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Final Dissertation

GRAVITATIONAL WAVES FROM PRIMORDIAL BLACK HOLE FORMATION IN ALTERNATIVE THEORIES OF GRAVITY

Thesis supervisor

Prof. Sabino Matarrese

Thesis co-supervisor

Dr. Yashar Akrami

Candidate

Anna Negro

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Abstract

Primordial black holes provide important information about the early Universe complementary to large-scale cosmological observations.
In the standard picture, primordial black holes are formed, during the epoch of radiation domination at early times, by (non-standard) enhancements in the power spectrum of primordial perturbations generated during inflation over some scale ranges. An alternative mechanism for describing the generation of primordial black holes is currently under active development. It is based on non-standard gravitational effects at sub-millimetre scales in the early Universe, for which we expect deviations from general relativity to become significant. In this approach to the primordial black hole formation, the power spectrum of primordial fluctuations is assumed to be of the standard, nearly scale-invariant form set by inflation, while the evolution of the fluctuations after inflation results in enhancements of the power spectrum, providing a new mechanism for generating primordial black holes. In this project we study the latter scenario: a non-inflationary mechanism for producing primordial black holes. The aim of this work is to investigate the physical consequences of this scenario in order to be able to set constraints on alternative theories of gravity. In particular, we study the production of scalar-induced gravitational waves from primordial black hole formation in this newly proposed framework for the production of primordial black holes.

The study of the production of scalar-induced gravitational waves, which is the main original contribution of this thesis work, could help to better understand and constrain the non-inflationary mechanism proposed in a work in progress [1] to which I have been exposed, for producing primordial black holes and, as a consequence, modifications of general relativity on small scales and early times.

Contents

Chapter 1

Introduction

Primordial black holes (PBHs) are black holes that are assumed to have been formed in the primordial Universe. They could have been naturally formed from features in the primordial curvature power spectrum and they may constitute today a dominant component of dark matter in the Universe ([2]). PBHs are interesting, not only because they are a cold dark matter candidate, but also because of their implications for astronomy. It has been pointed out that primordial black holes can be taken as the possible source of some astronomical events and some of the LIGO events [3]. Furthermore, as summarised in Ref. [4], by studying their formation and evolution, we can place interesting constraints on the early Universe. The primordial black hole formation requires non-standard enhancements in the power spectrum of primordial curvature perturbations. Since measurements of the cosmic microwave background (CMB) observations of the large-scale structure constrain the curvature perturbations on large scales, in order to generate a sizable amount of primordial black holes we need to enhance the small-scale curvature perturbations, as shown in **Figure 1.1**. When these enhanced curvature perturbations re-enter the horizon, if they are large enough, they collapse and result in the formation of black holes. This means that, by studying the observational constraints on the abundance of primordial black hole, we can constrain the proposed formation model.

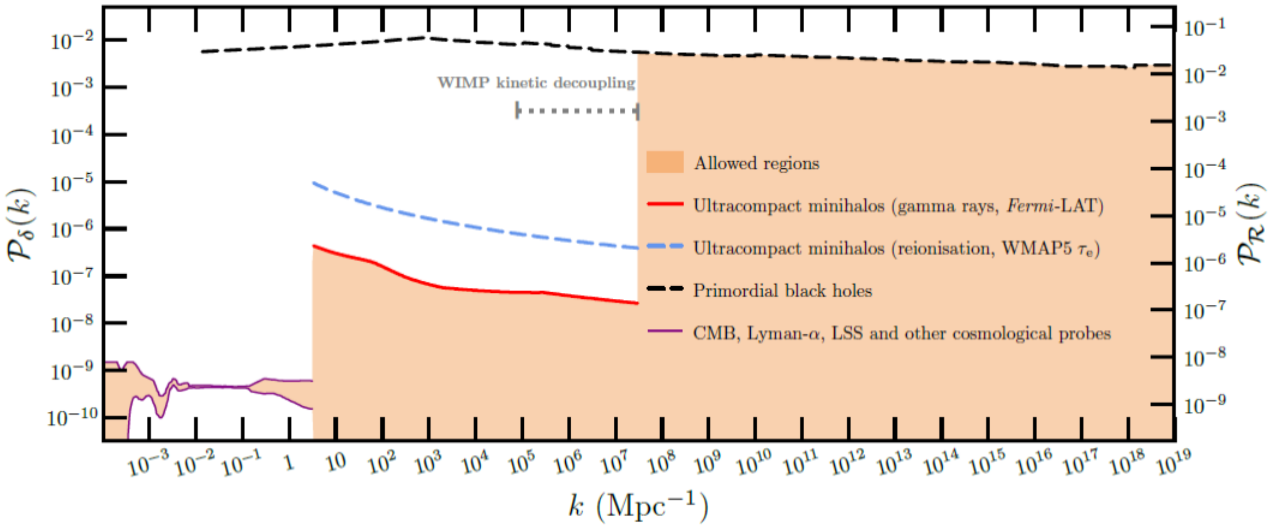

Figure 1.1: *Observational constraints on the amplitudes of primordial density and curvature perturbations P_δ and $P_\mathcal{R}$ at different scales. This figure is taken from Ref.*[5].

We are particularly interested in this latter aspect of primordial black holes: the formation mechanism. In this thesis work we study a new formation mechanism that explains the production of very large scalar metric perturbations during the post-inflationary epoch. This non-standard enhancement of the power spectrum, proposed in Ref. [1], will be called the *new scenario*, to be distinguished from the *standard scenario* for producing primordial black holes.

In the standard scenario, primordial curvature perturbations are produced during the inflation era. There are several possibilities to explain the formation of a peak-like feature in the curvature power spectrum during the inflationary phase. We could consider a multi-field inflationary model, such that one of the fields acts like the inflaton and the other one triggers either a phase transition or a fast evolution, resulting in an excess in the spectrum of curvature fluctuations. Depending on when this phenomenon occurs during inflation, we may have a narrow or a broad spectrum of masses for the primordial black holes which form during the radiation domination era upon horizon re-entry. We could also generate the same effect in a single-inflation model, by imposing an inflection point in the inflaton potential. Thanks to a period of evolution during which the single field slows down, we can achieve the enhancement of the curvature power spectrum as in the multi-field inflation, as we will see in the next section. Alternatively, models exist which that explain primordial black hole formation by studying phase transitions or the reheating phase. However, the common point of these production mechanisms is that the enhancement of the curvature power spectrum occurs before the radiation-dominated epoch. In other words, as shown in **Figure 1.2**, the power spectrum is enhanced before (or, at most, during) the reheating phase. It is then frozen in the super-horizon phase, and in the radiation-dominated epoch, after the horizon re-entry, overdense regions become causally connected and collapse to form primordial black holes.

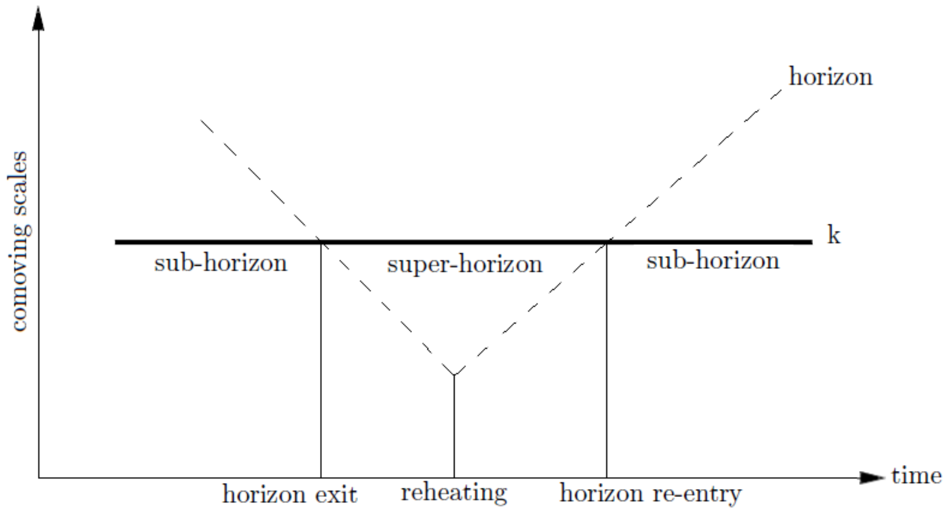

Figure 1.2: *Evolution of the horizon scale as a function of time. At first, during the inflationary epoch, the horizon decreases until the reheating phase, after which it starts increasing during the radiation-dominated epoch.* k *shows a comoving scale of interest.*

In the new scenario, the curvature perturbations sourced during the inflationary phase are scale-invariant and are of the order of quantum fluctuations. The enhancement of curvature perturbations needed for production of primordial black holes is achieved in the post-inflationary epoch, due to hypothetical coupling of a dynamical scalar degree of freedom to radiation. Thus, in reference to **Figure 1.2**, in the new scenario the power spectrum is almost scale-invariant up to the reheating phase, and it is not frozen anymore on super-horizon scales, during the radiation domination epoch. In other words, super-horizon curvature perturbations are enhanced in this later phase of radiation domination and they produce a broad peak in the power spectrum. In this way, after the horizon re-entery, overdense regions are large enough to form primordial black holes. The formation mechanism presented in this thesis work is indeed conceptually different from the previous ones: we are modifying the evolution of density perturbations instead of modifying the initial conditions.

As we will see in the next sections, this formation mechanism is well-motivated in the context of extra-dimensional theories of gravity in which the dynamical scalar field is the volume modulus of the compactified subspace. Thus, the discussion of this scenario can be considered as an inevitable consequence of extra-dimensional theories of gravity.

In conclusion, the new scenario is interesting in the context of both primordial black hole formation and alternative theories of gravity. In this thesis work after reviewing this new model, we investigate in depth one of its important implications by studying scalar-induced *gravitational waves*.

Scalar perturbations couple to tensor perturbations at second order in perturbation theory, although they decouple to each other at the linear level. These second order gravitational waves sourced by first order scalars perturbations are called *scalar induced gravitational waves*. Inevitably, the large-scalar metric perturbations become a significant source of gravitational waves and generate abundant gravitational waves signals via the second-order effect. The study of the concomitant induced gravitational waves provides a way to constrain, verify or rule out the proposed new scenario. Therefore, as we will see in the last section, the generated gravitational wave signals are expected to be strongly constrained in the near future, thanks to the incoming gravitational waves surveys. Thus, the detection of gravitational waves could provide a complementary way to constrain the newly proposed model of primordial black hole formation, and consequently modifications of gravity.

Goal

To study the production of gravitational waves in a newly proposed framework for the production of primordial black holes.

Thesis structure

In Section **2** we review the main topics we will refer to during the thesis work. In Section **2.1** we review primordial black holes: we present an example of primordial black hole formation mechanism in the standard scenario in Section **2.1.1**, we explain the standard procedure to link the curvature power spectrum to the primordial black hole abundance and mass in **2.1.2** and we give an overview of the current observational constraints of the primordial black hole abundance in Section **2.1.3**. We then review the perturbation theory in Section **2.2**: we geometrically define the *gauge problem* in Section **2.2.1** and we study the first order Einstein equations in Section **2.2.2**. Finally, in Section **2.3** we review the behaviour of first order tensor perturbations in the standard-inflationary model to better understand the nature of stochastic gravitational waves.

In Section **3** we present the model that describes the new scenario and in Section **4** we study the production of primordial black holes in this model. In Section **4.1** we explain the formalism we use to study scalar perturbations, and in Section **4.2** we derive and study first order scalars perturbtions equations of motion.

In Section **5** we study second order scalar-induced gravitational waves in the new scenario. In Section **5.1** we go back to our discussion of the gauge problem: we specify how to change the gauge up to the second order in Section **5.1.1** and in Section **5.1.2** we give an overview of the gauge dependence of second order gravitational waves. In Section **5.2** we derive the main results of this thesis work: we obtain the scalar-induced gravitational waves equation of motion in Section **5.2.1**, using the Mathematica coding reported in **Appendix**, and we study the power spectrum in Section **5.2.2**.

We conclude and present possibles future directions in Sections **6** and **7**, respectively.

Chapter 2

Preliminaries

In this section we review the basics of the main topics we will address in this thesis work: PBHs, perturbation theory and stochastic gravitational waves. In Section **2.1** we review the standard mechanism to produce PBHs and the ways we compute the PBH abundance in the monochromatic case; we summarize current experimental constraints on the PBH abundance. In Sections **2.2** and **2.3** we review the cosmological perturbation theory and stochastic gravitational waves, respectively.

2.1 Primordial black holes

As mentioned in the introduction, PBHs are black holes that have been formed in the primordial Universe. In contrast to black holes formed via astrophysical processes, that could have been formed only if their mass is greater than $\sim 3M_{\odot}$ (where the solar mass is $M_{\odot} \approx 2 \cdot 10^{33}\, g$), PBHs in principle can be arbitrarily light. There are no theoretical constraints on the mass of PBHs, but PBHs lighter than $\sim 10^{15}\, g$ cannot exist in the present Universe: such light PBHs should have already evaporated due to Hawking mechanism.

In the following subsections we present the formation of PBHs via inflation (the standard scenario), how we can compute the abundance of PBHs as a function of their masses and finally which experimental constraints we must satisfy if we want to introduce a new mechanism that explains the PBH production.

2.1.1 PBH formation

The most frequently studied PBH formation scenario is the gravitational collapse of overdense regions in the early Universe. During the radiation-dominated Universe, while the Hubble horizon grows, overdense regions, generated by density fluctuations, enter the Hubble horizon. If a density fluctuation is sufficiently large so that the gravitational force is stronger than pressure forces, then the fluctuation collapses and forms a PBH with a mass equal to the horizon mass.

In the simplest inflationary scenario, at the end of inflation a Harrison-Zel'dovich power spectrum with a small tilt is generated. Such a power spectrum, which is in agreement with CMB observations, does not allow PBHs to be formed. The generated overdense regions are not sufficiently large and massive to collapse. Thus, in order to produce PBHs, we need to modify the simplest inflationary scenario by changing the potential of the inflaton to both enhance density fluctuations and satisfy the CMB constraints. We briefly present an example of how this can occur in the case of the single field inflation, following [6]. This is just one example of how we can explain the enhancement of density fluctuations during inflation, and there are several models that could produce this effect (a few examples are: Higgs inflation [7], hybrid inflation [8], and reheating mechanism [9]).

How it is possible to generate a peak in the curvature power spectrum of single-field inflation is shown in [6]. The authors propose an inflation model where the potential of the inflaton field ϕ is:

$$V(\phi) = \left(\frac{1}{2}m^2\phi^2 - \frac{1}{3}\alpha v\phi^3 + \frac{1}{4}\lambda\phi^4\right)\left(1+\xi\phi^2\right)^{-2}, \tag{2.1}$$

where m, α, v, λ and ξ are free parameters (that must be tuned for the model to satisfy the CMB constraints). The potential can be rewritten as

$$V(x) = \frac{\lambda v^4}{12}\frac{x^2\left(6-4ax+3x^2\right)}{\left(1+bx^2\right)^2}, \tag{2.2}$$

where we define $x = \frac{\phi}{v}$, $m^2 = \lambda v^2$, $a = \frac{\alpha}{\lambda}$ and $b = \xi v^2$. The extrema of this potential are given by some values of the parameters (a,b) which solve the third order equation

$$1 - ax + (1-b)x^2 + \frac{ab}{3}x^3 = 0. \tag{2.3}$$

By studying this equation and the equation $\frac{\partial V(\phi)}{\partial \phi} = 0$, we find that the potential varies as shown in **Figure 2.1**, where the model parameters are: $a = 1$, $b = b_c(1) - \tilde{b}$, $\tilde{b} = 10^{-4}$ and $v = 0.108$. In particular, we see that the potential has an inflection point at a real value of x whenever the parameter b acquires a critical value $b_c(a)$.

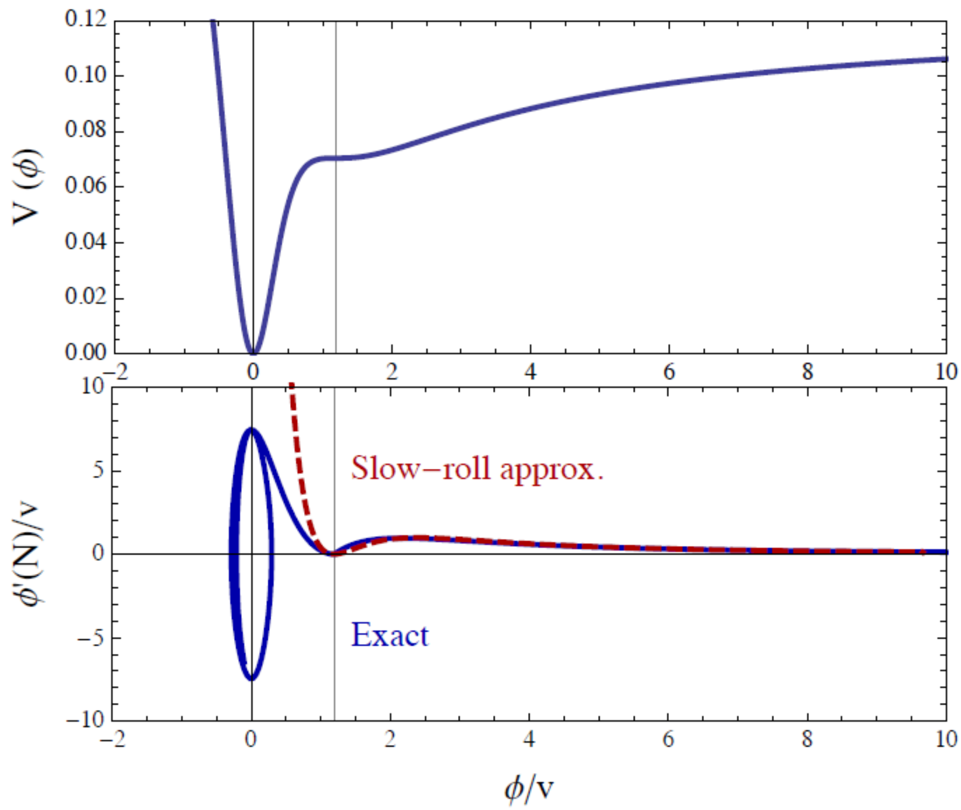

Figure 2.1: *The upper panel shows the potential of the inflaton $V(\phi)$ with an inflection point at $\phi = 1.191v$ and an asymptotically flat plateau. The lower panel shows the exact and approximated evolutions of the inflaton field in phase space. This figure is taken from* [6].

With this choice for the potential, the slow-roll parameter in the slow-roll approximation[1] becomes

$$\epsilon_{\rm SR} = \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{8}{v^2}\frac{(3-3ax+3(1-b)x^2+abx^3)^2}{x^2\,(6-4ax+3x^2)^2\,(1+bx^2)^2} \tag{2.4}$$

and the number of e-foldings $\mathrm{N_{SR}}$ is given by

$$\begin{aligned} \mathrm{N_{SR}} = \int \frac{d\phi}{\sqrt{2\epsilon_{\rm SR}}} &= \frac{v^2}{4}\int \frac{x\,(6-4ax+3x^2)\,(1+bx^2)}{(3-3ax+3(1-b)x^2+abx^3)}dx \\ &= \frac{v^2}{4ab}\int \frac{x\,(6-4ax+3x^2)\,(1+bx^2)}{(x-x_1)\left((x-x_0)^2+y_0^2\right)}dx, \end{aligned} \tag{2.5}$$

where we consider $M_{\rm Pl} = 1$ for the Planck mass. We notice that if we choose the critical parameters, $\frac{d\mathrm{N}}{dx}$ diverges in the slow-roll approximation. However, if we select values sufficiently close to the critical case, we avoid this problem of eternal inflation and the potential has a near-inflection point. In such a case we can still produce a significant peak

[1] In the slow-roll approximation we assume that the potential is almost flat ($\dot\phi \ll V(\phi)$, $\ddot\phi \ll H\dot\phi$) so that: $\epsilon \ll 1$. Furthermore, plugging these conditions into Friedman equations and the inflaton equation of motion we obtain $H^2 \approx V(\phi)$ and $3H\dot\phi + \frac{\partial V(\phi)}{\partial\phi} = 0$.

in the power spectrum.

Once we choose the values of the free parameters we can find the evolution of the inflaton field ϕ by integrating the system[2] (Einstein equations and the inflaton equation of motion):

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0 \\ H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \\ \dot{H} = -\frac{1}{2}\dot{\phi}^2 \end{cases} \tag{2.6}$$

Figure 2.1 shows the numerical result of equations (2.6) with $a = 1$, $b = b_c(1) - \tilde{b}$, $\tilde{b} = 10^{-4}$ and $v = 0.108$.

Now that we have the dynamic of the inflaton, we can compute the curvature power spectrum.[3] The exact expression of the primordial curvature power spectrum is given by

$$P_{\mathcal{R}}(\mathrm{k}) = \frac{H^2(\phi)}{8\pi^2 \epsilon(\phi)}, \tag{2.8}$$

which, in the slow-roll approximation, in our case results in

$$P_{\mathcal{R}}^{\mathrm{SR}}(\mathrm{k}) = \frac{V(x)}{24\pi^2 \epsilon_{\mathrm{SR}}(x)} = \frac{\lambda \kappa^6 v^6}{96 \times 24\pi^2} \frac{\left(6 - 4ax + 3x^2\right)^3 x^4}{\left(3 - 3ax + 3(1-b)x^2 + abx^3\right)^2}. \tag{2.9}$$

We observe that in order to produce a peak in $P_{\mathcal{R}}(\mathrm{k})$ we need to choose a potential with an inflection point, as anticipated. In that case, at the inflection point $\epsilon \sim V' \longrightarrow 0$ and thus $P_{\mathcal{R}}(\mathrm{k}) \longrightarrow \infty$.

Both the exact and approximated numerical results of the curvature power spectrum are shown in **Figure 2.2** with $a = 1$, $b = b_c(1) - \tilde{b}$, $\tilde{b} = 10^{-4}$ and $v = 0.108$.

[2]We represent time derivatives with respect to conformal time as ϕ' while we represent time derivatives with respect to the cosmic time as $\dot{\phi}$.

[3]The power spectrum of a generic random field $g(x,t) = \int \frac{d^3\mathrm{k}}{(2\pi)^{3/2}} e^{i\mathrm{k}\cdot x} g_{\mathrm{k}}(t,\mathrm{k})$ is defined as

$$\left\langle g_{\mathrm{k}_1}, g^*_{\mathrm{k}_2} \right\rangle \equiv \frac{2\pi^2}{\mathrm{k}^3} P_{\mathrm{g}}(\mathrm{k}) \delta^{(3)}\left(\mathrm{k}_1 - \mathrm{k}_2\right), \tag{2.7}$$

where angle brackets denote ensemble average.

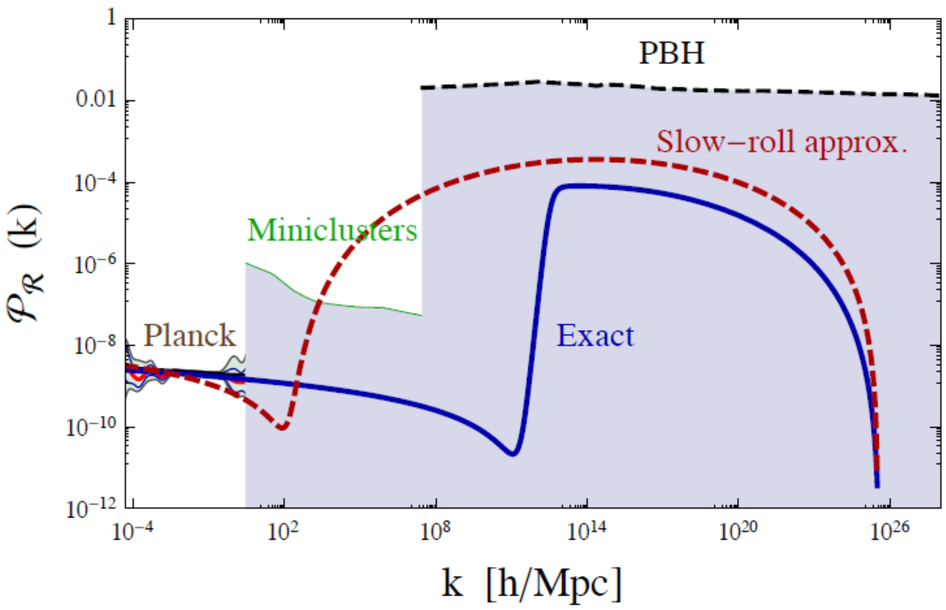

Figure 2.2: *Curvature power spectrum as a function of* k*. The dashed red line shows the result in the slow-roll approximation while the blue line is the exact solution. The figure also shows the range of values allowed by CMB constraints (Plank 2015), mini-halos and PBH abundance constraints. This figure is taken from* [6].

Even without going through the analytical solution (see [6] for more details), we can see from **Figure 2.2** that thanks to the choice of $V(\phi)$ we obtain a peak in the power spectrum. Furthermore, we notice that on the one hand we satisfy the CMB constraints recovering an almost scale-invariant power spectrum at large scales, and on the other hand we successfully enhance the power spectrum at small scales. We will see in the next section that, given the curvature power spectrum, we can compute the PBH abundance. Thus, an enhancement in the power spectrum for a specific scale becomes a peak in the PBH abundance with a specific mass.

In conclusion, we underline that in this scenario we manage to modify the initial conditions of curvature perturbations. Indeed, instead of having almost scale-invariant initial curvature perturbations (which means an almost Harrison Zel'dovich power spectrum at the end of inflation), we introduce a peak on the curvature power spectrum. This is how we explain PBH formation in what we previously called the standard scenario: in this scenario the goal is to find a mechanism which modifies the initial power spectrum. This is, as we will see in detail in the next sections, different from the approach we have in the new scenario. In the latter we start from an almost scale-invariant initial power spectrum and modify the evolution of curvature perturbations during the radiation domination epoch.

2.1.2 PBH abundance

We now want to link the curvature power spectrum to PBH abundance. After inflation, the horizon starts growing[4] and regions that were not causally connected during

[4] After inflation the Universe is radiation dominated, thus the horizon goes as $R_{\rm H} \equiv H^{-1} \sim t$.

inflation become so. Thus, we expect that sufficiently overdense regions, that correspond to enhancements in $P_{\mathcal{R}}(\mathrm{k})$, collapse and form PBHs. In order to link $P_{\mathcal{R}}(\mathrm{k})$ to PBH abundance we use the Press-Schechter formalism following [10]. For a different approach see [11].

The Press-Schechter formalism, as shown in [12], provides a simple analytical description of the evolution of gravitational structures in a hierarchical universe. The core idea is that, starting from perturbations in the mean density of the early Universe, as the horizon grows a larger portion of the mass becomes bounded into larger and larger condensations. Furthermore, if these bounded regions are massive enough, the expansion of the Universe will become negligible with respect to the gravitational attraction. These regions will form galaxies, groups of galaxies, clusters, or, in our case, they will collapse to form PBHs.

To form a PBH, a collapsing overdense region must be large enough to overcome the pressure force resisting its collapse, as it falls within its Schwarzschild radius. Following [13], we consider a spherically symmetric region with mean energy density $\tilde{\rho}$ greater than that of the background. $\tilde{\rho}$ is governed by the positive curvature Friedman equation in a matter-dominated Universe,

$$\tilde{H}^2 = \frac{8\pi}{3M_{\mathrm{Pl}}^2}\tilde{\rho} - \frac{\mathrm{k}}{a^2}, \tag{2.10}$$

while the background ρ_{b} evolves in a radiation-dominated Universe with $\mathrm{k} \approx 0$: $H^2 = \frac{8\pi}{3M_{\mathrm{Pl}}^2}\rho_{\mathrm{b}}$. The perturbed region stops expanding at time t_{c} when $\tilde{H} = 0$. At this time the region has a size of $R_{\mathrm{c}} \approx \delta_{\mathrm{i}}^{-1/2} R_{\mathrm{i}}$, where R_{i} is the radius at some initial time and δ_{i} is the initial density perturbation. If $R_{\mathrm{c}} \geq R_{\mathrm{Jeans}} = c_{\mathrm{s}} t_{\mathrm{c}}$, where $c_{\mathrm{s}} = 1/\sqrt{3}$ is the sound speed, the perturbed region contains enough matter to overcome any pressure forces and it will continue to contract. This condition can be seen as a lower bound on initial density fluctuations that can form PBHs: only regions with $\delta \geq \delta_{\mathrm{c}}$ will produce PBHs. The simplest estimate for the critical density is $\delta_{\mathrm{c}} = 1/3$ but, as shown in [14], one could obtain better estimates for this parameter.

When a perturbation satisfying the above condition crosses the horizon, a PBH will be formed with mass equal to the horizon mass

$$M_{\mathrm{f}} = \gamma M_{\mathrm{H}} = \gamma \frac{4\pi}{3} \rho_{\mathrm{f}} H_{\mathrm{f}}^{-3}, \tag{2.11}$$

where γ is a fixed number related to δ_{c}, M_{H} is the horizon mass and the subscript f means that the quantity is evaluated at the time of formation (i.e., at the horizon crossing). We can make explicit the $k = a_{\mathrm{f}} H_{\mathrm{f}}$ dependence by rewriting equation (2.11) as

$$M_{\mathrm{f}} = \gamma M_{\mathrm{eq}} \left(\frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{eq}}}\right)^{\frac{1}{2}} \left(\frac{a_{\mathrm{f}}}{a_{\mathrm{eq}}}\right)^2 \left(\frac{\mathrm{k_{eq}}}{\mathrm{k}}\right)^2 \tag{2.12}$$

and using $\rho \sim g(T)T^4$ and entropy conservation: $g_*(T)T^3a^3 = const$, where $g_*(T)$ is the temperature dependent effective number of degrees of freedom. In that way we find

$$M_{\mathrm{f}} = \gamma M_{\mathrm{eq}} \left(\frac{g_{\mathrm{eq}}}{g_{\mathrm{f}}}\right)^{\frac{1}{3}} \left(\frac{\mathrm{k_{eq}}}{\mathrm{k}}\right)^2, \tag{2.13}$$

where we have assumed $g_*(T) \sim g(T)$. Thus, given the enhanced mode k in the curvature power spectrum, we can estimate the mass of the generated PBHs with equation (2.13). As an example,[5] by enhancing the wavenumber $\mathrm{k} = 1.4 \cdot 10^{14} Mpc^{-1}$, corresponding to the comoving scale $\lambda = 4.5 \cdot 10^{-14} Mpc$, we produce PBHs with mass $\sim 10^{-16} M_{\odot}$.

To compute the abundance of PBHs we assume that the probability distribution of density fluctuations is Gaussian and that its variance depends on the curvature power spectrum (see [15] and [16] for constraining non-Gaussianity with PBHs). Furthermore, we will assume a monochromatic production of PBHs (for examples in which this assumption is relaxed see [17] or [18]). We then define the PBH abundance evaluated at time of formation as the integral of the probability distribution of the fluctuations, over the fluctuations greater than δ_c

$$\beta \equiv \frac{\rho_{\mathrm{f,PBH}}}{\rho_{\mathrm{f,TOT}}} \approx \int_{\delta_c}^{\infty} p(\delta) d\delta, \tag{2.14}$$

which, under the assumptions of Gaussian density fluctuations, becomes

$$\beta = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\delta^2}{2\sigma^2}} d\delta. \tag{2.15}$$

σ is the mass variance at horizon crossing of the fluctuations

$$\sigma^2 = \int_{\delta_c}^{\infty} \delta^2 p(\delta) d\delta = \frac{16}{81} \int_0^{\infty} \left(\frac{\tilde{\mathrm{k}}}{\mathrm{k}}\right)^4 P_{\mathcal{R}}(\tilde{\mathrm{k}}) W^2(\tilde{\mathrm{k}}, R) \frac{d\tilde{\mathrm{k}}}{\tilde{\mathrm{k}}}, \tag{2.16}$$

where the second equality is true for the Gaussian approximation and $W^2(\mathrm{k}, R)$ is the Gaussian window function, $W^2(\mathrm{k}, R) = e^{-\frac{\mathrm{k}^2 R^2}{2}}$.

In conclusion, we underline that β grows until t_{eq}. Indeed, $\rho_{\mathrm{PBH}} \sim a^{-3}$ and $\rho_{\mathrm{TOT}} \sim a^{-4}$ so $\beta \sim a$. Thus, at the end of the radiation era PBHs with low masses, formed earlier, contribute more importantly to the total energy density than more massive ones, given identical values of β at time time of formation.

