ALGANT Master Program

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Streamlining Scheme Theory with Topoi

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Abstract

Streamlining Scheme Theory With Topoi
Francesco Tognetti, 2024.

The aim of the script is to act as a course on Scheme theory from the internal perspective of the topos $\mathbf{Sh}(X)$, therefore showing that the internal logic of sheaf topoi is a strong enough foundation to build the whole theory on without necessarily referring back to the usual methods.

In this thesis we define what it means to work from the internal perspective:

We define elementary topoi and how to build and interpret formulas in the internal logic.

After, we move to the specific case that is the category of sheaves on either a topological space or a locale, and explicit the semantics of that language.

We show that the logic is intuitionistically solid and prove some results about geometric formulas that apply to later constructions.

When that is done, we procede to rebuild some theory of schemes from this perspective:

First we define abelian groups, rings, local rings and modules over sheaves, and some special cases.

Then we build the basics of scheme theory by defining affine schemes, general schemes, coherent modules, and some special classes of morphism of schemes,

In the end we attempt to talk about relative schemes from this perspective and what is needed to build the theory, then procede to show that it is well suited for a synthetic approach to schemes through some exercises from Hartshorne's Algebraic Geometry chapter II.

Contents

Introduction

Robin Hartshorne's own "Algebraic Geometry"[Har77] occupies a special place in every reader's heart. Love it or hate it, it serves as a first digestion of Grothendieck's famously tough EGA[1] and its exercises are fundamental to learning the subject.

The textbook serves as a solid base for any algebraic geometry course, so much that in fact the whole structure of the thesis is based on professor Adrian Iovita's lectures on algebraic geometry at Concordia university in tha academic year 2022/2023 (which I had the pleasure to attend), and those are strongly based on Hartshorne's textbook, even though the order is changed.

This work aims to use Topos Theory and the fact that the category of sheaves has an internal logic to transform the theory of schemes into theory of modules internal to the category of sheaves, making the definitions, the theorems and all much more resembling to the usual set theoretical module theory.

We will venture into the basic theory, the relative theory, and we'll talk about differentials.

As the comparison between internal and external methods has already been explored, my work instead translates the theory of schemes from the usual external language to the internal one, building the theory from scratch using topos-theoretic knowledge.

I'll assume that readers are discretely familiar with at least some commutative algebra, basic category theory and some logic, but not necessarily with schemes, which is the level at which most first year master's are (or at which anyone not particularly interested in algebraic geometry is).

I encourage the readers to pick up the forementioned book and try to solve some exercises using the theory provided in this thesis, especially the readers who already have at least some familiarity with scheme theory: it is quite a different experience, and one I hope you'd consider peleasant.

One may notice that, at times, I will tend to make jokes and stray from the formal setting of books and papers. This is deliberate, as I believe it makes for a more pleasant, as I believe it allows readers to build a colloquial understanding of what the theory is trying to do before diving into the pure mathematical point of view (and to be honest it makes for a pleasant draft, I really enjoy writing informally).

[1] Éléments de Géométrie Algébrique

The importance of "Dilly-Dally" reasoning

I like to call "Dilly-Dally" a kind of explaination that doesn't hold much mathematical rigor but helps understanding the reasoning behind a definition or a theorem.

Some examples of "Dilly Dally" are giving an example in low dimension when reasoning with general n-dimensional objects, or assigning an intuitive (even if sometimes slightly wrong) meaning to an object, for example the simplification $\frac{\mathrm{d}y}{\cancel{\mathrm{d}x}}\frac{\cancel{\mathrm{d}x}}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}z}$ One of the reasons I found this topic for a thesis particularly interesting is because it justifies a lot of the usual Dilly-Dally we do on sheaves and schemes, making it more intuitive and malleable though sacrificing some principles.

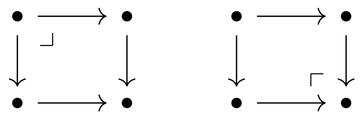
I believe that the understanding of a subject is based on how accurately one can dilly-dally with it.

One may think that this approach is "wrong". Maybe it doesn't solve as many problems as the usual ones, maybe the gain in simplicity is not worth the extra theory, I don't know. I won't pretend to be an expert, or to know enough about revolutionary mathematical research to substantiate this claim, but I do strongly believe that being wrong in an interesting way is always better than being boringly right[2] and I feel that the theory presented deserves the appellative "interesting".

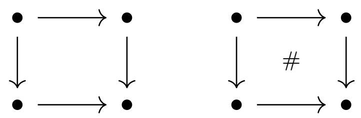
You'll find throughout the script many blue boxes with this description that aim to do just that.

[2]Of course one may notice that I'm leaving out the much harder "being right in an interesting way", but being interestingly right may come after being interestingly wrong

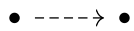
Notation



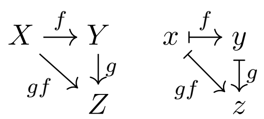
The $\lrcorner$ on the left means "The diagram is a pullback" and the one on the right means "The diagram is a pushout".



Both mean "The diagram is commutative".



The dashed arrow means it exists and it's unique.



the diagram on the right shows where the elements are mapped in the diagram on the left

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$U \subseteq^{\mathrm{o}} X$	U open subset of X
$U \subseteq^{\mathsf{aff}} X$	U affine open of X
$U \mathrel{\bar{\subseteq}} X$	U closed subset of X
$f_{\to} \quad f^{\leftarrow}$	(respectively) image and preimage of f
U_x	U open neighborhood of the point x
$H \lesssim G$	H is isomorphic to a subgroup of G
$A \trianglelefteq_{\mathsf{p}} B$	A prime ideal of B
$\lvert X \rvert$	Topological space of a scheme X
$\sqrt{R}$	Radical ideal of R
$C(A, B)$	The hom-set of maps between A and B in the category C
$\mathrm{Lim}^{\to} \quad \mathrm{Lim}^{\leftarrow}$	Inverse and Direct Limit

1 Topoi and internal Logics

In this chapter we shall define the objects that will serve as a basis for this whole deal.

Topoi are categories that "behave like **Set**", and, most importantly, have a defined concept of internal logic and internal semantics.

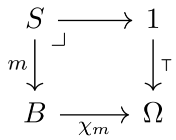
1.1 Elementary Topoi

Definition 1.1.1 (Elementary Topos)**.** *[Sau93]*
*An **Elementary Topos** is a category $\mathcal{E}$ with*

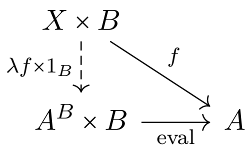
(i) All pullbacks

(ii) A Terminal object 1

*(iii) An object called **subobject classifier** or **truth values object** Ω and a monic arrow $\top : 1 \to \Omega$ such that for any monic $m : S \to B$ there is a unique arrow χ_m such that*



*(iv) For each pair of objects A, B an object A^B called **exponential object** together with a map $\mathrm{eval} : A^B \times B \to A$ such that for all object X and arrow $f : X \times B \to A$ there exists a unique map $\lambda f : X \to A^B$ such that*



Dilly-Dally (Topoi have elements and subobjects)**.**
(i) and (ii) are quite simple to understand, while (iii) and (iv) are annoying.

*The Dilly-Dally explaination for (iii) is that if we think of Ω as $\{0, 1\}$ in **Set** and thinking of m as an inclusion, we get that $\chi_S(x) = 1$ for all $x \in S$. The pullback condition says that S is the biggest subset fulfilling this diagram, meaning χ_S is the usual characteristic function.*

In other words we have a way to say "$X \subseteq Y$" in such a way that subsets

correspond with formulas.

(iv) is needed to have an internal equivalent to the hom-set

An useful property is that

Proposition 1.1.2.
*in a topos **a map that is both monic and epic is an isomorphism**.*

Proof. [Sau93, page 197] □

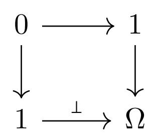
Definition 1.1.3 (Power Object)**.**
*In a topos, the combination of exponentials and subobject classifier allows us to define for every object A an unique associated **Power Object***

$$P(A) = \Omega^A$$

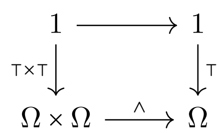
Every object has subobjects and the uniqueness of the arrows in $(1.1.1)$ gives us that $\mathrm{Sub}(X)$ is a Poset, i.e. a preorder with reflexivity and antisymmetricity.

Definition 1.1.4 (Structure of Ω)**.**
We will define some important maps:

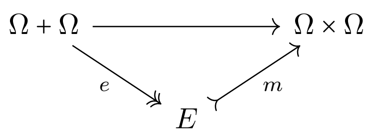
1. *We define $\perp$ as the characteristic of the $0-$object:*



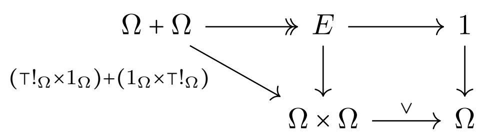
2. *We define $\wedge : \Omega \times \Omega \to \Omega$ as the characteristic of $\top \times \top$, i.e.*



3. *$\vee : \Omega \times \Omega \to \Omega$ is a little harder: Consider the maps $(\top \circ !_\Omega) \times 1_\Omega : 1 \times \Omega = \Omega \to \Omega \times \Omega$ and $1_\Omega \times (\top \circ !_\Omega) : \Omega \times 1 = \Omega \to \Omega \times \Omega$*
and their coproduct as a map $\Omega + \Omega \to \Omega \times \Omega$, and let $m \circ e$ be its mono-epi factorization



We define $\vee : \Omega \times \Omega \to \Omega$ to be the characteristic map of the m.



4. *We define the subobject* $\leq \hookrightarrow \Omega \times \Omega$ *as the equalizer of* $\wedge, \pi_1 : \Omega \times \Omega \rightrightarrows \Omega$.

5. *We define* $\Rightarrow : \Omega \times \Omega \to \Omega$ *as the characteristic of* $\leq$.

$$\begin{array}{ccc} \leq & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \Omega \times \Omega & \xrightarrow{\Rightarrow} & \Omega \end{array}$$

6. *We define equality = in a type* X *as the characteristic of the diagonal morphism*

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ {\scriptstyle \Delta_X}\downarrow & & \downarrow \\ X \times X & \xrightarrow{=_X} & \Omega \end{array}$$

7. *We define the membership to a subobject* $:$ *as the swapped of the evaluation map*

$$\Omega^X \times X \longrightarrow X \times \Omega^X \xrightarrow{ev} \Omega, \qquad \Omega^X \times X \xrightarrow{\;:\;} \Omega$$

Moreover we can define quantifiers: Recall that for any topos $\mathcal{E}$, *the power object functor* $X \mapsto \Omega^X$ *is a self adjoint (contravariant) endofunctor of* $\mathcal{E}$ *and it possesses both left and right adjoint.[Sau93].*

This means that in particular, for $x : X \to 1$ *we have three induced morphisms*

$$\Omega^X \underset{\forall_x}{\overset{\exists_x}{\leftleftarrows}} \;\; \Omega^X \xrightarrow{\Omega^x} \Omega$$

Suppose $\phi : X \times Y \to \Omega$ *is a formula, by the product-exponential adjunction it's a term* $\widetilde{\phi} : Y \to \Omega^X$.

then we have two formulas $\exists x : X\ \phi : Y \to \Omega$ *and* $\forall x : X\ \phi : Y \to \Omega$, *defined as*

$$Y \xrightarrow{\widetilde{\phi}} \Omega^X \underset{\forall_x}{\overset{\exists_x}{\rightrightarrows}} \Omega, \qquad \exists x{:}X\ \phi = \exists_x \circ \widetilde{\phi}, \quad \forall x{:}X\ \phi = \forall_x \circ \widetilde{\phi}$$

Later in our application we will see that in the internal language Ω is a Heyting Algebra, i.e. a lattice with implication, meet and join such that

1. $\wedge, \vee$ are distributive and commutative

2. $\bot \leq x \leq 1$ for all $x : \Omega$

3. $z \wedge x \leq y \iff z \leq x \Rightarrow y$

In a topos we have a type-theoretical internal language.

A **Type-Theoretical language**[Uni13][1] consists of

- Types,
- Terms,
- Formulas.

The syntax of the language is dealt through the Mitchell-Bénabou language of the topos:

Definition 1.1.5 (Mitchell-Bénabou language).
*The **Mitchell-Bénabou language** of an elementary topos $\mathcal{E}$ is a type theoretical language where*

- *A **Type** A is an object of $\mathcal{E}$;*
- *A **Variable** x of a given type A is interpreted as the identity morphism $1_A : A \to A$ (we write $x : A$)*
- *A **Term** $t(x_1, ..., x_m)$ of a given type A in variables $x_i : X_i$ is interpreted as a morphism $t : \prod_{i=1}^{m} X_i \to A$*
- *A **Formula** is a term of type Ω*

This means that given some atomic formulas ϕ_i we have a way of building well-formulated-formulae through $\wedge, \vee, \Rightarrow$ (plus parentheses, comma and other clarifiers) by composition:

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{(\phi_1,\phi_2)} & \Omega \times \Omega \\ & \searrow^{\phi_1 \star \phi_2} & \downarrow{-\star-} \\ & & \Omega \end{array}$$

(where $\star$ is one of the forementioned symbols.)

This means that we have, by any means, an internal language with a truth value object in which we can "evaluate the truth" of any proposition.

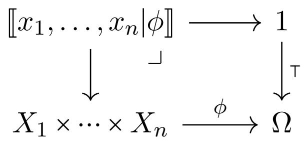
[1]The definition is not contained in the book but I believe the first chapter gives a nice introduction to type theory

1.2 Kripke-Joyal Semantics

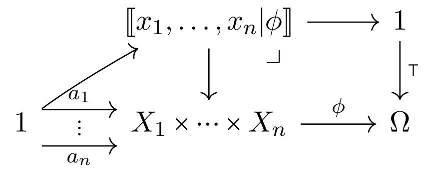
[Sau93]
We took care of defining a syntax, i.e. a way to reliably build formulas, now we want to have a way to assign truth values to said formulas.

We can obtain the Kripke-Joyal semantics by looking at formulas ϕ. Provided that any variable comes with an attached type $x_i : X_i$

We can construct $[\![x_1, \ldots, x_n | \phi]\!]$ as the pullback

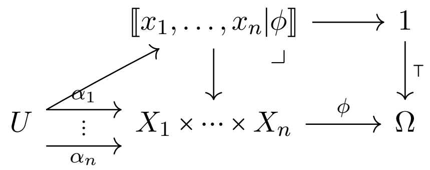


And we can say that $a_1, \ldots, a_k$ satisfy $\phi[x_1/a_1, \ldots, x_n/a_n]$ if



The **Kripke-Joyal semantics** are obtained by extending this to a generalized element U

Definition 1.2.1 (Forcing relation)**.**
A generalized element $\alpha : U \to X$ is said to satisfy $\phi[\alpha_1/x_1, , \alpha_n/x_n]$ when there exists a map $m : U \to [\![x_1, \ldots, x_n|\phi]\!]$ such that



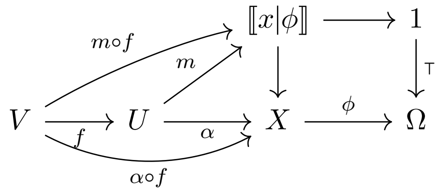
In that case we say that if $\phi = \phi(x_1, \ldots, x_k)$[2] *$U$ **forces** $\phi(\alpha_1, \ldots, \alpha_k)$, with notation $U \vDash \phi(\alpha(1, \ldots, \alpha_k))$*

We can study a bit of the properties of this forcing relation:

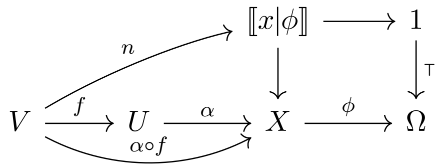
Proposition 1.2.2.
Suppose ϕ depends on one variable $x : X$ for simplcity
Let $f : V \to U$ and $U \vDash \phi(\alpha)$, then $V \vDash \phi(\alpha \circ f)$.
Conversely if $f : V \to U$ is epic and $V \vDash \phi(\alpha \circ f)$, then $U \vDash \phi(\alpha)$.

[2] ϕ generally needn't depend on all of the variables we defined!

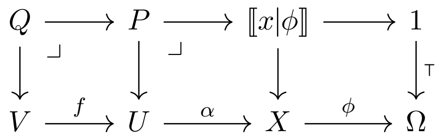
Proof.
For the first part the proof is contained in this diagram



For the converse we start with this diagram

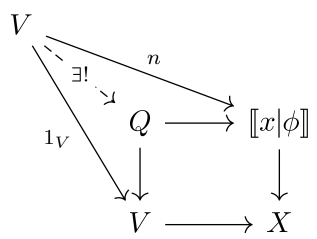


And try to find a suitable $m : U \to [\![x|\phi]\!]$: Consider the pullbacks

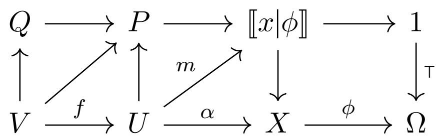


Q is a pullback so parallel arrows mantain their mono-epi properties: $[\![x|\phi]\!] \to X$ is monic, so is $P \to U$, so is $Q \to V$ and $Q \to P$ is epic.

Moreover $Q \to V$ is split epic as



hence $Q \to V$ an isomorphism. This means that $Q \to U$ is epic and so is $P \to U$, meaning that it is an isomorphism as well, getting



□

Dilly-Dally. *The idea is to "cancel out" image and preimage,* $\phi(\alpha \circ f)_{|f^{-1}(x)}$ *can be seen* $\phi(\alpha \circ f \circ f^{-1}x) = \phi(\alpha)_{|x}$ *and the converse is true only when f is surjective.*

This means that we have a way to transfer the forcing properties between objects.

Theorem 1.2.3 (Interactions of the forcing with logic operators)**.**

1) $U \vDash \phi(\alpha) \wedge \psi(\alpha) \iff U \vDash \phi(\alpha) \wedge U \vDash \psi(\alpha)$

2) $U \vDash \phi(\alpha) \vee \psi(\alpha) \iff \exists g_1 : U_1 \to U, g_2 : U_2 \to U$ *with* $g_1 + g_2$[3] *epic and such that* $U_1 \vDash \phi(\alpha \circ g_1), U_2 \vDash \psi(\alpha \circ g_2)$

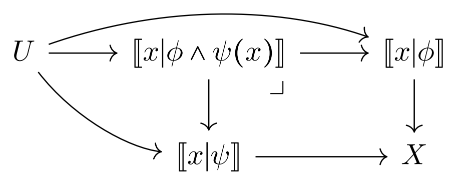
3) $U \vDash \phi(\alpha) \Rightarrow \psi(\alpha) \iff \forall g : V \to U, V \vDash \phi(\alpha \circ g) \implies V \vDash \psi(\alpha \circ g)$

4) $U \vDash \exists y \phi(\alpha, y) \iff \exists g : V \to U$ *epic and* $h : V \to Y$ *such that* $V \vDash \phi(\alpha \circ g, h)$

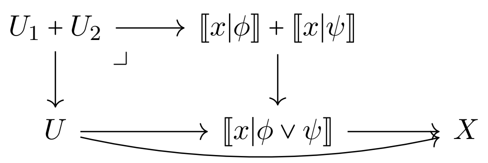
5) $U \vDash \forall y \phi(\alpha, y) \iff \forall V, g : V \to U, h : V \to Y, V \vDash \phi(\alpha \circ g, h)$

Sketch of proof. (I'll mute the (α)'s)

1) $[\![x|\phi \wedge \psi]\!]$ can be seen as the pullback of $\{x|\phi\} \to X \leftarrow \{x|\psi\}$ and we can construct the maps as follows.



2) $[\![x|\phi \vee \psi]\!]$ can be seen as the coproduct of $\{x|\phi\}$ and $\{x|\psi\}$ and we can construct the maps as follows.



□

We can find a complete proof on [Sau93]

[3] Here the + means the coproduct

Dilly-Dally. *We want to see how to bring the logic operators from the internal (being forced on U) to external perspective. $\wedge$ is a sort of intersection, meaning we can stay inside the same subset, otherwise we need to find the forcing inside something bigger for everything else*

1.3 Logical Functors

Of course every object comes with its own flavour of morphisms, and we are looking for the kind of morphisms that preserve internal logic.

That is the case with logical functors

Definition 1.3.1 (Logical functor)**.**
*A functor $F : \mathcal{E} \to \mathcal{E}'$ is a **Logical functor** whenever it*

1. *preserves finite limits,*
2. *preserves the exponential objects, i.e. $F(A^B) = F(A)^{F(B)}$,*
3. *preserves the subobject classifier.*

Since the logical structure of a topos is given by these elements it's quite trivial to see how these functors preserve the internal structure.

Sadly, especially in the applications that we are going to see, not many logical functors arise naturally.

We will think of morphism of topoi from now on based on "how logical" they are, i.e. how closely they preserve the internal logic of the topos.

The most important class among those is the one of geometric functors:

Definition 1.3.2 (Geometric functors)**.**
*A pair of **geometric functors** is an adjoint pair $R : \mathcal{E} \to \mathcal{E}'$, $L : \mathcal{E}' \to \mathcal{E}$ such that $L \dashv R$ and L preserves finite limits.*

Of course both preserve finite limits, but they needn't necessarily preserve the truth value object.

Let's enunciate another theorem that is regarded as fundamental

Theorem 1.3.3 (Fundamental theorem of Topos Theory)**.**
Let $\mathcal{E}$ be a topos and A an object. Then the slice category $\mathcal{E}/A$ is itself a topos and for all $f : A \to B$ the canonical pullback $\mathcal{E}/B \to \mathcal{E}/A$ is geometric.

Proof. [Sau93, p. IV.7] □

1.4 Modalities

The logic in a generic topos, and in particular in a sheaf topos, is usually not classic, meaning that in general ϕ and $\neg\neg\phi$ are not equivalent formulas.

We have that $(\neg\neg)(\neg\neg)\phi$ and $\neg\neg\phi$ are equivalent, meaning that we have a translation $\phi \mapsto \neg\neg\phi$ where classical proofs are valid, paying the price of actually proving weaker formulas.

Dilly-Dally (Intuitionistic logic)**.**
The usual way to make sense of this is to treat "truth" as "constructibility": If you lose the keys inside your house, proving that they're not outside of your house is not sufficient to make it to work in time.

A more fitting alternative would be to treat "truth" as "truth everywhere": something that is not not true everywhere is not false everywhere, meaning it's somewhere true.

Notice how both of these interpretations are a weakening of the formula, meaning that $\phi \Rightarrow \neg\neg\phi$ always, but generally $\neg\neg\phi \not\Rightarrow \phi$.

Definition 1.4.1 (Modal operator)**.**
*A **modal operator** $\square$ is a map $\square : \Omega \to \Omega$ such that the following diagrams commute:*

1.
$$\begin{array}{ccc} 1 & \xrightarrow{\top} & \Omega \\ & \searrow^{\top} & \downarrow{\scriptstyle \square} \\ & & \Omega \end{array}$$

2.
$$\begin{array}{ccc} \Omega & \xrightarrow{\square} & \Omega \\ & \searrow^{\square} & \downarrow{\scriptstyle \square} \\ & & \Omega \end{array}$$

3.
$$\begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ {\scriptstyle \square \times \square}\downarrow & & \downarrow{\scriptstyle \square} \\ \Omega \times \Omega & \xrightarrow[\wedge]{} & \Omega \end{array}$$

With the forementioned structure of Heyting algebra for Ω in the internal language, thus the three diagram conditions are equivalent to

1. $\phi \implies \square\phi$
2. $\square\square\phi \implies \square\phi$
3. $\square(\phi \wedge \psi) \iff \square\phi \wedge \square\psi$

We have that

Lemma 1.4.2 (□ is monotonic)**.**

$$\phi \Rightarrow \psi \textit{ implies } \Box\phi \Rightarrow \Box\psi.$$

Moreover

$$\Box\phi, \phi \Rightarrow \Box\psi \textit{ implies } \Box\psi.$$

Proof.
$\phi \Rightarrow \psi$ is equivalent to $\phi \wedge \psi \Leftrightarrow \phi$.

This means that

$$(\phi \Rightarrow \psi) \simeq (\phi \wedge \psi \Leftrightarrow \phi) \text{ implies } \Box\phi \wedge \Box\psi \Leftrightarrow \Box(\phi \wedge \psi) \Leftrightarrow \Box\phi.$$

Moreover

$$\phi \Rightarrow \Box\psi \text{ implies } \Box\phi \Rightarrow \Box\Box\psi \Rightarrow \Box\psi.$$

□

Those properties are to be expected if we go back to the dilly-dally interpretation of the symbol.

