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A modern approach to Feynman Integrals and Differential Equations

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Abstract

In this Thesis we discuss recent ideas concerning the evaluation of multi-loop Feynman Integrals in the context of Dimensional Regularization.

In the first part we study relations fulfilled by Feynman Integrals, with a particular focus on Integration By Parts Identities (IBPs). We present the latter both in the standard *momentum space* representation, where we essentially we integrate a set of denominators over the loop momenta, and in Baikov representation, in which denominators are promoted to integration variables, and the *Gram determinant* of the whole set of loop and external momenta, referred to as *Baikov Polynomial*, emerges as a leading object.

IBPs in Baikov representation naturally lead to the study and the implementation of concepts and algorithms developed in Computational Algebraic Geometry, such as *Sygyzies*. We present a `Mathematica` code devoted to IBPs generation in Baikov representation.

In the second part we focus on the Method of *Differentil Equations* for Feynman Integrals, with a particular emphasis on the algorithm based on the *Magnus Exponential* to achieve the *Canonical Form*: in both of them, an underlying algebraic structure arises. We present applications relevant to phenomenology: namely we compute the Mis for the 1-loop box which appear in the $\mu e \to \mu e$ scattering and we obtain the Canonical Form for a 2-loops *non plaanar* 3-points function, which is part of a wider task regarding the calculation of the 2-loops *non planar* box which is needed for the $q\bar{q} \to t\bar{t}$ process.

In the last part we analyze the role of Cut Integrals as solutions of *homogeneous* Differential Equations, and their implementation in Baikov representation. Working on an explicit example, we show how different IBPs-compatible integration regions lead to different solutions for a higher order Differential Equation.

Contents

Introduction

Scattering Amplitudes constitute one of the key elements in QFT, and undoubtedly play an essential role for Particle Physics Phenomenology, since they are related to the transition probability between an initial state $|i\rangle$ and a finial state $|f\rangle$. Working in an appropriate energy regime, Perturbation Theory allows us to expand the Amplitude as a Series in the small coupling constant; each term in the Series can be represented through Feynman Diagrams, and the associated analytic expression is dictated by Feynman Rules. The Leading Order term, the so called Tree Level term, is, by far, the easiest to compute; the other Diagrams in the series involve *loops*, namely closed lines with virtual particles circulating inside. Feynman Rules force us to sum or, better, to integrate over all the possible momenta carried by virtual particles. On the one hand, dealing with these multivariate integrals, referred to as *multiloop* integrals, constitutes one of the most challenging part in the calculation, but on the other hand its mandatory in order to make reliable predictions, and fully exploit the possibilities offered by the experiments, surely the LHC but also several project in the so called Physics Beyond Colliders, [67], [68]. Moreover it is worth stressing that *multiloop integrals* are almost ubiquitous in Physics: they are needed for QCD scattering processes, Top physics, Higgs physics, QED corrections to lepton form factors, static parameters, forward-backward asymmetry, as well as for Supergravity Theories and Critical Exponents.

As a remarkable fact, integrals potentially involved in the calculation turn to be not independent.

Among others, Integration By Parts Identities (IBPs) [69], which arise from the vanishing of a total derivative under the integral sign, provide a huge set of relations between integrals. Then, the Laporta Algorithm [2], allows us to identify a minimal set of integrals, the so called Master Integral (MIs), which constitute a *basis*for the whole set of integrals (we stress that just the *size* of the basis is *dictated by the problem*, while the choice of the basis elements is arbitrary). Thus, the remaining ones can be expressed as combination of these carefully chosen Mis; the coefficients in these relations turn to be rational functions in d, namely the space time dimensions, and in the kinematics variables. Thanks to this strategy we can impressively reduce the number of *hard multiloop* integrals by several order of magnitudes, and so, not surprisingly, the Laporta algorithm has been implemented in various public (and private) codes [52]-[59]. Despite this fact, there is still room for improvement; it is not clear which representation for multiloop integrals is the most natural and suitable one to perform the IBPs. The Baikov representation (which appears several times through this work) seems to be very promising in order to implement ideas and algorithms related to Computational Algebraic Geometry, such as Sygyzy Equations [9], [11] or module-intersection computations [48]-[70], needed in order trim and handle the tremendous number of equations.

Once the MIs have been identified, we are left with the problem concerning their evaluation. Among others, a very powerful strategy is the method of *Differential Equations for Feynman Integrals*. In fact, we can consider the unknown MIs as functions of internal

masses, as originally advocated by Kotikov [6], or kinematic invariants, as proposed by Remiddi [7] and Gherman and Remiddi [8] (see [1] and [71] for a review); deriving the MIs w.r.t. the latter variables, and rearranging the output through IBPs, we obtain a suitable system of 1$^{\text{st}}$ order Differential Equations (DEQs system) and we eventually recast the latter as a single higher order DEQ for *one* of the MI involved. Solving this *unique* DEQ, we determine *all* the MIs, avoiding a direct integration over the loop momenta (the other MIs are related to the one we are considering via differential operators). Over recent years, triggered by Henn [15], a more systematic and, in a certain sense, *algebraic* picture has emerged. Instead of obtaining an higher order DEQ, the system itself is solved in terms of *iterated integrals*, by means of the *Dyson Series*. More in detail, choosing an appropriate basis of MIs, the so called *Canonical Basis*, ϵ, namely the dimensional regularization parameter, results completely factorized from the kinematics and the system turns to have *simple poles*. Given these conditions, the solution naturally arise as a *Taylor Expansion* in ϵ involving Generalized Polylogarithms (GPLs). Even if it is not clear *if* the *Canonical Form* can be found for any process, and, a fortiori, no *completely general* algorithm is known to achieve such a basis, several criteria have been proposed, starting from different *ansatz*. Among others, the one based on the *Magnus Exponential* [23], intensively used in this work, requires a DEQs sysetm *linear* in ϵ, and provides a *rotation matrix* in the space of Master Integrals, which is very closely related to the solution of the DEQ system itself at $\epsilon = 0$, that reabsorbs the $\mathcal{O}(\epsilon)$ term, leading to the Canonical Form.
The importance of Cut Integrals, i.e Integrals in which we impose virtual particles to be *on shell* was understand long ago; since their appearance in the pioneering works by Cutkosky [72], they have been a formidable tool in the study of Scattering Amplitudes. Amazingly, Cut Integrals turn to have a key role also in the context of Differential Equations. In fact, as it was shown for the first time in [74] and then generalized in [43], [44], Maximal Cut Integrals (i.e. integrals in which the whole set of denominators is cut) solve the *homogeneous* part of the DEQ. This is a remarkable fact since, for example, once the homogeneous solution is known, the non homogeneous one can be recovered via standard techniques. Considering an higher order DEQ, *no general strategy*is known to solve its homogeneous part, and having this *physical input* (i.e. Cut Integrlas) prevent us from a case-by-case analysis, which is heavily limited by the classification of Differential Equations and their solutions present in the Mathematical Literature.
As it was shown by many Authors [45]-[47], the Baikov representation turns to be a very powerful tool for computing Cut Integrals. This representation does not only make the *Cut procedure* almost straightforward, but it provides a better understanding of the whole picture. Thanks to the Lee-Pomeransky [65] criterium, we can determine *a priori* the number of MIs (for a given *topology*), and so the order of the corresponding DEQ. Moreover it naturally suggests the different, IBPs-compatible, integrations regions for the *residual integration variables* (i.e. variables that are not fixed by the Cut; the latter are always present in 2-loop cases). Recalling that, a m^{th}-order DEQ requires m independent homogeneous solutions, the the integration over the different regions mentioned above, provides a whole set of homogeneous solutions.