2.1.3 Observational constraints on PBHs

In this section we briefly review the observational constraints on PBH abundance. This overview is based on [4] and it is divided into two sections. In the first one we present the way we can constrain the abundance of PBHs via electromagnetic signals. In the second one we explain which improvements we gain with the detection of gravitational signals. Indeed, the former are useful to place constraints on the PBH abundance, while the latter could also give us an evidence for the existence of PBHs.

Electromagnetic signals

In this section we briefly recall some direct and indirect electromagnetic signals[6] and we show in **Figure 2.3** some constraints produced by the non-observation of these signals.

[5] We take $\gamma = 0.2$, $M_{\mathrm{eq}} = 1.3 \cdot 10^{49} \frac{g}{(\Omega_0 h^2)^2}$, $g_{\mathrm{eq}} = 3$, $g_{\mathrm{f}} = 106.75$ and $\mathrm{k}_{\mathrm{eq}} = 0.07 \frac{\Omega_0 h^2}{Mpc}$ in equation (2.13).
[6] For more details we refer the reader to the literature for each described effect.

First of all, let us define what we mean by direct/indirect constraints. Direct constraints are derived from investigating the effects that PBHs directly trigger by their gravitational potential. These effects are independent from the mechanism that formed PBHs and, because of the Hawking evaporation, these signals have to come from PBHs with mass larger than $\sim 10^{15}\, g$. Indirect constraints are those that can be obtained by observational effects that are not caused directly by the PBHs' gravitational field.

For each mentioned effect we specify a PBHs mass range. The reported mass range is qualitatively the mass range for which the signal is observable. Thus, a few examples of electromagnetic signals that constrain the PBH abundance are (see [4] for more examples)

- **Gravitational lensing.** If PBHs are present in the Universe, they cause gravitational lensing on background objects such as stars. By the non-detection of lensing effects we can put an upper limit on the PBHs fraction. Gravitational lensing is a very powerful method to constrain PBHs. Indeed, it is based on gravitational physics and thus it does not suffer from the uncertainties that exist in studying electromagnetic signals. (millilensing [19] $10^6\, M_\odot < M_{\rm PBH}$, microlensing [20] $10^{-13}\, M_\odot < M_{\rm PBH} < 10^{-4}\, M_\odot$, femtolensing [21] $10^{-16}\, M_\odot < M_{\rm PBH} < 10^{-13}\, M_\odot$).

- **Disruption of white dwarfs.** White dwarfs are compact objects that have a mass of one solar mass, a size comparable to the size of the Earth and are supported by electron degeneracy pressure. The passage of a PBH through a white dwarf well below the Chandrasekhar limit, can ignite the thermonuclear runaway that eventually makes the white dwarf explode. It is then possible to constrain the range of PBHs masses by observing the abundance of white dwarfs of known masses: if there are too many PBHs above a certain mass, then white dwarfs with the corresponding mass should not exist in the present Universe. Thus, for the white dwarf abundance, we can fix the "rareness" of heavier PBHs in order for white dwarfs not to encounter many heavy PBHs. (white dwarfs disruption [22] $10^{-14}\, M_\odot < M_{\rm PBH} < 10^{-12}\, M_\odot$, neutron stars disruption [23] $10^{-14}\, M_\odot < M_{\rm PBH} < 10^{-8}\, M_\odot$).

- **Dynamical friction on PBHs.** If we assume that the galactic halo is entirely or partially composed of massive PBHs, some of them must be in the region near the galactic centre. Such PBHs would receive strong dynamical friction from stars and the dark matter in the form of lighter PBHs or elementary particles, they would lose their kinetic energy, and spiral into the centre. Then, if this in-fall time scale is shorter than the age of the Universe, accumulation of PBHs continues in the central region. As a result, we will have that the galactic centre would be dominated by a dense cluster of PBHs. Since we have an upper limit on the mass in the galactic centre, we can translate this limit into the fraction of PBHs in the galactic halo for some PBH mass range. ([24], $10^4\, M_\odot < M_{\rm PBH}$).

- **Disk heating.** If we assume that PBHs are moving randomly in the galactic halo, they will have a chance to pass through the galactic disk. Thus, the stars in the disk are pulled by gravitational attraction, acquiring velocity. Assuming that the direction of the velocity that a star gains for an individual PBH passage is random, the time evolution of the velocity of the disk stars is described by a random walk. We then expect that the variance of the star's velocity increases and the disk becomes hotter and hotter as a function of time. We can constrain the PBH abundance by requiring that the increase of the velocity due to the nearby passage of PBHs does not exceed the observed velocity. ([25], $10^6\, M_\odot < M_{\rm PBH}$).

- **Accretion effects on the CMB.** We can study the accretion of gas into PBHs to constrain the abundance of PBH; however, we have to keep in mind that the physics is much more involved in the case of the accretion constraint compared to the lensing and dynamical constraints. In particular, we can use CMB to constrain accretion effects that occur in the early Universe. The gravity of the PBHs attracts the baryonic gas around PBHs. The gas falling into the central region is compressed, it increases its density and temperature and so it can be fully ionized by internal collisions of gas particles or by the outgoing radiation. The outward radiation due to the ionized gas near the PBH horizon modifies the spectrum of the CMB photons by heating or ionizing the gas filling the Universe. ([26], $10^2\, M_\odot < M_{\rm PBH} < 10^4\, M_\odot$).

- **CMB spectral distortions.** We can constrain the PBH abundance using the fact that we do not observe μ-distortion[7] in the CMB spectrum. The μ-type distortion is useful to quantify the energy injected into the CMB. Then, if PBHs exist, we should see in the CMB distortion the sign of the PBH production mechanism. Indeed, when perturbations re-enter the Hubble horizon they are damped by Silk damping[8] and their energy is transferred to the background homogeneous plasma. As a result, we should observe μ-type distortion produced by the PBH formation mechanism. ([27], $10^4\, M_\odot < M_{\rm PBH} < 10^{13}\, M_\odot$).

- **Big bang nucleosynthesis.** The Silk damping does not leave any spectral feature of small scales perturbations in the CMB. Thus, to constrain primordial perturbations with $k > 10^4\, Mpc^{-1}$, we can study the effects in the Big-Bang nucleosynthesis. The energy injected due to the Silk damping results in the increase of the temperature of the plasma and, because of the baryon number ($n_{\rm b}$) conservation, only the photon number density (n_γ) is increased by this process. Therefore $\eta \equiv \frac{n_{\rm b}}{n_\gamma}$ decreases through this effect. This means that the η at the Big-Bang nucleosynthesis is larger than the η at the CMB formation ($\eta_{\rm BBN} > \eta_{\rm CMB}$). By measuring these two values of η we are able to quantify the Silk damping and so we can constrain the abundance of small perturbations. ([28], $M_{\rm PBH} < 10^4\, M_\odot$).

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[7] We assume that in the early Universe the spectrum of photons is the Bose-Einstein spectrum and, if the Compton interaction is effective, the equilibrium electron temperature $T_{\rm e}$ is exactly equal to the radiation temperature T. We can then expand the Bose-Einstein spectrum, $n_{\rm BE} = \frac{1}{exp[\mu + xT/T_{\rm e}]-1}$, where $x = \frac{h\nu}{k_{\rm B}T}$, for small distortions of the chemical potential μ. In that way we obtain the μ-type distortion, which, thanks to the fact that the chemical potential $\mu \sim \frac{\Delta E}{E}$ depends on the energy injected into the CMB, is strongly correlated with the injected energy.

[8] The Silk damping, or diffusion damping, is due to the photon diffusion: the anisotropies of the primordial plasma are damped by the photons diffused from the hot, overdense regions of plasma, to the cold, underdense ones. Photons drag along the protons and electrons in the following way: by Compton scattering photons push electrons which, thanks to the Coulomb force, pull on protons. This causes the temperatures and densities of the hot and cold regions to be averaged and the Universe to become less anisotropic.

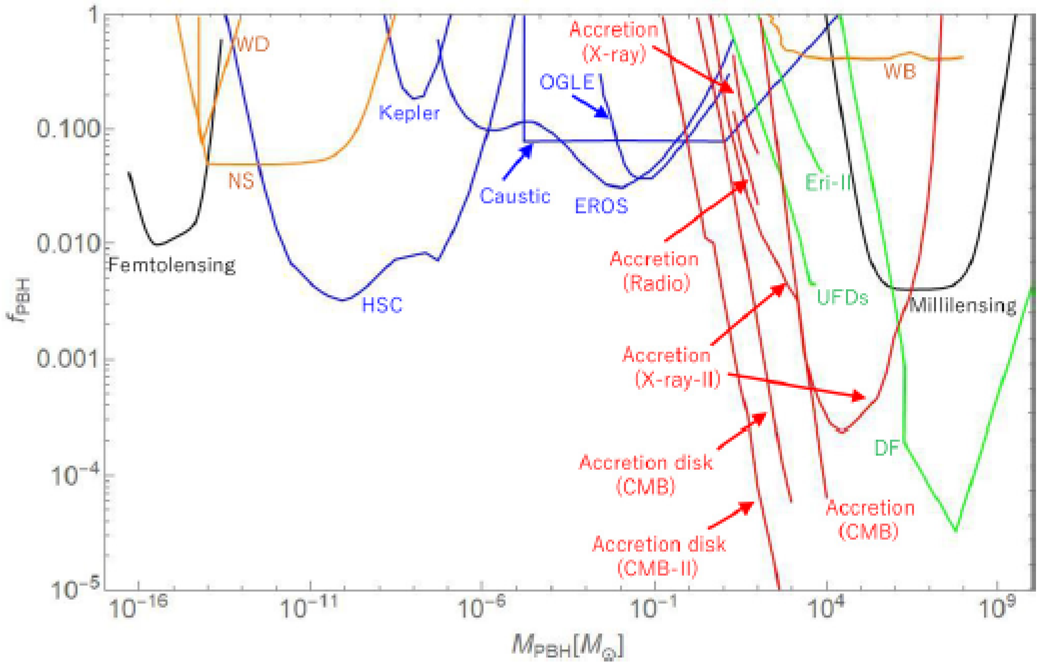

Figure 2.3: *Upper limit on* $f_{\rm PBH} = \frac{\Omega_{\rm PBH}}{\Omega_{\rm DM}}$ *versus PBH mass. Blue curves represent lensing constraints, black curves represent constraints by millilensing and femtolensing, orange curves represent dynamical constraints obtained by requiring the existence of the observed compact objects' abundance, green curves represent constraints due to dynamical friction on PBHs and red curves represent constraints due to the accretion onto the PBHs. This figure is taken from* [4].

To study the constraints on the PBH abundance, we introduce the quantity $f_{\rm PBH} = \frac{\Omega_{\rm PBH}}{\Omega_{\rm DM}}$: the fraction of PBHs compared to the total dark matter component. Indeed, by studying the direct/indirect constraints, we can find some upper bounds on $f_{\rm PBH}$ as a function of the PBHs mass. The constraints derived from the signals discussed above are shown in **Figure 2.3**.

Before ending this section, we briefly mention the constraints on PBHs lighter than $\approx 10^{15} g$ that have already evaporated, or are in the final stage of evaporation, via Hawking radiation [7]. Although those PBHs do not exist or are fading at the present epoch, high-energy particles emitted from PBHs leave certain signals from which we can place upper limits on the PBH abundance. These include production of the lightest supersymmetric particles (if they exist) ($10^4\, g < M_{\rm PBH} < 10^9\, g$), entropy production in the early Universe ($10^6\, g < M_{\rm PBH} < 10^9\, g$), change of the abundance of the light elements produced by the big bang nucleosynthesis ($10^9\, g < M_{\rm PBH} < 10^{13}\, g$), extragalactic photon background ($10^{14}\, g < M_{\rm PBH} < 10^{15}\, g$), and damping of the CMB temperature anisotropies on small scales by modifying the cosmic ionization history ($10^{13}\, g < M_{\rm PBH} < 10^{13}\, g$).

Gravitational wave detection

In the previous section we briefly presented some of the currently known constraints on the PBH abundance. As underlined, the detection of electromagnetic signals does not give us an indisputable evidence for the existence or non-existence of PBHs. In this section we want to focus on future gravitational wave detections which could eventually demonstrate the existence of PBHs. This overview is based on [4].

First of all, we remark that in this section we discuss only one type of gravitational wave signals: gravitational waves generated by merging events. There is another important gravitational waves signal that we could detect to have an indisputable evidence for the existence or non-existence of PBHs: stochastic gravitational waves. As we will see, while the former is not always related to the PBH formation mechanism, the latter could be an evidence to confirm a specific mechanism for PBH formation. We will discuss in detail stochastic gravitational waves from primordial density fluctuations in Sections **2.3** and **5**.

To understand how we can use the detection of merging events to constrain the existence of PBH we first briefly explain how we can model the formation of PBH binaries. There are two possible ways by which we can explain the formation of PBH binaries: PBH binary formation in the early Universe and PBH binary formation in the present Universe.

The former occurs in the radiation-dominated epoch, when we assume that PBHs were randomly distributed (Poisson distribution). In that way PBHs were sparsely distributed just after their formation and the mean distance was larger than the Hubble horizon. In the radiation-dominated epoch the mean distance between PBHs ($l_{\mathrm{PBH}}(t)$) grew as the scale factor $a(t) \propto t^{1/2}$, while the Hubble horizon grew as $H(t)^{-1} \propto t$. The mean distance relative to the Hubble horizon decreases as $t^{-1/2}$ and therefore there is a moment in the radiation-dominated epoch at which we had more than one PBH in the Hubble horizon. It is then possible for the binary formation to occur.

The latter, which is not in contrast with the binary formation in the early Universe, is the result of an accidental near-miss. We can assume that a PBH that travels in the space, accidentally becomes sufficiently close to another PBH so that it becomes gravitationally bounded. Indeed, the condition that we have to impose in order to form a binary system, is that the amount of the emitted gravitational waves is greater than the kinetic energy of PBHs. In this case PBHs cannot escape to infinity anymore and so a binary system is formed (see [29] for more details).

In conclusion, PBHs can be considered as a reasonable candidate scenario for the observed black hole merging events and, in principle, we can explain some of the detected LIGO events as merging of PBHs.

However, PBH binary merging is not the only possible option: we have to explain how we can distinguish this scenario from the astrophysical one.

The first effect that could distinguish the PBH scenario from the astrophysical one is the dependence of the merger rate on redshift. We know that, if they exist, PBHs were formed in the radiation-dominated era while astrophysical black holes are the result of the death of super-massive stars. This means that astrophysical black holes appear in the low-redshift Universe. Because of this difference, the redshift evolution of the number density of PBHs will be different from the one of astrophysical black holes. We expect that this difference will condition the merger rate. In a qualitative way, as shown in **Figure 2.4**, we expect that the merger rate of astrophysical black holes drops as z is bigger than the formation redshift, while we expect that the merger rate of PBHs is not negligible also at high-redshift (see [30] for a quantitative analysis of this effect).

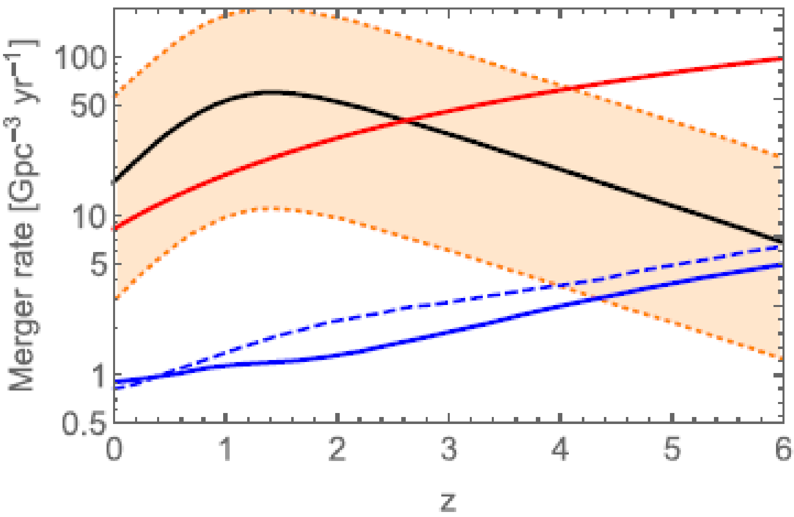

Figure 2.4: *Redshift evolution of the merger rate per unit source time and unit comoving volume* ($\frac{1}{Gpc^3\, yr}$) *as a function of redshift z. The black curve represents the merger rate of astrophysical black holes, the orange curves show its upper and lower limits, the blue curve represents the merger rate of PBH binaries formed in the present Universe with $f_{\rm PBH} = 1$ and the red curve represents the merger rate of PBH binaries formed in the early Universe with $f_{\rm PBH} = 10^{-3}$. This figure is taken from* [30].

A second effect that could rule out the existence of PBHs is the mass distribution of black holes that compose binary systems. If we plot the merger events in the 2D mass plane we obviously expect that, with a greater statistic, we can reach some information about the mass distribution of black holes that compose binary systems. We can then link this information to the binary formation mechanism and so we can discriminate different formation scenarios. In other words, if PBHs exist in the Universe, the merger events that we detect will be a mixture of PBH binaries and the ones produced by astrophysical effects. We expect that the 2D mass plane should be influenced by this eventuality.
However, the main problem of this approach is that we have a large number of uncertainties at the theoretical level: we do not have well known physics that describes the PBH abundance, the mass function and the formation of black hole binaries by astrophysical processes.

In conclusion, testing the PBH scenario with gravitational waves detection opens a large number of interesting and challenging questions both from the observational point of view (we need a greater statistic to have meaningful constraints) and from the theoretical point of view. Furthermore, gravitational waves detection could give us a smoking gun for the existence of PBHs.

2.2 Theory of cosmological perturbations

In this section we give an overview of perturbation theory based on [31], [32], [33] and [34]. We first introduce the gauge problem in Section **2.2.1**, and then in Section **2.2.2** we go through the study of the first order Einstein equations in the case of a Universe dominated by a single perfect fluid.

The Cosmological Principle is valid for large scales but we need to relax the assumption of homogeneity and isotropy at small scales to explain the CMB anisotropies and the cosmological structure formation. The present Universe is not homogeneous: we observe clusters of matter and void. Furthermore, we know that the CMB radiation is not perfectly isotropic: we observe small fluctuations in the temperature ($\frac{\Delta T}{T} \approx 10^{-5}$). Thus, to explain this characteristic of the observed Universe, we need to go beyond the Cosmological Principle and study the cosmological perturbation theory.

2.2.1 The gauge problem

Now that we explained the importance of studying the perturbed Einstein equations to model the Universe, we need to adapt the invariance of general relativity (GR) under diffeomorphism to the perturbation theory. Unfortunately, this invariance makes the definition of perturbations gauge-dependent. This means that, in general, perturbation of a tensor is not well defined.

To study the evolution of perturbations, we need to map the homogeneous and isotropic background space-time $\mathcal{M}$ onto the physical perturbed space-time $\tilde{\mathcal{M}}$ as represented in **Figure 2.5**.

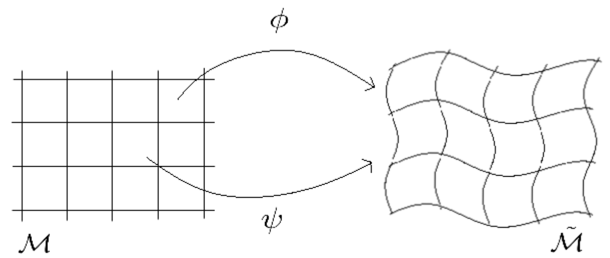

Figure 2.5: *Two different gauge maps ϕ and ψ between the background space-time $\mathcal{M}$ and the perturbed space-time $\tilde{\mathcal{M}}$.*

This identification between points of the background and perturbed space-times is called gauge choice and in general perturbations are not invariant under gauge transformations.

To understand how to deal with this gauge problem, we need to geometrically define

it. Then, once we formalize how a tensor field behaves under a change of gauge, we can define gauge-independent quantities (see [35] for more details) or we can work with gauge-dependent quantities being able to change our results from a gauge to another. If we choose to work with gauge-dependent quantities we have to be aware that our results would have been different if we had chosen a different the map between $\mathcal{M}$ and $\tilde{\mathcal{M}}$.
We now formalize this problem from a geometrical point of view. Moreover, we will study in deeper detail the gauge dependence of second order quantities and how to go change the gauge in Section **5**.

The main problem of perturbing a tensor is that a Taylor expansion is essentially a convenient way to express the value of the function at a point p in terms of its value, as well as the value of all its derivatives, at another point q. This is not anymore a convenient way to express a tensor field T on the m dimensional manifold $\mathcal{M}$, because $T(p)$ and $T(q)$ belong to different spaces $\mathbb{R}^m$. Thus, to write a Taylor expansion, we need to introduce a map between tensors at different points of $\mathcal{M}$. We follow [32] to understand the case in which such a mapping arises from one-parameter family of diffeomorphisms of $\mathcal{M}$.

We start recalling by that taking a *diffeomorphism* between two manifolds $\mathcal{M}$ and $\mathcal{N}$ ($\varphi : \mathcal{M} \longrightarrow \mathcal{N}$), for each point p in $\mathcal{M}$, defines the push-forward between the tangent spaces $\varphi_*(p) : T_p\mathcal{M} \longrightarrow T_{\varphi(p)}\mathcal{N}$ and the pull-back between the cotangent spaces $\varphi^*(p) : T^*_{\varphi(p)}\mathcal{N} \longrightarrow T^*_p\mathcal{M}$. We can also define a push-forward $T^*_p\mathcal{M} \longrightarrow T^*_{\varphi(p)}\mathcal{N}$ and a pull-back $T_{\varphi(p)}\mathcal{N} \longrightarrow T_p\mathcal{M}$ using φ^{-1}. In that way the pull-back and the push-forward are maps that are well-defined for tensors of arbitrary type.

We can extend the previous definitions to tensors of higher rank. Indeed, (see [34] for more details) the action of the pull-back and of the push-forward on a $(0,l)$ tensor N on $\mathcal{N}$, and on a $(k,0)$ tensor M on $\mathcal{M}$, is respectively given by:

$$\begin{aligned} (\varphi^* N)\left(V^1, .., V^l\right) &= N\left(\varphi_* V^1, ..., \varphi_* V^l\right), \\ (\varphi_* M)\left(\omega^1, .., \omega^k\right) &= M\left(\varphi^* \omega^1, ..., \varphi^* \omega^k\right), \end{aligned} \tag{2.17}$$

where V^i are vectors on $\mathcal{M}$ and ω^i are one-forms on $\mathcal{N}$. Then, to define a gauge transformation, we need to introduce the *flow* $\phi : \mathbb{R} \times \mathcal{M} \longrightarrow \mathcal{M}$ generated by the vector field ξ on the manifold $\mathcal{M}$. ϕ is defined such that $\phi(0,p) = p$, $\forall p \in \mathcal{M}$ and for any $\lambda \in \mathbb{R}$ we write $\phi(\lambda, p) := \phi_\lambda(p)$, $\forall p \in \mathcal{M}$. Thus, the pull-back of ϕ defines a new field $\phi^*_\lambda T$ on $\mathcal{M}$ which is a function of λ. The pull-back $\phi^*_\lambda T$ can be expanded around $\lambda = 0$ and the expansion results in (see [32] for the proof):

$$\phi^*_\lambda T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k}\bigg|_{\lambda=0} \phi^*_\lambda T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds^k_\xi T, \tag{2.18}$$

where $\pounds_\xi$ is the *Lie derivative* along the vector field ξ, defined as:

$$f_\xi T = \lim_{\lambda \to 0} \frac{1}{\lambda}\left(\phi^*_\lambda T - T\right) = \frac{d}{d\lambda}\bigg|_0 \phi^*_\lambda T. \tag{2.19}$$

Now we can geometrically define the perturbation of a tensor, a gauge transformation and how to change a gauge, following [33].

The basic assumptions of perturbation theory are: the existence of a parametric family of solutions of the field equations and that the $\lambda = 0$ solution is the unperturbed background solution $\mathcal{M}_0$. Thus, in cosmology, we are dealing with a one-parameter family of models $\mathcal{M}_\lambda$ where on each $\mathcal{M}_\lambda$ we can define tensor fields T_λ representing the physical and geometrical quantities (i.e., the metric and the stress-energy tensor). The aim of perturbation theory is to construct an approximated solution for $\mathcal{M}_\lambda$.

However, we do not have a unique choice for the one-to-one correspondence between points of the background field $\mathcal{M}_0$ and the physical field $\mathcal{M}_\lambda$. We can choose two flows ϕ_λ, ψ_λ that link the unperturbed manifold and the physical manifold as shown in **Figure 2.5**. Then, if we take some coordinates x^μ on $\mathcal{M}_0$, a one-to-one correspondence, ϕ_λ or ψ_λ, carries these coordinates over $\mathcal{M}_\lambda$. This defines a choice of gauge: we call the correspondence itself a *gauge* and a change in this correspondence, keeping the background fixed, is a gauge transformation.

Now that we geometrically defined a gauge, we want to understand how to change a gauge and how this changing of the gauge influences the definition of the perturbed tensors T_λ. Let p be a point on $\mathcal{M}_0$ where we defined the coordinates x^μ, then the gauge ψ_λ "evolves" this point to $\mathcal{M}_\lambda$, mapping p to $P = \psi_\lambda(p) \in \mathcal{M}_\lambda$. However, choosing a different gauge ϕ_λ, the point P on the perturbed manifold could correspond to a different point q on the background $\mathcal{M}_0$, as shown in **Figure 2.6**.

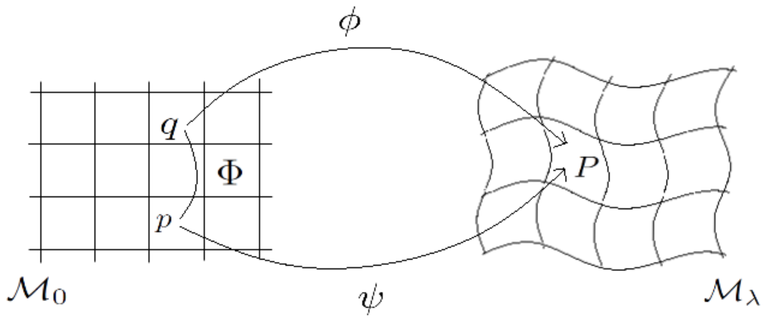

Figure 2.6: *Two different gauge maps ϕ and ψ between the background space-time $\mathcal{M}_0$ and the perturbed space-time $\mathcal{M}_\lambda$ and the action of $\Phi_\lambda : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$.*

Thus, the change of the correspondence, i.e., the gauge transformation, may actually be seen as a one-to-one correspondence between different points in the background $(y^\mu(x^\mu, \lambda))$. This is indeed called *an active coordinate transformation.*

If we consider a tensor field T_λ on each $\mathcal{M}_\lambda$, using ϕ_λ and ψ_λ we can define in two different ways a representation of T_λ on the background manifold $\mathcal{M}_0$. We denote these two different representations by T_λ and $\tilde{T}_\lambda$. These are tensor fields on $\mathcal{M}_0$ and thus they can be compared to the background tensor field T_0 to define the total perturbations

$$\begin{aligned} \Delta T_\lambda &= T_\lambda - T_0 \quad \text{in the first gauge,} \\ \Delta \tilde{T}_\lambda &= \tilde{T}_\lambda - T_0 \quad \text{in the second gauge,} \end{aligned} \tag{2.20}$$

where T_λ and $\tilde{T}_\lambda$ are the pull-back of the tensor field on $\mathcal{M}_0$ using ϕ^*_λ and ψ^*_λ, respectively.

The first term of the RHS of the above equations can be Taylor expanded to get

$$\begin{aligned} \Delta T_\lambda &= \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \delta^k T, \\ \Delta \tilde{T}_\lambda &= \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \delta^k \tilde{T}, \end{aligned} \tag{2.21}$$

where $\delta^k T = \frac{d^k \phi_\lambda^* T}{d\lambda^k}\Big|_{\lambda=0}$ and $\delta^k \tilde{T} = \frac{d^k \psi_\lambda^* \tilde{T}}{d\lambda^k}\Big|_{\lambda=0}$. This non–uniqueness of the definition of ΔT is the gauge dependence of the perturbations.

To understand how the representation of a tensor field T_λ on $\mathcal{M}_0$ behaves under a gauge transformation, we notice that the one-to-one correspondence between the points p and q in the background manifold can be seen as a gauge transformation itself. Indeed, we can define the one-parameter diffeomorphism $\Phi_\lambda : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ as

$$\Phi_\lambda = \phi_\lambda^{-1}(\psi_\lambda), \tag{2.22}$$

so that $q = \Phi_\lambda(p) = \phi_\lambda^{-1}(\psi_\lambda(p))$.

We then introduce the two vector fields that generate the gauge transformations ϕ_λ and ψ_λ, X and Y, respectively. In that way we can use the expansion (2.18) to expand the pull-back in the RHS of the definition (2.20). Using (2.18) we can write

$$\begin{aligned} T_\lambda^X &= \phi_\lambda^* T|_{\lambda=0} = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \delta^k T^X = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_X^k T \Bigg|_{\lambda=0} = T_0 + \Delta T_\lambda, \\ T_\lambda^Y &= \psi_\lambda^* T|_{\lambda=0} = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \delta^k T^Y = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_Y^k T \Bigg|_{\lambda=0} = T_0 + \Delta \tilde{T}_\lambda. \end{aligned} \tag{2.23}$$

In conclusion, we can find a relation between T_λ^X and T_λ^Y using the map Φ_λ,

$$T_\lambda^Y = \Phi_\lambda^* T_\lambda^X. \tag{2.24}$$

Expanding the above relation and inserting the result in the definition (2.20) we can write the relation between the first and the second order perturbations in the two different gauges (see [32] for more details),

$$\begin{aligned} \delta T^Y - \delta T^X &= \pounds_{\xi 1} T_0 \\ \delta^2 T^Y - \delta^2 T^X &= \pounds_{\xi_2} T_0 + \pounds_{\xi_1}^2 T_0 + 2\pounds_{\xi_1} \delta T^X, \end{aligned} \tag{2.25}$$

where ξ_1 and ξ_2 are the associated vector fields with the one parameter diffeomorphism Φ_λ: $y^\mu \approx x^\mu + \lambda \xi_1^\mu + \frac{\lambda^2}{2}\left(\partial_\nu \xi_1^\mu \xi_1^\nu + \xi_2^\mu\right)$.

We finally notice that from equation (2.25) we can say that T_λ is gauge-invariant to first order *iff* $\pounds_{\xi 1} T_0 = 0$ for any vector field ξ on $\mathcal{M}$ (see [35] for more details on gauge-invariant quantities).

2.2.2 Cosmological perturbations

We now perturb the metric tensor and the stress-energy tensor and we study the first order Einstein equations in the case in which the matter content of the Universe is a perfect fluid. We follow [31] and [33].

To perturb Einstein equations we need to perturb the metric and the stress-energy tensor. The perturbed metric can be written as:[9]

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.26}$$

where all the entries in $\delta g_{\mu\nu}$ have to be small with respect to the background metric $g^{(0)}_{\mu\nu}$. We consider perturbations about the flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ds^2 = g^{(0)}_{\mu\nu} = a^2(-d\eta^2 + \delta_{ij}dx^i dx^j). \tag{2.27}$$

The components of $\delta g_{\mu\nu}$ then are

$$\begin{aligned} g_{00} &= -a^2\left(1 + 2\sum_{r=1}^{\infty}\frac{1}{r!}\Psi^{(r)}\right), \\ g_{0i} &= a^2\sum_{r=1}^{\infty}\frac{1}{r!}\omega_i^{(r)}, \\ g_{ij} &= a^2\left[\left(1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\Phi^{(r)}\right)\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}h_{ij}^{(r)}\right], \end{aligned} \tag{2.28}$$

where $\Psi^{(r)}$, $\omega_i^{(r)}$, $\Phi^{(r)}$ and $h_{ij}^{(r)}$ represent the r-th order perturbation of the metric. We further split $\Psi^{(r)}$, $\omega_i^{(r)}$, $\Phi^{(r)}$ and $h_{ij}^{(r)}$ into the so-called scalar, vector and tensor parts. Scalars are those related to a scalar potential, vectors are those related to transverse (divergence-free) vector fields, and tensor parts to transverse and trace-free tensors. Thus we write

$$\begin{aligned} \omega_i^{(r)} &= \partial_i\omega^{(r)\parallel} + \omega_i^{(r)\perp}, \\ h_{ij}^{(r)} &= D_{ij}h^{(r)\parallel} + \partial_i h_j^{(r)\perp} + \partial_j h_i^{(r)\perp} + h_{ij}^{(r)\top}, \end{aligned} \tag{2.29}$$

where $\omega_i^{(r)\perp}$ satisfies $\partial^i\omega_i^{(r)\perp} = 0$, $\omega^{(r)\parallel}$ and $\chi^{(r)\parallel}$ are scalar functions, $h_j^{(r)\perp}$ satisfies $\partial^i h_j^{(r)\perp} = 0$, $h_{ij}^{(r)\top}$ satisfies $\partial^i h_{ij}^{(r)\top} = 0$ and the trace free operator D_{ij} is defined as:

$$D_{ij} = \partial_i\partial_j - \frac{1}{3}\nabla_k\nabla^k\delta_{ij}. \tag{2.30}$$

To perturb the stress-energy tensor $T_{\mu\nu}$ we recall that it can be written as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \Pi_{\mu\nu} \tag{2.31}$$

where ρ and p are the energy density and the pressure, respectively, u^μ is the 4-velocity normalized to one ($u^\mu u_\mu = -1$) and $\Pi_{\mu\nu}$ is the anisotropic stress tensor. We consider

[9] Greek indices run from 0 to 3 while Latin indices from 1 to 3.