2 Sheaves and the Zariski Topos

In this chapter we will introduce our main category, the category of sheaves on a topological space (or a locale). It is important to note that there is a more general notion, i.e. that of sheaves on a site.

A site is a category endowed with a Grothendieck topology, a structure that allows us to talk about restrictions and coverings, meaning it behaves similarly to a topological space when it comes to building sheaves.

We will stick to topological spaces (or locales): This is supposed to be a guide to simplify reasoning for Hartshorne's book which did not use general sites.

2.1 Presheaves and Sheaves

It's easier to start by defining presheaves.

Definition 2.1.1 (Presheaf).
Let X be a topological space. A ***presheaf*** *(of sets) $\mathcal{F}$ on X is a functor $X^{op} \to$* ***Set****, i.e.*

I) A set $\mathcal{F}U$ for each $U \subseteq^{o} X$

II) A morphism $\rho_{UV} : \mathcal{F}U \to \mathcal{F}V$ for each inclusion $V \hookrightarrow U \subseteq^{o} X$

With properties

i) $\rho_{UU} = 1_{\mathcal{F}U}$

ii) $\rho_{VU} \circ \rho_{WV} = \rho_{WU}$

In this we can see any topological space as a preorder, saying that the objects are open subsets of X and the morphisms are the inclusions $U \hookrightarrow V$ (they are unique between any couple $U \subseteq V$).

We will also use $\rho_{UV}(s) = s_{|V}$ when it's clear enough that $s \in U$ and $\rho^{\mathcal{F}}_{UV}$ when we need to specify which sheaf we're working on.

Sheaves are presheaves that behave well on covers.

Remark (Open cover).
Remember that an ***open cover*** *(or just* ***cover*** *since we will never use other types of cover) of an open subset U of a topological space X is a collection of open subsets $\{U_i\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$*

Definition 2.1.2 (Sheaf).
A presheaf $\mathcal{F}$ is a ***sheaf*** *if section agreeing on intersections lift.*

This means that, given $U \subseteq^{o} X$ and $\{U_i\}_{i\in I}$ a cover,

1. *if* $s, t \in \mathcal{F}(U)$ *and* $\rho_{UU_i}(s) = \rho_{UU_i}(t)$ *for all* $i \in I$*, then* $s = t$*;*
2. *if for any family of elements* s_i *in* $\mathcal{F}(U_i)$ *(for* $i \in I$*) we have that* $\rho_{U_iU_i \cap U_j}(s_i) = \rho_{U_jU_i \cap U_j}(s_j)$*, then there exists an unique* $s \in \mathcal{F}U$ *such that* $\rho_{UU_i}(s) = s_i$

In this we note that ($\mathcal{F}\varnothing = \{*\}$)

Dilly-Dally. *The idea is to have an object that behaves like the set of regular functions on a space, for example the functor mapping* U *to the set of continuous functions* $U \to \mathbb{R}$ *is a sheaf.*

We like this because it allows us to work locally instead of globally on the topological space without losing any data.

Presheaves are way more flexible than sheaves, for example we can relax the definition just a tad from presheaves over X *to presheaves over a general category* C *as functors* $C^{op} \to$ ***Set*** *and use the Yoneda embedding to embed any category into a presheaf category.*

Any category of presheaves is an elementary topos, giving us access to The Kripke-Joyal semantics and potentially leading to some silly shenanigans.

Definition 2.1.3 (Morphism of presheaves)**.**
A ***morphism of presheaves*** $\alpha : \mathcal{F} \to \mathcal{G}$ *is a natural transformation* $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$*. In other words a map* $\alpha_U : \mathcal{F}U \to \mathcal{G}U$ *for any* $U \subseteq^{\mathrm{o}} X$ *such that*

$$\begin{array}{ccc} \mathcal{F}U & \xrightarrow{\rho^{\mathcal{F}}_{UV}} & \mathcal{F}V \\ {\scriptstyle \alpha_U}\downarrow & & \downarrow{\scriptstyle \alpha_V} \\ \mathcal{G}U & \xrightarrow[\rho^{\mathcal{G}}_{UV}]{} & \mathcal{G}V \end{array}$$

Definition 2.1.4 (Morphism of sheaves)**.**
A ***morphism of sheaves*** $\alpha : \mathcal{F} \to \mathcal{G}$ *is the underlying morphism of presheaves.*

We can note without proving that both **Psh**(X) are **Sh**(X) are categories (**Psh**(X) is the functor category $\textbf{Set}^{X^{op}}$ and **Sh**(X) is a full subcategory of it since only the objects are a subset.)

Now we want a way to talk about sheaves at points of the topological space, but all we have access to is open sets.

Definition 2.1.5 (Stalks)**.**
Let $\mathcal{F}$ *be a presheaf on* X *and let* $x \in X$*. The* ***stalk*** *of* $\mathcal{F}$ *at* x *is the set*

$$\mathcal{F}_x = \overrightarrow{\mathrm{Lim}}_{x \in U}\, \mathcal{F}U$$

Explicitly

$$\mathcal{F}_x := \frac{\{[U, s] : x \in U \subseteq^{\mathrm{o}} X, s \in \mathcal{F}U\}}{\sim_x}$$

where $[U, s_x] \sim_x [V, t]$ *if and only if there exists an open set* $W \subseteq^{\mathrm{o}} U \cap V$ *such that* $x \in W$ *and* $\rho_{UW}(s) = \rho_{VW}(t)$

We call the elements $[U, s]_{\sim_x} =: [U_x, S_x]$*.*

The two definitions are equivalent by definition of limit in **Set**.

Dilly-Dally. *The basic idea behind stalks is to consider a point "equivalently all neighborhoods of the point", or to consider the sheaf as a contravariant continuous functor:* $\mathcal{F}(\overleftarrow{\mathrm{Lim}}_{x\in U} U) = \overrightarrow{\mathrm{Lim}}_{x\in U} \mathcal{F} U$.

Visually we can think of it as a cone where each circular section is a "zoom in" of the neighborhood of the point.

To get ahead of ourselves, this makes more sense if you think of what happens with localizations: We can find a nice visual example in [EH00] at page 53.

This allows us to define a primitive sheafification, i.e. a functor $\widetilde{-} : \mathbf{Psh}(X) \to \mathbf{Sh}(X)$ defined as

$$\widetilde{\mathcal{F}}(U) = \prod_{x\in U} \mathcal{F}_x$$ [1]

With restriction maps

$$\rho_{UV}^{\widetilde{\mathcal{F}}} : \prod_{x\in U} \mathcal{F}_x \to \prod_{x\in V} \mathcal{F}_x$$

the canonical projections. And sending natural transformations $\mathcal{F} \to \mathcal{G}$ to the obvious product of transformations of stalks.

Proposition 2.1.6.
Let $\mathcal{F}$ be a presheaf, then $\widetilde{\mathcal{F}}$ is a sheaf.

Proof.
We'll prove the two conditions in inverse order:

Let $s_i \in \widetilde{\mathcal{F}}(U_i)$ be a family of elements such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$.

Recall that $s_i = [U_x^i, s_x^i]_{x\in U_i}$. Define $s = [V_x, t_x]_{x\in U}$ as $[U_x^i, s_x^i]$ if $x \in U_i$.

Since they agree on intersections and they form a cover this is well defined. Moreover if we restrict at any U_i we get back the original element by definition.

Now let s, t be two elements of $\widetilde{\mathcal{F}}(U)$ such that they agree on a cover.

Since they agree on a cover, this means that if $s = [U_x, s_x]_{x\in U},\ t = [V_x, t_x]_{x\in U}$ then $[U_x, s_x] = [V_x, t_x]$ for all $x \in U$, thus they agree everywhere. □

This is a way to assign sheaves to any presheaf, but sadly it has too many elements to preserve all the information we care about.

Definition 2.1.7 (Sheafification)**.**
*We define the **sheafification** functor* $-^+ : \textbf{\textit{Psh}}(X) \to \textbf{\textit{Sh}}(X)$ *as*

$$\begin{aligned}\mathcal{F}^+(U) = \{[U_x, s_x]_{x\in U} \in \widetilde{\mathcal{F}}(U) | \forall y \in U\ \exists V \subseteq^{o} U \\ \text{with } y \in V \text{ and } s^y \in \mathcal{F}(V) \\ \text{such that } \forall z \in V,\ [V, s^y] \in \mathcal{F}_z\}\end{aligned}$$

[1] please note that the empty product is always a singleton!

with restriction maps defined as the one making the diagram

$$\begin{array}{ccc}
\mathcal{F}U & \xrightarrow{\rho^{\mathcal{F}}_{UV}} & \mathcal{F}V \\
{\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} \\
\mathcal{F}^+U & \overset{\rho^{\mathcal{F}^+}_{UV}}{\dashrightarrow} & \mathcal{F}^+V \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{F}}U & \xrightarrow[\rho^{\widetilde{\mathcal{F}}}_{UV}]{} & \widetilde{\mathcal{F}}V
\end{array}$$

commute. Of course for any morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$*,* $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ *is the one satisfying the diagram*

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{F}^+ & \overset{\varphi^+}{\dashrightarrow} & \mathcal{G}^+ \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{F}} & \xrightarrow[\widetilde{\varphi}]{} & \widetilde{\mathcal{G}}
\end{array}$$

Dilly-Dally. *This is as intuitive as it looks, but don't worry, we won't use the definition very often.*

The basic explaination is that the elements of $\mathcal{F}^+_U$ *are equivalence classes* $[U, s]_{\sim_x}$ *that all come from the same section.*

This is needed to glue back together the sections: every element in $\widetilde{\mathcal{F}}(U)$ *is a collection* $[V_x, s_x]_{x \in U}$ *ranging on the points of* U*.*

If we consider a section $[V, s]$*,* V *could be in principle a neighborhood of any point of itself. Doing the sheafification means selecting just one.*

We are left to prove that the dashed arrow exists unique:

Proof. We only need to show that $\rho^{\widetilde{\mathcal{F}}}_{UV}(\alpha) \in \mathcal{F}^+V$ for any $\alpha \in \mathcal{F}^+U$.

$\alpha \in \mathcal{F}^+U$ means that $\alpha = (\alpha_x)_{x \in U}$ and for all $y \in U$ we can extract an open subset W_y and a section $s^y \in \mathcal{F}W_y$ such that for any $z \in W_y$ we have $(s^y)_z = \alpha_z$

$\rho^{\widetilde{F}}_{UV}(\alpha) = (\alpha_x)_{x \in V}$.

Let $y \in V$ we can pick $W'_y = W_y \cap V$ with $y \in W'_y$ and a section $s'^y = s^y_{|W'_y}$.

For any $z \in W'_y$ $(s'^y)_z = (s^y)_z = \alpha_z$. ∎

This has the nice properties we were looking for, that is, it's adjoint to the inclusion $\mathbf{Sh}(X) \hookrightarrow \mathbf{Psh}(X)$

Theorem 2.1.8 (The sheafification is left adjoint to the inclusion)**.**
Written in diagram

$$\textbf{\textit{Sh}}(X) \underset{j}{\overset{-^+}{\underset{\perp}{\leftrightarrows}}} \textbf{\textit{Psh}}(X)$$

sketch of proof. We can prove that if we have a sheaf $\mathcal{G}$ and a presheaf $\mathcal{F}$, then any map $\mathcal{F} \to \mathcal{G}$ factors uniquely as a map $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{G}$ It's proven by noticing that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ & \longrightarrow & \widetilde{\mathcal{F}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ & \longrightarrow & \widetilde{\mathcal{G}} \end{array}$$

and noticing that if $\mathcal{G} \in \mathbf{Sh}(X)$ then $\mathcal{G} \simeq \mathcal{G}^+$ as sheaves.
With these we obtain $\mathbf{Sh}(X)(\mathcal{F}^+, \mathcal{G}) \simeq \mathbf{Psh}(X)(\mathcal{F}, \mathcal{G})$. □

Lastly we need a simple fact

Theorem 2.1.9 ($\mathbf{Psh}(X)$ is complete and cocomplete)**.**
***Psh**(X) is complete and cocomplete, i.e. it has all limits and colimits of small diagrams.*

sketch of proof. We only need to prove that $\mathbf{Psh}(X)$ has all products, all equalizers, all coproducts and all coequalizers. It's immediate to see that they are

1. $\prod_i \mathcal{F}_i : U \mapsto \prod_i (\mathcal{F}_i U)$
2. $\mathrm{eq}(\alpha, \beta) : U \mapsto \mathrm{eq}(\alpha_u, \beta_u)$
3. $\coprod_i \mathcal{F}_i : U \mapsto \coprod_i (\mathcal{F}_i U)$
4. $\mathrm{coeq}(\alpha, \beta) : U \mapsto \mathrm{coeq}(\alpha_u, \beta_u)$

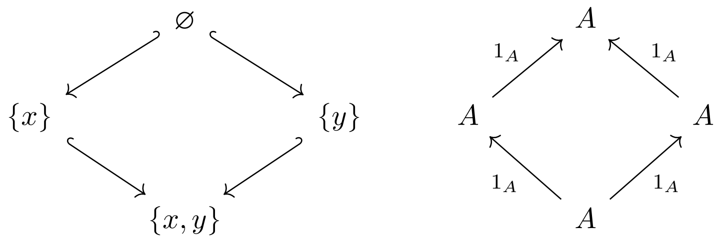
□

An useful example on why we need the sheafification is when we talk about constant sheaves:

Definition 2.1.10. *Given a set A the **constant sheaf** $\underline{A}$ is defined as*

$$(U \mapsto A)^+$$

I.e. the sheafification of the constant presheaf (which is actually constant.) This is needed because because the constant presheaf is usually pretty far from being a sheaf. Take the easy topological space $\{x, y\}$ with the discrete topology.



It is not a sheaf since if we take the cover of ∅ given by an empty family of sets, thus the condition $\rho_{\varnothing,U_i}(s) = s'$ *is vacuously true for any pair of sections, giving us that any two elements of* A *are equal, and this is false for any set other than* $\{*\}$ *or* ∅.

Even if we take the relaxed version of it by substituting the section of ∅ with $\{0\}$ we still don't have a sheaf: any two elements of $\{x\} \mapsto A$ and $\{y\} \mapsto A$ get mapped to 0 but they do not lift to an unique element of $\{x, y\} \mapsto A$.

2.2 Locales

A locale is something that behaves like the set of opens of a topological space.

Dilly-Dally (Locales).
Our usual point-set topology usually defines a set (thus a collection of points), then selects from the subsets which ones we can call "open".

Localic theory does the opposite, giving us an approach much closer to the one of physical observation:

A locale is something that behaves like the frame of open subsets of a topological space, and we call "points" the inverse limits of converging open subsets.

This is indeed closer to observing a physical space: we see areas and we can "zoom in" closer and closer and eventually hypotesize the existence of an indivisible point (what the greek called an a-tomos, or what we call an atom[a]).

This means that every Topological space is a Locale but the converse isn't necessary since we may have locale-theoretical points that do not correspond to any set-theoretical ones.

When we define sheaves we define them on open subsets, the only notion that mentions points is the stalk, but it uses the locale-theoretical notion too, thus, we shall forget that the underlying space may not be a topology.

[a] I guess technically there are smaller things than atoms but I'm bad at physics so let's leave it at that

Definition 2.2.1 (Frame).
A ***Frame*** *is a poset* $(X, \subseteq^{o})$ *with all joins* $(\bigcup)$ *and finite meets* $(\cap)$ *meeting the infinitary distributive law*

$$U \cap (\bigcup_i V_i) = \bigcup_i (U \cap V_i)$$

A ***morphism of frames*** *is a map* $f : X \to Y$ *of posets such that*

1. $f(\bigcup_i U_i) = \bigcup_i f(U_i)$*, for arbitrary joins and*

2. $f(\bigcap_j V_j) = \bigcap_j f(V_j)$ *for finite meets.*

Definition 2.2.2 (Category of locales). *The* ***category of locales*** *is the opposite of the category of frames.*

Dilly-Dally. *Loosely, again, we will refer to the frame as "the opens" and the opposite category means we are working with continuous preimages.*

Theorem 2.2.3 (Too trivial for a proof).
The topology of a topological space is a frame and the preimage through a continuous function is a morphism of locales.

2.3 Internal perspective

We want to specify the Kripke-Joyal semantics on the category **Sh**(X) of sheaves on a topological space X . First we need to verify that **Sh**(X) is actually a topos.

Theorem 2.3.1 (**Sh**(X) is a topos)**.**
*The category **Sh**(X) of sheaves on a topological space X is an elementary topos.*

Proof.

(i) (existence of a terminal object): A terminal object is given by the sheaf sending all open subsets to $\{0\}$. This is a presheaf since it's just a constant functor, and the sheaf conditions are trivial.

(ii) (existence of all pullbacks): With the standard notation U is an open, U_i an open cover of U.

Note that given two sheaves $\mathcal{F}, \mathcal{G}$, then $\mathcal{F} \times \mathcal{G}$ defined as $\mathcal{F} \times \mathcal{G}(U) = \mathcal{F}(U) \times \mathcal{G}(U)$ is a sheaf: It's obviously a presheaf and

- if $s = (s_1, s_2), t = (t_1, t_2) \in \mathcal{F}(U) \times \mathcal{G}(U)$ agree on every element of an open cover U_i then in particular s_1, t_1 agree on the open cover and s_2, t_2 agree on the open cover, thus s_1, t_1 agree on U and s_2, t_2 agree on U, meaning s, t agree on U.
- if the family $s^i = (s_1^i, s_2^i) \in \mathcal{F} \times G(U_i)$ agrees on intersections, this means that the family s_1^i and the family s_2^i agree on intersection, thus there exists s_1, s_2 such that $s_{1|U_i} = s_1^i$ and $s_{2|U_i} = s_2^i$, thus (s_1, s_2) is the desired lift.

Then to show that the equalizer presheaf $\mathrm{eq}(f,g)(U) = \mathrm{eq}(f_U, g_U)$ for $f, g : \mathcal{F} \rightrightarrows \mathcal{G}$ is a sheaf as well:

- Let $s, t \in \mathrm{eq}(f,g)(U)$ such that $s_{|U_i} = t_{|U_i}$ for all $i \in I$. Recall that $e : \mathrm{eq}(f,g) \to \mathcal{F}$ is injective.

  $s_{U_i} = t_{U_i} \implies e_{U_i}(s_{|U_i}) = e_{U_i}(t_{|U_i})$ for all U_i, thus since $\mathcal{F}$ is a sheaf we get that $e_U(s) = e_U(t)$, and by injectivity this implies $s = t$.
- Let $s_i \in \mathrm{eq}(f_{U_i}, g_{U_i})$ be a family agreeing on intersections of opens.

  Then $e_{U_i}(s_i)$ agree on intersections of opens as well, thus there exists a unique $t \in \mathcal{F}(U)$ such that $t_{|U_i} = e_{U_i}(s_i)$.

  We need to note that $f_{U_i}(t_{|U_i}) = f_{U_i}(e_{U_i}(s_{u_i})) = g_{U_i}(t_{|U_i})$, thus $f_U(t) = g_U(t)$, meaning that $t \in \mathrm{eq}(f,g)(U)$, i.e.

  $t = e_U(s)$ for some s and $e_{U_i}(s_{|U_i}) = e_{U_i}(s_i)$ thus $s_{|U_i} = s_i$.

This means that **Sh**(X) has all finite limits, in particular pullbacks.[2]

[2] A nicer way to do this is to consider **Psh**(X) as sheaves over the trivial site on X and see that **Sh**$(X) \hookrightarrow$ **Psh**(X) is geometric.

(iii) (existence of a subobject classifier): Ω is the sheaf of open subsets: $\Omega(U) = \{\text{open subsets of } U\}$ This works since if we have an injective homomorphism $m : S \to B$, meaning an injective map $m_U : S(U) \to B(U)$ such that

$$\begin{array}{ccc} S(U) & \longrightarrow & \{0\} \\ \big\downarrow {\scriptstyle m_U} \lrcorner & & \big\downarrow {\scriptstyle \top} \\ B(U) & \dashrightarrow & \Omega(U) \end{array}$$

Then we can define a map $\operatorname{char} m_U : B(U) \to \Omega(U)$ as

$$b \mapsto \{V \subseteq^{\mathrm{o}} U \text{ such that } \rho_{UV}(b) \in m_V(S(V))\}$$

It exists and it's unique. It's obviously a presheaf and given an open cover U_i of U then $\rho^{\Omega}_{UU_i}(V) = V'$ for all i implies that $V_{|U_i} = V'$ for all i means that V lies in the intersection of all of the covering subsets, therefore $V = V'$ (we are restricting to a subset in which V is contained), and if we have $\rho_{U_iU_i\cap U_j}(V_i) = \rho_{U_jU_i\cap U_j}(V_j)$ for all i, j we can construct $V \in \Omega(U)$ as $\bigcup_i V_i$ and $\rho_{UU_i}(V) = V_i$

(iv) (existence of exponential objects): We have that the sheaf defined as

$$\mathcal{G}^{\mathcal{F}}(U) = \textbf{Sh}(X)(yU^+ \times \mathcal{F}, \mathcal{G})$$

(where y is the yoneda embedding) trivially satisfies the exponential condition.

□

Dilly-Dally. *I believe that the subobject classifier needs a little bit of dilly-dallying: If we want to interpret formulas, the fact that Ω is the set of open subsets gives us the interpretation that the truth value of a formula is given by the open subsets on which it holds.*

This is a topology by the way, meaning that it will generally not be a boolean algebra, therefore the logic we will have will not have the excluded middle.

Now we have a topos $\textbf{Sh}(X)$. We shall build the diagram

$$X \xrightarrow{\;y\;} \textbf{Psh}(X) \underset{j}{\overset{\_^+}{\rightleftarrows}}_{\bot} \textbf{Sh}(X) = \mathscr{X}$$

Where y is the yoneda embedding.

If $\mathcal{F}$ is a sheaf we have that

$$\textbf{Psh}(X)(yU, \mathcal{F}) \simeq \textbf{Sh}(yU^+, \mathcal{F})$$

meaning that we can see any open of X as an object of $\textbf{Sh}(X)$.

We say that in particular

$$\begin{aligned} U \vDash \phi(\alpha) &\text{ if and only if } \alpha \in [\![x \mid \phi]\!](U) \\ &\text{ if and only if } \alpha : yU \to \mathcal{F} \text{ factors through the mono } [\![x \mid \phi]\!] \to \mathcal{F} \\ &\text{ if and only if } \alpha^+ : yU^+ \to \mathcal{F} \text{ factors through the mono } [\![x \mid \phi]\!] \to \mathcal{F} \end{aligned}$$

The semantics that arises is a special case of the general one we described earlier where we only look at open subsets as generalized elements and we use the gluing as we discussed in the definition of sheaf.

In particular, since **Sh**(X) is complete and cocomplete[3] , we have acess to the infinitary sums and products of Ω, thus to infinitary conjunctions and disjunctions.

Since both are associative, we can generally take the $\wedge$ and $\vee$ arbitrary many times.