This work is organized as follows. In the first Chapter we introduce the key objects of this work, namely scalar Feynman Integrals. After briefly describing some alternative representations of Feynman Integrals (with respect to the standard momentum space representation, in which we integrate Denominators and ISPs over the loop momenta) we re-derive the Baikov representation, following [3]. In this representation, Denominators and ISPs are promoted to integration variables, and the so-called Baikov Polynomial, namely the Gram determinant of the *loop* and *independent external* momenta, naturally emerges as the leading element of the integrand. Then, we discuss the sources of relations among integrals, Lorentz Invariance Identities (LIs), Sector Symmetries (SecSym) and Integration by Parts Identities (IBPs), and we briefly introduct the Laporta algorithm. We finally move to IBPs in Baikov representation. As anticipated above, the latter requires a careful study of particular relations among determinants, and, more in general, among polynomials, known in Algebraic Geometry as *Sygyzies Equations*. Strongly inspired by [48] [70], we furnish an independent proof concerning Sygyzies Equations for determinants and their relation with the Laplace Expansion; moreover we presented a new algorithm for computing Sygyzies based on the Euler's theorem on homogeneous functions. Finally we present an extended version of an existing `Mathematica` code [49] devoted to IBPs generation in Baikov representation, which required the Software Singular [13] (for Sygyzies generation) as well as the code `Reduze 2` [58] (as a starting point to obtain SecSym relations in Baikov representation and to discard vanishing integrals from the huge system of identities).

In the Second Chapter we discuss the method of *Differential Equations for Mis*. We focus on the *Canonical Form* and the algorithm based on the *Magnus Exponential*, developed to achieve such a form. We present the special classes of functions which appear in the solution of DEQs system, once the latter is solved in terms of iterated integrals over rational kernel, namely Harmonic Polylogarithms (HPLs) [31] and Goncharov Polylogarithms (GPLs) [30] [33]. We present (some of) the properties fulfilled by these functions, which we implement in `Mathematica`. Moreover we discuss the problem concerning the *fixing of the Boundary Conditions* (BCs): once the general solution has been determined, we have to fix the integration constants in order the match the *"physical* value of the integrals. Along the lines suggested in [27] we present, and apply to an explicit one-loop example, a strategy in order the infer the BCs for massive integrals from massive ones, and vice-versa.
We propose the calculation of the MIs for the 1-loop 4-point function topology in the full massless case, and for the 1-loop 4 point function for the $\mu e \to \mu e$ scattering. Finally we obtain the Canonical System for a 2-loop *non planar* 3 point topology, which is needed in order to complete the calculation of the Mis for the $q\bar{q} \to t\bar{t}$ process, which are currently known just numerically.

In the Third Chapter we reconsider the algorithm based on the Magnus Exponential, and we apply it to the so-called *"QED Sunrise"* Integral Family, starting from a basis of MIs which fulfills a DEQs system *linear* in ϵ. In particular we emphasize its connection with the solution of the DEQs system at $\epsilon = 0$: obtaining different solutions for the system at $\epsilon = 0$ we build a matrix, which is similar, and, on practical grounds, fully equivalent to the one obtained through the Magnus algorithm.
Then we consider another basis of MIs, namely the *Laporta basis*, related to the previous one via IBPs and we show how Cut Integrals, and their natural implementation in Baikov Representation, together with a careful analysis of *different IBP-compatible integration domains*, provide the *whole* set of solutions for the *homogeneous* part of an higher order DEQ [47].
Finally, starting from the full set of homogeneous solutions obtained through Cuts and using IBPs, we obtain another matrix, for the original basis of MIs, similar but, on practical grounds, equivalent, to the one obtained through the Magnus algorithm.

Chapter 1

Feynman Integrals Representations and Properties

In this Chapter we review the decomposition of the Amplitude in terms of scalar Feynman Integrals. After briefly introducing some alternative representations of Feynman Integrals, we present the Baikov representation, in which Denominators and Irreducible Scalar Products (ISPs) are promoted to integration variables. The so-called Baikov Polynomial, namely the Gram determinant of the *loop* and *external* momenta expressed in terms of Denominators and ISPs, turns to have a key role in this representation; thus we review and derive some useful properties concerning determinants.
Then, we focus on relations fulfilled by Feynman Integrals, in particular Integration By Parts Identities (IBPs). As a direct application (but this is not the only one), IBPs relate different Integrals in a given *Integral Family*, thus just a finite number of them, the so called Master Integrals (MIs) are needed in order to evaluate the Amplitude.
We exploit IBPs in Baikov representation, both analytically and numerically, extending an existing `Mathematica` code dedicated to IBPs generation in this representation. As mentioned above, this approach requires the study of determinants, their algebra, and more generally concepts and ideas from Algebraic Geometry, such as *Sygyzies Polynomials*.

1.1 From Feynman Diagrams to Feynman Integrals

The *Amplitude*, $\mathcal{M}$, associated to (one of) the Feynman Diagram(s) for a process involving $n+1$ external particles, and $\mathbf{p} = \{p_i\}_{i=1,\ldots,n}$ *independent* external momenta is dictated by the *Feynman Rules*; it can be represented as:

$$\mathcal{M}(\mathbf{p}) = \epsilon_{\mu_1} \ldots \epsilon_{\mu_{n_b}} \mathcal{M}^{\mu_1 \cdots \mu_{n_b}}(\mathbf{p}), \quad \mathbf{p} = \{p_1, \ldots, p_n\}; \tag{1.1}$$

where we pull out the set of polarization vectors associated to the $n_b \leq n+1$ external bosons. Then, $\mathcal{M}^{\mu_1 \cdots \mu_{n_b}}$, can be further decomposed as (but this is neither the unique nor the most powerful way):

$$\mathcal{M}^{\mu_1 \cdots \mu_{n_b}}(\mathbf{p}) = \sum_j \mathcal{T}^{\mu_1 \cdots \mu_{n_b};\, j}(\mathbf{p}) f_j(\mathbf{p}), \quad \mathbf{p} = \{p_1, \ldots, p_n\}. \tag{1.2}$$

where the $\mathcal{T}^{\mu_1 \cdots \mu_{n_b};\, j}$ capture the Lorentz and Dirac structure of the Amplitude; the remainder f_j, usually called *Form Factors* contain *all* the *scalar loop integrals*, referred to as *Feynman Integrals*, we are interested in. A generic *l-loop Feynman Integral*, $\mathcal{F}$, reads:

$$\mathcal{F}^{(\ell,n)}(\mathbf{p}) = \int \prod_{j=1}^{\ell} d^d\, k_j \frac{\mathcal{N}(\mathbf{k},\mathbf{p})}{\prod_{k=1}^{t'} \mathcal{D}_k^{r_k}}, \quad \mathbf{k} = \{k_1,\ldots,k_\ell\}, \;\; \mathbf{p} = \{p_1,\ldots,p_n\}. \tag{1.3}$$

The denominators, $\{\mathcal{D}_k\}_{k=1,\ldots,t'}$, are inherited from the original Feynman Diagram(s), namely: $\mathcal{D}_k = (q_k^2 + m_k^2) = ((\sum_{i=1}^n a_{ki} p_i + \sum_{j=1}^{\ell} b_{kj} k_j)^2 + m_k^2)$.
The numerator, $\mathcal{N}(\mathbf{k},\mathbf{p})$, often involves the scalar products formed by either one of the (independent) *external momenta* and one of the *loop momenta*, or by two *loop momenta*. Given *n independent external momenta*, and *l loop momenta* the number of scalar products, n_{SP}, ($n_{SP} \neq n$) is:

$$n_{SP} = \underbrace{\frac{\ell(\ell+1)}{2}}_{\text{internal-internal}} + \underbrace{\ell\, n}_{\text{internal- independent external}} . \tag{1.4}$$

Whenever it is possible, it is preferable to express the scalar products in the numerator in terms of denominators, in order to obtain simplifications in (1.3). Beyond one loop, the number of scalar products, n_{SP} always exceedes the number of denominators, t, forcing us to identify $n_{ISP} = n_{SP} - t$, *Irreducible Scalar Products*, ISPs: namely the scalar products which cannot be re-expressed in terms of denominators. Therfore we deal with integral of the form:

$$\mathcal{I}^{d\,(\ell,n)}(\mathbf{p}) = \int \prod_{j=1}^{\ell} d^d\, k_j \frac{\prod_{h=1}^{n_{ISP}} \mathcal{S}_h^{-s_h}}{\prod_{k=1}^{t} \mathcal{D}_k^{r_k}}, \tag{1.5}$$

where $\{\mathcal{S}_h\}_{h=1,\ldots,n_{ISP}}$ is the set of ISPs, t is the number of *different* denominators. Integrals of the form (1.5) identify an *Integral Family*.

