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On Esteves' Compactification of The Relative Jacobian

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Contents

Introduction

We divide the contents of our thesis into four broad parts. In the introduction we try to introduce and motivate some part of the basic theory of what will follow.

Moduli Spaces

The first concept of interest is that of *moduli spaces.* Moduli problems deal with one of the most fundamental questions in mathematics, namely that of classifying a family of geometric objects. A moduli space can be thought of as a space where each point corresponds to a unique isomorphism class of a certain type of object. However, it's not just that; if all we required was a bijection, then the sets would simply need to be of equal cardinality. What we need more is that the geometry of the space to somehow reflect the intrinsic nature of the objects in a some kind of a natural way.

The subject has its origins in the theory of elliptic functions, where one shows that there is a continuous family of such functions parameterised by the complex numbers. The word *moduli* is due to Riemann, who showed in his celebrated paper of 1857 on abelian functions that an isomorphism class of Riemann surfaces of genus p "hängt.. von 3p-3 stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen" (depends on 3p-3 continuous variables, which shall be called the modules of this class) [New12]. Much work was done on the subject while trying to study moduli of curves. Teichmüller was the first to define a *fine moduli space* and the term *coarse moduli space* first appears in Mumford's Geometric Invariant Theory. However, it was only at the Cartan seminar in 1960-61 that Grothendieck precisely formulated moduli problems in terms of categories and functors. [Ji15]

Not only have moduli problems been deeply intertwined with theoretical physics but also have a wide range of application to string theory, computer science and surprisingly to biochemistry as well [Pen16]. Mathematical objects often arise in families, and studying the families can often lead to a better understanding of the individual objects.

Quasi-Coherent Sheaves and Line Bundles

The second objects we consider are *coherent sheaves* and *line bundles.* Let us recall some definitions. Suppose $(X, \mathcal{O}_X)$ is a ringed space. A sheaf $\mathcal{F}$ is called *free* if it is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X =: \mathcal{O}_X^I$ for some index set I as an $\mathcal{O}_X$-module. It is said to have *rank* n if $\mathcal{F} \cong \bigoplus_{i=1}^{n} \mathcal{O}_X =: \mathcal{O}_X^n$. It is *locally free* (of rank n) if there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i}$ is free of rank n as an $\mathcal{O}_{U_i}$-module for every i. A *line bundle* $\mathcal{L}$ on X is a locally free sheaf of rank 1.

Just for this definition, assume that X is a scheme. We define a *vector bundle of rank* n on X to be a locally free $\mathcal{O}_X$-module of rank n. Actually, one can first define a (geometric) vector bundle of rank n on a scheme by taking an open cover $X = \bigcup_i U_i$ and an X-scheme V such that $V|_{U_i} \xrightarrow[\sim]{f_i} \mathbb{A}^n_{U_i}$ as U_i-schemes for all i such that the automorphism $f_i \circ f_j^{-1}$ of $\mathbb{A}^n_{U_i \cap U_j}$ are linear maps for all i, j. One can show that this construction agrees with the classical definition in differential geometry. Then, one can construct a contravariant functor which gives an equivalence between the locally free $\mathcal{O}_X$-modules of rank n and vector bundles of rank n as we have defined. A complete proof is given in Chapter 11 of [GW20]. Informally, a vector bundle V on a space X (such as a manifold, or a scheme) is a family of vector spaces continuously parameterized by points of X. In other words, for each point p of X, there is a vector space, and these vector spaces are glued into a space V so that as p varies, the vector space above p varies continuously. The notion of locally free sheaves, in some sense, generalizes that of a vector bundle [Vak17].

A *quasi-coherent sheaf* $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is an $\mathcal{O}_X$-module that has a local presentation, that is, every point in X has an open neighborhood U in which there is an exact sequence

$$\mathcal{O}_X^{\oplus I}|_U \to \mathcal{O}_X^{\oplus J}|_U \to \mathcal{F}|_U \to 0$$

for some (possibly infinite) sets I and J. A *coherent sheaf* $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is an $\mathcal{O}_X$-module satisfying the following two properties:

- $\mathcal{F}$ is of finite type over $\mathcal{O}_X$, that is, every point in X has an open neighborhood U in X such that there is a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$ for some n.
- For any open set $U \subseteq X$, any $n \in \mathbb{N}$, and any morphism $\varphi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$ of $\mathcal{O}_X$-modules, the kernel of φ is of finite type.

Quasi-coherent sheaves form an abelian subcategory containing the locally free sheaves that is much smaller than the full category of $\mathcal{O}_X$-modules which is also abelian. One may think that the way modules generalize free modules, in much the same manner, quasi-coherent sheaves generalize free sheaves. Coherent sheaves can be thought of as a finite rank version of quasi-coherent sheaves, which form an abelian category containing locally free sheaves of finite rank (or, finite rank vector bundles).

Line bundles are locally free sheaves of rank 1 as defined above which means, heuristically one can think that the associated vector space at each point of our space X is one dimensional. As an example, think of a curve in the plane having a tangent line at each point determines a varying line, the tangent bundle is a way of organising these. Line bundles are also referred to as *invertible sheaves* as they have an inverse with respect to tensor product of $\mathcal{O}_X$-modules where the identity element is $\mathcal{O}_X$ itself. Let $\mathcal{L}$ be a line bundle, then it satisfies the following three equivalent conditions:

1. There exists a line bundle $\mathcal{M}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \cong \mathcal{O}_X$.
2. $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$ where $\mathcal{L}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$
3. the functor $\mathcal{F} \longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ defines an equivalence of categories.

In fact, with the tensor product as the operation, and $\mathcal{O}_X$ as the identity element, the isomorphism classes of line bundles on X forms an abelian group called the *Picard Group of* X, $\mathrm{Pic}(X)$. Now, in case X is an integral scheme, one can define Cartier divisors ([GW20, p. 301]) and to each line bundle associate a Cartier divisor. This association induces an isomorphism between line bundles and Cartier divisors. Moroever, if X is Noetherian, integral, and locally factorial, then one has another notion of divisors called Weil divisors which are elements of the free abelian group on the set of codimension 1 subschemes, denoted $Z^1(X)$. This notion agrees with Cartier divisors [Har77, p. 141]. Weil divisors have the form $\sum_Z n_Z[Z]$ where $n_Z \in \mathbb{Z}$ and Z is a closed subscheme of codimension 1 such that at most finitely many $n_Z \neq 0$. One also has a degree map $\deg\colon Z^1(X) \to \mathbb{Z}$ which maps $\sum_Z n_Z[Z] \longmapsto \sum_Z n_Z$.

Let $\mathcal{F}$ be a coherent sheaf and X be a proper scheme over $\mathrm{Spec} k$. We define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F})$$

where $h^i(X, \mathcal{F})$ denotes the k-dimension of the i-th sheaf cohomology group of $\mathcal{F}$. The sum is finite due to Grothendieck's Vanishing Theorem [Har77, p. 208]. Now, in case X is a proper curve, then from the Riemann-Roch Theorem [Har77, p. 294], one gets a homomorphism $\deg\colon \mathrm{Pic}(X) \to \mathbb{Z}$ which agrees with our previous definition of the degree map and so we can talk about the degree of a line bundle $\mathcal{L}$ on X. We obtain,

$$\deg(\mathcal{L}) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}).$$

This gives us the motivation to define the *Jacobian*.

The Picard and Jacobian Schemes

On any ringed space X, the isomorphism classes of invertible sheaves form a group denoted by $\mathrm{Pic}(X)$ called the (absolute) Picard group. Suppose X is an integral scheme that is projective over an algebraically closed field k (i.e. X is a projective variety), then, $\mathrm{Pic}(X)$ underlies a natural k-scheme which is a disjoint union of quasi-projective schemes called the *Picard Scheme*. The Picard scheme, $\mathbf{Pic}_X$ was introduced in 1962 by Alexander Grothendieck in two Bourbaki talks (nos. 232 and 236) which were later published in [Gro62]. The interested reader can read more about the history of the Picard scheme which eventually led to Grothendieck's definition of it in much detail in the first section of [Kle05].

The connected component of the Picard scheme that contains the identity, $\mathbf{Pic}^0_X$ is called the *Jacobian scheme of X*. Just as the motivation for the Picard functor came from the Picard group and line bundles, the motivation for the Jacobian came from the *Jacobian variety* defined for non-singular curves. The Jacobian of a curve C is the variety that parameterizes its degree 0 line bundles. It is the connected component of the identity in the Picard group, and hence, naturally an abelian variety [EGM]. The theory of the Jacobian variety stems from that of abelian varieties. The abstract theory of abelian varieties over arbitrary fields, can be said to have begun with Weil's famous announcement of the proof of the Riemann hypothesis for function fields. Parts of the projected proof can best be understood in terms of intersection theory on the Jacobian variety of the curve, and Weil was to spend the next six years developing the foundational material necessary for making his proof rigorous. Unable in 1941 to construct the Jacobian as a projective variety, Weil was led to introduce the notion of an abstract variety. In 1944 Weil completed his book *Foundations of Algebraic Geometry* which laid the necessary foundations in algebraic geometry, and in 1946 he completed two books in which abelian varieties are defined and Jacobian varieties are constructed, but it was not shown that the Jacobian could be defined over the same field as the curve. Chow in 1950, and 1954 gave a construction of the Jacobian variety which realized it as a projective variety defined over the same ground field as the original curve [Mil08].

Compatification of Picard Scheme and the Jacobian

A good motivation for compactifying Jacobians comes from asking "how to take limits of line bundles?" A simple example can be constructed from plane cubics. Let X_0 be a plane cubic curve in $\mathbb{P}^2$ with a node at the origin $p_0 = [0, 0, 1]$ and X_∞ be a general cubic curve that passes through p_0. Consider the pencil $\{X_t\}$ spanned by these curves. For $t \neq 0$, we can define a line bundle $\mathcal{L}_t$ by setting $\mathcal{L}_t = \mathcal{O}_{X_t}(-p_0)$ (the ideal sheaf of p_0). It is natural to ask "what is the limit"

$$\lim_{t \to 0} \mathcal{L}_t = ?$$

This limit is supposed to be the ideal sheaf of p_0 in X_0, but this sheaf is not a line bundle. One can state this question in a more formal way. Let $X \to \mathbb{P}^1_t$ be the pencil in question. There is a scheme $\mathrm{Pic}_{X/\mathbb{P}^1} \to \mathbb{P}^1$, called the *relative Picard scheme* that parametrizes families of line bundles on the given pencil of curves. Let $\Delta = \mathrm{Spec}(k[[t]])$ be a formal disc around the origin of $\mathbb{P}^1$ and $\Delta^* = \mathrm{Spec}(\mathrm{Frac}(k[[t]]))$ the formal punctured disc. The line bundles $\mathcal{L}_t, t \neq 0$, fit together to form a family of line bundles and hence induce a morphism $\Delta^* \to \mathrm{Pic}_{X/\mathbb{P}^1}$. This morphism fits into the diagram below:

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \mathbf{Pic}_{X/\mathbb{P}^1} \\ \downarrow & \nearrow & \downarrow \\ \Delta & \longrightarrow & \mathbb{P}^1 \end{array}$$

Because the ideal sheaf $\mathcal{I}_{p_0}$ of X_0 is not a line bundle, it follows that there is no extension of $\Delta^* \to \mathrm{Pic}_{X/\mathbb{P}^1}$ to a morphism $\Delta \to \mathrm{Pic}_{X/\mathbb{P}^1}$. This shows that $\mathrm{Pic}_{X/\mathbb{P}^1}$ does not satisfy the valuative criteria of properness. The question "how to take limits of line bundles?" can be restated as "how to compactify the relative Picard scheme?" there is no canonical answer to this question [Kas07].

The problem of finding a natural relative compactification of the relative Jacobian over a family of curves has drawn a lot of attention since Igusa's pioneering work [Igu56] in the fifties. Igusa was probably the first to consider the problem of compactifying the (generalized) Jacobian variety of a reduced curve X. His method was to construct a compactification as the limit of the Jacobians of smooth curves approaching X, and he applied his method to the case where X was nodal and irreducible. He also showed that this compactification, in spite of the construction, did not depend on the family of approaching smooth curves. Later, Meyer and Mumford [MM64] found an intrinsic characterization of Igusa's compactification by means of torsion-free, rank 1 sheaves.

D'Souza used the ideas of Meyer and Mumford in his thesis [DSo79], to construct a compactification of the Jacobian variety using G.I.T., when X is irreducible, and then showed that there is a universal torsion-free, rank-1 sheaf over it. In the case where X is reducible and nodal, Oda and Seshadri [OS79] used G.I.T. to construct several compactifications of disjoint unions of copies of the Jacobian variety. Finally, Seshadri used G.I.T. to deal with a general reduced curve X in [Ses82] (where he considered also the higher rank case).

In the case where the curves in the family are geometrically integral, a very satisfactory solution has been found by Altman and Kleiman [AK80]. Their relative compactification is a fine moduli space; that is, it admits a universal family after an étale base change.

Esteves' article [Est01] which we present in this thesis aims at constructing a natural relative compactification of the relative Jacobian over a projective, flat family of geometrically reduced and connected curves. In contrast to earlier relative compactifications, his admits a universal object, after an étale base change. The points of the compactification correspond to simple, torsion-free, rank-1 sheaves that are semistable with respect to a given polarization. It must be said that the relative compactification given by Esteves is an algebraic space, rather than a scheme but it becomes a scheme, after an étale base change.

Main Results

Let S be a locally Noetherian scheme, $f\colon X \to S$ be a projective, flat map whose geometric fibers are connected, reduced curves. Let

$$\mathrm{Pic}_{X/S}\colon (\mathbf{Sch}_S)^{op} \to (\mathbf{Sets})$$

be the relative Picard functor defined on an S-scheme T as isomorphism classes of line bundles on $X \times_S T$ modulo the Picard group of T where the isomorphism between two line bundles is described in detail in the first chapter. The relative Jacobian functor, $\mathrm{Pic}^0_{X/S}$ is defined as the subfunctor of $\mathrm{Pic}_{X/S}$ parameterizing degree 0 line bundles. Let $\mathbf{P} := \mathrm{Pic}_{X/S(et)}$ be the étale sheaf associated to $\mathrm{Pic}_{X/S}$. Artin showed that $\mathbf{P}$ is represented by an algebraic space P locally of finite type over S [Art70].

In order to compactify P over S in general, it is natural following Mayer and Mumford [MM64] to consider the functor

$$\mathbf{F}^* \colon (\mathbf{Sch}_S)^{op} \to (\mathbf{Sets})$$

defined on an S-scheme T as the set of T-flat, coherent sheaves on $X \times_S T$ whose fibers over T are torsion-free, rank-1 sheaves. $\mathbf{F}$ be the étale sheafification of $\mathbf{F}^*$, it is clear that $\mathbf{F}$ contains $\mathbf{P}$ as an open subfunctor. It is easy to show that $\mathbf{F}$ meets the existence condition of the valuative criterion of properness. In other words, it contains enough degenerations.

However, the functor $\mathbf{F}$ is not representable by an algebraic space in general since it admits non-trivial endomorphisms. But considering the subfunctor $\mathbf{J} \subseteq \mathbf{F}$, parameterizing sheaves with simple fibers, it was shown by Altman and Kleiman in [AK80] that $\mathbf{J}$ is represented by an algebraic space J. Since the geometric fibers of f are reduced and connected, J contains P as an open subspace. Esteves [Est01] found that J does contain enough degenerations over S. It turns out that J is in fact a fine moduli space but since J is, as Esteves calls, "awkwardly big", we need to decompose it into simpler pieces. For this, we use polarizations as defined by Seshadri in [Ses82]. For us, a polarization on X/S is a vector bundle $\mathcal{E}$ on X of rank r and relative degree, $\deg(\mathcal{E}/S) = -rd$ over S for a certain fixed integer d.

Associated to a polarization $\mathcal{E}$ on X/S, let $J^s_{\mathcal{E}}, J^{\sigma}_{\mathcal{E}}, J^{\mathrm{qs}}_{\mathcal{E}}, J^{\mathrm{ss}}_{\mathcal{E}}$ denote the subspaces of J parameterizing sheaves with stable, σ-quasi-stable, quasi-stable, and semi-stable sheaves respectively. The main theorem of our interest from Esteves' article is the following:

Theorem 0.1. *The algebraic space $J^{ss}_{\mathcal{E}}$ is of finite type over S. In addition, $J^{ss}_{\mathcal{E}}$ and $J^{qs}_{\mathcal{E}}$ are universally closed over S.*

Outline of The Thesis

Chapter 1 begins with discussing the basics of moduli problems following the text by Newstead [New12] and lecture notes by Lucia Caporaso [Cap10] and Johannes Schmitt [Sch20]. We then move to studying the Picard scheme in detail following Steven L. Kleiman's note on The Picard Scheme [Kle05] for most of the material.

The main theorems we prove in this chapter are about the existence of the Picard scheme [Thm 1.29] and the existence of a Poincaré sheaf for the relative Jacobian functor [Cor 1.35].

In Chapter 2, we start dealing with the case where X is a curve, i.e. X is a geometrically reduced, projective scheme of pure dimension 1. We define torsion-free, rank-1 sheaves, and a polarization E on X as described earlier and describe the various notions of stability with respect to E. The rest of the chapter is devoted to various lemmas about torsion-free, rank-1 sheaves on X and their behaviour with respect to subcurves $Y \subseteq X$. We also define the Jordan-Hölder filtration of a torsion-free, rank-1 sheaf and see how to construct such a sheaf with a prescribed Jordan-Hölder filtration. This chapter is based on the first section of [Est01].

With all the necessary background developed in the first two chapters, we begin Chapter 3 by proving a number of lemmas and propositions to prove Theorem 0.1 stated above. We then look at a few examples to better understand the applications of the theorem. We follow Sections 2, 3 and 5 of [Est01] for this chapter.

We have added an appendix about basic notions from algebraic geometry we have used without stating or proving throughout the thesis to assist fellow students in reading it. Most of the material in this section can be found in standard algebraic geometry textbooks.

Chapter 1

Moduli Problems

The basic items required to construct a classification problem are a collection of objects A and an equivalence relation $\sim$ on A. The problem is to describe the set of equivalence classes $A/_\sim$. In this chapter we review the basics of moduli problems and spaces through classical cases as done in [New12]. We then move on to the more categorical definitions as in [Sch20], and construction of the Picard scheme following [Cap10] and [Kle05]

Families

Definition 1.1. Let $\mathcal{A}$ be a collection of (algebro-geometric) objects and $\sim$ be an equivalence relation on $\mathcal{A}$. A *family* of objects of $\mathcal{A}$ parameterised by a variety S is a collection $X = (X_s)_{s\in S}$ such that $X_s \in \mathcal{A}$ satisfying the following:

1. A family parameterised by $S = \{*\}$ is a single object of $\mathcal{A}$
2. If X and X' are two families parameterised by S, then there is a notion of equivalence between X and X' which reduces to $\sim_{\mathcal{A}}$ on $\mathcal{A}$ when $S = \{*\}$ (we denote this equivalence by $\sim$ too).
3. For any morphism of varieties $\phi\colon S' \to S$ and any family X parameterised by S, there is an induced family, $\phi^* X$ parameterised by S'. This construction is functorial i.e.

   $$\text{for } S'' \xrightarrow{\phi'} S' \xrightarrow{\phi} S,\ (\phi \circ \phi')^* = {\phi'}^* \circ \phi^* \qquad 1_S^* = \mathrm{Id}$$

   and respects equivalences,

   $$X \sim X' \implies \phi^* X \sim \phi^* X'$$

We now try to understand this more concretely through some examples.

Example 1.2. Let $\mathcal{A}$ consist of all complete varieties and let $\sim$ be given by isomorphism of varieties. A family of objects of $\mathcal{A}$ parameterised by S consists of a variety X and a proper flat morphism $\alpha\colon X \to S$. The object X_s corresponding to a point s of S is the fibre $\alpha^{-1}(s)$. This is most naturally done by working in terms of schemes and scheme-theoretic fibres.

Example 1.3. Let X be a fixed variety and $\mathcal{A}$ consist of all vector bundles over X and let $\sim$ be given by isomorphism of bundles. A family of objects of $\mathcal{A}$ parameterised by S is a vector bundle $\mathcal{E}$ over $S \times X$. The object $\mathcal{E}_s$ corresponding to a point $s \in S$ is the vector bundle $\mathcal{E}$ restricted to $\{s\} \times X$. For any morphism $\phi\colon S' \to S$, the induced family is given by $(\phi \times \mathrm{id}_X)^*\mathcal{E}$. To extend $\sim$ one might consider isomorphism of vector bundles over $S \times X$, however, this is too weak. At the very least, bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over $S \times X$ should define equivalent families if $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi_S^*\mathcal{L}$ for some line bundle $\mathcal{L}$ over S.

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Moduli Spaces

Now given a moduli problem, we would like to impose on the set ${\mathcal{A}}/{\sim_X}$, the structure of a variety in a way that reflects the structure of families of objects of $\mathcal{A}$. To do this, suppose that M is a variety whose underlying set is $\mathcal{A}/\sim$. Next, for any family $\mathcal{X}$ parameterised by S, consider the map $\nu_{\mathcal{X}}\colon S \to M$ given by

$$s \mapsto [\mathcal{X}_s]$$

where $[\mathcal{X}_s]$ denotes the equivalence class of the fiber $\mathcal{X}_s$ of $\mathcal{X}$. It is reasonable to ask that this map be a morphism. The best possible case would be that ν defines a bijective correspondence between families parameterised by S, and morphisms $S \to M$.

