom{c_1 - b_1}{c_3}"'] & P^0 \arrow[d, "b_2 - c_0"] \arrow[dl, dashed, "\binom{x}{y}"'] \\ P^{-1} \oplus A \arrow[r, "(-p \ \ e)"'] & P^0. \end{tikzcd}$$

i.e. such that:

$$\begin{cases} xp = c_1 - b_1 \\ yp = c_3 \\ -px + ey = b_2 - c_0. \end{cases}$$

Define $a := c_4 + ye$ and $t := c_2 + xe$. With this choice of a and t we get the claim. First of all we have that t satisfies $pt = ea - b_2 e$. To show it, consider the chain map:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "\binom{0}{1}"'] & P^0 \arrow[d, "1"] \\ P^{-1} \oplus A \arrow[r, "(-p \ \ e)"'] & P^0 \end{tikzcd}$$

which, composed with c, yields the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "\binom{c_2}{c_4}"'] & P^0 \arrow[d, "c_0"] \\ P^{-1} \oplus A \arrow[r, "(-p \ \ e)"'] & P^0 \end{tikzcd}$$

so that we have $c_0 e = ec_4 - pc_2$. Using this fact and the conditions we found before we get:

$$pt = pc_2 + pxe = pc_2 + c_0 e + eye - b_2 e = ec_4 + eye - b_2 e = e(c_4 + ye) - b_2 e = ea - b_2 e$$

as we wanted. Let $\bar{c} := \left(\begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix}, b_2 \right) = \left(\begin{pmatrix} b_1 & c_2 + xe \\ 0 & c_4 + ye \end{pmatrix}, b_2 \right)$. It is left to show that c and $\bar{c}$ are homotopic. We have:

$$\bar{c} - c = \left(\begin{pmatrix} b_1 - c_1 & xe \\ -c_3 & ye \end{pmatrix}, b_2 - c_0 \right) = \left(\begin{pmatrix} -xp & xe \\ -yp & ye \end{pmatrix}, -px + ey \right) = \left(\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -p & e \end{pmatrix}, \begin{pmatrix} -p & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

so that the diagram:

$$\begin{tikzcd} P^{-1} \oplus A \arrow[r, "(-p \ \ e)"] \arrow[d, "\bar{c}^{-1} - c^{-1}"'] & P^0 \arrow[d, "\bar{c}^0 - c^0"] \arrow[dl, "\binom{x}{y}"'] \\ P^{-1} \oplus A \arrow[r, "(-p \ \ e)"'] & P^0 \end{tikzcd}$$

commutes and thus $\bar{c}$ and c are homotopic via $\binom{x}{y}$. Hence, $\phi_{\mathbf{P}}$ is surjective. Now we want to give a description of $\ker \phi_{\mathbf{P}}$. Define:

$$I := \left\{ \begin{matrix} v\alpha u \mid u \in \mathsf{Hom}_{\mathcal{D}^b(A)}(A, \mathbf{P}_1), \alpha \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}_1, \mathbf{P}_2[-1]), \\ v \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}_2[-1], A) \text{ where } \mathbf{P}_1, \mathbf{P}_2 \in \mathsf{add}\,\mathbf{P} \end{matrix} \right\}.$$

$$A \xrightarrow{u} \mathbf{P}_1 \xrightarrow{\alpha} \mathbf{P}_2[-1] \xrightarrow{v} A$$

We will show that $\ker \phi_{\mathbf{P}} = I$, so the kernel is the family of endomorphisms of A that factors through two complexes of $\mathsf{add}\,\mathbf{P}$, with one shifted. We first show $\ker \phi_{\mathbf{P}} \subseteq I$. Take $a \in \ker \phi_{\mathbf{P}}$. Then $\phi_{\mathbf{P}}(a) = (b, c) = 0$ in $K^b(\mathsf{add}\,\mathbf{P})$, i.e. it is null-homotopic. Thus there exists $d = ((d_1, d_2), w) : \mathbf{P}'' \to \mathbf{P}'$ such that $b = df$ and $c = fd$ in $K^b(\mathsf{add}\,\mathbf{P})$:

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \\ {\scriptstyle b}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle c} \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''. \end{array}$$

In particular we have b homotopic to df:

$$(b_1, b_2) \sim ((d_1, d_2), w) \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, 1 \right) = (-d_1, w).$$

So there exists $\delta : P^0 \to P^{-1}$ homotopy:

$$\begin{array}{ccc} P^{-1} & \xrightarrow{p} & P^0 \\ {\scriptstyle b_1}\downarrow\downarrow{\scriptstyle -d_1} & \overset{\delta}{\swarrow} & {\scriptstyle b_2}\downarrow\downarrow{\scriptstyle w} \\ P^{-1} & \xrightarrow[p]{} & P^0 \end{array}$$

such that:

$$\begin{cases} \delta p = b_1 + d_1 \\ p\delta = b_2 - w. \end{cases}$$

On the other hand, we have c homotopic to fd:

$$\left(\begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix}, b_2 \right) \sim \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, 1 \right) ((d_1, d_2), w) = \left(\begin{pmatrix} -d_1 & -d_2 \\ 0 & 0 \end{pmatrix}, w \right).$$

So there exists $\binom{\varepsilon}{\theta} : P^0 \to P^{-1} \oplus A$ homotopy:

$$\begin{array}{ccc} P^{-1} \oplus A & \xrightarrow{(-p \;\; e)} & P^0 \\ {\scriptstyle \left(\begin{smallmatrix} b_1 & t \\ 0 & a \end{smallmatrix}\right)}\downarrow\downarrow{\scriptstyle \left(\begin{smallmatrix} -d_1 & -d_2 \\ 0 & 0 \end{smallmatrix}\right)} & \overset{\binom{\varepsilon}{\theta}}{\swarrow} & {\scriptstyle b_2}\downarrow\downarrow{\scriptstyle w} \\ P^{-1} \oplus A & \xrightarrow[(-p \;\; e)]{} & P^0 \end{array}$$

such that:

$$\begin{cases} \begin{pmatrix} b_1+d_1 & t+d_2 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \varepsilon \\ \theta \end{pmatrix} (-p \;\; e) = \begin{pmatrix} -\varepsilon p & \varepsilon e \\ -\theta p & \theta e \end{pmatrix} \\ b_2 - w = (-p \;\; e) \begin{pmatrix} \varepsilon \\ \theta \end{pmatrix} = -p\,\varepsilon + e\theta. \end{cases}$$

In particular we have $\theta p = 0$ and $\theta e = a$. Consider now the map $\begin{pmatrix} \delta+\varepsilon \\ \theta \end{pmatrix} : P^0 \to P^{-1} \oplus A$ and the diagram:

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow{\scriptstyle e} & & \\ P^{-1} & \xrightarrow{p} & P^0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \binom{\delta+\varepsilon}{\theta}} & & \downarrow \\ 0 & \longrightarrow & P^{-1} \oplus A & \xrightarrow[(-p \;\; e)]{} & P^0 \\ & & \downarrow{\scriptstyle (0 \;\; 1)} & & \\ & & A. & & \end{array}$$

Using the conditions above, we have that:

$$p(\delta+\varepsilon) = p\delta + p\,\varepsilon = b_2 - w - b_2 + w + e\theta = e\theta$$

and

$$(\delta+\varepsilon)p = \delta p + \varepsilon\, p = b_1 + d_1 - b_1 - d_1 = 0.$$

These two, together with $\theta p = 0$, ensure that $\begin{pmatrix} \delta+\varepsilon \\ \theta \end{pmatrix}$ is a chain map. Finally, we have that $(0 \;\; 1) \begin{pmatrix} \delta+\varepsilon \\ \theta \end{pmatrix} e = \theta e = a$, so the vertical map in the diagram compose to a and thus $a \in I$ as it factors through P^0 and $P^{-1} \oplus$.