We define a *Topology* (or Sector) as a set of *different denominators* which can be represented as a graph satisfying momentum conservation at each vertex; a *Subtpology* (or Subsector) is simply a subset of the latter denominators, which again can be represented as a graph, satisfying momentum conservation at each vertex, as well.
As a final remark we notice that we can identify an integral as a string, specifying the set of indices for denominators and ISPs, namely:

$$\mathcal{I}[\{r_1,\ldots,r_t,s_1,\ldots,s_{n_{ISP}}\}] \quad \text{or} \quad \mathcal{I}_{r_1,\ldots,r_t,s_1,\ldots,s_{n_{ISP}}}, \tag{1.6}$$

and we can compute two useful quantities, namely:

$$r = \sum_{k=1}^{t} r_k, \quad s = -\sum_{h=1}^{n_{ISP}} s_h \tag{1.7}$$

Moreover, given a set of indices $\{k_1,\ldots k_p\} \subseteq \{1,\ldots t\}$ with $p \leq t$, we can define the *Identification Number*, ID,:

$$\text{ID} = \sum_{n=1}^{p} 2^{k_n - 1}, \tag{1.8}$$

which uniquely identify a (sub)topology.

1.2 Feynman Integrals in $d = 4 - 2\epsilon$ euclidean dimensions, an invitation

As it is well known, it is possible to deal with integrals like (1.5), either in the *minkowskian* metric or in the *euclidean* one; we will assume the *former* as the *default* choice through this thesis. However, not surprisingly, working with the *latter*, i.e.: the *euclidean* one, could offer some advantages, in particoular a clear and intuitive geometrical interpretation for *angles* and *scalar products* among vectors. In virtue of that, we will consider in this section the *euclidean metric*, and then, eventually, we would perform *analytic continuation* on our results.
Up to now, we made no assumption on the particular *regularization scheme* adopted; a common and well established pattern, consists in promoting the *four loop momenta* to a *d-dimensional one*, while different choices are possible regarding the *non divergent parts*, such as γ-algebra, metric tensor, external momenta and polarization vectors [37].

For example, we can consider the *external momenta* and the *polarizaton vectors*, as strictly *four dimensional objects*. More in detail, we can split the d-dimensional metric tensor, $g^{\alpha\beta}$, as:

$$g^{\alpha\beta} = \begin{pmatrix} g^{\alpha\beta}_{[4]} & \mathbb{0} \\ \mathbb{0} & g^{\alpha\beta}_{[-2\epsilon]} \end{pmatrix}. \tag{1.9}$$

where $g^{\alpha\beta}_{[4]}$ part of the metric tensor associated to the *physical* four dimensional part, while $g^{\alpha\beta}_{[-2\epsilon]}$ is dictated by the necessity of regularize divergences. We assume:

$$g^{\alpha\beta}_{[4]}(g_{[4]})_{\alpha\beta} = 4, \qquad g^{\alpha\beta}_{[-2\epsilon]}(g_{[-2\epsilon]})_{\alpha\beta} = -2\epsilon. \tag{1.10}$$

Moreover we naturally split a *vector*, v^α:

$$v^\alpha = v^\alpha_{[4]} + v^\alpha_{[-2\epsilon]}, \tag{1.11}$$

and we define *external momenta* and *polarization vectors* such that:

$$p^\alpha_i \equiv p^\alpha_{[4]\,i}, \quad \epsilon^\alpha_i \equiv \epsilon^\alpha_{[4]\,i}. \tag{1.12}$$

Beyond that, we regard at a generic *loop momentum*, k^α_j as:

$$k^\alpha_j = k^\alpha_{[4]\,j} + k^\alpha_{[-2\epsilon]\,j} = k^\alpha_{[4]\,j} + \mu^\alpha_j, \tag{1.13}$$

where μ^α_j denotes the $[-2\epsilon]$ dimensional part. Thus, the scalar product among a *loop momentum*, k^α_j, and an *external one*, p^β_i, reads:

$$k_j \cdot p_i = k^\alpha_j \, g_{\alpha\,\beta} \, p^\beta_i \equiv k^\alpha_{[4]\,j} \, (g_{[4]})_{\alpha\,\beta} \, p^\beta_i = k_{[4]\,j} \cdot p_i. \tag{1.14}$$

Similarly, we have:

$$\begin{aligned} &k_j \cdot \epsilon_i \equiv k_{[4]\,j} \cdot \epsilon_i, \\ &k_j \cdot k_l = k_{[4]\,j} \cdot k_{[4]\,l} + \mu_j \cdot \mu_l \equiv k_{[4]\,j} \cdot k_{[4]\,l} + \mu_{jl}. \end{aligned} \tag{1.15}$$

Then, looking at a generic *scalar feynman integrand* expressed in terms of the latter decomposition, turns to be illuminating. In fact a *denominator*, D_j, reads:

$$D_j = l^2_{j\,[4]} + \sum_{i,k} \alpha_{i\,j}\alpha_{k\,j}\mu_{i\,j} + m^2_j, \quad l^\alpha_{[4]\,j} = \sum_i \alpha_{i\,j} k^\alpha_{[4]\,i} + \sum_i \alpha_{i\,j} p^\alpha_{[4]\,i}. \tag{1.16}$$

and, in general, *numerators* depend on $k^{\alpha}_{[4]\,i}$ and on the *scalar products* $\mu_{i\,j}$.
Moreover, it is worth recalling that in the *4 dimensional* part of the *loop momenta*, the standard *canonical basis* is assumed, namely:

$$k^{\alpha}_{[4]\,j} = \sum_{j=1}^{4} x_{ji}\, \hat{e}^{\alpha}_{i}, \quad \hat{e}^{\alpha}_{i} = \delta^{\alpha}_{i}. \tag{1.17}$$

Thus, we argue that a generic *ℓ-loop feynman integrand* depends only on a *finite number of variables*, namely $\frac{\ell(\ell+9)}{2}$, wich are:

$$\mathbf{v} = \{x_{j1}, x_{j2}, x_{j3}, x_{j4}, \mu_{ij}\}, \quad 1 \leq i \leq j \leq \ell. \tag{1.18}$$

The latter correspond to the 4ℓ *four dimensional* components of the *loop momenta* and the $\frac{\ell(\ell+1)}{2}$ scalar product among the (-2ϵ) dimensional part of the *loop momenta* themselves. This analysis suggests that the integration over these $\frac{\ell(\ell+9)}{2}$ variables, which are, in a certain sense, *"dictated by the integrand"*, could offer more advantages than the usual integration over the $l \cdot d$ *individual loop momenta components.* Starting from this observations, the underlying concepts can be pushed even forward in the concept of the so called *Adaptive Integrand Decomposition* [38] and the *Baikov representation* [39], [40], [41].

Let's sketch how a Scalar Feynman Integral transforms in these variables. For the sake of simplicity we will consider a 2-loop integral (for a more general derivation see [3]).

2-loop Integral Measure in $d = 4 - 2\epsilon$ dimensions
The Integral Meausre reads:

$$\mathcal{M}^{d\,(\ell,n)}_{\mathcal{I}} = \int \prod_{j=1}^{\ell=2} d^d k_j = \prod_{j=1}^{\ell=2} d^4 k_{[4]\,j} d^{[-2\epsilon]} \mu_j. \tag{1.19}$$

Then, we introduce *spherical coordinates* in the $[-2\epsilon]$-dimensional space:

$$\int d^{[-2\epsilon]}\mu_j = \frac{1}{2}\int_0^{+\infty} d\mu_{jj}\,(\mu_{jj})^{\frac{[-2\epsilon]-2}{2}} \int d\Omega_{[-2\epsilon]-1\,j}, \quad 1 \leq j \leq \ell = 2, \tag{1.20}$$

where:

$$d\Omega_{[-2\epsilon]-1\,j} = (\sin\theta_{1j})^{[-2\epsilon]-3} d\cos\theta_{1j}\,(\sin\theta_{2j})^{[-2\epsilon]-4} d\cos\theta_{2j} \ldots d\theta_{[-2\epsilon]-1\,j}, \tag{1.21}$$

with:

$$\theta_{i\,j} \in [0,\pi], \;\; i = 1, \ldots, [-2\epsilon]-2, \quad \theta_{[-2\epsilon]-1\,j} \in [0, 2\pi[. \tag{1.22}$$