One can express this more conveniently in terms of categories. For this, let $\mathcal{F}(S)$ denote the set of equivalence classes of families parameterised by S. By condition 3 from Definition 1.1, $\mathcal{F}$ is a contravariant functor from the category of algebraic varieties to sets. Moreover, we have natural maps $\Phi(S)\colon \mathcal{F}(S) \to Hom(S, M)$ given by

$$\Phi(S)(X) = \nu_X$$

which define a natural transformation

$$\Phi\colon \mathcal{F} \implies Hom(-, M)$$

i.e. $\mathcal{F}$ is represented by (M, Φ).

Definition 1.4. A *fine moduli space* for a given moduli problem is a pair (M, Φ) which represents $\mathcal{F}$.

Note that in the definition, we need not insist a priori that $M = \mathcal{A}/\sim$ for if (M, Φ) represents $\mathcal{F}$, then we have a natural bijection

$$\Phi(\{*\})\colon \mathcal{A}/\sim = \mathcal{F}(\{*\}) \xrightarrow{\sim} Hom(\{*\}, M) = M$$

Moreover, for any s in S, the inclusion map induces a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\Phi(S)} & Hom(S, M) \\ {\scriptstyle [X]\mapsto[X_s]}\big\downarrow & & \big\downarrow{\scriptstyle \phi\mapsto\phi_s} \\ \mathcal{F}(\{*\}) & \xrightarrow[\Phi(\{*\})]{} & M \end{array}$$

Hence, $\Phi(S)(X) = \Phi(\{*\}) \circ \nu_X = \nu_X^{'}$ (say). Furthermore, the identity morphism 1_M determines a family U parameterised by M, and for any family X parameterised by S, the families X, and ${\nu_X^{'}}^*U$ correspond to the same morphism. Thus one has an equivalence $X \sim {\nu_X^{'}}^*U$, which leads to the following alternate definition:

Definition 1.5. A *fine moduli space* consists of a variety M and a family U parameterised by M such that for every family X parameterised by S, there is a unique morphism $\phi\colon S \to M$ with $X \sim \phi^*U$. Such a family U is called a universal family for the given problem.

Unfortunately, there are only a very few cases where one can hope for a fine moduli space. Thus, we try to find some weaker conditions which nevertheless determine a unique structure of a variety on M. The solution is to drop the requirement that M satisfy a universal property for families, and rather ask that Φ satisfies a universal property for natural transformations $\mathcal{F} \implies Hom(-, M)$.

Definition 1.6. A *coarse moduli space* for a given moduli problem is a variety M together with a natural transformation $\Phi\colon \mathcal{F} \implies Hom(-, M)$ such that

1. $\Phi(\{*\})$ is bijective,
2. for any variety N and any natural transformation $\Psi\colon \mathcal{F} \implies Hom(-, N)$, there exists a unique natural transformation

   $$\Omega\colon Hom(-, M) \implies Hom(-, N)$$

   such that $\Psi = \Omega \circ \Phi$

This is a nice definition in categorical terms but the intuition behind it is rather less clear. Note however that $\Phi(S)(X) = \Phi(\{*\}) \circ \nu_X$ and for any natural transformation

$$\psi \colon \mathcal{F} \implies Hom(-, N)$$

we have a map $\mu := \psi(*) \circ \Phi(*)^{-1} \colon M \to N$. It is easy to see from Definition 1.5.2 that $\mu = \Omega(*)$ coincides with the morphism $\Omega(M)(1_M)$. In fact, more generally,

$$\Omega(S)(\phi) = \mu \circ \phi$$

for any $\phi \in Hom(S, M)$. Conversely, if μ is a morphism, Ω can be defined using the formula above. This yields another definition:

Definition 1.7. A *coarse moduli space* consists of a variety M and a bijection $\alpha \colon \mathcal{A}/\sim \to M$ such that

1. for any family X parameterised by a variety S, $\alpha \circ \nu_X$ is a morphism
2. for any variety N and any natural transformation $\psi \colon \mathcal{F} \implies Hom(-, N)$, the map
$$\mu = \psi(*) \circ \alpha^{-1} \colon M \to N$$
is a morphism.

Proposition 1.8. (M_1, α_1) *and* (M_2, α_2) *be two coarse moduli spaces as in Definition 1.7. Then there exists an isomorphism* $\mu \colon M_1 \to M_2$ *such that* $\mu \circ \alpha_1 = \alpha_2$.

Proof: This uniqueness is direct from Definition 1.7.2. □

It is clear from Definition 1.7 and the fact that $\psi(S)(X) = \psi(*) \circ \nu_X$ that the construction of a coarse moduli space is independent of the choice of $\sim$ for families. This, however is not true in general for fine moduli spaces.

Proposition 1.9. *A coarse moduli space is a fine moduli space if and only if*

1. *there exists a family U parameterised by M such that $\forall m \in M$, U_m belongs to the equivalence class $\Phi(*)^{-1}(m)$*
2. *for any families parameterised by a variety S*
$$\nu_X = \nu_{X'} \iff X \sim X'$$

Proof: Notice that 1. holds if an only if Φ is surjective, and 2. holds if and only if Φ is injective. Hence, the claims follow from Definitions 1.4, and 1.6. □

Before we move on to more concrete examples, we note that the definitions translate nicely for the category of schemes.

Definition 1.10. A *moduli functor* h is a functor $h\colon \mathbf{(Sch)}^{op} \to \mathbf{(Set)}$, which means one needs to specify the following:

- for every scheme S, a set $h(S)$ i.e. the families parameterised over S.
- for every morphism $\varphi\colon S' \to S$, a map $h(\varphi)\colon h(S) \to h(S')$, the pullback of families under φ.
- and h must satisfy $h(id_S) = id_{h(S)}$, and for $S'' \xrightarrow{\psi} S' \xrightarrow{\varphi} S$, the composition $h(S) \xrightarrow{h(\varphi)} h(S') \xrightarrow{h(\psi)} h(S'')$ equals $h(S) \xrightarrow{h(\varphi\circ\psi)} h(S'')$.

If h is a representable moduli functor and H the scheme which represents it, then H is called a *fine moduli scheme*. Similarly, if a moduli functor is coarsely represented by a scheme H, then it is called a *coarse moduli scheme*.

Definition 1.11. Given a moduli functor h, and a scheme M which represents h, we define the *universal family* $U \in h(M)$ as the element corresponding to the canonical element $id_M \in \mathrm{Hom}(M, M)$.

—

Further Examples

Example 1.12. We begin with a familiar example, namely that of the projective space, showing that it is a fine moduli space for "lines through origin" in $\mathbb{A}^{n+1}$. Let us also, for the time being, assume that we work over an algebraically closed field k. Define a moduli functor $h\colon \mathbf{(Sch)}^{op} \to \mathbf{(Set)}$ as follows:

- for every k-scheme S, the set $h(S)$ is given by

$$\left\{ \begin{array}{ccc} L & \xrightarrow{\;i\;} & S \times k^{n+1} \\ \downarrow & \swarrow & \\ S & & \end{array} \right\} / \sim_S$$

  where $L \to S$ is a line bundle on S which is a subbundle of the trivial bundle $S \times k^{n+1} \to S$ and $(L \to \ S \times k^{n+1}) \sim_S (L' \to S \times k^{n+1})$ if and only if there is an isomorphism $L \xrightarrow{\sim} L'$ of line bundles on S making the obvious diagram commute.

- for every morphism $f\colon S' \to S$, define the pullback $h(f)\colon h(S) \to h(S')$ by

$$(L \xrightarrow{i} S \times k^{n+1}) \longmapsto (f^*L \xrightarrow{f^*i} f^*(S \times k^{n+1}) = S' \times k^{n+1})$$

It is easy to see that the compatibility conditions of the pullback are satisfied. Finally, we want to show that h is representable by $\mathbb{P}^n_k$. To do this, we need, for every scheme S, a bijection $h(S) \to \mathrm{Hom}(S, \mathbb{P}^n)$ such that for $f\colon S' \to S$, the diagram

$$\begin{array}{ccc} h(S) & \longrightarrow & \mathrm{Hom}(S, \mathbb{P}^n) \\ {\scriptstyle h(f)}\downarrow & & \downarrow \\ h(S) & \longrightarrow & \mathrm{Hom}(S', \mathbb{P}^n) \end{array}$$

commute. Now, given $s \in S$, the element $(i\colon L \to S \times k^{n+1}) \in h(S)$ should be mapped to the morphism $S \to \mathbb{P}^n$ which sends s to the class $[i(L_s)] \in \mathbb{P}^n$ since $i(L_s) \subset \{s\} \times k^{n+1}$ is a line through the origin. So, a priori it is only clear what to do at closed points $s \in S$. To make this more algebraically rigorous, we first fix a scheme S, and $L \xrightarrow{i} S \times k^{n+1} \in h(S)$. This map i corresponds to a short exact sequence of locally free sheaves on S

$$0 \to \mathcal{L} \xrightarrow{\iota} \mathcal{O}_S^{n+1} \to \mathcal{Q} \to 0$$

Taking the dual, this is equivalent to an exact sequence,

$$0 \leftarrow \mathcal{L}^\vee \xleftarrow{\iota^\vee} \mathcal{O}_S^{n+1} \leftarrow \mathcal{Q}^\vee \leftarrow 0$$

Since the kernel of a map of locally free sheaves is locally free, this is equivalent to specifying the surjection $\iota^\vee\colon \mathcal{O}_S^{n+1} \to \mathcal{L}^\vee$ which can be done by specifying sections $s_0, \ldots, s_n \in H^0(S, \mathcal{L}^\vee)$ without a common zero. Furthermore, since any map $\varphi\colon S \to \mathbb{P}^n$ is equivalent to the data of a line bundle $\mathcal{K}$ $(:\ = \mathcal{L}^\vee)$ on S, and $n+1$ sections without a common zero, we have the desired equivalence.

An easy computation shows that the universal family for the moduli functor in question is the tautological line bundle given by

$$\begin{array}{ccc} \{(l, v) \in \mathbb{P}^n \times k^{n+1} \colon v \in l\} = L & \xrightarrow{\ i\ } & \mathbb{P}^n \times k^{n+1} \\ \downarrow & \swarrow & \\ \mathbb{P}^n & & \end{array}$$

which is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Example 1.13. "Moduli functor for smooth curves"
Consider the functor $\mathcal{M}_g\colon \textbf{(Sch)} \to \textbf{(Sets)}$ which maps a scheme B (over $S = \mathrm{Spec}\ \mathbb{Z}$) to equivalence classes of families of smooth curves of genus g over B,

$$B \longmapsto \{\mathcal{C} \to B \text{ family of smooth curves of genus } g\} /_{\cong_B}$$

where $\{\mathcal{C} \to B\} \cong_B \{\mathcal{C}' \to B\}$ if there exists an isomorphism $\mathcal{C} \to \mathcal{C}'$ such that one has the following

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\cong} & \mathcal{C}' \\ & \searrow \ \circlearrowleft & \downarrow \\ & & B \end{array}$$

Note: A classical result states that there exists a coarse moduli scheme M_g for $\mathcal{M}_g$. [DM69]

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The Picard Scheme

We now look at the Picard functor and the Picard scheme as more concrete examples of moduli problems. The Picard scheme is a coarse moduli space, in the sense that it coarsely represents a certain functor. Fix a flat projective morphism $f\colon X \to B$, let X_T denote the fiber product $X \times_B T$ and $f_T\colon X_T \to T$ be the base change morphism.

Definition 1.14. The Picard functor, $\mathcal{P}ic_f$ associated to $f\colon X \to B$ goes from the category of B-schemes, $(\mathbf{Sch}_B)$ to $\mathbf{(Sets)}$ and associates to any B-scheme T the set,

$$\mathcal{P}ic_f(T) = \{\text{line bundles on } X_T\} /_{\cong}$$

where two line bundles $\mathcal{L}$ and $\mathcal{L}'$ on X_T are equivalent if there exists a line bundle $\mathcal{K}$ on X_T if $\mathcal{L} \cong \mathcal{L}' \otimes f_T^*\mathcal{K}$ (note that $\mathcal{P}ic_f$ is a group under tensor product). Now, to say that $\mathcal{P}ic_f$ is coarsely representable, it is enough to say that there exists a B-scheme $\mathbf{Pic}_f$ such that

1. for all B-schemes T and any line bundle $\mathcal{L}$ on X_T, there exists a unique morphism $\mu_{\mathcal{L}}\colon T \to \mathbf{Pic}_f$ called the *moduli map of* $\mathcal{L}$ which maps any point $t \in T$ to the isomorphism class of the restriction of $\mathcal{L}$ to the fiber f_t. More concretely, one can require that there exists a map
$$\mu^T\colon \mathcal{P}ic_f(T) \to \mathrm{Hom}_B(T, \mathbf{Pic}_f); \quad \mathcal{L} \mapsto \mu_{\mathcal{L}} \tag{1.1}$$
2. for every algebraically closed field k, the map μ^{Speck} is a bijection i.e., the closed points of $\mathbf{Pic}_f$ are in bijection with isomorphism classes of line bundles on the fibers of f over the closed points of B.
3. the moduli scheme $\mathbf{Pic}_f$ is uniquely determined up to isomorphism.

The following theorem of Grothendieck summarizes this result,

Theorem 1.15 (Grothendieck)**.** *$f\colon X \to B$ be a flat, projective morphism with integral geometric fibers. Then*

- *There exists a group scheme $\boldsymbol{Pic}_f$ over B which coarsely represents $\mathcal{P}ic_f$*
- *For all B- schemes T, the natural map $\mathcal{P}ic_f(T) \to Hom_B(T, \boldsymbol{Pic}_f)$; $\mathcal{L} \mapsto \mu_{\mathcal{L}}$ is an injection.*

- *If f admits a section, then $\boldsymbol{Pic}_f$ is a fine moduli scheme for $\mathcal{P}ic_f$. In other words, the above map is an isomorphism.*

Notice, if $B = Spec\, k$ where k is algebraically closed, and X is an integral projective variety over k, the above construction for $\boldsymbol{Pic}_f$ yields the classical Picard group, $Pic(X)$.

Remark: The injectivity of the map $\mathcal{P}ic_f \to \mathrm{Hom}_B(B, \mathbf{Pic}_f)$ implies that if $\mathcal{L}$ and $\mathcal{L}'$ are two line bundles on X_T which agree on every fiber of f_T, then $\mathcal{L}$ and $\mathcal{L}'$ are equivalent in the sense that they only differ by the pullback of some line bundle of T. We will come back to proving this theorem after having defined some more technical tools that go into proving it.

Example 1.16. What prevents $\mathcal{P}ic_f$ from having a fine moduli space is the existence of non-trivial maps $T \to \mathbf{Pic}_f$. In other words, a continuously varying family of line bundles on the fibers of f_T does not necessarily "glue together" to a line bundle on the total space X_T. The theorem says that such a gluing exists if f has a section. Suppose that $\mathcal{P}ic_f$ is represented by a scheme $\mathbf{Pic}_f \to B$, we would like to know what the universal element, $\mathcal{U} = \mathcal{U}_{\mathcal{P}ic_f} = \mathcal{P}ic_f(\mathbf{Pic}_f)$ is. By definition of the Picard funtor, $\mathcal{U}$ is a line bundle on $\mathcal{X}_{\mathbf{Pic}_f}$ such that for any B-scheme T, and any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_T)$, the moduli map $\mu_{\mathcal{L}} \colon \mathcal{X}_T \to \mathbf{Pic}_f$ lifts to a map $\widehat{\mu_{\mathcal{L}}} \colon \mathcal{X}_T \to \mathcal{X}_{\mathbf{Pic}_f}$. Now, the pullback $\widehat{\mu_{\mathcal{L}}}^*\mathcal{U}$ is a line bundle on $\mathcal{X}_T$ whose moduli map must coincide with $\mu_{\mathcal{L}}$. Thus, by Grothendieck's theorem, we have that $\mathcal{L}$ and $\widehat{\mu_{\mathcal{L}}}^*\mathcal{U}$ are isomorphic up to tensoring with the pullback of a line bundle on T.

A line bundle $\mathcal{U} \in \mathrm{Pic}(\mathcal{X} \times_B \mathbf{Pic}_f)$ with this universal property is called a *Poincaré line bundle.*

We now turn our attention to the relative Picard, and relative Jacobian functors.

Definition 1.17. Given two S-schemes X, and T where $f \colon X \to S$ is a map of finite type and a fiber product diagram,

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

we define the *relative Picard functor* denoted by $\mathcal{P}ic_{X/S}(T)$ as

$$\mathcal{P}ic_{X/S}(T) \colon = \mathcal{P}ic_f(T) \Big/ \mathcal{P}ic(T) = \mathcal{P}ic(X_T) \Big/ \mathcal{P}ic(T)$$

where $\mathcal{P}ic(T)$ denotes the Picard group of T.

We denote the associated sheaf of it in the étale topology by $\mathrm{Pic}_{(X/S)(\acute{e}t)}$. Note that the (absolute) Picard functor $\mathcal{P}ic_f$ is never a separated presheaf and hence, $\mathcal{P}ic_{X/S}$ is not a priori a sheaf. It is remarkable that it is representable so often in practice.

At this juncture, we state an important theorem and a lemma without proof which prove useful in later parts of the chapter. The proofs can be found in [Kle05]

Theorem 1.18 (Comparison)**.** *Assume* $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ *holds universally (i.e., for any* S*-scheme* T*, the comorphism of* f_T *is an isomorphism,* $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$*). Then* $\mathcal{P}ic_{X/S} \hookrightarrow Pic_{(X/S)(\acute{e}t)}$*. This map is an isomorphism if* f *also has a section.*

Lemma 1.19.

1. *Assume* $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$*. Then the functor* $\mathcal{N} \mapsto f^*\mathcal{N}$ *is fully faithful from the category* $\mathcal{C}$ *of locally free sheaves of finite rank on* S *to those on* X*. The essential image is formed by the sheaves* $\mathcal{M}$ *on* X *such that the image* $f_*\mathcal{M}$ *is in* $\mathcal{C}$ *and the natural map* $f^*f_*\mathcal{M} \to \mathcal{M}$ *is an isomorphism.*

2. *Let* $f: X \to S$ *be proper and flat, and its geometric fibers be reduced and connected. Then* $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ *holds universally.*

Now to prove Theorem 1.15, Grothendieck constructed the Picard scheme by taking a suitable family of effective divisors and forming the quotient modulo linear equivalence. We develop the basic theory of these notions in the following pages.

Definition 1.20 (Effective divisors)**.** A closed subscheme $D \subset X$ is called an *effective (Cartier) divisor* if its ideal $\mathcal{I}$ is ivertible. Given an $\mathcal{O}_X$-module $\mathcal{F}$ and $n \in \mathbb{Z}$, set

$$\mathcal{F}(nD) \colon = \mathcal{F} \otimes \mathcal{I}^{\otimes -n}.$$

In particular, $\mathcal{O}_X(-D) = \mathcal{I}$.

So, the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$ yields an inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$, via tensor product with $\mathcal{O}_X(D)$, which in turn corresponds to a global section of $\mathcal{O}_X(D)$. These sections, which correspond to injections, are called *regular.*

Conversely, given an arbitrary invertible sheaf $\mathcal{L}$ on X, let $H^0(X,\mathcal{L})_{\text{reg}}$ denote the subset of regular sections corresponding to injections $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$ in $H^0(X,\mathcal{L})$. And, let $|\mathcal{L}|$ be the set of effective divisors D such that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{L}$. We call $|\mathcal{L}|$ the *complete linear system* associated to $\mathcal{L}$. With these definitions, it is easy to establish a canonical isomorphism

$$H^0(X,\mathcal{L})_{\text{reg}} \Big/ H^0(X,\mathcal{O}_X^*) \cong |\mathcal{L}|.$$

Definition 1.21. A *relative effective divisor* on X/S is an effective divisor $D \subset X$ that is S-flat.

Lemma 1.22. *Let* $D \subset X$ *be a closed subscheme,* $x \in D$ *a point, and* $s \in S$ *its image. Then the following are equivalent:*

1. *The subscheme* $D \subset X$ *is a relative effective divisor at* x *(i.e., in a neighborhood of* x*).*

2. *The schemes* X *and* D *are* S*-flat at* x*, and the fiber* D_s *is an effective divisor on* X_s *at* x*.*

3. *The scheme X is S-flat at x, and the subscheme $D \subset X$ is cut out at x by one element that is regular on the fiber X_s.*

Proof. For the ease of notation, let us set $A := \mathcal{O}_{S,s}$, $B := \mathcal{O}_{X,x}$, and $C := \mathcal{O}_{D,x}$. Then $B \otimes_A k = \mathcal{O}_{X_s,x}$.