$\Pi_{\mu\nu} = 0$ as we want to study a perfect fluid. Thus, to perturb $T_{\mu\nu}$ we need to perturb the energy density, the pressure and the 4-velocity.

The perturbed energy density can be expressed as

$$\rho = \rho_0 + \sum_{r=1}^{\infty} \frac{1}{r!}\delta\rho^{(r)}, \tag{2.32}$$

where ρ_0 is the unperturbed energy density and it is a function of the only. To write the perturbed pressure we assume that it is a function of the energy density only (adiabatic perturbation),

$$\delta p = \frac{\partial p}{\partial \rho}\delta\rho = c_{\mathrm{s}}^2 \delta\rho, \tag{2.33}$$

where c_{s}^2 is the sound speed. Finally, the perturbed velocity can be written as

$$u^\mu = \frac{1}{a}\left(\delta_0^\mu + \sum_{r=1}^{\infty} \frac{1}{r!} v^{\mu(r)}\right), \tag{2.34}$$

where the first term $u^\mu = \frac{1}{a}\delta_0^\mu$ is the background velocity and the second term represents the perturbation. In particular, the background velocity represents a motion comoving with the cosmic expansion, while the peculiar velocity $v^{\mu(r)}$ represents the motion with respect to the general expansion. Imposing that equation (2.34) satisfies the constraint $u^\mu u_\mu = -1$, we find a relation between $v^{0(r)}$ and $\Psi^{(r)}$. At first order we have

$$v^{0(1)} = -\Psi^{(1)} \tag{2.35}$$

so that the first order perturbed 4-velocity results in

$$\begin{aligned} u^\mu &= \left[\frac{1}{a}(1 - \Psi^{(1)}),\ \frac{v^{(1)i}}{a}\right], \\ u_\mu &= \left[-a(1 + \Psi^{(1)}),\ a v^{(1)i}\right]. \end{aligned} \tag{2.36}$$

Furthermore, as we did for $\delta g_{\mu\nu}$, we can split the velocity perturbation into scalar and vector parts

$$v^i_{(r)} = \partial_i v^{(r)\parallel} + v_i^{(r)\perp}, \tag{2.37}$$

where $v_i^{(r)\perp}$ satisfies $\partial^i v_i^{(r)\perp} = 0$. In conclusion, the components of (2.31) are[10]

$$\begin{aligned} T^0_0 &= -\left(\rho_0 + \delta\rho\right), \\ T^0_i &= -\left(1 + w\right)\rho_0 v_i, \\ T^i_0 &= \rho_0\left(1 + w\right)\left(v_i - \omega_i\right), \\ T^i_j &= c_{\mathrm{s}}^2 \delta\rho\delta^i_j, \end{aligned} \tag{2.38}$$

where we have used the equation of state $p = w\rho$ to express the pressure as a function of the energy density.

[10] From now on we drop the notation $^{(r)}$ because in this section we always refer to first order perturbation quantities.

We now have both the perturbed tensors and we need to study the first order Einstein equations

$$\delta R^{\mu}_{\nu} - \frac{1}{2}\delta g^{\mu}_{\nu} R - \frac{1}{2} g^{\mu}_{\nu}\delta R = 8\pi G \delta T^{\mu}_{\nu}, \tag{2.39}$$

where G is the Newtonian constant and R^{μ}_{ν} and R are the Ricci tensor and Ricci scalar, respectively. However, as explained in the previous section, we first need to fix a gauge. Indeed, both the metric and the stress-energy tensor are not gauge-invariant quantities.

We recall a few gauges used in cosmology before proceeding with the study of first-order Einstein equations:

- **Newtonian gauge.** This gauge is defined by the choice $\omega^{\parallel} = h^{\parallel} = 0$ and this is the gauge that in general relativity has the most direct link with analogous quantities appearing in Newtonian physics.

- **Poisson gauge.** This gauge is defined by the choice $\omega^{\parallel} = h^{\parallel} = h^{\perp}_i = 0$ and generalizes the aforementioned Newtonian gauge.

- **Comoving gauge.** This gauge is defined by the choice $v^{\parallel} = v^{\perp}_i = 0$. We are then imposing that the 3-peculiar velocity of the fluid vanishes. If we also require orthogonality of the time-constant hypersurfaces to the 4-velocity this gives $v^{\parallel} + \omega^{\parallel} = 0$.

- **Synchronous gauge.** This gauge is defined by the choice $\Psi = 0$ and it leaves the freedom of choosing one further scalar or vector perturbation. If we also require $\omega^{\parallel} = \omega^{\perp}_i = 0$, it is called Synchronous and time-orthogonal gauge. In this gauge the proper time for observers at fixed spatial coordinates coincides with cosmic time in the FLRW background.

- **Uniform density gauge.** This gauge is defined by the choice $\delta\rho = 0$ and thus it selects spatial hypersurfaces at $\eta = const$. Where the energy density of the fuid is left unperturbed. Also this gauge choice leaves the freedom of choosing one further scalar or vector perturbation.

We now go back to the study of equations (2.39). We note that at first order the vector and tensor perturbations decouple completely from the scalar terms and thus $\omega^{\perp}_i$ and $h^{\top}_{ij}$ can be treated separately. The terms which are intrinsically vectorial couple to pure rotational modes, while tensorial terms represent gravitational waves, coupled to matter only for anisotropic perturbations. Furthermore, it can be shown that if vorticity modes are initially zero, they remain zero. Since we are now interested in studying the scalar part of equations (2.39) for a perfect fluid, we will not further take into account the contribution of $\omega^{\perp}_i$ and $h^{\top}_{ij}$. We will study the behaviour of $h^{\top}_{ij}$ in the next section.

To fix the gauge we choose the Newtonian gauge. With this choice, the perturbed metric becomes

$$ds^2 = a^2\left[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right], \tag{2.40}$$

so that the (00), $(0i)$, (ij) and (ii) components of equations (2.39) result in[11] (see [31] for

[11] $\mathcal{H} = \frac{a'(\eta)}{a(\eta)} = a(t)H$ is the Hubble constant in terms of conformal time.

more details)

$$
\begin{aligned}
&3\mathcal{H}\left(\mathcal{H}\Psi+\Phi'\right)-\nabla^2\Phi=-4\pi Ga^2\delta\rho,\\
&\nabla_i\nabla^i\left(\Phi'+\mathcal{H}\Psi\right)=-4\pi Ga^2(1+w)\rho_0\theta,\\
&\Psi=\Phi,\\
&\Phi''+2\mathcal{H}\Phi'+\mathcal{H}\Psi'+\left(\mathcal{H}^2+2\mathcal{H}'\right)\Psi=4\pi Ga^2c_{\mathrm{s}}^2\delta\rho,
\end{aligned}
\tag{2.41}
$$

where θ is defined as: $\theta=\nabla_i v^{i\perp}$. From the (0) and ($i$) components of the continuity equation $\nabla_\mu T^{\mu\nu}=0$, we obtain

$$
\begin{aligned}
&(\delta\rho)'+3\mathcal{H}\delta\rho(1+w)=-\rho(1+w)\left(\theta-3\Phi'\right),\\
&\delta q'+3\mathcal{H}\delta q=-aw\delta\rho-a\rho(1+w)\Psi,
\end{aligned}
\tag{2.42}
$$

where $\delta q=a\rho(1+w)v^{\parallel}$. Using the 0^{th} order dynamics and defining $\delta=\frac{\delta\rho}{\rho}$, equations (2.42) become

$$
\begin{aligned}
&\delta'+3\mathcal{H}\left(c_{\mathrm{s}}^2-w\right)\delta=-(1+w)\left(\theta-3\Phi'\right),\\
&\theta'+\left[\mathcal{H}(1-3w)+\frac{w'}{1+w}\right]\theta=-\nabla_i\nabla^i\left(\frac{c_{\mathrm{s}}^2}{1+w}\delta+\Psi\right).
\end{aligned}
\tag{2.43}
$$

We now move to the Fourier space and we define all the perturbation quantities as the Fourier expansions

$$
\begin{aligned}
\Phi&=\int e^{i\mathrm{k}\cdot x}\Phi_{\mathrm{k}}\mathrm{d}^3\mathrm{k},\\
\Psi&=\int e^{i\mathrm{k}\cdot x}\Psi_{\mathrm{k}}\mathrm{d}^3\mathrm{k},\\
\delta&=\int e^{i\mathrm{k}\cdot x}\delta_{\mathrm{k}}\mathrm{d}^3\mathrm{k},\\
\theta&=\int e^{i\mathrm{k}\cdot x}\theta_{\mathrm{k}}\mathrm{d}^3\mathrm{k},
\end{aligned}
\tag{2.44}
$$

where subscript k represents a Fourier mode for each comoving wavenumber k; however, in what follows we drop the subscript k. To solve equations (2.41) in Fourier space, we assume that the modes k are decoupled. Furthermore, since we are just interested in the direction-averaged equations (the equations that depend only on the modulus k), we consider all variables of the form $\Psi_{\mathrm{k}}(\eta)e^{i\mathrm{k}\cdot x}$. In practice, this means that we can substitute in equations (2.41) and (2.43) the quantities

$$
\begin{aligned}
&\phi(x,\eta)\to e^{i\mathrm{k}\cdot x}\phi(\eta),\\
&\nabla\phi(x,\eta)\to ie^{i\mathrm{k}\cdot x}\mathrm{k}\phi(\eta),\\
&\nabla_i\nabla^i\phi(x,\eta)\to -e^{i\mathrm{k}\cdot x}\mathrm{k}^2\phi(\eta),\\
&\theta\to ie^{i\mathrm{k}\cdot x}\mathrm{k}\cdot v^{\perp}(\eta).
\end{aligned}
\tag{2.45}
$$

Thus, equations (2.41) and (2.43) become

$$
\begin{aligned}
&\mathrm{k}^2\Phi + 3\mathcal{H}\left(\Phi' + \mathcal{H}\Psi\right) = -4\pi G a^2 \rho_0 \delta, \\
&\mathrm{k}^2\left(\Phi' + \mathcal{H}\Psi\right) = 4\pi G a^2 (1+w)\rho_0 \theta, \\
&\Psi = \Phi, \\
&\Phi'' + 2\mathcal{H}\Phi' + \mathcal{H}\Psi' + \left(\mathcal{H}^2 + 2\mathcal{H}'\right)\Psi = 4\pi G a^2 c_\mathrm{s}^2 \rho_0 \delta, \\
&\delta' + 3\mathcal{H}\left(c_\mathrm{s}^2 - w\right)\delta = -(1+w)\left(\theta - 3\Phi'\right), \\
&\theta' + \left[\mathcal{H}(1-3w) + \frac{w'}{1+w}\right]\theta = \mathrm{k}^2\left(\frac{c_\mathrm{s}^2}{1+w}\delta + \Psi\right).
\end{aligned}
\tag{2.46}
$$

Let us just remark that the above equations are not independent, they are valid just for a universe composed of a single perfect fluid and the third equation reduces the scalars perturbations of the metric to only one function. This last result will no longer be true in the model we will present in Section **3**.

We now follow [31] to briefly go through the solution of equations (2.46) in the limit $\frac{a''}{a} \gg \mathrm{k}^2$ and assuming that w is a constant (valid for both matter and radiation). Indeed, as we will see in Section **4.2**, this behaviour is different from the one of the model presented in Section **3**.

We first find an equation for Φ and $\delta^* = \delta + 3\mathcal{H}(w+1)\frac{\theta}{\mathrm{k}^2}$ alone

$$
\begin{aligned}
&\Phi'' + 3\mathcal{H}\left(1 + c_\mathrm{s}^2\right)\Phi' + \left(c_\mathrm{s}^2\mathrm{k}^2 + 3\mathcal{H}^2 c_\mathrm{s}^2 + 2\mathcal{H}' + \mathcal{H}^2\right)\Phi = 0, \\
&(\delta^*)'' + \mathcal{H}\left(1 + 3c_\mathrm{s}^2 - 6w\right)(\delta^*)' - \left[\frac{3}{2}\mathcal{H}^2\left(1 - 6c_\mathrm{s}^2 - 3w^2 + 8w\right) - c_\mathrm{s}^2\mathrm{k}^2\right]\delta^* = 0.
\end{aligned}
\tag{2.47}
$$

Then, in order to solve the above equations in the super-horizon limit, we neglect those terms that vary as k and we use the relation $\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2$. In that way the equation for Φ reduces to

$$
\Phi'' + 3\mathcal{H}\left(1 + c_\mathrm{s}^2\right)\Phi' = 0 \tag{2.48}
$$

and it has a solution $\Phi' = 0$. Inserting this solution in the first equation of (2.46) we obtain

$$
3\mathcal{H}^2\Phi = -4\pi G a^2 \rho_0 \delta \tag{2.49}
$$

from which it follows that[12]

$$
\delta = -2\Phi. \tag{2.50}
$$

In conclusion, in standard general relativity in the limit $\frac{a''}{a} \gg \mathrm{k}^2$ we have that Φ and δ are frozen. It is indeed possible to show that the solution $\Phi' = 0$ is a growing mode (a dominating solution) and thus that the gravitational potential and the density fluctuations do not evolve on super-horizon scales.

2.3 Stochastic gravitational waves

In the previous section we showed that in order to describe the Universe as we observe it we need to perturb Einstein equations. In this section we focus our attention

[12] We need to use the background equation $3\mathcal{H}^2 = 8\pi G \rho_0 a^2$.

on tensorial perturbations: space-produced fluctuations in the metric tensor result in a stochastic background of relic gravitational waves. Indeed, the evolution of the first order tensorial perturbation $h_{ij}^{(1)\top}$ encodes an isotropic set of gravitational waves characterized by a smoothly-varying power spectrum as a function of frequency.

This is particularly interesting in the study of PBH production: as we mentioned in Section **2.1.3**, in addition to gravitational waves produced by localized sources, we also have the possibility of detecting stochastic gravitational-wave backgrounds. The detection of this background of gravitational waves could help us better understand the early Universe cosmology.

In this section we follow [36] to review the dynamics of stochastic gravitational waves predicted by the standard inflationary scenario.

In standard single-field slow-roll inflation, primordial tensor fluctuations of the metric are characterized by a nearly scale-invariant power spectrum on super-horizon scales. The action that describes the dynamics of the inflationary epoch is

$$S = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R - \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\nabla_\mu\zeta\nabla_\nu\zeta + V(\zeta)\right], \tag{2.51}$$

where R is the Ricci scalar and ζ is the inflaton field. From the action (2.51) we can see that tensor fluctuations are not constrained by equations derived from the stress-energy continuity equation.[13] Thus, the evolution of stochastic gravitational waves is only regulated by the traceless spatial part of the perturbed Einstein equations.

By perturbing equation (2.51) up to second order[14] in h_{ij}, we obtain the action for the tensor perturbations[15] h_{ij},

$$S_h = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x\, a^2(t)\left[\dot{h}_{ij}\dot{h}_{ij} - \frac{1}{a^2}\left(\nabla_k h_{ij}\right)^2\right]. \tag{2.53}$$

From the above action we can see that in the presence of perfect fluids the dynamics of h_{ij} does not contain any direct influence from the energy content of the Universe, except for the underlying background solution. This will no longer be true in the discussions of Section **2.3** where we study the dynamics of second order gravitational waves.

h_{ij} is gauge-invariant, so we do not need to fix the gauge to study its dynamics (see Section **5** for more details). Varying (2.53) with respect to h_{ij} we obtain the equation of motion

$$\nabla^2 h_{ij} - a^2\ddot{h}_{ij} - 3a\dot{a}\dot{h}_{ij} = 0. \tag{2.54}$$

[13] We recall that from the Lagrangian density $\mathcal{L}$ we can find the stress-energy tensor,

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}, \tag{2.52}$$

and that the scalar field ζ behaves like a perfect fluid.

[14] To study the dynamics of the first order tensor perturbations $h_{ij}^{(1)\top}$ we need to expand the action up to second order. However, this should not be confused with the study of second order tensorial perturbations.

[15] From now on in this section we drop the notation $h_{ij}^{(1)\top}$ and we simply use h_{ij} to label the first order transverse and traceless tensorial perturbations.

To solve the above equation we split the tensorial part of h_{ij}. Recalling that it is symmetric, transverse and traceless, we can rewrite h_{ij} as

$$h_{ij}(x,t) = h(t) e_{ij}^{(+,\times)}(x), \tag{2.55}$$

where $e_{ij}^{(+,\times)}$ is the polarization tensor which satisfies the conditions: $e_{ij} = e_{ji}$, $\mathrm{k}^i e_{ij} = 0$ and $e_{ii} = 0$, and $(+,\times)$ are the two gravitational waves polarization states. Thus, we can solve equation (2.54) for the scalar field $h(t)$ and then write the most general solution for h_{ij} as

$$h_{ij}(x,t) = \sum_{\lambda=(+,\times)} h^{(\lambda)}(t) e_{ij}^{(\lambda)}(x). \tag{2.56}$$

To get the solution of the equation of motion it is useful to perform the transformation $v_{ij} = \frac{a\, M_{\mathrm{Pl}}}{\sqrt{2}} h_{ij}$. Thus, going from cosmic time to conformal time and from $h(t)$ to $v(t)$, the equation of motion of the scalar field $v_{\mathrm{k}}^{(\lambda)}$ in Fourier space, reads

$$v_{\mathrm{k}}^{(\lambda)} + \left(\mathrm{k}^2 - \frac{a''}{a}\right) v_{\mathrm{k}}^{(\lambda)} = 0. \tag{2.57}$$

Equation (2.57) is a wave equation the qualitative behaviour of which can be split into two main regimes depending on the relative magnitude of the second and third terms

- If $\frac{a''}{a} \ll \mathrm{k}^2$ (sub-horizon limit), we obtain that equation (2.57) is the equation of a free harmonic oscillator. Thus, h_{ij} oscillates with a damping factor $\frac{1}{a}$ due to the expansion of the Universe.

- If $\frac{a''}{a} \gg \mathrm{k}^2$ (super-horizon limit), there are two possible solutions for equation (2.57): $v(\eta) \propto a$ and $v(\eta) \propto \frac{1}{a^2}$. In terms of h_{ij} these solutions are $h \propto const$. And a solution decreasing in time, respectively.

To find the general solution of equation (2.57) we perform the standard quantization of the field,

$$v_{\mathrm{k}}^{(\lambda)} = v_{\mathrm{k}}(\eta) \hat{a}_{\mathrm{k}}^{(\lambda)} + v_{\mathrm{k}}^*(\eta) \hat{a}_{-\mathrm{k}}^{(\lambda)\dagger}, \tag{2.58}$$

where the modes are such that $v_{\mathrm{k}}^* v_{\mathrm{k}}' - v_{\mathrm{k}} v_{\mathrm{k}}'^* = -i$ and $\hat{a}_{-\mathrm{k}}^{(\lambda)\dagger}$ and $\hat{a}_{\mathrm{k}}^{(\lambda)}$ are the creation and annihilation operators, respectively. Then, assuming the easiest initial conditions, in which the Universe is in the vacuum state $\hat{a}_{\mathrm{k}}^{(\lambda)} \mid 0 >= 0$, equation (2.57) is a Bessel equation. In the case of a de Sitter Universe it has the exact solution (see [37] for more details)

$$v_{\mathrm{k}}(\eta) = \sqrt{-\eta}\left[C_1 H_\nu^{(1)}(-\mathrm{k}\eta) + C_2 H_\nu^{(2)}(-\mathrm{k}\eta)\right], \tag{2.59}$$

where C_1 and C_2 are integration constants, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of first and second order and $\nu \approx \frac{3}{2} + \epsilon$, where ϵ is the slow-roll parameter.

To choose the two constants, C_1 and C_2, we impose that the solution (2.59) matches the plane-wave solution $e^{-i\mathrm{k}\eta}/\sqrt{2\mathrm{k}}$ in the $\frac{a''}{a} \ll \mathrm{k}^2$ limit. In this limit the Hankel functions read

$$\begin{aligned} H_\nu^{(1)}(z \gg 1) &\sim \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)}, \\ H_\nu^{(2)}(z \gg 1) &\sim \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)}, \end{aligned} \tag{2.60}$$

so that we have to impose $C_2 = 0$ and $C_1 = \frac{\sqrt{\pi}}{2} e^{\frac{i\pi}{2}(\nu+1/2)}$. The exact solution of (2.59) becomes

$$v_\mathrm{k}(\eta) = \frac{\sqrt{\pi}}{2} e^{\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)} \sqrt{-\eta} H_\nu^{(1)}(-\mathrm{k}\eta). \tag{2.61}$$

In particular, taking the limit $\frac{a''}{a} \gg \mathrm{k}^2$ of the Hankel function $H_\nu^{(1)}$, the solution in the super-horizon limit becomes:

$$v_\mathrm{k} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2\mathrm{k}}} (-\mathrm{k}\eta)^{\frac{1}{2}-\nu}, \tag{2.62}$$

where Γ is the Euler function.

To find the power spectrum we can use the definition (2.58) to compute $\left\langle v_{ij}(\mathrm{k}_1), v_{ij}^*(\mathrm{k}_2)\right\rangle$. Using $[a_\mathrm{k}, a_{\mathrm{k}'}] = 0$ and $[a_\mathrm{k}, a_{\mathrm{k}'}^\dagger] = \delta^3\left(\mathrm{k} - \mathrm{k}'\right)$ we obtain

$$\left\langle v_{ij}(\mathrm{k}_1), v_{ij}^*(\mathrm{k}_2)\right\rangle = \frac{|v_\mathrm{k}|^2}{a^2} \delta^{(3)}\left(\mathrm{k}_1 - \mathrm{k}_2\right). \tag{2.63}$$

Thus, the power spectrum of stochastic gravitational waves P_T can be computed as

$$P_\mathrm{T}(\mathrm{k}) = \frac{\mathrm{k}^3}{2\pi^2} \sum_\lambda \left| h_\mathrm{k}^{(\lambda)} \right|^2. \tag{2.64}$$

In the case of the super-horizon limit the power spectrum becomes

$$P_\mathrm{T}(\mathrm{k}) = \frac{8}{M_\mathrm{Pl}^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{\mathrm{k}}{aH}\right)^{-2\epsilon}, \tag{2.65}$$

which, as anticipated, is almost scale-invariant. This means that the stochastic gravitational waves produced in the standard inflationary scenario are almost frozen on super-horizon scales. We will see that in Section **5** we obtain a different result for second order gravitational waves.

Let us briefly mention that to observationally constrain the amplitude of the gravitational waves signal, the tensor-to-scalar ratio r is normally used. We define r as the ratio between the tensor and scalar power spectrum amplitudes, at a given pivot scale k^*,

$$r(\mathrm{k}^*) = \frac{A_\mathrm{T}(\mathrm{k}^*)}{A_\mathrm{S}(\mathrm{k}^*)}. \tag{2.66}$$

At present we have only an upper bound on r, $r(0.05Mpc^{-1}) < 0.07$ at the 95% C.L. (see [38] for more details). The important point is that different inflationary scenarios predict different values for r. Thus, the study of observational signatures of primordial gravitational waves provides us with a way not only to test the general inflationary paradigm, but also to distinguish between specific models.

Furthermore, the detection of stochastic gravitational waves provides evidence for what happened in the early Universe. It is not only strictly related to the inflationary phase, but also more in general to what happened in the early times. Indeed, as we will show in Section **5**, the production mechanism for PBHs acts as a source of second order gravitational waves which modifies the resulting power spectrum.

Chapter 3

The model

In this section we discuss the model presented in [1] and particularly go through the compactification of the action following the computations of [39]. In this way we find the action that describes the new scenario. We will use the results of this section to study the production of PBHs and gravitational waves in Sections **4** and **5**.

We start from a higher-dimensional gravity theory minimally coupled to higher-dimensional radiation fluid. The latter is modeled with the help of a $P(\chi)$ theory which is explained in more detail in Section **3.1**. The higher dimensional theory under consideration is a 6D manifold which has the topology $\mathbb{R}^4 \times S_2$. We therefore start with the action:

$$S = \frac{M_{(6)}^4}{2} \int \mathrm{d}^6 X \sqrt{G} R^{(6)} + \int \mathrm{d}^6 X \sqrt{G} \left(-\frac{1}{2} G^{AB} \nabla_A \chi \nabla_B \chi \right)^2 , \tag{3.1}$$

where $M_{(6)}$ and $R^{(6)}$ are respectively the Plank mass and the Ricci scalar in the 6-dimensional space-time and $\sqrt{G}$ is the determinant of the 6-dimensional metric.

In the 6-dimensional manifold we then take coordinates $X^A = \{x^\mu, y^a\}$ where x^μ are 4-dimensional coordinates of a flat FLRW Universe and y^a are coordinates on the 2-sphere. The metric is parametrised as

$$ds^2 = G_{AB} dX^A dX^B = \hat{g}_{\mu\nu} dx^\mu dx^\nu + b^2 \gamma_{ab} dy^a dy^b, \tag{3.2}$$

where b is the radius of the 2-sphere, $\hat{g}_{\mu\nu}$ is the metric of the 4-dimensional manifold and γ_{ab} is the metric of the 2-sphere.

Now we perform the Kaluza-Klein reduction to obtain the dimensionally reduced 4-dimensional action. Parameterising the 6-dimensional manifold as in (3.2) the action becomes

$$\begin{aligned} S =& \frac{M_{(6)}^4}{2} \int \mathrm{d}^4 x \sqrt{-\hat{g}} \int \mathrm{d}^2 y b^2 \sqrt{\gamma} \left(R^{(4)}(\hat{g}) + R^{(2)}(\gamma) \right) \\ &+ \int \mathrm{d}^4 x \sqrt{-\hat{g}} \int \mathrm{d}^2 y b^2 \sqrt{\gamma} \left(-\frac{1}{2} \hat{g}^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi \right)^2 , \end{aligned} \tag{3.3}$$

where $R^{(4)}(\hat{g})$ and $R^{(2)}(\gamma)$ are, respectively, the Ricci scalars of the 4-dimensional and 2-dimensional metrics.

We assume that the radiation field does not depend on the extra dimensions and we integrate equation (3.3) over the 2-sphere. Based on the results of [39], after the integrating over the extra dimensions the action takes the form

$$\begin{aligned} S = &\int \mathrm{d}^4x 4\pi b^2 \sqrt{-\hat{g}} \left\{ \frac{M_{(6)}^4}{2} \left[R^{(4)}(\hat{g}) - \frac{2}{b^2}\hat{g}^{\mu\nu}\nabla_\mu b \nabla_\nu b + \frac{2}{b^2} \right] - \frac{f^2}{2} \right\} \\ &+ \int d^4x \sqrt{-\hat{g}} 4\pi b^2 \left(-\frac{1}{2}\hat{g}^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi \right)^2 . \end{aligned} \tag{3.4}$$

The f-dependent term, as shown in [39], comes from the quantization of the extra-dimensional magnetic flux which implies $f = \frac{n}{2eb^2}$ with n an integer and e the gauge coupling of the 6-dimensional abelian gauge field.

We define $M_{\rm Pl}^2 = 4\pi M_{(6)}^4 b_*^2$ and the radius $b = b_* e^{\frac{\phi}{2M_{\rm Pl}}}$. In the latter definition the field ϕ parametrises the radius of the compactified sub-manifold and b_* denotes the radius at some fiducial epoch (for example the present epoch). With these definitions the action becomes

$$\begin{aligned} S = &\frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-\hat{g}} e^{\frac{\phi}{M_{\rm Pl}}} R^{(4)}(\hat{g}) - \frac{1}{4} \int \mathrm{d}^4x \sqrt{-\hat{g}} e^{\frac{\phi}{M_{\rm Pl}}} \hat{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ &+ \int \mathrm{d}^4x \sqrt{-\hat{g}} \left(\frac{M_{\rm Pl}^2}{b_*^2} - 2\pi f^2 b_*^2 e^{\frac{\phi}{M_{\rm Pl}}} \right) + 4\pi b_*^2 \int d^4x \sqrt{-\hat{g}} e^{\frac{\phi}{M_{\rm Pl}}} \left(-\frac{1}{2}\hat{g}^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi \right)^2 . \end{aligned} \tag{3.5}$$

We then perform the conformal transformation: $g_{\mu\nu} = \hat{g}_{\mu\nu} e^{\frac{\phi}{M_{\rm Pl}}}$ to move to the Einstein frame. In the Einstain frame we have

$$\begin{aligned} &\sqrt{-g} = e^{\frac{2\phi}{M_{\rm Pl}}} \sqrt{-\hat{g}}, \\ &R^{(4)}(g) = e^{\frac{-\phi}{M_{\rm Pl}}} \left[R^{(4)}(\hat{g}) - 2(d-1) \left(\nabla^\mu\nabla_\mu \frac{\phi}{2M_{\rm Pl}^2} \right) - (d-1)(d-2) \left(\hat{g}^{\mu\nu}\nabla_\mu \frac{\phi}{2M_{\rm Pl}} \nabla_\nu \frac{\phi}{2M_{\rm Pl}} \right) \right] . \end{aligned} \tag{3.6}$$

Thus, using $\hat{g}^{\mu\nu} = g^{\mu\nu} e^{\frac{\phi}{M_{\rm Pl}}}$ and neglecting the $\nabla^\mu\nabla_\mu \frac{\phi}{2M_{\rm Pl}^2}$ term,[16] we can substitute

$$\begin{aligned} &\sqrt{-\hat{g}} = e^{\frac{-2\phi}{M_{\rm Pl}}} \sqrt{-g}, \\ &R^{(4)}(\hat{g}) = e^{\frac{\phi}{M_{\rm Pl}}} R^{(4)}(g) + \frac{6}{4M_{\rm Pl}^2} e^{\frac{\phi}{M_{\rm Pl}}} \left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \right), \end{aligned} \tag{3.8}$$

in equation (3.5). In that way the action becomes

$$\begin{aligned} S = &\frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-g} R^{(4)}(g) + \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ &+ \int \mathrm{d}^4x \sqrt{-g} \left(\frac{M_{\rm Pl}^2}{b_*^2} e^{\frac{-2\phi}{M_{\rm Pl}}} - 2\pi f^2 b_*^2 e^{-\frac{\phi}{M_{\rm Pl}}} \right) + 4\pi b_*^2 \int d^4x \sqrt{-g} e^{\frac{\phi}{M_{\rm Pl}}} \left(-\frac{1}{2} g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi \right)^2 . \end{aligned} \tag{3.9}$$

[16] Replacing $R^{(4)}(\hat{g})$ in S we obtain

$$[...] + \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-g} e^{\frac{-\phi}{M_{\rm Pl}}} \left(6 e^{\frac{\phi}{M_{\rm Pl}}} g^{\mu\nu}\nabla_\mu\nabla\nu \frac{\phi}{2M_{\rm Pl}^2} \right) = [...] + \frac{3}{2} \int \mathrm{d}^4x \sqrt{-g} g^{\mu\nu}\nabla_\mu\nabla\nu\phi, \tag{3.7}$$

which, integrating by parts, gives a total derivative and the term: $(\nabla_\mu g^{\mu\nu}) \cdot (...)$.