We have to prove now that $I \subseteq \ker \phi_{\mathbf{P}}$. Let $a \in I$, then we can write $a = v\alpha u$ with $u \in \mathsf{Hom}_{\mathcal{D}^b(A)}(A, \mathbf{P}_1)$, $\alpha \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}_1, \mathbf{P}_2[-1])$ and $v \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}_2[-1], A)$ where $\mathbf{P}_1, \mathbf{P}_2[-1] \in \mathsf{add}\,\mathbf{P}$. Since $e : A \to \mathbf{P}'$ is an $\mathsf{add}\,\mathbf{P}$-preenvelope, there exists $u' : \mathbf{P}' \to \mathbf{P}_1$ such that $u'e = u$. Similarly, since $g : \mathbf{P}'' \to A[1]$ is an $\mathsf{add}\,\mathbf{P}$-precover, there exists $v' : \mathbf{P}_2[-1] \to \mathbf{P}'[-1]$ such that $g[-1]v' = v$. Let $\beta : \mathbf{P}' \to \mathbf{P}''[-1]$ defined by $\beta := v'\alpha u'$. Then we have $a = g[-1]\beta e$:

$$\begin{array}{ccccccc} & \nearrow^{e} & \mathbf{P}' & \xrightarrow{\beta} & \mathbf{P}''[-1] & \searrow^{g[-1]} & \\ & & \vdots{\scriptstyle u'} & & \vdots{\scriptstyle v'} & & \\ A & \xrightarrow{u} & \mathbf{P}_1 & \xrightarrow{\alpha} & \mathbf{P}_2[-1] & \xrightarrow{v} & A \\ & & & \underbrace{\qquad\qquad}_{a} & & & \end{array}$$

Let β be represented by:

$$\begin{array}{ccccc} P^{-1} & \xrightarrow{p} & P^0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \binom{\beta_1}{\beta_2}} & & \downarrow \\ 0 & \longrightarrow & P^{-1} \oplus A & \xrightarrow[(-p \;\; e)]{} & P^0. \end{array}$$

Thus, we have:

$$\begin{cases} \beta_1 p = 0 \\ \beta_2 p = 0 \\ p\beta_1 = e\beta_2 \end{cases}$$

and since $g[-1] = (0\ \ 1)$ we also have:

$$a = g[-1]\beta_2 e = (0\ \ 1)\left(\begin{smallmatrix}\beta_1 \\ \beta_2\end{smallmatrix}\right) e = \beta_2 e.$$

Consider the map $\psi \in \mathsf{End}_{K^b(\mathsf{add}\,\mathbf{P})}(\hat{Q})$, which is homotopic to $\phi_{\mathbf{P}}(a)$, given by:

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\ f\ } & \mathbf{P}'' \\ {\scriptstyle (0,\, e\beta_2)}\downarrow & & \downarrow{\scriptstyle \left(\left(\begin{smallmatrix}0 & 0 \\ 0 & \beta_2 e\end{smallmatrix}\right),\, e\beta_2\right)} \\ \mathbf{P}' & \xrightarrow{\ f\ } & \mathbf{P}''. \end{array}$$

ψ is null-homotopic in $\mathsf{add}\,\mathbf{P}$. In fact, $(0, e\beta_2)$ is null-homotopic in $K^b(\mathsf{proj}\,A)$:

$$\begin{array}{ccc} P^{-1} & \xrightarrow{\ p\ } & P^0 \\ {\scriptstyle 0}\downarrow & \swarrow{\scriptstyle \beta_1} & \downarrow{\scriptstyle e\beta_2} \\ P^{-1} & \xrightarrow{\ p\ } & P^0 \end{array}$$

since, by before, $\beta_1 p = 0$ and $e\beta_2 = p\beta_1$. Also $\left(\left(\begin{smallmatrix}0 & 0 \\ 0 & \beta_2 e\end{smallmatrix}\right), e\beta_2\right)$ is null-homotopic in $K^b(\mathsf{proj}\,A)$:

$$\begin{array}{ccc} P^{-1}\oplus A & \xrightarrow{\ (-p\ \ e)\ } & P^0 \\ {\scriptstyle \left(\begin{smallmatrix}0 & 0 \\ 0 & \beta_2 e\end{smallmatrix}\right)}\downarrow & \swarrow{\scriptstyle \left(\begin{smallmatrix}0 \\ \beta_2\end{smallmatrix}\right)} & \downarrow{\scriptstyle e\beta_2} \\ P^{-1}\oplus A & \xrightarrow[\ (-p\ \ e)\ ]{} & P^0 \end{array}$$

since $(-p\ \ e)\left(\begin{smallmatrix}0 \\ \beta_2\end{smallmatrix}\right) = e\beta_2$ and:

$$\begin{pmatrix}0 \\ \beta_2\end{pmatrix}(-p\ \ e) = \begin{pmatrix}0 & 0 \\ -\beta_2 p & \beta_2 e\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & \beta_2 e\end{pmatrix}.$$

Thus, $a \in \ker\phi_{\mathbf{P}}$ and so $\ker\phi_{\mathbf{P}} = I$.

It is left to prove that $\ker\phi_{\mathbf{P}} = 0$ if and only if $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$. Obviously, if we have no map between $\mathbf{P}$ and $\mathbf{P}[-1]$, by the description of $\ker\phi_{\mathbf{P}}$ we have $\ker\phi_{\mathbf{P}} = 0$. Conversely, let $0 \neq \eta \in \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}[-1])$. η is a chain map of the form:

$$\begin{array}{ccccc} P^{-1} & \xrightarrow{\ p\ } & P^0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow \\ 0 & \longrightarrow & P^{-1} & \xrightarrow{\ p\ } & P^0. \end{array}$$

Since η is not zero, there exists P_i and P_j projective indecomposable direct summands of P^0 and P^{-1} respectively, such that η restricted to P_i and P_j is non-zero. As they are projective indecomposable, they are also direct summands of A, so η induces a map $a_\eta \in \mathsf{End}_A(A)$ which factors through η. Thus $\ker\phi_{\mathbf{P}} \neq 0$ and so we get the claim. $\square$

Note that the last statement of the previous proposition means exactly that $\phi_{\mathbf{P}}$ is an isomorphism if and only if $\mathbf{P}$ is a tilting complex. In this sense we can see the proposition as a generalization of what happens in the tilting case.

Corollary 3.3.2. *If $\boldsymbol{P}$ is tilting, then also $\boldsymbol{Q}$ is tilting.*

Proof. To show that $\mathbf{Q} : \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P},f)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'')$ is tilting we show that $\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathbf{Q}[-1]) = 0$. So, let $\alpha \in \mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathbf{Q}[-1])$. Then it is of the form:

$$\begin{array}{ccccc}
\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P},f)} & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'') & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
0 & \longrightarrow & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P},f)} & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'')
\end{array}$$

and such that $\alpha \, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f) = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)\alpha = 0$ since α is a chain map.

