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Dipartimento di Fisica e Astronomia "Galileo Galilei"

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Parity violation in inflationary models

Relatore

Prof. Nicola Bartolo

Correlatore

Dr. Giorgio Orlando

Laureando

Luca Caloni

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Abstract

Parity symmetry is a fundamental property of the gravitational interaction in the description provided by General Relativity. Therefore its breaking could be a signal of deviation from such a standard model. Since inflation involves very high energy scales, it provides a unique window to test fundamental physics. In this thesis we study inflation within new parity violating theories of gravity that have been recently proposed. These theories generalize Chern-Simons gravity by including in the action coupling terms to gravity involving first and second derivatives of a scalar field (the inflaton field, in the context of inflation). Besides breaking parity, these theories also break Lorentz invariance. Our purposes are to see whether the new couplings lead to detectable parity breaking signatures in the power spectrum of primordial gravitational waves and whether they modify the tensor power spectrum, possibly leading to a blue power spectrum of primordial gravitational waves. Having a blue power spectrum means that the amplitude of tensor perturbations increases going to lower scales. This could be an important feature, because it could enhance the spectrum of primordial gravitational waves at scales relevant for interferometers. We find that the two circular polarization states evolve following different dynamical equations. Moreover, the speed of propagation of tensor modes is modified and aquires different values for the two polarization states during inflation. In the case of the theory with second derivatives of the scalar field we find that the term which corrects the tensor speed is however highly suppressed, working in the regime where we avoid the production of ghost fields. Hence the conclusions are analogous to the case with Chern-Simons gravity, both for the chirality in the tensor power spectrum and for the tensor spectral index. In the case of the theory with only first derivatives of the scalar field there is instead an additional contribution that modifies the tensor speed and which can be relevant if the couplings of the new operators have large time derivatives during inflation. We comment on possible interesting phenomenological signatures arising from this feature.

Contents

Introduction

Inflation is a phase of accelerated expansion of the early Universe, originally introduced to provide a solution to the shortcomings of the standard Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model [1–4]. Indeed, the latter requires very fine-tuned initial conditions to let the Universe evolve towards its current state. The inflationary phase provides a dynamical mechanism that leads the Universe to these conditions when the standard radiation era begins.

Remarkably, it was soon realized that inflation can also produce the primordial perturbations that represent the seeds for the subsequent formation of the Large Scale Structures in the Universe [5–7]. These are produced by quantum fluctuations that are stretched to cosmological sizes by the almost exponential expansion, thus freezing after horizon exit and becoming "classical" fluctuations. When these perturbations reenter the horizon, during the radiation or matter era, they give rise to matter and temperature perturbations, that can be observed through the Cosmic Microwave Background (CMB) [8, 9] and which then start growing to give rise to the structures we observe today.

Another fundamental prediction of all inflationary models is the production of a stochastic background of primordial gravitational waves (PGWs) [10] (see [11] for a review). These are transverse and traceless tensor perturbations of the metric that are produced by the vacuum fluctuations of the metric itself and which, as for scalar perturbations, are then amplified by the inflationary expansion.

The simplest models of inflation consist of a scalar field, called the inflaton, which slowly rolls on a very flat potential. During such a phase, the scalar field mimics a cosmological constant and drives the almost exponential inflationary expansion. Then inflation ends when the inflaton approaches the minimum of the potential, where it oscillates to decay into ultra-relativistic particles. After this phase, called reheating, the standard radiation era begins.

In these single field slow-roll inflationary models both the spectra of scalar and tensor perturbations are predicted to be almost scale-invariant [6, 7, 12], with a tiny level of non-Gaussianity [13, 14]. The first statement means that the amplitude of the fluctuations varies slightly with the cosmological scale and is a consequence of the slow-roll dynamics (see Subsection 3.6.1 for a discussion of this point). The amplitude of tensor perturbations can be directly related to the Hubble parameter during inflation, which is a measure of the energy scale at which inflation occurred [9, 11]. Moreover, in these models a consistecy relations holds: this relates the so-called tensor-to-scalar perturbation ratio (that is, the ratio between the amplitude of tensor and scalar perturbations) to the tensor spectral index. We have not observed primordial gravitational waves yet, but we have an upper bound on the tensor-to-scalar ratio coming from the measurements of the CMB made by the Planck satellite (and also other CMB experiments) [9]. If the consistency relation really holds, given this limit it will be extremely difficult to measure the scale dependence of tensor perturbations. Moreover, the spectrum of the primordial gravitational waves produced in these models is below the range of the future space interferometer LISA [15].

Given that inflation involves energy scales far beyond the ones that can be reached on Earth (e.g. by particle accelerators), it provides a window into processes that could not be tested in any other way, and thus also provides a powerful tool to constrain fundamental physics. In this thesis we will focus on studying how inflation can be used to test whether the theory of gravity deviates from General Relativity (GR). Modifying the theory of gravity in general can leave some signatures in the primordial perturbations from inflation. In particular we will focus on theories that modify GR by introducing parity breaking effects. The breaking of parity symmetry manifests itself in the polarization of PGWs into left and right circular polarization states, which have a different dynamical evolution during inflation and thus acquire different power spectra on super-horizon scales. It is then also extremely interesting to study if the spectrum of PGWs can be modified in such a way to obtain a blue power spectrum, where the amplitude of the PGWs increases going to smaller cosmological scales. This could enahance the spectrum of PGWs at scales possibly relevant for interferometers.

We will first review inflation with Chern-Simons modified gravity [16–18]. This predicts a tiny level of chirality in the power spectrum of PGWs, and the spectral index of PGWs is not modified with respect to the standard case, at leading order in slow-roll. The consistency relation is modified by a contribution which is even more suppressed than the chirality itself. We will study also the parity breaking effects in the primordial bispectra (i.e. the Fourier transform of the three-point functions), both of tensor perturbations and mixed correlators. The only correlator where the parity breaking signatures are not suppressed is the bispectrum between one scalar and two tensor perturbations [17].

Then we will study inflation within new parity breaking theories of gravity that have been recently proposed in the literature and which generalize Chern-Simons gravity [19]. These include in the action new operators containing first and second derivatives of the scalar (inflaton) field respectively (see Eqs. (6.1)-(6.3)). Besides the breaking of parity symmetry, it is important to stress that these theories also break Lorentz invariance. First of all we will comment on the fact that these theories do not modify the background dynamics of inflation and the power spetcrum of scalar perturbations. We will then study how the dynamical evolution of PWGs is modified by the new operators introduced in the action. With respect to the Chern-Simons case, these introduce an additional contribution that modifies the speed of propagation of tensor modes during inflation. The speed of tensor modes now varies with time during inflation and, remarkably, one of the two polarization states acquires a superluminal speed, with the other one propagating instead with a subluminal speed. The superluminal propagation of one of the two polarization states is a phenomenological manifestation of the breaking of Lorentz invariance, while the fact that one of the two states have superluminal speed and the other one subluminal is a manifestation of the breaking of parity symmetry.

In the case of the theory with second derivatives of the scalar field (described by the action (6.9)) there is only one term that modifies the tensor speed (see Eq. (6.88)). Working in the regime where we avoid the production of ghost fields, this term is highly suppressed, hence we can take the speed of tensor modes to be equal to the speed of light apart from tiny corrections. This brings us back to an equation of motion for tensor modes analogous to the one which occurs with Chern-Simons gravity. The conclusions, both for the chirality in the power spetcrum and for the tensor spectral index, are therefore the same.

In the case of the theory with only first derivatives of the scalar field (described by the action (6.8)) there is instead an additional contribution that modifies the tensor speed (see Eq. (6.59)). This becomes relevant if the time derivative of the couplings of the new operators with the inflaton are large during inflation, and could leave interesting signatures both at level of chirality in the power spectrum of PGWs and at the level of the tensor spctral index (of one or both of the polarization states). The study of these possible features, however, requires a numerical study to solve the equation of motion of tensor modes and then compute the power spectum. We leave this for a possile future work.

The thesis is organized as follows.

In **Chapter 1** we review the standard cosmological (Hot Big Bang) model with its shortcomings, and introduce the concept of inflation as a solution for them. Then, we study the background dynamics of the simplest models of inflation, namely single field slow-roll inflationary models, and we briefly classify them into three different categories.

In **Chapter 2** we introduce the theory of cosmological perturbations, which represents a powerful tool in GR to go beyond the zero-order homogeneous and isotropic solution described by the FLRW models. We explain the issues concerning the gauge transformations and we outline the most common gauges used in cosmology. Finally, we explain how the metric and the stress-energy tensors can be perturbed and we introduce the ADM decomposition of the metric tensor.

In **Chapter 3** we study the primordial perturbations produced during inflation. We characterize both the scalar and tensor perturbations arising from single field slow-roll models in terms of their power spectra, and then we comment on the resulting observable predictions with the corresponding experimental constraints.

In **Chapter 4** we introduce the concept of primordial non-Gaussianity, which is related to the interactions between the field(s) present during inflation. We focus on the bispectrum of primordial perturbations and we outline the main shapes of primordial non-Gaussianity, linking them to the different inflationary models from which they can arise. Then we introduce the in-in formalism and we exploit it to compute non-Gaussianities arising from single field slow-roll inflationary models. At the end of the chapter we summarize the main predictions coming from single field slow-roll models.

In **Chapter 5** we study inflation with Chern-Simons modified gravity. We begin by studying the parity breaking signatures in the power spectrum of primordial tensor perturbations. Then we consider the bispectra, both of tensor perturbations and mixed correlators.

In **Chapter 6** we introduce chiral scalar-tensor theories that generalize Chern-Simons gravity. As the original contribution of this thesis, we study how the new operators affect the evolution of primordial gravitational waves from inflation. New features arise in this regard, which hint at possible interesting results that however require a further numerical analysis. This leaves open the possibility for possible future works.

Chapter 1

The paradigm of inflation

1.1 The Friedmann-Lemaître-Robertson-Walker metric

Cosmology is the study of the composition and the evolution of the Universe as a whole. The Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model, also known as the Hot Big Bang model, explains extraordinarily well the evolution of our Universe from its early stages (corresponding to some fraction of a second) to its current state, about 13.8 billion years later. The FLRW model is set within a theoretical framework based on General Relativity, while from the observational side it is supported by three pillars: the first one is the observation of the expansion of the Universe, discovered in 1929 by Hubble [20], who found that galaxies are receding from us with a velocity proportional to their distance from us[1]. The second one is the Big Bang Nucleosynthesis (BBN), namely the explanation of the relative abundance of light elements made by Alpher, Bethe and Gamow in 1948 [21]. The third one is the detection of the Cosmic Microwave Background (CMB) by Penzias and Wilson in 1965 [22]. This is an extremely isotropic radiation emitted when the Universe, while cooling down, reached a temperature low enough to allow the formation of neutral atoms; after that, photons decoupled from matter and free-streamed to us.

In order to have success in the beautiful but complex task which cosmology proposes, we need to make some simplifying assumptions. Usually in physics this is done by assuming that the physical system under consideration obeys some particular symmetry. In the case of cosmology, the FLRW model is based upon the so-called *Cosmological Principle*, which tells us that:

"Each *comoving* observer sees the Universe around him, at a fixed time, as *homogeneous* and *isotropic*, on sufficiently large scales."

Let us clarify what this sentence means. First of all, a *comoving* observer is one which has no motion with respect to the cosmic fluid; in practice, a comoving observer sees the CMB as isotropic, apart from its tiny anisotropies. *Isotropy* means that the space looks the same no matter in what direction we look, namely we have rotational invariance. *Homogeneity* means that the space looks the same at each point, namely it is independent of position and we have translational invariance. When we say "large scales" we refer to distances bigger than about 100 Mpc.

Homogeneity and isotropy are not necessarily related to each other. Indeed, a manifold can be homogeneous but nowhere isotropic, or it can be isotropic around a point without being homogeneous. However, if it is isotropic around any point, then it is also homogeneous. Likewise, if it is isotropic around a point and also homogeneous, then it is isotropic around any point. Since we have ample observational evidence for isotropy around us, like the isotropy of the CMB [23]

[1] Actually, we are not the center from which the Universe is expanding. There is no center of the expansion, each galaxy is receding from any other one with a velocity proportional to the relatve distance between them.

and the isotropy in the statistical properties of clustering of galaxies, and we assume that we are not in a special place in the Universe (the so-called *Copernican principle*), then it follows that the Universe is homogeneous and isotropic around each point. Notice that homogeneity has to be assumed and it cannot be verified if not on small distance scales.

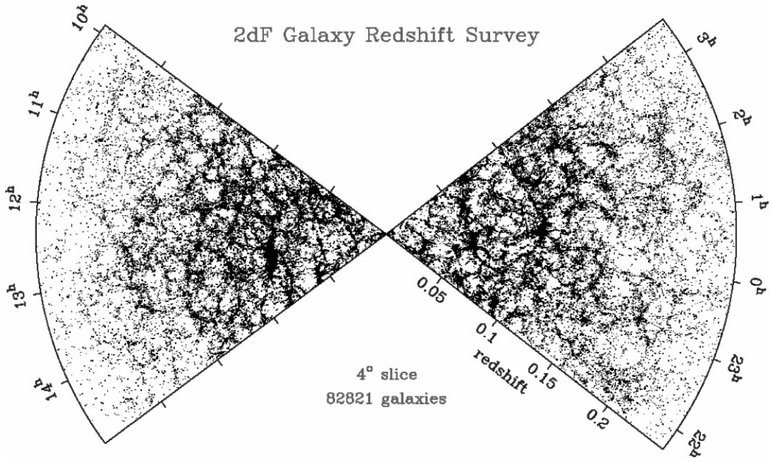

Figure 1.1: 2dF Galaxy Redshift Survey: the distribution of galaxies becomes uniform on large scales (from [24]).

As we have already said, we know that our Universe is expanding. Thus, we can say that the Universe is *spatially* homogeneous and isotropic, but it evolves with time. In General Relativity (GR) this translates into the statement that the Universe can be foliated into spacelike slices, such that each three-dimensional slice is maximally symmetric. We therefore assume our spacetime to be $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ represents the time direction and Σ is a maximally symmetric three-manifold. The FLRW metric can thus be written as[2]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.1}$$

where (r, θ, ϕ) are comoving polar coordinates and $a(t)$ is the *scale factor*, which determines proper distances in terms of the comoving coordinates (see Fig. 1.2). By an appropriate rescaling of the coordinates, k can be set equal to +1, -1, or 0 for spaces of constant positive, negative, or zero spatial curvature respectively. In this case, r is dimensionless, while $a(t)$ has dimensions of length. The time coordinate t is called *cosmic time* and represents the proper time measured by comoving observers.

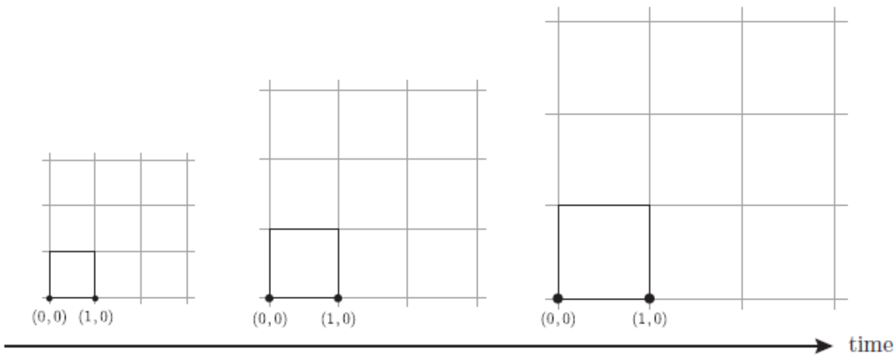

Figure 1.2: The comoving distance between points on an imaginary coordinate grid remains constant as the Universe expands. The physical distance, which is proportional to the comoving distance through the scale factor $a(t)$, gets larger as time evolves (from [25]).

[2]Here, and almost everything in the following, we work in units with $c = 1$.

The FLRW metric can also be rewritten as

$$ds^2 = -dt^2 + a^2(t)\,\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right], \tag{1.2}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{1.3}$$

is the metric of a 2-sphere, and we have defined

$$d\chi \equiv \frac{dr}{\sqrt{1-kr^2}}, \quad S_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1. \end{cases} \tag{1.4}$$

We will mainly focus in the case with $k = 0$ since, by observations, our (observable) Universe is practically indistinguishable from a Universe with vanishing spatial curvature [26].

In order to study the causal structure in a FLRW Universe it is useful to introduce the *conformal time* τ, defined through the relation

$$d\tau = \frac{dt}{a(t)}. \tag{1.5}$$

Hence, the FLRW metric factorizes into a static Minkowski metric, multiplied by a time-dependent conformal factor $a(\tau)$:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dr^2 + r^2 d\Omega^2\right]. \tag{1.6}$$

1.2 The dynamics of the expanding Universe

The dynamics of the expansion of the Universe is encoded in the time dependence of the scale factor $a(t)$. In order to find it, we must resort to the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.7}$$

where on the left hand side we have the Einstein's tensor, which is a measure of the curvature of the spacetime and is constructed by the metric and its (second) derivatives, while on the right hand side we have the stress-energy tensor, which is a measure of the matter/energy content of the Universe.

To be consistent with the symmetries of the metric, the stress-energy tensor must be diagonal and the spatial components must be equal, due to isotropy. The simplest realization of this is given by the stress-energy tensor of a perfect fluid, which has the form

$$T^\mu{}_\nu = (\rho+p)u^\mu u_\nu + p\delta^\mu_\nu = \mathrm{diag}(-\rho, p, p, p), \tag{1.8}$$

where ρ is the energy density, p is the pressure, and $u^\mu = (1, 0, 0, 0)$ is the four-velocity of the fluid in comoving coordinates, with respect to which the fluid is at rest.

The stress-energy tensor satisfies a continuity equation

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.9}$$

whose time component, in the case of a perfect fluid, yields

$$\dot\rho = -3H(\rho+p), \tag{1.10}$$

where $H = \dot{a}/a$ is the Hubble parameter. One usually refers also to Eq. (1.10) as the *continuity equation*.

By specifying the Einstein's equations to the case of a Universe described by a FLRW metric and filled with a perfect fluid, one can obtain (from the 00- and the ij-components) the so-called *Friedmann equations*:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.11}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1.12}$$

which have to be considered together with the continuity equation. However, because of the Bianchi identities, these three equations are not independent; only two of them are so[3].

As we will soon see, if we know from the continuity equation how the energy density ρ evolves with the scale factor a during a given era, we can integrate the first Friedmann equation (1.11) to obtain the dependence of the scale factor on the cosmic time t. The second Friedmann equation (1.12) gives us the acceleration of the expansion. We will see that "ordinary components", like radiation and matter, give negative contributions to $\ddot{a}$ since $\rho + 3p > 0$, leading to attractive gravity. A cosmological constant or a slowly-rolling scalar field (which is the simplest realization of an inflationary phase) obeys instead $\rho + 3p < 0$, hence leading to repulsive gravity and to an accelerated expansion.

If we define the *critical density*

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1.13}$$

and the *density parameter*

$$\Omega = \frac{\rho}{\rho_c}, \tag{1.14}$$

we can rewrite the first Friedmann equation as

$$\Omega - 1 = \frac{k}{a^2 H^2}. \tag{1.15}$$

Since $a^2H^2 > 0$, it follows that the sign of k is equal to the sign of $\Omega - 1$. In particular, we have:

$$\begin{array}{lllllll} \rho < \rho_c & \Leftrightarrow & \Omega < 1 & \Leftrightarrow & k < 0 & \Leftrightarrow & \text{open Universe}, \\ \rho = \rho_c & \Leftrightarrow & \Omega = 1 & \Leftrightarrow & k = 0 & \Leftrightarrow & \text{flat Universe}, \\ \rho > \rho_c & \Leftrightarrow & \Omega > 1 & \Leftrightarrow & k > 0 & \Leftrightarrow & \text{closed Universe}. \end{array} \tag{1.16}$$

This means that the density parameter tells us which of the three possible geometries describes our Universe.

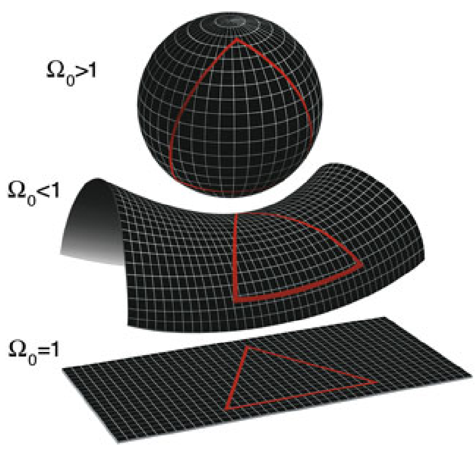

Figure 1.3: The three possible geometries of our Universe in a 2-D analogy (from [27]).

[3]In Section 3.3 we will see that Eq. (1.11) is a (Hamiltonian) constraint, rather than a true dynamical equation.

Since we have two independent equations in three variables, $a(t)$, $\rho(t)$ and $p(t)$, we need to introduce a third equation to close the system. This can be done by specifying the equation of state of the cosmic fluid. We assume our cosmic fluid to be described by a barotropic equation of state, which means that the pressure is a function only of the energy density, $p = p(\rho)$. Moreover, we assume this relation to be linear:

$$p = w\rho, \tag{1.17}$$

where w is a dimensionless constant. Plugging this ansatz into the continuity equation, it is easy to solve it, finding

$$\rho \propto a^{-3(1+w)}. \tag{1.18}$$

Therefore we have a relation which tells us how the energy density varies with the scale factor. Depending on the value of w, we can classify different sources:

- **Matter**: w=0. In this case $p = 0$, and consequently

  $$\rho \propto a^{-3}. \tag{1.19}$$

  This is a simple consequence of the expansion of the Universe, which implies that volumes scale as $V \propto a^3$. This is the case of *Dark Matter* and, in general, non relativistic matter.

- **Radiaton**: $w = 1/3$. In this case $p = \frac{1}{3}\rho$, and

  $$\rho \propto a^{-4}. \tag{1.20}$$

  This is the case of radiation and, more generally, relativistic particles. In addition to the a^{-3} factor due to the expansion of the Universe there is a contribution due to the redshifting of the energy, $E \propto a^{-1}$.

- **Cosmological Constant**: $w = -1$. In this case $p = -\rho$, and

  $$\rho = \text{const.} \tag{1.21}$$

  Since the energy density is constant, the energy has to increase while the Universe is expanding. More in general, *Dark Energy* has $w < -1/3$. From the second Friedmann equation (1.12), this last condition leads to an accelerated expansion of the Universe, $\ddot{a} > 0$.

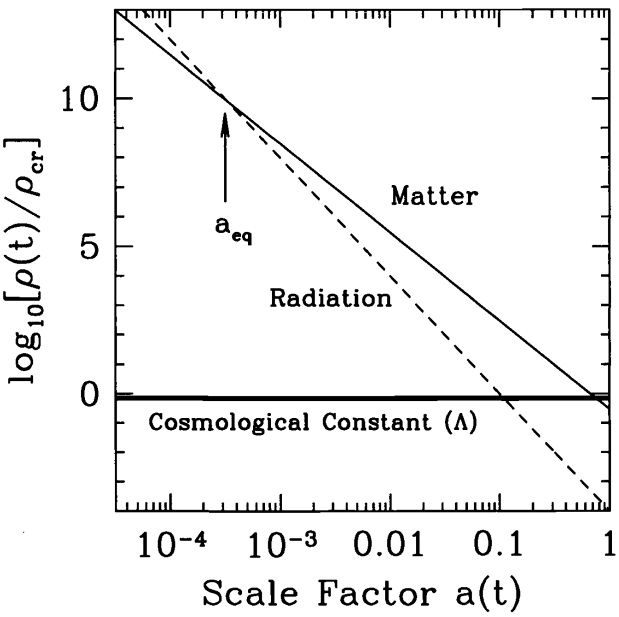

Figure 1.4: Evolution of the energy density with the scale factor for different components in a flat Universe (from [28]).

For a flat Universe we can integrate the Friedmann equation (1.11) together with the continuity equation, thus obtaining the time evolution of the scale factor

$$a(t) \propto \begin{cases} t^{2/3(1+w)} & w \neq -1, \\ e^{Ht} & w = -1. \end{cases} \tag{1.22}$$

It follows that

$$a(t) \propto \begin{cases} t^{1/2} & \text{radiation domination,} \\ t^{2/3} & \text{matter domination,} \\ e^{Ht} & \text{cosmological constant.} \end{cases} \tag{1.23}$$

We summarize the solutions for a flat FLRW Universe dominated by radiation, matter or a cosmological constant in the box below, where we report also the evolution of the scale factor in terms of the conformal time τ.

	w	$\rho(t)$	$a(t)$	$a(\tau)$
RD	1/3	a^{-4}	$t^{1/2}$	τ
MD	0	a^{-3}	$t^{2/3}$	τ^2
Λ	-1	a^0	e^{Ht}	$-\tau^{-1}$

1.3 Brief thermal history of our Universe

We now want to briefly outline the main features of the thermal evolution of our Universe. We know that, going back in time, the Universe becomes hotter and denser. If we push this argument to the exterme, we reach a state of infinite density and temperature, a singularity, which we identify as the initial time $t = 0$. However, when we go beyond the Planck epoch, $t_{Pl} \sim 10^{-43}$ (corresponding to $T_{Pl} \sim 10^{19}$ GeV), the quantum gravity effects cannot be neglected anymore and become relevant. Since we do not have any succesful theory of quantum gravity yet, the first moments of life of our Universe are at the moment highly speculative. Therefore we start our description well after the Planck time.

- **Inflation and reheating**: as we will soon see in detail, there are many reasons to believe that the early Universe, at a time $t \sim 10^{-35}$ s, underwent a phase of almost exponential expansion, called *inflation*, which provided the Universe with the primordial perturbations that then led to the formation of the structures we can observe today. Inflation ended with the so-called *reheating* phase, after which the standard radiaton era started.

- **Baryogenesis**: when we observe the Universe, we have a great number of evidences that it is (almost) entirely composed of matter, with no antimatter. However, we would not expect this from particle physics: indeed, the CPT theorem (valid for each local, relativistic QFT) tells us that for any particle species there exist a corresponding antiparticle with the same mass, decay width and opposite charge. Thus, we would expect the Universe to be symmetric with respect to the content of matter and antimatter. Since particles and (their respective) antiparticles annhilate between themselves, if the Universe was initially filled with equal amounts of matter and antimatter then these annihilations should have led to a Universe dominated by radiation. This means that we need to introduce in the early Universe some mechanism that generates dinamically a baryon asymmetry starting from an initially symmetric state. This is called *baryogenesis*. We do not know which is this mechanism, and we do not know when it exactly occured, but it is plausible that it happened during the reheating phase, when the ultra-relativistic particles that then filled the Universe were created.

- **Electroweak phase transition**: at a temperature $T \sim 100$ GeV ($t \sim 10^{-10}$ s) the electroweak phase transitions occured and particles obtained their masses through the Higgs mecchanism. The symmetry $SU(2)_L \otimes U(1)_Y$ of the electroweak interaction was broken to $U(1)_{EM}$ and the weak interaction became short-range, since the gauge bosons $W^{\pm}$ and Z^0 became massive ($m_W \simeq 80$ GeV, $m_Z \simeq 91$ GeV).

- **QCD phase transition**: at $T \sim 100$ MeV ($t \sim 10^{-5}$ s) a second phase transition occurred, namely the QCD phase transition. After that, quarks were not “free” anymore, but formed bound states of (color singlet) quarks triplets, called *baryons*, and quark-antiquark states, called *mesons*.

- **Neutrino decoupling**: at $T \sim 2$ MeV, the interaction rate of neutrinos became of order of the expansion rate H. As a consequence, neutrinos decoupled from the primordial plasma and from there on they freely-streamed through the Universe[4].

- **Big Bang Nucleosynthesis**: at $T \sim 1$ MeV, soon after neutrino decoupling, the light elements (^{1}H, ^{4}He, with small amounts of ^{2}H and ^{3}He, and with a tiny component of ^{7}Li) formed. This process is known as *Big Bang Nucleosynthesis* and it is the farthest observational window we have in our Universe.

- **Electron-positron annihilation**: at $T \sim 511$ KeV electrons and positrons became non relativistic, hence their numerical density dropped due to annihilation processes like $e^+e^- \to 2\gamma$. Indeed, the energy of the thermal bath was not sufficient anymore to allow the production of e^-e^+ couples via the inverse process from photons. The energy coming from these annihlations was then transferred to the thermal plasma, but not to neutrinos, which were already decoupled. As a consequence, the temperature of the neutrino background today ($T_{\nu,0} = \simeq 1.9$ eV) is expected to be lower than the corresponding temperature of the photons ($T_{\gamma,0} \simeq 2.7$ eV)[5].

- **Radiation-matter equivalence**: at $T \sim 0.75$ eV the energy density of matter became equal to that of radiation; after that, the matter era started.

- **Recombination**: at $T \sim 0.3$ eV electrons and nuclei became bound to form the first neutral atoms through the process $e^- + p \to H + \gamma$. This is called *recombination* (even though they actually combined for the first time).

- **Photon decoupling**: soon after recombination, photons decoupled from the primordial plasma and from there on they freely-streemed to us. Indeed, after recombinaton the numerical density of free electrons rapidly decreased, and photons interact much less efficiently with neutral atoms. Today we can observe them as a background radiation (the CMB) with a temperature $T_{\gamma,0} \simeq 2.7$ eV. The energy density they had at the time of decoupling has indeed been red-shifted by the expansion of the Universe.

- **Structure formation**: much later, the primordial perturbations generated during inflation grew via gravitational instability to form the structures we can observe today.

[4] At sufficiently high temperatures in the early Universe neutrinos were in thermal equilibrium with the primordial plasma. This was maintained by processes like neutrino scattering off electrons, positrons and themselves, and neutrino-antineutrino annihilation into electron-positron pairs, plus inverse processes.

[5] The cosmic neutrino has not been observed yet, since neutrinos interact only weakly with ordinary matter.

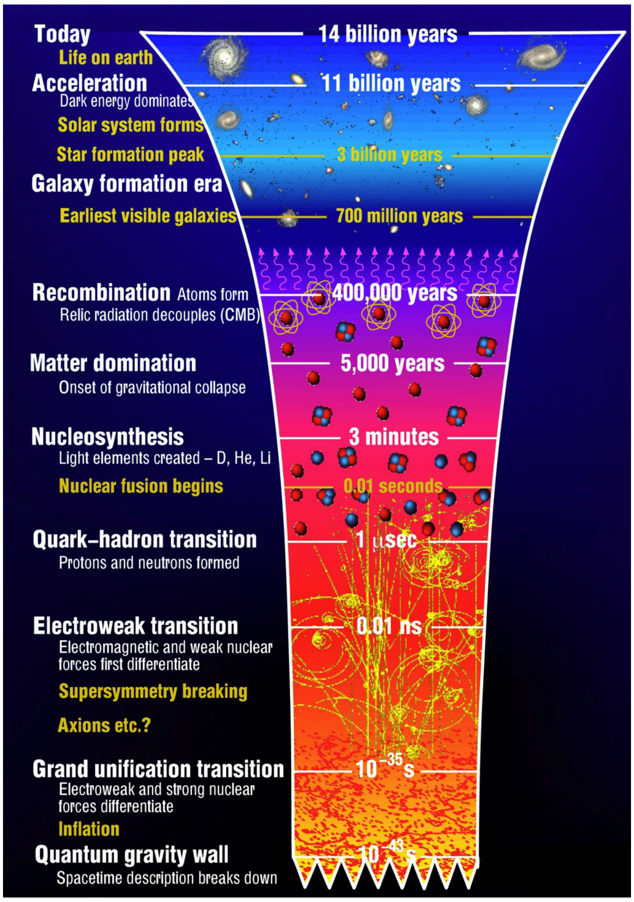

Figure 1.5: Thermal history of the Universe (from [29]).

1.4 The particle horizon and the Hubble radius

In this section we introduce two important quantities, particularly useful when studying inflationary cosmology. The *particle horizon* is defined as

$$d_H(t) = a(t) \int_0^t \frac{c\, dt'}{a(t')}, \tag{1.24}$$

where we have reintroduced for the moment the speed of light c. Let us consider a sphere of radius $d_H(t)$ centered on us at the time t. This sphere contains all the points in the Universe which have been in causal contact with us from the time $t = 0$ to the time t. Then, in general, if two points are separated by a distance greater than the particle horizon, they have *never* been in causal contact between each other. For standard FLRW models, in which

$$p = w\rho, \quad a(t) \propto t^\alpha, \quad \text{with} \quad \alpha = \frac{2}{3(1+w)}, \tag{1.25}$$

we have

$$d_H(t) = \frac{3(1+w)}{1+3w} ct \sim \frac{1}{H(t)}. \tag{1.26}$$

Notice that the integral (1.24) with the scale factor given in (1.25) converges only if $\alpha < 1$. This condition implies that $w > -1/3$ and hence, from the second Friedmann equation, $\ddot{a} < 0$. If the integral diverges it means that there is no horizon and we are in causal contact with each point in the Universe. From what just said, this does not happen in the case of a Universe undergoing a decelerated expansion, and we have a finite horizon.

Now let us define another important concept, namely that of the *Hubble radius*:

$$R_c(t) = \frac{c}{H(t)}. \tag{1.27}$$

Since the *Hubble time*, $\tau_H = H^{-1}(t)$, is the characteristic expansion time of the Universe, the Hubble radius is a measure of the distance travelled by light in a Hubble time. For standard FLRW models, we have

$$R_c(t) = \frac{3(1+w)}{2} ct = \frac{1+3w}{2} d_H(t) \sim d_H(t). \tag{1.28}$$

Hence, for standard FLRW models the particle horizon and the Hubble radius are nearly equal, up to some numerical factor. But there is a crucial difference between these two concepts which is particularly important in the case of inflation. The particle horizon, as we have said, takes into account all the past history of the observer, while the Hubble radius describes properties of causal connections only related to a Hubble time. In other words, two particles which are separated by a distance greater than the Hubble radius have not been in causal contact during the last Hubble time, but it is possible that they were causally connected in the past; two particles separated by a distance greater than the particle horizon have never been in causal contact.

We can then define the *comoving Hubble radius* as

$$r_H(t) = \frac{R_c(t)}{a(t)} = \frac{1}{aH} = \frac{1}{\dot{a}}, \tag{1.29}$$

where we have set again $c = 1$. For FLRW models, we have:

$$\text{Radiation dominated era:} \qquad a(t) \propto t^{1/2} \quad \Rightarrow \quad r_H(t) \propto t^{1/2}, \tag{1.30}$$

$$\text{Matter dominated era:} \qquad a(t) \propto t^{2/3} \quad \Rightarrow \quad r_H(t) \propto t^{1/3}. \tag{1.31}$$

Notice that $r_H(t)$ is an increasing function of time both in a Universe dominated by radiation and in a Universe dominated by matter. We can see more precisely the relation between the particle horizon and the Hubble radius in the following way. The comoving particle horizon (which is nothing else than the conformal time) is

$$\tau_H(t) \equiv \frac{d_H(t)}{a(t)} = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'} \frac{1}{\dot{a}'} = \int_0^a d\ln a' r_H. \tag{1.32}$$

Thus, it is the logarithmic integral of the comoving Hubble radius. This means that, in order to obtain τ_H, we need to integrate r_H over the past interactions.

Our next aim is to see how these two concepts of the Hubble radius and the particle horizon are crucial to understand what is known as the *horizon problem*, and how it can be solved by introducing an inflationary phase in the early stage of the Universe.

1.5 The Hot Big-Bang shortcomings and the inflationary solutions

The role of physics is to predict the evolution of a system (in the case of cosmology, the whole Universe) given some initial conditions. We will now see that the standard cosmological model requires very fine-tuned initial conditions to let the Universe evolve to the one we observe today. Inflation provides an attractor mechanism that brings *dinamically* the Universe towards these conditions at the beginning of the standard radiation era.

1.5.1 The horizon problem

In the previous section we have introduced the crucial concept of the comoving particle horizon τ_H, which is basically the comoving distance travelled by light since the beginning of the Universe. Objects separated by a comoving distance greater than τ_H have never been in causal contact. We have then seen that in standard FLRW models τ_H is an increasing function of time. Let us now think at the implications of this statement. The comoving particle horizon increases with the expansion of the Universe, but comoving wavelengths stays constant. This means that all cosmological modes had comoving wavelengths λ much greater than τ_H in the past, hence no causal physical processes could have affected them. At every instant of time new regions that had never been in causal contact before come into contact for the first time. Hence, we should expect them to be very different from each other, but this is not what we see: on the contrary, the Universe appears extremely homogeneous, even if we look at those scales that have just come into causal contact for the first time. How can this be possible?

This can be understood even more clearly if we think at the CMB. We know that the CMB originated when the Universe had a temperature of about 0.3 eV, corresponding to a redshift $z \sim 1100$. At this epoch, photons decoupled from the primordial plasma and from there on they free-streamed to us. Therefore the CMB is a snapshot of the Universe when it was only 380,000 years old. Since the conformal time (i.e. the comoving particle horizon) elapsed between $t_i = 0$ and the time when the CMB formed t_{rec} is finite, this implies that most of the spots in the CMB have non-overlapping past light cones, hence they have never been in causal contact (see Fig. 1.6). How is it possible that they have almost exactly the same temperature?

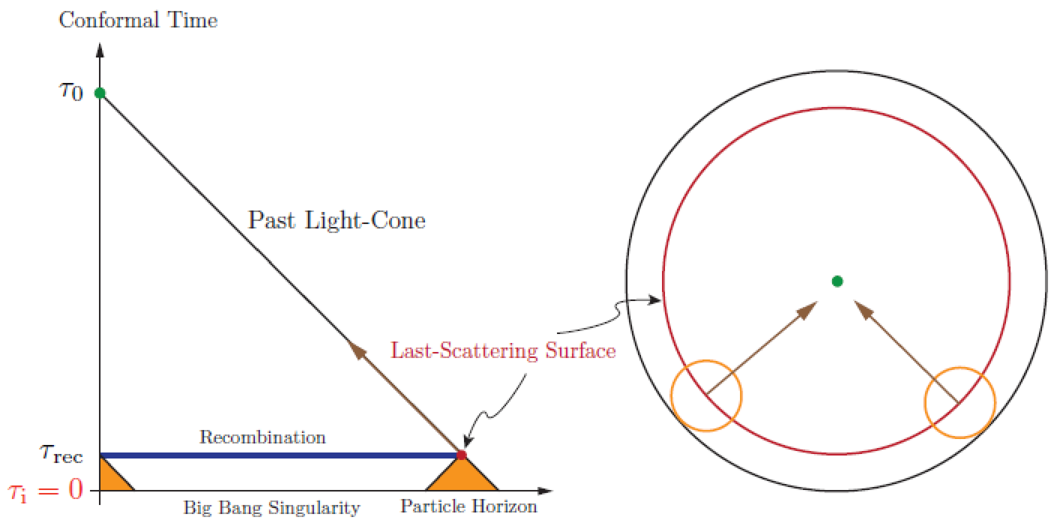

Figure 1.6: Conformal diagram for a standard FRW Universe and horizon problem (from [30]).

To see this "problem" more quantitatively, let us consider the comoving particle horizon at the last scattering surface, which is given by

$$\tau_H^{ls} = \tau_0 - \tau_{ls}. \tag{1.33}$$

If we neglect curvature effects, the angular projection on the last-scattering surface of a given comoving scale λ is

$$\theta \simeq \frac{\lambda}{\tau_0 - \tau_{ls}}. \tag{1.34}$$

In particular we consider the comoving sound horizon at the time of last-scattering, namely $\lambda \sim c_s \tau_{ls}$, where $c_s \simeq 1/\sqrt{3}$ is the sound speed at which photons propagate in the plasma at the last scattering. This corresponds to

$$\theta_{hor} \simeq \frac{c_s \tau_{ls}}{\tau_0 - \tau_{ls}} \simeq c_s \frac{\tau_{ls}}{\tau_0}, \tag{1.35}$$

where in the last step we have used the fact that $\tau_0 \gg \tau_{ls}$. Since the Universe is matter dominated from the time of last scattering onwards, the scale factor grows like

$$a(\tau) \propto \tau^2, \tag{1.36}$$

and we also know that $a \propto T^{-1}$. Using these relations, we find

$$\theta_{hor} \simeq c_s \left(\frac{T_0}{T_{ls}}\right)^{1/2} \sim 1^\circ, \tag{1.37}$$

where we have used $T_{ls} \simeq 0.3$ eV and $T_0 \simeq 10^{-4}$ eV. In terms of multipoles, this corresponds to

$$l_{hor} = \frac{\pi}{\theta_{hor}} \simeq 200. \tag{1.38}$$

This means that two photons which on the last-scattering surface are separated by an angle larger than $\theta_{hor} \sim 1^\circ$, corresponding to multipoles smaller than $l_{hor} \sim 200$, were not in causal contact at decoupling. However, we know that the CMB is extremely isotropic, and moreover its power spectrum (Fig. 1.7) shows us that tiny temperature anisotropies, of the same order of magnitude $\delta T/T \sim 10^{-5}$, are present even at $l \ll 200$.

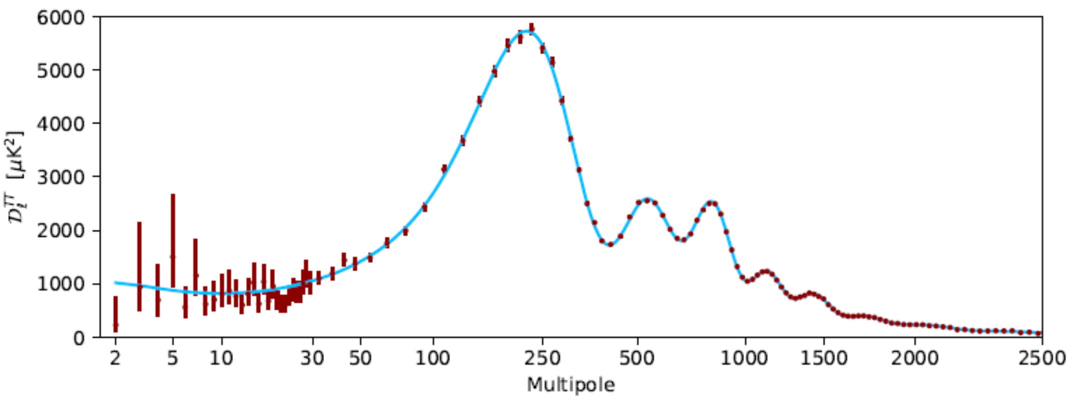

Figure 1.7: Power spectrum of temperature fluctuations in the CMB (from [8])

To understand a possible solution to the horizon problem, let us recall the relation between the comoving particle horizon and the comoving Hubble radius, which is given by

$$\tau_H(t) = \int_0^a d\ln a' r_H. \tag{1.39}$$

The fact that r_H is an increasing function of time in FLRW models implies that τ_H is dominated by the contributions coming from late times. This means that, as we have already said, at every instant of time new regions that had never been in causal contact before come into contact for the first time.

However, if τ_H could be much greater than r_H now, this would mean that particles which cannot communicate today were instead in causal contact in the past. This happens if the comoving Hubble radius was in the past much greater than it is now, so that τ_H gets the most contributions from early times. This leads us to assume that the early Universe, before the standard FLRW phases dominated by radiation and matter, underwent a phase during which the comoving Hubble radius decreased. Thus, particles separated by many Hubble radii today were instead in causal contact during this stage.

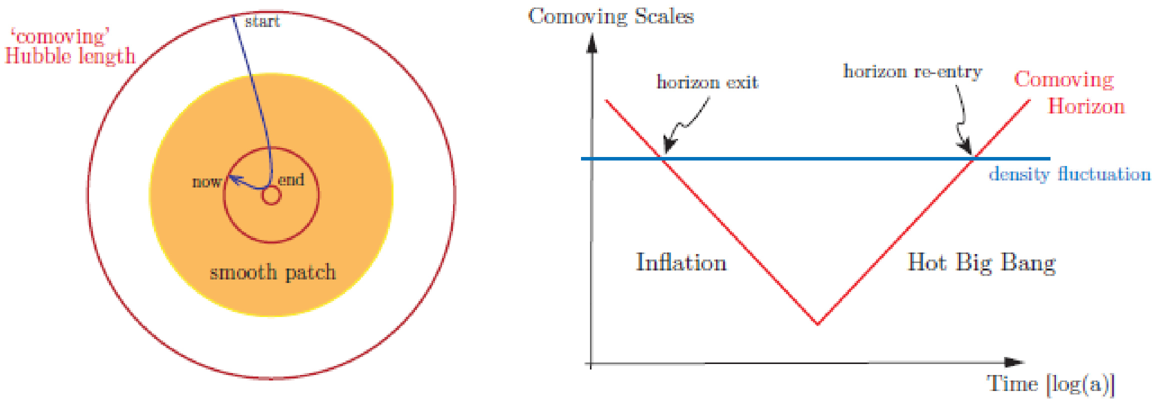

Figure 1.8: *Left*: The comoving Hubble radius shrinks during inflation and then increases. Hence, our entire observable Universe is inside a region (the one indicated as "smooth patch") which was well inside the Hubble radius at the beginning of inflation. *Right*: The scales relevant to cosmological observations were in the early Universe smaller than the Hubble radius. During inflation they "crossed" the Hubble radius and became causally disconnected. Eventually, they reentered the Hubble radius in the radiation or in the matter era (from [30]. For a similar picture see also [31]).

How must the scale factor evolve to solve the horizon problem? The condition of decreasing comoving Hubble radius means that

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \quad \Rightarrow \quad \ddot{a} > 0. \tag{1.40}$$

Therefore, the Universe must have gone through a phase of accelerated expansion: this is what is called *inflation*. From the continuity equation, this translates into a condition on the equation of state of the energy/matter sourcing inflation:

$$p < -\frac{1}{3}\rho, \tag{1.41}$$

meaing that we need a negative pressure to realize an inflationary phase.

Finally, we want to see how inflation solves the horizon problem in terms of the conformal diagram shown in Fig. 1.6. For the moment we assume that inflation corresponds to an exact de Sitter phase, were H = const. (in this case $w = -1$.), and hence the scale factor grows exponentially with time. As a function of conformal time, the scale factor is given by

$$a(\tau) = -\frac{1}{H\tau}, \tag{1.42}$$

and we can see that the initial singularity $a = 0$, which in FLRW models is at $\tau_i = 0$, is instead pushed to the infinite far past, $\tau_i \to -\infty$. At $\tau = 0$ the scale factor instead diverges[6]. This means that $\tau = 0$ is not the initial singularity anymore; rather, it is the end of inflation (what will be called reheating). Therefore the light cones can extend through the "apparent Big Bang" at $\tau = 0$, so that points which would be otherwise disconnected had instead enough conformal time to get in causal contact before recombination. This is well described in the conformal diagram shown in Fig. 1.9.

[6]This happens because a de Sitter phase is characterized by an exponential expansion which lasts forever. However, since inflation has to end in order to let the structure in the Universe form, this can be an approximation valid only at early times. We will come back soon to this point, but for the moment let us stick with this.

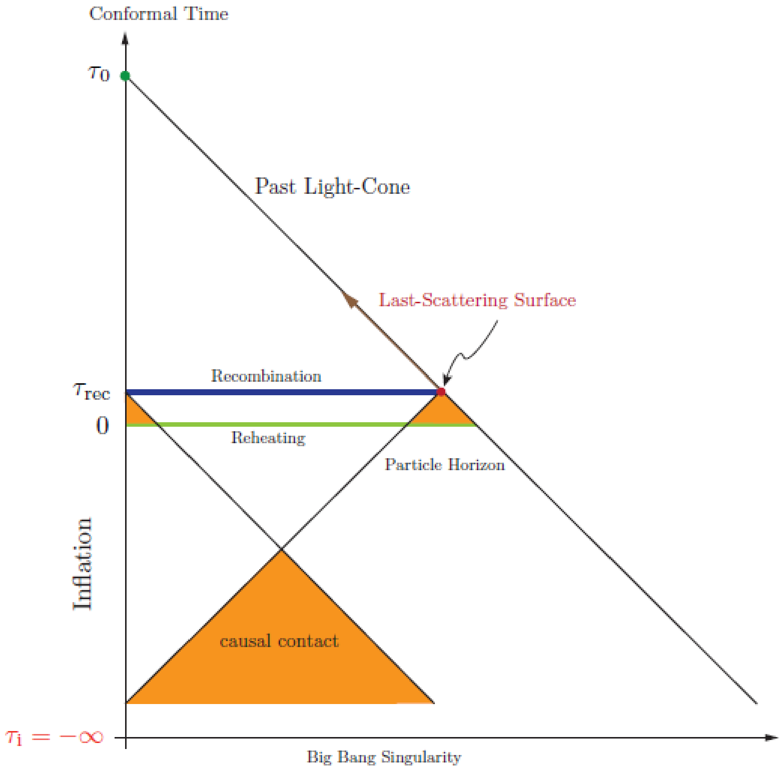

Figure 1.9: Conformal diagram on inflationary cosmology. Inflation extends conformal time to negative values, hence light cones intersect in the past (from [30]).

1.5.2 The flatness problem

The second of the shortcomings of the Hot Big Bang model is the so-called flatness problem. To understand what it concerns, let us recall the first Friedmann equation, which can be written as

$$|\Omega(t) - 1| = \frac{|k|}{a^2 H^2} = |k|\, r_H^2(t). \tag{1.43}$$

If the Universe is perfectly flat, i.e. $k = 0$, then $\Omega = 1$ at all times[7]. If this is not the case, the situation is radically different. We know that in the case of standard FLRW models the Hubble radius is an increasing function of time. This implies that the departure of $\Omega(t)$ from 1 also increases with time and it must eventually diverge (see Fig. 1.10). In other words, $\Omega = 1$ is an unstable point. A small deviation of Ω from unity in the early Universe leads to a Unverse rapidly recollapsing (in the case of a close geometry), or too rapidly expanding, resulting in an empty Universe in which gravity doesn't succeed in forming structures (in the case of a open geometry). This requires an extreme fine-tuning of the value of Ω, which must be enormously close to 1 in the early Universe. From the observed value of Ω today

$$|\Omega(t_0) - 1| < 0.005 \quad (95\%\ \text{C.L.}), \tag{1.44}$$

it is possible to show that the deviation from $\Omega = 1$ at the Planck epoch t_{Pl} (which corresponds to a temperature of the Universe of $T_{Pl} \sim 10^{19}$ GeV) and at the epoch of nucleosynthesis t_{nuc} ($T_{nuc} \sim 1$ MeV) must be given respectively by (see e.g. [31])

$$\Omega(t_{Pl}) \simeq 1 + (\Omega_0 - 1) \cdot 10^{-60}, \tag{1.45}$$

[7] Actually the case with $k = 0$ is represented by a set of zero measure, namely a point in the set of all possible values of the curvature of the Universe. Thus, the probability of our Universe having exactly zero curvature vanishes.

$$\Omega(t_{nuc}) \simeq 1 + (\Omega_0 - 1) \cdot 10^{-16}. \tag{1.46}$$

While this extremely flatness of the early Universe has no explanation and must be assumed in standard FLRW models, inflation provides a solution for this fine-tuning problem, since the exponential expansion leads dinamically the density parameter towards unity. Indeed, during the inflationary phase the scale factor grows like $a(t) \approx e^{Ht}$ hence, from (**??**), we have

$$|\Omega(t) - 1| \propto e^{-2Ht}. \tag{1.47}$$

This means that, independently on the initial value of Ω, a (long enough) inflationary phase drives its value towards unity, thus solving the flatness problem (see again Fig. 1.10).

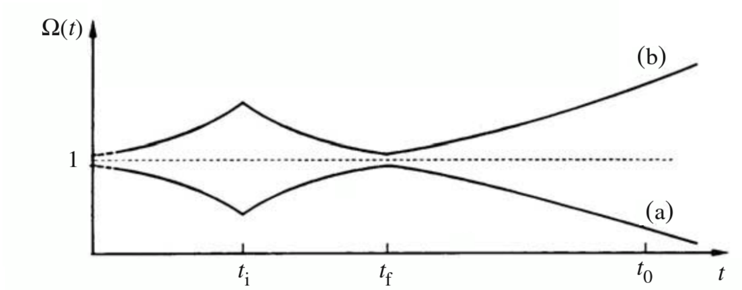

Figure 1.10: Evolution of $\Omega(t)$ for an open Universe (a) and a closed Universe (b), characterized by three periods $(0, t_i)$, (t_i, t_f) and (t_f, t_0). The first and last phases are standard FLWR, while the second is an inflationary phase, during which $\Omega(t)$ gets led towards one. If inflation lasts enough time, the later divergence from $\Omega = 1$ can be delayed well beyond our current time t_0 (from [31]).

We want to stress that inflation does not change the global geometry of the spacetime. If the Universe is flat, closed or open, it remains so with or without inflation. Inflation only increases the radius of curvature, so that *locally* (i.e. in our observable patch) the Universe looks flat with a great precision.

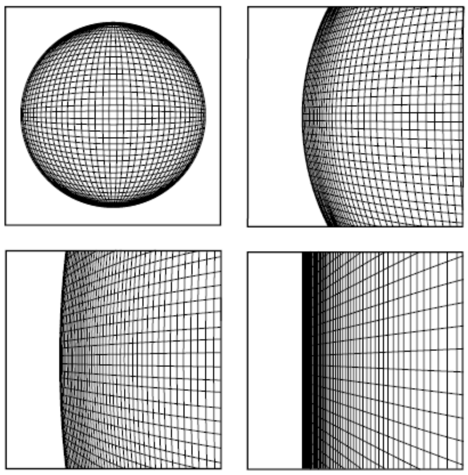

Figure 1.11: The exponential expansion during inflation magnifies the radius of curvature of the Universe, so that it locally looks flat. Here we report the (3-dimensional) analogy with the surface of an inflating sphere (from [32]).

1.5.3 The "unwanted relics" problem

We briefly mention another problem of the standard cosmological model which is successfully solved by inflation. Historically, this is the problem that initially led to the introduction of the inflationary paradigm.

It is likely that the early Universe underwent a series of phase transitions which, depending on the broken symmetries, may have produced different topological defects like domain walls, monopoles and cosmic strings. If produced, it is easy to see that their energy density tends to become the dominant one in the Universe, leading to $\Omega_{TD} \gg 1$. This is obviously in contrast with observations, which tell us that $\Omega_{\rm tot} \simeq 1$.

Inflation naturally solves also this problem, since the accelerated expansion strongly dilutes the energy density contribution of these topological defects.

1.5.4 How much did the Universe inflate?

Introducing an early stage of accelerated expansion is not sufficient to reproduce the Universe we observe today. We need also to require for such a phase to last enough to solve the horizon and the flatness problems. For the former, we have to impose that the inflationary phase was long enough to allow a region initially smaller than the size of the horizon (and hence causally connected) to grow and become larger than the observable Universe today. In practice, we need to require that the comoving Hubble radius at the beginning of inflation was at least equal or larger than its value today

$$r_H(t_{in}) \geq r_H(t_0). \tag{1.48}$$

This means that a comoving size which is entering the horizon today had already been sub-horizon in the far past.

