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Effects of disorder on bosonic superfluids

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Introduction

The theory of dilute Bose gas has been studied for a long time and now is a standard condensed-matter physics subject, well known and well understood [1]. However, this field has seen an increased interest since the observation, for the first time, of Bose-Einstein condensation in rubidium-87 atoms in 1995 by E. Cornell and C. Weiman at JILA [2].
The theory, within the framework of path integrals, has also been extended to treat systems that share some properties with Bose gases while being strongly interacting, such as Helium-4 and Helium-3, in which the condensation involves Cooper pairs of two atoms. So far, the field of pure bosonic superfluids has been extensively studied.
However, much less focus and effort has been put in understanding how quenched disorder, even very weak one, affects the correlation functions of a system and ultimately the physical behavior, in particular of quantities such as condensate fraction and normal-fluid particle density. The first efforts in this direction have been in somewhat recent times, starting from the work of Huang and Meng [3].
The problems of the effects of quenched impurities on bosonic systems, apart from the fact that irregularities are an unavoidable feature of real-life systems, is interesting for many reasons. The first one is that disorder can actually be controlled using laser speckles [4]. Moreover it is well known that cold atoms are could be used to simulate and reproduce other physical systems (Feynman's quantum simulator [5]).
Another reason to study disordered Bose fluids is the fact that the formalism can be extended to include the study of magnetic flux lines in type-II superconductors with high critical temperature T_c, exploiting a formal analogy between the dynamic of two dimensional bosons and the dynamic of (2+1) dimensional directed lines [6]. Starting from this analogy it is possible to realize a mapping between the descriptions of the two systems. In this context, the mass m of the bosons can be mapped onto the line tension η, the density n and chemical potential μ onto the magnetic flux density B and externally applied field H. One relevant difference that has to be dealt with while work-

ing with magnetic flux lines is that the periodic boundary condition is too artificial for these systems; therefore we shall impose free boundary condition and, to find averaged values, we shall integrate over them.

So, for the all mentioned reasons, in this work we shall present a review of a modern and general approach to disordered bosonic systems in the path-integral approach. The first chapter shall present basic notions about pure bosonic systems, the definition of the relevant quantities (such as the normal-fluid particle density n_n considered as transport coefficient of a response function) and the discussion of the approximations we shall use (mainly harmonic approximation).

In the second chapter we shall present the formalism that can be used to study disorder and find the impurities contribution to the quantities defined in the first chapter. We shall do this in a general way, not specifying the shape of the disorder self-correlator $\overline{U_D(\mathbf{r},\tau)U_D(\mathbf{r}',\tau')}$, where $U_D(\mathbf{q},\omega)$ is the Fourier transform of the disorder potential $U(\mathbf{r},\tau)$.

In the third chapter we shall present the results for relevant kind of disorders, showing explicitly the results for condensate depletion and normal-fluid particle density, along with leading-order corrections in finite temperature.

In particular, in section 3.3 we shall extend some results of the previous sections to the case of finite-range interaction potential, meaning that $V(q) = g_0 + g_2 q^2 + \mathcal{O}(q^4)$. Moreover, we will also consider the case of a Lorentzian correlator with finite-range interaction. This is an interesting situation because this kind of correlation can be created experimentally through laser speckles. This analysis of finite-range interaction in disordered systems is, at the best of our knowledge, a new result.

Finally, in the fourth chapter we shall briefly show how the formalism can be extended to treat magnetic flux lines, comparing the analogies and differences with the case of real bosonic superfluids.

Chapter 1

Weakly interacting Bose gas

Consider the Hamiltonian for N bosons of mass m, in D spatial dimensions, interacting through a two body potential $V(\mathbf{r})$:

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\boldsymbol{\nabla}_i^2 + \frac{1}{2}\sum_{i\neq j} V(|\mathbf{r_i} - \mathbf{r_j}|) \tag{1.0.1}$$

In order to study the statistical properties of many-particle systems at finite temperature we need to compute the partition function of the system, in the grand canonical ensamble [7]:

$$\mathcal{Z} = Tr[e^{-\beta(H-\mu N)}] = \sum_{\text{bosonic states n}} \langle n|e^{-\beta(H-\mu N)}|n\rangle \tag{1.0.2}$$

where μ is the chemical potential and $\beta = \frac{1}{k_B T}$, where T is the temperature. This trace can be evaluated within the functional integral approach.
This approach uses coherent bosonic fields as dynamical variables and computes the partiton function of the system, $\mathcal{Z}$, via integration on all possible field configurations, with each configuration weighted by a factor proportional to the action.
The fact that we are integrating on all possible configurations means we are taking into account the quantum nature of the system. More specifically we have

$$\mathcal{Z} = \int_{\substack{\psi(\mathbf{r},\beta\hbar)=\psi(\mathbf{r},0),\\ \psi^*(\mathbf{r},\beta\hbar)=\psi^*(\mathbf{r},0)}} \mathcal{D}\psi(\mathbf{r},\tau)\mathcal{D}\psi^*(\mathbf{r},\tau)e^{-\frac{S(\psi,\psi^*)}{\hbar}} \equiv e^{-\beta\Omega} \tag{1.0.3}$$

Where Ω is called grand potential and is the fundamental object to compute physical quantities. The action is given by

$$S = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\mathcal{L} = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\{\hbar\psi^*(\mathbf{r},\tau)\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} + \frac{\hbar^2}{2m}\left|\boldsymbol{\nabla}\psi(\mathbf{r},\tau)\right|^2 - \mu\left|\psi(\mathbf{r},\tau)\right|^2 + \frac{1}{2}\int d^D\mathbf{r}'U(|\mathbf{r}-\mathbf{r}'|)\left|\psi(\mathbf{r},\tau)\right|^2\left|\psi(\mathbf{r}',\tau)\right|^2\} \quad (1.0.4)$$

where $\mathcal{L}$ is the Euclidean lagrangian density and ψ is the complex scalar field such that

$$N = \int d^D\mathbf{r}\left|\psi(\mathbf{r},\tau)\right|^2 \quad (1.0.5)$$

where N is the total number of bosons, and the field is a function of the space coordinates $\mathbf{r}$ in $\mathbb{R}^D$ and imaginary time τ.
This equation can be obtained from Eq (1.0.1) after rewriting the Hamiltonian in second quantization language and then performing Wick rotation to obtain imaginary-time action (for a complete introduction to coherent state path-integral see [8]).
While working within this framework, it is possible to evaluate correlation functions by functional derivatives. In fact, adding a source term to the action we get

$$\mathcal{Z}[J] = \mathcal{Z}[0]\int_{\substack{\psi(\mathbf{r},\beta\hbar)=\psi(\mathbf{r},0),\\ \psi^*(\mathbf{r},\beta\hbar)=\psi^*(\mathbf{r},\tau)}} \mathcal{D}\psi(\mathbf{r},\tau)\mathcal{D}\psi^*(\mathbf{r},\tau)e^{\int d\tau\int d^D\mathbf{r}J(\mathbf{r},\tau)\psi^*(\mathbf{r},\tau)+J(\mathbf{r},\tau)^*\psi(\mathbf{r},\tau)} \quad (1.0.6)$$

which let us write, for example

$$\langle\psi(\mathbf{r},\tau)\rangle \equiv \frac{1}{\mathcal{Z}[J]}\int_{\substack{\psi(\mathbf{r},\beta\hbar)=\psi(\mathbf{r},0),\\ \psi^*(\mathbf{r},\beta\hbar)=\psi^*(\mathbf{r},0)}} \mathcal{D}\psi(\mathbf{r},\tau)\mathcal{D}\psi^*(\mathbf{r},\tau)\psi(\mathbf{r},\tau)e^{-\frac{S(\psi,\psi^*)}{\hbar}} = \frac{\delta\ln\mathcal{Z}[J]}{\delta J(\mathbf{r},\tau)}|_{J=0} \quad (1.0.7)$$

Of course it is, in general, impossible to compute the action in closed form, due to the presence of the two-body potential. There are, however, many interesting cases that can be evaluated, such as the free field case and the contact interaction approximation.
We shall use these easy cases as a device to show how to find relevant physical quantities starting from the partition function sum and to introduce correlation functions and quantities such as superfluid fraction that will be affected

by disorder. Moreover, for simplicity sake, we shall work in the superfluid phase, in which the $U(1)$ simmetry is broken. This allows us to write

$$\psi(\mathbf{r},\tau) = \psi_0 + \eta(\mathbf{r},\tau) \tag{1.0.8}$$

where $\eta(\mathbf{r},\tau)$ describes fluctuations around the order parameter ψ_0, which describes the condensate. We assume that ψ_0 is constant in space and time and real.

1.1 Non interacting Bose gas

In the non interacting Bose gas we assume the interparticle two-body potential to be zero. Therefore we have the following lagrangian density

$$\mathcal{L} = -\mu\psi_0^2 + \eta^*(\mathbf{r},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 - \mu\right]\eta(\mathbf{r},\tau) \tag{1.1.1}$$

We can cast the action into a more manageable by using Fourier decomposition:

$$\eta(\mathbf{r},\tau) = \frac{1}{\sqrt{\hbar\beta\mathcal{V}}}\sum_Q \eta_Q e^{i(\mathbf{qr}-\omega\tau)} \tag{1.1.2}$$

where Q is a short hand notation for a $D{+}1$ vector: $Q = (\mathbf{q}, i\omega_n)$, where $\omega_n = \frac{2\pi n}{\beta\hbar}$ are the so-called bosonic Matsubara frequencies [9] and $\mathcal{V}$ is the volume of the system. These frequencies form a discrete spectrum as a consequence of the periodicity of $\eta(r,\tau)$. The action is then:

$$S = -\mu\psi_0^2\hbar\beta\mathcal{V} + \sum_Q \hbar\lambda_Q\left|\eta_Q\right|^2 \tag{1.1.3}$$

where

$$\lambda_Q = \beta(-i\omega_n + \frac{\hbar^2}{2m}q^2 - \mu). \tag{1.1.4}$$

This allows us to write the partition function as

$$Z = Z_0\int \mathcal{D}[\eta,\eta^*]e^{-\sum_Q \hbar\lambda_Q\left|\eta_Q\right|^2} = \frac{1}{\Pi_Q\lambda_Q} \tag{1.1.5}$$

where Z_0 is the partition function related to the condensate. Note that we were able to carry out the partition function sum because the action is at most quadratic in the bosonic field.
Considering that

$$\sum_{n=-\infty}^{+\infty}\ln(-in + a) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\ln(n^2 + a^2) \tag{1.1.6}$$

We can finally conclude that the grand potential is

$$\Omega = -\frac{1}{\beta}\ln(Z) = -\mu\psi_0^2\mathcal{V} + \frac{1}{2\beta}\sum_{\mathbf{q}}\sum_{n=-\infty}^{n=+\infty}\ln[\beta^2(\hbar^2\omega_n^2 + \xi_q^2)] \tag{1.1.7}$$

where ξ_q is the shifted free-particle energy:

$$\xi_q = \frac{\hbar^2 q^2}{2m} - \mu \tag{1.1.8}$$

To evaluate the summation over Matsubara frequency we can first take the derivative with respect to ξ_q, which gives us (see [10])

$$\frac{1}{\beta}\sum_{n=-\infty}^{n=+\infty}\frac{\xi_q}{\omega_n^2 + \xi_q^2} = \frac{1}{2} + \frac{1}{e^{\beta\xi_q} - 1} \tag{1.1.9}$$

This, finally, leads to

$$\Omega = \Omega_0 + \Omega^{(0)} + \Omega^{(T)} \tag{1.1.10}$$

where the first term $\Omega_0 = -\mu\psi_0^2\mathcal{V}$ is the grand potential of the condensate,

$$\Omega^{(0)} = \frac{1}{2}\sum_{\mathbf{q}}\xi_q \tag{1.1.11}$$

is the zero point energy of the bosonic single-particle excitations and

$$\Omega^{(T)} = \frac{1}{\beta}\sum_{\mathbf{q}}\ln(1 - e^{-\beta\xi_q}) \tag{1.1.12}$$

We still need to carry out the momentum sum. To do this we can consider the continuum limit, in which the integrals take the form:

$$\frac{\Omega^{(0)}}{\mathcal{V}} = \frac{1}{2^D(\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2})}\int_0^\infty dq q^{D-1}\left(\frac{\hbar^2 q^2}{2m} - \mu\right) \tag{1.1.13}$$

This integral is clearly ultraviolet divergent when D=1,2,3. However it can be proven that, for an ideal Bose gas, dimensional regularization completely cancels this term. For this reason, we are left with the following contribution

$$\frac{\Omega}{\mathcal{V}} = -\mu\psi_0 + \frac{1}{\beta(2\pi)^D}\int d^D\mathbf{q}\ln(1 - e^{-\beta\xi_q}) \tag{1.1.14}$$

At this point we can notice what ψ_0 is not a free parameter, but is rather determined by the following relation

$$\left(\frac{\partial\Omega_0}{\partial\psi_0}\right)_{\mu,T,\mathcal{V}} = 0, \tag{1.1.15}$$

which leads to

$$\psi_0 = \begin{cases} 0 & \text{if } \mu \neq 0 \\ \text{any value} & \text{if } \mu = 0 \end{cases} \tag{1.1.16}$$

Remembering the well known thermodynamic relation

$$n = \frac{N}{\mathcal{V}} = -\frac{1}{\mathcal{V}}\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,\mathcal{V},\psi_0} \tag{1.1.17}$$

gives us (after passing to the continuum limit and setting μ to zero, since we are in the condensate phase)

$$n = \psi_0^2 + \int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{1}{e^{\frac{\hbar^2 q^2}{2mk_BT}} - 1} = n_0^2 + \int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{1}{e^{\frac{\hbar^2 q^2}{2mk_BT}} - 1} \tag{1.1.18}$$

From this equation we can find the condensation temperature T_c by setting $\psi_0 = 0$ in the previous equation. This leads to the following well known result [7]

$$k_BT_c = \begin{cases} \frac{1}{2\pi\zeta(\frac{3}{2})^{\frac{2}{3}}}\frac{\hbar^2}{m}n^{\frac{2}{3}} & \text{for D } = 3 \\ 0 & \text{for D } = 2 \\ \text{no solution} & \text{for D } = 1 \end{cases} \tag{1.1.19}$$

The fact that, when D=2, there is no condensation at finite temperature for a non interacting system can be generalized: Mermin and Wagner proved that this statement holds true even for interacting systems, provided they are homogeneous and with short-range interaction. From Eq (1.1.17), taking D=3, we can express che condensate fraction $\frac{n_0}{n}$ as a function of the temperature

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}} \tag{1.1.20}$$

which is the same result derived by Einstein and Bose, as it should be [7].

1.2 Interacting Bose gas in Bogoliubov Approximation

Next, we focus our attention to the simplest non trivial interaction: contact interaction.

This is carried out by considering $V(|\mathbf{r}-\mathbf{r}'|) = g_0\delta(\mathbf{r}-\mathbf{r}')$. Clearly, the Fourier transform of this potential reads:

$$\tilde{V}(q) = g_0 \tag{1.2.1}$$

The constant g_0 can be expressed in terms of measurable quantities through scattering theory. For example, when D=3

$$g_0 = \frac{4\pi\hbar^2 a_s}{m} \tag{1.2.2}$$

where a_s is the s-wave scattering length. This greatly simplifies Eq (1.0.4), leading to the following lagrangian density:

$$\mathcal{L} = \hbar\psi^*(\mathbf{r},\tau)\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} + \frac{\hbar^2}{2m}\left|\boldsymbol{\nabla}\psi(\mathbf{r},\tau)\right|^2 - \mu\left|\psi(\mathbf{r},\tau)\right|^2 + \frac{1}{2}g_0\left|\psi(\mathbf{r},\tau)\right|^4 \tag{1.2.3}$$

Considering the mean-field approximation, we can write the partition function as

$$\mathcal{Z} = e^{-\beta\Omega_0} \tag{1.2.4}$$

where

$$\Omega_0 = \mathcal{V}(-\mu\psi_0^2 + \frac{1}{2}g_0\psi_0^4) \tag{1.2.5}$$

Fig. 1.1 shows the two shapes of the grand potential, according to the value of μ. Minimising the grand potential with respect to the order parameter we find that in the superfluid phase the chemical potential is positive

$$\mu = g_0\psi_0^2 \tag{1.2.6}$$

We can then express the grand potential as a function of μ

$$\Omega_0 = -\frac{\mathcal{V}\mu^2}{2g_0} \tag{1.2.7}$$

In this phase the mean field value of the order parameter is

$$|\psi_0| = \sqrt{\frac{\mu}{g_0}} \tag{1.2.8}$$

which gives us the expression for the condensate particle density

$$n_0 = |\psi_0|^2 = \frac{\mu}{g_0} \tag{1.2.9}$$

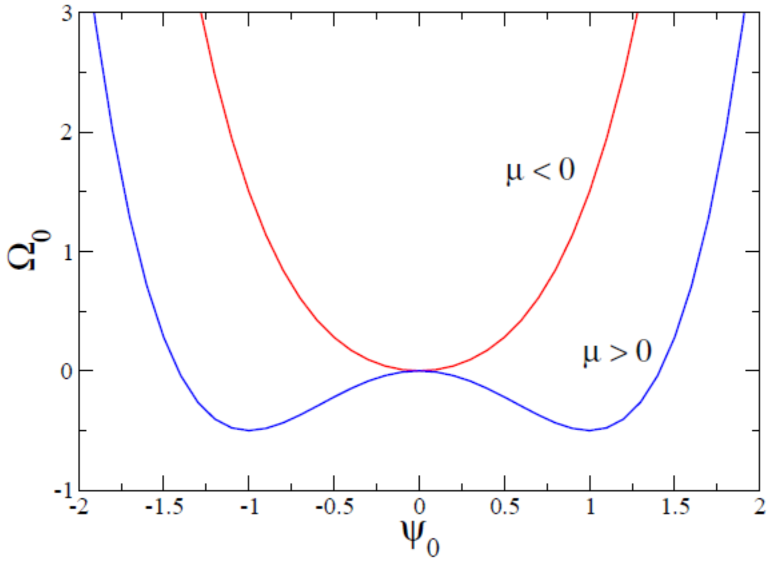

Figure 1.1: Mean-field potential for interacting bosons. We can the effect of the sign of the chemical potential on the shape of Ω_0, see Eq (1.2.5).

The next step is to take into account fluctuations around the mean-field value. To do this we can switch to the polar parametrization, using real fields:

$$\psi(\mathbf{r},\tau) = \sqrt{n_0 + \pi(\mathbf{r},\tau)}e^{i\Theta(\mathbf{r},\tau)} \tag{1.2.10}$$

From this equation follow immediately that the field $\pi(\mathbf{r},\tau)$ describes density fluctuations:

$$n(\mathbf{r},\tau) = \left|\psi(\mathbf{r},\tau)\right|^2 = n_0 + \pi(\mathbf{r},\tau) \tag{1.2.11}$$

where n_0 is the boson order parameter in the superfluid phase in the mean-field approximation. We see that, if we allow the boson field to fluctuate, i.e. not to be constant in space and time, then not all the particles enter the condensate and $n_0 \neq n$.
We can then define the superfluid velocity $\mathbf{v}_s$

$$\mathbf{v}_s(\mathbf{r},\tau) = \hbar\boldsymbol{\nabla}\Theta(\mathbf{r},\tau)/m \tag{1.2.12}$$

and the mass current $\mathbf{g}(\mathbf{r},\tau)$

$$\mathbf{g}(\mathbf{r},\tau) = m\mathbf{j}(\mathbf{r},\tau) = \frac{\hbar}{2i}\left[\psi^*(\mathbf{r},\tau)\boldsymbol{\nabla}\psi(\mathbf{r},\tau) - \psi(\mathbf{r},\tau)\boldsymbol{\nabla}\psi^*(\mathbf{r},\tau)\right] \tag{1.2.13}$$

In the context of superfluid bosons, it is often useful to imagine the system as composed by two kind of fluids (two-fluid model): one is in the superfluid state and the other is the normal fluid.
These two fluids have in general different velocities and particle densities $n_s = \rho_s/m$ and $n_n = \rho_n/m$ such as $n = n_s + n_n$. It is important to evaluate this quantites, since disorder will affect them: we expect that, in presence of impurities, the superfluid fraction will decrease.