By Lemma 3.2.2, since $\mathbf{P}', \mathbf{P}'' \in \mathsf{add}\,\mathbf{P}$, we have an isomorphism:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', \mathbf{P}') \cong \mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}''), \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'))$$

so there exists $h : \mathbf{P}'' \to \mathbf{P}'$ such that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, h) = \alpha$. Moreover $hf = fh = 0$ since h is a chain map:

$$\begin{array}{ccccc}
\mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\
0 & \longrightarrow & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''.
\end{array}$$

We thus have the following diagram:

$$\begin{array}{ccccccc}
 & & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\
 & & & & & & \downarrow{\scriptstyle h} & & \\
\mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''. & &
\end{array}$$

Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', -)$ to the second triangle, we obtain the exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', A) \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'',e)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', \mathbf{P}') \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'',f)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', \mathbf{P}'') \to \cdots.$$

Now, $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', f)(h) = fh = 0$, so $h \in \ker \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', f)$ and thus, by exactness, $h \in \operatorname{im} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', e)$. Hence there exists $h_1 : \mathbf{P}'' \to A$ such that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', e)(h_1) = eh_1 = h$:

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\
 & & & & {\scriptstyle h_1 f}\,\vdots\downarrow & {\scriptstyle h_1}\swarrow & \downarrow{\scriptstyle h} & & \\
\mathbf{P}'[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''. & &
\end{array}$$

Similarly as before, we have $eh_1f = 0$ and so, applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', -)$ to the second triangle, we get the exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', \mathbf{P}''[-1]) \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', -g[-1])} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', A) \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', e)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', \mathbf{P}') \to \cdots .$$

We have $h_1f \in \ker \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', e)$ since $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', e)(h_1f) = eh_1f = 0$. So by exactness, $h_1f \in \operatorname{im} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', -g[-1])$, i.e. there exists $h_2 : \mathbf{P}' \to \mathbf{P}''[-1]$ such that $-g[-1]h_2 = h_1f$:

$$\begin{array}{ccccccc}
 & & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\
 & \swarrow^{h_2} & & & \downarrow h_1f & & \downarrow h & & \\
\mathbf{P}'[-1] & \xrightarrow{-g[-1]} & & & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''.
\end{array}$$

Now, $h_2 : \mathbf{P}' \to \mathbf{P}''[-1]$ and we have $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', \mathbf{P}''[-1]) = 0$ since $\mathbf{P}$ is tilting. So $h_2 = 0$ and $h_1f = 0$. Applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(-, A)$ to the first triangle we obtain the exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(A)}(A[1], A) \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(g, A)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', A) \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(f, A)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', A) \to \cdots .$$

Since $h_1f = 0$, $h_1 \in \ker \mathsf{Hom}_{\mathcal{D}^b(A)}(f, A)$. But $\mathsf{Hom}_{\mathcal{D}^b(A)}(A[1], A) = 0$ and so, by exactness, $\mathsf{Hom}_{\mathcal{D}^b(A)}(f, A)$ is monic. Thus $h_1 = 0$ and so $h = eh_1 = 0$, which implies $\alpha = 0$. So $\mathbf{Q}$ is tilting. □

We built an epimorphism $\phi_{\mathbf{P}} : A \to \mathsf{End}_{\mathcal{D}^b(B)}(\mathbf{Q}) = \bar{A}$. This induces an inclusion functor between the categories of modules $\phi_* : \mathsf{mod}\, \bar{A} \hookrightarrow \mathsf{mod}\, A$.

Theorem 3.3.3. *We have $\phi_*(\mathcal{X}(\mathbf{Q})) = \mathcal{T}(\mathbf{P})$ and $\phi_*(\mathcal{Y}(\mathbf{Q})) = \mathcal{F}(\mathbf{P})$*

Proof. We prove that $\phi_*(\mathcal{Y}(\mathbf{Q})) = \mathcal{F}(\mathbf{P})$ as the other statement is similar. We have the following chain of equivalences:

$$\begin{aligned}
\mathcal{Y}(\mathbf{Q}) &= \mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathcal{T}(\mathbf{Q})) = \mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathcal{X}(\mathbf{P})) = \\
&= \mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{F}(\mathbf{P})[1]))
\end{aligned}$$

where the first and last equalities follows from the definitions of $\mathcal{T}(\mathbf{Q})$ and $\mathcal{X}(\mathbf{P})$, while the second one follows from Proposition 3.2.5.

We want to show that there exists an A-module isomorphism between Y and

$$\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1]))$$

for each $Y \in \mathcal{F}(\mathbf{P})$. Indeed from the previous chain we have that $\mathcal{Y}(\mathbf{Q})$ is just the family of $\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1]))$ with $Y \in \mathcal{F}(\mathbf{P})$.

There exists a triangle:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'') \to \mathbf{Q} \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}')[1]$$

since $\mathbf{Q}$ is the cone of $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)$. Applying $\mathsf{Hom}_B(-, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1]))$ to the triangle, since $\mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}')[1], \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1])) = 0$, we get that

$$\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1]))$$

is the kernel of the map

$$\mathsf{Hom}_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f), \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1])).$$

By Lemma 3.2.2 this is isomorphic to:

$$\mathsf{Hom}_{\mathcal{D}^b(A)}(f, Y[1]) : \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}'', Y[1]) \to \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', Y[1]).$$

Since $Y \in \mathcal{F}(\mathbf{P})$, $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}', Y) = 0$, so by applying $\mathsf{Hom}_{\mathcal{D}^b(A)}(-, Y[1])$ to the triangle $\Delta_{\mathbf{P}}$ we get that $\mathsf{Hom}_{\mathcal{D}^b(A)}(A, Y) \cong \mathsf{Hom}_A(A, Y) \cong Y$ is the kernel of $\mathsf{Hom}_{\mathcal{D}^b(A)}(f, Y[1])$. So we obtain the vector space isomorphism:

$$\varphi : \mathsf{Hom}_A(A, Y) \cong \mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1])).$$

For any map $v \in \mathsf{Hom}_A(A, Y)$, the morphism $\varphi(v)$ is given by the chain map:

$$\begin{array}{ccc} \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, f)} & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'') \\ \downarrow & & \downarrow {\scriptstyle \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, v[1]g)} \\ 0 & \longrightarrow & \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y[1]) \end{array}$$

So it is given by the shifted post-composition to $g : \mathbf{P}'' \to A[1]$. We now prove that φ is an A-module map. To show it, let $a \in \mathsf{End}_A(A)$ and $(b, c) = \phi_{\mathbf{P}}(a)$. We have:

$$\begin{aligned} \phi_{\mathbf{P}}(a)\varphi(v) &= (0, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, v[1]g)) \circ \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, (b, c)) = (0, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, v[1]gc)) \\ &= (0, \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, v[1]a[1]g)) = \varphi(va) \end{aligned}$$

where the third equality follows from the fact that (a, b, c) is a map of triangles. So φ is an A-module isomorphism and we conclude the proof since $Y \cong \mathsf{Hom}_A(A, Y)$. □