As regarding the flatness problem, we need to require that

$$\frac{1-\Omega_i^{-1}}{1-\Omega_0^{-1}} \geq 1, \tag{1.49}$$

which means that the density parameter differed from unity at the beginning of inflation much than it does today.

These two requirements lead to a similar condition for the duration of inflation. This can be expressed in terms of the number of *e-folds* N, which is defined as

$$N_\lambda \equiv \int_{t_\lambda}^{t_f} H dt = \ln\left[\frac{a(t_f)}{a(t_\lambda)}\right], \tag{1.50}$$

where λ is a given scale which left the horizon during inflation at the time $t = t_\lambda$, while t_f represents the time at which inflation ended. Thus, N_λ is a measure of the expansion of the Universe during the time interval between t_λ and t_f. For the cosmological scales that we can observe through the CMB anisotropies one finds (see e.g. [31])

$$N_{\rm CMB} \simeq 60, \tag{1.51}$$

meaning that we need to require that inflation lasted for at least about 60 e-folds. Usually inflationary models lead to $N \gg N_{\rm CMB}$, and it would not be natural to ask that inflation lasted just long enough for a region with size equal to our observable Universe to start sub-horizon during inflation. What we can say is that the cosmological scales which can be proved through the CMB had to leave the horizon 60 e-folds before the end of inflation (this corresponds to the so-called "observable window", see Fig. 1.14).

1.5.5 The entropy problem

In this section we want to reformulate the flatness problem in a different way. This is useful since it makes more clear which is the physical reason behind the extremely small numbers we have obtained, and thus also suggests a possible solution, leading again to the idea of introducing an inflationary phase in the early Universe. In explaining this argument we follow the discussion of [33].

The key point is that the flatness problem is related to the assumption of adiabatic expansion of the Universe made in the FLRW model. To show this, let us rewrite Eq. (1.43) as

$$|\Omega(t) - 1| = \frac{|k| M_{Pl}^2}{T^2 S^{2/3}}, \tag{1.52}$$

where $S \sim a^3 T^3$ is the entropy in a comoving volume[8]. If our Universe undergoes an adiabatic expansion, then S is constant and we can evaluate it at the present time. Hence, we have

$$|\Omega(t) - 1| = \frac{|k| M_{Pl}^2}{T^2 S_0^{2/3}}, \tag{1.53}$$

with S_0 denoting the present value of the entropy, which can be estimated as

$$S_0 \sim T_0^3 H_0^{-3} \simeq 10^{90}. \tag{1.54}$$

Let us now evaluate the expression (1.53) at the Planck time:

$$|\Omega(t_{Pl}) - 1| = \frac{|k| M_{Pl}^2}{T_{Pl}^2 S_0^{2/3}} \simeq 10^{-60}. \tag{1.55}$$

Notice that this value is of the same order of the one obtained when studying the flatness problem (see (1.45)). However, this reformulation suggests us that the flatness problem is due to the huge value of the entropy of our Universe or, equivalently, to the fact that our Universe contains a huge number of particles.

Put in this way, we can see that a possible solution to this problem is to assume that our Universe underwent a phase of non-adiabatic expansion during its early stage. This phase of non-adiabatic expansion is the transition phase between inflation and the standard radiation era[9] (the so-called *reheating* phase) which, as we will see in more detail, produce a huge amount of entropy and hence reheats the Universe. Let us thus assume that from the beginning to the end of the inflationary+reheating phase the entropy changed by

$$S_f = Z^3 S_i, \tag{1.56}$$

where Z is a numercal factor. Let us also assume that at the beginning of inflation the entropy in a comoving volume was of order unity, meaning that there was about one particle per horizon. If the expansion of the Universe after this phase is adiabatic, it follows that $S_f = S_0$ and hence $Z \sim 10^{30}$. Since $S \sim a^3 T^3$, from (1.56) we find

$$\frac{a_f}{a_i} \equiv e^N \simeq 10^{30} \, \frac{T_i}{T_f}, \tag{1.57}$$

where T_i and T_f are the temperature at the beginning and at the end of the inflationary+reheating phase respectively. From the last expression we can find the value of N during such a phase, which is given by

$$N \sim 70, \tag{1.58}$$

up to a factor $\ln(T_i/T_f)$ which does not modify considerably the result. This is really similar to the values of N required to solve also the horizon (and the flatness, in the previous formulation) problem.

[8] In this estimation we are neglecting the effective number of relativistic degrees of freedom g_{*s}. In principle this quantity varies during the expansion of the Universe, but for our purposes we can neglect it without commiting any relevant error.

[9] Indeed, during the inflationary phase the Universe still expands adiabatically.

1.5.6 A problem of initial conditions

It is important to stress that the Hot Big Bang "problems" are not inconsistencies within the model itself. Indeed, if we assume an initial state of the Universe with Ω extremely close to one and with homogeneity on super-horizon scales, but with just the tiny fluctuations required to explain the formation of the large scale structures, the Universe will evolve towards the state we can observe today. Rather, they are shortcomings in the predictive power of the standard cosmological model, since the peculiar fine-tuned initial conditions we have seen must be assumed without any kind of explanation. As we have seen, inflation provides instead a dynamical mechanism which leads the Universe towards them when the radiation era begins.

1.6 The dynamics of inflation and the slow-roll approximation

We have seen that inflation is a phase of accelerated expansion sourced by a negative pressure, $w = \frac{p}{\rho} < -\frac{1}{3}$. The simplest way to realize such an inflationary phase is through vacuum energy, which is characterized by an equation of state with $w = -1$ and which can be introduced in the Einstein's equations through a cosmological constant. As we have already seen, vacuum energy leads to a de Sitter phase, where the scale factor grows exponentially with time, $a(t) \propto e^{Ht}$. The problem with this realization is that vacuum energy does not provide any mechanism through which inflation ends, and the exponential expansion never stops. However we know that our Universe went through a long period of decelerated expansion, when first radiation and then matter dominated its energy content. During such phases, the primordial perturbations generated during inflation[10] re-entered the horizon to give rise, via gravitational instability, to the structures we can observe today. This can not happen in a Universe undergoing a never-ending exponential expansion. Moreover, there are other important evidences supporting this expansion history, like the CMB and the Big Bang Nucleosynthesis. For these reasons we need to introduce some kind of "clock", which regulates the amount of inflation and controls its end. This can be simply realized by a slowly-rolling scalar field.

Let us consider a scalar field, which we call the *inflaton*, and slightly tilt (with respect to a constant value) the potential under which it evolves. If the potential is sufficiently flat, the scalar field can mimic an effective cosmological constant and drives inflation. Then, inflation could be ended by adding some mechanism triggered by some particular value of the field. In the case of the potential in Fig. 1.12, when the potential is no longer flat and the inflaton starts seeing the minimum, it begins to oscillate and then decays into ultra-relativistic particles. This is called the *reheating* phase. After that, it follows the standard radiation-dominated era.

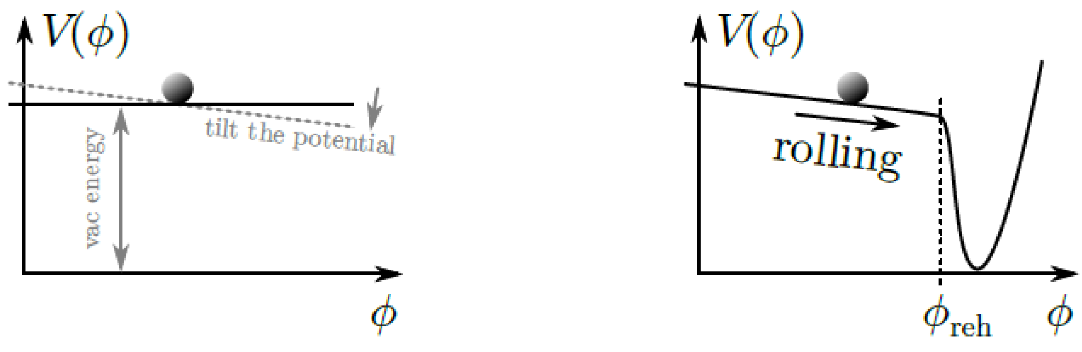

Figure 1.12: Slowly-rolling scalar field (from [34]).

[10]We will talk about primordal perturbations from inflation in detail in the next chapters.

The dynamics of a scalar field minimally coupled to gravity is encoded in the action

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.59}$$

where $M_{Pl}^2 \equiv (8\pi G)^{-1}$ is the (squared) reduced Planck mass, $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor and $V(\phi)$ is the potential term, which contains the self-interactions of the scalar field. The first term is the Einstein-Hilbert action, while the second term describes a scalar field minimally coupled to gravity through the metric $g_{\mu\nu}$. By varying this action with respect to the scalar field ϕ, one can obtain the equation of motion for the latter. This is given by the Klein-Gordon equation

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right) = V'(\phi), \tag{1.60}$$

where $V'(\phi) = \frac{\partial V(\phi)}{\partial\phi}$. In the case of a Universe described by a FLRW metric, this becomes

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) = 0. \tag{1.61}$$

Notice in particular that the second term is a friction term which is due to the expansion of the Universe (indeed, it vanishes if $a(t) = \text{const}$) and which leads to the red-shifting of the momentum of the scalar field. One can then compute the stress-energy tensor $T_{\mu\nu}$ of the scalar field, which is given by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right). \tag{1.62}$$

We can then split the inflaton field as

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t), \tag{1.63}$$

where $\phi_0(t)$ is the "classical" value of the field, i.e. its vacuum expectation value (vev) on the initial homogeneous and isotropic state, $\phi_0(t) = \langle\phi(\mathbf{x}, t)\rangle$, and $\delta\phi(\mathbf{x}, t)$ are the quantum fluctuations around it[11]. For the moment we focus only on $\phi_0(t)$, namely we look at the background dynamics of the classical field; in the next chapters we will study the perturbations on top of this background solution. The components of the stress-energy tensor become

$$T^0{}_0 = -\left[\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)\right], \quad T^i{}_j = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0). \tag{1.64}$$

We can see that the stress-energy tensor of the scalar field is that of a perfect fluid with[12]

$$\rho_{\phi_0} = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0), \tag{1.65}$$

$$p_{\phi_0} = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0). \tag{1.66}$$

[11] This splitting is justified since $\langle\delta\phi^2\rangle \ll \langle\phi_0^2\rangle$. Notice that, by definition of fluctuation, $\langle\delta\phi\rangle = 0$.

[12] If we include also the inhomogeneities in the scalar field, we get:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2},$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}\frac{(\nabla\phi)^2}{a^2}.$$

The inflationary expansion smooths out the spatial variations, since $a^{-2} \propto e^{-2Ht}$. Thus, this term becomes quickly subdominant and the two previous expressions for ρ_ϕ and p_ϕ reduce to (1.65) and (1.66).

Therefore, the equation of state for the scalar field can be written as

$$w_{\phi_0} = \frac{p_{\phi_0}}{\rho_{\phi_0}} = \frac{\frac{1}{2}\dot{\phi}_0^2 - V(\phi_0)}{\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)}. \tag{1.67}$$

Now notice that, if the potential term dominates over the kinetic one

$$\frac{1}{2}\dot{\phi}_0^2 \ll V(\phi_0), \tag{1.68}$$

the equation of state reduces to $w_{\phi_0} \simeq -1$, and the Universe undergoes a *quasi-de Sitter phase*, $a(t) \simeq a_i e^{Ht}$. The condition (1.68) is known as the *slow-roll condition*, since it tells us that the scalar field moves slowly with respect to its potential.

Notice also that, if before the flat plateau characterizing the slow-roll phase the Universe starts kinetic energy dominated, from (1.67) it follows that $w_{\phi_0} \simeq 1$. Then, from the continuity equation, we have that

$$\rho_{\phi_0} \propto a^{-3(1+w_{\phi_0})} \propto a^{-6}. \tag{1.69}$$

This means that the kinetic energy of the inflaton field gets redshifted away quickly, and the potential energy starts dominating. In this sense inflation is a dynamical attractor solution.

To formalize more precisely the slow-roll approximations, let us rewrite the equation of motion for the (homogeneous) scalar field and the first Friedmann equation:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0, \tag{1.70}$$

$$3M_{Pl}^2 H^2 = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0). \tag{1.71}$$

The equation of motion for the scalar field is the same as the equation for a ball rolling down its potential. As we have already said, the term $3H\dot{\phi}_0$ is a friction term due to the expansion of the Universe. Indeed, in the case $V'(\phi_0) = 0$, the two solutions of the equation are $\phi_0 = \text{const}$ and $\dot{\phi}_0 \propto a^{-3}$. The first one is the case in which the ball is at rest. The second solution means that if there is no external driving force, the initial velocity decreases as a consequence of the friction term.

The slow-roll approximation consists in requiring that the potential term dominates over the kinetic term and that the potential is sufficiently flat so that the acceleration of the field is negligible

$$\frac{1}{2}\dot{\phi}_0^2 \ll V(\phi_0), \quad |\ddot{\phi}_0| \ll |3H\dot{\phi}_0|. \tag{1.72}$$

The first condition leads to an inflationary phase since implies that $w_{\phi_0} \simeq -1$, as we have already seen. The second condition requires $\dot{\phi}_0$ to change slowly such that the first condition can be satisfied for a long enough time. This corresponds to the attractor solution in which the friction force balances the external force provided by the potential, so that the acceleration of the field is nearly equal to zero[13].

With these two conditions, we can rewrite the equations (1.71) and (1.70) respectively as

$$H \simeq \sqrt{\frac{V}{3M_{Pl}^2}}, \tag{1.73}$$

$$\dot{\phi}_0 \simeq -\frac{V'}{3H}. \tag{1.74}$$

[13] To give an analogy, let us consider the case of a ball subjected to the force of gravity and a friction force. As the ball accelerates, the friction term increases (being proportional to the velocity of the ball). At a certain time this equates the external force and the ball undergoes a uniform motion with constant speed.

We can now use them to translate the conditions (1.72) into two conditions for the potential of the inflaton field. Let us first define the so-called *potential slow-roll parameters*:

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \simeq \frac{1}{2}\frac{\dot{\phi}_0^2}{H^2 M_{Pl}^2}, \tag{1.75}$$

$$\eta_V \equiv M_{Pl}^2 \frac{V''}{V} \simeq -\frac{\ddot{\phi}_0}{\dot{\phi}_0 H} + \frac{1}{2}\frac{\dot{\phi}_0^2}{M_{Pl}^2 H^2}. \tag{1.76}$$

The slow-roll conditions can now be written as

$$\epsilon_V \ll 1, \quad |\eta_V| \ll 1. \tag{1.77}$$

At leading order in the slow-roll parameters, ϵ_V and η_V can be considered constant, since

$$\frac{\dot{\epsilon}_V}{H} = \mathcal{O}(\epsilon_V^2, \eta_V^2), \quad \frac{\dot{\eta}_V}{H} = \mathcal{O}(\epsilon_V^2, \eta_V^2). \tag{1.78}$$

We can also introduce another set of slow-roll parameters, which are not directly related to the shape of the potential, but rather to the dynamics of the scale factor:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}_0^2}{2H^2 M_{Pl}^2}, \tag{1.79}$$

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H} = 2\epsilon + 2\frac{\ddot{\phi}_0}{H\dot{\phi}_0}. \tag{1.80}$$

These are called *Hubble slow-roll parameters*. The condition $\epsilon \ll 1$ requires the energy driving inflation to be dominated by the potential, exactly as the first of (1.77). The condition $|\eta| \ll 1$ requires that

$$\frac{\ddot{\phi}_0}{H\dot{\phi}_0} = -\epsilon + \frac{\eta}{2} \ll 1, \tag{1.81}$$

namely tells us, analogously to the second condition of (1.77), that the (background) evolution of the inflaton is determined by the attractor solution (1.74).

When the slow-roll conditions (1.77) are satisfied, the relations between the two sets of slow-roll parameters are

$$\epsilon = \epsilon_V + \mathcal{O}\left(\epsilon_V^2, \eta_V^2\right), \quad \eta = 4\epsilon_V - 2\eta_V + \mathcal{O}\left(\epsilon_V^2, \eta_V^2\right). \tag{1.82}$$

The conditions defined through the Hubble slow-roll parameters ϵ and η are more general than the ones defined through ϵ_V and η_V. Indeed in some cases, while the first ones are still necessary to ensure a prolonged enough inflationary phase, the other ones are not satisfied anymore. To give an example, the shape of the potential can be steeper, therefore not satisfying the first of (1.72), or the inflationary energy can be dominated by the kinetic energy, rather than by the potential.

Notice that ϵ can also be regarded as a parameter that characterizes the departure from an exact de Sitter phase, in which H is constant and therefore $\epsilon = 0$. In particular

$$\ddot{a} = (aH)^{\cdot} = a(H^2 + \dot{H}) = aH^2(1-\epsilon). \tag{1.83}$$

Therefore an accelerated expansion, for which $\ddot{a} > 0$, requires $\epsilon < 1$. Inflation ends when $\epsilon \sim 1$, namely when the inflaton begins to “feel” the potential well.

1.7 Single field slow-roll models of inflation

Inflationary models are defined by specifying the action for the inflaton field. If one assumes a canonical kinetic term (as it is the case for slow-roll models), $X \equiv -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and a minimal coupling to gravity, the dynamics of inflation is then fully specified by the potential $V(\phi)$.

1.7.1 Large-field and small-field models

Single field slow-roll models are tipically divided in two different classes, depending on the excursion of the scalar field during inflation:

1. *Large-field models*: these are characterized by an excursion of the inflaton field greater than the Planck mass, $\Delta\phi > M_{Pl}$.

2. *Small-field models*: these are instead characterized by a field excursion lower than the Planck mass, $\Delta\phi < M_{Pl}$.

As we will see later on, the fact of having a field excursion greater or lower than M_{Pl} has both theoretical and phenomenological fundamental consequences. We now want to better clarify what we mean by the field excursion $\Delta\phi$ and provide some examples of models which fall into both the two classes.

Large-field models

A tipical example of large-field model is provided by *Chaotic Inflation*, which is described by a potential for the scalar field with a single polynomial term:

$$V(\phi) = \lambda_p \left(\frac{\phi}{\mu}\right)^p, \quad \text{with } \; p > 0, \;\; \mu < M_{Pl}. \tag{1.84}$$

The scale μ is the relevant scale for the higher-dimensional terms in this effective potential, and corresponds to the mass of heavy states which have been integrated out. See [35] for a discussion of this point and to see why one needs to require the condition $\mu < M_{Pl}$. To obtain a small amplitude of density perturbations (consistently with the CMB observations) one needs to require that $\lambda_p \ll 1$. This implies that the potential energy density is sub-Planckian, $V \ll M_{Pl}^4$.

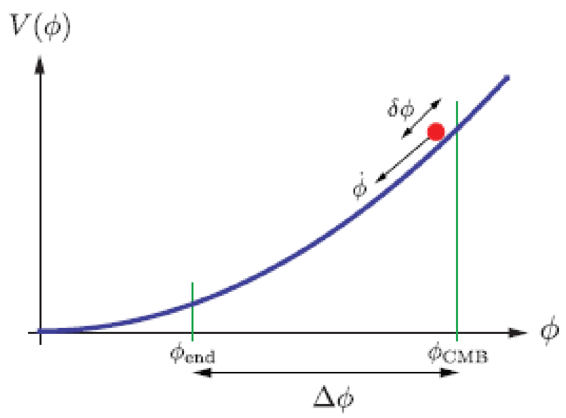

Figure 1.13: Large-field inflation (from [30]).

We can now compute the slow-roll parameters for this potential. These are given by

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{p^2}{2}\left(\frac{M_{Pl}}{\phi}\right)^2, \tag{1.85}$$

$$\eta_V \equiv M_{Pl}^2 \left(\frac{V''}{V}\right) = p(p-1)\left(\frac{M_{Pl}}{\phi}\right)^2. \tag{1.86}$$

It is clear that, in order to have the slow-roll conditions satisfied, $\epsilon_V \ll 1$ and $|\eta_V| \ll 1$, we must require that

$$\phi \gg M_{Pl}. \tag{1.87}$$

Thus, to ensure a prolonged enough phase of inflation, the value of the inflaton field has to be greater than the Planck mass.

It is also interesting to consider the field excursion during inflation. The excursion of the field between two generic instants of time t_1 and t_2 is given by

$$\Delta\phi = \int_{t_1}^{t_2} \dot{\phi}\, dt. \tag{1.88}$$

By taking H and $\dot{\phi}$ to be constant as a first approximation, this can be rewritten as

$$\Delta\phi \approx \frac{\dot{\phi}}{H}\int_{t_1}^{t_2} H\, dt \approx \epsilon_V^{1/2} M_{Pl} N_{1,2}, \tag{1.89}$$

where

$$N_{1,2} = \int_{t_1}^{t_2} H\, dt \tag{1.90}$$

is the number of e-folds between the times t_1 and t_2. In particular, we are interested in the value of $\Delta\phi$ referring to the so-called "observable window": this is the variation of ϕ between the time instant when the present cosmological horizon (i.e., our observable Universe) went above the horizon during inflation, and the time at which inflation ended. In other words, this corresponds to the maximum cosmological scale observable through the CMB.

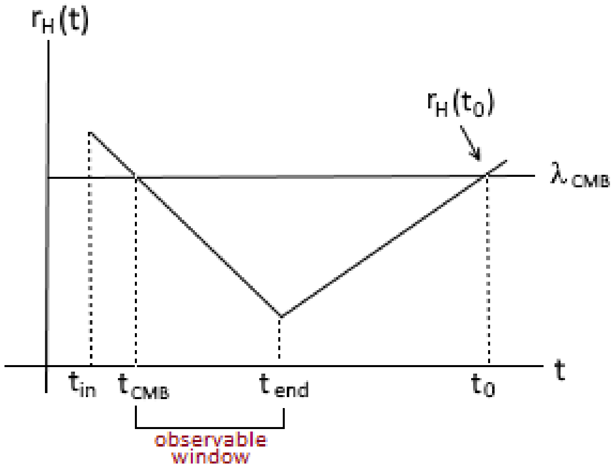

Figure 1.14: The "observable window".

Large-field models tipically produce $\epsilon_V \sim \frac{1}{N_{\rm CMB}}$. From Eq. (1.89), this impies that

$$\Delta\phi \gtrsim M_{Pl}. \tag{1.91}$$

This is why this model falls into the class of large-field models.

Small-field models

A typical small-field potential has the following form

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^p + \ldots\right], \quad \text{with } \phi < \mu \ll M_{Pl}, \quad p > 2. \tag{1.92}$$

This is shown in Fig. 1.15. The requirement of having $p > 2$ is necessary to have a small-field model; indeed, in the case $p = 2$ one recovers a large-field model. This is the prototype of a potential arising from a spontaneous symmetry breaking. The dots represent higher-order terms that become important at the end of inflation.

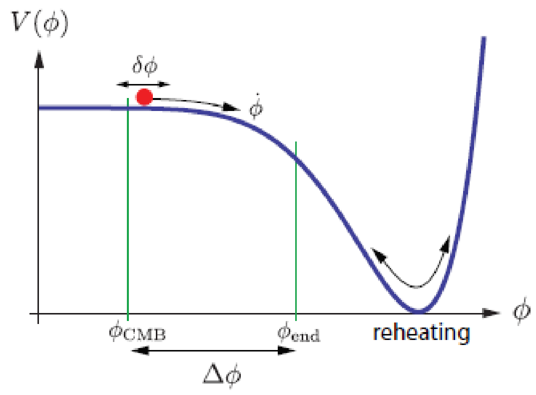

Figure 1.15: Small-field inflation (from [30]).

The slow-roll parameters in this case are given by

$$\epsilon_V = \frac{1}{2}p^2 \left(\frac{M_{Pl}}{\phi}\right)^2 \left(\frac{\phi}{\mu}\right)^{2p} \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]^{-2}, \tag{1.93}$$

$$\eta_V = -p(p-1) \left(\frac{M_{Pl}}{\phi}\right)^2 \left(\frac{\phi}{\mu}\right)^{p} \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]^{-1}. \tag{1.94}$$

For $p > 2$, if $\phi < \mu \ll M_{Pl}$ the slow-roll conditions are satisified. Thus, the inflaton field has sub-Planckian values and can assume also very small values. Notice that, for $\phi \to 0$, also $\epsilon_V \to 0$. This means that, if the value of the inflaton field is sufficiently close to zero, also the slow-roll parameter ϵ_V is very small. From (1.89), it follows that $\Delta\phi \to 0$. More in general we have $\Delta\phi \ll M_{Pl}$, which characterizes small-field models.

1.7.2 Hybrid models

There is also a third class of inflationary models. These are called *Hybrid models*, since they share some features with both large and small-field models. Hybrid models are embedded in high energy physics frameworks like supersymmetry and supegravity, and in the simplest version are described by potentials with two fields, ϕ and ψ. A common example is given by

$$V(\phi, \psi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}(\psi^2 - M^2)^2 + \frac{\lambda'}{2}\phi^2\psi^2. \tag{1.95}$$

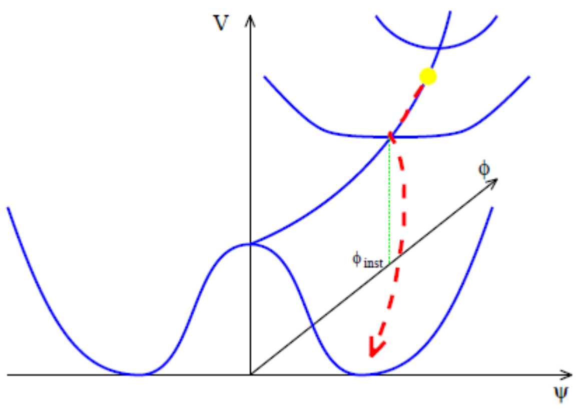

Figure 1.16: Hybrid inflation (from [36]).

We have a large-field potential for the field ϕ and a small-field potential for the field ψ. During the inflationary phase ϕ assumes large values and the minimum of the potential is for $\psi = 0$. Thus, the field ψ is initially trapped at $\psi = 0$ (i.e., it is not dynamical) by the interactions with ϕ, while the latter rolls in the false vacuum along the valley $\psi = 0$. During this phase therefore we have a single field model with an effective (cahotic-like) potential for ϕ, given by

$$V_{\text{eff}}(\phi) \equiv V(\phi, \psi = 0) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} M^4, \tag{1.96}$$

where

$$V_0 \equiv \frac{\lambda}{4} M^4 \tag{1.97}$$

is the vacuum energy during inflation, which is provided by the field ψ. When the field ϕ reaches the critical value[14]

$$\phi = \phi_c \equiv \frac{\lambda}{\lambda'} M, \tag{1.99}$$

then $\psi = 0$ becomes unstable and the fields roll towards one of the true minima of the system, $\phi = 0$ and $\psi = \pm M$. The oscillations around the true minimum give rise to reheating.

These models are an hybrid between large-field and small-field models also because the field ϕ starts from a high value, which is typical of large-field models, but then goes towards a minimum of the potential where $\phi \neq 0$, as occurs in small-field models (see again Fig. 1.15).

1.8 Reheating

The reheating phase represents the transition between inflation and the standard radiation-dominated era. The discussion follows [37]. At the end of inflation the Universe is in a low-entropy state with few degrees of freedom, very much unlike the Universe we see. We need then to explain how the transition between the two phases might arise. Here the reheating phase comes into play: when the inflaton reaches the minimum of the potential, it starts oscillating and acquires a mass; then it decays into ultra-relativistic particles, producing entropy and hence reheating the Universe.

[14]This can be simply obtained by

$$\frac{\partial^2 V(\phi, \psi)}{\partial \psi^2}\Big|_{\psi=0, \phi=\phi_c} = \lambda' \phi_c^2 - \lambda M^2 = 0 \quad \Rightarrow \quad \phi_c = \frac{\lambda}{\lambda'} M. \tag{1.98}$$

During the ocillations regime we have $V'' \gg H^2$, and the inflaton evolves rapidly on the expansion time scale. Once it reaches the minimum of the potential ($\phi = \sigma$), it begins to oscillate with a frequency $\omega^2 = V''(\sigma)$. The equation describing the dynamics of the inflaton is

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} = -V'(\phi), \tag{1.100}$$

where Γ_ϕ is the decay rate of ϕ into the other particles to which it couples, and represents an additional damping term besides the one due to the expansion of the Universe. For simplicity of notation we have neglected the subscript 0 to denote the background value of the inflaton field. Since $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, Eq. (1.100) can be rewritten as

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\,\dot{\phi}^2 = 0. \tag{1.101}$$

As ϕ is rapidly oscillating around σ, $\dot{\phi}^2$ can be replaced by its average over a period of oscillation:

$$\dot{\phi}^2 \to \langle\dot{\phi}^2\rangle_{period} = 2\langle V\rangle_{period} = \rho_\phi. \tag{1.102}$$

Notice that this implies that $\langle P_\phi\rangle = \langle\dot{\phi}^2/2 - V(\phi)\rangle = 0$, which means that, while oscillating around the minimum, the inflaton behaves like non-relativistic matter. With the substitution (1.102), Eq. (1.101) becomes

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi. \tag{1.103}$$

As expected from what just said, this is the equation describing the decay of a massive particle species; if we neglect Γ_ϕ, it is exactly the continuity equation for non-relativistic matter. Assuming that the decay products of ϕ are relativistic particles, the equations we need to describe the reheating process are

$$\begin{cases} \dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi, \\ \dot{\rho}_r + 4H\rho_r = +\Gamma_\phi \rho_\phi, \\ H^2 = \frac{8}{3}\pi G\left(\rho_\phi + \rho_r\right). \end{cases} \tag{1.104}$$

The solution of the equation for ρ_ϕ is

$$\rho_\phi = M^4 \left(\frac{a}{a_{osc}}\right)^{-3} e^{-\Gamma_\phi (t - t_{osc})}, \tag{1.105}$$

where t_{osc} labels the epoch when the oscillations begin and $\rho_\phi\left(t_{osc}\right) = M^4$ is the vacuum energy of the inflaton at that time. If we assume that from t_{osc} to $t_{dec} \simeq \Gamma_\phi^{-1}$ the Universe is dominated by non relativistic particles ($a(t) \propto t^{2/3}$), then $H = \frac{2}{3t}$ and we have

$$\begin{cases} t_{osc} \simeq H_{inf}^{-1} \\ H_{inf}^2 \simeq \frac{8}{3}\frac{\pi}{M_{Pl}^2}V = \frac{8}{3}\frac{\pi}{M_{Pl}^2}V \end{cases} \quad \Rightarrow \quad t_{osc} \simeq \sqrt{\frac{3}{8\pi}}\frac{M_{Pl}}{M^2}. \tag{1.106}$$

By using these relations together with (1.105), the equation for ρ_R becomes

$$\dot{\rho}_r + \frac{8}{3}\frac{\rho_r}{t} = \frac{3}{8\pi}\Gamma_\phi M_{Pl}^2 \frac{1}{t^2}. \tag{1.107}$$

The solution, with the initial condition $\rho_r\left(t_{osc}\right) = 0$, is given by

$$\rho_r \simeq \frac{0.6}{\sqrt{\pi}}\Gamma_\phi M_{Pl} M^2 \left(\frac{a}{a_{osc}}\right)^{-3/2} \left[1 - \left(\frac{a}{a_{osc}}\right)^{-5/2}\right]. \tag{1.108}$$

Thus ρ_r rapidly increases until $a = a_{osc}$ and reaches a maximum value of

$$\rho_r^{max} \simeq \Gamma_\phi M_{Pl} M^2. \tag{1.109}$$

Then it decreases as $a^{-3/2}$ due to the expansion of the Universe, even though radiation gets still created. Assuming that the radiation has yet thermalized, such that $\rho_R = \frac{\pi^2}{30} g_* T^4$, the maximum temperature corresponding to (1.109) is given by

$$T^{max} \simeq g_*^{-1/4} M^{1/2} \left(\Gamma_\phi M_{Pl}\right)^{1/4}. \tag{1.110}$$

Regarding the entropy, we have

$$s \propto T^3 \propto \rho_r^{3/4} \Rightarrow S \propto a^3 \rho_r^{3/4} \propto a^{15/8}, \tag{1.111}$$

where s is the entropy density and S the entropy, with $S = a^3 s$. Notice that, even though ρ_r is decreasing, the entropy increases because the scalar field, while decaying, produces radiation.

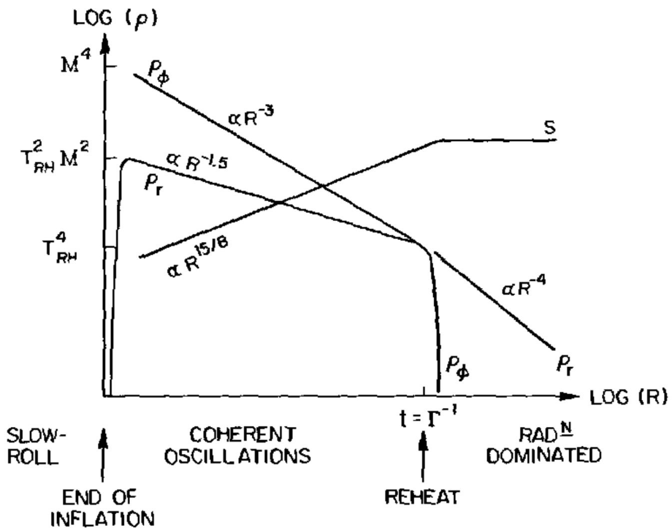

Figure 1.17: Evolution of ρ_ϕ, ρ_r and S. Here $a \equiv R$ (from [37]).

At the end of reheating, and then at the beginning of the radiation-dominated phase, the temperature of the Universe is given by $T_{rh} = T\left(t_{dec} = \Gamma_\phi^{-1}\right)$. In order to find it, we make a match with the radiation dominated phase, in which $a(t) \propto a^{1/2}$:

$$H^2 = \frac{1}{4t^2}\Big|_{t_{dec}} = \frac{8}{3}\frac{\pi}{M_{Pl}^2}\rho_r\Big|_{t_{dec}} = \frac{8}{3}\frac{\pi}{M_{Pl}^2} g_* T^4\Big|_{t_{dec}}. \tag{1.112}$$

Since $t_{dec} = \Gamma_\phi^{-1}$, we find

$$T_{rh} \simeq 0.55\, g_*^{-1/4} \left(\Gamma_\phi M_{Pl}\right)^{1/2}. \tag{1.113}$$

Notice that T_{rh} is determined by the decay rate Γ_ϕ of the inflaton field into the other particles to which it couples, hence this mechanism depends on the specific model of particle physics we adopt. This means also that the reheating temperature is not simply equal to the vacuum energy the inflaton field has during inflation, but it depends on the efficiency with which it decays into radiation; during the oscillations, indeed, much of the initial vacuum energy gets red-shifted. Only if the inflaton decays as soon as inflation ends ($\Gamma_\phi \gg H_{inf}$), then the reheating temperature is given by the vacuum energy of the inflaton field ($T_{rh} \sim M$).

1.9 Cosmic no-hair "theorem"

We have seen that inflation is an attractive theory because it allows us to explain the current state of the Universe without assuming any special, fine-tuned initial condition. But is it really true that, starting from the most general initial conditions (e.g. with curvature and anisotropies), the Universe tends towards an attractor solution and thus towards a FLRW metric? Indeed, it is not obvious that cosmological models with non-FLRW initial conditions will even enter an inflationary epoch; if this is the case, it is then also not obvious that the initial inhomogeneities and anisotropies would be smoothed out. A complete proof of this fact, which is known as *cosmic no-hair theorem*, does not exist, so we will consider a specifc example to show that this idea is well motivated [37].

Let us consider a homogeneous but not isotropic Universe. These are known as *Bianchi models*, where the metric has the form

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3(t)^2dz^2. \tag{1.114}$$

The scale factor is different along the three axes, which thus expand at different rates. One can show that [37]

$$\bar{H}^2 \equiv \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \frac{8\pi G}{3}\rho + F(a_1, a_2, a_3), \tag{1.115}$$

where

$$\bar{a} \equiv (a_1 a_2 a_3)^{1/3} \tag{1.116}$$

is the mean scale factor, $\bar{H}$ is the mean Hubble parameter and ρ is the total energy density, which includes the scalar field ϕ and any other component present, like radiation and matter. The function F accounts for the anisotropic expansion on the mean expansion rate. Its functional form depends on the specific model and can be very complicated. However it has the crucial property that for all the Bianchi models it decreases at least as $\bar{a}^{-2}$. Once inflation starts, this term gets exponentially suppressed, at most as $F \propto e^{-2Ht}$ for $H \simeq$ const. Thus the anisotropies are smoothed out, like the curvature term, $\rho_k \propto e^{-2Ht}$, and the other contributions to the energy density, $\rho_m \propto e^{-3Ht}$ and $\rho_r \propto e^{-4Ht}$. This means that, if inflation starts, the only component that gets not diluted after a while is the scalar field, which becomes the dominant contribution to the energy density of the Universe. Therefore, starting with anisotropies in the initial conditions, the Universe gets attracted towards an inflationary phase.

To demonstrate that this is still true in more general cases, one can start by exploiting the so-called *Wald theorem* [38]: this tells us that, in the case of a Universe dominated by an exact cosmological constant $\rho_\phi = \Lambda$, the Universe rapidly reaches a homogeneous and isotropic state with a de Sitter expansion. This is valid with the only exception of the Bianchi IX model, where the Universe recollapses so rapidly that the vacuum energy has no enough time to become the dominant component. One can then try to extend this to the case of a dynamcal (slowly-rolling) scalar field. The question now is: does the Universe evolve into a (quasi) de Sitter state before the scalar field reaches the minimum of its potential? Indeed, strong anisotropies could in principle push the field rapidly towards the potential well. This is not what happens, rather the field keeps on slowly-rolling. This occurs because the anisotropies lead to an increased expansion rate and thus have the effect of increasing the friction term ($3H\dot{\phi}$) in the equation of motion for the scalar field. This ensures that the scalar field moves little during the time required for the vacuum energy to become dominant, and consequently that the vacuum energy provided by the scalar field during the slow-roll does not "disappear" before the Universe begins to inflate.

Analogous results are still valid if one starts even from more general initial conditons, like negatively curved models with inhomogeneities [39], or models with inhomogeneous spacetimes where the metric allows four compltely arbitrary functions of space and time to be specified [40].

To summarize, we have seen that many spacetimes with different initial conditions (of anisotropies, inhomogeneities and curvature) will eventually inflate. This is an important feature, since we have introduced inflation to solve problems of fine-tuned initial conditions of the standard FLRW model, hence it is a good feature that inflation does not require particular conditions to begin.

1.10 Towards the primordial perturbations from inflation

We have seen that inflation was initially introduced to solve the shortcomings of the standard model of cosmology. Soon enough it became clear that an inflationary phase can do more than that. Indeed the most important feature of inflation, which makes it a *predictive* theory, is that it naturally provides a mechanism to generate the first primordial perturbations, which are the seeds for the following formation of the large scale structures and which can be observed through the temperature anisotropies in the CMB[15] (see Fig. 1.18). These are an immediate consequence of introducing quantum mechanics into the game. Indeed, the quantum fluctuations of the inflaton field are stretched outside the horizon by the exponential expansion (remember that during inflation the comoving Hubble radius decreases) and hence the amplitude of the fluctuations is "frozen" at a non-zero value, since no causal process can act anymore. These fluctuations of the scalar field induce fluctuations in the energy density and hence in the curvature, which, after the fluctuations reenter the horizon during the radiation or matter era (when the Hubble radius increases faster than the scale factor), give rise to matter and temperature perturbations through the Poisson equation.

Moreover, inflation also predicts the production (in a similar manner as for scalar perturbations) of a stochastic background of gravitational waves. This is the "smoking gun" of inflation, in a sense which will be clarified in the next chapters.

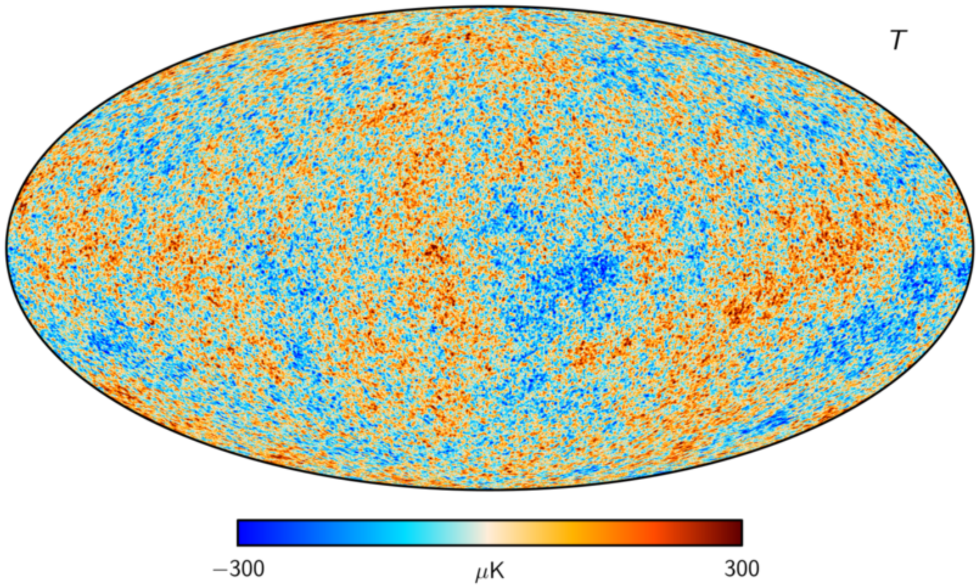

Figure 1.18: CMB map from Planck (from [41]).

[15]Note that when it was realized that inflation could source primordial perturbations, the CMB anisotropies were not yet observed.

Chapter 2

Cosmological perturbation theory

Until now we have described the properties and the dynamics of a spatially homogeneous and isotropic Universe. Of course, this can not be the entire story. If we look around us we can see a Universe which is neither homogeneous nor isotropic. A great complexity of different structures, from planets and stars to galaxies and galaxy clusters, appears to us. If we want to understand how these structures form and evolve we have to introduce some inhomogeneities on top of our homogeneous and isotropic description of the Universe.

As we have seen, the theoretical framework of cosmology is strongly based on General Relativity (GR). It is well known that the Einstein's field equations are really difficult to solve because of their non-linearity, and the solutions are often based on too idealized assumptions to represent in a proper way the physical phenomena. A fundamental method in GR to go beyond the zero-order background solution is provided by the perturbative approach. The idea underlying perturbation theory in GR is the same as in any perturbative approach: we try to find approximate solutions of the field equations, treating them as "small" deviations from a known exact background solution. In our case the background solution is given by the FLRW metric. However, there is a fundamental difference between the perturbations in GR and the perturbations in other field theories, where the underlying spacetime is fixed. In the latter case one can define the perturbation of a given tensor field T as

$$\Delta T(x) = T(x) - T_0(x), \tag{2.1}$$

where T_0 is the unperturbed field and x is any point in spacetime. In GR spacetime must also be perturbed if matter is perturbed. In the context of inflation, for example, any perturbation in the inflaton field translates into a perturbation in the stress-energy tensor which then implies, through Einstein's equations, a perturbation in the metric tensor. Moreover, the perturbations in the metric enter the perturbed Klein-Gordon equation, thus affecting the evolution of the inflaton fluctuations. Hence, the perturbations in the inflaton field and the perturbations in the metric are tightly coupled and must be studied together. As a consequence, the unperturbed field T_0 and the perturbed field T are defined in different spacetimes, the *background spacetime* $\mathcal{M}_0$ and the *perturbed spacetime* $\mathcal{M}$. In order to compare the two fields we need a prescription to identify points of $\mathcal{M}_0$ with points of $\mathcal{M}$: this is a *gauge choice*, which is mathematically provided by any diffeomorphism between the two spacetimes. A change of this diffeomorphism is what is called a *gauge transformation*. Since GR is invariant under diffeomorphisms, that is, two solutions of the Einstein's field equations are physically equivalent if they are diffeomorphic to each other, the definition of perturbations is in general gauge dependent (unless the tensor T is gauge-invariant, as we will see). This is an extremely important issue, since it is possible that some of the perturbations in a given quantity are not real physical perturbations, but rather are gauge artefacts which can be removed by applying a gauge transformation. Likewise, it is possible that a real perturbation in some physical quantity does not appear in a certain gauge.

There are two approaches to face this problem: the first possibility is to fix a gauge from the beginning and then perform all the calculations in this gauge; alternatively, one can work with gauge invariant quantities. In the rest of this thesis we will follow the first approach.

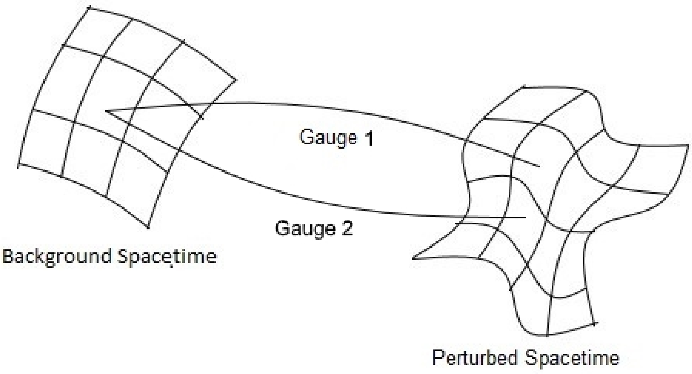

Figure 2.1: Two different "gauge maps" between the background and the perturbed spacetime.

2.1 Some basics of differential geometry

To give a more formal and geometrical description of the gauge transformations, we start by recalling some basic definitions of differential geometry. In our discussion in the next sections we mainly follow [42]. Let us consider a diffeomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ between two manifolds $\mathcal{M}$ and $\mathcal{N}$. For each $p \in \mathcal{M}$, φ naturally induces a map between the tangent space to $\mathcal{M}$ at the point p and the tangent space to $\mathcal{N}$ at the point $\varphi(p)$. This is called the *push-forward*:

$$\varphi_*|_p : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}. \tag{2.2}$$

Analogously, φ can also be used to define a map between the cotangent spaces, the so-called *pull-back*:

$$\varphi^*|_{\varphi(p)} : T^*_{\varphi(p)}\mathcal{N} \to T^*_p\mathcal{M}. \tag{2.3}$$

We can extend the previous definitions to tensors of higher rank. For example, the action of the pull-back and of the push-forward respectively on a $(0, l)$ tensor W on $\mathcal{N}$, and on a $(k, 0)$ tensor S on $\mathcal{M}$, is given by (see [43])

$$(\varphi^* W)(V^{(1)}, ..., V^{(l)}) = W(\varphi_* V^{(1)}, ..., \varphi_* V^{(l)}), \tag{2.4}$$

$$(\varphi_* S)(\omega^{(1)}, ..., \omega^{(k)}) = S(\varphi^* \omega^{(1)}, ..., \varphi^* \omega^{(k)}), \tag{2.5}$$

where the $V^{(i)}$'s are vectors on $\mathcal{M}$ and the $\omega^{(i)}$'s are one-forms on $\mathcal{N}$. Since φ is a diffeomorphism, we can also use φ^{-1} to define the push-forward of $T^*_p\mathcal{M}$ to $T^*_{\varphi(p)}\mathcal{N}$, and the pull-back of $T_{\varphi(p)}\mathcal{N}$ to $T_p\mathcal{M}$. Thus, we can pull-back and push-forward tensors of arbitrary type. For example, the push-forward of a (k, l) tensor on $\mathcal{M}$ is defined by

$$(\varphi_* T)(\omega^{(1)}, ..., \omega^{(k)}, V^{(1)}, ..., V^{(l)}) = T(\varphi^* \omega^{(1)}, ..., \varphi^* \omega^{(k)}, [\varphi^{-1}]_* V^{(1)}, ..., [\varphi^{-1}]_* V^{(l)}), \tag{2.6}$$

where the $V^{(i)}$'s and the $\omega^{(i)}$'s are respectively vectors and one-forms on $\mathcal{N}$. We can then extend the above definitions to tensor fields by requiring that, $\forall p \in \mathcal{M}$,

$$(\varphi_* T)(\varphi(p)) := \varphi_*|_p(T(p)), \tag{2.7}$$

$$(\varphi^* T)(p) := \varphi^*|_{\varphi(p)}(T(\varphi(p))). \tag{2.8}$$

Let us consider a vector field ξ on $\mathcal{M}$, which generates a *flow* $\phi : \mathbb{R}\times\mathcal{M} \to \mathcal{M}$, with $\phi(0,p) = p$, $\forall p \in \mathcal{M}$. For any $\lambda \in \mathbb{R}$, we define $\phi_\lambda(p) := \phi(\lambda, p)$, $\forall p \in \mathcal{M}$. Intuitively, we start from the point p and move along the integral curve of ξ for a value of the parameter which parametrizes the integral curves equal to λ. The pull-back of the tensor field T by ϕ_λ, which is a function of λ, can be expanded in the following way around $\lambda = 0$ (see [42] for the proof):

$$\phi_\lambda^* T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k}\Big|_0 \phi_\lambda^* T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_\xi^k T, \tag{2.9}$$

where

$$\pounds_\xi T := \lim_{\lambda\to 0} \frac{1}{\lambda}(\phi_\lambda^* T - T) = \frac{d}{d\lambda}\Big|_0 \phi_\lambda^* T \tag{2.10}$$

is the *Lie derivative* of T with respect to ξ, which characterizes the variation of T along the integral curves of ξ. Notice that ϕ_λ forms a one-parameter group, i.e. it satisfies $\phi_{-\lambda} = {\phi_\lambda}^{-1}$ and $\phi_{\lambda+\sigma} = \phi_\lambda \circ \phi_\sigma$. The first condition just tells us that moving for a value equal to $-\lambda$ along an integral curve of ξ is equal to "moving back" by λ; the second condition tells us that moving a first time by λ and then moving a second time by σ is equal to moving once by $\lambda + \sigma$.

The Lie derivative of a tensor field T of rank (k, l) can be written as

$$\begin{aligned} \pounds_\xi T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = {} & \xi^\sigma \nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \\ & - (\nabla_\alpha \xi^{\mu_1}) T^{\alpha\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - \ldots - (\nabla_\alpha \xi^{\mu_k}) T^{\mu_1\cdots\alpha}{}_{\nu_1\cdots\nu_l} + \\ & + (\nabla_{\nu_1} \xi^\alpha) T^{\mu_1\cdots\mu_k}{}_{\alpha\cdots\nu_l} + \ldots + (\nabla_{\nu_l} \xi^\alpha) T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\alpha}. \end{aligned} \tag{2.11}$$

2.2 Gauge transformations

A basic assumption in perturbation theory is the existence of a parametric family of solutions of the field equations, to which the background spacetime belongs. Thus, let us consider a one-parameter family of spacetimes $\{(\mathcal{M}, g_\lambda, \tau_\lambda)\}$, where $\mathcal{M}$ is a m-dimensional manifold, while the metric g_λ and the matter fields τ_λ satisfy the field equations

$$\varepsilon\,[g_\lambda, \tau_\lambda] \equiv R^{(\lambda)}_{\mu\nu} - \frac{1}{2} R^{(\lambda)} g^{(\lambda)}_{\mu\nu} - 8\pi G T^{(\lambda)}_{\mu\nu} = 0, \tag{2.12}$$

with $\lambda \in \mathbb{R}$. The case with $\lambda = 0$ corresponds to the background solution, and we assume that g_λ and τ_λ depend smoothly on the parameter λ, so that the latter can be regarded as a measure of how much a particular solution $\mathcal{M}_\lambda \equiv (\mathcal{M}, g_\lambda, \tau_\lambda)$ differs from the background $\mathcal{M}_0 \equiv (\mathcal{M}, g_0, \tau_0)$.

In order to describe more naturally this situation, let us introduce a $(m + 1)$-dimensional manifold $\mathcal{N}$, foliated by submanifolds diffeomorphic to $\mathcal{M}$, such that $\mathcal{N} = \mathcal{M} \times \mathbb{R}$. Both the background spacetme $\mathcal{M}_0$ and the perturbed spacetimes $\mathcal{M}_\lambda$ ($\forall\lambda \in \mathbb{R}$) are 4-dimensional submanifolds of $\mathcal{N}$. By how it is defined, $\mathcal{N}$ has the differentiable structure which is the product of those of $\mathcal{M}$ and $\mathbb{R}$. Hence, we can introduce charts such that x^μ, with $\mu = 0, 1, ..., m - 1$, are coordinates on each leave $\mathcal{M}_\lambda$, and $x^m \equiv \lambda$. We then define $x^A \equiv (x^\mu, \lambda)$. If a tensor field T_λ is defined on each $\mathcal{M}_\lambda$, we can also naturally define a tensor field T on $\mathcal{N}$ through $T(p, \lambda) := T_\lambda(p)$, $\forall p \in \mathcal{M}$. This is in particular also true for g_λ and τ_λ, which are then defined on $\mathcal{N}$. Also the field equations (2.12), which are satisfied on each $\mathcal{M}_\lambda$, are then extended to $\mathcal{N}$.

We now come back to the issue of defining the perturbation of a generic tensor field T. As we have said in the introduction, we need a prescription for identifying points of $\mathcal{M}_\lambda$ with points of $\mathcal{M}_0$. To do so, let us introduce a diffeomorphism $\varphi_\lambda : \mathcal{N} \to \mathcal{N}$, such that

$$\varphi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_\lambda. \tag{2.13}$$

We can also see φ_λ as the member of a flow φ on $\mathcal{N}$, generated by some vector field X. The points which lie on the same integral curve γ of X can be regarded as the same point.

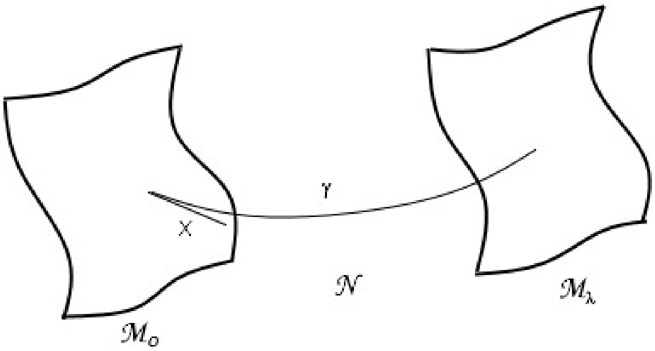

Figure 2.2: A vector field X generates a gauge map between $\mathcal{M}_0$ and $\mathcal{M}_\lambda$.

From now on, we will refer also to such a vector field with the term "gauge".