Choosing θ_{12} as the relative orientation between μ^{α}_1 and μ^{α}_2, namely:

$$\mu_{12} = \sqrt{\mu_{11}\,\mu_{22}}\cos\theta_{12}, \tag{1.23}$$

and performing the irrelevant angular integration:

$$\mathcal{M}^{d\,(\ell=2,n)}_{\mathcal{I}} = \frac{\Omega_{[-2\epsilon]-1,1}\,\Omega_{[-2\epsilon]-2,2}}{4} \int \prod_{j=1}^{\ell=2} d^4 k_{[4]\,j} \int \prod_{j=1}^{\ell=2} d\mu_{jj}\,(\mu_{jj})^{[-2\epsilon]-2} \int_{-1}^{+1} d\cos\theta_{12}\,(\sin\theta_{12})^{\frac{[-2\epsilon]-3}{2}}. \tag{1.24}$$

Moreover, inverting (1.23) we have:

$$\begin{aligned} \sin^2(\theta_{12}) &= \frac{\mu_{11}\mu_{22}-\mu_{12}^2}{\mu_{11}\,\mu_{22}}, \\ d\cos(\theta_{12}) &= \frac{d\,\mu_{12}}{\sqrt{\mu_{11}\,\mu_{22}}}, \end{aligned} \tag{1.25}$$

and so (1.24), is equivalent to:

$$\mathcal{M}_{\mathcal{I}}^{d\,(\ell=2,n)} = <\Omega>_{[-2\epsilon]}^{\ell=2} \int \prod_{j=1}^{\ell=2} d^4 k_{[4]\,j} \int \prod_{j=1}^{\ell=2} d\,\mu_{jj} \int_{-\sqrt{\mu_{11}\,\mu_{22}}}^{+\sqrt{\mu_{11}\,\mu_{22}}} d\,\mu_{12}\,(G(\mu_1,\mu_2))^{\frac{[-2\epsilon]-3}{2}}, \tag{1.26}$$

where $G(\mu_1, \mu_2)$ is the *Gram determinant* associated to the (-2ϵ)-dimensional part of the 2-loop momenta, and for convenience we introduced

$$<\Omega>_{[-2\epsilon]}^{\ell=2} = \prod_{j=1}^{\ell=2} \frac{\Omega_{[-2\epsilon]-j,\,j}}{2}. \tag{1.27}$$

ℓ-lopp Integral Measure in $d = 4 - 2\epsilon$ dimensions
Then (1.26) can be generalized to higher loops (see [3]):

$$\mathcal{M}_{\mathcal{I}}^{d\,(\ell,n)} = <\Omega>_{[-2\epsilon]}^{\ell} \int \prod_{j=1}^{\ell} d^4 k_{[4]\,j} \int \prod_{1\le i\le j\le \ell} d\,\mu_{ij}\,(G(\mu_1,\ldots,\mu_\ell))^{\frac{d-5-\ell}{2}}, \tag{1.28}$$

where:

$$<\Omega>_{[-2\epsilon]}^{\ell} = \prod_{j=1}^{\ell} \frac{\Omega_{[-2\epsilon]-j,\,j}}{2}, \tag{1.29}$$

and $G(\mu_1, \ldots, \mu_\ell)$ is the Gram determinant associated to the (-2ϵ)-dimensional part of the ℓ-loop momenta.

1.3 Feynman Integrals in $d = d_{||} + d_{\perp}$ euclidean dimensions, an invitation

Roughly speaking, one of the key observations contained in [38], consits in the aim of maximizing the number of vectors in the *four* dimensional basis *orthogonal* to the *external momenta*. In fact we can split the whole d-dimensional space, into a *longitudinal* space, namely the one spanned by the *external momenta*, and its *orthogonal* and *complementary* part, denoted by *transverse* space.
Considering a generic integral, associated to a diagram with $n+1$ external legs, recalling *momentum conservation*, the dimension of the *longitudinal* space is:

$$d_{||} = \min(4, n). \tag{1.30}$$

It is worth stressing that, if $n < 4$, the *longitudinal* space covers just a *subspace* of the whole *four* dimensional space and the *orthogonal* space *"eats"* $4 - n$ dimensions *physical* directions. Obviously, in the case in which $n \geq 5$, the *orthogonal* space collapses on the

(-2ϵ)-dimensional space introduced above.
Then, essentially in the spirit of (1.9), we decompose the metric tensor as:

$$g^{\alpha\beta} = \begin{pmatrix} g^{\alpha\beta}_{[d_{||}]} & \mathbb{0} \\ \mathbb{0} & g^{\alpha\beta}_{[d_\perp]} \end{pmatrix}, \tag{1.31}$$

with:

$$g^{\alpha\beta}_{[d_{||}]}(g_{[d_{||}]})_{\alpha\beta} = d_{||}, \quad \text{and} \quad g^{\alpha\beta}_{[d_\perp]}\,(g_{[d_\perp]})_{\alpha\beta} = d_\perp. \tag{1.32}$$

For a l-loop feynman integral with $n \leq 4$ independent external legs, we can introduce a set of $(4-d_{||})$ basis elements, namely $\{e^\alpha_{d_{||}+1}, \ldots, e^\alpha_4\}$, and eventually complete the latter to a basis for the whole *orthogonal* space, such that:

$$\begin{aligned} &e_i \cdot p_j = 0, \quad j \leq n, \quad i > d_{||}, \\ &e_i \cdot e_k = \delta_{ik}, \quad i, k > d_{||}. \end{aligned} \tag{1.33}$$

Thus, we can split a d-dimensional loop momenta, as:

$$k^\alpha_i = k^\alpha_{||\,i} + \lambda^\alpha_i, \tag{1.34}$$

where $= k^\alpha_{||\,i}$ lies on the *longitudinal* space, namely:

$$k^\alpha_{||\,i} = \sum_{j=1}^{d_{||}} x_{ij} p^\alpha_j, \tag{1.35}$$

and:

$$\lambda^\alpha_i = \sum_{j=d_{||}+1}^{4} x_{ij} e^\alpha_j + \mu^\alpha_i, \tag{1.36}$$

where in the last equality we completed the set of $(4-d_{||})$ vectors to a whole basis for the *orthogonal* space.
Working in this basis, the scalar products involving *loop* momenta read:

$$k_i \cdot p_j = k^\alpha_i g_{\alpha\beta} p^\beta_j = k^\alpha_{||\,i}\,(g_{[d_{||}]})_{\alpha\beta}\, p^\beta_j = k_{||\,i} \cdot p_j, \tag{1.37a}$$

$$k_i \cdot k_j = k^\alpha_i\, g_{\alpha\beta} k^\beta_k = k^\alpha_{||\,i}\,(g_{[d_{||}]})_{\alpha\beta} k^\beta_{||\,j} + \lambda^\alpha_i\,(g_{[d_\perp]})_{\alpha\beta}\lambda^\beta_j = k_{||\,i} \cdot k_{||\,j} + \lambda_{ij}; \tag{1.37b}$$

where in the last line we introduce, similarly to the previous section:

$$\lambda_{ij} \equiv \lambda_i \cdot \lambda_j = \sum_{k=d_{||}+1}^{4} x_{ik} x_{jk} + \mu_{ij}. \tag{1.38}$$

Thus, denoting collectively by $\mathbf{x}_{||\,i}$ the projection of k^α_i on the *longitudinal* space, and by $\mathbf{x}_{\perp\,i}$ the projection of the same *loop* momentum (or, better the *four dimensional* part) on the *orthogonal* space, we notice that in a generic ℓ-loop *integrand*, *denominators* depends explicitly on the $\frac{\ell(\ell+1)}{2}$ scalar products λ_{ij} and on the $d_{||} \cdot \ell$ components $\mathbf{x}_{||\,i}$, wich is only a *subset* of the $\frac{\ell(\ell+9)}{2}$ identified in the previous section[1].
On the other hand, $\mathbf{x}_{\perp\,i}$ may survive in the *numerator* of the integrand, due to the presence of *four dimensional* vectors orthogonal to external momenta. In fact, recalling the

[1]This statement holds iff $d_{||} < 4$

orthogonality condition between the *polarization vector* of a *massless* particle, say ϵ_i^α, and its momentum p_i^α, we should consider:

$$k_j \cdot \epsilon_i \propto \lambda_j \cdot \epsilon_i = \sum_{k=d_{||}+1}^{4} x_{jk}(e_k \cdot \epsilon_i). \tag{1.39}$$

Anyway, even if this analysis goes beyond the scope of this thesis, we notice that this *"residual"* dependence on $\mathbf{x}_{\perp\, j}$, namely the $(4 - d_\perp)$ transverse components for each loop, is a *polynomial* dependence, and the related integration can be carried out in terms of *Gegenbauer polynomials*; we refer the interested readers to [3] and [38] for detailed discussions and derivations.