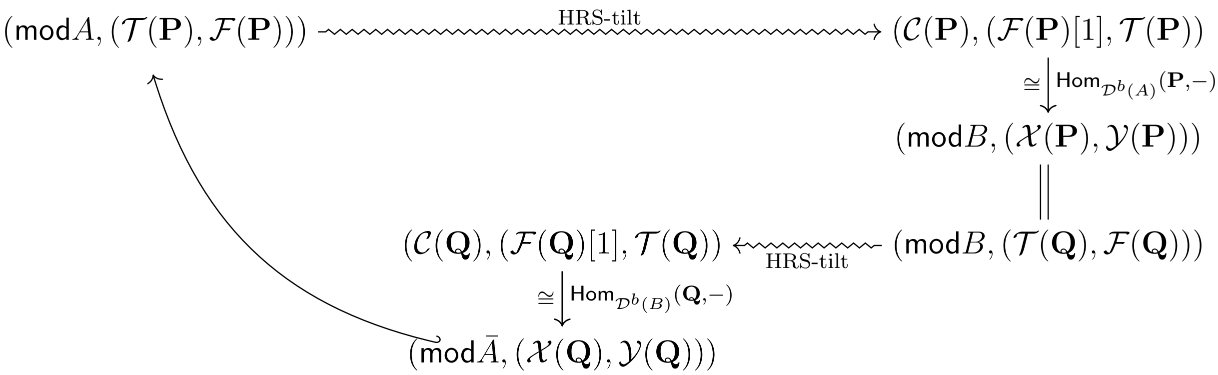
Let us recap what we have done in this chapter, following the diagram below. Our setting is $\mathcal{D}^b(A)$ and we take a two-term silting complex $\mathbf{P}$. By Proposition 3.1.12 we have that $\mathbf{P}$ induces a torsion pair in $\mathsf{mod}\, A$: $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$. In Proposition 3.1.12 we also saw that the via the HRS-tilting on this torsion pair we obtain the t-structure $(\mathcal{D}^{\leq 0}(\mathbf{P}), \mathcal{D}^{\geq 0}(\mathbf{P}))$. In its heart $\mathcal{C}(\mathbf{P})$ we have the torsion pair $(\mathcal{F}(\mathbf{P})[1], \mathcal{T}(\mathbf{P}))$. By Theorem 3.1.14 we have that $\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ is an equivalence of abelian categories between $\mathcal{C}(\mathbf{P})$ and $\mathsf{mod}\, B$, where $B = \mathsf{End}_{\mathcal{D}^b(A)}(\mathbf{P})$. Moreover, by Corollary 3.1.15 we have that $\mathcal{X}(\mathbf{P})$ and $\mathcal{Y}(\mathbf{P})$ are equivalent to $\mathcal{F}(\mathbf{P})[1]$ and $\mathcal{T}(\mathbf{P})$ respectively. So $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ forms a torsion pair in $\mathsf{mod}\, B$.

Now in $\mathcal{D}^b(B)$ we define the complex $\mathbf{Q}$ using the morphism $P' \xrightarrow{f} P''$ coming from the triangle $\Delta_{\mathbf{P}}$. From Proposition 3.2.5 we have that $\mathbf{Q}$ is a silting complex and so, as before with $\mathbf{P}$, $\mathbf{Q}$ induces a torsion pair $(\mathcal{T}(\mathbf{Q}), \mathcal{F}(\mathbf{Q}))$ in $\mathsf{mod}\, B$. From the same proposition we also have that the torsion pair induced by $\mathbf{Q}$ coincides with the torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$. Thus, we have passed from $\mathsf{mod}\, A$, with the torsion pair induced by $\mathbf{P}$, to $\mathsf{mod}\, B$, with the torsion pair induced by $\mathbf{Q}$.

Repeating the same process done with $\mathbf{Q}$ we obtain firstly, as a result of the HRS-tilting, the torsion pair $(\mathcal{F}(\mathbf{Q})[1], \mathcal{T}(\mathbf{Q}))$ and then, via the equivalence $\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, -)$, the torsion pair $(\mathcal{X}(\mathbf{Q}), \mathcal{Y}(\mathbf{Q}))$ in $\mathsf{mod}\, \bar{A}$, where $\bar{A} := \mathsf{End}_{\mathcal{D}^b(B)}(\mathbf{Q})$. In Proposition 3.3.1 we show that $\bar{A}$ is an epimorphic image of A. So, in particular, passing to the categories of modules, we have a natural inclusion $\mathsf{mod}\, \bar{A} \hookrightarrow \mathsf{mod}\, A$. Finally, by Theorem 3.3.3 we have that the torsion pair $(\mathcal{X}(\mathbf{Q}), \mathcal{Y}(\mathbf{Q}))$ coincides with $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$.

So, if we start from a silting complex $\mathbf{P}$, and apply the previous process twice, we don't return back to the whole category of modules over the original algebra A, but to a subcategory of it. However, we still get an equivalence if we restrict to the torsion pair induced by $\mathbf{P}$.

It is interesting to notice that if $\mathbf{P}$ is tilting, then by Proposition 3.3.1 we have $\bar{A} = A$ and so we obtain the same results found in Chapter 2.



3.4 Silted algebras

In this section we will be working in the setting of hereditary algebras or hereditary abelian categories. We will study endomorphism algebras of 2-term silting complexes and characterize them with some nice homological properties.

We recall that an algebra is called *hereditary* if it has global dimension equal to 1, i.e. if $\mathsf{Ext}^n(-,-)$ vanishes for all $n \geq 2$. Similarly an abelian category is said to be hereditary if $\mathsf{Ext}^n(-,-)$ vanishes for all $n \geq 2$.

We now introduce the main object that we will study in this section: silted algebras and shod algebras.

Definition 3.4.1. Let B be a finite-dimensional K-algebra, with K field. We say that B is *silted* if there exists a finite-dimensional hereditary algebra A and a 2-term silting complex $\mathbf{P} \in K^b(\mathsf{proj}\, A)$ such that $B \cong \mathsf{End}_{\mathcal{D}^b(A)}(\mathbf{P})$.

The algebra is called *quasi-silted* if there exists an Ext-finite hereditary abelian category $\mathcal{H}$ and a 2-term silting complex $\mathbf{P} \in \mathcal{D}^b(\mathcal{H})$ such that $B \cong \mathsf{End}_{\mathcal{D}^b(\mathcal{H})}(\mathbf{P})$.

So a silted algebra is simply the endomorphism algebra of a silting complex coming from a finite-dimensional hereditary algebra, whereas in the case of quasi-silted we start from hereditary abelian categories. In particular we obviously have that all silted algebras are quasi-silted, since we can just take $\mathcal{H} = \mathsf{mod}\, A$.

Definition 3.4.2. Let A be a finite-dimensional algebra. We say that the category $\mathsf{mod}\, A$ has the *shod-property* if for every indecomposable module X we have $\mathsf{pd}X \leq 1$ or $\mathsf{id}X \leq 1$. We call algebras with the shod-property *shod algebras*.

Note that having the shod-property means that every module is close to be either projective or injective. The name shod in fact stands for **s**mall **ho**mological **d**imension. With this property we are also able to give a bound on the global dimension, thanks to a result in [HRS96].

Proposition 3.4.3. *Let A be a shod algebra. Then $\mathsf{gl.dim}A \leq 3$.*

Proof. Let $M \in \mathsf{mod}\, A$, and consider the beginning of its minimal projective resolution:

$$P_1 \to P_0 \to M \to 0.$$

Let K be the kernel of the map $P_1 \to P_0$, so that we obtain the exact sequence:

$$0 \to K \to P_1 \to P_0 \to M \to 0.$$

Let X be an indecomposable direct summand of K. Since there exists the previous exact sequence we have that $\mathsf{Ext}^2_A(M, X) \neq 0$. Since $\mathsf{id}X = \mathsf{sup}\{n | \mathsf{Ext}^n_A(-, X) \neq 0\}$, we get that $\mathsf{id}X \geq 2$. Since A is shod, we must have $\mathsf{pd}X \leq 1$. Repeating this argument for all the indecomposable direct summands of K, we have that also K has projective dimension not greater than one. So there exists a projective resolution of K:

$$0 \to P_1^K \to P_0^K \to 0.$$

So we get that:

$$0 \to P_1^K \to P_0^K \to P_1 \to P_0 \to 0$$

is a projective resolution of M, so that $\mathsf{pd}M \leq 3$. Being M arbitrary, we have that $\mathsf{gl.dim}A \leq 3$. □

Definition 3.4.4. We call a shod algebra *strictly shod* if it has global dimension equal to three.