Now we have all the ingredients to define the perturbation of the tensor T as

$$\Delta T_\lambda \equiv \varphi_\lambda^* T|_{\mathcal{M}_0} - T_0. \tag{2.14}$$

Notice that ΔT_λ is defined in the background spacectime $\mathcal{M}_0$; this is what one refers to, when saying that "perturbations are fields which live in the background". We can make a Taylor expansion (around $\lambda = 0$) of the first term on the rhs of (2.14), as in Eq. (2.9):

$$\varphi_\lambda^* T|_{\mathcal{M}_0} = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k(\varphi_\lambda^* T)}{d\lambda^k}\Big|_{\lambda=0,\mathcal{M}=0}. \tag{2.15}$$

Plugging this in Eq. (2.14), we find

$$\Delta T_\lambda = \sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k(\varphi_\lambda^* T)}{d\lambda^k}\Big|_{\lambda=0,\mathcal{M}=0}. \tag{2.16}$$

Since our main goal is to see how a generic tensor T transforms under a gauge transformation, let us introduce another vector field Y, which generates a different gauge ψ. We can now use both the fields X and Y to pull-back T and to define, on $\mathcal{M}_0$,

$$T_\lambda^X \equiv \varphi_\lambda^* T|_0, \quad T_\lambda^Y \equiv \psi_\lambda^* T|_0, \tag{2.17}$$

where with the suffix 0 we refer to the restriction to $\mathcal{M}_0$. These are the representations of the perturbed tensor on $\mathcal{M}_0$ according to the two different gauges, φ and ψ respectively. Using (2.15), we can rewrite them as

$$T_\lambda^X = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k(\varphi_\lambda^* T)}{d\lambda^k}\Big|_0 = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_X^k(T)|_0, \tag{2.18}$$

$$T_\lambda^Y = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k(\psi_\lambda^* T)}{d\lambda^k}\Big|_0 = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_Y^k(T)|_0. \tag{2.19}$$

If $T_\lambda^X = T_\lambda^Y$ for any pair of gauges X and Y, then T is said to be *totally gauge-invariant*. From (2.18) and (2.19), this implies that $\delta T^X = \delta T^Y$, so it is a very strong condition. Considering perturbations to a fixed order n, one can weaken the previous definition, saying that T is *gauge-invariant to order n* if and only if (iff) $\delta^k T^X = \delta^k T^Y$, for any two gauges X and Y, and $\forall k \leq n$. Then, one can show (see [42]) that T is gauge-invariant to order $n \geq 1$ iff $\pounds_\xi \delta^k T = 0$, for any vector field ξ on $\mathcal{M}$, and $\forall k < n$.

If T is not gauge-invariant, we want to know how its representation on the background spacetime $\mathcal{M}_0$ behaves under a gauge transformation. To this end notice that, correspondingly to the two gauges X and Y, there are two *different* points in $\mathcal{M}_0$, P and Q, which are mapped to the *same* point O in $\mathcal{M}_\lambda$ (see Fig. 2.3). Thus, the gauge transformation can also be seen as one-to-one correspondence between different points in the background manifold. This is the so-called *active view*. We then define, $\forall \lambda \in \mathbb{R}$, the diffeomorphism $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ as

$$\Phi_\lambda := \varphi_{-\lambda} \circ \psi_\lambda, \tag{2.20}$$

whose action is represented in Fig. 2.3.

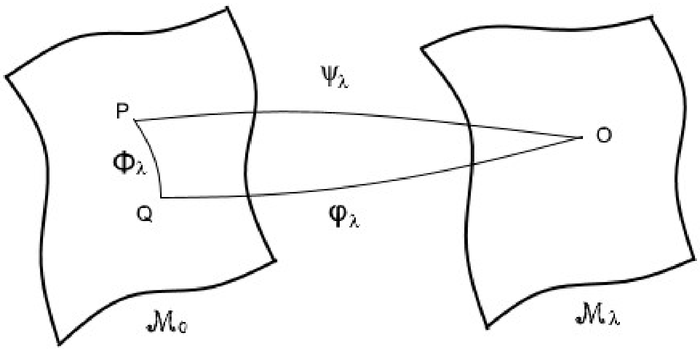

Figure 2.3: The action of the gauge transformations Φ_λ.

Notice that Φ_λ is not a one-parameter group of diffeomorphisms on $\mathcal{M}_0$, even though φ_λ and ψ_λ are so. This is due to the fact that $\Phi_\lambda \neq \Phi_\lambda{}^{-1}$ and $\Phi_{\lambda+\sigma} \neq \Phi_\lambda \circ \Phi_\sigma$, basically because X and Y in general don't commute.

The relation between T_λ^X and T_λ^Y is now easy to find:

$$T_\lambda^Y = \psi_\lambda^* T|_0 = (\psi_\lambda^* \phi_{-\lambda}^* \phi_\lambda^*) T|_0 = \Phi_\lambda^* (\phi_\lambda^*) T|_0 = \Phi_\lambda^* T_\lambda^X. \tag{2.21}$$

Therefore, the representation of the perturbed tensor T on $\mathcal{M}_0$ through the gauge ψ is given by the representation of T through the gauge φ, pulled-back (to the point P) by Φ. Expanding the above equation (to third order), one finds [42]

$$T_\lambda^Y = T_\lambda^X + \lambda \pounds_{\xi_1} T_\lambda^X + \frac{\lambda^2}{2}(\lambda_{\xi_1}^2 + \pounds_{\xi_2}) T_\lambda^X + \frac{\lambda^3}{3!}(\pounds_{\xi(1)}^3 + 3\pounds_{\xi(1)}\pounds_{\xi(2)} + \pounds_{\xi(3)}) T_\lambda^X + \mathcal{O}(\lambda^4), \tag{2.22}$$

where ξ_1 and ξ_2 are the first two generators of the gauge transformation. Finally, given the tensor T, we can write the relation between the first, second, and third order perturbations in the two different gauges:

$$\delta T^Y - \delta T^X = \pounds_{\xi_1} T_0, \tag{2.23}$$

$$\delta^2 T^Y - \delta^2 T^X = (\pounds_{\xi_2} + \pounds_{\xi_1}^2) + 2\pounds_{\xi_1} \delta T^X, \tag{2.24}$$

$$\delta^3 T^Y - \delta^3 T^X = (\pounds_{\xi(1)}^3 + 3\pounds_{\xi(1)}\pounds_{\xi(2)} + \pounds_{\xi(3)}) T_0 + 3(\pounds_{\xi(2)} + \pounds_{\xi(1)}^2)\delta T + 3\pounds_{\xi(1)} \delta^2 T^X. \tag{2.25}$$

Before proceeding to study the perturbations of the metric and of the stress-energy tensor, we want to stress an important point: gauge transformations are not the same as coordinate transformations! We have seen that a gauge is a map between the background and the perturbed spacetime, $\mathcal{M}_0$ and $\mathcal{M}$ respectively. This assigns the coordinates of a point P in $\mathcal{M}_0$ to a point Q in $\mathcal{M}$; when we make a gauge transformation we assign the coordinates of the point P to a different point Q' in $\mathcal{M}$ (namely we change the coordinates on $\mathcal{M}$), but the coordinates of the background spacetime are kept fixed. A coordinate transformation affects instead both the background and the perturbed spacetime. Thus, fixing a gauge induces a coordinates fixing (only) in the perturbed spacetime. A coordinate choice in GR defines:

- a *threading* of the spacetime into lines, corresponding to fixed spatial coordinates;
- a *slicing* of the spacetime into spatial hypersurfaces at $\tau = \text{const}$.

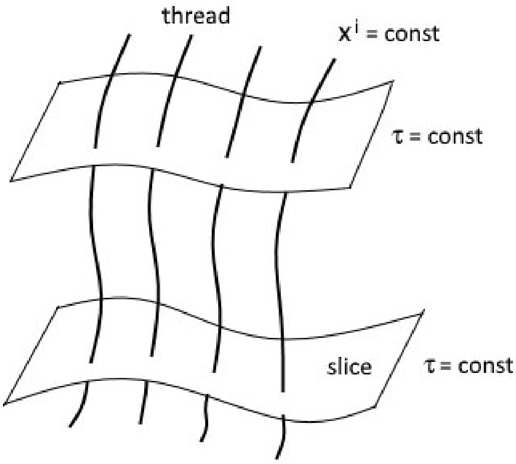

Figure 2.4: Slicing and threading of the spacetime.

In un unperturbed spacetime there is a unique "preferred" threading and slicing. The preferred threading consists in the worldlines of comoving observers, which move with the cosmic fluid. These are geodesics and correspond to a uniform expansion with no shear and vorticity. The preferred slicing is orthogonal to this threading. Within it, all the quantities like the energy density and the spatial curvature are homogeneous. In particular the latter is assumed to vanish, in accordance with the observations. We can thus say that the preferred slicing is comoving (i.e., orthogonal to the worldlines of comoving observers), flat and with uniform energy density.

In a perturbed spacetime there is instead no preferred slicing and threading. Fixing a gauge means fixing the slicing and threading of the perturbed spacetime.

2.3 Perturbations of the metric tensor

The components of a perturbed spatially flat FLRW metric can be written as

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{r=1}^{\infty}\frac{1}{r!}\Psi^{(r)}\right), \tag{2.26}$$

$$g_{0i} = a^2(\tau)\sum_{r=1}^{\infty}\frac{1}{r!}\omega_i^{(r)}, \tag{2.27}$$

$$g_{ij} = a^2(\tau)\left[\left(1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\Phi^{(r)}\right)\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}h_{ij}^{(r)}\right], \tag{2.28}$$

where $\Psi^{(r)}$, $\omega_i^{(r)}$, $\Phi^{(r)}$ and $h_{ij}^{(r)}$ are the rth-order perturbations of the metric, with $h_{ij}^{(r)}$ being traceless, i.e. $h_i^{(r)i} = 0$.

We can further decompose the perturbations in the so-called scalar (or longitudinal), vector (solenoidal), and tensor parts, which are decoupled at linear order. Thus, the shift vector $\omega_i^{(r)}$ can be decomposed as

$$\omega_i^{(r)} = \partial_i \omega^{(r)\parallel} + \omega_i^{\perp}, \tag{2.29}$$

with $\partial^i \omega_i^{\perp} = 0$. Analogously, the traceless part of the spatial metric can be decomposed as

$$h_{ij}^{(r)} = D_{ij} h^{(r)\parallel} + \partial_i h_j^{(r)\perp} + \partial_j h_i^{(r)\perp} + h_{ij}^{(r)T}, \tag{2.30}$$

where

$$D_{ij} := \partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2 \tag{2.31}$$

is a trace-free operator, $h^{(r)}$ and $h_i^{(r)}$ are respectively the scalar and vector part, and $h_{ij}^{(r)T}$ is the tensor part, which is symmetric, transverse and traceless:

$$\partial^i h_{ij}^{(r)T} = 0, \quad h_i^{(r)i\,T} = 0. \tag{2.32}$$

As we have seen, at linear order a gauge transformation is determined by a vector field ξ_1. Its time and space components can also be decomposed in scalar and vector parts as

$$\xi_1^0 = \alpha, \quad \xi_1^i = \partial^i \beta_1 + d_1^i, \tag{2.33}$$

with $\partial_i d^i = 0$. Therefore fixing a gauge is equivalent to fixing two scalar and one vector perturbations. Moreover, since ξ does not contain any tensor component, it is clear that at linear order in perturbation theory tensor perturbations are gauge-invariant.

The transverse, traceless tensor perturbations $h_{ij}^{(r)T}$ are the gravitational waves, which carry two degrees of freedom. Indeed, $h_{ij}^{(r)T}$ has in principle six components, being symmetric; however, the conditions (2.32) reduce the d.o.f. by four, leaving us with only two.

From Eq. (2.23) it follows that, at linear order, the perturbations of the metric tensor transform under a gauge transformation as

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \pounds_{\xi_1} g_{\mu\nu}^{(0)}, \tag{2.34}$$

where $g_{\mu\nu}^{(0)}$ is the background FLRW metric. Using Eq. (2.11), it is easy to find that the quantities appearing in Eqs. (2.26)-(2.28) transform as

$$\tilde{\Psi} = \Psi + \alpha' + \frac{a'}{a}\alpha, \tag{2.35}$$

$$\tilde{\omega}_i = \omega_i - \partial_i \alpha + \partial_i \beta' + d_i', \tag{2.36}$$

$$\tilde{\Phi} = \Phi - \frac{1}{3}\nabla^2\beta - \frac{a'}{a}\alpha, \tag{2.37}$$

$$\tilde{h}_{ij} = h_{ij} + 2D_{ij}\beta + \partial_j d_i + \partial_i d_j. \tag{2.38}$$

where the prime denotes a derivative with respect to the conformal time and, for simplicity of notation, we have neglected the superscripts denoting the first order perturbation of the corresponding quantity.

2.4 Perturbations of the stress-energy tensor

The stress-energy tensor for a fluid can be written as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \Pi_{\mu\nu}, \tag{2.39}$$

where ρ is the energy density, p the (isotropic) pressure, u^μ the 4-velocity and $\Pi_{\mu\nu}$ is the anisotropic stress tensor. The latter vanishes in the case of a perfect fluid or a minimally coupled scalar field, and is subjected to the constraints $u^v \Pi_{\mu\nu} = 0 = \Pi_\mu{}^\mu$.

Let us now perturb the energy density, the isotropic pressure and the velocity:

- As regarding the energy density, we have

$$\rho = \rho_0 + \sum_{r=1}^{\infty} \frac{1}{r!}\delta\rho^{(r)}, \tag{2.40}$$

where $\rho_0 = \rho_0(t)$ is the unpertrubed energy density in FLRW.

- Viewing the pressure as a function of the energy density and the entropy through the equation of state, $p = p(\rho, S)$, we can perturb it as

$$\delta p = \left.\frac{\partial p}{\partial \rho}\right|_S \delta\rho + \left.\frac{\partial p}{\partial S}\right|_\rho \delta S = c_s^2 \delta\rho + \left.\frac{\partial p}{\partial S}\right|_\rho \delta S, \tag{2.41}$$

where in the second equality we have introduced the sound speed

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_S. \tag{2.42}$$

The first term in (2.41) represents an adiabatic perturbation, while the second term is an entropy perturbation.

- The velocity can be perturbed as

$$u^\mu = \frac{1}{a}\left(\delta_0^\mu + \sum_{r=1}^{\infty} \frac{1}{r!} v^\mu_{(r)}\right), \tag{2.43}$$

where the first term

$$\bar{u}^\mu \equiv \frac{1}{a}\delta_0^\mu \tag{2.44}$$

is the background velocity in FLRW, which represents a motion comoving with the cosmic expansion. The second term in (2.43) is instead a peculiar velocity. Since the velocity obeys the constraint $u_\mu u^\mu = -1$, we can relate at each perturbative order $v^0_{(r)}$ to the scalar perturbation $\Psi^{(r)}$ in the 0-0 component of the metric tensor (2.26). At first order, we have

$$v^0_{(1)} = -\Psi^{(1)}. \tag{2.45}$$

The spatial part of the velocity perturbation, $v^i_{(r)}$, can be split into a scalar and a vector perturbation:

$$v^i_{(r)} = \partial^i v^{\|}_{(r)} + v^i_{(r)\perp}, \tag{2.46}$$

with $\partial_i v^i_{(r)\perp} = 0$.

The components of the stress-energy tensor (2.39) can now be perturbed at first order as

$$T^0{}_0 = -(\rho_0 + \delta\rho), \tag{2.47}$$
$$T^i{}_i = 3(p_0 + \delta p) = 3p_0(1 + \Pi_L), \tag{2.48}$$
$$T^i{}_0 = -(\rho_0 + p_0)v_i, \tag{2.49}$$
$$T_0{}^i = (\rho_0 + p_0)(v_i - \omega_i), \tag{2.50}$$
$$T^i{}_j = p\,[(1 + \Pi_L)\delta^i_j + \Pi^{T\,i}{}_j], \tag{2.51}$$

where Π_L is the amplitude of an isotropic pressure perturbation and Π^T_{ij} is the tensor part of the anisotropic stress tensor. We neglect the T superscript on the latter for simplicity of notation.

At linear level, $\delta\rho$. δp, v^0 and v^i transform under a gauge transformation following (2.23). We obtain

$$\tilde{\delta\rho} = \delta\rho + \alpha\rho_0', \tag{2.52}$$

$$\tilde{\delta p} = \delta p - \alpha p_0', \tag{2.53}$$

$$\tilde{v}^0 = v^0 - \frac{a'}{a}\alpha - \alpha', \tag{2.54}$$

$$\tilde{v}^i = v^i - \partial^i\beta' - d^{i\prime}. \tag{2.55}$$

2.5 Some of the most commonly used gauges in cosmology

We now give a brief overview of the main gauges used in cosmology, working at linear order in perturbation theory. As we have seen, fixing a gauge is equivalent to fix two scalar and one vector perturbations.

2.5.1 Poisson gauge

The Poisson gauge is defined by requiring

$$\omega^{\|} = 0, \quad h^{\|} = 0, \quad h_i^{\perp} = 0. \tag{2.56}$$

This is an extension of the longitudinal gauge (or conformal Newtonian gauge), which is defined by $\omega^{\|} = 0 = h^{\|}$ and is the gauge that in GR has the more direct link with analogous quantities appearing in Newtonian physics. The Poisson gauge is also known as *orthogonal zero-shear gauge* for the reason we are now going to eplain. Let us introduce a unit time-like vector field orthogonal to $\tau = \text{const}$ hypersurfaces. Within this gauge, this can be written as

$$n^\mu = \frac{1}{a}(1 - \Psi, -\omega^i). \tag{2.57}$$

It is possible to decompose the covariant derivative of n^μ as (see e.g. [43])

$$\nabla_\nu n_\mu = \frac{1}{3}\theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu N_\nu, \tag{2.58}$$

where:

- $P_{\mu\nu}$ is the projector on hypersurfaces perpendicular to n^μ, defined as

$$P_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu. \tag{2.59}$$

- θ defines the local expansion rate of the fluid and is defined as

$$\theta \equiv \nabla^\mu n_\mu. \tag{2.60}$$

In FLRW the expansion θ is given by the Hubble parameter, $\theta = H(t)$.

- $\sigma_{\mu\nu}$ is the shear tensor:

$$\sigma_{\mu\nu} \equiv \frac{1}{2} P_\mu{}^\alpha P_\nu{}^\beta (\nabla_\beta n_\alpha + \nabla_\alpha n_\beta) - \frac{1}{3}\theta P_{\mu\nu}. \tag{2.61}$$

This is symmetric and traceless, and represents the distorsion in the shape of a collection of test particles from an initial sphere into an ellipsoid. The symmetry property means that the elongation along the x-direction is the same as the elongation along the -x-direction. The tracelessness property means that the volume is not changed.

- $\omega_{\mu\nu}$ is the vorticity tensor:

$$\omega_{\mu\nu} \equiv \frac{1}{2} P_\mu{}^\alpha P_\nu{}^\beta (\nabla_\beta n_\alpha - \nabla_\alpha n_\beta). \tag{2.62}$$

This is antisymmetric and represent rotations; for example, the xy component describes a rotation around the z-axis, while the yx component describes a rotation still around the z-axis, but in the opposite sense (as a reflection of the antisymmetry property).

- a^μ is the “acceleration”, and it is given by

$$a^\mu \equiv n^\nu \nabla_\nu n^\mu. \tag{2.63}$$

This is different from zero if the integral curves of n^μ are not geodesics.

From the expression of n^μ given in (2.57) it follows that $\omega_{\mu\nu} = 0$. As regarding the scalar perturbations in the shear tensor, we have

$$\sigma_{0\mu} = 0, \quad \sigma_{ij} = \partial_i\partial_j\sigma - \frac{1}{3}\delta_{ij}\nabla^2\sigma, \quad \sigma \equiv -\omega^{\|} + \frac{1}{2}h^{\|\prime}. \tag{2.64}$$

From the conditions (2.56) it follows that in the Poisson gauge $\sigma = 0$. Thus the Poisson gauge is characterized by the fact that the shear of n^μ vanishes for scaar perturbations. This is why it is also called orthogonal zero-shear gauge.

2.5.2 Synchronous gauge

The synchronous gauge is defined by

$$\Psi = 0. \tag{2.65}$$

If we require also the condition

$$\omega^{\|} = 0 = \omega_i^{\perp}, \tag{2.66}$$

we have the so-called synchronous and time-orthogonal gauge. This second statement means that in this gauge the vector orthogonal to space-like hypersurfaces at $\tau = \text{const}$ has only a time component:

$$n^\mu = \frac{1}{a}(1, 0). \tag{2.67}$$

Let us now consider the proper time of obervers at fixed spatial coordinates. This is given by[1]

$$d\tau = \left[a^2(1+2\Psi)d\eta^2\right]^{1/2} \simeq a(1+\Psi)d\eta\,, \tag{2.68}$$

[1] In this case, just not to make confusion, we label with τ the proper time and with η the conformal time. We will immediately come back to our usual conventions, where τ denotes the conformal time and η one of the slow-roll parameters.

where in the second equality we have expanded for small values of Ψ. From the defining condition of the comoving gauge, $\Psi = 0$, it follows that all the observers at fixed spatial coordinates have the same proper time

$$d\tau = a(\eta)d\eta = dt, \tag{2.69}$$

which corresponds to the comsic time in a FLRW Universe. The synchronous gauge is thus the gauge in which all the commoving observers have a proper time which coincides with the cosmic time, exacctly as it happens in the unperturbed FLRW case.

Notice that the synchronous gauge is not uniquely fixed, there is still the freedom of fixing one further scalar and vector perturbations (as it is done in (2.66)).

2.5.3 Comoving gauge

This is defined by requiring that the velocity of the fluid vanishes:

$$v^i = 0 \quad \Rightarrow \quad v^{\|} = 0 = v^i_{\perp}. \tag{2.70}$$

This means that we are comoving with the cosmic fluid. If we also require that space-like hypersurfaces are orthogonal to the 4-velocity of the fluid, this yields

$$v^{\|} + \omega^{\|} = 0. \tag{2.71}$$

Indeed, from

$$u_\mu = a\left[-(1+\Psi), v_i + \omega_i\right], \quad n_\mu = -a(1+\Psi, 0), \tag{2.72}$$

it follows that, of for the scalar part we impose the condition (2.71), we identify u_μ with the vector orthogonal to space-like hypersurfaces at τ = const, n_μ. In this case we refer to this gauge as *orthogonal comoving gauge*.

2.5.4 Spatially flat gauge

This is defined by the condition of selecting spatial hypersurfaces at τ = const where the induced metric is left unperturbed by scalar and vector pertrubations:

$$\Phi = 0, \quad h^{\|} = 0, \quad h_i^{\perp} = 0. \tag{2.73}$$

It can be shown that the Ricci scalar on space-like hypersurfaces at τ = const is given by

$${}^{(3)}R = \frac{6k}{a^2} + \frac{12k}{a^2}\hat{\Phi} + \frac{4}{a^2}\nabla^2\hat{\Phi}, \quad \hat{\Phi} \equiv \Phi + \frac{1}{6}\nabla^2 h^{\|}, \tag{2.74}$$

where k is the spatial curvature parameter in FLRW. If $k = 0$ we are left with

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\hat{\Phi}. \tag{2.75}$$

Due to this last relation, $\hat{\Phi}$ is known as the *curvature perturbation.* By imposing the conditions (2.73) we obtain $\hat{\Phi} = 0$, namely we select a gauge where the curvature perturbation vanishes. For this reason this is also called *uniform curvature gauge.*

2.5.5 Uniform energy density gauge

This is defined by

$$\delta\rho = 0. \tag{2.76}$$

This selects spatial hypersurfaces at τ = const where the energy density of the fluid is left unperturbed. Analogously to the synchronous gauge, also the uniform energy density gauge leaves the freedom of choosing one further scalar and vector perturbation.

2.6 Gauge-invariant scalar perturbations

In this section we will introduce some fundamental quantities that will be used throughout the course of this thesis to describe scalar perturbations. We have seen that the intrinsic spatial curvature on hypersurfaces of constant conformal time is given (at linear order) by

$$^{(3)}R = \frac{4}{a^2}\nabla^2\hat{\Phi}, \quad \hat{\Phi} \equiv \Phi + \frac{1}{6}\nabla^2 h^{\parallel}. \tag{2.77}$$

The curvature perturbation $\hat{\Phi}$ is however not gauge-invariant, since under a gauge transformation it transforms as

$$\hat{\Phi} \to \tilde{\hat{\Phi}} = \hat{\Phi} - \frac{a'}{a}\alpha, \tag{2.78}$$

as can be seen using Eqs. (2.37) and (2.38). Therefore we need a gauge-invariant scalar quantity that reduces to the curvature perturbation in some particular gauge. To this end, let us introduce the so-called *curvature perturbation on uniform energy-density hypersurfaces*, which is defined as

$$\zeta \equiv -\hat{\Phi} - \mathcal{H}\frac{\delta\rho}{\rho_0'}, \tag{2.79}$$

where $\mathcal{H} \equiv a'/a$. Geometrically, ζ can be thought of as a measurement of the spatial curvature of constant energy density ($\delta\rho = 0$) hypersurfaces, given Eq. (2.77). This is a gauge-invariant quantity, indeed from (2.78) and (2.52) we have

$$\zeta \to \tilde{\zeta} = -\tilde{\hat{\Phi}} - \mathcal{H}\frac{\tilde{\delta\rho}}{\rho_0'} = -\hat{\Phi} + \mathcal{H}\alpha - \mathcal{H}\left(\frac{\delta\rho}{\rho_0'} + \alpha\right) = \zeta \tag{2.80}$$

A second important quantity which we introduce is the so-called *comoving curvature perturbation*, which is defined as

$$\mathcal{R} \equiv \hat{\Phi} + \frac{\mathcal{H}}{\phi_0'}\delta\phi. \tag{2.81}$$

Geometrically, $\mathcal{R}$ is a measurement of the spatial curvature of comoving ($\delta\phi = 0$) hypersurfaces. This is related to ζ by [11]

$$-\zeta = \mathcal{R} + \frac{2\rho}{9(\rho_0 + p_0)}\left(\frac{k}{aH}\right)^2\Psi. \tag{2.82}$$

From this relation we can see that on super-horizon scales, $k \ll aH$, the comoving curvature pertubation and the curvature perturbation on uniform energy-density hypersurfaces are equal, $\mathcal{R} \simeq -\zeta$.

Another important gauge-invariant quantity is the so-called Sasaki-Mukhanov gauge-invariant variable, defined as

$$\mathcal{Q} \equiv \delta\phi + \frac{\phi_0'}{\mathcal{H}}\hat{\Phi}. \tag{2.83}$$

This represents the inflaton perturbations on spatially flat slices, since

$$\mathcal{Q}|_{\hat{\Phi}=0} = \delta\phi, \tag{2.84}$$

and can be related to $\mathcal{R}$ by

$$\mathcal{Q} = \frac{\phi_0'}{\mathcal{H}}\mathcal{R}. \tag{2.85}$$

2.6.1 The curvature perturbation on uniform energy-density hypersurfaces

As we have seen, ζ is a gauge-invariant quantity. In the uniform energy density gauge it takes the form

$$\zeta|_{\delta\rho=0} = -\hat{\Phi}. \tag{2.86}$$

A crucial property of ζ, which makes it a fundamental quantity to relate what happens during inflation to the post-inflationary evolution of the Universe, is that it is constant on super-horizon scales in single field models with no non-adiabatic perturbations. This can be shown starting from the perturbed continuity equation, which can be written as

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) = 3(\rho_0 + p_0)\hat{\Phi}' - (\rho_0 + p_0)\nabla^2(V + \sigma), \tag{2.87}$$

where $V = v^{\|} + \omega^{\|}$ and σ is as in Eq. (2.64). In the uniform energy density gauge, where ζ has the form given by Eq. (2.86) and $\delta\rho = 0$, Eq. (2.87) becomes

$$3\mathcal{H}\delta p + 3(\rho_0 + p_0)\zeta' + (\rho_0 + p_0)\nabla^2(V + \sigma) = 0. \tag{2.88}$$

On super-horizon scales the term with the laplacian is negligible, hence we are left with

$$\zeta' \simeq -\frac{\mathcal{H}}{\rho_0 + p_0}\delta p\bigg|_{\delta\rho=0}. \tag{2.89}$$

Let us now recall that the perturbations in the isotropic pressure can be in generral written as the sum of an adiabatic and an entropy perturbation:

$$\delta p = c_s^2\delta\rho + \delta p_{\text{non-ad}}. \tag{2.90}$$

Since we are in the comoving gauge, Eq. (2.89) reduces to

$$\zeta'|_{\delta\rho=0} \simeq -\frac{\mathcal{H}}{\rho_0 + p_0}\delta p_{\text{non-ad}} \tag{2.91}$$

As we will explain in the next chapter, non-adiabatic perturbations arise when the energy densities of the different components of the Universe are perturbed in a different way, but these do not arise in single field slow-roll models (for more details on this, refer to Section 3.1 and Section 3.9). In this case, from (2.91) it follows that

$$\zeta' = 0 \;\Rightarrow\; \zeta = \text{const}, \tag{2.92}$$

which proves our previous assertion.

We know that a given mode k exits the horizon at a time $t_{\mathrm{H}}^{(1)}(k)$ during inflation and then reenters the horizon at a time $t_{\mathrm{H}}^{(2)}(k)$ during the radiation or matter era. The constance of ζ during the time when the mode is outside the horizon means that we can relate the perturbations in the inflaton field during inflation to the perturbations in the energy density in the standard (radiation or matter) era. For example, if the mode k reenters the horizon during radiation dominance, we have

$$\zeta_{t_{\mathrm{H}}^{(2)}(k)} = H\frac{\delta\rho}{\dot{\rho}_0}\bigg|_{t_{\mathrm{H}}^{(2)}(k)} = \frac{1}{4}\frac{\delta\rho}{\rho_0}\bigg|_{t_{\mathrm{H}}^{(2)}(k)} = \zeta_{t_{\mathrm{H}}^{(1)}(k)} = H\frac{\delta\phi}{\dot{\phi}_0}\bigg|_{t_{\mathrm{H}}^{(1)}(k)}, \tag{2.93}$$

where the second equality follows since $\rho_r \propto a^{-4}$. Notice that this also allows us to relate the perturbations in the inflaton field during inflation to the temperature fluctuations in the CMB, since

$$\rho_{0\gamma} \propto T^4 \;\Rightarrow\; \frac{\Delta T}{T} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_{0\gamma}}. \tag{2.94}$$

2.7 ADM decomposition of the metric

The aim of this section is to introduce a different way of parametrizing the spacetime metric, in such a way that this parametrization is adapted to a given choice of foliation of the spacetime by constant time, spacelike hypersurfaces. In particular, this parametrizaton of the metric has the advantage of making it explicit which are the true dynamical degrees of freedom of the system. In this discussion we mainly follow [44].

Before doing that, we only mention a few facts about which spacetimes admit a foliation, following [45]; for a more detailed discussion see [46]. A *Cauchy surface* is a spacelike hypersurface Σ in the spacetime $\mathcal{M}$ such that each causal (i.e. timelike or null) curve without endpoints intersect Σ only once. Not all spacetimes admit a Cauchy surface; for instance, those which have closed timelike curves do not. A spacetime that admits a Cauchy surface is said to be *globally hyperbolic*. Any globally hyperbolic spacetime can be foliated by a family of spacelike hypersurfaces.

To achieve this we assume that the spatial hypersurfaces are level sets of some time function $t(x^\mu)$:

$$\Sigma_{t_o} = \{x^\mu \,;\, t(x^\mu) = t_0\}, \tag{2.95}$$

with timelike and future-oriented normal vector n^μ and $n_\mu \sim \partial_\mu t$. We can introduce coordinates (t, y^i) on $\mathcal{M}$ via

$$x^\mu = x^\mu(t, y^i) \tag{2.96}$$

in the following way:

- For any fixed $t = t_0$,
  $$x^\mu_{t_0}(y^i) := x^\mu(t_0, y^i) \tag{2.97}$$
  provides us with the embedding of the hypersurface Σ with coordinates y^i as the hypersurface Σ_{t_0} in $\mathcal{M}$:
  $$x_{t_0} : \Sigma \to \Sigma_{t_0} \subset \mathcal{M}. \tag{2.98}$$
- We define the curves
  $$x^\mu_{y_0}(t) := x^\mu(t, y^i_0) \tag{2.99}$$
  that connect points on different hypersurfaces with the same values of the spatial coordinates $y^i = y^i_0$. These provide us with a notion of "time evolution" from one hypersurface to the next.

The tangent vectors to the surfaces Σ_t (or, better, their components) are[2]:

$$E^\mu_i = \left(\frac{\partial x^\mu}{\partial y^i}\right)_t, \tag{2.100}$$

while the components of the time-evolution vector field ∂_t are

$$(\partial_t)^\mu = \left(\frac{\partial x^\mu}{\partial t}\right)_y. \tag{2.101}$$

We can think of the vectors defined in (2.101) as representing the "flow of time" throughout spacetime. As we "move forward in time" by a parameter t from the $t = 0$ surface, $\Sigma_{t=0}$, we reach the hypersurface Σ_t. Thus we reach Σ_t by following the integral curves generated by the vector field $(\partial_t)^\mu$, and we can think of "moving forward in time" as resulting in the changing of the induced spatial metric on Σ from $h_{ij}(0)$ to $h_{ij}(t)$. This suggests us that we can view

[2] These are the pushforward of the vectors ∂_{y^i} from Σ to $\mathcal{M}$ by the embedding x^μ. Thus, they are linearly independent tangent vectors to the image of Σ in $\mathcal{M}$.

a globally hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu})$ as representing the time evolution of a Riemannian metric on a fixed three-dimensional manifold. Indeed, in a canonical formulation of General Relativity one can see that the metric induced on a spacelike hypersurface is really a dynamical variable. The canonical variables are in this case h_{ij} and its conjugate momentum, which is related to the extrinsic curvature K_{ij} of the embedded hypersurface. The dynamical equations evolve the metric h_{ij} and the extrinsic curvature K_{ij} forward in time from one hypersurface to the next.

Coming back to our discussion, it is important to notice that ∂_t is not necessarily normal to the hypersurface. We can decompose it into normal and tangential components in the following way

$$(\partial_t)^\mu = N n^\mu + E^\mu_i N^i, \tag{2.102}$$

where N is the so-called *lapse function* and N^i is the *shift vector field*. The lapse function measures the proper time between adjacent hyperurfaces and captures the fact that the coordinate time can pass more quickly in some regions of the spacetime than in others; the shift vector measures the change of coordinates (relative to the normal) from one hypersurface to the subsequent one, namely describes how spatial coordinates are propagated between two adjacent hypersurfaces.

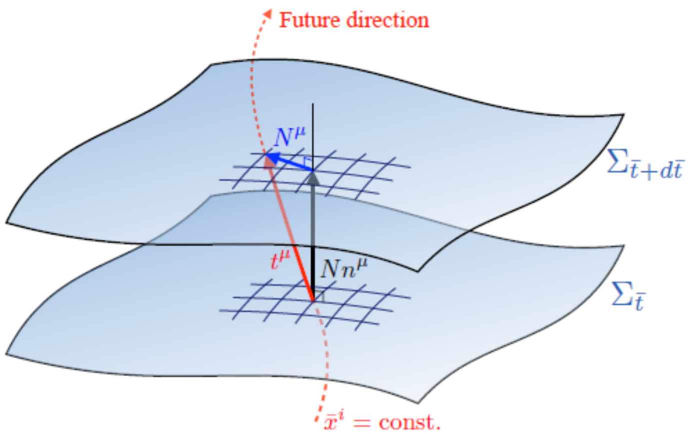

Figure 2.5: ADM decomposition of the spacetime (from [47]).

Using the previous decomposition, we have

$$dx^\mu = (Nn^\mu + E^\mu_i N^i)dt + E^\mu_i dy^i = Nn^\mu dt + E^\mu_i (dy^i + N^i dt). \tag{2.103}$$

Given that the normal vector is normalized and is a timelike vector, i.e. $g_{\mu\nu}n^\mu n^\nu = -1$, the line element can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dy^i + N^i dt)(dy^j + N^j dt), \tag{2.104}$$

where

$$h_{ij} = E^\mu_i E^\nu_j g_{\mu\nu} \tag{2.105}$$

is the induced metric, given by the pullback of the metric g. This is the *ADM decomposition* of the metric.

We can now write the metric and its inverse as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & N_i \\ N_j & h_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^j/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix}, \tag{2.106}$$

while the determinant of the metric is given by

$$\sqrt{-g} = N\sqrt{h}. \tag{2.107}$$

The normalized timelike normal vector field to the surfaces of constant t is

$$n_\mu = -N\partial_\mu t. \tag{2.108}$$

Thus, in the coordinates (t, y^i), one finds

$$n_\mu = (-N, 0, 0, 0), \quad n^\mu = (1/N, -N^i/N), \tag{2.109}$$

and

$$E^t_i = 0, \quad E^j_i = \delta^j_i. \tag{2.110}$$

From the decomposition (2.104) one can see that the lapse function N and the shift vector N_i contain the same information as the metric perturbations Ψ and ω_i in Eqs. (2.26)-(2.27), but the former are chosen in such a way that they appear as non-dynamical Lagrange multipliers in the action. This means that their equations of motion are purely algebraic, namely they are constraints. The six degrees of freedom contained in the 3-dimensional metric h_{ij} are thus reduced by the four contraints provided by the equations for the lapse function and the shift vector. Therefore we are left with a total of two dynamical degrees of freedom.

Chapter 3

The primordial perturbations from inflation

3.1 General argument for the production of primordial perturbations

In the first chapter we have studied the background dynamics of inflation, i.e. we have considered the dynamics associated with the homogeneous value of the scalar field driving inflation. It is the purpose of this and the next chapters that of studying the effects of introducing quantum fluctuations in the inflaton field. Before entering in the details, we want to give an intuitive argument to understand how the quantum fluctuations can be transformed into curvature and then density perturbations by the inflationary expansion. For the moment we consider only the perturbations in the scalar field, neglecting those in the metric tensor. This is not strictly consistent since, have we have already said, a perturbation in the scalar field induces a perturbation in the stress-energy tensor which then implies, through Einstein's equations, a pertubation in the metric tensor. The perturbations in the metric tensor will be included in the next sections, where we will compute the effect of the primordial perturbations in a rigorous way.

We have seen the inflaton can be split as

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t), \tag{3.1}$$

and obeys the Klein-Gordon equation

$$\ddot{\phi}(\mathbf{x}, t) + 3H\dot{\phi}(\mathbf{x}, t) - \frac{\nabla^2\phi(\mathbf{x}, t)}{a^2} = -V'(\phi). \tag{3.2}$$

For the "classical" value of the field, $\phi_0(t)$, the Klein-Gordon equation reduces to

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 = -V'(\phi_0). \tag{3.3}$$

By perturbing the Klein-Gordon equation (3.2) around $\phi_0(t)$ and then linearizing it, we can obtain an equation for the fluctuations $\delta\phi$, which reads

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} = -V''(\phi_0)\delta\phi. \tag{3.4}$$

Now we notice that, if we take the time derivative of the equation for the classical field (here we assume H to be constant), we get

$$(\dot{\phi}_0)^{\cdot\cdot} + 3H(\dot{\phi}_0)^{\cdot} = -V''(\phi_0)\dot{\phi}_0, \tag{3.5}$$

which is different from the equation obeyed by $\delta\phi$ due to the absence of the term $\nabla^2\delta\phi/a^2$. Going to Fourier space, this last term takes the form

$$\frac{\nabla^2\delta\phi}{a^2} \quad \xrightarrow{\text{F.S.}} \quad -k^2\,\frac{\delta\phi_{\mathbf{k}}(t)}{a^2}. \tag{3.6}$$

If we focus on super-Hubble scales, corresponding to $k/a \ll H$, this last term is negligible, hence $\dot{\phi}_0$ and $\delta\phi$ obey the same equation, and we have

$$\begin{cases} (\dot{\phi}_0)^{\cdot\cdot} + 3H(\dot{\phi}_0)^{\cdot} = -V''(\phi_0)\dot{\phi}_0, \\ \delta\ddot{\phi} + 3H\delta\dot{\phi} = -V''(\phi_0)\delta\phi. \end{cases} \tag{3.7}$$

To see whether the solutions are linearly dependent or not, let us consider the Wronskian, which is defined in our case by

$$W(\delta\phi, \dot{\phi}_0) = \delta\dot{\phi}\dot{\phi}_0 - \delta\phi\ddot{\phi}_0. \tag{3.8}$$

If this is equal to zero, the solutions are linearly dependent. It is easy to find that $\dot{W} = -3HW$, from which it follows that $W \propto e^{-3Ht}$. This implies that after a while the Wronskian goes to zero, and the two solutions become related by a constant of proportionality which depends on space only[1]:

$$\delta\phi(\mathbf{x}, t) = -\delta t(\mathbf{x})\dot{\phi}_0(t). \tag{3.9}$$

Then it follows that

$$\phi(\mathbf{x}, t) = \phi_0(t - \delta t(\mathbf{x})), \tag{3.10}$$

as can be easily seen by expanding to first order the latter relation. The physical meaning of this result can be understood in two equivalent ways:

- At a fixed time t, the inflaton assumes different values in different regions.
- The inflaton assumes the same value in different regions, but at slightly different times.

This means that the quantum fluctuations introduce a time shift in the evolution of the field with respect to the classical value, and different regions in the Universe undergo the same history, but at slightly different times[2].

Let us consider two different regions R_1 and R_2 where the inflaton assumes two different values, $\phi_1 \neq \phi_2$, at the time t. If, for example, the value ϕ_1 is a bit farther from the minimum of the potential than the value ϕ_2, this implies that in the region R_1 the inflationary expansion ends slightly later than in R_2. As a result of this process, at the end of inflation regions with slightly different energy densities emerge in the Universe. This is what then leads, via gravitational instability, to the formation of the structures we can observe today.

3.2 Gauge fixing at linear order

Now we want to study in more detail the perturbations around the background solution. As we have anticipated, if we perturb the inflaton field we are inevitably led to introduce also perturbations in the metric tensor:

$$\phi(t) \to \phi(t) + \delta\phi(\mathbf{x}, t), \tag{3.11}$$

[1] The constant of proportionality depends on $\mathbf{x}$ because we have neglected the gradient term. In other words, neglecting that term, which can be done if we cosider regions which are outside the horizon, is like making a smoothing, a coarse graining of the field. Thus, there is a spatial dependence when passing from a (coarse grained) region to another one.

[2] This allows us to think once again at the inflaton as a kind of "clock", which controls the amount of inflation.

$$g_{\mu\nu}(t) \to g_{\mu\nu}(t) + \delta g_{\mu\nu}(\mathbf{x}, t). \tag{3.12}$$

We expect to have three dynamical degrees of freedom, two coming from gravity and one from the scalar field. In order to see that, we decompose the metric following the ADM formalism, which has the nice property of making it clear which are the true dynamical degrees of freedom of the theory. As we have seen in Section 2.7, the line element can be written as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{3.13}$$

Let us recall the starting action, which is that of a scalar field minimally coupled to gravity

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \tag{3.14}$$

Setting for the moment $M_{Pl} = 1$ and decomposing the metric as in (3.13), this action becomes [13]

$$S = \frac{1}{2} \int d^3x dt \sqrt{h} \left[N R^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i \partial_i \phi)^2 - N h^{ij} \partial_i \phi \partial_j \phi \right], \tag{3.15}$$

with

$$E_{ij} = \frac{1}{2}(\dot{h}_{ij} - D_i N_j - D_j N_i) = N K_{ij}, \quad E = h^{ij} E_{ij}, \tag{3.16}$$

where h_{ij} is the metric on the three.dimensional spacelike hypersurfaces and K_{ij} is the extrinsic curvature, which is related with how the spatial hypersurfaces are embedded in the full spacetime. We have denoted with $R^{(3)}$ and D_i the Ricci scalar and the covariant derivative calculated using the metric h_{ij}. Notice that in the action N^i and N do not have any time derivative acting on them, hence they are Lagrange multipliers instead of true dynamical variables. Varying the action with respect to them, we obtain the so-called *Hamiltonian* and *Momentum constraints*. In order to find the action for the dynamical degrees of freedom we need to solve these constraints for N^i and N and plug the solutions back into the action. Before doing that, however, we have to fix a gauge, which allows us to eliminate some others unphysical gauge modes.

In this thesis we consider two different gauges: the *comoving gauge* and the *spatially flat gauge*. In the former, at linear order we have

$$\delta\phi = 0, \quad h_{ij} = a^2[(1 + 2\zeta)\delta_{ij} + \gamma_{ij}], \tag{3.17}$$

with

$$\partial_i \gamma_j{}^i = 0, \quad \gamma_i{}^i = 0. \tag{3.18}$$

This is a comoving gauge in the sense that the observer moves with the cosmic fluid and hence measures no matter flux. Notice that in this gauge the inflaton field has no fluctuations, while the metric tensor contains one scalar and one tensor perturbation. The transverse and traceless tensor perturbations γ_{ij} in the spatial metric are the *gravitational waves*, which carry two degrees of freedom. Indeed, γ_{ij} is a 3×3 symmetric tensor which in principle has six independent components; the conditions (3.18) represent four constraints, which reduce the independent components by four, hence leaving two physical degrees of freedom. This implies that, as anticipated, we have three dynamical degrees of freedom, one coming from the scalar perturbation ζ and two coming from γ_{ij}.

The scalar perturbation ζ is the so-called *curvature perturbation on uniform energy-density hypersurfaces*, which has been defined in the previous chapter. This tells us how much the spatial directions have expanded and it is conserved above the horizon in the case of single field inflationary models with adiabatic perturbations. For this reason, this quantity can be directly connected with the fluctuations in the CMB and thus allows us to connect what happened during inflation to the standard FLRW phase[3]. Moreover, when studying super-Hubble processes it is also more intuitive to think about the perturbations as ζ instead of $\delta\phi$.

The spatially flat gauge is instead defined by the requirements

$$\delta\phi(\mathbf{x},t) \equiv \varphi(\mathbf{x},t), \quad h_{ij} = a^2(\delta_{ij} + \gamma_{ij}), \tag{3.19}$$

with (3.18) still being satisfied. The spatial part of the scalar sector of the metric is flat, $\zeta = 0$, but we have fluctuations in the inflaton field. This implies that we still have three dynamical degrees of freedom, but the scalar one has been transferred from the curvature perturbation in the metric to the inflaton sector, φ. This gauge is particularly useful when considering sub-Hubble processes, and has also the advantage that the calculations are much simpler than in the comoving gauge.

From what just said, one can do the calculations in the comoving gauge to obtain directly an equation for ζ (which is the conserved quantity above the horizon), or start in the spatially flat gauge and then pass form φ to ζ after horizon crossing. The two are related at linear order by (see [13, 48])

$$\zeta = -\frac{H}{\dot{\phi}}\varphi. \tag{3.20}$$

Here we choose the first of the two options. The Hamiltonian and momentum constraints in the comoving gauge are

$$R^{(3)} - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi} - 2V = 0, \tag{3.21}$$

$$D_j[N^{-1}(E^j{}_i - \delta^j_i E)] = 0. \tag{3.22}$$

3.3 Constraint equations

As we have seen, N^i and N are non-dynamical fields, hence we can solve their equations of motion in terms of the dynamical degrees of freedom and then plug the solutions back into the action. Since these equations cannot be solved exactly, what we can do is to solve them perturbatively order by order. We now want to show, following the discussion of [49], that by solving the constraint equations to the n-th order we can obtain the action to the $(2n+1)$-th order. Thus, in order to find the action at the cubic order (which will be needed to compute the bispectra), it will be sufficient to solve the constraint equations to first order.

Let us consider the Lagrangian $\mathcal{L}(h, \bar{N}, \bar{N}_i)$, where we have suppressed the indices in the spatial metric for simplicity of notation. We can obtain $\bar{N}(h)$ and $\bar{N}_i(h)$ by solving the equations

$$\frac{\delta\mathcal{L}}{\delta N}(h, N, N_i)\Big|_{\bar{N}} = \left[\frac{\partial\mathcal{L}}{\partial N} - \partial_i \frac{\partial\mathcal{L}}{\partial(\partial_i N)}\right]_{\bar{N}} = 0, \tag{3.23}$$

$$\frac{\delta\mathcal{L}}{\delta N_i}(h, N, N_i)\Big|_{\bar{N}} = \left[\frac{\partial\mathcal{L}}{\partial N_i} - \partial_j \frac{\partial\mathcal{L}}{\partial(\partial_j N_i)}\right]_{\bar{N}} = 0. \tag{3.24}$$

Notice that, since N and N_i are non-dynamical, there are no time derivatives acting on them. Solving these equations to n-th order means obtaining

$$\bar{N} = N^{(n)} + \mathcal{O}(\zeta^{n+1}), \quad \bar{N}_i = N_i^{(n)} + \mathcal{O}(\zeta^{n+1}). \tag{3.25}$$

[3]This is extremely important since we don't know anything about the history of the Universe from the end of inflation until the time at which Big Bang Nucleosynthesis occured, during the radiation era.

Now let us Taylor expand the full action in terms of the approximate solutions (3.25) to the constraint equations:

$$\begin{aligned}\int d^4x\,\mathcal{L}(h,\bar{N},\bar{N}_i) = &\int d^4x\,\mathcal{L}(h,N^{(n)},N_i^{(n)}) + \int d^4x\,(\bar{N}-N^{(n)})\frac{\delta\mathcal{L}}{\delta N}(h,N^{(n)},N_i^{(n)})+\\ &+\int d^4x\,(\bar{N}_i-N_i^{(n)})\frac{\delta\mathcal{L}}{\delta N_i}(h,N^{(n)},N_i^{(n)})+\\ &+\int d^4x\,\partial_j\left[(\bar{N}-N^{(n)})\frac{\partial\mathcal{L}}{\partial(\partial_j N)}(h,N^{(n)},N_i^{(n)})\right]+\\ &+\int d^4x\,\partial_j\left[(\bar{N}_i-N_i^{(n)})\frac{\partial\mathcal{L}}{\partial(\partial_j N_i)}(h,N^{(n)},N_i^{(n)})\right]+\\ &+\mathcal{O}\left((\bar{N}-N^{(n)})^2,(\bar{N}_i-N_i^{(n)})^2\right).\end{aligned} \tag{3.26}$$

The fourth and fifth terms are total derivatives, hence can be neglected. The second and third terms are the constraint equations (3.23)-(3.24), which have been solved to the n-th order in perturbation theory. Thus, the error in the Taylor expansion of the full action is of order $2(n+1)$: one $n+1$ comes from the error in $\bar{N}-N^{(n)}$ and the second one from the error in $\delta\mathcal{L}/\delta N$. The last term in Eq. (3.26) gives an error of the same order. It follows that, using the n-th order solutions to the constraint equations, we make an error of order $2(n+1)$ in the fluctuations in the full action. This means that we can obtain a perturbative expansion of the action up to order $2n+1$.

Let us now expand the lapse function and the shift vector as

$$N = N^{(0)} + N^{(1)} + N^{(2)} + \dots, \quad N_i = N_i^{(0)} + N_i^{(1)} + N_i^{(2)} + \dots \tag{3.27}$$

The shift vector can further be decomposed into a scalar and a vector perturbation

$$N_i^{(n)} = \partial_i\psi^{(n)} + \beta_i^{(n)}, \tag{3.28}$$

with $\partial^i\beta_i^{(n)} = 0$. We can now solve the Hamiltonian and the momentum constarints order by order and then plug the solutions back into the action.

At 0-th order, the solution is given by $N^{(0)} = 1$ and $N_i^{(0)} = 0$. The momentum constaint (3.22) vanishes, reflceting the isotropy of the background solution, while the Hamiltonian constraint (3.21) gives us the first Friedmann equation.

Now we go to the next order. We first concentrate on scalar perturbations, which are the density perturbations that then grow to form the large scale structures in the Universe and we can observe through the temperature anisotropies in the CMB. Then, we will study the tensor perturbations, i.e. the primordial gravitational waves from inflation.

3.4 Scalar perturbations from inflation

Since we now focus on scalar peturbations, we can set $\gamma_{ij} = 0$. Notice also that, for the same reason, the vector perturbation β_i in the shift vector can only be the gradient of a function F of the scalar perturbation ζ itself and its derivatives, $\beta_i = \partial_i F$. Therefore the condition $\partial^i\beta_i = 0$ implies that $\partial^2 F = 0$. If we assume that F goes to zero at infinity, this is solved by $F = 0$. Hence also $\beta_i = 0$.