We can then rewrite the partition function using the new degrees of freedom:

$$\mathcal{Z} = \int \mathcal{D}\pi(\mathbf{r},\tau)\mathcal{D}\Theta(\mathbf{r},\tau)e^{-S_0[\pi,\Theta]+S_{int}[\pi,\Theta]/\hbar} \tag{1.2.14}$$

The new fields still satisfy periodic boundary conditions. Here S_0 denotes the quadratic or gaussian part of the action:

$$S_0[\pi,\Theta] = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\; i\hbar\pi(\mathbf{r},\tau)\frac{\partial\Theta(\mathbf{r},\tau)}{\partial\tau} + \frac{\hbar^2}{8mn_0}[\boldsymbol{\nabla}\pi(\mathbf{r},\tau)]^2 + \frac{\hbar^2 n_0}{2m}[\boldsymbol{\nabla}\Theta(\mathbf{r},\tau)]^2 + \frac{g_0}{2}\pi(\mathbf{r},\tau)^2 \tag{1.2.15}$$

While $S_{int}[\pi,\Theta]$ is the anharmonic part

$$S_{int}[\pi,\Theta] = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\frac{-\hbar^2}{8mn_0}\frac{\pi(\mathbf{r},\tau)}{n_0+\pi(\mathbf{r},\tau)}[\boldsymbol{\nabla}\pi(\mathbf{r},\tau)]^2 + \frac{\hbar^2}{2m}\pi(\mathbf{r},\tau)[\boldsymbol{\nabla}\Theta(\mathbf{r},\tau)]^2 \tag{1.2.16}$$

The physical meaning behind this splitting is the fact the first part of the action describes single-quasiparticle excitations, while the second one multi-quasiparticle ones [11].
In the Bogoliubov approximation we neglect the interacting part and consider just S_0. By doing this we can derive exact results, since the resulting action is gaussian in the fields.
Passing in the reciprocal space gives us:

$$S_0[\pi,\Theta] = \frac{1}{2\beta\hbar\mathcal{V}}\sum_{\mathbf{q},\omega_n}\Big(\Theta(-\mathbf{q},-\omega_n),\pi(-\mathbf{q},-\omega_n)\Big)\mathbf{A}(\mathbf{q},\omega_n)\begin{pmatrix}\Theta(\mathbf{q},\omega_n)\\ \pi(\mathbf{q},\omega_n)\end{pmatrix} - \frac{1}{2}n_0^2\beta\hbar\mathcal{V}V_0 \tag{1.2.17}$$

where

$$\mathbf{A}(\mathbf{q},\omega_n) = \begin{pmatrix} n_0\hbar^2q^2/m & -\hbar\omega_n \\ \hbar\omega_n & g_0+\hbar^2q^2/4mn_0\end{pmatrix} \tag{1.2.18}$$

The inverse of this matrix is usually called Feynman propagator. The poles of the propagator give us the energy spectrum (remember that in the imaginary-time formalism the excitations correspond to $i\omega_n = \epsilon(q)$) .

$$\Delta_F(\mathbf{q},\omega_n) = \hbar\mathbf{A}^{-1}(\mathbf{q},\omega_n) = \frac{1}{\omega_n^2 + E_B(q)^2/\hbar^2}\begin{pmatrix} mE_B(q)^2/n_0\hbar^3 q^2 & \omega_n \\ -\omega_n & n_0\hbar q^2/m \end{pmatrix} \tag{1.2.19}$$

with $E_B(q)$ the Bogoliubov quasi-particle spectrum

$$E_B(q) = \sqrt{n_0 g_0 \hbar^2 q^2/m + (\hbar^2 q^2/2m)^2} = \sqrt{\epsilon^2(q) + 2\mu\epsilon(q)} \tag{1.2.20}$$

where $\epsilon(q) = \frac{\hbar^2 q^2}{2m}$ is the free particle energy.
In the long wavelenght limit we find a linear spectrum $E_B(q) = \hbar\sqrt{\mu/m}\; q = \hbar c q$, where c is called sound speed. This spectrum is gapless and describes a frictionless fluid (following a famous argument from Landau).
We can then evaluate the grand potential after integration on the quadratic degrees of freedom. We get

$$\Omega = \frac{1}{2\beta}\sum_{\mathbf{q},n=-\infty}^{n=+\infty} \log\left[\beta^2(\hbar^2 + E_B(q)^2)\right] \tag{1.2.21}$$

Summing over Matsubara frequencies we get

$$\Omega = \Omega_0 + \Omega_g^{(0)} + \Omega_g^{(T)} \tag{1.2.22}$$

where Ω_0 is the mean-field contribution and

$$\Omega_g^{(0)} = \frac{1}{2}\sum_{\mathbf{q}} E_B(q) \tag{1.2.23}$$

$$\Omega_g^{(T)} = \frac{1}{\beta}\sum_{\mathbf{q}} \log\left(\exp^{-\beta E_B(q)}\right) \tag{1.2.24}$$

The first term represent the zero-point energy, while the second one the termal fluctuations. In the following we shall use the Feynman propagator to compute correlation functions.
Before doing that, however, we shall briefly touch on the topic of the dimensional regularization of Bogoliubov theory.

1.3 Dimensional regularization

In the continuum limit the gaussian fluctuation contribution to the grand potential is

$$\frac{\Omega_g^{(0)}}{\mathcal{V}} = \frac{S_D}{2(2\pi)^D}\int_0^\infty dq q^{D-1}\sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m}+2\mu\right)} \tag{1.3.1}$$

which is clearly ultraviolet divergent for any D.
There are a couple ways to regularize (i.e. remove the divergent behavior) these kind of expressions. We shall show the dimensional regularization approach that was used for the first time by t'Hooft and Veltman in the context of quantum field theory [12].

In this approach we treat D as a parameter that can assume any value, not necessarily integer. We can introduce Euler's gamma function

$$\Gamma(z) = \int_0^\infty dt t^{z-1}e^{-z} \tag{1.3.2}$$

that is convergent for z>0. However we can extend the domain via analytic continuation, at the price of developing poles for negative integers. Moreover, we can define Euler's beta function

$$B(x,y) = \int_0^\infty dt \frac{t^x - 1}{(1+t)^{x+y}} \tag{1.3.3}$$

which is well defined for Re(x), Re(y)>0. This function can be continued to

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.3.4}$$

Using these functions we can write [13]

$$\frac{\Omega_g^{(0)}}{\mathcal{V}} = \frac{S_D(2\mu)^{D/2+1}}{4(2\pi)^D}\left(\frac{2m}{\hbar}\right)^{D/2} B(\frac{D+1}{2}, -\frac{D+2}{2}) \tag{1.3.5}$$

These expression can be evaluated for D=3, which leads to

$$\frac{\Omega_g^{(0)}}{\mathcal{V}} = \frac{8m^{3/2}}{15\pi^2\hbar^{3/2}}\mu^{5/2} \tag{1.3.6}$$

For D=2 however, we run into some issues, since $\Gamma(-2)$ is ill-defined. The workaround consists into considering $\Gamma(2-\epsilon)$ and then take the limit $\epsilon \to 0$ at the end of the calculation. Considering that

$$\Gamma(2-\epsilon) = \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \tag{1.3.7}$$

we have that

$$\frac{\Omega_g^{(0)}}{\mathcal{V}} = -\frac{m\mu^2}{4\pi\hbar^2 k^\epsilon}\Gamma(2-\epsilon) = -\frac{m\mu^2}{4\pi\hbar^2 \epsilon k^\epsilon} \tag{1.3.8}$$

where we have introduced an arbitrary ave-number regulator k, for dimensional consistency. This leads to a renormalized interaction constant, meaning

$$\frac{1}{g_r}k^\epsilon\left(\frac{1}{g} + \frac{m}{2\pi\hbar^2 k^\epsilon \epsilon}\right) \tag{1.3.9}$$

The renormalization procedure led us to a running coupling constant g_r. We can obtain the differential flow equation

$$k\frac{dg_r}{dk} = \frac{m}{2\pi\hbar^2}g_r^2 \tag{1.3.10}$$

which leads to

$$\frac{1}{g_r(k_0)} - \frac{1}{g_r(k)} = -\frac{m}{2\pi\hbar^2}\log\frac{k_0}{k} \tag{1.3.11}$$

We can set the Landau pole (i.e. the energy at which the coupling diverges) at the energy $E_0 = \hbar^2 k_0^2/2m$ (for a more extensive analysis of this issues, see [13]).
This leads to

$$\frac{1}{g_r(k)} = \frac{m}{4\pi\hbar^2}\log\frac{E_0}{\mu} \tag{1.3.12}$$

when k is such as $\hbar^2 k^2/2m = \mu$.
These results are in agreement with the ones found by Popov [14] using the T-matrix cutoff approach.

1.4 Correlation functions in Bogoliubov approximation, condensate depletion and normal-fluid particle density

In this section we shall show how to derive condensate depletion and normal-fluid particle density using the functional approach in the Bogoliubov approximation. This will result in the evaluation of specific correlation functions, built upon the fundamental correlators between the phase field $\Theta(\mathbf{r}, \tau)$ and the density field $\pi(\mathbf{r}, \tau)$.
We shall closely follow the steps of [15]. The first correlator we can evaluate is the density-density correlation function

$$S(\mathbf{r}, \tau, \mathbf{r}', \tau') = \langle n(\mathbf{r}, \tau)n(\mathbf{r}', \tau')\rangle - n^2 \approx \langle \pi(\mathbf{r}, \tau)\pi(\mathbf{r}', \tau')\rangle \tag{1.4.1}$$

This can be quickly evaluated by adding a density source term $\int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\pi(\mathbf{r},\tau)J(\mathbf{r},\tau)$ in the action and taking functional derivatives:

$$S(\mathbf{r},\tau,\mathbf{r}',\tau') = \hbar^2 \frac{\delta^2 \ln \mathcal{Z}[J]}{\delta J(\mathbf{r},\tau)\delta J(\mathbf{r}',\tau')}|_{J=0} \tag{1.4.2}$$

In the reciprocal space, assuming our system to be invariant for space and time translation we get:

$$S(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) = \beta\hbar S(\mathbf{q},\omega_m)(2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\delta_{m,-m'} \tag{1.4.3}$$

whit

$$S(\mathbf{q},\omega_m) = \frac{n_0\hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2}. \tag{1.4.4}$$

The Matsubara frequencies summation can be performed in a standard fashion by continuing the variable $i\omega_n$ to a generic complex value z and using contour integration. This yields:

$$S(\mathbf{q},\tau) = \frac{1}{\beta\hbar}\sum_m S(\mathbf{q},\omega_m)e^{-i\omega_m\tau} = \frac{n_0\hbar^2 q^2}{2mE_B(q)}\frac{e^{(\beta-\tau/\hbar)E_B(q)} + e^{E_B(q)\tau/\hbar}}{e^{\beta E_B(q)}-1} \tag{1.4.5}$$

We can take the static limit: $\tau = 0$

$$S(\mathbf{q},0) = \frac{n_0\hbar^2 q^2}{2mE_B(q)}\coth\frac{\beta E_B(q)}{2} \tag{1.4.6}$$

which is the form factor we can find in the Bogoliubov quasi-particle approximation [16].
Going back from imaginary to real time, we get the dynamic structure factor

$$S(\mathbf{q},\omega) = \int dt S(\mathbf{q},t)e^{i\omega t} = \frac{2\pi n_0\hbar^2 q^2}{2mE_B(q)\big(e^{\beta E_B(q)}-1\big)}\left[e^{\beta E_B(q)}\delta\big(\omega - E_B(q)/\hbar\big) + \delta(\omega + E_B(q)/\hbar)\right] \tag{1.4.7}$$

This is a very important relation, because the dynamic structure factor can br directly measured in scattering experiments. In particular, we expect to see peaks that follow the dispersion relation of the Bogoliubov spectrum. These kind of experiments can also give some insight on the interparticle potential $V(q)$. For example, for strongly interacting liquids, such as Helium 4 we would see a spectrum like that of Fig. 1.2.
In these spectra we have both a linear, phonon-like part, for small values of

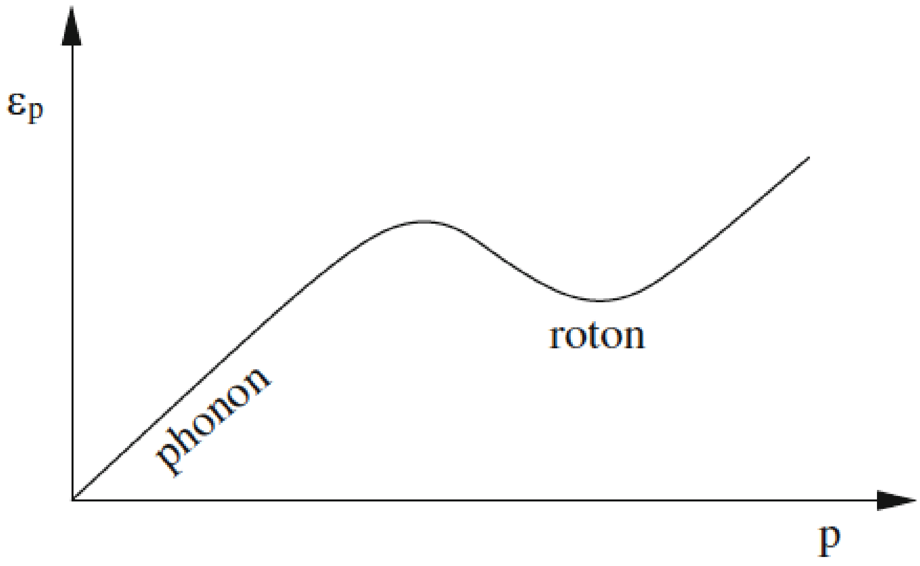

Figure 1.2: Elementary excitation spectrum in Helium-4. Note the local minimum that corresponds to roton excitations.

q and a local minimum that corresponds to rotons. This kind of excitations can be pictured as almost free particles surrounded by a cloud of phonons [17]. The second correlation function we want to compute is the mass current one, which is defined as:

$$C_{ij}(\mathbf{r},\tau;\mathbf{r}',\tau') = \langle g_i(\mathbf{r},\tau)g_j(\mathbf{r}',\tau')\rangle - \langle g_i(\mathbf{r},\tau)\rangle\langle g_j(\mathbf{r}',\tau')\rangle \tag{1.4.8}$$

where

$$\mathbf{g}(\mathbf{r},\tau) = m\mathbf{j}(\mathbf{r},\tau) = \frac{\hbar}{2i}\Big[\psi^*(\mathbf{r},\tau)\boldsymbol{\nabla}\psi(\mathbf{r},\tau) - \psi(\mathbf{r},\tau)\boldsymbol{\nabla}\psi^*(\mathbf{r},\tau)\Big] \tag{1.4.9}$$

Using Wick theorem we can evaluate a four-point correlation function in terms of lower order ones. In this case, though, we don't have any three-point function, due to the fact the action is quadratic in the fields. This leads to:

$$\begin{aligned} C_{ij}(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) &= -\hbar^2 n_0^2 q_i q_j' \langle\Theta(\mathbf{q},\omega_m)\Theta(\mathbf{q}',\omega_{m'})\rangle \\ -\frac{1}{(\beta\mathcal{V})^2}\sum_{\mathbf{p},\omega_n}\sum_{\mathbf{p}',\omega_n'} p_i p_j' &\Big[\langle\pi(\mathbf{q}-\mathbf{p},\omega_{m-n})\pi(\mathbf{q}'-\mathbf{p}',\omega_{m'-n'})\rangle\langle\Theta(\mathbf{p},\omega_n)\Theta(\mathbf{p}',\omega_{n'})\rangle \\ &+ \langle\pi(\mathbf{q}-\mathbf{p},\omega_{m-n})\Theta(\mathbf{p}',\omega_{n'})\rangle\langle\Theta(\mathbf{p},\omega_n)\pi(\mathbf{q}'-\mathbf{p}',\omega_{m'-n'})\rangle\Big] \end{aligned} \tag{1.4.10}$$

We can then define the transverse current response function $\chi_\perp$:

$$\chi_\perp(\mathbf{q},\omega_m) \equiv \frac{1}{(D-1)\hbar}\sum_{ij} P^T_{ij}(\mathbf{q})C_{ij}(\mathbf{q},\omega_m) \tag{1.4.11}$$

where $P^T_{ij}(\mathbf{q}) = \delta_{ij} - q_i q_j/q^2$ is the transverse projector.
We then find:

$$\chi_\perp(\mathbf{q},\omega_m) = \frac{1}{(D-1)\beta\mathcal{V}}\sum_{\mathbf{p},\omega_n}\frac{q^2p^2-(\mathbf{q}\mathbf{p})^2}{q^2p^2} \frac{(\mathbf{q}-\mathbf{p})^2E_B(p)^2/\hbar^2 + p^2\omega_{m-n}\omega_n}{\left[\omega^2_{m-n}+E_B(|\mathbf{q}-\mathbf{p}|)^2/\hbar^2\right]\left[\omega_n^2+E_B(p)^2/\hbar^2\right]} \tag{1.4.12}$$

Had we not included the nonlinear terms of Eq (1.4.10) the transverse response function would have vanished.
We can finally define the normal-fluid particle density n_n by taking the limit $q\to 0$ for χ_T after ω_m has been set to zero. The reason for this is that the normal-fluid particle density is defined as the transport coefficient related to the reaction to transverse motion [18]. We find that

$$n_n = \frac{1}{m}\lim_{q\to 0}\chi_T(\mathbf{q},0) = \frac{1}{mD\beta\mathcal{V}}\sum_{\mathbf{p},\omega_n} p^2\frac{E_B(p)^2/\hbar^2-\omega_n^2}{\left[E_B(p)^2/\hbar^2+\omega_n^2\right]^2} \tag{1.4.13}$$

We can then sum over the Matsubara frequencies, obtaining

$$n_n = \frac{\beta\hbar^2}{4mD}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{q}{\sinh\beta E_B(q)/2}\right)^2 \tag{1.4.14}$$

We shall explicitly evaluate this integral later on, after showing an alternative derivation of the normal density that uses Green's functions.
The last important correlation function for superfluid bosons systems is the vorticity correlation function. This is defined, in two and three dimensions as:

$$V_{ij}(\mathbf{r},\tau;\mathbf{r}',\tau') \equiv= \frac{1}{m^2}\left\langle\left[\boldsymbol{\nabla}\times\mathbf{g}(\mathbf{r},\tau)\right]_i\left[\boldsymbol{\nabla}\times\mathbf{g}(\mathbf{r}',\tau')\right]_j\right\rangle \tag{1.4.15}$$

After using again Wick's theorem, we get in Fourier space:

$$V_{ij}(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) = \sum_{k,l,k',l'}\epsilon_{ikl}\epsilon_{jk'l'}\frac{1}{(m\beta\mathcal{V})^2}\sum_{\mathbf{p},\omega_n}\sum_{\mathbf{p}',\omega_{n'}}(q_k-p_k)p_l(q'_k-p'_k)p'_{l'} \left[\langle\pi(\mathbf{q}-\mathbf{p},\omega_{m-n})\pi(\mathbf{q}'-\mathbf{p}',\omega_{m'-n'})\rangle\langle\Theta(\mathbf{p},\omega_n)\Theta(\mathbf{p}',\omega'_n)\rangle \langle\pi(\mathbf{q}-\mathbf{p},\omega_{m-n})\Theta(\mathbf{p}',\omega_{n'})\rangle\langle\Theta(\mathbf{p},\omega_n)\pi(\mathbf{q}'-\mathbf{p}',\omega m'-n')\rangle\right] \tag{1.4.16}$$

It is possible to show, through brute force, that the longitudinal component of this correlation function vanishes: $V_{\parallel} = \sum_{ij} P^L_{ij} V_{ij} = 0$.
Moreover, after plugging in the density-phase correlators, we find that:

$$V_{\perp}(\mathbf{q}, \omega_m) = \hbar q^2 \chi_{\perp}(\mathbf{q}, \omega_m) \tag{1.4.17}$$

This relation shows that the intrinsic vortex-vortex correlations make the transverse response function, and therefore the normal-fluid particle density, non zero ad finite temperatures.
In fact, considering an effective theory in which the density degree of freedom have been integrated out (phase-only approximation) one would find that n_n is zero even at finite temperature. This is the case for Kosterlitz-Thouless superfluid films.
To understand and calculate quantites suchs as condensate depletion it is useful to define the following Green functions [8]:

$$G(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle \psi(\mathbf{r}, \tau) \psi^*(\mathbf{r}', \tau') \rangle - n_0 \tag{1.4.18}$$

$$G_{12}(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle \psi(\mathbf{r}, \tau) \psi(\mathbf{r}', \tau') \rangle - n_0 \tag{1.4.19}$$

To compute these Green functions we can expand ψ, as defined in Eq (1.2.10), in powers of the density and phase fields:

$$\psi(\mathbf{r}, \tau) \approx \sqrt{n_0} \left[1 - \frac{\pi(\mathbf{r}, \tau)}{2n_0} + i\Theta(\mathbf{r}, \tau) - \frac{\pi(\mathbf{r}, \tau)^2}{8n_o^2} + \frac{i}{2n_0} \pi(\mathbf{r}, \tau)\Theta(\mathbf{r}, \tau) - \frac{1}{2}\Theta(\mathbf{r}, \tau)^2 \right] \tag{1.4.20}$$

To leading order this gives, in Fourier space

$$\begin{aligned} G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) \approx & \frac{1}{4n_0} \langle \pi(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle - \frac{i}{2} \langle \pi(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle \\ & + \frac{i}{2} \langle \Theta(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle + n_0 \langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle \end{aligned} \tag{1.4.21}$$

$$\begin{aligned} G_{12}(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) \approx & \frac{1}{4n_0} \langle \pi(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle + \frac{i}{2} \langle \pi(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle \\ & + \frac{i}{2} \langle \Theta(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle - n_0 \langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle \end{aligned} \tag{1.4.22}$$

After inserting the gaussian two-point correlators we get:

$$G(\mathbf{q}, \omega_m) = \frac{i\omega_m + \hbar q^2/2m + n_0 g_0/\hbar}{\omega_m^2 + E_B(q)^2/\hbar^2}, \tag{1.4.23}$$

$$G_{12}(\mathbf{q}, \omega_m) = \frac{-n_0 g_0/\hbar}{\omega_m^2 + E_B(q)^2/\hbar^2} \tag{1.4.24}$$

It is useful to rewrite this functions in terms of poles and residues:

$$G(\mathbf{q},\omega_m) = \frac{|u(q)|^2}{-i\omega_m + E_B(q)/\hbar} + \frac{|v(q)|^2}{i\omega_m + E_B(q)\hbar}, \tag{1.4.25}$$

$$G_{12}(\mathbf{q},\omega_m) = -u(q)v(q)\left(\frac{1}{-i\omega_m + E_B(q)/\hbar} + \frac{1}{i\omega_m + E_B(q)\hbar}\right) \tag{1.4.26}$$

Here we introduced the weight functions $u(q)$ and $v(q)$ defined as

$$\left|u(q)\right|^2 = \frac{1}{2}\left(\frac{n_0 g_0 + \hbar^2 q^2/2m}{E_B(q)} + 1\right), \tag{1.4.27}$$

$$\left|v(q)\right|^2 = \frac{1}{2}\left(\frac{n_0 g_0 + \hbar^2 q^2/2m}{E_B(q)} - 1\right) \tag{1.4.28}$$

Note that these are exactly the coefficients of the Bogoliubov transformation. This proves once again the equivalence between the classic transormation approach and the gaussian approximation. We are now ready to evaluate relevant physical quantities.