Finally, we define $\psi \equiv (\pi b_*^2)^{1/4}\chi$ and $f \equiv \frac{\hat{f}}{b^2}$ in order to find the result with the same notation of [1],

$$S = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R^{(4)}(g) + \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) + \int d^4x\sqrt{-g}e^{\frac{\phi}{M_{\rm Pl}}}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi\right)^2, \tag{3.10}$$

where $V(\phi) = -\frac{M_{\rm Pl}^2}{b_*^2}e^{\frac{-2\phi}{M_{\rm Pl}}} + \frac{2\pi\hat{f}^2}{b_*^2}e^{\frac{-3\phi}{M_{\rm Pl}}}$.

3.1 P(χ) models

An essential ingredient of the extra-dimensional model presented above is the coupling of the radion field ϕ to the radiation field ψ. This is easy to obtain with the help of a P(χ) theory where the radiation is represented by the field ψ.

In our model we neglect dissipative phenomena so that we can approximate the matter content as a sum of perfect fluids. As first shown in [40], the physics of a perfect fluid can be derived from a unique scalar field. We will follow the notation of [41] in order to better understand why we choose the above reported Lagrangian to describe radiation.

A perfect and barotropic fluid is defined to have a stress-energy tensor of the form:

$$T_{\mu\nu} = (\rho + p(\rho))u_\mu u_\nu + p(\rho)g_{\mu\nu}, \tag{3.11}$$

where u^μ is the 4-velocity of the fluid and the pressure p is a function only of the energy density ρ. We will also assume that the fluid is irrotational. Indeed, it is possible to show that if the initial vorticity is zero it does not evolve. While, if we assume a non zero initial vorticity, it is diluted by the expansion of the Universe and thus it is set to zero by inflation. Under these constraints the fluid is naturally described by a single scalar function.

We then start by assuming that the scalar field σ describes a perfect fluid and we derive the form of the Lagrangian that gives a stress-energy tensor of the form of equation (3.11). We start with the Lagrangian density

$$\mathcal{L} = P(\chi), \quad \chi \equiv -\nabla_\mu\sigma\nabla^\mu\sigma, \tag{3.12}$$

so that the stress-energy tensor of this field reads

$$T_{\mu\nu} = 2P'(\chi)\nabla_\mu\sigma + P(\chi)g_{\mu\nu}\nabla^\mu\sigma. \tag{3.13}$$

In order to recover the form of (3.11) we need to identify

$$\rho = 2P'(\chi)\chi - P(\chi), \quad p = P(\chi), \quad u_\mu = \frac{\nabla_\mu\Psi}{\sqrt{\chi}}, \tag{3.14}$$

and we need to impose that $\nabla_\mu\Psi$ is everywhere timelike and future directed. Then, if $P(\chi)$ satisfies the constraints (3.14), we have that the Lagrangian density (3.12) describes the behaviour of a perfect fluid: we can recover the conservation of energy and the Euler equation, the number particle density and the adiabatic speed of sound as a function of σ. As an example we explicitly find the particle number density n in the $P(\chi)$ formalism. In order to find n we can start from the equation of motion

$$\nabla_\mu \left[P'(\chi)\nabla^\mu\right] = 0 \tag{3.15}$$

to define the conserved current:[17]

$$J^\mu = 2\sqrt{\chi}P'(\chi)u^\mu. \tag{3.16}$$

J^μ can be interpreted as the conserved particle flux $J^\mu = nu^\mu$. Thus, we can identify the particle density as:

$$n = 2\sqrt{\chi}P'(\chi). \tag{3.17}$$

Going back to our case, we can find the explicit form of $P(\chi)$: we want to use this formalism to describe radiation. In our case the equation of state is $p = w\rho$. Thus, solving the first two equations in (3.14), we find

$$P(\chi) = \chi^{\frac{1+w}{2w}}. \tag{3.18}$$

In order to find the Lagrangian density for the radiation fluid we only have to consider $w = 1/3$. In that way $\mathcal{L}$ reduces to:

$$\mathcal{L} = \chi^2 = (-\nabla_\mu\sigma\nabla^\mu\sigma)^2 \tag{3.19}$$

and, in an expanding flat FLRW background, the homogeneous solution satisfies

$$n = P'(\chi)\sigma' \propto a^{-3}. \tag{3.20}$$

In conclusion, with this formalism we are able to describe the dynamics of radiation in a flat FLRW universe within a Lagrangian formalism. This formalism, as we will see in the next sections, is useful for studies of cosmological perturbations and it allows the coupling between radiation and extra dimensions to be introduced.

Furthermore, the recovered Lagrangian is a particular case of the so-called k-essence; perturbation theory of k-essence models has already been studied extensively (as an example see [42]).

[17] The conservation is a consequence (by Noether's theorem) of the invariance of the action under shift of σ.

Chapter 4

Primordial black hole formation in the new scenario

Our objective in this chapter is to show that in the presence of extra dimensions, super-horizon curvature perturbations can be sufficiently enhanced during radiation domination to produce PBHs.

Following the preliminary version of [1] in this chapter we study the new scenario: we derive the evolution of first order scalar perturbations to show that in the new scenario the super-horizon modes are not frozen anymore. The enhancement of curvature perturbations is achieved during the post-inflationary epoch: even assuming an almost Harrison-Zel'dovich initial power spectrum, we show that we can explain the PBH production.
Here we use the ADM formalism, presented in Section **4.1**. We derive and analyse the dynamics of scalar perturbations in Section **4.2**.

4.1 ADM decomposition of the metric

To study the evolution of scalar perturbations in the model presented in Section **3**, we will use the ADM formalism. In particular, this parametrisation of the metric has the advantage of making explicit the true dynamical degrees of freedom of the system. To find the ADM decomposition of the metric we need a parametrisation of the space-time metric adapted to a given choice of foliation of the space-time by constant time hypersurfaces. We go through the ADM decomposition of the metric following [43].

We first assume that the spatial hypersurfaces Σ of this foliation are hypersurfaces of constant time. This means that Σ are the level sets of a time function $\tau(y^\alpha)$,

$$\Sigma_{\tau_0} = \{y^\alpha \ s.t. \ \tau(y^\alpha) = \tau_0\}, \tag{4.1}$$

with future-oriented normal vector $\nabla_\alpha \tau$.

We then define the coordinates (τ, x^i) on the manifold via a coordinate transformation

$$y^\alpha = y^\alpha(\tau, x^i). \tag{4.2}$$

Given the choice of the foliation and time evolution $y^\alpha = y^\alpha(\tau, x^i)$, we define the tangent vector E_i^α of the spatial surface at time τ,

$$E_i^\alpha = \left(\frac{\partial y^\alpha}{\partial x^i}\right)_\tau . \tag{4.3}$$

We then decompose ∂_τ into a normal and a tangential part as

$$(\partial_\tau)^\alpha = N N^\alpha + E_i^\alpha \mathcal{N}^i , \tag{4.4}$$

where N^α is the normal vector, and the function N and the vector field $\mathcal{N}^i$ are called *lapse function* and *shift vector field.* The lapse function and shift vector field encode the freedom in the choice of $(\partial_\tau)^\alpha$. Inserting this into the definition of the chosen coordinates we have

$$dy^\alpha = (N N^\alpha + E_i^\alpha \mathcal{N}^i) d\tau + E_i^\alpha dx^i . \tag{4.5}$$

Finally, inserting equation (4.5) into the line element of the space-time metric, we find:

$$ds^2 = g_{\alpha\beta} dy^\alpha dy^\beta = -N^2 d\tau^2 + h_{ij}(dx^i + \mathcal{N}^i dt)(dx^j + \mathcal{N}^j d\tau), \tag{4.6}$$

where $h_{ij} = g_{\alpha\beta} E_i^\alpha E_j^\beta$ is called *induced metric.* Equation (4.6) is the so-called *ADM decomposition of the metric.*

This formalism is usually adopted for developing the Hamiltonian formulation of general relativity and it is often useful to develop numerical solutions.

Before inserting the metric (4.6) into the action (3.10), we recall that in terms of these variables the 4-dimensional volume element $\sqrt{-g}$ takes the form $\sqrt{-g} = N\sqrt{h}$. We also recall another useful definition: the extrinsic curvature. The extrinsic curvature tensor of the surfaces of constant τ is defined as

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - L_{\mathcal{N}} h_{ij}\right), \tag{4.7}$$

where $\dot{h}_{ij} = \partial_\tau h_{ij}$ and the Lie derivative of h_{ij} can be written as

$$L_{\mathcal{N}} h_{ij} = \nabla_i \mathcal{N}_j + \nabla_j , \mathcal{N}_i \tag{4.8}$$

where ∇_i is the induced covariant derivative.

Now that we introduced the ADM formalism, we have all the tools we need to write the action (3.10) in terms of the ADM decomposition. Inserting the metric (4.6) into the action (3.10) we find:

$$S = \int d^4x \sqrt{h}\left[\frac{M_{\rm Pl}^2}{2}\mathcal{L}_{\rm G} + \mathcal{L}_\phi + \mathcal{L}_\psi\right], \tag{4.9}$$

with

$$\begin{aligned} \mathcal{L}_{\rm G} &= \left[N R^{(3)} + \frac{1}{N}\left(E^{ij}E_{ij} - E^2\right)\right], \\ \mathcal{L}_\phi &= \left[\frac{1}{2N}\left(\dot{\phi} - \mathcal{N}^i \nabla_i \phi\right)^2 - \frac{N}{2} h^{ij} \nabla_i \phi \nabla_j \phi - N V(\phi)\right], \\ \mathcal{L}_\psi &= N e^{\phi/M_{\rm p1}}\left[\frac{1}{N^2}\left(\dot{\psi} - \mathcal{N}^i \nabla_i \psi\right)^2 - h^{ij} \nabla_i \psi \nabla_j \psi\right]^2 . \end{aligned} \tag{4.10}$$

In (4.10) we have introduced the notation

$$\begin{aligned} E_{ij} &= NK_{ij} = \frac{1}{2}\left[\dot{h}_{ij} - \nabla_i \mathcal{N}_j - \nabla_j \mathcal{N}_i\right], \\ E &= E^i_i, \end{aligned} \tag{4.11}$$

and $R^{(3)}$ is the induced Ricci scalar.

4.2 First order scalar perturbation behaviour

In Section **3** we defined a specific 6-dimensional model of gravity in which the extra dimensions are coupled to the radiation field. We now intend to study the dynamics during radiation domination to show that, starting from a standard scale-invariant power spectrum, we can enhance the curvature perturbations in the super-horizon modes. We follow the computations of the preliminary version of [1].

We first study the background dynamics. Varying the action (4.9) we find the equations of motion

$$\begin{aligned} 3H^2 &= \frac{1}{2}\left(\dot{\phi}_0\right)^2 + 3e^{\phi_0}\left(\dot{\psi}_0\right)^4 + V\left(\phi_0\right), \\ \ddot{\phi}_0 &= -3H\dot{\phi}_0 - \frac{\partial V(\phi_0)}{\partial \phi} + e^{\phi_0}\left(\dot{\psi}_0\right)^4, \\ \ddot{\psi}_0 &= -\dot{\psi}_0\left(H + \frac{\dot{\phi}_0}{3}\right), \end{aligned} \tag{4.12}$$

where ϕ_0 and ψ_0 are the background radion field, and the background radiation field respectively. The last equation can be integrated to give the background evolution of the radiation

$$\dot{\psi}_0 = \dot{\psi}_0\left(t = t_{\text{in}}\right) e^{\frac{-\phi_0(t=t_{\text{in}})}{3M_{\text{Pl}}}} \left(\frac{e^{\frac{-\phi_0}{3M_{\text{Pl}}}}}{a}\right). \tag{4.13}$$

Once we have the background dynamics, we can study the dynamics of scalar perturbations.

To study the dynamics of first order scalars perturbations (see Section **2.2** for more details about perturbation theory) we need to impose a gauge. In this section we work with the so-called *equipotential gauge* which assumes a space-time foliation such that the radiation fluctuations are gauged away

$$\begin{aligned} \phi(t,x) &= \phi_0(t) + \varphi(t,x), \\ \psi(t,x) &= \psi_0(t), \\ h_{ij}(t,x) &= a^2(t)e^{2\mathcal{R}(t,x)}\delta_{ij}. \end{aligned} \tag{4.14}$$

Thus, in this gauge the first order scalar perturbations are the curvature perturbations $\mathcal{R}$ and the radion perturbations φ. As usual, we assume a homogeneous and isotropic background such that the background fields, ϕ_0 and ψ_0 are functions of time only.

Once we impose the gauge, we can solve the equation of motion for the shift function

N and the lapse vector $\mathcal{N}_i$ and we substitute the solutions back into the action to obtain the action for the perturbations to a given order. As shown in [44], it turns out that we only need to solve the dynamics of N and $\mathcal{N}_i$ up to linear order in the perturbations to obtain the action up to the cubic order. This is because the terms in the action that come from solving the constraints to quadratic order multiply the zeroth order constraint equations, which vanish by the background equations of motion.
In the equipotential gauge the momentum and Hamiltonian constraint equations (the equations of motion for N and $\mathcal{N}_i$) become

$$\begin{aligned} &\nabla_i\left[N^{-1}\left(E^i_k - E\delta^i_k\right)\right] = \frac{\dot\phi_0}{NM_{\rm Pl}^2}\nabla_k\varphi, \\ &\frac{R^{(3)}}{2} - \frac{1}{2N^2}\left(E^{ij}E_{ij}-E^2\right) - \frac{1}{M_{\rm Pl}^2}\left\{\frac{\dot\phi^2}{2N^2} + V(\phi) + \frac{3e^{\frac{\phi}{M_{\rm Pl}}}}{N^4}\dot\psi_0^4\right\} = 0. \end{aligned} \tag{4.15}$$

We solve equations (4.15) defining

$$\begin{aligned} N &= 1+\alpha_1, \\ N^i &= \nabla_i\Theta + N^i_{\rm T}, \end{aligned} \tag{4.16}$$

where α_1, Θ and $N^i_{\rm T}$ are all first order quantities. Integrating equations (4.15) we find the solutions

$$\begin{aligned} &\alpha_1 = \frac{\dot{\mathcal{R}}}{H} + \frac{\dot\phi_0}{2HM_{\rm Pl}^2}\varphi, \\ &\nabla_i\nabla^i\Theta = -\frac{\nabla_i\nabla^i\mathcal{R}}{a^2H} + \frac{\epsilon}{c_{\rm s}^2}\dot{\mathcal{R}} + \left(-3+\frac{\epsilon}{c_{\rm s}^2}\right)\frac{\dot\phi_0}{2M_{\rm Pl}^2}\varphi - \frac{3e^{\phi_0/M_{\rm Pl}}\dot\psi_0^4}{2M_{\rm Pl}^2H}\frac{\varphi}{M_{\rm Pl}} - \frac{\dot\phi_0\dot\varphi}{2HM_{\rm Pl}^2} - \frac{\partial V}{\partial\varphi}\frac{\varphi}{2HM_{\rm Pl}^2}, \\ &N^i_{\rm T} = 0, \end{aligned} \tag{4.17}$$

where $c_{\rm s}^2$ is defined as

$$c_{\rm s}^2 = \left(\frac{\dot\phi_0^2}{2H^2M_{\rm Pl}^2} + \frac{2\dot\psi_0^4e^{\phi_0/M_{\rm Pl}}}{H^2M_{\rm Pl}^2}\right)\Big/\left(\frac{\dot\phi_0^2}{2H^2M_{\rm Pl}^2} + \frac{6\dot\psi_0^4e^{\phi_0/M_{\rm Pl}}}{H^2M_{\rm Pl}^2}\right) \tag{4.18}$$

and the Hubble slow-roll parameter $\epsilon = -\frac{\dot H}{H^2}$ is given via the background equations of motion (4.12),

$$\epsilon = \frac{\dot\phi_0^2}{2H^2M_{\rm Pl}^2} + \frac{2\dot\psi_0^4e^{\phi_0/M_{\rm Pl}}}{H^2M_{\rm Pl}^2}. \tag{4.19}$$

Note that $\frac{1}{3} \le c_{\rm s}^2 \le 1$ as one goes from radiation domination to radion domination.

We can now insert the solution for the lapse N and the shift $\mathcal{N}_i$ back into the action and expand the latter up to second order in perturbation variables. After integrating by part, imposing the background equations of motion and switching to conformal time, the quadratic action becomes

$$S^{(2)} = S_1^{(2)} + S_2^{(2)} + S_3^{(2)}, \tag{4.20}$$

where

$$\begin{aligned} S_1^{(2)} &= M_{\rm Pl}^2 \int {\rm d}^4x\, a^2\epsilon \left[-\frac{\mathcal{R}'^2}{c_{\rm s}^2} - \delta^{ij}\nabla_i\mathcal{R}\nabla_j\mathcal{R}\right], \\ S_2^{(2)} &= \int {\rm d}^4x \frac{a^2}{2}\left[\varphi'^2 - \delta^{ij}\nabla_i\varphi\nabla_j\varphi - \mu^2\varphi^2\right], \\ S_3^{(2)} &= \int {\rm d}^4x\, a^2\left[-\frac{\phi_0'}{\mathcal{H}}\varphi'\mathcal{R}' + \frac{\phi_0'}{\mathcal{H}}\delta^{ij}\nabla_i\mathcal{R}\nabla_j\varphi + \omega^2\mathcal{R}'\varphi\right]. \end{aligned} \tag{4.21}$$

In the second order action (4.21) we defined ω and μ as

$$\begin{aligned} \omega^2(\eta) &= \phi_0'\left(\frac{\epsilon}{c_{\rm s}^2} - 3\right) - \frac{a^2}{\mathcal{H}}\frac{\partial V}{\partial\phi} - \frac{3\psi_0'^4 e^{\phi_0/M_{\rm Pl1}}}{a^2\mathcal{H}M_{\rm Pl}}, \\ \mu^2(\eta) &= \frac{\phi_0^2}{M_{\rm Pl}^2}\left(3 - \frac{\epsilon}{2c_{\rm s}^2} - \frac{\epsilon}{2}\right) + a^2\frac{\partial^2 V}{\partial\phi\partial\phi} + \frac{2\phi_0'}{\mathcal{H}}\left(\frac{a^2}{M_{\rm Pl}^2}\frac{\partial V}{\partial\phi} + \frac{\psi_0'^4 e^{\phi_0/M_{\rm Pl}}}{a^2M_{\rm Pl}^3}\right) - \frac{\psi_0'^4 e^{\phi_0/M_{\rm Pl}}}{a^2M_{\rm Pl}^2}. \end{aligned} \tag{4.22}$$

From the second order action we then obtain the equations of motion for the perturbation variables, which result in

$$\begin{aligned} \mathcal{R}'' + \left(2\mathcal{H} + \frac{\epsilon'}{\epsilon} - \frac{2c_{\rm s}'}{c_{\rm s}}\right)\mathcal{R}' - c_{\rm s}^2\nabla_i\nabla^i\mathcal{R} &= \frac{c_{\rm s}^2\phi_0'}{2\mathcal{H}M_{\rm Pl}^2\epsilon}\left[\varphi'' + 2\mathcal{H}\varphi' - \nabla_i\nabla^i\varphi\right] \\ &\quad - \frac{c_{\rm s}^2}{2M_{\rm Pl}^2\epsilon}\left[\omega^{2'} + 2\mathcal{H}\omega^2\right]\varphi + \frac{4c_{\rm s}^2\psi_0'^4 e^{\phi_0/M_{\rm Pl}}}{a^2\epsilon\mathcal{H}M_{\rm Pl}^3}\left(1 - \frac{\phi_0'}{\mathcal{H}M_{\rm Pl}}\right)\varphi', \\ \varphi'' + 2\mathcal{H}\varphi' - \nabla_i\nabla^i\varphi + \mu^2\varphi &= \frac{\phi_0'}{\mathcal{H}}\left[\mathcal{R}'' + 2\mathcal{H}\mathcal{R}' - \nabla_i\nabla^i\mathcal{R}\right] + \mathcal{R}'\left[\omega^2 + \phi_0'(\epsilon - 1) + \frac{\phi_0''}{\mathcal{H}}\right]. \end{aligned} \tag{4.23}$$

The above equations of motion describe how the curvature and radion perturbations interact in a background where the radion motion is arbitrary, coupled to a thermal fluid. Before solving equations (4.23) numerically, we simplify their form assuming that the background radion evolution is stabilised by some potential whose value at the minimum is vanishing. This means that there is no cosmological constant like contribution and thus $\phi_0' = V(\phi_0) = \partial_{\phi_0}V = 0$. In this case ϵ, $c_{\rm s}^2$, ω, μ and the first equation of the background equations of motion (4.12) become

$$\begin{aligned} \mathcal{H}^2 &= \frac{\psi_0'^4}{a^2M_{\rm Pl}^2}e^{\frac{\phi_0}{M_{\rm Pl}}}, \\ \epsilon &= 2, \\ c_{\rm s}^2 &= \frac{1}{3}, \\ \omega^2 &= -3\mathcal{H}M_{\rm Pl}, \\ \mu^2 &= a^2\frac{\partial^2 V}{\partial\phi\partial\phi} - \mathcal{H}^2. \end{aligned} \tag{4.24}$$

Furthermore, we find that during radiation domination $a(\eta) \propto \eta$ and $\mathcal{H} = \frac{1}{\eta}$. With this simplification the equations of motion in Fourier space simply become

$$\begin{aligned}\mathcal{R}_{\mathrm{k}}'' + \frac{2}{\eta}\mathcal{R}_{\mathrm{k}}' + \frac{\mathrm{k}^2}{3}\mathcal{R}_{\mathrm{k}} &= \frac{\widetilde{\varphi}_{\mathrm{k}}}{4\eta^2} + \frac{\widetilde{\varphi}_{\mathrm{k}}'}{3\eta},\\ \widetilde{\varphi}_{\mathrm{k}}'' + \frac{2}{\eta}\widetilde{\varphi}_{\mathrm{k}}' + \left(\mu^2 + \mathrm{k}^2\right)\varphi_{\mathrm{k}} &= -\frac{3}{\eta}\mathcal{R}_{\mathrm{k}}',\end{aligned} \tag{4.25}$$

where $\widetilde{\varphi} = \frac{\varphi}{M_{\mathrm{Pl}}}$ and we can identify the mass of the radion field $m = \frac{\partial^2 V}{\partial\phi\partial\phi}$.

Now that we have the equations of motion that describe how the curvature and radion perturbations evolve, we can numerically integrate them. Following the computations of [1], in this section we show that it is possible to enhance super-horizon initially scale-invariant perturbations.

We first rewrite the equations of motion in the form

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{R}}{\mathrm{dN}^2} - \alpha\beta\frac{\mathrm{d}^2\varphi}{\mathrm{dN}^2} &= \mathcal{S}_{\mathcal{R}}\left(\mathcal{R}, \frac{\mathrm{d}\mathcal{R}}{\mathrm{dN}}, \varphi, \frac{\mathrm{d}\varphi}{\mathrm{dN}}\right),\\ \frac{\mathrm{d}^2\varphi}{\mathrm{dN}^2} - \beta\frac{\mathrm{d}^2\mathcal{R}}{\mathrm{dN}^2} &= \mathcal{S}_{\varphi}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{dN}}, \mathcal{R}, \frac{\mathrm{d}\mathcal{R}}{\mathrm{dN}}\right),\end{aligned} \tag{4.26}$$

where we consider the e-foldings N as the time variable and we define $\alpha = \frac{c_{\mathrm{s}}^2}{2\epsilon}$ and $\beta = \frac{d\phi_0}{dN}$. The introduced source terms are

$$\begin{aligned}S_{\mathcal{R}} &= S_{\mathcal{R}}^{(1)}\mathcal{R} + \mathcal{S}_{\mathcal{R}}^{(2)}\frac{\mathrm{d}\mathcal{R}}{\mathrm{dN}} + \mathcal{S}_{\mathcal{R}}^{(3)}\varphi + \mathcal{S}_{\mathcal{R}}^{(4)}\frac{\mathrm{d}\varphi}{\mathrm{dN}},\\ \mathcal{S}_{\varphi} &= \mathcal{S}_{\varphi}^{(1)}\varphi + \mathcal{S}_{\varphi}^{(2)}\frac{\mathrm{d}\varphi}{\mathrm{dN}} + \mathcal{S}_{\varphi}^{(3)}\mathcal{R} + \mathcal{S}_{\varphi}^{(4)}\frac{\mathrm{d}\mathcal{R}}{\mathrm{dN}},\end{aligned} \tag{4.27}$$

with the individual terms

$$\begin{aligned}\mathcal{S}_{\mathcal{R}}^{(1)} &= -c_{\mathrm{s}}^2\frac{\mathrm{k}^2}{H^2a^2}\mathcal{R},\\ \mathcal{S}_{\mathcal{R}}^{(2)} &= -\left((\epsilon-3) - \frac{1}{\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{dN}} + \frac{2}{c_{\mathrm{s}}}\frac{\mathrm{d}c_{\mathrm{s}}}{\mathrm{dN}}\right),\\ \mathcal{S}_{\mathcal{R}}^{(3)} &= \frac{\mathrm{k}^2}{H^2a^2}\alpha\beta - 2\frac{\omega}{aH}\alpha\left[\frac{\mathrm{d}\omega}{\mathrm{dN}} + \omega\right],\\ \mathcal{S}_{\mathcal{R}}^{(4)} &= (3-\epsilon)\alpha\beta + \alpha\frac{4\left(\dot{\psi}_0\right)^4 e^{\phi_0/M_{\mathrm{Pl}}}}{H^2}(1-\beta),\\ \mathcal{S}_{\varphi}^{(1)} &= -\frac{\mathrm{k}^2+\mu^2}{H^2a^2}, \quad \mathcal{S}_{\varphi}^{(2)} = \epsilon - 3, \quad \mathcal{S}_{\varphi}^{(3)} = \frac{\mathrm{k}^2}{H^2a^2}\beta,\\ \mathcal{S}_{\varphi}^{(4)} &= (\epsilon-3)\beta + \frac{\omega^2}{aH} - \frac{1}{H^2}\frac{\partial V}{\partial\phi} + \frac{\left(\dot{\psi}_0\right)^4 e^{\phi_0/M_{\mathrm{Pl}}}}{H^2M_{\mathrm{Pl}}}.\end{aligned} \tag{4.28}$$

Equations (4.26) are not explicitly in a form of a well-posed initial value problem. To have a well-posed system we rewrite them in the form

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{R}}{\mathrm{dN}^2} &= \frac{1}{1-\alpha\beta^2}\left[\beta\mathcal{S}_{\mathcal{R}} + \mathcal{S}_{\varphi}\right],\\ \frac{\mathrm{d}^2\varphi}{\mathrm{dN}^2} &= \frac{1}{1-\alpha\beta^2}\left[\mathcal{S}_{\mathcal{R}} + \alpha\beta\mathcal{S}_{\varphi}\right].\end{aligned} \tag{4.29}$$

To solve equations (4.29) we first numerically solve the background dynamics (equations (4.12)). Then we use the background solutions to integrate the system (4.29). The system (4.29) can become singular if $1-\alpha\beta^2$ crosses zero but in [1] the authors have verified that this does not occur in any point of the integration domain.

For the choice of the initial conditions we assume initial scale-invariant curvature perturbations ($\mathcal{R}_{\rm ini} \approx 10^{-5}$) and spiky initial values for the radion perturbations, enhanced on a certain pivot scale $\rm k^2_{piv}$ as shown in **Figure 4.1**.

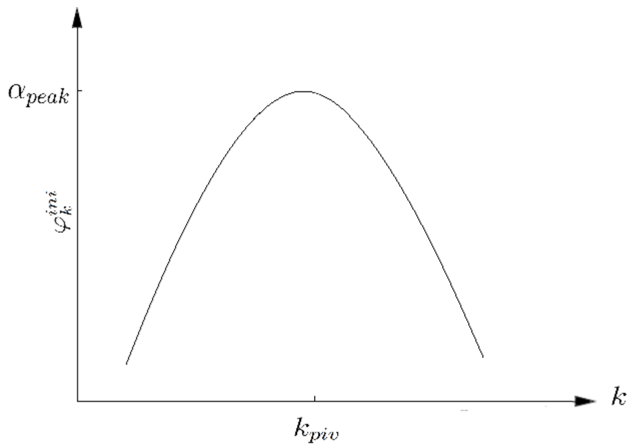

Figure 4.1: *Scale dependence of initial radion perturbations in Fourier space. Here it is assumed that initially radion perturbations peak at a pivot scale $k_{\rm piv}$ with an amplitude $\alpha_{\rm peak}$.*

The initial value for the curvature perturbations $\mathcal{R}_{\rm ini} \approx 10^{-5}$ is consistent with simple inflationary predictions. As we assume the inflationary phase to be standard, we can take $\mathcal{R}_{\rm ini} \approx 10^{-5}$ and vanishing initial velocities for curvature perturbations. For the radion perturbations initial conditions we take a spiky initial value, enhanced on a certain pivot scale $\rm k^2_{piv}$ as shown in **Figure 4.1**. The enhancement of the radion's initial conditions can be achieved by the coupling of the radion field ϕ to the inflaton field ζ, but this is an aspect that is currently being developed by the authors of [1].

The results of numerical integration of the system (4.12) are shown in **Figure 4.2** and **Figure 4.3**. We can see from these figures that for modes k close to $\rm k_{piv}$, which is the position of the peak in the $\varphi_{\rm k}$ initial conditions, we have a substantial super-horizon growth of the curvature perturbations. Indeed, looking at the red line, we can see that $\mathcal{R}$ is enhanced by a factor of 10^4 after 5 e-foldings.

This shows that the model described in Section **3** can produce an enhancement of curvature perturbations required for generating PBHs even if we start with scale-invariant initial conditions generated by standard inflationary scenarios. Furthermore, we can see that the enhancement depends on $\rm k^2_{piv}$: by choosing $\rm k^2_{piv}$ we can choose the modes we want to enhance and thus we can set the mass of the produced PBHs.

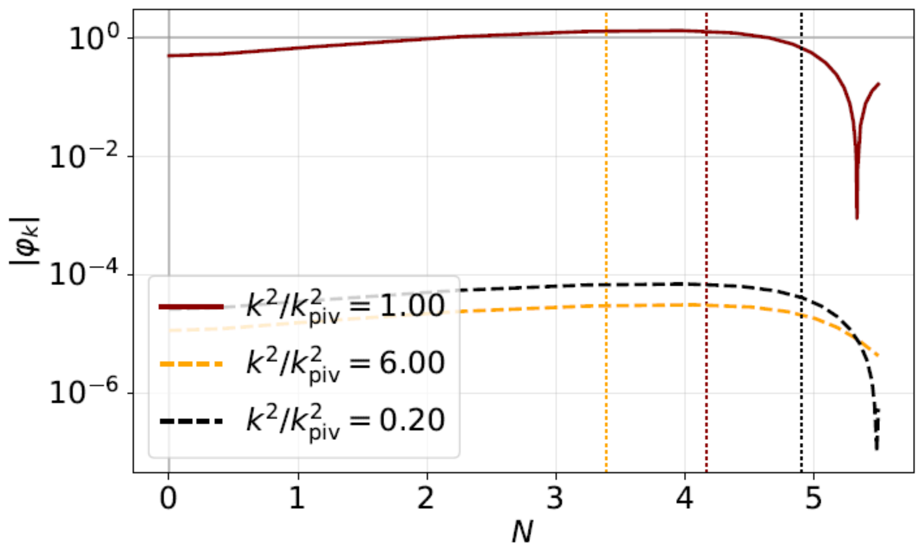

Figure 4.2: *Time evolution for three different modes of the radion perturbations $\varphi_{\rm k}$ as a function of the e-foldings N ($k^2_{\rm piv}$ is the position of the peak in the $\varphi_{\rm k}$ initial conditions). The vertical dotted lines indicate the horizon re-entry of the corresponding mode. This figure is provided by Valeri Vardanyan, an author of* [1].