Let $X, Y \in \mathsf{mod}\, A$ be indecomposable modules. We say that X is a *predecessor* of Y and that Y is a *successor* of X if there exists a sequence

$$X \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \ldots \to X_n \xrightarrow{f_n} Y$$

with X_i indecomposable and f_i non-zero for $i = 0, \ldots, n$.

Recall that in the previous chapter, starting from a finite-dimensional algebra A and a silting complex $\mathbf{P} \in K^b(\mathsf{proj}\, A)$, we built a torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ in $\mathsf{mod}\, B$, where $B = \mathsf{End}_{\mathcal{D}^b(A)}(\mathbf{P})$. We now see that we can say more about that torsion pair in the case that A is hereditary.

Proposition 3.4.5. *Let A be a hereditary algebra, $\boldsymbol{P} \in K^b(\mathsf{proj}\, A)$ a 2-term silting complex and $B = \mathsf{End}_{\mathcal{D}^b(A)}(\boldsymbol{P})$. We have:*

1) The torsion pair $(\mathcal{X}(\boldsymbol{P}), \mathcal{Y}(\boldsymbol{P}))$ is split in $\mathsf{mod}\, B$.

2) $\mathcal{X}(\boldsymbol{P})$ is closed under successors and $\mathcal{Y}(\boldsymbol{P})$ is closed under predecessors.

3) For any $X \in \mathcal{X}(\boldsymbol{P})$ we have $\mathsf{id} X \leq 1$ and for any $Y \in \mathcal{Y}(\boldsymbol{P})$ we have $\mathsf{pd} Y \leq 1$.

Proof. 1) Using the isomorphism of Corollary 3.1.16 and the definition of $\mathcal{X}(\mathbf{P})$ and $\mathcal{Y}(\mathbf{P})$ we get the following equalities:

$$\begin{aligned}\mathsf{Ext}^2_A(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P})) &\cong \mathsf{Ext}^1_B(\mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{T}(\mathbf{P})), \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{F}(\mathbf{P})[1])) \\ &= \mathsf{Ext}^1_B(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P})).\end{aligned}$$

Since A is hereditary, $\mathsf{Ext}^2_A(-,-)$ vanishes, we obtain $\mathsf{Ext}^1_B(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P})) = 0$, so the torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ is split since it has no extensions.

2) We show only that $\mathcal{X}(\mathbf{P})$ is closed under successors, as the other statement is dual. Let $X \in \mathcal{X}(\mathbf{P})$ and take Y to be a successor of X. So there exists a sequence $X \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \dots \to X_n \xrightarrow{f_n} Y$ with X_i indecomposable and f_i non-zero for $i = 0, \dots, n$. We will show the claim by induction on n. For $n = 0$, we have $X \xrightarrow{f_0} Y$. Since f_0 is not zero and X is torsion, Y cannot be in $\mathcal{Y}(\mathbf{P})$. Since, by 1), we have that $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ is split, we have that $Y \in \mathcal{X}(\mathbf{P})$. Now let n be arbitrary and suppose that the claim holds for $i = 0, \dots, n-1$. So we have the sequence $X \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \dots \to X_n \xrightarrow{f_n} Y$. By induction hypothesis $X_n \in \mathcal{X}(\mathbf{P})$ and so, by the reasoning of the case $n = 0$, also $Y \in \mathcal{X}(\mathbf{P})$. So we get the claim.

3) We show that for any $Y \in \mathcal{Y}(\mathbf{P})$ we have $\mathsf{pd}\, Y \leq 1$, the other is similar. Let $Y \in \mathcal{Y}(\mathbf{P}) = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{T}(\mathbf{P}))$, so there exists $M \in \mathcal{T}(\mathbf{P})$ such that $Y = \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, M)$. By Proposition 3.1.11 2), there exists a short exact sequence

$$0 \to L \to T_0 \to M \to 0$$

with $L \in \mathcal{T}(\mathbf{P})$ and $T_0 \in \mathsf{add}\, H^0(\mathbf{P})$.

For any $N \in \mathcal{T}(\mathbf{P})$, applying $\mathsf{Hom}_A(-, N)$ to the short exact sequence we obtain the exact sequence:

$$\cdots \to \mathsf{Ext}^1_A(T_0, N) \to \mathsf{Ext}^1_A(L, N) \to \mathsf{Ext}^2_A(M, N) \to \cdots .$$

Notice that $\mathsf{Ext}^2_A(M, N) = 0$ since A is hereditary and $\mathsf{Ext}^1_A(T_0, N) = 0$ since $T_0 \in \mathsf{add}\, H^0(\mathbf{P})$ and so, by Proposition 3.1.11, T_0 is Ext-projective in $\mathcal{T}(\mathbf{P})$. So, by exactness, also $\mathsf{Ext}^1_A(L, N) = 0$. This holds for any $N \in \mathcal{T}(\mathbf{P})$, so L is Ext-projective in $\mathcal{T}(\mathbf{P})$ and again from Proposition 3.1.11, $L \in \mathsf{add}\, H^0(\mathbf{P})$.

Since A is hereditary, for every complex $\mathbf{X}$ we have:

$$\mathbf{X} \cong \bigoplus_{i \in \mathbb{Z}} H^i(\mathbf{X})[-i].$$

In particular we have that $H^0(\mathbf{P}) \in \operatorname{add} \mathbf{P}$. Combining these two results we get that $L \in \operatorname{add} \mathbf{P}$.

So we have $T_0, L \in \operatorname{add} \mathbf{P}$. Applying $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ to both we get that $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0)$, $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L) \in \operatorname{add} B = \operatorname{proj} B$. So $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0)$ and $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L)$ are projectives. Now, applying $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, -)$ to the short exact sequence we obtain the short exact sequence:

$$0 \to \operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L) \to \operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, T_0) \to \operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, M) \to 0$$

since $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, L[1]) = 0$ as $L \in \mathcal{T}(\mathbf{P})$. So we get that $\operatorname{pd} Y \leq 1$ as this short exact sequence is a projective resolution of $\operatorname{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, M) = Y$.

□

Corollary 3.4.6. *Every silted algebra is shod.*

Proof. It follows from the previous proposition by the fact that every module in $\mathcal{X}(\mathbf{P})$ or $\mathcal{Y}(\mathbf{P})$ has either the projective dimension or injective dimension not greater than one and since the torsion pair is splitting. □

We define *tilted algebras* and *quasi-tilted algebras* in the same way as we defined silted and quasi-silted algebras but for tilting objects. We clearly have that tilted algebras are quasi-tilted and, moreover, since all tilting objects are silting, that (quasi-)tilted algebras are (quasi-)silted. It was shown in [HRS96] that quasi-tilted algebras are shod. They were able to prove something more by giving a characterization of quasi-tilted algebras in terms of shod algebras:

quasi-tilted algebras are exactly the shod algebras of global dimension at most two.

In the paper [BZ16a], Buan and Zhou studied (quasi-)silted algebras and gave the following characterization of them in terms of shod algebras:

Theorem 3.4.7 ([BZ16a])**.** *Let A be a connected finite-dimensional K-algebra with K algebraically closed field. Then we have:*

1) A is a quasi-silted algebra if and only if it is a shod algebra.

2) A is strictly shod if and only if it is a silted but not tilted algebra.

So we have that quasi-silted algebras are exactly the algebras with the shod property. Moreover, we can see that we have a clear distinction between silted algebras and quasi-tilted algebras by their global dimension (of course without considering tilted algebras which lie exactly in the intersection of the two).