Neglecting tensor perturbations, the three-dimensional metric and its inverse reduce to

$$h_{ij} = a^2 e^{2\zeta}\delta_{ij}, \quad h^{ij} = a^{-2}e^{-2\zeta}\delta^{ij}, \tag{3.29}$$

while their time derivatives are given by

$$\dot{h}_{ij} = 2a^2(H+\dot{\zeta})e^{2\zeta}\delta_{ij}, \quad \dot{h}^{ij} = -2a^{-2}(H+\dot{\zeta})e^{-2\zeta}\delta^{ij}. \tag{3.30}$$

The second relation in (3.30) comes from

$$h^{im}h_{jm} = \delta^i_j \quad \Rightarrow \quad \dot{h}^{im}h_{jm} = -h^{im}\dot{h}_{jm} \quad \Rightarrow \quad \dot{h}^{ik} = -h^{im}h^{jk}\dot{h}_{jm}. \tag{3.31}$$

Hence

$$\dot{h}^{ik} = -h^{im}h^{jk}2a^2(H+\dot{\zeta})e^{2\zeta}\delta_{jm} = -2a^{-2}(H+\dot{\zeta})e^{-2\zeta}\delta^{ik}. \tag{3.32}$$

From the metric we can then compute the connection coefficients, which are given by

$$\Gamma^k_{ij} = \frac{1}{2}h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) = \delta^{kl}(\partial_i\zeta\delta_{jl} + \partial_j\zeta\delta_{il} - \partial_l\zeta\delta_{ij}). \tag{3.33}$$

As regarding the extrinsic curvature (remember that $E_{ij} = NK_{ij}$), we have

$$\begin{aligned} E_{ij} &= \frac{1}{2}(\dot{h}_{ij} - D_iN_j - D_jN_i) = \frac{1}{2}\left[2a^2(H+\dot{\zeta})e^{2\zeta}\delta_{ij} - 2\partial_{(i}N_{j)} + 2\Gamma^k_{ij}N_k\right] \\ &= a^2(H+\dot{\zeta})e^{2\zeta}\delta_{ij} - \partial_{(i}N_{j)} + 2N_{(i}\partial_{j)}\zeta - N^k\partial_k\zeta\delta_{ij}, \end{aligned} \tag{3.34}$$

$$E^{ij} = h^{ik}h^{jl}E_{kl} = a^{-4}e^{-4\zeta}\delta^{ik}\delta^{jl}E_{kl}, \tag{3.35}$$

$$E = h^{ij}E_{ij} = a^{-2}e^{-2\zeta}\delta^{ij}E_{ij} = 3(H+\dot{\zeta}) - a^{-2}e^{-2\zeta}(\partial^iN_i + N^i\partial_i\zeta). \tag{3.36}$$

Thus, we find

$$\begin{aligned} E^{ij}E_{ij} - E^2 = &- 6(H+\dot{\zeta})^2 + 4e^{-2\zeta}a^{-2}(H+\dot{\zeta})(\partial^iN_i + N^i\partial_i\zeta) + \\ &- e^{-4\zeta}a^{-4}\Big\{(\partial^iN_i)^2 + 2(\partial^iN_i\zeta)^2 - \left[\partial_{(i}N_{j)} - (\partial^iN_i + N^i\partial_i\zeta)\right]^2\Big\}. \end{aligned} \tag{3.37}$$

The three-dimensional Ricci scalar is instead given by

$$\begin{aligned} R^{(3)} &= h^{ij}R^m{}_{imj} = h^{ij}\left(\partial_m\Gamma^m_{ij} - \partial_j\Gamma^m_{im} + \Gamma^m_{mk}\Gamma^k_{ij} - \Gamma^m_{jk}\Gamma^k_{im}\right) \\ &= a^{-2}e^{-2\zeta}\delta^{ij}\Big\{\partial_m\left[\delta^{ml}(\partial_i\zeta\delta_{jl} + \partial_j\zeta\delta_{il} - \partial_l\zeta\delta_{ij})\right] - \partial_j\left[\delta^{ml}(\partial_i\zeta\delta_{ml} + \partial_m\zeta\delta_{il} - \partial_l\zeta\delta_{im})\right] + \\ &\quad + \delta^{ml}(\partial_m\zeta\delta_{kl} + \partial_k\zeta\delta_{ml} - \partial_l\zeta\delta_{mk})\delta^{kr}(\partial_i\zeta\delta_{jr} + \partial_j\zeta\delta_{ir} - \partial_r\zeta\delta_{ij}) + \\ &\quad - \delta^{ml}(\partial_j\zeta\delta_{kl} + \partial_k\zeta\delta_{jl} - \partial_l\zeta\delta_{jk})\delta^{kr}(\partial_i\zeta\delta_{mr} + \partial_m\zeta\delta_{ir} - \partial_r\zeta\delta_{im})\Big\} \\ &= a^{-2}e^{-2\zeta}\delta^{ij}\left(-\partial_j\partial_i\zeta - \partial_l\partial_l\zeta\delta_{ij} - \delta_{ij}\partial_l\zeta\partial_l\zeta + \partial_i\zeta\partial_j\zeta\right) = -2a^{-2}e^{-2\zeta}\left[2\partial^2\zeta + (\partial_i\zeta)^2\right]. \end{aligned} \tag{3.38}$$

Plugging these expressions into the Hamiltonian and momentum constraints we can solve them, finding[4]

$$N^{(1)} = \frac{\dot{\zeta}}{H}, \quad \psi^{(1)} = -\frac{\zeta}{H} + \chi, \quad \partial^2\chi = a^2\frac{\dot{\phi}^2}{2H^2}\dot{\zeta}, \tag{3.39}$$

where we have used the background equations of motion to get rid of the zero-th order quantities.

We can then substitute these solutions into the action (3.15) and expand the latter to second order. By using again the background equations of motion and integrating by parts some of the terms, we finally get

$$S_2 = \epsilon_V M_{Pl}^2 \int d^4x \left[a^3\dot{\zeta}^2 - a\,(\partial_i\zeta)^2\right], \tag{3.40}$$

where we have also reintroduced the Planck mass. We now make a field redefinition by defining the so-called *Mukhanov-Sasaki variable*:

$$v \equiv z\zeta, \quad \text{with } z \equiv \sqrt{2\epsilon_V}aM_{Pl}. \tag{3.41}$$

[4] The solution of the momentum constraint is $N^{(1)} = \frac{\dot{\zeta}}{H} + \text{const}$. We can then set the constant to zero in order to recover the background metric once the perturbations are removed.

Switching to conformal time, the action (3.40) can be rewritten as

$$
\begin{aligned}
S_2 &= \frac{1}{2}\int d^3x\, d\tau\, az^2 \left[a\left(\frac{1}{a}\frac{d}{d\tau}\frac{v}{z}\right)^2 - \frac{1}{az^2}(\partial_i v)^2 \right] \\
&= \frac{1}{2}\int d^3x\, d\tau \left[(v')^2 + v^2\left(\frac{z'}{z}\right)^2 - 2(v^2)'\frac{z'}{z} - (\partial_i v)^2 \right],
\end{aligned}
\tag{3.42}
$$

where $' \equiv \frac{d}{d\tau}$. Integrating by parts the third term, we get

$$
S_2 = \frac{1}{2}\int d^3x\, d\tau \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2 \right]. \tag{3.43}
$$

If we then expand v in Fourier modes

$$
v(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3} v_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.44}
$$

and vary the action (3.43), we find the equations of motion for $v_{\mathbf{k}}$, also known as the *Mukhanov-Sasaki equation*:

$$
v''_{\mathbf{k}} + \left(k^2 - \frac{z''}{z}\right) v_{\mathbf{k}} = 0. \tag{3.45}
$$

This is the equation of an harmonic oscillator with a time-dependent frequency. This time dependence of the frequency accounts for the interaction of the scalar field with the gravitational background. In order to compute the second term, notice that

$$
\frac{z''}{z} = \frac{d}{d\tau}\left(\frac{z'}{z}\right) + \left(\frac{z'}{z}\right)^2, \tag{3.46}
$$

and

$$
\frac{z'}{z} = \frac{a}{2}\frac{(z^2)^{\cdot}}{z^2} = \dot{a} + \frac{a}{2}\frac{\dot{\epsilon}_V}{\epsilon_V} = aH\left(1 + \frac{1}{2}\frac{\dot{\epsilon}_V}{\epsilon_V H}\right) \simeq -\frac{1}{\tau(1-\epsilon_V)}(1 + 2\epsilon_V - \eta_V) \simeq -\frac{(1 + 3\epsilon_V - \eta_V)}{\tau} \tag{3.47}
$$

to lowest order in the slow-roll parameters. Therefore, using Eq. (3.46) we find

$$
\frac{z''}{z} \simeq \frac{1 + 3\epsilon_V - \eta_V}{\tau^2} + \frac{1 + 6\epsilon_V - 2\eta_V}{\tau^2} \simeq \frac{2 + 9\epsilon_V - 3\eta_V}{\tau^2}. \tag{3.48}
$$

The equationss of motion (3.45) can then be rewritten as

$$
v''_{\mathbf{k}} + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) v_{\mathbf{k}} = 0, \tag{3.49}
$$

with

$$
\nu^2 = \frac{9}{4} + 9\epsilon_V - 3\eta_V. \tag{3.50}
$$

At leading order in the slow-roll parameters, this gives

$$
\nu \simeq \frac{3}{2} + 3\epsilon_V - \eta_V. \tag{3.51}
$$

We can now canonically quantize the fields $v_{\mathbf{k}}$ by expanding them in terms of the creation and annihilation operators as

$$
\hat{v}_{\mathbf{k}}(\tau) = v_k(\tau)\hat{a}(\mathbf{k}) + v_k^*(\tau)\hat{a}^\dagger(-\mathbf{k}). \tag{3.52}
$$

The creation and annihilation operators satisfy the equal time commutation relations

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')], \tag{3.53}$$

and act on the vacuum state obeying

$$\hat{a}|0\rangle = 0, \quad \langle 0|\hat{a}^\dagger = 0. \tag{3.54}$$

The mode functions $v_k(\tau)$ are instead normalized such that[5]

$$W[v_k, v_k^*] \equiv v_k' v_k^* - v_k v^{*\prime} = 2i, \tag{3.55}$$

and, from (3.49), obey the equations of motion

$$v_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) v_k = 0. \tag{3.56}$$

The normalization condition (3.55) provides one of the boundary conditions required to solve Eq. (3.56). The second one required to fix the mode functions comes from the choice of the vacuum state, as we will now see.

3.4.1 Sub-horizon and super-horizon limits

Before solving exactly the equations of motion, it is useful and instructive to consider the two limits corresponding to sub-horizon and super-horizon scales.

Sub-horizon limit

In the limit $-k\tau \to \infty$ the modes have wavelengths much smaller than the horizon, and the second term inside the parentheses in the equations of motion (3.56) is negligible with respect to the first one. Thus we are left with

$$v_k'' + k^2 v_k = 0. \tag{3.57}$$

This is exactly the same equation obeyed by a scalar field in Minkowski spacetime, and the general solution is given by a linear combination of plane waves

$$v_k = c_1 e^{ik\tau} + c_2 e^{-ik\tau}. \tag{3.58}$$

This means that the fluctuations with wavelengths much smaller than the horizon oscillate, as in a flat Minkowski spacetime. Physically we can think that, when the modes are well inside the horizon, they are not affected by gravity (and they do not feel the expansion of the Universe), and thus behave just like in a flat, static spacetime. This allows us to require that the solutions of (3.56) reduce to the mode functions of Minkowski in the sub-horizon limit. In practice this means that we have to solve the Mukhanov-Sasaki equation (3.56) with the initial condition

$$\lim_{-k\tau\to\infty} v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{3.59}$$

This is the so-called *Bunch-Davies vacuum*[6], and provides the second boundary condition required to solve Eq. (3.56).

[5] This normalization condition is required to make the commutation relations (3.53) compatible with the canonical commutation relations between the field and its conjugate momentum. See [50] for more details on these technical issues.

[6] Notice that in a time-varying background there is no unique way to define the vacuum state, since the Hamiltonian of the system depends explicitly on time and thus does not possess time-independent eigenstates. There are different possible mode functions that solve (3.56) still obeying (3.55) and these are related to each other by the so-called *Bogolyubov transformations*. Defining the vacuum state allows us to completely fix the solutions $v_k(\tau)$. See again [50] for more details on this.

Super-horizon limit

In the limit $-k\tau \to 0$ the modes have instead wavelengths much larger than the horizon. In this case the equation of motion reduces to

$$v_k'' - \frac{\nu^2 - \frac{1}{4}}{\tau^2} v_k = 0, \tag{3.60}$$

whose solution can be written as the linear combination of a growing and a decaying mode:

$$v_k = B_+(k)a + B_-(k)a^{-2}. \tag{3.61}$$

When coming back to our original variable, $\zeta_k \propto v_k a^{-1}$, the solution is given by a constant plus the decaying mode that goes like a^{-1}. The latter goes to zero since $a^{-1} \propto \tau \to 0$, and we are left only with the constant solution. This means that ζ becomes constant on super-horizon scales and the amplitude of the fluctuations gets frozen. This happens because no causal process can affect the evolution of the fluctuations when their wavelength is outside the horizon.

3.4.2 Exact solution of the equation of motion

Let us now solve exactly the equations of motion (3.56). In order to do that, we make a change of variable by defining

$$v_k \equiv (-\tau)^{1/2} y. \tag{3.62}$$

The first and second derivatives of v_k are given by

$$v_k' = -\frac{1}{2}(-\tau)^{-1/2} y + (-\tau)^{1/2} y', \tag{3.63}$$

$$v_k'' = (-\tau)^{1/2} y'' - (-\tau)^{-1/2} y' - \frac{1}{4}(-\tau)^{-3/2} y. \tag{3.64}$$

Substituting these results in the equations of motion (3.56), we find

$$\tau^2 y'' + \tau y' + \left(k^2\tau^2 - \nu^2\right) y = 0. \tag{3.65}$$

In terms of the variable

$$x \equiv -k\tau, \tag{3.66}$$

this can be rewritten in the form of a *Bessel equation*:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right) y = 0. \tag{3.67}$$

The general solution of the Bessel equation is given by

$$y(x) = c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x), \tag{3.68}$$

where $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are the *Hankel functions* of first and second species, which satisfy

$$H_\nu^{(2)}(x) = H_\nu^{(1)*}(x), \tag{3.69}$$

while c_1 and c_2 are constants which have to be determined by imposing the initial condition. Coming back to our original variables, we can rewrite the general solution of the equation of motion as

$$v_k(\tau) = \sqrt{-\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right]. \tag{3.70}$$

This solution of the Bessel equation is valid for constant values of ν. Since during inflation we can consider ϵ_V and η_V to be constant at leading order in the slow-roll parameters, it follows from (3.51) that also ν can be taken constant. From the asymptotic expansion of the Hankel functions

$$H_\nu^{(1)}(-k\tau) \overset{-k\tau\gg 1}{\sim} \sqrt{\frac{2}{\pi}}(-k\tau)^{-1/2} e^{i\left(-k\tau-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}, \tag{3.71}$$

$$H_\nu^{(2)}(-k\tau) \overset{-k\tau\gg 1}{\sim} \sqrt{\frac{2}{\pi}}(-k\tau)^{-1/2} e^{-i\left(-k\tau-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}, \tag{3.72}$$

and imposing the Bunch-Davies vacuum initial condition, we can fix the values of c_1 and c_2, which then read

$$c_2(k) = 0, \quad c_1(k) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)}. \tag{3.73}$$

Therefore the solution for the mode function of scalar perturbations is

$$v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)} \sqrt{-\tau}\, H_\nu^{(1)}(-k\tau). \tag{3.74}$$

3.5 Tensor perturbations from inflation

With a similar mechanism to that just seen for scalar perturbations, inflation generates also transverse and traceless tensor perturbations of the metric, namely gravitational waves. As we will discuss in detail in Section 3.8, this is one of the most important predictions of inflation. Remarkably, the amplitude of primordial gravitational waves (PGWs) is uniquely determined by the Hubble parameter H during inflation. This means that a measurement of PGWs would give direct information about the most important inflationary parameter, namely the energy scale at which inflation occurred.

As we know, scalar, vector and tensor perturbations are decoupled at linear order, hence, when studying tensor perturbations, we can set the scalar and vector ones to zero. Therefore we have

$$\zeta = 0, \quad N_i = 0, \quad N = 1. \tag{3.75}$$

By expanding the Einstein-Hilbert action at second order in tensor perturbations one finds

$$S_{\gamma\gamma} = \frac{M_{Pl}^2}{8} \int d\tau d^3x a^2 \left[(\gamma_{ij}')^2 - (\partial_l \gamma_{ij})^2\right]. \tag{3.76}$$

Going to Fourier space we can expand the tensor perturbations in the $+, \times$ polarization basis (see Appendix C) as

$$\gamma_{ij}(\mathbf{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+,\times} \epsilon_{ij}^{(s)}(\mathbf{k}) \gamma_s(\mathbf{k}, \tau) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.77}$$

The polarization tensor satisfies the following properties

$$\epsilon_i{}^i = 0, \quad k^i \epsilon_{ij} = 0, \quad \epsilon_{ij}^{(s)}(\mathbf{k}) \epsilon^{(s')ij}(\mathbf{k}) = 2\delta^{ss'}. \tag{3.78}$$

The fields $\gamma_s(\mathbf{k}, \tau)$ are the two polarization modes of tensor perturbations, and each of them behaves like a massless scalar field. Indeed, defining

$$A_T^2 \equiv \frac{M_{Pl}^2}{2} a^2, \tag{3.79}$$

we can rewrite the action (3.76) as

$$S_{\gamma\gamma} = \frac{1}{2} \sum_{s=+,\times} \int \frac{d^3k}{(2\pi)^3} d\tau A_T^2 \left[\gamma_s'^2(\mathbf{k}, \tau) - k^2 \gamma_s^2(\mathbf{k}, \tau)\right], \tag{3.80}$$

which is the action for two massless scalar fields. By making the field redefinition

$$\mu_s \equiv A_T \gamma_s, \tag{3.81}$$

and integrating once by parts, the action (3.80) takes the following form

$$S_{\gamma\gamma} = \frac{1}{2} \sum_{s=+,\times} \int \frac{d^3k}{(2\pi)^3} d\tau A_T^2 \left[\mu_s'^2(\mathbf{k},\tau) - k^2 \mu_s^2(\mathbf{k},\tau) + \frac{A_T''}{A_T} \mu_s^2(\mathbf{k},\tau) \right]. \tag{3.82}$$

We can then obtain the equations of motion for the fields μ_s by varying the action with respect to the fields themselves. These read

$$\mu_s'' + \left(k^2 - \frac{A_T''}{A_T} \right) \mu_s = 0. \tag{3.83}$$

In order to calculate $\frac{A_T''}{A_T}$, notice that

$$\frac{A_T''}{A_T} = \frac{d}{d\tau} \left(\frac{A_T'}{A_T} \right) + \left(\frac{A_T'}{A_T} \right)^2, \tag{3.84}$$

with

$$\frac{A_T'}{A_T} = \frac{a}{2} \frac{(A_T^2)^{\cdot}}{A_T^2} = aH = -\frac{1}{\tau}(1+\epsilon) + \mathcal{O}(\epsilon^2). \tag{3.85}$$

It follows from Eq. (3.84) that, at leading order in the slow-roll parameters, we have

$$\frac{A_T''}{A_T} \simeq \frac{2+\epsilon}{\tau^2}. \tag{3.86}$$

Therefore the equations of motion (3.83) become

$$\mu_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} \right) \mu_s = 0, \tag{3.87}$$

where we have introduced the parameter ν_T, defined through

$$\nu_T^2 = \frac{9}{4} + 3\epsilon. \tag{3.88}$$

At leading order in the slow-roll parameters, this leads to

$$\nu_T \simeq \frac{3}{2} + \epsilon. \tag{3.89}$$

We can now canonically quantize the fields μ_s by expanding them in terms of the creation and annihilation operators as

$$\hat{\mu}_s(\mathbf{k},\tau) = u_s(k,\tau)\hat{a}_s(\mathbf{k}) + u_s^*(k,\tau)\hat{a}_s^\dagger(-\mathbf{k}). \tag{3.90}$$

The creation and annihilation operators satisfy the equal time commutation relations

$$[\hat{a}_s(\mathbf{k}), \hat{a}_{s'}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')], \tag{3.91}$$

and act on the vacuum state obeying

$$\hat{a}_s|0\rangle = 0, \quad \langle 0|\hat{a}_s^\dagger = 0. \tag{3.92}$$

The mode functions $u_s(k,\tau)$ are normalized (as the scalar modes) such that

$$W[u_s(k), u_s(k)^*] \equiv u_s'(k)u_s(k)^* - u_s(k)u_s(k)^{*\prime} = 2i, \tag{3.93}$$

and obey the equations of motion

$$u_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2}\right) u_s = 0. \tag{3.94}$$

Thus each of the two polarization states obeys the same equation of motion as the scalar modes, as expected. The only difference between the two cases lies in the normalization factors present in the field redefinitions (3.41) and (3.81), that have allowed us to rewrite the equations of motion in the form of that of a free harmonic oscillator with a time-varying effective frequency. Therefore, with the initial condition of the Bunch-Davies vacuum state

$$\lim_{-k\tau\to\infty} u_s(k,\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}, \tag{3.95}$$

the solution of the equations of motion for tensor modes is given by

$$u_s(k,\tau) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}\left(\nu_T + \frac{1}{2}\right)} \sqrt{-\tau}\, H^{(1)}_{\nu_T}(-k\tau). \tag{3.96}$$

3.6 Power spectrum of perturbations

The power spectrum of a perturbation field is a measure of the amplitude of the fluctuations at a given scale k. Given a certain random field $f(\mathbf{x},t)$, we can expand it in Fourier space as

$$f(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\, f_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.97}$$

The *power spectrum* $P(k)$ is defined through the relation

$$\langle f_{\mathbf{k}_1}(t) f_{\mathbf{k}_2}(t)\rangle = (2\pi)^3 P(k_1)\, \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2), \tag{3.98}$$

where the angular brackets denote an average over the statistical ensemble. Notice that the power spectrum depends only on the modulus of $\mathbf{k}_1$ as a consequence of isotropy. Similarly, the delta function comes from the assumption of homogeneity (i.e., invariance under translations), and implies that different modes are uncorrelated. We can then show the validity of the following theorem:

Wiener-Khinchin theorem
The power spectrum $P(k)$ is the Fourier transform of the two-point correlation function.
Proof
From the definition of the two-point correlation function in the real space, we have:

$$\begin{aligned}
\xi(\mathbf{r}) \equiv \langle f(\mathbf{x}) f(\mathbf{x}+\mathbf{r})\rangle &= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle f_{\mathbf{k}} f_{\mathbf{k}'}\rangle e^{i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})} \\
&= \int \frac{d^3k}{(2\pi)^3} \int d^3k' P(k) \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})} \\
&= \int \frac{d^3k}{(2\pi)^3} P(k) e^{-i\mathbf{k}\cdot\mathbf{r}},
\end{aligned} \tag{3.99}$$

where to pass from the first to the second line we have used (3.98). Therefore, we find

$$P(k) = \int \frac{d^3r}{(2\pi)^3} \xi(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3.100}$$

□

If $\mathbf{r} = 0$, the two-point function is the variance of the random field, namely

$$\sigma^2 \equiv \langle f^2(\mathbf{x}, t)\rangle = \int \frac{d^3k}{(2\pi)^3} P(k) = \frac{1}{2\pi^2} \int_0^{+\infty} dk\, k^2 P(k) = \int_0^{+\infty} \frac{dk}{k} \Delta(k), \tag{3.101}$$

where

$$\Delta(k) \equiv \frac{k^3}{2\pi^2} P(k) \tag{3.102}$$

is the so-called *dimensionless power spectrum*. As can be seen from (3.101), this is the contribution to the variance per unit logarithmic interval at the wavenumber k.

In the case of a scalar field we can canonically quantize the latter in the usual way:

$$\hat{f}_{\mathbf{k}}(\tau) = v_k(\tau)\hat{a}(\mathbf{k}) + v_k^*(\tau)\hat{a}^\dagger(-\mathbf{k}), \tag{3.103}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the creation and annihilation operators, which satisfy the equal time commutation relations

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')], \tag{3.104}$$

and act on the vacuum state obeying

$$\hat{a}|0\rangle = 0, \quad \langle 0|\hat{a}^\dagger = 0. \tag{3.105}$$

In order to evaluate the power spectrum we need to compute

$$\begin{aligned} \langle f_{\mathbf{k}_1} f_{\mathbf{k}_2}\rangle &= \langle 0|[v_{k_1}\hat{a}(\mathbf{k}_1) + v_{k_1}^*\hat{a}^\dagger(-\mathbf{k}_1)][v_{k_2}\hat{a}(\mathbf{k}_2) + v_{k_2}^*\hat{a}^\dagger(-\mathbf{k}_2)]|0\rangle \\ &= v_{k_1} v_{k_2}^* \langle 0|\hat{a}(\mathbf{k}_1)\hat{a}^\dagger(-\mathbf{k}_2)|0\rangle \\ &= v_{k_1} v_{k_2}^* \langle 0|\big([\hat{a}(\mathbf{k}_1), \hat{a}^\dagger(-\mathbf{k}_2)] + \hat{a}^\dagger(-\mathbf{k}_2)\hat{a}(\mathbf{k}_1)\big)|0\rangle \\ &= (2\pi)^3 |v_{k_1}|^2 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2). \end{aligned} \tag{3.106}$$

Combining this result with the definition of the power spectrum given in (3.98), it follows that

$$P(k_1) = |v_{k_1}|^2. \tag{3.107}$$

The dimensionless power spectrum can thus be written as

$$\Delta(k, \tau) \equiv \frac{k^3}{2\pi^2} |v_k(\tau)|^2. \tag{3.108}$$

In the case of tensor perturbations we can proceed in the same way, with the only difference of an extra factor 2 coming from the contraction of the polarization tensors. With the notations of the previous section, the dimensionless power spectrum is thus given by

$$\Delta_T(k, \tau) \equiv \frac{k^3}{\pi^2} |u(k, \tau)|^2. \tag{3.109}$$

Another important quantity which we now define is the *spectral index*, which tells us how $\Delta(k)$ varies with the cosmological scale $k \sim 2\pi/\lambda$. For scalar perturbations this is defined through the relation

$$n_s(k) - 1 \equiv \frac{d\ln\Delta(k)}{d\ln k}. \tag{3.110}$$

For a constant $n_s(k)$, the power spectrum is a power law

$$\Delta(k) = \Delta(k_0)\left(\frac{k}{k_0}\right)^{n_s - 1}, \tag{3.111}$$

where k_0 is a given pivot scale. In the particular case with $n_s = 1$, the amplitude of the perturbations does not depend on the cosmological scale, $\Delta(k) = \text{const.}$, and we have a scale-invariant power spectrum. This is also known as the *Harrison-Zel'dovich spectrum.*

In the case of tensor perturbations the spectral index is instead defined as

$$n_T(k) \equiv \frac{d\ln\Delta_T(k)}{d\ln k}. \tag{3.112}$$

From this definition, a scale-invariant spectrum of tensor perturbations has $n_T = 0$.

3.6.1 Power spectrum of scalar perturbations

We are now interested in computing the power spectrum of scalar perturbations on super-horizon scales. To do so, we need to find the solution for the mode function v_k given in (3.74) in the limit $-k\tau \ll 1$. In this limit, the Hankel function of first species has the asymptotic expression

$$H_\nu^{(1)}(-k\tau) \overset{-k\tau \ll 1}{\sim} \sqrt{\frac{2}{\pi}}\, e^{-i\frac{\pi}{2}}\, 2^{\nu - \frac{3}{2}}\, \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}(-k\tau)^{-\nu}, \tag{3.113}$$

from which one finds that

$$v_k(\tau)|_{-k\tau \ll 1} = e^{i\frac{\pi}{2}(\nu + \frac{1}{2})} 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2} - \nu}. \tag{3.114}$$

By recalling the field redefinition (3.41), one can then compute the dimensionless power spectrum of curvature perturbations at leading order in the slow-roll parameters. This reads

$$\Delta_\zeta(k) \simeq \frac{1}{\pi^2 \epsilon} \frac{H^2}{M_{Pl}^2} \left(\frac{k}{aH}\right)^{3 - 2\nu}, \tag{3.115}$$

with

$$3 - 2\nu \simeq 2\eta - 6\epsilon. \tag{3.116}$$

From this last relation it is immediate to compute the spectral index of scalar perturbations, which is given by

$$n_s - 1 = \frac{d\ln\Delta_\zeta(k)}{d\ln k} = 2\eta - 6\epsilon. \tag{3.117}$$

Since in slow-roll inflation $\epsilon, |\eta| \ll 1$, the spectral index predicted by slow-roll inflationary models is slightly lower than unity. This corresponds to an almost scale-invariant, *red spectrum* ($n_s < 1$), where the amplitude of the fluctuations decreases for decreasing cosmological scales. This feature can be understood in the following way. We know that different modes leave the horizon at different times, and after that they are frozen on super-horizon scales. The amplitude of the fluctuation of a given mode k is roughly given by the Hubble parameter H at the corresponding horizon-crossing time. Since the Hubble parameter is slightly decreasing during slow-roll infation, the longest modes, which leave the horizon before the shortest ones, have a slightly higher amplitude of fluctuations. On the contrary, in an exact de Sitter phase the Hubble parameter is exactly constant, hence the amplitude of the fluctuations is the same for each mode. This leads to an Harrison-Zel'dovich power spectrum.

The scale-invariance of the power spectrum of perturbations in de Sitter is related to the symmetry of the de Sitter geometry under dilations. This is explained in Appendix A.

3.6.2 Power spectrum of tensor perturbations

By proceeding in the same way as for scalar perturbations, the mode function of tensor perturbations on super-horizon scales takes the form

$$u(k,\tau)|_{-k\tau\ll 1} = e^{i\frac{\pi}{2}(\nu+\frac{1}{2})}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu}. \tag{3.118}$$

By recalling the field redefinition (3.81), we can then obtain the power spectrum of tensor perturbations at leading order in the slow-roll parameters, which is given by

$$\Delta_T(k) \simeq \frac{16}{\pi}\frac{H^2}{M_{Pl}^2}\left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{3.119}$$

Therefore the spectral index of tensor perturbations is given by

$$n_T = -2\epsilon. \tag{3.120}$$

This means that slow-roll inflationary models predict an almost scale-invariant, *red* ($n_T < 0$) spectrum of primordial gravitational waves, analogously to what happens for scalar perturbations.

3.7 Observable predictions from inflation

As we have seen, the power spectra of scalar and tensor perturbations are respectively given by

$$\Delta_\zeta(k) = \left(\frac{H^2}{2\pi\dot\phi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}, \tag{3.121}$$

$$\Delta_T(k) = \frac{16H^2}{\pi M_{Pl}^2}\left(\frac{k}{aH}\right)^{3-2\nu_T}. \tag{3.122}$$

These can be rewritten in terms of two amplitudes and two spectral indices as

$$\Delta_\zeta(k) = \Delta_\zeta(k_*)\left(\frac{k}{k_*}\right)^{n_s-1}, \quad \Delta_\zeta(k_*) = \left(\frac{H^2}{2\pi\dot\phi}\right)^2\Big|_{t_H^{(1)}(k_*)}, \quad n_s - 1 = 2\eta_V - 6\epsilon_V, \tag{3.123}$$

$$\Delta_T(k) = \Delta_T(k_*)\left(\frac{k}{k_*}\right)^{n_T}, \quad \Delta_T(k_*) = \frac{16}{\pi}\frac{H^2}{M_{Pl}^2}\Big|_{t_H^{(1)}(k_*)}, \quad n_T = -2\epsilon_V, \tag{3.124}$$

where k_* is a fixed pivot scale. This means that in principle we have four observables: $\Delta_\zeta(k_*)$, $\Delta_T(k_*)$, n_s and n_T. However, one of the two amplitudes can be eliminated by measuring the overall normalization of the angular spectrum of the CMB. We can then define the *tensor-to-scalar ratio*

$$r(k_*) \equiv \frac{\Delta_T(k_*)}{\Delta_\zeta(k_*)}, \tag{3.125}$$

which is a measure of the amplitude of tensor perturbations with respect to that of scalar perturbations at the pivot scale k_*. The tensor-to-scalar ratio can be directly related to the slow-roll parameter ϵ. Indeed, we have

$$\begin{cases} H^2 = \frac{8\pi G}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right] \\ \ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \end{cases} \Rightarrow \dot{H} = -4\pi G\dot{\phi}^2 \Rightarrow \epsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G\frac{\dot{\phi}^2}{H^2}, \tag{3.126}$$

and hence, from the two amplitudes in (3.123) and (3.124), the tensor-to-scalar ratio can be written as

$$r = \frac{\Delta_T(k_*)}{\Delta_\zeta(k_*)} = \frac{16H^2}{\pi M_{Pl}^2} \bigg/ \frac{H^2}{\pi M_{Pl}^2 \epsilon} = 16\epsilon. \tag{3.127}$$

This relation tells us that single field slow-roll inflationary models produce gravitational waves with an amplitude much smaller than the amplitude of scalar perturbations. Moreover, using the expression of the tensor spectral index found in the previous section, $n_T = -2\epsilon$, one obtains the so-called *consistency relation*:

$$r = -8n_T. \tag{3.128}$$

This relates the amplitude of tensor perturbations to their spectral index. It is a difficult relation to check because it requires a measurement of the full tensor power spectrum, namely both its amplitude and its spectral index. At the moment we have only an upper bound on r, from measurements of the CMB temperature anisotropies by the Planck satellite combined with the measurements by BICEP [9]:

$$r < 0.064 \quad (95\% \text{ C.L.}). \tag{3.129}$$

Notice that, if the consistency relation is really true, it will be very difficult to measure the scale dependence of tensor perturbations, given the small upper bound we have on r.

If we assume the consistency relation to hold, we can remove one further observable, remaining with only two of them: n_s and r. Inflationary models are usually classified in the (r, n_s) plane (see [51] for more details), as shown in Fig. 3.1.

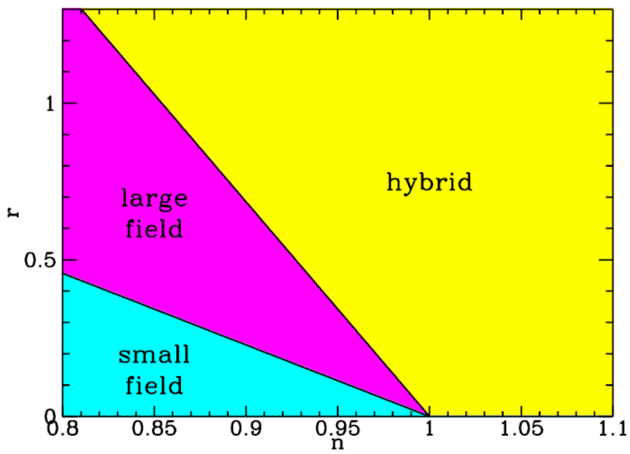

Figure 3.1: Inflationary models in the (r, n_s) plane (from [51]).

Depending on the values of the slow-roll parameters, we can include single field slow-roll models in the different classes we have introduced in the first chapter:

- **Large-field models**: $0 < \eta_V < 2\epsilon_V$. Typical examples are "Chaotic Inflation" [52] and "Power-law Inflation" [53]. These models have $V''(\phi) > 0$, which implies that $\eta_V > 0$.

- **Small-field models**: $\eta_V < 0$. These typically arise from spontaneous symmetry breaking, such as the original models of "new" inflation, or from Nambu-Goldstone modes, like "Natural Inflation" [54]. The condition $\eta_V < 0$ is due to the fact that small field models are characterized by $V''(\phi) < 0$.
- **Hybrid models**: $\eta_V > 2\epsilon_V$. These consist in models which incorporate inflation into supersymmetry and supergravity. In a typical hybrid inflationary model, the scalar field responsible for inflation evolves toward a minimum with non zero vacuum energy and the end of inflation arises as a result of the instability in a second field.

An important phenomenological (and also theoretical, as we will briefly comment in the next section) difference between small-field and large-field models is that the latter produce a higher tensor-to-scalar ratio, hence a higher amplitude of tensor perturbations. We will see the relation between the excursion of the field during inflation and r in the next section, but for the moment we can justify this assertion in the following way. From $r = 16\epsilon_V$ and $n_s - 1 = 2\eta_V - 6\epsilon_V$, one can easily find that

$$r = \frac{8}{3}(1 - n_s) + \frac{16}{3}\eta_V. \tag{3.130}$$

The first term is always positive, while the second one is negative for small-field models and positive for large-field ones, which thus predict higher values of r (see Fig. 3.1).

It is then important to stress that constraining our observables means constraining the values of the slow-roll parameters, and hence the shape of the potential. Ultimately, this means that we can constrain the different inflationary models. Fig. 3.2 shows the predictions of various slow-roll models for n_s and r, as well as the latest constraints from the measurements made by the Planck satellite.

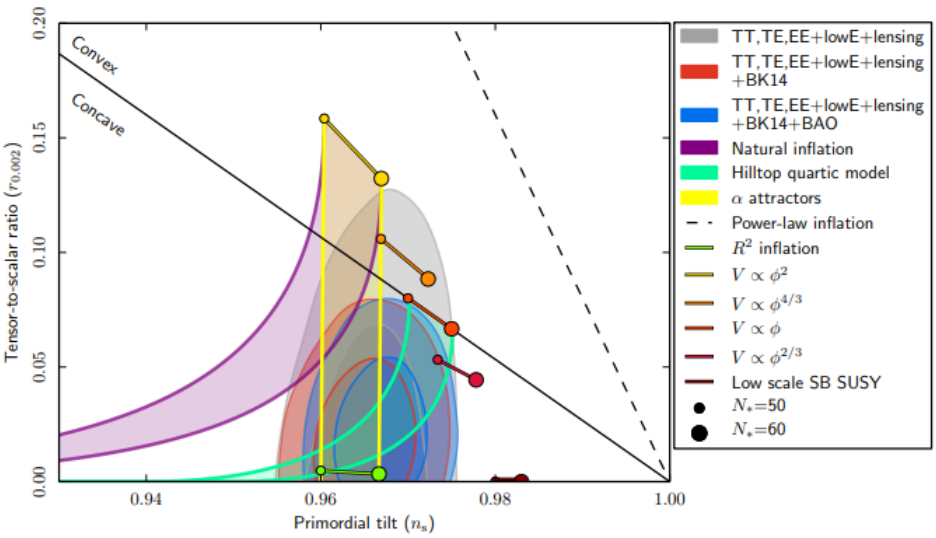

Figure 3.2: Constraints on inflationary models from Planck [9]

As regarding the scalar spectral index, from the latest Planck data release we have [9]

$$n_s = 0.9649 \pm 0.0042 \quad (68\% \text{ C.L.}). \tag{3.131}$$

This rules out the possibility of having a scale-invariant (Harrison-Zel'dovich) spectrum of scalar perturbations. As we have already discussed, this is consistent with slow-roll models.

The latest constraints on the slow-roll parameters are instead given by [9]

$$\epsilon_V < 0.0097 \quad (95\% \text{ C.L.}), \qquad \eta_V = -0.010^{+0.007}_{-0.011} \quad (68\% \text{ C.L.}). \tag{3.132}$$

These favour small-field models, which have concave potentials, $V''(\phi) < 0$, resulting in $\eta_V < 0$.

3.8 The importance of primordial gravitational waves

Primordial gravitational waves (PGWs) are a fundamental tool to study the physics of the early Universe and to constrain inflationary models. For a review of the subject, see [11]. PGWs are often said to be the "smoking gun" of inflation. Indeed, they are a key and general prediction of all inflationary models, while alternative models to inflation, which have been proposed in the latest years, tend to produce an extremely small amplitude of PGWs, practically unobservable. Thus, if observed, PGWs would represent a strong evidence supporting the inflationary paradigm.

An extremely important property of PGWs is that their amplitude is directly related to the energy scale of inflation, which instead cannot be determined uniquely from a measurement of the amplitude of scalar perturbations. Indeed the latter depends also on the slow-roll parameter ϵ_V, which depends on the specific model of inflation. A measurement of the amplitude of scalar perturbations would provide a measurement of H in terms of ϵ_V. For tensor perturbations instead we have

$$\begin{cases} \Delta_T = \frac{16H^2}{\pi M_{Pl}^2} \\ H^2 \simeq \frac{8}{3}\pi G V(\phi) \sim \frac{V(\phi)}{M_{Pl}^2} \end{cases} \implies \quad \Delta_T \propto V(\phi), \tag{3.133}$$

with $V(\phi)$ determining the energy scale of inflation, $E_{inf} = V^{1/4}$. To give a more precise relation, let us recall that the power spectrum of scalar perturbations is given by

$$\Delta_\zeta(k) = \frac{1}{2M_{Pl}^2\epsilon_V}\left(\frac{H_*}{2\pi}\right)^2\left(\frac{k}{aH_*}\right)^{n_s-1}. \tag{3.134}$$

This is related to the Hubble parameter evaluated during inflation when the pivot scale k_* leaves the horizon and to the slow-roll parameter ϵ_V. By using the Friedmann equation in the slow-roll approximation, $H^2 = V/3M_{Pl}^2$, we can relate the energy scale of inflation at the time when k_* leaves the horizon directly to ϵ_V [11]:

$$V_* = 24\pi^2 M_{Pl}^4 \Delta_\zeta \epsilon_V. \tag{3.135}$$

Using the relation (3.127) between r and ϵ_V, this becomes

$$V_* = \frac{3}{2}\pi^2 \Delta_\zeta M_{Pl}^4\, r. \tag{3.136}$$

Then, from the amplitude of scalar perturbations obtained by the Planck satellite [55], one gets the following relation between the energy scale of inflation at the time when k_* leaves the horizon and the tensor-to-scalar ratio [11]:

$$V_* = \left(1.88 \cdot 10^{16}\,\text{GeV}\right)^4 \frac{r}{0.10}. \tag{3.137}$$

With the bound (3.129) on the tensor-to-scalar ratio, one can obtain an upper bound on V_* [9]:

$$V_* < (1.7 \cdot 10^{16}\ \text{GeV})^4 \quad (95\%\ \text{C.L.}). \tag{3.138}$$

Equivalently, this implies a bound on the Hubble parameter during inflation [9]:

$$\frac{H_*}{M_{Pl}} < 2.7 \cdot 10^{-5} \quad (95\%\ \text{C.L.}). \tag{3.139}$$

Another important property of primordial gravitational waves is that there is a relation between their amplitude and the excursion of the scalar field, $\Delta\phi$, during inflation. The latter is given by

$$\Delta\phi = \int_{\phi(t_*)}^{\phi(t_f)} d\phi = \int_{t_*}^{t_f} \dot{\phi}\, dt \simeq \frac{\dot{\phi}}{H}\int_{t_*}^{t_f} H dt, \tag{3.140}$$

where $t_* = t_H^{(1)}(k_*)$ is the time when the pivot scale crosses the horizon, t_f is the time corresponding to the end of inflation and we have taken $\dot{\phi}/H$ to be constant. If we take as a pivot scale the one relevant for the CMB, then

$$\int_{t_{CMB}}^{t_f} H dt \equiv N_{\rm CMB} \simeq 60. \tag{3.141}$$

This corresponds to the observable window (see again Fig. 1.14). As can be seen from (3.126), $\dot{\phi}/H = \sqrt{2} M_{Pl}(\epsilon_V)^{1/2}$, and thus, from (3.127) and (3.140), we can find the so-called *Lyth bound*:

$$\left(\frac{r}{0.01}\right)^{1/2} \simeq \frac{\Delta\phi}{M_{Pl}}. \tag{3.142}$$

This explicitly relates the tensor-to-scalar ratio to the excursion of the inflaton field, and tells us that:

Large-field models: $\Delta\phi/M_{Pl} > 1 \;\Rightarrow\; r > 0.01$,

Small-field models: $\Delta\phi/M_{Pl} < 1 \;\Rightarrow\; r < 0.01$.

As we have already seen in the previous section, large-field models produce a higher value of the tensor-to-scalar ratio with respect to small-field ones, and hence a higher value of the amplitude of tensor perturbations.

Knowing if the excursion of the inflaton field was sub-Planckian or super-Planckian is extremely important also from the point of view of fundamental physics. Indeed, the excursion of the inflaton is sensitive to the symmetry properties of the UV completion of the theory of gravity, and thus its measurement (via the measurement of r) could provide the first experimental hint about the nature of quantum gravity [35].

3.9 Adiabatic and isocurvature perturbations

As we have seen in the first section of this chapter, the fluctuations of the inflaton field on large scales can be identified with a local shift backwards or forwards along the trajectory of the homogeneous background field. This affects the total energy density in different parts of the Universe after inflation, but cannot give rise to variations in the relative density between different components. Hence, single field slow-roll inflationary models produce purely adiabatic primordial density perturbations, characterized by an overall curvature perturbation. This means that all the perturbations in the different components of the Universe (photons, neutrinos, baryons and cold dark matter) originate from the same curvature perturbation and satisfy the adiabaticity property, which can be defined as follows [56]. Let us consider a given species, labelled by the subscript X, and define the perturbation in its number and energy densities as

$$\delta n_X = \bar{n}_X - n_X, \quad \delta\rho_X = \bar{\rho}_X - \rho_X, \tag{3.143}$$

where the bar refers to the unperturbed background value of the corresponding quantity. *Adiabatic perturbations* are defined as perturbations affecting all the cosmological species such that the relative ratios in the number densities remain unperturbed:

$$\delta\left(\frac{n_X}{n_Y}\right) = 0, \tag{3.144}$$

for any pair of species X and Y. Since these correspond to a global perturbation of the energy content, they are related to a perturbation in the intrinsic spatial curvature through the Einstein's equations. If we define

$$\delta_X \equiv \frac{\delta\rho_X}{\rho_X}, \tag{3.145}$$

the condition (3.144) can be rewritten as

$$\frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu = \frac{1}{3}\delta_b = \frac{1}{3}\delta_{cdm}, \tag{3.146}$$

where the different numerical coefficients arise from the equation of state for the corresponding species, and where the subsrcipt *cdm* labels cold dark matter.

It is also possible to perturb the different components without perturbing the underlying geometry. These are known as *isocurvature perturbations*[7] and are characterized by variations in the particle number ratios but with vanishing curvature perturbation. The variation in the relative particle number densities between two different species X and Y can be quantified by introducing the so-called *entropy perturbation*

$$S_{X,Y} = \frac{\delta n_X}{n_X} - \frac{\delta n_Y}{n_Y}, \tag{3.147}$$

which, by exploiting the equation of state $p = w\rho$ for the two species, can be written as

$$S_{X,Y} = \frac{\delta_X}{1+w_X} - \frac{\delta_Y}{1+w_Y}. \tag{3.148}$$

One can also define the entropy perturbations relatively to a given species. Choosing photons as a species of reference, we have

$$S_b \equiv \delta_b - \frac{3}{4}\delta_\gamma, \quad S_{cdm} \equiv \delta_{cdm} - \frac{3}{4}\delta_\gamma, \quad S_\nu \equiv \frac{3}{4}\delta_\nu - \frac{3}{4}\delta_\gamma. \tag{3.149}$$

These are respectively the *baryon, cold dark matter* and *neutrino isocurvature modes.* In terms of these entropy perturbations the adiabatic mode is characterized by $S_b = S_{cdm} = S_\nu = 0$.

One can decompose a general perturbation into adiabatic and isocurvature modes. However, this decomposition is not time-invariant since, e.g., one can have a primordial pure isocurvature mode which then generates an adiabatic perturbation. This happens if the energy densities of the various species evolve differently, so that the balance that provided an unperturbed total energy density at the beginning is then lost.

The best way to study the isocurvature and adiabatic behaviour of primordial perturbations is through the CMB, since the two different types of perturbations produce distinctive signatures on the CMB anisotropies [56]. Whereas an adiabatic initial perturbation generates a cosine oscillatory mode in the photon-baryon fluid, leading to an acoustic peak at $l \simeq 220$ (in the case of a flat universe) in the angular power spectrum of the CMB, a pure isocurvature initial perturbation generates a sine oscillatory mode, which leads to a first peak at $l \simeq 330$. From the observation of the first peak at $l \simeq 220$ we know that the dominant component must be adiabatic, even though subdominant isocurvature contributions may be present.

[7] Isocurvature perturbations are tipically generated in inflationary models with multiple fields.

Chapter 4

Non-Gaussianity from inflation

4.1 Gaussian random fields

The time dependence of each perturbation is determined by the laws describing the dynamical evolution of our physical system, as we have seen in the previous chapter. However, if we look at the perturbations as a function of position at a fixed time, as it is the case when we make cosmological observations, they have random distributions. Therefore, through observations we aim to obtain information about the statistical properties of these distributions, while from a theoretical point of view we want to relate them to a model which describes the origin of the perturbations themselves. Dealing with statistical properties means that, e.g., rather than trying to predict things like the precise location of a particular galaxy we should care about the typical distance between galaxies.

To describe these stochastic properties we introduce the concept of a *random field* (see [57]). Let us consider a generic perturbation at a fixed time, $\phi(\mathbf{x})$. Having a random field means that there is a set of functions $\phi_n(\mathbf{x})$, each with an associated probability P_n. This set of functions is called the *ensemble*, and each individual function is a *realization* of the ensemble. In practical applications we have a continuous set of functions and a continuous probability distribution.

We can expand the field $\phi(\mathbf{x})$ in the Fourier space as

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.1}$$

and parametrize the Fourier coefficients with two real functions

$$\phi(\mathbf{k}) = a_{\mathbf{k}} + ib_{\mathbf{k}}, \tag{4.2}$$

with amplitude $|\phi(\mathbf{k})| = \sqrt{a_{\mathbf{k}}^2 + b_{\mathbf{k}}^2}$. Reality of $\phi(\mathbf{x})$ implies that $\phi^*(\mathbf{k}) = \phi(-\mathbf{k})$, hence $a_{\mathbf{k}} = a_{-\mathbf{k}}$ and $b_{\mathbf{k}} = -b_{-\mathbf{k}}$. This means that any given field configuration $\phi(\mathbf{x})$ in the real space is fully described by a set of real numbers $(a_{\mathbf{k}}, b_{\mathbf{k}})$. In order to determine $\phi(\mathbf{x})$ is sufficient to assign a probability distribution function (PDF) for the set $(a_{\mathbf{k}}, b_{\mathbf{k}})$.

Let us now focus on a single mode $\mathbf{k}$. A *Gaussian random field* is such that the PDF is [58]

$$P(a_{\mathbf{k}}, b_{\mathbf{k}}) = \frac{1}{\pi\sigma_k^2}\exp\left(-\frac{a_{\mathbf{k}}^2 + b_{\mathbf{k}}^2}{\sigma_k^2}\right). \tag{4.3}$$

This distribution, which has zero mean, is normalized such that

$$\int_{-\infty}^{\infty} da_{\mathbf{k}} \int_{-\infty}^{\infty} db_{\mathbf{k}}\, \frac{1}{\pi\sigma_k^2}\exp\left(-\frac{a_{\mathbf{k}}^2 + b_{\mathbf{k}}^2}{\sigma_k^2}\right) = 1. \tag{4.4}$$

Notice that the integral is over all the possible values of $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ for the mode $\mathbf{k}$.

In order to generalize what we have just seen to all the modes, we define the PDF via the functional

$$P\left[\phi(\mathbf{k})\right] = \frac{1}{\pi\sigma_k^2}\exp\left(-\frac{|\phi(\mathbf{k})|^2}{\sigma_k^2}\right). \tag{4.5}$$

This just tells us that all the fields $\phi(\mathbf{k})$ are drawn from a distribution that obeys (4.3). This also means that in the case of a Gaussian random field there is no correlation between the different modes, since each of them has an independent probability distribution. Thanks to the central limit theorem, this implies that the PDF of $\phi(\mathbf{x})$ at any given point is Gaussian[1]. Notice also that, in accordance with the cosmological principle, we have assumed that the variance σ_k depends only on the modulus of $\mathbf{k}$ (statistical isotropy).

Coming back to what we have said at the beginning of this section, in order to compare observations with theoretical predictions we have to consider the expectation value of some observable Q, namely we have to average it over the ensemble of all the possible realizations. Here come into play two fundamental concepts in cosmology, namely the *fair sample* assumption and the *Ergodic hypothesis*. Since we have only one Universe to observe, which is just one realization of the ensemble, we assume that samples from well separated part of the Universe are independent realizations of the same physical phenomenon, and that, in the observable part of the Universe, there are enough independent samples to be representative of the statistical ensemble. Then the Ergodic hypothesis says that averaging over the different realizations of the ensemble is equivalent to averaging over a large enough volume of the Universe [59].

The expectation value of a generic observable $Q\left[\phi(\mathbf{k})\right]$, functional of the field $\phi(\mathbf{k})$, is given by

$$\langle Q\left[\phi(\mathbf{k})\right]\rangle = \int \mathcal{D}\phi\, Q\left[\phi(\mathbf{k})\right] P\left[\phi(\mathbf{k})\right], \tag{4.6}$$

where the integral is over all the possible field configurations in the $\mathbf{k}$-space. In the case of a Gaussian PDF, this becomes

$$\langle Q\left[\phi(\mathbf{k})\right]\rangle = \prod_{\mathbf{k}} \int da_{\mathbf{k}} \int db_{\mathbf{k}} Q\left[\phi(\mathbf{k})\right] \frac{1}{\pi\sigma_k^2}\exp\left(-\frac{a_{\mathbf{k}}^2 + b_{\mathbf{k}}^2}{\sigma_k^2}\right), \tag{4.7}$$

since $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ parametrize the different configurations.

Let us consider the simple example $Q\left[\phi(\mathbf{k})\right] = a_{\mathbf{q}}a_{\mathbf{q}'}$:

$$\langle a_{\mathbf{q}}a_{\mathbf{q}'}\rangle = \prod_{\mathbf{k}} \int da_{\mathbf{k}} \int db_{\mathbf{k}}\, a_{\mathbf{q}}a_{\mathbf{q}'}\, \frac{1}{\pi\sigma_k^2}\exp\left(-\frac{a_{\mathbf{k}}^2 + b_{\mathbf{k}}^2}{\sigma_k^2}\right) = \frac{\sigma_q}{2}\delta^{(3)}(\mathbf{q}+\mathbf{q}'). \tag{4.8}$$

This comes from the fact that, since the Gaussian distribution is even under both $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$, all odd products of a and b vanish; we have also used the condition $a_{\mathbf{k}} = a_{-\mathbf{k}}$. Having obtained this result, we can then compute

$$\langle \phi(\mathbf{q})\phi(\mathbf{q}')\rangle = \langle a_{\mathbf{q}}a_{\mathbf{q}'}\rangle - \langle b_{\mathbf{q}}b_{\mathbf{q}'}\rangle = \sigma_q^2\delta^{(3)}(\mathbf{q}+\mathbf{q}'), \tag{4.9}$$

since the cross terms (which are odd products of a and b) vanish, and $b_{\mathbf{k}} = -b_{-\mathbf{k}}$. This tells us that the Fourier transform of the two-pont function is given by the variance of the Gaussian distribution. This is a well known result; indeed, if we repeat a similar calculation in the configuration space, we find

$$\begin{aligned}
\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\, e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}\left(\langle a_{\mathbf{k}_1}a_{\mathbf{k}_2}\rangle - \langle b_{\mathbf{k}_1}b_{\mathbf{k}_2}\rangle\right) \\
&= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\, e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}\sigma_{k_1}^2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2) \\
&= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\frac{\sigma_k^2}{(2\pi)^3}.
\end{aligned} \tag{4.10}$$

[1] Equivalently, the Fourier transform of a Gaussian is still a Gaussian.

This is nothing else than the two-point correlation function in real space, which we have seen in Eq. (3.99), with $P(k) \equiv \sigma_k^2/(2\pi)^3$. Thus, σ_k is a measure of the amplitude of the correlations at a given mode k.

Since the Gaussian PDF is even around the mean 0, any expectation value of an odd product of fields vanishes, while any expectation value of an even product of fields can be written in terms of two-point functions. Thus, for example, we have:

$$\langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\rangle = 0, \tag{4.11}$$

$$\langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\phi(\mathbf{k}_4)\rangle = \sigma_{k_1}^2\sigma_{k_3}^2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4) + 1 \leftrightarrow 3 + 1 \leftrightarrow 4. \tag{4.12}$$

This is an example of the *Wick's Theorem*, which we will encounter again in a while. A Gaussian distribution, as one can see from its definition, is fully characterized by its mean and its variance, so these results are expected.

However, if $\phi(\mathbf{x})$ is *non-Gaussian*, then the odd correlation functions do not vanish. In this case all the higher-order correlation functions are required to fully characterize the distrbution. The lowest oder of non-Gaussianity is given by the 3-point correlation function or, equivalently, by its Fourier transform, the *bispectrum*.

Before proceeding on characterizing non-Gaussianity, we want to stress that Gaussianity is not a statement about two-point correlation functions. Indeed, let us consider the following example [58]: if we take a Gaussian CMB map and rearrange the cold and hot spots in such a way to create a clear distinguishable picture, then the map will be highly correlated in a specific way, but it will still be completely Gaussian. In other words, what we have changed after the rearrangement of the spots is the power spectrum, not the underlying distribution $P[\phi(\mathbf{k})]$.