Thanks to the previous analysis on the integr*and*, we are ready to recast the integr*al* in terms of the new, carefully chosen, set of variables. Let's consider, for the sake of simplicity, a 2-loop integral measure (for a more general derivation see again [3]) with $n \leq 4$ external legs.

2-loop Integral Measure in $d = d_{||} + d_\perp$ dimensions
Our starting point is:

$$\mathcal{M}_{\mathcal{I}}^{d\,(\ell=2,n)} = \int \prod_{j=1}^{\ell=2} d^d k_j \tag{1.40}$$

We stress that in (1.40), we are integrating over the *individual components* of the *loop momenta*, expressed in the *"standard" canonical basis*, $\mathcal{E}$, namely:

$$k_j = \sum_{\alpha=1}^{d} k_j^\alpha \hat{e}_\alpha, \qquad (\hat{e}_\alpha)_j = \delta_{\alpha\, j}, \quad \hat{e}_\alpha \cdot \hat{e}_\beta = \delta_{\alpha\beta}. \tag{1.41}$$

Thus, we can identify another basis in which the first elements are given by the n *independent* external legs; we denote $d_{||} \equiv n$,

$$\mathcal{B} = \{p_1, \ldots, p_{d_{||}}, e_{d_{||}+1}, \ldots, e_d\}, \tag{1.42}$$

with:

$$\begin{aligned} &p_i \cdot e_j = 0, \quad i \leq d_{||},\, j \geq d_{||}+1, \\ &e_j \cdot e_k = \delta_{jk}, \quad j, k \geq d_{||}. \end{aligned} \tag{1.43}$$

Thus, we can regard at the same vector, k_j, expressed in the basis $\mathcal{B}$:

$$k_j = \sum_{i=1}^{d_{||}} x_j^i p_i + \sum_{k=d_{||}+1}^{d} \lambda_j^{k-d_{||}} e_k; \tag{1.44}$$

Then, re-expressing (1.40) within the basis $\mathcal{B}$, we have to take in to account the *determinant* of the *jacobian* associated to change of basis $\mathcal{G} : \mathcal{B} \to \mathcal{E}$, namely $\det(\mathcal{J}_\mathcal{G})$[2]

$$\det(\mathcal{J}_\mathcal{G}) = \sqrt{G(p_1, \ldots, p_{d_{||}})}\,. \tag{1.45}$$

[2] Let's consider, for the sake of simplicity, a *linear change of basis* $\mathcal{F} : \mathcal{B} \to \mathcal{B}'$, and a vector, $\mathbf{v}$, expressed in the *different* basis:

$$\mathbf{v} = \sum_{i=1}^{m} x^i \mathbf{b}_i = \sum_{j=1}^{m} y^j \mathbf{b}'_j;$$

Then (1.40), can be rewritten as:

$$\mathcal{M}_{\mathcal{I}}^{d\,(\ell=2,n)} = \Big(G(p_1,\ldots,p_{d_{||}})\Big)^{(\ell=2)/2} \int \prod_{j=1}^{\ell=2} d^{d_{||}}x_j \, d^{d_\perp}\lambda_j \tag{1.46}$$

Introducing *spherical coordinates* we can rewrite the integral measure as:

$$\int d^{d_\perp}\lambda_j = \frac{1}{2}\int_0^\infty d\lambda_{jj}\,(\lambda_{jj})^{(d_\perp-2)/2}\int d\Omega_{d_\perp-1\,j} \quad 1\leq j\leq \ell=2. \tag{1.47}$$

and:

$$d\Omega_{d_\perp-1\,j} = (\sin\theta_{1j})^{d_\perp-3} d\cos\theta_{1j}\,(\sin\theta_{2j})^{d_\perp-4} d\cos\theta_{2j}\ldots d\theta_{d_\perp-1\,j}, \tag{1.48}$$

with:

$$\theta_{i\,j}\in[0,\pi],\;\; i=1,\ldots,d_{\perp-2}, \quad \theta_{d_{\perp-1}}\in[0,2\pi[. \tag{1.49}$$

Choosing *spherical coordinates* for λ_2 w.r.t λ_1, i.e.: considering θ_{12} as the relative orientation between λ_1 and λ_2, we can write:

$$\lambda_{12} = \sqrt{\lambda_{11}\lambda_{22}}\,\cos\theta_{12}. \tag{1.50}$$

Then, performing the irrelevant angular integrations[3]: we obtain:

$$\begin{aligned}\mathcal{M}_{\mathcal{I}}^{d\,(\ell=2,n)} &= \Big(G(p_1,\ldots,p_{d_{||}})\Big)^{(\ell=2)/2} \frac{\Omega_{d_\perp-1,1}\Omega_{d_\perp-2,2}}{4}\times \\ &\int\prod_{j=1}^{\ell=2} d^{d_{||}}x_j \int_0^{+\infty}\prod_{j=1}^{\ell=2} d\lambda_{jj}\,(\lambda_{jj})^{(d_\perp-2)/2}\int_{-1}^{+1} d\cos\theta_{12}(\sin\theta_{12})^{d_\perp-3}.\end{aligned} \tag{1.51}$$

Then, inverting (1.50) we have:

$$\begin{aligned}\sin^2(\theta_{12}) &= \frac{\left(\lambda_{11}\lambda_{22}-\lambda_{12}^2\right)}{\lambda_{11}\lambda_{22}}, \\ d\cos(\theta_{12}) &= \frac{d\lambda_{12}}{\sqrt{\lambda_{11}\lambda_{22}}}\end{aligned} \tag{1.52}$$

Plugging the latter relations in (1.51) we obtain:

$$\begin{aligned}\mathcal{M}_{\mathcal{I}}^{d\,(\ell=2,n)} &= \Big(G(p_1,\ldots,p_{d_{||}})\Big)^{(\ell=2)/2} \quad <\Omega>_{d_\perp}^{\ell=2}\int\prod_{j=1}^{\ell=2} d^{d_{||}}x_j\int_0^{+\infty}\prod_{j=1}^{\ell=2} d\lambda_{jj}\times \\ &\int_{-\sqrt{\lambda_{11}\lambda_{22}}}^{+\sqrt{\lambda_{11}\lambda_{22}}} d\lambda_{12}\,(G(\lambda_1,\lambda_2))^{(d_\perp-3)/2},\end{aligned} \tag{1.53}$$

---

then, the *jacobian* associated to the transformation, $\mathcal{J}_{\mathcal{F}}$, simply reads:

$$\mathcal{J}_{\mathcal{F}} = (\mathbf{b}_1\cdots\mathbf{b}_m),$$

namely the *columns* of the *jacobian* are the *bais vectors* of $\mathcal{B}$ expressend in the other basis $\mathcal{B}'$. Then, it is sufficient to recall the identity:

$$\det(\mathcal{J}_{\mathcal{F}}) = \sqrt{\det{}^2(\mathcal{J}_{\mathcal{F}})} = \sqrt{\det(\mathcal{J}_{\mathcal{F}}^t)\det(\mathcal{J}_{\mathcal{F}})} = \sqrt{\det(\mathcal{J}_{\mathcal{F}}^t\mathcal{J}_{\mathcal{F}})}.$$