1.5 Explicit evaluation of condensate depletion

For the condensate depletion, we can write

$$N - N_0 = \int d^D\mathbf{r}\big(\langle\psi(\mathbf{r},\tau-\eta)\psi^*(\mathbf{r}',\tau+\eta)\rangle - n_0\big) =$$
$$\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{1}{(\beta\hbar)^2}\sum_{m,m'} e^{-i(\omega_m+\omega_{m'})\tau} e^{'(\omega_m-\omega_{m'})\eta} G(\mathbf{q},\omega_m;-\mathbf{q},\omega'_{\mathbf{m}}) \tag{1.5.1}$$

Here, the η factor, which is crucial for convergence sake, comes from the time ordering of the fields. We shall take the limit $\eta \to 0$ at the end.
Supposing time and space translation invariance we have

$$n - n_0 = \int \frac{d^D q}{(2\pi)^D}\frac{1}{\beta\hbar}\sum_m e^{i\omega_m\eta} G(\mathbf{q},\omega_m) \tag{1.5.2}$$

Using eq (1.4.23) and then performing the Matsubara frequency summation we arrive to

$$n - n_0 = \int \frac{d^D q}{(2\pi)^D}\left[v(q)^2 + \frac{u(q)^2 + v(q^2)}{e^{\beta E_B(q)} - 1}\right] \tag{1.5.3}$$

From this equation we can easily read that the first term describes the zero-temperature depletion, caused by interaction between particles and quantum

fluctuations, while the second term describes the depletion caused by thermal fluctuations.
Since we have assumed contact interaction, we can easily evaluate this integral at $T = 0$:

$$n(T=0) - n_0 = \frac{1}{2}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{n_0 g_0 + \hbar^2 q^2/2m}{E_B(q)} - 1 \quad (1.5.4)$$

Defining

$$x^2 = \frac{\hbar^2 q^2}{4mn_0 g_0}$$

We get

$$\frac{(mn_0 g_0)^{\frac{D}{2}}}{2\hbar^D \pi^{D/2}\Gamma(\frac{D}{2})}\int_0^\infty dx\; x^{D-2}\left[\sqrt{1+x^2} - 2x - \frac{x^2}{\sqrt{1+x^2}}\right] \quad (1.5.5)$$

As we expected from Mermin-Wagner theorem, this integral diverges as a logarithm at the lower limit for D=1, meaning that there is no condensation. In D=2 the final result is

$$n(T=0) - n_0 = \frac{mn_0 g_0}{4\pi\hbar^2} \quad (1.5.6)$$

while for D=3 we have

$$n(T=0) - n_0 = \frac{(mn_0 g_0)^{3/2}}{3\hbar^3\pi^2} \quad (1.5.7)$$

We can evaluate the lowest order finite-temperature corrections by taking the phonon limit for the Bogoliubov spectrum $E_B(q) \approx \hbar cq$. Within this limit we have $v(q)^2 + u(q)^2 \approx mc/\hbar q$. This leads us to

$$n(T) - n_0 \equiv \Delta n(T) = \int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{mc/\hbar q}{e^{\beta E_B(q)} - 1} \quad (1.5.8)$$

defining $t = \beta\hbar cq$ we get

$$\Delta n(T) = \frac{1}{2^{D-1}\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})\hbar^D c^{D-2}}\beta^{1-D}\int dt\frac{t^{D-2}}{e^t - 1} = \frac{\Gamma(D-1)\zeta(D-1)}{2^{D-1}\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})\hbar^D c^{D-2}}\beta^{1-D} \quad (1.5.9)$$

In two dimensions the ζ function diverges. This means that a non zero order parameter is possible only at zero temperature. The interesting fact is that,

as we shall see below, at finite temperature in D=2 the superfluid particle density n_s will be non zero.
The last equation, in D=3, gives:

$$\Delta n(T) = \frac{m(k_B T)^2}{12\hbar^3 c} \tag{1.5.10}$$

To keep track of all the different contributions, from here on, we shall use the following conventions: n_0 denotes the condensate particle density, as given by Eq (1.2.9) while all the other contributions (quantum, thermal, and eventually disorder) will be denoted as n_{out}.
For example, at finite temperature and in D=3 we would write

$$n = n_0 + n_{out} \tag{1.5.11}$$

where

$$n_{out} = n_Q + n_{th} \tag{1.5.12}$$

In this case n_Q is given by Eq (1.5.7) and n_{th} by Eq (1.5.10).

1.6 Explicit evaluation of normal-fluid particle density

As we have said, it is possible to derive Eq (1.4.13) using Green functions.
Let's imagine a cylindrical pipe whose walls are moving with velocity $\mathbf{v}$ with respect to the superfluid. In this case the superfluid velocity is zero and the normal-phase fluid velociy is $\mathbf{v}$. This is because, in the two-fluid picture, only normal fluid can be dragged by the moving walls. This leads to:

$$\langle g(\mathbf{r}, \tau)\rangle = m\langle \mathbf{j}(\mathbf{r}, \tau)\rangle = m n_n \mathbf{v} \tag{1.6.1}$$

We now have to compute $\mathbf{g}(\mathbf{r},\tau)$. Remembering the definiton in terms of the bosonic field, we have

$$\langle g(\mathbf{r}, \tau)\rangle = \int \frac{d^D\mathbf{q}}{(2\pi)^D}\hbar\mathbf{q}\frac{1}{(\beta\hbar)^2}\sum_{m,m'} e^{-i(\omega_m+\omega_{m'})\tau}e^{-i\omega_{m'}\eta}G(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'})_{-\mathbf{v}} \tag{1.6.2}$$

Here, the subscript is put to remember that we need the Green function of a system with relative velocity between the walls and the superfluid.
Moreover, assuming we have time translation invariance, we get the reduced formula

$$\langle g(\mathbf{r}, \tau)\rangle = \int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\mathbf{q}}{\beta}\sum_{m} e^{i\omega_m\eta}G(\mathbf{q}, \omega_m)_{-\mathbf{v}} \tag{1.6.3}$$

We can obtain $G(\mathbf{q},\omega_m)_{-\mathbf{v}}$ by performing a Galilean transformation with velocity **-v** from $G(\mathbf{q},\omega_m)$, i.e. the Green functions for a still system. This results in a shift in the Matsubara frequencies. In fact we have

$$\psi'(\mathbf{r}',\tau') = \psi(\mathbf{r}+i\mathbf{v}\tau,\tau) = \frac{1}{\beta\hbar V}\sum_{\mathbf{q},\omega_m}\psi(\mathbf{q},\omega_m)\exp^{i\mathbf{q}\mathbf{r}-i(\omega_m-i\mathbf{q}\mathbf{v})\tau} = \tag{1.6.4}$$

$$\sum_{\mathbf{q}',\omega_m'}\psi'(\mathbf{q}',\omega_m')\exp^{i\mathbf{q}'\mathbf{r}'-i\omega_m'\tau'} \tag{1.6.5}$$

This means that, under Galilean transformation we have

$$\mathbf{q}\to\mathbf{q}' = \mathbf{q} \tag{1.6.6}$$

$$\omega_m\to\omega_m' = \omega_m - i\mathbf{q}\mathbf{v} \tag{1.6.7}$$

By plugging in the shifted Matsubara frequencies in Eq (1.4.23) we get

$$G(\mathbf{q},\omega_m)_{-\mathbf{v}} = \frac{1-|v(q)|^2}{-i\omega_m + E_B(q)/\hbar - \mathbf{q}\mathbf{v}} + \frac{|v(q)|^2}{i\omega_m + E_B(q)/\hbar + \mathbf{q}\mathbf{v}} \tag{1.6.8}$$

After summing over ω_m we arrive at Landau's formula

$$\langle\mathbf{g}(\mathbf{r},\tau)\rangle = \int\frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\hbar\mathbf{q}}{\exp^{\beta[E_B(q)-\hbar\mathbf{q}\mathbf{v}]}-1} \tag{1.6.9}$$

We see that the moving walls resulted in a reduction in energy eigenvalues of the quasiparticles. This equation reduces to the Eq. (1.4.13) once we expand up to linear order in **v**

To evaluate this integral we restrict ourselves to phonon spectrum i.e. $E_B(q)\approx\hbar cq$.

Defining $t=\hbar\beta cq/2$, in the above limit, Eq. (1.4.13) becomes

$$n_n(T) = \frac{2(k_BT)^{D+1}}{mD\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})\hbar^Dc^{D+1}}\int dt\frac{t^{D+1}}{\sinh(t)} = \tag{1.6.10}$$

$$\frac{\Gamma(D+2)\zeta(D+1)(k_BT)^{D+1}}{m2^{D-1}D\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})\hbar^Dc^{D+1}} \tag{1.6.11}$$

This equation, then, gives for D=2

$$n_{n,th} = \frac{3\zeta(3)(k_BT)^3}{2\pi\hbar^2c^4m} \tag{1.6.12}$$

and for D=3

$$n_{n,th} = \frac{2\pi^2(k_BT)^4}{45\hbar^3c^5m} \tag{1.6.13}$$

Here we have introduced the subscript "th" to remark the fact that this normal-fluid particle density is caused by thermal fluctuations. After introducing disorder, we shall see that also impurities contribute to the normal-fluid particle density. It is very important to note, once again, that these results are derived taking into account both phase and amplitude fluctuations. While dealing with low-energy modes, it is customary to integrate out the π fields, since the most relevant degrees of freedom are the Θ fields and depending on what features one is interest in this could work.
However we have already seen that, in this context, neglecting amplitude fluctuations would result in having no vorticity in the system and, as a result, no normal-fluid particle density at $T>0$.

Chapter 2

Disorder in bosonic systems

In the first chapter we introduced superfluid bosons systems and discussed their properties. Moreover we derived the expression of physical quantities of interest in harmonic approximation, using Green functions. We shall retain this approach while discussing the role of disorder on these kind of systems. This section follows the approach to disorder of reference [15].
To account for the role of defects in the system, we add an extra term to the action in Eq (1.0.4), obtaining

$$S = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\mathcal{L} = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\Big\{\hbar\psi^*(\mathbf{r},\tau)\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} + \frac{\hbar^2}{2m}\big|\boldsymbol{\nabla}\psi(\mathbf{r},\tau)\big|^2 - \Big[\mu + U_D(\mathbf{r},\tau)\Big]\big|\psi(\mathbf{r},\tau)\big|^2 + \frac{1}{2}\int d^D\mathbf{r}V(|\mathbf{r}-\mathbf{r}'|)\big|\psi(\mathbf{r},\tau)\big|^2\big|\psi(\mathbf{r}',\tau)\big|^2\Big\} \tag{2.0.1}$$

where $U_D(\mathbf{r},\tau)\big|\psi(\mathbf{r},\tau)\big|^2$ describes the coupling between the disorder potential and the bosons density field and can be thought as a local ridefinition of the chemical potential: $\tilde{\mu(\mathbf{r},\tau)} = \mu + U_D(\mathbf{r},\tau)$.
The disorder potential satisfies the following relations:

$$\overline{U_D(\mathbf{r},\tau)U_D(\mathbf{r}',\tau)} = kR(\mathbf{r}-\mathbf{r}')H(\tau-\tau') \tag{2.0.2}$$

$$\overline{U_D(\mathbf{r},\tau)} = 0 \tag{2.0.3}$$

Disorder affects even the mean-field approximation. In fact it shifts the transition point according to:

$$n_0 = 0 \quad \text{for} \quad \mu + \overline{U}_D \leq 0 \tag{2.0.4}$$

$$n_0 = \frac{\mu + \bar{U}_D}{g} \quad \text{for} \quad \mu + \overline{U}_D \geq 0 \tag{2.0.5}$$

Here $\overline{U}_D$ denotes the configurational average for the disorder potential. For now, the space and time self-correlation functions will be left general. After developing a general formalism, we shall consider some concrete cases.
We can now take the same steps we followed in the previous chapter, to get to the action in Bogoliubov approximation, this time with disorder.

$$S_0[\pi, \Theta] = \frac{1}{2\beta\hbar\mathcal{V}} \sum_{\mathbf{q},\omega_n} \Bigg[\Big(\Theta(-\mathbf{q}, -\omega_n), \pi(-\mathbf{q}, -\omega_n) \Big) \mathbf{A}(\mathbf{q}, \omega_n) \begin{pmatrix} \Theta(\mathbf{q}, \omega_n) \\ \pi(\mathbf{q}, \omega_n) \end{pmatrix} - 2 \Big(\Theta(-\mathbf{q}, \omega_m), \pi(-\mathbf{q}, -\omega_m) \Big) \begin{pmatrix} 0 \\ U_D(\mathbf{q}, \omega_n) \end{pmatrix} \Bigg] - \frac{1}{2} n_0^2 V_0 \beta\hbar\mathcal{V} \quad (2.0.6)$$

We can define the disorder correlator in Fourier space as the two point function:

$$\Lambda(\mathbf{q}, \omega_m; \mathbf{q'}, \omega_{m'}) = \overline{U_D(\mathbf{q}, \omega_m) U_D(\mathbf{q'}, \omega_{m'})} \quad (2.0.7)$$

we shall work to first order in this correlator.
Let us call, for simplicity sake, $X = \begin{pmatrix} \Theta \\ \pi \end{pmatrix}$.
To obtain the correlation matrix $\langle X(\mathbf{q}, \omega_m) X^T(\mathbf{q'}, \omega'_m) \rangle$ in Fourier space we use the following change of variables

$$\begin{pmatrix} \tilde{\Theta}(\mathbf{q}, \omega_m) \\ \tilde{\pi}(\mathbf{q}, \omega_m) \end{pmatrix} = \begin{pmatrix} \Theta(\mathbf{q}, \omega_m) \\ \pi(\mathbf{p}, \omega_m) \end{pmatrix} - \mathbf{A}(\mathbf{q}, \omega_\mathbf{m})^{-1} \begin{pmatrix} 0 \\ U_D(\mathbf{q}, \omega_m) \end{pmatrix} \quad (2.0.8)$$

Expressed in these fields, the action is purely Gaussian and can be readily evaluated.
Using the source method, we can find the correlation matrix for the shifted fields:

$$\langle \tilde{X}(\mathbf{q}, \omega_m) \tilde{X}^T(\mathbf{q'}, \omega_{m'}) \rangle = \hbar \mathbf{A}^{-}1(\mathbf{q}, \omega_m) \beta\hbar\delta(\mathbf{q} + \mathbf{q'}) \delta_{m,-m'} \quad (2.0.9)$$

Switching back to the original fields, using the fact the single-field correlation functions are vanishing, the fact that $\mathbf{A}^T(\mathbf{q}, \omega_m) = \mathbf{A}(-\mathbf{q}, -\omega_m)$ and taking the quenched average, we finally arrive at

$$\overline{\langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q'}, \omega_{m'}) \rangle} = \frac{m E_B(q)^2 / n_0 \hbar^3 q^2}{\omega_m^2 + E_B(q)^2/\hbar^2} (2\pi)^D \delta(\mathbf{q} + \mathbf{q'}) \beta \delta_{m,-m'} + \frac{\omega_m \omega_{m'} / \hbar^2}{\left[\omega_m^2 + E_B(q^2/\hbar^2)\right] \left[\omega_{m'}^2 + E_B(q)^2/\hbar^2\right]} \Lambda(\mathbf{q}, \omega_m; \mathbf{q'}, \omega_{m'}) \quad (2.0.10)$$

$$\overline{\langle\Theta(\mathbf{q},\omega_m)\pi(\mathbf{q}',\omega_{m'})\rangle} = \frac{\omega_m}{\omega_m^2 + E_B(q)^2/\hbar^2}(2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\beta\delta_{m,-m'}+ \frac{n_0 q'^2\omega_m/\hbar m}{\left[\omega_m^2 + E_B(q^2/\hbar^2)\right]\left[\omega_{m'}^2 + E_B(q)^2/\hbar^2\right]}\Lambda(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) \quad (2.0.11)$$

$$\overline{\langle\pi(\mathbf{q},\omega_m)\pi(\mathbf{q}',\omega_{m'})\rangle} = \frac{\hbar n_0 q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2}(2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\beta\delta_{m,-m'}+ \frac{n_0^2 q^2 q'^2\omega_m/m^2}{\left[\omega_m^2 + E_B(q^2/\hbar^2)\right]\left[\omega_{m'}^2 + E_B(q)^2/\hbar^2\right]}\Lambda(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) \quad (2.0.12)$$

It is not restrictive to assume that, on average, the system will be invariant under space and time translations. Therefore

$$\Lambda(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) = \Lambda(\mathbf{q},\omega_m)\hbar\beta(2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\delta_{m,-m'} \quad (2.0.13)$$

This leads to, for the structure factor

$$\overline{S(\mathbf{q},\omega_m)} = \frac{n_0\hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2} + \left(\frac{n_0 q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2}\right)^2\Lambda(\mathbf{q},m) \quad (2.0.14)$$

As a result of impurities being added in the system, the structure factor has an extra Lorentzian term.
The static structure factor then becomes

$$\overline{S(\mathbf{q},\tau=0)} = \frac{\hbar^2 q^2 n_0}{2mE_B(q)}\coth\frac{\beta E_B(q)}{2} + \left(\frac{n_0 q^2}{m}\right)^2\frac{1}{\beta\hbar}\sum_m\frac{\Lambda(\mathbf{q},\omega_m)}{\left[\omega_m^2 + E_B(q)^2/\hbar^2\right]} \quad (2.0.15)$$

Before moving on to the evaluation of the correlation functions with impurities we want to show how the fundamental correlators Eqs (2.0.10)-(2.0.12) can be derived using the **replica trick**.

2.1 Replica method for disordered systems

The replica trick is a mathematical device that is commonly used in disorder theory to evaluate the partition function of a system in presence of impurities (For a more extensive introduction and implementation of the replica trick see [19]).
In general, we can define observable $\mathcal{O}$ through functional differentiation

$$\mathcal{O} = -\frac{\delta}{\delta J} \log \mathcal{Z}|_{J=0} \tag{2.1.1}$$

If the system is disordered, however, this is very hard to do.
We can, however, use the following identity:

$$\mathcal{O} = -\frac{\delta}{\delta J} \log \mathcal{Z}|_{J=0} = -\frac{\delta}{\delta J} \lim_{R\to 0} \left(e^{R \log \mathcal{Z}} - 1\right)|_{J=0} = -\frac{\delta}{\delta J} \lim_{R\to 0} \mathcal{Z}^R|_{J=0} \tag{2.1.2}$$

We can then average over the disorder to obtain an easier expression to evaluate. Here, the Rth power of the partition function can be thought as the partition function of R identical copies of the system.
To obtain the final result we then have to implement analytical continuation $R \to 0$. In general there is no way to tell if this limit exists. This shortcoming reflects the fact that, in general $\overline{\mathcal{Z}^R} \neq \overline{\mathcal{Z}}^R$.
This fact may undermine the trust we can put in this method; however in the very large majority of situations in which there is an alternative solution, the replica trick gives the right answer, as in the case we are about to present.
We shall derive the expression for the structure factor $S(\mathbf{q}, \tau)$ working with δ- function self-correlated disorder, i.e.

$$R(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{2.1.3}$$

$$H(\tau - \tau') = \delta(\tau - \tau') \tag{2.1.4}$$

In the replica trick scheme, we have that

$$\overline{\langle \pi(\mathbf{r}, \tau) \pi(\mathbf{r}', \tau') \rangle} = \lim_{R\to 0} \sum_{\alpha=1}^{R} \langle \pi_\alpha(\mathbf{r}, \tau) \pi_\alpha(\mathbf{r}', \tau') \rangle \tag{2.1.5}$$

where the ensamble average on the right side is evaluated with respect to the replicated action:

$$S[\psi, \psi^*]_R = \sum_{\alpha}^{R} S_0[\psi_\alpha, \psi_\alpha^*] - \frac{\Lambda}{2} \int d^D\mathbf{r} \int d\tau \sum_{\alpha,\beta} |\psi_\alpha|^2 |\psi_\beta|^2 \tag{2.1.6}$$

where Λ is the disorder coupling constant. This action leads to a replicated partition function in the form of

$$\overline{Z^R} = \int \prod_{\alpha=1}^{m} \mathcal{D}\psi_\alpha \mathcal{D}\psi_\alpha^* e^{-S[\psi,\psi^*]_R/K_B T} \tag{2.1.7}$$

Using the usual expansion in terms of π and θ, and defining $X_\alpha = \begin{pmatrix} \Theta_\alpha \\ \pi_\alpha \end{pmatrix}$
We see that the quadratic form that defines the action takes the recursive form [20]

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 & \cdots & \mathbf{A}_2 \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{A}_2 & \mathbf{A}_2 & \cdots & \mathbf{A}_1 \end{pmatrix} \tag{2.1.8}$$

where $\mathbf{A}_1$ $\mathbf{A}_2$ are 2x2 matrices

$$\mathbf{A}_1(\mathbf{q}, \omega_n) = \begin{pmatrix} n_0\hbar^2 q^2/m & -\hbar\omega_n \\ \hbar\omega_n & g_0 + \hbar^2 q^2/4mn_0 - \Lambda \end{pmatrix} \tag{2.1.9}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\Lambda \end{pmatrix} \tag{2.1.10}$$

We can put $\tilde{\mathbf{A}}$ in block diagonal form using the transformation

$$Y_a = \sum_{\alpha=1}^{R} e^{2\pi i a(\alpha/R)} X_\alpha \tag{2.1.11}$$

Note that the zero mode is the sum of the original vector : $Y_0 = \sum_\alpha X_\alpha$. This way we obtain

$$\tilde{\mathbf{A}}' = \begin{pmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_0 & 0 & \cdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \mathbf{A}_0 \end{pmatrix} \tag{2.1.12}$$

where $\mathbf{A}_0$ is the matrix that describes the pure system.
Using the fact that $\sum_a e^{2\pi i a(\alpha/R)} e^{2\pi i a(\beta/R)} = R\delta_{\alpha\beta}$ we find that

$$\overline{\langle \pi(\mathbf{q}, \omega_m)\pi(\mathbf{q}', \omega_{m'})\rangle} = \lim_{n\to 0} \frac{1}{n} \sum_{a\leq 1}^{R} \langle \pi_n(\mathbf{q}, \omega_m)\pi_n(-\mathbf{q}, -\omega_m)\rangle =$$

$$\lim_{R\to 0} \frac{1}{R} \left[(n-1) \frac{n_0\hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2} + \frac{n_0\hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2 - R\Lambda n_0\hbar q^2/m} \right] \tag{2.1.13}$$

In the limit $R \to 0$ we can expand the denominator of the second term and find the same result we had already found.