Theorem 2.3.2 (Rules inside **Sh**(X)).

$$\begin{aligned}
U \vDash s = t : \mathcal{F} \quad &:\Longleftrightarrow s = t \text{ as elements of } \mathcal{F}U \text{ for } U \subseteq^{\mathrm{o}} X \\
U \vDash s : \mathcal{G} \quad &:\Longleftrightarrow s \in \mathcal{G}U \text{ for } U \subseteq^{\mathrm{o}} X \ (\mathcal{G} \leq \mathcal{F}, s \in \mathcal{F}U) \\
U \vDash \top \quad &:\Longleftrightarrow U = U \text{ i.e. the trivial truth} \\
U \vDash \bot \quad &:\Longleftrightarrow U = \varnothing \\
U \vDash \phi \wedge \psi \quad &:\Longleftrightarrow U \vDash \phi \text{ and } U \vDash \psi \\
U \vDash \bigwedge_{j \in J} \phi_j \quad &:\Longleftrightarrow \text{for all } j \in J,\ U \vDash \phi_j \\
U \vDash \phi \vee \psi \quad &:\Longleftrightarrow \text{there exists a covering } \{U_i\}_{i \in I} \text{ of } U \text{ such that} \\
&\qquad \text{for all } i \in I,\ U_i \vDash \phi \text{ or } U_i \vDash \psi \\
U \vDash \bigvee_{j \in J} \phi_j \quad &:\Longleftrightarrow \text{there exists a covering } \{U_i\}_{i \in I} \text{ of } U \text{ such that} \\
&\qquad \text{for all } i \in I \text{ exists } j \in J \text{ such that } U_i \vDash \phi_j \\
U \vDash \phi \Rightarrow \psi \quad &:\Longleftrightarrow \text{for all } V \subseteq^{\mathrm{o}} U,\ V \vDash \phi \text{ implies } V \vDash \psi \\
U \vDash \forall s : \mathcal{F}, \phi(s) \quad &:\Longleftrightarrow \text{for all } s \in \mathcal{F}V \text{ on an open subset } V \subseteq^{\mathrm{o}} U,\ V \vDash \phi(s) \\
U \vDash \exists s : \mathcal{F}, \phi(s) \quad &:\Longleftrightarrow \text{there exists a covering } \{U_i\}_{i \in I} \text{ of } U \text{ such that} \\
&\qquad \text{for all } i \in I \text{ exists an } s_i \in \mathcal{F}U_i \text{ such that } U_i \vDash \phi(s_i)
\end{aligned}$$

Sketch of proof.
This is easily obtained by looking at **Theorem 1.3.3** and rememebering that the only maps of X are the inclusions $V \hookrightarrow U$ whenever $V \subseteq U$.

For example if we assume the General Kripke-Joyal semantics statement
$U \vDash \phi \vee \psi$ if and only if we have epimorphisms $g_1 : U_1 \to U, g_2 : U_2 \to U$ with $g_1 + g_2$ epi and $U_1 \vDash \phi$ and $U_2 \vDash \psi$.

This means that $U_1 + U_2 = U_1 \cup U_2$ is a cover of U and ϕ holds on U_1, ψ holds on U_2, thus we have a cover such that for each element of the cover either ϕ or ψ holds.

[3] It's been proven in a lot of textbooks, [BP94] to cite one

Conversely if we assume the Sheaf semantics statement
$U \vDash \phi \vee \psi$ if and only if on a cover U_i $U_i \vDash \phi$ or $U_i \vDash \psi$. Call U_1 the union of the elements of the cover where ϕ holds and U_2 the union of the elements where ψ holds to get the General Kripke-Joyal statement. □

Dilly-Dally (Idea behind the proof)**.**
Basically what we try to do is split U into smaller subsets (a cover) and then gluing them back together.

An alternative notation that we'll sometime use is $\phi/U, \frac{\phi}{U}$ "ϕ holds over U".

Of course we define $\neg\phi$ as $\phi \Rightarrow \bot$.

We shall note that every restriction is epic, thus if $V \subseteq^o U$, $V \vDash \phi(\alpha) \iff U \vDash \phi(\alpha_{|V})$

We can see that $[\![s : \mathcal{F} \mid \phi(s)]\!]$ is the sheaf $U \mapsto \{s \in \mathcal{F}U \mid U \vDash \phi(s)\}$

Proposition 2.3.3 (Locality of the language)**.**
Let U_i be an open cover of U. Then

$$U \vDash \phi \iff U_i \vDash \phi \quad \textit{for all } i$$

Proof. Note that all constructions are stable by restrictions, thus we can prove this inductively on the complexity of ϕ. □

This whole thing is made useful by the following lemma

Lemma 2.3.4 (Soundness of the internal language)**.**
If a formula ψ follows from ϕ in intuitionistic logic, ($\phi \vdash \psi$), then

$$U \vDash \psi \textit{ follows from } U \vDash \phi$$

Proof. We need to show that the Kripke-Joyal semantics we defined satisfies the rules of intuitionistic logic. Those are

1. (ID): $\dfrac{}{\phi \vdash \phi}$ (Sub): $\dfrac{\phi \vdash \psi}{\phi[s/x] \vdash \psi[s/x]}$ (Tr): $\dfrac{\phi \vdash \psi \qquad \psi \vdash \eta}{\phi \vdash \eta}$
2. ($\top$): $\dfrac{}{\phi \vdash \top}$ ($\wedge$L): $\dfrac{}{\phi \wedge \psi \vdash \phi}$ ($\wedge$R): $\dfrac{}{\phi \wedge \psi \vdash \psi}$ ($\wedge$): $\dfrac{\phi \vdash \psi \qquad \phi \vdash \eta}{\phi \vdash \psi \wedge \eta}$
3. ($\bot$): $\dfrac{}{\bot \vdash \phi}$ ($\vee$L): $\dfrac{}{\phi \vdash \phi \vee \psi}$ ($\vee$R): $\dfrac{}{\psi \vdash \phi \vee \psi}$ ($\vee$): $\dfrac{\phi \vdash \eta \qquad \psi \vdash \eta}{\phi \vee \psi \vdash \eta}$
4. ($\bigwedge$L): $\dfrac{}{\bigwedge_{i \in I} \phi_i \vdash \phi_j \text{ for all } j \in I}$ ($\bigwedge$R): $\dfrac{\phi \vdash \psi_j \text{ for all } j \in I}{\phi \vdash \bigwedge_{i \in I} \psi_i}$
5. ($\bigvee$R): $\dfrac{}{\phi_j \vdash \bigvee_{i \in I} \phi_i \text{ for all } j \in I}$ ($\bigvee$L): $\dfrac{\phi_j \vdash \psi \text{ for all } j \in I}{\bigvee_{i \in I} \phi_i \vdash \psi}$

6. (⇒): $$\frac{\phi \wedge \psi \vdash \eta}{\phi \vdash \psi \Rightarrow \eta}$$

7. (∃x): $$\frac{\phi \vdash \psi}{\exists x : X\ \phi \vdash \psi}$$ (∀x): $$\frac{\phi \vdash \psi}{\phi \vdash \forall x : X \psi}$$

8. (=): $$\frac{}{\top \vdash x = x}$$ (=Sub): $$\frac{}{(x = y) \wedge \phi \vdash \phi[y/x]}$$

Where the double line means that the rule can be read up-to-down or down-to-up.

This is going to be a long proof.

1. (ID) and (Tr) are trivial.

2. (Sub): Suppose that for any U $U \vDash \phi$ implies $U \vDash \psi$ and for any U, $U \vDash \phi[s/x]$. Then interpreting ϕ via the language will give us a set-theoretical formula with s substituted on any instance of x. In this the rule is valid, and thus implies an alternative version of ψ with s substituting x at any instance, that can be reinternalized.

3. ($\top$): This says that if any $U \vDash \phi$ then $U \vDash \top$, i.e. $U = U$. (∧R), (∧L) and (∧) are true since for any U $U \vDash \phi \wedge \psi \iff U \vDash \phi$ and $U \vDash \psi$.

4. ($\bot$): $U \vDash \bot$ means that $U = \varnothing$. $\varnothing \vDash \phi$ is always true as the only sheaf on $\varnothing$ is a singleton, thus any equality condition is trivially satisfied, every cover is the identity and there are no nonidentical open subset, thus we can prove recursively that any formula is true.

   (∨L): if $U \vDash \phi$ then by locality $U_i \vDash \phi$ for any open cover U_i of U, thus $U_i \vDash \phi$ or $U_i \vDash \psi$ on an open cover, meaning $U \vDash \phi \vee \psi$. The same goes for (∨R).

   (∨): suppose that for any U $U \vDash \phi$ implies $U \vDash \eta$, $U \vDash \psi$ implies $U \vDash \eta$ and $U \vDash \phi \vee \psi$. Then there exists an open cover U_i of U such that $U_i \vDash \phi$ or $U_i \vDash \psi$.

   Since the above condition is true for all U, this implies that $U_i \vDash \eta$ or $U_i \vDash \eta$. Since this is true for all U_i, by locality of the language we can say that $U \vDash \eta$.

5. (⋀L) and (⋀R) follow immediately from the rule.

6. (⋁R): suppose ϕ_j holds over any U, then fixing a given U it holds over a cover U_i of U, in particular the cover realizing $U_i \vDash \phi_j$ for all i.

   (⋁L): Suppose that for any U and for all $j \in I$, $U \vDash \phi_j$ implies $U \vDash \psi$ and $U \vDash \bigvee_{i \in I} \phi_i$. Then on a cover of U $U_i \vDash \phi_j$ for some ϕ_j and for all i,

   Thus $U_i \vDash \psi$ for all i, meaning $U \vDash \psi$ for the locality of the language.

7. (⇒): Suppose that for any U, $U \vDash (\phi \wedge \psi)$ implies $U \vDash \eta$ and suppose that for any U, $U \vDash \phi$.

   We want to show that for any U and all $V \subseteq^{o} U$ $V \vDash \phi$ implies $V \vDash \psi$:

   For all $V \subseteq^{o} U$, if $V \vDash \psi$, then we already know that $V \vDash \phi$, thus $V \vDash \phi \wedge \psi$, therefore $V \vDash \eta$.

Conversely, if for any U, $U \vDash \phi$ implies $U \vDash \psi \Rightarrow \eta$ and for any U $U \vDash \phi \wedge \psi$, we want to show that $U \vDash \eta$. Note that $U \vDash \psi \Rightarrow \eta$ is translated into the fact that for all $V \subseteq^{o} U$, $V \vDash \psi$ implies $V \vDash \eta$.

Thus since $U \vDash \phi \wedge \psi$ and $V \vDash \phi \wedge \psi$, we have that $U \vDash \phi$, thus $V \vDash \psi$ implies $V \vDash \eta$, but since $V \vDash \psi$ also we have that $V \vDash \eta$ for any open subset of U.

Since U is general and V is any open subset, this means that $U \vDash \eta$ for any U.

8. (∃x): suppose that for any U, $U \vDash \phi$ implies $U \vDash \psi$ and $U \vDash \exists x : X\ \phi$. This means that given a cover $\{U_i\}_{i \in I}$ of U, for all $i \in I$ there exists $s_i \in XU_i$ such that $U_i \vDash \phi(s_i)$.

   Choose one such s_i, then $U_i \vDash \phi(s_i)$, for all $i \in I$, therefore $U_i \vDash \psi$ for all $i \in I$, thus $U \vDash \psi$.

   Conversely suppose that for any U, $U \vDash \exists x : X\phi$ implies $U \vDash \psi$, and for any U $U \vDash \phi$.

   $U \vDash \exists x : X\phi$ means that there exists a cover $\{U_i\}_{i \in I}$ of U such that for all $i \in I$ exists $s_i \in XU_i$ such that $U_i \vDash \phi(s_i)$.

   Since for all U $U \vDash \phi$, then in particular for U_i we have that $U_i \vDash \phi(s_i)$. This means that $U \vDash \exists x : X\phi$, implying $U \vDash \psi$. (∀x): suppose that for any U, $U \vDash \phi$ implies $U \vDash \psi$ and for any U, $U \vDash \phi$. Then for all $V \subseteq^{o} U$ $V \vDash \phi$ implies $V \vDash \psi$. Thus for all $s : XV$ we have that $V \vDash \phi$, thus $V \vDash \psi$. Internalizing, this means that $U \vDash \forall x : X\psi$.

   Conversely Suppose that for any U $U \vDash \phi$ implies $U \vDash \forall x : X\psi$, and for any U $U \vDash \phi$. Then we know that for all $V \subseteq^{o} U$ and for all $x : X$, $V \vDash \psi$. In particular $U \vDash \psi$.

9. (=) and (=Sub) are trivial.

□

Dilly-Dally (What's happening?)**.**
The most common way of interpreting intuitionistic logic is the constructivist point of view: something is true if you can directly prove that it is.

In that sense, given a formula ϕ of course ϕ and $\neg\neg\phi$ are not equivalent, as there are (at least a priori) theorems you can prove by contraddiction but not directly.

While this is a interesting and useful point of view I don't personally believe this is the right way to think of the logic of ***Sh***(X)*.*

While being indeed intuitionistic, we are not really interested in constructivism: What we're doing here is handing over the sheaf to the logical evaluation.

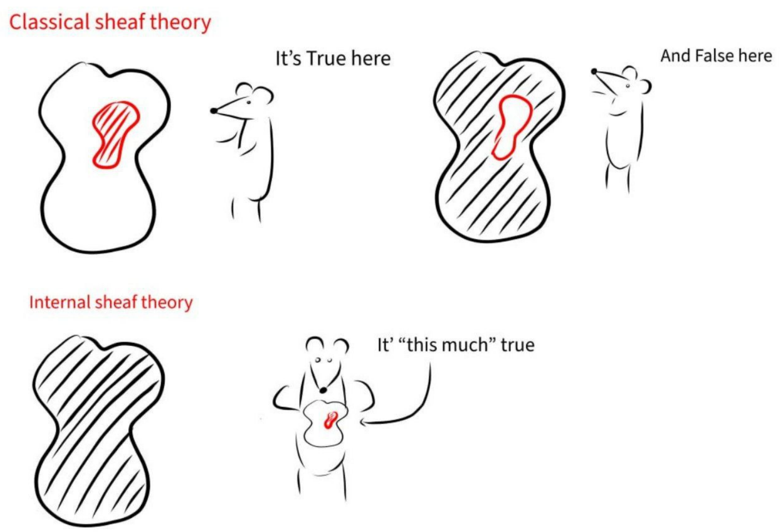
Usually a sheaf is a map that associates a set to an open subset of a topological space, thus saying that a formula regarding $\mathcal{F}$ is true is saying that every formula regarding $\mathcal{F}U$ is true for all U;

Here the truth value of the formula is the subspace[a] *where the formula is true, so for example if ϕ is true just for the right half of the space we would classically say that ϕ is true there and false on the left half. This approach is equivalent to say that "the right half of the space" is the measure of how much ϕ is true.*

In this sense the intuitionistic logic arises naturally, since if we say that something is true only if it's true everywhere $\neg\neg$true means "not false everywhere", i.e. "true somewhere".

I guess this could be expressed in modal logic terms, but I don't want to create confusion with the internal modal logic that we will employ.

Anyway, here's a cute little drawing that I hope will aid the explaination.



[a]sublocale, whatever

2.4 Modalities part 2

We want to see how $\Box$ interacts with formulas on a category of sheaves.

We can denote $V \vDash U$ whenever $V \subseteq U$. Let's also introduce the notation $V \vDash !x$ whenever $x \notin V$ (x here is a point of X)

Those are indeed formulas: Every $U \subseteq^{o} X$ defines a morphism $U : 1 \to \Omega$ defined as $U(V) = (U \cap V)$

With this the formula $!x$ is the one associated to the space $(X \setminus \{x\})^{o}$

Definition 2.4.1 (Nucleus)**.**
*Given a Modality $\Box$ we define the **Nucleus** of $\Box$ as*

$$j_{\Box}(U) = \bigcup_{V \vDash \Box U} V \subseteq^{o} X$$

Or, alternatively

$$j_{\Box}(U) = \mathrm{Lim}^{\rightarrow}_{V \vDash \Box U} V$$

Recall that the truth value object in **Sh**(X) is "made of open subsets" (since $\Omega(U) = \{V \subseteq^{o} U\}$). This means that defining the behaviour of $\Box$ and other relative data on each open subset of X is enough to describe it on all Ω.

We have properties that derive from modality:

Proposition 2.4.2 (Properties of $j_{\Box}$)**.**
We'll call $j_{\Box}$ just j here and whenever it's clear what operator we're referring to.

1. $U \subseteq j(U)$
2. $j(j(U)) \subseteq j(U)$
3. $j(U \cap V) = j(U) \cap j(V)$

Proof.

1. $V \vDash U \implies V \vDash \Box U$ thus $V \subseteq U \implies V \subseteq j(U)$;
2. $V \vDash \Box\Box U \implies V \vDash \Box U$ thus $V \subseteq \Box\Box U \implies V \subseteq j(U)$;
3. $V \vDash \Box U \wedge \Box U' \iff V \vDash \Box(U \cap U')$ thus $j(U \cap U') = j(U) \cap j(U')$.

$\square$

In particulare we are interested in the modal operator $\Box\phi = ((\phi \Rightarrow !x) \Rightarrow !x)$.

In particular $U \vDash \Box\phi$ means that

$$\forall V \subseteq^{o} U (\forall W \subseteq^{o} V, W \vDash \phi \Rightarrow x \notin W) \implies x \notin V$$

Thus either $x \notin U$ or $x \in W \vDash \phi$ for some open neighborhood W.

The associated nucleus $j(U)$ is $X \setminus \overline{\{x\}}$ if $x \notin V$ or X if $x \in V$.

Definition 2.4.3 (Nucleic Sublocale)**.**
Given a modality $\Box$ *with nucleus* j*, we call*

$$X_\Box = X_j := \{U \subseteq^{o} X \mid j(U) = U\}\,.$$

This means that we can define the category of sheaves on $X_\Box$ (which in general is a locale) like we did for our usual topological spaces.
There is another related perspective:

Definition 2.4.4 ($\Box$–sheaves)**.**
We will give a series of definitions: A set S *is a* ***subsingleton*** *if*

$$\forall x, y : S \; x = y$$

and it's a ***singleton*** *if it's also inhabited, i.e.*

$$\exists x : S \; \top$$

Now a set T *is* $\Box$*–**separated** if*

$$\forall x, y : T \;\; \Box(x = y) \Rightarrow x = y$$

and it's a $\Box$*–**sheaf** if also*

$$\forall S \leq T, \; \Box(S \text{ “singleton''}) \Rightarrow \exists x : T \;\; \Box(x : S)$$

We may see that $\Box$–sheaves of **Sh**(X) and sheaves over $X_\Box$ are equivalent as categories:

Theorem 2.4.5 ($\Box$–sheaves and sheaves over $\Box$)**.**

$$\textbf{Sh}(X_\Box) \simeq \textbf{Sh}_\Box(\textbf{Sh}(X))$$

The syntax in the language of $\textbf{Sh}_\Box(\textbf{Sh}(X))$ is given by the usual syntax and a $\Box : \Omega \to \Omega$, its semantics are the usual with the rules for $\Box$ we defined above.
We can define the $\Box$–translation of formulas recursively by placing a $\Box$ before every subformula:

Definition 2.4.6 ($\Box$-translation)**.**

$$
\begin{aligned}
(f = g)^\Box &:= \Box(f = g) \\
(x : \mathcal{F})^\Box &:= \Box(x : \mathcal{F}) \\
\top^\Box &:= \Box\top & \bot^\Box &:= \Box\bot \\
(\phi \wedge \psi)^\Box &:= \Box(\phi^\Box \wedge \psi^\Box) & (\bigwedge_i \phi_i)^\Box &:= \Box(\bigwedge_i \phi_i^\Box) \\
(\phi \vee \psi)^\Box &:= \Box(\phi^\Box \vee \psi^\Box) & (\bigvee_i \phi_i)^\Box &:= \Box(\bigvee_i \phi_i^\Box) \\
(\phi \Rightarrow \psi)^\Box &:= \Box(\phi^\Box \Rightarrow \psi^\Box) \\
(\forall x : \mathcal{F}\, \phi)^\Box &:= \Box(\forall x : \mathcal{F}\, \phi^\Box) & (\exists x : \mathcal{F}\, \phi)^\Box &:= \Box(\exists x : \mathcal{F}\, \phi^\Box)
\end{aligned}
$$

Theorem 2.4.7 (□–translation)**.**

$$X \vDash \phi^{\square} \textit{ in } \textbf{\textit{Sh}}(X) \iff X_{\square} \vDash \phi \textit{ in } \textbf{\textit{Sh}}(X_{\square})$$

Proof. A proof of both theorems can be found in [Ble17] □

Lastly we will talk about sheafification:

Definition 2.4.8 (#-construction for sets.)**.**
Let F be a set. We construct

$$F^{\#} := \{S \subseteq F \mid \square(S \text{" singleton"})\} / \sim$$

where $S \sim T \iff \square(S = T)$

*We define the □–**sheafification** of a set as*

$$F^{\square} = F^{\#\#}.$$

Note that $-^{\square}$ is a functor **Set** → $\textbf{Sh}_{\square}(\textbf{Set})$.

This means that for sheaves we have

Definition 2.4.9 (□–sheafification)**.**
Let X be a topological space, and $\mathcal{F} \in \textbf{\textit{Sh}}(X)$

$$\begin{array}{ccc} X & \searrow & \\ \mathcal{F}\downarrow & & \\ \textbf{\textit{Set}} & \xrightarrow[-^{\square}]{} & \textbf{\textit{Sh}}_{\square}(\textbf{\textit{Set}}) \end{array} \qquad \begin{array}{ccc} U & \searrow & \\ \mathcal{F}\downarrow & & \\ \mathcal{F}(U) & \longmapsto & \mathcal{F}(U)^{\square} \end{array}$$

We define the □–sheafification of $\mathcal{F}$ as the composite $\mathcal{F} \circ -^{\square}$.

Note that $-^{\square} : \textbf{Sh}(X) \to \textbf{Sh}_{\square}(\textbf{Sh}(X)) \simeq \textbf{Sh}(X_{\square})$ This sheafification coincides with the translation and it's left adjoint to the inclusion $\textbf{Sh}(X_{\square}) \hookrightarrow \textbf{Sh}(X)$:

We could write Theorem 2.3.7 as $\mathrm{Hom}(X, \Omega^{\square}) \simeq \mathrm{Hom}(X_{\square}, \Omega)$ to evidentiate the adjunction.

One last theorem we should throw in is that

Proposition 2.4.10 (Stalks through modality)**.**
Let $x \in X$, $\mathcal{F} \in \textbf{\textit{Sh}}(X)$ and $\square \equiv ((- \Rightarrow !x) \Rightarrow !x)$.

Then $\mathcal{F}_x = \mathcal{F}^{\square}$ and any for any formula ϕ, $\phi^{\square}$ is obtained by substituting any occurring sheaf or morphism of sheaves with its stalk at x.

Proof.
We know that

$$X \vDash \phi^{\square} \iff X_{\square} \vDash \phi$$

$X_{\square}$ is defined as the set of open subsets U of X such that $j_{\square}(U) = U$.

We previously calculated that $j(U) = X \setminus \overline{\{x\}}$ if $x \notin U$ and X if $x \in U$

The internal locale of a subspace A is defined as

$$j_A(U) = \bigcup \{V \subseteq U : U \cap A = V \cap A\}$$

It's easy to verify that the nucleus for the space $\{x\}$ is the same as the nucleus for the sheafification.

Recall that for the inclusion $i : \{x\} \to X$

$$_*i\, \mathcal{F} = (U \mapsto \mathrm{Lim}^{\to}_{U \in i^{\leftarrow} V}\, \mathcal{F} V)^+$$

which is exactly the definition of stalk. □

The most famous and arguably one of the most important modal operators is the $\neg\neg$. We get

Proposition 2.4.11 ($X_{\neg\neg}$)**.**
$X_{\neg\neg}$ is the smallest dense sublocale of X.

Proof. We get that $j_{\neg\neg}(V) = (\overline{V})^o$, $j_{\neg\neg}(V) = V \iff (\overline{V})^o = V$

This means that any open of $X_{\neg\neg}$ is equal to the interior of its closure. It's easy to see that this is true for the smallest dense sublocale. □

In particular, if we have a generic point ξ such that $\overline{\xi} = X$ taking the stalk at ξ is the same as $\neg\neg$–sheafifying.

Moreover the pushout of a set A through $X_{\neg\neg} \to X$ is the same as taking the constant sheaf.

2.5 Geometric constructions

A special class of formulas are geometric formulas

Definition 2.5.1 (Geometric formulas and implications).
*A formula is **geometric** if and only if it only consists of*

$$=,\ :,\ \top,\ \bot,\ \wedge,\ \vee,\ \bigvee,\ \exists$$

*A **geometric implication** is a formula of the type*

$$\forall x_1 \cdots \forall x_n\ \phi(x_1, ..., x_n) \Rightarrow \psi(x_1, ..., x_n)$$

where ϕ and ψ are geometric formulas.

We say that ϕ **holds on a point** x whenever ϕ holds with all of its terms and types substituted with the stalk at x.

for example $z = y : \mathcal{F} \Rightarrow f(z) = f(y) : \mathcal{G}$ holds at x whenever $z = y : \mathcal{F}_x \Rightarrow f_x(z) = f_x(y) : \mathcal{G}_x$.

we write $x \vDash \phi$

Theorem 2.5.2 (Geometric modality).
Let ϕ be a geometric formula, then for all modalities $\Box$,

$$\phi^{\Box} \iff \Box\phi$$

Proof. Can be found in [Ble17] □

Lemma 2.5.3 (geometric formulas extend from points).
Let $x \in X$ be a point. Let ϕ be a geometric formula (over some neighborhood V of x).