Finally it is sufficient to regard at the *row-column* matrix multiplication, as *scalar products* among vectors, and so:

$$\det(\mathcal{J}_{\mathcal{F}}) = \sqrt{G(\mathbf{b}_1\cdots\mathbf{b}_m)}.$$

In order to obtain (1.45), we need to recall the *block diagonal* structure of the *Gram matrix*, due to the orthogonality conditions:

$$\mathbf{p}_i\cdot\mathbf{e}_j = 0, \quad \mathbf{e}_k\cdot\mathbf{e}_j = \delta_{ij}.$$

[3]The integrand depends just on λ_{11}, λ_{22} and λ_{12}.

being $G(\lambda_1, \lambda_2)$ the *Gram determinant* associated to the *transverse* components of *loop momenta*, and for convenience we introduced:

$$< \Omega >_{d_\perp}^{\ell=2} = \prod_{j=1}^{\ell=2} \frac{\Omega_{d_\perp - j, j}}{2}. \tag{1.54}$$

ℓ-lopp Integral Measure in $d = d_{||} + d_\perp$ dimensions
Thus, generalizing (1.53) to a generic ℓ-loop integral see [3]:

$$\mathcal{M}_{\mathcal{I}}^{d\,(\ell,n)} = < \Omega >_{d_\perp}^{\ell} (G(p_1, \ldots p_n))^{\ell/2} \int \prod_{j=1}^{\ell} d^{||} x_j \int \prod_{1 \le i \le j}^{\ell} d\lambda_{ij} \, (G(\lambda_1, \ldots, \lambda_\ell))^{(d_\perp - 1 - \ell)/2}, \tag{1.55}$$

where:

$$< \Omega >_{d_\perp}^{\ell} = \prod_{j=1}^{\ell} \frac{\Omega_{d_\perp - j,\, j}}{2}, \tag{1.56}$$

and $G(\lambda_1, \ldots \lambda_\ell)$ is the Gram determinant of the transverse components of loop momenta.

1.4 Baikov representation

A key observation is that the *number of integration variables* in (1.53) matches exactly the *number of possible scalar products* involving at least *one loop momentum*:

$$s_{ji} = k_j \cdot p_i, \quad j \le \ell, i \le n, \tag{1.57a}$$
$$\tilde{s}_{jk} = k_j \cdot k_k \quad j, k \le \ell; \tag{1.57b}$$

in this section we will denote the whole set of n_{sp} scalar product involving at least one loop momentum as $\mathbf{s} = \{s_{ji}, \tilde{s}_{jk}\}$, and we will reintroduce $d_{||} = n$ and $d_\perp = d - n$ for the sake of clarity. We stress again that n is the number of *independent external momenta.*
Not surprisingly the scalar products defined just above, constitute a *"natural"* set of integration variables, as well. As a first step, we work out the relation between $\{x_j^i\}_{i \le n, j \le \ell}$, defined in (1.44), and the set of *scalar products* $\{s_{ji}\}_{j \le \ell, i \le n}$ (1.57*a*). From the very definition we have ($e_m \cdot p_i = 0$):

$$\begin{aligned} s_{ji} &= \left(\sum_{k=1}^{n} x_k^j p_k + \sum_{m=n+1}^{d} \lambda_j^{m-n} e_m \right) \cdot p_i = \\ &= \sum_{k=1}^{n} x_k^j \, p_i \cdot p_k \end{aligned} \tag{1.58}$$

wich leads to:

$$\begin{pmatrix} s_{j1} \\ s_{j2} \\ \vdots \\ s_{jn} \end{pmatrix} = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \cdots & p_1 \cdot p_n \\ p_2 \cdot p_1 & p_2 \cdot p_2 & \cdots & p_2 \cdot p_n \\ \vdots & \vdots & \vdots & \vdots \\ p_n \cdot p_1 & p_n \cdot p_2 & \cdots & p_n \cdot p_n \end{pmatrix} \begin{pmatrix} x_1^j \\ x_2^j \\ \vdots \\ x_n^j \end{pmatrix}, \tag{1.59}$$

and finally:

$$\prod_{i=1}^{n} ds_{ji} = G(p_1, \ldots p_n) \prod_{i=1}^{n} dx_i^j, \quad j = 1, \ldots, \ell. \tag{1.60}$$

where $G(p_1, \dots p_n)$ is the *Gram determinant* associated to the n *independent external momenta*.
The change of variables from $\{\lambda_{ij}\}_{1\le i\le j\le \ell}$ to $\{\tilde{s}_{ij}\}_{1\le i\le j\le \ell}$ is even more straightforward, since the relation, encoded in (1.37b), is essentially a *linear shift*, and so:

$$\prod_{1\le i\le j\le \ell} \tilde{s}_{ij} = \prod_{1\le i\le j\le \ell} \lambda_{ij}. \tag{1.61}$$

Moreover, we can introduce the *Gram determinant* of the *whole* set of *loop* and *independent external* momenta, namely $G(k_1, \dots, k_\ell, p_1, \dots, p_n)$. The latter will play a crucial role in the following. We notice that the *longitudinal* component of *each loop momentum* drops out, since it is linear dependent from the *external kinematics*. Thus, we have:

$$G(k_1, \dots, k_\ell, p_1, \dots, p_n) = G(\lambda_1, \dots, \lambda_\ell, p_1, \dots, p_n). \tag{1.62}$$

In addition, due to the trasversality condition $\lambda_j \cdot p_i = 0$, the *Gram matrix* $\mathbb{G}$, is *block-diagonal*:

$$\mathbb{G}(\lambda_1, \dots, \lambda_\ell, p_1, \dots, p_n) = \begin{pmatrix} \mathbb{G}(\lambda_1, \dots, \lambda_\ell) & \mathbb{0} \\ \mathbb{0} & \mathbb{G}(p_1, \dots, p_n) \end{pmatrix}. \tag{1.63}$$

So, we have[4]

$$G(\lambda_1, \dots, \lambda_\ell, p_1, \dots, p_n) = G(\lambda_1, \dots, \lambda_\ell) G(p_1, \dots, p_n). \tag{1.64}$$

Then, (1.53) results:

$$\begin{aligned} \mathcal{M}_{\mathcal{I}}^{d\,(\ell,n)} = &< \Omega >_{d-n}^{\ell} \left(G(p_1, \dots, p_n)\right)^{(n+1-d)/2} \int \prod_{j=1}^{\ell} \prod_{i=1}^{n} ds_{ji} \times \\ &\prod_{1\le i\le j}^{\ell} d\tilde{s}_{ij} \left(G(k_1, \dots, k_\ell, p_1, \dots, p_n)\right)^{(d-n-1-\ell)/2}. \end{aligned} \tag{1.65}$$

Thus, we can rewrite an integral with n independent in terms of *scalar products* as:

$$\begin{aligned} \mathcal{I}^{d\,(l,n)} = &< \Omega >_{d-n}^{l} \left(G(p_1, \dots, p_n)\right)^{(n+1-d)/2} \int \prod_{j=1}^{\ell} \prod_{i=1}^{n} ds_{ji} \times \\ &\prod_{1\le i\le j}^{\ell} d\tilde{s}_{ij} \left(G(k_1, \dots, k_\ell, p_1, \dots, p_n)\right)^{(d-n-1-\ell)/2} \frac{\mathcal{N}(s_{ji}, \tilde{s}_{ij})}{\prod_{k=1}^{t} D_k^{r_k}(s_{ji}, \tilde{s}_{ij})}. \end{aligned} \tag{1.66}$$

Finally, we can recall the *one to one* correspondence between the whole set of scalar products involving at least *one loop momentum*, $\mathbf{s} = \{s_{ij}, \tilde{s}_{jk}\}$, and the set of denominators and ISPs, collectively denoted by $\mathbf{z} = \{z_i\}_{i=1,\dots,n_{SP}}$, with $n_{SP} = t + n_{ISP}$. Once again, the relation between $\mathbf{z}$ and $\mathbf{s}$ is linear:

$$\mathbf{z} = \mathbb{A}\,\mathbf{s} + \mathbf{c}, \tag{1.67}$$

where $\mathbb{A}$ is an invertible matrix, and $\mathbf{c}$ is a constant vector wich depends on *masses* and *external kinematics*.
Therefore, (1.66) can be rewritten, with a minimal effort, as:

$$\mathcal{I}^{d\,(\ell,n)} = C_d^{\ell} \left(G(p_1, \dots, p_n)\right)^{(n+1-d)/2} \int \prod_{i=1}^{n_{SP}} dz_i \left(F(\mathbf{z})\right)^{(d-n-1-\ell)/2} \frac{\mathcal{N}(\mathbf{z})}{\prod_k^t z_k^{r_k}}. \tag{1.68}$$