2.2 Disorder contribution to condensate depletion

From the fundamental correlators we can derive the expression on disorder-averaged Green functions:

$$\overline{G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})} = \frac{i\omega_m + \hbar q^2/2m + n_0 g_0/\hbar}{\omega_m^2 + E_B(q)^2/\hbar^2} (2\pi)^D \beta\hbar\delta(\mathbf{q} + \mathbf{q}')\delta_{m,-m'} + \frac{n_0}{\hbar^2} \frac{\big(i\omega_m + \hbar q^2/2m\big)\big(-i\omega_{m'} + \hbar q'^2/2m\big)}{\big[\omega_m^2 + E_B(q)^2/\hbar^2\big]\big[\omega_{m'}^2 + E_B(q')^2/\hbar^2\big]} \Lambda(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) \quad (2.2.1)$$

Remembering that

$$N - N_0 = \int d^D\mathbf{r} \big(\langle\psi(\mathbf{r}, \tau - \eta)\psi^*(\mathbf{r}', \tau + \eta)\rangle - n_0\big) \quad (2.2.2)$$

we can evaluate the contribution of disorder to the condensate depletion.

$$\overline{n(\tau)} - n_0 = \int \frac{d^D\mathbf{q}}{(2\pi)^D} \left[|v(q)|^2 + \frac{1 + 2|v(q)|^2}{e^{\beta E_B(q)} - 1}\right] + \frac{n_0}{\hbar^4\beta^2\mathcal{V}} \sum_{m,m'} \Bigg(e^{-i(\omega_m + \omega_{m'})\tau} \frac{\big(i\omega_m + \hbar q^2/2m\big)\big(-i\omega_{m'} + \hbar q'^2/2m\big)}{\big[\omega_m^2 + E_B(q)^2/\hbar^2\big]\big[\omega_{m'} + E_B(q')^2/\hbar^2\big]} \Lambda(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'})\Bigg) \quad (2.2.3)$$

We can see that this expression is not time independent in the general case. If, however, we assume time translation invariance to be true, we find a simplified disorder contribution:

$$\overline{n_\Lambda} = \frac{n_0}{\beta\hbar^3} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \sum_m \frac{(\hbar q^2/2m)^2 - \omega_m^2}{\big[E_B(q)^2/\hbar^2 + \omega_m^2\big]} \Lambda(\mathbf{q}, \omega_m) \quad (2.2.4)$$

Assuming static disorder this further reduces to

$$\overline{n_\Lambda} = \frac{n_0}{4m^2} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \left(\frac{\hbar q}{E_B(q)}\right)^4 \Lambda(\mathbf{q}) \quad (2.2.5)$$

We can clearly see that, under the assumption of static impurities, the disorder contribution to condensate depletion is temperature-independent.

2.3 Disorder contribution to normal-fluid particle density

We can now repeat the steps that lead us to the evaluation of the current and vorticity correlators, considering now the extra disorder term.
Assuming the system is, on average, isotropic we can define again the transverse and longitudinal parts of the mass flow correlator.
Even with disorder, the relation between the transverse vorticity correlator and the transverse current response function holds true

$$
\begin{aligned}
m^2\overline{V_\perp(\mathbf{q},\omega_m)} = \hbar q^2\overline{\chi_\perp(\mathbf{q},\omega_m)} = \\
\frac{\hbar}{(D-1)\mathcal{V}\beta}\sum_{\mathbf{q},\omega_m}\frac{\left[q^2p^2-(\mathbf{qp})^2\right]\left[(\mathbf{q}-\mathbf{p})^2E_B(q^2)/\hbar^2+p^2\omega_{m-n}\omega_n\right]}{p^2\left[\omega_{m-n}^2+E_B(|\mathbf{q}-\mathbf{p}|)/\hbar^2\right]\left[\omega_n^2+E_B(p)^2/\hbar^2\right]} \\
+\frac{n_0}{(D-1)m\beta\mathcal{V}}\sum_{\mathbf{p},\omega_n}\frac{q^2p^2-(\mathbf{qp})^2}{\left[\omega_{m-n}^2+E_B(|\mathbf{q}-\mathbf{p}|)/\hbar^2\right]\left[\omega_n^2+E_B(p)^2/\hbar^2\right]} \\
\left[\frac{(\mathbf{q}-\mathbf{p})^2}{p^2}\frac{(\mathbf{q}-\mathbf{p})^2E_B(p)^2/\hbar^2+p^2\omega_{m-n}\omega_n}{\omega_{m-n}^2+E_B(|\mathbf{q}-\mathbf{p}|)^2/\hbar^2}\Lambda(\mathbf{q}-\mathbf{p},\omega_{m-n})\right. \\
\left.-\frac{(\mathbf{q}-\mathbf{p}^2)\omega_n^2-p^2\omega_{m-n}\omega_n}{\omega_n^2+E_B(q)^2/\hbar^2}\Lambda(\mathbf{p},\omega_n)\right]
\end{aligned}
\tag{2.3.1}
$$

This is clearly a very cumbersome expression. Luckily, to evaluate the normal-fluid particle density we need to take the limit $q\to 0$. This yields

$$
\begin{aligned}
\overline{n_{n\Lambda}} = \frac{1}{m}\lim_{q\to 0}\overline{\chi_\perp(\mathbf{q},\mathbf{0})} = \frac{1}{m\mathcal{V}D\beta}\sum_{\mathbf{p},\omega_n}p^2\frac{E_B(p)^2\hbar^2-\omega_n^2}{\left[E_B(p)^2/\hbar^2+\omega_n^2\right]^2}+ \\
\frac{n_0}{Dm\beta\hbar\mathcal{V}}\sum_{\mathbf{p},\omega_n}p^4\frac{E_B(p)^2/\hbar^2-3\omega_n^2}{\left[E_B(p)^2/\hbar^2+\omega_n^2\right]^3}\Lambda(\mathbf{p},\omega_n)
\end{aligned}
\tag{2.3.2}
$$

Assuming static disorder, i.e. $\Lambda(\mathbf{q},\omega_m)=\beta\hbar\Lambda(\mathbf{q})\delta_{m,0}$ we find in the continuum limit:

$$
\overline{n_n} = \frac{\hbar^2\beta}{4mD}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{q}{\sinh\beta E_B(q)/2}\right)^2+\frac{n_0}{Dm^2}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{\hbar q}{E_B(q)^4}\Lambda(\mathbf{q})\right)
\tag{2.3.3}
$$

This equations shows two interesting facts.
The first one is that, assuming static disorder, the second term, that describes the impurities contribution to normal-fluid particle density is temperature-independent.

The second one is that there is a close relation between condensate depletion and normal-fluid particle density in isotropic systems:

$$\overline{n_\Lambda} = \frac{D}{4}\overline{n_{n\Lambda}} \tag{2.3.4}$$

This is the same relation derived by Huang and Meng in [3] using Bogoliubov transformation approach and by Schakel in [17] using the path-integral approach.
To conclude the general treatment of disorder in harmonic approximation, we shall derive the effects of impurities on the normal-fluid particle density using Landau's argument in the two-fluid model.
As we have seen in the previous section, the relative motion between fluid and walls can te taken into account by shifting the Matsubara frequencies according to $\omega_m \to \omega_m - i\mathbf{qv}$. In the comoving reference frame the disorder correlator is

$$\Lambda(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})_{\mathbf{v}} = \Lambda(\mathbf{q}, \omega_{\mathbf{m}} + \mathbf{iqv}; \mathbf{q}', \omega_{\mathbf{m'}} + \mathbf{iq'v}) \tag{2.3.5}$$

This leads to the following comoving Green's function:

$$\begin{aligned}
&\overline{G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})}_{-\mathbf{v}} = \\
&\qquad \frac{i\omega_m + \hbar q^2/2m + n_0 g_0/\hbar + \mathbf{qv}}{(\omega_m^- ß\mathbf{qv})^2 + E_B(q)^2/\hbar^2}(2\pi)^D \beta\hbar\delta(\mathbf{q}+\mathbf{q}')\delta_{m,-m'} + \\
&\frac{n_0}{\hbar}\frac{\big(i\omega_m + \hbar q^2/2m + \mathbf{qv}\big)\big(-i\omega_{m'} + \hbar q'^2/2m - \mathbf{q}'\mathbf{v}\big)}{\big[(\omega_m - i\mathbf{qv})^2 + E_B(q)^2/\hbar^2\big]\big[(\omega_{m'} - i\mathbf{q}'\mathbf{v})^2 + E_B(q')^2/\hbar^2\big]}\Lambda(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})
\end{aligned} \tag{2.3.6}$$

Expanding this equation in the velocity terms, and comparing with the relation $\overline{\langle \mathbf{g}(\mathbf{r}, \tau)\rangle} = m\overline{n_n}\mathbf{v}$ we can find the normal-fluid particle density. The effects of disorder, in general, turn the normal density into a tensor $\overline{n_{n,ij}}$. Assuming static disorder we find:

$$\overline{n_{n,ij}} = \frac{\beta\hbar^2}{4mD}\delta_{ij}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{q}{\sinh\beta E_B(q)/2}\right)^2 + \frac{n_0}{m^2}\int\frac{d^D\mathbf{q}}{(2\pi)^D}q_i q_j\left(\frac{\hbar^2 q}{E_B(q)^2}\right)^2\Lambda(\mathbf{q}) \tag{2.3.7}$$

Chapter 3

Explicit results for various kind of disorder

We shall now start our analysis of the effects of various kind of disorder on bosonic systems, in particular regarding response functions, ground-state depletion and normal-fluid density.
As we have said, this is equivalent to choosing a specific form for the disorder correlator, either in real or reciprocal space.

3.1 Point, linear and planar defects

The first kind of impurities we are considering are static and spatially uncorrelated, either point-like or extended. Here the words point and line are used to indicate the dimension of the disorder in D-dimensional space.
We shall begin our analysis starting from spatially-uncorrelated, static disorder to work up to more complicated situations.

3.1.1 Spatially-uncorrelated, static disorder

Under our assumptions, the disorder correlator takes the form

$$\Lambda(\mathbf{r},\tau;\mathbf{r}',\tau') = \Lambda\delta(\mathbf{r}-\mathbf{r}') \tag{3.1.1}$$

$$\Lambda(\mathbf{q},\omega_m) = \Delta\hbar\beta\delta_{m,0} \tag{3.1.2}$$

The density correlation function then reads

$$\overline{S(\mathbf{q},\omega_m)} = \frac{n_0\hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2} + \Lambda\left(\frac{n_0\hbar q^2/m}{E_B(q)^2}\right)^2 \beta\hbar\delta_{m,0} \tag{3.1.3}$$

This yields, for the static structure factor

$$\overline{S(\mathbf{q},\tau=0)} = \frac{n_0\hbar^2q^2}{2mE_B(q)}\coth\frac{\beta E_B(q)}{2} + \Lambda\left(\frac{n_0\hbar^2q^2}{2mE_B(q)}\right)^2 \tag{3.1.4}$$

For the condensate depletion we find

$$\overline{n_\Lambda} = \frac{n_0\Lambda}{4m^2}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{\hbar q}{E_B(q)}\right)^4 \tag{3.1.5}$$

Assuming that $V(q) \approx g_0 =$ we find

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{4m^2(2\pi)^D}\int d^D\mathbf{q}\frac{\hbar^4q^4}{\left(\hbar^2n_0g_0q^2/m + \left(\hbar^2q^2/2m\right)^2\right)^2} \tag{3.1.6}$$

Setting $x^2 = \hbar^2q^2/4mn_0g_0$ we can solve the integral. For D=2 this leads to

$$\overline{n_\Lambda} = \frac{\Lambda m}{4\pi\hbar^2g_0} \tag{3.1.7}$$

while for D=3 we have

$$\overline{n_\Lambda} = \frac{\Lambda m^{3/2}n_0^{1/2}}{4\hbar^3\pi g_0^{1/2}} \tag{3.1.8}$$

These are the results for point-like impurities. Putting all the pieces together, we have that at zero temperature the total number density can be written as

$$n = n_0 + n_{out} \tag{3.1.9}$$

where

$$n_{out} = n_Q + n_{th} + \overline{n_\Lambda} \tag{3.1.10}$$

Here n_Q describes the particles that have been expelled by the ground state due to quantum fluctuations and is given by Eqs. (1.5.6)-(1.5.7),n_{th} describes the thermal depletion and is given by Eq. (1.5.10) and the disorder contribution is given by the previous equations.
Another way to summarize these results is by expressing the ratio $\frac{n_0}{n}$ in terms of the above quantities. For D=2, at zero temperature (remember that the thermal corrections diverge when D=2, see Eq (1.5.9)) this leads to

$$\frac{n_0}{n} = 1 - \frac{mn_0g_0}{4\pi\hbar^2n} - \frac{\Lambda m}{4\pi\hbar^2g_0n} \approx 1 - \frac{mg_0}{4\pi\hbar^2} - \frac{\Lambda m}{4\pi\hbar^2g_0n} \tag{3.1.11}$$

While for D=3 we have

$$\frac{n_0}{n} = 1 - \frac{(mn_0g_0)^{3/2}}{3\hbar^3\pi^2 n} - \frac{m(k_BT)^2}{12\hbar^3 cn} - \frac{\Lambda m^{3/2} n_0^{1/2}}{4\hbar^3\pi g_0^{1/2} n} \approx$$
$$1 - \frac{(mg_0)^{3/2} n^{1/2}}{3\hbar^3\pi^2} - \frac{m(k_BT)^2}{12\hbar^3 cn} - \frac{\Lambda m^{3/2}}{4\hbar^3\pi g_0^{1/2} n^{1/2}} \quad (3.1.12)$$

On the right hand side we have used the approximation $n_0 \approx n$, which, even though it is not exact, gives a useful expression of the condensate fraction in terms of the total particle density, which is a controllable parameter. Since we are dealing with static disorder, we can immediately find the normal-fluid particle density caused by the impurities using Eq (2.3.4). This means that when D=2 we have

$$\overline{n_{n\Lambda}} = \frac{\Lambda m}{2\pi\hbar^2 g_0} \quad (3.1.13)$$

while for D=3 we have

$$\overline{n_{n\Lambda}} = \frac{\Lambda m^{3/2} n_0^{1/2}}{3\hbar^3\pi g_0^{1/2}} \quad (3.1.14)$$

In this case we can write

$$n_n = n_{n,th} + \overline{n_{n\Lambda}} \quad (3.1.15)$$

where $n_{n,th}$ is the thermal contribution and is given by Eqs. (1.6.12)-(1.6.13), and the disorder contribution follows from the previous equations.
In the same vein of before, remembering that $n = n_n + n_s$, where n_s is the superfluid particle density, we can write, when D=2

$$\frac{n_s}{n} = 1 - \frac{3\zeta(3)(k_BT)^3}{2\pi\hbar^2 c^4 mn} - \frac{\Lambda m}{2\pi\hbar^2 g_0 n} \quad (3.1.16)$$

When D=3 we have

$$\frac{n_s}{n} \approx 1 - \frac{2\pi^2(k_BT)^4}{45\hbar^3 c^5 mn} - \frac{\Lambda m^{3/2}}{3\hbar^3\pi g_0^{1/2} n^{1/2}} \quad (3.1.17)$$

We can easily generalize them to the case of static parallel defects, extended alond a $d_\parallel$ supsbace $\mathbf{r}_\parallel$.
If $d_\parallel = 1$ we have line-like defects, for $d_\parallel = 2$ we have planar defects and so on.
Therefore, the disorder is uncorrelated only in the $d_\perp$ transverse spatial directions

$$\Lambda(\mathbf{r}, \tau; \mathbf{r}', \tau') = \Lambda\delta(\mathbf{r}_\perp - \mathbf{r}'_\perp) \quad (3.1.18)$$

$$\Lambda(\mathbf{q}, \omega_m) = \Lambda(2\pi)^{D_\parallel}\delta(\mathbf{q}_\parallel)\beta\hbar\delta_{m,0} \quad (3.1.19)$$

This leads to a different density correlation function

$$\overline{S(\mathbf{q},\omega_m)} = \frac{n_0\hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2} + \Lambda\left(\frac{n_0\hbar q_\perp^2/m}{E_B(q_\perp)^2}\right)^2 (2\pi)^{d_\parallel}\delta(\mathbf{q}_\parallel)\beta\hbar\delta_{m,0} \quad (3.1.20)$$

then the structure factor becomes

$$\overline{S(\mathbf{q},\tau=0)} = \frac{n_0\hbar^2 q^2}{2mE_B(q)}\coth\frac{\beta E_B(q)}{2} + \Lambda\left(\frac{n_0\hbar^2 q_\perp^2}{2mE_B(q_\perp)}\right)^2 (2\pi)^{d_\parallel}\delta(\mathbf{q}_\parallel) \quad (3.1.21)$$

and finally we get, for the disorder-caused depletion

$$\overline{n_\Lambda} = \frac{n_0\Lambda}{4m^2}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{\hbar q}{E_B(q)}\right)^4 = \Lambda\frac{m^{D_\perp/2}n_0^{D_\perp/2-1}g_0^{D_\perp/2-2}}{2\pi^{D_\perp/2}\Gamma(D_\perp/2)\hbar^{D_\perp}}\int_0^\infty dx\frac{x^{D_\perp-1}}{(1+x^2)^2} \quad (3.1.22)$$

The presence of extended defects makes the system non isotropic, as we have already said.
Therefore the normal-fluid particle density, defined as the transport coefficient describing response to movement with respect to its walls, will depend on the direction relative to the disorder.
This means that, for motion happening along the disorder direction, the impurities will have no effect and their contribution to the normal-fluid particle density will vanish: $\overline{n_{n\|\Lambda}} = 0$. This can be obtained from Eq (2.2.12) setting $q_i = q_{\|,i}$
However,considering the perpendicular direction to the defects, we find a result similar to that of point-like defects, only scaled to the number of perpendicular dimension.
So, for planar disorder three dimensions, or line-like disorder in two dimension (in both cases we have $D_\perp = 1$)

$$\overline{n_\Lambda} = \frac{\Lambda m^{1/2}}{8\hbar n_0 g_0^{3/2}} \quad (3.1.23)$$

while for line impurities in D=3 we have, i.e. when $D_\perp = 2$

$$\overline{n_\Lambda} = \frac{\Lambda m}{4\hbar^2\pi g_0} \quad (3.1.24)$$

In this case the total particle density is still described by Eq. (3.1.10), but with the disorder contribution given by Eqs. (3.1.19)-(3.1.20).
The same considerations apply to the normal-fluid particle density. We have that, when $D_\perp = 1$

$$\overline{n_{n\Lambda}} = \frac{\Lambda m^{1/2}}{4\hbar n_0 g_0^{3/2}} \quad (3.1.25)$$

while for $D_\perp = 2$ we have

$$\overline{n_{n\Lambda}} = \frac{\Lambda m}{3\hbar^2 \pi g_0} \tag{3.1.26}$$

Also in this case the total normal-fluid particle particle density is describes by Eq (3.1.13), with the disorder contribution given by the previous equations. It is worth noting that in the Bogoliubov approximation, for $d_\perp < 4$ the consdensate depletion caused by the impurities is smaller than the normal-fluid density enrichment. This means that, when $n_{s,\perp} \to 0$ the bosons will become localized. This new phase is called Bose glass (for a detailed analysis of the properties of this phase, see for example [22]).
Within this classification of disorders falls another interesting case.
Let's consider a Helium film on a substrate with impurities such as lines (deposited with lithography, for example) randomly positioned and randomly oriented. We can think that lines are grouped in clusters, each described by a two-dimensional vector, perpendicular to each lines within the same cluster. Inside a cluster, the disorder correlator in reciprocal space is

$$\Lambda_{\hat{n}}(\mathbf{q}, \omega_m) = 2\Lambda\delta(\hat{\mathbf{n}}\mathbf{q})\beta\hbar\delta_{m,0} \tag{3.1.27}$$