Let's assume 1., then by hypothesis, D is an effective divisor at x. So, there is a regular element $b \in B$ that generates the ideal of D, and multiplication by b defines a short exact sequence

$$0 \to B \to B \to C \to 0$$

which in turn induces the exact sequence given by

$$\mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(C, k) \to B \otimes k \to B \otimes k.$$

Now, since D is S-flat at x by hypothesis, $\mathrm{Tor}_1^A(C, k) = 0$. So, the map $B \otimes k \to B \otimes k$ is injective and its image is the ideal of D_s. Thus, D_s is an effective divisor. As $\mathrm{Tor}_1^A(C, k) = 0$, the map $\mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(B, k)$ is surjective. This map is given by multiplication by b, and b lies in the maximal ideal of B. Also, $\mathrm{Tor}_1^A(B, k)$ is a finitely generated B-module. Thus, by Nakayama's lemma, $\mathrm{Tor}_1^A(B, k) = 0$, and hence, by the local criterion of flatness, B is A-flat i.e. X is S-flat at x which shows 2.

Now we assume 2. Let's call the ideal of D in B by I, and that of D_s in $B \otimes k$ by I'. Consider an element $b \in I$ whose image $b' \in B \otimes k$ generates I'. Such a b exists because D_s is an effective divisor at x as assumed, and for the same reason, b' is regular. Remains to show that b generates I. Consider now the short exact sequence

$$0 \to I \to B \to C \to 0.$$

By hypothesis, C is A-flat. Hence, the map $I \otimes k \to B \otimes k$ is injective and its image I' is generated by b'. So the image of b in $I \otimes k$ generates it, and hence by Nakayama's lemma, b generates I. Thus, 3. holds.

Finally, let's assume 3. Again, let I be the ideal of D in B. By hypothesis, I is generated by an element b whose image b' in $B \otimes k$ is regular. We need to show that b is regular and C is A-flat. Now, the exact sequence $0 \to I \to B \to C \to 0$, gives

$$\mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(C, k) \to I \otimes k \to B \otimes k.$$

Since, $I = Bb$, multiplication by b induces a surjection $B \to I$, and hence a surjection $B \otimes k \to I \otimes k$. Consider now, the composition $B \otimes k \to I \otimes k \to B \otimes k$ given by multiplication by b', and since b' is regular, this map is injective. Thus, $B \otimes k \xrightarrow{\sim} I \otimes k$, and hence the last map in the exact sequence above is injective. Also, since B and C are A-flat, and $0 \to I \to B \to C \to 0$ is exact, I is also A-flat.

Now, define K by the exact sequence $0 \to K \to B \to I \to 0$. Then the sequence $0 \to K \otimes k \to B \otimes k \to I \otimes k \to 0$ is also exact because I is A-flat. But, $B \otimes k \xrightarrow{\sim} I \otimes k$ and thus $K \otimes k = 0$. Hence, $K = 0$ by Nakayama's lemma. But since K is the kernel of the multiplication by b map, this means that b is regular. This shows 1. □

Now suppose that we are given a relative effective divisor D on X_T/T and an arbitrary S-map $p\colon T' \to T$, one might ask if the T'-flat closed subscheme $D_{T'} \subset X_{T'}$ is an effective divisor. Let $\mathcal{I}$ denote the ideal of D. Since, D is T-flat, $p^*_{X_T}\mathcal{I}$ is equal to the ideal of $D_{T'}$, but since $\mathcal{I}$ is invertible, so is $p^*_{X_T}\mathcal{I}$. Thus, $D_{T'}$ is indeed a (relative) effective divisor. This motivates the following definition and theorem.

Definition 1.23. Given an S-scheme X, define a functor $\mathrm{Div}_{X/S}$ on the category of S-schemes by the formula

$$\mathrm{Div}_{X/S}(T) := \{\text{relative effective divisors } D \text{ on } X_T/T\}$$

Theorem 1.24. *Assume X/S is flat and projective. Then $Div_{X/S}$ is representable by an open subscheme $\boldsymbol{Div}_{X/S}$ of the Hilbert scheme $\boldsymbol{Hilb}_{X/S}$.*

Proof. Let H denote the Hilbert scheme $\mathbf{Hilb}_{X/S}$, $W \subset X \times H$ be the universal closed subscheme, and $q\colon W \to H$ the projection map. Suppose V denotes the set of points $w \in W$ at which W is an effective divisor, then V is open in W. Set $Z := q(W - V)$, then Z is closed because q is proper. Set $U := H - Z$ then U is open and $q^{-1}U$ is an effective divisor in $X \times U$. In fact, since q is flat, $q^{-1}U$ is a relative effective divisor in $X \times U\,/_U$. We wish to show that U represents $\mathrm{Div}_{X/S}$. To do this, let T be an S-scheme and $D \subset X_T/T$ be a relative effective divisor. By the universal property of the pair (H, W), there exists a unique map $g\colon T \to H$ such that $g_X^{-1}W = D$. We have to show that g factors through U.

Now, for each $t \in T$, the fiber D_t is an effective divisor by Lemma 1.21. But $D_t = W_{g(t)} \otimes k_t$ (where k_t is the residue field of t). So, $W_{g(t)}$ too is a divisor since a field extension is faithfully flat. Therfore, since $X \times H$ and W are H-flat, by Lemma 1.21, W is a relative effective divisor along the fiber of $g(t)$. Thus, $g(t) \in U$, and since U is open, g factors through U. □

Definition 1.25. Let $\mathcal{L}$ be an invertible sheaf on X. Define a subfunctor $\mathrm{LinSys}_{\mathcal{L}/X/S}$ of $\mathrm{Div}_{X/S}$ as follows

$$\mathrm{LinSys}_{\mathcal{L}/X/S}(T) := \{\text{relative effective divisors } D \text{ on } X_T/T \text{ such that } \mathcal{O}_{X_T}(D) \cong \mathcal{L}_T \otimes f_T^*\mathcal{N}\}$$

for some invertible sheaf $\mathcal{N}$ on T.

We now look at an important $\mathcal{O}_S$-module called the 'module $\mathcal{Q}$' following [7.7.6] from [GD63], which acts as an important technical tool in proving Theorem 1.15.

Definition 1.26. Let $f\colon X \to S$ be proper and $\mathcal{F}$ be a coherent $\mathcal{O}_X$ module flat over S. There exists a coherent $\mathcal{O}_S$-module $\mathcal{Q}$ and an isomorphism of fuctors in the quasi-coherent $\mathcal{O}_S$-module $\mathcal{N}$

$$q\colon\ \mathrm{Hom}(\mathcal{Q}, \mathcal{N}) \xrightarrow{\sim} f_*(\mathcal{F} \otimes f^*\mathcal{N}).$$

The pair $(\mathcal{Q}, q)$ is unique up to unique isomorphism and its construction commutes with base change, and localization.

Fix $s \in S$ and let $S = \mathrm{Spec}(\mathcal{O}_{S,s})$, then it is not hard to check that the following conditions are equivalent:

1. $\mathcal{Q}$ is a free $\mathcal{O}_S$-module (or equivalently, projective);
2. $\mathcal{N} \longmapsto f_*(\mathcal{F} \otimes f^*\mathcal{N})$ is a right exact functor;
3. for all $\mathcal{N}$, the natural map $f_*(\mathcal{F}) \otimes \mathcal{N} \to f_*(\mathcal{F} \otimes f^*\mathcal{N})$ is an isomorphism;
4. the natural map, $H^0(X, \mathcal{F}) \otimes k_s \to H^0(X_s, \mathcal{F}_s)$ is a surjection.

In addition, all these are implied by the following condition:

5. the first cohomology group of the fiber vanishes, $H^1(X_s, \mathcal{F}_s)$

Before, we move on, we look at the construction of a functor which helps us in proving some theorems. Let $\mathcal{E}$ be an arbitrary quasi-coherent sheaf on a scheme S. Define a functor $P(\mathcal{E})$ as follows: for each S-scheme T, $P(\mathcal{E})(T)$ is the set of invertible quotients $\mathcal{L}$ of the pullback $\mathcal{E}_T$. Invertible means that $\mathcal{L}$ is the sheaf of sections of a line bundle. The functor $P(\mathcal{E})$ is representable by an S-scheme $\mathbf{P}(\mathcal{E})$.

It is clear from our construction that $\mathbf{P}(\mathcal{E})$ carries a universal invertible quotient of $\mathcal{E}_{\mathbf{P}(\mathcal{E})}$ denoted by $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Namely, each invertible quotient $\mathcal{L}$ of $\mathcal{E}_T$ defines a unique S-map $\varphi\colon T \to \mathbf{P}(\mathcal{E})$ with $\varphi^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Moreover, given any S-scheme T, $\mathbf{P}(\mathcal{E}) \times_S Y = \mathbf{P}(\mathcal{E}_Y)$. Also, if X is projective over S, then, X can be embedded in $\mathbf{P}(\mathcal{E})$ for some coherent sheaf $\mathcal{E}$ on S.

Theorem 1.27. *Assume X/S is flat and proper, and its geometric fibers are integral. Let $\mathcal{L}$ be an invertible sheaf on X, and $\mathcal{Q}$ be the $\mathcal{O}_S$-module attached to it as defined above. Let $L := \boldsymbol{P}(\mathcal{Q})$, then L represents $LinSys_{\mathcal{L}/X/S}$.*

Proof. Let $D \in \mathrm{LinSys}_{\mathcal{L}/X/S}(T)$. Say, $\mathcal{O}_{X_T}(D) \cong \mathcal{L}_T \otimes f_T^*\mathcal{N}$, then $\mathcal{N}$ is determined up to isomorphism. Let, $\mathcal{N}'$ be a second choice. Then

$$\mathcal{L}_T \otimes f_T^*\mathcal{N} \cong \mathcal{L}_T \otimes f_T^*\mathcal{N}'.$$

So, $f_T^*\mathcal{N} \cong f_T^*\mathcal{N}'$ as $\mathcal{L}$ is invertible, and hence, $\mathcal{N} \cong \mathcal{N}'$ by Lemma 1.19.

Say D is defined by $\sigma \in H^0(X_T, \mathcal{L}_T \otimes f_T^*\mathcal{N})$, now as constructing $\mathcal{Q}$ commutes with changing base, we get

$$Hom(\mathcal{Q}_T, \mathcal{N}) \xrightarrow{\sim} f_{T_*}(\mathcal{L}_T, f_T^*\mathcal{N})$$

so, σ corresponds to a map $u\colon \mathcal{Q}_T \to \mathcal{N}$.

Consider an arbitrary $t \in T$. Since D is a relative effective divisor on X_T/T, its fiber D_t is a divisor on X_t by Lemma 1.22. Also, since D_t is defined by $\sigma_t \in H^0(X_t, \mathcal{L}|X_t)$, one must have $\sigma_t \neq 0$. But, σ_t corresponds to $u \otimes k_t$, so $u \otimes k_t \neq 0$. Now, $\mathcal{N}$ is invertible, so $\mathcal{N} \otimes k_t$ is a k_t-vector space of dimension 1, thus, $u \otimes k_t$ is surjective and by Nakayama's lemma, u is surjective at t but since t was arbitrary, u is surjective everywhere. Therefore, $u\colon \mathcal{Q}_T \to \mathcal{N}$ defines an S-map $p\colon T \to L$ by [4.2.3] of [GD61a] and since $(\mathcal{N}, u)$ is determined up to isomorphism, so is p. Plainly, this construction is functorial in T and yields a map of functors

$$\Lambda\colon \mathrm{LinSys}_{\mathcal{L}/X/S}(T) \to L(T).$$

We now need to show that Λ is an isomorphism.

Let $p \in L(T)$, so $p\colon T \to L$ is an S-map. Then p arises from a surjection $u\colon \mathcal{Q}_T \to \mathcal{N}$, namely, $u = p^*\alpha$ where $\alpha\colon \mathcal{Q}_L \to \mathcal{O}(1)$ is the canonical map. Moreover, there is only one such pair $(\mathcal{N}, u)$ up to isomorphism. Now, via the isomorphism obtained by base change above, the surjection u corresponds to a global section $\sigma \in H^0(X_T, \mathcal{L}_T \otimes f_T^*\mathcal{N})$. Let $t \in T$, then $u \otimes k_t$ is surjective, so it is non-zero. But $u \otimes k_t$ corresponds to $\sigma_t \in H^0(X_t, \mathcal{L}|X_t)$, so $\sigma_t \neq 0$. But, by hypothesis the geometric fibers of X/S are integral, so X_t is integral, and hence, σ_t is regular. The section σ defines a map $(\mathcal{L}_T \otimes f_T^*\mathcal{N})^{-1} \to \mathcal{O}_{X_T}$ and its image is the ideal of a closed subscheme $D \subset X$ which is cut out locally by one element. Moreover, on the fiber X_t, this element corresponds to σ_t, so is regular. Hence, D is a relative effective divisor on X_T/T by Lemma 1.21, and thus, $D \in \mathrm{LinSys}_{\mathcal{L}/X/S}(T)$. In particular, D is the only such divisor corresponding to $(\mathcal{N}, u)$ and maps to p under Λ which shows that Λ is an isomorphism. □

Definition 1.28. The *Abel map* is the natural map of functors

$$A_{X/S}(T)\colon \mathrm{Div}_{X/S}(T) \to \mathcal{P}ic_{X/S}(T)$$

defined by sending a relative effective divisor D on X_T/T to the sheaf $\mathcal{O}_{X_T}(D)$.

The target $\mathcal{P}ic_{X/S}(T)$ may be replaced by any of its associated sheaves. If the Picard scheme, $\mathbf{Pic}_{X/S}$ exists, then the term "Abel map" may refer to the corresponding map of schemes

$$\mathbf{A}_{X/S}\colon \mathbf{Div}_{X/S} \to \mathbf{Pic}_{X/S}.$$

We now have all the necessary tools to prove Grothendieck's theorem.

Theorem 1.29 (Grothendieck)**.** *Assume $f\colon X \to S$ is projective locally over S, and is flat with integral geometric fibers.*

1. *then $\boldsymbol{Pic}_{X/S}$ exists, is separated and locally of finite type over S, and represents $Pic_{(X/S)(\acute{e}t)}$.*
2. *If S is Noetherian and X/S is projective, then $\boldsymbol{Pic}_{X/S}$ is a disjoint union of open subschemes, each an increasing union of open quasi-projective S-schemes.*

Proof. By [GD60, p. 106], it is a local matter on S to represent a Zariski sheaf on the category of S-schemes. Moreover, it is also a local matter on S to prove that an S-scheme is separated and locally of finite type. Thus, to prove (1), we can assume that S is Noetherian and X/S is projective. Also, an S-scheme is separated if it is a disjoint union of separated open subschemes, or if it is an increasing union of separated open subschemes. Hence, (1) follows from (2).

Now to prove (2), by means of the Yoneda lemma, we may view the category of schemes as a full subcategory of the category of functors by identifying a scheme T with its functor of points. To lighten the notation, we denote this functor too by T and say that the functor is a scheme as well as that it is representable. Also, set $P := \mathrm{Pic}_{(X/S)(\acute{e}t)}$ (note that $P(T) = \mathrm{Hom}(T, P)$).

Given a polynomial $\phi \in \mathbb{Q}[n]$, let $P^\phi \subset P$ denote the étale subsheaf associated to the presheaf whose T-points are represented by invertible sheaves $\mathcal{L}$ on X_T such that

$$\chi(X_t, \mathcal{L}_t^{-1}(n)) = \phi(n) \tag{1.2}$$

for all $t \in T$. This presheaf is well defined, because the previous equation remains valid after base change because given any base change $p\colon T' \to T$, and any i, n, we have

$$H^i(X_{t'}, \mathcal{L}_{t'}^{-1}(n)) = H^i(X_{p(t')}, \mathcal{L}_{p(t')}^{-1}(n)) \otimes_{k_t} k_{t'} \tag{1.3}$$

because cohomology commutes with flat base change. Thus, P^ϕ is also well defined.

Fix a map $T \to P$ and represent it by means of an étale covering $p\colon T' \to T$ and an invertible sheaf $\mathcal{L}'$ on $X_{T'}$. Consider then the subset $T'^\phi \subset T'$ defined by

$$T'^\phi := \{t' \in T' | \chi(X_{t'}, \mathcal{L}_{t'}^{-1}(n)) = \phi(n)\}.$$

Then T'^ϕ is open by [7.9.11] of [GD63]. Set $T^\phi := p(T'^\phi)$, then $T^\phi \subset T$ is open as $T'^\phi \subset T'$ is open and p is étale. Moreover, $T'^\phi = p^{-1}(T^\phi)$. To see this, let $t' \in p^{-1}(T^\phi)$ and say $p(t') = p(t'_1)$ where $t'_1 \in T'^\phi$. Now, there is an étale covering $T'' \to T' \times_T T'$ such that the two pullbacks of $\mathcal{L}'$ to $X_{T''}$ are isomorphic. Let $t'' \in T''$ have image t' under the first map $T'' \to T'$ and $t'_1 \in T'$ under the second. Then

$$\chi(X_{t'}, \mathcal{L}_{t'}^{-1}(n)) = \chi(X_{t''}, \mathcal{L}_{t''}^{-1}(n)) = \chi(X_{t'_1}, \mathcal{L}_{t'_1}^{-1}(n)) = \phi(n) \tag{1.4}$$

and hence, $t' \in T'^\phi$ which shows our claim. Furthermore, T^ϕ is (represents) the fiber product of functors $P^\phi \times_P T$, one could prove this essentially by showing that they have the same R-points for $R \to T$.

Let ϕ vary, plainly, the T'^ϕ are disjoint and cover T'. So, the T^ϕ are disjoint and cover T. So, by [GD60, p. 103], if P^ϕ are representable by schemes, then P is representable by their disjoint union. Thus, it remains to represent each P^ϕ by an increasing union of open quasi-projective S-schemes.

Fix ϕ, given $m \in \mathbb{Z}$, let $P_m^\phi \subset P^\phi$ be the étale subsheaf associated to the presheaf whose T-points are represented by $\mathcal{L}$ on X_T such that in addition to equation (1.2), one also has

$$R^i f_{T_*}\mathcal{L}(n) = 0 \text{ for all } i \geq 1 \text{ and } n \geq m. \tag{1.5}$$

We now show that this presheaf is also well defined as equation (1.5) remains valid even after base change:

Let $p\colon T' \to T$ be a base change. First, note that (1.5) is equivalent to saying

$$H^i(X_t, \mathcal{L}_t(n)) = 0 \text{ for all } i \geq 1,\, n \geq m,\, t \in T \tag{1.6}$$

because for any given i, t, and n, if $H^i(X_t, \mathcal{L}_t(n)) = 0$, then $R^i f_{T_*}(\mathcal{L}(n) \otimes f_{T_*}\mathcal{N})_t = 0$ for all quasi-coherent sheaves $\mathcal{N}$ on T by [7.5.3] of [GD63]. Conversely, if $R^i f_{T_*}\mathcal{L}(n) = 0$, fix t, n. Then $H^i(X_t, \mathcal{L}_t(n))$ vanishes for $i \gg 1$ by Serre's theorem (2.2.2, [GD61b]). Now suppose that it vanishes for some $i \geq 2$, then since $R^i f_{T_*}(\mathcal{L}(n) \otimes f_{T_*}\mathcal{N})_t$ vanishes as just noted, so $R^{i-1} f_{T_*}(\mathcal{L}(n) \otimes f_{T_*}\mathcal{N})_t$ is right exact in $\mathcal{N}$ owing to the long exact sequence of higher direct images. Therefore, there is a natural isomorphism of functors

$$R^{i-1}f_{T_*}(\mathcal{L}(n))_t \otimes \mathcal{N}_t \xrightarrow{\sim} R^{i-1}f_{T_*}(\mathcal{L}(n) \otimes f_{T_*}\mathcal{N})_t.$$

Since (1.2) holds, both the source and the target vanish. Taking $\mathcal{N} := k_t$ gives the vanishing of $H^{i-1}(X_t, \mathcal{L}_t(n))$. Finally, for any $t' \in T'$, any i, and any n, we have

$$H^i(X_{t'}, \mathcal{L}_{t'}(n)) = H^i(X_{p(t')}, \mathcal{L}_{p(t')}(n)) \otimes_{k_t} k_{t'}$$

as cohomology commutes with flat base change and so the presheaf is well defined, and so P^{ϕ}_m is well defined too.

Arguing as we did for $P^{\phi} \times_P T$, we find that given a map $T \to P^{\phi}$, as m varies, the products $P^{\phi}_m \times_{P^{\phi}} T$ form a nested sequence of open subschemes of T, whose union is T. The key change in the argument is proving openness wherein instead of [7.9.11] in [GD63], we use Serre's theorem [2.2.2], [GD61b]. Therefore, again by [GD61b, p. 103], it suffices to represent each P^{ϕ}_m by a quasi-projective S-scheme. Now, fix ϕ and m and define $\phi_0(n) := \phi(m+n)$, then there is an isomorphism of functors $P^{\phi}_m \to P^{\phi_0}_0$ defined as follows. Begin by defining an automorphism ϵ of $\mathcal{P}ic_{X/S}$ by sending an invertible sheaf $\mathcal{L}$ on some X_T to $\mathcal{L}(m)$. ϵ induces an automorphism ϵ^+ of the associated sheaf P. Plainly, ϵ^+ carries P^{ϕ}_m onto $P^{\phi_0}_0$, and thus it suffices to represent $P^{\phi_0}_0$ by a quasi-projective S-scheme.