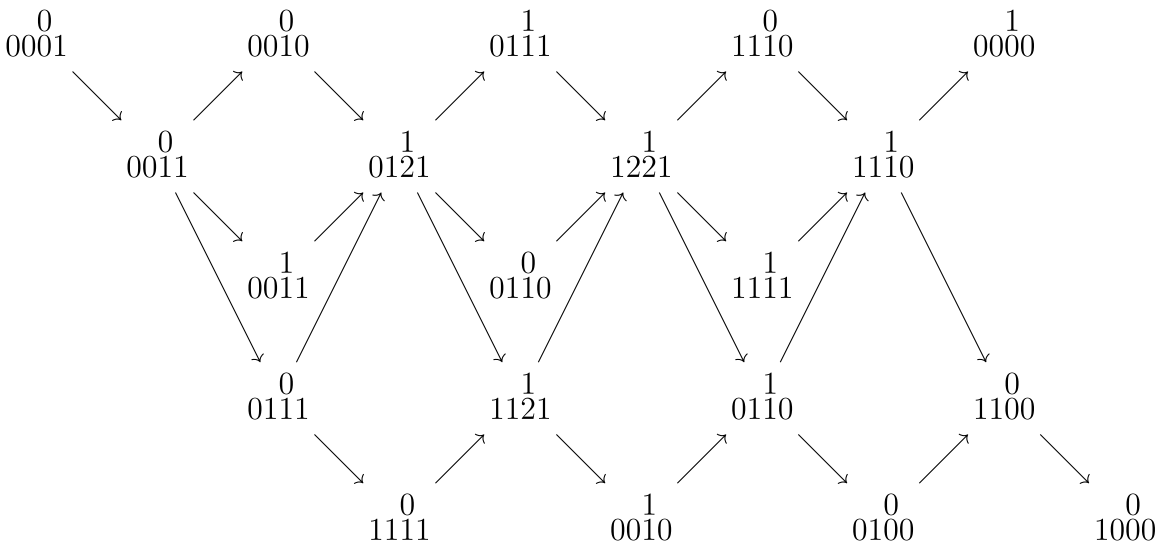
Chapter 4

An example

In this last part of the thesis we present a complete example to show all the main results of Chapter 3. We begin by considering the following quiver Q:

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & \downarrow & & \\ 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5. \end{array}$$

We have that Q is finite, acyclic and connected, so $A := KQ$ is hereditary. The Auslander–Reiten quiver of $\mathsf{mod}\, A$ is:



Now, denote by P_i the projective module relative to the vertex i. Let $\mathbf{P}$ be the complex in $\mathcal{D}^b(\mathsf{proj}\, A)$ given by:

$$\begin{aligned} \mathbf{P} &= P_2[1] \oplus P_3[1] \oplus (P_4 \to P_1) \oplus P_1 \oplus P_5 \\ &= \substack{0\\1111}[1] \oplus \substack{0\\0111}[1] \oplus \substack{1\\0000} \oplus \substack{1\\0011} \oplus \substack{0\\0001}. \end{aligned}$$

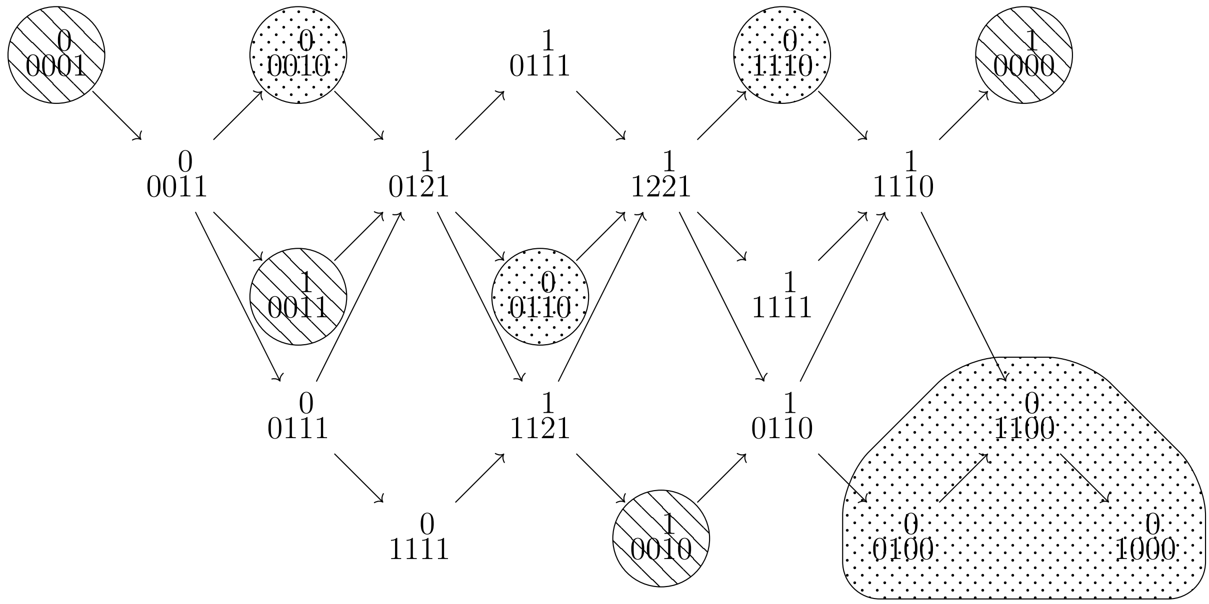
It is easy to check that we have $\mathsf{Hom}_{\mathcal{K}^b(\mathsf{proj}\, A)}(\mathbf{P}, \mathbf{P}[1]) = 0$. Moreover, since all the projectives appear in $\mathbf{P}$, we have that $\mathsf{thick}(\mathbf{P}) = \mathcal{K}^b(\mathsf{proj}\, A)$. So $\mathbf{P}$ is a silting complex.

By Theorem 2.2.11, $\mathbf{P}$ induces a torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ in $\mathsf{mod}\, A$ given by:

$$\mathcal{T}(\mathbf{P}) = \{X \in \mathsf{mod}\, A \mid \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, X[1]) = 0\}$$

$$\mathcal{F}(\mathbf{P}) = \{Y \in \mathsf{mod}\, A \mid \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, Y) = 0\}.$$

We can look at the torsion pair in the Auslander–Reiten quiver. $\mathcal{T}(\mathbf{P})$ is denoted with the lines and $\mathcal{F}(\mathbf{P})$ by the dots:



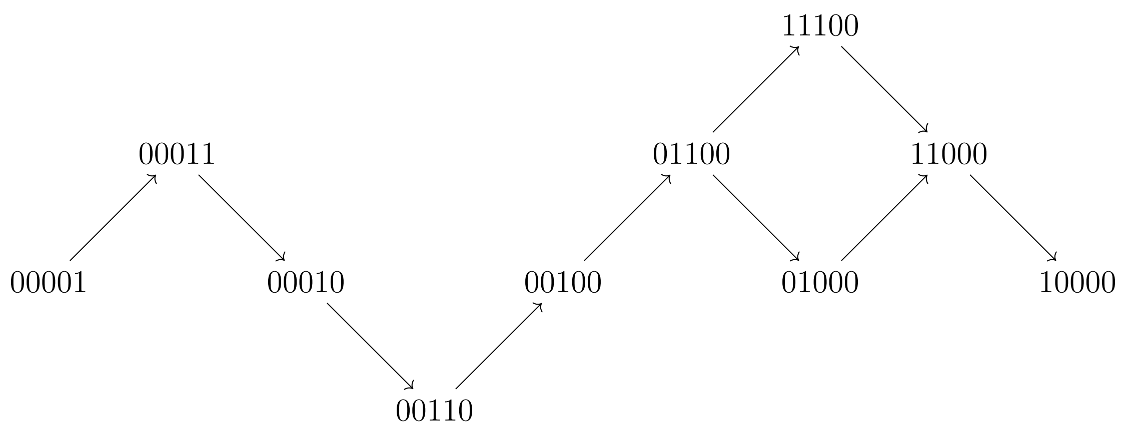
The endomorphism algebra of $\mathbf{P}$ is given by:

$$B = \mathsf{End}_{\mathcal{K}^b(\mathsf{proj}\, A)} \mathbf{P} = \begin{pmatrix} K & K & K & 0 & 0 \\ 0 & K & K & 0 & 0 \\ 0 & 0 & K & K & 0 \\ 0 & 0 & 0 & K & K \\ 0 & 0 & 0 & 0 & K \end{pmatrix}$$