Writing the delta function in integral form $2\pi\delta(x) = \int_{-\infty}^{+\infty} e^{ixs} ds$, we can then average over all the possible directions, obtaining

$$\begin{aligned} \Lambda(\mathbf{q}.\omega_m) = \frac{\Lambda\beta\hbar\delta_{m,0}}{4\pi^2} \int_0^{2\pi} d\phi \int_{-1}^{1} d\theta \sin\theta \int_{-\infty}^{+\infty} ds e^{iq\cos\theta s} = \\ \frac{\Lambda\beta\hbar\delta_{m,0}}{2\pi i q} \int_{-\infty}^{\infty} ds \frac{e^{iqs} - e^{-iqs}}{s} = \frac{\Lambda}{q}\beta\hbar\delta_{m,0} \end{aligned} \tag{3.1.28}$$

Hence, we have a slight variation of the disorder contribution

$$\overline{S(\mathbf{q}, \omega_m)} = \frac{n_0 \hbar q^2/m}{\omega_m^2 + E_B(q)^2/\hbar^2} + \frac{\Lambda}{q}\left(\frac{n_0\hbar q_\perp^2/m}{E_B(q_\perp)^2}\right)^2 (2\pi)^{d_\parallel}\delta(\mathbf{q}_\parallel)\beta\hbar\delta_{m,0} \tag{3.1.29}$$

$$\overline{S(\mathbf{q}, \tau = 0)} = \frac{n_0\hbar^2 q^2}{2mE_B(q)} \coth\frac{\beta E_B(q)}{2} + \frac{\Lambda}{q}\left(\frac{n_0\hbar^2 q_\perp^2}{2mE_B(q_\perp)}\right)^2 (2\pi)^{d_\parallel}\delta(\mathbf{q}_\parallel) \tag{3.1.30}$$

For the condensate depletion we have, similarly

$$\begin{aligned} \overline{n_\Lambda} = \frac{n_0\Lambda}{4m^2} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{q} \left(\frac{\hbar q}{E_B(q)}\right)^4 \approx \\ \Lambda \frac{n_0^{(D-3)/2} g_0^{(D-5)/2} m^{(D-1)/2}}{4\hbar^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty dx \frac{x^{D-2}}{(1+x^2)^2} \end{aligned} \tag{3.1.31}$$

This time the integral diverges as a logarithm when D=1.
For D=2 this leads to

$$\overline{n_\Lambda} = \frac{\Lambda m^{1/2}}{16\hbar n_0^{1/2} g_0^{3/2}} \tag{3.1.32}$$

while for D=3 we have

$$\overline{n_\Lambda} = \frac{\Lambda m}{4\hbar^2\pi^2 g_0} \tag{3.1.33}$$

We see that, even though the correlator is divergent as for $q \to 0$, the disorder contribution is finite and the disorder effect on condensate depletion is larger than the one caused by point-like disorder.
We can again summarize the results saying that the total number density in two dimensions at zero temperature is given by

$$\frac{n_0}{n} \approx 1 - \frac{mg_0}{4\pi\hbar^2} - \frac{\Lambda m^{1/2}}{16\hbar g_0^{3/2} n^{3/2}} \tag{3.1.34}$$

While for D=3 we have

$$\frac{n_0}{n} \approx 1 - \frac{(mg_0)^{3/2} n^{1/2}}{3\hbar^3\pi^2} - \frac{m(k_B T)^2}{12\hbar^3 cn} - \frac{\Lambda m}{4\hbar^2\pi^2 g_0 n} \tag{3.1.35}$$

The same considerations apply to the total normal-fluid particle density. we can therefore write, when D=2

$$\frac{n_s}{n} \approx 1 - \frac{3\zeta(3)(k_B T)^3}{2\pi\hbar^2 c^4 mn} - \frac{\Lambda m^{1/2}}{8\hbar g_0^{3/2} n^{3/2}} \tag{3.1.36}$$

While in three dimensions we have

$$\frac{n_s}{n} \approx 1 - \frac{2\pi^2(k_B T)^4}{45\hbar^3 c^5 mn} - \frac{\Lambda m}{3\hbar^2\pi^2 g_0 n} \tag{3.1.37}$$

3.1.2 Spatially and time-uncorrelated disorder

We shall now focus on extending the results for space-uncorrelated disorder to the case of impurities that are not correlated both in $\mathbf{r}$ and τ. In this case we have that

$$\Lambda(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) = \Lambda(2\pi)^D \delta(\mathbf{q} + \mathbf{q}')\beta\hbar\delta_{m,-m'} \tag{3.1.38}$$

These kind of disorder can be found in cuprate superconductors and describes the oxygen vacancies in these materials. In this case the disorder-dependent structure factor becomes

$$\overline{S_\Lambda(\mathbf{q})} = \frac{\Lambda}{\hbar E_B(q)} \left(\frac{n_0\hbar^2 q^2}{2mE_B(q)}\right)^2 \coth\frac{\beta E_B(q)}{2} \left(1 + \frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right) \tag{3.1.39}$$

where we have already summed over the Matsubara frequencies. In this case the condensate depletion yields (remembering Eq (2.2.7)):

$$\overline{n_\Lambda} = \frac{n_0}{4}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\Lambda}{\hbar E_B(q)}\coth\frac{\beta E_B(q)}{2}$$
$$\left[\left(\frac{\hbar^2 q^2}{2,E_B(q)}\right)^2\left(1+\frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right)-\left(1-\frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right)\right] \quad (3.1.40)$$

We can evaluate the zero-temperature limit $\beta\hbar\to\infty$:

$$\overline{n_\Lambda} = \frac{n_0}{4}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\Lambda}{\hbar E_B(q)}\left[\left(\frac{\hbar^2 q^2}{2mE_B(q)}\right)^2-1\right] = -\Lambda\frac{n_0^2\hbar}{4m}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\frac{q^2V(q)}{E_B(q)^3} \quad (3.1.41)$$

Assuming contact interaction we get

$$\overline{n_\Lambda} = -\frac{\Lambda n_0^{D/2}g_0^{(D-2)/2}m^{D/2}}{4\hbar^{D+1}\pi^{D/2}\Gamma(D/2)}\int_0^\infty\frac{x^{D-2}}{(1+x^2)^{3/2}}dx \quad (3.1.42)$$

This yields for D=2

$$\overline{n_\Lambda} = -\frac{\Lambda m n_0}{4\pi\hbar^3} \quad (3.1.43)$$

and, for D=3,

$$\overline{n_\Lambda} = -\frac{\Lambda n_0^{3/2}g_0^{1/2}m^{3/2}}{2\hbar^4\pi^2} \quad (3.1.44)$$

Since we are considering a zero-temperature case, we have that, when D=2

$$\frac{n_0}{n}\approx 1-\frac{mg_0}{4\pi\hbar^2}+\frac{\Lambda m}{4\pi\hbar^3} \quad (3.1.45)$$

While for D=3

$$\frac{n_0}{n}\approx 1-\frac{(mg_0)^{3/2}n^{1/2}}{3\hbar^3\pi^2}+\frac{\Lambda n^{1/2}g_0^{1/2}m^{3/2}}{2\hbar^4\pi^2} \quad (3.1.46)$$

We can also evaluate the leading order finite temperature corrections in the phonon approximation. Manipulation of Eq. (3.1.25) gives

$$\overline{n_\Lambda(T)} = -\Lambda\frac{n_0}{4\hbar}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\frac{n_0\hbar^2q^2V(q)}{mE_B(q)^3}\left(\coth\frac{\beta E_B(q)}{2}-1\right)$$
$$-\frac{\beta/2}{\sinh^2\beta E_B(q)/2}\left[1+\left(\frac{\hbar^2q^2}{2mE_B(q)}\right)^2\right] \quad (3.1.47)$$

Using the fact that $E_B(q) \approx \hbar cq$, setting $x = \beta\hbar cq/2$ and using the following identity

$$\int_0^\infty dx\left[x^{D-2}(\coth x - 1) - \frac{x^{D-1}}{\sinh^2 x}\right] = -\delta_{D,2} - \left[\frac{D-2}{D-1}\right]\int_0^\infty dx\frac{x^{D-1}}{\sinh^2 x} \tag{3.1.48}$$

We finally get

$$\overline{n_\Lambda(T)} \approx \frac{\Lambda n_0 (k_B T)^{D-1}}{4\pi^{D/2}\Gamma(D/2)\hbar^{D+1}c^D}\left[\delta_{D,2} + \frac{D-2}{D-1}\int_0^\infty dx\frac{x^{D-1}}{\sinh^2 x} + \frac{(k_B T)^2}{m^2c^4}\int_0^\infty dx\frac{x^{D+1}}{\sinh^2 x}\right] \tag{3.1.49}$$

When evaluated for D=2 this leads to

$$\overline{n_\Lambda(T)} = \frac{\Lambda n_0 k_B T}{4\pi\hbar^3c^2} + \mathcal{O}(T^3) \tag{3.1.50}$$

while for D=3 we have

$$\overline{n_\Lambda(T)} = \frac{\Lambda n_0 (k_B T)^2}{24\hbar^4c^3} \tag{3.1.51}$$

At finite temperature, in D=3 (remember that according to Eq (1.4.36) thermal corrections diverge when D=2) we can write

$$\frac{n_0}{n} \approx 1 - \frac{(mg_0)^{3/2}n^{1/2}}{3\hbar^3\pi^2} - \frac{m(k_BT)^2}{12\hbar^3cn} + \frac{\Lambda n^{1/2}g_0^{1/2}m^{3/2}}{2\hbar^4\pi^2} - \frac{\Lambda(k_BT)^2}{24\hbar^4c^3} \tag{3.1.52}$$

. We point out the fact that both the thermal depletion ad the disorder depletion are quadratic in the absolute temperature.
In a similar fashion we can derive the disorder contribution to normal-fluid particle density. Since we are dealing with isotropic disorder we have

$$\overline{n_{n\Lambda}} = \frac{1}{m}\lim_{q\to 0}\overline{\chi_\perp(\mathbf{q},\mathbf{0})} = \frac{1}{m\mathcal{V}D\beta}\sum_{\mathbf{p},\omega_n} p^2\frac{E_B(p)^2\hbar^2 - \omega_n^2}{\left[E_B(p)^2/\hbar^2 + \omega_n^2\right]^2} + \frac{n_0}{Dm^2\beta\hbar\mathcal{V}}\sum_{\mathbf{p},\omega_n} p^4\frac{E_B(p)^2/\hbar^2 - 3\omega_n^2}{\left[E_B(p)^2/\hbar^2 + \omega_n^2\right]^3}\Lambda(\mathbf{p},\omega_n) \tag{3.1.53}$$

which leads, after summing over Matsubara frequencies

$$\overline{n_{n\Lambda}} = \frac{n_0\beta\hbar^4}{8Dm^2}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\Lambda q^4}{\hbar E_B(q)}\frac{\cosh\beta E_B(q)/2}{\sinh^3\beta E_B(q)/2} \tag{3.1.54}$$

It is easy to see that in this case

$$\overline{n_{n\Lambda}(T=0)} = 0 \tag{3.1.55}$$

We can find the leading-order temperature corrections in phonon approximation by using the fact that

$$\int_0^\infty dx x^{D+2} \frac{\cosh x}{\sinh^3 x} = \frac{D+2}{2} \int_0^\infty dx \frac{x^{D+1}}{\sinh^2 x} \tag{3.1.56}$$

This leads to

$$\overline{n_{n\Lambda}(T)} = \frac{\Gamma(D+3)\zeta(D+1)}{2^{D+1}\pi^{D/2}\Gamma(1+D/2)} \frac{\Lambda n_0 (k_B T)^{D+1}}{m^2 \hbar^{D+1} c^{D+4}} \tag{3.1.57}$$

when D=2 this yields

$$\overline{n_{n\Lambda}(T)} = \frac{3\zeta(3)\Lambda n_0 (k_B T)^3}{\pi \hbar^3 c^6 m^2} \tag{3.1.58}$$

while for D=3 we have

$$\overline{n_{n\Lambda}(T)} = \frac{\Lambda n_0 (k_B T)^4 \pi^2}{9 m \hbar^4 c^7 m^2} \tag{3.1.59}$$

Again, the total normal-fluid particle density takes into account the thermal contribution given by Eqs (1.6.12)-(1.6.13). However, for this kind of disorder, there is no normal-fluid particle density contribution for T=0, as can be seen by eq (3.1.55). Therefore qe have that, when D=2

$$\frac{n_s}{n} \approx 1 - \frac{3\zeta(3)(k_B T)^3}{2\pi \hbar^2 c^4 m n} - \frac{3\zeta(3)\Lambda (k_B T)^3}{\pi \hbar^3 c^6 m^2} \tag{3.1.60}$$

While for D=3

$$\frac{n_s}{n} \approx 1 - \frac{2\pi^2 (k_B T)^4}{45 \hbar^3 c^5 m n} - \frac{\Lambda (k_B T)^4 \pi^2}{9 m \hbar^4 c^7 m^2} \tag{3.1.61}$$

Also, in this case, the thermal and impurities contribution are of the same order in the temperature.

3.2 Correlated, nearly isotropic splay

This kind of disorder models the situation in which columnar defects are inserted in the system. Assuming the z direction as a preferred direction, we can parametrize the impurities as $\mathbf{r_i}(z) = \mathbf{R_i} + \mathbf{v_i} z$.

Supposing a gaussian distribution for the velocities $\mathbf{v_i}$, meaning $P[\mathbf{v_i}] \propto \prod_i e^{-v_i^2/2v_D^2}$, we obtain in Fourier space

$$\Lambda(\mathbf{q}, q_z) = \frac{\Lambda}{\sqrt{2\pi} v_d q} e^{-q_z^2/2v_D^2 q^2} \tag{3.2.1}$$

where v_D is a dispersion parameter and $\mathbf{q}$ is a two-dimensional vector, describing the motion in the plane perpendicular to the z axis.
To get a more manageable expression we can take the limit $v_D \to \infty$ while keeping the ratio $\frac{\Lambda}{v_D}$ fixed, obtaining

$$\Lambda(\mathbf{q}, q_z) \propto 1/q \tag{3.2.2}$$

an almost isotropic situation. This is not exactly isotropic because we have started from a system with a preferred direction. A truly isotropic correlator would be in the form of

$$\Lambda(\mathbf{q}, q_z) \propto \frac{1}{\sqrt{\mathbf{q^2} + q_z^2}} \tag{3.2.3}$$

We shall discuss the case of nearly isotropic splay, since it makes the calculations easier.
With this said, the impurities correlator is

$$\Lambda(\mathbf{q}, \omega_m; \mathbf{q}', \omega_m') = \frac{\Lambda}{q} (2\pi)^D \delta(\mathbf{q} + \mathbf{q}') \delta_{m,-m'} \tag{3.2.4}$$

We can use the results from the previous section, since the Matsubara summation is untouched. This leads to

$$\overline{S_\Lambda(\mathbf{q})} = \frac{\Lambda}{\hbar q E_B(q)} \left(\frac{n_0 \hbar^2 q^2}{2m E_B(q)} \right)^2 \coth \frac{\beta E_B(q)}{2} \left(1 + \frac{\beta E_B(q)}{\sinh \beta E_B(q)} \right) \tag{3.2.5}$$

For the condesate depletion we get

$$\overline{n_\Lambda} = \frac{n_0}{4} \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{\Lambda}{\hbar q E_B(q)} \coth \frac{\beta E_B(q)}{2} \left[\left(\frac{\hbar^2 q^2}{2, E_B(q)} \right)^2 \left(1 + \frac{\beta E_B(q)}{\sinh \beta E_B(q)} \right) - \left(1 - \frac{\beta E_B(q)}{\sinh \beta E_B(q)} \right) \right] \tag{3.2.6}$$

Taking the zero temperature limit we find

$$\overline{n_\Lambda} = \frac{n_0}{4} \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{\Lambda}{\hbar q E_B(q)} \left[\left(\frac{\hbar^2 q^2}{2m E_B(q)} \right)^2 - 1 \right] = -\Lambda \frac{n_0^2 \hbar}{4m} \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{q V(q)}{E_B(q)^3} \tag{3.2.7}$$

Assuming contact interaction $V(q) \approx g_0$this yields

$$\overline{n_\Lambda} = -\frac{\Lambda n_0^{(D-1)/2} g_0^{(D-3)/2} m^{(D-1)/2}}{8\hbar^D \pi^{D/2} m\Gamma(D/2)} \int_0^\infty \frac{x^{D-3}}{(1+x^2)^{3/2}} dx \tag{3.2.8}$$

This expression is logarithmically divergent when D=2. Since we can write $n - \overline{n_0} = n - \overline{|\psi|^2}$, this fact means that $\overline{|\psi|^2}$ becomes unbound from above. One possible explanation for this is the idea that, in this case, the motion of the superfluid becomes superdiffusve, leading to the divergence.
In three dimension the correction is finite and is

$$\overline{n_\Lambda} = -\frac{\Lambda n_0 m}{4\hbar^3 \pi^2} \tag{3.2.9}$$

Therefore, at zero temperature and for D=3 we have

$$\frac{n_0}{n} \approx 1 - \frac{(mg_0)^{3/2} n^{1/2}}{3\hbar^3\pi^2} + \frac{\Lambda m}{4\hbar^3\pi^2} \tag{3.2.10}$$

Repeating the same steps of the previous section we find expressions for finite temperature corrections

$$\overline{n_\Lambda(T)} \approx \frac{\Lambda n_0 (k_B T)^{D-2}}{8\pi^{D/2}\Gamma(D/2)\hbar^D c^{D-1}} \left[\delta_{D,3} + \frac{D-3}{D-2} \int_0^\infty \frac{x^{D-2}}{\sinh^2 x} dx + \frac{(k_B T)^2}{m^2 c^4} \int_0^\infty \frac{x^D}{\sinh^2 x} dx \right] \tag{3.2.11}$$

This yields for D=3

$$\overline{n_\Lambda(T)} = \frac{\Lambda n_0 k_B T}{4\pi^2\hbar^3 c^2} + \mathcal{O}(T^3) \tag{3.2.12}$$

Therefore we can write

$$\frac{n_0}{n} \approx 1 - \frac{(mg_0)^{3/2} n^{1/2}}{3\hbar^3\pi^2} + \frac{\Lambda m}{4\hbar^3\pi^2} - \frac{m(k_B T)^2}{12\hbar^3 cn} - \frac{\Lambda k_B T}{4\pi^2\hbar^3 c^2} \tag{3.2.13}$$

In this case the normal-fluid density can be written as

$$\overline{n_{n\Lambda}} = \frac{\Gamma(D+2)\zeta(D)}{2^{D+1}\pi^{D/2}\Gamma(1+D/2)} \frac{\Lambda n_0 (k_B T)^D}{m^2\hbar^D c^{D+3}} \tag{3.2.14}$$

This expression, finally, gives in D=2

$$\overline{n_{n\Lambda}} = \frac{\Lambda n_0 \pi (k_B T)^2}{8m^2\hbar^2 c^5} \tag{3.2.15}$$

and when D=3

$$\overline{n_{n\Lambda}} = \frac{2\Lambda\zeta(3)n_0(k_BT)^3}{\pi^2m^2\hbar^3c^6} \tag{3.2.16}$$

These gives, when D=2

$$\frac{n_s}{n} \approx 1 - \frac{3\zeta(3)(k_BT)^3}{2\pi\hbar^2c^4mn} - \frac{\Lambda\pi(k_BT)^2}{8m^2\hbar^2c^5} \tag{3.2.17}$$

while for D=3 we have

$$\frac{n_s}{n} \approx 1 - \frac{2\pi^2(k_BT)^4}{45\hbar^3c^5mn} - \frac{2\Lambda\zeta(3)(k_BT)^3}{\pi^2m^2\hbar^3c^6} \tag{3.2.18}$$

Unlike other cases, there is no easy relation between the normal-fluid density and the condensate depletion.
In all these calculations we have always assumed that the disorder only couples to the particle density. This can be thought as local variations in the chemical potential.
While this coupling is the most relevant one, it is still possible to consider another disorder field $\mathbf{H}(\mathbf{r},\tau)_D$ that couples to the mass current $\mathbf{g}(\mathbf{r},\tau)$.