[4]The standard notation $G := \det(\mathbb{G})$ is assumed.

where:

$$C_d^\ell = \det(\mathbb{A}^{-1}) < \Omega >_{d-n}^\ell, \quad < \Omega >_{d-n}^\ell = \prod_{j=1}^{\ell} \frac{\Omega_{d-n-j,\, j}}{2}, \tag{1.69}$$

and $F(\mathbf{z})$, which is often referred to as *Baikov polynomial*, is nothing that the *Gram determinant* of the *loop* and *external* momenta expressed in terms of the $\{z_i\}_{i=1,\dots,n_{SP}}$ variables:

$$F(\mathbf{z}) = G(k_1, \dots, k_l, p_1, \dots, p_n)|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}. \tag{1.70}$$

Looking at (1.68), it is worth noting that $F(\mathbf{z})$ carries the whole dependence on d, namely the *space time* dimensions in the integr*and*.

1.5 Identities for determinants

As we can realize from (1.68) and (1.70), and we will extensively exploit in the following, the *Gram determinant* associated to the *loop* and *independent external* momenta plays a key role in this representation. We present here some of its properties, which will be extremely important in the context of *Integration By Parts Identities* (IBPs) for Feynman Integrals, see Section (1.7) and (1.11).

- First of all, we cast the *whole* set of momenta as:

$$\{k_1, \dots, k_\ell, p_1, \dots, p_n\}, \tag{1.71}$$

  then we define s_{ij} as the *scalar product* among the i^{th} and j^{th} elemnts in the list (1.71)[5]. We cearly have: $s_{ij} \equiv s_{ji}$, thus the *Gram matrix*, $\mathbb{G}$, *expressend in terms of scalar products*, s_{ij}, turns to be symmetric.

Then, given a *generic matrix* $\mathbb{A} = \{a_{ij}\}_{1\le i,j\le m}$:

- $\det(\mathbb{A})$ can be obtained through the *Laplace expansion* w.r.t. the i^{th} row; it reads:

$$\det(\mathbb{A}) = \sum_{k=1}^{m} a_{ik}\, C_{ik}, \tag{1.72}$$

  where C_{ik}, referred to as (i,k)-*cofactor*, is: $C_{ik} := (-1)^{i+k} \det(M_{ik})$, being M_{ik} the (i,k)-*minor*;

- $\det(\mathbb{A})$ is a *homogeneous function* of degree n w.r.t. its entries. Then, a classical result by Euler guarantees that $\det(\mathbb{A})$ is a homogeneous function of degree m, if and only if:

$$\sum_{\{ij\}} a_{ij} \frac{\partial \det(\mathbb{A})}{\partial a_{ij}} = m \det(\mathbb{A}). \tag{1.73}$$

Let's consider a generic matrix, in which the entries depend on a parametr: t, namely $\mathbb{A}(t) = \{a_{ij}(t)\}_{1\le i,j\le m}$, then:

[5] We think that the notation $\mathbf{s} = \{s, \tilde{s}\}$ is useful just in Section (1.4, but we will not adopt it in the following.

- The derivative of the determinant w.r.t. t reads:

$$\frac{d\,\det(\mathbb{A}(t))}{dt} = \sum_{i=1}^{m} \det\big(\mathbb{A}'_i(t)\big), \tag{1.74}$$

where $\mathbb{A}'_i(t)$ means that we derive the i^{th}-row with respect to t.

Proof. It is sufficient to recall the expression for the determinant in terms of the *Levi-Civita tensor*:

$$\det(\mathbb{A}(t)) = \sum_{i_1,\ldots,i_n=1}^{m} \epsilon_{i_1,\ldots,i_m}\, a_{1i_1}(t)\ldots a_{mi_m}; \tag{1.75}$$

Thus, we immediatly have:

$$\begin{aligned}\frac{d\,\det(\mathbb{A}(t))}{dt} &= \sum_{i_1,\ldots,i_m=1}^{m} \epsilon_{i_1,\ldots,i_m}\, a'_{1i_1}(t)\ldots a_{mi_m}(t) + \cdots + \epsilon_{i_1,\ldots,i_m}\, a_{1i_1}(t)\ldots a'_{mi_m}(t) = \\ &= \det\big(\mathbb{A}'_1(t)\big) + \cdots + \det\big(\mathbb{A}'_m(t)\big) = \\ &= \sum_{i=1}^{m} \det\big(\mathbb{A}'_i(t)\big),\end{aligned} \tag{1.76}$$

as stated just above. □

Let's consider a symmetric matrix $\mathbb{S} = \{s_{ij}\}_{1\le i\le j\le m}$, with $s_{ij} = s_{ji}$, then:

- The following relation holds [48]:

$$\sum_{k=1}^{m}(1+\delta_{jk})s_{ik}\frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = 2\,\delta_{ij}\det(\mathbb{S}). \tag{1.77}$$

We provide here an independent proof.

Proof. Let's consider (1.74) with $t \to s_{jk}$, then for $j \neq k$ we have:

$$\frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = C_{jk} + C_{kj} = 2\,C_{jk}, \quad j \neq k; \tag{1.78}$$

where in the first equality we use the fact that, being the matrix symmetric, $s_{jk} = s_{kj}$ and we have *two* non vanishing derivatives, namely the derivative in the j^{th}-row and the one in the k^{th}-row. The last equality comes from the symmetry condition as well, namely $C_{kj} = (-1)^{k+j}\det(M_{kj}) = (-1)^{k+j}\det\Big(M^t_{kj}\Big) = (-1)^{j+k}\det(M_{jk}) = C_{jk}$. For $j = k$, we simply have:

$$\frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = C_{jj}, \quad k = j. \tag{1.79}$$

Thus, we can multiply (1.78) by s_{jk} and sum over k, with $k \neq j$:

$$\sum_{k=1,\,k\neq j}^{m} s_{jk}\frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = 2\sum_{k=1,\,k\neq j}^{m} s_{jk}C_{jk} \tag{1.80}$$

Then, adding the term: $2 s_{jj} C_{jj}$, we obtain:

$$\sum_{k=1,\, k\neq j}^{m} s_{jk} \frac{\partial \det(\mathbb{S})}{\partial s_{jk}} + 2 s_{jj} C_{jj} = 2 \sum_{k=1}^{m} s_{jk} C_{jk} = 2 \det(\mathbb{S}), \tag{1.81}$$

where in the last equality we used the Laplace expansion (1.72) w.r.t. the j^{th}-row. We can recast the l.h.s. in (1.81) as:

$$\sum_{k=1}^{m} (1 + \delta_{jk}) s_{jk} \frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = 2 \det(\mathbb{S}). \tag{1.82}$$

Repeating the last steps with $s_{jk} \to s_{ik}$, namely considering:

$$\sum_{k=1,\, k\neq j}^{m} s_{ik} \frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = \sum_{k=1,\, k\neq j}^{m} s_{ik} C_{jk}, \tag{1.83}$$

and adding the term: $2 s_{ij} C_{jj}$, we obtain:

$$\sum_{k=1}^{m} (1 + \delta_{jk}) s_{ik} \frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = 2 \sum_{k=1}^{m} s_{ik} C_{jk} = 0, \tag{1.84}$$

where the last equality is due to the fact that we recognize the Laplace expansion of the determinant (1.72) for a matrix with *two equal rows* (i.e.: the j^{th} row in the original matrix is replaced by the i^{th}-row, thus the latter appears *twice* in the *"modified"* matrix). Thus, (1.81) and (1.84) can be combined in:

$$\sum_{k=1}^{m} (1 + \delta_{jk}) s_{ij} \frac{\partial \det(\mathbb{S})}{\partial s_{jk}} = 2\, \delta_{ij} \det(\mathbb{S}), \tag{1.85}$$

as stated above. □

- Finally, we think that identities concerning determinants and determinants algebras already known in mathematical literature, such as the Cailey identity [86], (and it's generalization, namely *Bernstein-Sato* identities)

$$\det(\partial)(\det \mathbb{A})^{s} = s(s+1)\ldots(s+m-1)(\det \mathbb{A})^{s-1}, \tag{1.86}$$

  where: $\partial = \{\partial / \partial a_{ij}\}_{1 \leq i,j \leq m}$ will be important in order to understand special relations between integral in Baikov representation.