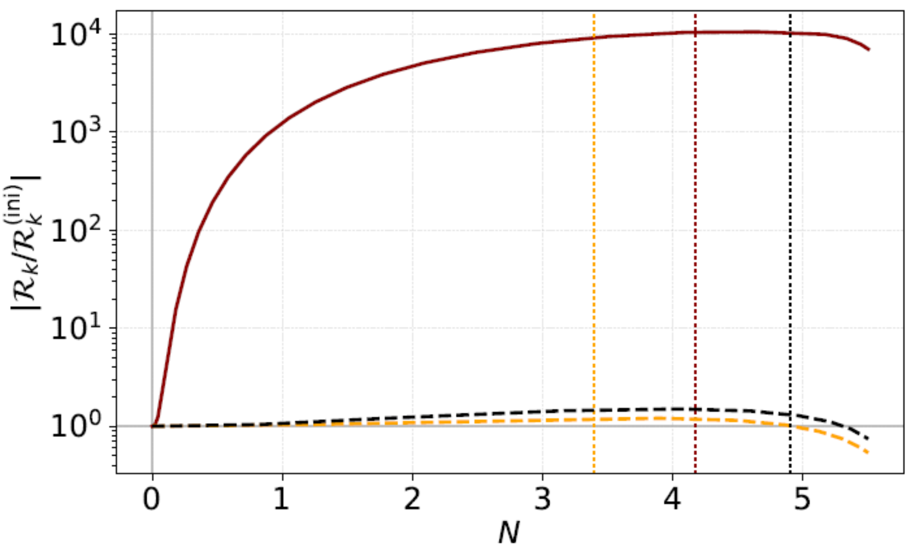

Figure 4.3: *Time evolution for three different modes of the curvature perturbations $\mathcal{R}$ as a function of the e-foldings N ($k^2_{\rm piv}$ is the position of the peak in the $\varphi_{\rm k}$ initial conditions). The vertical dotted lines indicate the horizon re-entry of the corresponding mode. This figure is provided by Valeri Vardanyan, an author of* [1].

Now that we presented the numerical solution for the dynamics of first order scalar perturbations ($\mathcal{R}$ and φ) we can estimate the fraction of the generated PBHs. We can indeed compute the PBH abundance using the numerical solutions provided by [1].

In order to estimate $f_{\text{PBH}} = \frac{\Omega_{\text{PBH}}}{\Omega_{\text{DM}}}$ we refer to our discussions of Section **2.1.2**. As mentioned in Section **2.1.2**, this is the most basic way to estimate the abundance of PBH. Note that in Section **2.1.2** we assumed a monochromatic mass function while, as shown in **Figure 4.3**, in our case we do not necessarily obtain a narrow peak in the power spectrum. However, as a first approximation, we assume that the monochromatic assumption is still valid. Thus, after computing the power spectrum:

$$\left\langle \mathcal{R}(\mathrm{k})\mathcal{R}(\tilde{\mathrm{k}}) \right\rangle = (2\pi)^3 P_{\mathcal{R}}(\mathrm{k})\delta^3\left(\mathrm{k} - \tilde{\mathrm{k}}\right) \tag{4.30}$$

with the numerical evolution of $\mathcal{R}$ shown in **Figure 4.3**, we estimate σ with equation (2.16),

$$\sigma^2 = \frac{16}{81}\int_0^\infty \left(\frac{\tilde{\mathrm{k}}}{\mathrm{k}_{\text{piv}}}\right)^4 P_{\mathcal{R}}(\tilde{\mathrm{k}})W^2(\tilde{\mathrm{k}}, R)\frac{d\tilde{\mathrm{k}}}{\tilde{\mathrm{k}}}. \tag{4.31}$$

To estimate σ we take $R = \frac{1}{\mathrm{k}_{\text{piv}}}$ and we assume that $W^2(\mathrm{k}, R)$ is a Gaussian window function.

Having obtained the variance we can compute the PBH abundance β using equation (2.15) and finally compute f_{PBH}

$$f_{\text{PBH}} = \frac{\Omega_{\text{PBH}}}{\Omega_{\text{DM}}} = 2.7 \times 10^8 \left(\frac{\gamma}{0.2}\right)^{1/2} \left(\frac{g_{*,\,\text{form}}}{10.75}\right)^{-1/4} \left(\frac{M_{\text{PBH}}}{M_\odot}\right)^{-1/2} \beta, \tag{4.32}$$

where M_{PBH} is estimated using equation (2.13).

Due to the natural dependence of f_{PBH} on the initial conditions of the radion, and thus on k_{piv} and α_{peak}, we can choose to produce PBHs with masses within an unconstrained range. This model can generate PBHs with arbitrarily small or arbitrarily large masses by tuning the initial conditions of φ_{k}.

Chapter 5

Gravitational wave production

In this section we present the main original contribution of this work: the study of second order gravitational waves induced by scalar perturbations in the model presented in Section **3**.

Before presenting how we computed the gravitational waves equation of motion in Section **5.2.1** and the resulting power spectrum in Section **5.2.2**, we go back to the gauge problem discussed in Section **2.2.1**. In Section **5.1** we study in more detail how to change the gauge at first and second orders, while in section **5.1.2** we review the gauge dependence of second order tensor perturbations.

5.1 Gauge changing

In this section we review, following [33] and [45], the method with which we can change the gauge in the perturbed metric up to second order. We then explicitly compute the relation between the Newtonian gauge mentioned in Section **2.2.2** and the equipotential gauge used in Section **4.2**. Finally we discuss the gauge dependence of the second order scalar-induced gravitational waves.

5.1.1 First and second order gauge changing

Up to now we did not need to use equation (2.25) to change the gauge choice. In the study of the dynamics of scalar perturbations, both in the previous section and in Section **2.2.2**, we fixed the gauge while being aware of the gauge dependence in our results. We now explicitly adapt equation (2.25) for the metric and we find how to write the scalar perturbations defined in the Newtonian gauge as a function of the perturbations in the equipotential gauge (equation (4.14)).

From equation (2.25) it follows that in order to write the first and second order perturbation of the metric in a different gauge we need to find ξ_1 and ξ_2 such that

$$\begin{aligned} \delta\tilde{g}_{\mu\nu} &= \delta g_{\mu\nu} + \pounds_{\xi_1} g^{(0)}_{\mu\nu}, \\ \delta^2\tilde{g}_{\mu\nu} &= \delta^2 g_{\mu\nu} + \pounds_{\xi_2} g^{(0)}_{\mu\nu} + \pounds^2_{\xi_1} g^{(0)}_{\mu\nu} + 2\pounds_{\xi_1}\delta g^{(0)}_{\mu\nu}. \end{aligned} \tag{5.1}$$

Thus, extending the definition of the Lie derivative (2.19) for a generic tensor field T of rank (k,l),

$$\begin{aligned}\mathcal{L}_\xi T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} &= \xi^\sigma\nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \\ &+ (\nabla_{\nu_1}\xi^\alpha)\, T^{\mu_1\cdots\mu_k}{}_{\alpha\cdots\nu_l} + \ldots + (\nabla_{\nu_l}\xi^\alpha)\, T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\alpha} \\ &- (\nabla_\alpha\xi^{\mu_1})\, T^{\alpha\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - \ldots - (\nabla_\alpha\xi^{\mu_k})\, T^{\mu_1\cdots\alpha}{}_{\nu_1\cdots\nu_l},\end{aligned} \tag{5.2}$$

we find that the quantities appearing in the definition (2.28) transform at first order as[18]

$$\begin{aligned}\widetilde{\Psi} &= \Psi + \alpha_1' + \mathcal{H}\alpha_1, \\ \widetilde{\omega}_i^\perp &= \omega_i^\perp + d_{i,1}', \\ \widetilde{\omega}^\parallel &= \omega^\parallel - \alpha_1 + \beta_1', \\ \widetilde{\Phi} &= \Phi - \frac{1}{3}\nabla_i\nabla^i\beta_1 - \mathcal{H}\alpha_1, \\ \widetilde{h}_{ij}^\top &= h_{ij}^\top, \\ \widetilde{h}_i^\perp &= h_i^\perp + d_{i,1}, \\ \widetilde{h}\| &= h\| + 2\beta_1.\end{aligned} \tag{5.3}$$

In the above equations we split the temporal and spatial parts of ξ_1 as

$$\begin{aligned}\xi_r^0 &= \alpha_r, \\ \xi_r^i &= \nabla^i\beta_r + d_r^i,\end{aligned} \tag{5.4}$$

where $d_{i,r}$ satisfies $\partial^i d_{i,r}$. Moreover at second order they transform as

$$\begin{aligned}\tilde{\Psi}^{(2)} = \;& \Psi^{(2)} + \alpha_1\left[2\left(\Psi'^{(1)} + 2\tfrac{a'}{a}\Psi^{(1)}\right) + \alpha_1'' + 5\tfrac{a'}{a}\alpha_1' + \left(\tfrac{a''}{a} + \tfrac{a'^2}{a^2}\right)\alpha_1\right] \\ & + \xi_1^i\left(2\Psi^{(1)}_{,i} + \alpha'_{,i1} + \tfrac{a'}{a}\alpha_{,i1}\right) + 2\alpha_1'\left(2\Psi^{(1)} + \alpha_1'\right) \\ & + \xi_1^{i\prime}\left(\alpha_{,i1} - \xi_{i1}' - 2\omega_i^{(1)}\right) + \alpha_2' + \tfrac{a'}{a}\alpha_2, \\ \tilde{\omega}_i^{(2)} = \;& \omega_i^{(2)} - 4\Psi^{(1)}\alpha_{,i1} + \alpha_1\left[2\left(\omega_i^{(1)\prime} + 2\tfrac{a'}{a}\omega_i^{(1)}\right) - \alpha'_{,i1} + \xi''_{i1} - 4\tfrac{a'}{a}\left(\alpha_{,i1} - \xi'_{i1}\right)\right] \\ & + \xi_1^j\left(2\omega^{(1)}_{i,j} - \alpha_{,ij1} + \xi'_{i,j1}\right) + \alpha_1'\left(2\omega_i^{(1)} - 3\alpha_{,i1} + \xi'_{i1}\right) \\ & + \xi_1^{j\prime}\left(-4\Phi^{(1)}\delta_{ij} + 2h_{ij}^{(1)} + 2\xi_{j,i1} + \xi_{i,j1}\right) + \xi^j_{1,i}\left(2\omega_j^{(1)} - \alpha_{,j1}\right) - \alpha_{,i2} + \xi'_{i2}, \\ \tilde{\Phi}^{(2)} = \;& \Phi^{(2)} + \alpha_1\left[2\left(\Phi'_{(1)} + 2\tfrac{a'}{a}\Phi_{(1)}\right) - \left(\tfrac{a''}{a} + \tfrac{a'^2}{a^2}\right)\alpha_1 - \tfrac{a'}{a}\alpha_1'\right] + \xi_1^i\left(2\Phi^{(1)}_{,i} - \tfrac{a'}{a}\alpha^{(1)}_{,i}\right) \\ & - \tfrac{1}{3}\left(-4\Phi_{(1)} + \alpha_1\partial_0 + \xi^i_{(1)}\partial_i + 4\tfrac{a'}{a}\alpha_1\right)\nabla_i\nabla^i\beta_1 - \tfrac{1}{3}\left(2\omega^i_{(1)} - \alpha_1^i + \xi_1^{i\prime}\right)\alpha_{,i1} \\ & - \tfrac{1}{3}\left(2h_{ij}^{(1)} + \xi_{i,j1} + \xi_{j,i1}\right)\xi_1^{j,i} - \tfrac{a'}{a}\alpha_2 - \tfrac{1}{3}\nabla_i\nabla^i\beta_2,\end{aligned} \tag{5.5}$$

[18]In equation (5.5) we make explicit the order of perturbations as we mix the first and second order perturbations, while in equation (5.3) we do not specify the order because we are only referring to first order perturbation quantities.

$$\begin{aligned}\tilde{h}_{ij}^{(2)} = \; & h_{ij}^{(2)} + 2\left(h_{ij}^{(1)\prime} + 2\frac{a'}{a}h_{ij}^{(1)}\right)\alpha_1 + 2h_{ij,k}^{(1)}\xi_1^k \\ & +2\left(-4\Phi_{(1)} + \alpha_1\partial_0 + \xi_1^k\partial_k + 4\frac{a'}{a}\alpha_1\right)\left(d_{(i,j)1} + \mathrm{D}_{ij}\beta_1\right) \\ & +2\left[\left(2\omega_{(i}^{(1)} - \alpha_{,(i1} + \xi'_{(i1}\right)\alpha_{jj1} - \frac{1}{3}\delta_{ij}\left(2\omega_{(1)}^k - \alpha_1^k + \xi_1^{k\prime}\right)\alpha_{,k1}\right] \\ & +2\left[\left(2h_{(i|k|}^{(1)} + \xi_{k,(i1} + \xi_{(i,|k|}^{(1)}\right)\xi_{,j)1}^k - \frac{1}{3}\delta_{ij}\left(2h_{lk}^{(1)} + \xi_{k,l1} + \xi_{l,k1}\right)\xi_1^{k,l}\right]\right] \\ & +2\left(d_{(i,j)2} + \mathrm{D}_{ij}\beta_2\right),\end{aligned} \tag{5.6}$$

where, as we take this result from [33], we keep a consistent notation: $\Phi_{,\mu} = \partial_\mu\Phi$, $\xi_{(i}\omega_{j)} = \frac{1}{2}(\xi_i\omega_j + \xi_j\omega_i)$ and we do not split the quantities appearing in the the definition (2.28) into scalar, vector and tensor parts as we did in (5.3).

It is interesting to notice that from (5.3) what we assumed in Section **2.3** automatically follows: $h_{ij}^{\top}$ is a gauge-invariant quantity at first order. However, it is not true for the first order scalar and vector perturbations and it is no longer true for the second order tensor perturbations, as shown in equation (5.5).

We do not explicitly report the first and second order gauge transformations of the stress-energy tensor because in the model presented in Section **3** we used the Lagrangian formalism to describe the matter content of the Universe. Thus, in order to change the gauge of the radiation field, we just need to adapt equation (2.25) to the case of a scalar field. To see how the stress-energy tensor changes under gauge transformations see [45].

From Newtonian gauge to equipotential gauge

We now explicitly use equations (5.3) to link two different gauges. We want to find the vector ξ_1 such that the perturbed metric

$$ds^2 = -a^2(1+2\Phi)d\eta^2 + a^2(1-2\Psi)\delta_{ij}dx^i dx^j \tag{5.7}$$

can be mapped onto the metric we used in Section **4.2**,

$$ds^2 = -N^2dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right). \tag{5.8}$$

In the latter metric we recall that h_{ij} is defined as $h_{ij} = a^2 e^{2\mathcal{R}}\delta_{ij}$, and N and N_i depend on $\mathcal{R}$ and φ, as shown in equations (4.16) and (4.17). The perturbed metric (5.7) is written in the Newtonian gauge: it is indeed equal to the metric (2.40) except that we change the notation ($\Phi \longrightarrow \Psi$ and $\Psi \longrightarrow \Phi$). This is because, as we will see in Section **5.2.1**, in this section we are maintaining the notation of the Mathematica package we used to compute the equation of motion of the tensor perturbations. Furthermore, as we have shown in Section **2.2.2**, in standard general relativity we have $\Psi = \Phi$, thus, in principle, we cannot distinguish them. In addition, in (5.7) and (5.8) we are not considering tensor perturbations, as we did in (2.40). However, in the latter case we justified this assumption by saying that in order to study the dynamics of scalar perturbations we are allowed to neglect tensor perturbations, while in this case we are neglecting $h_{ij}^{\perp}$ because they do not change under first order gauge transformations. In both (5.7) and (5.8) we assume that vector perturbations are switched off (this is not a gauge choice, we just neglect them as we did in Section **2.2.2**).

To find ξ_1, we need to solve (5.3) by inserting (5.7) and (5.8): the LHS refers to the Newtonian gauge, Φ and Ψ, while in the RHS we have to insert the perturbations of (5.8), $\mathcal{R}$ and φ. Solving the system for $\xi = (\alpha, \nabla^i\beta + d^i)$ we find that from tensor and vector gauge transformations we obtain that the vector part of ξ must be zero, while from scalars transformations we find α, Ψ and Φ as functions of $\mathcal{R}$ and φ. The result is

$$\begin{aligned} d_i &= \beta = 0, \\ \alpha &= a\Theta, \\ \Phi &= \alpha_1 + 2a'\Theta + a\Theta', \\ \Psi &= -\mathcal{R} - a'\Theta, \end{aligned} \tag{5.9}$$

where α_1 and Θ are defined in equations (4.17).

This relation between Ψ, Φ and $\mathcal{R}$, φ will be useful in the next section where we first study the evolution of second order tensor perturbations in the Newtonian gauge and then we intend to express our result in the equipotential gauge.

Let us also mention how to deal with gauge changing of radiation perturbations in the $P(\chi)$ formalism. We first label the radiation field ψ in the action (3.10) as σ. This is done to avoid confusion between the radiation field and the perturbation Ψ. In that way the action becomes

$$\begin{aligned} S =& \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R^{(4)}(g) + \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) \\ &+ \int d^4x\sqrt{-g}e^{\frac{\sigma}{M_{\rm Pl}}}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma\right)^2. \end{aligned} \tag{5.10}$$

Then we recall that in order to change the gauge for a generic scalar field $\Upsilon = \Upsilon_0 + \delta\Upsilon$ we have to find the α that satisfies

$$\delta\widetilde{\Upsilon} = \delta\Upsilon + \Upsilon_0'\alpha. \tag{5.11}$$

Thus, using (5.11), we find that the radiation perturbation in the Newtonian gauge $\sigma = \sigma_0 + \varsigma$ as a function of the unperturbed radiation (4.14) in the equipotential gauge ($\sigma = \sigma_0$) becomes

$$\varsigma = \sigma_0' a\Theta = \frac{\sigma_{0,\rm in}'}{a}e^{\frac{-\phi_0}{3M_{\rm Pl}}}a\Theta. \tag{5.12}$$

In the second equality of (5.12) we use the background solution (4.13) and in this formalism $\sigma_{0,\rm in}'$ is the initial condition.

Before discussing in more detail the gauge invariance of second order perturbations, we mention a substantial difference which is easily understood in the Newtonian gauge, between our model and the general relativity case. In Section **2.2.2**, from equation (2.46) we can see that the traceless and transverse component of the Einstein equations yields the so-called *slip relation*: $\Phi = \Psi$ [46]. This equality is not longer true in our model; in fact, from (5.9) it is easy to see that $\Phi \neq \Psi$. However, it is possible to show that in the absence of the coupling between the radion and the radiation fluid we recover the general relativity result $\Phi = \Psi$. This difference in the slip relation is a general result for modified gravity models and it is well-studied in the literature both from the theoretical and observational points of view (see [47] for an overview of modified gravity theories and observational constraints).

5.1.2 Second order perturbations and gauge dependence

In this section we intend to study the gauge dependence of second order gravitational waves induced by scalar perturbations. Even if, as previously shown, second order tensor perturbations are not gauge-independent, in [48] and [49] the authors show that the observed scalar-induced gravitational waves are identical in Newtonian, synchronous and uniform curvature gauges.

We now briefly present how to calculate the energy density spectra of scalar-induced gravitational waves in the Newtonian, synchronous and uniform curvature gauges following [48]. Since in this section we are only interested in the results presented in [48] and [49], we will present the full derivation and solution of the dynamics of scalar induced tensor perturbations in the next section.

The energy density spectrum of scalar-induced gravitational waves Ω_{gw} is defined as:

$$\Omega_{\mathrm{gw}}(f) = \frac{1}{\rho_{\mathrm{c}}} \frac{d\rho_{\mathrm{gw}}}{d\log(f)}, \tag{5.13}$$

where $\rho_{\mathrm{c}} = \frac{3H_0^2}{8\pi G}$ is the critical density, ρ_{gw} is the gravitational waves' energy density and $f = 2\pi\mathrm{k}$ is the frequency. ρ_{gw} can be expressed as a function of the gravitational waves amplitude (see [50] for more details),

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G} \left\langle \dot{h}_{ij}^{(2)\top} \dot{h}_{ij}^{(2)\top} \right\rangle. \tag{5.14}$$

Then, using the same procedure we used in Section **2.3** and in particular equation (2.56), we can write $h_{ij}^{(2)\top}$ using spherical coordinates as (see [51] for more details)

$$h_{ij}^{(2)\top}(x,t) = \sum_{\lambda=(+,\times)} \int_{-\infty}^{+\infty} df \int_{S^2} d\hat{\Omega} h_{(\lambda)}^{(2)\top}(f,\hat{\Omega}) e^{i2\pi f(t-\hat{\Omega}\cdot x)} e_{ij}^{(\lambda)}(\hat{\Omega}). \tag{5.15}$$

In that way ρ_{gw} becomes

$$\rho_{\mathrm{gw}} = \frac{32\pi^5}{G} \int_0^{+\infty} df\, f^2 P(f), \tag{5.16}$$

where we have used the definition of the power spectrum

$$\left\langle h_{(\lambda)}^{(2)\top}(f,\hat{\Omega}) h_{(\lambda)}^{(2)\top}(f',\hat{\Omega}') \right\rangle = (2\pi)^3 \delta(f+f') \delta(\hat{\Omega},\hat{\Omega}') P(f) \tag{5.17}$$

to rewrite $\left\langle \dot{h}_{ij}^{(2)\top} \dot{h}_{ij}^{(2)\top} \right\rangle$. Thus, Ω_{gw} can now be written as a function of the power spectrum:

$$\Omega_{\mathrm{gw}}(f) = \frac{256\pi^3}{3H_0^2} f^3 P(f). \tag{5.18}$$

In conclusion, in order to estimate Ω_{gw} we need to find the amplitude of the induced gravitational waves. As we will see in the next section, the evolution of gravitational waves is governed by an equation of motion of the form:

$$(h_{ij}^{(2)\top})'' + 2\mathcal{H}(h_{ij}^{(2)\top})' - \nabla_k \nabla^k h_{ij}^{(2)\top} = -4\mathcal{T}_{ij}^{lm} S_{lm}, \tag{5.19}$$

where S_{lm} is a source term and it is a quadratic function of first order scalar perturbations and $\mathcal{T}_{ij}^{lm}$ is the projection operator onto transverse and traceless tensors.

As we will see in Section **5.2.2**, in order to solve (5.19) we can use the Green's function method so that the amplitude becomes

$$h^{(2)\top}(\eta, \mathrm{k}) = \frac{1}{a(\eta)} \int_0^\eta g_{\mathrm{k}}\left(\eta; \eta'\right) a\left(\eta'\right) S\left(\eta', \mathrm{k}\right) \mathrm{d}\eta', \tag{5.20}$$

where g_{k} is the Green's function and $S\left(\eta', \mathrm{k}\right)$ the source term in Fourier space. Thus, evaluating $h^{(2)\top}(\eta, \mathrm{k})$ with the above equation, we can compute the power spectrum and the resulting energy density spectra of scalar-induced gravitational waves Ω_{gw}. This is the procedure followed by C. Yuan *et al.* in [48] where they start from the source term written in three different gauges, Newtonian, synchronous and uniform curvature and then they compute Ω_{gw} in the three different cases. The source terms become respectively,

$$\begin{aligned}
S_{ij}^N = {} & 3\Psi\nabla_i\nabla_j\Psi - \tfrac{2}{\mathcal{H}}\nabla_i\Psi\nabla_j\Psi - \tfrac{1}{\mathcal{H}^2}\nabla_i\Psi\nabla_j\Psi, \\
S_{ij}^S = {} & -3\Phi'\nabla_i\nabla_j(h^{\parallel\prime} - \omega^{\parallel}) + \Phi\nabla_i\nabla_j\Phi - \tfrac{1}{\mathcal{H}^2}\left(\nabla_i\Phi'\right)\left(\nabla_j\Phi'\right) \\
& + \left(\nabla_k\nabla^k(h^{\parallel\prime} - \omega^{\parallel})\right)\left(\nabla_i\nabla_j(h^{\parallel\prime} - \omega^{\parallel})\right) - \left(\nabla^k\nabla_i(h^{\parallel\prime} - \omega^{\parallel})\right)\left(\nabla_k\nabla_j(h^{\parallel\prime} - \omega^{\parallel})\right), \\
S_{ij}^U = {} & \left(\nabla_k\nabla^k\omega^{\parallel}\right)\left(\nabla_i\nabla_j\omega^{\parallel}\right) - \left(\nabla_j\nabla_i\omega^{\parallel}\right)\left(\nabla_i\nabla_j\omega^{\parallel}\right) + 4\mathcal{H}\Psi\nabla_i\nabla_j\omega^{\parallel} + \phi\nabla_i\nabla_j\omega^{\parallel} \\
& + 2\Psi\nabla_i\nabla_j\omega^{\parallel\prime} + 2\Psi\nabla_i\nabla_j\Psi,
\end{aligned} \tag{5.21}$$

where we keep the notation of the perturbed metric (2.28) defined in Section **2.2.2** as we are working with standard general relativity and we drop the $^{(1)}$ as the perturbation quantities on the RHS are all first order perturbations. The resulting Ω_{gw} has the same form in the three different cases,

$$\begin{aligned}
\Omega_{\mathrm{gw}}(\mathrm{k}) &= \frac{1}{24}\left(\frac{\mathrm{k}}{\mathcal{H}}\right)^2 \overline{P_h(\mathrm{k})} \\
&= \left(\frac{\mathrm{k}}{\mathcal{H}}\right)^2 \frac{\mathrm{k}^3}{48\pi^2 a(\eta)^2} \int \mathrm{d}\tilde{\eta}_1 \mathrm{d}\tilde{\eta}_2 g_{\mathrm{k}}\left(\eta; \tilde{\eta}_1\right) g_{\mathrm{k}'}\left(\eta; \tilde{\eta}_2\right) a\left(\tilde{\eta}_1\right) a\left(\tilde{\eta}_2\right) \left\langle S\left(\mathrm{k}, \tilde{\eta}_1\right) S\left(\mathrm{k}', \tilde{\eta}_2\right)\right\rangle \\
&= \frac{1}{6}\int_0^\infty \mathrm{d}u \int_{|1-u|}^{1+u} \mathrm{d}v \frac{v^2}{u^2}\left[1 - \left(\frac{1+v^2-u^2}{2v}\right)^2\right]^2 P_\Phi(u\mathrm{k}) P_\Phi(v\mathrm{k}) \overline{I_{\mathrm{S}}^2(u, v, x \to \infty)},
\end{aligned} \tag{5.22}$$

where u and v are integration variables defined from the wavenumbers k and p that results from the convolution of the source term, P_Φ is the power spectrum of scalar perturbations, the overline denotes the oscillating average and $\overline{I_{\mathrm{S}}}$ is defined in [48] and encodes the evolution of the primordial power spectrum taking into account the evolution of scalar perturbations through the radiation- and matter-dominated epochs.

We do not go through the full computations of [48] because in this section we just want to stress the fact that, even if the source terms (5.21) are explicitly gauge-dependent, the observable Ω_{gw} is gauge-independent. This possibility of gauge invariance of Ω_{gw}, confirmed by [49], is an important result for the the computations presented in the next section. In Section **5.2** we compute the power spectrum of scalar-induced gravitational waves in the non-GR model presented in Section **3**. The result is in principle gauge-dependent, but, as mentioned in this section, it could be instead general and well defined.

5.2 Gravitational wave production

In this section we study second order tensor perturbations in the model presented in Section **3**.

We essentially follow the procedure presented in Section **2.3**. We start from the action (3.10), we find the equations of motion for the metric and we perturb them up to second order in the Newtonian gauge. We then isolate the equation of motion for second order tensor perturbations and we solve them with the Green's function method. However, the resulting equation of motion, as we will see in Section **5.2.1**, is different from (2.54). In (2.54) we do not have any source term and the RHS is null. This is in general true at first order because at first order the evolutions of scalar, vector and tensor perturbations independent.

However, the most important phenomenon of second-order perturbation theory is mode mixing: at second order the scalar perturbations couple to the tensor perturbations. Thus, large scalar perturbations also produce the so-called scalar-induced gravitational waves ([33]). In this section we study gravitational waves induced by the curvature and radion perturbations studied in Section **4.2**. We expect that the described enhancement of curvature perturbations, not only allows the formation of PBHs, but also acts as a source for tensor modes, producing a detectable gravitational waves power spectrum.

As mentioned in Section **5.1**, the specific form of these waves is gauge-dependent, as tensor modes are no longer gauge-invariant beyond the linear level. We will work in a gauge different from the equipotential gauge, the gauge used in Section **4.2**. Thus, before proceeding with the computation of the power spectrum in Section **5.2.2**, we will use the results of Section **5.1.1** to change the gauge and work in the equipotential gauge. However, as mentioned in the previous section, it is not clear whether the result is indeed gauge-dependent.

5.2.1 Equation of motion

To study the dynamics of second order tensor perturbations we do not start perturbing directly the action, as we did for scalar perturbations; we first find the equation of motion. Thus, varying the action (5.10),

$$\begin{aligned} S = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R^{(4)}(g) + \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) \\ + \int d^4x\sqrt{-g}e^{\frac{\sigma}{M_{\rm Pl}}}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma\right)^2, \end{aligned} \tag{5.23}$$

we find the equation of motion

$$\begin{aligned} -\frac{1}{2}M_{\rm Pl}^2R_{\mu\nu} + \frac{1}{4}M_{\rm Pl}^2g_{\mu\nu}R + \frac{1}{4}M_{\rm Pl}^2\nabla_\beta\phi\nabla^\beta\phi g_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}e^{\frac{\sigma}{M_{\rm Pl}}}\left(\nabla_\beta\sigma\nabla^\beta\sigma\right)^2 g_{\mu\nu} \\ - 2e^{\frac{\sigma}{M_{\rm Pl}}}\nabla_\beta\sigma\nabla^\beta\sigma\nabla_\mu\sigma\nabla_\nu\sigma + \left(\frac{M_{\rm Pl}^2}{2b_*^2}e^{-\frac{2\sigma}{M_{\rm Pl}}} - \frac{\pi\hat{f}^2}{b_*^2}e^{-\frac{3\sigma}{M_{\rm Pl}}}\right)g_{\mu\nu} = 0. \end{aligned} \tag{5.24}$$

Derivation of the above equation of motion and the following computations to expand the equation of motion up to the second order in perturbation theory have been done with Mathematica, using the xAct package[19] (more precisely the application package 'xPand').

We then choose the Newtonian gauge ($\omega^{(r)\|} = h^{(r)\|} = 0$, $r = (1, 2)$) and neglect first order vector and tensor perturbations. With these choices the perturbed metric takes the form

$$ds^2 = a^2 \Big[-(1 + 2\Phi^{(1)} + 2\Phi^{(2)}) d\eta^2 - \omega_i^{(2)\perp} d\eta dx^i + (1 - 2\Psi^{(1)} - 2\Psi^{(2)}) \delta_{ij} dx^i dx^j + \frac{1}{2} h_{ij}^{(2)\top} dx^i dx^j \Big], \tag{5.25}$$

where we use the same notation of (5.7) for the scalars perturbations in order to be consistent with the results in **Appendix**, while, for tensor and vector perturbations, we keep the notation of (2.40) in order to be consistent with the results in the literature (i.e., [45] and [53]). Indeed, in the perturbed metric given in **Appendix** we have $h_{ij}^{(2)\top} = 4E_{ij}^{(2)}$ and $\omega_i^{(2)\perp} = B_i^{(2)}$.

We then expand the equation of motion (5.24) up to the second order, finding the result called **SecondOrdEqMotion** reported in **Appendix**. Since we want to study the dynamics of scalar induced second order tensor perturbations $h_{ij}^{(2)\top}$, the result **SecondOrdEqMotion** can be simplified. Indeed, as we want to study $h_{ij}^{(2)\top}$, which is transverse and traceless, we can neglect the trace part of the (i, j) components of the perturbed equation of motion. Furthermore, we drop those terms that depend on second order perturbation quantities as we want to study the second order gravitational waves induced by first order perturbation quantities. After these simplifications we obtain the result called **FinalResult** in **Appendix**,

$$\begin{aligned} h_{ij}'' + 2\mathcal{H} h_{ij}' - \nabla_k \nabla^k h_{ij} = &- 4\hat{\mathcal{T}}_{ij}^{lm} \Big(4\Phi \nabla_l \nabla_m \Phi - 4\Psi \nabla_l \nabla_m \Phi + 8\Psi \nabla_l \nabla_m \Psi + 2\nabla_l \Phi \nabla_m \Phi \\ &- 2\nabla_l \Phi \nabla_m \Psi - 2\nabla_l \Psi \nabla_m \Phi + 6\nabla_l \Psi \nabla_m \Psi + \frac{2}{M_{\rm Pl}^2} \nabla_l \varphi \nabla_m \varphi \\ &- \frac{8}{a^2 M_{\rm Pl}^2} e^{\frac{\phi_0}{M_{\rm Pl}}} \sigma_0'^2 \nabla_l \varsigma \nabla_m \varsigma \Big), \end{aligned} \tag{5.26}$$

where $\mathcal{T}_{ij}^{lm}$ is the projection operator onto transverse and traceless tensors and from now on we denote $h_{ij}^{(2)\top}$ simply as h_{ij}.