Then ϕ holds at x if and only if exists an open neighborhood $U \subseteq^{o} V$ of x such that ϕ holds on U.

Proof. Consider the modality $\Box \equiv (- \Rightarrow !x) \Rightarrow !x$.

$X \vDash \Box\phi$ iff $X \vDash (\phi \Rightarrow !x) \Rightarrow !x$
iff for all $U \subseteq^{o} X,\ U \vDash \phi \Rightarrow !x$ implies $x \notin U$
iff for all $U \subseteq^{o} X,\ ($ for all $V \subseteq^{o} U,\ V \vDash \phi$ implies $x \notin V)$ implies $x \notin U$

Since the external logic is classic we can say that this is equivalent to

for all $U \subseteq^{o} X,\ \neg($for all $V \subseteq^{o} U,\ V \vDash \phi$ implies $x \notin V)$ or $x \notin U$

Suppose $x \in U$, this means that

$\neg($ for all $V \subseteq^{o} U,\ V \vDash \phi$ implies $x \notin V)$

Holds, equivalently

$$\begin{aligned}
&\text{there exists a } V \subseteq^{o} U \text{ such that } \neg(V \vDash \phi \text{ implies } x \notin V)\\
\text{iff } &\text{there exists a} V \subseteq^{o} U \text{ such that } \neg(\neg(V \vDash \phi) \text{ or } x \notin V)\\
\text{iff } &\text{there exists a} V \subseteq^{o} U \text{ such that } \neg\neg(V \vDash \phi) \text{ and } x \in V\\
\text{iff } &\text{there exists a} V \subseteq^{o} U \text{ such that } V \vDash \phi \text{ and } x \in V
\end{aligned}$$

That is, ϕ holds on a neighborhood of x.

As ϕ is geometric, $\Box\phi \iff \phi^{\Box}$ which we discussed is the formula at the stalk at x. □

Lemma 2.5.4 (Locality/Globality of geometric implications).
A geometric formula holds on X if and only if it holds at every point $x \in X$.

Proof. Suppose $X \vDash \phi$. Note that for all $x \in X$, X is a neighborhood of x, meaning we can use the lemma above.

Now suppose $x \vDash \phi$ for all x. This means that for all x ϕ holds on a neighborhood U_x.

This means that $\{U_x\}_x$ form a cover of X and $U_x \vDash \phi$ for all x. By locality $X \vDash \phi$. □

2.6 Change of space

An important question to ask ourselves is what happens when we want to transport information from **Sh**(X) to **Sh**(Y) from X to a different topological space Y

We may notice that if we have a continuous function $f : X \to Y$,
then $f^{\leftarrow}(U)$ is open in X for all U open in Y. We can define a functor

Definition 2.6.1 (Direct image functor).
Given a continuous map $f : X \to Y$ we can define the ***Direct image functor***

$$f_* : \textbf{Sh}(X) \to \textbf{Sh}(Y)$$

as $(f_* \mathcal{F})(U) = \mathcal{F}(f^{\leftarrow}(U))$.

Proposition 2.6.2 ($f_* \mathcal{F}$ is a sheaf).
Given a sheaf $\mathcal{F} \in \textbf{Sh}(X)$, $f_* \mathcal{F} \in \textbf{Sh}(Y)$

Proof.
The fact that it's a presheaf is trivial (it's the composition of two functors).

Note that if V_i is a cover of $V \subseteq^{o} Y$, then $f^{\leftarrow}(V_i)$ is an open cover of $f^{\leftarrow}(V)$.

Since $\rho^{f_* \mathcal{F}}_{UV} = \rho^{\mathcal{F}}_{f^{\leftarrow}(U) f^{\leftarrow}(V)}$ the sheaf structure is inherited from $\mathcal{F}$. □

In **Psh**(X) we have a left adjoint

Definition 2.6.3 (Left adjoint of f_*).
Let $f : X \to Y$ be a continuous function and let $I(U) = \{V \subseteq^{o} Y : f(U) \subseteq V\}$. We call ${}_*f : \textbf{Psh}(Y) \to \textbf{Psh}(X)$[4] *the functor such that*

$$({}_*f\, \mathcal{F})U = \underrightarrow{\mathrm{Lim}}_{V \in I(U)}\, \mathcal{F} V$$

Proof. We have to prove that ${}_*f \dashv f_*$, i.e. **Psh**$(X)({}_*f\, \mathcal{G}, \mathcal{F}) = $ **Psh**$(Y)(\mathcal{G}, f_* \mathcal{F})$. The natural isomorphism is given by

$$\textbf{Psh}(X)({}_*f\, \mathcal{G}, \mathcal{F}) \longrightarrow \textbf{Psh}(Y)(\mathcal{G}, f_* \mathcal{F})$$

$$\psi :{}_* f\, \mathcal{G} \to \mathcal{F} \longmapsto f_*\psi \circ \phi : \mathcal{G} \to f_{**}f\, \mathcal{G} \to f_* \mathcal{F}$$

where $\phi : G \to f_{**}f\, \mathcal{G}$ is defined as $GV \mapsto \underrightarrow{\mathrm{Lim}}_{f_{\to} f^{\leftarrow} V \subseteq V' \subseteq^{o} Y}\, GV'$ □

[4] I was not sure whether to use the classical notation f^* for this left adjoint, since this symbol is often reserved to another important map of sheaves of modules. Professor Iovita used f^{-1} in his class but I'm personally not fond of it since it suggests that the functor is an inverse of some kind, which it isn't. The ultimate decision was for ${}_*f$, since it's consistent and if you read $f : X \to Y$ you can remember that "the map with the asterisk on the right points to the right and the map with the asterisk on the left points to the left".

Since we have an adjoint pair **Psh**$(X) \leftrightarrows$ **Sh**(X) we compose to have a left adjoint in **Sh**(X), by defining ${}_*f/\,$**Sh**(X) as $({}_*f/\,\mathbf{Psh}(X))^+$

We can note that

$$({}_*f\,\mathcal{G})_x = \mathrm{Lim}^{\to}_{U_x \subseteq^{o} X}\, {}_*f\,\mathcal{G}\,U_x = \mathrm{Lim}^{\to}_{U_x \subseteq^{o} X}\, \mathrm{Lim}^{\to}_{f_{\to}(U) \subseteq^{o} V_y \subseteq^{o} Y}\, \mathcal{G}\,V \simeq \mathcal{G}_y$$

In other words $({}_*f\,\mathcal{G})_x = \mathcal{G}_{f(x)}$

Proposition 2.6.4 (Geometric Morphisms)**.**
f_* and ${}_*f$ defined above are geometric morphisms.

Proof. We have proven that they are an adjoint pair, now we only need to show that ${}_*f$ preserves finite limits: To show that we only need to prove that it preserves binary products and binary equalizers.

Note that preserving products and equalizers are geometric implications fixed a morphism of sheaves $\varphi :_* f(\mathcal{F}\times\mathcal{G}) \to_* f\,\mathcal{F} \times_* f\,\mathcal{G}$ and $\Psi :_* f(\mathrm{eq}(\varphi_1,\varphi_2)) \to \mathrm{eq}({}_*f\varphi_1, {}_*f\varphi_2)$

$$X \vDash \forall x :_* f(\mathcal{F}\times\mathcal{G})\ \top \Rightarrow \exists\ y :_* f\,\mathcal{F}\times_* f\,\mathcal{G}\ \ |\varphi(x) = y$$

$$X \vDash \forall x, y :_* f(\mathcal{F}\times\mathcal{G})\varphi(x) = \varphi(y) \Rightarrow x = y$$

(and equivalently for Ψ)

And thus locally we have $\mathcal{F}_{f(x)} \times \mathcal{G}_{f(x)} = (\mathcal{F}\times\mathcal{G})_{f(x)}$ and $\mathrm{eq}(\varphi_{1f(x)}, \varphi_{2f(x)}) = \mathrm{eq}(\varphi_1,\varphi_2)_{f(x)}$ which are true by definition and uniqueness of the colimit.

All is left is to define the maps φ and Ψ called above, and actually we only need to define it on presheaves, since we defined ${}_*f\,\mathcal{F}$ as a sheafification.

Precisely we get

$$\varphi_U : \mathrm{Lim}^{\to}_{V\in I(U)}(\mathcal{F}\,V \times \mathcal{G}\,V) \to \mathrm{Lim}^{\to}_{V\in I(U)}\mathcal{F}\,V \times \mathrm{Lim}^{\to}_{V\in I(U)}\mathcal{G}\,V$$

through

$$\begin{array}{ccccc}
\mathcal{F}\,V & \longleftarrow & (\mathcal{F}\times\mathcal{G})V & \longrightarrow & \mathcal{G}\,V \\
\downarrow & \swarrow & \downarrow{\scriptstyle \varphi} & \searrow & \downarrow \\
\mathrm{Lim}^{\to}_{V\in I(U)}\mathcal{F}\,V & \longrightarrow & \mathrm{Lim}^{\to}_{V\in I(U)}\mathcal{F}\,V \times \mathrm{Lim}^{\to}_{V\in I(U)}\mathcal{F}\,V & \longleftarrow & \mathrm{Lim}^{\to}_{V\in I(U)}\mathcal{G}\,V
\end{array}$$

for all $V \in I(U)$ thus an unique map from the direct limit, and

$$\Psi_U : \mathrm{Lim}^{\to}[\![x : \mathcal{F}\,V \mid \phi_{1V}(x) = \phi_{2V}(x)]\!] \to [\![x : \mathrm{Lim}^{\to}\mathcal{F}\,V \mid \mathrm{Lim}^{\to}\varphi_{1V}(x) = \mathrm{Lim}^{\to}\varphi_{2V}(x)]\!]$$

given by the restriction. □

Note that the we took the easy route by finding the existence of a morphism a priori, since the formula

$$X \vDash \exists\varphi :_* f(\mathcal{F}\times\mathcal{G}) \to_* f\,\mathcal{F}\times_* f\,\mathcal{G}\ \mid \forall x[\cdots]$$

is not a geometric implication.

To complete what we were saying about logical functors, $f_*\Omega_X(U) = \Omega_X(f^{\leftarrow}(U)) = \left\{V \subseteq^o f^{\leftarrow(U)}\right\}$ is usually not $\Omega_Y(U)$:

For example $f : \{0,1\} \to \{0\}$ with the discrete topology,

$$\Omega_{\{0\}}(\{0\}) = \{\{0\}, \varnothing\}, \text{ while } f_*\Omega_{\{0,1\}}(\{0\}) = \Omega_{\{0,1\}}(\{0,1\}) = \{\{0,1\}, \{0\}, \{1\}, \varnothing\}$$

We generally have $\Omega_Y \leq f_*\Omega_X$.

So f_* is not a logical functor, since it doesn't preserve the truth values object, we can easily deduce that neither is ${}_*f$.

Lemma 2.6.5 (Geometric morphisms are almost-logical)**.**
Let f be a geometric morphism, then both direct image and inverse image preserve geometric formulas and geometric implications.

Proof.
A proof can be found in [BP94] at page 365, For coherent formulas (i.e. excluding infinitary disjunctions).

Having trivial access in both domain and codomain to infinitary disjunctions it holds for them too as a limiting case of finitary disjunctions. □

2.7 Barr's theorem

In this section we will discuss Barr's Theorem, a theorem that lets us prove geometric implications with classical logic.

In [Ble17] this is avoided since -at least at the time of writing- it wasn't proven intuitionistically and the author wanted his whole work to be intuitionistically sound.

We don't have any such inspirations and will use anything that makes our work easier as long as it makes sense.

The theorem states

Theorem 2.7.1 (Barr)**.**
For every Grothendieck topos $\mathcal{E}$ there exists a boolean algebra B and a geometric morphism $p_* : \textbf{Sh}(B) \to \mathcal{E}$ such that ${}_*p : \mathcal{E} \to \textbf{Sh}(B)$ is faithful.

Proof.
A proof can be found in [Bar74] but I believe the sketch found in [Rey77] is clearer. □

Since we won't prove this I believe it's important that we discuss what it means for us.

There are many different definitions of a boolean topos. One of these is

$$B \vDash 1 + 1 \simeq \Omega_B$$

, in other words, "the set of truth values (from the internal perspective) has two elements."

This means that, in particular, $B \vDash \phi \vee \neg\phi$ for any formula ϕ, allowing us to use the usual tricks of classical logic.

Having a geometric pair $p_*, {}_*p : \textbf{Sh}(B) \leftrightarrows \textbf{Sh}(X)$ means that any geometric implication in $\textbf{Sh}(X)$ is preserved in $\textbf{Sh}(B)$, where it can be proven classically.

This means that geometric implications can be proven classically inside $\textbf{Sh}(X)$ without loss of generality.

This justifies a lot of the "weirder" definitions we will give in the following pages: we want everything to be as geometric as possible.

3 (Re)defining classical objects

Now that we have a working model for formulas in a category of sheaves we can define mathematical objects in it. We have a way to build definitions for classes of objects and morphisms.

We need to be a little careful with our definitions and proofs since the logic we are working with is intuitionistic, so the statement ϕ and $\neg\neg\phi$ are generally not equivalent.

Definition 3.0.1 (Injective and surjective maps)**.**
A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is ***injective*** *if*

$$X \vDash \forall x, y : \mathcal{F},\ f(x) = f(y) \Rightarrow x = y$$

and is ***surjective*** *if*

$$X \vDash \forall y : \mathcal{G}\ \exists x : \mathcal{F}\ \mid f(x) = y$$

Recall that a subsheaf $\mathcal{F} \leq \mathcal{G}$ is a sheaf such that $\mathcal{F}U \subseteq \mathcal{G}U$ for all $U \subseteq^{o} X$ We can define the image and preimage of a map as

Definition 3.0.2.
Given a morphism $f : \mathcal{F} \to \mathcal{G}$, we define the ***image*** *of $X \leq \mathcal{F}$ in $\mathcal{G}$ as*

$$f_{\to}(X) := [\![y : \mathcal{G} | \exists x : X | f(x) = y]\!]$$

and the ***preimage*** *of $Y \leq \mathcal{G}$ in $\mathcal{F}$ as*

$$f^{\leftarrow}(Y) := [\![x : \mathcal{F} | f(x) : Y]\!]$$

Remember that we can define a power object $P(\mathcal{F})$ as $\Omega^{\mathcal{F}}$, and $f_{\to}$ and $f^{\leftarrow}$ can be internalized as maps $P(\mathcal{F}) \leftrightarrows P(\mathcal{G})$.

It's also important to remember that those are not sets but the objects $[\![x | \phi(x)]\!]$ defined previously, hence the double square bracket notation.

3.1 Abelian Groups

Definition 3.1.1 (Sheaves of groups)**.**
A ***Sheaf of groups*** *in* ***Sh***(X) *is a sheaf G together with a map $* : G \times G \to G$ and global element e such that*

$$X \vDash \forall x, y, z : G\ x * (y * z) = (x * y) * z,$$

$$X \vDash \forall x : G\ \exists y : G \mid x * y = y * x = e$$

and

$$X \vDash \forall x : G\ x * e = e * x = x.$$

*We will also call this a **group over** X or **group over Sh**(X).*

*It is called a **sheaf of abelian groups** (resp. **abelian group over** X **or Sh**(X)) if moreover*

$$X \vDash \forall x, y : G,\ x * y = y * x.$$

*A **(homo)morphism of groups** given two groups* $\dfrac{(G, *_G), (H, *_H)}{X}$ *is a morphism of sheaves* $f : G \to H$ *such that*

$$X \vDash f(x *_G y) = f(x) *_H f(y).$$

We'll make a standard notation decision to omit the operation inside the groups, and call it $\cdot$ (or omit it completely) when the group is not abelian and call it + when the group is abelian.

Meanwhile the identity element e will be called 1 for nonabelian groups and 0 for abelian groups.

Another way to phrase these is to say $X \vDash$"G is a group/abelian group" or respectively $X \vDash$"ϕ is a group homomorphism"

We can define kernels and images as

Definition 3.1.2 (Kernels and images)**.**
Given two groups $\dfrac{G, H}{X}$ *and a group homomorphism* $\dfrac{f : G \to H}{X}$ *we can define The **kernel** of* f *as*

$$\ker(f) := [\![x : G | f(x) = e]\!] = f^{\leftarrow}([\![x : H | x = e]\!]) =: f^{\leftarrow}(e)$$

*and the **Image** of* f *as*

$$\mathrm{im}(f) := f_{\to}(G)$$

We shall note that a lot of the usual results also apply.

Proposition 3.1.3.
Given a group homomorphism $\dfrac{f : G \to H}{X}$*, then* $X \vDash f(1) = 1$

Proof.
$X \vDash f(1 \cdot x) = f(1) \cdot f(x) \wedge f(1 \cdot x) = f(x)$
$\iff X \vDash f(x) = f(1) \cdot f(x)$
Remember that $X \vDash \forall y : H\ \exists y^{-1} : H \mid yy^{-1} = 1$
$\iff \forall y$ section of HU, $U \vDash\ (\exists y^{-1} : H \mid yy^{-1} = 1)$
$\iff$ exists a cover U_i of U such that $\forall i\ \exists y_i^{-1} : HU_i \mid y_{|U_i} y_i^{-1} = 1_{U_i}$

That means that any element has a local inverse.

Since $y_{|U_i \cap U_j} y^{-1}_{i|U_i \cap U_j} = 1_{U_i \cap U_j} = y_{|U_i \cap U_j} y^{-1}_{j|U_i \cap U_j}$ and the inverse is unique we can say that

$y^{-1}_{i|U_i\cap U_j} = y^{-1}_{j|U_i\cap U_j}$

That means we can use the sheaf property (2) to say that exists $y^{-1} : U$ such that $y^{-1}_{|U_i} = y^{-1}_i$

Using the sheaf property (1) we can deduce that $\forall y : HU \ \exists y^{-1} : U \mid yy^{-1} = 1$

This means that $X \vDash 1 \cdot f(x) = f(1) \cdot f(x) \iff X \vDash 1 = f(1)$. □

Note that the usual proof of the fact is intuitionistically valid, thus we could avoid the sheaf mechanics.

I believe, anyway, that showing that we could prove things directly is a nice exercise when dealing with easy enough proofs.

Theorem 3.1.4 (Characterization of injective and surjective homomorphisms)**.**

1. *A morphism* $\dfrac{f : G \to H}{X}$ *of groups over* X *is injective if and only if*

$$\ker(f) = [\![x : G | x = 1]\!]$$

2. *A morphism* $\dfrac{f : G \to H}{X}$ *of groups over* X *is surjective if and only if*

$$\operatorname{im}(f) = H$$

Proof.

1. $X \vDash$ "f is injective"
   $\iff X \vDash \forall x, y : G,\ f(x) = f(y) \Rightarrow x = y$
   $\iff$ for all $x, y : GU$ we have that $, U \vDash [f(x) = f(y) \Rightarrow x = y]$
   $\iff$ for all open subsets $V \subseteq^{o} U$, $(V \vDash f(x)_{|V} = f(y)_{|V} \implies V \vDash x_{|V} = y_{|V})$
   $\implies [\![x : G | f(x) = 1]\!](V) = [\![x : G | f(x) = f(1)]\!](V)$ and on V this implies that $x = 1$
   therefore $\ker(f) = [\![x : G | x = 1]\!] = 1$

   For the other implication we can get that if $U \vDash [f(x) = 1 \Rightarrow x = 1] \wedge U \vDash f(x) = f(y)$ then
   for any open subset $V \subseteq^{o} U$, $f(x)_{|V} = 1_{|V} \implies x_{|V} = 1_{|V} \wedge f(x)_{|V} = f(y)_{|V}$.
   This means that $f(x)_{|V} f(y)^{-1}_{|V} = 1 \implies xy^{-1}_{|V} = 1_{|V}$
   Therefore $x_{|V} = y_{|V}$, meaning $V \vDash x = y$ for all open subsets. Since this is true for every subset, it's particularly true for a cover and on the intersections, meaning $X \vDash x = y$.

2. $X \vDash$ "f is surjective"
   $\iff X \vDash \forall y : H \ \exists x : G \mid f(x) = y$
   $\iff$ for all $y : HU$ we have that $U \vDash \exists x : G \mid f(x) = y$
   Therefore $\operatorname{im}(f)U = HU$, therefore $\operatorname{im}(f) = H$.

□

Definition 3.1.5 (Exact sequence)**.**
A sequence

$$\cdots \longrightarrow G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \longrightarrow \cdots$$

is called a ***complex*** *if* $f_i f_{i-1} = 0$ *and it's called* ***exact*** *if* $\ker(f_{i+1}) = \mathrm{im}(f_i)$ *for all* $i \in I$.

Note that the condition $X \vDash \ker(f_{i+1}) = \mathrm{im}(f_i)$ is given by two geometric implications:

$$X \vDash \forall y : G_{i+1}\ \exists x : G_i \mid f_i(x) = y \Rightarrow f_{i+1}(y) = 0$$

and

$$X \vDash \forall y : G_{i+1}\ f_{i+1}(y) = 0 \Rightarrow \exists x : G_i \mid f_i(x) = y$$

Thus a sequence is exact if and only if it's exact on all the stalks.

3.2 Rings and Modules

Definition 3.2.1 (Sheaf of rings).
*A **Sheaf of rings** is a ring over **Sh**(X), i.e. an object R together with a sum + and a product · such that*

$$X \vDash \text{"}(R, +, \cdot) \text{ is a ring"}.$$

We define, to give consistency to the classical notation a Ringed space as

Definition 3.2.2 (Ringed Space).
*A **Ringed Space** is a pair composed of a topological space X and a ring $\mathcal{O}_X$ over X.*
We will sometimes refer to either terms of the pair as the ringed space itself.

Of course a morphism of ringed space is a morphism that arises from a map of the underlying topological spaces.

Definition 3.2.3 (Morphism of ringed spaces).
A morphism of ringed spaces is -equivalently- a pair consisting of a continuous maps $f : X \to Y$ and

1. *a morphism of rings over Y $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$*
2. *a morphism of rings over X $f^\flat :_* f \mathcal{O}_Y \to \mathcal{O}_X$*

Where f_* and $_*f$ are the image and preimage of a geometric morphism as defined in 2.6

We should show that $f_* \mathcal{O}_X$ and $_*f \mathcal{O}_Y$ are sheaves of rings and the two conditions are equivalent. For the equivalence just notice that $_*f$ is left adjoint to f_*, thus

$$\textbf{Sh}(X)(_*fR, S) \simeq \textbf{Sh}(Y)(R, f_*S)$$

For the two objects being actually sheaves of rings we can notice that "being a ring" is composed of geometric conditions (it will be shown explicitly in section 4), thus they are maintained through f_* and $_*f$

Given a ring we can of course define modules over it.

Definition 3.2.4.
*Given a ringed space $\mathcal{O}_X$, a **sheaf of (left or right)**$\mathcal{O}_X$**−modules** is an object M such that*

$$X \vDash \text{"}M \text{ is a (left or right) } \mathcal{O}_X\text{−module"}.$$

*A **morphism of** $\mathcal{O}_X$ **−modules** is a morphism of the underlying abelian groups respecting the ring action.*

Of course if $\mathcal{O}_X$ is commutative we have that left modules are right modules and vice versa.

We want to define operations with those modules as we'd usually do:

Note that given a ring over X $\mathcal{O}_X$ we have a category $\mathrm{Mod}_{\mathcal{O}_X}$ of modules over $\mathcal{O}_X$. All of the following diagrams are inside $\mathrm{Mod}_{\mathcal{O}_X}$.

Definition 3.2.5 (Direct sum)**.** *Given a collection of $\mathcal{O}_X$ −modules $\{M_i\}_{i\in I}$ we define $\bigoplus_{i\in I} M_i$ as the object such that for all i*

$$\begin{array}{ccc} M_i & \longrightarrow & \bigoplus_{i\in I} M_i \\ & \searrow & \downarrow \\ & & N \end{array}$$

(This is the coproduct in the (internal) category of $\mathcal{O}_X$ −modules)

Definition 3.2.6 (Direct product)**.** *Given a collection of $\mathcal{O}_X$ −modules $\{M_i\}_{i\in I}$ we define $\prod_{i\in I} M_i$ as the object such that for all i*

$$\begin{array}{ccc} M_i & \longleftarrow & \prod_{i\in I} M_i \\ & \nwarrow & \uparrow \\ & & N \end{array}$$

(This is the product in the (internal) category of $\mathcal{O}_X$ −modules)

Definition 3.2.7 (Tensor product)**.** *Given two $\mathcal{O}_X$ −Modules M and N, we define the tensor product $M \otimes N$ as*

$$\frac{\langle x \otimes y | x : M, y : N \rangle_{\mathcal{O}_X}}{R}$$

where R is the submodule generated by the linearity conditions, i.e.