3.3 Disorder corrections with finite range interaction

The next situation we are going to discuss is the effects of disorder in systems with finite range interactions. In ultracold, dilute boson gases it is customary to consider contact interaction i.e. $V(\mathbf{r}) \propto \delta(\mathbf{r})$. To take into account the effects of the finite range interatomic potential, however, we can consider the following low-momentum expansion in Fourier space

$$V(q) \approx g_0 + g_2q^2 + \mathcal{O}(q^4) \tag{3.3.1}$$

The first order term is not present for symmetry reasons. In general it is useful to connect the two parameters that describe the expansion to measurable quantities, such as the scattering length a_s and the effective range r_e.
This connection is given by scattering theory. For example, in D=1 [23], we can describe the scattering amplitude for an even scattering wave function in terms of the phase shift [24]

$$f_0(q) = qe^{i\delta_0(q)}\sin\delta_0(q) \tag{3.3.2}$$

The phase shift function is useful beacuse we can define the scattering length and the effective range as

$$q\tan\delta_0(q) = \frac{1}{a_s} + \frac{1}{2}r_eq^2 + \mathcal{O}(q^4) \tag{3.3.3}$$

The T matrix can then be written as

$$T_0(q) = \frac{-2\hbar^2}{m} f_0(q) \tag{3.3.4}$$

On the other hand, the T matrix can be found as the solution of [24]

$$T_0(q) = \left[\frac{1}{V(q)} - \frac{m}{2\pi\hbar^2}\int \frac{dp}{p^2 - q^2 + i\epsilon}\right]^{-1} \tag{3.3.5}$$

Inserting the low momentum expansion of V(q) in the previous equation we get

$$T_0(q) = \left[\frac{1}{g_0} - \frac{g_2 q^2}{g_0^2 + \frac{im}{2\hbar^2 q}}\right]^{-1} \tag{3.3.6}$$

Confronting this expression with Eq.(3.3.4) and with some manipulation we find

$$g_0 = -\frac{2\hbar^2}{m a_s} \tag{3.3.7}$$

and

$$g_2 = -\frac{\hbar^2 r_e}{m} \tag{3.3.8}$$

For a more thorough derivation of these results and their consequences see [23]. For the analysis of a similar problem in D=3 see [25].

3.3.1 Uncorrelated static disorder

We shall start our discussion with the most simple case. We shall assume uncorrelated static disorder. This means that the disorder correlator is

$$\Lambda(\mathbf{q}, \omega_m) = \Delta\hbar\beta\delta_{m,0} \tag{3.3.9}$$

For this reason, the impurities contribution to condensate depletion is

$$\overline{n_\Lambda} = \frac{n_0}{4m^2}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{\hbar q}{E_B(q)}\right)^4 \Lambda(\mathbf{q}) \tag{3.3.10}$$

Assuming finite range for the interatomic potential, $V(q) = g_0 + g_2 q^2 + \mathcal{O}(q^4)$, the Bogoliubov excitations assume the form of

$$E_B(q) = \sqrt{\alpha_0\hbar^2 q^2 + \alpha_2\hbar^2 q^4} \tag{3.3.11}$$

where

$$\alpha_0 = \frac{n_0 g_0}{m} \tag{3.3.12}$$

and

$$\alpha_2 = \frac{n_0 g_2}{m} + \frac{\hbar^2}{4m^2} \tag{3.3.13}$$

Therefore we can rewrite (3.3.10) as

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{2^{D+1} m^2 \pi^{D/2} \Gamma(D/2)} \int_0^\infty dq \frac{q^{D-1}}{(\alpha_0 + \alpha_2 q^2)^2} \tag{3.3.14}$$

At this point we can change variables as $t = \frac{\alpha_2}{\alpha_0} q^2$ we get

$$\overline{n_\Lambda} = \frac{\Lambda n_0 \alpha_0^{D/2-2}}{2^{D+2} m^2 \pi^{D/2} \Gamma(D/2) \alpha_2^{D/2}} \int_0^\infty dt \frac{t^{D/2-1}}{(1+t)^2} = \frac{\Lambda n_0 \alpha_0^{D/2-2}}{2^{D+2} m^2 \pi^{D/2} \alpha_2^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} \tag{3.3.15}$$

where we used the properties of Euler beta function. This leads to, for D=2

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{16 m^2 \pi \alpha_0 \alpha_2} \tag{3.3.16}$$

When D=3, on the other hand, we get

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{32 m^2 \pi \alpha_2^{3/2} \alpha_0^{1/2}} \tag{3.3.17}$$

Since we are considering isotropic disorder, the normal-fluid particle density is related to the disorder-induced condensate depletion by

$$\overline{n_{n\Lambda}} = \frac{4}{D} \overline{n_\Lambda} \tag{3.3.18}$$

We can now summarize the obtained results in terms of the quotient $\frac{n_0}{n}$. When D=2, at zero temperature we have

$$\frac{n_0}{n} = 1 - \frac{m n_0 g_0}{4\pi\hbar^2 n} - \frac{\Lambda n_0}{16 m^2 \pi \alpha_0 \alpha_2 n} \approx 1 - \frac{m g_0}{4\pi\hbar^2} - \frac{\Lambda}{16 m^2 \pi (n g_0/m)(n g_2/m + \hbar^2/4m^2)} \tag{3.3.19}$$

While for D=3

$$\frac{n_0}{n} = 1 - \frac{(m n_0 g_0)^{3/2}}{3\hbar^3 \pi^2 n} - \frac{m (k_B T)^2}{12 \hbar^3 c n} - \frac{\Lambda n_0}{32 m^2 \pi \alpha_0^{1/2} \alpha_2^{3/2} n} \approx 1 - \frac{(m g_0)^{3/2} n^{1/2}}{3\hbar^3 \pi^2} - \frac{m (k_B T)^2}{12 \hbar^3 c n} - \frac{\Lambda}{32 m^2 \pi (n g_0/m)^{1/2} (n g_2/m + \hbar^2/4m^2)^{3/2}} \tag{3.3.20}$$

For the superfluid fraction, we can write, when D=2

$$\frac{n_s}{n} \approx 1 - \frac{3\zeta(3)(k_BT)^3}{2\pi\hbar^2 c^4 mn} - \frac{\Lambda}{8m^2\pi(ng_0/m)(ng_2/m + \hbar^2/4m^2)} \tag{3.3.21}$$

And when D=3

$$\frac{n_s}{n} \approx 1 - \frac{2\pi^2(k_BT)^4}{45\hbar^3 c^5 mn} - \frac{\Lambda}{24m^2\pi(ng_0/m)^{1/2}(ng_2/m + \hbar^2/4m^2)^{3/2}} \tag{3.3.22}$$

In the above expression, on the right hand side, we have approximated the condensate fraction with the total particle density, which is a fixed parameter and we have used the definition of the coefficients α_0 and α_2 (see Eqs. (3.3.12) and (3.3.13)). This, albeit non exact, is very useful to compare experimental data with the calculation results. The first term describes the depletion caused by quantum fluctuations, while the second one the disorder contribution.
We can now extend the above analysis to the case with disorder uncorrelated in both space and time. This contribution is, in the general case, very complicated to evaluate, since it is given by

$$\begin{aligned}\overline{n_\Lambda} = \frac{n_0}{4}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\Lambda}{\hbar q E_B(q)}\coth\frac{\beta E_B(q)}{2} \\ \left[\left(\frac{\hbar^2q^2}{2, E_B(q)}\right)^2\left(1+\frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right) - \left(1-\frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right)\right]\end{aligned} \tag{3.3.23}$$

This is a very complicated expression. In the previous sections we were able to find explicit results at finite temperature at the price of approximating the dispersion relation with the phonon limit. However, when studying the finite range corrections, we can not use this approximation. To obtain exact results, therefore, we shall focus on the zero temperature limit, in which the condensate depletion takes the form of

$$\overline{n_\Lambda} = -\frac{\Lambda n_0^2\hbar}{2^{D+1}m\pi^{D/2}\Gamma(D/2)}\int_0^\infty dq\frac{q^{D+1}(g_0+g_2q^2)}{\left(\alpha_0\hbar^2q^2+\alpha_2\hbar^2q^4\right)^{3/2}} \tag{3.3.24}$$

Therefore, after some manipulations, we have two terms that contribute to disorder-induced condensate depletion

$$\begin{aligned}\overline{n_\Lambda} = -\frac{\Lambda n_0^2}{2^{D+2}\hbar^2\alpha_0^{3/2}m\pi^{D/2}\Gamma(D/2)}\left[g_0\left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{D-1}{2}}\int_0^\infty dt\frac{t^{(D-3)/2}}{(1+t)^{3/2}}\right. \\ \left. + g_2\left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{D+1}{2}}\int_0^\infty dt\frac{t^{(D-1)/2}}{(1+t)^{3/2}}\right]\end{aligned} \tag{3.3.25}$$

The first term of the sum in square brackets in Eq. (3.3.21) gives a finite result. However, the second term is clearly ultraviolet-divergent. Using Euler beta function we can regularize the divergence with ease when D=3. Therefore we can write

$$\overline{n_\Lambda} = -\frac{\Lambda n_0^2}{16\hbar^2 m\pi^2\alpha_0^{3/2}}\left[2g_0\frac{\alpha_0}{\alpha_2} + g_2\left(\frac{\alpha_0}{\alpha_2}\right)^2\frac{\Gamma(2)\Gamma(1/2)}{\Gamma(3/2)}\right] = \\ -\frac{\Lambda n_0^2}{8\hbar^2 m\pi^2\alpha_0^{3/2}}\left[g_0\frac{\alpha_0}{\alpha_2} + g_2\left(\frac{\alpha_0}{\alpha_2}\right)^2\right] \quad (3.3.26)$$

The case with D=2, however, needs more attention. In this case, in fact, even using beta function regularization does not neutralize the divergence. Let us focus on this contribution and write

$$I = \left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{D+1}{2}}\int_0^\infty dt\frac{t^{(D-1)/2}}{(1+t)^{3/2}} \quad (3.3.27)$$

If we let D be a complex parameter, we can then consider the quantity $\epsilon = 2 - D$ and the integral becomes

$$I = \left(\frac{\alpha_0}{\alpha_2}\right)^{3/2}\left(\frac{\alpha_2\kappa^2}{\alpha_0}\right)^{\frac{\epsilon}{2}}\frac{\Gamma((3-\epsilon)/2)\Gamma(\epsilon/2)}{\Gamma(3/2)} \quad (3.3.28)$$

Here κ is a parameter introduced for dimensional reasons and describes a high energy scale for the system. We now wish to evaluate the limit $\epsilon \to 0$ and isolate the divergent contribution. Remembering that

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} + \tilde{\psi}(1) + \mathcal{O}(\epsilon) \quad (3.3.29)$$

where $\tilde{\psi}(z)$ is the digamma function

$$\tilde{\psi}(z) = \frac{d}{dx}\log\Gamma(z) \quad (3.3.30)$$

When $z = 1$ we have $\tilde{\psi}(1) = -\gamma$, where γ is the Eulero-Mascheroni constant. Moreover we can write

$$\left(\frac{\alpha_2\kappa^2}{\alpha_0}\right)^{\frac{\epsilon}{2}} = e^{\frac{\epsilon}{2}\log\left(\frac{\alpha_2\kappa^2}{\alpha_0}\right)} = 1 + \frac{\epsilon}{2}\log\left(\frac{\alpha_2\kappa^2}{\alpha_0}\right) + \mathcal{O}(\epsilon^2) \quad (3.3.31)$$

Putting all together we find

$$I = \left(\frac{\alpha_0}{\alpha_2}\right)^{3/2}\left(\frac{2}{\epsilon} + \log\frac{e^{-\gamma}\alpha_2\kappa^2}{\alpha_0} + \mathcal{O}(\epsilon)\right) \quad (3.3.32)$$

We have successfully isolated the divergent part of the integral. Therefore we can remove it and, taking the $\epsilon \to 0$, putting all the pieces together, we find that the disorder-induced condensate depletion when D=2 is

$$\overline{n_\Lambda} = -\frac{\Lambda n_0^2}{8\hbar^2 \alpha_0^{3/2}\pi}\left[g_0\sqrt{\frac{\alpha_0}{\alpha_2}} + \frac{2}{\sqrt{\pi}}\left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{3}{2}} g_2 \log\frac{e^{-\gamma}\alpha_2\kappa^2}{\alpha_0}\right] \tag{3.3.33}$$

We can, again, summarize the results as

$$\frac{n_0}{n} = 1 - \frac{mn_0g_0}{4\pi\hbar^2 n} + \frac{\Lambda n_0^2}{8\hbar^2\alpha_0^{3/2}\pi^2 n}\left[g_0\sqrt{\frac{\alpha_0}{\alpha_2}} + \frac{2}{\sqrt{\pi}}\left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{3}{2}} \log\frac{e^{\gamma}\alpha_2\kappa^2}{\alpha_0}\right] \approx$$
$$1 - \frac{mg_0}{4\pi\hbar^2} + \frac{\Lambda n}{8\hbar^2(ng_0/m)^{3/2}\pi^2}\left[g_0\sqrt{\frac{ng_0/m}{ng_2/m + \hbar^2/4m^2}} + \right.$$
$$\left.\frac{2}{\sqrt{\pi}}\left(\frac{ng_0/m}{ng_2/m+\hbar^2/4m^2}\right)^{\frac{3}{2}} \log\frac{e^{-\gamma}\alpha_2\kappa^2}{\alpha_0}\right] \tag{3.3.34}$$

This is valid when D=2.
When D=3 we have

$$\frac{n_0}{n} = 1 - \frac{(mn_0g_0)^{3/2}}{3\hbar^3\pi^2 n} + \frac{\Lambda n_0^2}{8\hbar^2 m\pi^2\alpha_0^{3/2} n}\left[g_0\,\frac{\alpha_0}{\alpha_2} + g_2\left(\frac{\alpha_0}{\alpha_2}\right)^2\right] \approx$$
$$1 - \frac{(mg_0)^{3/2}n^{1/2}}{3\hbar^3\pi^2} + \frac{\Lambda n}{8\hbar^2 m\pi^2(ng_0/m)^{3/2}}\left[g_0\frac{ng_0/m}{ng_2/m+\hbar^2/4m^2} + \right.$$
$$\left. g_2\left(\frac{ng_0/m}{ng_2/m+\hbar^2/4m^2}\right)^2\right] \tag{3.3.35}$$

In this limit, we have that the normal-fluid particle density is still given by Eq. (3.1.54), with a different expression for the Bogoliubov spectrum. Therefore, at zero temperature, this contribution is zero.

3.3.2 Lorentzian correlation

In this section we calculate the effects of a Lorentzian disorder correlator both for contact and finite range interaction. This is motivated by the fact that this kind of disorder can be artificially implement in a system by using laser speckle techniques.
In this case, the disorder correlator is

$$\Lambda(\mathbf{q}) = \frac{\Lambda}{1+\sigma q^2} \tag{3.3.36}$$

where Λ describes the strength of the disorder and σ describes the correlation length. We shall calculate the disorder contribution with contact interaction first

$$\overline{n_\Lambda} = \frac{n_0}{4m^2} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \left(\frac{\hbar q}{E_B(q)} \right)^4 \Lambda(\mathbf{q}). \tag{3.3.37}$$

After the substituting for the specific case, and remembering that $V(q) \approx g_0$ we get

$$\overline{n_\Lambda} = \frac{m^{D/2} n_0^{(D-2)/2} g_0^{(D-4)/2}}{2\hbar^D \pi^{D/2} \Gamma(D/2)} \int_0^\infty dt \frac{t^{D-1}}{(1+t^2)^2(1+\tilde{\sigma} t^2)} \tag{3.3.38}$$

Where

$$\tilde{\sigma} = \frac{4 n_0 g_0 m}{\hbar^2} \sigma \tag{3.3.39}$$

Finally, we can set D=2 and we find

$$\overline{n_\Lambda} = \frac{\Lambda m}{2\pi\hbar^2 g_0} f_2(\tilde{\sigma}) \tag{3.3.40}$$

where

$$f_2(\tilde{\sigma}) = \frac{1 - \tilde{\sigma} + \tilde{\sigma} \log \tilde{\sigma}}{2(1-\tilde{\sigma})^2} \tag{3.3.41}$$

When D=3 we have

$$\overline{n_\Lambda} = \frac{\Lambda n_0^{1/2} m^{3/2}}{\pi \hbar^3 g_0^{1/2}} f_3(\tilde{\sigma}) \tag{3.3.42}$$

where

$$f_3(\tilde{\sigma}) = \frac{1}{4(1+\sqrt{\tilde{\sigma}})^2} \tag{3.3.43}$$

These equations describe the **static** impurities contribution to condensate depletion in case of Lorentzian-correlated disorder with contact interaction. We can generalize the analysis to the case of finite range interaction.
In fact, remembering Eqs. (3.3.12)-(3.3.13), we can write the disorder contribution as

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{2^{D+1} m^2 \pi^{D/2} \Gamma(D/2)} \int_0^\infty dq \frac{q^{D-1}}{(\alpha_0 + \alpha_2 q^2)^2} \frac{1}{1+\sigma q^2} \tag{3.3.44}$$

By changing variables $t = \sqrt{\frac{\alpha_2}{\alpha_0}} q$ and by defining

$$\overline{\sigma} = \frac{\alpha_0}{\alpha_2} \sigma \tag{3.3.45}$$

we get

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{2^{D+1} m^2 \pi^{D/2} \Gamma(D/2) \alpha_0^2} \left(\frac{\alpha_0}{\alpha_2}\right)^{D/2} f_D(\overline{\sigma}) \tag{3.3.46}$$

Therefore we get, for D=2,

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{8 m^2 \pi \alpha_0 \alpha_2} f_2(\overline{\sigma}) \tag{3.3.47}$$

While for D=3 we get

$$\overline{n_\Lambda} = \frac{\Lambda n_0}{16 m^2 \pi \alpha_0^{1/2} \alpha_2^{3/2}} f_3(\overline{\sigma}) \tag{3.3.48}$$

We can summarize these results in the following equations.
When D=2 we have at zero temperature

$$\frac{n_0}{n} \approx 1 - \frac{m g_0}{4\pi\hbar^2} - \frac{\Lambda}{8 m^2 \pi (n g_0/m)(n g_2/m + \hbar^2/4m^2)} f_2(\overline{\sigma}) \tag{3.3.49}$$

When D=3 we have

$$\frac{n_0}{n} \approx 1 - \frac{(m g_0)^{3/2} n^{1/2}}{3\hbar^3\pi^2} - \frac{m (k_B T)^2}{12 \hbar^3 c n} - \frac{\Lambda}{16 m^2 \pi (n g_0/m)^{1/2} (n g_2/m + \hbar^2/4m^2)^{3/2}} f_3(\overline{\sigma}) \tag{3.3.50}$$

We want to stress the fact that, since these results have been found assuming static disorder, Eq. (3.3.18) holds true, so there is an easy relation between condensate depletion and normal-fluid particle density.
Finally, we wish to find results for time-uncorrelated Lorentzian disorder. This means that the correlator function in Fourier space is

$$\Lambda(\mathbf{q}, \omega_m) = \Lambda \frac{\hbar\beta}{1 + \sigma q^2} \delta_{m,-m'} \tag{3.3.51}$$

This, as we have already showed, leads to the following equation for the condensate depletion.

$$\overline{n_\Lambda} = \frac{n_0}{4} \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{\Lambda}{\hbar E_B(q)(1 + \sigma q^2)} \coth \frac{\beta E_B(q)}{2} \left[\left(\frac{\hbar^2 q^2}{2, E_B(q)}\right)^2 \left(1 + \frac{\beta E_B(q)}{\sinh \beta E_B(q)}\right) - \left(1 - \frac{\beta E_B(q)}{\sinh \beta E_B(q)}\right) \right] \tag{3.3.52}$$

Taking the zero temperature limit, this leads to

$$\overline{n_\Lambda(T=0)} = -\Lambda \frac{n_0^2 \hbar}{4m} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{q^2 V(q)}{E_B(q)^3} \frac{1}{1+\sigma q^2} \tag{3.3.53}$$

Remembering that

$$E_B(q) = \sqrt{\alpha_0 \hbar^2 q^2 + \alpha_2 \hbar^2 q^4} \tag{3.3.54}$$

And that

$$V(q) = g_0 + g_2 q^2 + \mathcal{O}(q^4) \tag{3.3.55}$$

We can rewrite the previous integral as

$$\overline{n_\Lambda(T=0)} = -\frac{\Lambda n_0^2}{2^{D+1} m \hbar^2 \pi^{D/2} \Gamma(D/2) \alpha_0^{3/2}}$$
$$\left[\left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{D-1}{2}} g_0 \int_0^\infty \frac{t^{D-2} dt}{(1+t^2)^{3/2}(1+\overline{\sigma} t^2)} + \left(\frac{\alpha_0}{\alpha_2}\right)^{\frac{D+1}{2}} g_2 \int_0^\infty \frac{t^D dt}{(1+t^2)^{3/2}(1+\overline{\sigma} t^2)} \right] \tag{3.3.56}$$

where

$$t = \sqrt{\frac{\alpha_2}{\alpha_0}} q \tag{3.3.57}$$

and $\overline{\sigma}$ is given by Eq. (3.3.29). This leads to the following results. When D=2 we have

$$\overline{n_\Lambda(T=0)} = -\frac{\Lambda n_0}{8 m \hbar^2 \pi \alpha_0^{3/2}} \left[\sqrt{\frac{\alpha_0}{\alpha_2}} g_0 f_2(\overline{\sigma}) + \left(\frac{\alpha_0}{\alpha_2}\right)^{3/2} g_2 h_2(\overline{\sigma}) \right] \tag{3.3.58}$$