4.2 Non-Gaussian statistics

4.2.1 Why studying non-Gaussianity?

Non-Gaussianity is an important tool to understand and constrain the physics of the early Universe (see [60]). In order to make progress in our comprehension of inflation, we need to understand how the quantum fields evolve and interact in the early stages of the Universe. As we have seen in the previous chapters, the power spectrum is determined by the inflationary expansion rate and its time dependence, hence it is related to the evolution of the energy density which drives inflation. However, the power spectrum does not allow us to strongly constrain the interactions of the field (or fields) associated with this energy density, being also strongly constrained by the symmetries of the underlying background geometry (e.g., the scale invariance of the power spectrum in the de Sitter case is due to the symmetry of the de Sitter geometry under dilations). This means that models of inflation with different field interactions can lead to very similar predictions for the power spectrum. Non-Gaussianity can help us to break this degeneracy, since it is sensitive to the interactions between the field(s) driving inflation. Moreover, as we will see, different classes of inflationary models lead to different and specific imprints in the shape of non-Gaussianity. For these reasons, the role non-Gaussianity plays for the early Universe is similar to the role colliders play for particle physics.

Another point we want to mention is that some potential scenarios which are studied as alternatives to inflation tend to generate large non-Gaussianity (see e.g. [61]). A null or strongly constrained detection of non-Gaussianity would then rule out all of these models.

4.2.2 Primordial bispectrum

The leading non-Gaussian signature is the three-point correlation function, or its Fourier transform, the bispectrum:

$$\langle \zeta(\mathbf{k}_1), \zeta(\mathbf{k}_2), \zeta(\mathbf{k}_3)\rangle \equiv B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \tag{4.13}$$

If we consider perturbations around a FLRW background, the momentum dependence of the bispectrum is remarkably simplified by the symmetries of the underlying geometry. Because of homogeneity (invariance under translations), the bispectrum is proportional to a delta function of the sum of the momenta; in other words, the three momenta form a closed triangle in the Fourier space. Because of isotropy (invariance under rotations), the bispectrum depends only on the magnitude of the momenta. Thus, we have

$$B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_\zeta(k_1, k_2, k_3). \tag{4.14}$$

This implies that the bispectrum depends on three parameters only. This fact can be understood also in the following way. In principle the bispectrum is a function of the three momenta $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$, for a total of 9 parameters. The delta function due to homogeneity implies that one of the three momenta is fixed by the other two, leaving us with 6 parameters. Rotational invariance allows us to rotate the axes in the two dimensional plane defined by the remaining two momenta, such that one of them is along one of the axes. This fixes other 3 parameters, hence we are left with a total of 3 parameters. These can be chosen as two of the angles of the triangle formed by the three momenta and the overall scale of this triangle. If the power spectrum is scale invariant, as it is approximately the case in slow-roll inflationary models, the bispectrum is a homogeneous function of degree -6, namely

$$B_\zeta(\lambda k_1, \lambda k_2, \lambda k_3) = \lambda^{-6} B_\zeta(k_1, k_2, k_3). \tag{4.15}$$

This fixes how the bispectrum depends on the overall scale of the triangle, leaving only the two angles as free parameters. Usually, one uses as the two parameters the ratios between the magnitudes of the comoving momenta, as $x_2 \equiv k_2/k_1$ and $x_3 \equiv k_3/k_1$. In this case, a scale invariant power spectrum leads to

$$B_\zeta(k_1, k_2, k_3) = k_1^{-6} B_\zeta(1, x_2, x_3). \tag{4.16}$$

More in general, the bispectrum can be parametrized as [62]

$$B_\zeta(k_1, k_2, k_3) = \frac{S(k_1, k_2, k_3)}{(k_1 k_2 k_3)^2}\, \Delta_\zeta^2(k_*), \tag{4.17}$$

where $\Delta_\zeta^2(k_*) = k_*^3 P_\zeta(k_*)$ is the dimensionless power spectrum at a pivot scale k_*. The function S is dimensionless and, for scale-invariant bispectra, is invariant under rescaling of all the three momenta. There are two types of dependence of S on momentum:

- the *shape* of the bispectrum is the dependence of S on the momentum ratios k_2/k_1 and k_3/k_2, while keeping fixed the overall momentum $K \equiv \frac{1}{3}(k_1 + k_2 + k_3)$;
- the *running* of the bispectrum is the dependence of S on the overall momentum K, while keeping fixed the momentum ratios k_2/k_1 and k_3/k_2.

We can then define the *amplitude* of non-Gaussianity as the size of the bispectrum in the equilateral configuration[2]

$$f_{\rm NL}(K) = \frac{5}{18} S(K, K, K). \tag{4.18}$$

For scale invariant bispectra $f_{\rm NL}$ is constant, and we can extract it from the shape function. The bispectrum can then be rewritten as

$$B_\zeta(k_1, k_2, k_3) = \frac{18}{5} f_{\rm NL} \frac{S(k_1, k_2, k_3)}{(k_1 k_2 k_3)^2} \Delta_\zeta^2(k_*), \tag{4.19}$$

having also normalized the shape function as $S(K, K, K) = 1$.

[2]The equilateral configuration correponds to the case in which the three momenta are equal (in absolute value).

4.2.3 Shapes of non-Gaussianity

The shape of non-Gaussianity is an important tool to distinguish among different models of inflation. Indeed, different inflationary models lead to different and specifc signatures in the shape function S. The latter, which depends on two parameters only, contains however a lot of information about the source of non-Gaussianity. In the following subsections we give a brief overview of the main shape functions, and also shortly characterize the various models of inflation depending on the shape where they give the maximum contribution. For more details on this, see [62–66].

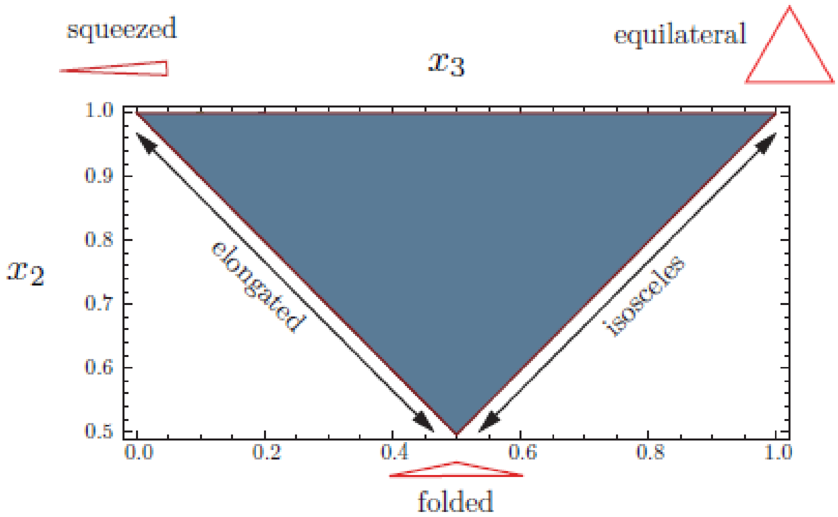

Figure 4.1: Momentum configurations of the bispectrum, with $x_2 \equiv k_2/k_1$ and $x_3 \equiv k_3/k_1$ (from [62]).

Local shape

One of the first ways introduced to parametrize non-Gaussianity was through a non-linear correction to a Gaussian perturbation:

$$\zeta(\mathbf{x}) = \zeta_G(\mathbf{x}) + \frac{3}{5} f_{\rm NL}^{loc} \left[\zeta_G(\mathbf{x})^2 - \langle \zeta_G(\mathbf{x})^2 \rangle\right] , \tag{4.20}$$

where the 3/5 factor is due the fact that (4.20) was initially defined for the Bardeen gravitational potential Φ [67]:

$$\Phi(\mathbf{x}) = \Phi_G(\mathbf{x}) + f_{\rm NL}^{loc} \left[\Phi_G(\mathbf{x})^2 - \langle \Phi_G(\mathbf{x})^2 \rangle\right] . \tag{4.21}$$

During the matter dominated era this is related to ζ by a factor of 3/5, $\Phi = \frac{3}{5}\zeta$. In practice, we split the field into its linear Gaussian part and its non-linear part, which is the square of the former minus its variance. This definition is local in real space, hence the name *local non-Gaussianity.*

The local ansatz has the following physical meaning: we are assuming that non-Gaussianities are generated independently at different spatial points. In the context of inflation, typically this implies that non-Gaussianity is generated on super-Hubble scales. This happens for all the models in which the fluctuations of an additional field contribute to the curvature perturbations we observe. In this case non-linearities come from the evolution of this field outside the horizon and from the conversion mechanism which transforms the fluctuations of this field into density perturbations [64]. Indeed, as we have seen, we need $\epsilon \ll 1$, $|\eta| \ll 1$ in order to have a sufficiently long period of inflation, and this implies that self-interaction terms in the inflaton potential (which are proportional to higher derivatives of $V(\phi)$) and the gravitational coupling must be very small. Hence, non-linearities are suppressed too. However, if there is another scalar field whose contribution to the total energy density is negligible, its self-interactions are not constrained by any slow-roll conditions, and large non-Gaussianities can be generated. If then this second scalar field is coupled to the inflaton field, it is possible that non-Gaussianities are transferred to the latter.

In position space, we can characterize this non-Gaussianity by the probability distribution function of ζ, which is plotted in Fig. 4.2. The tail at large and positive ζ indicates where structures are formed in the Universe. Thus, positive $f_{\rm NL}^{loc}$ means more galaxies are formed given a fixed power spectrum. For the CMB map, positive $f_{\rm NL}^{loc}$ means that there are more very cold spots than very hot spots, and more modestly hot spots than modestly cold spots [34].

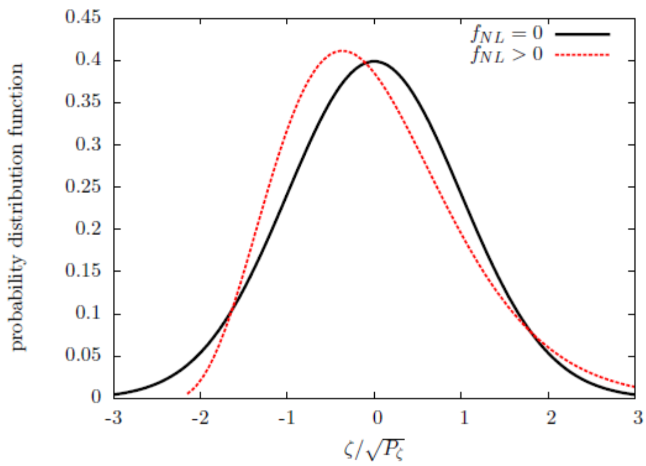

Figure 4.2: Probability distribution of ζ for different f_{NL}^{loc} parameters (from [34]).

In momentum space, the bispectrum of local non-Gaussianity is[3]

$$\begin{aligned} B_\zeta(k_1,k_2,k_3) &= \frac{6}{5} f_{\rm NL}^{loc}\left[P_\zeta(k_1)P_\zeta(k_2)+P_\zeta(k_2)P_\zeta(k_3)+P_\zeta(k_3)P_\zeta(k_1)\right] \\ &= \frac{6}{5} f_{\rm NL}^{loc}\frac{\Delta_\zeta^2}{(k_1k_2k_3)^2}\left(\frac{k_1^2}{k_2k_3}+\frac{k_2^2}{k_1k_3}+\frac{k_3^2}{k_1k_2}\right), \end{aligned} \tag{4.22}$$

where, to obtain the second line, we have assumed a scale-invariant power spectrum, $P_\zeta(k) = \Delta_\zeta k^{-3}$. Then, the local shape function is

$$S_{loc}(k_1,k_2,k_3) = \frac{1}{3}\left(\frac{k_3^2}{k_1k_2} + 2 \text{ perms.}\right). \tag{4.23}$$

Let us now assume, without loss of generality, that $k_1 \le k_2 \le k_3$. The bispectrum for local non-Gaussianity is largest when $k_1 \ll k_2 \sim k_3$, where the momenta k_2 and k_3 are nearly equal due to momentum conservation. This is the so-called *squeezed limit* (see Fig. 4.1). Physically, when one of the momenta is much smaller than the other two, the longest mode (which is the one with the smallest momentum) is already super-horizon, and hence is "frozen", when the other two modes cross the horizon. Thus we can consider this constant mode as a shift of the background, which has the effect of antcipating the horizon exit of the short modes, and the three point function reduces to the background shift of the two point function. From (4.23), the local shape function in the squeezed limit is

$$\lim_{k_1\ll k_2\sim k_2} S_{loc}(k_1,k_2,k_3) = \frac{2}{3}\frac{k_2}{k_1}. \tag{4.24}$$

[3]The 6/5 factor is responsible for the 5/18 in the definition of $f_{\rm NL}$ given in (4.18).

As we show in Appendix B, there is a consistency relation for the bispectrum of scalar pertubations in the squeezed limit for single-field models (having a single dynamical field is the only assumption made in deriving this result). This tells us that the bispectrum is very suppressed in this configuration, hence a convincing detection of non-Gaussianity in the squeezed limit would rule out all classes of single field inflationary models [68].

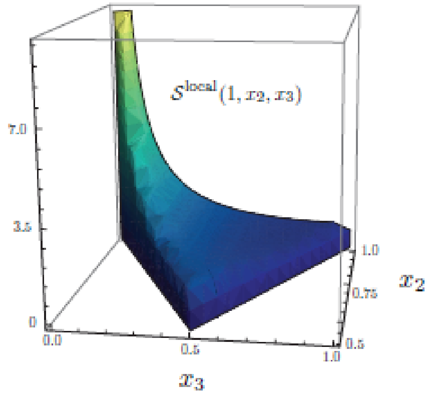

Figure 4.3: Local shape (from [62])

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Equilateral shape

In the case of the *equilateral shape* the shape function peaks at the equilateral triangle limit, $k_1 = k_2 = k_3$. The template for the shape function used in the CMB analysis is [62]

$$S_{equil}(k_1, k_2, k_3) = \left(\frac{k_1}{k_2} + 5 \text{ perms.}\right) - \left(\frac{k_1^2}{k_2 k_3} + 2 \text{ perms.}\right) - 2. \tag{4.25}$$

Physically this means that the correlation is among modes with comparable wavelengths, which go out of the horizon nearly at the same time. The equilateral shape is typically generated when considering theories with higher-derivative corrections. This happens because derivative interactions are suppressed when any individual mode is far outside the horizon, because both time and spatial derivatives become small. This implies that the bispectrum is maximal when all the three modes have wavelenghts approximately equal to the horizon size.

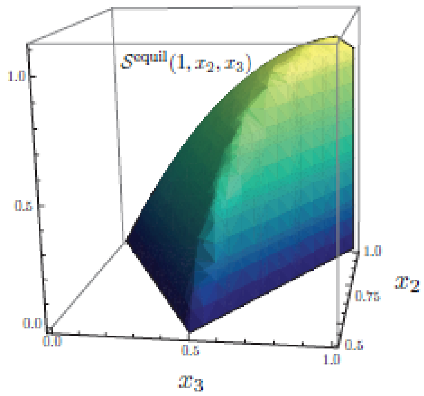

Figure 4.4: Equilateral shape (from [62])

Folded shape

The *folded shape* has the maximum signal at the flattened configuration, $k_1 \sim 2k_2 \sim 2k_3$, and typically arises when one consider a non-Bunch-Davies vacuum state as initial state, i.e. when the modes are deep inside the horizon. Without going into details we briefly explain why this happens, for more details see [63] and [69]. The usual mode function of the Bunch-Davies vacuum has the positive energy mode $\sim e^{-ik\tau}$. Let us consider a non-Bunch-Davies vacuum by adding a small component of the negative energy mode, which is instead proportional to $e^{ik\tau}$. The three point function, as we will see, involves an integration of the product of three mode functions with momenta k_1, k_2 and k_3. Hence, the leading correction to the Bunch-Davies results is to replace one of the momentum, say k_3, with $-k_3$. The usual term $k_T = k_1 + k_2 + k_3$ present in $e^{-ik_T\tau}$ becomes $\tilde{k} = k_1 + k_2 - k_3$ (and its cyclic permutations). After the integration over τ this term appears in the denominator as $\tilde{k}^{-3}$, and the effect is to enhance non-Gaussianity in the folded triangle limit, $\tilde{k} = 0$. An ansatz for this shape has been proposed in [69]:

$$S_{fold}(k_1, k_2, k_3) = 6\left(\frac{k_1^2}{k_2 k_3} + 2 \text{ perms.}\right) - 6\left(\frac{k_1}{k_2} + 5 \text{ perms.}\right) + 18. \tag{4.26}$$

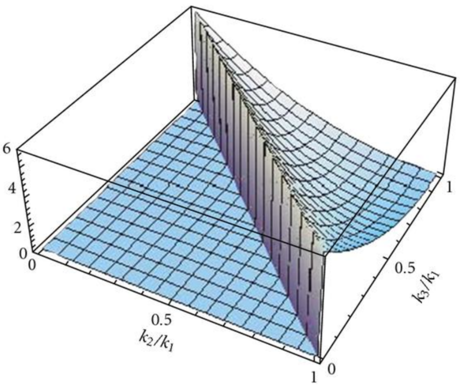

Figure 4.5: Folded shape (from [63])

Orthogonal shape

The *orthogonal shape* is a phenomenological shape that is orthogonal to both the local and equilateral templates, and which arises in in inflationary models with higher-derivative interactions. A template for this shape function is [62]

$$S_{ortho}(k_1, k_2, k_3) = -3.84\left(\frac{k_1^2}{k_2 k_3} + 2 \text{ perms.}\right) + 3.94\left(\frac{k_1}{k_2} + 5 \text{ perms.}\right) - 11.10. \tag{4.27}$$

4.2.4 Experimental constraints

From the measurements of the CMB temperature anisotropies made by the Planck satellite, we have the following constraints on the local, equilateral, and orthogonal bispectrum amplitudes [55]:

$$\begin{aligned} f_{\rm NL}^{loc} &= 2.5 \pm 5.7 \quad [68\% \text{ C.L.}], \\ f_{\rm NL}^{equil} &= -16 \pm 70 \quad [68\% \text{ C.L.}], \\ f_{\rm NL}^{ortho} &= -34 \pm 33 \quad [68\% \text{ C.L.}]. \end{aligned} \tag{4.28}$$

Combining temperature and polarization data, the constraints are [55]:

$$\begin{aligned} f_{\rm NL}^{loc} &= 0.8 \pm 5.0 \quad [68\%\ \text{C.L.}], \\ f_{\rm NL}^{equil} &= -4 \pm 43 \quad [68\%\ \text{C.L.}], \\ f_{\rm NL}^{ortho} &= -26 \pm 21 \quad [68\%\ \text{C.L.}]. \end{aligned} \tag{4.29}$$

Thus, the Planck results on primordial non-Gaussianity are consistent with single field slow-roll inflationary models, which, as we will see, predict almost Gaussian primordial fluctuations, with a tiny deviation from Gaussianity.

4.3 In-in formalism

The *in-in formalism* is used to compute n-point correlation functions of cosmological perturbations [58,63,70]. This issue is different from the corresponding analysis of QFT. In the latter case, indeed, we are interested in calculating S-matrix elements describing the transition probability from a state in the far past to a state in the far future, thus we impose asymptotic conditions at both very early and very late times. These asymptotic states are assumed to correspond to free (i.e., non interacting) fields, namely they are eigenstates of the free Hamiltonian, since the particles involved in the process are assumed to be faraway from the interaction region.

In cosmology we are instead interested in evaluating expectation values of product of fields at a fixed time. Hence, boundary conditions are not imposed on the fields at both very early and very late times, but only at very early times, when the wavelength is well below the horizon and, according to the Equivalence Principle, the interaction picture fields should have the same form as in Minkowski spacetime.

Let us consider a generic system described by an action

$$S = \int d^4x\, \mathcal{L}(\phi_a(\mathbf{x},t), \dot{\phi}_a(\mathbf{x},t)) = \int dt\, L, \tag{4.30}$$

where a ranges over all the fields in the theory (e.g. the metric, matter scalar fields, etc.). The conjugate momenta are defined by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \tag{4.31}$$

and the Hamiltonian is

$$H\left[\phi_a(t), \pi_a(t)\right] = \int d^3x\, \dot{\phi}_a \pi_a - L. \tag{4.32}$$

The fields satisfy the equal-time commutation relations

$$[\phi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)] = i\delta_{ab}\, \delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi_a(\mathbf{x},t), \phi_b(\mathbf{y},t)] = 0, \quad [\pi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)] = 0, \tag{4.33}$$

and the Heisenberg equations of motion are given by

$$\begin{cases} \dot{\phi}_a(\mathbf{x},t) = i\left[H\left[\phi(t),\pi(t)\right], \phi_a(\mathbf{x},t)\right], \\ \dot{\pi}_a(\mathbf{x},t) = i\left[H\left[\phi(t),\pi(t)\right], \pi_a(\mathbf{x},t)\right]. \end{cases} \tag{4.34}$$

Now we split the fields into a time-dependent "classical" background and spacetime-dependent "quantum fluctuations":

$$\phi_a(\mathbf{x},t) = \bar{\phi}_a(t) + \delta\phi_a(\mathbf{x},t), \quad \pi_a(\mathbf{x},t) = \bar{\pi}_a(t) + \delta\pi_a(\mathbf{x},t), \tag{4.35}$$

where the classical parts obey the classical equations of motion

$$\dot{\bar{\phi}}_a(\mathbf{x},t) = \frac{\delta H\left[\bar{\phi}(t), \bar{\pi}(t)\right]}{\delta \bar{\pi}_a(\mathbf{x},t)}, \quad \dot{\bar{\pi}}_a(\mathbf{x},t) = -\frac{\delta H\left[\bar{\phi}(t), \bar{\pi}(t)\right]}{\delta \bar{\phi}_a(\mathbf{x},t)}. \tag{4.36}$$

Notice that what we are doing in eq. (4.36) is varying the Hamiltonian density, namely

$$\dot{\bar{\phi}}_a(\mathbf{x},t) = \int d^3y \, \frac{\delta\mathcal{H}\left[\bar{\phi}(\mathbf{y},t),\bar{\pi}(\mathbf{y},t)\right]}{\delta\bar{\pi}(\mathbf{y},t)}\, \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{4.37}$$

Since the classical parts are just c-number valued functions, their commutator with everything vanishes, hence the perturbations satisfy the same commutation relations as the total variables

$$[\delta\phi_a(\mathbf{x},t), \delta\pi_b(\mathbf{y},t)] = i\delta_{ab}\, \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{4.38}$$

Thus, we can think of the system as a theory of quantized perturbations living on the classical (time-dependent) background. In practice, we are assuming that the background fields are sitting in their "vacuum expectation values", $\langle\phi_a(\mathbf{x},t)\rangle = \phi_a(t)$, and then we consider fluctuations around them.

The next step is to expand the Hamiltonian around the classical background as

$$\begin{aligned} H\left[\phi(t),\pi(t)\right] =& H\left[\bar{\phi}(t),\bar{\pi}(t)\right] + \sum_a \frac{\delta H\left[\bar{\phi}(t),\bar{\pi}(t)\right]}{\delta\bar{\phi}_a(\mathbf{x},t)}\,\delta\phi_a(\mathbf{x},t) + \\ &+ \sum_a \frac{\delta H\left[\bar{\phi}(t),\bar{\pi}(t)\right]}{\delta\bar{\pi}_a(\mathbf{x},t)}\,\delta\pi_a(\mathbf{x},t) + \tilde{H}\left[\delta\phi(t),\delta\pi(t);t\right], \end{aligned} \tag{4.39}$$

where $\tilde{H}\left[\delta\phi(t),\delta\pi(t);t\right]$ contains all the terms of second and higher order in $\delta\phi(\mathbf{x},t)$ and/or $\delta\pi(\mathbf{x},t)$, and has a time-dependence which arises both from the time dependence of the fluctuations and from the explicit time-dependence of the background fields; the latter is denoted by $;t$.

Plugging this expansion into the Heisenberg equations of motion (4.34), we get

$$\begin{aligned} \dot{\bar{\phi}}_a(\mathbf{x},t) + \delta\dot{\phi}_a(\mathbf{x},t) =& i\Bigg[H\left[\bar{\phi}(t),\bar{\pi}(t)\right] + \sum_b \int d^3y \left\{ \frac{\delta\mathcal{H}\left[\bar{\phi}(\mathbf{y},t),\bar{\pi}(\mathbf{y},t)\right]}{\delta\bar{\phi}_b(\mathbf{y},t)}\,\delta\phi_b(\mathbf{y},t) + \right. \\ &\left. + \frac{\delta\mathcal{H}\left[\bar{\phi}(\mathbf{y},t),\bar{\pi}(\mathbf{y},t)\right]}{\delta\bar{\pi}_b(\mathbf{y},t)}\,\delta\pi_b(\mathbf{y},t)\right\} + \tilde{H}, \bar{\phi}_a(\mathbf{x},t) + \delta\phi_a(\mathbf{x},t)\Bigg]. \end{aligned} \tag{4.40}$$

Using the fact that the background quantities commute with everything, we find

$$\begin{aligned} \dot{\bar{\phi}}_a(\mathbf{x},t) + \delta\dot{\phi}_a(\mathbf{x},t) =& i\Bigg[\sum_b \int d^3y \left\{ \frac{\delta\mathcal{H}\left[\bar{\phi}(\mathbf{y},t),\bar{\pi}(\mathbf{y},t)\right]}{\delta\bar{\phi}_b(\mathbf{y},t)}\,\delta\phi_b(\mathbf{y},t) + \right. \\ &\left. + \frac{\delta\mathcal{H}\left[\bar{\phi}(\mathbf{y},t),\bar{\pi}(\mathbf{y},t)\right]}{\delta\bar{\pi}_b(\mathbf{y},t)}\,\delta\pi_b(\mathbf{y},t)\right\} + \tilde{H}, \delta\phi_a(\mathbf{x},t)\Bigg]. \end{aligned} \tag{4.41}$$

Focusing on the first commutator in the rhs, we have $[\delta\phi_b(\mathbf{x},t), \delta\phi_a(\mathbf{y},t)] = 0$, and the second contribution gives

$$\left[\sum_b \int d^3y \frac{\delta\mathcal{H}}{\delta\bar{\pi}_b(\mathbf{y},t)}\,\delta\pi_b(\mathbf{y},t), \delta\phi_a(\mathbf{x},t)\right] = \dot{\bar{\phi}}_a(\mathbf{x},t), \tag{4.42}$$

where we have used the background equations of motion and the commutation relations between the perturbations. This term cancels out the equal term in the lhs of Eq. (4.41). Repeating an analogous calculation for $\delta\dot{\pi}_a$, we end up with

$$\begin{cases} \delta\dot{\phi}_a(\mathbf{x},t) = i\left[\tilde{H}\left[\delta\phi(t),\delta\pi(t);t\right], \delta\phi_a(\mathbf{x},t)\right], \\ \delta\dot{\pi}_a(\mathbf{x},t) = i\left[\tilde{H}\left[\delta\phi(t),\delta\pi(t);t\right], \delta\pi_a(\mathbf{x},t)\right]. \end{cases} \tag{4.43}$$

Thus, we have shown that the perturbations evolve with the Hamiltonian $\tilde{H}$. The equations (4.43) are solved by

$$\begin{cases} \delta\phi_a(\mathbf{x},t) = U^{-1}(t,t_0)\delta\phi_a(\mathbf{x},t_0)U(t,t_0), \\ \delta\pi_a(\mathbf{x},t) = U^{-1}(t,t_0)\delta\pi_a(\mathbf{x},t_0)U(t,t_0), \end{cases} \tag{4.44}$$

where the unitary operator U satisfies

$$\frac{d}{dt}U(t,t_0) = -i\tilde{H}\left[\delta\phi(t),\delta\pi(t);t\right]U(t,t_0), \tag{4.45}$$

with the initial condition $U(t_0,t_0) = 1$.

To describe the time evolution of the perturbations in the presence of interactions, we split the perturbed Hamiltonian into a quadratic part H_0 and an interacting part H_{int}:

$$\tilde{H} = H_0 + H_{int}. \tag{4.46}$$

The H_0 piece corresponds to the "free" part of the theory, namely include those terms that are quadratic in the fields and that contain no more than two derivatives. H_{int} contains everything else, namely all of the higher order self-interactions and the interactions amongst (eventual) different fields.

We then define the so-called *interaction picture* fields $\delta\phi_a^I(\mathbf{x},t)$ and $\delta\pi_a^I(\mathbf{x},t)$ by requiring that, at some initial time t_0, they equate the full theory fields

$$\delta\phi_a^I(\mathbf{x},t_0) = \delta\phi_a(\mathbf{x},t_0), \quad \delta\pi_a^I(\mathbf{x},t_0) = \delta\pi_a(\mathbf{x},t_0), \tag{4.47}$$

and such that they evolve with the free Hamiltonian H_0:

$$\begin{cases} \delta\dot{\phi}_a^I(\mathbf{x},t) = i\left[H_0\left[\delta\phi^I(t),\delta\pi^I(t);t\right],\delta\phi_a^I(\mathbf{x},t)\right], \\ \delta\dot{\pi}_a^I(\mathbf{x},t) = i\left[H_0\left[\delta\phi^I(t),\delta\pi^I(t);t\right],\delta\pi_a^I(\mathbf{x},t)\right]. \end{cases} \tag{4.48}$$

In these equations we can choose the time at which evaluate $H_0\left[\delta\phi^I(t),\delta\pi^I(t);t\right]$. This is due to the fact that of course H_0 commutes with itself, hence we can evolve it anytime we want inside the commutator. In particular, let us evolve it to t_0, so that

$$H_0\left[\delta\phi^I(t),\delta\pi^I(t);t\right] \to H_0\left[\delta\phi^I(t_0),\delta\pi^I(t_0);t\right]. \tag{4.49}$$

Notice that the intrinsic time-dependence due to the time evolution of the background still remains.

The solution of Eqs. (4.48) can again be written as a unitary transformation

$$\begin{cases} \delta\phi^I(\mathbf{x},t) = U_0^{-1}(t,t_0)\delta\phi^I(\mathbf{x},t_0)U_0(t,t_0), \\ \delta\pi^I(\mathbf{x},t) = U_0^{-1}(t,t_0)\delta\pi^I(\mathbf{x},t_0)U_0(t,t_0), \end{cases} \tag{4.50}$$

where U_0 satisfies

$$\frac{d}{dt}U_0(t,t_0) = -iH\left[\delta\phi(t_0),\delta\pi(t_0);t\right]U_0(t,t_0), \tag{4.51}$$

with the initial condition $U_0(t_0,t_0) = 1$.

Now we have all the ingredients to understand how to calculate n-point correlation functions like

$$\langle Q(t)\rangle \equiv \langle\Omega|Q(t)|\Omega\rangle, \tag{4.52}$$

where $Q(t)$ is an operator given by the product of field perturbations $\delta\phi_a$ and $\delta\pi_a$ evaluated at a fixed time t, and $|\Omega\rangle$ is the vacuum state of the interacting theory at some time t_0 in the far past. More precisely, we consider the time t_0 at which the fluctuations are supposed to behave like free fields as $t_0 = -\infty$, which is appropriate for cosmology since at very early times the fluctuation wavelengths are well below the horizon[4]. In order to evaluate $\langle Q(t)\rangle$, we first use the operator $U(t,t_0)$ to evolve $Q(t)$ back to $Q(t_0)$

$$\langle Q(t)\rangle = \langle\Omega|Q\left[\delta\phi_a(t),\delta\pi_a(t)\right]|\Omega\rangle = \langle\Omega|U^{-1}(t,t_0)Q\left[\delta\phi_a(t_0),\delta\pi_a(t_0)\right]U(t,t_0)|\Omega\rangle. \tag{4.53}$$

Then we insert the identity operator $1 = U_0(t,t_0)U_0^{-1}(t,t_0)$, obtaining

$$\langle Q(t)\rangle = \langle\Omega|F^{-1}(t,t_0)U_0^{-1}(t,t_0)Q\left[\delta\phi_a(t_0),\delta\pi_a(t_0)\right]U_0(t,t_0)F(t,t_0)|\Omega\rangle, \tag{4.54}$$

with $F(t,t_0) = U_0^{-1}(t,t_0)U(t,t_0)$. Since $\delta\phi_a(t_0) = \delta\phi_a^I(t_0)$, $\delta\pi_a(t_0) = \delta\pi_a^I(t_0)$ and $U_0(t,t_0)$ determines the time-evolution of the interaction picture fields, (4.54) becomes

$$\langle Q(t)\rangle = \langle\Omega|F^{-1}(t,t_0)Q\left[\delta\phi_a^I(t),\delta\pi_a^I(t)\right]F(t,t_0)|\Omega\rangle. \tag{4.55}$$

Moreover, using (4.44) and (4.51), we find

$$\begin{aligned}\frac{d}{dt}F(t,t_0) &= \frac{d}{dt}\left(U_0^{-1}(t,t_0)U(t,t_0)\right) = -U_0^{-2}\frac{dU_0}{dt}U + U_0^{-1}\frac{dU}{dt}\\ &= iU_0^{-1}\Big(H_0\left[\delta\phi(t_0),\delta\pi(t_0);t\right] - \tilde{H}\left[\delta\phi(t_0),\delta\pi(t_0);t\right]\Big)U\\ &= -iU_0^{-1}H_{int}\left[\delta\phi(t_0),\delta\pi(t_0);t\right]U_0U_0^{-1}U\\ &= -iH_{int}\left[\delta\phi^I(t),\delta\pi^I(t);t\right]F(t,t_0).\end{aligned} \tag{4.56}$$

Thus, $F(t,t_0)$ satisfies

$$\frac{d}{dt}F(t,t_0) = -iH_{int}^I\left[\delta\phi^I(t),\delta\pi^I(t);t\right]F(t,t_0), \tag{4.57}$$

with $F(t_0,t_0) = 1$. This tells us that $F(t,t_0)$ is the unitary evolution operator associated with H_{int}, with the interaction Hamiltonian constructed out of the interaction picture fields. The solution of (4.57) is

$$F(t,t_0) = T\exp\left[-i\int_{t_0}^{t} dt' H_{int}^I(t')\right], \tag{4.58}$$

where T is the time-ordering operator, defined by

$$T\left\{\chi(t)\chi(t')\right\} = \begin{cases}\chi(t)\chi(t') & t > t'\\ \chi(t')\chi(t) & t' > t.\end{cases} \tag{4.59}$$

Therefore, from (4.55) we have

$$\langle\Omega|Q(t)|\Omega\rangle = \langle\Omega|\left[\bar{T}\exp\left(i\int_{t_0}^{t} dt' H_{int}^I(t')\right)\right]Q^I(t)\left[T\exp\left(-i\int_{t_0}^{t} dt' H_{int}^I(t')\right)\right]|\Omega\rangle, \tag{4.60}$$

where $\bar{T}$ is the anti time-ordering operator.

We now turn to the issue of the vacuum state, following [71], [34]. As before we use $|\Omega\rangle$ to denote the vacuum state of the interacting theory and $|0\rangle$ for the one of the free theory. We start by expanding $|0\rangle$ using a complete set of energy eigenstates of the full theory. In particular, we consider the energy eigenstates at an initial time for the perturbation modes[5]. Strictly speaking, this requires all the modes to be sub-horizon; however, during inflation there are horizon-crossing and super-horizon modes. For $t_0 \to -\infty$, we have

$$e^{-iH(t-t_0)}|0\rangle = \sum_n e^{-iH(t-t_0)}|n\rangle\langle n||0\rangle = \sum_n e^{-iE_n(t-t_0)}|n\rangle\langle n||0\rangle, \tag{4.61}$$

[4]Now we should interpret t as the conformal time, such that the limit $t \to -\infty$ makes sense.
[5]Remember that we are in an expanding background, thus energy is not conserved.

where E_n are the eigenvalues of H. We then assume that $|\Omega\rangle$ has some overlap with $|0\rangle$, that is $\langle\Omega||0\rangle \neq 0$, otherwise H_{int} would not be a small perturbation. We can then write

$$e^{-iH(t-t_0)}|0\rangle = e^{-iE_\Omega(t-t_0)}|\Omega\rangle\langle\Omega||0\rangle + \sum_{n'} e^{-iE_{n'}(t-t_0)}|n'\rangle\langle n'||0\rangle, \tag{4.62}$$

where n' denotes the states excluding the ground state, $E_\Omega \equiv \langle\Omega|H|\Omega\rangle$, and $E_n > E_\Omega$. Now we add a small imaginary part to time

$$t \to \tilde{t} = t(1 - i\epsilon) \tag{4.63}$$

For $t_0 \to -\infty$, the $i\epsilon$ factor introduces a term $e^{-\infty\cdot\epsilon E_n}$. Then $e^{-iE_\Omega(\tilde{t}-\tilde{t_0})}$ dies slower than $e^{-iE_{n'}(\tilde{t}-\tilde{t_0})}$, and we can solve for $|\Omega\rangle$

$$e^{-iH(\tilde{t}-\tilde{t_0})}|\Omega\rangle = \frac{e^{-iH(\tilde{t}-\tilde{t_0})}|0\rangle}{\langle\Omega|0\rangle} \quad \Rightarrow \quad F(\tilde{t},\tilde{t_0})|\Omega\rangle = \frac{F(\tilde{t},\tilde{t_0})|0\rangle}{\langle\Omega|0\rangle}. \tag{4.64}$$

In practice, we use the $i\epsilon$ prescription to turn off the interactions in the infinitely far past, and that brings us to identify the interacting vacuum with the free-field vacuum. From now on, we will neglect the tilde on t, assuming that t includes the small imaginary part.

Using (4.64), the in-in expectation value can be rewritten as

$$\langle\Omega|Q(t)|\Omega\rangle = \frac{\langle 0|\left[\bar{T}e^{i\int_{t_0}^t H_{int}^I(t')dt'}\right] Q^I(t)\left[Te^{-i\int_{t_0}^t H_{int}^I(t')dt'}\right]|0\rangle}{|\langle 0|\Omega\rangle|^2}. \tag{4.65}$$

Notice that in the anti time-ordering the $i\epsilon$ comes with a minus sign; this is due to the fact that the $\bar{T}$ part is the Hermitian conjugate of the T part. In order to evaluate the denominator of (4.65), we take $Q(t)$ to be the identity matrix. Thus, we can write it as

$$|\langle 0|\Omega\rangle|^2 = \frac{\langle 0|\left[\bar{T}e^{i\int_{t_0}^t H_{int}^I(t')dt'}\right]\left[Te^{-i\int_{t_0}^t H_{int}^I(t')dt'}\right]|0\rangle}{\langle\Omega|\Omega\rangle} = 1. \tag{4.66}$$

To get the previous result, we have to require both the states $|\Omega\rangle$ and $|0\rangle$ to be normalized. The $\bar{T}$ part cancels the T part because they are unitary operators and one is the Hermitian conjugate of the other. Finally, we can write the *in-in master formula*:

$$\langle\Omega|Q(t)|\Omega\rangle = \langle 0|\left[\bar{T}e^{i\int_{t_0}^t H_{int}^I(t')dt'}\right] Q^I(t)\left[Te^{-i\int_{t_0}^t H_{int}^I(t')dt'}\right]|0\rangle. \tag{4.67}$$

In practice one expands the above formula order by order in perturbation theory. As previously said, the coordinate t is usually taken to be the conformal time and $t_0 = -\infty$. For example, the time ordered exponential gives

$$Te^{-i\int_{-\infty_-}^t H_{int}^I(t')dt'} = 1 - i\int_{-\infty_-}^t H_{int}^I(t')dt' + \frac{i^2}{2} + \int_{-\infty_-}^t dt' \int_{-\infty_-}^{t'} H_{int}^I(t')H_{int}^I(t'')dt'' + ..., \tag{4.68}$$

where $-\infty_\pm = -\infty(1 \pm i\epsilon)$, and the anti-time ordered exponential is simply the conjugate. Plugging these expansions in (4.67), one finds:

$$\langle\Omega|Q(t)|\Omega\rangle = \langle 0|\left(1 + i\int_{-\infty_+}^t H_{int}^I(t')dt' + ...\right) Q^I(t)\left(1 - i\int_{-\infty_-}^t H_{int}^I(t')dt' + ...\right)|0\rangle. \tag{4.69}$$

If we stop at first order and consider the case of the bispectrum (i.e. $Q^I(t)$ is the product of three fields), the first term in the above expression vanishes, since we have the product of an odd number of creation and annihilation operators acting on the vacuum. Thus, we have:

$$\begin{aligned}\langle Q(t)\rangle &= \langle 0| \left(i \int_{-\infty_+}^{t} H^I_{int}(t')dt' \right) Q^I(t) - Q^I(t) \left(i \int_{-\infty_-}^{t} H^I_{int}(t')dt' \right) |0\rangle \\ &= 2\,\mathrm{Im} \int_{-\infty_-}^{t} dt' \, \langle 0|Q^I(t) H^I_{int}(t')|0\rangle .\end{aligned} \tag{4.70}$$

The next step for calculating correlators is to expand $Q^I(t)$ and H_{int} in terms of interaction picture fields, and then contract them by using the Wick's theorem.

4.4 Non-Gaussianity in single field slow-roll inflation

The in-in formalism introduced in the previous section is a powerful tool to compute non-Gaussianities from inflationary models. In the rest of this chapter we will do this for single field slow-roll models: we will compute explicitly the bispectra of scalar and tensor perturbations, and then give the results for the mixed correlators. In doing so, we will follow [13]. See also [14] for a different derivation of this result.

4.4.1 Gauge fixing at non-linear order

The first step needed to compute non-Gaussianities is to fix the gauge we are working in. We can extend at the non-linear level the definitions (3.17) and (3.19) of the comoving and the spatially flat gauge. These are respectively given by [13]:

$$\begin{aligned}&\phi(\mathbf{x},t) = \phi(t), \quad h_{ij} = a^2 e^{2\zeta} [\exp \gamma]_{ij}, \quad \gamma_i{}^i = 0, \quad \partial^i \gamma_{ij} = 0, \\ &[\exp \gamma]_{ij} = \delta_{ij} + \gamma_{ij} + \frac{1}{2!}\gamma_{ik}\gamma^k{}_j + \dots, \quad e^{2\zeta} = +2\zeta + \frac{1}{2!}(4\zeta^2) + \dots,\end{aligned} \tag{4.71}$$

and

$$\begin{aligned}\phi(\mathbf{x},t) = \phi(t) + \varphi(\mathbf{x},t), \quad h_{ij} = a^2 [\exp \tilde\gamma]_{ij}, \quad \tilde\gamma_i{}^i = 0, \quad \partial^i \tilde\gamma_{ij} = 0, \\ [\exp \gamma]_{ij} = \delta_{ij} + \tilde\gamma_{ij} + \frac{1}{2!}\tilde\gamma_{ik}\tilde\gamma^k{}_j + \dots .\end{aligned} \tag{4.72}$$

At linear level, we have seen that we can pass from φ to ζ through the linear relation

$$\zeta_1 = -\frac{H}{\dot\phi}\varphi, \tag{4.73}$$

while tensor perturbations are gauge-invariant. At the non-linear level this last assertion is not true anymore, and we have to work out the transformations that relate both scalar and tensor perturbations in the two different gauges. These are given by [13] (see also [48])

$$\begin{aligned}\zeta =& \zeta_1 + \frac{1}{2}\frac{\ddot\phi}{\dot\phi H}\zeta_1^2 + \frac{1}{4}\frac{\dot\phi^2}{H^2}\zeta_1^2 + \frac{1}{H}\dot\zeta_1\zeta_1 - \frac{1}{4}\frac{a^{-2}}{H^2}(\partial_i\zeta_1)(\partial^i\zeta_1) - \frac{1}{4}\frac{a^{-2}}{H^2}\partial^{-2}\partial_i\partial_j(\partial^i\zeta_1\partial^j\zeta_1) + \\ &+ \frac{1}{2H}\partial_i\psi\partial^i\zeta_1 - \frac{1}{2H}\partial^{-2}\partial_i\partial_j(\partial^i\psi\partial^j\zeta_1),\end{aligned} \tag{4.74}$$

$$\gamma_{ij} = \tilde\gamma_{ij} + \frac{1}{H}\dot{\tilde\gamma}_{ij}\zeta_1 - \frac{a^{-2}}{H^2}\partial_i\zeta_1\partial_j\zeta_1 + \frac{1}{H}\left(\partial_i\psi\partial_j\zeta_1 + \partial_j\psi\partial_i\zeta_1\right), \tag{4.75}$$

where ψ is the scalar perturbation in the shift vector. On super-horizon scales only the first three terms in (4.74) and the first one in (4.75) are non-vanshing. It is indeed possible to show that ζ and γ, defined via (4.71), are constant above the horizon. To show this fact we follow the demonstration provided in [13]. We need to expand the action to all orders in powers of the fields, but only to first order in the derivatives of the fields. It will be soon clear why this is sufficient. We assume that $N = 1 + \delta N$, where δN has an expansion in derivatives that starts with a first order term, while N_i is of zero-th order in derivatives, so that $D_i N_j$ is of first order in derivatives. By expanding the Hamiltonian constraint to first order in derivatives and solving for δN, one obtains

$$2V\delta N = 2H(3\dot{\zeta} - D_i N^i). \tag{4.76}$$

The action to first order in derivatives on a solution of the Hamiltonian constraint is given by

$$\begin{aligned} S &= \int d^3x\, dt\, \sqrt{h} N (R^{(3)} - 2V) = \int d^3x\, dt\, \sqrt{h}(-2V - 2V\delta N) \\ &= \int d^3x\, dt\, a^3 e^{3\zeta}(-6H^2 + \dot{\phi}^2 - 6\dot{\zeta}H) = -2\int d^3x\, dt\, \partial_t(a^3 e^{3\zeta} H), \end{aligned} \tag{4.77}$$

where we have neglected the contributions coming from $R^{(3)}$, which are of second order in the derivatives, and we have used (4.76) and the equations of motion (1.70)-(1.71). Therefore, the only contribution to the action (4.77) is given by a total derivative term, which can be neglected. This result tells us that the action to all orders in powers of the fields contains no terms with first derivatives of the fields themseleves. Since outside the horizon we can neglect all the spatial derivatives, it follows that the only derivative terms present in the action are time derivative terms of higher than the first order. This implies that ζ and γ_{ij} are contant outside the horizon to all orders. From the relation (4.75) between the tensor perturbations in the two gauges, γ_{ij} and $\tilde{\gamma}_{ij}$, it follows that these have the same expression on super-horizon scales.

To compute the bispectra for single field slow-roll models we choose to work in the spatially flat gauge. This is convenient since it prevents us from doing a lot of integrations by parts and the calculations are much simpler. Moreover, working in this gauge makes it more clear which are the terms dominant in the slow-roll parameters. In order to pass from the variable φ to ζ, which is the one that stays constant above the horizon, we will then use the relation (4.74).

One can repeat the same steps we have done in the comoving gauge to find the first order expressions of N and N_i in the spatially flat gauge, finding [13]:

$$N^{(1)} = \frac{\dot{\phi}^2}{2H}\varphi, \quad N_i^{(1)} = \partial_i \psi, \quad \partial^2 \psi = -\frac{\dot{\phi}^2}{2H^2}\partial_t\left(\frac{H}{\dot{\phi}}\varphi\right), \tag{4.78}$$

Notice that both $N^{(1)}$ and $N_i^{(1)}$ are subleading in the slow-roll parameter ϵ_V with respect to φ, since

$$N^{(1)} \sim \sqrt{\epsilon_V}\varphi, \quad N_i^{(1)} \sim \sqrt{\epsilon_V}\varphi. \tag{4.79}$$

4.4.2 Bispectrum of scalar perturbations

To compute the bispectrum of scalar perturbations we need to expand the action (3.15) to cubic order in scalar perturbations. This is given by [13]

$$\begin{aligned} S_3 = \int d^3x\, dt\, a^3 \Bigg[&-\frac{\dot{\phi}}{4H}\varphi\dot{\varphi}^2 - a^{-2}\frac{\dot{\phi}}{4H}\varphi(\partial_i\varphi)(\partial^i\varphi) - \dot{\varphi}\partial_i\psi\partial^i\varphi + \\ &+\frac{3\dot{\phi}^3}{8H}\varphi^3 - \frac{\dot{\phi}^5}{16H^3}\varphi^3 - \frac{\dot{\phi}V''}{4H}\varphi^3 - \frac{V'''}{6}\varphi^3 + \frac{\dot{\phi}^3}{4H^2}\varphi^2\dot{\varphi} + \frac{\dot{\phi}^2}{4H}\varphi^2\partial^2\psi + \\ &+\frac{\dot{\phi}}{4H}\left(-\varphi\partial_i\partial_j\psi\partial^i\partial^j\psi + \varphi\partial^2\psi\partial^2\psi\right)\Bigg]. \end{aligned} \tag{4.80}$$

Only the first line contributes at leading order in slow-roll and, by using the first order relation (4.73) between φ and ζ, it is easy to see that the leading order terms in ζ are of order ϵ_V^2. This is another convenience of this gauge choice. What is instead not obvious in this gauge is that there is some quantity which is constant outside the horizon, since the φ^3 couplings could lead to some evolution of the perturbations. To see that ζ stays indeed constant, it is more convenient to work in the comoving gauge. See again [13] for a detailed discussion of this issue.