- $(x + x') \otimes y - x \otimes y - x' \otimes y$,
- $x \otimes (y + y') - x \otimes y - x \otimes y'$,
- $(\alpha x) \otimes y - \alpha(x \otimes y)$,
- $x \otimes (\alpha y) - \alpha(x \otimes y)$.

The quotient is well defined and it works as usual since the category of $\mathcal{O}_X$ −modules is abelian.

3.3 Finiteness, Coherentness

We have some special classes of modules:

Definition 3.3.1 (Classes of modules)**.**
Let M be an $\mathcal{O}_X$ −module over X. M is said

1. *"Finite locally free" if*

$$X \vDash \bigvee_{n \geq 0} \text{"}M \simeq (\mathcal{O}_X)^n\text{"}$$

or, more elementary

$$X \vDash \bigvee_{n\geq 0} \exists x_1, ..., x_n : M \mid \forall x : M \ \exists! a_1, ..., a_n : \mathcal{O}_X \ \mid x = \sum_i a_i x_i,$$

2. *"Finite free" if all x_i were to be found in $M(X)$,*
3. *"Of finite type" if it's finitely generated, i.e.*

$$X \vDash \bigvee_{n\geq 0} \exists x_1,...,x_n : M \mid \forall x : M \; \exists a_1,...,a_n : \mathcal{O}_X \; \mid x = \sum_i a_i x_i,$$

4. *"Of finite presentation" if*

$$X \vDash \bigvee_{n,m\geq 0} \mathcal{O}_X^m \to \mathcal{O}_X^n \to M \to 0 \text{ exact },$$

5. *"Flat" if*

$$X \vDash M \text{ is flat.}$$

i.e.

$$X \vDash \forall a : \mathcal{O}_X^p, m : M^p, a \cdot m = 0 \Rightarrow \exists n : M^q, A : M_{p,q}(\mathcal{O}_X) \; | A \cdot n = m \wedge a^T \cdot A = 0$$ [1]

There are two odd ones among these:

First the definition of "finite free" is not formulated internally: That is because from the point of view of **Sh**(X) there is no difference between a finite free module and a finite locally free module, thus we need to see what happens with the sheaf:

As these are geometric definitions we interpret them as "on every point", i.e. "on every stalk". Since stalks usually contain more information than global sections (for example $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z}),0} \simeq \mathbb{Q}, \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}(\mathrm{Spec}(\mathbb{Z})) \simeq \mathbb{Z}$) there is no need for a module that is finite free on the stalks to be finite free on global sections as well ($\mathbb{Q}$ is finite free on $\mathbb{Q}$ but not on $\mathbb{Z}$)

Second, the definition of flat is less pretty and less intuitive than the others. As you may imagine, that is because we want it to be a geometric implication.

We want to show that it's equivalent to having that $M \otimes -$ is an exact functor.

Proposition 3.3.2.
The condition shown above is (intuitionistically) equivalent to the usual definition of flat module.

Proof.
First we need to show that if we have $\mathcal{O}_X$ a ring and M an $\mathcal{O}_X$ −module of finite type generated by $m_1,...,m_p$ and a generic $\mathcal{O}_X$ −module N, then

$$X \vDash \sum_i m_i \otimes n_i = 0 \Leftrightarrow \exists \{r_{i,j}\}_{i,j} : \mathcal{O}_X, \{c_j\}_j : N | (n_i = \sum_j r_{i,j} c_j) \wedge (\sum_j m_i r_{i,j} = 0)$$

[1] This formula is slightly incorrect: the correct one would be

$$X \vDash \bigvee_{p\geq 0} \forall a[\cdots] \Rightarrow \bigvee_{q\geq 0} \exists n[\cdots]$$

But we avoid the two $\bigvee$s for simplicity (and because they keep the formula geometric). From now on keep in mind that whenever there are unspecified dimensions we mentally add a $\bigvee$ in front.

Of course if the two conditions on the right hold, then

$$\sum_i m_i \otimes n_i = \sum_i m_i \otimes (\sum_j r_{i,j} c_j) = \sum_{i,j} m_i r_{i,j} \otimes c_j = 0$$

Conversely let $\sum_i m_i \otimes n_i = 0$. Let F be a locally free module of rank p and let $f : F \to M$ be an injective map with kernel K.

$$F \otimes N \longrightarrow M \otimes N$$

$$\textstyle\sum_i f_i \otimes n_i \longmapsto \sum_i m_i \otimes n_i = 0$$

So $\sum_i f_i \otimes n_i$ is an element of $K \otimes B$, meaning we can write it as

$$\sum_i f_i \otimes n_i = \sum_j (\sum_i f_i r_{i,j}) \otimes c_j$$

Thus we have $n_i = \sum_j r_{i,j} c_j$ and since $\sum_i f_i r_{i,j}$ are elements of the kernel $\sum_i m_i r_{i,j} = 0$.

Now onto proving the main theorem:

Let M such that $M \otimes -$ is exact, i.e. given a short exact sequence of modules $0 \to A \to B \to C \to 0$, then $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact as well.

Let $a : \mathcal{O}_X^p$, $m : M^p$ such that $a \cdot m = 0$. Denote $a = (a_1, ..., a_p), m = (m_1, ..., m_p)$.

Let I be the ideal generated by the a_i's. The map $i : I \otimes M \to \mathcal{O}_X \otimes M$ is injective and $i(\sum_i a_i m_i) = 0$ thus $\sum_i a_i \otimes m_i = 0$.

Since I is an ideal it's finitely generated, thus we can apply the result above to get that M is indeed flat.

Conversely let M be flat.

Indeed to reverse the proof we did above we only need to prove that every map $\iota : I \otimes M \to \mathcal{O}_X \otimes M$ is injective.

Suppose $\iota(\sum_i a_i \otimes m_i) = 0 : M$, then $\sum_i r_i \otimes m_i = 0$, meaning there exists $m'_j : M$,$r_{i,j} : \mathcal{O}_X$ such that $m_i = \sum_j r_{i,j} m'_j$ and $\sum_j a_i r_{i,j} = 0$

Then

$$\sum_i a_i \otimes m_i = \sum_{i,j} a_i \otimes r_{i,j} m'_j = \sum_{i,j} a_i r_{i,j} \otimes m'_j = 0$$

□

3.4 Concrete categories

We shall note that all of these objects we described form a category (they have well defined objects, arrows, identities and compositions). Since **CRing** and **Ab** are concrete categories, we have a functor $C \to$ **Set**.

We can think of the categories of "Rings/Abelian Groups" over X as sheaves on **CRing** / **Ab** via this diagram, which is also how they are classically defined.

$$\begin{array}{ccc} X^{op} & & \\ \downarrow & \searrow & \\ \mathbf{Set} & \xleftarrow[F]{} & C \end{array}$$

This should look very similar to what we usually define as "concrete category".

Remark (Concrete category)**.** *A category C is called **Concrete** if there exists a faithful functor $\Phi : C \to$ **Set**, which we call the **Forgetful functor**.*

Since all we are doing is just generalizing set-theoretical concepts over a specific topos it's natural to define

Definition 3.4.1 (Relatively concrete category)**.**
*Let $\mathcal{E}$ be a topos and C a category. We say that C is $\mathcal{E}$-**concrete** when there exists a faithful functor $C \to \mathcal{E}$*

Dilly-Dally (proportions)**.** *The idea is that we are doing a change of base:*

$$\mathbf{Ab} : \mathbf{Set} = \mathbf{Ab}_{\mathcal{E}} : \mathcal{E}$$

If we think of a concrete category as "endowing a set with structure" we can do the same to objects of a topos.

In this sense we can think of (small) concrete categories as **internal categories** of **Set**.

We can generalize this notion by extending it to internal categories of **Sh**(X) (or any topos really).

In particular, we can build

$$\begin{array}{ccc} C_{/X} & \longrightarrow & \mathbf{Sh}(X) \\ \downarrow{\scriptstyle -(U)} & & \downarrow{\scriptstyle -(U)} \\ C & \longrightarrow & \mathbf{Set} \end{array}$$

Where $-(U)$ is the functor sending a sheaf to its sections over U, meaning that the sections of $C_{/X}$ are actually elements of C.

For example if we have an abelian group A over X then its section $A(U)$ is an abelian group (in the usual sense.)

4 Schemes

This is the main matter of the thesis, and of the theory for that matter[1].

From now on all rings we will consider are commutative rings with identity.

4.1 Affine Schemes

Affine schemes are the building block of the theory: We define them as the spectrum of a ring (actual ring, not over X!)

Definition 4.1.1 ($\mathrm{Spec}(R)$ as a set)**.**
Let R be a ring. We define the ***Spectrum*** *of R as the set of all prime ideals $P \trianglelefteq_{\mathfrak{p}} R$*

We usually see it endowed with a topology

Definition 4.1.2 ($\mathrm{Spec}(R)$ as a topological space)**.**
We define the ***Zariski topology*** *on $\mathrm{Spec}(R)$ as the topology whose base of closed subsets is given by*

$$\{V(f) : f \in R\}$$

where $V(f) = \{P \trianglelefteq_{\mathfrak{p}} R \mid f \in P\}$ This means that if we call $D(f) = \mathrm{Spec}(R) \smallsetminus V(f)$ the set

$$\{D(f) : f \in R\}$$

is a base of open subsets.

This V acts as the "set of roots" functor, that is

Lemma 4.1.3 (Properties of V)**.**
Let E be a subset of the ring A, $X = \mathrm{Spec}(A)$ and define $V(E) = \{x \in X \mid E \subseteq X\}$, Then:

1. $V(\{1\}) = \varnothing,\ V(\{0\}) = X$,
2. $V(\bigcup_i E_i) = \bigcap_i V(E_i)$,
3. $V(E \cdot E') = V(E) \cup V(E')$,
4. $E \subseteq E' \implies V(E') \subseteq V(E)$,
5. $V(E) = V(\sqrt{E})$ *where* $\sqrt{E} = \bigcap_{x \in E} x$,
6. $V(\mathfrak{p}) \simeq \mathrm{Spec}(A/\mathfrak{p})$ *for any prime ideal* $\mathfrak{p} \trianglelefteq_{\mathfrak{p}} A$.

[1] no pun intended

Proof. 1. $V(\{1\}) = \{x \in X : 1 \in x\} = \varnothing, V(\{0\}) = \{x \in X : 0 \in x\} = X$;

2. $x \in V(\bigcup_i E_i) \iff \bigcup_i E_i \subseteq x \iff$ for all i $E_i \subseteq x \iff x \in \bigcap_i V(E_i)$;

3. $x \in V(E \cdot E') \iff E \cdot E' \subseteq x \iff E \subseteq X$ or $E' \subseteq x$ (x is a prime ideal) $\iff x \in V(E) \cup V(E')$;

4. $x \in V(E') \iff E' \subseteq x \implies E \subseteq E' \subseteq x \implies x \in V(E)$;

5. $E \subseteq \sqrt{E}$ thus $V(\sqrt{E}) \subseteq V(E)$, moreover $E \subseteq x \implies$ for any element $\alpha \in \sqrt{E}$, $\alpha^n \in E$, thus $\alpha \in x$ since it's a prime ideal, meaning $\sqrt{E} \subseteq x$;

6. there is a natural bijective correspondence of prime ideals of $A/\mathfrak{p}$ and prime ideals of A containing $\mathfrak{p}$

□

Proposition 4.1.4 (Alternative description)**.**
The frame of open subsets of the spectrum is isomorphic to the frame of radical ideals.

Proof.
We have two maps $\Phi : \tau(\mathrm{Spec}(A)) \to \{a \trianglelefteq A : a = \sqrt{a}\}$ mapping $U \mapsto \{h \in A : D(h) \subseteq U\}$ and its inverse $\Psi : a \mapsto \bigcup_{h \in a} D(h)$ They are actually inverse of one another: on the base of opens

$$D(f) \mapsto \{g \in A : D(g) \subseteq D(f)\} = \sqrt{f} \mapsto \bigcup_i D(f^i) = D(f)$$

and

$$a \mapsto \bigcup_{h \in a} D(h) \mapsto \left\{ g \in A : D(g) \subseteq \bigcup_{h \in A} D(h) \right\} = \sqrt{a} = a$$

□

We also have an associated local ring $\mathcal{O}_{\mathrm{Spec}(R)}$ over **Sh**$(\mathrm{Spec}(R))$ but we need more work to define it.

First we define localizations:

Definition 4.1.5 (Localization of a Ring)**.**
*Let R be a ring. We call $S \subseteq R$ a **multiplicative set** when*

- $1 \in S$,
- $x, y \in S \iff xy \in S$,

*Moreover, we call S a **filter** whenever*

- $0 \notin S$,
- $x + y \in S \implies x \in S$ *or* $y \in S$.

*If S is a multiplicative set (or a filter) of R we define the **localization of** R **at** S*

$$R[S^{-1}] := [\![\frac{r}{s} \mid r : R, s : S]\!]/\sim$$

where $\frac{r_1}{s_1} \sim \frac{r_2}{s_2} :\iff r_1 s_2 - r_2 s_1 = 0.$

The localization of a ring is made into a ring in the usual way, meaning

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \text{ and } \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

One may check that both the operations are well defined through the equivalence but the proof is the same to what we usually do when computing the fraction field of a ring.

Dilly-Dally (Localization)**.** *This is in fact a weakening of the fraction field construction, but in general this doesn't give us a field. If we localize at every point of the ring $R[R^{\times -1}]$ we obtain $\mathrm{Frac}(R)$*

If we want to get ahead of ourselves once again, through the machine of schemes we get that fields have a single point, meaning that localizing everywhere outside of a point trivializes the scheme everywhere outside of the point, allowing us to "zoom in" near the point.

In particular if $\mathfrak{p}$ is a prime ideal of r we call $R_{\mathfrak{p}} := R[(R \setminus \mathfrak{p})^{-1}]$

The standard notation wants that if $f \in R$ we denote $R_f := R[(f)^{-1}] \simeq \frac{R[x]}{(xf - 1)}$. To avoid confusion we will use $R[1/f]$ instead, resembling the field extension notation.[2]

If we take the constant sheaf $\underline{R}$ on $\mathrm{Spec}(R)$ we may easily notice that it is a ring over $\mathrm{Spec}(R)$ with sum and product being the usual ones.

Definition 4.1.6 (Generic filter)**.**
*The **generic filter** $\mathcal{S}$ on $\underline{R}$ is defined as the filter such that for all $f : R$*

$$D(f) \vDash x \in \mathcal{S} \text{ if and only if } f \in \sqrt{(x)}$$

Remember that externally $\underline{R}$ is the sheaf $(U \mapsto R)^+$ for all U, thus we can see the generic filter as

$$U \mapsto \{f : U \to R \mid f(\mathfrak{p}) \notin \mathfrak{p} \;\; \forall\, \mathfrak{p} \in U\}$$

Definition 4.1.7 (Structure Sheaf)**.**
Let R be a Ring, $\mathrm{Spec}(R)$ what we defined above.

$$\mathcal{O}_{\mathrm{Spec}(R)} = \underline{R}[\mathcal{S}^{-1}]$$

*is called the **structure sheaf** of $\mathrm{Spec}(R)$.*

[2] Off topic, I don't know how good an idea it is to use such similar notation to denote two objects built in such different ways, Robin please fix.

With this definition if we have $f \in R$ and we want to calculate

$$\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R[\mathcal{S}(D(f))^{-1}]$$

By definition $D(f) \vDash x : \mathcal{S}$ if and only if $x \notin \mathfrak{p}$ for all prime ideals with $f \notin \mathfrak{p}$.

This means that if x is an element, either x is invertible or $x \in (f)$, therefore $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R[1/f]$.

This lets us calculate the stalks, since

$$\overrightarrow{\mathrm{Lim}}_{\mathfrak{p} \in U} \mathcal{O}_X(U) = \overrightarrow{\mathrm{Lim}}_{f \notin \mathfrak{p}} R[1/f] = R_{\mathfrak{p}}.$$

We actually want to prove that it is a local ring, but locality is not as easy as the other properties: We usually define a local ring as a ring with a unique maximal ideal. To write it down as a formula it would be

$$X \vDash \exists! M : \forall I \trianglelefteq R(M \leq I \implies I = R)$$

This is not geometric since, for instance, it quantifies over the ideals of R and not over elements.

We want to write a geometric locality condition that is classically equivalent to this.

Definition 4.1.8 (Local Ring)**.**
A Ring is called ***Local*** *if* $1 \neq 0$ *and*

$$\forall x, y \in R \ (x + y \textit{ invertible} \implies x \textit{ invertible} \vee y \textit{ invertible})$$

Note that this is geometric, thus if the stalks are local rings, then $X \vDash$ "$\mathcal{O}_X$ is local".

Proposition 4.1.9 (The definition is classically equivalent to the usual one.)**.**
Having a single maximal ideal is classically equivalent to being local

Proof.
The idea is that the maximal ideal is the one formed by the non-invertible elements. Formally:

Let R be a commutative ring with identity. First suppose that it has a unique maximal ideal $M \trianglelefteq R$.

The condition is classically equivalent to saying "x not invertible $\wedge$ y not invertible $\implies$ $x + y$ not invertible"

Suppose x and y are not invertible, then (x) is a proper ideal, thus $(x) \leq M$, and the same goes for (y). This means that $(x) + (y)$ is contained in M, thus a proper ideal, meaning $x + y \in (x) + (y)$ must not be a unit, since otherwise $(x + y) = R$

Vice versa, suppose that the sum of non-unit is not a unit, call N the set of non-units of R.

By assumption it's additively closed, and let $n \in N, x \in R$. $xn \in N$ since if $xn \notin N$ then xn would be a unit. Therefore $yxn = 1$ for some $y \in R$ but this would mean that $(yx)n = 1$, meaning n is a unit. It's unique by definition and maximal since adding any element $u \in R \setminus N$ to form a bigger ideal M would get us $R = (u) \subseteq M$. □

Now it's just a little bit more work to say

Proposition 4.1.10 ($\mathcal{O}_{\mathrm{Spec}(R)}$ is a local Ring over $\mathrm{Spec}(R)$.)**.**
$\mathrm{Spec}(R) \vDash$ "$\mathcal{O}_{\mathrm{Spec}(R)}$ *is a local ring*".

Proof.
$\mathfrak{p}\, R_{\mathfrak{p}}$ is the only maximal ideal of $R_{\mathfrak{p}}$. Thus it's a local ring in the geometric sense above, meaning that we can extend the property to the whole $\mathrm{Spec}(R)$. □

The ringed spaces that arise from the spectrum of a ring are called **Affine Schemes**

Dilly-Dally (Why are we doing this?)**.**
The connection with classical algebraic geometry is very non trivial:
If we have a ring R we can consider a prime ideal $\mathfrak{p} \trianglelefteq_{\mathfrak{p}} R$ and build out this commutative square:

$$\begin{array}{ccc} R & \twoheadrightarrow & R/\mathfrak{p} \\ \downarrow & & \downarrow \\ R_{\mathfrak{p}} & \longrightarrow & \frac{R_{\mathfrak{p}}}{\mathfrak{p}\, R_{\mathfrak{p}}} \simeq \mathrm{Frac}(R/\mathfrak{p}) \end{array}$$

Thus we can think of every element of R as acting on a prime ideal of R, thus as a function associating a point $\mathfrak{p}$ of $\mathrm{Spec}(R)$ to its image in $k(\mathfrak{p}) = \dfrac{R_{\mathfrak{p}}}{\mathfrak{p}\, R_{\mathfrak{p}}}$
The domain and codomain of the map would be $X = \mathrm{Spec}(R)$ and some $\coprod_{x\in X} k(x)$.
If R is a polynomial ring with n variables over a field, $\mathrm{Spec}(R)$ will be the n–dimensional affine space over that field and every polynomial can be evaluated at a point.

Since we can construct a ringed space out of a ring a natural question to ask ourselves is whether we can build modules over $\mathrm{Spec}(R)$ given a module over R.
It turns out that indeed we can.

Definition 4.1.11.
Let R be a ring and M be a R–module. Moreover let $\mathrm{Spec}(R) = (X, \mathcal{O}_X)$ We define $\widetilde{M}$ the $\mathcal{O}_X$–module given by

$$\widetilde{M} = \underline{M}[\mathcal{S}^{-1}] = \underline{M} \otimes_{\underline{R}} \mathcal{O}_X$$

We have a strong theorem that says that every quasicoherent $\mathcal{O}_X$–module over an affine scheme $\mathrm{Spec}(R)$ arises from an R–module.
The reason why we value affine schemes is that they are the scheme-theoretical equivalent of affine spaces, in the sense that

Example 4.1.12 (Affine complex line)**.**
*The points of $\mathrm{Spec}(\mathbb{C}[x])$ are the prime ideals of $\mathbb{C}[x]$, i.e. the $(x-\alpha)$ for $\alpha \in \mathbb{C}$ and (0). This means that $\mathrm{Spec}(\mathbb{C}[x]) \simeq \mathbb{C} \sqcup \{\xi\}$ where $\overline{\{\xi\}} = \mathrm{Spec}(\mathbb{C}[x])$ itself (this called the **generic point** of the scheme).*

The sheaf of rings is $\mathbb{C}[x]/(x-\alpha) \simeq \mathbb{C}$ *at each closed point and* $\mathbb{C}(x)$ *at the generic, thus it's the function field.*

Another example is

Example 4.1.13 (Affine complex plane)**.**
The points of $\mathrm{Spec}(\mathbb{C}[x,y])$ *are the Irreducible polynomials in two complex variables, thus all the* $(x-\alpha, y-\beta)$ *for* $\alpha, \beta \in \mathbb{C}$ *and a point for any irreducible curve* η*, plus a generic point* (0)*. This means that the points are* $\mathbb{C}^2 \sqcup \{\xi\} \sqcup \{$ *a generic point for each algebraic curve*$\}$
The sheaf of rings is the field of functions at each point

and so on.

4.2 Gluing

Building maps of sheaves is generally not easy. In a previous example we needed to figure out the morphism prior to any other operations.

For example, if $X \vDash \forall x : X \exists f_x : \mathcal{F}_x \simeq \mathcal{G}_x$ we couldn't guarantee that all f_x lift to an unique morphism $f : \mathcal{F} \to \mathcal{G}$.

We can -given enough information- define maps on open subsets and then glue them together.

This operation is (unsurprisingly) called Gluing.

Lemma 4.2.1 (Gluing schemes)**.**
Let X_i *be a family of schemes. For each* $i \neq j$ *suppose there exists an open subset* U_{ij} *and let it have the induced scheme structure.*

Suppose also that for each $i \neq j$ *we have an isomorphism of schemes* $\varphi_{ij} : U_{ij} \to U_{ji}$ *and* $\varphi_{ji} = \varphi_{ij}^{-1}$.

Moreover $\varphi_{ij\to}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ *and* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ *on* $U_{ij} \cap U_{ik}$.

Then there exists a scheme X *together with morphisms* $f_i : X_i \to X$ *for each* i *such that*

1. $f_{i\to}(X_i)$ *form an open cover of* X
2. $f_{i\to}(U_{ij}) = f_{i\to}(X_i) \cap f_{j\to}(X_j)$ *and*
3. $f_i = f_j \circ \varphi_{ij}$ *on* U_{ij}

Proof.
The topological part is easy:

$$X = \frac{\coprod_{i\in I} X_i}{\sim} \text{ for the equivalence } x \sim \phi_{ij}(x)\ \forall i,j$$

Thus also the maps $X_i \to X$ are well defined as $X_i \to \coprod X_i \to X$. and they form an open cover.
$f_{i\to}(U_{ij}) = \{[x]_\sim : x \in U_{ij}\} \subseteq f_{i\to}(X_i)$,
$f_{i\to}(U_{ij}) = \{[x]_\sim : x \in U_{ij}\} = \{[\varphi_{ji}(x)] : x \in U_{ji}\} \subseteq f_{j\to}(X_j)$

$[x] \in (f_{i\to}(X_i) \cap f_{j\to}(X_j))$ means that $[x] = f_i(\xi) = f_j(\eta)$ thus $\varphi_{ij}(x) = \eta$ for some i, j, thus $x \in U_{ij}$.