In conclusion, the equation of motion of second order tensor perturbation is of the form (5.19):

$$h_{ij}'' + 2\mathcal{H} h_{ij}' - \nabla_k \nabla^k h_{ij} = -4\mathcal{T}_{ij}^{lm} S_{lm}, \tag{5.27}$$

as we anticipated in Section **5.1.2**. In this case the source term S_{lm} becomes

$$\begin{aligned} S_{lm} =& 4\Phi \nabla_l \nabla_m \Phi - 4\Psi \nabla_l \nabla_m \Phi + 8\Psi \nabla_l \nabla_m \Psi + 2\nabla_l \Phi \nabla_m \Phi - 2\nabla_l \Phi \nabla_m \Psi - 2\nabla_l \Psi \nabla_m \Phi \\ &+ 6\nabla_l \Psi \nabla_m \Psi + \frac{2}{M_{\rm Pl}^2} \nabla_l \varphi \nabla_m \varphi - \frac{8}{a^2 M_{\rm Pl}^2} e^{\frac{\phi_0}{M_{\rm Pl}}} \sigma_0'^2 \nabla_l \varsigma \nabla_m \varsigma. \end{aligned} \tag{5.28}$$

[19] http://www.xact.es/

This result is in agreement with the source term presented in [45] up to a rescaling: the first seven terms are the standard general relativity result reported in the literature, while the last two terms are the new contributions due to the modifications of general relativity introduced in Section **3**.

5.2.2 Power spectrum

We now solve the equation of motion (5.26) in a radiation-dominated Universe and estimate the power spectrum of scalar induced second order gravitational waves.

We start by defining the Fourier transform of second order tensor metric perturbations as

$$h_{ij}(x,\eta) = \int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^{3/2}} e^{i\mathrm{k}\cdot x}\left[h_{\mathrm{k}}(\eta)\mathrm{e}_{ij}(\mathrm{k}) + \bar{h}_{\mathrm{k}}(\eta)\bar{\mathrm{e}}_{ij}(\mathrm{k})\right], \tag{5.29}$$

where e_{ij} and $\bar{\mathrm{e}}_{ij}$ are the two polarization tensors, and $h_{\mathrm{k}}(\eta)$ and $\bar{h}_{\mathrm{k}}(\eta)$ are the amplitudes. We then rewrite the RHS of equation (5.26) as

$$\hat{\mathcal{T}}_{ij}^{lm}\mathcal{S}_{lm}(x,\eta) = \int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^{3/2}} e^{i\mathrm{k}\cdot x}\left[\mathrm{e}_{ij}(\mathrm{k})\mathrm{e}^{lm}(\mathrm{k}) + \bar{\mathrm{e}}_{ij}(\mathrm{k})\bar{\mathrm{e}}^{lm}(\mathrm{k})\right]\mathcal{S}_{lm}(\mathrm{k}), \tag{5.30}$$

where $\mathcal{S}_{lm}(\mathrm{k})$ is the Fourier transformed source term

$$\mathcal{S}_{lm}(\mathrm{k}) = \int \frac{\mathrm{d}^3x'}{(2\pi)^{3/2}} e^{-i\mathrm{k}\cdot x'}\mathcal{S}_{lm}(x'). \tag{5.31}$$

Thus, in Fourier space the equation of motion for the gravitational waves amplitude $h_{\mathrm{k}}(\eta)$ for both polarizations becomes

$$h''(\mathrm{k},\eta) + 2\mathcal{H}h'(\mathrm{k},\eta) + \mathrm{k}^2 h(\mathrm{k},\eta) = \mathcal{S}(\mathrm{k},\eta). \tag{5.32}$$

$\mathcal{S}(\mathrm{k},\eta)$ is the convolution of the two first-order scalar perturbations at different wavenumbers k and p,[20]:

$$\begin{aligned}\mathcal{S}(\mathrm{k},\eta) =& -4\mathrm{e}^{lm}\mathcal{S}_{lm}(\mathrm{k},\eta)\\ =&4\int \frac{d^3\mathrm{p}}{(2\pi)^{\frac{3}{2}}}\mathrm{e}^{lm}\mathrm{p}_l\mathrm{p}_m\Big[2\Phi(\mathrm{p},\eta)\Phi(\mathrm{k}-\mathrm{p},\eta) + 2\Psi(\mathrm{p},\eta)\Psi(\mathrm{k}-\mathrm{p},\eta)\\ &-\frac{2}{M_{\mathrm{Pl}}^2}\varphi(\mathrm{p},\eta)\varphi(\mathrm{k}-\mathrm{p},\eta) + \frac{8\sigma_{0,\mathrm{in}}'^2}{a^4M_{\mathrm{Pl}}^2}e^{\frac{\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\varsigma(\mathrm{p},\eta)\varsigma(\mathrm{k}-\mathrm{p},\eta)\Big].\end{aligned} \tag{5.33}$$

We notice that equation (5.33) is in agreement with the results of [53], where the authors use the first order Einstein equations in order to rewrite the perturbation of radiation ς as a function of the metric perturbations Φ and Ψ. Because of this substitution the correspondence is not immediate; however, we have checked that the standard-GR terms of (5.33) correspond to the known results. However, as specified in the equation of motion, we have in addition the radion contribution that adds non-standard terms to the source term (5.33).

[20]We recall that by definition $\mathrm{k}^i\mathrm{e}_{ij} = 0$.

We can now easily solve (5.32) with the Green's function method. We define g_{k} such that it solves

$$g_{\mathrm{k}}'' + \left(\mathrm{k}^2 - \frac{a''}{a}\right) g_{\mathrm{k}} = \delta(\eta - \tilde{\eta}). \tag{5.34}$$

Then, multiplying both the RHS and the LHS by the source term (5.33), we find that the amplitude becomes

$$h(\mathrm{k}\eta) = \frac{1}{a(\eta)} \int \mathrm{d}\tilde{\eta} g_{\mathrm{k}}(\eta, \tilde{\eta})[a(\tilde{\eta})\mathcal{S}(\mathrm{k}, \tilde{\eta})]. \tag{5.35}$$

The solution of equation (5.34) in a radiation-dominated universe (where $a \approx \eta$) results in (see [45] for the full computations and the result in a matter-dominated universe)

$$g_{\mathrm{k}}(\eta, \tilde{\eta}) = \frac{1}{\mathrm{k}}[\sin(\mathrm{k}, \eta)\cos(\mathrm{k}, \tilde{\eta}) - \sin(\mathrm{k}, \tilde{\eta})\cos(\mathrm{k}, \eta)], \quad \eta < \eta_{\mathrm{eq}}. \tag{5.36}$$

In conclusion, we can express the power spectrum of the second order gravitational waves[21] as a function of first order quantities. Indeed, we obtain

$$\langle h(\mathrm{k}, \eta) h(\mathrm{K}, \eta)\rangle = \frac{1}{a^2(\eta)} \int_{\eta_0}^{\eta} \mathrm{d}\bar{\eta}_2 \int_{\eta_0}^{\eta} \mathrm{d}\bar{\eta}_1 a\left(\bar{\eta}_1\right) \mathrm{a}\left(\bar{\eta}_2\right) g_{\mathrm{k}}\left(\eta, \bar{\eta}_1\right) g_{\mathrm{K}}\left(\eta, \bar{\eta}_2\right) \left\langle \mathcal{S}\left(\mathrm{k}, \bar{\eta}_1\right) \mathcal{S}\left(\mathrm{K}, \bar{\eta}_2\right)\right\rangle . \tag{5.38}$$

To obtain a more explicit expression for the power spectrum we should know the analytical behaviour of first order scalar perturbations so that we are able to estimate the (k, η) dependence of the source term in equation (5.38). As discussed in **4.2**, up to now the linear order is solved numerically.

However, even if we cannot proceed with the estimation of the power spectrum, we now try to explicitly write the dependence of the power spectrum on dynamics and initial conditions of first order scalar perturbations, so that once we obtain an analytical solution of the first order equations of motion, we can easily estimate the power spectrum.

We start writing the source term (5.33) in the equipotential gauge. Thus, using the relations found in Section **5.1.1**

$$\begin{aligned} \Phi &= \alpha_1 + 2a'\Theta + a\Theta', \\ \Psi &= -\mathcal{R} - a'\Theta, \\ \varsigma &= \sigma_0' a\Theta = \frac{\sigma_{0,\mathrm{in}}'}{a} e^{\frac{-\phi_0}{3M_{\mathrm{Pl}}}} a\Theta, \end{aligned} \tag{5.39}$$

[21] We recall that the power spectrum $P_h(\mathrm{k}, \eta)$ of $h(\mathrm{k}, \eta)$ is defined as

$$\langle h(\mathrm{k}, \eta) h(\mathrm{K}, \eta)\rangle = \frac{2\pi^2}{\mathrm{k}^3} \delta(\mathrm{k} + \mathrm{K}) P_h(\mathrm{k}, \eta). \tag{5.37}$$

we find that (5.33) in the equipotential gauge becomes

$$\begin{aligned}\mathcal{S}(\mathrm{k},\eta) = \ & 4\int \tfrac{d^3\mathrm{p}}{(2\pi)^{\frac{3}{2}}}\mathrm{e}^{lm}\mathrm{p}_l\mathrm{p}_m\Bigg[\tfrac{2}{\mathcal{H}^2}\mathcal{R}'(\mathrm{p},\eta)\mathcal{R}'(\mathrm{k}-\mathrm{p},\eta)+2\mathcal{R}(\mathrm{p},\eta)\mathcal{R}(\mathrm{k}-\mathrm{p},\eta)\\ &+4a'\mathcal{R}(\mathrm{p},\eta)\Theta(\mathrm{k}-\mathrm{p},\eta)+\tfrac{2\phi_0'(\eta)}{\mathcal{H}^2M_{\mathrm{Pl}}^2}\mathcal{R}'(\mathrm{p},\eta)\varphi(\mathrm{k}-\mathrm{p},\eta)\\ &+\left(\tfrac{\phi_0'^2(\eta)}{2\mathcal{H}^2M_{\mathrm{Pl}}^4}-\tfrac{2}{M_{\mathrm{Pl}}^2}\right)\varphi(\mathrm{p},\eta)\varphi(\mathrm{k}-\mathrm{p},\eta)+\left(10a'^2+\tfrac{8\sigma_{0,\mathrm{in}}'^4}{a^4M_{\mathrm{Pl}}^2}e^{\frac{-\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\right)\Theta(\mathrm{p},\eta)\Theta(\mathrm{k}-\mathrm{p},\eta)\\ &+2a^2\Theta'(\mathrm{p},\eta)\Theta'(\mathrm{k}-\mathrm{p},\eta)+8aa'\Theta(\mathrm{p},\eta)\Theta'(\mathrm{k}-\mathrm{p},\eta)\\ &+4\left(\tfrac{\mathcal{R}'(\mathrm{p},\eta)}{\mathcal{H}}+\tfrac{\phi_0'}{2\mathcal{H}M_{\mathrm{Pl}}^2}\varphi(\mathrm{p},\eta)\right)\Big(a\Theta'(\mathrm{k}-\mathrm{p},\eta)+2a'\Theta(\mathrm{k}-\mathrm{p},\eta)\Big)\Bigg].\end{aligned} \tag{5.40}$$

Then, we want to split the source term $\mathcal{S}(\mathrm{k},\eta)$ into a background part and a transfer function. In that way we explicitly have (5.38) depend on the primordial perturbations, and the solution of the first order equation of motion (4.23). We write $\mathcal{R}(\mathrm{p},\eta)=\widetilde{R}(\mathrm{p},\eta)\mathcal{R}^{\mathrm{pr}}$ and $\varphi(\mathrm{p},\eta)=\widetilde{\varphi}(\mathrm{p},\eta)\varphi^{\mathrm{pr}}$. $\widetilde{R}(\mathrm{p},\eta)$ and $\widetilde{\varphi}(\mathrm{p},\eta)$ are the transfer functions of $\mathcal{R}$ and φ and thus the solutions of (4.23). $\mathcal{R}^{\mathrm{pr}}$ and φ^{pr} are the primordial perturbations. In that way we can split the source term as

$$\mathcal{S}(\mathrm{k},\eta)=\mathcal{S}_{\mathcal{R}}(\mathrm{k},\eta)+\mathcal{S}_{\varphi}(\mathrm{k},\eta)+\mathcal{S}_{\mathcal{R}\varphi}(\mathrm{k},\eta), \tag{5.41}$$

where

$$\begin{aligned}\mathcal{S}_{\mathcal{R}}(\mathrm{k},\eta) &= \int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k},\tilde{\mathrm{k}})f_{\widetilde{R}}(\mathrm{k},\tilde{\mathrm{k}},\eta)\mathcal{R}^{\mathrm{pr}}_{\mathrm{k}-\tilde{\mathrm{k}}}\mathcal{R}^{\mathrm{pr}}_{\tilde{\mathrm{k}}}\,,\\ \mathcal{S}_{\varphi}(\mathrm{k},\eta) &= \int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k},\tilde{\mathrm{k}})f_{\widetilde{\varphi}}(\mathrm{k},\tilde{\mathrm{k}},\eta)\varphi^{\mathrm{pr}}_{\mathrm{k}-\tilde{\mathrm{k}}}\varphi^{\mathrm{pr}}_{\tilde{\mathrm{k}}}\,,\\ \mathcal{S}_{\mathcal{R}\varphi}(\mathrm{k},\eta) &= \int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k},\tilde{\mathrm{k}})f_{\widetilde{R},\widetilde{\varphi}}(\mathrm{k},\tilde{\mathrm{k}},\eta)\mathcal{R}^{\mathrm{pr}}_{\mathrm{k}-\tilde{\mathrm{k}}}\varphi^{\mathrm{pr}}_{\tilde{\mathrm{k}}}\end{aligned} \tag{5.42}$$

and $\mathrm{e}(\mathrm{k},\tilde{\mathrm{k}})\equiv\mathrm{e}^{ij}(k)\tilde{k}_i\tilde{k}_j=\tilde{k}^2\left[1-\mu^2\right],\ \mu\equiv\frac{\mathrm{k}\cdot\tilde{\mathrm{k}}}{k\tilde{k}}$.

In order to find the transfer functions $f_{\widetilde{R}}(\mathrm{k},\tilde{\mathrm{k}},\eta)$, $f_{\widetilde{\varphi}}(\mathrm{k},\tilde{\mathrm{k}},\eta)$ and $f_{\widetilde{R},\widetilde{\varphi}}(\mathrm{k},\tilde{\mathrm{k}},\eta)$ we use the definition of Θ as given in (4.17) and we Fourier transform to obtain:

$$\Theta_{\mathrm{k}}=-\frac{\mathcal{R}_{\mathrm{k}}}{a\mathcal{H}}-\frac{\epsilon}{a\mathrm{k}^2c_{\mathrm{s}}^2}\mathcal{R}'_{\mathrm{k}}-\left(-3+\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'}{2a\mathrm{k}^2M_{\mathrm{Pl}}^2}\varphi_{\mathrm{k}}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4}{2\mathrm{k}^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}\varphi_{\mathrm{k}}+\frac{\phi_0'}{2a\mathrm{k}^2\mathcal{H}M_{\mathrm{Pl}}^2}\varphi'_{\mathrm{k}}+\frac{aV_{,\varphi}}{2\mathrm{k}^2\mathcal{H}M_{\mathrm{Pl}}^2}\varphi_{\mathrm{k}}. \tag{5.43}$$

We can now substitute this form for Θ into the source term to obtain

$$\begin{aligned}f_{\widetilde{R}}(\mathrm{k},\tilde{\mathrm{k}},\eta) =\ & 2\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}+\tfrac{2}{\mathcal{H}}\widetilde{R}'_{\tilde{\mathrm{k}}}\widetilde{R}'_{\mathrm{k}-\tilde{\mathrm{k}}}-\left(4a'\widetilde{R}_{\tilde{\mathrm{k}}}+\tfrac{8a'}{\mathcal{H}}\widetilde{R}'_{\tilde{\mathrm{k}}}\right)\left(\tfrac{\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\mathrm{k}-\tilde{\mathrm{k}}}}{a(\mathrm{k}-\tilde{\mathrm{k}})^2c_{\mathrm{s}}^2}\right)-\tfrac{4a}{\mathcal{H}}\widetilde{R}'_{\tilde{\mathrm{k}}}\times\\ &\left(\tfrac{\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\mathrm{k}-\tilde{\mathrm{k}}}}{a(\mathrm{k}-\tilde{\mathrm{k}})^2c_{\mathrm{s}}^2}\right)'+\left(10a'^2+\tfrac{8\sigma_{0,\mathrm{in}}'^4}{a^4M_{\mathrm{Pl}}^2}e^{\frac{-\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\right)\left(\tfrac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\tilde{\mathrm{k}}}}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)\left(\tfrac{\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\mathrm{k}-\tilde{\mathrm{k}}}}{a(\mathrm{k}-\tilde{\mathrm{k}})^2c_{\mathrm{s}}^2}\right)\\ &+2a^2\left(\tfrac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\tilde{\mathrm{k}}}}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)'\left(\tfrac{\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\mathrm{k}-\tilde{\mathrm{k}}}}{a(\mathrm{k}-\tilde{\mathrm{k}})^2c_{\mathrm{s}}^2}\right)'+8aa'\left(\tfrac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\tilde{\mathrm{k}}}}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)\left(\tfrac{\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{a\mathcal{H}}+\tfrac{\epsilon\widetilde{R}'_{\mathrm{k}-\tilde{\mathrm{k}}}}{a(\mathrm{k}-\tilde{\mathrm{k}})^2c_{\mathrm{s}}^2}\right)',\end{aligned} \tag{5.44}$$

$$
\begin{aligned}
f_{\widetilde{\varphi}}(\mathrm{k},\tilde{\mathrm{k}},\eta) &= \left(\frac{\phi_0'^2(\eta)}{2\mathcal{H}^2M_{\mathrm{Pl}}^4}-\frac{2}{M_{\mathrm{Pl}}^2}\right)\widetilde{\varphi}_{\tilde{\mathrm{k}}}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}+\left[\frac{2a\phi_0'}{\mathcal{H}M_{\mathrm{Pl}}^2}\widetilde{\varphi}_{\tilde{\mathrm{k}}}+2a^2\left(\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2a\tilde{\mathrm{k}}^2M_{\mathrm{Pl}}^2}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2\tilde{\mathrm{k}}^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\tilde{\mathrm{k}}}'}{2a\tilde{\mathrm{k}}^2\mathcal{H}M_{\mathrm{Pl}}^2}\right.\right.\\
&\left.\left.+\frac{V_{,\varphi}\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2\tilde{\mathrm{k}}^2\mathcal{H}M_{\mathrm{Pl}}^2}\right)'\right]\times\left[\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^2}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}'}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}+\frac{V_{,\varphi}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}\right]'\\
&+\left[\frac{4a'\phi_0'}{\mathcal{H}M_{\mathrm{Pl}}^2}\widetilde{\varphi}_{\tilde{\mathrm{k}}}+8aa'\left(\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2a\tilde{\mathrm{k}}^2M_{\mathrm{Pl}}^2}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2\tilde{\mathrm{k}}^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\tilde{\mathrm{k}}}'}{2a\tilde{\mathrm{k}}^2\mathcal{H}M_{\mathrm{Pl}}^2}+\frac{V_{,\varphi}\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2\tilde{\mathrm{k}}^2\mathcal{H}M_{\mathrm{Pl}}^2}\right)'\right.\\
&\left.+\left(10a'^2+\frac{8\sigma_{0,\mathrm{in}}'^4}{a^4M_{\mathrm{Pl}}^2}e^{\frac{-\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\right)\left(\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2a\tilde{\mathrm{k}}^2M_{\mathrm{Pl}}^2}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2\tilde{\mathrm{k}}^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\tilde{\mathrm{k}}}'}{2a\tilde{\mathrm{k}}^2\mathcal{H}M_{\mathrm{Pl}}^2}+\frac{V_{,\varphi}\widetilde{\varphi}_{\tilde{\mathrm{k}}}}{2\tilde{\mathrm{k}}^2\mathcal{H}M_{\mathrm{Pl}}^2}\right)\right]\\
&\times\left[\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^2}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}'}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}+\frac{V_{,\varphi}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}\right],
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
f_{\widetilde{R},\widetilde{\varphi}}(\mathrm{k},\tilde{\mathrm{k}},\eta) &= \left[4a'\widetilde{R}_{\tilde{\mathrm{k}}}-2\left(10a'^2+\frac{8\sigma_{0,\mathrm{in}}'^4}{a^4M_{\mathrm{Pl}}^2}e^{\frac{-\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\right)\left(\frac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\frac{\epsilon\widetilde{R}_{\tilde{\mathrm{k}}}'}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)-8aa'\left(\frac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\frac{\epsilon\widetilde{R}_{\tilde{\mathrm{k}}}'}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)'+\frac{8a'\widetilde{R}_{\tilde{\mathrm{k}}}}{\mathcal{H}}\right]\\
&\times\left[\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^2}+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}'}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}+\frac{V_{,\varphi}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}\right]\\
&+\left[\frac{4a\widetilde{R}_{\tilde{\mathrm{k}}}'}{\mathcal{H}}-4a'\left(\frac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\frac{\epsilon\widetilde{R}_{\tilde{\mathrm{k}}}'}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)'-8aa'\left(\frac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\frac{\epsilon\widetilde{R}_{\tilde{\mathrm{k}}}'}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)\right]\times\left[\left(3-\frac{\epsilon}{c_{\mathrm{s}}^2}\right)\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^2}\right.\\
&\left.+\frac{3e^{\phi_0/M_{\mathrm{Pl}}}\sigma_0'^4\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2M_{\mathrm{Pl}}^3a^3\mathcal{H}}+\frac{\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}'}{2a(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}+\frac{V_{,\varphi}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{2(\mathrm{k}-\tilde{\mathrm{k}})^2\mathcal{H}M_{\mathrm{Pl}}^2}\right]'+\frac{2\phi_0'\widetilde{R}_{\tilde{\mathrm{k}}}'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{\mathcal{H}^2M_{\mathrm{Pl}}^2}\\
&-\frac{2\phi_0'\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}}{\mathcal{H}M_{\mathrm{Pl}}^2}\left(a\left(\frac{\widetilde{R}_{\tilde{\mathrm{k}}}}{a\mathcal{H}}+\frac{\epsilon\widetilde{R}_{\tilde{\mathrm{k}}}'}{a\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right)'+2\widetilde{R}_{\tilde{\mathrm{k}}}+\frac{2\mathcal{H}\epsilon\widetilde{R}_{\tilde{\mathrm{k}}}'}{\tilde{\mathrm{k}}^2c_{\mathrm{s}}^2}\right),
\end{aligned}
\tag{5.46}
$$

where we do not express anymore the time dependence of the transfer functions.

Now that we have the first order quantities split into the contribution of the first order dynamics and the primordial fluctuations, we can express $P_h(\mathrm{k})$ as a function of the primordial power spectrum of the curvature perturbations $\langle\mathcal{R}_{\mathrm{k}}\mathcal{R}_{\tilde{\mathrm{k}}}\rangle=\frac{2\pi^2}{\mathrm{k}^3}P_{\mathcal{R}}(\mathrm{k})\delta(\mathrm{k}+\tilde{\mathrm{k}})$, the primordial power spectrum of the radion $\langle\varphi_{\mathrm{k}}\varphi_{\tilde{\mathrm{k}}}\rangle=\frac{2\pi^2}{\mathrm{k}^3}P_{\varphi}(\mathrm{k})\delta(\mathrm{k}+\tilde{\mathrm{k}})$ and a mixed primordial power spectrum $\langle\mathcal{R}_{\mathrm{k}}\varphi_{\tilde{\mathrm{k}}}\rangle=\frac{2\pi^2}{\mathrm{k}^3}P_{\mathcal{R}\varphi}(\mathrm{k})\delta(\mathrm{k}+\tilde{\mathrm{k}})$. The term of $\langle\mathcal{S}(\mathrm{k},\bar{\eta}_1)\,\mathcal{S}(\mathrm{K},\bar{\eta}_2)\rangle$ in equation (5.38) includes several contributions,

$$
\begin{aligned}
\langle\mathcal{S}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}(\mathrm{K},\tilde{\eta}_2)\rangle =& \langle\mathcal{S}_{\mathcal{R}}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_{\mathcal{R}}(\mathrm{K},\tilde{\eta}_2)\rangle+\langle\mathcal{S}_{\varphi}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_{\varphi}(\mathrm{K},\tilde{\eta}_2)\rangle\\
&+\langle\mathcal{S}_{\mathcal{R}\varphi}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_{\mathcal{R}\varphi}(\mathrm{K},\tilde{\eta}_2)\rangle+2\,\langle\mathcal{S}_{\mathcal{R}}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_{\mathcal{R}\varphi}(\mathrm{K},\tilde{\eta}_2)\rangle\\
&+2\,\langle\mathcal{S}_{\varphi}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_{\mathcal{R}\varphi}(\mathrm{K},\tilde{\eta}_2)\rangle+2\,\langle\mathcal{S}_{\mathcal{R}}(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_{\varphi}(\mathrm{K},\tilde{\eta}_2)\rangle\,,
\end{aligned}
\tag{5.47}
$$

and each contribution can be written as

$$\begin{aligned}\langle \mathcal{S}_\alpha(\mathrm{k},\tilde{\eta}_1)\,\mathcal{S}_\beta(\mathrm{K},\tilde{\eta}_2)\rangle &= \int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k},\tilde{\mathrm{k}})f_\alpha\left(\mathrm{k},\tilde{\mathrm{k}},\tilde{\eta}_1\right)\int \mathrm{d}^3\tilde{\mathrm{K}}\mathrm{e}(\mathrm{K},\tilde{\mathrm{K}})f_\beta\left(\mathrm{K},\tilde{\mathrm{K}},\tilde{\eta}_2\right)\langle \alpha_{\mathrm{k}-\tilde{\mathrm{k}}}\alpha_{\tilde{\mathrm{k}}}\beta_{\mathrm{K}-\tilde{\mathrm{K}}}\beta_{\tilde{\mathrm{K}}}\rangle \\ &= \delta(\mathrm{k}+\mathrm{K})\int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k},\tilde{\mathrm{k}})^2 f_\alpha\left(\mathrm{k},\tilde{\mathrm{k}},\tilde{\eta}_1\right)\left[f_\beta\left(\mathrm{k},\tilde{\mathrm{k}},\tilde{\eta}_2\right)+f_\beta\left(\mathrm{k},\mathrm{k}-\tilde{\mathrm{k}},\tilde{\eta}_2\right)\right]\times \\ &\quad \frac{P_\alpha(|\mathrm{k}-\tilde{\mathrm{k}}|)}{|\mathrm{k}-\tilde{\mathrm{k}}|^3}\frac{P_\beta(\tilde{k})}{\tilde{k}^3},\end{aligned} \tag{5.48}$$

where α and β can be $\mathcal{R}$, φ or the mixed term. In conclusion, we can express the power spectrum of scalar-induced gravitational waves as:

$$P_h(\mathrm{k},\eta) = \sum_{\alpha,\beta}\int_0^\infty \mathrm{d}\tilde{k}\int_{-1}^1 \mathrm{d}\mu P_\alpha(|\mathrm{k}-\tilde{\mathrm{k}}|)P_\beta(\tilde{k})\mathcal{F}_{\alpha\beta}(k,\tilde{k},\mu;\eta), \tag{5.49}$$

where $\mathcal{F}_{\alpha\beta}$ is defined in terms of the Green's function g_k and the transfer functions $\widetilde{R}(\mathrm{p},\eta)$ and $\widetilde{\varphi}(\mathrm{p},\eta)$. P_h is defined fully in terms of the Green's function, the dynamics of first order scalars perturbations and the primordial power spectrum of first order scalar fluctuations.

We now want to better understand the non-GR contributions to the source term in terms of observable quantities. To do that, we compare our results to the general relativity case. We first follow [53] and make the approximation that the generation of gravitational waves induced by $\mathcal{S}(\mathrm{k},\eta)$ is instantaneous when the relevant mode enters the horizon. In that way, we expect that especially those modes that enters the horizon during the radiation-dominated epoch, and thus form PBHs, produce scalar-induced gravitational waves that are influenced by non-GR contributions.
We define the transfer function for scalar-induced gravitational waves, $t(\mathrm{k},\eta)$, as

$$h_\mathrm{k}(\eta) \equiv t(\mathrm{k},\eta)h_\mathrm{k}^{(\mathrm{i})}, \tag{5.50}$$

where $h_\mathrm{k}^{(\mathrm{i})}$ is the value of the amplitude just after the instantaneous generation of gravitational waves, after horizon entry. Then, given $h_\mathrm{k}^{(\mathrm{i})}$, we can estimate the power spectrum of gravitational waves at horizon crossing

$$P_h^{(\mathrm{i})}(\mathrm{k},\eta_\mathrm{i}(\mathrm{k})) \equiv \frac{\mathrm{k}^3}{2\pi^2}\left\langle\left(h_\mathrm{k}^{(\mathrm{i})}\right)^2\right\rangle, \tag{5.51}$$

where $\eta_i \sim \mathrm{k}^{-1}$ is the conformal time at which a comoving scale k enters the horizon. This means that in order to estimate the power spectrum, we need to find the form of the transfer function $t(\mathrm{k},\eta)$ and the amplitude just after horizon entry, $h_\mathrm{k}^{(\mathrm{i})}$.

We assume that our model is particularly effective in the super-horizon limit when high energy corrections of general relativity become important for the evolution of scalar perturbations, as shown in Section **4.2**. Furthermore, we are especially interested in those scales that enter the horizon during the radiation-dominated epoch because those scales form PBHs. Under these assumptions we have that:

- The transfer function $t(\mathrm{k},\eta)$ is the standard one, as it refers to the dynamics of scalar perturbations in the sub-horizon limit.

- The amplitude just after horizon entry $h_{\rm k}^{(\rm i)}$ is the standard one if the considered scale re-enters the horizon during the matter-dominated epoch. It is different if horizon re-entry happens during the radiation-dominated epoch.

Thus, the transfer function

$$t(\mathrm{k},\eta)=\frac{h(\mathrm{k},\eta)}{h_{\rm k}^{(\rm i)}} \tag{5.52}$$

can be estimated following [53]. To estimate the amplitude $h(\mathrm{k},\eta)$ in equation (5.52) we need to estimate the time evolution of the standard-GR source term,

$$\begin{aligned}\mathcal{S}_{\rm GR}(\mathrm{k},\eta)=&4\int\frac{\mathrm{d}^3\tilde{\mathrm{k}}}{(2\pi)^{3/2}}\mathrm{e}^{lm}(\mathrm{k})\tilde{k}_l\tilde{k}_m\left[\left\{\frac{7+3w}{3(1+w)}-\frac{2c_{\rm s}^2}{w}\right\}\Phi_{\tilde{\rm k}}(\eta)\Phi_{\mathrm{k}-\tilde{\rm k}}(\eta)\right.\\&+\left(1-\frac{2c_{\rm s}^2\tilde{k}^2}{3w\mathcal{H}^2}\right)\Psi_{\tilde{\rm k}}(\eta)\Psi_{\mathrm{k}-\tilde{\rm k}}(\eta)+\frac{2c_{\rm s}^2}{w}\left(1+\frac{\tilde{k}^2}{3\mathcal{H}^2}\right)\Phi_{\tilde{\rm k}}(\eta)\Psi_{\mathrm{k}-\tilde{\rm k}}(\eta)\\&+\left\{\frac{8}{3(1+w)}+\frac{2c_{\rm s}^2}{w}\right\}\frac{1}{\mathcal{H}}\Phi_{\tilde{\rm k}}(\eta)\Psi'_{\mathrm{k}-\tilde{\rm k}}(\eta)-\frac{2c_{\rm s}^2}{w\mathcal{H}}\Psi_{\tilde{\rm k}}(\eta)\Psi'_{\mathrm{k}-\tilde{\rm k}}(\eta)\\&\left.+\frac{4}{3(1+w)\mathcal{H}^2}\Psi'_{\tilde{\rm k}}(\eta)\Psi'_{\mathrm{k}-\tilde{\rm k}}(\eta)\right],\end{aligned} \tag{5.53}$$

where $\Phi=\Psi$ in the GR case. The time dependence of Φ can be evaluated by solving the first order Einstein equations (2.41), which gives

$$\Phi(k\eta)=\begin{cases}\frac{1}{1+k^2\eta^2} & \eta<\eta_{\rm eq}\\ \frac{1}{1+k^2\eta_{\rm eq}^2} & \eta>\eta_{\rm eq}\end{cases} \tag{5.54}$$

Thus, inserting this expression into the definition of the source term and then back into the solution (5.35), we obtain the transfer function (5.52),

$$t(\mathrm{k},\eta)=\begin{cases}1 & \mathrm{k}<\mathrm{k}_{\rm eq}\\ \left(\frac{\mathrm{k}}{\mathrm{k}_{\rm eq}}\right)^{-\gamma} & \mathrm{k}_{\rm eq}<\mathrm{k}<\mathrm{k}_{\rm c}(\eta)\end{cases} \tag{5.55}$$

where $\mathrm{k}_{\rm c}(\eta)$ is the horizon scale at the time η. Scales greater than the horizon will evolve as a^{-1} and, considering that we want to evaluate the power spectrum at low redshift, these scales will be too large for the simplifications considered above.