The function $s \mapsto \chi(X_s, \mathcal{O}_{X_s}(n))$ is locally constant on S by [7.9.11], [GD63]. Hence, we can assume it is constant by replacing S by an open subset. Set $\psi(n) := \chi(X_s, \mathcal{O}_{X_s}(n))$. Consider the Abel map $A_{X/S} \colon \mathrm{Div}_{X/S} \to P$, recall by Theorem 1.24, we know that $\mathrm{Div}_{X/S}$ is an open subscheme of the Hilbert scheme. Form the fiber product $Z := P^{\phi_0}_0 \times_P \mathrm{Div}_{X/S}$. From what we proved above, we can conclude that it is an open subscheme of $\mathrm{Div}_{X/S}$. Set $\theta(n) := \psi(n) - \phi_0(n)$, then clearly, Z lies in $\mathbf{Hilb}^{\theta}_{X/S}(n)$ which is projective over S, thus Z is quasi-projective over S.

Now, we prove that the projection $\alpha \colon Z \to P^{\phi_0}_0$ is a surjection of étale sheaves i.e., given a T and a $\lambda \in P^{\phi_0}_0(T)$, we have to find an étale covering $T_1 \to T$ and a $\lambda_1 \in Z(T_1)$ such that $\alpha(\lambda_1) \in P^{\phi_0}_0(T_1)$ is equal to the image of λ. For this, represent λ by means of an étale covering $p \colon T' \to T$ and an invertible sheaf $\mathcal{L}'$ on $X_{T'}$. Virtually, by definition, $T' \times_{P^{\phi_0}_0} Z$ is equal to $\mathrm{LinSys}_{\mathcal{L}/X/S}$, so by Theorem 1.26, it is equal to $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ is the $\mathcal{O}_{T'}$-module attached to $\mathcal{L}'$. Now, $m = 0$, so $H^1(X_t, \mathcal{L}_t) = 0$ for all $t \in T'$ owing to (1.5). And as noted in the equivalent conditions after Definition 1.25, $\mathcal{Q}$ is locally free. Thus, $\mathbf{P}(\mathcal{Q})$ is smooth over T'. So, there exists an étale covering $T_1 \to T'$ and a T'-map $T_1 \to \mathbf{P}(\mathcal{Q})$ by [17.16.3(ii)] from [GD67]. Then the composition $T_1 \to \mathbf{P}(\mathcal{Q}) \to Z \to P^{\phi_0}_0$ is equal to the composition $T_1 \to T' \to T \to P^{\phi_0}_0$ i.e., the map $T_1 \to Z$ is some $\lambda_1 \in Z(T_1)$ such that $\alpha(\lambda_1) \in P^{\phi_0}_0(T_1)$ is equal to the image of λ, and since the composition $T_1 \to T' \to T$ is an étale covering, α is a surjection of étale sheaves. Plainly, the map $\alpha \colon Z \to P^{\phi_0}_0$ is defined by the invertible sheaf associated to the universal relative effective divisor on X_Z/Z, so taking $T := Z$, and $T' := T$, we can conclude that the product $Z \times_{P_0^{\phi_0}} Z$ is a smooth projective Z-scheme. The theorem then follows from the following general lemma. □

Lemma 1.30. *$\alpha\colon Z \to P$ be a map of étale sheaves and let $R := Z \times_P Z$. Assume α is a surjection and Z is representable by a quasi-projective scheme S-scheme, and R is representable by a smooth and proper Z-scheme. Then P is representable by a quasi-projective S-scheme, and α is representable by a smooth map.*

Proof. Using [8.11.5] from [GD66], one can show that the map $R \to Z \times_S Z$ is a closed embedding. Thm 2.9 from [AK80, p. 70] shows that there exists a quasi-projective S-scheme Q and a faithfully flat and projective map $Z \to Q$ such that $R = Z \times_Q Z$. In fact, R defines a map from Z to the Hilbert scheme $\mathbf{Hilb}_{Z/S}$. Also $Z \to Q$ is smooth, since it is a flat map, it is smooth if and only if its fibers are smooth and these fibers up to extension of the ground field are the same as those of $R \to Z$ which is smooth by hypothesis. To see that Q represents P, we first set $A := Z \times_Q T$, and make use of [17.16.3] from [GD67] to conclude that $Z \to Q$ is a surjection of étale sheaves. Now, a map of étale sheaves $F \to G$ is a surjection if and only if G is the coequalizer of the pair of maps $F \times_G F \rightrightarrows F$, which implies that Q is the coequalizer of the pair of maps $R \rightrightarrows Z$ in the category of étale sheaves. By the same argument, so is P, and since the coequalizer in any category is unique up to unique isomorphism, Q indeed represents P. □

Mumford in [Mum66] proved the following generalization of Theorem 1.29 which we state here without proof:

Theorem 1.31 (Mumford)**.** *Assume X/S is projective and flat, and its geometric fibers are reduced and connected; assume the irreducible components of its ordinary fibers are geometrically irreducible. Then $\boldsymbol{Pic}_{X/S}$ exists.*

Recall that an invertible sheaf $\mathcal{P}$ on $X \times \mathbf{Pic}_{X/S}$ is called a Poincaré sheaf or a universal sheaf if for any S-scheme T and any invertible sheaf $\mathcal{L}$ on X_T, there exists a unique S-map $h\colon T \to \mathbf{Pic}_{X/S}$ such that for some invertible sheaf N on T,

$$\mathcal{L} \cong (1 \times h)^*\mathcal{P} \otimes f_T^*\mathcal{N}.$$

It can be shown that a universal sheaf exists if and only if $\mathbf{Pic}_{X/S}$ exists and represents $\mathrm{Pic}_{X/S}$. We now state a lemma which helps in proving the existence of a Poincaré family later.

Lemma 1.32. *1. Assume $f\colon X \to S$ is proper and flat, and its geometric fibers are reduced and connected. Then, $\mathcal{O}_S \cong f_*\mathcal{O}_X$ holds universally, that is, for any S-scheme T, $\mathcal{O}_T \cong f_T*\mathcal{O}_{X_T}$.*

2. A universal sheaf $\mathcal{P}$ exists if and only if $\boldsymbol{Pic}_{X/S}$ represents $Pic_{X/S}$. Assume $\mathcal{O}_S \cong f_*\mathcal{O}_X$ holds universally. Then, if $\mathcal{P}$ exists, it is unique up to tensor product with the pullback of a unique invertible sheaf on $\boldsymbol{Pic}_{X/S}$. Also, if f has a section, then a universal sheaf $\mathcal{P}$ exists.

3. *Assume X/S is proper and flat with integral geometric fibers. Assume $\boldsymbol{Pic}_{X/S}$ exists, and denote it by P. View $\boldsymbol{Div}_{X/S}$ as a P-scheme via the Abel map. Assume a universal sheaf $\mathcal{P}$ exists, and let $\mathcal{Q}$ be the sheaf on P associated to $\mathcal{P}$ as in Definition 1.26. Then, $\boldsymbol{P}(\mathcal{Q}) = \boldsymbol{Div}_{X/S}$ as P -schemes.*

4. *Assume X/S is projective and flat, with integral geometric fibers, and S is Noetherian. Let $Z \subseteq \boldsymbol{Pic}_{X/S}$ be a subscheme of finite type. Then Z is quasi-projective.*

Proof. 1. Let $s \in S$. Let K be the algebraic closure of k_s, and set $A := H^0(X_K, \mathcal{O}_{X_K})$. Since f is proper, A is finite dimensional as a K-vector space; hence, A is an Artin ring. Given that X_K is connected, A is not a product of two nonzero rings by [7.8.6.1] of [GD63]; so A is an Artin local ring. Because X_K is reduced, A is reduced; hence, A is a field, which is a finite extension of K. As K is algebraically closed, therefore $A = K$. Given that cohomology commutes with flat base change, we get that $k_s \cong H^0(X_s, \mathcal{O}_{X_s})$ and the isomorphism $k_s \cong H^0(X, \mathcal{O}_X)$ factors through $f_*(\mathcal{O}_X) \otimes k_s$:

$$k_s \to f_*(\mathcal{O}_X) \otimes k_s \to H^0(X_s, \mathcal{O}_{X_s}).$$

Therefore, the second map is surjective. Thus, this map is an isomorphism by the implication (iv) $\Longrightarrow$ (iii) stated after Definition 1.26 with $\mathcal{F} := \mathcal{O}_X$ and $\mathcal{N} := k_s$. Consequently, the first map is an isomorphism as well.

It follows that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is surjective at s. Let $\mathcal{G}$ denote its cokernel. Given that the tensor product is right exact and since $k_s \to f_*(\mathcal{O}_X) \otimes k_s$ is an isomorphism, $\mathcal{G} \otimes k_s = 0$. By Nakayama's lemma, the stalk $\mathcal{G}_s$ vanishes, as claimed.

Let $\mathcal{Q}$ be the $\mathcal{O}_S$-module associated with $\mathcal{F} := \mathcal{O}_X$ as in Definiton 1.26. $\mathcal{Q}$ is free at s by the implication (iv) $\Longrightarrow$ (i) after Definition 1.26. And the rank of $\mathcal{Q}_s$ is 1 due to the isomorphism in Definition 1.26 with $\mathcal{N} := k_s$. But, with $\mathcal{N} := \mathcal{O}_S$, the isomorphism becomes $\mathrm{Hom}(\mathcal{Q}, \mathcal{O}_X) \cong f_*\mathcal{O}_X$. Hence, $f_*\mathcal{O}_X$ is free of rank 1 at s. Therefore, the surjection $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism at s. As s is arbitrary, $\mathcal{O}_S \cong f_*\mathcal{O}_X$ everywhere.

Finally, let T be an arbitrary S-scheme. Then $f_T : X_T \to T$ is proper and flat, and its geometric fibers are reduced and connected. Hence, by what we just proved, $\mathcal{O}_T \cong f_{T*}\mathcal{O}_{X_T}$.

2. By Yoneda's Lemma, a universal sheaf $\mathcal{P}$ exists if and only if $\mathrm{Pic}_{X/S}$ represents $\mathbf{Pic}_{X/S}$. Set $P := \mathbf{Pic}_{X/S}$. Assume $\mathcal{P}$ exists. Then for any invertible sheaf $\mathcal{N}$ on P, clearly $\mathcal{P} \otimes f_P^*\mathcal{N}$ is also a universal sheaf. Moreover, if $\mathcal{P}'$ is also a universal sheaf, then $P' \simeq \mathcal{P} \otimes f^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on $\mathcal{P}$ by defining $h := \mathrm{id}_P$. Assume $\mathcal{O}_S \simeq f_*\mathcal{O}_X$ holds universally. If $\mathcal{P} \otimes f^*\mathcal{N} \simeq \mathcal{P} \otimes f^*\mathcal{N}'$ for some invertible sheaves $\mathcal{N}$ and $\mathcal{N}'$ on P, then $\mathcal{N} \simeq \mathcal{N}'$ by Lemma 1.19. By Theorem 1.18, if also f has a section, then $\mathrm{Pic}_{X/S}$ does represent $\mathrm{Pic}_{X/S}$; so then $\mathcal{P}$ exists.

3. An S-map $h : T \to \mathbf{Div}_{X/S}$ corresponds to a relative effective divisor D on X_T. So the composition $A_{X/sh} : T \to P$ corresponds to the invertible sheaf $\mathcal{O}_{X_T}(D)$. Hence $\mathcal{O}_{X_T}(D) \cong (1 \times A_{X/sh})^* P \otimes f_T^* \mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Therefore, if T is viewed as a P-scheme via $A_{X/sh}$, then D defines a T-point η of $\mathrm{LinSys}_{P/X\times P/P}$. Clearly, the assignment $h \mapsto \eta$ is functorial in T. Thus if $\mathbf{Div}_{X/S}$ is viewed as a P-scheme via $A_{X/sh}$, then there is a natural map Λ from its functor of points to $\mathrm{LinSys}_{P/X\times P/P}$.

   Furthermore, Λ is an isomorphism. Indeed, let T be a P-scheme. A T-point η of $\mathrm{LinSys}_{P/X\times P/P}$ is given by a relative effective divisor D on X_T such that $\mathcal{O}_{X_T}(D) \cong P_T \otimes f_T^* \mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Then $\mathcal{O}_{X_T}(D)$ and P_T define the same S-map $T \to P$. But P_T defines the structure map. And $\mathcal{O}_{X_T}(D)$ defines the composition $A_{X/sh}$ where $h : T \to \mathbf{Div}_{X/S}$ is the map defined by D. Thus $\eta = \Lambda(h)$. And h is determined by η; hence, Λ is an isomorphism.

   In other words, $\mathbf{Div}_{X/S}$ represents $\mathrm{LinSys}_{P/X\times P/P}$. But $P(Q)$ too represents $\mathrm{LinSys}_{P/X\times P/P}$ by Theorem 1.27. Therefore, $P(Q) = \mathbf{Div}_{X/S}$ as P-schemes.

4. Theorem 1.29 implies each connected component Z' of Z lies in an increasing union of open quasi-projective subschemes of $\mathbf{Pic}_{X/S}$. So Z' lies in one of them since it is quasi-compact. So Z' is quasi-projective. But Z has only finitely many components Z'. Therefore, Z is quasi-projective.

□

Having treated the existence of the Picard scheme, we now turn to its structure, more specifically, the union of the connected components of the identity element, $\mathbf{Pic}^0_{X/S}$.

Definition 1.33. The subfunctor of $\mathcal{P}ic_{X/S}$ parameterizing the line bundles of degree 0 is called the *Jacobian functor* and is denoted by $\mathcal{P}ic^0_{X/S}$ or $J_{X/S}$.

Theorem 1.34. *Let S be the spectrum of a field k. Assume X/k is projective, and X is geometrically integral. Then $\boldsymbol{Pic}^0_{X/k}$ exists and is quasi-projective. Also, if X is geometrically normal, then $\boldsymbol{Pic}^0_{X/k}$ is projective.*

Proof. Theorem 1.29 implies that $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)(\acute{e}t)}$, and is locally of finte type. Hence, $\mathbf{Pic}^0_{X/k}$ exists and is locally of finite type, and from Lemma 1.32.4, $\mathbf{Pic}^0_{X/k}$ is also quasi-projective.

Suppose X is also geometrically normal, then since $\mathbf{Pic}^0_{X/k}$ is quasi-projective, it suffices to prove it is proper to show that it is, in fact, projective. By Lemma 4.16, forming $\mathbf{Pic}^0_{X/k}$ commutes with extending k, and by [2.7.1] of [GD65], a k-scheme is complete if and only if it is after extending k. So, we assume k is algebraically closed. Now, by the structure theorem for algebraic groups, and the Lie-Kolchin theorem, it suffices to show that if T denotes the affine line minus the origin, then every k-map, $t\colon T \to (\mathbf{Pic}^0_{X/k})_{\mathrm{red}}$ is constant.

Since k is algebraically closed, X/k has a section. So, t arises from an invertible sheaf $\mathcal{L}$ on X_T by the comparison theorem. Also, since X_T is integral, there exists a divisor D such that $\mathcal{O}(D) = \mathcal{L}$ by Exercise 6.15 in [Har77, p. 145]. Now, form the projection $p\colon X_T \to X$ and restrict $\mathcal{L}$ to its generic fiber. This restriction is trivial as T is an open subscheme of the affine line. So, there exists a rational function ϕ on X_T such that $(\phi) + D$ restricts to the trivial divisor. Let $s\colon X \to X_T$ be a section. Set $E := s^*((\phi)+D)$, then E is well-defined as a divisor on X. Clearly, p^*E and $(\phi) + D$ coincide as cycles, and hence as divisors as X_T is normal. Therefore, $\mathcal{L} = p^*\mathcal{O}(E)$, and hence, $t\colon T \to \mathbf{Pic}_{X/k}$ is constant. □

Remark: More generally, Theorem 5.4 holds whenever X/k is proper, whether X is integral or not. The proof is essential the same, but requires the associated sheaf in the fppf topology in place of Theorem 1.29 ([4.18.3] in [Kle05]).

Corollary 1.35. *Assume S is the spectrum of an algebraically closed field k and X is projective and integral. Set $P := \boldsymbol{Pic}^0_{X/S}$, and let $\mathcal{P}$ be the restriction of a Poincaré sheaf to X_P. Then a Poincaré family W exists. By definition, W is a relative effective divisor on X_P/P such that*

$$\mathcal{O}_{X_P}(W - (W_0 \times P)) \cong \mathcal{P} \otimes f_P^*\mathcal{N}$$

where W_0 is the fiber over $0 \in P$ and $\mathcal{N}$ is an invertible sheaf on P.

Proof. By the previous theorem, P exists and is quasi-projective, and $\mathcal{P}$ exists by Lemma 1.32.1, and 1.32.2. Since P is Noetherian, Serre's theorem (2.2.1, [GD61b]) implies that there exists $N \in \mathbb{N}$ such that $R^i f_{P*}\mathcal{P}(n) = 0$ for all $i > 0$ and $n \geq N$. Since equation (1.5) implies equation (1.6), $H^i(\mathcal{P}_t(n)) = 0$ for all $t \in P$. Now, fix $n \geq N$ such that $h^0(\mathcal{O}_X(n)) > \dim P$. Let $\lambda \in \mathbf{Pic}_{X/k}$ represent $\mathcal{O}_X(n)$. Consider the automorphism of $\mathbf{Pic}_{X/k}$ given by multiplication by λ and let $q\colon P \xrightarrow{\sim} P'$ be the induced isomorphism. Also, let $\mathcal{P}'$ be the restriction to $X_{P'}$ of a Poincaré sheaf. Clearly, $(1 \times q)^*\mathcal{P}' \cong \mathcal{P}(n) \otimes f_P^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$.

By Lemma 1.32.3, there is a coherent sheaf $\mathcal{Q}$ on P such that $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$. Moreover, $\mathcal{Q}|P'$ is locally free of rank $h^0(\mathcal{O}_X(n))$, so $\mathcal{Q}|P'$ is of rank at least $1 + \dim P$. Now, there is an m such that the sheaf $\underline{\mathrm{Hom}}(\mathcal{Q}|P', \mathcal{O}_P)(m)$ is generated by finitely many global sections, a linear combination of which vanishes by a lemma attributed to Serre [Mum66, p. 148], hence, there is a surjection $\mathcal{Q}|P' \twoheadrightarrow \mathcal{O}_P(m)$. Correspondingly, there is a P'-map $h'\colon P' \to \mathbf{P}(\mathcal{Q}|P')$ i.e. h' is a section of the restriction over P' of the Abel map, $\mathbf{A}_{X/S}|P'$.

Now, let $W' \subset X_{P'}$ be the pullback of the universal effective divisor under $1 \times h'$, then $\mathcal{O}_{X_{P'}}(W') = \mathcal{P}'$ since h' is a section of $\mathbf{A}_{X/S}|P'$. So, in particular, $\mathcal{O}_X(W'_\lambda) = \mathcal{O}_X(n)$. Set $W := (1 \times q)^{-1}W'$, then clearly, W is a Poincaré family as desired. □

Remark: The existence of a Poincaré family shows that $J_{X/S}$ serves as a fine moduli scheme for degree 0 line bundles on X/S. This means that every family of degree 0 line bundles on X/S parameterized by another scheme factors uniquely through the $J_{X/S}$. This is an important result as it provides a concrete geometric object that represents the moduli problem.

Chapter 2

Semi-Stable Sheaves

Beginning from this chapter, we will specialize to the case where X is a *curve*, i.e. X (over S, where S is a locally Noetherian scheme over Spec$k$) is a geometrically reduced, projective scheme of pure dimension 1 over a field k. Suppose $X_1, \cdots, X_n$ are the irreducible components of X, throughout this chapter, we assume that they are geometrically integral. This is a mild assumption as there always is an extension $k' \supseteq k$ such that the irreducible components of $X(k')$ are geometrically integral. The material in this chapter is based on the first section of [Est01].

By a *subcurve* of X, we will mean a reduced closed subscheme $Y \subseteq X$ of pure dimension 1. The empty set is a subcurve of X. If $Y, Z \subseteq X$ are subcurves, set $Y \wedge Z$ to be the maximal subcurve of X contained in $Y \cap Z$ and $Y - Z$ to be the minimal subcurve containing $Y \setminus Z$. Also set $Y^c := X - Y$.

We also set some notation at this point which we shall use in the later chapter too. Suppose $X \to S$ is a flat morphism of schemes. Then for any S-flat, coherent sheaf $\mathcal{F}$ on X, we will denote by $\chi(\mathcal{F}/S)$ the relative Euler characteristic of $\mathcal{F}$. If in addition, X is flat over S and $\mathcal{E}$ is a vector bundle over X of rank r, set $\deg(\mathcal{E}/S) := \chi(\mathcal{E}/S) - r\chi(\mathcal{O}_X/S)$. By flatness, both $\chi(\mathcal{F}/S)$, and $\deg(\mathcal{E}/S)$ are locally constant on S.

Definition 2.1. Let I be a coherent sheaf on X. I is said to be *torsion-free* if I has no embedded components. We say that I is *rank-1* if I has generic rank 1 at every irreducible component of X. Lastly, I is said to be *simple* if $\text{End}_X(I) = k$.

Basic examples of torsion-free, rank-1 sheaves include line bundles, non-zero ideal sheaves.