Now for the rings: $U_{ij} \vDash$ "$\mathcal{O}_{X_i}$ is a Ring".

$$\mathcal{O}_X = \frac{\prod_{i\in I} {}_*f_i \, \mathcal{O}_{X_i}}{\sim} \text{ for the equivalence } (x_i)_i \sim ({}_*\phi_{ij}(x_i))_j$$

Is clearly a ring over X. □

Theorem 4.2.2 (Gluing morphisms)**.**
Let X and Y be schemes, U_i cover of $|X|$, and $f_i : U_i \to Y$ morphisms of schemes such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$ for all i, j.

Then it exists an unique morphism $f : X \to Y$ such that $f_{|U_i} = f_i$.

Sketch of proof.
A way to prove this is to apply the reasoning done above to the scheme/ sheaf $\mathrm{Hom}(X, Y)$ as a gluing of $\mathrm{Hom}(U_i, Y)$. □

This gives us access to a useful theorem

Theorem 4.2.3 (Ring-Scheme correspondence)**.**
Let A be a ring, $S = \mathrm{Spec}(A)$ and X a scheme.

$$\textbf{\textit{Sch}}(X, S) \simeq \textbf{\textit{CRing}}(A, \mathcal{O}_X(X))$$

Proof.
Note that any map $\alpha : X \to S$ gives rise to a map $f^\sharp : \mathcal{O}_S \to f_* \mathcal{O}_X$, If we take the global sections we get $f^\sharp_S A = \mathcal{O}_S(S) \to \mathcal{O}_X(f^\leftarrow(S)) = \mathcal{O}_X(X)$

We want to show that this map is a bijection.

To do this, first we look at the case $X = \mathrm{Spec}(B)$.

If we have a morphism of rings $u : A \to B$ we can take the map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $f(\mathfrak{p}) = u^\leftarrow(\mathfrak{p})$

This is continuous since $f^\leftarrow(D_S(a)) = D_X(u_\to(a))$ and the map of sheaves of rings is $f^\sharp : \mathcal{O}_S \to f_* \mathcal{O}_B$ is the map obtained by taking the map $\underline{u} : \underline{A} \to f_*\underline{B}$ and localizing it at the principal filter:

$f^\sharp : \underline{A}[\mathcal{S}^{-1}] \to \underline{B}[u^\leftarrow(\mathcal{S}^{-1})]$.

This is indeed the inverse since $\underline{A}[\mathcal{S}^{-1}(S)] = A \to \underline{B}[u^\leftarrow(\mathcal{S}^{-}1)(S)] = B$ is u itself.

To extend it to the general case we just glue on the affine open subsets of X. □

4.3 Schemes

A general scheme is composed by gluing together affine schemes.

Definition 4.3.1 (Scheme)**.** *A **scheme** X is a ringed space $(|X|, \mathcal{O}_X)$ such that for every point $x \in |X|$ exists an open neighborhood U_x such that $(U, \mathcal{O}_U)$ is isomorphic to an affine scheme.*

*We call those open subsets **Affine open subsets** and we denote $U \subseteq^{\text{aff}} X$*

It's standard to refer to both the scheme and the underlying topological space as X. We will use $|X|$ to denote the topological space when necessary.

Note that schemes are locally ringed spaces since every $\mathcal{O}_{X,x}$ is R_x for some ring R and locality is geometric.

Definition 4.3.2 (Morphism of schemes)**.**
*Given a morphism of ringed spaces $(f, f^\flat)$, it is a **morphism of schemes** if it's local. That is whenever*

$$X \vDash f(x) \text{ invertible } \Rightarrow x \text{ invertible}$$

Remember that the sections of the sheaf of truth values are the lattices of open subsets, meaning that topological properties of the space get translated into logical properties of the forcing relation.

Let's define some class of schemes

Definition 4.3.3 (Some classes of schemes)**.**
A scheme X is

1. ***Irreducible** whenever*

$$X \vDash \neg(\phi \wedge \psi) \text{ implies that either} X \vDash \neg\phi \text{ or } X \vDash \neg\psi,$$

and not $X \vDash \bot$.

2. ***Quasi-Compact (QK)** whenever given a family of monotonic formulas $\{\phi_i\}_{i \in I}$, i.e. a family of formulas such that for all $i \geq j$ in I $X \vDash \phi_i \Rightarrow \phi_j$*

$$X \vDash \bigvee_{i \in I} \phi_i \text{ implies that } X \vDash \phi_i \text{ for some } i \in I.$$

One interesting phenomenon is that

Theorem 4.3.4 (Affine Schemes are QK)**.** *Affine schemes are quasi-compact.*

Proof. Let $\operatorname{Spec}(R)$ be an affine scheme and let ϕ_i be a family of monotonic formulas such that $\operatorname{Spec}(R) \vDash \bigvee_{i \in I} \phi_i$. With the Kripke-Joyal semantics, this means that exists a covering $\{U_j\}_j$ of $\operatorname{Spec}(A)$ such that for all j, $U_j \vDash \phi_i$ for some i.

The condition $X \vDash \phi_i \Rightarrow \phi_j$ means that for all $U \subseteq^{\text{o}} X$, $U \vDash \phi_i$ implies $U \vDash \phi_j$.

Note that any U_j is union of some (possibly infinitely many) $D(f_i)$, meaning that its complementary is intersection of $V(f_i)$ meaning that it's $V(g_i)$ for some appropriate g.

This means that $\mathrm{Spec}(R) = D(\sum_i g_i) = V(1)$. since R is a ring we can refine the sum of the g_i to a finite one.

Therefore we can consider the finite cover $\{U_1, ..., U_n\}$. For each of these one ϕ_i holds. Let k be the smallest i such that ϕ_i holds on one of the elements of the cover

On every element of the cover $U_j \vDash \phi_{j_i} \implies U_j \vDash \phi_k$, thus ϕ_k holds on every element of the cover, by locality ϕ_k holds on X. □

We want to prove that $X = \mathrm{Spec}(B)$ is Irreducible if and only if $\mathcal{O}_X$ is an integral domain but we run into a small issue:

The usual definition of "integral domain" is that we don't have non-zero zero divisors.

Of course, since we are doing intuitionistic logic, this branches into several non-equivalent definitions:

Definition 4.3.5 (Integral domain)**.**
Let R be a ring over X such that $X \vDash 1 \neq 0 : R$.

*We say that R is a **weak integral domain** if*

$$X \vDash \forall x, y : R,\ xy = 0 \Rightarrow (x = 0) \vee (y = 0),$$

*And we say that it's a **strong integral domain** if*

$$X \vDash \forall x : R,\ x = 0 \vee (\forall y : R,\ xy = 0 \Rightarrow y = 0).$$

Theorem 4.3.6.
If $\mathrm{Spec}(R)$ is irreducible, then $\mathcal{O}_{\mathrm{Spec}(R)}$ is a weak integral domain.

Proof.
If $\mathrm{Spec}(R)$ is irreducible, then we can show that any filter is the complement of a prime ideal, thus (0) is a prime ideal of $\mathcal{O}_{\mathrm{Spec}(R)}$, meaning it is an integral domain in the weak sense. □

Proposition 4.3.7.
A scheme is integral if it's both reduced and irreducible

We end this section with the following theorem:

Lemma 4.3.8 (Nilpotency)**.**
Let X be a scheme, then

$$X \vDash \forall s : \mathcal{O}_X,\ \neg\text{"}s \textit{ invertible}\text{"} \Rightarrow \text{"}s \textit{ nilpotent}\text{"}$$

Proof. The language is local, thus the proposition is true if and only if it's true on every element of the affine cover.

This means that we can -without loss of generality- reduce to the case where X is affine.

The proposition $\neg$"s invertible" via the Kripke-Joyal semantics is equivalent to say that $U \vDash \text{"}s\,\mathrm{inv.}\text{"} \Rightarrow U = \varnothing$.

In particular this means that $s_{|U} \in \mathcal{O}_X(U)$ is invertible only when $U = \varnothing = D(0)$, thus $s_{|U}$ is an element of $\sqrt{0}(U)$, meaning $s : \sqrt{0} \leq \mathcal{O}_X$. □

4.4 Some classes of morphisms

In this section we will define some classes of morphisms and some of their properties (that will be useful later.)

Definition 4.4.1 (Quasi-compact morphism)**.**
*We say that a morphism $f : X \to Y$ is **Quasi-compact** if the inverse image of every affine open of Y is quasi-compact.*

Definition 4.4.2 (Morphism locally of finite type)**.**
*We say that a morphism $f : X \to Y$ is **locally of finite type** if with the underlying algebra structure given by $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$, $f_* \mathcal{O}_X$ is a finitely generated $\mathcal{O}_Y$ −module.*

Definition 4.4.3 (Morphism of finite type)**.**
*We say that a morphism $f : X \to Y$ is **of finite type** if it's locally of finite type and quasicompact.*

Definition 4.4.4 (Morphism locally of finite presentation)**.**
*We say that a morphism $f : X \to Y$ is **locally of finite type** if with the underlying algebra structure given by $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$, $f_* \mathcal{O}_X$ is a finitely presented $\mathcal{O}_Y$ −module.*

Proposition 4.4.5 (Stability under composition)**.**
The composition of two morphism of the same type (between the ones listed above) yields another morphism of that type.

Proof.
locally finite type and locally finte presentation are easy: Let $gf : X \to Y \to Z$ be two composable morphisms.

Then if $f_* \mathcal{O}_X$ is finitely generated over $\mathcal{O}_Y$, then $g_* f_* \mathcal{O}_X$ is finitely generated over $g_* \mathcal{O}_Y$, which is fintely generated over $\mathcal{O}_Z$. Thus $(gf)_* \mathcal{O}_X$ is finitely generated over $\mathcal{O}_Z$ as well, and the same goes for finite presentation.

Quasi-compactness is a little trickier since it's a metaproperty of **Sh**(X) and not something in the internal language.

The advantage is that metaproperties are expressed in the usual logic, thus we can make proofs by contraddiction:

Suppose $g \circ f$ is not quasicompact, i.e. $(gf)^{\leftarrow} U \vDash \bigvee_{i \in I} \phi_i$ but not $(gf)^{\leftarrow} U \vDash \phi_i$ for some $i \in I$.

Transporting it through g_* onto $V := g_{\to}(gf)^{\leftarrow} U \subseteq f^{\leftarrow}(U)$ we would get that simultaneously $V \vDash \phi_i$ (as it's a subset of $f^{\leftarrow} U$) and $V \nvDash \phi_i$ since if it did then taking the geometric morphism in the other direction would give us that $(gf)^{\leftarrow} U \vDash \phi_i$. □

4.5 Quasi-Coherent $\mathcal{O}_X$ −modules

A special class of modules is the one given by Quasi-Coherent modules.
First a remark:

Remark (Locally constant sheaves)**.**
*A sheaf $\mathcal{F} \in$ **Sh**(X) is called **locally constant** whenever*

for all $x \in X$ there exists $U_x \ni x$ such that $U_x \vDash$ "$\mathcal{F}_{|U_x}$ is a constant sheaf"

Definition 4.5.1 (Quasi-Coherent $\mathcal{O}_X$ −modules)**.** *Let M be a $\mathcal{O}_X$ −module. It is called **quasi-coherent** if and only if*

$$X \vDash \exists I, J \text{ locally constant : "} \mathcal{O}_X^{(J)} \to \mathcal{O}_X^{(I)\ 3} \to M \to 0 \text{ exact"}$$

*And it's called **coherent** if moreover I and J are finite.*

We will now give a series of equivalent characterizations, and giving a basic proof for each one of them

Theorem 4.5.2 (Quasi-coherentness through localizations)**.**
Let X be a scheme, $U \subseteq^{\text{aff}} X$, M an $\mathcal{O}_X$ −module.
then M is quasi-coherent if and only if for every base element of the topology

$$U \vDash M_{|D(f)} \simeq M[1/f]$$

Proof.
Suppose M is a quasi-coherent module. This means that we have the short exact sequence shown above.

$$D(f) \vDash \underline{R}[1/f]^{(J)} \to \underline{R}[1/f]^{(I)} \to M \to 0$$

This means that we have an isomorphism $M_{|D(f)} \simeq M[1/f]$ (they are both final elements of a s.e.s.)
Conversely, suppose that the isomorphism holds.
We know that every module is quotient of a free module, thus localizing

$$D(f) \vDash \underline{R}[1/f]^{(I)} \to M[1/f] \to 0$$

We get that $\mathcal{O}_X^{(J)} = \ker(\underline{R}[1/f]^{(I)} \to M)$. □

[3] Here we use what I believe is standard notation:

$$R^I = \prod_{i \in I} R \quad R^{(I)} = \bigoplus_{i \in I} R$$

Theorem 4.5.3 (Modal quasi-coherentness).
Let X be a scheme, M an $\mathcal{O}_X$ −module and $\Box_f$ the modal operator ("f invertible" ⇒ −).
Then M is quasi-coherent if and only if $M[1/f]$ is a $\Box_f$ −sheaf for any $f : \mathcal{O}_X$.
This means that for all $f : \mathcal{O}_X$

$$\forall s : M[1/f]\ (\text{"}f\,\mathrm{inv.}\text{"} \Rightarrow s = 0) \Rightarrow s = 0$$

and for any subsingleton $S \le M[1/f]$

$$(\text{"}f\,\mathrm{inv.}\text{"} \Rightarrow S \textit{ inhabited}) \implies \exists s : M[1/f] \mid (\text{"}f\,\mathrm{inv.}\text{"} \Rightarrow s : S).$$

Proof.
The separatedness condition is equivalent to

$$\forall f : \mathcal{O}_X, \forall s : M,\ (\text{"}f\,\mathrm{inv.}\text{"} \Rightarrow s = 0 : M) \implies \bigvee_{n\ge 0} f^n s = 0 : M$$

And the inhabitedness condition is equivalent to

$$\forall f : \mathcal{O}_X, \forall S \le M,\ (\text{"}f\,\mathrm{inv.}\text{"} \Rightarrow \text{"}S \text{ singleton"}) \implies \bigvee_{n\ge 0} \exists s : M \mid \text{"}f\,\mathrm{inv.}\text{"} \Rightarrow f^{-n} s : S$$

In a ring every element is either invertible or a zero divisor and we have none of both, so the "invertible/non invertible" partition is intuitionistically valid.

$$X \vDash \forall r : R\ \top \Rightarrow \exists r^{-1} \mid rr^{-1} = 1 \vee \exists s : R \mid sr = 0$$

and

$$X \vDash \forall r : R\ \text{"}r\,\mathrm{inv.}\text{"} \wedge \text{"}r \text{ zero divisor"} \Rightarrow \bot$$

This means that if r is a zero divisor, then $r\,\mathrm{inv.} \Rightarrow \bot$ i.e. $\neg\text{"}r\,\mathrm{inv.}\text{"}$:
(from the rules of intuitionistic logic)

$$\frac{\text{"}f\text{zd"} \wedge \text{"}f\,\mathrm{inv.}\text{"} \vdash \bot \qquad \vdash \text{"}f \text{ zd"}}{\text{"}f\text{zd"} \vdash \text{"}f\,\mathrm{inv.}\text{"} \Rightarrow \bot}$$

This means that -invoking lemma 4.2.8- f zero divisor implies f nilpotent.
Now we can work by cases:
Suppose f is invertible, then we can reduce the two conditions to

- $\forall f : \mathcal{O}_X, \forall s : M, s = 0 : M \Rightarrow \bigvee_{n\ge 0} f^n s = 0 : M$
- $\forall f : \mathcal{O}_X, \forall S \le M, \text{"}S \text{ singleton"} \Rightarrow \bigvee_{n\ge 0} \exists s : M \mid f^{-n} s : S$

now suppose f is a zero divisor, then we can reduce them to

$$M \text{ is inhabited.}$$

when f is invertible the first condition is trivial, the second one means that on an open affine cover

$$D(f) \vDash \forall t : M\ \bigvee_{n\ge 0} t = \frac{s}{f^n}$$

Since M is inhabited, this is equivalent to saying $M_{|D(f)} \simeq M[1/f]$.
The converse is straightforward. □

Note that this is a theorem that does not exist in classical logic, since classically every sheaf is quasi-coherent.

And finally, the most important characterization:

Theorem 4.5.4 (Quasi-coherent modules and constant sheaves of modules)**.**
Let X be a scheme, M an $\mathcal{O}_X$ −module, then
M is quasi-coherent if and only if $M_{|U} \simeq \widetilde{N}$,
where if $U \simeq \mathrm{Spec}(R)$ then N is a R−module.

Proof. An easy consequence of theorem 4.5.2 □

4.6 The Projective scheme

We showed that Spec is an analogous construction to the affine space. We want to build the scheme-theoretical analogous of the projective space.

Remark (Graded Ring).
*Recall that a **Graded ring** $S_\bullet$ is a direct sum of abelian groups $\bigoplus_{n\in\mathbb{N}} S_n$ equipped with a grade-compatible product, i.e. $S_n \cdot S_m \subseteq S_{n+m}$.*

*The elements of $S_n \smallsetminus \{0\}$ are called **homogeneous**, and an **homogeneous ideal** $I \trianglelefteq^{\mathcal{H}} S_\bullet$ is an ideal generated by homogeneous elements.*

*Moreover we call $S_+ = \bigoplus_{n\geq 1} S_n \trianglelefteq S_\bullet$ the **Irrelevant ideal** of $S_\bullet$.*

Definition 4.6.1 ($\mathrm{Proj}(S_\bullet)$ as a set).
*Let $S_\bullet$ be a graded ring. We define the **Projective** on $S_\bullet$ as the set of homogenous prime ideals $\mathfrak{p}$ such that $S_+ \nsubseteq \mathfrak{p}$ and $\{0\}$.*

And we usually see it endowed with a topology

Definition 4.6.2 ($\mathrm{Proj}(S_\bullet)$ as a topological space).
We define the topology on $\mathrm{Proj}(S_\bullet)$ as the one whose base is

$$\{D_+(f) : f \in S_+\}$$

Note that $\mathfrak{p} \trianglelefteq^{\mathcal{H}}{}_{\mathfrak{p}} S_\bullet \implies \mathfrak{p} \trianglelefteq_{\mathfrak{p}} S_\bullet$, thus we have a map $\mathrm{Proj}(S) \to \mathrm{Spec}(S)$.

Definition 4.6.3 (homogeneous generic filter).
*Given a graded ring $S_\bullet$ we define the **homogeneous generic filter** $\mathcal{P}$ as the pullback of the generic filter through this map.*

Now we can finally define

Definition 4.6.4 (Structure Sheaf).
Let $S_\bullet$ be a graded ring, $\mathrm{Proj}(S_\bullet)$ what we defined above.

$$\mathcal{O}_{\mathrm{Proj}(S_\bullet)} = \underline{S_\bullet}[\mathcal{P}^{-1}]_0$$

i.e. the zero-degree subring of $\underline{S_\bullet}[\mathcal{P}^{-1}]$

We have to show that it is a scheme.

Proposition 4.6.5 ($\mathrm{Proj}(S_\bullet)$ is a scheme).
$D_+(f) \simeq \mathrm{Spec}(S[1/f]_0)$ as topological spaces and $\mathcal{O}_{D_+(f)} \simeq \underline{S[1/f]_0}[\mathcal{P}^{-1}]$

Sketch of proof.
$D_+(f) \to \mathrm{Spec}(S[1/f]_0)$ is the map $x \mapsto x \cap S_0$ and the inverse $\mathrm{Spec}(S[1/f]_0) \to D_+(f)$ is the map $\mathfrak{p} \mapsto \sqrt{pS_\bullet}$. The homeomorphiosm is the intersection with S_0 thus the sheaf of rings is clearly isomorphic. □

We want to show that this is the correct generalization of the projective space: Let M be an $S_\bullet$–module.

Theorem 4.6.6.
$\widetilde{M}$ *is a quasi-coherent* $\mathcal{O}_{\mathrm{Proj}(S_\bullet)}$*-Module.*

Proof.
$\widetilde{M}[1/f]$ as a $\mathcal{O}_{\mathrm{Proj}(S_\bullet)}$ −module is a module over $\underline{\mathsf{S}}[\mathcal{P}^{-1}]_0 \simeq S_0[\mathcal{F}^{-1}]$.
This means -by the previous lemma- that it's a $\square_f$ −sheaf, thus it's quasi-coherent.
In particular, this means that $\mathcal{O}_{\mathrm{Proj}(S_\bullet)} \simeq \widetilde{S}_\bullet$. □

Dilly-Dally. *This is a generalization of how the projective space behaves.*
The graded ring of classic algebraic geometry is the homogeneous polyonomials, you can build out the generalization from there.

4.7 Subschemes

Given a variety, we expect to have subvarieties: a simple example would be circles seen as subvarieties of the sphere. For schemes we can use modalities to describe intuitively the resulting schemes.

Definition 4.7.1 (Open subscheme)**.**
Let $U \subseteq^{o} X$ *and* $\square \equiv (U \Rightarrow -)$.

We define the ***open subscheme*** $(U, \mathcal{O}_U)$ *as the image of* $(X, \mathcal{O}_X)$ *through the* $\square$*−sheafification* ***Sh***$(X) \to$ ***Sh***$(X_\square) =$ ***Sh***(U)

Dilly-Dally. *In other words, we are restricting every formula to* U.

We also want to take a look at closed subschemes:

Definition 4.7.2 (Closed subscheme)**.**
Let $A \overline{\subseteq} X$ *and* $\square \equiv (- \vee A^c)$.

We define the ***closed subscheme*** $(A, \mathcal{O}_A)$ *as the image of* $(X, \mathcal{O}_X)$ *through the* $\square$*−sheafification* ***Sh***$(X) \to$ ***Sh***$(X_\square) =$ ***Sh***(A)

Dilly-Dally. *In classical logic*

$$A \to B \equiv \neg A \vee B.$$

We don't have access to A *as a subset, so this is the next best thing, even though the equivalence is not intuitionistically sound.*

There's a major lemma that characterizes closed subschemes: First we define the support of a module over $\mathcal{O}_X$

Definition 4.7.3 (Support)**.**
Let M *be an* $\mathcal{O}_X$*−module in* ***Sh***(X). *We define* $\operatorname{Supp}(M)$ *as the subset of* X *such that* $U = (X \setminus \operatorname{Supp}(M))^o$ *is the largest open subset of* X *such that*

$$U \vDash \forall s : M \;\; s = 0 \text{ (if and only if } U \vDash \text{"}M = 0\text{")}$$

Lemma 4.7.4 (Closed schemes are quasi-coherent ideals)**.**
Let Y *be a closed subscheme of* X, *with* $\theta : \mathcal{O}_X \to i_* \mathcal{O}_Y$ *be the canonical surjection for* $i : A \hookrightarrow X$.

Then

a) $I = \ker(\theta)$ *is a Q-C ideal of* $\mathcal{O}_X$, $i_* \mathcal{O}_Y \simeq \mathcal{O}_X / I$ *and* $|Y| = \operatorname{Supp}(i_* \mathcal{O}_Y)$, *Moreover* ${}_*i(\mathcal{O}_X / I) \simeq \mathcal{O}_Y$.

b) If $I \trianglelefteq \mathcal{O}_X$ *is a Q-C ideal, then* $(\operatorname{Supp}(\mathcal{O}_X / I), {}_* i(\mathcal{O}_X / I))$ *is a closed subscheme of* X.

Proof.
a) First note that the essential image of i_* consists of the sheaves on X that are $\square$–sheaves. This means that we can write that

$$X \vDash (A^c \Rightarrow \text{"}i_* \mathcal{O}_Y = 0\text{"})$$

This is equivalent to $i_* \mathcal{O}_Y$ being a $\square$–sheaf. Moreover $i_{**}i \mathcal{F} \simeq \mathcal{F}$ for all sheaves of this type, since sheafifying does nothing and forgetting the $\square$–sheaf structure once again does nothing.

Then we need to show that we have a canonical epimorphism $\theta : \mathcal{O}_X \to i_* \mathcal{O}_A$:

We know that $-^{\square}$ is simply $_*i$ for $i : X_{\square} \hookrightarrow X$.

We have the unit of the adjunction

$$\theta : \mathcal{O}_X \to i_{**}i \mathcal{O}_X = i_* \mathcal{O}_X^{\square} = i_* \mathcal{O}_A .$$

This is an epimorphism since

$$\begin{aligned} X \vDash \text{"}\theta : \mathcal{O}_X \to i_* \mathcal{O}_A \text{ epi"} &\iff X_{\square} \vDash \text{"}\theta^{\flat} :_* i \mathcal{O}_X \to \mathcal{O}_A = \mathcal{O}_X^{\square} \text{ epi"} \\ &\iff X \vDash \text{"}\theta \text{ epi"}^{\square} \\ &\iff X \vDash \square \text{"}\theta \text{ epi"} = \text{"}\theta \text{ epi"} \vee A^c \end{aligned}$$

Where the last "if and only if" is valid since being an epimorphism is geoemtric.