Keeping only the leading terms in the action (4.80) and using Eqs. (4.78), the cubic interaction Lagrangian can be written as

$$L_{\rm int}^{\zeta_1\zeta_1\zeta_1} = \epsilon_V^2 M_{Pl}^2 \int d^3x \left[a\zeta_1\zeta_1'^2 + a\zeta_1(\partial_i\zeta_1)(\partial^i\zeta_1) - 2a\zeta_1'(\partial_i\partial^{-2}\zeta_1')(\partial^i\zeta_1)\right], \tag{4.81}$$

where we have reintroduced the Planck mass by dimensional analysis. We have also switched to the conformal time τ, with the prime denoting a derivative with respect to it. Our strategy is to first compute the bispectrum of ζ_1, and then find that of ζ by exploiting the non-linear relation (4.74) between the two variables, where we have seen that only the first line contributes on super-horizon scales. As regarding this point, it is useful to notice that we can rewrite the super-horizon relation between ζ_1 and ζ as

$$\zeta = \zeta_1 + \lambda\zeta_1^2, \tag{4.82}$$

with

$$\lambda \equiv \frac{1}{2}\frac{\ddot\phi}{\dot\phi H} + \frac{1}{4}\frac{\dot\phi^2}{H^2} \simeq \frac{1}{4}\eta, \tag{4.83}$$

where η is the Hubble slow-roll parameter, defined as $\eta \equiv \frac{\dot\epsilon}{\epsilon H}$. This introduces an additonal contributon to the three-point function of ζ, since in real space we have [13]

$$\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\rangle = \langle\zeta_1(\mathbf{x}_1)\zeta_1(\mathbf{x}_2)\zeta_1(\mathbf{x}_3)\rangle + 2\lambda[\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_3)\rangle + \text{perm}]. \tag{4.84}$$

Let us now start the computation of the bispectrum of ζ_1. As usual, we expand the field ζ_1 in the Fourier space as

$$\zeta_1(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3}\zeta_1(\mathbf{k},\tau)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.85}$$

The interaction Lagrangian can then be written as

$$\begin{aligned} L_{\rm int}^{\zeta_1\zeta_1\zeta_1} &= \frac{\epsilon_V^2 M_{Pl}^2}{(2\pi)^9}\int d^3x\, d^3k\, d^3p\, d^3q \Big[a\zeta_1(\mathbf{k},\tau)\zeta_1'(\mathbf{p},\tau)\zeta_1'(\mathbf{q},\tau) - a(\mathbf{p}\cdot\mathbf{q})\zeta_1(\mathbf{k},\tau)\zeta_1(\mathbf{p},\tau)\zeta_1(\mathbf{q},\tau) + \\ &\qquad -2a\frac{(\mathbf{p}\cdot\mathbf{q})}{p^2}\zeta_1'(\mathbf{k},\tau)\zeta_1'(\mathbf{p},\tau)\zeta_1(\mathbf{q},\tau)\Big] e^{i(\mathbf{k}+\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\ &= \frac{\epsilon_V^2 M_{Pl}^2}{(2\pi)^6}\int d^3k\, d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}+\mathbf{p}+\mathbf{q}) \Big[a\zeta_1(\mathbf{k},\tau)\zeta_1'(\mathbf{p},\tau)\zeta_1'(\mathbf{q},\tau) + \\ &\qquad -a(\mathbf{p}\cdot\mathbf{q})\zeta_1(\mathbf{k},\tau)\zeta_1(\mathbf{p},\tau)\zeta_1(\mathbf{q},\tau) - 2a\frac{(\mathbf{p}\cdot\mathbf{q})}{p^2}\zeta_1'(\mathbf{k},\tau)\zeta_1'(\mathbf{p},\tau)\zeta_1(\mathbf{q},\tau)\Big], \end{aligned} \tag{4.86}$$

where in the second line we have integrated over $\mathbf{x}$. We can now compute the bispectrum of scalar perturbations by exploiting the in-in formalism introduced in the previous section. We are interested in evaluating the bispectrum in the super-horizon limit, corresponding to $\tau \to 0$. Since in our case

$$H_{\rm int}^{\zeta_1\zeta_1\zeta_1} = -L_{\rm int}^{\zeta_1\zeta_1\zeta_1}, \tag{4.87}$$

we have

$$\langle\zeta_1(\mathbf{k}_1,0)\zeta_1(\mathbf{k}_2,0)\zeta_1(\mathbf{k}_3,0)\rangle = -2\frac{\epsilon_V^2 M_{Pl}^2}{(2\pi)^6}\,\mathrm{Im}\Bigg\{\int d^3k\, d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\int_{-\infty_-}^{0} d\tau' a\,\cdot \\ \cdot\left[S_1(\tau')+S_2(\tau')+S_3(\tau')\right]\Bigg\}, \tag{4.88}$$

where we have defined

$$S_1(\tau') \equiv a\langle 0|\zeta_1(\mathbf{k}_1,0)\zeta_1(\mathbf{k}_2,0)\zeta_1(\mathbf{k}_3,0)\zeta_1(\mathbf{k},\tau')\zeta_1(\mathbf{p},\tau')\zeta_1(\mathbf{q},\tau')|0\rangle, \tag{4.89}$$

$$S_2(\tau') \equiv -a(\mathbf{p}\cdot\mathbf{q})\langle 0|\zeta_1(\mathbf{k}_1,0)\zeta_1(\mathbf{k}_2,0)\zeta_1(\mathbf{k}_3,0)\zeta_1(\mathbf{k},\tau')\zeta_1(\mathbf{p},\tau')\zeta_1(\mathbf{q},\tau')|0\rangle, \tag{4.90}$$

$$S_3(\tau') \equiv -2a\frac{(\mathbf{p}\cdot\mathbf{q})}{p^2}\langle 0|\zeta_1(\mathbf{k}_1,0)\zeta_1(\mathbf{k}_2,0)\zeta_1(\mathbf{k}_3,0)\zeta_1(\mathbf{k},\tau')\zeta_1(\mathbf{p},\tau')\zeta_1(\mathbf{q},\tau')|0\rangle. \tag{4.91}$$

To evaluate these terms we can use the Wick theorem: this consists in contracting the couples of fields evaluated at different times in all the possible ways, namely considering all the possible permutations of the momenta. In doing so, we need to use the following relations

$$\langle\zeta_1(\mathbf{k},\tau)\zeta_1(\mathbf{k}',\tau')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')u(k,\tau)u^*(k',\tau'), \tag{4.92}$$

$$\langle\zeta_1(\mathbf{k},\tau)\zeta_1'(\mathbf{k}',\tau')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')u(k,\tau)\frac{d}{d\tau}u^*(k',\tau'), \tag{4.93}$$

which can be easily obtained by expanding the field ζ_1 in terms of the creation and annihilation operators, exactly as we have done in (3.106). Thus, we find

$$S_1 = (2\pi)^9 a\Bigg[\delta^{(3)}(\mathbf{k}+\mathbf{k}_1)u(k_1,0)u^*(k,\tau')\delta^{(3)}(\mathbf{p}+\mathbf{k}_2)u(k_2,0)\frac{d}{d\tau}u^*(p,\tau')\cdot \\ \cdot\delta^{(3)}(\mathbf{q}+\mathbf{k}_3)u(k_3,0)\frac{d}{d\tau}u^*(q,\tau')\Bigg] + \mathrm{perm}(k_i), \tag{4.94}$$

$$S_2 = -(2\pi)^9 a(\mathbf{k}_1\cdot\mathbf{k}_2)\Big[\delta^{(3)}(\mathbf{k}+\mathbf{k}_1)u(k_1,0)u^*(k,\tau')\delta^{(3)}(\mathbf{p}+\mathbf{k}_2)u(k_2,0)u^*(p,\tau')\cdot \\ \cdot\delta^{(3)}(\mathbf{q}+\mathbf{k}_3)u(k_3,0)u^*(q,\tau')\Big] + \mathrm{perm}(k_i), \tag{4.95}$$

$$S_3 = -(2\pi)^9 2a\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k_1^2}(\mathbf{k}_1\cdot\mathbf{k}_2)\Bigg[\delta^{(3)}(\mathbf{k}+\mathbf{k}_1)u(k_1,0)\frac{d}{d\tau}u^*(k,\tau')\delta^{(3)}(\mathbf{p}+\mathbf{k}_2)u(k_2,0)u^*(p,\tau')\cdot \\ \cdot\delta^{(3)}(\mathbf{q}+\mathbf{k}_3)u(k_3,0)\frac{d}{d\tau}u^*(q,\tau')\Bigg] + \mathrm{perm}(k_i). \tag{4.96}$$

Plugging these exprssions into Eq. (4.88), we find

$$\langle\zeta_1(\mathbf{k}_1,0)\zeta_1(\mathbf{k}_2,0)\zeta_1(\mathbf{k}_3,0)\rangle = 2\epsilon_V^2 M_{Pl}^2\,(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot \\ \cdot\,\mathrm{Im}\left[-I_1+(\mathbf{k}_1\cdot\mathbf{k}_2)I_2-2\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k_1^2}I_3\right]+\mathrm{perm}(k_i), \tag{4.97}$$

with

$$I_1 \equiv u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty_-}^{0} d\tau' a^2\left[u^*(k_1,\tau')\left(\frac{d}{d\tau}u^*(k_2,\tau')\right)\left(\frac{d}{d\tau}u^*(k_3,\tau')\right)\right], \tag{4.98}$$

$$I_2 \equiv u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty_-}^{0} d\tau' a^2\left[u^*(k_1,\tau')u^*(k_2,\tau')u^*(k_3,\tau')\right], \tag{4.99}$$

$$I_3 \equiv u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty_-}^{0} d\tau' a^2\left[\left(\frac{d}{d\tau}u^*(k_1,\tau')\right)u^*(k_2,\tau')\left(\frac{d}{d\tau}u^*(k_3,\tau')\right)\right]. \tag{4.100}$$

To compute these integrals we need the analytic expression of the mode function $u(k,\tau)$. From the solution we have found in the previous chapter and taking the expression at leading order in ϵ_V, we have (see e.g. [63])

$$u(k,\tau) = \frac{iH}{M_{Pl}\sqrt{4\epsilon_V k^3}}(1+ik\tau)e^{-ik\tau}, \tag{4.101}$$

$$\frac{d}{d\tau}u^*(k,\tau) = \frac{iH}{M_{Pl}\sqrt{4\epsilon_V k^3}}k^2\tau e^{ik\tau}. \tag{4.102}$$

To keep only the lowest order terms in slow-roll, we use the de Sitter expression for the scale factor, $a(\tau) \simeq -1/H\tau$, with $H \simeq$ const. Indeed, keeping the time variation of the Hubble parameter gives a contribution of order ϵ_V, as can be seen by expanding H around a constant time value t_*:

$$H(t) = H(t_*) + \dot{H}(t)|_{t=t_*}(t-t_*) + \ldots \simeq H(t_*) - \epsilon_V H^2(t_*)(t-t_*) + \ldots \tag{4.103}$$

We can minimize the error due to these approximations by taking the value of H at the horizon crossing time of the total momentum $k_T = k_1+k_2+k_3$. We label this time as t_* and $H_* \equiv H(t_*)$. The first integral (4.98) thus gives

$$I_1 = -H_*^4 k_2^2 k_3^2 \left(\prod_{i=1}^{3} \frac{1}{4M_{Pl}^2\epsilon_V k_i^3}\right) \int_{-\infty_-}^{0} d\tau'\tau'(1-k_1\tau')e^{ik_T\tau'}, \tag{4.104}$$

and the integral in this last expression can be easily computed as

$$\begin{aligned}
\int_{-\infty_-}^{0} d\tau'\tau'(1-k_1\tau')e^{ik_T\tau'} &= \int_{-\infty_-}^{0} d\tau' e^{ik_T\tau'} - ik_1\int_{-\infty_-}^{0} d\tau'\tau' e^{ik_T\tau'} \\
&= -\frac{i}{k_T}e^{ik_T\tau'}\Big|_{-\infty_-}^{0} - ik_1\left[-\frac{i}{k_T}\tau' e^{ik_T\tau'}\Big|_{-\infty_-}^{0} + \frac{i}{k_T}\int_{-\infty_-}^{0} d\tau' e^{ik_T\tau'}\right] \\
&= -\frac{i}{k_T} - i\frac{k_1}{k_T^2}.
\end{aligned} \tag{4.105}$$

Notice that the $i\epsilon$ prescription cancels out the contributions at far past infinity in the last step. Thus, we have

$$I_1 = iH_*^4\left(\prod_{i=1}^{3}\frac{1}{4M_{Pl}^2\epsilon_V k_i^3}\right)\left(\frac{k_2^2k_3^2}{k_T} + \frac{k_2^2k_3^2k_1}{k_T^2}\right). \tag{4.106}$$

As regarding I_2, we have

$$I_2 = -H_*^4\left(\prod_{i=1}^{3}\frac{1}{4M_{Pl}^2\epsilon_V k_i^3}\right)\int_{-\infty_-}^{0}\frac{d\tau'}{\tau'^2}(1-ik_1\tau')(1-ik_2\tau')(1-ik_3\tau')e^{ik_T\tau'}. \tag{4.107}$$

The last integral can be written as

$$\begin{aligned}
\int_{-\infty_-}^{0}\frac{d\tau'}{\tau'^2}(1-ik_1\tau')(1-ik_2\tau')(1-ik_3\tau')e^{ik_T\tau'} &= \int_{-\infty_-}^{0}\frac{d\tau'}{\tau'^2}e^{ik_T\tau'} - ik_T\int_{-\infty_-}^{0}\frac{d\tau'}{\tau'}e^{ik_T\tau'} + \\
-\sum_{i\neq j}k_ik_j\int_{-\infty_-}^{0}d\tau' e^{ik_T\tau'} + ik_1k_2k_3\int_{-\infty_-}^{0}d\tau'\tau' e^{ik_T\tau'}.
\end{aligned} \tag{4.108}$$

The first term on the right hand side is given by

$$\int_{-\infty_-}^{0} \frac{d\tau'}{\tau'^2} e^{ik_T\tau'} = -\frac{1}{\tau'} e^{ik_T\tau'}\Big|_{-\infty_-}^{0} + ik_T \int_{-\infty_-}^{0} \frac{d\tau'}{\tau'} e^{ik_T\tau'}, \tag{4.109}$$

and the second term cancels out with the second one in the right hand side of (4.108). As regarding the first term, we have

$$\lim_{\tau'\to 0} \left(-\frac{1}{\tau'} e^{ik_T\tau'} \right) = -ik_T. \tag{4.110}$$

The other two terms present in (4.108) can be evaluated analogously and yield

$$\sum_{i\neq j} k_i k_j \int_{-\infty_-}^{0} d\tau' e^{ik_T\tau'} = -\sum_{i\neq j} k_i k_j \frac{i}{k_T}, \tag{4.111}$$

$$ik_1k_2k_3 \int_{-\infty_-}^{0} d\tau' \tau' e^{ik_T\tau'} = i\frac{k_1k_2k_3}{k_T^2}. \tag{4.112}$$

Putting all these results together, we find

$$\int_{-\infty_-}^{0} \frac{d\tau'}{\tau'^2}(1-ik_1\tau')(1-ik_2\tau')(1-ik_3\tau')e^{ik_T\tau'} = i\left(-k_T + \frac{\sum_{i>j} k_i k_j}{k_T} + \frac{k_1k_2k_3}{k_T^2} \right). \tag{4.113}$$

The final expression of I_2 thus reads

$$I_2 = iH_*^4 \left(\prod_{i=1}^{3} \frac{1}{4M_{Pl}^2 \epsilon_V k_i^3} \right) \left(k_T - \frac{\sum_{i>j} k_i k_j}{k_T} - \frac{k_1k_2k_3}{k_T^2} \right). \tag{4.114}$$

Finally we need to evaluate I_3. The computation is exactly the same as for I_1, and the final result is

$$I_3 = iH_*^4 \left(\prod_{i=1}^{3} \frac{1}{4M_{Pl}^2 \epsilon_V k_i^3} \right) \left(\frac{k_1^2k_2^2}{k_T} + \frac{k_2k_1^2k_3^2}{k_T^2} \right). \tag{4.115}$$

To compute the bispectrum we then need to add all the contributions just computed in the expression (4.97), taking into account all the possible permutations of the k_i. To this last end, it is useful to notice that

$$(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)^2 = 0 = k_1^2 + k_2^2 + k_3^2 + 2\mathbf{k}_1\cdot\mathbf{k}_2 + 2\mathbf{k}_1\cdot\mathbf{k}_3 + 2\mathbf{k}_2\cdot\mathbf{k}_3, \tag{4.116}$$

which implies that

$$\mathbf{k}_1\cdot\mathbf{k}_2 + \mathbf{k}_1\cdot\mathbf{k}_3 + \mathbf{k}_2\cdot\mathbf{k}_3 = -\frac{1}{2}(k_1^2 + k_2^2 + k_3^2). \tag{4.117}$$

When summing up all the permutations of the last term in (4.97), it is useful also the following relation

$$\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1^2} + \frac{\mathbf{k}_1\cdot\mathbf{k}_3}{k_1^2} + \frac{\mathbf{k}_2\cdot\mathbf{k}_3}{k_2^2} + \frac{\mathbf{k}_2\cdot\mathbf{k}_1}{k_2^2} + \frac{\mathbf{k}_3\cdot\mathbf{k}_1}{k_3^2} + \frac{\mathbf{k}_3\cdot\mathbf{k}_2}{k_3^2} = -\frac{1}{2}\left(\frac{k_3^2}{k_1^2} + \frac{k_2^2}{k_1^2} + \frac{k_1^2}{k_2^2} + \frac{k_3^2}{k_2^2} + \frac{k_2^2}{k_3^2} + \frac{k_1^2}{k_3^2} \right). \tag{4.118}$$

Therefore, taking into account all the permutations in (4.97), we find (see [48] for more details about the computation)

$$\begin{aligned} \langle \zeta_1(\mathbf{k}_1,0)\zeta_1(\mathbf{k}_2,0)\zeta_1(\mathbf{k}_3,0)\rangle =& (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{H_*^4}{M_{Pl}^4}\frac{1}{(k_1k_2k_3)^3}\frac{1}{32\epsilon_V}\cdot \\ &\cdot\left(\sum_i k_i^3 + \sum_{i\neq j} k_i k_j^2 + \frac{8}{k_T}\sum_{i>j} k_i^2 k_j^2 \right). \end{aligned} \tag{4.119}$$

The contribution coming from the non-linear relation between ζ_1 and ζ gives

$$\frac{\eta}{2}[\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_3)\rangle + \text{perm}] = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{H_*^4}{32\epsilon^2}\frac{\eta}{(k_1k_2k_3)^3}(k_1^2+k_2^2+k_3^2). \tag{4.120}$$

Thus, summing the two previous contributions, the final expression for the (super-horizon) bispectrum of scalar perturbations reads [13]

$$\begin{aligned}\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{H_*^4}{M_{Pl}^4}\frac{1}{(k_1k_2k_3)^3}\frac{1}{4\epsilon^2}\cdot\\ &\cdot\left[\frac{\eta}{8}\sum_i k_i^3 + \frac{\epsilon}{8}\left(\sum_i k_i^3 + \sum_{i\neq j}k_ik_j^2 + \frac{8}{k_T}\sum_{i>j}k_i^2k_j^2\right)\right].\end{aligned} \tag{4.121}$$

In order to obtain the shape function $S(k_1,k_2,k_3)$ we have to normalize this result to the power spectrum (see the definiton given in Subsection 4.2.2). This cancels out the ϵ^{-2} factor, and the shape function can be written as the superposition of local and equilateral shapes [72]:

$$S(k_1,k_2,k_3) \simeq (6\epsilon-2\eta)S^{local}(k_1,k_2,k_3) + \frac{5}{3}\epsilon S^{equil}(k_1,k_2,k_3). \tag{4.122}$$

This expression tells us that non-Gaussianity of scalar perturbations in single field slow-roll models is suppressed by the slow-roll parameters,

$$f_{\text{NL}} \sim \mathcal{O}(\epsilon,\eta). \tag{4.123}$$

In [72] it has been shown that the dominant contribution to $S(k_1,k_2,k_3)$ comes from the local shape. Therefore, like the local shape, the shape function of scalar perturbations in single field slow-roll inflation peaks in the squeezed limit. This can also be seen directly from (4.121), since the factor $(k_1k_2k_3)^{-3}$ gives its maximum contribution when one of the momenta goes to zero.

4.4.3 Bispectrum of tensor perturbations

To compute the bispectrum of primordial gravitational waves we need to expand the action at third order in tensor perturbations. This is given by

$$S_{\gamma\gamma\gamma} = \frac{M_{Pl}^4}{4}\int d\tau\, d^3x\, a^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\partial_k\partial_l\gamma_{ij}. \tag{4.124}$$

Then, as usual, we expand the tensor perturbations in the Fourier space as

$$\gamma_{ij}(\mathbf{x},\tau) = \int\frac{d^3k}{(2\pi)^3}\sum_{\lambda'=+,\times}\epsilon_{ij}^{\lambda'}(\mathbf{k})\gamma_{\mathbf{k}}^{\lambda'}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.125}$$

After integrating over $\mathbf{x}$, the Lagrangian at third order in tensor perturbations takes the form

$$\begin{aligned}L_{\text{int}}^{\gamma\gamma\gamma} =& \frac{M_{Pl}^4}{4}\int d^3k\, d^3p\, d^3q\frac{1}{(2\pi)^6}\delta^{(3)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\, a^2\left[-q_kq_l\gamma_{\mathbf{k}}^{\lambda_1'}\gamma_{\mathbf{p}}^{\lambda_2'}\gamma_{\mathbf{q}}^{\lambda_3'}\epsilon_{ik}^{\lambda_1'}(\mathbf{k})\epsilon_{jl}^{\lambda_2'}(\mathbf{p})\epsilon_{ij}^{\lambda_3'}(\mathbf{q})+\right.\\ &\left.+\frac{1}{2}q_kq_l\gamma_{\mathbf{k}}^{\lambda_1'}\gamma_{\mathbf{p}}^{\lambda_2'}\gamma_{\mathbf{q}}^{\lambda_3'}\epsilon_{ij}^{\lambda_1'}(\mathbf{k})\epsilon_{kl}^{\lambda_2'}(\mathbf{p})\epsilon_{ij}^{\lambda_3'}(\mathbf{q})\right],\end{aligned} \tag{4.126}$$

where we have neglected the sum over the polarization indices for simplicity of notation. As in the case of scalar perturbations, the interaction Hamiltonian is related to the interaction

Lagrangian simply by $H_{\rm int}^{\gamma\gamma\gamma} = -L_{\rm int}^{\gamma\gamma\gamma}$, and the super-horizon (i.e. $k\tau \to 0$) bispectrum of tensor perturbations can be written as

$$\langle \gamma_{\mathbf{k}_1}^{\lambda_1}(0)\gamma_{\mathbf{k}_2}^{\lambda_2}(0)\gamma_{\mathbf{k}_3}^{\lambda_3}(0)\rangle = 2{\rm Im}\left\{\frac{M_{Pl}^4}{4}\frac{1}{(2\pi)^6}\int d^3k\, d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}{+}\mathbf{p}{+}\mathbf{q})\int_{-\infty_-}^{0} d\tau' a^2\left[T_1(\tau')+T_2(\tau')\right]\right\}, \tag{4.127}$$

where we have defined

$$T_1(\tau') \equiv q_k q_l \epsilon_{ik}^{\lambda_1'}(\mathbf{k})\epsilon_{jl}^{\lambda_2'}(\mathbf{p})\epsilon_{ij}^{\lambda_3'}(\mathbf{q})\,\langle 0|\gamma_{\mathbf{k}_1}^{\lambda_1}(0)\gamma_{\mathbf{k}_2}^{\lambda_2}(0)\gamma_{\mathbf{k}_3}^{\lambda_3}(0)\gamma_{\mathbf{k}}^{\lambda_1'}(\tau')\gamma_{\mathbf{p}}^{\lambda_2'}(\tau')\gamma_{\mathbf{q}}^{\lambda_3'}(\tau')|0\rangle, \tag{4.128}$$

$$T_2(\tau') \equiv -\frac{1}{2}q_k q_l \epsilon_{ij}^{\lambda_1'}(\mathbf{k})\epsilon_{kl}^{\lambda_2'}(\mathbf{p})\epsilon_{ij}^{\lambda_3'}(\mathbf{q})\,\langle 0|\gamma_{\mathbf{k}_1}^{\lambda_1}(0)\gamma_{\mathbf{k}_2}^{\lambda_2}(0)\gamma_{\mathbf{k}_3}^{\lambda_3}(0)\gamma_{\mathbf{k}}^{\lambda_1'}(\tau')\gamma_{\mathbf{p}}^{\lambda_2'}(\tau')\gamma_{\mathbf{q}}^{\lambda_3'}(\tau')|0\rangle. \tag{4.129}$$

In order to evaluate these two contributions we exploit the Wick's theorem, exactly as we did in the case of scalar perturbations. Since

$$\langle \gamma_{\mathbf{k}_1}^{\lambda_1}(0)\gamma_{\mathbf{k}}^{\lambda_1'}(\tau')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k})\delta^{\lambda_1\lambda_1'}u_{\lambda_1}(k_1,0)u_{\lambda_1'}^*(k,\tau'), \tag{4.130}$$

we find

$$\begin{aligned}T_1(\tau') =&(2\pi)^9 q_k q_l \epsilon_{ik}^{\lambda_1'}(\mathbf{k})\epsilon_{jl}^{\lambda_2'}(\mathbf{p})\epsilon_{ij}^{\lambda_3'}(\mathbf{q})\Big[\delta^{(3)}(\mathbf{k}_1+\mathbf{k})\delta^{\lambda_1\lambda_1'}u_{\lambda_1}(k_1,0)u_{\lambda_1'}^*(k,\tau')\delta^{(3)}(\mathbf{k}_2+\mathbf{p})\cdot\\ &\cdot\delta^{\lambda_2\lambda_2'}u_{\lambda_2}(k_2,0)u_{\lambda_2'}^*(p,\tau')\delta^{(3)}(\mathbf{k}_3+\mathbf{q})\delta^{\lambda_3\lambda_3'}u_{\lambda_3}(k_3,0)u_{\lambda_3'}^*(q,\tau')+{\rm perm}(k_i)\Big],\end{aligned} \tag{4.131}$$

$$\begin{aligned}T_2(\tau') =&-\frac{(2\pi)^9}{2}q_k q_l \epsilon_{ij}^{\lambda_1'}(\mathbf{k})\epsilon_{kl}^{\lambda_2'}(\mathbf{p})\epsilon_{ij}^{\lambda_3'}(\mathbf{q})\Big[\delta^{(3)}(\mathbf{k}_1+\mathbf{k})\delta^{\lambda_1\lambda_1'}u_{\lambda_1}(k_1,0)u_{\lambda_1'}^*(k,\tau')\delta^{(3)}(\mathbf{k}_2+\mathbf{p})\cdot\\ &\cdot\delta^{\lambda_2\lambda_2'}u_{\lambda_2}(k_2,0)u_{\lambda_2'}^*(p,\tau')\delta^{(3)}(\mathbf{k}_3+\mathbf{q})\delta^{\lambda_3\lambda_3'}u_{\lambda_3}(k_3,0)u_{\lambda_3'}^*(q,\tau')+{\rm perm}(k_i)\Big].\end{aligned} \tag{4.132}$$

Plugging these expressions into (4.127), this becomes

$$\begin{aligned}\langle \gamma_{\mathbf{k}_1}^{\lambda_1}(0)\gamma_{\mathbf{k}_2}^{\lambda_2}(0)\gamma_{\mathbf{k}_3}^{\lambda_3}(0)\rangle =&(2\pi)^3\frac{M_{Pl}^2}{2}\,\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,{\rm Im}\bigg\{\Big[k_{3k}k_{3l}\epsilon_{ik}^{\lambda_1}(\mathbf{k}_1)\epsilon_{jl}^{\lambda_2}(\mathbf{k}_2)\epsilon_{ij}^{\lambda_3}(\mathbf{k}_3)+\\ &-\frac{1}{2}k_{3k}k_{3l}\epsilon_{ij}^{\lambda_1}(\mathbf{k}_1)\epsilon_{kl}^{\lambda_2}(\mathbf{k}_2)\epsilon_{ij}^{\lambda_3}(\mathbf{k}_3)\Big]\,I+{\rm perm}(k_i)\bigg\},\end{aligned} \tag{4.133}$$

with

$$I \equiv u_{\lambda_1}(k_1,0)u_{\lambda_2}(k_2,0)u_{\lambda_3}(k_3,0)\int_{-\infty_-}^{0} d\tau' u_{\lambda_1}^*(k_1,\tau')u_{\lambda_2}^*(k_2,\tau')u_{\lambda_3}^*(k_3,\tau'). \tag{4.134}$$

To compute explicitly the integral I we need the analytic expression of the mode function $u_\lambda(k,\tau)$. Since we are interested in obtaining the dominant contribution in the slow-roll parameters, we can take the zero-th order value of the mode function:

$$u_\lambda(k,\tau) = \frac{iH}{M_{Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau}. \tag{4.135}$$

For the same exact reason we also take

$$a(\tau) \simeq -\frac{1}{H\tau}, \tag{4.136}$$

with $H \simeq$ const. As we did for scalar perturbations, to minimize the error which comes from taking the Hubble parameter as a constant, we evaluate it at the horizon crossing time of the total momentum $k_T = k_1 + k_2 + k_3$, and denote it H_*. The integral I can now be written as

$$I = \frac{H_*^4}{M_{Pl}^6} \frac{1}{(k_1k_2k_3)^3} \int_{-\infty_-}^0 \frac{d\tau'}{\tau'^2}(1 - ik_1\tau')(1 - ik_2\tau')(1 - ik_3\tau')e^{ik_T\tau'}. \tag{4.137}$$

This is the same integral as in (4.113), hence we find

$$I = i\frac{H_*^4}{M_{Pl}^6} \frac{1}{(k_1k_2k_3)^3} \left(-k_T + \frac{\sum_{i>j} k_ik_j}{k_T} + \frac{k_1k_2k_3}{k_T^2}\right). \tag{4.138}$$

We can then write the bispectrum (4.133) as

$$\begin{aligned}\langle\gamma^{\lambda_1}_{\mathbf{k}_1}(0)\gamma^{\lambda_2}_{\mathbf{k}_2}(0)\gamma^{\lambda_3}_{\mathbf{k}_3}(0)\rangle =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left(\frac{H_*}{M_{Pl}}\right)^4 \frac{1}{2(k_1k_2k_3)^3} \left(-k_T + \frac{\sum_{i>j} k_ik_j}{k_T} + \frac{k_1k_2k_3}{k_T^2}\right)\cdot\\ &\cdot \left[k_{3k}k_{3l}\epsilon^{\lambda_1}_{ik}(\mathbf{k}_1)\epsilon^{\lambda_2}_{jl}(\mathbf{k}_2)\epsilon^{\lambda_3}_{ij}(\mathbf{k}_3) - \frac{1}{2}k_{3k}k_{3l}\epsilon^{\lambda_1}_{ij}(\mathbf{k}_1)\epsilon^{\lambda_2}_{kl}(\mathbf{k}_2)\epsilon^{\lambda_3}_{ij}(\mathbf{k}_3) + \mathrm{perm}(k_i)\right].\end{aligned} \tag{4.139}$$

Taking into account all the possible permutations of the momenta, this can be written as [13] (see also [73]):

$$\begin{aligned}\langle\gamma^{\lambda_1}_{\mathbf{k}_1}(0)\gamma^{\lambda_2}_{\mathbf{k}_2}(0)\gamma^{\lambda_3}_{\mathbf{k}_3}(0)\rangle =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left(\frac{H_*}{M_{Pl}}\right)^4 \frac{1}{2(k_1k_2k_3)^3} \left(-k_T + \frac{\sum_{i>j} k_ik_j}{k_T} + \frac{k_1k_2k_3}{k_T^2}\right)\cdot\\ &\cdot \left(-\epsilon^{\lambda_1}_{ii'}(\mathbf{k}_1)\epsilon^{\lambda_2}_{jj'}(\mathbf{k}_2)\epsilon^{\lambda_3}_{ll'}(\mathbf{k}_3)t_{ijl}t_{i'j'l'}\right),\end{aligned} \tag{4.140}$$

where t_{ijl} is given by

$$t_{ijl} = k_{1l}\delta_{ij} + k_{2i}\delta_{jl} + k_{3j}\delta_{il}. \tag{4.141}$$

Notice that the bispectrum of tensor modes is subdominant in the slow-roll parameters with respect to the bispectrum of scalar perturbations (4.121). Analogously to the scalar case, however, also the tensor bispectrum peaks in the squeezed limit ($k_1 \ll k_2, k_3$), where it takes the form [13, 74]

$$\langle\gamma^{\lambda_1}_{\mathbf{k}_1}(0)\gamma^{\lambda_2}_{\mathbf{k}_2}(0)\gamma^{\lambda_3}_{\mathbf{k}_3}(0)\rangle_{k_1\ll k_2,k_3} = (2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\frac{3}{2}P_T(k_2)P_T(k_1)\delta^{\lambda_2\lambda_3}\frac{k_{2i}k_{2j}\epsilon^{\lambda_1}_{ij}(\mathbf{k_1})}{k_2^2}. \tag{4.142}$$

4.4.4 Mixed correlators

Analogously to what we have done for scalar and tensor perturbations, one can also compute the mixed correlators between the primordial perturbations. Without entering in the details of the calculations, we give the final results [13]:

$$\langle\gamma^{\lambda}_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1{+}\mathbf{k}_2{+}\mathbf{k}_3) \left(\frac{H_*}{M_{Pl}}\right)^4 \frac{1}{2(k_1k_2k_3)^3}\frac{1}{\epsilon} \left(-k_T + \frac{\sum_{i>j} k_ik_j}{k_T} + \frac{k_1k_2k_3}{k_T^2}\right)\epsilon^{\lambda}_{ij}k_2^ik_3^j, \tag{4.143}$$

$$\langle\zeta_{\mathbf{k}_1}\gamma^{\lambda_2}_{\mathbf{k}_2}\gamma^{\lambda_3}_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1{+}\mathbf{k}_2{+}\mathbf{k}_3) \left(\frac{H_*}{M_{Pl}}\right)^4 \frac{1}{2(k_1k_2k_3)^3} \left(-\frac{1}{16}k_1^3 + \frac{1}{8}k_1(k_2^2+k_3^2) + \frac{k_2^2k_3^2}{k_T}\right)\epsilon^{\lambda_2}_{ij}\epsilon^{\lambda_3}_{ij}. \tag{4.144}$$

The correlator between one tensor and two scalar perturbations is of the same order in the slow-roll parameters as the correlator between scalar perturbations (4.121). The correlator between one scalar and two tensor perturbations is instead subdominant in the slow-roll parameters, like the correlator between tensor perturbations (4.140).

All the correlators are dominated by the squeezed limit, since all of them contain the factor $(k_1 k_2 k_3)^{-3}$, which gives its maximum contribution when one of the momenta goes to zero.

The experimental constraints on the scalar bispectrum are those presented in Subsection 4.2.4, and come from the measurements of the temperature anisotropies and polarization of the CMB made by the Planck satellite. Since we have not detected primordial gravitational waves yet, we have no measurements of the other correlators.

4.5 Summary of the predictions from single field slow-roll inflation

In the previous chapter we have studied the dynamics of the perturbations at the quadratic level (namely, we have studied the evolution of the free fields) in single field slow-roll inflationary models. Then, in this chapter we have studied non-Gaussianity, which is determined by the interactions among the dynamical degrees of freedom of the theory. It is now time to summarize which are the fundamental predictions arising in single field slow-roll models of inflation:

- The primordial perturbations are *adiabatic*: this is a consequence of the fact that the effect of the quantum fluctuations is that of making different regions of the Universe inflate for a slightly different amount of time. This produces curvature perturbations, which then affect all the different components of the Universe in the same way and cannot give rise to variations in the relative energy densities between them.

- The spectrum of the primordial perturbations is *almost scale-invariant*: this is due to the fact that the Hubble parameter is a slowly decreasing function of time during slow-roll inflation. We have seen that the amplitude of the fluctuations of a given mode k is roughly given by the Hubble parameter at the horizon-crossing time of the corresponding mode. Thus the shortest modes, which leave the horizon after the longest modes, have a slightly lower ampltude. This results in a almost scale-invariant, red-tilted power spectrum.

- The primordial perturbations are very nearly *Gaussian*: this reflects the fact that in single field slow-roll models the self-interaction terms for the inflaton field are suppressed by the requirement of having a sufficiently flat potential to guarantee a long enough period of inflation.

- Tensor perturbations of the metric (i.e., primordial gravitational waves) are produced: like for the scalar perturbations, they have a red-tilted, almost scale-invariant and nearly Gaussian spectrum. Their amplitude is directly related to the energy scale at which inflation occurred.

Chapter 5

Parity violation in inflation with Chern-Simons gravity

5.1 Modified gravity and inflation

Standard slow-roll models of inflation are based on General Relativity (GR) as the theory of gravity. There are however different reasons to believe that GR does not provide the full complete description of the gravitational interaction. From the point of view of fundamental physics, a quantum description of the gravitational interaction lacks, and cannot be included within the framework of GR. From the cosmological side, we have seen in depth that the standard cosmological model requires the introduction of (at least) an additional field to explain the accelerated expansion of the early Universe, in order to solve the shortcomings of the FLRW model. Likewise, there is solid observational evidence that our Universe is currently undergoing a second phase of accelerated expansion. The most simple way to explain this feature is by introducing the so-called Dark Energy through a cosmological constant in the field equations. However, it is well known that the theoretical prediction for the vacuum energy is many order of magnitudes greater than the value required by the observations. Moreover, the so-called "coincidence problem" ("why do we live in the precise period of the cosmic history in which the energy density of Dark Energy and Dark Matter are of the same order of magnitude?") seems to suggest a dynamical explanation which goes beyond a simple cosmological constant. For all these reasons, many models which go beyond GR have been studied during the last years.

In GR the gravitational interaction is mediated by a rank-2 tensor field, which in the quantum view is represented by a massless spin-2 particle. One common way to modify GR is to include additional fields besides the metric tensor, like a scalar field in the case of scalar-tensor theories.

A second way to modify the theory of gravity is to include in the gravitational action higher-order curvature invariants. Indeed, it is well known that the Einstein's field equations with a cosmological constant are the only second order equations for a pure metric theories in four dimensions. These are obtained by a (diffeomorphism invariant) action constructed with only the metric tensor and its second derivatives. Many theories that aim at unifying all the fundamental interactions (including gravity), like string theory, lead to effective actions which contain non-minimal coupling with the geometry or terms with higher-order curvature invariants. These tipically lead to higher-order field equations.

Higher-order curvature invariants can be included in the action following an effective field theory approach, as we will soon see. These new terms should become important at high energies, but the theory of gravity should then reduce to GR in the limits where it is experimentally tested and confirmed. Since inflation involves very high energy scales, it is possible that signatures of modification to GR are left imprinted in the fluctuations coming from inflation. In this sense inflation can be used to test and constrain fundamental physics.

One of the most important features of inflationary models which include modifications to the theory of gravity is the production of primordial non-Gaussianities which, as we have seen, are highly suppressed in the "standard models" with GR. Modifying the theory of gravity usually leads to the introduction of additional degrees of freedom in the theory and also new interaction terms. Both of these can lead to an enhancement of non-Gaussianity. An example of this is given in Appendix D, where a brief review of $f(R)$ theories of gravity and their application to inflation is presented.

A second interesting issue to consider when studying inflationary models within modified theories of gravity is the possible modification to the spectrum of primordial gravitational waves. This is particularly interesting in view of the future experiments with space interferometers, like LISA. We have seen that PGWs are a fundamental prediction of all inflationary models; however, single field slow-roll models of inflation predict a spectrum of PGWs below the range of current ground-based GW detectors, and also below the range of future space interferometers like LISA. Modifying the theory of gravity could vary the amplitude and the tilt of PGWs, possibly enhancing their spectrum at scales accessible to LISA (see e.g. [15]).

Another important feature which can arise as a consequence of some modifications to the theory of gravity is the breaking of parity symmetry. Also this feature is particularly important because, if observed, would give indication of deviations from the standard models. The breaking of parity manifests itself through the polarization of PGWs into left and right circular polarization states. These propagate following different dynamical equations and thus acquire different power spectra. This will be one of the main subject of study of the rest of this thesis.

5.2 Modified gravity from an effective field theory approach

The Einstein-Hilbert action is the most general diffeomorphism invariant action constructed with covariant terms containing no more than two derivatives of the metric tensor. Thus, it is built only through the Ricci scalar R, and the resulting Lagrangian density is

$$\mathcal{L} = \sqrt{-g}\left(\frac{M_{Pl}^2}{2}R\right), \tag{5.1}$$

where $g \equiv det\, g_{\mu\nu}$ and $M_{Pl}^2 = (8\pi G)^{-1}$ is the reduced Planck mass.

To modify Einstein's theory of gravity we can assume that (5.1) is just the first term in a generic effective field theory (EFT) and admit terms containing higher numbers of derivatives of the metric tensor, such that they are suppressed by negative powers of some mass scale M which characterizes whatever fundamental theory underlies this effective field theory [75]. The first correction is constructed with the most general covariant terms built with four derivatives of the metric, and can be written as

$$\Delta\mathcal{L} = \sqrt{-g}\left(f_1 R^2 + f_2 R_{\mu\nu}R^{\mu\nu} + f_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) + f_4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta}, \tag{5.2}$$

where f_n are dimensionless coefficients and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor density. It is useful (for reasons that we will see) to rewrite the Lagrangian (5.2) in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$, which is defined by taking the Riemann tensor and removing from it all of its contractions:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\nu}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{R}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right). \tag{5.3}$$

The Weyl tensor is defined so that all the possible contractions of $C_{\mu\nu\rho\sigma}$ vanish and at the same time the simmetries of the Riemann tensor still hold. An important property of the Weyl tensor, which will be used later on, is that it is conformally invariant, namely it is invariant under *conformal transformations*. These act as a local rescaling of the metric:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \omega(x) g_{\mu\nu}(x). \tag{5.4}$$

Moreover, using the definition (5.3), one can prove that the following relations hold [17]:

$$\epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\alpha\beta} C_{\rho\sigma\alpha\beta} = \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta}, \tag{5.5}$$

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2. \tag{5.6}$$

By using them, we can rewrite the Lagrangian (5.2) as

$$\Delta\mathcal{L} = \sqrt{-g}\left[f_1 R^2 + f_2 R_{\mu\nu}R^{\mu\nu} + f_3\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right] + f_4 \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\alpha\beta} C_{\rho\sigma\alpha\beta}. \tag{5.7}$$

If we now define

$$h_1 \equiv f_1 - \frac{1}{3}f_3, \quad h_2 \equiv f_2 + 2f_3, \quad h_3 \equiv f_3, \tag{5.8}$$

the Lagrangian (5.7) becomes

$$\Delta\mathcal{L} = \sqrt{-g}\left(h_1 R^2 + h_2 R_{\mu\nu}R^{\mu\nu} + h_3 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right) + f_4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta}. \tag{5.9}$$

Finally, renaming h_n as f_n, we have

$$\Delta\mathcal{L} = \sqrt{-g}\left(f_1 R^2 + f_2 R_{\mu\nu}R^{\mu\nu} + f_3 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right) + f_4 \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\alpha\beta} C_{\rho\sigma\alpha\beta}. \tag{5.10}$$

The usefulness of rewriting the Lagrangian in terms of the Weyl tensor will become evident when we will study inflation with Chern-Simons gravity in the next section. Indeed, this will allow us to understand more easily some properties of the Chern-Simons operator and if/how it modifies the dynamics (both of the background and of the perturbations) of inflation.

Until now we have included only gravity, namely only the metric tensor in the Lagrangian. If we introduce an additional scalar field ϕ, which in the case of inflation is represented by the inflaton itself, we have to include also all the covariant terms made with up to four spacetime derivatives of ϕ. Therefore, we obtain the full Lagrangian density [17, 75]

$$\begin{aligned}\mathcal{L} =& \sqrt{-g}\left[\frac{1}{2}M_{Pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) + f_1(\phi)\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right)^2 + f_2(\phi)g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\,\Box\phi + \right.\\ &+ f_3(\phi)\left(\Box\phi\right)^2 + f_4(\phi)R^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + f_5(\phi)Rg^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + f_6(\phi)R\,\Box\phi + f_7(\phi)R^2 + \\ &\left. + f_8(\phi)R^{\mu\nu}R_{\mu\nu} + f_9(\phi)C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right] + f_{10}(\phi)\epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\alpha\beta} C_{\rho\sigma\alpha\beta},\end{aligned} \tag{5.11}$$

where now the dimensionless coefficients f_n depend on ϕ. In principle these coupling functions must be fixed by the underlying fundamental theory of quantum gravity. However, since we haven't succeeded in constructing a theory of quantum gravity yet, we can treat them as free parameters and try to fix or constrain them through cosmological observations.

5.3 Inflation with Chern-Simons modified gravity

From a phenomenological point of view it is particularly interesting to study which is the effect on inflation of the last term in the Lagrangian (5.11). This is the so-called Chern-Simons term [76] and it is a parity-breaking term, being constructed with the Levi-Civita tensor. Has we have already said, considering a theory that breaks parity has the effect of polarizing PGWs into left and right circular polarization states. The "new" Chern-Simons term is coupled to the inflaton field through a generic coupling function $f(\phi)$. Therefore our model is described by the following Lagrangian:

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}M_{Pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + f(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\alpha\beta}C_{\rho\sigma\alpha\beta}. \tag{5.12}$$

The first thing we want to emphasize is that the Chern-Simons term does not modify the background dynamics of inflation, since it vanishes when computed on the background metric. Indeed the background metric, which is the FLRW metric, is conformally flat, and the Weyl tensor vanishes in Minkowski spacetime. Since the Weyl tensor is conformally invariant, this implies that it has to vanish also when computed in FLRW.

Another important point is that one can prove that the Chern-Simons term is a total derivative term [77]. Therefore the coupling $f(\phi)$ with the inflaton field is fundamental, otherwise the new operator would give a vanishing contribution at the level of the equations of motion.

In the following we work in the spatially-flat gauge, which leaves us with the 3-metric h_{ij} that, at the non-linear level, has the form

$$h_{ij} = a^2\left[\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma^k{}_j + \dots\right], \quad \gamma_i{}^i = 0, \quad \partial^i\gamma_{ij} = 0, \tag{5.13}$$

plus the scalar perturbation in the inflaton field $\delta\phi$. In order to pass from the inflaton perturbation to the curvature perturbation on comoving hypersurfaces ζ, which is conserved above the horizon and directly related to the physics of the CMB, we will then use the non-linear relation which we have used also in the previous chapter:

$$\begin{aligned}\zeta =&\zeta_1 + \frac{1}{2}\frac{\ddot\phi}{\dot\phi H}\zeta_1^2 + \frac{1}{4}\frac{\dot\phi^2}{H^2}\zeta_1^2 + \frac{1}{H}\dot\zeta_1\zeta_1 - \frac{1}{4}\frac{a^{-2}}{H^2}\partial^{-2}\partial_i\partial_j(\partial^i\zeta_1\partial^j\zeta_1) + \frac{1}{2H}\partial_i\psi\partial^i\zeta_1 + \\ &- \frac{1}{2H}\partial^{-2}\partial_i\partial_j(\partial^i\psi\partial^j\zeta_1) - \frac{1}{4H}\dot\gamma_{ij}\partial^i\partial^j\zeta_1,\end{aligned} \tag{5.14}$$

where ϕ is the background value of the inflaton, ψ is the scalar perturbation to the shift vector and ζ_1 is the value of ζ at linear level:

$$\zeta_1 = -\frac{H}{\dot\phi}\delta\phi. \tag{5.15}$$

On super-horizon scales only the first three terms in Eq. (5.14) are non-vanishing.

Another issue one has to consider when introducing modifications to the theory of gravity is that these can introduce additional degrees of freedom in theory, or make dynamical some of the fields which were not in the theory with GR. We have seen that the lapse function N and the shift vector N_i are auxiliary fields in standard gravity, so that their equations of motion are actually constraints which can be solved order by order in perturbation theory. Then, plugging back the solutions into the action, we can obtain the action for the dynamical degrees of freedom. Introducing the Chern-Simons term in the Lagrangian certainly does not modify the behaviour of N, which is a scalar field and hence does not receive contributions from parity breaking terms. The same happens for the scalar perturbation in the shift vector. The vector perturbation of the shift vector can instead in principle receive a contribution, possibly becoming a dynamical field. In [19] it has indeed been shown that the Einstein-Hilbert action plus the Chern-Simons term propagate 5 degrees of freedom. It is not our aim that of considering this additional degree of freedom and its interactions with the degrees of freedom of the standard theory, hence we can set it to zero. In this case we are left with the ususal 3 degrees of freedom, 2 of which represent the transverse and traceless tensor perturbations of the metric, i.e. the gravitational waves, and with the other one represented by the inflaton field (in the spatially flat gauge).

We can now start to study the parity breaking signatures introduced by the Chern-Simons coupling. The power spectrum of scalar perturbations does not receive any contributions by the Chern-Simons term being a scalar quantity, as already explained for the scalar perturbations in N and N_i. Therefore, we start our analysis by considering the power spectrum of tensor perturbations; then we will study the parity breaking signatures in the primordial bispectra, both tensor bispectrum and mixed correlators.

5.3.1 Power spectrum of tensor perturbations

When considering theories that break parity it is convenient to expand the tensor perturbations in the Fourier space in the chiral polarization basis (see Appendix C) as

$$\gamma_{ij}(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=L,R} \epsilon_{ij}^{(s)}(\mathbf{k})\gamma_s(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.16}$$

where

$$\epsilon_{ij}^L = \frac{\epsilon_{ij}^+ - i\epsilon_{ij}^\times}{\sqrt{2}}, \quad \epsilon_{ij}^R = \frac{\epsilon_{ij}^+ + i\epsilon_{ij}^\times}{\sqrt{2}} \tag{5.17}$$

and

$$\gamma_R(\mathbf{k},t) \equiv \frac{\gamma_+(\mathbf{k},t) - i\gamma_\times(\mathbf{k},t)}{\sqrt{2}}, \quad \gamma_L(\mathbf{k},t) \equiv \frac{\gamma_+(\mathbf{k},t) + i\gamma_\times(\mathbf{k},t)}{\sqrt{2}}. \tag{5.18}$$

The usefulness of this decomposition lies in the fact that under a parity transformation the right and the left modes get transformed into each other. The equations of motion for the two circular polarization modes are decoupled, while parity-violating terms mix the + and × components[1]. Then one can prove that the following relations hold (see e.g. [78]):

$$\epsilon_{ij}^L(\mathbf{k})\epsilon_L^{ij}(\mathbf{k}) = 0 = \epsilon_{ij}^R(\mathbf{k})\epsilon_R^{ij}(\mathbf{k}), \tag{5.19}$$

$$\epsilon_{ij}^L(\mathbf{k})\epsilon_R^{ij}(\mathbf{k}) = 2, \tag{5.20}$$

$$\epsilon_{ij}^R(-\mathbf{k}) = \epsilon_{ij}^L(\mathbf{k}), \tag{5.21}$$

$$\gamma_L(-\mathbf{k}) = \gamma_R^*(\mathbf{k}), \tag{5.22}$$

$$k_l\epsilon^{mlj}\epsilon_j^{(s)j}(\mathbf{k}) = -i\lambda_s k\epsilon^{(s)im}(\mathbf{k}), \tag{5.23}$$

where $\lambda_R = +1$ and $\lambda_L = -1$.

In order to calculate the power spectrum of PGWs we have to expand the Lagrangian (5.12) at second order in tensor perturbations. To do so, we need to find the expressions of the lapse function N and of the shift vector N_i only at first order. Since it is not possible to have first order perturbations in N and N_i including only tensor perturbations, we are left solely with their zero-order value, namely $N = 1$ and $N_i = 0$. Moreover, since the Weyl tensor vanishes on the background, the contribution of the Chern-Simons term at second order in tensor perturbations comes only from the term $f(\phi_0)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\alpha\beta}|_T^{(1)}C_{\rho\sigma\alpha\beta}|_T^{(1)}$, where ϕ_0 is the background value of the inflaton field and the indices (1) and T refer to the fact that the corresponding terms are evaluated at first order in tensor perturbations. The full action (GR+Chern-Simons) at second order in tensor perturbations takes the form [17]

$$S_{\gamma\gamma} = \sum_{s=L,R}\int d\tau \frac{d^3k}{(2\pi)^3} A_{T,s}^2 \left[|\gamma_s'(\mathbf{k},\tau)|^2 - k^2|\gamma_s(\mathbf{k},\tau)|^2\right], \tag{5.24}$$

[1] Indeed, the + and × modes are not eigenstates of the parity transformation.

where we have introduced the conformal time τ and

$$A_{T,s}^2 = \frac{M_{Pl}^2}{2} a^2 \left(1 - \lambda_s \frac{k_{phys}}{M_{CS}}\right), \tag{5.25}$$

with

$$M_{CS} \equiv \frac{M_{Pl}^2}{8\dot{f}(\phi)}. \tag{5.26}$$

The latter is the so-called Chern-Simons mass and it has been introduced for the following reason: since $\lambda_R = +1$, the right modes which have a physical wavenumber $k_{phys} = k/a$ greater than the Chern-Simons mass have a negative kinetic term, hence they are ghost fields (see e.g. [79, 80]). In order to be safe in this sense, let us introduce a UV cut-off Λ in theory such that $\Lambda \ll M_{CS}$. If we require that $k_{phys} < \Lambda$ at the beginning of inflation, this condition continues to be valid as inflation proceeds, given that k_{phys} decreases, and we are free from ghost instabilities. We have seen that an important feature of the inflationary expansion is that the modes started deep inside the horizon, before crossing it and thus becoming "classical perturbations". This means that at the beginning of inflation $k_{phys} \gg H$. From the two conditions $k_{phys} < \Lambda$ and $k_{phys} \gg H$ taken together, it follows that $M_{CS} \gg H$ during inflation.

By making the field redefinition

$$\mu_s \equiv A_{T,s}\gamma_s, \tag{5.27}$$

the action (5.24) becomes

$$S_{\gamma\gamma} = \sum_{s=L,R} \int d\tau \frac{d^3k}{(2\pi)^3} \left[|\mu_s'(\mathbf{k},\tau)|^2 - k^2 |\mu_s(\mathbf{k},\tau)|^2 + \frac{A_{T,s}''}{A_{T,s}} |\mu_s(\mathbf{k},\tau)|^2 \right], \tag{5.28}$$

where we have performed an integration by parts and neglected the total derivative term, which gives no contributions at the level of the equations of motion. By varying this action we find the equations of motions for the fields μ_s, which read

$$\mu_s'' + \left(k^2 - \frac{A_{T,s}''}{A_{T,s}}\right)\mu_s = 0. \tag{5.29}$$

Following [16], the second term inside the parentheses can be computed by exploiting the relation

$$\frac{A_{T,s}''}{A_{T,s}} = \frac{d}{d\tau}\left(\frac{A_{T,s}'}{A_{T,s}}\right) + \left(\frac{A_{T,s}'}{A_{T,s}}\right)^2, \tag{5.30}$$

with

$$\frac{A_{T,s}'}{A_{T,s}} = \frac{a}{2}\frac{(A_{T,s}^2)^{\cdot}}{A_{T,s}^2} = \frac{a}{2}\frac{\left(2H - \lambda_s \frac{k}{a}\frac{H}{M_{CS}}\right)}{\left(1 - \lambda_s \frac{k}{a}\frac{1}{M_{CS}}\right)} = -\frac{1+\epsilon}{\tau} + \lambda_s \frac{k}{2}\frac{H}{M_{CS}} + \mathcal{O}\left[\epsilon^2, \left(\frac{H}{M_{CS}}\right)^2, \epsilon\frac{H}{M_{CS}}\right]. \tag{5.31}$$

In this computation we have used

$$aH = -\frac{1}{(1-\epsilon)\tau} = -\frac{1+\epsilon}{\tau} + \mathcal{O}(\epsilon^2) \tag{5.32}$$

and we have assumed that the Chern-Simons mass stays constant during inflation. To check this last assumption, let us introduce a dimensionless parameter ξ which quantifies the rate of time variation of the Chern-Simons mass in a Hubble time:

$$\xi \equiv \frac{\dot{M}_{CS}}{HM_{CS}} = \eta_V - \epsilon_V - \sqrt{2\epsilon_V} M_{Pl} \frac{f''(\phi)}{f'(\phi)}. \tag{5.33}$$

In slow-roll inflation the slow-roll parameters are much smaller than unity. Thus, in order to neglect the time variation of M_{CS} during inflation, we need to require that

$$\sqrt{2\epsilon_V} M_{Pl} \frac{f''(\phi)}{f'(\phi)} \ll 1. \tag{5.34}$$

Coming back to the computation of the equations of motions, from Eq. (5.30) and (5.31) we find

$$\frac{A''_{T,s}}{A_{T,s}} = \frac{2+3\epsilon_V}{\tau^2} - \frac{\lambda_s k}{\tau} \frac{H}{M_{CS}} + \mathcal{O}\left[\epsilon_V^2, \left(\frac{H}{M_{CS}}\right)^2, \epsilon_V \frac{H}{M_{CS}}\right]. \tag{5.35}$$

The general expression for $A''_{T,s}/A_{T,s}$, taking into account the time dependence of the Chern-Simons mass, is given by [18]

$$\frac{A''_{T,s}}{A_{T,s}} = \frac{2}{\tau^2}\left(1 + \frac{\lambda_s}{2}\frac{k_{phys}}{H}\frac{H}{M_{CS}}\mathcal{A}\right), \tag{5.36}$$

with

$$\mathcal{A} = \frac{1}{(1-\lambda_s k_{phys}/M_{CS})^2}\left\{\left[1 - \xi + \frac{\omega}{2} - \frac{\xi}{2H\tau}\right]\left(1 - \lambda_s \frac{k_{phys}}{M_{CS}}\right) - \lambda_s \frac{k_{phys}}{2M_{CS}}\left[\frac{1}{2} + \xi + \frac{\xi^2}{2}\right]\right\}, \tag{5.37}$$

where we have neglected some slow-corrections and

$$\omega \equiv \frac{\ddot{M}_{CS}}{M_{CS}H^2}. \tag{5.38}$$

If we neglect the time dependence of the Chern-Simons mass, $\xi \ll 1$ and $\omega \ll 1$, and since we have assumed $k_{phys} \ll M_{CS}$ and $H \ll M_{CS}$ in order to prevent some modes of the fields from becoming ghost fields, we recover $\mathcal{A} \simeq 1$, which agrees with the result (5.35). With the previous assumptions, we can obtain the equations of motion for the fields μ_s, which are given by

$$\mu''_s + \left(k^2 - \frac{2+3\epsilon_V}{\tau^2} + \frac{\lambda_s k}{\tau}\frac{H}{M_{CS}}\right)\mu_s = 0. \tag{5.39}$$

If we introduce the parameter

$$\nu_T^2 = \frac{9}{4} + 3\epsilon_V, \tag{5.40}$$

we can rewrite the equations of motion (5.39) as

$$\mu''_s + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \frac{\lambda_s k}{\tau}\frac{H}{M_{CS}}\right)\mu_s = 0, \tag{5.41}$$

At leading order in the slow-roll parameters, from (5.40) we have

$$\nu_T = \frac{3}{2} + \epsilon_V + \mathcal{O}(\epsilon_V^2). \tag{5.42}$$

As we have done also in the previous chapters, by redefining the fields we have been able to reabsorb the damping term in the equations of motions, which have now the form of the equation of an harmonic oscillator with a time-dependent frequency accounting for the dynamics of the spacetime where the fields evolve. With respect to the standard slow-roll models of inflation, the equations of motions (5.41) have an additional term in the effective mass, which is proportional to τ^{-1} and depends on the polarization index λ_s. This means that the two polarization states have a different dynamical evolution and this is a consequence of the fact that the Chern-Simons term is not invariant under a parity transformation. Therefore the dynamics of right and left-circularly polarized gravitational waves is different and we expect also their power spectra to be different.