Where

$$f_2(\overline{\sigma}) = \frac{1}{1-\overline{\sigma}} + \frac{\overline{\sigma} \sec^{-1}(\sqrt{\overline{\sigma}})}{(-1+\overline{\sigma})^{3/2}} \tag{3.3.59}$$

and

$$h_2(\overline{\sigma}) = \frac{1}{-1+\overline{\sigma}} - \frac{\sec^{-1}(\sqrt{\overline{\sigma}})}{(-1+\overline{\sigma})} \tag{3.3.60}$$

Finally, when D=3 we get

$$\overline{n_\Lambda(T=0)} = -\frac{\Lambda n_0^2}{8 m \hbar^2 \pi^2 \alpha_0^{3/2}} \left[\frac{\alpha_0}{\alpha_2} g_0 f_3(\overline{\sigma}) + \left(\frac{\alpha_0}{\alpha_2}\right)^2 g_2 h_3(\overline{\sigma}) \right] \tag{3.3.61}$$

where

$$f_3(\overline{\sigma}) = \frac{1-\overline{\sigma} - \sqrt{\overline{\sigma}(1-\overline{\sigma})} \cos^{-1}(\sqrt{\overline{\sigma}})}{(\overline{\sigma}-1)^2} \tag{3.3.62}$$

and

$$h_3(\overline{\sigma}) = \frac{-1 + \overline{\sigma} + \sqrt{-1 + 1/\overline{\sigma}}\cos^{-1}(\sqrt{\overline{\sigma}})}{(\overline{\sigma} - 1)^2} \tag{3.3.63}$$

Therefore, we can write that, when D=2

$$\frac{n_0}{n} \approx 1 - \frac{mg_0}{4\pi\hbar^2} + \frac{\Lambda}{8m\hbar^2\pi(ng_0/m)^{3/2}}\left[\sqrt{\frac{ng_0/m}{ng_2/m + \hbar^2/4m^2}} g_0 f_2(\overline{\sigma}) + \left(\frac{ng_0/m}{ng_2/m + \hbar?4/4m^2}\right)^{3/2} g_2 h_2(\overline{\sigma})\right] \tag{3.3.64}$$

When D=3 we can write

$$\frac{n_0}{n} \approx 1 - \frac{(mg_0)^{3/2} n^{1/2}}{3\hbar^3\pi^2} + \frac{\Lambda n}{8m\hbar^2\pi^2(ng_0/m)^{3/2}}\left[\frac{ng_0/m}{ng_2/m + \hbar^2/4m^2} g_0 f_3(\overline{\sigma}) + \left(\frac{ng_0/m}{ng_2/m + \hbar^2/4m^2}\right)^2 g_2 h_3(\overline{\sigma})\right] \tag{3.3.65}$$

Chapter 4

Magnetic Flux Lines in type-II superconductors

Type-II superconductors are materials characterized by the formation of quantized vortices inside them, when a external magnetic field H_{c1} is applied. Moreover, there is also an upper limit for the external field H_{c2} over which the superconductive state is broken.
Another interesting feature of type-II superconductors is the absence of complete Meissner effect. This means that it is possible for magnetic fields to penetrate the bulk of the sample. As a relevant consequence, this kind of superconductive materials are subject to flux pinning when in the vortex state, i.e they can be pinned in space over a magnet. This is a very relevant feature, since there have been works that suggests that the right kind of disorder could actually increase flux pinning [26].
The first theory that tried to explain the nature of this behavior is the Ginzburg-Landau theory. In this theory, there are two relevant length scales, the coherence length ξ and the penetration length λ. The first one, ξ is related to the length over which the electron density correlation function relaxes to the mean-field value. The second one λ arises from the phenomenological London theory and describes the depth of the penetration of the magnetic field inside the bulk of a superconductor.
Working in the London limit (i.e. very short coherence length) the Gibbs free energy of a slab of superconductive material of lenght L in the z direction in

the vortex state, with N flux lines takes the form of [6]:

$$G_N(\mathbf{H}) = \int_0^L dz \sum_{i=1}^{N} \left(\alpha \sqrt{1 + \frac{m_\perp}{m_z} \left| \frac{d\mathbf{r}_i(z)}{dz} \right|^2} + U_D(\mathbf{r}_i(z)) \right) + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i(z) - \mathbf{r}_j(z)|) - \frac{\mathbf{H}}{4\pi} \int d^3 r \mathbf{b}(\mathbf{r}) \tag{4.0.1}$$

Here we have parametrized the planar coordinates of each flux line with the z coordinate. In the path integral formulation, z will become the imaginary time τ.
Here $\alpha = \alpha' \log \kappa$, where $\alpha' = \left(\frac{\phi_0}{4\pi\lambda}\right)^2$ where ϕ_0 is the vortex quantum and $\kappa = \frac{\lambda}{\xi}$ is an adimensional parameter that in the Ginzburg-Landau theory discriminates between type-I ($0 < \kappa < \sqrt{1/2}$) and type-II ($\kappa > \sqrt{1/2}$) materials.
We have taken into consideration eventual material anisotropies by defining an effective mass ratio $\frac{m_\perp}{m_z}$. Finally, the last term describes the interaction between the local magnetic field density $\mathbf{b}(\mathbf{r})$ and the external magnetic field.
We can assume that the lines are almost parallel to the z direction, so that the square root can be expanded. Moreover, we can express the last piece in terms of the number of lines and the flux quantum. This leads to

$$G_N(\mathbf{H}) = \mu N L + F_N \mathbf{r}_i(z) = \mu N L + \int_0^L dz \sum_{i=1}^{N} \left(\frac{\tilde{\alpha}}{2} \left| \frac{d\mathbf{r}_i(z)}{dz} \right|^2 + U_D(\mathbf{r}_i(z)) \right) + \frac{1}{2} \sum_{i,j} V(|\mathbf{r}_i(z) - \mathbf{r}_j(z)|) \tag{4.0.2}$$

where μ is the flux-line equivalent of the chemical potential and is defined as

$$\mu = \frac{\phi_0}{4\pi}(H - H_{c1}) \tag{4.0.3}$$

where H_{c1} is the critical field above which we have magnetic vortices and is given by $H_{c1} = \frac{4\pi\alpha}{\phi_0}$.
In Eq (4.0.2) we have introduced a new parameter $\tilde{\alpha} = \frac{m_\perp \alpha}{m_z}$. Looking at the problem from a statistical point of view, we would have to compute the grand canonical partition function (i.e summation over all the possible vertex trajectories)

$$\mathcal{Z}_{gr}^{fl} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\beta \mu N L} \mathcal{Z}_n^{fl} \tag{4.0.4}$$

where the canonical partition function for N flux lines them can be written in the functional form

$$\mathcal{Z}_N^{fl} = \prod_{i=1}^{N} \int \mathcal{D}[\mathbf{r}_i(z)] e^{\beta F_N[\mathbf{r_i}(z)]} \tag{4.0.5}$$

where $F_N[\mathbf{r}_i(z)]$ is given by Eq. (4.0.2). Eq. (4.0.5) can be interpreted as a quantum-mechanical partition function in the path integral representation. It would describe particles of effective mass $\tilde{\alpha}$ moving through imaginary time z. To further the analogy with the quantum bosonic case, we can compare the role of the external field to the that of the chemical potential. For example, when T=0, vortices will penetrate in the bulk of the sample when H>H_{c1}. The analogy with bosonic systems can be made more precise. The last partition sum can be written in terms of the transfer matrix [27] $e^{-L\beta H_N^{fl}}$, connecting neighboring slices at constant z. The Hamiltonian is given by

$$H_N^{fl} = \sum_{i=1}^{N} \left[\frac{-(k_B T)^2}{2\tilde{\alpha}} \boldsymbol{\nabla}_i^2 + U_D(\mathbf{r_i}) \right] + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r_i} - \mathbf{r_j}|) \tag{4.0.6}$$

This Hamiltonian is remarkably similar to Eq (1.0.1). This let us write

$$\mathcal{Z}_N^{fl} = \prod_{i,i'}^{N} \int d\mathbf{r}'_{\mathbf{i}} d\mathbf{r_i} \langle \mathbf{r}'_{\mathbf{1}} \cdots \mathbf{r}'_N | e^{-L\beta H_N^{fl}} | \mathbf{r_1} \cdots \mathbf{r}_N \rangle \tag{4.0.7}$$

here the states $\langle \mathbf{r}'_{\mathbf{1}} \cdots \mathbf{r}'_N|$ and $|\mathbf{r_1} \cdots \mathbf{r}_N\rangle$ describe the points of entry and exit of the vortices. Introducing energy eigenstates $|n\rangle$, and defining the zero-momentum state (this is a consequence of the orthogonality of position and momentum eigenvectors)

$$|\mathbf{p_1} = \mathbf{0} \cdots \mathbf{p_N} = \mathbf{0}\rangle = \prod_{i=1}^{n} \int d\mathbf{r_i} |\mathbf{r_i} \cdots \mathbf{r_N}\rangle \tag{4.0.8}$$

We can rewrite Eq (4.0,7) as

$$\mathcal{Z}_N^{fl} = \langle \mathbf{p_1} = \mathbf{0} \cdots \mathbf{p_N} = \mathbf{0} | e^{-\beta L H_N^{fl}} | \mathbf{p_1} = \mathbf{0} \cdots \mathbf{p_N} = \mathbf{0}\rangle$$
$$\sum_n |\langle n | \mathbf{p_1} = \mathbf{0} \cdots \mathbf{p_N} = \mathbf{0}\rangle|^2 e^{-\beta L E_N^{fl}} \tag{4.0.9}$$

We can interpret this matrix element as if we had a system in the ground-state $|\mathbf{p_1} = \mathbf{0} \cdots \mathbf{p_N} = \mathbf{0}\rangle$ of the ideal Bose gas at z=0 and we let it evolve under

the action of interaction and disorder up to time L and then we projected it onto the ideal Bose gas ground state.
The reason we have only bosonic state is that the zero-momentum state onto which we are projecting, and the Hamiltonian (4.0.6) are symmetrical under the exchange of any two fictitious particles. Therefore the states that contribute to the partition sum are symmetric themselves. We can therefore write

$$\mathcal{Z}_N^{fl} = \sum_{\text{bosonic states } |n\rangle} |\langle n|\mathbf{p_1} = \mathbf{0} \cdots \mathbf{p_N} = \mathbf{0}\rangle|^2 e^{-\beta L E_N^{fl}} \tag{4.0.10}$$

Comparing this partition function with the pure bosonic one (1.0.2) we can see that there is a difference in the weights of the projections on the ground state. However, if we consider the thermodynamical limit L $\to \infty$ then only the lowest energy state contributes to sum (4.0.10) and we have a complete mapping between the quantum behavior of superfluids bosons at T=0 and the statistical behavior of flux lines in the thermodynamical limit. If we impose periodic boundary conditions to the flux lines, for example considering a thoroidal geometry, this mapping survives even at finite temperature.
However, periodic boundary conditions are very artificial and don't suit many physical situations very well. Therefore it is better to assume opening boundary conditions in the flux lines picture, as in Eq (4.0.7), and integrate over the entry and exit points. This assumption is translated in the boson language using an ideal Bose gas with boundary conditions

$$\psi(\mathbf{r}, 0) = \psi(\mathbf{r}, L) = \sqrt{n} \tag{4.0.11}$$

Imposing this constraint, however, has consequences that need to be addressed. Eq. (4.0.11) implies that, while studying the flux lines problem using the boson fictitious particles formalism, the boson order parameter can not fluctuate at the edges of the sample. This means that, within the harmonic approximation in the path integral approach, the fields $\pi(\mathbf{r}, \tau)$ and $\Theta(\mathbf{r}, \tau)$ will have additional constraints. In the following section we shall discuss the properties of flux lines using the boson formalism, to provide a unified framework. The following table summarizes the correspondence of the quantities in the two pictures

Superfluid bosons	m	$\hbar$	$\beta\hbar$	n	μ
Vortex liquid	$\tilde{\alpha}$	T	L	B/ϕ_0	(H-H$_{c1}$)$\phi_0/4\pi$

In particular we can see how, in the flux lines picture, the size of the sample in the z-direction is equivalent to the temperature in the boson case. Therefore, when we shall talk about finite-temperature corrections in open

boundary conditions, we shall be describing the effects of the finite size of the superconductiong slab.
Before moving on, however, it is very important to understand the meaning of the bosonic order parameter n_0 and the normal-fluid particle density n_n in the flux line picture. A non-zero order parameter, associated to superfluidity, implies a long-range order for the system, which in turn implies non-vanishing correlation functions

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} \langle \psi(\mathbf{r},\tau)\psi^*(\mathbf{r}',\tau')\rangle = n_0 \neq 0 \tag{4.0.12}$$

Therefore, in the flux line picture, the order parameter describes the entanglement of the directed magnetic lines.
To better understand the meaning of n_n we have to go back at the definition in terms on the appropriate limit of a response function. In the case of bosonic superfluids it was the coefficient associated to the perturbation induced by a velocity field; in the flux lines picture, on the other hand, this transport coefficient is associated to a tilt in the external magnetic field. Therefore we have a $H_\perp(\mathbf{r}, z)$, perpendicular to the z-direction (the imaginary time in the bosonic picture). This tilt can be taken into consideration by adding an extra term to the lagrangian

$$\int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\ i\mathbf{v}(\mathbf{r},\tau)\mathbf{g}(\mathbf{r},\tau) - mv(\mathbf{r},\tau)^2 n(\mathbf{r},\tau)/2 \tag{4.0.13}$$

where

$$\mathbf{v}(\mathbf{r},\tau) = \frac{\phi_0}{4\pi\tilde{\alpha}}\mathbf{H}_\perp(\mathbf{r},\tau) \tag{4.0.14}$$

and $\mathbf{g}(\mathbf{r},\tau)$ si given by Eq. (1.2.13). We can then define

$$T_{ij}(\mathbf{r},\tau;\mathbf{r}',\tau') = \frac{\hbar^2}{m^2}\frac{\delta^2 \log \mathcal{Z}[\mathbf{v}]}{\delta v_i \delta v_j}|_{\mathbf{v}=0} \tag{4.0.15}$$

In Fourier space this can be expressed as

$$T_{ij}(\mathbf{q},\omega_m) = \frac{n\hbar}{m}\delta_{ij} - C_{ij}(\mathbf{q},\omega_m)/m^2 \tag{4.0.16}$$

where $C_{ij}(\mathbf{q},\omega_m)$ is given by Eq. (1.4.8). Therefore, evaluating this expression in the limit $\mathbf{q}\to 0$ and then $\omega_m \to 0$ we can extract a matrix of transport coefficients

$$(c^v_{ij})^{-1} = \frac{1}{n^2\hbar}\lim_{\omega_m\to 0} T_{ij}(0,\omega_m) = \frac{1}{nm}\left(\delta_{ij} - \frac{1}{nm\beta\mathcal{V}}\sum_{\mathbf{p},\omega_n} p_i p_j \frac{E_B(p)^2/\hbar^2 - \omega_n^2}{[E_(p)^2/\hbar^2 + \omega_n^2]^2}\right) \tag{4.0.17}$$

For an isotropic system the matrix reduces to the tilt modulus $(c^v_{ij})^{-1} = \tilde{c}^{-1}\delta_{ij}$ where

$$\tilde{c} = \frac{(nm)^2}{mn - mn_n} \tag{4.0.18}$$

This is the definition of normal-fluid particle density in the flux line picture and is related to the response of the system to a tilting external magnetic field.

4.1 Corrections for open boundary condtions

As we have discussed in the previous section, we can study the flux lines in superconductors with open boundary conditions using bosonic path integral with constraints on the fields, embodied by Eq. (4.0.11). This results in the following constraints for the fields $\pi(\mathbf{r}, \tau)$ and $\Theta(\mathbf{r}, \tau)$ introduces in Eq.(1.2.10)

$$\pi(\mathbf{r}, 0) = \pi(\mathbf{r}, \beta\hbar) = 0 \tag{4.1.1}$$

and

$$\Theta(\mathbf{r}, 0) = \Theta(\mathbf{r}, \beta\hbar) = 0 \tag{4.1.2}$$

These constraints can be included in the path-integral formulation. For notation clarity, we can introduce the vector

$$X(\mathbf{r}, \tau) = \begin{pmatrix} \Theta(\mathbf{r}, \tau) \\ \pi(\mathbf{r}\tau) \end{pmatrix} \tag{4.1.3}$$

Then, we can write the correlation matrix as

$$\langle X(\mathbf{r}, \tau) X^T(\mathbf{r}', \tau')\rangle = \frac{\int \mathcal{D}[X(\mathbf{r}, \tau)] X(\mathbf{r}, \tau) X^T(\mathbf{r}', \tau') e^{-S[X]/\hbar} \prod_{\mathbf{r}} \delta(X(\mathbf{r}, 0))}{\int \mathcal{D}[X(\mathbf{r}, \tau)] e^{-S[X]/\hbar} \prod_{\mathbf{r}} \delta(X(\mathbf{r}, 0))} =$$
$$\frac{\int \mathcal{D}[X(\mathbf{r}, \tau)] \mathcal{D}[\sigma(\mathbf{r})] X(\mathbf{r}, \tau) X^T(\mathbf{r}', \tau') e^{-S[X]/\hbar + i\int d^D\mathbf{r}\sigma(\mathbf{r})X(\mathbf{r})}}{\int \mathcal{D}[X(\mathbf{r}, \tau)] \mathcal{D}[\sigma(\mathbf{r})]^{-S[X]/\hbar + i\int d^D\mathbf{r}\sigma(\mathbf{r})X(\mathbf{r})}} \tag{4.1.4}$$

In the second line we have introduces an auxiliary field that realizes the constraints of Eqs. (4.1.1)-(4.1.2). To compute this functional integral we can rewrite it in the following form

$$\langle X(\mathbf{r}, \tau) X^T(\mathbf{r}', \tau')\rangle = \frac{\int \mathcal{D}[\sigma(\mathbf{r})] \langle X(\mathbf{r}, \tau) X^T(\mathbf{r}', \tau') e^{i\int d^D\mathbf{r}\sigma(\mathbf{r})X(\mathbf{r})}\rangle_0}{\int \mathcal{D}[\sigma(\mathbf{r})] \langle e^{i\int d^D\mathbf{r}\sigma(\mathbf{r})X(\mathbf{r})}\rangle_0} \tag{4.1.5}$$

where the average $\langle \cdots \rangle_0$ is taken with respect to the dynamical fields and the harmonic action. We can easily evaluate the gaussian averages in Fourier space. Doing so let us split the integration in two steps, averaging out the dynamical degrees of freedom at first, and then the static field [27]. In this way, we can write the correlation matrix as

$$\langle X(\mathbf{q},\omega_m)X^T(\mathbf{q}',\omega_{m'})\rangle = (2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\hbar\mathbf{A^{-1}}(\mathbf{q},\omega_m)\big[\beta\hbar\delta_{m,-m'}-\mathbf{A}(\mathbf{q})\mathbf{A^{-1}}(\mathbf{q},-\omega_{\mathbf{m'}})\big] \tag{4.1.6}$$

Here, we have that $\hbar\mathbf{A^{-1}}(\mathbf{q},\omega_m)$ is given by Eq.(1.2.19) and $\mathbf{A}(\mathbf{q})$ is the inverse of $\mathbf{A^{-1}}(\mathbf{q},\tau=0)=\sum_m \mathbf{A^{-1}}(\mathbf{q},\omega_m)$. The explicit form is then

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} n_0\hbar^2q^2/mE_B(q) & 0 \\ 0 & mE_B(q)/n_0\hbar^2q^2 \end{pmatrix} 2\hbar\tanh\beta E_B(q)/2 \tag{4.1.7}$$

By direct matrix multiplication we can find the correlation matrix, and in particular the density-density correlation function (under open boundary conditions)

$$S(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) = (2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\frac{n_0\hbar q^2/m}{\omega_m^2+E_B(q)^2/\hbar^2}\Bigg(\beta\hbar\delta_{m,m'} - \frac{2\hbar}{E_B(q)}\frac{\omega_m\omega_{m'}+E_B(q)^2\hbar^2}{\omega_m^2+E_B(q)^2/\hbar^2}\tanh\beta E_B(q)/2\Bigg) \tag{4.1.8}$$

We can repeat the same steps as in the first chapter to find the relevant physical quantities, starting from the structure factor. The result is

$$S(\mathbf{q},0) = \frac{n_0\hbar^2q^2}{2mE_B(q)}\coth\beta E_B(q)/2\Bigg(1-\frac{1}{\cosh^2\beta E_B(q)/2}\Bigg) = \frac{n_0\hbar^2q^2}{2mE_B(q)}\tanh\beta E_B(q)/2 \tag{4.1.9}$$

An interesting feature of the structure factor with open boundary conditions is that is proportional to the hyperbolic tangent, as if the quasi-particle excitations followed the Fermi-Dirac distribution, instead of the bosonic distribution implied by Eq.(1.4.6). Again, we can find the condensate depletion following the same steps of chapter 1. This leads to

$$n-n_0 = \int\frac{d^D\mathbf{q}}{(2\pi)^D}\Bigg(|v(q)|^2-\frac{|u(q)|^2+|v(q)|^2}{e^{\beta E_B(q)}+1}\Bigg) \tag{4.1.10}$$

Here, the order parameter n_0 can be thought as describing the entanglement of the flux lines in the superconducting sample.
Compared to what we found in the case of periodic boundary conditions, there are a couple relevant differences.