$A^c \vDash$ "θ epi", and on every other open subset we are in the case above, where $i_{**}i$ acts like the identity.

I is a quasi-coherent module since for every $f : \mathcal{O}_X$ we have that $I[1/f]$ is a $\square_f$–sheaf since it's the kernel of $\mathcal{O}_X[1/f] \to i_* \mathcal{O}_Y[1/f]$, which are both $\square_f$–sheaves as they are the structure rings of the scheme.

Thus $\mathrm{im}(\theta) = i_* \mathcal{O}_Y \simeq \mathcal{O}_X / I$.

The last point is given by the fact that $\mathbf{Sh}(Y)(_*i \mathcal{F}, \mathcal{G}) \simeq \mathbf{Sh}(X)(\mathcal{F}, i_* \mathcal{G})$ is a natural isomorphism, thus it maps isomorphisms to isomorphisms.

b) Let $\square \equiv (- \vee \mathrm{Supp}(\mathcal{O}_X / I)^c)$ and reverse the argument above.

□

Corollary 4.7.5.
If A is a closed subscheme of X, then $\mathcal{O}_X \to \mathcal{O}_A$ is surjective, and vice versa.

4.8 Noetherian schemes

There is a noetherianness property of schemes. This is usually done by claiming that the underlying topological space is noetherian. We do it in a different way.

Definition 4.8.1 (Locally Noetherian and Noetherian Schemes)**.**
A Scheme X is said to be ***locally noetherian*** *if any ascending chain of finitely generated ideals of $\mathcal{O}_X$ stabilizes.*
It's called ***Noetherian*** *if, moreover, it's quasi-compact.*

Let's enunciate and prove some properties:

Proposition 4.8.2.
X is locally noetherian if and only if for every affine open $U \subseteq^{\mathsf{aff}} X$ The ring $\mathcal{O}_X(U)$ is Noetherian.

Proof.
Since the internal logic of **Sh**(X) can be restricted at will and glued from coverings, we only need to prove that $\operatorname{Spec}(A)$ is locally noetherian if and only if A is noetherian.
This is true since Noetherianness is local, thus if A, therefore $\underline{A}$ is noetherian if and only if $\mathbb{A}[\mathcal{S}^{-1}]$ is. □

Proposition 4.8.3.
If X is locally Noetherian, then every stalk $\mathcal{O}_{X,x}$ is Notherian.

Sketch of proof. If X is locally Noetherian then $\mathcal{O}_X$ is processly Noetherian, then the stalks are Notherian. For the precise meaning follow [Ble17, pp.35-37] □

Corollary 4.8.4.
Any open subscheme of a locally Notherian scheme is locally Notherian.

Proposition 4.8.5.
Let X be a reduced scheme or a locally Noetherian scheme, then every ideal of $\mathcal{O}_X$ is not not finitely generated.

Proof. [Ble17, p.39] □

Proposition 4.8.6.
A morphism of schemes $f : X \to Y$ with with X noetherian is quasi-compact

Proof.
Follows from the fact that X is quasi-compact. □

Lemma 4.8.7 (Locally Noetherian Schemes are rich of closed points)**.**
Let X be a (nonempty) locally Noetherian scheme, then any nonempty closed subscheme of X has a closed point.

Proof. Let Z be a closed subscheme of X. Let $x \in Z$, if $\{x\}$ is closed then we're done.

If $\{x\}$ is open, then consider $\overline{\{x\}}$ as a closed subscheme. We will have an ideal I such that $\mathcal{O}_X \to \mathcal{O}_X / I$ is the corresponding morphism of rings.

Let's assume for a moment that the ideal is finitely generated (and not only not not fintely generated), then we can repeat the process, getting an ascending sequence of ideals of $\mathcal{O}_X$ that stabilizes.

This means that at some point the process of picking $x_i \in \overline{\{x_{i-1}\}}$ stops as well, meaning that we have a closed point.

The fact that the ideals are actually finitely generated is true: let's just restrict to the affine open in which a point is contained, we have that the structure sheaf of the affine open is actually noetherian, thus every ideal is actually finitely generated. □

5 Relative Schemes

A relative scheme is a generalized notion of an algebra over a ring.

Alexander Grothendieck[1] believed that the notion of "relative scheme" was the better generalization of the notion of algebraic variety.

Dilly-Dally (Relative POV)**.**
The relative point of view is way more natural than it seems: When we talk about varieties, we think of them over a certain space: riemannian manifolds are manifolds over $\mathbb{C}$, smooth manifolds are over $\mathbb{R}$, algebraic varieties are over a fixed closed field, yadda yadda.

It makes less sense to compare a variety over $\mathbb{R}$ with one over $\mathbb{Z}$ than to compare real varieties between themselves.

In fact when we want to compare-for example $\mathbb{R}-$varieties and $\mathbb{C}-$varieties, we do so by first moving the field of definition of the second ones to $\mathbb{R}$.

So, we define a relative scheme X over S as a morphism of schemes $X \to S$, i.e. a pair of maps

$$x : |X| \to |S|, \; x^\sharp : \mathcal{O}_S \to x_* \mathcal{O}_X \, .$$

Since we are always working inside **Sh**(S), we will omit the direct image functor, thus we will write $x^\sharp : \mathcal{O}_S \to \mathcal{O}_X$ unless absolutely necessary.

When we try to take the fiber products, it's not super clear what $\mathcal{O}_{X\times_S Y}$ would be.

It would be nice if the structure sheaf of the fiber product was the tensor product of the structure sheaves, but that is sadly not the case.

The tensor product of two local algebras generally doesn't give a local algebra[2], thus it cannot automatically be the structure sheaf of a scheme.

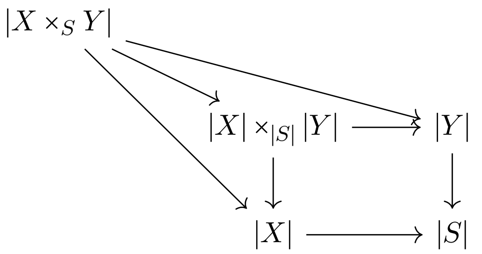
This means that we need to keep track of locality.

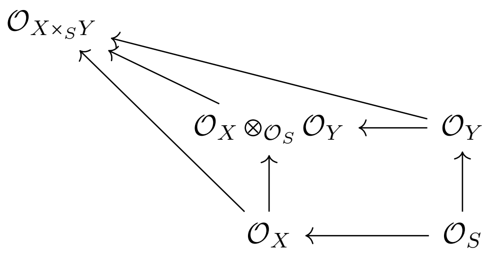
Generally we get that $|X \times_S Y| \not\simeq |X| \times_{|S|} |Y|$ but it's true that there exists a unique continuous function $|X \times_S Y| \to |X| \times_{|S|} |Y|$

[1]For the uninitiated, he's the main mind behind this machinery. It would be nice to cite his EGA or SGA in this thesis but I haven't read them yet, and if I were to trust the people who did, I don't hate myself enough to do it now.

[2]for example $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^2$, which is not local

If we look at commutative diagrams, we have that





The diagram on the right makes sense because the category of local rings is a non-full subcategory of commutative rings.

There's an important disclaimer here. As you may have noticed I'm following [Ble17]'s blueprints to build the theory, but this last chapter differs a lot from the approach of that document.

That employs a much smoother[3] way to work with relative schemes, called the Big Zariski topos. To explain it properly it would require a lot more theory and it would require us to abandon this way of doing things.

The jist of it is to consider the category of sheaves over **Sch**$/S$[4], then express notions in the internal logic and all of that. A major benefit is that base changes are super easy, as we can employ the fundamental theorem of topos theory to transport formulas from **Sch**$/S$ to **Sch**$/S'$ (or better, from **Sh**(**Sch**)$/y(S)$ to **Sh**(**Sch**)$/y(S')$). I encourage the readers to pick it up if interested.

[3] and frankly, way prettier

[4] Technically we employ a smaller site to disregard size issues, he suggests either the site of affine schemes over S or the one given by all schemes contained in a given Grothendieck universe.

5.1 Internalizations and externalizations

While the subcategory is strictly not full, we will show that it is reflective, meaning the inclusion has a left adjoint.

We will use this left adjoint to transport the pushout to the category of local rings.

The category **Sh**(S), internally to itself looks like the category of sets, thus S inside **Sh**(S) looks like a point.

That is because the singleton $[\![*]\!]_S$ is -externally- a sheaf $S^{op} \to \{*\}$, whose information is contained in the topology of S.

This means that $X \to S$ externally is equivalent to a map $I(X) \to *$ internally.

Definition 5.1.1 (Relative locales)**.**
Let $x : X \to S$ be a scheme over S.

*The **relative locale** $I(X)$ is given by*

$$I(X) := x_* \Omega_X \in \textbf{\textit{Sh}}(S)$$

.

Note that if we look at $1_S : S \to S$ we get that $I(S) := 1_{S*}\Omega_S = \Omega_S = P(1)$ i.e. the opens of the singleton space.

Since we can use **Sh**(S) as a substitute category of sets, we can have sheaves on them.

Definition 5.1.2 (Relative sheaves)**.**
*We call the category of **relative** S−**sheaves** **Sh**$(X|S)$ The category induced by the canonical geometric morphism **Sh**$_{\textbf{Sh}(S)}(I(X)) \to$ **Sh**(S).*

Dilly-Dally. *Recall that a presheaf is a functor $X^{op} \to$ **Set**. If we consider **Sh**(S) as a makeshift category of sets, a **Sh**(S)−presheaf is a map $X^{op} \to$ **Sh**(S) (where X is a locale internal to **Sh**(S)), and we can make it into a category of sheaves by the usual gluing conditions.*

Lemma 5.1.3 (Formulas over the relative sheaves)**.**
Let $x : X \to S$ be an S−scheme.

$$X \vDash \phi \iff S \vDash \text{"}I(X) \vDash \phi\text{"}$$

sketch of proof.
Let $X \vDash \phi$. Then we know that $S \vDash x_*\phi$ In other words for all $U \in x_*\Omega_X$ we have that $U \vDash \phi$ in the internal logic of **Sh**(S) and vice versa.

As usual a complete proof, that uses theory that I don't want to cover can be found in [Ble17] □

We have shown[5] that

$$\mathbf{Sch}(X, \mathrm{Spec}(A)) \simeq \mathbf{CRing}(A, \mathcal{O}_X(X))$$

In other words, the spectrum is the right adjoint to the global sections functor for (local) rings. We want to use this to build locally ringed topoi.

In particular we can say that

$$\mathbf{LocRing}_{/X}(\mathcal{O}_X, \mathcal{O}_{\mathrm{Spec}(A)}) \simeq \mathbf{CRing}(A, \mathcal{O}_X(X))$$

Thus, lifting it to sheaves over S

$$\mathbf{LocRing}_{/I(X)}(\mathcal{O}_{IX}, \mathcal{O}_{\mathrm{Spec}(A)}) \simeq \mathbf{CRing}_{/X}(A, \mathcal{O}_{IX}(IX))$$

For an opportune (intuitionistically valid) definition of the spectrum.

This means that the spectrum functor is right adjoint of the inclusion, thus it preserves pushouts.

This means that the spectrum of the tensor products of the internalized rings is the pushout- as local rings- of the structure sheaves.

Definition 5.1.4 (Internal Relative spectrum)**.**
*Let R be a ring over X, A an R–algebra. The **spectrum of** A **relative to** $R_1, ..., R_n$ $\mathrm{Spec}_X(A|R_1, ..., R_n)$ is the locale internal to X whose opens are the sets*

$$\{a \trianglelefteq A \mid a \text{ radical and } \forall f : R_1 \vee ... \vee R_n,\ \forall s : A(f\,\mathrm{inv.} \Rightarrow s \in a) \Rightarrow fs \in a\}$$

This comes equipped with a sheaf of rings over X $\mathcal{O}_{\mathrm{Spec}(A|R_1,...,R_n)}$.

Supposing, for the sake of notation that we have a single ring R,

*Of course we can reinternalize the logic of **Sh**$(X|S)$ and $\mathcal{O}_{\mathrm{Spec}(A|R)}$ there looks like a plain local ring.*

*We have the inclusion $i : \mathrm{Spec}(A|R) \to \mathrm{Spec}(A)$ as an inclusion of schemes over **Sh**(S) and we have that*

$$\mathcal{O}_{\mathrm{Spec}(A|R)} =_* i\,\mathcal{O}_{\mathrm{Spec}(A)}\,.$$

These are all internal to $\mathbf{Sh}(S)$, thus we need a way to externalize the notion properly.

Definition 5.1.5 (externalization of an internal locally ringed locale)**.**
*Let X be an internal locally ringed locale of **Sh**(S). The **Externalization** of X is defined as the global section $X(S)$, i.e. $E(|X|) = |X|(S)$ and $E(\mathcal{O}_X) = \mathcal{O}_X(S)$ as a functor **Sh**$(|X|(S)) \to$ **Set**.*

We have a canonical map $\xi : E(X) \to S$[6]. In particular we call

$$\mathrm{Spec}(R|S) \coloneqq E(\mathrm{Spec}(R|\,\mathcal{O}_S))$$

[5] in Theorem 4.2.3

[6] [Joh02]

Theorem 5.1.6 (Adjoint of the relative spectrum)**.**
*Let A be an $\mathcal{O}_X$ –algebra over X, Y a locally ringed locale over X. Inside **Sh**(X) $\mathrm{Spec}(-|\mathcal{O}_X) : {}_{\mathcal{O}_X}\textbf{\textit{Alg}}^{op} \to \textbf{\textit{LRL}}_{/X}$ is left adjoint, i.e.*

$$\textbf{\textit{LRL}}_{/X}(Y, \mathrm{Spec}(A|\mathcal{O}_X)) \simeq {}_{\mathcal{O}_X}\textbf{\textit{Alg}}^{op}(\mu_* \mathcal{O}_Y(\mathrm{Spec}(A|\mathcal{O}_X)), A)$$

Proof.
Let f be a morphism of locally ringed locales over X.

f is a pair $f : Y \to \mathrm{Spec}(A|\mathcal{O}_X), f^\sharp : \mathcal{O}_{\mathrm{Spec}(A|\mathcal{O}_X)} \to f_* \mathcal{O}_Y$.

In particular this gives a morphism (of $\mathcal{O}_X$ –algebras) of the global sections.

Conversely, if we have a morphism $\mathcal{O}_X \to A \to f_* \mathcal{O}_Y(\mathrm{Spec}(A|\mathcal{O}_X))$ we get a morphism $\mathrm{Spec}(A|\mathcal{O}_X) \to Y$ as the preimage of the filters that lay over the units of $\mathcal{O}_X$.

We have that $\mathcal{O}_Y \to \mathcal{O}_{\mathrm{Spec}(A|\mathcal{O}_X)}$ as $\mathcal{O}_Y \to \underline{A}[i^{\leftarrow}\mathcal{S}^{-1}]$

Those are trivially inverse of one another. □

Lastly

Lemma 5.1.7 (Characterization of the relative spectrum)**.**
Let S be a scheme, A an $\mathcal{O}_S$ –algebra and $f : \mathrm{Spec}(A|S) \to S$ the canonical projection.

Then the canonical morphism $A \to \mathcal{O}_{A|S}$ is an isomorphism of $\mathcal{O}_S$ –algebras if

1. *A is quasicoherent, or*

2. *A is local and $\mathcal{O}_S \to A$ is local.*

Proof.
If A is quasicoherent, then by the modal characterization we have that $A[1/f]$ is a $\Box_f$ –sheaf for all invertible elements of $\mathcal{O}_S$ Then, by definition of local spectrum, we have that the opens are exactly the opens of S, meaning that the map $A \to \mathcal{O}_{A|S}$ is an isomorphism on each open, thus it's an isomorphism.

If A is local and the morphism is local, then every stalk A_x is local and every homomorphism $\mathcal{O}_{S,x} \to A_x$ is local.

This means that $\mathcal{O}_{S,x} \to A_x \to \mathcal{O}_{A|S,x}$ is local. Since A_x and $\mathcal{O}_{A|S_x}$ are isomorphic as rings, we get that the same isomorphism is local. □

5.2 Relative schemes

From the internal point of view of **Sh**(S) the category of internal sheaves $\mathbf{Sh}_{\mathbf{Sh}(S)}(IX)$ is the category of sheaves (of sets) on a locale.

This is a full-fledged category of schemes, the only thing we discussed that we don't have access to is Barr's theorem, since as it is now its proof is not intuitionistic, thus it isn't valid in the internal logic of **Sh**(S)

Definition 5.2.1 (Relative scheme).
*A **Relative scheme** X over S is the externalization of a scheme in the category of relative sheaves **Sh**$(X|S)$*

Proposition 5.2.2 (Morphism of schemes are relative schemes).
*The slice category **LRL**$/S$ and $\mathbf{LRL}_{\mathbf{Sh}(S)}$ are equivalent.*

*In particular every morphism of schemes $X \to S$ is equivalent to a scheme over **Sh**(S)*

Proof.
[Joh02] proves that EI and IE are both isomorphic to the identity of the appropriate space.

This means that they are inverse of one another modulo isomorphism, i.e. for any morphism of schemes $X \to S$ the internalization $(IX, I\mathcal{O}_X)$ is a scheme in **Sh**(S) and for every scheme over **Sh**(S) X $(EX, E\mathcal{O}_X)$ has a morphism of schemes towards S, moreover $EIX \simeq X$ and $IEX \simeq X$. □

We can construct what we call the fiber product of schemes as follows:

Definition 5.2.3 (Fiber product).
Let $X \to S$ and $Y \to S$ be S–schemes. We can define $X \times_S Y$ as the scheme $E(\mathrm{Spec}(\mathcal{O}_X \otimes \mathcal{O}_Y)|\mathcal{O}_X, \mathcal{O}_Y)$ with the associated structure ring $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y[\mathcal{S}^{-1}]$.

Since this structure ring is effectively the local tensor product of two algebras we will denote it as

$$\mathcal{O}_X \otimes^L_{\mathcal{O}_S} \mathcal{O}_Y$$

Theorem 5.2.4 (Fiber products are pullbacks).
In the category of locally ringed locales (or in the category of schemes)

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & S \end{array}$$

Proof.
Consider the rings over S $\mathcal{O}_S, \mathcal{O}_X$ and $\mathcal{O}_Y$. We have the following pushout

$$\begin{array}{ccc} \mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_X & \longleftarrow & \mathcal{O}_Y \\ \uparrow & \lrcorner & \uparrow \\ \mathcal{O}_X & \longleftarrow & \mathcal{O}_S \end{array}$$

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_X) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_Y) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(\mathcal{O}_X) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_S) \end{array}$$

Since by definition we get that $\mathcal{O}_{\mathrm{Spec}(\mathcal{O}_X)|X} \simeq \mathcal{O}_X, \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_Y)|Y} \simeq \mathcal{O}_Y$ and $\mathcal{O}_{\mathrm{Spec}(\mathcal{O}_S)|S} \simeq \mathcal{O}_S$.

We get that from the internal perspective of $\mathrm{Spec}(\mathcal{O}_X \times \mathcal{O}_Y \,|X, Y)$ we have that it's the largest sublocale of $\mathrm{Spec}(\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y)$ such that the two morphisms

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y), \mathcal{O}_X \otimes^L_{\mathcal{O}_S} \mathcal{O}_Y & \longrightarrow & \{*\}, \mathcal{O}_Y \\ \downarrow & & \downarrow \\ \{*\}, \mathcal{O}_X & \longrightarrow & \{*\}, \mathcal{O}_S \end{array}$$

are morphisms of locally ringed locales from the internal perspective of $X \times_S Y$. □

Definition 5.2.5 (Fibers of a morphism)**.**
Let $f : X \to Y$ be a morphism of schemes, and y an element of Y, $k(y) := \frac{\mathcal{O}_{Y,y}}{m_y}$
*Then we call the **Fiber** of f at y the scheme $X_y = X \times_Y \mathrm{Spec}(k(y))$.*

Proposition 5.2.6 (Fibers are preimages of points)**.**

$$|X_y| \simeq f^{\leftarrow}(\{y\})$$

Proof.
$X_y \simeq \mathrm{Spec}(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_{\mathrm{Spec}(k(y))} \,|Y) = E(\mathrm{Spec}(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_{\mathrm{Spec}(k(y))} \,|\mathcal{O}_Y))$

Inside **Sh**(Y) this is $\mathrm{Spec}\left(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \dfrac{\mathcal{O}_{Y,y}}{m_y}\right)$

Recall

$$\mathcal{O}_X \otimes_{\mathcal{O}_Y} \frac{\mathcal{O}_{Y,y}}{m_y} \simeq (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{Y,y}} \frac{\mathcal{O}_{Y,y}}{m_y} \simeq \mathcal{O}_{X,f^{\leftarrow}(y)} \otimes_{\mathcal{O}_{Y,y}} \frac{\mathcal{O}_{Y,y}}{m_y} \simeq \frac{\mathcal{O}_{X,f^{\leftarrow}(y)}}{{}_*fm_y}$$

The internal locale of the spectrum is given by the filters that classically would correspond to prime ideals.

Those are the filters that lay over the maximal ideal of $\mathcal{O}_{Y,y}$. Since the locale associated to that is $\{y\}$ when externalized, this corresponds to the preimage of $\{y\}$. □

In the case of affine schemes, we have that this action is nonnecessary:

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(C) = \mathrm{Spec}(A \otimes_B C)$$

The first way to explain it is that $\mathrm{Spec}(-)$: **CRing** → **LRL** is left adjoint and contravariant, giving us the equality.

The second way, probably more consistent to the theory given until now is that

$$\mathcal{O}_{\mathrm{Spec}(A)} \otimes_{\mathcal{O}_{\mathrm{Spec}(B)}} \mathcal{O}_{\mathrm{Spec}(C)}[\mathcal{S}^{-1}] = A[\mathcal{S}_A^{-1}] \otimes_{B[\mathcal{S}_B^{-1}]} C[\mathcal{S}_C^{-1}][\mathcal{S}^{-1}] = A \otimes_B C[\mathcal{S}^{-1}]$$

As the generic filters of the three rings already sit under the generic filter of the tensor product

We expect the local tensor product to behave like a tensor product, i.e.

Proposition 5.2.7.

- $A \otimes_B^L (C \otimes_B^L D) \simeq (A \otimes_B^L C) \otimes_B^L D$
- $A \otimes_B^L (B \otimes_C^L D) \simeq A \otimes_C^L D$

Proof.
If we reduce to affine covers these all look like regular tensor products. This means that we have the equality on a cover of the fiber product, thus global equalities. □

5.3 Base change

This section is a bit weird. I'm going to present an idea on how to approach the problem but I can't quite iron it out yet.

Let $\sigma : S' \to S$ be a morphism of schemes. We want to show that

Theorem 5.3.1 (Stability under base change)**.**
Let X, Y be two S-schemes, $f : X \to Y$ a morphism and $\sigma : S' \to S$ a morphism of base schemes. Then the following properties are stable under base change:

1. *f is an isomorphism,*
2. *f is an open immersion,*
3. *f is a closed immersion,*
4. *f is an immersion,*
5. *f is locally of finite type,*
6. *f is locally of finite presentation,*
7. *f is of finite type,*
8. *f is injective,*
9. *f is surjective.*

Where "stable under base change" means that if $f : X \to Y$ has said property, then $f_{S'} : X_{S'} \to Y_{S'}$ has that property as well.

Dilly-Dally. *This is equivalent to saying that these are stable under extensions of scalars for morphism of algebras, which is true.*

As I mentioned in the beginning of the chapter, this is not the "proper" topos-theoretical way to talk about the subject.

Know and notice that the theorem is true and proven, as can be seen in [Har77],[Ble17] and numerous other resources, but not in a way that fits the theory we've built in the thesis.

The underlying idea is that we have previously seen that $\mathbf{LRL}/S \simeq \mathbf{LRL}_{\mathbf{Sh}(S)}$. This means that we have an induced morphism $\mathbf{LRL}/S \to \mathbf{LRL}/S'$, that translates to $\mathbf{LRL}_{\mathbf{Sh}(S)} \to \mathbf{LRL}_{\mathbf{Sh}(S')}$.