We can now canonically quantize the perturbations by expanding the field μ_s in terms of the creation and annihilation operators as

$$\hat{\mu}_s(\mathbf{k},\tau) = u_s(k,\tau)\hat{a}_s(\mathbf{k}) + u_s^*(k,\tau)\hat{a}_s^\dagger(-\mathbf{k}). \tag{5.43}$$

The creation and annihilation operators satisfy the equal time commutation relations

$$[\hat{a}_s(\mathbf{k}), \hat{a}_{s'}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')], \tag{5.44}$$

and act on the vacuum state obeying

$$\hat{a}_s|0\rangle = 0, \quad \langle 0|\hat{a}_s^\dagger = 0. \tag{5.45}$$

From Eq. (5.41), the equations of motion for the mode functions $u_s(k,\tau)$ read

$$u_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \frac{\lambda_s k}{\tau}\frac{H}{M_{CS}}\right)u_s = 0. \tag{5.46}$$

Assuming as initial condition the Bunch-Davies vacuum state

$$\lim_{-k\tau\to\infty} u_s(k,\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}, \tag{5.47}$$

the solution of Eq. (5.46) is given by [17]

$$u_s(k,\tau) = 2\sqrt{-k^2\tau^3}\, e^{-i(\frac{\pi}{4}-\frac{\pi}{2}\nu_T)} e^{-ik\tau}\, U\left(\frac{1}{2}+\nu_T - \lambda_s\frac{H}{M_{CS}}, 1+2\nu_T, 2ik\tau\right) e^{\frac{\pi}{4}\lambda_s\frac{H}{M_{CS}}}, \tag{5.48}$$

where U is the confluent hypergeometric function. We are interested in the expression of $u_s(k,\tau)$ on super-horizon scales, $-k\tau \ll 1$, which can be written as

$$u_s(k,\tau)|_{-k\tau\ll 1} = \sqrt{\frac{1}{2k^3\tau^2}} e^{-i(\frac{\pi}{4}-\frac{\pi}{2}\nu_T)}\frac{\Gamma(\nu_T)}{\Gamma(3/2)}\left(\frac{-k\tau}{2}\right)^{\frac{3}{2}-\nu_T} e^{\frac{\pi}{4}\lambda_s\frac{H}{M_{CS}}}. \tag{5.49}$$

We can now compute the power spectrum on super-horizon scales for both the polarization states, namely

$$P_T^L = 2\frac{|u_L(z)_{z\ll 1}|^2}{A_{T,L}^2}, \tag{5.50}$$

$$P_T^R = 2\frac{|u_R(z)_{z\ll 1}|^2}{A_{T,R}^2}, \tag{5.51}$$

where we have defined $z \equiv -k\tau$. At leading order in the slow-roll parameters, these are given by [17]

$$P_T^L = \frac{P_T}{2}e^{-\frac{\pi}{4}\frac{H}{M_{CS}}}, \tag{5.52}$$

$$P_T^R = \frac{P_T}{2}e^{+\frac{\pi}{4}\frac{H}{M_{CS}}}, \tag{5.53}$$

where

$$P_T = \frac{4}{k^3}\frac{H^2}{M_{Pl}^2}\left(\frac{z}{2}\right)^{3-2\nu_T} \tag{5.54}$$

is the total power spectrum of tensor modes in standard models with GR.

Let us now introduce a quantity which chacarterizes the level of parity-breaking in the power spectrum. This is defined as the difference between the (super-horizon) power spectra of right and left modes, normalized to the total power spectrum in order to have a dimensionless quantity. Since the ratio H/M_{CS} is much smaller than unity for the reasons already explained, we can expand in series the exponentials in Eqs. (5.52)-(5.53), and we find

$$\Theta \equiv \frac{P_T^R - P_T^L}{P_T^R + P_T^L} = \frac{\pi}{2} \frac{H}{M_{CS}}. \tag{5.55}$$

Therefore the signature of parity violation in the power spectrum of PGWs is suppressed by the ratio H/M_{CS}, namely it is suppressed by the requirement of avoiding the formation of ghost instabilities during inflation.

We want to briefly comment about the possibilities of not neglecting the time variation of the Chern-Simons mass, which has been studied in [18]. If one considers the case in which M_{CST} grows fastly with time during inflation[2], then the factor H/M_{CS} becomes even more suppressed as inflation proceeds (independently on how much ξ and ω are large), and thus the same happens for the parity breaking signatures both in (5.35) and in the tensor power spectrum. Moreover, the time dependence of the Chern-Simons mass would lead to different equations of motion for the fields μ_s.

From the power spectra (5.52)-(5.53) it is then possible to compute the correction arising in the tensor-to-scalar ratio, given that the amplitude of scalar perturbations is not modified. This is given by [17]

$$r_{CS} = r\left(1 + \frac{\Theta^2}{4}\right), \tag{5.56}$$

where r is the tensor-to-scalar ratio in the standard models with GR. Moreover, at leading order in the slow-roll parameters and taking into account that $H/M_{CST} \ll 1$, the tensor spectral index is not modified with respect to the standard case, $n_T \simeq -2\epsilon$ (see [17] for the complete expression). Then, by using the consistency relation (3.128), which holds for standard models, it is possible to obtain the modified consistency relation

$$r_{CS} \simeq -8n_T\left(1 + \frac{\Theta^2}{4}\right). \tag{5.57}$$

Notice that the correction to the standard relation goes like Θ^2, hence it is even more suppressed than the chirality itself.

The conclusions of this analysis of the power spectrum of PGWs are that, both at level of chirality and of the tensor-to-scalar ratio, the corrections introduced by the Chern-Simons operator to the standard results are highly suppressed, and thus extremely difficult to observe.

5.3.2 Parity breaking signatures in the primordial bispectra

In the previous section we have seen that the chirality in the power spectrum of PGWs is suppressed by the ratio H/M_{CS}. It is the purpose of the rest of this chapter that of studying if the new interaction terms coming from the Chern-Simons operator can lead to detectable parity breaking signatures in the bispectra of primordial perturbations. This is particularly important also because in [81] it has been shown (we will give more details about this in Section 5.4) that, at the level of the power spectrum, the CMB can only weakly constrain the chirality Θ of PWGs. Considering higher-order correlators, starting from the three-point function, is thus extremely

[2]If M_{CST} decreased during inflation, at a certain time the condition $k_{phys} < M_{CST}$ would not be satisfied anymore and some of the modes of the field would acquire a negative kinetic energy, thus becoming ghost fields.

important. Following the results of [17], we will see that the only unsuppressed contribution comes from the mixed correlator between two tensor and one scalar perturbations. For the same reason as for the power spectrum, the bispectrum of scalar perturbations does not get modified by the Chern-Simons term and hence remains the same as in standard models with GR.

5.3.2.1 Bispectrum of tensor perturbations

To compute the bispectrum of tensor perturbations we need to expand the action at third order in tensor perturbations. The contributions we are interseted in come from the Lagrangian

$$\mathcal{L}_{\text{int}}^{\gamma\gamma\gamma} = f(\phi)\epsilon^{\mu\nu\rho\sigma}\left[C_{\mu\nu}{}^{\alpha\beta}|_T^{(2)} C_{\rho\sigma\alpha\beta}|_T^{(1)} + C_{\mu\nu}{}^{\alpha\beta}|_T^{(1)} C_{\rho\sigma\alpha\beta}|_T^{(2)}\right]. \tag{5.58}$$

In Ref. [17] an estimate of this interaction vertex is given by using the fact that the Chern-Simons term is a total derivative in the case of a constant coupling f. Therefore one has

$$\mathcal{L}_{\text{int}}^{\gamma\gamma\gamma} = f(\phi)\nabla_\mu A^\mu, \tag{5.59}$$

where A^μ depends only on γ_{ij} and its derivatives. Evaluating the coupling $f(\phi)$ on the background, so that it depends only on time through the background inflaton field, we can integrate (5.59) by parts, obtaining

$$\mathcal{L}_{\text{int}}^{\gamma\gamma\gamma} = -f'(\phi)A^0, \tag{5.60}$$

where, as usual, we have neglected the total derivative term. By means of the in-in formalism introduced in the previous chapter, the bispectrum of tensor modes can then be computed as

$$\langle\gamma_{s_1}(\mathbf{k}_1,0)\gamma_{s_2}(\mathbf{k}_2,0)\gamma_{s_3}(\mathbf{k}_3,0)\rangle = i\int_{-\infty}^0 d\tau' a\dot{f}(\phi)\langle 0|\left[\gamma_{s_1}(\mathbf{k}_1,0)\gamma_{s_2}(\mathbf{k}_2,0)\gamma_{s_3}(\mathbf{k}_3,0), A^0\right]|0\rangle. \tag{5.61}$$

In doing this computation one can take the mode function $u_s(k,\tau)$ with vanishing values of the slow-roll parameters and of the ratio H/M_{CS}; this corresponds to the zero-th order value in a series expansion of the exact solution (5.48). In such a way, the mode function has the same expression as in standard models in a pure de Sitter phase, namely

$$u_s(k,\tau) = \frac{iH}{M_{Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau}. \tag{5.62}$$

Thus, Eq. (5.61) gives [17]

$$\langle\gamma_{s_1}(\mathbf{k}_1,0)\gamma_{s_2}(\mathbf{k}_2,0)\gamma_{s_3}(\mathbf{k}_3,0)\rangle = i\dot{f}_*\left(\prod_{i=1}^3 \frac{H_*^2}{M_{Pl}^2 k_i^3}\right)\frac{1}{H_*}\int_{-\infty}^0 d\tau'\left(-\frac{1}{\tau}\right)f(k_i,\tau')e^{ik_T\tau'}, \tag{5.63}$$

where $k_T = k_1+k_2+k_3$, $f(k_i,\tau')$ is a polynomial function of its arguments, while H and $\dot{f}(\phi)$ have been taken to be constant at leading order in slow-roll. In particular the latter have been evaluated at the horizon crossing time (labelled by $*$) of the overall momentum k_T. We have also assumed that $a(\tau) = -1/H\tau$, still neglecting a correction of order ϵ. From the definition (5.26) of the Chern-Simons mass it follows that

$$\dot{f}(\phi) = \frac{M_{Pl}^2}{8M_{CS}}, \tag{5.64}$$

thus one can get the following estimate for the bispectrum of tensor modes [17]:

$$\langle\gamma_{s_1}(\mathbf{k}_1,0)\gamma_{s_2}(\mathbf{k}_2,0)\gamma_{s_3}(\mathbf{k}_3,0)\rangle \sim \frac{H}{M_{CS}}\left(\sum_{i\neq j} P_T(k_i)P_T(k_j)\right)M(k_i), \tag{5.65}$$

where $M(k_i)$ is a dimensionless function of the momenta k_i, which is of order 1 and gives the shape of the bispectrum, while $P_T(k_i)$ is the power spectrum of tensor perturbations given in (5.54). Notice that, as anticipated, the parity breaking signature in the bispectrum of tensor perturbations is still suppressed by the ratio H/M_{CS}, as in the case of the power spectrum.

5.3.2.2 Two scalars and one tensor bispectrum

To compute the $\langle\delta\phi\delta\phi\gamma\rangle$ bispetrum we need to expand the action taking second order terms in scalar perturbations and first order terms in tensor perturbations. These contributions come from the following Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm int}^{\delta\phi\delta\phi\gamma} = \epsilon^{\mu\nu\rho\sigma}\Bigg[&\frac{\partial f(\phi)}{\partial\phi}\delta\phi\, C_{\mu\nu}{}^{\alpha\beta}|_S^{(1)} C_{\rho\sigma\alpha\beta}|_T^{(1)} + \frac{\partial f(\phi)}{\partial\phi}\delta\phi\, C_{\mu\nu}{}^{\alpha\beta}|_T^{(1)} C_{\rho\sigma\alpha\beta}|_S^{(1)} + \\ &+ f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_S^{(2)} C_{\rho\sigma\alpha\beta}|_T^{(1)} + f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_T^{(1)} C_{\rho\sigma\alpha\beta}|_S^{(2)} + f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_{ST}^{(2)} C_{\rho\sigma\alpha\beta}|_S^{(1)} + \\ &+ f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_S^{(1)} C_{\rho\sigma\alpha\beta}|_{ST}^{(2)}\Bigg].\end{aligned} \tag{5.66}$$

The scalar perturbations in the metric come from the (first-order) perturbations in the lapse function N and in the shift vector N_i, which, in the spatially flat gauge, are roughly given by

$$N \sim \sqrt{\epsilon_V}\delta\phi, \quad N_i \sim \sqrt{\epsilon_V}\delta\phi. \tag{5.67}$$

This means that the scalar perturbations in N and N_i are subdominant in the slow-roll parameter ϵ_V with respect to those in the inflaton field, $\delta\phi$. As a consequence, the terms in the second and in the third row of the Lagrangian (5.66) are subdominant in ϵ_V with respect to the two terms in the first row. These can be computed and give [17]

$$\mathcal{L}_{\rm int}^{\delta\phi\delta\phi\gamma} = -\frac{M_{Pl}^2}{M_{CS}\dot{\phi}}\sqrt{\epsilon_V}(\partial^l\delta\phi)\epsilon^{ijk}\left[(\partial_k\delta\phi)\partial_i\gamma'_{lj}\right], \tag{5.68}$$

where it has been used, from the definition of the Chern-Simons mass (5.26), that

$$\frac{\partial f(\phi)}{\partial\phi} = \frac{M_{Pl}^2}{8M_{CS}\dot{\phi}}. \tag{5.69}$$

Analogously to the case of the bispectrum of tensor perturbations, an estimate of this mixed correlator can be given as [17]

$$\langle\delta\phi(\mathbf{k}_1,0),\delta\phi(\mathbf{k}_2,0),\gamma_s(\mathbf{k}_3,0)\rangle \sim \frac{H}{M_{CS}}\left(\sum_{i\neq j}P_T(k_i)P_T(k_j)\right)F(k_i), \tag{5.70}$$

where $F(k_i)$ is still dimensionless and of order of unity. Thus, also in this case the bispectrum, and hence the parity breaking signature on it, are supprssed by the ratio H/M_{CS}.

5.3.2.3 One scalar and two tensors bispectrum

To compute this correlator we need to expand the action at first order in scalar perturbations and at second order in tensor perturbations. Thus, we are interested in the Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm int}^{\delta\phi\gamma\gamma} = \epsilon^{\mu\nu\rho\sigma}\Bigg[&\frac{\partial f(\phi)}{\partial\phi}\delta\phi\, C_{\mu\nu}{}^{\alpha\beta}|_T^{(1)} C_{\rho\sigma\alpha\beta}|_T^{(1)} + f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_S^{(1)} C_{\rho\sigma\alpha\beta}|_T^{(2)} + f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_T^{(2)} C_{\rho\sigma\alpha\beta}|_S^{(1)} + \\ &+ f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_T^{(1)} C_{\rho\sigma\alpha\beta}|_{ST}^{(2)} + f(\phi)C_{\mu\nu}{}^{\alpha\beta}|_{ST}^{(2)} C_{\rho\sigma\alpha\beta}|_T^{(1)}\Bigg].\end{aligned} \tag{5.71}$$

The $\langle \delta\phi\gamma\gamma \rangle$ bispectrum can be computed by exploiting the in-in formalism, exactly as we have done in the previous chapter for the standard model with GR. The final result of this computation is [17]

$$\langle \gamma_R(\mathbf{k}_1,0)\gamma_R(\mathbf{k}_2,0)\delta\phi(\mathbf{k}_3,0)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{\pi}{64}\frac{\dot{\phi}}{H}\left(\sum_{i\neq j}\Delta_T(k_i)\Delta_T(k_j)\right)\left(H^2\frac{\partial^2 f}{\partial\phi^2}\right)\cdot \\ \cdot\frac{(k_1+k_2)k_1k_2}{\sum_i k_i^3}\cos\theta(1-\cos\theta)^2, \tag{5.72}$$

$$\langle \gamma_L(\mathbf{k}_1,0)\gamma_L(\mathbf{k}_2,0)\delta\phi(\mathbf{k}_3,0)\rangle = -\langle \gamma_R(\mathbf{k}_1,0)\gamma_R(\mathbf{k}_2,0)\delta\phi(\mathbf{k}_3,0)\rangle, \tag{5.73}$$

where θ is the angle between the two momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ in the Fourier space, with (see [17] for more details)

$$\cos\theta = \frac{k_3^2-k_2^2-k_1^2}{2k_1k_2}. \tag{5.74}$$

The minus sign which differentiates the two correlators in (5.73) is again a consequence of the breaking of parity symmetry induced by the Chern-Simons operator.

We can now pass to the variable ζ (which stays constant above the horizon) through the relation (5.14). At leading order in slow-roll, we have

$$\langle \gamma_R(\mathbf{k}_1,0)\gamma_R(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle \simeq -(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{\pi}{64}\left(\sum_{i\neq j}\Delta_T(k_i)\Delta_T(k_j)\right)\left(H^2\frac{\partial^2 f}{\partial\phi^2}\right)\cdot \\ \cdot\frac{(k_1+k_2)k_1k_2}{\sum_i k_i^3}\cos\theta(1-\cos\theta)^2, \tag{5.75}$$

$$\langle \gamma_L(\mathbf{k}_1,0)\gamma_L(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle = -\langle \gamma_R(\mathbf{k}_1,0)\gamma_R(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle. \tag{5.76}$$

From the bispectrum (5.76) one can then obtain the shape function $S(k_1,k_2,k_3)$, which reads [17]

$$S(k_1,k_2,k_3) = \left(\sum_{i\neq j}\frac{1}{k_i^3k_j^3}\right)\frac{(k_1+k_2)k_1k_2}{\sum_i k_i^3}\cos\theta(1-\cos\theta)^2. \tag{5.77}$$

This shape function peaks in the squeezed limit, where the momentum of the scalar perturbation is much smaller than the momenta of the tensor perturbations, $k_3 \ll k_1 \sim k_2$.

The parity breaking signatures in the bispectrum (5.75) can be quantified by introducing a parameter defined as [17]

$$\Pi \equiv \frac{\langle \gamma_R(\mathbf{k})\gamma_R(\mathbf{k})\zeta(\mathbf{k})\rangle_{TOT} - \langle \gamma_L(-\mathbf{k})\gamma_L(-\mathbf{k})\zeta(-\mathbf{k})\rangle_{TOT}}{\langle \gamma_R(\mathbf{k})\gamma_R(\mathbf{k})\zeta(\mathbf{k})\rangle_{TOT} + \langle \gamma_L(-\mathbf{k})\gamma_L(-\mathbf{k})\zeta(-\mathbf{k})\rangle_{TOT}}, \tag{5.78}$$

where the bispectra are evaluated in the equilateral configuration and the subscript "TOT" means that it is included also the contribution to the bispecrum coming from standard gravity. From the results (5.75)-(5.76) and (4.144), with the latter being expressed in our present conventions, one finds [17]

$$\Pi = \frac{96\pi}{25}H^2\frac{\partial^2 f}{\partial\phi^2}. \tag{5.79}$$

Therefore the strength of the parity breaking signatures in the $\langle\zeta\gamma\gamma\rangle$ bispectrum depends on the second derivative of the coupling function $f(\phi)$, whereas we have seen that the chirality Θ

at the level of the power spectrum depends only on the first derivative of $f(\phi)$ (through the Chern-Simons mass, as can be seen in (5.69)). This means that only a non minimal coupling of the Chern-Simons operator to the inflaton field (where $f(\phi)$ is not simply proportional to ϕ) can leave such a parity-breaking signature in this correlator. Moreover, as we have already seen from (5.33), the assumption of taking the Chern-Simons mass to be constant during inflation constrains the values of the first and the second derivatives of $f(\phi)$. This implies that the parity breaking signatures in the bispectrum $\langle\zeta\gamma\gamma\rangle$ and in the tensor power spectrum are not really independent from each other. In particular we have seen that, in order to neglect the time variation of the Chern-Simons mass during inflation, we need to require that

$$\sqrt{2\epsilon_V} M_{Pl} \frac{f''(\phi)}{f'(\phi)} \ll 1. \tag{5.80}$$

From (5.55) and (5.69) we can then relate the first derivative of $f(\phi)$ to the chirality Θ and to the slow-roll parameter ϵ_V as

$$f'(\phi) = \frac{1}{4\sqrt{2}\pi} \frac{M_{Pl}^2}{H^2} \frac{\Theta}{\sqrt{\epsilon_V}}, \tag{5.81}$$

where we have also used Eq. (1.75). Plugging (5.81) into the condition (5.80), this can be rewritten as

$$H^2 f''(\phi) \ll \frac{1}{8\pi} \frac{\Theta}{\epsilon_V}. \tag{5.82}$$

Therefore, even if the chirality Θ in the tensor power spectrum is suppressed, we can still have a non-negligible value of Π if the ratio Θ/ϵ_V is sufficiently large, namely if ϵ_V is sufficiently smaller than Θ. However, in [18] it has been shown that the constraint (5.82) on the value of Π makes its detection not possible using the angular bispectra of the CMB. In the case of a time-varying Chern-Simons mass, instead, Π can be enhanced beyond the minimum detectable value. Indeed, it is possible to relate the parameter Π to the time variation of the Chern-Simons mass (encoded in ξ) as [18]

$$\xi = -\frac{25}{96\pi} \sqrt{2\epsilon_V} \frac{M_{Pl}}{H} \frac{\Pi}{H \partial f(\phi)/\partial\phi}, \tag{5.83}$$

where this expression holds at leading order in the slow-roll parameters. If we leave unconstrained the time variation of M_{CS}, and hence the parameter ξ, then also Π is not constrained anymore and can be enhanced, becoming detectable. Notice that in this way we could also use the observational constraints on Π to constrain the time variation of the Chern-Simons mass during inflation (see again [18] for more details).

5.4 Chirality of PGWs and observations

We have seen that the distinctive signature of parity-violating theories of gravity during inflation is the polarization of PGWs into left and right circular polarization states. These propagate differently from each other and thus acquire different super-horizon power spectra. In this section we want to give a brief summary about the possibilities of detecting such parity-violating signatures.

It is well known that the primordial tensor perturbations generate a B-mode (i.e., curl-like) polarization pattern in the CMB in addition to E-modes, which are also generated by scalar perturbations. E-modes and B-modes behave differently under a parity transformation: indeed, the former are even, as the temperature anisotropies T, while the latter are odd. Thus, if parity is preserved, the cross-correlators between temperature anisotropies and B-mode polarization, TB, and between E-modes and B-modes, EB, have to vanish. If these correlators could be measured to be different from zero, this would be a signature of parity violation from chiral gravitational waves. The possibility of testing chirality of PGWs with CMB data has been investigated in [81]. The result of this analysis is that the power spectrum of the CMB can only weakly constrain the chirality Θ. Indeed, this can be constrained only if the amplitude of PGWs is such that the tensor-to-scalar ratio r is high enough to induce a detectable signature in the cross-correlators TB and EB. Since we have not detected such effect, Θ cannot be constrained. The authors of [81] have also provided some forecasts for possible future CMB experiments (like COrE+). They have shown that even in the case in which $r \sim 0.05$ (the current upper limit for r is 0.064) the maximmal parity violation, $\Theta = \pm 1$, could be detected with a significance of 1.5σ at best. This tells us that it is not possible to constrain chirality with the two-point function. For this reason it is important to study higher-order correlators, starting with the bispectrum.

Chapter 6

Parity violation from more general chiral scalar-tensor theories

6.1 Chiral scalar-tensor theories and inflation

Chern-Simons gravity is the simplest scalar-tensor theory which include parity violating effects. As we have seen in the previous chapter, it can be considered as a low energy EFT in an expansion (at second order) in the curvature invariants. However, since it leads to equations of motion with higher order derivatives, it contains Ostrogradsky ghosts[1]. This has been shown in [19], where the authors have performed an Hamiltonian analysis of the theory. In [19] new parity-breaking theories which generalize the Chern-Simons action have been introduced, by including in the action terms with first and second derivatives of the scalar field. Following the original reference, we define $\phi_\mu \equiv \partial_\mu \phi$ and $\phi_{\mu\nu} \equiv \nabla_\mu \phi_\nu$.

With only first derivatives of the scalar field one has to construct a Lagrangian that is at least quadratic in the Riemann tensor. With two Riemann tensors there are four independent terms that can be constructed [19]:

$$\begin{aligned} \mathcal{L}_1 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} R_{\mu\nu}{}^{\rho}{}_{\lambda} \phi^\sigma \phi^\lambda, \qquad & \mathcal{L}_2 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} R_{\mu\lambda}{}^{\rho\sigma} \phi_\nu \phi^\lambda, \\ \mathcal{L}_3 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} R^{\sigma}{}_{\nu} \phi^\rho \phi_\mu, & \mathcal{L}_4 &\equiv XP, \end{aligned} \tag{6.1}$$

where $X \equiv \phi_\mu \phi^\mu$ is the kinetic term and $P \equiv \epsilon^{\mu\nu\rho\sigma} R_{\rho\sigma\alpha\beta} R^{\alpha\beta}{}_{\mu\nu}$ is the Pontryagin term, with which one constructs the Chern-Simons action.

If one includes also terms with second derivatives of the scalar field, there is only a Lagrangian which is linear in both the Riemann tensor and the second derivatives of ϕ, while there are six independent Lagrangians that are linear in the Riemann tensor but quadratic in the second derivatives of ϕ (up to quadratic order in the first derivatives of the scalar field) [19]:

$$\tilde{\mathcal{L}}_1 \equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi^\rho \phi_\mu \phi^\sigma_\nu, \tag{6.2}$$

$$\begin{aligned} \tilde{\mathcal{L}}_2 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi^\rho_\mu \phi^\sigma_\nu, & \tilde{\mathcal{L}}_3 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi^\sigma \phi^\rho_\mu \phi^\lambda_\nu \phi_\lambda, \\ \tilde{\mathcal{L}}_4 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi_\nu \phi^\rho_\mu \phi^\sigma_\lambda \phi^\lambda, & \tilde{\mathcal{L}}_5 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\alpha\rho\sigma\lambda} \phi^\rho \phi_\beta \phi^\sigma_\mu \phi^\lambda_\nu, \\ \tilde{\mathcal{L}}_6 &\equiv \epsilon^{\mu\nu\alpha\beta} R_{\beta\gamma} \phi_\alpha \phi^\gamma_\mu \phi^\lambda_\nu \phi_\lambda, & \tilde{\mathcal{L}}_7 &\equiv (\Box\phi) \tilde{\mathcal{L}}_1. \end{aligned} \tag{6.3}$$

For simplicity we do not consider Lagrangians with higher order terms in the derivatives of the scalar field and in the Riemann tensor. Therefore, we have the two following actions:

$$S^{1\,\mathrm{der}} = \sum_{A=1}^{4} \int d^4x \sqrt{-g}\, a_A \mathcal{L}_A, \tag{6.4}$$

[1] See [80] for a review about Ostrogradsky ghosts.

$$S^{2\,\mathrm{der}} = \sum_{A=1}^{7} \int d^4x \sqrt{-g}\, b_A \tilde{\mathcal{L}}_A, \tag{6.5}$$

where a priori $a_A = a_A(\phi, X)$ and $b_A = b_A(\phi, X)$. Notice that now we are working in units with $M_{Pl} = 1$; we will reintroduce the Planck mass later on by dimensional analysis.

In the reference [19] the authors have performed an Hamiltonian analysis of both of these theories and have found that they are expected to contain Ostrogradsky ghosts in their fully covariant version, as in the case of Chern-Simons gravity. One way to overcome this problem is to restrict the actions (6.4) and (6.5) to the unitary gauge, where the scalar field depends on time only. By tuning the functions a_A and b_A, which become functions of ϕ only, it is possible to obtain two actions which are ghost-free, since they do not contain any higher order time derivative of the metric anymore (from which the Ostrogradsky ghosts arise), but only higher order space derivatives. Therefore in this form they can be considered as Lorentz-breaking theories (like, e.g., Horava-Lifshitz gravity) with a parity violating sector[2]. These new theories are described, within the ADM formalism, by the following actions [19]:

$$\begin{aligned} S^{1\,\mathrm{der}}_{UG} = \frac{2\dot{\phi}^2}{N} \epsilon^{ijl} \Big[&(4a_1 + 2a_2 + 8a_4) \left(K K_{mi} D_l K^m_j + {}^{(3)}R_{mi} D_l K^m_j - K_{mi} K^{mn} D_j K_{jn} \right) + \\ &-(a_2 + a_4) \left(2 K_{mi} K^n_j D_n K^m_l + {}^{(3)}R_{jlm}{}^n D_n K^m_i \right) \Big] , \end{aligned} \tag{6.6}$$

$$\begin{aligned} S^{2\,\mathrm{der}}_{UG} = \frac{\dot{\phi}^3}{N^4} \epsilon^{ijl} \Big\{ &2N \left[b_1 N K_{mi} D_k K^m_j + (b_4 + b_5 - b_3) \dot{\phi} K_{mi} k^n_j D_n K^m_l \right] + \\ &+ \dot{\phi} \left[b_3\, {}^{(3)}R_{jlm}{}^n K^m_i D_n N - 2(b_4 + b_5)\, {}^{(3)}R_{ml} K^m_j D_i N \right] \Big\}, \end{aligned} \tag{6.7}$$

where K_{ij} is the extrinsic curvature (see Eq. (3.16)), while D_i, ${}^{(3)}R^i{}_{jkl}$, and ${}^{(3)}R_{ij}$ are respectively the covariant derivative, the Riemann tensor and the Ricci tensor calculated with the three-dimensional metric h_{ij}. According to the analysis of [19], both these theories propagate 3 degrees of freedom, like the Einstein-Hilbert action plus the Chern-Simons term in the unitary gauge, but, due to the presence of higher order derivatives of the scalar field, the action (6.7) contains also space derivatives of the lapse function. Notice in particular that, like in standard gravity, the lapse function and the shift vector are auxiliary fields, since the actions (6.6) and (6.7) do not contain any time derivative of N and N_i. Therefore one can solve their equations of motion, which are actually constraints, and plug the solutions back into the action, thus obtaining an action for the dynamical degrees of freedom.

The operators introduced in these theories are constructed with a higher number of derivatives. Indeed, it is well known that the action of GR contains two derivatives of the metric tensor[3], while the action of Chern-Simons gravity contains four derivatives, being constructed by contracting the Levi-Civita tensor with two Riemann tensors. In this case, the action (6.6) is made of terms constructed with six derivatives, while the action (6.7) contains also terms with 8 derivatives (actually, all the terms except the first one, which has six derivatives).

[2] At this point, one may ask: "how is it possible that, fixing a gauge, the conclusions about possible ghost instabilities in the theory are changed?". Indeed, if one performs an Hamiltonian analysis of the original "complete" theory and finds that ghost instabilities are expected to be present, then the same conclusions shoud be reached also repeating the Hamiltonian analysis after having fixed a gauge. In other words, the results of the analysis should be gauge-independent. However, it turns out that by fixing the unitary gauge one is actually introducing additional constraints in the theory, thanks to which the Ostrogradsky ghosts are removed. Anyway, in our discussion we follow the approach of Ref. [19]: we "forget" about our starting actions (i.e., those in Eqs. (6.4)-(6.5)) and take the actions in Eqs. (6.6)-(6.7) as desccribing "new", Lorentz-breaking, different theories.

[3] The Ricci tensor is constructed with second derivatives of the metric tensor, hence the same is true also for the Ricci scalar.

In the rest of this thesis we want to study some features of inflation within these new theories. In particular, we consider the two following actions:

$$S_{\rm tot}^{1\,\rm der} = \int d^4x\sqrt{-g}\left[\mathcal{L}_\phi + \mathcal{L}_{\rm GR} + \mathcal{L}_{UG}^{1\,\rm der}\right], \tag{6.8}$$

$$S_{\rm tot}^{2\,\rm der} = \int d^4x\sqrt{-g}\left[\mathcal{L}_\phi + \mathcal{L}_{\rm GR} + \mathcal{L}_{UG}^{2\,\rm der}\right], \tag{6.9}$$

where $\mathcal{L}_{UG}^{1\,\rm der}$ and $\mathcal{L}_{UG}^{2\,\rm der}$ are the Lagrangian densities corresponding to the actions (6.6) and (6.7), while $\mathcal{L}_\phi$ and $\mathcal{L}_{\rm GR}$ are respectively the Lagrangian density for the scalar (inflaton) field and the Einstein-Hilbert Lagrangian density, which are given by

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \quad \mathcal{L}_{\rm GR} = \frac{M_{Pl}^2}{2}R, \tag{6.10}$$

where $V(\phi)$ is the potential for the scalar field (which includes its self interactions) and R is the Ricci scalar. These are fundamental to ensure a background slow-roll inflationary phase. In this regard, an important point we want to outline is that, like in the case of Chern-Simons gravity, the operators introduced by these new theories in the actions (6.6) and (6.7) do not modify the background dynamics of inflation. Indeed, due to the homogeneity property of the FLRW solution, all the 3-dimensional derivatives present in each term of (6.6) and (6.7) (arising from the covariant 3-dimensional derivatives) vanish when evaluated in correspondence with the background quantities. The 3-dimensional Levi-Civita connection vanishes as well in the background, and we are left with no background contributions.

What is interesting to study is how the dynamics of the perturbations is modified. First of all, it is important to remark that the new operators do not provide any contribution to the scalar perturbations in the inflaton field, since a scalar field is invariant under a parity transformation and hence receives no contributions from parity-breaking terms. Thus the power spectrum of scalar perturbations is not modified with respect to the standard case with only GR.

What is instead modified is the dynamics of tensor perturbations, i.e. the dynamics of primordial gravitational waves. As we have seen in the case of Chern-Simons gravity, having a theory that breaks parity implies that the two circular polarization states have a different dynamical evolution, and therefore acquire different super-horizon power spectra.

6.2 PGWs from chiral scalar-tensor theories

In the rest of this thesis we want to study how the new operators introduced in the actions (6.6) and (6.7) affect the propagation of primordial gravitational waves produced during inflation. A first interesting aspect to study is whether these new terms lead to detectable parity breaking signatures in the power spectrum of tensor perturbations, which, as we have seen in the previous chapter, are suppressed in the case of inflation with Chern-Simons gravity. A second interesting issue is to see whether these theories are able to modify the tensor spectral index with respect to the standard models, possibly generating a blue power spectrum, where the amplitude of PGWs increases with k and hence for smaller and smaller cosmological scales. This could be an important feature in view of future experiments with space interferometers [15, 82, 83].

6.2.1 Action with only first derivatives of the scalar field

Let us consider first the action made up with only first derivatives of the scalar field, which in the unitary gauge (and within the ADM formalism) takes the following form

$$\begin{aligned} S_{UG}^{1\,\rm der} = \frac{2\dot\phi^2\epsilon^{ijl}}{N}\Big[&(4a_1+2a_2+8a_4)\Big(KK_{mi}D_lK_j^m + {}^{(3)}R_{mi}D_lK_j^m - K_{mi}K^{mn}D_lK_{jn}\Big) + \\ &-(a_2+4a_4)\Big(2K_{mi}K_j^nD_nK_l^m + {}^{(3)}R_{jlm}{}^nD_nK_i^m\Big)\Big]. \end{aligned} \tag{6.11}$$

We now want to expand the previous action at second order in tensor perturbations. We know that, if one is interested in expanding the action up to third order, it is sufficient to find the expressions of N and N_i only at first order. However, being interested in expanding the action in tensor perturbations and since it is not possible to have first order perturbations in N and N_i including only tensor perturbations, we are left solely with their zero-order value, namely $N = 1$ and $N_i = 0$. We also recall that

$$h_{ij} = a^2 \left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma^k{}_j + \dots \right), \quad h^{ij} = a^{-2} \left(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{ik}\gamma_k{}^j + \dots \right). \tag{6.12}$$

The extrinsic curvature has the following expression up to second order

$$K_{ij} = \frac{1}{2}\dot{h}_{ij} = \dot{a}a\delta_{ij} + \dot{a}a\gamma_{ij} + \frac{1}{2}a^2\dot{\gamma}_{ij} + \frac{1}{2}\dot{a}a\gamma_{ik}\gamma^k{}_j + \frac{1}{2}a^2\dot{\gamma}_{ik}\gamma^k{}_j + \frac{1}{2}a^2\gamma_{ik}\dot{\gamma}^k{}_j, \tag{6.13}$$

while its trace has only the zero-order value (since we are considering only tensor perturbations)

$$K = h^{ij}K_{ij} = 3\frac{\dot{a}}{a}. \tag{6.14}$$

The extrinsic curvature with upper indices and with one upper and one lower index is

$$K^{mn} = h^{mp}h^{nr}K_{pr} = \frac{\dot{a}}{a^3}\delta^{mn} - \frac{\dot{a}}{a^3}\gamma^{mn} + \frac{1}{2}a^{-2}\dot{\gamma}^{mn} + \frac{1}{2}\frac{\dot{a}}{a^3}\gamma^{nk}\gamma_k{}^m - \frac{1}{4}a^{-2}\dot{\gamma}^m{}_k\gamma^{kn} - \frac{1}{4}\gamma^m{}_k\dot{\gamma}^{kn}, \tag{6.15}$$

$$K^m_j = h^{lm}K_{jl} = \frac{\dot{a}}{a}\delta^m_j + \frac{1}{2}\dot{\gamma}_j{}^m + \frac{1}{4}\gamma_{jk}\dot{\gamma}^{km} - \frac{1}{4}\dot{\gamma}_{jk}\gamma^{km}. \tag{6.16}$$

The 3-dimensional connection coefficients are given by

$$\begin{aligned} {}^{(3)}\Gamma^i_{jk} = &\frac{1}{2}(\partial_j\gamma_k{}^i + \partial_k\gamma_j{}^i - \partial^i\gamma_{jk}) + \frac{1}{2}\gamma^{il}(\partial_l\gamma_{jk}) - \frac{1}{4}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl}) + \frac{1}{4}\gamma_{kr}(\partial_j\gamma^{ri})+ \\ &+ \frac{1}{4}\gamma_{jr}(\partial_k\gamma^{ri}) - \frac{1}{4}(\partial^i\gamma_{jr})\gamma^r{}_k - \frac{1}{4}\gamma_{jr}(\partial^i\gamma^r{}_k), \end{aligned} \tag{6.17}$$

from which one can compute the 3-dimensional Riemann and Ricci tensors. The 3-dimensional covariant derivative of the extrinsic curvature is

$$\begin{aligned} D_lK_{jn} = &\frac{1}{2}a^2(\partial_j\dot{\gamma}_{jn}) + \frac{1}{4}a^2\gamma^k{}_n(\partial_l\dot{\gamma}_{jk}) + \frac{1}{4}a^2\gamma_{jk}(\partial_l\dot{\gamma}^k{}_n) + \frac{1}{4}a^2\dot{\gamma}_{kn}(\partial^k\gamma_{lj}) + \frac{1}{4}a^2\dot{\gamma}_{jk}(\partial^k\gamma_{ln})+ \\ &- \frac{1}{4}a^2\dot{\gamma}_{kn}(\partial_j\gamma_l{}^k) - \frac{1}{4}a^2\dot{\gamma}_{jk}(\partial_n\gamma_l{}^k), \end{aligned} \tag{6.18}$$

$$\begin{aligned} D_lK^m_j = &\frac{1}{2}\partial_l\dot{\gamma}_j{}^m + \frac{1}{4}\gamma_{jk}(\partial_l\dot{\gamma}^{km}) - \frac{1}{4}(\partial_l\dot{\gamma}_{jk})\gamma^{km} + \frac{1}{4}(\partial_k\gamma_l{}^m)\dot{\gamma}_j{}^k - \frac{1}{4}(\partial^m\gamma_{lk})\dot{\gamma}_j{}^k - \frac{1}{4}(\partial_j\gamma_l{}^k)\dot{\gamma}_k{}^m+ \\ &+ \frac{1}{4}(\partial^k\gamma_{lj})\dot{\gamma}_k{}^m. \end{aligned} \tag{6.19}$$

Putting all these results together we can compute the various terms in the action (6.11) up to second order in tensor perturbations:

$$\begin{aligned} KK_{mi}D_lK^m_j = &\frac{3}{2}\dot{a}^2(\partial_l\dot{\gamma}_{ij}) + \frac{3}{4}\dot{a}^2\gamma_{jk}(\partial_l\dot{\gamma}^k{}_i) + \frac{3}{4}\dot{a}^2\gamma_{ik}(\partial_l\dot{\gamma}^k{}_j) + \frac{3}{4}\dot{a}^2\dot{\gamma}_j{}^k(\partial_k\gamma_{li}) - \frac{3}{4}\dot{a}^2\dot{\gamma}_j{}^k(\partial_i\gamma_{lk})+ \\ &- \frac{3}{4}\dot{a}^2\dot{\gamma}_{ki}(\partial_j\gamma_l{}^k) + \frac{3}{4}\dot{a}^2\dot{\gamma}_{ki}(\partial^k\gamma_{lj}) + \frac{3}{4}\dot{a}a(\partial_l\dot{\gamma}_j{}^k)\dot{\gamma}_{ki}, \end{aligned} \tag{6.20}$$

$$ {}^{(3)}R_{mi}D_l K^m_j = -\frac{1}{4}(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m), \tag{6.21} $$

$$ \begin{aligned} K_{mi}K^{mn}D_lK_{jn} = &\frac{1}{2}\dot{a}^2(\partial_l\dot{\gamma}_{ji}) + \frac{1}{4}\dot{a}^2\gamma^k{}_i(\partial_l\dot{\gamma}_{jk}) + \frac{1}{4}\dot{a}^2\gamma_{jk}(\partial_l\dot{\gamma}^k{}_i) - \frac{1}{4}\dot{a}^2\dot{\gamma}_{ki}(\partial_j\gamma_l{}^k)+\\ &-\frac{1}{4}\dot{a}^2\dot{\gamma}_{jk}(\partial_i\gamma_l{}^k) + \frac{1}{4}\dot{a}^2\dot{\gamma}_{ki}(\partial^k\gamma_{lj}) + \frac{1}{4}\dot{a}^2\dot{\gamma}_{jk}(\partial^k\gamma_{li}) + \frac{1}{2}\dot{a}a\dot{\gamma}^n{}_i(\partial_l\dot{\gamma}_{jn}), \end{aligned} \tag{6.22} $$

$$ \begin{aligned} K_{mi}K^n_jD_nK^m_l = &\frac{1}{2}\dot{a}^2(\partial_j\dot{\gamma}_{li}) + \frac{1}{4}\dot{a}^2\gamma_{lk}(\partial_j\dot{\gamma}^k{}_i) + \frac{1}{4}\dot{a}^2\gamma^k{}_i(\partial_j\dot{\gamma}_{lk}) + \frac{1}{4}\dot{a}^2\dot{\gamma}_l{}^k(\partial_k\gamma_{ji}) - \frac{1}{4}\dot{a}^2\dot{\gamma}_l{}^k(\partial_i\gamma_{jk})+\\ &-\frac{1}{4}\dot{a}^2\dot{\gamma}_{ki}(\partial_l\gamma_j{}^k) + \frac{1}{4}\dot{a}^2\gamma_{ki}(\partial^k\gamma_{lj}) + \frac{1}{4}\dot{a}a\dot{\gamma}_j{}^k(\partial_k\dot{\gamma}_{li}) + \frac{1}{4}\dot{a}a\dot{\gamma}_{ki}(\partial_j\dot{\gamma}_l{}^k), \end{aligned} \tag{6.23} $$

$$ \begin{aligned} {}^{(3)}R_{jlm}{}^nD_nK^m_i = &\frac{1}{4}(\partial_m\partial_l\gamma^n{}_j)(\partial_n\dot{\gamma}_i{}^m) - \frac{1}{4}(\partial_m\partial_j\gamma_l{}^n)(\partial_n\dot{\gamma}_i{}^m) + \frac{1}{4}(\partial^n\partial_j\gamma_{lm})(\partial_n\dot{\gamma}_i{}^m)+\\ &-\frac{1}{4}(\partial^n\partial_l\gamma_{jm})(\partial_n\dot{\gamma}_i{}^m). \end{aligned} \tag{6.24} $$

Therefore the action (6.11) can be written as

$$ \begin{aligned} S^{1\,\mathrm{der}}_{\gamma\gamma} = \frac{2\dot{\phi}^2\epsilon^{ijl}}{M^4_{Pl}}\Big[&\left(a_1+\frac{a_2}{2}+2a_4\right)\dot{a}a(\partial_l\dot{\gamma}_{jk})\dot{\gamma}^k{}_i - \left(a_1+\frac{a_2}{2}+2a_4\right)(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m)+\\ &-\left(\frac{a_2}{2}+2a_4\right)\dot{a}a\dot{\gamma}_{ik}(\partial_j\dot{\gamma}_l{}^k) - \left(\frac{a_2}{4}+a_4\right)(\partial_k\partial_l\gamma^n{}_j)(\partial_n\dot{\gamma}_i{}^k) + \left(\frac{a_2}{4}+a_4\right)(\partial_k\partial_j\gamma_l{}^n)(\partial_n\dot{\gamma}_i{}^k)+\\ &-\left(\frac{a_2}{4}+a_4\right)(\partial^n\partial_j\gamma_{lk})(\partial_n\dot{\gamma}_i{}^k) + \left(\frac{a_2}{4}+a_4\right)(\partial^n\partial_l\gamma_{jk})(\partial_n\dot{\gamma}_i{}^k)\Big], \end{aligned} \tag{6.25} $$

where we have reintroduced the Planck mass by dimensional analysis. The fourth and the fifth terms give no contributions after integrating by parts, since $\partial_n\gamma^n{}_j = 0$. The last two terms are antisymmetric in j and l, hence they can be summed when contracted with ϵ^{ijl}. Thus the action (6.25) takes the form

$$ S^{1\,\mathrm{der}}_{\gamma\gamma} = \frac{2\dot{\phi}^2\epsilon^{ijl}}{M^4_{Pl}}\left[f\dot{a}a(\partial_l\dot{\gamma}_{jk})\dot{\gamma}^k{}_i - f\,(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m) - g\dot{a}a\dot{\gamma}_{ik}(\partial_j\dot{\gamma}_l{}^k) + g\,(\partial^n\partial_l\gamma_{jk})(\partial_n\dot{\gamma}_i{}^k)\right], \tag{6.26} $$

where we have defined

$$ f \equiv a_1 + \frac{a_2}{2} + 2a_4, \quad g \equiv \frac{a_2}{2} + 2a_4. \tag{6.27} $$

Let us now focus on the second term in the action. We have

$$ \begin{aligned} \frac{2\dot{\phi}^2}{M^4_{Pl}}f\epsilon^{ijl}(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m) &= \frac{\dot{\phi}^2}{M^4_{Pl}}f\epsilon^{ijl}\left[(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m) + (\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m)\right]\\ &= \frac{\dot{\phi}^2}{M^4_{Pl}}f\epsilon^{ijl}\left[(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m) - (\partial_l\gamma_{im})(\partial_k\partial^k\dot{\gamma}_j{}^m)\right]\\ &= \frac{\dot{\phi}^2}{M^4_{Pl}}f\epsilon^{ijl}\left[(\partial_k\partial^k\gamma_{im})(\partial_l\dot{\gamma}_j{}^m) + (\partial_k\partial^k\dot{\gamma}_{im})(\partial_l\gamma_j{}^m)\right]\\ &= \frac{2\dot{\phi}^2}{M^4_{Pl}}\frac{f}{2}\epsilon^{ijl}\partial_t\left[(\partial_k\partial^k\gamma_{im})(\partial_l\gamma_j{}^m)\right]. \end{aligned} \tag{6.28} $$

To pass from the first to the second line we have integrated by parts three times in the spatial derivatives and we have neglected the total derivative term (which gives no contributions at the level of the equations of motion), while in the third equality we have used the antisymmetry property of the Levi-Civita tensor in i and j. We can repeat an analogous computation also for the last term in the action (6.26), obtaining

$$\frac{2\dot{\phi}^2}{M_{Pl}^4} g\epsilon^{ijl}(\partial^n\partial_l\gamma_{jk})(\partial_n\dot{\gamma}_i{}^k) = \frac{2\dot{\phi}^2}{M_{Pl}^4}\frac{g}{2}\epsilon^{ijl}\partial_t\left[(\partial^n\partial_l\gamma_{jk})(\partial_n\gamma_i{}^k)\right]. \tag{6.29}$$

If we then integrate by parts (in the time derivative) both these terms, the action (6.26) becomes

$$\begin{aligned} S_{\gamma\gamma}^{1\,\mathrm{der}} = \frac{2\dot{\phi}^2\epsilon^{ijl}}{M_{Pl}^4}\Bigg[& f\dot{a}a(\partial_l\dot{\gamma}_{jk})\dot{\gamma}^k{}_i + \left(\frac{\dot{f}}{2} + f\frac{\ddot{\phi}}{\dot{\phi}}\right)(\partial_k\partial^k\gamma_{im})(\partial_l\gamma_j{}^m) - g\dot{a}a\dot{\gamma}_{ik}(\partial_j\dot{\gamma}_l{}^k) + \\ & - \left(\frac{\dot{g}}{2} + g\frac{\ddot{\phi}}{\dot{\phi}}\right)(\partial^n\partial_l\gamma_{jk})(\partial_n\gamma_i{}^k)\Bigg]. \end{aligned} \tag{6.30}$$

The terms proportional to the second derivative of the scalar field are of higher order in the slow-roll parameters. Indeed, from the definitions

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \epsilon_V \simeq \frac{\dot{\phi}^2}{2H^2M_{Pl}^2}, \tag{6.31}$$

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H} = 2\epsilon + 2\frac{\ddot{\phi}}{H\dot{\phi}}, \tag{6.32}$$

it follows that

$$\frac{\dot{\phi}^2}{M_{Pl}^2}\frac{\ddot{\phi}}{\dot{\phi}} \simeq 2H^3\epsilon\left(\frac{\eta}{2} - \epsilon\right) \simeq \mathcal{O}(\epsilon^2, \eta\cdot\epsilon). \tag{6.33}$$

Since we work at leading order in the slow-roll parameters, we can neglect both these terms. Hence we are left with the action

$$S_{\gamma\gamma}^{1\,\mathrm{der}} = \frac{2\dot{\phi}^2\epsilon^{ijl}}{M_{Pl}^4}\left[f\dot{a}a(\partial_l\dot{\gamma}_{jk})\dot{\gamma}^k{}_i + \frac{\dot{f}}{2}(\partial_k\partial^k\gamma_{im})(\partial_l\gamma_j{}^m) - g\dot{a}a\dot{\gamma}_{ik}(\partial_j\dot{\gamma}_l{}^k) + \frac{\dot{g}}{2}(\partial^n\partial_l\gamma_{jk})(\partial_n\gamma_i{}^k)\right]. \tag{6.34}$$

Let us now expand the tensor perturbations in the Fourier space in the chiral polarization basis as

$$\gamma_{ij}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\sum_{s=L,R}\epsilon_{ij}^{(s)}(\mathbf{k})\gamma_{\mathbf{k}}^s(t)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.35}$$

As we have already seen, this decomposition into left and right modes is useful when considering theories that break parity. We can now rewrite the action (6.34) in the Fourier space as

$$\begin{aligned} S_{\gamma\gamma}^{1\,\mathrm{der}} = \sum_{s_1,s_2}\frac{2}{M_{Pl}^4}\int d^4x\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\dot{\phi}^2\epsilon^{ijl}\Bigg[& if\dot{a}ak_l\dot{\gamma}_{\mathbf{k}}^{s_1}\dot{\gamma}_{\mathbf{q}}^{s_2}\epsilon_{jk}^{(s_1)}(\mathbf{k})\epsilon_i^{(s_2)k}(\mathbf{q}) + \\ & -i\frac{\dot{f}}{2}k^2q_l\gamma_{\mathbf{k}}^{s_1}\gamma_{\mathbf{q}}^{s_2}\epsilon_{im}^{(s_1)}(\mathbf{k})\epsilon_j^{(s_2)m}(\mathbf{q}) - ig\dot{a}aq_j\dot{\gamma}_{\mathbf{k}}^{s_1}\dot{\gamma}_{\mathbf{q}}^{s_2}\epsilon_{ik}^{(s_1)}(\mathbf{k})\epsilon_l^{(s_2)k}(\mathbf{q}) + \\ & +i\frac{\dot{g}}{2}k^nk_lq_n\gamma_{\mathbf{k}}^{s_1}\gamma_{\mathbf{q}}^{s_2}\epsilon_{jk}^{(s_1)}(\mathbf{k})\epsilon_i^{(s_2)k}(\mathbf{q})\Bigg]e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}. \end{aligned} \tag{6.36}$$

We can first integrate over $\mathbf{x}$ and $\mathbf{q}$, obtaining

$$\begin{aligned} S_{\gamma\gamma}^{1\,\mathrm{der}} = \sum_{s_1,s_2}\frac{2}{M_{Pl}^4}\int dt\int\frac{d^3k}{(2\pi)^3}\dot{\phi}^2\epsilon^{ijl}\Bigg[& if\dot{a}ak_l\dot{\gamma}_{\mathbf{k}}^{s_1}\dot{\gamma}_{-\mathbf{k}}^{s_2}\epsilon_{jk}^{(s_1)}(\mathbf{k})\epsilon_i^{(s_2)k}(-\mathbf{k}) + \\ & +i\frac{\dot{f}}{2}k^2k_l\gamma_{\mathbf{k}}^{s_1}\gamma_{-\mathbf{k}}^{s_2}\epsilon_{im}^{(s_1)}(\mathbf{k})\epsilon_j^{(s_2)m}(-\mathbf{k}) + ig\dot{a}ak_j\dot{\gamma}_{\mathbf{k}}^{s_1}\dot{\gamma}_{-\mathbf{k}}^{s_2}\epsilon_{ik}^{(s_1)}(\mathbf{k})\epsilon_l^{(s_2)k}(-\mathbf{k}) + \\ & -i\frac{\dot{g}}{2}k^2k_l\gamma_{\mathbf{k}}^{s_1}\gamma_{-\mathbf{k}}^{s_2}\epsilon_{jk}^{(s_1)}(\mathbf{k})\epsilon_i^{(s_2)k}(-\mathbf{k})\Bigg]. \end{aligned} \tag{6.37}$$

For circular polarization modes we have seen that the following relation holds [78]

$$k_l \epsilon^{mlj} \epsilon_j^{(s)j}(\mathbf{k}) = -i\lambda_s k \epsilon^{(s)im}(\mathbf{k}), \tag{6.38}$$

where $\lambda_R = +1$ and $\lambda_L = -1$. In the case of the terms in the action (6.37), we have

$$\epsilon^{ijl} k_l \epsilon_{jk}^{(s_1)}(\mathbf{k}) = -\epsilon^{ilj} k_l \epsilon_{jk}^{(s_1)}(\mathbf{k}) = i\lambda_{s_1} k \epsilon^{(s_1)i}{}_k(\mathbf{k}), \tag{6.39}$$

$$\epsilon^{ijl} k_l \epsilon_{im}^{(s_1)}(\mathbf{k}) = \epsilon^{jli} k_l \epsilon_{im}^{(s_1)}(\mathbf{k}) = -i\lambda_{s_1} k \epsilon^{(s_1)j}{}_m(\mathbf{k}), \tag{6.40}$$

$$\epsilon^{ijl} k_j \epsilon_{ik}^{(s_1)}(\mathbf{k}) = -\epsilon^{lji} k_j \epsilon_{ik}^{(s_1)}(\mathbf{k}) = i\lambda_{s_1} k \epsilon^{(s_1)l}{}_k(\mathbf{k}), \tag{6.41}$$

from which the action (6.37) becomes

$$\begin{aligned}
S_{\gamma\gamma}^{1\,\mathrm{der}} &= \sum_{s_1,s_2} \frac{2}{M_{Pl}^4} \int dt \int \frac{d^3k}{(2\pi)^3} \dot{\phi}^2 \epsilon^{ijl} \Bigg[-\lambda_{s_1} f \dot{a} a k \dot{\gamma}_{\mathbf{k}}^{s_1} \dot{\gamma}_{-\mathbf{k}}^{s_2} \epsilon^{(s_1)i}{}_k(\mathbf{k}) \epsilon_i^{(s_2)k}(-\mathbf{k}) + \\
&\quad + \lambda_{s_1} \frac{\dot{f}}{2} k^3 \gamma_{\mathbf{k}}^{s_1} \gamma_{-\mathbf{k}}^{s_2} \epsilon^{(s_1)j}{}_m(\mathbf{k}) \epsilon_j^{(s_2)m}(-\mathbf{k}) - \lambda_{s_1} g \dot{a} a k \dot{\gamma}_{\mathbf{k}}^{s_1} \dot{\gamma}_{-\mathbf{k}}^{s_2} \epsilon^{(s_1)l}{}_k(\mathbf{k}) \epsilon_l^{(s_2)k}(-\mathbf{k}) + \\
&\quad + \lambda_{s_1} \frac{\dot{g}}{2} k^3 \gamma_{\mathbf{k}}^{s_1} \dot{\gamma}_{-\mathbf{k}}^{s_2} \epsilon^{(s_1)i}{}_k(\mathbf{k}) \epsilon_i^{(s_2)k}(-\mathbf{k}) \Bigg] \\
&= \sum_{s=L,R} \frac{4}{M_{Pl}^4} \int dt \int \frac{d^3k}{(2\pi)^3} \dot{\phi}^2 \left[-(f+g) \dot{a} a \lambda_s k |\dot{\gamma}_{\mathbf{k}}^s|^2 + \left(\frac{\dot{f}+\dot{g}}{2} \right) \lambda_s k^3 |\gamma_{\mathbf{k}}^s|^2 \right],
\end{aligned} \tag{6.42}$$

where we have also used

$$\epsilon_{ij}^L(\mathbf{k}) \epsilon_L^{ij}(\mathbf{k}) = 0 = \epsilon_{ij}^R(\mathbf{k}) \epsilon_R^{ij}(\mathbf{k}), \quad \epsilon_{ij}^L(\mathbf{k}) \epsilon_R^{ij}(\mathbf{k}) = 2, \quad \gamma_{-\mathbf{k}}^R = (\gamma_{\mathbf{k}}^L)^*, \quad \gamma_{-\mathbf{k}}^L = (\gamma_{\mathbf{k}}^R)^*. \tag{6.43}$$

Switching from the cosmic time to the conformal time, this becomes

$$S_{\gamma\gamma}^{1\,\mathrm{der}} = \sum_{s=L,R} \frac{4}{M_{Pl}^4} \int d\tau \int \frac{d^3k}{(2\pi)^3} \frac{\phi'^2}{a} \left[-(f+g) H \lambda_s k |\gamma_{\mathbf{k}}^{s\prime}|^2 + \left(\frac{f'+g'}{2} \right) \lambda_s \frac{1}{a} k^3 |\gamma_{\mathbf{k}}^s|^2 \right]. \tag{6.44}$$

If we now add the contribution from standard gravity

$$S_{\gamma\gamma}^{\mathrm{GR}} = \sum_{s=L,R} \frac{M_{Pl}^2}{2} \int d\tau \int \frac{d^3k}{(2\pi)^3} a^2 \left[|\gamma_{\mathbf{k}}^{s\prime}|^2 - k^2 |\gamma_{\mathbf{k}}^s|^2 \right], \tag{6.45}$$

we can write the total action for the tensor perturbations as

$$S_{\gamma\gamma} \equiv S_{\gamma\gamma}^{1\,\mathrm{der}} + S_{\gamma\gamma}^{\mathrm{GR}} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} \left[A_{T,s}^2 |\gamma_{\mathbf{k}}^{s\prime}|^2 - B_{T,s}^2 k^2 |\gamma_{\mathbf{k}}^s|^2 \right], \tag{6.46}$$

where we have defined

$$A_{T,s}^2 \equiv \frac{M_{Pl}^2}{2} a^2 \left[1 - \frac{8}{M_{Pl}^6} \frac{\phi'^2}{a^3} (f+g) \lambda_s H k \right] = \frac{M_{Pl}^2}{2} a^2 \left(1 - \lambda_s \frac{k_{phys}}{M_{CST}} \right), \tag{6.47}$$

with

$$M_{CST} \equiv \frac{M_{Pl}^6}{8} \frac{a^2}{\phi'^2} \frac{1}{(f+g)H}, \tag{6.48}$$

and

$$B_{T,s}^2 \equiv \frac{M_{Pl}^2}{2} a^2 \left[1 - \frac{4}{M_{Pl}^6} \frac{\phi'^2}{a^4} (f'+g') \lambda_s k \right]. \tag{6.49}$$

The right modes ($\lambda_R = +1$) with a physical wavenumber, $k_{phys} = k/a$, larger than M_{CST} have a negative kinetic term, thus they are ghost fields[4] [79]. From this we can see that M_{CST} plays the same role as the Chern-Simons mass in the case with Chern-Simons modified gravity [17]. Hence, in analogy with the Chern-Simons case, we introduce a UV cut-off Λ in the theory such that $\Lambda \ll M_{CST}$. If we then require that $k_{phys} < \Lambda$ at the beginning of inflation[5], we work in a regime where we are free from ghosts. We have also to require that at the beginning of inflation the modes were deep inside the horizon, and this leads to the condition $k_{phys} \gg H$. From these two conditions taken together it follows that $M_{CST} \gg H$. In the following we take M_{CST} to be constant. To justify this, in analogy with the Chern-Simons case we introduce a dimensionless parameter which quantifies the rate of variation of M_{CST} in a Hubble time:

$$\xi \equiv \frac{\dot{M}_{CST}}{M_{CST}H} \simeq 2\eta - \epsilon - \frac{\dot{f}+\dot{g}}{H(f+g)}. \tag{6.50}$$

If we assume that f and g depend "weakly" on the inflaton field[6] ϕ, the last term in (6.50) gives a negligible contribution. Since the slow-roll parameters obey $\epsilon \ll 1$, $|\eta| \ll 1$, it follows that $\xi \ll 1$.