In order to estimate $h_{\rm k}^{(\rm i)}$ for those scales that re-enter the horizon during the matter-dominated era we can again take the results of [53]. Having the analytical behaviour of $\mathcal{S}_{\rm GR}(\mathrm{k},\eta)$, it is possible to estimate $\langle h(\mathrm{k},\eta)h(\mathrm{K},\eta)\rangle$ using (5.38). Then the power spectrum becomes

$$\begin{aligned}P_h^{(\rm i)}(\mathrm{k},\eta_{\rm i}(\mathrm{k}))\sim\frac{\Delta_{\mathcal{R}}^4}{\mathrm{k}}\int_0^\infty\mathrm{d}\tilde{\mathrm{k}}\int_{-1}^1\mathrm{d}\mu\left(1-\mu^2\right)^2&\left[\frac{\tilde{\mathrm{k}}^3}{(\mathrm{k}^2+\tilde{\mathrm{k}}^2-2\mathrm{k}\tilde{\mathrm{k}}\mu)^{3/2}}\frac{1}{[1+(\tilde{\mathrm{k}}/\mathrm{k}_{\rm eq})^2]^2}\right.\\&\left.\frac{1}{[1+(\mathrm{k}^2+\tilde{\mathrm{k}}^2-2\mathrm{k}\tilde{\mathrm{k}}\mu)/\mathrm{k}_{\rm eq}^2]^2}\right].\end{aligned} \tag{5.56}$$

To estimate $h_{\rm k}^{(\rm i)}$ for the scales that re-enter the horizon during radiation-dominated era we consider our source term (5.41). In this case, as previously mentioned, we do not have the analogous of equation (5.54), thus we are not able to find the explicit (k, η) dependence of $h_{\mathrm{k}}^{(\mathrm{i})}$ as for the matter-dominated case. However, we estimate $h_{\mathrm{k}}^{(\mathrm{i})}$ by simplifying the general solution (5.38). We simplify the equation of motion (5.27) dropping the time derivatives,[22]

$$h_{\mathrm{k}}^{(\mathrm{i})} \sim \frac{1}{\mathrm{k}^2}\mathcal{S}^{(\mathrm{i})}. \tag{5.57}$$

We neglect in the transfer functions (5.44), (5.45) and (5.46) those terms that are time derivatives and those terms that vary as $\frac{1}{\mathrm{k}^2}$. In this way the transfer functions become

$$\begin{aligned} f_{\widetilde{R}}(\mathrm{k}, \tilde{\mathrm{k}}, \eta) &= -2\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}} + \left(10a'^2 + \frac{8\sigma'^4_{0,\mathrm{in}}}{a^4 M_{\mathrm{Pl}}^2} e^{\frac{-\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\right)\frac{\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{(a\mathcal{H})^2}, \\ f_{\widetilde{\varphi}}(\mathrm{k}, \tilde{\mathrm{k}}, \eta) &= \left(\frac{\phi_0'^2(\eta)}{2\mathcal{H}^2 M_{\mathrm{Pl}}^4} - \frac{2}{M_{\mathrm{Pl}}^2}\right)\widetilde{\varphi}_{\tilde{\mathrm{k}}}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}, \\ f_{\widetilde{R},\widetilde{\varphi}}(\mathrm{k}, \tilde{\mathrm{k}}, \eta) &= -\frac{4\phi_0'^2(\eta)}{\mathcal{H}M_{\mathrm{Pl}}^2}\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}. \end{aligned} \tag{5.58}$$

We notice that the assumptions we made to simplify the transfer functions, i.e., neglecting time derivatives and terms that go as $\frac{1}{\mathrm{k}^2}$, are not both in agreement with the results of Section **4.2**. Indeed, if assuming that those terms varying as $\frac{1}{\mathrm{k}^2}$ are negligible is a good approximation for scales that enter the horizon during the radiation-dominated epoch, the fact that we consider the time derivative of first order perturbation quantities equal to zero is not justified in our model. In Section **4.2** we explicitly showed that in the model described in Section **3** the scalar perturbations are not frozen on super-horizon scales. Thus, the latter assumption is well justified only in the general relativity case where the time derivative of scalar perturbations in the super-horizon limit is indeed zero. However, we maintain this assumption as it does not cancel all the non-GR terms (we still have radion dependent terms in (5.58)).

In conclusion, for scales that re-enter the horizon in a radiation-dominated universe we have

$$\begin{aligned} h_{\mathrm{k}}^{(\mathrm{i})} &\sim \frac{1}{\mathrm{k}^2}\int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k}, \tilde{\mathrm{k}})\left[f_{\widetilde{R}}(\mathrm{k}, \tilde{\mathrm{k}}, \eta)\mathcal{R}_{\mathrm{k}-\tilde{\mathrm{k}}}\mathcal{R}_{\tilde{\mathrm{k}}} + f_{\widetilde{\varphi}}(\mathrm{k}, \tilde{\mathrm{k}}, \eta)\varphi_{\mathrm{k}-\tilde{\mathrm{k}}}\varphi_{\tilde{\mathrm{k}}} + f_{\widetilde{R},\widetilde{\varphi}}(\mathrm{k}, \tilde{\mathrm{k}}, \eta)\mathcal{R}_{\mathrm{k}-\tilde{\mathrm{k}}}\varphi_{\tilde{\mathrm{k}}}\right] \\ &\sim \frac{1}{\mathrm{k}^2}\int \mathrm{d}^3\tilde{\mathrm{k}}\mathrm{e}(\mathrm{k}, \tilde{\mathrm{k}})\left[\left(-2\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}} + \left(10a'^2 + \frac{8\sigma'^4_{0,\mathrm{in}}}{a^4 M_{\mathrm{Pl}}^2} e^{\frac{-\phi_0(\eta)}{3M_{\mathrm{Pl}}}}\right)\frac{\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{R}_{\mathrm{k}-\tilde{\mathrm{k}}}}{(a\mathcal{H})^2}\right)\mathcal{R}_{\mathrm{k}-\tilde{\mathrm{k}}}\mathcal{R}_{\tilde{\mathrm{k}}}\right. \\ &\left.+\left(\frac{\phi_0'^2(\eta)}{2\mathcal{H}^2 M_{\mathrm{Pl}}^4} - \frac{2}{M_{\mathrm{Pl}}^2}\right)\varphi_{\mathrm{k}-\tilde{\mathrm{k}}}\varphi_{\tilde{\mathrm{k}}} - \left(\frac{4\phi_0'^2(\eta)}{\mathcal{H}M_{\mathrm{Pl}}^2}\widetilde{R}_{\tilde{\mathrm{k}}}\widetilde{\varphi}_{\mathrm{k}-\tilde{\mathrm{k}}}\right)\mathcal{R}_{\mathrm{k}-\tilde{\mathrm{k}}}\varphi_{\tilde{\mathrm{k}}}\right]. \end{aligned} \tag{5.59}$$

From this $h_{\mathrm{k}}^{(\mathrm{i})}$ we then obtain a power spectrum of the form (5.49), which is again the easiest form we can reach without knowing the analytical behaviour of the first order scalar perturbations $\widetilde{R}$ and $\widetilde{\varphi}$.

In particular, this result should be compared to the standard power spectrum at

[22] After horizon entry $\mathrm{k}\eta > 1$.

horizon crossing during the radiation-dominated epoch (see [53] for more details),

$$P_h^{(\mathrm{i})}\left(k,\eta_{\mathrm{i}}(k)\right) \sim \frac{\Delta_{\mathcal{R}}^4}{k}\int_0^\infty \mathrm{d}\tilde{k}\int_{-1}^{1}\mathrm{d}\mu\left(1-\mu^2\right)^2\frac{\tilde{k}^3}{\left(k^2+\tilde{k}^2-2k\tilde{k}\mu\right)^{3/2}}\frac{1}{\left[1+(\hat{k}/k)^2\right]^2} \times \frac{1}{\left[1+\left(k^2+\tilde{k}^2-2k\tilde{k}\mu\right)/k^2\right]^2}. \tag{5.60}$$

Finally, as mentioned in Section **5.1.2**, we can process the two power spectra at the horizon crossing, (5.56) and (5.59), using the transfer function $t(\mathrm{k},\eta)$ in (5.55) in order to compute the power spectrum of scalar-induced gravitational waves at any time. With these results we can estimate the relative energy density of scalar-induced gravitational waves,

$$\begin{aligned}\Omega_{\mathrm{gw}}^{(2)}(\mathrm{k},\eta) &= \frac{1}{6\pi^2\mathcal{H}^2(\eta)}\mathrm{k}^2 t^2(\mathrm{k},\eta)P_h^{(\mathrm{i})}(\mathrm{k})\\ &= \frac{a(\eta)\mathrm{k}^2}{a_{\mathrm{eq}}\mathrm{k}_{\mathrm{eq}}^2}t^2(\mathrm{k},\eta)P_h^{(\mathrm{i})}(\mathrm{k}),\end{aligned} \tag{5.61}$$

as a function of the first and zeroth order dynamics, the Green's function and the primordial power spectra of the curvature perturbations and the radion.

Even if up to now we have not been able to achieve the analytical behaviour of $\Omega_{\mathrm{gw}}^{(2)}(\mathrm{k},\eta)$, we expect that the enhancement of scalar perturbations shown in **4.2** will enhance also the relative energy density of scalar-induced gravitational waves for a specific frequency range. This effect, modeled in this work for the first time, is particularly relevant for the upcoming gravitational wave surveys such as LISA. As shown in **Figure 5.1**, in the standard-GR scenario we do not expect to detect a signal from scalar-induced gravitational waves. On the contrary, in our model we could in principle properly set the initial conditions in order to make $\Omega_{\mathrm{gw}}^{(2)}(\mathrm{k},\eta)$ possible to be observed in any range of frequencies. In other words, in case $\Omega_{\mathrm{gw}}^{(2)}(\mathrm{k},\eta)$ will be detected by Laser Interferometer Space Antenna (LISA) [52], this model will be a possible way to explain the observed data.

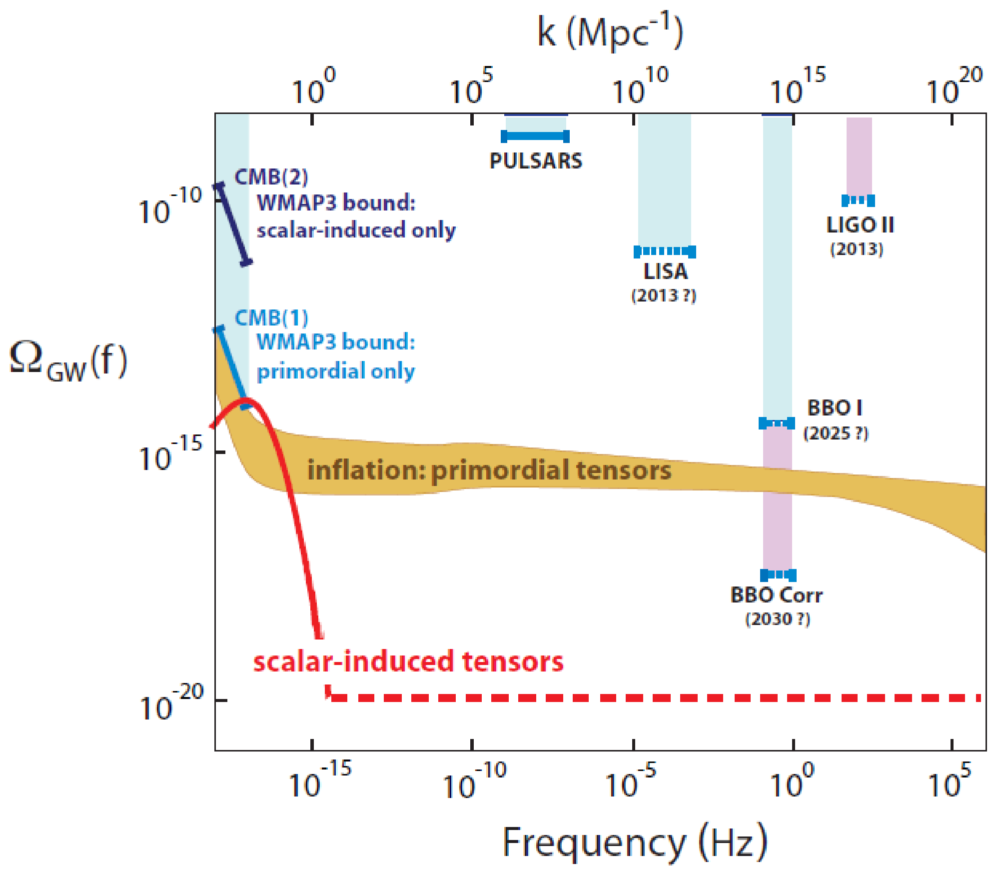

Figure 5.1: *Observational prospects upcoming gravitational wave surveys. In this figure we can see the theoretical predictions for the present value of relative energy density of primordial inflationary first order gravitational waves (see **2.3**) and scalar-induced gravitational waves in the standard-inflationary scenario, as well as current (solid bars) and future (dashed bars) observational bounds. This figure is taken from* [53].

Chapter 6

Conclusions

In this thesis work we reviewed the new mechanism for generating PBHs proposed in [1] and studied scalar-induced gravitational waves produced as a consequence of PBH formation in this new scenario.

The main new feature of the higher-dimensional gravity model described in Section **3** is that it allows a different behaviour of scalar perturbations on super-horizon scales to be described. As discussed in Section **4.2**, an enhancement in the initial conditions of the radion perturbations causes the curvature perturbations $\mathcal{R}$ to grow on super-horizon scales for a specific range of k modes. This is different from the super-horizon behaviourin the standard general relativity scenario where, as shown in Section **2.2.2**, both scalar and density perturbations are frozen on scales larger than the horizon. This difference in the dynamics of super-horizon modes is the key element of this new model of PBH formation. Indeed, even starting from almost scale-invariant primordial curvature perturbations, we are able to produce enhancements in the density fluctuations required for generating PBHs. This is a new approach in theoretical modeling of PBH formation: we do not generate enhancements in the primordial power spectrum of curvature fluctuations in a specific range of scales during the inflationary phase, we instead modify the evolution of these curvature perturbations during the radiation-dominated epoch.

After studying scalar perturbations we presented the main original result of this work: derivation of tensor perturbations, their properties and evolution in this newly proposed mechanism for PBH formation. In Section **5** we studied the equation of motion and power spectrum of scalar induced gravitational waves. We explicitly derived the second order power spectrum as a function of first order perturbations quantities and initial conditions, and studied the main differences from the general relativity case. Indeed, both in general relativity and in our scenario we predict that at second order in perturbation theory we have mode-mixing and therefore production of scalar-induced gravitational waves. However, in the case of general relativity we expect an almost scale-invariant relative energy density at frequencies probed by upcoming gravitational wave surveys, whereas, due to the scalar enhancement described in Section **4.2**, we expect additional features in this relative energy density that can potentially be detected by these surveys.

Chapter 7

Future directions

A natural continuation of this work consists at least of three directions: modeling and studying initial conditions, analytically solving first order Einstein equations andnumerically studying the relative energy density of scalar-induced gravitational waves.

Indeed, as have shown in Section **4.2**, the mass and the abundance of the produced PBHs are strongly dependent on the initial conditions of the radion perturbations. We need to better understand, from both the theoretical and observational points of view, the exact shape of radion's initial conditions. We can proceed with the theoretical study, as in Section **4.2**, of the radion's dynamics in the inflationary phase, or we can observationally constrain the initial conditions using the constraints on the PBH abundance presented in Section **2.1.3**. Furthermore, we naturally expect the relative energy density of scalar-induced gravitational waves to also depend on the radion's initial conditions, thus, as soon as we have more constraints from gravitational wave detections, we can use this information to observationally constrain the initial conditions of radion perturbations.

We also need to better understand the analytical behaviour of first order curvature and radion perturbations. This will help us to analytically estimate the (k, η) dependence of the scalar induced power spectrum using the results of Section **5**. However, in order to better understand the implications of the super-horizon enhancement of curvature perturbations for second order gravitational waves, we can proceed with the numerical analysis of our results. Using the results of Sections **4.2** and **5.2.2** we can numerically estimate the the relative energy density of scalar-induced gravitational waves.

Appendix - Mathematica codes

In[∘]:=
```
<< xAct`xPand`
```

In[∘]:=
```
DefManifold[M, 4, {α, β, χ, λ, μ, ν}]
```

```
** DefManifold: Defining manifold M.

** DefVBundle: Defining vbundle TangentM.
```

In[∘]:=
```
DefMetric[-1, g[-α, -β], CD, {";", "𝒟"}, PrintAs → "g"]
```

In[∘]:=
```
$PrePrint = ScreenDollarIndices;
```

In[∘]:=
```
$CovDFormat = "Prefix";
```

In[∘]:=
```
org[expr_] :=
 NoScalar@Collect[ContractMetric[expr], $PerturbationParameter,
```

In[∘]:=
```
ToCanonical] SetSlicing[g, n, h, cd, {"|", "∇"}, "FLFlat"]
```

In[∘]:=
```
DefMetricFields[g, dg, h]
```

In[∘]:=
```
$FirstOrderTensorPerturbations = False;
```

In[∘]:=
```
$FirstOrderVectorPerturbations = False;
```

In[∘]:=
```
VisualizeTensor[dg[LI[1], α, β] /. SplitMetric[g, dg, h, "NewtonGauge"], h]
```

Out[∘]=

	n	h
n	$-2\left({}^{(1)}\phi\right)$	0
h	0	$-2\,\bar{h}^{\alpha\beta}\left({}^{(1)}\psi\right)$

In[∘]:=
```
VisualizeTensor[dg[LI[2], α, β] /. SplitMetric[g, dg, h, "NewtonGauge"], h]
```

Out[∘]=

	n	h
n	$-2\left({}^{(2)}\phi\right)$	$-\left({}^{(2)}B^{\beta}\right)$
h	$-\left({}^{(2)}B^{\alpha}\right)$	$2\left({}^{(2)}E^{\alpha\beta}\right)-2\,\bar{h}^{\alpha\beta}\left({}^{(2)}\psi\right)$

In[∘]:=
```
DefConstantSymbol[Mplank, PrintAs → "Mpl"]
```

```
** DefConstantSymbol: Defining constant symbol Mplank.
```

In[∘]:=
```
DefConstantSymbol[b, PrintAs → "b*"]
DefConstantSymbol[f, PrintAs → "f"]
```

```
** DefConstantSymbol: Defining constant symbol b.

** DefConstantSymbol: Defining constant symbol f.
```

In[∘]:=
```
DefProjectedTensor[G[], h] (* this is the radion *)
```

```
** DefTensor: Defining tensor
 G[LI[xAct`xPand`Private`p$5660], LI[xAct`xPand`Private`q$5660]].
```

In[∘]:=
```
DefProjectedTensor[L[], h](* this is the radiation *)
```

```
** DefTensor: Defining tensor
 L[LI[xAct`xPand`Private`p$5666], LI[xAct`xPand`Private`q$5666]].
```

In[∘]:=
```
Lgr = Mplank^2 Sqrt[-Detg[]]/2 RicciScalarCD[] +
  Mplank^2 Sqrt[-Detg[]]/b^2 Exp[-2 G[] / Mplank] -
  Sqrt[-Detg[]] 2 π f^2/b^2 Exp[-3 G[] / Mplank] + 1/2 Sqrt[-g] g^{αβ} (D_α G) (D_β G) +
  Sqrt[-Detg[]] g[α, β] × CD[-α][L[]] × CD[-β][L[]] ×
   g[μ, ν] × CD[-μ][L[]] × CD[-ν][L[]] Exp[G[] / Mplank]
```

Out[∘]=

$$\frac{e^{-\frac{2G}{M_{pl}}} M_{pl}^2 \sqrt{-\tilde{\tilde{g}}}}{b_*^2} - \frac{2 e^{-\frac{3G}{M_{pl}}} f^2 \pi \sqrt{-\tilde{\tilde{g}}}}{b_*^2} + \frac{1}{2} M_{pl}^2 \sqrt{-\tilde{\tilde{g}}}\, R[\mathcal{D}] + \frac{1}{2} \sqrt{-\tilde{\tilde{g}}}\, \tilde{g}^{\alpha\beta} (\mathcal{D}_\alpha G)(\mathcal{D}_\beta G) + e^{\frac{G}{M_{pl}}} \sqrt{-\tilde{\tilde{g}}}\, \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} (\mathcal{D}_\alpha L)(\mathcal{D}_\beta L)(\mathcal{D}_\mu L)(\mathcal{D}_\nu L)$$

In[∘]:=
```
ExpandPerturbation@Perturbation[Lgr, 1] // org // Simplify;
```

In[∘]:=
```
VarD[dg[LI[1], -α, -β], CD][%/Sqrt[-Detg[]]] // org // Simplify;
```

In[∘]:=
```
% /. delta[-LI[1], LI[1]] → 1;
```

In[∘]:=
```
EqMotion = ContractMetric[% g[-β, -μ]]
```

Out[∘]=

$$\frac{e^{-\frac{2(G)}{M_{pl}}} M_{pl}^2 \delta_\mu{}^\alpha}{2 b_*^2} - \frac{e^{-\frac{3(G)}{M_{pl}}} f^2 \pi \delta_\mu{}^\alpha}{b_*^2} - \frac{1}{2} M_{pl}^2 R[\mathcal{D}]^\alpha{}_\mu + \frac{1}{4} M_{pl}^2 \delta_\mu{}^\alpha R[\mathcal{D}] + \frac{1}{4} \delta_\mu{}^\alpha (\mathcal{D}_\beta G)(\mathcal{D}^\beta G) + \frac{1}{2} e^{\frac{(G)}{M_{pl}}} \delta_\mu{}^\alpha (\mathcal{D}_\beta L)(\mathcal{D}^\beta L)(\mathcal{D}_\lambda L)(\mathcal{D}^\lambda L) - \frac{1}{2} (\mathcal{D}^\alpha G)(\mathcal{D}_\mu G) - 2 e^{\frac{(G)}{M_{pl}}} (\mathcal{D}^\alpha L)(\mathcal{D}_\beta L)(\mathcal{D}^\beta L)(\mathcal{D}_\mu L)$$

In[∘]:=
```
ExpandPerturbation@Perturbation[Conformal[g, gah2][EqMotion], 2] // org // Simplify;
```

** DefTensor: Defining tensor ChristoffelCDCDah2[α, -β, -λ].

In[∘]:=
```
SplitPerturbations[%, SplitMetric[g, dg, h, "NewtonGauge"], h] // Simplify // org;
```

The Splitting of $\frac{1}{8}$ +... was performed in 71.4274 seconds.

In[∘]:=
```
SecondOrdEqMotion = %
```

Out[∘]=

$$-\frac{M_{pl}^2 \left({}^{(2)}E''^\alpha{}_\mu\right)}{2a^2} + \frac{2 e^{-\frac{2(G)}{M_{pl}}} \delta^\alpha{}_\mu \left({}^{(1)}G\right)^2}{b_*^2} - \frac{9 e^{-\frac{3(G)}{M_{pl}}} f^2 \pi \delta^\alpha{}_\mu \left({}^{(1)}G\right)^2}{b_*^2 M_{pl}^2} - \frac{\delta^\alpha{}_\mu \left({}^{(1)}G'\right)^2}{2a^2} - \frac{e^{-\frac{2(G)}{M_{pl}}} M_{pl} \delta^\alpha{}_\mu \left({}^{(2)}G\right)}{b_*^2} + \frac{3 e^{-\frac{3(G)}{M_{pl}}} f^2 \pi \delta^\alpha{}_\mu \left({}^{(2)}G\right)}{b_*^2 M_{pl}} - \frac{\delta^\alpha{}_\mu G' \left({}^{(2)}G'\right)}{2a^2} - \frac{M_{pl}^2 \left({}^{(2)}E'^\alpha{}_\mu\right) \mathcal{H}}{a^2} + \frac{e^{\frac{(G)}{M_{pl}}} \delta^\alpha{}_\mu \left({}^{(1)}G\right)^2 L'^4}{2 M_{pl}^2 a^4} + \frac{e^{\frac{(G)}{M_{pl}}} \delta^\alpha{}_\mu \left({}^{(2)}G\right) L'^4}{2 M_{pl} a^4} + \frac{4 e^{\frac{(G)}{M_{pl}}} \delta^\alpha{}_\mu \left({}^{(1)}G\right) L'^3 \left({}^{(1)}L'\right)}{M_{pl} a^4} + \frac{6 e^{\frac{(G)}{M_{pl}}} \delta^\alpha{}_\mu L'^2 \left({}^{(1)}L'\right)^2}{a^4} + \frac{2 e^{\frac{(G)}{M_{pl}}} \delta^\alpha{}_\mu L'^3 \left({}^{(2)}L'\right)}{a^4} + \frac{\left({}^{(2)}B^\alpha\right) G'^2 \bar{n}_\mu}{2a^2} + \frac{M_{pl}^2 \left({}^{(2)}B^\alpha\right) \mathcal{H}^2 \bar{n}_\mu}{a^2} - \frac{M_{pl}^2 \left({}^{(2)}B^\alpha\right) \mathcal{H}' \bar{n}_\mu}{a^2} - \frac{2 e^{\frac{(G)}{M_{pl}}} \left({}^{(2)}B^\alpha\right) L'^4 \bar{n}_\mu}{a^4} - \frac{\left({}^{(1)}G'\right)^2 \bar{n}^\alpha \bar{n}_\mu}{a^2} - \frac{G' \left({}^{(2)}G'\right) \bar{n}^\alpha \bar{n}_\mu}{a^2} +$$

$$\frac{2\, e^{\frac{G}{M_{\rm Pl}}} \left({}^{(1)}G\right)^2 L'^4\, \bar{n}^{\alpha}\, \bar{n}_{\mu}}{M_{\rm Pl}^2\, a^4} + \frac{2\, e^{\frac{G}{M_{\rm Pl}}} \left({}^{(2)}G\right) L'^4\, \bar{n}^{\alpha}\, \bar{n}_{\mu}}{M_{\rm Pl}\, a^4} + \frac{16\, e^{\frac{G}{M_{\rm Pl}}} \left({}^{(1)}G\right) L'^3 \left({}^{(1)}L'\right) \bar{n}^{\alpha}\, \bar{n}_{\mu}}{M_{\rm Pl}\, a^4} +$$

$$\frac{24\, e^{\frac{G}{M_{\rm Pl}}} L'^2 \left({}^{(1)}L'\right)^2 \bar{n}^{\alpha}\, \bar{n}_{\mu}}{a^4} + \frac{8\, e^{\frac{G}{M_{\rm Pl}}} L'^3 \left({}^{(2)}L'\right) \bar{n}^{\alpha}\, \bar{n}_{\mu}}{a^4} + \frac{2\, \delta^{\alpha}{}_{\mu}\, G' \left({}^{(1)}G'\right) \left({}^{(1)}\phi\right)}{a^2} -$$

$$\frac{4\, e^{\frac{G}{M_{\rm Pl}}} \delta^{\alpha}{}_{\mu} \left({}^{(1)}G\right) L'^4 \left({}^{(1)}\phi\right)}{M_{\rm Pl}\, a^4} - \frac{16\, e^{\frac{G}{M_{\rm Pl}}} \delta^{\alpha}{}_{\mu}\, L'^3 \left({}^{(1)}L'\right) \left({}^{(1)}\phi\right)}{a^4} + \frac{4\, G' \left({}^{(1)}G'\right) \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)}{a^2} -$$

$$\frac{16\, e^{\frac{G}{M_{\rm Pl}}} \left({}^{(1)}G\right) L'^4\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)}{M_{\rm Pl}\, a^4} - \frac{64\, e^{\frac{G}{M_{\rm Pl}}} L'^3 \left({}^{(1)}L'\right) \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)}{a^4} - \frac{2\, \delta^{\alpha}{}_{\mu}\, G'^2 \left({}^{(1)}\phi\right)^2}{a^2} +$$

$$\frac{4\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H}^2 \left({}^{(1)}\phi\right)^2}{a^2} + \frac{8\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H}' \left({}^{(1)}\phi\right)^2}{a^2} + \frac{12\, e^{\frac{G}{M_{\rm Pl}}} \delta^{\alpha}{}_{\mu}\, L'^4 \left({}^{(1)}\phi\right)^2}{a^4} - \frac{4\, G'^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)^2}{a^2} -$$

$$\frac{8\, M_{\rm Pl}^2\, \mathcal{H}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)^2}{a^2} + \frac{8\, M_{\rm Pl}^2\, \mathcal{H}'\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)^2}{a^2} + \frac{48\, e^{\frac{G}{M_{\rm Pl}}} L'^4\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right)^2}{a^4} +$$

$$\frac{8\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(1)}\phi\right) \left({}^{(1)}\phi'\right)}{a^2} + \frac{8\, M_{\rm Pl}^2\, \mathcal{H}\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right) \left({}^{(1)}\phi'\right)}{a^2} + \frac{\delta^{\alpha}{}_{\mu}\, G'^2 \left({}^{(2)}\phi\right)}{2\, a^2} - \frac{M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H}^2 \left({}^{(2)}\phi\right)}{a^2} -$$

$$\frac{2\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H}' \left({}^{(2)}\phi\right)}{a^2} - \frac{2\, e^{\frac{G}{M_{\rm Pl}}} \delta^{\alpha}{}_{\mu}\, L'^4 \left({}^{(2)}\phi\right)}{a^4} + \frac{G'^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(2)}\phi\right)}{a^2} + \frac{2\, M_{\rm Pl}^2\, \mathcal{H}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(2)}\phi\right)}{a^2} -$$

$$\frac{2\, M_{\rm Pl}^2\, \mathcal{H}'\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(2)}\phi\right)}{a^2} - \frac{8\, e^{\frac{G}{M_{\rm Pl}}} L'^4\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(2)}\phi\right)}{a^4} - \frac{M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(2)}\phi'\right)}{a^2} - \frac{M_{\rm Pl}^2\, \mathcal{H}\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(2)}\phi'\right)}{a^2} +$$

$$\frac{12\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(1)}\phi\right) \left({}^{(1)}\psi'\right)}{a^2} - \frac{4\, M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(1)}\phi\right) \left({}^{(1)}\psi'\right)}{a^2} + \frac{3\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu} \left({}^{(1)}\phi'\right) \left({}^{(1)}\psi'\right)}{a^2} -$$

$$\frac{M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu} \left({}^{(1)}\phi'\right) \left({}^{(1)}\psi'\right)}{a^2} + \frac{3\, M_{\rm Pl}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi'\right) \left({}^{(1)}\psi'\right)}{a^2} - \frac{12\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(1)}\psi\right) \left({}^{(1)}\psi'\right)}{a^2} +$$

$$\frac{4\, M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(1)}\psi\right) \left({}^{(1)}\psi'\right)}{a^2} - \frac{M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu} \left({}^{(1)}\psi'\right)^2}{a^2} - \frac{3\, M_{\rm Pl}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\psi'\right)^2}{a^2} + \frac{6\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu} \left({}^{(1)}\phi\right) \left({}^{(1)}\psi''\right)}{a^2} -$$

$$\frac{2\, M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu} \left({}^{(1)}\phi\right) \left({}^{(1)}\psi''\right)}{a^2} + \frac{6\, M_{\rm Pl}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\phi\right) \left({}^{(1)}\psi''\right)}{a^2} - \frac{6\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu} \left({}^{(1)}\psi\right) \left({}^{(1)}\psi''\right)}{a^2} +$$

$$\frac{2\, M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu} \left({}^{(1)}\psi\right) \left({}^{(1)}\psi''\right)}{a^2} - \frac{6\, M_{\rm Pl}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(1)}\psi\right) \left({}^{(1)}\psi''\right)}{a^2} - \frac{3\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(2)}\psi'\right)}{a^2} +$$