The first one is that the excitations are explicitly distributed according to a Fermi-Dirac distribution with vanishing chemical potential. The second one is that the finite-temperature corrections have the opposite sign to what we had previously found. Choosing the phonon branch of the Bogoliubov spectrum, we can evaluate in close form Eq.(4.1.10).
When D=2 we find

$$n - n_0 = -\frac{\log 2mk_BT}{2\pi\hbar^2} \tag{4.1.11}$$

which is a finite correction, in sharp contrast to the pure bosonic case. In two dimensions we can have long-range order because the static field strengthens the order parameter, favoring condensation in the bulk of the material. When D=3 we get

$$n - n_0 = -\frac{m(k_BT)^2}{24\hbar^3 c} \tag{4.1.12}$$

Following the same Landau argument as in the first chapter we can find the normal-fluid particle density. We find

$$n_n(\tau) = \frac{\hbar}{mD}\int \frac{d^D\mathbf{q}}{(2\pi)^D} q^2 \left(2\tau \frac{\sinh[(\beta\hbar - 2\tau)E_B(q)/\hbar]}{\sinh \beta E_B(q)} - \frac{\beta\hbar}{4\sinh^2 \beta E_B(q)/2}\right) \tag{4.1.13}$$

We can evaluate this equation in the center of the sample , when $\tau = \beta\hbar/2$

$$n_n(\tau = \beta\hbar/2) = -\frac{\beta\hbar^2}{4mD}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{q}{\sinh \beta E_B(q)/2}\right)^2 \tag{4.1.14}$$

which is the same result we found for the case with periodic boundary conditions but with the opposite sign. For the finite-temperature / finite thickness situation,we can use the phonon approximation to find the following results.
When D=2 we have

$$n_n(T) = -\frac{3\zeta(3)(k_BT)^3}{2\pi m\hbar^2 c^4} \tag{4.1.15}$$

and when D=3 we get

$$n_n(T) = -\frac{2\pi^2(k_BT)^4}{45m\hbar^3 c^5} \tag{4.1.16}$$

4.2 Disorder with open boundary conditions

So far we have derived the effects of thermal fluctuations on the physical properties of the flux lines inside superconducting samples. We now want to show how to include disorder, through an extra term in the lagrangian density describing the system. With respect to the analysis shown in chapter

2, we now have to take into account the extra static field $\sigma(\mathbf{r})$. This section will closely follow [15]. To begin, we define the vector

$$\gamma(\mathbf{q},\omega_m) = \begin{pmatrix} 0 \\ U_D(\mathbf{q},\omega_m) \end{pmatrix} + i\hbar\sigma(\mathbf{q}) \tag{4.2.1}$$

Using this convention, the harmonic effective action can be cast into the following form

$$S_0[\tilde{\sigma}, X] = \frac{1}{2\beta\hbar\mathcal{V}} \sum_{\mathbf{q},\omega_m} \Big[X^T(-\mathbf{q},-\omega_m)\mathbf{A}(\mathbf{q},\omega_m)X(\mathbf{q},\omega_m) \\ - 2X^T(-\mathbf{q},-\omega_m)\gamma(\mathbf{q},\omega_{\mathbf{m}})\Big] \tag{4.2.2}$$

Using the change of variable

$$\tilde{X} = X - \mathbf{A^{-1}}\gamma \tag{4.2.3}$$

the action becomes purely quadratic and can be easily integrated. Therefore we can easily evaluate the original fields correlation matrix

$$\langle X^T(\mathbf{q},\omega_m)X(\mathbf{q}',\omega_{m'})\rangle = \hbar^2\beta\mathcal{V}\mathbf{A^{-1}}(\mathbf{q},\omega_m)\delta\mathbf{q} + \mathbf{q}'\delta_{m,-m'} \\ + \mathbf{A^{-1}}(\mathbf{q},\omega_m)\langle\gamma(\mathbf{q},\omega_m)\gamma^T(\mathbf{q}',\omega_{m'})\rangle_\sigma \mathbf{A^{-1}}(\mathbf{q}',-\omega_{\mathbf{m}'}) \tag{4.2.4}$$

where

$$\langle\gamma(\mathbf{q},\omega_m)\gamma^T(\mathbf{q}',\omega_{m'})\rangle_\sigma = \frac{\int \mathcal{D}[\sigma(\mathbf{q})]\gamma(\mathbf{q},\omega_m)\gamma^T(\mathbf{q}',\omega_{m'})e^{S_\gamma[\sigma]/\hbar}}{\int \mathcal{D}[\sigma(\mathbf{q})]e^{-S_\gamma[\sigma]/\hbar}} \tag{4.2.5}$$

and

$$S_\gamma[\sigma] = -\frac{1}{2\beta\hbar\mathcal{V}} \sum_{\mathbf{q},\omega_m} \gamma^T(-\mathbf{q},-\omega_m)\mathbf{A}^{-1}(\mathbf{q},\omega_m)\gamma(\mathbf{q},\omega_m) \tag{4.2.6}$$

To evaluate more easily this expression we can define

$$a(\mathbf{q}) = \frac{1}{\beta\hbar} \sum_m \mathbf{A}^{-1}(\mathbf{q},\omega_{\mathbf{m}}) \begin{pmatrix} 0 \\ U_D(\mathbf{q},\omega_m) \end{pmatrix} \tag{4.2.7}$$

and then switch to shifted fields

$$\tilde{\sigma(\mathbf{q})} = \sigma(\mathbf{q}) - i\mathbf{A}(\mathbf{q})a(\mathbf{q})/\hbar \tag{4.2.8}$$

where $\mathbf{A}(\mathbf{q})$ is defined in Eq. (4.1.7). This allows us to perform a second gaussian integration in σ. The final result for the quenched-disorder averaged correlation matrix

$$
\begin{aligned}
&\overline{\langle X(\mathbf{q},\omega_m)X^T(\mathbf{q}',\omega_{m'})\rangle} \\
&\quad = (2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\big[\beta\hbar\delta_{m,-m'} - \mathbf{A}^{-1}(\mathbf{q},\omega_m)\mathbf{A}(\mathbf{q})\big]\hbar\mathbf{A}^{-\mathbf{1}}(\mathbf{q},-\omega_{m'})\big]+ \\
&\qquad \frac{1}{(\beta\hbar)^2}\sum_{n,n'}\big[\beta\hbar\delta_{m,n} - \mathbf{A}^{-\mathbf{1}}(\mathbf{q},\omega_m)\mathbf{A}(\mathbf{q})\big]\mathbf{A}^{-1}(\mathbf{q},\omega_n)\begin{pmatrix}0\\1\end{pmatrix} \\
&\quad \left\{[\beta\hbar\delta_{m',n'} - \mathbf{A}^{-1}(\mathbf{q}',\omega_{m'})\mathbf{A}(\mathbf{q}')]\mathbf{A}^{-\mathbf{1}}(\mathbf{q}',\omega_{n'})\begin{pmatrix}0\\1\end{pmatrix}^T\right\}\Lambda(\mathbf{q},\omega_m;\mathbf{q}',\omega_{n'})
\end{aligned}
\tag{4.2.9}
$$

This is, of course, a very unwieldy expression. From this expression we can derive the structure factor in the center of the material , when $\tau = \beta\hbar/2$

$$
\begin{aligned}
\overline{S(\mathbf{q},\tau=\frac{\beta\hbar}{2})} &= \frac{n_0\hbar^2q^2}{2mE_B(q)}\tanh\beta E_B(q)/2+ \\
&\quad \frac{1}{(\beta\hbar)^2\mathcal{V}}\sum_{n,n'}\frac{(n_0q^2/m)^2\Lambda(\mathbf{q},\omega_n;-\mathbf{q},\omega_{n'})}{\big[\omega_n^2+E_B(q)^2/\hbar^2\big]\big[\omega_{n'}^2+E_B(q)^2/\hbar^2\big]} \\
&\left(e^{-i\omega_n\beta\hbar/2} - \frac{1}{\cosh\beta E_B(q)/2}\right)\left(e^{-i\omega_{n'}\beta\hbar/2} - \frac{1}{\cosh\beta E_B(q)/2}\right)
\end{aligned}
\tag{4.2.10}
$$

If we consider the static impurities

$$
\Lambda(\mathbf{q},\omega_m) = \Lambda(\mathbf{q})\beta\hbar\delta_{m,0} \tag{4.2.11}
$$

and remembering that $e^{-i\omega_n\beta\hbar/2} = (-1)^n$ we can find the static structure factor in presence of disorder

$$
\begin{aligned}
&\overline{S(\mathbf{q},\tau=\beta\hbar/2)} = \\
&\quad \frac{n_0\hbar^2q^2}{2mE_B(q)}\tanh\beta E_B(q)/2 + \Lambda(\mathbf{q})\left(\frac{n_0\hbar^2q^2}{mE_B(q)^2}\right)^2\left(1-\frac{1}{\cosh\beta E_B(q)/2}\right)^2
\end{aligned}
\tag{4.2.12}
$$

In the same vein of chapter 2 , we can use the disorder averaged correlation matrix to find condensate depletion. However, since the expression are very lengthy and not transparent at all, we shall show the explicit formulas only for static disorder in the bulk of the material. Therefore, we find that

$$
\overline{\langle n_\Lambda(\tau=\beta\hbar/2)\rangle} = \frac{n_0}{4m^2}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\left(\frac{\hbar q}{E_B(q)}\right)^4\Lambda(\mathbf{q})\left(1-\frac{1}{\cosh\beta E_B(q)/2}\right)^2
\tag{4.2.13}
$$

and, for the normal-fluid particle density we have

$$\overline{n_{n,ij,\Lambda}(\tau=\beta\hbar/2)} = \frac{n_0}{m^2}\int \frac{d^D\mathbf{q}}{(2\pi)^D} q_i q_j \left(\frac{\hbar^2 q}{E_B(q)^2}\right)^2 \Lambda(\mathbf{q}) \left(1 - \frac{1}{\cosh\beta E_B(q)/2} - \frac{\beta E_B(q)/2}{\sinh\beta E_B(q)/2} + \frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right) \quad (4.2.14)$$

4.3 Uncorrelated disorder

We have already discussed this kind of disorder for bosonic superfluids in subsection (3.1.2). Here we shall show the difference of the results assuming open boundary conditions instead of periodic ones. The disorder correlator is

$$\Lambda(\mathbf{q},\omega_m;\mathbf{q}',\omega_{m'}) = \Lambda(2\pi)^D\delta(\mathbf{q}+\mathbf{q}')\delta_{m,-m'}\beta\hbar \quad (4.3.1)$$

We shall evaluate the condensate depletion and the normal-fluid particle density in the bulk of the sample. when $\tau = \beta\hbar/2$. For the disorder contribution to condensate depletion we find

$$\overline{n_\Lambda} = \frac{n_0}{4}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\Lambda}{\hbar E_B(q)} \tanh\beta E_B(q)/2 \left[\left(\frac{\hbar^2 q^2}{2mE_B(q)}\right)\left(1-\frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right)^2 - 1 - \frac{\beta E_B(q)}{\sinh\beta E_B(q)}\right] \quad (4.3.2)$$

Considering the phonon branch of the Bogoliubov spectrum, we can rewrite the integral as

$$\overline{n_\Lambda(T)} \approx \frac{\Lambda n_0 (k_B T)^{D-1}}{4\pi^{D/2}\Gamma(D/2)\hbar^{D+1}c^D}\left[\frac{D-2}{D-1}\int_0^\infty \frac{x^{D-1}}{\cosh^2 x}dx + \frac{(k_B T)^2}{m^2c^4}\int_0^\infty \frac{x^{D+1}}{\cosh^2 x}\right] \quad (4.3.3)$$

where $x = \beta\hbar c q/2$. When D=2 this leads to

$$\overline{n_\Lambda(T)} = -\mathcal{O}(T^3) \quad (4.3.4)$$

while for D=3 we have

$$\overline{n_\Lambda(T)} = -\frac{\Lambda n_0 (k_B T)^2}{48\hbar^4 c^3} \quad (4.3.5)$$

We can clearly see that both of this corrections have the opposite sign compared to what we find for periodic boundary conditions (Eqs. (3.1.50)-(3.1.51)). In a similar fashion we can derive the normal-fluid particle density. We find that

$$\overline{n_{n\Lambda}} = -\frac{n_0\beta^2\hbar^4}{8Dm^2}\int \frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\Lambda q^4}{\hbar E_B(q)}\frac{1}{\sinh \beta E_B(q)} \tag{4.3.6}$$

We can see that there is no infinite thickness correction. In fact

$$\overline{n_{n\Lambda}(T=0)} = 0 \tag{4.3.7}$$

In the phonon branch, we find for D=2

$$\overline{n_{n\Lambda}(T)} = -\frac{93\zeta(5)\Lambda n_0(k_BT)^3}{64\pi m^2\hbar^3c^6} \tag{4.3.8}$$

while for D=3 we find

$$\overline{n_{n\Lambda}(T)} = -\frac{\pi^4\Lambda n_0(k_BT)^4}{192m^2\hbar^4c^7} \tag{4.3.9}$$

Again, we can see that these finite thickness corrections, which are the equivalent of finite temperature corrections in the bosonic picture, have the opposite sign to what we had found in chapter 3.

Chapter 5

Conclusions

In this work we showed and reviewed a formalism to deal with disorder in bosonic systems using the path-integral approach. This method extends the previous results by Huang and Meng [3] and by Giorgini et al. [21] because allows the systematic evaluation of the quantities of interest.
In particular, in the first chapter we defined the condensate depletion and the normal-fluid particle density n_n, the transport coefficient associated with transverse motion of the system, in terms of simple two-point correlation functions. The evaluation of these correlators, at least in harmonic approximation, is straightforward and yielded explicit formulas both at zero and finite temperature. In the second chapter, following [15], we showed how to take into consideration the effects of a disorder potential coupled to the particle density $U_D(\mathbf{r},\tau)|\psi(\mathbf{r},\tau)|^2$. The machinery presented in the first chapter easily allowed to take into account this effects, for a general potential. Therefore we showed how to derive general formulas for the response function, the condensate depletion and the normal-fluid particle density.
In the third chapter we then discussed in detail some particular kinds of disorder, starting from the static one up to tilted and extended defects, highlighting the physical situations associated with those kind of impurities distributions.
Moreover,in the third chapter, we extended the analysis of the effects of disorder to the case of finite-range interaction. This is a very important topic, because the finite-range represents a more realistic situation than contact interaction. Taking into account these corrections led to an interesting result: in fact, we showed that, when considering zero-temperature corrections in D=2 the results are divergent and it is necessary to introduce a energy scale to regularize the calculations.
In the same section, we considered the effects of finite range in case of Lorentzian-correlated disorder. Finally, in the last chapter, we discussed

the application of the boson superfluids formalism to type-II superconductors. This application stems from the fact that the statistical mechanics of magnetic flux lines in (D+1) dimensions, where the extra dimension corresponds to the imaginary time in the boson picture, is equivalent to the quantum mechanical problem of superfluid bosons in D spatial dimensions. We have showed how to argue this equivalence and in particular how only states symmetric under the exchange of any pair of particles contribute to the partition function of the superconducting system (see Eqs. (4.0.8)-(4.0.10)). We also discussed the meaning of the order parameter and the normal-fluid particle density in this case: the first one describes how much the flux lines are entangled or correlatd, while the second one characterizes the response of the system to a tilt in the external magnetic field. However, there is a difference between the two picture: while periodic boundary conditions are good to describe bosonic superfluids, in the case of magnetic lines are quiet artificial. Therefore we adopted open boundary conditions, summing over all the possible entry and exit points of the flux lines. In order to implement this different constraint in the path integral formulation, we introduced an extra auxiliary fields. Moreover, open boundary conditions, that ensure that the fluctuations of the bosonic order parameter are vanishing at the surfaces of the sample, greatly change the physics of the system. In particular thermal fluctuations result in an enhancement of the order parameter of the system.

Appendix A

Matsubara Frequency summations

The idea behind performing Matsubara frequency summation is to switch this infinite sum with a contour integral that involves the evaluation of a finite number of residues [19]. The following formulas apply to the bosonic case, i.e. with even integers.

$$\frac{1}{\beta\hbar}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n + E_B(q)/\hbar} = \frac{e^{\tau E_B(q)/\hbar}}{e^{\beta E_B(q)} - 1} \tag{A.0.1}$$

$$\frac{-1}{\beta\hbar}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - E_B(q)/\hbar} = \frac{e^{(\beta-\tau/\hbar)E_B(q)}}{e^{\beta E_B(q)} - 1} \tag{A.0.2}$$

$$\frac{1}{\beta\hbar}\sum_n \frac{e^{-i\omega_n\tau}}{\omega_n^2 + E_B(q)^2/\hbar^2} = \frac{\hbar\cosh[(\beta\hbar - 2\tau)E_B(q)/2\hbar]}{2E_B(q)\sinh\beta E_B(q)/2} \tag{A.0.3}$$

$$\frac{1}{\beta\hbar}\sum_n \frac{i\omega_n e^{-i\omega_n\tau}}{\omega_n^2 + E_B(q)^2/\hbar^2} = \frac{\sinh[(\beta\hbar - 2\tau)E_B(q)/2\hbar]}{2\sinh\big(\beta E_B(q)/2\big)} \tag{A.0.4}$$

Moreover, we can take derivatives with respect to $E_B(q)$. This yields

$$\frac{1}{\beta\hbar}\sum_n \frac{e^{-i\omega_n\tau}}{(\omega_m^2 + E_B(q)^2/\hbar^2)^2} = \frac{\hbar^2}{4E_B(q)^3\sinh\beta E_B(q)/2}\Bigg\{\cosh\frac{(\beta\hbar - 2\tau)E_B(q)}{2\hbar} - \frac{\tau E_B(q)}{\hbar}\sinh\frac{(\beta\hbar - 2\tau)E_B(q)}{2\hbar} + \frac{\beta E_B(q)}{2}\frac{\cosh\tau E_B(q)/\hbar}{\sinh\beta E_B(q)/2}\Bigg\} \tag{A.0.5}$$

and

$$\frac{1}{\beta\hbar}\sum_n \frac{i\omega_n e^{-i\omega_n\tau}}{(\omega_n^2 + E_B(q)^2/\hbar^2)^2} = \frac{\hbar}{4E_B(q)\sinh\beta E_B(q)/2}\left\{\tau\cosh\frac{(\beta\hbar - 2\tau)E_B(q)}{2\hbar} - \frac{\beta\hbar\sinh\tau E_B(q)/\hbar}{2\sinh\beta E_B(q)/2}\right\} \quad \text{(A.0.6)}$$

Bibliography

[1] A.L. Fetter, J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, 1971).

[2] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science **269**, 5221 (1995).

[3] K. Huang and H.-F. Meng, Phys. Rev. Lett. **69**, 644 (1992).

[4] J.C. Dainty, *Laser Speckles and Related Phenomena* (Springer-Verlag, 1984).

[5] R.P. Feynman, Int. J. Theor. Phys. **21**, 467 (1982).

[6] D.R. Nelson, H.S. Seung, Physical Review B **39** (1989).

[7] K. Huang, *Statistical Mechanics* (Wiley, 1987).

[8] J.W. Negele, J. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, 1988).

[9] N. Nagaosa, *Quantum Field Theory in Condensed Matter Physics* (Springer, 1999).

[10] M. Lebellac, *Thermal Field Theory* (Cambridge University press, 1996).

[11] A. Griffin, *Excitations in a Bose-Condensed Liquid* (Cambridge Univeristy Press, 1993).

[12] G. t'Hooft and M. Veltman, Nucl. Phys B **44**, 189 (1972).

[13] L. Salasnich, F. Toigo, Phys Rep. **640**, 1 (2016).

[14] V.N. Popov, Theor. and Math. Phys. A **11**, 565 (1972).

[15] U.C. Täuber and D.R. Nelson, Phys. Rep. **289**, 157 (1997).

[16] P- Nozières, D. Pines, *The Theory of Quantum Liquids*, Volume II (Addison-Wesley, 1990).

[17] A.M.J Schakel, *Boulevard of Broken Symmetries*, (World Scientific, 2008).

[18] V.N. Popov, *Functional Integrals and Collective excitations* (Cambridge University Press, 1981).

[19] A. Atland, B. Simons,*Condensed matter field theory* (Cambridge University Press, 2010).

[20] D.R. Nelson and P. Le Doussal, Phys. Rev. B **42**, 10113 (1990).

[21] S. Giorgini, L. Pitaevskii, and S. Stringari, Phys. Rev. B **49**, 12938 (1994).

[22] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989).

[23] A. Cappellaro and L. Salasnich, Physical Review A **96**, 063610 (2017).

[24] H.T.C. Stoof, D.B.M. Dickerscheid, K. Gubbels, *Ultracold Quantum Fields*, (Springer, 2009).

[25] A. Cappellaro and L. Salasnich, Phys. Rev. A **95**, 033627 (2017).

[26] T. Hwa, P. Le Doussal, D.R. Nelson, and V.M. Vinokur, Phys. Rev. Let. **71**, 3545 (1993).

[27] D.R. Nelson and H.S. Seung, Phys. Rev. B **39**, 9153 (1989).