Let I be a torsion-free, rank-1 sheaf on X. If $Y \subseteq X$ of is a subcurve, we denote by I_Y, the maximum torsion-free quotient of $I|_Y$. Of course, there is a canonical surjection $I \twoheadrightarrow I_Y$, and one may view I_Y as the unique quotient of I that has Y as support and is torsion-free on it.

Definition 2.2. I be a torsion-free, rank-1 sheaf on X. We say that I is *decomposable* if there are proper subcurves $Y, Z \subsetneq X$ such that the canonical injection $I \to I_Y \oplus I_Z$ is an isomorphism. In this case, one says that 'I decomposes at Y (or Z)'.

Proposition 2.3. *Let I be a torsion-free, rank-1 sheaf on X. Then I is simple if and only if I is not decomposable.*

Proof. It is clear that if I is decomposable, then I is not simple. Assume now that I is not simple. Then there is an endomorphism $h\colon I \to I$ that is not a multiple of the identity morphism. Let $Y \subseteq X$ be the curve such that $I_Y \cong \mathrm{im}(h)$, and let $h'\colon I_Y \hookrightarrow I$ be the induced injection. Since h is non-zero, the subcurve Y is not empty. Since I_W is simple for every irreducible component $W \subseteq X$ by Lemma 5.4 in [AK80] up to subtracting a multiple of identity from h, we may further assume that $Y \neq X$. The map h' then factors through $J := \ker(I \twoheadrightarrow I_Z)$, where $Z := Y^c$. Moreover, since h' and the composition $J \hookrightarrow I \twoheadrightarrow I_Y$ are injective, h' is, in fact, an isomorphism onto J. So, $\chi(I) = \chi(I_Y) + \chi(I_Z)$, and hence, $I = I_Y \oplus I_Z$. □

The above proposition does not hold in higher rank, even if we assume X is smooth. In fact, if X is smooth and not rational, then any vector bundle E fitting in the middle of a non-split short exact sequence of the form

$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$

is neither simple nor decomposable. The above proposition is the key reason as to why we are able to get a fine moduli space in the rank-1 case.

The next lemma is useful to determine when a torsion-free, rank-1 sheaf is simple.

Lemma 2.4. *Let $Y, Z \subset X$ be non-empty subcurves covering X. Let M be a torsion-free, rank-1 sheaf on X. Then the following statements hold.*

1. *If $Y \wedge Z \neq \emptyset$, and both M_Y and M_Z are simple, then M is simple.*

2. *If there is an exact sequence of the form*

$$0 \to I \to M \to J \to 0,$$

where I and J are simple, torsion-free, rank-1 sheaves on Y and Z respectively, then M is simple if and only if the sequence is not split.

Proof. Let's assume that there are subcurves $X_1, X_2 \subseteq X$ such that $M = M_{X_1} \oplus M_{X_2}$. Furthermore, assume that there is a surjection $\mu\colon M \twoheadrightarrow J$, where J is a simple, torsion-free, rank-1 sheaf on a subcurve $Z \subseteq X$. Clearly, μ is the direct sum of two maps, $\mu_1\colon M_{X_1} \to J$ and $\mu_2\colon M_{X_2} \to J$. Since, μ is surjective, $\mathrm{im}(\mu_i) = M_{Z_i}$, where $Z_i := Z \wedge X_i$ for $i = 1, 2$. So, $J = M_{Z_1} \oplus M_{Z_2}$ and since, J is simple, either $Z \subseteq X_1$ or $Z \subseteq X_2$.

Now, to prove (1), we apply the above reasoning twice to both $J := M_Y$ and $J := M_Z$. Without loss of generality, either $Y \subseteq X_1$ and $Z \subseteq X_2$, or $Y \cup Z \subseteq X_1$. By hypothesis, $Y \wedge Z \neq \emptyset$. So, $Y \cup Z \subseteq X_1$. Since, $Y \cup Z = X$, we have $X_1 = X$, thus, by Proposition 2.3, the sheaf M is simple.

We prove (2) now. The only if part is trivial, we show the if part. Assume by contradiction that $M = M_{X_1} \oplus M_{X_2}$ for proper subcurves $X_1, X_2 \subsetneq X$. Applying the initial reasoning to the surjection $\mu\colon M \twoheadrightarrow J$, we may assume without loss of generality $Z \subseteq X_1$. So, $\mu_2 = 0$, and hence $I = \ker(\mu_1) \oplus M_{X_2}$. Since I is simple and X_2 is non-empty, $\ker(\mu_1) = 0$. So, $J = M_{X_1}$, and thus, the sequence is split. A contradiction. □

Definition 2.5. Let $d \in \mathbb{Z}$, and E be a vector bundle on X of rank $r > 0$ and degree $-rd$. We say that E is *polarization on X*. For every subcurve $Y \subseteq X$, let $e_Y := -\deg(E|_Y)$, $E|_Y$ is a polarization on Y if $r|e_Y$.

Observe that if I is a torsion-free, rank-1 sheaf on X, and F is a vector bundle on X of rank m and degree f, then $\chi(I \otimes F) = m\chi(I) + f$.

Let us now suppose I is a torsion-free, rank-1 sheaf such that $\chi(I) = d$ and E be as in Definition 2.3, then by our observation above $\chi(I \otimes E) = 0$. This leads to the following definitions.

Definition 2.6. Let I be a torsion-free, rank-1 sheaf on X. We say that I is *stable* (resp. *semi-stable*) with respect to E if for every non-empty, proper subcurve $Y \subsetneq X$,

$$\chi(I_Y) > e_Y/r \ \ \text{(resp. } \chi(I_Y) \geq e_Y/r\text{)},$$

or equivalently,

$$\chi(I_Y \otimes E) > 0 \ \ \text{(resp. } \chi(I_Y \otimes E) \geq 0\text{)}.$$

If X is irreducible, then any torsion-free, rank-1 sheaf I on X with $\chi(I) = d$ is stable with respect to E.

Let I be a torsion-free, rank-1 sheaf on X with $\chi(I) = d$. For every subcurve $Y \subseteq X$, let us define $\beta_I(Y) := \chi(I_Y) - e_Y/r$. Of course, I is stable (resp. semi-stable) if and only if $\beta_I(Y) > 0$ (resp. $\beta_I(Y) \geq 0$) for every non-empty proper subcurve $Y \subsetneq X$. Furthermore, if I is semi-stable and $Y \subseteq X$ is a non-empty subcurve, then $\beta_I(Y) = 0$ if and only if I_Y is semi-stable with respect to $E|_Y$.

Lemma 2.7. *Let I be a torsion-free, rank-1 sheaf on X with $\chi(I) = d$. If $Y, Z \subseteq X$ are subcurves, then*

$$\chi(I_{Y \cup Z}) + \chi(I_{Y \wedge Z}) \leq \chi(I_Y) + \chi(I_Z),$$

or equivalently,

$$\beta_I(Y \cup Z) + \beta_I(Y \wedge Z) \leq \beta_I(Y) + \beta_I(Z).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} I_{Y \cup Z} & \xrightarrow{\ \ u\ \ } & I_Y \oplus I_{Z-Y} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ I_Z & \xrightarrow{\ \ v\ \ } & I_{Y \wedge Z} \oplus I_{Z-Y} \end{array}$$

where the maps are the natural restriction maps and the sums thereof. Note that f and g are surjective, and u and v are injective with cokernels of finite length.

Therefore, coker(u) maps onto coker(v). Comparing the Euler characteristics, one gets

$$\chi(I_Y) + \chi(I_{Z-Y}) - \chi(I_{Y\cup Z}) \geq \chi(I_{Y\wedge Z}) + \chi(I_{Z-Y}) - \chi(I_Z)$$

which proves the lemma. □

Now suppose that $W \subseteq X$ is an irreducible component and I is a strictly semi-stable sheaf on X (i.e., I is semi-stable but not stable). It follows from the previous lemma that there is a minimal subcurve $Z \subseteq X$ containing W such that $\beta_I(Z) = 0$ which leads to the following definition.

Definition 2.8. Let $W \subseteq X$ be an irreducible component. A strictly semi-stable sheaf I is called *W-quasi-stable with respect to E* if $\beta_I(Y) > 0$ for all proper subcurves $Y \subsetneq X$ containing W. A semi-stable sheaf I on X is stable if and only if I is W-quasi-stable for every irreducible component $W \subseteq X$. I is said to be *quasi-stable with respect to E* if there exists is an irreducible component $W \subseteq X$ such that I is W-quasi-stable with respect to E.

Lemma 2.9. *Let $Y, Z \subsetneq X$ be proper subcurves covering X such that $Y \wedge Z = \emptyset$, but $Y \cap Z \neq \emptyset$. Let I (resp. J) be a torsion-free, rank-1 sheaf on Y (resp. Z). Then there is a non-split exact sequence of the form*

$$0 \to J \to M \to I \to 0.$$

Proof. To show the existence of a non-split exact sequence as above, we need to show that $\mathrm{Ext}^1_X(I, J) \neq 0$. Since I and J are torsion-free sheaves supported on Y and Z, and $Y \wedge Z = \emptyset$, we have $\underline{\mathrm{Hom}}_X(I, J) = 0$. Thus,

$$\mathrm{Ext}^1_X(I, J) = H^0(X, \underline{\mathrm{Ext}}^1_X(I, J)).$$

Also, it is clear that the topological support of $\underline{\mathrm{Ext}}^1_X(I, J)$ is contained in $Y \cap Z$. Since $Y \cap Z$ is non-empty, fix $p \in Y \cap Z$ and let $\mathcal{O}_p$ denote the local ring of X at p with maximal ideal $\mathfrak{m}_p$. Since p is arbitrary, it is enough to show that $\mathrm{Ext}^1_{\mathcal{O}_p}(I_p, J_p) \neq 0$. Let $\mathcal{M}_Y \subseteq \mathcal{O}_p$ and respectively, $\mathcal{M}_Z \subseteq \mathcal{O}_p$ be the ideals of Y and Z at p. By hypothesis, $\mathcal{M}_Y \cap \mathcal{M}_Z = 0$ and $\mathcal{M}_Y + \mathcal{M}_Z$ is a primary ideal of $\mathfrak{m}_p$.

Now, let

$$({}^{\mathcal{O}_p}/_{\mathcal{M}_Y})^{\oplus s_1} \xrightarrow{\phi} ({}^{\mathcal{O}_p}/_{\mathcal{M}_Y})^{\oplus s_0} \to I_p \to 0$$

be a presentation of I_p. Applying the functor $\mathrm{Ext}^1_{\mathcal{O}_p}(-, J_p)$ to the above sequence, we obtain

$$0 \to \mathrm{Ext}^1_{\mathcal{O}_p}(I_p, J_p) \to H^{\oplus s_0} \xrightarrow{\phi^* \otimes H} H^{\oplus s_1}$$

where $H := \mathrm{Ext}^1_{\mathcal{O}_p}({}^{\mathcal{O}_p}/_{\mathcal{M}_Y}, J_p)$ and ϕ^* is the dual of ϕ. Now, since $\mathrm{Hom}_{\mathcal{O}_p}(K, J_p) = 0$ for all $\mathcal{O}_p$-modules K with $\mathcal{M}_Y K = 0$. Thus, we only need to show that $\phi^* \otimes H$ is not injective. Let us assume that it is. Since H has finite length, it follows that $\phi^* \otimes k(p)$ is injective. As ϕ^* is a map of free modules over the local ring ${}^{\mathcal{O}_p}/_{\mathcal{M}_Y}$, it follows that ϕ^* is also injective. Since the first sequence above is exact, $\mathrm{Hom}_{\mathcal{O}_p}(I_p, {}^{\mathcal{O}_p}/_{\mathcal{M}_Y}) = 0$. Since I is rank-1, torsion-free on Y, we get $I_p = 0$, a contradiction. □

We look at an example.

Example 2.10. Let X be a curve and $X_1, X_2, X_3 \subseteq X$ be connected subcurves covering X such that $X_i \wedge X_j = \emptyset$ for $i \neq j$ but $X_1 \cap X_2 \neq \emptyset$ and $X_1 \cap X_3 \neq \emptyset$. Let I_1, I_2, I_3 be simple, semi-stable sheaves on X_1, X_2, X_3 respectively. By Lemma 2.9, there is a non-split short exact sequence of the form

$$0 \to I_2 \oplus I_3 \to I \to I_1 \to 0$$

whose pushout to I_2 (and resp. I_3) is a non-split exact sequence of the form

$$0 \to I_2 \to I_{X_1 \cup X_2} \to I_1 \to 0 \quad (\text{resp.} 0 \to I_3 \to I_{X_1 \cup X_3} \to I_1 \to 0).$$

Since I_1, I_2, I_3 are semi-stable, then so are $I, I_{X_1 \cup X_2}$ and $I_{X_1 \cup X_3}$, and hence I is not quasi-stable. Moreover, since the latter sequences are not split, the second part of Lemma 2.4 implies that $I_{X_1 \cup X_2}$ and $I_{X_1 \cup X_3}$ are simple. So, by the first part of Lemma 2.4, I is simple. Thus, by means of Lemma 2.9 we have produced a simple, semi-stable sheaf I on X which is not quasi-stable.

Lemma 2.9 allows us to construct torsion-free, rank-1 sheaves on X with prescribed Jordan-Hölder filtrations which we define below.

Definition 2.11 (Jordan-Hölder filtrations)**.** E be a polarization on X. Let I be a semi-stable sheaf on X with respect to E. We now construct a filtration of I. To begin with, let $I_0 := I$ and $Z_0 := X$. Let $Y_0 \subseteq X$ be a non-empty subcurve such that I_{Y_0} is stable with respect to $E|_{Y_0}$. Set $I_1 := \ker(I \to I_{Y_0})$, then clearly, if I_1 is non-zero, then it is a torsion-free, rank-1 sheaf on $Z_1 := Y_0^c$ and semi-stable with respect to $E|_{Z_1}$. Repeating the above procedure with I_1, we end up with filtrations

$$0 = I_{q+1} \subsetneq I_q \subseteq \cdots \subsetneq I_1 \subsetneq I_0 = I$$

$$\emptyset = Z_{q+1} \subsetneq Z_q \subsetneq \cdots \subsetneq Z_1 \subsetneq Z_0 = X$$

which have the following properties:

1. for $i = 0, \cdots, q$, the sheaf I_i is torsion-free, rank-1 on the subcurve $Z_i \subseteq X$, and is semi-stable with respect to $E|_{Z_i}$,
2. for $i = 0, \cdots, q$, the quotient $I_i \big/ I_{i+1}$ is torsion-free, rank-1 on the subcurve $Y_i = Z_i - Z_{i+1}$ and is stable with respect to $E|_{Y_i}$

The above is called a *Jordan-Hölder filtration* of I and depends on the choices made in the construction. However,

$$\text{Gr}(I) := I_0/I_1 \oplus I_1/I_2 \oplus \cdots \oplus I_q/I_{q+1}$$

depends only on I by the Jordan-Hölder theorem. In particular, the collection of subcurves $\{Y_0, \cdots, Y_q\}$ covering X depends only on I.

It is clear from our construction that Gr(I) is torsion-free, rank-1, and semi-stable with respect to E. Moreover, one also has Gr(Gr(I)) = Gr(I), and if I is stable, then Gr(I) = I. And, one says that two semi-stable sheaves I and J are *Jordan-Hölder equivalent* (or, JH-equivalent) if Gr(I) = Gr(J).

Proposition 2.12. *I be a semi-stable sheaf on X. Let*

$$0 = I_{q+1} \subsetneq I_q \subseteq \cdots \subsetneq I_1 \subsetneq I_0 = I$$

$$\emptyset = Z_{q+1} \subsetneq Z_q \subsetneq \cdots \subsetneq Z_1 \subsetneq Z_0 = X$$

be a Jordan-Hölder filtration of I. Let $W \subseteq X$ be an irreducible component. Then the following statements are equivalent.

1. *I is W-quasi-stable.*
2. *I_i is W-quasi-stable for $0 \leq i \leq q$.*
3. *$W \subseteq Z_q$, and the short exact sequence $0 \to I_{i+1} \to I_i \to I_i \big/ I_{i+1} \to 0$ is not split for $0 \leq i \leq q-1$.*

Proof. The proof is similar to that of Proposition 5 in [Est97]. One can replace the notion of ϵ-quasi-stability used in the original proof with W-quasi-stability. □

Theorem 2.13. *Assume X is connected. Let $Y_0, \ldots, Y_q \subseteq X$ be subcurves covering X with $Y_i \wedge Y_j = \emptyset$ for $i \neq j$, and $J_0, \ldots, J_q$ stable sheaves on $Y_0, \ldots, Y_q$ respectively. Let $W \subseteq X$ be an irreducible component. Then there is a W-quasi-stable sheaf I on X such that Gr(I) $\cong J_0 \oplus \cdots \oplus J_q$.*

Proof. Without loss of generality, let $W \subseteq Y_q$. Since X is connected, we may also assume $(Y_1 \cup \cdots \cup Y_i) \cap Y_{i-1} \neq \emptyset$ for $1 \leq i \leq q$. Let $Z_i := Y_i \cup \cdots \cup Y_q$ for $0 \leq i \leq q$ and $I_q := J_q$. We recursively construct a torsion-free, rank-1 sheaf I_i on Z_i for $i = q-1, \cdots, 0$. Firstly, assume we have $I_q, \ldots I_i$ for a certain i, $1 \leq i \leq q$. Then let I_{i-1} be the middle term in a non-split exact sequence of the form

$$0 \to I_i \to I_{i-1} \to J_{i-1} \to 0$$

the existence of which is guaranteed by lemma 4. Let $I_0 := I$, we then have a JH-filtration of I

$$0 = I_{q+1} \subsetneq I_q \subseteq \cdots \subsetneq I_1 \subsetneq I_0 = I$$

$$\emptyset = Z_{q+1} \subsetneq Z_q \subsetneq \cdots \subsetneq Z_1 \subsetneq Z_0 = X$$

such that Gr(I) = $J_0 \oplus \cdots \oplus J_q$ and I is W-quasi-stable by previous proposition. □

Our main object of interest in the next chapter would be *families of curves*, so before moving on, we define what these are and how our notions of torsion-free, rank-1, stability, etc. for coherent sheaves on a curve translate to the case of families.

Definition 2.14. Let $f : X \to S$ be a flat, projective map whose geometric fibers are curves, then we say that X/S is a *family of curves.* Let $\mathcal{I}$ be an S-flat, coherent sheaf on X. We say that $\mathcal{I}$ is *torsion-free* (resp. *rank-1*, *simple*) if $\mathcal{I}(s)$ is torsion-free (resp. rank-1, simple) for every geometric point s of S.

Let $\mathcal{E}$ be a vector bundle of rank $r > 0$ on X such that $r | \deg(\mathcal{E}/S)$ then we say that $\mathcal{E}$ is a *polarization on* X/S.

The notion of stability, semi-stability, quasi-stability translate in the same way. Namely, a torsion-free, rank-1 sheaf $\mathcal{I}$ on a family of curves X/S is stable with respect to $\mathcal{E}$ over S if $\mathcal{I}(s)$ is stable with respect to $\mathcal{E}(s)$ for every geometric point s of S.

However, the notion of W-quasi-stability is not so easy to manage when dealing with families of curves. We shall replace it with the equivalent, but more suitable notion of 'p-quasi-stability' for a non-singular point $p \in X$. A semi-stable sheaf I on a **curve** X is *p-quasi-stable with respect to E* if $\beta_I(Y) > 0$ for all proper subcurves $Y \subsetneq X$ containing p.

Definition 2.15. Let X/S be a family of curves and $\sigma \colon S \to X$ be a section of f through the S-smooth locus of X. A torsion-free, rank-1 sheaf $\mathcal{I}$ on a family of curves X/S is *σ-quasi-stable with respect to $\mathcal{E}$ over S* if $\mathcal{I}(s)$ is $\sigma(s)$-quasi-stable with respect to $\mathcal{E}(s)$ for every geometric point s of S.

Chapter 3

The Main Theorem of Esteves

Let $f : X \to S$ be a flat, projective map whose geometric fibers are curves. Let ω be a relative dualizing sheaf for f. Let $\mathbf{J}^*$ denote the contravariant functor from the category of locally Noetherian S-schemes to sets, defined on an S-scheme T by

$$\mathbf{J}^*(T) := \{\text{simple, torsion-free, rank-1 sheaves on } X \times T/T\}/\sim,$$

where $\mathcal{I}_1 \sim \mathcal{I}_2$ if there is an invertible sheaf M on T such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes p^*M$, for $p : X \times T \to T$ the projection. Let $\mathbf{J}$ be the étale sheaf associated to $\mathbf{J}^*$. By [Alt-Klei, Thm. 7.4, p. 99], the functor $\mathbf{J}$ is represented by an algebraic space J, locally of finite type over S. Note that the formation of $\mathbf{J}$ commutes with base change. For every integer d, let $J_d \subseteq \mathbf{J}$ be the subspace parameterizing simple, torsion-free, rank-1 sheaves $\mathcal{I}$ on X/S with $\chi(\mathcal{I}/S) = d$. It is clear that J_d is an open subspace of $\mathbf{J}$, and that $\mathbf{J}$ is the disjoint union of the J_d, for d ranging through all the integers. The formation of J_d commutes also with base change.