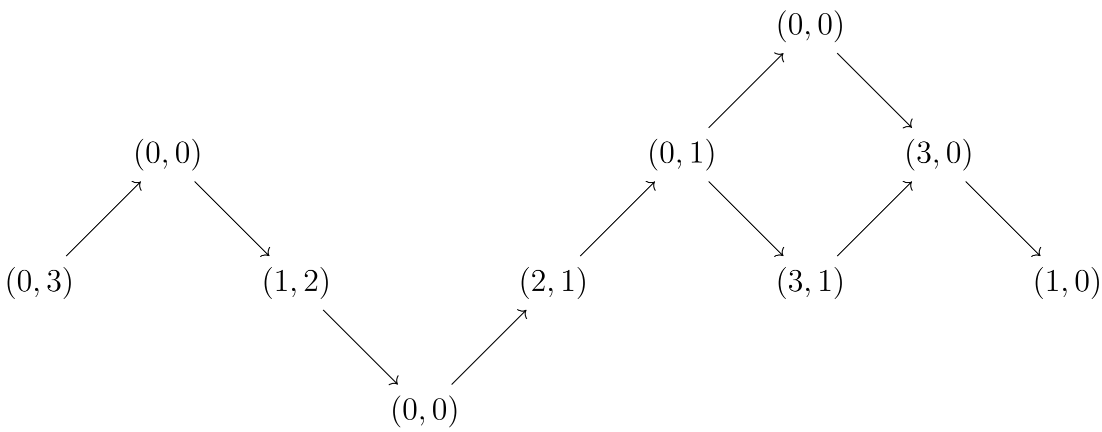
that is equal to $KQ'/\mathcal{I}$ where Q' is by the quiver:

$$6 \xrightarrow{\alpha} 7 \xrightarrow{\beta} 8 \xrightarrow{\gamma} 9 \xrightarrow{\delta} 10$$

and $\mathcal{I}$ is the ideal generated by the relations $\beta\gamma$ and $\gamma\delta$. The Auslander–Reiten quiver of B is:



By Corollary 3.4.6, B is a shod algebra as it is the endomorphism algebra of a silting complex. We can directly see that every indecomposable has either projective dimension or injective dimension less or equal than one from the following diagram. It is the Auslander–Reiten quiver where in every vertex corresponding to the indecomposable module X we write the pair $(\mathsf{pd}\,X, \mathsf{id}\,X)$.



The global dimension of B is 3. Indeed, for example, the minimal projective resolution of the module $S(7) = 01000$ is given by:

$$(0 \to P(10) \to P(9) \to P(8) \to P(7) \to S(7) \to 0) = \\ = (0 \to 00001 \to 00011 \to 00110 \to 01100 \to 01000 \to 0).$$

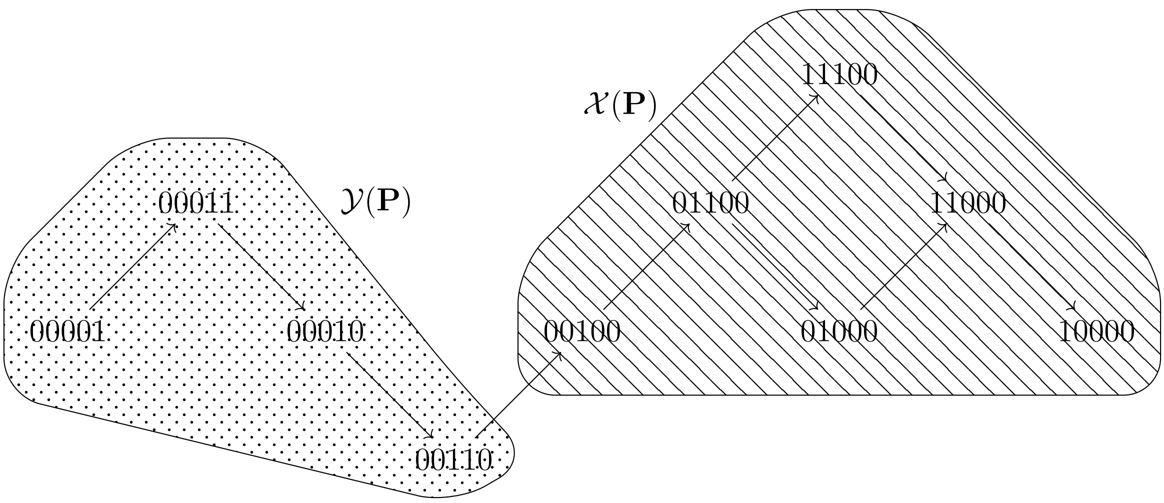
So B is a silted algebra that is strictly shod.

By Corollary 3.1.15, in $\mathsf{mod}\,B$ we have the torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ given by:

$$\mathcal{X}(\mathbf{P}) := \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{F}(\mathbf{P})[1])$$

$$\mathcal{Y}(\mathbf{P}) := \mathsf{Hom}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathcal{T}(\mathbf{P})).$$

Namely, in the Auslander–Reiten quiver they are:



We now want to find the complex $\mathbf{Q}$ constructed in the Chapter 3. We have to start from the triangle:

$$\Delta_{\mathbf{P}} : A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1]$$

where $\mathbf{P}', \mathbf{P}'' \in \operatorname{\mathsf{add}} \mathbf{P}$. Moreover, e is an $\operatorname{\mathsf{add}} \mathbf{P}$ preenvelope, so we get that:

$$\mathbf{P}' = P_1 \oplus P_5 \oplus P_1$$

and

$$\mathbf{P}'' = P_2[1] \oplus P_3[1] \oplus (P_4 \to P_1).$$

The map f goes from P_1 to $(P_4 \to P_1)$. So the complex $\mathbf{Q}$ is:

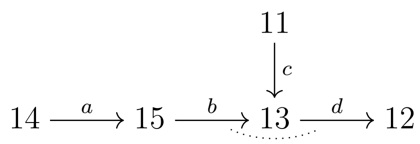
$$\begin{aligned}\mathbf{Q} &= \operatorname{\mathsf{Hom}}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}') \xrightarrow{\operatorname{\mathsf{Hom}}_{\mathcal{D}^b(A)}(\mathbf{P}, f)} \operatorname{\mathsf{Hom}}_{\mathcal{D}^b(A)}(\mathbf{P}, \mathbf{P}'') \\ &= 00011[1] \oplus 00001[1] \oplus (00011 \to 00110) \oplus 11100 \oplus 01100.\end{aligned}$$

By Proposition 3.2.5, $\mathbf{Q}$ is a silting complex. We now want to compare the endomorphism algebra of $\mathbf{Q}$ with A.