This is given exactly by the fiber product as we discussed before, i.e., if X is a sheaf over S

$$X \mapsto X \times_S S' =: X_{S'}$$

I'd like to make use of classifying topoi([Sau93],[Car18]):

Suppose $\mathcal{L}$ is the classifying topos of the theory of locally ringed locales.

By definition we have the following diagram

$$\begin{array}{ccc} \mathbf{Geom}(\mathbf{Sh}(S), \mathcal{L}) & \xrightarrow{\simeq} & \mathbf{LRL}_{/\,\mathbf{Sh}(S)} \\ \downarrow & & \downarrow \\ \mathbf{Geom}(\mathbf{Sh}(S'), \mathcal{L}) & \xrightarrow[\simeq]{} & \mathbf{LRL}_{/\,\mathbf{Sh}(S')} \end{array}$$

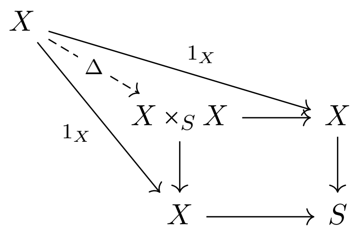
If $\mathbf{Geom}(-, \mathcal{L})$ acts as a representable functor, it should yield a geometric morphism $\mathbf{Sh}(S') \to \mathbf{Sh}(S)$ that acts as the forementioned pullback, meaning that the base change is actually geometric and preserves all the listed properties.[7]

[7] If someone has an idea on how to make this work please let me know, everybody reading should have my email already.

5.4 Diagonal and separation

An interesting morphism is the diagonal morphism:

Definition 5.4.1 (Diagonal morphism)**.**
*Let X be an S−scheme. We define the **diagonal morphism** Δ as*



Definition 5.4.2 (Separated scheme)**.** *We say that an S-scheme X or equivalently a morphism of schemes $x : X \to S$ is **separated** whenever Δ is a closed immersion*

In a trivial (previously proven) way whenever the map $\Delta^* : \mathcal{O}_X \otimes^L_{\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X$ is surjective.

Proposition 5.4.3 (Properties of separation)**.**

1. *separation is stable by base change;*
2. *composition of separated morphism is separated;*
3. *any mono is separated;*
4. *any affine scheme is separated over $\mathrm{Spec}(\mathbb{Z})$;*
5. *any projective scheme is separated over $\mathrm{Spec}(\mathbb{Z})$.*

Proof.

1. base change of a closed immersion is a closed immersion
2. let $f : X \to Y$ separated, $g : Y \to Z$ separated.
   $\mathcal{O}_X \otimes^L_{\mathcal{O}_Z} \mathcal{O}_X \simeq \mathcal{O}_X \otimes^L_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes^L_{\mathcal{O}_Z} \mathcal{O}_X)$. Since being an epimorphism is stable by base change we get that
   $\mathcal{O}_X \otimes^L_{\mathcal{O}_Z} \mathcal{O}_X \to \mathcal{O}_X$ is surjective because $\mathcal{O}_X \otimes^L_{\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{O}_X$ is.
3. If $X \to Y$ is a mono, then Δ is an isomorphism (this is true for any pullback), thus it's separated.
4. Recall that the local fiber product for spectra is the spec of the tensor product. Over $\mathrm{Spec}(\mathbb{Z})$ means that we are tensoring as abelian groups, thus the surjectivity of the maps of structure rings is trivial.
5. A similar argument goes for the projective scheme.

□

5.5 Differentials

We can define the sheaf of differentials on a (relative) scheme.

> **Dilly-Dally** (Tangent and Cotangent space).
> *When working with differential geometry we usually define the tangent space and from that the cotangent space, i.e. the space of differentials as its dual. In this context there is an easier description for differentials than for tangent vectors.*

Definition 5.5.1 (Kähler differentials).
Let R be an $S-$algebra and M an $R-$module.
*We call a map $d : R \to M$ a $R-$**linear derivation** if*

1. *d is an $S-$module homomorphism,*
2. *$d(fg) = fd(g) + gd(f)$, and*
3. *$d(s) = 0$ for all $s \in S$.*

*We call the **module of Kähler differentials***

$$\Omega^1_{R/S} := \frac{\bigoplus_{f \in R} R[df]}{\langle d(a+b) = da + db, d(ab) = adb + bda, ds = 0 \rangle_{a,b \in R, s \in S}}$$

We can easily internalize the construction:

Definition 5.5.2 (Relative differentials).
Let X be an $S-$scheme.
*We call the **module of relative differentials** the (internal) module of Kähler differentials*

$$\Omega^1_{X|S} := \Omega^1_{\mathcal{O}_X / \mathcal{O}_S}$$

We have different characterizations of the relative differentials.

Theorem 5.5.3 (Alternative description 1).
$R \to \Omega_{R/S}$ is the left adjoint of the inclusion $\mathrm{Mod}_R \hookrightarrow \mathrm{Der}_S$

Proof.
Trivially from the description of the module we have

$$\mathrm{Mod}_R(\Omega_{R/S}, M) \simeq \mathrm{Der}_S(R, M)$$

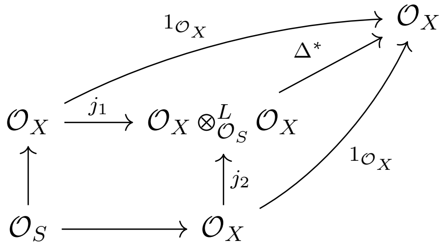
obtained by the composition with d. □

Theorem 5.5.4 (Alternative description 2).
Let X be an S-scheme. I be the kernel of the map $\Delta^* : \mathcal{O}_X \otimes^L_{\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X$ Then

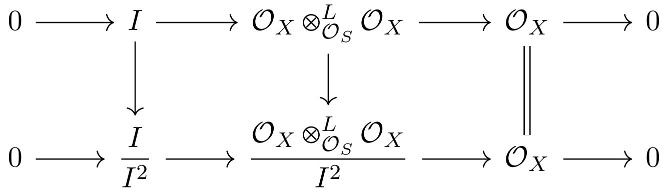
$$\Omega^1_{X|S} \simeq \Delta^* \frac{I}{I^2}$$

Proof. We need to show that

$$\mathrm{Mod}_{\mathcal{O}_X}(\Delta^* I/I^2, M) \simeq \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, M)$$



If we define $d_{X|S} := j_1 - j_2$ We get that $j_1\Delta^* = j_2\Delta^* = 1_{\mathcal{O}_X}$, meaning that $j1\Delta^* - j_2\Delta^* = (j_1 - j_2)\Delta^* = 1_{\mathcal{O}_X} - 1_{\mathcal{O}_X} = 0$ thus $\mathrm{im}(j_1 - j_2) \leq \ker \Delta^* = I$.
Since Δ^* is also the coequalizer of j_1 and j_2 we get that it's actually an equality.
We have the exact sequence



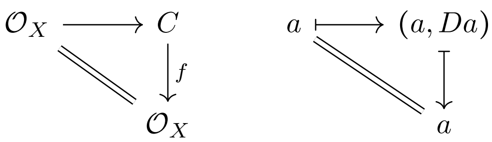
Let u be a module homomorphism $\Delta^* I/I^2 \to M$. Then $ud_{X|S}$ is an $\mathcal{O}_S$–derivation $\mathcal{O}_X \to M$.

1. for $s \in \mathcal{O}_S$, $d_{X|S}(s) = j_1 - j_2(s) = 0$
2. for $f, g \in \mathcal{O}_X$, $d_{X|S}(fg) = j_1(fg) - j_2(fg)$. On stalks we have

$$\begin{aligned}\frac{fg \otimes 1 - 1 \otimes fg}{*} &= \frac{fg \otimes 1 - f \otimes g + f \otimes g - i \otimes fg}{*} \\ &= \frac{(f \otimes 1)(g \otimes 1 - 1 \otimes g) + (1 \otimes g)(f \otimes 1 - 1 \otimes f)}{*} \\ &= f d_{X|S}(g) + d_{X|S}(f) g\end{aligned}$$

It holds on stalks and it's geometric, thus it holds globally as well.

Conversely if D is a derivation $\mathcal{O}_X \to M$, then exists $u : \Delta^* I/I^2 \to M$ such that $D = ud_{X|S}$ Let's define the $\mathcal{O}_X$–algebra $A := \mathcal{O}_X \oplus M$ with the product defined as $(a, m)(a', m') = (aa', am' - a'm)$
This means that $M \trianglelefteq A$ with $(0, m)(0, m') = 0$, thus $M^2 = 0$.



Thus we have an induced morphism

$$\begin{array}{ccc} \mathcal{O}_X \otimes^L_{\mathcal{O}_S} \mathcal{O}_X & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow \\ \mathcal{O}_X & = & \mathcal{O}_X \end{array}$$

We get that $\pi(I) \subseteq M$, thus $\pi(I^2) \subseteq M^2 = 0$.

This means that π induces a morphism $I/I^2 \to M$ and $u(d_{X|S}(b)) = \pi(j_1 b - j_2 b) = \pi j_1 b - \pi j_2 b = (b,0)j_1(1) - (1,0)j_2(b) = -(0, D(b))$. Thus $-u$ is the morphism we're looking for.

Moreover this is unique since if $ud_{X|S} = u'd_{X|S}$, $u - u'(d_{X|S}) = 0$. Thus $(u - u')d_{X|S\to}(\mathcal{O}_X) = 0$, therefore $u - u' = 0$. □

This makes the corollary trivial

Corollary 5.5.5.
$\Omega^1_{X|S}$ *is quasicoherent.*

We can easily change base schemes:
Suppose we have a base change

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Theorem 5.5.6 (Base change of differentials)**.**
There exists a canonical $\mathcal{O}'_X$ *−module homomorphism* $\Omega^1_{X|S} \to \Omega^1_{X'|S'}$

sketch of proof.

$$\Omega^1_{X'|S'} \simeq \Delta'^* I'/I'^2 \simeq \Omega^1_{X|S} \otimes^L_{\mathcal{O}_S} \mathcal{O}_{S'}$$

□

This gives us the two fundamental exact sequences:

Proposition 5.5.7 (First fundamental exact sequence)**.**
Let $f : X \to Y \to S$ *be a morphism of* S*−schemes. Then*

$$f^*\Omega^1_{Y|S} \xrightarrow{\alpha} \Omega^1_{X|S} \xrightarrow{\beta} \Omega^1_{X|Y} \longrightarrow 0$$

is exact

Proposition 5.5.8 (Second fundamental sequence)**.**
Let $i : X \to Z \to S$ be a closed immersion of S–schemes. Then

$$_*i(I/I^2) \longrightarrow i^*\Omega^1_{Z|S} \longrightarrow \Omega^1_{X|S} \longrightarrow 0$$

is exact.

Proof. This is true for the commutative ring counterpart, thus it's true in the internal logic of **Sh**(S). □

6 Exercises

I feel it's useful to show how this theory works in practice, and what better way to do this than solving some Hartshorne's exercises?

We will take some exercises from [Har77] chapter II and solve them using internal reasoning.

Exercise (1.19).
Let X be a topological space, Z a closed subset and U its complementary, $i : Z \hookrightarrow X, j : U \hookrightarrow X$.

(a) Let $\mathcal{F}$ be a sheaf on Z. show that the stalk $(i_* \mathcal{F})_p$ of the direct image is $\mathcal{F}_p$ if $p \in Z$ and 0 otherwise.

(b) Now let $\mathcal{F}$ be a sheaf on U. Let $j_!$ be the sheaf associated to $\mathcal{F}(V)$ if $V \subseteq^{\circ} U$ and 0 otherwise. Show that the stalk $(j_! \mathcal{F})_p$ is equal to $\mathcal{F}_p$ if $p \in U$ and 0 otherwise.

(c) Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves $0 \to j_! \mathcal{F}_{|U} \to \mathcal{F} \to i_* \mathcal{F}_{|Z} \to 0$

Solution.
The explicit descriptions of the first two sheaves are

$$[\![x : \mathcal{F} \mid (x = 0) \vee Z]\!] \text{ and } [\![x : \mathcal{F} \mid (x = 0) \vee U]\!]$$

respectively, making the first two point obvious.

As for the last point we just need to take the stalks and we get that it's all identities or 0–morphisms, making it exact in every case. □

Exercise (2.1).
Let A be a ring, $X = \mathrm{Spec}(A)$, $f \in A$ and $D(f)$ as defined above.

Show that $(D(f), \mathcal{O}_{X|D(f)})$ is isomorphic to $\mathrm{Spec}(A[1/f])$

Solution.
Remember that $D(f) = \{x \in X \mid f \notin x\}$. The map $\varphi : D(f) \to \mathrm{Spec}(A[1/f])$ sending $\mathfrak{p} \to \mathfrak{p}\, A[1/f]$ is well defined:

if $f \notin \mathfrak{p}$ then $\mathfrak{p}\, A[1/f]$ is a prime ideal of $A[1/f]$ since $xy \in \mathfrak{p}\, A[1/f] \implies xy = \dfrac{pa}{f^k}$, $f \not| \; xy$ thus $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$

its inverse is given by the preimage of $\psi : A \to A[1/f]$ $a \mapsto \frac{a}{1}$.

They are both open since $\varphi_{\to}(D(fg)) = \{x \in \mathrm{Spec}(A) : fg \notin x\}$ and $\psi_{\to}(U) = U$.

We have two open maps one inverse of the other, thus an homeomorphism.

Now for the morphism of the rings, take $\psi^\flat :_* \psi(\mathcal{O}_{\mathrm{Spec}(A[1/f])}) \to \mathcal{O}_{X|D(f)}$

Note that the condition of being an isomorphism is geometric and that $\psi^\flat$ acts as the identity on the stalks. □

Exercise (Reduced schemes)**.**
A scheme X is ***reduced*** *if and only if $\mathcal{O}_X$ has no nilpotent elements.*

(a) Show that X is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent elements for all $x \in X$.

(b) Let X be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf of rings $\frac{\mathcal{O}_X}{\sqrt{0}}$. Show that $X_{\text{red}} = (X, (\mathcal{O}_X)_{\text{red}})$ is a scheme and that there is a morphism of schemes $X_{\text{red}} \to X$ which is a homeomorphism of the underlying topological spaces.

(c) Let $f : X \to Y$ be a morphism of schemes and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f factors through g and the map $Y_{\text{red}} \to Y$.

Solution.

(a) "having no nilpotent elements" can be written as

$$X \vDash \forall x : \mathcal{O}_X,\ (\bigvee_n x^n = 0) \Rightarrow x = 0$$

which is a geometric implication. This holds if and only if it holds on every stalk.

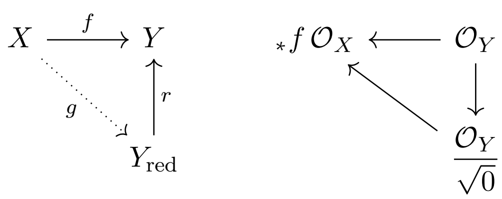
(b) X is a scheme thus we have a cover of affine opens, we only need to check that $U \vDash (\mathcal{O}_X)_{\text{red}} \simeq \mathcal{O}_{\text{Spec}(R)}$ for each element of the cover.

$D(f) \vDash (\mathcal{O}_{\text{Spec}(A)})_{\text{red}} = \frac{\mathcal{O}_{\text{Spec}(A)}}{\sqrt{0}} = \mathcal{O}_{\text{Spec}(\frac{A}{\sqrt{0}})}$ is geometric, thus it's true if and only if it is on the stalks, but localizations commute with quotients, thus

$$\left(\frac{A}{\sqrt{0}A}\right)_{\mathfrak{p}} = \frac{A_{\mathfrak{p}}}{\sqrt{0}A_{\mathfrak{p}}}.$$

For the second part the obvious choice for a map of topological spaces is the identity 1_X and we have the quotient map $\mathcal{O}_X \to \frac{\mathcal{O}_X}{\sqrt{0}}$.

(c)



$g^\flat : \mathcal{O}_Y / \sqrt{0}$ is well defined since for any element $s : \sqrt{0}\mathcal{O}_Y$, $g^\flat(s^n) = g^\flat(s)^n = 0 \implies g^\flat(s) = 0$. This means that g is the kernel-decompositon of f.

□

Exercise (3.6)**.**
*Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field, called the **function field** of X and denoted $K(X)$.*

Also show that if $U = \mathrm{Spec}(A) \subseteq^{\text{aff}} X$ then $K(X)$ is isomorphic to the to the quotient field of A.

Solution.
Recall that a scheme is integral if and only if it's both reduced and irreducible.

If the scheme is reduced, then we have that $\mathcal{O}_X$ has no nilpotent element.

It's also true that every non-invertible element of $\mathcal{O}_X$ is nilpotent.

Being irreducible means that

$$X \vDash \neg(\phi \wedge \psi) \implies X \vDash \neg\phi \vee X \vDash \neg\psi$$

Thus we have the condition

$$X \vDash \forall s : \mathcal{O}_X \, \neg\text{"}s \,\text{inv.}\text{"} \Rightarrow s = 0$$

Taking the stalk at the generic point is equivalent to sheafifying through the $\neg\neg$ modality, thus the condition on the stalk is equivalent to the condition of being a field.

Moreover if $U = \mathrm{Spec}(A)$ we get that $K(X) \simeq A_0 \simeq \mathrm{Frac}(A)$. □

Exercise (Criterion for affineness)**.**
Let $f : X \to Y$ be a morphism of schemes and let V_i be a cover of Y such that $f^{\leftarrow}(V_i) \to V_i$ is an isomorphism, then f is an isomorphism.

Then show that a scheme X is affine if and only if there exists a finite set of elements $f_1, ..., f_n \in \mathcal{O}_X(X) =: A$ such that $X_{f_i} := \{x \in |X| \mid f_{ix} \notin m_x\}$ is affine and $\langle f_1, ..., f_n \rangle = A^\times$

Solution.
being an isomorphism of schemes can be formulated in the internal language, and once true on a cover it's true globally.

The "only if" implication of the second part is trivial, so let's focus on the "if":

Recall that $f_x \notin m_x \iff f_x \in \mathcal{O}_{X,x}^\times$. $\mathrm{Spec}(A)$ as a topological space is covered by $D(f_i) = X_{f_i}$.

The sheaf of rings is the localization of $\mathcal{O}_X(X)$ at the filter lying over the filter of units, thus the generic filter, meaning $\mathcal{O}_X \simeq \mathcal{O}_{\mathrm{Spec}(A)}$ □

Exercise (5.2)**.**
Let M, N be $\mathcal{O}_X$ −modules such that M is of finite presentation. Then for any $x \in X$

$$\mathrm{Mod}_{\mathcal{O}_X}(M, N) \simeq \mathrm{Mod}_{\mathcal{O}_{X,x}}(M_x, N_x)$$

Solution.
Recall that a module is of finite presentation if we have a Short right exact sequence $\mathcal{O}_X^m \to \mathcal{O}_X^n \to M \to 0$

If that's the case we can assume that M is the cokernel of a presentation matrix $a_{i,j}$. Then

$$\mathrm{Mod}_{\mathcal{O}_X}(M,N) \simeq [\![x : N^n \mid \sum_{i=1}^{m} a_{i,j}x_i = 0 : N]\!]$$

This means that $\mathrm{Mod}_{\mathcal{O}_X}(M,-)$ is geometric, thus we have an isomorphism when passing at the stalks. □

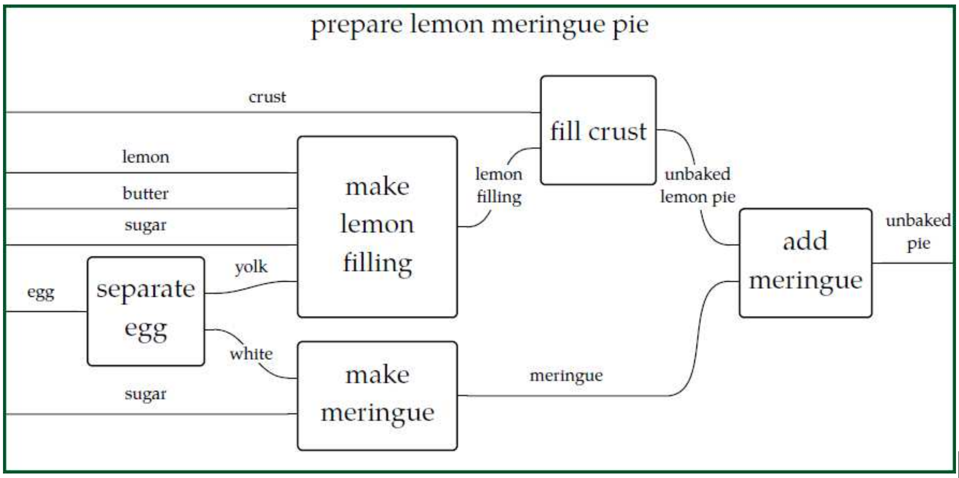
Some other exercises become extremely trivial in the internal logic, for example $II.5.1$

Appendix: Cooking with Topoi.

[WARNING: This is one of the forementioned silly shenanigans.]

Some authors already explored the idea of using monoidal categories to encode the making of a Lemon meringue pie.

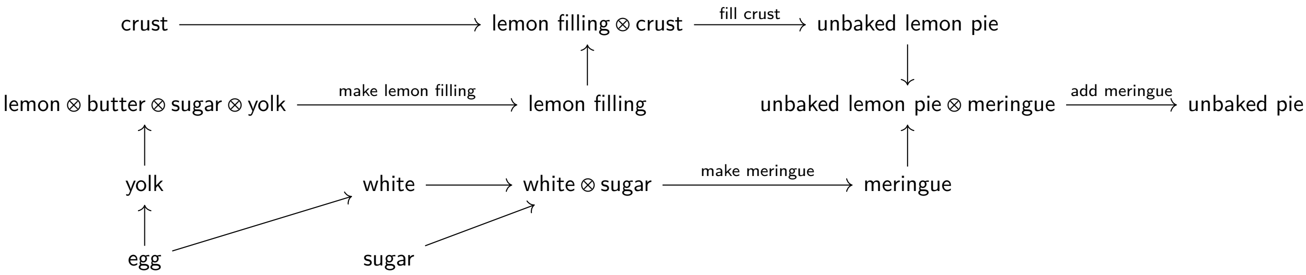
The data is given by this diagram



[Spi19]

First let's give an usual interpretation of the diagram, since -as you may see- it's not the usual "object and arrows" representation.

Basically, the wires are objects and the square are morphisms. We can turn this into the usual diagram:



This is a symmetric monoidal preorder but most of all is a small (finite) category that we'll call C (for cooking).

We can consider the topos $\mathbf{Set}^{C^{op}}$ of presheaves on cooking.

We can have a language on that topos, meaning we have that each type is a functor $C^{op} \to \mathbf{Set}$.

A functor from a finite preorder to **Set** just gives us a set of sets (WLOG) contained in each other with the inverse relationship of that imposed by the preorder.

We can identify sets with their cardinalities (modulo isomorphism), meaning that we can loosely think of these as the quantities of each ingredient with the relation "of which":

for example $F : C^{op} \to \mathbf{Set}$ could be

$$F(\text{crust}) = 100g, \; F(\text{lemon filling} \otimes \text{crust}) = 200g$$

$$F(\text{crust} \to \text{lemon filling} \otimes \text{crust}) = 200g \textbf{ of lemon filling of which crust } 100g.$$

Ω in particular in a presheaf is the functor sending any object to the set of sieves on it.

Definition 6.0.1 (Sieve).
*A **Sieve** S on an object X is a subfunctor of $\mathrm{Hom}(-,X)$, i.e. a collection of arrows $* \to X$ such that for any $g : Y \to X \in S$ and for any other arrow $g' : Y' \to Y$, $gg' \in S$.*

In our case a sieve on an ingredient is just the chain behind some of its composing ingredients, or (WLOG) some of its composing ingredients.

For example

$$\Omega(\text{lemon filling}) \simeq P(\{\text{lemon,butter,sugar,yolk}\})$$

(where P indicates the power set.)

This means that a formula in variable $x : X$ is a morphism $X \to \Omega$, which is a natural transformation associating to each preorder which quantities correspond to which ingredient.

This gives us a pretty reliable way -given some recipe- of proving theorems via cooking.

If you're wondering: yes, this does extend to Burritos[1][2].

[1] [Mor15]

[2] I think I have spent enough time on this -frankly pointless- idea but anyone is welcome to do various lemon meringue pies corresponding to different proofs, and maybe find a preorder on a language based on how tasty the final recipe becomes.

You're welcome.

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