Fix an integer d. Fix a vector bundle $\mathcal{E}$ on X of rank r and $\deg(\mathcal{E}/S) = -rd$. We consider $\mathcal{E}$ our polarization on X/S. Let $J_{\mathcal{E}}^{\mathrm{s}}$ (resp. $J_{\mathcal{E}}^{\mathrm{qs}}$, resp. $J_{\mathcal{E}}^{\mathrm{ss}}$) denote the subspace of J_d parameterizing simple, torsion-free, rank-1 sheaves on X/S that are stable (resp. semi-stable, resp. quasi-stable) with respect to $\mathcal{E}$. If $\sigma : S \to X$ is a section of f through the S-smooth locus of X, let $J_{\mathcal{E}}^{\sigma}$ denote the subspace of J_d parameterizing simple, torsion-free, rank-1 sheaves on X/S that are σ-quasi-stable with respect to $\mathcal{E}$. It is clear from the definitions in Section 2 that

$$J_{\mathcal{E}}^{s} \subseteq J_{\mathcal{E}}^{\sigma} \subseteq J_{\mathcal{E}}^{\mathrm{qs}} \subseteq J_{\mathcal{E}}^{\mathrm{ss}} \subseteq J_d.$$

The formations of all the above spaces commute with base change.

Henceforth, let S denote the spectrum of R, a DVR. π be the generator of its maximal ideal. And, let s and η be the special and generic points of S respectively. X/S be a family of curves, and assume that the irreducible components of the special fiber $X(s)$ are geometrically integral.

$\mathcal{I}$ be a torsion-free, rank-1 sheaf on X and $Y \subseteq X(s)$ be a subcurve and $\mathcal{I}^Y$ denote the kernel of the canonical surjection $\mathcal{I} \twoheadrightarrow \mathcal{I}(s)_Y$. Clearly, the inclusion $\mathcal{I}^Y \hookrightarrow \mathcal{I}$ is an isomorphism on $X - Y$. Moreover, $\mathcal{I}^Y$ is torsion-free, rank-1 on X/S.

We begin this section by proving some lemmas, and propositions which act as technical tools for the proof of Esteves' main theorem.

Theorem 3.1 (Esteves)**.** *The algebraic space $J^{ss}_{\mathcal{E}}$ is of finite type over S. In addition, $J^{ss}_{\mathcal{E}}$ and $J^{qs}_{\mathcal{E}}$ are universally closed over S.*

Lemma 3.2. *Let $\mathcal{I}$ be a torsion-free, rank-1 sheaf on X/S. Let $Y \subseteq X(s)$ be a subcurve and $Z := Y^c$. Then the following statements hold*

1. $(\mathcal{I}^Y)^Z = \mathcal{I}^Y \cap \mathcal{I}^Z = \pi\mathcal{I}$
2. *There exist short exact sequences*

$$0 \longrightarrow \mathcal{I}^Y(s)_Z \longrightarrow \mathcal{I}(s) \longrightarrow \mathcal{I}(s)_Y \longrightarrow 0 \tag{3.1}$$

$$0 \longrightarrow \mathcal{I}(s)_Y \longrightarrow \mathcal{I}^Y(s) \longrightarrow \mathcal{I}^Y(s)_Z \longrightarrow 0 \tag{3.2}$$

Proof. Since $\mathcal{I}/\mathcal{I}^Y$ is supported on Y, the natural map $\mathcal{I}^Y(s)_Z \to \mathcal{I}(s)_Z$ is injective. Therefore, $(\mathcal{I}^Y)^Z = \mathcal{I}^Y \cap \mathcal{I}^Z$. Additionally, because the natural map $\mathcal{I}(s) \to \mathcal{I}(s)_Y \oplus \mathcal{I}(s)_Z$ is injective, it follows that $\mathcal{I}^Y \cap \mathcal{I}^Z = \pi\mathcal{I}$. To prove (2), given that the natural map $\mathcal{I}(s) \to \mathcal{I}(s)_Y \oplus \mathcal{I}(s)_Z$ is injective, the kernel H of the map $\mathcal{I}(s) \to \mathcal{I}(s)_Y$ is contained within $\mathcal{I}(s)_Z$. Thus, H is torsion-free and rank-1 on Z. According to the definition of $\mathcal{I}^Y$, H is the image of $\mathcal{I}^Y(s) \to \mathcal{I}(s)$. Consequently, $\mathcal{I}^Y(s)_Z \cong H$, establishing the exactness of the first exact sequence. Since $\mathcal{I} \cong (\mathcal{I}^Y)^Z$ by (1), the exactness of the second sequence follows in the same manner. □

The following existence lemma is the main technical tool that will be used in the proof of the aforementioned theorem.

Lemma 3.3. *$Y \subseteq X(s)$ be a subcurve. Let*

$$\cdots \subseteq \mathcal{I}^i \subseteq \mathcal{I}^{i-1} \subseteq \cdots \subseteq \mathcal{I}^1 \subseteq \mathcal{I}^0 := \mathcal{I}$$

be an infinite filtration of $\mathcal{I}$ with quotients

$$\mathcal{I}^i / \mathcal{I}^{i+1} = \mathcal{I}^i(s)_Y$$

for every $i \geq 0$. If R is complete, and $\mathcal{I}^i(s)$ decomposes at Y for each $i \geq 0$, then there exists an S-flat quotient $\mathcal{F}$ of $\mathcal{I}$ on X such that $\mathcal{F}(s) = \mathcal{I}^i(s)_Y$.

Proof. For every $i \geq 0$, let $S_i := \mathrm{Spec}(R/\pi^{i+1})$ and $X_i := X \times S_i$. For every coherent sheaf $\mathcal{H}$ on X, let $H_i := \mathcal{H}|_{X_i}$ for every $i \geq 0$. Let $Z := Y^c$. We claim first that

$$\mathrm{im}(\mathcal{I}^i_l \to \mathcal{I}^k_l) = \mathrm{im}(\mathcal{I}^j_l \to \mathcal{I}^k_l) \qquad (*)$$

if $i \geq j \geq k \geq 0$ and $l \geq 0$, as long as $j - k > l$. Indeed, we argue by induction on l. Since $\mathcal{I}^i(s)$ decomposes at Y for every $i \geq 0$, the inclusion $\mathcal{I}^{i+1} \hookrightarrow \mathcal{I}^i$ induces an isomorphism $\mathcal{I}^{i+1}(s)_Z \to \mathcal{I}^i(s)_Z$ for every $i \geq 0$. Thus, our claim holds for $l = 0$. Now assume our claim holds for l. Let i, j, k be integers with $i \geq j \geq k \geq 0$ and

$j - k > l + 1$. For every $m \geq 0$, there is a natural and functorial isomorphism $H_0 \cong \pi^m H_m$ for every coherent sheaf $\mathcal{H}$ on X. Therefore, we obtain a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_0^i & \longrightarrow & \mathcal{I}_{l+1}^i & \longrightarrow & \mathcal{I}_l^i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_0^j & \longrightarrow & \mathcal{I}_{l+1}^j & \longrightarrow & \mathcal{I}_l^j & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_0^{k+1} & \longrightarrow & \mathcal{I}_{l+1}^{k+1} & \longrightarrow & \mathcal{I}_l^{k+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_0^k & \longrightarrow & \mathcal{I}_{l+1}^k & \longrightarrow & \mathcal{I}_l^k & \longrightarrow & 0 \end{array} \tag{3.3}$$

with exact rows. By induction hypothesis, $\mathrm{im}(\mathcal{I}_l^i \to \mathcal{I}_l^{k+1}) = \mathrm{im}(\mathcal{I}_l^j \to \mathcal{I}_l^{k+1})$ and $\mathrm{im}(\mathcal{I}_0^i \to \mathcal{I}_0^k) = \mathrm{im}(\mathcal{I}_0^{k+1} \to \mathcal{I}_0^k)$. Chasing the above diagram, we get

$$\mathrm{im}(\mathcal{I}_{l+1}^i \to \mathcal{I}_{l+1}^k) = \mathrm{im}(\mathcal{I}_{l+1}^j \to \mathcal{I}_{l+1}^k)$$

and thus the first claim $(*)$ holds.

Now, for every $i \geq 0$, set $F_i := \mathrm{coker}(\mathcal{I}_i^{i+1} \to \mathcal{I}_i)$. We claim that

$$F_i|_{X_j} = F_j \qquad \text{if } i \geq j \geq 0 \quad (\dagger)$$

. For this, consider the natural commutative diagram,

$$\begin{array}{ccccccc} \mathcal{I}_i^{i+1} & \longrightarrow & \mathcal{I}_i^{j+1} & \longrightarrow & \mathcal{I}_i & \longrightarrow & F_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_j^{i+1} & \longrightarrow & \mathcal{I}_j^{j+1} & \longrightarrow & \mathcal{I}_j & \longrightarrow & F_j \end{array} \tag{3.4}$$

Of course, $F_i \to F_j$ is surjective. By our first claim, the images of $\mathcal{I}^{i+1}$ and $\mathcal{I}^{j+1}$ in $\mathcal{I}_j$ are equal. Chasing diagram (3.4), we get that $\ker(\mathcal{I}_i \to \mathcal{I}_j) \to \ker(F_i \to F_j)$ is surjective. It follows that $\ker(F_i \to F_j) = \pi^{j+1} F_i$, proving our second claim.

For integers i, j with $i, j \geq 0$, let $\mu_i^j : F_i \to F_i$ denote the multiplication-by-π^j map. We claim now that F_i is S_i-flat for every $i \geq 0$. Indeed, we argue by induction on i. If $i = 0$, there is nothing to prove. Let i, j be integers with $i \geq j \geq 1$. We need to show that $\ker(\mu_i^j) = \pi^{i+1-j} F_i$. Assume F_{i-1} is S_{i-1}-flat. So, $\ker(\mu_{i-1}^j) = \pi^{i-j} F_{i-1}$. It follows that $\ker(\mu_i^j) \subseteq \pi^{i-j} F_i$. Thus, we only need to show that

$$\ker(\mu_i^1) = \pi F_i \qquad (\ddagger)$$

for every $i \geq 0$.

Let $U \subset X$ be an affine open subset and $\tau \in \mathcal{I}(U)$. For every $i \geq 0$ let $\tau_i \in I_i(U)$ denote the restriction of τ, and $\overline{\tau_i}$ be the class of τ_i in $F_i(U)$. Assume $\overline{\tau_i} \in \ker(\mu_i^1)$. So, there is $\mu \in \mathcal{I}^{i+1}(U)$ such that $\pi^i\tau - \mu \in \pi^{i+1}\mathcal{I}(U)$. Since $\pi^{i+1}\mathcal{I} \subset \mathcal{I}^{i+1}$, we have $\pi^i\tau \in \mathcal{I}^{i+1}(U)$. From (1) of Lemma 3.2 we get that

$$\pi\mathcal{I}^j \cap \mathcal{I}^{j+2} = \pi\mathcal{I}^{j+1} \tag{3.5}$$

for every $j \geq 0$. Since π is a non-zero-divisor in $\mathcal{I}^j$ for every $j \geq 0$, applying (3.5) repeatedly, we get that $\pi^i\mathcal{I} \cap \mathcal{I}^{i+1} = \pi^i\mathcal{I}^1$. So $\tau \in \mathcal{I}^1(U)$. Since $\mu_0 \in \operatorname{im}(\mathcal{I}_1^{i+1} \to \mathcal{I}_0)$ as well. Thus there is $\gamma \in \mathcal{I}^{i+1}(U)$ such that $\tau - \gamma \in \pi\mathcal{I}$. So, $\overline{\tau_i} \in \pi F_i$, finishing the proof of (‡) and hence (†). Then by Grothendieck's existence theorem [14, III-1, 5.1.7], since R is complete, there is a quotient $\mathcal{F}$ of $\mathcal{I}$ on X such that $\mathcal{F}$ is the inverse limit of the F_i. Since each F_i is S_i-flat, $\mathcal{F}$ is S-flat. Moreover, $\mathcal{F}(s) = F_0 = \mathcal{I}(s)_Y$. □

Lemma 3.4. *Let $\mathcal{I}$ be a semi-stable sheaf on X/S and $Y \subseteq X(s)$ be a subcurve. Then $\mathcal{I}^Y$ is semi-stable on X/S if and only if $\mathcal{I}(s)_Y$ is semi-stable with respect to $\mathcal{E}(s)|_Y$. Moreover, in this case, $\mathcal{I}^Y(s)$ is JH-equivalent to $\mathcal{I}(s)$.*

Proof. Firstly, $\mathcal{I}^Y$ is semi-stable over S if and only if $\mathcal{I}^Y(s)$ is semi-stable. Let $Z := Y^c$. If $\mathcal{I}^Y(s)$ is semi-stable, then so is $\mathcal{I}(s)_Y$ since $\mathcal{I}(s)_Y = \operatorname{coker}(\mathcal{I}^Y(s) \to \mathcal{I}(s))$ by the short exact sequence in equation (3.1). Secondly, if $\mathcal{I}(s)_Y$ is semi-stable, then so is $\mathcal{I}^Y(s)_Z$, again by equation (3.1). Now, since by the short exact sequence in equation (3.2), $\mathcal{I}^Y(s)$ is an extension of semi-stable sheaves, it is semi-stable as well. In which case,

$$\operatorname{Gr}(\mathcal{I}^Y(s)) \cong \operatorname{Gr}(\mathcal{I}(s)_Y) \oplus \operatorname{Gr}(\mathcal{I}^Y(s)_Z) \cong \operatorname{Gr}(\mathcal{I}(s))$$

□

Lemma 3.5. *$\mathcal{I}$ be a torsion-free, rank-1 sheaf on X/S. Let $Y \subseteq X(s)$ be a subcurve such that $\mathcal{I}(s)$ decomposes at Y. Then for any subcurve $Z \subseteq X$, if $\mathcal{I}^Y(s)$ decomposes at Z, then so does $\mathcal{I}(s)$.*

Proof. Restricting the second exact sequence (3.2) to Z, and discarding the torsion part, we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(s) & \longrightarrow & \mathcal{I}^Y(s) & \longrightarrow & \mathcal{I}^Y(s)_{Y^c} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}(s)_{Y\wedge Z} & \longrightarrow & \mathcal{I}^Y(s)_Z & \longrightarrow & \mathcal{I}^Y(s)_{Y^c\wedge Z} & \longrightarrow & 0 \end{array} \tag{3.6}$$

where the second row is exact at the middle term. Combining the above diagram with that of Z^c, we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(s)_Y & \longrightarrow & \mathcal{I}^Y(s) & \longrightarrow & \mathcal{I}^Y(s)_{Y^c} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J' & \longrightarrow & J & \longrightarrow & J'' & \longrightarrow & 0 \end{array} \tag{3.7}$$

where $J := \mathcal{I}^Y(s)_Z \oplus \mathcal{I}^Y(s)_{Z^c}$, $J' := \mathcal{I}(s)_{Y\wedge Z} \oplus \mathcal{I}(s)_{Y\wedge Z^c}$, and $J'' := \mathcal{I}^Y(s)_{Y^c\wedge Z} \oplus \mathcal{I}^Y(s)_{Y^c\wedge Z^c}$

Now assume that $\mathcal{I}^Y(s)$ decomposes at Z. Then applying the snake lemma to the commutative diagram (3.7), it follows that $\mathcal{I}(s)_Y$ decomposes at $Y \wedge Z$ and $\mathcal{I}^Y(s)_{Y^c}$ decomposes at $Y^c \wedge Z$. Since $\mathcal{I}(s)$ decomposes at Y, the sequence

$$0 \longrightarrow \mathcal{I}^Y(s)_Z \longrightarrow \mathcal{I}(s) \longrightarrow \mathcal{I}(s)_Y \longrightarrow 0$$

splits, and so the natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^Y(s)_{Y^c} & \longrightarrow & \mathcal{I}(s) & \longrightarrow & \mathcal{I}(s)_Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J'' & \longrightarrow & \mathcal{I}(s)_Z \oplus \mathcal{I}(s)_{Z^c} & \longrightarrow & J' & \longrightarrow & 0 \end{array}$$

has exact rows. Now since $\mathcal{I}(s)_Y = J'$ and $\mathcal{I}^Y(s)_{Y^c} = J''$, we get that $\mathcal{I}(s)$ decomposes at Z.

□

Definition 3.6. Let $\mathcal{I}$ be a torsion-free, rank-1 sheaf on X/S. If $Y, Z \subseteq X(s)$ are subcurves with $Y \wedge Z = \emptyset$, define

$$\delta_{\mathcal{I}}(Y, Z) := \chi(\mathcal{I}_Y) + \chi(\mathcal{I}_Z) - \chi(\mathcal{I}_{Y\cup Z}).$$

If $Z' \subseteq Z$ is a subcurve, then $\delta_{\mathcal{I}}(Y, Z') \leq \delta_{\mathcal{I}}(Y, Z)$ by Lemma 2.7. In particular, $\delta_{\mathcal{I}}(Y, Z) \geq 0$ where equality holds if and only if $\mathcal{I}_{Y\cup Z} = \mathcal{I}_Y \oplus \mathcal{I}_Z$.

Lemma 3.7. *$\mathcal{I}$ be a torsion-free, rank-1 sheaf on X/S with $\chi(\mathcal{I}/S) = d$. Let $Y, Z \subseteq X(s)$ be subcurves. Then*

$$\beta_{\mathcal{I}^Y(s)}(Z) + \beta_{\mathcal{I}(s)}(Y) \geq \beta_{\mathcal{I}(s)}(Y \wedge Z) + \beta_{\mathcal{I}(s)}(Y \cup Z)$$

with equality if and only if $\delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c) = \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c \wedge Z)$.

Proof. Restricting the exact sequences in Lemma 3.2.2 to $Y \cup Z$ and $Y^c \cup Z$ respectively, and removing torsion, we get exact sequences

$$0 \to \mathcal{I}^Y(s)_{Y^c\wedge Z} \to \mathcal{I}(s)_{Y\cup Z} \to \mathcal{I}(s)_Y \to 0,$$
$$0 \to \mathcal{I}(s)_{Y\wedge Z} \to \mathcal{I}^Y(s)_{Y^c\cup Z} \to \mathcal{I}^Y(s)_{Y^c} \to 0.$$

It follows from the above exact sequences that

$$\beta_{\mathcal{I}^Y(s)}(Y^c \wedge Z) = \beta_{\mathcal{I}(s)}(Y^c \wedge Z) - \delta_{\mathcal{I}(s)}(Y^c \wedge Z, Y),$$
$$\beta_{\mathcal{I}^Y(s)}(Y \wedge Z) = \beta_{\mathcal{I}(s)}(Y \wedge Z) + \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c).$$

Hence,

$$\begin{aligned}\beta_{\mathcal{I}^Y(s)}(Z) &= \beta_{\mathcal{I}^Y(s)}(Y \wedge Z) + \beta_{\mathcal{I}^Y(s)}(Y^c \wedge Z) - \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c \wedge Z)\\ &= \beta_{\mathcal{I}(s)}(Y \wedge Z) + \beta_{\mathcal{I}(s)}(Y^c \wedge Z) - \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c \wedge Z)\\ &\quad + \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c) - \delta_{\mathcal{I}(s)}(Y^c \wedge Z, Y)\\ &\geq \beta_{\mathcal{I}(s)}(Y \wedge Z) + \beta_{\mathcal{I}(s)}(Y^c \wedge Z) - \delta_{\mathcal{I}(s)}(Y^c \wedge Z, Y)\\ &= \beta_{\mathcal{I}(s)}(Y \wedge Z) - \beta_{\mathcal{I}(s)}(Y) + \beta_{\mathcal{I}(s)}(Y \cup Z),\end{aligned}$$

with equality if and only if $\delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c) = \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c \wedge Z)$. □

Lemma 3.8. *In the setting of the previous lemma, let $Y \subseteq X(s)$ be the maximal subcurve among the subcurves $W \subseteq X(s)$ with minimal $\beta_{\mathcal{I}(s)}(W)$. Then $\beta_{\mathcal{I}^Y(s)}(Z) \geq \beta_{\mathcal{I}(s)}(Y)$ with equality only if $Z \subseteq Y$. Moreover, $\beta_{\mathcal{I}^Y(s)}(Y) = \beta_{\mathcal{I}(s)}(Y)$ if and only if $\mathcal{I}^Y(s)$ decomposes at Y.*

Proof. Let $Z \subseteq X(s)$ be a subcurve. Since $\beta_{\mathcal{I}(s)}(Y)$ is minimal, by Lemma 3.7,

$$\beta_{\mathcal{I}^Y(s)}(Z) \geq \beta_{\mathcal{I}(s)}(Y \wedge Z) - \beta_{\mathcal{I}(s)}(Y) + \beta_{\mathcal{I}(s)}(Y \cup Z) \geq \beta_{\mathcal{I}(s)}(Y),$$

with equality if and only if

$$\begin{aligned}\delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c) &= \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c \wedge Z),\\ \beta_{\mathcal{I}(s)}(Y \cup Z) &= \beta_{\mathcal{I}(s)}(Y),\\ \beta_{\mathcal{I}(s)}(Y \wedge Z) &= \beta_{\mathcal{I}(s)}(Y).\end{aligned}$$