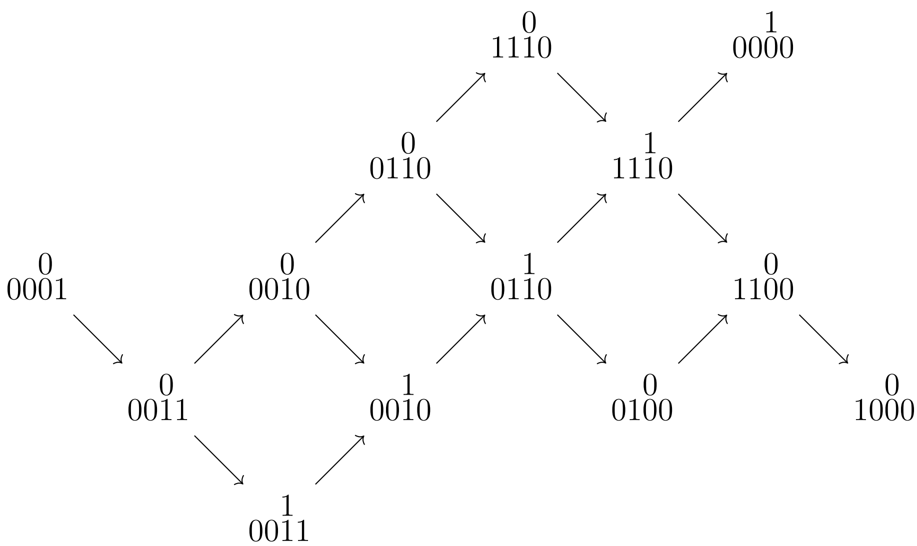
The endomorphism algebra of $\mathbf{Q}$ is:

$$\bar{A} = \operatorname{\mathsf{End}}_{K^b(\operatorname{\mathsf{proj}} B)} = \begin{pmatrix} K & K & K & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ 0 & K & K & 0 & 0 \\ 0 & 0 & K & K & K \\ 0 & 0 & K & 0 & K \end{pmatrix}.$$

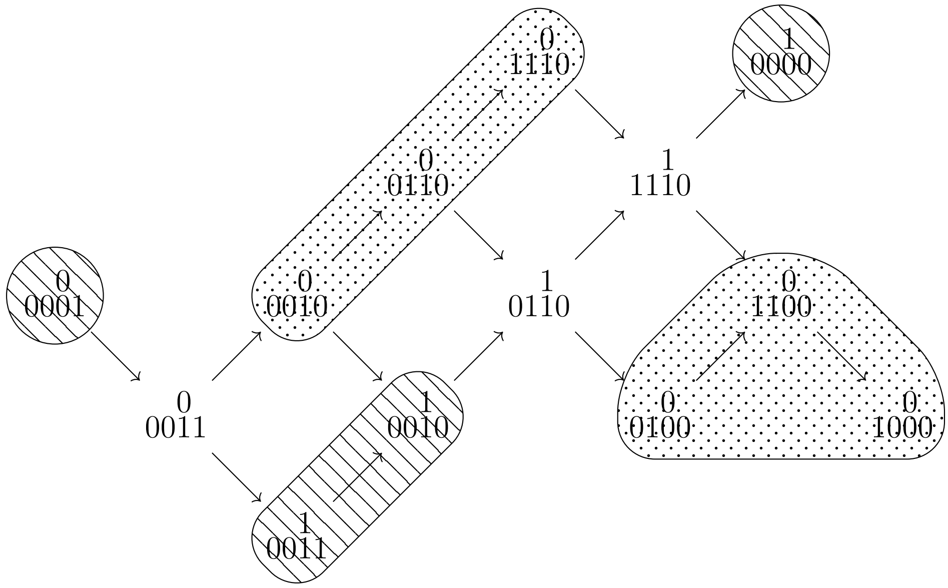
It can be represented by the quiver:



with relation $bd = 0$. Note that $\bar{A}$ is a quotient of the algebra A. The Auslander–Reiten quiver of $\operatorname{\mathsf{mod}} \bar{A}$ is:

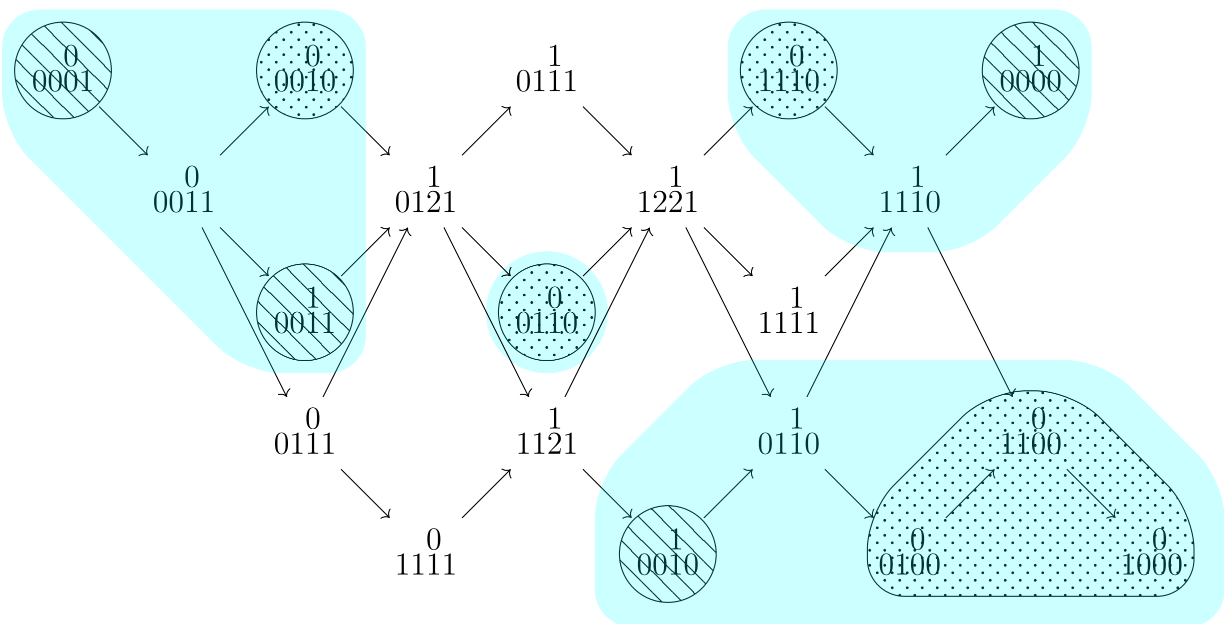


As before, $\mathbf{Q}$ induces a torsion pair $(\mathcal{X}(\mathbf{Q}), \mathcal{Y}(\mathbf{Q}))$ in $\mathsf{mod}\, B$. From Proposition 3.2.5 we have that $\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P})$ and $\mathcal{F}(\mathbf{Q}) = \mathcal{Y}(\mathbf{P})$. So the torsion pair $(\mathcal{X}(\mathbf{Q}), \mathcal{Y}(\mathbf{Q})) = (\mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathcal{F}(\mathbf{Q})[1]), \mathsf{Hom}_{\mathcal{D}^b(B)}(\mathbf{Q}, \mathcal{T}(\mathbf{Q})))$ is:



where $\mathcal{X}(\mathbf{Q})$ is denoted with the lines and $\mathcal{Y}(\mathbf{Q})$ with dots.

Finally, we can see how $\mathsf{mod}\, \bar{A}$ is a subcategory of $\mathsf{mod}\, A$. Indeed in the Auslander–Reiten quiver of $\mathsf{mod}\, A$ we can find the Auslander–Reiten quiver of $\mathsf{mod}\, \bar{A}$. In the following picture we have the Auslander–Reiten quiver of $\mathsf{mod}\, A$ and in blue we have $\mathsf{mod}\, \bar{A}$. Note also that the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ and $(\mathcal{X}(\mathbf{Q}), \mathcal{Y}(\mathbf{Q}))$ coincide, as from Theorem 3.3.3.



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