Let us then rewrite the action (6.46) as

$$S_{\gamma\gamma} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} A^2_{T,s}\left[|\gamma^{s\prime}_{\mathbf{k}}|^2 - C^2_{T,s}k^2|\gamma^s_{\mathbf{k}}|^2\right], \tag{6.51}$$

where we have defined

$$C^2_{T,s} \equiv \frac{B^2_{T,s}}{A^2_{T,s}}. \tag{6.52}$$

By making the field redefinition

$$\mu_s \equiv A_{T,s}\gamma_s, \tag{6.53}$$

the action (6.51) becomes

$$S_{\gamma\gamma} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3}\left[|\mu_s'(\mathbf{k},\tau)|^2 - C^2_{T,s}k^2|\mu_s(\mathbf{k},\tau)|^2 + \frac{A''_{T,s}}{A_{T,s}}|\mu_s(\mathbf{k},\tau)|^2\right]. \tag{6.54}$$

Varying this action yields the equations of motion for the fields μ_s, which read

$$\mu_s'' + \left(C^2_{T,s}k^2 - \frac{A''_{T,s}}{A_{T,s}}\right)\mu_s = 0. \tag{6.55}$$

The second term inside the parentheses is given by

$$\frac{A''_{T,s}}{A_{T,s}} = \frac{d}{d\tau}\left(\frac{A'_{T,s}}{A_{T,s}}\right) + \left(\frac{A'_{T,s}}{A_{T,s}}\right)^2 = \frac{2+3\epsilon}{\tau^2} - \frac{\lambda_s k}{\tau}\frac{H}{M_{CST}} + \mathcal{O}\left(\epsilon^2, \frac{H^2}{M^2_{CST}}, \epsilon\frac{H}{M_{CST}}\right). \tag{6.56}$$

The first term is the usual term present also in the standard models (see Eq. (3.87)), while the second term is analogous to the one which arises with the Chern-Simons coupling (see (5.39)),

[4] Here and in the following we assume that M_{CST} is positive. If it was negative, as it could occur if f and g (or even only one of them) were negative, we would reach the same conclusions, but the role of the right and the left modes would be exchanged. For example, in this case the left modes with a physical wavenumber greater than M_{CST} would be ghost fields.

[5] Notice that if this condition is satisfied at the beginning of inflation it continues to be satisfied as inflation proceeds, since k_{phys} decreases during the inflationary expansion.

[6] This is equivalent to requiring that the same condition holds for the parameters a_1, a_2, a_3, a_4.

with the Chern-Simons mass M_{CS} being replaced by M_{CST}. Thus the equations of motion for the fields μ_s are

$$\mu_s'' + \left(C_{T,s}^2 k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau} \frac{H}{M_{CST}} \right) \mu_s = 0, \tag{6.57}$$

with

$$\nu_T \simeq \frac{3}{2} + \epsilon \tag{6.58}$$

at leading order in the slow-roll parameters. From these equations of motion we can see that the speed of propagation of tensor perturbations is modified in these theories with respect to the standard GR case and to the Chern-Simons cases, since $C_{T,s}^2 \neq 1$. This is indeed already evident in Eq. (6.51). Two additional interesting features are worth to note. Firstly, the two circular polarization states propagate with a different speed during inflation, since this depends on the polarization index s. Secondly, the speed of tensor perturbations varies during inflation. We will comment about the possible phenomenological consequences of these later on. Now let us write $C_{T,s}^2$ as

$$C_{T,s}^2 = \left[1 - \frac{4}{M_{Pl}^6} \frac{\phi'^2}{a^4} (f' + g') \lambda_s k \right] \left(1 - \lambda_s \frac{k}{a} \frac{1}{M_{CST}} \right)^{-1} \simeq \frac{1 + \frac{1}{2} \lambda_s \frac{\dot{f} + \dot{g}}{f + g} k \frac{H}{M_{CST}} \frac{\tau}{H}}{1 + \lambda_s k \frac{H}{M_{CST}} \tau}, \tag{6.59}$$

where to obtain the final expression we have used the definition of M_{CST} and $a(\tau) \simeq -1/(H\tau)$. Notice that the standard slow-roll models with GR are recovered by setting the couplings of the new terms in the action to zero, namely $f = g = 0$. In this case $M_{CST} \to \infty$, as can be seen from (6.48), and therefore we recover $C_{T,s}^2 = 1$, as expected. This is a consistency check of our calculations, since it shows us that if we set to zero the new terms we recover the standard results.

There is a further important feature which arises from the expression (6.59) of $C_{T,s}^2$. Let us rewrite this as

$$C_{T,s}^2 = \frac{1 + \lambda_s a}{1 + \lambda_s b}, \tag{6.60}$$

where we have defined

$$b \equiv k \frac{H}{M_{CST}} \tau, \quad a \equiv \frac{1}{2H} \frac{\dot{f} + \dot{g}}{f + g}\, b. \tag{6.61}$$

The UV cut-off we have introduced (to avoid ghosts) through the condition $k/a \ll M_{CST}$ implies that

$$-1 \ll k \frac{H}{M_{CST}} \tau < 0 \tag{6.62}$$

holds during inflation. This means that $|b| \ll 1$ in the regime where we avoid ghosts. From the expression (6.60) of a, it follows that if the couplings f and g are slowly-varying functions[7] of time during inflation, then also a is much smaller than one. This implies that, apart from tiny corrections, we can take the speed of tensor modes to be equal to 1 during inflation. In this case the equations of motion reduce to the equations of motion we have with Chern-Simons gravity, with the Chern-Simons mass M_{CS} being replaced by M_{CST}:

$$\mu_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau} \frac{H}{M_{CST}} \right) \mu_s = 0. \tag{6.63}$$

Therefore the conclusions we reach are the same. The chirality in the tensor power spectrum (at leading order in the slow-roll parameters) is given by

$$\Theta \equiv \frac{P_T^R - P_T^L}{P_T^R + P_T^L} = \frac{\pi}{2} \frac{H}{M_{CST}}, \tag{6.64}$$

[7] Notice that the same considerations have allowed us to neglect the time variation of M_{CST}.

and it is suppressed since $H/M_{CST} \ll 1$. As regarding the tensor spectral index, working at leading order in the slow-roll parameters and taking into account that $H/M_{CST} \ll 1$, we get the same result as in the standard case with GR, namely

$$n_T \simeq -2\epsilon. \tag{6.65}$$

More interesting features could instead arise by allowing the time derivatives of the couplings f and g to be large during inflation. Let us consider the case with[8]

$$\left| \frac{1}{2H} \frac{\dot{f}+\dot{g}}{f+g} \right| = \left| \frac{a}{b} \right| > 1, \tag{6.66}$$

Notice that b is always negative during inflation, while the sign of a depends on the sign of the ratio $(\dot{f}+\dot{g})/(f+g)$, and hence on the fact that f and g are decreasing or increasing during inflation. Therefore in principle we can consider two cases:

1. in the first case, both a and b are negative: this happens if $\frac{\dot{f}+\dot{g}}{f+g} > 0$;

2. in the second case, b is negative, while a is positive: this instead happens if $\frac{\dot{f}+\dot{g}}{f+g} < 0$.

In the first case, it is easy to see from (6.61) with (6.66) that the left modes ($\lambda_L = -1$) acquire a superluminal speed during inflation, $C^2_{T,L} > 1$. The right modes ($\lambda_R = +1$) have instead $C^2_{T,R} < 1$, hence they are subluminal. For both the polarization states $C^2_{T,s} > 0$.

In the second case instead the right modes have a superluminal speed of propagation, while the left modes are subluminal. The condition $C^2_{T,s} > 0$ is still satisfied.

Therefore we can say that, depending on the sign of the ratio $(\dot{f}+\dot{g})/(f+g)$, one of the two polarization states of GWs is superlumianl during inflation, while the other one is subluminal. The superluminal propagation of one of the two polarization states is a phenomenological manifestation of the breaking of Lorentz invariance arising in these theories. It is also important to note that in the super-horizon limit ($-k\tau \to 0$) we recover $C^2_{T,s} = 1$ for both right and left modes. All the terms in the action introduced by the new operators contribute in modifying the tensor speed and thus having $C^2_{T,s} > 1$ for one of the two modes during inflation, as can be seen from (6.52) or (6.59). Since the new terms contain lower powers of the scale factor with respect to the standard terms (see (6.47) and (6.49)), they become less and less important as inflation proceeds (since the scale factor grows almost exponentially), and are eventually dominated by the standard terms. This explains why we recover $C^2_{T,s} = 1$ in the super-horizon limit.

The fact of having $C^2_{T,s} > 0$ is also extremely important, since it tells us that we do not have any gradient instability during inflation. A gradient instability occurs when the square of the speed of propagation of the perturbations becomes negative. In this case we do not have any wave-like solution, but rather the perturbations grow exponentially. This might be a problem from an observational point of view, since we have a (small) upper bound on the amplitude of primordial tensor perturbations (as we have seen in Chapter 3). An exponential growth of tensor modes during inflation might probably produce a signal which we should have observed. Notice that the fact of having $C^2_{T,s} > 0$ follows from the condition required in the footnote 8 and also from the UV cut-off that we have introduced to avoid ghosts. In the absence of the latter, the terms a and b in $C^2_{T,s}$ would be in principle unconstrained (since the condition (6.62) would not occur anymore), thus allowing the possibility of having $C^2_{T,s} < 0$.

Let us recap the results we have found until now: the two polarization states propagate with a different and time-varying speed from each other during inflation; depending on the sign of

[8] In the following we assume that the relation (6.66) holds but with $|a| < 1$, in order to prevent one of the two polarization states from having $C^2_{T,s} < 0$, leading to a gradient instability.

$(\dot{f}+\dot{g})/(f+g)$, one of the two polarization states propagates with a superluminal speed during inflation, while the other one propagates with a subluminal speed[9]. In the super-horizon limit the speed of tensor modes becomes again equal to the speed of light, for both left and right modes. No gradient instabilities occur during inflation.

We can now canonically quantize the fields μ_s by expanding them in terms of the creation and annihilation operators as

$$\hat{\mu}_s(\mathbf{k},\tau) = u_s(k,\tau)\hat{a}_s(\mathbf{k}) + u_s^*(k,\tau)\hat{a}_s^\dagger(-\mathbf{k}). \tag{6.67}$$

The creation and annihilation operators satisfy the equal time commutation relations

$$[\hat{a}_s(\mathbf{k}), \hat{a}_{s'}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')], \tag{6.68}$$

and act on the vacuum state obeying

$$\hat{a}_s|0\rangle = 0, \quad \langle 0|\hat{a}_s^\dagger = 0. \tag{6.69}$$

The equations of motion for the mode functions u_s are then

$$u_s'' + \left(C_{T,s}^2 k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau}\frac{H}{M_{CST}}\right) u_s = 0. \tag{6.70}$$

In the sub-horizon limit, these reduce to

$$u_s'' + C_{T,s}^2|_{k\gg aH}\, k^2 u_s \simeq 0, \tag{6.71}$$

where[10]

$$C_{T,s}^2|_{k\gg aH} \equiv \lim_{-k\tau\to\infty} C_{T,s}^2. \tag{6.72}$$

One can then impose the Bunch-Davies initial condition

$$\lim_{-k\tau\to\infty} u_s(k,\tau) = \frac{1}{\sqrt{2C_{T,s}|_{k\gg aH}k}} e^{-iC_{T,s}|_{k\gg aH}k\tau}. \tag{6.73}$$

At this point we have to notice that the equations of motion (6.70) have in general no analytical solutions, due to time variation of $C_{T,s}^2$. Interesting features could arise from the time variation of the tensor speed, which in this case would also be different from the speed of light and different for the two helicity modes. Indeed, we know that the perturbations get frozen when they cross the horizon corresponding to their speed of propagation, namely when $kC_{T,s} \simeq aH$. If the two helicity modes propagate with a different speed, they get frozen at different times during inflation. This could enhance the chirality Θ in the power spetcrum of PGWs with respect to the Chern-Simons case. A varying tensor speed could also modify the spectral index of one or both of the polarization states, possibly leading to a blue power spectrum of tensor perturbations. As we have said, having a blue power spectrum means that the amplitude of PGWs increases going to lower scales. This could be an important feature because it could enhance the spectrum of PGWs at scales accessible to interferometers, thus allowing us to test experimentally the theoretical predictions .

[9] Notice that, since we have assumed (without loss of generality) that M_{CST} is positive, this means that also $f+g$ is positive. Hence the sign of $(\dot{f}+\dot{g})/(f+g)$ depends only on whether $f(\phi)$ and $g(\phi)$ are increasing or decreasing during inflation.

[10] Actually, we never reach the limit $k\tau \to -\infty$, due to the UV cut-off we have imposed in the theory to avoid ghosts. Indeed, the cut-off in the physical wavenumber, $k_{phys} = k/a$, induces a cut-off in the conformal time τ (see e.g. (6.62)). Therefore with the limit in Eq. (6.72) we mean that we are going in the far past, but not far enough to reach energies higher than our cut-off.

In order to proceed in analysing these possible signatures, a numerical study of the equations of motion (6.70) is required. We leave this for a potential future work, with the following comment as a possible starting point. We have seen that, in order to have a non-negligible contribution to the tensor speed, we need to require that the time derivatives of the couplings $f(\phi)$ and $g(\phi)$ are large during inflation. In this regard, given the expression (6.48) of M_{CST} it is safer to assume that the couplings decrease during inflation. If they increased, M_{CST} would instead decrease, hence the condition $k_{phys} < M_{CST}$ required to avoid ghosts could fail to be satisfied as inflation proceeds.

6.2.2 Action with second derivatives of the scalar field

Now we want to repeat an analysis analogous to the one of the previous section for the action which includes also second derivatives of the scalar field. Its expression is given by

$$S_{UG}^{2\,\mathrm{der}} = \frac{\dot{\phi}^3}{N^4}\epsilon^{ijl}\Big\{2N\Big[b_1 N K_{mi} D_l K_j^m + (b_4+b_5-b_3)\dot{\phi} K_{mi} K_j^n D_n K_l^m\Big] + \\ + \dot{\phi}\Big[b_3\, {}^{(3)}R_{jlm}{}^n K_i^m D_n N - 2(b_4+b_5)\, {}^{(3)}R_{ml} K_j^m D_i N\Big]\Big\}. \tag{6.74}$$

As we have done in the previous case, we expand this action at second order in tensor perturbations. Since, as we have explained at the beginning, we take for the lapse function and the shift vector the zero order values, $N = 1$ and $N_i = 0$, the terms in the second row of (6.74) vanish. Reintroducing the Planck mass by dimensional analysis, the action at second order in tensor perturbations can be written as

$$S_{\gamma\gamma}^{2\,\mathrm{der}} = \frac{\dot{\phi}^3}{M_{Pl}^5}\epsilon^{ijl}\left[\frac{1}{2} b a^2 \dot{\gamma}_{ki}(\partial_l \dot{\gamma}_j{}^k) + \frac{1}{2}\tilde{b}\frac{\dot{\phi}}{M_{Pl}^3}\dot{a} a \dot{\gamma}_{ki}(\partial_j \dot{\gamma}_l{}^k)\right], \tag{6.75}$$

where we have defined

$$b \equiv b_1, \quad \tilde{b} \equiv b_4 + b_5 - b_3. \tag{6.76}$$

Notice that, with respect to the case with only first derivatives of the scalar field, we do not have any term like the second and the fourth ones in (6.26), which were then integrated by parts, thus leading to the contributions proportional to the derivatives of the couplings (f and g in the previous case).

Repeating the same steps as in the previous case, we can write the total action (including that of GR) for tensor perturbations as

$$\tilde{S}_{\gamma\gamma} = \sum_{s=L,R}\int d\tau \int \frac{d^3k}{(2\pi)^3}\left[\tilde{A}_{T,s}^2|\gamma_{\mathbf{k}}^{s\prime}|^2 - \frac{M_{Pl}^2}{2}a^2k^2|\gamma_{\mathbf{k}}^s|^2\right], \tag{6.77}$$

where we have defined

$$\tilde{A}_{T,s}^2 \equiv \frac{M_{Pl}^2}{2}a^2\left[1 + 2\frac{\phi'^3}{a^4}\frac{k}{M_{Pl}^7}\lambda_s\left(\frac{\phi'\tilde{b}a'}{M_{Pl}^3 a^3} - b\right)\right] = \frac{M_{Pl}^2}{2}a^2\left(1 - \lambda_s\frac{k}{a}\frac{1}{\tilde{M}_{CST}}\right), \tag{6.78}$$

with

$$\tilde{M}_{CST} \equiv \frac{M_{Pl}^7 a^3}{2\phi'^3}\left(b - \frac{\phi'\tilde{b}a'}{M_{Pl}^3 a^3}\right)^{-1}. \tag{6.79}$$

We have introduced $\tilde{M}_{CST}$ for the same reason as for M_{CST} in the case with only first derivatives of the scalar field. It represents the energy scale at which some modes of the fields acquire a negative kinetic term, thus becoming ghost fields[11]. Proceeding as in the previous case, we introduce a UV cut-off $\Lambda << \tilde{M}_{CST}$ by requiring that $k_{phys} < \Lambda$. By requiring also that the modes started deep below the horizon, it follows that $H/\tilde{M}_{CST} \ll 1$.

Let us now rewrite the action (6.77) as

$$\tilde{S}_{\gamma\gamma} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} \tilde{A}^2_{T,s} \left[|\gamma^{s\prime}_{\mathbf{k}}|^2 - \tilde{C}^2_{T,s} k^2 |\gamma^s_{\mathbf{k}}|^2 \right], \tag{6.80}$$

with

$$\tilde{C}^2_{T,s} \equiv \frac{M^2_{Pl} a^2}{2\tilde{A}^2_{T,s}}. \tag{6.81}$$

Making the field redefinition

$$\mu_s \equiv \tilde{A}_{T,s} \gamma_s, \tag{6.82}$$

the action (6.80) becomes

$$\tilde{S}_{\gamma\gamma} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} \left[|\mu_s'(\mathbf{k},\tau)|^2 - \tilde{C}^2_{T,s} k^2 |\mu_s(\mathbf{k},\tau)|^2 + \frac{\tilde{A}''_{T,s}}{\tilde{A}_{T,s}} |\mu_s(\mathbf{k},\tau)|^2 \right]. \tag{6.83}$$

By varying this action we obtain the equations of motion for the fields μ_s

$$\mu_s'' + \left(\tilde{C}^2_{T,s} k^2 - \frac{\tilde{A}''_{T,s}}{\tilde{A}_{T,s}} \right) \mu_s = 0, \tag{6.84}$$

where

$$\frac{\tilde{A}''_{T,s}}{\tilde{A}_{T,s}} \simeq \frac{2+3\epsilon}{\tau^2} - \lambda_s \frac{k}{\tau} \frac{H}{\tilde{M}_{CST}}. \tag{6.85}$$

at leading order in the slow-roll parameters and in the ratio $H/\tilde{M}_{CST}$. The equations of motion (6.84) can now be rewritten as

$$\mu_s'' + \left(\tilde{C}^2_{T,s} k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau} \frac{H}{\tilde{M}_{CST}} \right) \mu_s = 0, \tag{6.86}$$

with

$$\nu_T \simeq \frac{3}{2} + \epsilon. \tag{6.87}$$

As in the case with only first derivatives of the scalar field, the speed of propagation $\tilde{C}_{T,s}$ of GWs is modified. At leading order in the slow-roll parameters and in the ratio $H/\tilde{M}_{CST}$, we have

$$\tilde{C}^2_{T,s} \simeq \frac{1}{1 + \lambda_s k \frac{H}{\tilde{M}_{CST}} \tau}, \tag{6.88}$$

However, notice that with respect to the previous case we do not have (in the numerator) the terms proportional to the time derivatives of the coupling functions (which are not present in the action (6.75)), that can potentially lead to interesting signatures. The term in the denominator is still suppressed since we work in a regime where we are free from ghost fields. Therefore, apart from tiny corrections, we can take

$$\tilde{C}^2_{T,s} \simeq 1 \tag{6.89}$$

[11] In this regard, see again the footnote 4 at page 112.

to hold during inflation.

We can now canonically quantize the fields μ_s by expanding them in terms of the creation and annihilation operators as

$$\hat{\mu}_s(\mathbf{k},\tau) = u_s(k,\tau)\hat{a}_s(\mathbf{k}) + u_s^*(k,\tau)\hat{a}_s^\dagger(-\mathbf{k}). \tag{6.90}$$

The creation and annihilation operators satisfy the equal time commutation relations

$$[\hat{a}_s(\mathbf{k}), \hat{a}_{s'}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')], \tag{6.91}$$

and act on the vacuum state obeying

$$\hat{a}_s|0\rangle = 0, \quad \langle 0|\hat{a}_s^\dagger = 0. \tag{6.92}$$

The equations of motion for the mode functions u_s are

$$u_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau}\frac{H}{\tilde{M}_{CST}}\right) u_s = 0. \tag{6.93}$$

These are the same equations as in the Chern-Simons case, with the only difference that the Chern-Simons mass is replaced by $\tilde{M}_{CST}$. In the sub-horizon limit the equations of motion (6.93) reduce to

$$u_s'' + k^2 u_s \simeq 0. \tag{6.94}$$

and we can impose the Bunch-Davies initial condition

$$\lim_{-k\tau\to\infty} u_s(k,\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{6.95}$$

Therefore, as in the Chern-Simons case, the chirality in the power spectrum of PGWs is suppressed by the ratio $H/\tilde{M}_{CST}$, and the tensor spectral index remains the same as in standard models, at leading order in the slow-roll parameters and using the fact that $H/\tilde{M}_{CST} \ll 1$.

Conclusions

The inflationary paradigm is the leading mechanism capable of solving the shortcomings of the standard cosmological model, namely the horizon, the flatness and the unwanted relics problems [1–4]. All the observations are in agreement with its basic predictions. As we have studied in this thesis, the most important feature of the inflationary phase is that this provides also a way of producing the primordial perturbations which are the seeds for the subsequent formation of the Large Scale Structures of the Universe [5–7]. A stochastic background of gravitational waves is another fundamental and general prediction of all inflationary models [11]. The simplest models of inflation, namely single field slow-roll models, predict an almost scale-invariant power spectrum of both scalar and tensor perturbations, with a tiny level of non-Gaussianity. For scalar perturbations these predictions have been extraordinarily confirmed by the measurements of the temperature anisotropies of the CMB made by the Planck sateltite [8,9,41,55], while primordial gravitational waves have not been observed yet.

The main aim of this thesis was to study inflation within new parity violating theories of gravity which have been recently proposed [19], and which generalize Chern-Simons gravity [16–18,76] by including in the action coupling terms to gravity involving first and second derivatives of a scalar field (the inflaton field, in the context of inflation). Our motivations for doing this are at least twofold: from the point of view of fundamental physics, inflation allows us to test physics at very high energies, far beyond the ones that can be reached on Earth. In the context of our interest, modifying the theory of gravity could leave some signatures in the primordial perturbations originated during the inflationary phase, thus possibly allowing us to test whether the theory of gravity deviates from General Relativity. Moreover, it is well known that primordial gravitational waves are the "smoking gun" of inflation, being a general prediction of all inflationary models. However, they have not been observed yet. Modifying the theory of gravity could enhance the spectrum of primordial gravitational waves at scales accessible to interferometers, thus allowing us to detect them [15, 83]. This would be a strong evidence supporting the inflationary paradigm.

First of all we have seen that the background dynamics of inflation is not modified with respect to the standard models with GR. Also the power spectrum of scalar perturbations remains the same, as can be seen by symmetry arguments. Then we have focused on studying how the dynamics of primordial gravitational waves is modified.

In the case of the theory with only first derivatives of the scalar field, described by the action (6.8), the resulting equations of motion (at leading order in slow-roll) for the two helicity modes are

$$u_s'' + \left(C_{T,s}^2 k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau} \frac{H}{M_{CST}} \right) u_s = 0, \tag{6.96}$$

where M_{CST} is the energy scale at which some modes of the fields become ghost fields, namely it is the analogous of the Chern-Simons mass [17, 18] in these new theories. We have thus introduced a UV cut-off $\Lambda \ll M_{CST}$ to avoid the presence of ghosts. Notice that, in deriving Eq. (6.96), M_{CST} has been assumed constant. If we compare Eqs. (6.96) with the corresponding equations of motion arising in the case with Chern-Simons gravity [16, 17],

$$u_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau} \frac{H}{M_{CS}}\right) u_s = 0, \tag{6.97}$$

we can see that the speed of propagation of tensor modes during inflation is modified, since $C_{T,s}^2 \neq 1$. Its expression, at leading order in slow-roll, is now given by

$$C_{T,s}^2 = \frac{1 + \frac{1}{2}\lambda_s \frac{\dot{f}+\dot{g}}{f+g}\, k \frac{H}{M_{CST}} \frac{\tau}{H}}{1 + \lambda_s k \frac{H}{M_{CST}} \tau}. \tag{6.98}$$

Thus the speed of tensor modes varies with time during inflation and is different for the two polarization states. In particular, one of the two polarization states propagates with a superluminal speed, while the other one is subluminal (which one is superluminal and which one is subluminal depends on wheter the couplings f and g increase or decrease during inflation). As can be seen from Eq. (6.98), in the super-horizon limit ($k\tau \to 0$) the tensor speed approaches the speed of light for both the polarization states. The term in the denominator of (6.98) if suppressed, $k(H/M_{CST})\tau \ll 1$, if we work in a regime of energies where we avoid the presence of ghost fields. The term in the numerator can instead give a non-negligible contribution if the coupling functions $f(\phi)$ and $g(\phi)$ have large time derivatives during inflation. This could leave some interesting phenomenological signatures both at the level of chirality in the power spectrum and in the tensor spectral index. For the former, we recall that the chirality Θ is defined as

$$\Theta \equiv \frac{P_T^R - P_T^L}{P_T^R + P_T^L}, \tag{6.99}$$

where P_T^R and P_T^L are the super-horizon tensor power spectra for right and left modes respectively. If the two helicity modes propagate with a different speed, they cross the horizon and hence get frozen at different times during inflation. This could enhance the chirality Θ in the power spectrum of primordial gravitational waves with respect to the Chern-Simons case. The fact that the tensor speed varies during inflation and depends on the wavenumber k could also modify the spectral index of one or both of the polarization states, possibly leading to a blue power spectrum of tensor perturbations. This could be interesting beacause the spectrum of PGWs could be enhanced at scales detectable by interferometers [15, 83]. In order to study both these features we need to solve numerically the equations of motion (6.96), which have no analytical solutions due to the time dependence of $C_{T,s}^2$. We leave this for a possible future work. Another possible interesting future work could involve the study of non-Gaussianities, which could be unsuppressed due to some interaction terms introduced by the new operators in the action, with the aim of seeing whether parity-breaking signatures could be detectable in the bispectra of primordial perturbations [17, 18, 82].

In the case of the theory with second derivatives of the scalar field, described by the action (6.9), the equations of motion for the two helicity modes are instead given by

$$u_s'' + \left(\tilde{C}_{T,s}^2 k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau} \frac{H}{\tilde{M}_{CST}}\right) u_s = 0, \tag{6.100}$$

where $\tilde{M}_{CST}$ has the same role in this theory as M_{CST} in the previous one. The tensor speed is now given by

$$\tilde{C}_{T,s}^2 \simeq \frac{1}{1 + \lambda_s k \frac{H}{\tilde{M}_{CST}} \tau}. \tag{6.101}$$

With respect to the previous case we do not have in the numerator the term proportional to the time derivatives of the coupling functions, that can potentially lead to interesting signatures.

The term in the denominator is still suppressed given that we work in a regime where we are free from ghost fields, since $k(H/\tilde{M}_{CTS})\tau \ll 1$. Therefore, apart from tiny corrections, the tensor speed is equal to the speed of light during inflation,

$$\tilde{C}^2_{T,s} \simeq 1. \tag{6.102}$$

The equations of motion have now the same form of the ones which arise in the case with Chern-Simons gravity, and the conclusions are therefore the same: the chirality in the power spectrum is suppressed, since

$$\Theta \sim \frac{H}{\tilde{M}_{CST}} \ll 1, \tag{6.103}$$

and the tensor spectral index is the same as in standard models with GR ($n_T \simeq -2\epsilon$), at leading order in the slow-roll parameters and since $H/\tilde{M}_{CST} \ll 1$. As for the other model, an interesting extension of this work could be the study of non-Gaussianities, considering in particular the parity-breaking signatures in the primordial bispectra.

Appendix A

De Sitter spacetime: symmetry under dilations and scale-invariance of the power spectrum

Studying the symmetries of the inflating spacetime is important to better understand the properties of the perturbations from inflation. We now want to see that the scale-invariance of the spectrum of the perturbations in a pure de Sitter phase is related to the symmetry of the underlying geometry under dilations. In this discussion we follow [33], see also [84].

Dilations are transformations that act as

$$x^\mu \to x'^\mu = \lambda x^\mu \quad \Rightarrow \quad \begin{cases} x^i \to x'^i = \lambda x^i, \\ \tau \to \tau' = \lambda \tau \end{cases} \tag{A.1}$$

and are a symmetry of the de Sitter geometry, together with spatial translations and rotations on $\tau = \text{const}$ slicings. This transformation acts on a scalar field ϕ (in our case, the inflaton) as

$$\phi(x) \to \phi_\lambda(x) = \phi(\lambda x). \tag{A.2}$$

In the Fourier space, we have

$$\phi(\lambda \mathbf{x}) = \int d^3k\, \phi(\mathbf{k}) e^{i\mathbf{k}\cdot\lambda\mathbf{x}} = \lambda^{-3} \int d^3p\, \phi(\mathbf{p}/\lambda) e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{A.3}$$

where in the second equality we have changed the variable of integration from $\mathbf{k}$ to $\mathbf{p} = \lambda\mathbf{k}$. Thus, in the Fourier space dilations act on ϕ as

$$\phi_{\mathbf{k}} \to \phi'_{\mathbf{k}} = \lambda^{-3} \phi_{\mathbf{k}/\lambda}. \tag{A.4}$$

Due to the symmetry under dilations, the power spectrum of perturbations must have the form

$$\langle \phi_{\mathbf{k_1}} \phi_{\mathbf{k_2}} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2}) \frac{F(k_1\tau)}{k_1^3}, \tag{A.5}$$

where F is a generic function. Indeed, in this way[1]

$$\langle \phi_{\mathbf{k_1}} \phi_{\mathbf{k_2}} \rangle \to \lambda^{-6} \langle \phi_{\mathbf{k_1}/\lambda} \phi_{\mathbf{k_2}/\lambda} \rangle = \lambda^{-6} (2\pi)^3 \delta^{(3)} \left(\frac{\mathbf{k_1}}{\lambda} + \frac{\mathbf{k_2}}{\lambda} \right) \frac{F(k_1\tau/\lambda)}{(k_1/\lambda)^3} = (2\pi)^3 \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2}) \frac{F(k_1\tau)}{k_1^3}. \tag{A.7}$$

If on super-Hubble scales the perturbations become time-independent, the function F must be constant in this limit. This leads to a scale-invariant power spectrum.

[1] In the last step we have used the following property of the three-dimensional Dirac delta:

$$\delta^{(3)}(\lambda \mathbf{x}) = |\lambda|^{-3} \delta^{(3)}(\mathbf{x}). \tag{A.6}$$

Appendix B

Consistency relation for single-field models

In this Appendix we show the existence of a consistency relation for the three-point function of scalar perturbations, with the *unique* assumption that the inflaton is the only dynamical field during inflation. It is not based on any slow-roll approximation and it is independent on other details like the shape of the potential, the form of the kinetic term and the initial vacuum state. Therefore, if proved wrong by experimental results, this consistency relation would rule out in a model independent way the possibility that the inflaton is the only dynamical field. The general proof is given in [68]; see also [62]

This consistency relation relates the squeezed limit of the three-point function of curvature perturbations to the power spectrum and the tilt of the two-point function:

$$\lim_{k_1 \to 0} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)(1 - n_s) P_\zeta(k_1) P_\zeta(k_3). \tag{B.1}$$

The curvature perturbation ζ is such that the metric, once a mode is outside the horizon and hence is frozen, can be written as

$$ds^2 = -dt^2 + a^2(t)\, e^{2\zeta(\mathbf{x},t)} d\mathbf{x}^2. \tag{B.2}$$

Hence ζ acts as a local rescaling of the spatial coordinates within a given Hubble patch. In particular, we are considering the squeezed limit of the three-point function, which correlates one long wavelength mode, which we take to be k_1, to two short wavelength modes, k_2 and k_3 (with $k_2 \approx k_3$). Therefore, the long mode will be already frozen outside the horizon when the shorter ones freeze and acts as a background field for them (it locally rescales the spatial coordinates). It is then quite intuitive that, if the power spectrum of the short modes is scale invariant, then there is no correlation with the long mode.

In order to show it explicitly, we start by calculating the power spectrum of the short modes; then we will correlate this result with the long background mode. We define the long wavelength mode as $k_L = k_1$ and the two short modes as $k_S = k_2 \approx k_3$. In position space, we can expand the two-point function around its value when $\zeta_L \to 0$ (i.e., in the absence of the long mode):

$$\langle \zeta_S \zeta_S \rangle_{\zeta_L}(\Delta x) = \langle \zeta_S \zeta_S \rangle_0(\Delta x) + \zeta_L \frac{\partial}{\partial \zeta_L} \langle \zeta_S \zeta_S \rangle \Big|_0 + \dots \tag{B.3}$$

For what said, the derivative with respect to the long mode is equal to the derivative with respect to the logarithm of the distance Δx between the points where the short modes are evaluated. Hence, we have

$$\langle \zeta_S \zeta_S \rangle_{\zeta_L} = \langle \zeta_S \zeta_S \rangle_0 + \zeta_L (1 - n_s) \langle \zeta_S \zeta_S \rangle_0. \tag{B.4}$$

To get the three-point function we multiply by ζ_L and take the average, finding

$$\langle\langle\zeta_S\zeta_S\rangle_{\zeta_L}\zeta_L\rangle = (1-n_s)\langle\zeta_L\zeta_L\rangle\langle\zeta_S\zeta_S\rangle_0. \tag{B.5}$$

Going to Fourier space, we find the final result:

$$B(k_S, k_S, k_L \to 0) = (1-n_s)P_\zeta(k_L)P_\zeta(k_S). \tag{B.6}$$

Appendix C

Polarization operators

In this section we construct the two different polarization basis used within the thesis and discuss their properties, following [82].

First of all, we expand the tensor perturbations in the Fourier space as

$$\gamma_{ij}(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_s \epsilon_{ij}^{(s)}(\hat{k})\gamma_s(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{C.1}$$

where $\epsilon_{ij}^{(\lambda)}(\hat{k})$ is an arbitrary polarization basis. One of the most commonly used is the $\{+,\times\}$ one. To construct it, we start by expressing $\hat{k}$ in the $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ basis in polar coordinates:

$$\hat{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{C.2}$$

and then define the vectors

$$\hat{u} = (\sin\phi, -\cos\phi, 0), \quad \hat{v} = \hat{k}\times\hat{u} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{C.3}$$

such that $\{\hat{k},\hat{u},\hat{v}\}$ form an orthonormal basis. Thus we define the $+$, $\times$ polarization tensors as

$$\epsilon_{ij}^{(+)} = \frac{\hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j}{\sqrt{2}}, \quad \epsilon_{ij}^{(\times)} = \frac{\hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j}{\sqrt{2}}, \tag{C.4}$$

which are real and satisfy the properties

$$\epsilon_{ij}^{(+)}\epsilon_{ij}^{(+)} = \epsilon_{ij}^{(\times)}\epsilon_{ij}^{(\times)} = 1, \quad \epsilon_{ij}^{(+)}\epsilon_{ij}^{(\times)} = 0. \tag{C.5}$$

We can then define the chiral polarization tensor basis as

$$\epsilon_{ij}^{(R)} = \frac{\epsilon_{ij}^{(+)} + i\epsilon_{ij}^{(\times)}}{\sqrt{2}}, \quad \epsilon_{ij}^{(L)} = \frac{\epsilon_{ij}^{(+)} - i\epsilon_{ij}^{(\times)}}{\sqrt{2}}, \tag{C.6}$$

which satisfy the properties

$$\epsilon_{ij}^{(\lambda)*}(\hat{k}) = \epsilon_{ij}^{(\lambda)}(-\hat{k}), \quad \epsilon_{ij}^{(L)*}(\hat{k}) = \epsilon_{ij}^{(R)}(\hat{k}), \quad \epsilon_{ij}^{(R)*}(\hat{k}) = \epsilon_{ij}^{(L)}(\hat{k}), \quad \epsilon_{ij}^{(\lambda)*}(\hat{k})\epsilon_{ij}^{(\lambda')*}(\hat{k}) = \delta^{\lambda\lambda'}. \tag{C.7}$$

Since the orientation of the $\{\hat{u},\hat{v}\}$ vectors within the plane orthogonal to $\hat{k}$ is arbitrary, we can rotate them by an angle α around the axis defined by $\hat{k}$, obtaining a new set $\{\hat{u}(\alpha),\hat{v}(\alpha),\hat{k}\}$, where $\hat{u}(0)\equiv\hat{u}$ and $\hat{v}(0)\equiv\hat{v}$:

$$\begin{pmatrix}\hat{u}(\alpha)\\ \hat{v}(\alpha)\end{pmatrix} = R[\alpha]\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}, \tag{C.8}$$

where

$$R[\alpha] = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{C.9}$$

is the rotation matrix around the $\hat{k}$-axis. By expressing the components of new vectors $\hat{u}(\alpha)$, $\hat{v}(\alpha)$ in terms of the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ it is easy to show that

$$|\hat{u}(\alpha)|^2 = 1 = |\hat{v}(\alpha)|^2, \quad \hat{u}(\alpha) \times \hat{v}(\alpha) = \hat{k}, \tag{C.10}$$

meaning that the new set of vectors still form an orthonormal basis, as expected. We can then construct the polarization tensors in the new basis, starting with

$$\epsilon_{ij}^{(+)}(\alpha) = \frac{\hat{u}_i(\alpha)\hat{u}_j(\alpha) - \hat{v}_i(\alpha)\hat{v}_j(\alpha)}{\sqrt{2}}, \quad \epsilon_{ij}^{(\times)}(\alpha) = \frac{\hat{u}_i(\alpha)\hat{v}_j(\alpha) + \hat{v}_i(\alpha)\hat{u}_j(\alpha)}{\sqrt{2}}. \tag{C.11}$$

Using (C.8) we can easily show that the $+, \times$ polarization tensors transform as the basis vectors themselves, namely

$$\begin{pmatrix} \epsilon_{ij}^{(+)}(\alpha) \\ \epsilon_{ij}^{(\times)}(\alpha) \end{pmatrix} = R[\alpha] \begin{pmatrix} \epsilon_{ij}^{(+)} \\ \epsilon_{ij}^{(\times)} \end{pmatrix}. \tag{C.12}$$

This means that if we want to expand the tensor perturbations in terms of the new polarization tensors, also the tensor modes should be rotated by the same matrix R

$$\begin{pmatrix} h_{(+)}(\alpha) \\ h_{(\times)}(\alpha) \end{pmatrix} = R[\alpha] \begin{pmatrix} h_{(+)} \\ h_{(\times)} \end{pmatrix}. \tag{C.13}$$

If we then define the chiral polarization tensors in the new basis

$$\epsilon_{ij}^{(R)}(\alpha) = \frac{\epsilon_{ij}^{(+)}(\alpha) + i\epsilon_{ij}^{(\times)}(\alpha)}{\sqrt{2}}, \quad \epsilon_{ij}^{(L)}(\alpha) = \frac{\epsilon_{ij}^{(+)}(\alpha) - i\epsilon_{ij}^{(\times)}(\alpha)}{\sqrt{2}}, \tag{C.14}$$

we can see that they are related to the ones in the old basis through the relation

$$\epsilon_{ij}^{(R,L)}(\alpha) = e^{\mp i2\alpha}\epsilon_{ij}^{(R,L)}. \tag{C.15}$$

Thus, if we want to expand the tensor perturbations in terms of the new chiral polarzation tensors, the tensor modes have to be rescaled by an opposite phase, namely

$$\gamma_{R,L}(\alpha) = e^{\pm i2\alpha}\gamma_{R,L}. \tag{C.16}$$

This transformation law reflects the spin-2 nature of the tensor field γ_{ij}.

Appendix D

$f(R)$ theories of gravity and non-Gaussianity from inflation

The term with R^2 in the expansion (5.11) is nothing else than the first term in an expansion in powers of the Ricci scalar of a more general $f(R)$ theory. Thus, we consider the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{D.1}$$

We now want to show that the $f(R)$ term brings to the introduction of an additional degree of freedom in the theory. In other words, a $f(R)$ theory plus a scalar field (which in our case is the inflaton) is equivalent to a theory with two scalar fields with a specific metric, and a generic potential for the two fields. To do so, we start by showing how our theory can be recast in the form of a scalar-tensor theory; then, by applying a Weyl (conformal) transformation, we go from the Jordan frame to the Einstein frame, obtaining the desired result.

We only mention that here we work in the so-called *metric formalism*, in which the connection is assumed a priori to be the Levi-Civita connection. When studying theories of gravity, one can use more general approaches: in the *Palatini formalism* the metric and the connection are assumed to be independent variables and one varies the action with respect to both of them, under the important assumption that the matter action does not depend on the connection[1]; in the *metric-affine formalism* one abandons also the latter assumption. Clearly, this last one is the most general of these theories (see [85, 86] for more details).

Coming back to our problem, the first step is to rewriting the Lagrangian in the form

$$\mathcal{L} = \sqrt{-g}\left[f(\chi) + f'(\chi)(R-\chi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{D.2}$$

where χ is a new auxiliary (i.e., non dynamical) field, and the prime denotes the derivative with respect to it. Setting the variation with respect to χ equal to zero, we find

$$f''(\chi)(R-\chi) = 0, \tag{D.3}$$

so that $\chi = R$, for each χ such that $f''(\chi) \neq 0$. Substituting this result in (D.2) we recover the original Lagrangian (D.1), showing that they are equivalent. Moreover, the case $f''(\chi) = 0$ gives us back the Einstein-Hilbert action.

Now, let us define a new scalar field

$$\psi \equiv \frac{2}{M_{Pl}^2} f'(\chi), \tag{D.4}$$

[1] In the case of Einstein gravity, the variation wrt to the connection gives the condition that the latter is the Levi-Civita connection; hence, the two formalisms are equivalent.

and, assuming that $\psi(\chi)$ is invertible, let us define the potential

$$\Lambda(\psi) \equiv f(\chi(\psi)) - \frac{M_{Pl}^2}{2}\psi\chi(\psi). \tag{D.5}$$

The Lagrangian (D.2) can then be rewritten as

$$\mathcal{L} = \sqrt{-g}\left[\frac{M_{Pl}^2}{2}\psi R + \Lambda(\psi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{D.6}$$

which is the Lagrangian of a scalar-tensor theory with vanishing kinetic term for ψ[2].

The second step, as anticipated, is to perform a Weyl transformation to move from the Jordan frame to the Einstein frame. In the *Jordan frame* there is a direct coupling between the scalar field ψ and the Ricci scalar, and the matter action doesn't depend on ψ; as a consequence, the stress-energy tensor $T_{\mu\nu}$ is still covariantly conserved and test particles follow geodesics of the metric. In the *Einstein frame* the gravitational part of the Lagrangian becomes that of Einstein gravity plus a kinetic term for ψ, and it is also present a direct coupling between the scalar field ψ and the matter fields (the inflaton, in our case); this implies that $T_{\mu\nu}$ is not covariantly conserved anymore and test particles don't follow geodesics of the metric.

Coming back to our discussion, let us now perform the conformal transformation

$$g_{\mu\nu} \to e^{-2\omega}g_{\mu\nu}, \tag{D.7}$$

with $e^{2\omega} = \psi$. By defining the fields

$$\varphi_1 \equiv \sqrt{6}M_{Pl}\omega, \quad \varphi_2 \equiv \phi, \tag{D.8}$$

the Lagrangian takes the form [87]

$$\begin{aligned}\tilde{\mathcal{L}} &= \sqrt{-g}\left[\frac{1}{2}M_{Pl}^2R - \frac{1}{2}g^{\mu\nu}\gamma_{ab}\partial_\mu\varphi^a\partial_\nu\varphi^b - U_1(\varphi_1) - e^{-4\varphi_1/\sqrt{6}M_{Pl}}U_2(\varphi_2)\right]\\ &= \sqrt{-g}\left[\frac{1}{2}M_{Pl}^2R - \frac{1}{2}g^{\mu\nu}\gamma_{ab}\partial_\mu\varphi^a\partial_\nu\varphi^b - V(\varphi_1,\varphi_2)\right],\end{aligned} \tag{D.9}$$

where $a, b = 1, 2$ and we have defined

$$V(\varphi_1,\varphi_2) = U_1(\varphi_1) + e^{-4\varphi_1/\sqrt{6}M_{Pl}}U_2(\varphi_2), \quad U_1(\varphi_1) = e^{-4\varphi_1/\sqrt{6}M_{Pl}}\Lambda(\psi(\omega(\varphi_1))) \tag{D.10}$$

and the metric

$$\gamma_{ab} = \begin{pmatrix}1 & 0\\ 0 & e^{-2\varphi_1/\sqrt{6}M_{Pl}}\end{pmatrix}. \tag{D.11}$$

This is the Lagrangian of a two-field model with a specific metric and the two-field potential $V(\varphi_1,\varphi_2)$.

Having one more degree of freedom, i.e. one more scalar field, it is possible that the interactions between the two fields lead to some observable effects, possibly enhancing local non-Gaussianity to an observable level. If both fields contribute to the background inflationary dynamics, we have in principle to impose slow-roll conditions on both of them. However, if the new scalar field is subdominant during inflation, we can relax this condition, leaving its background dynamics unconstrained; then, it is possible for non-Gaussianity to be transferred to the inflaton field.

[2] In the "language" of scalar-tensor theories, the so-called Brans-Dicke parameter ω_0 is equal to zero. One can show that the equivalence between $f(R)$ theories and scalar-tensor theories still holds also in the Palatini formalism; however, in this case $\omega_0 = -3/2$ [86].

In [87] it has been considered the case

$$f(R) = \frac{M_{Pl}^2}{2} R + \frac{R^2}{12M^2}, \tag{D.12}$$

which is the leading-order expansion of a $f(R)$ theory in powers of the Ricci scalar R. In this case, the two-field potential is given by

$$V(\varphi_1, \varphi_2) = \frac{3}{4} M^2 M_{Pl}^2 \left(1 - e^{-2\varphi_1/\sqrt{6}M_{Pl}}\right)^2 + e^{-4\varphi_1/\sqrt{6}M_{Pl}} U(\varphi_2). \tag{D.13}$$

By expanding the Lagrangian at second order (neglecting the metric perturbations for simplicity), one finds [87]

$$\delta\mathcal{L}_2 = \frac{2}{\sqrt{6}M_{Pl}} e^{-2\bar{\varphi}_G/\sqrt{6}M_{Pl}} \dot{\bar{\varphi}}_I \delta\varphi_G \delta\dot{\varphi}_I, \tag{D.14}$$

where $\varphi_I \equiv \varphi_2$ denotes the inflaton field, and $\varphi_G \equiv \varphi_1$ the isocurvature field, which is those that describes the modifcations of gravity. The bar refers to the background value of the corresponding quantity. At third order one finds [87]

$$\delta\mathcal{L}_3 = -\frac{1}{6} U_1'''(\bar{\varphi}_I) \delta\varphi_G^3. \tag{D.15}$$

As we have already said, since the field φ_G is subdominant during inflation, it is not subjected to any slow-roll condition. Thus the isocurvature potential U_1''' is not negligible and gives a contribution to $\delta\mathcal{L}_3$. In [87] an estimate of the level of non-Gaussianity has been obtained:

$$f_{\rm NL} \simeq \alpha(\nu) \left(\hat{\delta\mathcal{L}_2}\right)^3 \hat{\delta\mathcal{L}_3} P_\zeta^{-1/2}, \tag{D.16}$$

where $\hat{\delta\mathcal{L}_2}$ and $\hat{\delta\mathcal{L}_3}$ are the vertices of the interactions (namely the coefficients in) (D.14)-(D.15), and $\alpha(\nu)$ is a numerical factor (function of $M_{\rm eff}/H$, with $M_{\rm eff}$ being the effectve mass of φ_I) which can range from 0.2 to about 300. As shown in [87], it is possible to obtain $f_{\rm NL} \approx -1$ to -30, with a characteristic shape of non-Gaussianity which is intermediate between an equilateral and a local shape.

For further developements on primordial non-Gaussianities from modifications to the theory of gravity, see also [88].

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