Since Y is maximal among the subcurves $W \subseteq X(s)$ with minimal $\beta_{\mathcal{I}(s)}(W)$, the middle equality above occurs if and only if $Z \subseteq Y$. The first statement of the lemma is proved. If $Z = Y$, then the last two equalities above are obviously satisfied, whereas the first equality is satisfied if and only if $\delta_{\mathcal{I}^Y(s)}(Y, Y^c) = 0$. □

Lemma 3.9. *$\mathcal{I}$ be a semi-stable sheaf on X/S with respect to $\mathcal{E}$. Fix an irreducible component $W \subseteq X(s)$, and $Y \subseteq X(s)$ be the minimal subcurve containing W such that $\beta_{\mathcal{I}(s)}(Y) = 0$. Then $\mathcal{I}^Y$ is also semi-stable on X/S with respect to $\mathcal{E}$. Moreover, if $Z \subseteq X(s)$ is the minimal subcurve containing W such that $\beta_{\mathcal{I}^Y(s)}(Z) = 0$, then $Y \subseteq Z$ with equality if and only if $\mathcal{I}^Y(s)$ decomposes at Y.*

Proof. The first statement follows directly from Lemma 3.4. As for the second one, since $\mathcal{I}(s)$ is semi-stable, $\beta_{\mathcal{I}(s)}(Y) = 0$ and $\beta_{\mathcal{I}^Y(s)}(Z) = 0$, by Lemma 3.7,

$$0 = \beta_{\mathcal{I}^Y(s)}(Z) \geq \beta_{\mathcal{I}(s)}(Y \wedge Z) + \beta_{\mathcal{I}(s)}(Y \cup Z) \geq 0.$$

So, $\beta_{\mathcal{I}(s)}(Y \wedge Z) = 0$ and, by Lemma 3.7 again,

$$\delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c) = \delta_{\mathcal{I}^Y(s)}(Y \wedge Z, Y^c \wedge Z). \tag{3.8}$$

Since $Z \supseteq W$, and Y is the minimal subcurve containing W with $\beta_{\mathcal{I}(s)}(Y) = 0$, it follows that $Z \supseteq Y$. The rest of the second statement follows now from equation (3.8). □

Theorem 3.10. *I_η be a torsion-free, rank-1 sheaf on $X(\eta)$. Then the following statements hold.*

1. *There exists an extension $\mathcal{I}$ of I_η which is a torsion-free, rank-1 sheaf on S.*
2. *If I_η is simple, then there is an extension $\mathcal{I}$ of I_η which is simple over S.*
3. *If I_η is (simple and) semi-stable with respect to $\mathcal{E}(\eta)$, then there is an extension $\mathcal{I}$ of I_η that is (simple and) semi-stable over S with respect to $\mathcal{E}$*
4. *Let $\sigma\colon S \to X$ be a section through the smooth locus of X/S. If I_η is $\sigma(\eta)$-quasi-stable with respect to $\mathcal{E}(\eta)$, then there is an extension $\mathcal{I}$ of $I(\eta)$ which is σ-quasi-stable over S with respect to $\mathcal{E}$.*
5. *If I_η is quasi-stable with respect to $\mathcal{E}(\eta)$, then there is an extension $\mathcal{I}$ of I_η that is quasi-stable over S with respect to $\mathcal{E}$.*

Proof. Statement (1) follows immediately from the same argument used in Lemma 7.8 (i) in [AK80, p. 100].

We prove (2) now. By (1), we may pick an extension $\mathcal{I}$ of I_η. If $\mathcal{I}(s)$ is simple, then we are done. If not, it follows from Prop. 1 that there is a non-empty, proper subcurve $Z \subsetneq X(s)$ such that $\mathcal{I}(s)$ decomposes at Z. In this case, let

$$\mathcal{I}^1 := \ker(\mathcal{I} \to \mathcal{I}(s)_Z) \quad \text{and} \quad \mathcal{I}^{-1} := \ker(\mathcal{I} \to \mathcal{I}(s)_{Z^c}).$$

By Lemma 3.5, the set $\mathcal{C}^1$ (resp. $\mathcal{C}^{-1}$) of subcurves $Y \subseteq X(s)$ such that $\mathcal{I}^1(s)$ (resp. $\mathcal{I}^{-1}(s)$) decomposes at Y is contained in the set $\mathcal{C}$ of subcurves $Y \subseteq X(s)$ such that $\mathcal{I}(s)$ decomposes at Y. If C^1 (or C^{-1}) is strictly contained in C, then we replace $\mathcal{I}$ by $\mathcal{I}^1$ (or $\mathcal{I}^{-1}$) and start the above procedure again, but now with a "better" extension. If not, then both $\mathcal{I}^1(s)$ and $\mathcal{I}^{-1}(s)$ decompose at Z. In this case, let

$$\mathcal{I}^2 := \ker(\mathcal{I}^1 \to \mathcal{I}^1(s)_Z) \quad \text{and} \quad \mathcal{I}^{-2} := \ker(\mathcal{I}^{-1} \to \mathcal{I}^{-1}_{Z^c}),$$

and apply the argument used above for $\mathcal{I}^1$ and $\mathcal{I}^{-1}$ to both $\mathcal{I}^2$ and $\mathcal{I}^{-2}$. Applying the above procedure repeatedly, it is clear that we either obtain an extension $\mathcal{I}$ of $\mathcal{I}_n$ that is relatively simple over S, or we end up with two infinite filtrations of a certain extension $\mathcal{I}$,

$$\cdots \subseteq \mathcal{I}^i \subseteq \cdots \subseteq \mathcal{I}^1 \subseteq \mathcal{I}^0 := \mathcal{I},$$
$$\cdots \subseteq \mathcal{I}^{-i} \subseteq \cdots \subseteq \mathcal{I}^{-1} \subseteq \mathcal{I}^0 := \mathcal{I},$$

with quotients

$$\frac{\mathcal{I}^i}{\mathcal{I}^{i+1}} = \mathcal{I}^i(s)_Z \quad \text{and} \quad \frac{\mathcal{I}^{-i}}{\mathcal{I}^{-i-1}} = \mathcal{I}^{-i}(s)_{Z^c}$$

for $i \geq 0$, where $Z \nsubseteq X(s)$ is a non-empty, proper subcurve such that $\mathcal{I}^i(s)$ decomposes at Z for every integer i. We will show by contradiction that the latter situation is not possible. We may assume that R is complete. (If not, just extend the sheaves $\mathcal{I}^i$ over the completion of R.) By Lemma 3.3, there are S-flat quotients $\mathcal{F}$ and $\mathcal{G}$ of

$\mathcal{I}$ such that $\mathcal{F}(s) = \mathcal{I}(s)_Z$ and $\mathcal{G}(s) = \mathcal{I}(s)_{Z^c}$. Consider the induced homomorphism $\phi : \mathcal{I} \to \mathcal{F} \oplus \mathcal{G}$. By assumption, $\phi(s)$ is an isomorphism. Since being an isomorphism is an open property, then ϕ is an isomorphism. Thus, $\mathcal{I}_\eta \cong \mathcal{I}(\eta)$ is not simple, a contradiction. The proof of (2) is complete.

To prove (3), note that by (1), we may pick an extension $\mathcal{I}$ of $\mathcal{I}_n$. Consider the infinite filtration,

$$\cdots \subseteq \mathcal{I}^i \subseteq \cdots \subseteq \mathcal{I}^1 \subseteq \mathcal{I}^0 := \mathcal{I},$$

with quotients

$$\frac{\mathcal{I}^i}{\mathcal{I}^{i+1}} = \mathcal{I}^i(s)_{Z_i},$$

where $Z_i \subseteq X(s)$ is the maximal subcurve among the subcurves $W \subseteq X(s)$ with minimal $\beta_{\mathcal{I}^i(s)}(W)$, for each $i \geq 0$. We claim that $\mathcal{I}^i(s)$ is semistable with respect to $\mathcal{E}(s)$ for some $i \geq 0$. Suppose by contradiction that our claim is false. We may assume that R is complete. (If not, just extend $\mathcal{E}$ and the sheaves $\mathcal{I}^i$ over the completion of R.) Since $\mathcal{I}^i(s)$ is not semistable, then Z_i is a non-empty, proper subcurve of $X(s)$ with $\beta_{\mathcal{I}^i(s)}(Z_i) < 0$, for every $i \geq 0$. By Lemma 3.2, we may assume that both Z_i and $\beta_{\mathcal{I}^i(s)}(Z_i)$ do not depend on i, and $\mathcal{I}^i(s)$ decomposes at Z_i for every $i \geq 0$. Let $Z := Z_i$ and $\beta := \beta_{\mathcal{I}^i(s)}(Z_i)$ for every $i \geq 0$. By Lemma 3.3, there is an S-flat quotient $\mathcal{F}$ of $\mathcal{I}$ such that $\mathcal{F}(s) = \mathcal{I}(s)_Z$. Since $\mathcal{F}$ is S-flat and $\chi(\mathcal{F}(s) \otimes \mathcal{E}(s)) = r\beta < 0$, then also $\chi(\mathcal{F}(\eta) \otimes \mathcal{E}(\eta)) < 0$. Thus, $\mathcal{I}_\eta \cong \mathcal{I}(\eta)$ is not semistable with respect to $\mathcal{E}(\eta)$. This contradiction shows that there is an extension $\mathcal{I}$ of $\mathcal{I}_n$ that is relatively semistable with respect to $\mathcal{E}$. Suppose now that $\mathcal{I}_n$ is simple. It is not necessarily true that $\mathcal{I}$ is relatively simple. Nevertheless, we can apply the construction in the proof of (2) to $\mathcal{I}$ to get a relatively simple sheaf that will still be relatively semistable with respect to $\mathcal{E}$, by Lemma 3.4. The proof of (3) is complete.

We prove (4) now. By (3), there is a relatively semistable sheaf $\mathcal{I}$ on X over S with respect to $\mathcal{E}$ such that $\mathcal{I}(\eta) \cong \mathcal{I}_n$. Consider the infinite filtration

$$\cdots \subseteq \mathcal{I}^i \subseteq \cdots \subseteq \mathcal{I}^1 \subseteq \mathcal{I}^0 := \mathcal{I},$$

with quotients

$$\frac{\mathcal{I}^i}{\mathcal{I}^{i+1}} = \mathcal{I}^i(s)Z_i,$$

where $Z_i \subseteq X(s)$ is the minimal subcurve of $X(s)$ containing $\sigma(s)$ such that $\beta_{\mathcal{I}^i(s)}(Z_i) = 0$, for $i \geq 0$. We claim that $\mathcal{I}^i(s)$ is $\sigma(s)$-quasistable with respect to $\mathcal{E}(s)$ for some $i \geq 0$. In fact, it follows from Lemma 3.9 that

$$Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_i \subseteq \cdots .$$

Thus, up to replacing $\mathcal{I}$ with $\mathcal{I}^j$ for some j we may assume that Z_i does not depend on i. Let $Z := Z_i$ for every $i \geq 0$. We will show that $Z = X(s)$. It follows from Lemma 3.9 that $\mathcal{I}^i(s)$ decomposes at Z for every $i \geq 0$. We may now assume that R is complete. (If not, just extend σ, $\mathcal{E}$ and the sheaves $\mathcal{I}^i$ over the completion of R.) By Lemma 3.3, there is an S-flat quotient $\mathcal{F}$ of $\mathcal{I}$ such that $\mathcal{F}(s) = \mathcal{I}(s)_Z$. Since $\beta_{\mathcal{I}(s)}(Z) = 0$, and $\mathcal{F}$ is S-flat, then $\chi(\mathcal{F}(\eta) \otimes \mathcal{E}(\eta)) = 0$.

Suppose now that $Y \subseteq X(\eta)$ is the maximal subcurve of $X(\eta)$ contained in the support of $\mathcal{F}(\eta)$. Since $\chi(\mathcal{F}(\eta) \otimes \mathcal{E}(\eta)) = 0$, and $\mathcal{I}(\eta)$ is semistable, then $\mathcal{F}(\eta) = \mathcal{I}(\eta)$ and $\beta_{\mathcal{I}(\eta)}(Y) = 0$. It follows that $\mathcal{F}$ is relatively torsion-free on X over S. So, $\mathcal{F}^*$ is free. Since $\sigma^*\mathcal{F}(s) \neq 0$, then also $\sigma^*\mathcal{F}(\eta) \neq 0$. Thus, Y contains $\sigma(\eta)$. Since $\mathcal{I}(\eta) \cong \mathcal{I}_n$ is $\sigma(\eta)$-quasistable, then $Y = X(\eta)$. It follows that $\mathcal{F}(\eta) = \mathcal{I}(\eta)$, and thus $\mathcal{F}(s) = \mathcal{I}(s)$. So, $Z = X(s)$, and thus $\mathcal{I}^i(s)$ is $\sigma(s)$-quasi-stable. The proof of (4) is complete. The proof of (5) consists of applying n times the argument in the proof of (4), where n is the number of irreducible components of $X(s)$.

□

Remark. Note that the previous theorem is not just a proof of existence in the sense that it also establishes a method to produce an extension of I_η with the same properties (semi-stability, quasi-stability, etc.) as I_η, given any extension $\mathcal{I}$. We construct a filtration

$$\cdots \subseteq \mathcal{I}^i \subseteq \cdots \subseteq \mathcal{I}^1 \subseteq \mathcal{I}^0 = \mathcal{I}$$

of $\mathcal{I}$ with quotients of the form $\mathcal{I}^i / \mathcal{I}^{i+1} = \mathcal{I}^i(s)_{Y_i}$ where $Y_i \subseteq X(s)$ is a subcurve, suitably chosen for each $i \geq 0$ as described in the proof. Then $\mathcal{I}^i$ for some $i \geq 0$ will be the extension of I_η we needed, with the same nice properties. However, the minimum of such i's will depend on the original extension $\mathcal{I}$.

At this stage we look at a theorem which describes the relative cohomological characterizations of semi-stability and quasi-stability.

Theorem 3.11. *Let T be an S-scheme, $t \in T$, and $m \geq 2$ be a fixed integer. We say that the maps $U \to S$ and $V \to T \times U$ form a neighbourhood of t in T/S if the induced map $V \to T$ contains t in its image. For short, we say that V/U is a neighbourhood of t in T/S. The neighborhood is étale if $U \to S$ and $V \to T \times U$ are étale.*

1. *$\mathcal{I}$ be a torsion-free, rank-1 sheaf on X_T/T with $\chi(\mathcal{I}/T) = d$. then $\mathcal{I}(t)$ is semi-stable with respect to $\mathcal{E}(t)$ if and only if there is a neighbourhood V/U of t in T/S and a vector bundle $\mathcal{F}$ on X_U of rank mr and $\det \mathcal{F} \cong (\det \mathcal{E}_U)^{\otimes m}$ such that $R^i f_{V*}(\mathcal{I}_V \otimes \mathcal{F}_V) = 0$ for every i.*
2. *$\mathcal{I}$ be a semi-stable sheaf on X_T/T with respect to $\mathcal{E}_T$. Then $\mathcal{I}(t)$ is quasi-stable if and only if there is a neighbourhood V/U of t in T/S, a section $\sigma : U \to X_U$ through the smooth locus of X_U/U, and a vector bundle $\mathcal{F}$ on X_U of rank mr and $\det \mathcal{F} \cong (\det \mathcal{E}_U)^{\otimes m} \otimes \mathcal{O}_{X_U}(-\sigma(U))$ such that*
   - *$f_{V*}(\mathcal{I}_V \otimes \mathcal{F}_V) = 0$ and $R^1 f_{V*}(\mathcal{I}_V \otimes \mathcal{F}_V)$ is invertible.*
   - *the natural map $\mathcal{I}_V \to \mathcal{F}_V^* \otimes \omega_V \otimes f_V^* R^1 f_{V*}(\mathcal{I}_V \otimes \mathcal{F}_V)$ is injective with V-flat cokernel*

In both cases, one can take the neighborhood V/U of t in T/S to be étale.

Similar characterizations exist for σ-quasi-stability, and stability too. (§2 [Est01])

Proposition 3.12. *The subspaces* $J^s_{\mathcal{E}}, J^g_{\mathcal{E}}, J^{qs}_{\mathcal{E}}, J^{ss}_{\mathcal{E}} \subseteq J_d$ *are open.*

Proof. Let T be an S-scheme and $\mathcal{I}$ a torsion-free, rank-1 sheaf on X_T/T with $\chi(\mathcal{I}/T) = d$. Suppose there is $t \in T$ such that $\mathcal{I}(t)$ is semi-stable with respect to $E(t)$. By Theorem 3.12, there are an étale map $h : V \to T$ containing t in its image and a vector bundle $\mathcal{F}$ on X_V of rank $2r$ and $\det \mathcal{F} \cong (\det \mathcal{E}_V)^{\otimes 2}$ such that $R^1 f_{V*}(\mathcal{I}_V \otimes \mathcal{F}) = 0$. Let $U := h(V)$. Since h is étale , $U \subseteq T$ is open. By Theorem 3.11.1, $\mathcal{I}_U$ is semi-stable on X_U/U with respect to $\mathcal{E}_U$. So $J^{\mathrm{ss}}_{\mathcal{E}}$ is open. Similarly, using Theorem 3.11.2, we can show that $J^{\mathrm{qs}}_{\mathcal{E}}$ is also open. □

Now before we prove Theorem 3.1, we define the notion of "r-regularity" for a coherent sheaf over a scheme X which will be used in the proof.

Definition 3.13. $\mathcal{F}$ be a coherent sheaf over a scheme X. $\mathcal{F}$ is said to be *r-regular* if

$$H^i(X, \mathcal{F}(r-i)) = 0$$

for every $i > 0$.

Proof of Theorem 3.1. We may assume that S is Noetherian. Fix an ample sheaf $\mathcal{O}_X(1)$ on X/S. For every integer m, let $\Sigma_m \subseteq J^{\mathrm{ss}}_{\mathcal{E}}$ denote the open subspace parameterizing m-regular sheaves on X/S with respect to $\mathcal{O}_X(1)$. The subspaces Σ_m cover $\overline{J^{\mathrm{ss}}_{\mathcal{E}}}$ and are of finite type over S by [Alt-Klei, 7.3, 7.4], so it suffices to show that $\Sigma_m = \overline{J^{\mathrm{ss}}_{\mathcal{E}}}$ for some m. Since S is Noetherian, there is an integer N such that $\chi(\omega(k)_Y) \leq N$ and $\deg \mathcal{E}(k)|_Y \leq N$ for every $s \in S$, every field extension $k \supseteq k(s)$ and every subcurve $Y \subseteq X(k)$. Let $s \in S$, consider a field extension $k \supseteq k(s)$ such that $X(k) =: C$ has geometrically integral irreducible components. Let I be a semi-stable sheaf on C with respect to $\mathcal{E}(k) =: E$. Then for every integer m, and every subcurve $Y \subseteq C$,

$$\chi(I(m)_Y) = \chi(I_Y) + m \deg \mathcal{O}_C(1)|_Y \geq -(\deg E|_Y)/r + m \geq -N/r + m$$

If $m > N(r+1)/r$ then $\chi(I(m)_Y) > \chi(\omega(k)_Y)$ for every regular subcurve $Y \subseteq C$. By duality, $h^1(C, I(m)) = 0$, and hence I is m-regular with respect to $\mathcal{O}_C(1)$. And thus, $\Sigma_m = \overline{J^{\mathrm{ss}}_{\mathcal{E}}}$ for every $m > M(r+1)/r$. Now since $J^{\mathrm{ss}}_{\mathcal{E}}$ is of finite type over S, using Theorem 3.10, one can conclude that $J^{\mathrm{ss}}_{\mathcal{E}}$, and $J^{\mathrm{qs}}_{\mathcal{E}}$ are universally closed over S. □

Remark: In contrast to earlier relative compactifications mentioned in the introduction, the one by Esteves we presented here admits a Poincaré sheaf, after an étale base change ([Est01, p. 3084]) but proving it requires building the theory of θ-functions and θ-line bundles as in [Est01], [Est97]. The points of this compactification correspond to simple, torsion-free, rank-i sheaves that are semi-stable with respect to a given polarization.

Some Special Cases

For purposes of this section, let X be a connected curve over an algebraic closed field k and let g be the arithmetic genus of X. Furthermore, let J be the algebraic space parameterizing simple, torsion-free, rank-1 sheaves on X.

Example 3.14 (Joining Two Curves)**.** Assume that there exist subcurves $Y, Z \subseteq X$ covering X such that Y and Z intersect transversally at one unique point such that this point is smooth on both Y, and Z. Let J_X, J_Y, and J_Z be the scheme parameterizing simple, torsion-free, rank-1 sheaves on X, Y, and Z respectively. Since every simple, torsion-free, rank-1 sheaf on X must be invertible along $Y \cap Z$, we have a map $J_X \to J_Y \times J_Z$ defined by restriction of sheaves on X to Y and Z. It is easy to show that, in fact, $J_X \xrightarrow{\sim} J_Y \times J_Z$.

Let us now return to the Abel map to study the case of genus 1 curves.

Definition 3.15 (Abel Maps)**.** Let $\delta_X := \min_{Y \subsetneq X} \chi(\mathcal{O}_{Y \cap Y^c})$. If X is irreducible, let $\delta_X := \infty$. Since we assume X to be connected, $\delta_X > 0$.