$$\frac{M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu}\, \mathcal{H} \left({}^{(2)}\psi'\right)}{a^2} - \frac{3\, M_{\rm Pl}^2\, \delta^{\alpha}{}_{\mu} \left({}^{(2)}\psi''\right)}{2\, a^2} + \frac{M_{\rm Pl}^2\, \bar{h}^{\alpha}{}_{\mu} \left({}^{(2)}\psi''\right)}{2\, a^2} - \frac{3\, M_{\rm Pl}^2\, \bar{n}^{\alpha}\, \bar{n}_{\mu} \left({}^{(2)}\psi''\right)}{2\, a^2} +$$

$$\frac{M_{\rm Pl}^2\, \mathcal{H} \left(\bar{\nabla}^{\alpha}\, {}^{(2)}B_{\mu}\right)}{2\, a^2} + \frac{M_{\rm Pl}^2 \left(\bar{\nabla}^{\alpha}\, {}^{(2)}B'_{\mu}\right)}{4\, a^2} + \frac{\left({}^{(1)}G'\right) \bar{n}_{\mu} \left(\bar{\nabla}^{\alpha}\, {}^{(1)}G\right)}{a^2} + \frac{2\, G'\, \bar{n}_{\mu} \left({}^{(1)}\psi\right) \left(\bar{\nabla}^{\alpha}\, {}^{(1)}G\right)}{a^2} +$$

$$\frac{G'\, \bar{n}_{\mu} \left(\bar{\nabla}^{\alpha}\, {}^{(2)}G\right)}{2\, a^2} - \frac{4\, e^{\frac{G}{M_{\rm Pl}}} \left({}^{(1)}G\right) L'^3\, \bar{n}_{\mu} \left(\bar{\nabla}^{\alpha}\, {}^{(1)}L\right)}{M_{\rm Pl}\, a^4} - \frac{12\, e^{\frac{G}{M_{\rm Pl}}} L'^2 \left({}^{(1)}L'\right) \bar{n}_{\mu} \left(\bar{\nabla}^{\alpha}\, {}^{(1)}L\right)}{a^4} +$$

$$\frac{8\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^3\,\bar{n}_\mu\left({}^{(1)}\phi\right)\left(\bar\nabla^\alpha\,{}^{(1)}L\right)}{a^4}-\frac{8\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^3\,\bar{n}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla^\alpha\,{}^{(1)}L\right)}{a^4}-\frac{2\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^3\,\bar{n}_\mu\left(\bar\nabla^\alpha\,{}^{(2)}L\right)}{a^4}-$$

$$\frac{4\,M_{\rm Pl}^2\,\mathcal{H}\,\bar{n}_\mu\left({}^{(1)}\phi\right)\left(\bar\nabla^\alpha\,{}^{(1)}\phi\right)}{a^2}+\frac{4\,M_{\rm Pl}^2\,\mathcal{H}\,\bar{n}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla^\alpha\,{}^{(1)}\phi\right)}{a^2}-\frac{2\,M_{\rm Pl}^2\,\bar{n}_\mu\left({}^{(1)}\psi'\right)\left(\bar\nabla^\alpha\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{M_{\rm Pl}^2\,\mathcal{H}\,\bar{n}_\mu\left(\bar\nabla^\alpha\,{}^{(2)}\phi\right)}{a^2}+\frac{4\,M_{\rm Pl}^2\,\bar{n}_\mu\left({}^{(1)}\psi'\right)\left(\bar\nabla^\alpha\,{}^{(1)}\psi\right)}{a^2}+\frac{8\,M_{\rm Pl}^2\,\bar{n}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla^\alpha\,{}^{(1)}\psi'\right)}{a^2}+$$

$$\frac{M_{\rm Pl}^2\,\bar{n}_\mu\left(\bar\nabla^\alpha\,{}^{(2)}\psi'\right)}{a^2}-\frac{M_{\rm Pl}^2\,\bar{n}_\mu\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}B^\alpha\right)}{4\,a^2}-\frac{M_{\rm Pl}^2\,\bar{n}^\alpha\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}B_\mu\right)}{4\,a^2}+\frac{M_{\rm Pl}^2\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}E^\alpha{}_\mu\right)}{2\,a^2}+$$

$$\frac{2\,M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left({}^{(1)}\phi\right)\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+\frac{2\,M_{\rm Pl}^2\,\bar{n}^\alpha\,\bar{n}_\mu\left({}^{(1)}\phi\right)\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{2\,M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-$$

$$\frac{2\,M_{\rm Pl}^2\,\bar{n}^\alpha\,\bar{n}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}\phi\right)}{2\,a^2}-\frac{M_{\rm Pl}^2\,\bar{n}^\alpha\,\bar{n}_\mu\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}\phi\right)}{2\,a^2}+$$

$$\frac{8\,M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(1)}\psi\right)}{a^2}-\frac{4\,M_{\rm Pl}^2\,\bar{h}^\alpha{}_\mu\left({}^{(1)}\psi\right)\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(1)}\psi\right)}{a^2}+\frac{M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}\psi\right)}{a^2}-$$

$$\frac{M_{\rm Pl}^2\,\bar{h}^\alpha{}_\mu\left(\bar\nabla_\beta\bar\nabla^\beta\,{}^{(2)}\psi\right)}{2\,a^2}+\frac{\delta^\alpha{}_\mu\left(\bar\nabla_\beta\,{}^{(1)}G\right)\left(\bar\nabla^\beta\,{}^{(1)}G\right)}{2\,a^2}-\frac{2\,e^{\frac{(G)}{M_{\rm Pl}}}\,\delta^\alpha{}_\mu\,L'^2\left(\bar\nabla_\beta\,{}^{(1)}L\right)\left(\bar\nabla^\beta\,{}^{(1)}L\right)}{a^4}-$$

$$\frac{4\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^2\,\bar{n}^\alpha\,\bar{n}_\mu\left(\bar\nabla_\beta\,{}^{(1)}L\right)\left(\bar\nabla^\beta\,{}^{(1)}L\right)}{a^4}+\frac{M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\phi\right)\left(\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{\rm Pl}^2\,\bar{n}^\alpha\,\bar{n}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\phi\right)\left(\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\psi\right)\left(\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{M_{\rm Pl}^2\,\bar{h}^\alpha{}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\psi\right)\left(\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{\rm Pl}^2\,\bar{n}^\alpha\,\bar{n}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\psi\right)\left(\bar\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{3\,M_{\rm Pl}^2\,\delta^\alpha{}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\psi\right)\left(\bar\nabla^\beta\,{}^{(1)}\psi\right)}{a^2}-\frac{M_{\rm Pl}^2\,\bar{h}^\alpha{}_\mu\left(\bar\nabla_\beta\,{}^{(1)}\psi\right)\left(\bar\nabla^\beta\,{}^{(1)}\psi\right)}{a^2}+\frac{M_{\rm Pl}^2\,\mathcal{H}\left(\bar\nabla_\mu\,{}^{(2)}B^\alpha\right)}{2\,a^2}+$$

$$\frac{M_{\rm Pl}^2\left(\bar\nabla_\mu\,{}^{(2)}B'^\alpha\right)}{4\,a^2}+\frac{\left({}^{(1)}G'\right)\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(1)}G\right)}{a^2}-\frac{2\,G'\,\bar{n}^\alpha\left({}^{(1)}\phi\right)\left(\bar\nabla_\mu\,{}^{(1)}G\right)}{a^2}-\frac{\left(\bar\nabla^\alpha\,{}^{(1)}G\right)\left(\bar\nabla_\mu\,{}^{(1)}G\right)}{a^2}+$$

$$\frac{G'\,\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(2)}G\right)}{2\,a^2}-\frac{4\,e^{\frac{(G)}{M_{\rm Pl}}}\left({}^{(1)}G\right)L'^3\,\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(1)}L\right)}{M_{\rm Pl}\,a^4}-\frac{12\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^2\left({}^{(1)}L'\right)\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(1)}L\right)}{a^4}+$$

$$\frac{16\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^3\,\bar{n}^\alpha\left({}^{(1)}\phi\right)\left(\bar\nabla_\mu\,{}^{(1)}L\right)}{a^4}+\frac{4\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^2\left(\bar\nabla^\alpha\,{}^{(1)}L\right)\left(\bar\nabla_\mu\,{}^{(1)}L\right)}{a^4}-\frac{2\,e^{\frac{(G)}{M_{\rm Pl}}}\,L'^3\,\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(2)}L\right)}{a^4}-$$

$$\frac{8\,M_{\rm Pl}^2\,\mathcal{H}\,\bar{n}^\alpha\left({}^{(1)}\phi\right)\left(\bar\nabla_\mu\,{}^{(1)}\phi\right)}{a^2}-\frac{2\,M_{\rm Pl}^2\,\bar{n}^\alpha\left({}^{(1)}\psi'\right)\left(\bar\nabla_\mu\,{}^{(1)}\phi\right)}{a^2}-\frac{M_{\rm Pl}^2\left(\bar\nabla^\alpha\,{}^{(1)}\phi\right)\left(\bar\nabla_\mu\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{M_{\rm Pl}^2\left(\bar\nabla^\alpha\,{}^{(1)}\psi\right)\left(\bar\nabla_\mu\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{\rm Pl}^2\,\mathcal{H}\,\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(2)}\phi\right)}{a^2}+\frac{4\,M_{\rm Pl}^2\,\bar{n}^\alpha\left({}^{(1)}\psi'\right)\left(\bar\nabla_\mu\,{}^{(1)}\psi\right)}{a^2}+$$

$$\frac{M_{\rm Pl}^2\left(\bar\nabla^\alpha\,{}^{(1)}\phi\right)\left(\bar\nabla_\mu\,{}^{(1)}\psi\right)}{a^2}-\frac{3\,M_{\rm Pl}^2\left(\bar\nabla^\alpha\,{}^{(1)}\psi\right)\left(\bar\nabla_\mu\,{}^{(1)}\psi\right)}{a^2}-\frac{4\,M_{\rm Pl}^2\,\bar{n}^\alpha\left({}^{(1)}\phi\right)\left(\bar\nabla_\mu\,{}^{(1)}\psi'\right)}{a^2}+$$

$$\frac{4\,M_{\rm Pl}^2\,\bar{n}^\alpha\left({}^{(1)}\psi\right)\left(\bar\nabla_\mu\,{}^{(1)}\psi'\right)}{a^2}+\frac{M_{\rm Pl}^2\,\bar{n}^\alpha\left(\bar\nabla_\mu\,{}^{(2)}\psi'\right)}{a^2}-\frac{2\,M_{\rm Pl}^2\left({}^{(1)}\phi\right)\left(\bar\nabla_\mu\bar\nabla^\alpha\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{2 M_{pl}^2 \left({}^{(1)}\psi\right)\left(\bar\nabla_\mu\bar\nabla^{\alpha\,(1)}\phi\right)}{a^2} + \frac{M_{pl}^2\left(\bar\nabla_\mu\bar\nabla^{\alpha\,(2)}\phi\right)}{2\,a^2} - \frac{4 M_{pl}^2\left({}^{(1)}\psi\right)\left(\bar\nabla_\mu\bar\nabla^{\alpha\,(1)}\psi\right)}{a^2} - \frac{M_{pl}^2\left(\bar\nabla_\mu\bar\nabla^{\alpha\,(2)}\psi\right)}{2\,a^2}$$

In[∘]:= `Collect[SecondOrdEqMotion, {n[α], n[-μ], delta[α, -μ], h[α, -μ]}];`

In[∘]:= `% -` $\bar{n}_\mu \Bigg(\frac{\left({}^{(2)}B^\alpha\right)\acute{G}^2}{2\,a^2} + \frac{M_{pl}^2\left({}^{(2)}B^\alpha\right)\mathcal{H}^2}{a^2} - \frac{M_{pl}^2\left({}^{(2)}B^\alpha\right)\acute{\mathcal{H}}}{a^2} - \frac{2\,e^{\frac{(G)}{M_{pl}}}\left({}^{(2)}B^\alpha\right)\acute{L}^4}{a^4} + \frac{\left({}^{(1)}\acute{G}\right)\left(\bar\nabla^{\alpha\,(1)}G\right)}{a^2} +$

$$\frac{2\,\acute{G}\left({}^{(1)}\psi\right)\left(\bar\nabla^{\alpha\,(1)}G\right)}{a^2} + \frac{\acute{G}\left(\bar\nabla^{\alpha\,(2)}G\right)}{2\,a^2} - \frac{4\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)\acute{L}^3\left(\bar\nabla^{\alpha\,(1)}L\right)}{M_{pl}\,a^4} - \frac{12\,e^{\frac{(G)}{M_{pl}}}\acute{L}^2\left({}^{(1)}\acute{L}\right)\left(\bar\nabla^{\alpha\,(1)}L\right)}{a^4} +$$

$$\frac{8\,e^{\frac{(G)}{M_{pl}}}\acute{L}^3\left({}^{(1)}\phi\right)\left(\bar\nabla^{\alpha\,(1)}L\right)}{a^4} - \frac{8\,e^{\frac{(G)}{M_{pl}}}\acute{L}^3\left({}^{(1)}\psi\right)\left(\bar\nabla^{\alpha\,(1)}L\right)}{a^4} - \frac{2\,e^{\frac{(G)}{M_{pl}}}\acute{L}^3\left(\bar\nabla^{\alpha\,(2)}L\right)}{a^4} -$$

$$\frac{4 M_{pl}^2\mathcal{H}\left({}^{(1)}\phi\right)\left(\bar\nabla^{\alpha\,(1)}\phi\right)}{a^2} + \frac{4 M_{pl}^2\mathcal{H}\left({}^{(1)}\psi\right)\left(\bar\nabla^{\alpha\,(1)}\phi\right)}{a^2} - \frac{2 M_{pl}^2\left({}^{(1)}\acute{\psi}\right)\left(\bar\nabla^{\alpha\,(1)}\phi\right)}{a^2} + \frac{M_{pl}^2\mathcal{H}\left(\bar\nabla^{\alpha\,(2)}\phi\right)}{a^2} +$$

$$\frac{4 M_{pl}^2\left({}^{(1)}\acute{\psi}\right)\left(\bar\nabla^{\alpha\,(1)}\psi\right)}{a^2} + \frac{8 M_{pl}^2\left({}^{(1)}\psi\right)\left(\bar\nabla^{\alpha\,(1)}\acute{\psi}\right)}{a^2} + \frac{M_{pl}^2\left(\bar\nabla^{\alpha\,(2)}\acute{\psi}\right)}{a^2} - \frac{M_{pl}^2\left(\bar\nabla_\beta\bar\nabla^{\beta\,(2)}B^\alpha\right)}{4\,a^2}\Bigg) -$$

$$\bar{h}^\alpha{}_\mu\Bigg(-\frac{4 M_{pl}^2\mathcal{H}\left({}^{(1)}\phi\right)\left({}^{(1)}\acute{\psi}\right)}{a^2} - \frac{M_{pl}^2\left({}^{(1)}\acute{\phi}\right)\left({}^{(1)}\acute{\psi}\right)}{a^2} + \frac{4 M_{pl}^2\mathcal{H}\left({}^{(1)}\psi\right)\left({}^{(1)}\acute{\psi}\right)}{a^2} - \frac{M_{pl}^2\left({}^{(1)}\acute{\psi}\right)^2}{a^2} -$$

$$\frac{2 M_{pl}^2\left({}^{(1)}\phi\right)\left({}^{(1)}\acute{\acute{\psi}}\right)}{a^2} + \frac{2 M_{pl}^2\left({}^{(1)}\psi\right)\left({}^{(1)}\acute{\acute{\psi}}\right)}{a^2} + \frac{M_{pl}^2\mathcal{H}\left({}^{(2)}\acute{\psi}\right)}{a^2} + \frac{M_{pl}^2\left({}^{(2)}\acute{\acute{\psi}}\right)}{2\,a^2} -$$

$$\frac{4 M_{pl}^2\left({}^{(1)}\psi\right)\left(\bar\nabla_\beta\bar\nabla^{\beta\,(1)}\psi\right)}{a^2} - \frac{M_{pl}^2\left(\bar\nabla_\beta\bar\nabla^{\beta\,(2)}\psi\right)}{2\,a^2} - \frac{M_{pl}^2\left(\bar\nabla_\beta{}^{(1)}\psi\right)\left(\bar\nabla^{\beta\,(1)}\phi\right)}{a^2} - \frac{M_{pl}^2\left(\bar\nabla_\beta{}^{(1)}\psi\right)\left(\bar\nabla^{\beta\,(1)}\psi\right)}{a^2}\Bigg) -$$

$$\delta^\alpha{}_\mu\Bigg(\frac{2\,e^{-\frac{2\,(G)}{M_{pl}}}\left({}^{(1)}G\right)^2}{b_*^2} - \frac{9\,e^{-\frac{3\,(G)}{M_{pl}}}f^2\pi\left({}^{(1)}G\right)^2}{b_*^2 M_{pl}^2} - \frac{\left({}^{(1)}\acute{G}\right)^2}{2\,a^2} - \frac{e^{-\frac{2\,(G)}{M_{pl}}}M_{pl}\left({}^{(2)}G\right)}{b_*^2} +$$

$$\frac{3\,e^{-\frac{3\,(G)}{M_{pl}}}f^2\pi\left({}^{(2)}G\right)}{b_*^2 M_{pl}} - \frac{\acute{G}\left({}^{(2)}\acute{G}\right)}{2\,a^2} + \frac{e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)^2\acute{L}^4}{2 M_{pl}^2 a^4} + \frac{e^{\frac{(G)}{M_{pl}}}\left({}^{(2)}G\right)\acute{L}^4}{2 M_{pl} a^4} + \frac{4\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)\acute{L}^3\left({}^{(1)}\acute{L}\right)}{M_{pl}\,a^4} +$$

$$\frac{6\,e^{\frac{(G)}{M_{pl}}}\acute{L}^2\left({}^{(1)}\acute{L}\right)^2}{a^4} + \frac{2\,e^{\frac{(G)}{M_{pl}}}\acute{L}^3\left({}^{(2)}\acute{L}\right)}{a^4} + \frac{2\,\acute{G}\left({}^{(1)}\acute{G}\right)\left({}^{(1)}\phi\right)}{a^2} - \frac{4\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)\acute{L}^4\left({}^{(1)}\phi\right)}{M_{pl}\,a^4} -$$

$$\frac{16\,e^{\frac{(G)}{M_{pl}}}\acute{L}^3\left({}^{(1)}\acute{L}\right)\left({}^{(1)}\phi\right)}{a^4} - \frac{2\,\acute{G}^2\left({}^{(1)}\phi\right)^2}{a^2} + \frac{4 M_{pl}^2\mathcal{H}^2\left({}^{(1)}\phi\right)^2}{a^2} + \frac{8 M_{pl}^2\acute{\mathcal{H}}\left({}^{(1)}\phi\right)^2}{a^2} +$$

$$\frac{12\,e^{\frac{(G)}{M_{pl}}}\acute{L}^4\left({}^{(1)}\phi\right)^2}{a^4} + \frac{8 M_{pl}^2\mathcal{H}\left({}^{(1)}\phi\right)\left({}^{(1)}\acute{\phi}\right)}{a^2} + \frac{\acute{G}^2\left({}^{(2)}\phi\right)}{2\,a^2} - \frac{M_{pl}^2\mathcal{H}^2\left({}^{(2)}\phi\right)}{a^2} - \frac{2 M_{pl}^2\acute{\mathcal{H}}\left({}^{(2)}\phi\right)}{a^2} -$$

$$\frac{2\,e^{\frac{(G)}{M_{pl}}}\acute{L}^4\left({}^{(2)}\phi\right)}{a^4} - \frac{M_{pl}^2\mathcal{H}\left({}^{(2)}\acute{\phi}\right)}{a^2} + \frac{12 M_{pl}^2\mathcal{H}\left({}^{(1)}\phi\right)\left({}^{(1)}\acute{\psi}\right)}{a^2} + \frac{3 M_{pl}^2\left({}^{(1)}\acute{\phi}\right)\left({}^{(1)}\acute{\psi}\right)}{a^2} -$$

$$\frac{12\,M_{pl}^2\,\mathcal{H}\left({}^{(1)}\psi\right)\left({}^{(1)}\psi'\right)}{a^2}+\frac{6\,M_{pl}^2\left({}^{(1)}\phi\right)\left({}^{(1)}\psi''\right)}{a^2}-\frac{6\,M_{pl}^2\left({}^{(1)}\psi\right)\left({}^{(1)}\psi''\right)}{a^2}-\frac{3\,M_{pl}^2\,\mathcal{H}\left({}^{(2)}\psi'\right)}{a^2}-$$

$$\frac{3\,M_{pl}^2\left({}^{(2)}\psi''\right)}{2\,a^2}+\frac{2\,M_{pl}^2\left({}^{(1)}\phi\right)\left(\nabla_\beta\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{2\,M_{pl}^2\left({}^{(1)}\psi\right)\left(\nabla_\beta\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{M_{pl}^2\left(\nabla_\beta\nabla^\beta\,{}^{(2)}\phi\right)}{2\,a^2}+$$

$$\frac{8\,M_{pl}^2\left({}^{(1)}\psi\right)\left(\nabla_\beta\nabla^\beta\,{}^{(1)}\psi\right)}{a^2}+\frac{M_{pl}^2\left(\nabla_\beta\nabla^\beta\,{}^{(2)}\psi\right)}{a^2}+\frac{\left(\nabla_\beta\,{}^{(1)}G\right)\left(\nabla^\beta\,{}^{(1)}G\right)}{2\,a^2}-\frac{2\,e^{\frac{(G)}{M_{pl}}}\,L'^2\left(\nabla_\beta\,{}^{(1)}L\right)\left(\nabla^\beta\,{}^{(1)}L\right)}{a^4}+$$

$$\left.\frac{M_{pl}^2\left(\nabla_\beta\,{}^{(1)}\phi\right)\left(\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{pl}^2\left(\nabla_\beta\,{}^{(1)}\psi\right)\left(\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+\frac{3\,M_{pl}^2\left(\nabla_\beta\,{}^{(1)}\psi\right)\left(\nabla^\beta\,{}^{(1)}\psi\right)}{a^2}\right)-$$

$$\bar{n}^\alpha\left(-\frac{M_{pl}^2\left(\nabla_\beta\nabla^\beta\,{}^{(2)}B_\mu\right)}{4\,a^2}+\bar{n}_\mu\left(-\frac{\left({}^{(1)}G'\right)^2}{a^2}-\frac{G'\left({}^{(2)}G'\right)}{a^2}+\frac{2\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)^2 L'^4}{M_{pl}^2\,a^4}+\frac{2\,e^{\frac{(G)}{M_{pl}}}\left({}^{(2)}G\right)L'^4}{M_{pl}\,a^4}+\right.\right.$$

$$\frac{16\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)L'^3\left({}^{(1)}L'\right)}{M_{pl}\,a^4}+\frac{24\,e^{\frac{(G)}{M_{pl}}}\,L'^2\left({}^{(1)}L'\right)^2}{a^4}+\frac{8\,e^{\frac{(G)}{M_{pl}}}\,L'^3\left({}^{(2)}L'\right)}{a^4}+\frac{4\,G'\left({}^{(1)}G'\right)\left({}^{(1)}\phi\right)}{a^2}-$$

$$\frac{16\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)L'^4\left({}^{(1)}\phi\right)}{M_{pl}\,a^4}-\frac{64\,e^{\frac{(G)}{M_{pl}}}\,L'^3\left({}^{(1)}L'\right)\left({}^{(1)}\phi\right)}{a^4}-\frac{4\,G'^2\left({}^{(1)}\phi\right)^2}{a^2}-\frac{8\,M_{pl}^2\,\mathcal{H}^2\left({}^{(1)}\phi\right)^2}{a^2}+$$

$$\frac{8\,M_{pl}^2\,\mathcal{H}'\left({}^{(1)}\phi\right)^2}{a^2}+\frac{48\,e^{\frac{(G)}{M_{pl}}}\,L'^4\left({}^{(1)}\phi\right)^2}{a^4}+\frac{8\,M_{pl}^2\,\mathcal{H}\left({}^{(1)}\phi\right)\left({}^{(1)}\phi'\right)}{a^2}+\frac{G'^2\left({}^{(2)}\phi\right)}{a^2}+$$

$$\frac{2\,M_{pl}^2\,\mathcal{H}^2\left({}^{(2)}\phi\right)}{a^2}-\frac{2\,M_{pl}^2\,\mathcal{H}'\left({}^{(2)}\phi\right)}{a^2}-\frac{8\,e^{\frac{(G)}{M_{pl}}}\,L'^4\left({}^{(2)}\phi\right)}{a^4}-\frac{M_{pl}^2\,\mathcal{H}\left({}^{(2)}\phi'\right)}{a^2}+$$

$$\frac{3\,M_{pl}^2\left({}^{(1)}\phi'\right)\left({}^{(1)}\psi'\right)}{a^2}-\frac{3\,M_{pl}^2\left({}^{(1)}\psi'\right)^2}{a^2}+\frac{6\,M_{pl}^2\left({}^{(1)}\phi\right)\left({}^{(1)}\psi''\right)}{a^2}-\frac{6\,M_{pl}^2\left({}^{(1)}\psi\right)\left({}^{(1)}\psi''\right)}{a^2}-$$

$$\frac{3\,M_{pl}^2\left({}^{(2)}\psi''\right)}{2\,a^2}+\frac{2\,M_{pl}^2\left({}^{(1)}\phi\right)\left(\nabla_\beta\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{2\,M_{pl}^2\left({}^{(1)}\psi\right)\left(\nabla_\beta\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}-\frac{M_{pl}^2\left(\nabla_\beta\nabla^\beta\,{}^{(2)}\phi\right)}{2\,a^2}-$$

$$\left.\frac{4\,e^{\frac{(G)}{M_{pl}}}\,L'^2\left(\nabla_\beta\,{}^{(1)}L\right)\left(\nabla^\beta\,{}^{(1)}L\right)}{a^4}+\frac{M_{pl}^2\left(\nabla_\beta\,{}^{(1)}\phi\right)\left(\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{pl}^2\left(\nabla_\beta\,{}^{(1)}\psi\right)\left(\nabla^\beta\,{}^{(1)}\phi\right)}{a^2}\right)+$$

$$\frac{\left({}^{(1)}G'\right)\left(\nabla_\mu\,{}^{(1)}G\right)}{a^2}-\frac{2\,G'\left({}^{(1)}\phi\right)\left(\nabla_\mu\,{}^{(1)}G\right)}{a^2}+\frac{G'\left(\nabla_\mu\,{}^{(2)}G\right)}{2\,a^2}-\frac{4\,e^{\frac{(G)}{M_{pl}}}\left({}^{(1)}G\right)L'^3\left(\nabla_\mu\,{}^{(1)}L\right)}{M_{pl}\,a^4}-$$

$$\frac{12\,e^{\frac{(G)}{M_{pl}}}\,L'^2\left({}^{(1)}L'\right)\left(\nabla_\mu\,{}^{(1)}L\right)}{a^4}+\frac{16\,e^{\frac{(G)}{M_{pl}}}\,L'^3\left({}^{(1)}\phi\right)\left(\nabla_\mu\,{}^{(1)}L\right)}{a^4}-\frac{2\,e^{\frac{(G)}{M_{pl}}}\,L'^3\left(\nabla_\mu\,{}^{(2)}L\right)}{a^4}-$$

$$\frac{8\,M_{pl}^2\,\mathcal{H}\left({}^{(1)}\phi\right)\left(\nabla_\mu\,{}^{(1)}\phi\right)}{a^2}-\frac{2\,M_{pl}^2\left({}^{(1)}\psi'\right)\left(\nabla_\mu\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{pl}^2\,\mathcal{H}\left(\nabla_\mu\,{}^{(2)}\phi\right)}{a^2}+$$

$$\left.\frac{4\,M_{pl}^2\left({}^{(1)}\psi'\right)\left(\nabla_\mu\,{}^{(1)}\psi\right)}{a^2}-\frac{4\,M_{pl}^2\left({}^{(1)}\phi\right)\left(\nabla_\mu\,{}^{(1)}\psi'\right)}{a^2}+\frac{4\,M_{pl}^2\left({}^{(1)}\psi\right)\left(\nabla_\mu\,{}^{(1)}\psi'\right)}{a^2}+\frac{M_{pl}^2\left(\nabla_\mu\,{}^{(2)}\psi'\right)}{a^2}\right);$$

In[∘]:= `SecondOrdNoDiagonalPart = %`

Out[∘]=

$$-\frac{M_{p1}^2\left({}^{(2)}E''^{\alpha}{}_{\mu}\right)}{2\,a^2}-\frac{M_{p1}^2\left({}^{(2)}E'^{\alpha}{}_{\mu}\right)\mathcal{H}}{a^2}+\frac{M_{p1}^2\,\mathcal{H}\left(\overline{\nabla}^{\alpha}\,{}^{(2)}B_{\mu}\right)}{2\,a^2}+\frac{M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(2)}B'_{\mu}\right)}{4\,a^2}+$$

$$\frac{M_{p1}^2\left(\overline{\nabla}_{\beta}\overline{\nabla}^{\beta}\,{}^{(2)}E^{\alpha}{}_{\mu}\right)}{2\,a^2}+\frac{M_{p1}^2\,\mathcal{H}\left(\overline{\nabla}_{\mu}\,{}^{(2)}B^{\alpha}\right)}{2\,a^2}+\frac{M_{p1}^2\left(\overline{\nabla}_{\mu}\,{}^{(2)}B'^{\alpha}\right)}{4\,a^2}-\frac{\left(\overline{\nabla}^{\alpha}\,{}^{(1)}G\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}G\right)}{a^2}+$$

$$\frac{4\,e^{\frac{(G)}{M_{p1}}}\,L'^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}L\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}L\right)}{a^4}-\frac{M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\psi\right)}{a^2}-\frac{3\,M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\psi\right)}{a^2}-\frac{2\,M_{p1}^2\left({}^{(1)}\phi\right)\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)}{a^2}+$$

$$\frac{2\,M_{p1}^2\left({}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)}{a^2}+\frac{M_{p1}^2\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(2)}\phi\right)}{2\,a^2}-\frac{4\,M_{p1}^2\left({}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(1)}\psi\right)}{a^2}-\frac{M_{p1}^2\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(2)}\psi\right)}{2\,a^2}$$

In[∘]:= `SecondOrdNoDiagonalPart -`

$$\left(\frac{M_{p1}^2\,\mathcal{H}\left(\overline{\nabla}^{\alpha}\,{}^{(2)}B_{\mu}\right)}{2\,a^2}+\frac{M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(2)}B'_{\mu}\right)}{4\,a^2}+\frac{M_{p1}^2\,\mathcal{H}\left(\overline{\nabla}_{\mu}\,{}^{(2)}B^{\alpha}\right)}{2\,a^2}+\frac{M_{p1}^2\left(\overline{\nabla}_{\mu}\,{}^{(2)}B'^{\alpha}\right)}{4\,a^2}+\frac{M_{p1}^2\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(2)}\phi\right)}{2\,a^2}-\frac{M_{p1}^2\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(2)}\psi\right)}{2\,a^2}\right);$$

In[∘]:= `FinalResult = % // Simplify`

Out[∘]=

$$\frac{4\,e^{\frac{(G)}{M_{p1}}}\,L'^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}L\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}L\right)}{a^4}-$$

$$\frac{1}{2\,a^2}\Big(M_{p1}^2\left({}^{(2)}E''^{\alpha}{}_{\mu}\right)+2\,M_{p1}^2\left({}^{(2)}E'^{\alpha}{}_{\mu}\right)\mathcal{H}-M_{p1}^2\left(\overline{\nabla}_{\beta}\overline{\nabla}^{\beta}\,{}^{(2)}E^{\alpha}{}_{\mu}\right)+2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}G\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}G\right)+$$

$$2\,M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\phi\right)-2\,M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\phi\right)-2\,M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)\left(\overline{\nabla}_{\mu}\,{}^{(1)}\psi\right)+6\,M_{p1}^2\left(\overline{\nabla}^{\alpha}\,{}^{(1)}\psi\right)$$

$$\left(\overline{\nabla}_{\mu}\,{}^{(1)}\psi\right)+4\,M_{p1}^2\left({}^{(1)}\phi\right)\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)-4\,M_{p1}^2\left({}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(1)}\phi\right)+8\,M_{p1}^2\left({}^{(1)}\psi\right)\left(\overline{\nabla}_{\mu}\overline{\nabla}^{\alpha}\,{}^{(1)}\psi\right)\Big)$$

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