Let $I \subseteq \mathcal{O}_X$ be the ideal sheaf of a subscheme $D \subset X$ of finite length m, then for each non-empty, proper subcurve $Y \subsetneq X$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_Y \oplus I_{Y^c} & \longrightarrow & \mathcal{O}_Y \oplus \mathcal{O}_{Y^c} & \longrightarrow & \mathcal{O}_{D\cap Y} \oplus \mathcal{O}_{D \cap Y^c} & \longrightarrow & 0 \end{array}$$

has exact rows, and hence by the snake lemma,

$$\chi(I_Y) + \chi(I_{Y^c}) - \chi(I) \geq \chi(\mathcal{O}_{Y \cap Y^c}) - m$$

with equality if and only if $D \subseteq Y \cap Y^c$, and therefore if $m < \delta_X$, then I is simple. Also, if $D = Y \cap Y^c$ for a non-empty, proper subcurve $Y \subsetneq X$, then I is not simple. Therefore, one can say that there are subcurves $D \subset X$ of length δ_X whose ideal sheaves are not simple and δ_X is the minimum length where this occurs.

Now, let $0 \leq m < \delta_X$ and H_m denote the Hilbert scheme of X parameterizing the subschemes of length m of X. Clearly, $H_1 = X$. Let M be an invertible sheaf on X. For every subscheme $D \subseteq X$, let $\mathcal{I}_D \subseteq \mathcal{O}_X$ denote its ideal sheaf. Since $m < \delta_X$, we have a well-defined map

$$\begin{aligned} \alpha_M^m \colon H_M &\longrightarrow J_d \\ [D] &\longmapsto [\mathcal{I}_D \otimes M] \end{aligned}$$

where $d := \chi(M) - m$. α_M^m is called the *Abel map of X in degree m*. If $\delta_X > 1$, set $\alpha_M := \alpha_M^1$. In case X is irreducible, then α_M is a closed embedding [AK80, p. 108]. Furthermore, if X has genus 1 then α_M is an isomorphism ([AK80, p. 109]) which we treat in the next example.

Example 3.16 (Genus 1 Curves)**.** Using notation as defined in the section, and from the previous definition, let $\delta_X > 1$, and $g = 1$. Assume that $g = 1$ and $\delta_X > 1$. We assert that every non-empty, connected, proper subcurve of X has arithmetic genus 0. Specifically, if $Y \subseteq X$ is a subcurve, then $h^1(Y, \mathcal{O}_Y) \leq 1$. Assume equality and that Y is connected. Let $Z := Y^c$. Then $h^1(Z, \mathcal{O}_Z) = 0$ and Z has exactly $\chi(\mathcal{O}_Y \cap Z)$ connected components. Since X is connected, $\chi(\mathcal{O}_{W \cap Y^c}) = \chi(\mathcal{O}_{W \cap Y}) = 1$ for each connected component $W \subseteq Z$. Given $\delta_X > 1$, we have $Y = X$, proving our assertion.

We claim secondly that $\mathcal{O}_X$ is the dualizing sheaf on X. In fact, since $g = 1$, there exists a non-zero map $h : \mathcal{O}_X \to \omega$, where ω is the dualizing sheaf on X. Let $Y \subset X$ be the non-empty subcurve such that $\mathcal{O}_Y \cong \mathrm{im}(h)$ and define $Z := Y^c$. Given that the map $\mathcal{O}_Z \to \omega_Z$ induced by h is zero, h factors through $\Omega \subseteq \omega$, where Ω is the dualizing sheaf on Y. Therefore, $h^1(Y, \mathcal{O}_Y) \geq 1$. Hence, there is a connected component of Y with arithmetic genus 1. By our first assertion, $Y = X$. Since $\chi(\omega) = \chi(\mathcal{O}_X) = 0$ and h is injective, h is an isomorphism, proving our second claim.

Let d be an integer and M an invertible sheaf on X of degree $d+1$. Let $J_M \subseteq J_d$ denote the subset parameterizing simple, torsion-free, rank-1 sheaves I on X such that $\chi(I_Y) \geq \deg M|_Y$ for every non-empty, proper subcurve $Y \subseteq X$.

We claim that J_M is a complete, open subscheme of J_d, and α_M factors through J_M. Indeed, let $p \in X$ be any non-singular point, and put $E := M^* \otimes \mathcal{O}_X(p)$. By definition, a torsion-free, rank-1 sheaf I on X with $\chi(I) = d$ is p-quasi-stable with respect to E if and only if $\chi(I_Y) \geq \deg M|_Y$ for every non-empty, proper subcurve $Y \subseteq X$. Thus, $J_M = J_E^p$. It follows from Theorem 3.1 and Proposition 3.12 that J_M is a complete, open subscheme of J_d.

Furthermore, let $q \in X$ and $Y \subseteq X$ be a connected, proper subcurve. Given that the arithmetic genus of Y is 0, we have $\chi(I_{q,Y}) = 1$ if $q \notin Y$, and $\chi(M_{q,Y}) = 0$ otherwise. In any case, we have $\chi(I_Y) \geq \deg M|_Y$, where $I := I_q \otimes M$. Thus, α_M factors through J_M, proving our third claim. We now show that α_M is an isomorphism onto J_M. Indeed, we will construct the inverse map $\beta_M : J_M \to X$ as follows. Let I be a simple, torsion-free, rank-1 sheaf on X such that $\chi(I) = d$ and

$$\chi(I_Y) \geq \deg M|_Y \tag{3.9}$$

for every non-empty, proper subcurve $Y \subsetneq X$. We claim that

$$h^0(X, I \otimes M^*) = 0, \tag{3.10}$$

and the non-zero map $\lambda\colon I \otimes M^* \to \mathcal{O}_X$ is an isomorphism is an isomorphism onto the ideal sheaf $\mathcal{I}_q$ of a point $q \in X$. Supposing so, let q be the image under β_M of the point of J_M represented by I, then if defined, β_M is clearly the inverse to α_M.

To demonstrate (3.10), consider the map $\mu : \mathcal{O}_X \to I \otimes M^*$. Let $Y \subseteq X$ be the subcurve such that $\mathcal{O}_Y \cong \mathrm{im}(\mu)$. Then μ factors through $J \otimes M^*$, where $J := \ker(I \to I_{Y^c})$. Assume Y is non-empty. Since $\chi(I \otimes M^*) = -1$, it follows from (3.9) that $\chi(J \otimes M^*) \leq -1$. Since μ induces an injection $\mathcal{O}_Y \to J \otimes M^*$, we have $\chi(\mathcal{O}_Y) \leq -1$ as well. On the other hand, $h^1(Y, \mathcal{O}_Y) \leq 1$ because $g = 1$, and hence $\chi(\mathcal{O}_Y) \geq 0$, leading to a contradiction. Thus, $\mu = 0$, proving (3.10).

Given that $\mathcal{O}_X$ is the dualizing sheaf on X by our second assertion and $\chi(I \otimes M^*) = -1$, it follows from (3.10) and duality that there exists a unique (modulo k^*) non-zero map $\lambda : I \otimes M^* \to \mathcal{O}_X$. Let $Y \subseteq X$ be the subcurve such that $I_Y \otimes M^* \cong \text{im}(\lambda)$. If Y is non-empty, it follows from our first assertion that $h^1(Y^c, \mathcal{O}_{Y^c}) = 0$. So, the ideal sheaf I_{Y^c} of Y^c satisfies $\chi(I_{Y^c}) \leq 0$, with equality only if $Y^c = \emptyset$. Hence, $Y = X$. By (3.9), $\chi(I_Y \otimes M^*) \geq -1$ with equality only if $Y = X$. On the other hand, since there is an injection $I_Y \otimes M^* \to I_{Y^c}$, we have $\chi(I_{Y^c}) \geq \chi(I_Y \otimes M^*)$. Hence, $Y = X$ or, in other words, λ is injective. Since $\chi(I \otimes M^*) = -1$, the image of λ is the ideal sheaf of a point, finishing the proof of our last claim.

Note that we have defined β_M as a map of sets, but it is clearly possible to apply the above argument to a family of torsion-free, rank-1 sheaves on X, and this defines β_M as a map of schemes.

Example 3.17 (Two-component curves)**.** [Cap94] Assume X has only two irreducible components, X_1 and X_2. Let δ denote the length of $X_1 \cap X_2$. Let P denote the Jacobian of X. Let $\mathcal{E}$ be a polarization on X of rank r and degree $-rd$, for an integer d. For $i = 1, 2$, let $e_i := -\deg \mathcal{E}|_{X_i}$, and $J^{\text{qs}}_{\mathcal{E}}$ denote the moduli space of X_i-quasi-stable sheaves on X with respect to $\mathcal{E}$. There are two cases:

1. $r \nmid e_i$: In this case, $J^{\text{ss}}_{\mathcal{E}} = J^{\text{ss}}_{E}$, and $J^{\text{qs}}_{\mathcal{E}}$ is complete. A torsion-free, rank-1 sheaf I on X is stable only if $-e_i/r < \chi(I_{X_i}) < -e_i/r + \delta$ for $i = 1, 2$. If I is invertible, then the converse holds. Since there are δ integers in the interval $[-e_i/r, -e_i/r + \delta]$, there are exactly δ connected components of P contained in $J^{\text{ss}}_{\mathcal{E}}$. If X is locally planar, then P is dense in J, and thus $J^{\text{ss}}_{\mathcal{E}}$ has exactly δ irreducible components.

2. $r \mid e_i$: In this case, $J^{\text{ss}}_{\mathcal{E}} \subseteq J^{\text{qs}}_{\mathcal{E}}$, $J^1_{\mathcal{E}} \subset J^{\text{ss}}_{\mathcal{E}}$. Reasoning as in Case 1, if X is locally planar, then $J^{\text{ss}}_{\mathcal{E}}$ has $\delta + 1$ irreducible components, whereas $J^1_{\mathcal{E}}$ and $J^2_{\mathcal{E}}$ have δ components each, and $J^{\text{qs}}_{\mathcal{E}}$ has $\delta - 1$ components. As we observed in Example 5, we have $J^{\text{qs}}_{\mathcal{E}} = J^{\text{ss}}_{\mathcal{E}}$.

Case 1 corresponds to Caporaso's general case , whereas Case 2 corresponds to her special case. [Cap94, p. 646]

Assume now that X_1 and X_2 are smooth and intersect at two ordinary nodes. So, $\delta = 2$ and X is locally planar. Let $P_{d+1} \subset P$ be the open subscheme parameterizing invertible sheaves of Euler characteristic $d + 1$, and consider the (well-defined) map:

$$\alpha : X \times P_{d+1} \to J_d(q, M) \mapsto [I_q \otimes M]$$

One can show that α is surjective, and smooth with relative dimension 1 (cf. [cite]). In Case 1, we have that $J^{\text{ss}}_{\mathcal{E}}$ is the image under α of a connected component of $X \times P_{d+1}$. In Case 2, we have that $J^{\text{ss}}_{\mathcal{E}} = J^1_{\mathcal{E}} \cup J^2_{\mathcal{E}}$, and both $J^1_{\mathcal{E}}$ and $J^2_{\mathcal{E}}$ are images under α of different connected components of $X \times P_{d+1}$. In fact, $J^i_{\mathcal{E}} = \alpha(X \times P^i_{\mathcal{E}})$, where $P^i_{\mathcal{E}}$ is the connected component of P_{d+1} parameterizing invertible sheaves on X with Euler characteristic $e_i/r + 2$ on X_i, for $i = 1, 2$. The patching of $J^1_{\mathcal{E}}$ and $J^2_{\mathcal{E}}$ to produce $J^{\text{ss}}_{\mathcal{E}}$ occurs on $J^{\text{ss}}_{\mathcal{E}}$, which is the image under α of both $(X_1 \setminus X_2) \times P^1_{\mathcal{E}}$ and $(X_2 \setminus X_1) \times P^2_{\mathcal{E}}$.

Chapter 4

Appendix

We take a look at some basic concepts from algebraic geometry. One can find most of the concepts discussed here in all standard textbooks on algebraic geometry like [Har77], [GW20]. We base this section on the Stacks Project [TSP] which is an online open source textbook and reference work on algebraic geometry.

Let X be an S-scheme, and $f\colon X \to S$ be the associated structure morphism.

Definition 4.1. A morphism of schemes is called *quasi-compact* if the underlying map of topological spaces is quasi-compact. We say that a topological space X is quasi-compact if every open covering of X has a finite subcover.

Remark. One may point out that the definition above for quasi-compact topological spaces is taught as *compact* in a first course in topology which is true. However, in algebraic geometry, most texts such as the ones mentioned above follow the terminology introduced by Bourbaki. Most algebraic geometry texts use the term quasi-compact as defined above and compact to mean quasi-compact and Hausdorff. [htt]

Definition 4.2. f is of *finite type* at $x \in X$ if there exists an affine open neighbourhood Spec $(A) = U \subset X$ of x and an affine open Spec $(R) = V \subset S$ such that $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type. Furthermore, f is said to be *locally of finite type* if it is of finite type at every point of X. A morphism which is locally of finite type and quasi-compact is said to be of *finite type.*

Definition 4.3. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say that

1. f is *flat at a point* $x \in X$ if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{S,f(x)}$.
2. $\mathcal{F}$ is *flat over* S *at a point* $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module.
3. f is *flat* if f is flat at every point of X.
4. $\mathcal{F}$ is *flat over* S if $\mathcal{F}$ is flat over S at every point x of X.

We define f to be flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

Definition 4.4. For an S-scheme X, consider the *diagonal morphism* $\Delta_X \colon X \to X \times_S X$ which is the unique scheme morphism such that $\mathrm{pr}_1 \circ \Delta_X = \mathrm{id}_X$ and $\mathrm{pr}_1 \circ \Delta_X = \mathrm{id}_X$. Let $f \colon X \to S$ be the structure morphism. Then we say that

1. f is *separated* if Δ_X is a closed immersion.

2. f is *quasi-separated* if Δ_X is a quasi-compact morphism.

Definition 4.5. A closed morphism $f \colon X \to Y$ of S-schemes is *universally closed* if for every morphism $h \colon Z \to Y$ the pullback $h^*(f) \colon Z \times_Y X \to Z$ is a closed morphism.

Definition 4.6. f is *proper* if it is separated, of finite type, and universally closed. Where universally closed means that the

Remark. To understand the geometric intuition behind the above two definitions, one can think that separatedness, for the case of schemes, is a good analogue of the Hausdorff property of topological spaces. The notion of properness can be thought of as an analogue of compactness.

Definition 4.7. Let X be a scheme. We say X is *integral* if it is nonempty and for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is an integral domain.

Definition 4.8. Let X be a scheme. We say X is reduced if at each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced. Every scheme X has a reduced scheme underlying it, denoted by X_{red}.

Definition 4.9. We say X is *locally Noetherian* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is Noetherian. We say X is Noetherian if X is locally Noetherian and quasi-compact.

Definition 4.10. A scheme X is called locally factorial if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is factorial.

Definition 4.11. $f \colon X \to Y$ be a morphism of algebraic spaces over a scheme S.

1. f is said to satisfy the *uniqueness part of the valuative criterion* if given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array} \tag{4.1}$$

   where A is a valuation ring with field of fractions K, there exists a unique map $\mathrm{Spec}(A) \to X$ making the diagram commute.

2. f is said to satisfy the *existence part of the valuative criterion* if given a diagram like (4.1), there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ which dominates A and a morphism $\mathrm{Spec}(A') \to X$ such that

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array} \tag{4.2}$$

f is said to satisfy the *valuative criterion for properness* if it satisfies both the existence, and the uniqueness parts of the valuative criterion.

Lemma 4.12 (Valuative Criterion for Properness)**.** *$f\colon X \to Y$ be a morphism of S-schemes. Assume f is finite and quasi-separated. Then the following are equivalent*

1. *f is proper.*
2. *f satisfies the valuative criterion as defined in Definition 4.6*
3. *Given any commutative square*

$$\begin{array}{ccc} Spec(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Spec(A) & \longrightarrow & Y \end{array} \tag{4.3}$$

where A is a valuation ring with field of fractions K, there exists a unique map $Spec(A) \to X$ making the diagram commute.

One usually does not need to consider all possible diagrams while testing the valuative criterion, this fact is illustrated in the next lemma.

Lemma 4.13. *$f\colon X \to S$, and $h\colon U \to X$ be morphisms of schemes. Assume that f and h are quasi-compact and $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{array}{ccccc} Spec(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow \\ Spec(A) & & \longrightarrow & & S \end{array} \tag{4.4}$$

where A is a valuation ring and K its quotient field, there exists a unique map $Spec(A) \to X$, then f is universally closed. Moreover, if f is quasi-separated, then f is separated.

Definition 4.14. A *group scheme* over S is a pair (G, m), where G is a scheme over S and $m\colon G \times_S G \to G$ is a morphism of schemes over S such that for every S-scheme T the pair $(G(T), m)$ is a group, i.e., the T-points of G form a group under m.

Definition 4.15. A morphism $\phi\colon (G,m) \to (G',m')$ of group schemes over S is a morphism $\phi\colon G \to G'$ of schemes over S such that for every S-scheme T, the induced map $\phi(T)\colon G(T) \to G'(T)$ is a homomorphism of groups.

Lemma 4.16. *Let k be a field, and G be a group scheme locally of finite type. Let G^0 denote the connected component of the identity element e. Then*

1. *G is separated*
2. *G is smooth if it has a geometrically reduced open subscheme*
3. *G^0 is an open and closed group subscheme of finite type. It is geometrically irreducible and its construction commutes with extending k.*

Proof. The proof follows from standard arguments presented in [GD60], [GD65]. The entire proof can be found in [Kle05]. □

Definition 4.17. X be a proper scheme of dimension n over $S = \mathrm{Spec} k$. The *dualizing sheaf on* X is a coherent sheaf ω_X on X together with a map

$$t_X\colon H^n(X, \omega_X) \to k$$

which induces a natural isomorphism of vector spaces

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee$$

$$\varphi \longmapsto t_X \circ \varphi$$

for every coherent sheaf $\mathcal{F}$ on X.

t_X is called the *trace morphism*. If it exists, the pair (ω_X, t_X) is unique. In the language of categories, ω_X represents the contravariant functor $\mathcal{F} \mapsto H^n(X, \mathcal{F})^\vee$ from the category of coherent sheaves on X and to the category of k-vector spaces.

Definition 4.18 (The Hilbert Functor and Hilbert Scheme)**.** X be a projective scheme over a base scheme S. The *Hilbert functor* $\mathrm{Hilb}_{X/S}$ from the category of S-schemes to sets for any S-scheme T is defined as

$$\mathrm{Hilb}_{X/S}(T) := \{T\text{-flat closed subschemes } Y \text{ of } X \times_S T\}.$$

Moreover, for each polynomial $F \in \mathbb{Q}[\nu]$, $\mathrm{Hilb}_{X/S}$ has a subfuntor $\mathrm{Hilb}^F_{X/S}$. Namely, for all S-schemes T, the set $\mathrm{Hilb}^F_{X/S}(T)$ is the set of Y such that such that Y_t has Hilbert polynomial F for all t, i.e., $F(\nu) = \chi(\mathcal{O}_{Y_t}(\nu))$ where $\mathcal{O}_{Y_t}(\nu)$ is the pullback to Y_t of the νth tensor power of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ for some coherent sheaf $\mathcal{E}$ on S. (Recall the definition of $\mathbf{P}(\mathcal{E})$ from Chapter 1).

Grothendieck proved that $\mathrm{Hilb}_{X/S}$ is representable by a locally Noetherian S-scheme $\mathbf{Hilb}_{X/S}$ called the *Hilbert scheme*. In fact, $\mathbf{Hilb}_{X/S}$ is the disjoint union of projective S-schemes $\mathbf{Hilb}^F_{X/S}$ which represents the subfunctors $\mathrm{Hilb}^F_{X/S}$. Note that $\mathbf{Hilb}^F_{X/S}$ depends on the choice of embedding of X in some $\mathbf{P}(\mathcal{E})$ but $\mathbf{Hilb}_{X/S}$ is independent of this.

The following theorems and their proofs can be found in any standard textbook on algebraic groups. The interested reader may look at [Mil17]

Theorem 4.19 (Structure Theorem for Algebraic Groups (Chevalley))**.** *Let k be a perfect field and G an algebraic group over k. Then there exists a unique normal linear algebraic closed subgroup $H \leqslant G$ for which ${}^{G}/_{H}$ is an abelian variety. That is, there is a unique short exact sequence of algebraic groups*

$$1 \to H \to G \to A \to 1$$

with H linear algebraic and A an abelian variety. The formation of H commutes with base change to an arbitrary perfect field extension over k.

Theorem 4.20 (Lie-Kolchin Theorem)**.** *If G is a connected and solvable linear algebraic group defined over an algebraically closed field and $\rho\colon G \to GL(V)$is a representation on a nonzero finite-dimensional vector space V, then there is a one-dimensional linear subspace L of V such that $\rho(G)(L) = L$. That is, $\rho(G)$ has an invariant line L, on which G therefore acts through a one-dimensional representation. This is equivalent to the statement that V contains a nonzero vector v that is a simultaneous eigenvector for all $\rho(g)$, $g \in G$.*

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