1.6 Praeludium

Within an Integral Family (1.5), the various Integrals turn out to be not independent: finding relations among them, identifying a minimal set of integrlas, namely the *Master Integrals*, drammaticaly simplify the amount of calculations needed for the Amplitude. Before facing the real problem, let us stress the importance of finding relations among integrals with a pedagogical example, adapted from [1].
Let's consider the Integral Family:

$$I_n(\alpha) = \int_0^{+\infty} dx e^{-\alpha x^2} x^n, \quad n \in \mathbb{N}. \tag{1.87}$$

Then we claim that, in the ntegration region indicated just above, namely $x \in (0, +\infty)$ we have:

$$0 = \int_0^{+\infty} dx \frac{\partial}{\partial x} \left(e^{-\alpha x^2} x^n \right), \tag{1.88}$$

which holds when $\Re(\alpha) > 0$. Furthermore we can perform the derivative in (1.88), obtaining:

$$0 = -2\alpha \int_0^{+\infty} dx\, e^{-\alpha x^2} x^{n+1} + n \int_0^{+\infty} dx\, e^{-\alpha x^2} x^{n-1}; \tag{1.89}$$

after a minor rearrangement ($n \to n+1$) the latter reads:

$$\int_0^{+\infty} dx e^{-\alpha x^2} x^{n+2} = \frac{n+1}{2\alpha} \int_0^{+\infty} dx e^{-\alpha x^2} x^n, \tag{1.90}$$

or, better:

$$I_{n+2}(\alpha) = \frac{n+1}{2d} I_n(\alpha), \tag{1.91}$$

which is a recurrence relation among intgerals within (1.87). Thanks to (1.91), it is sufficient to compute $I_0(\alpha)$ and $I_1(\alpha)$, in order to recover all the integrals in (1.87).
We trivially have:

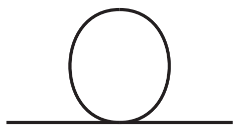
$$I_0(\alpha) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}, \quad I_1(\alpha) = \frac{1}{2\alpha}. \tag{1.92}$$

1.7 Integration By Parts Identities (IBPs)

First of all we have to *generate* identities among the various integrals within a given *Integral Family*; certainly a stunning procedure is the one offered by the *Integration By Parts Identities* (IBPs), [69] which can be interpred as *Gauss Theorem* in *d- dimensions*:

$$\int d^d k_j \frac{\partial}{\partial k_j^\mu} \left(v^\mu \frac{\prod_{h=1}^{n_{ISP}} \mathcal{S}_h^{-s_h}}{\prod_{k=1}^{t} \mathcal{D}_k^{r_k}} \right) = 0. \tag{1.93}$$

In the latter expression, v^μ is chosen among the *l-loop momenta* and the *n independent external momenta*, $v^\mu \in \{\{k_j^\mu\}_{j=1,\dots,l}, \{p_i^\mu\}_{i=1,\dots,n}\}$. We stress that, performing the algebra in (1.93), we obtained a linear combination of integrals with different exponents, but *no* new denominator can appear. Actually, $v^\mu \frac{\partial}{\partial k_j^\mu}$ could reproduce a *Reducible Scalar Product*, and so it could be possible to reconstruct one of the $\{\mathcal{D}_k\}_{k=1,\dots,t}$ in the numerator, leading to a remarkable simplification which ends with the appearance of an integral belonging to a *Sub-Topolgy*. Finally, it should be clear by the procedure described, that the coefficients among the various integrals are *rational functions* in the *kinematics invariants* (i.e.: scalar products among external momenta, and masses) and in *d*, namely the *space-time dimensions*.
We give here a few simple examples, in order to clarify the discussion.
Let's consider:

$\mathcal{T}^{(1,0)} =$  $= \int d^d k \frac{1}{(k^2 - m^2)^{r_1}}.$ (1.94)

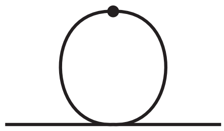
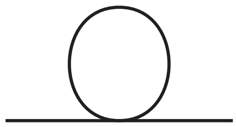
Then, for $r_1 = 1$ we have:

$$0 \equiv \int d^d k \frac{\partial}{\partial k^\mu} \frac{v^\mu}{(k^2 - m^2)}; \tag{1.95}$$

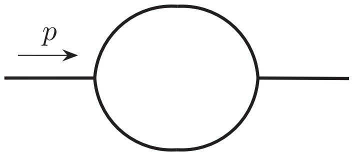
In this example, the only possible choice is $v^\mu \equiv k^\mu$, and so:

$$
\begin{aligned}
0 \equiv \int d^d k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - m^2)} &= d \int d^d k \frac{1}{(k^2 - m^2)} - 2 \int d^d \frac{k^\mu k_\mu}{(k^2 - m^2)^2} = \\
&= d \int d^d k \frac{1}{(k^2 - m^2)} - 2 \int d^d k \frac{(k^2 - m^2) + m^2}{(k^2 - m^2)^2}.
\end{aligned} \tag{1.96}
$$

Recasting the latter we find:

 $= \dfrac{d-2}{2m^2}$  . (1.97)

On the other hand, let's consider:

$\mathcal{B}^{(1,1)} =$  $= \displaystyle\int d^d k \frac{1}{((k-p)^2 - m^2)^{r_1} (k^2 - m^2)^{r_2}}.$ (1.98)

Choosing $r_1 = r_2 = 1$:

$$
0 \equiv \int d^d k \frac{\partial}{\partial k^\mu} \frac{v^\mu}{((k-p)^2 - m^2)(k^2 - m^2)}. \tag{1.99}
$$

Then, we could have $v^\mu = k^\mu$ or $v^\mu = p^\mu$. In the first case:

$$
\begin{aligned}
0 \equiv d \int d^d k \frac{1}{((k-p)^2 - m^2)(k^2 - m^2)} &- \int d^d k \frac{2k^2 - 2k \cdot p}{((k-p)^2 - m^2)^2 (k^2 - m^2)} + \\
- 2 \int d^d k \frac{k^2}{((k-p)^2 - m^2)(k^2 - m^2)^2};&
\end{aligned} \tag{1.100}
$$

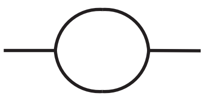
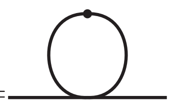
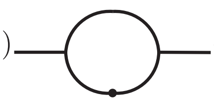
Then we can rearrange the numerators as:

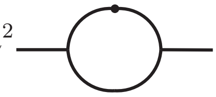
- $2k^2 - 2k \cdot p = (k^2 - m^2) + m^2 + ((k-p)^2 - m^2) - p^2 + m^2$;
- $k^2 = (k^2 - m^2) + m^2$.

Plugging the latter in (1.100), after some algebra, mostly *partial fractioning* and suitable *shift* of the loop momentum, namely $k \to k + p$:

$$
\int d^d k \frac{1}{((k-p)^2 - m^2)} = \int d^d k \frac{1}{(k^2 - m^2)}, \tag{1.101}
$$

we obtain:

$(d-3)$  $=$  $+ (2m^2 - p^2)$  $+$

$+ 2m^2$  . (1.102)

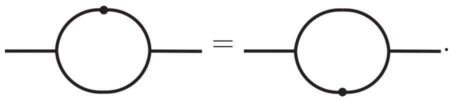
On the other hand, we could have $v^\mu = p^\mu$:

$$
\begin{aligned}
0 \equiv \int d^d k \frac{\partial}{\partial k^\mu} \frac{p^\mu}{((k-p)^2 - m^2)\,(k^2 - m^2)} &= \int d^d k \frac{(2p^2 - 2k \cdot p)}{((k-p)^2 - m^2)^2\,(k^2 - m^2)} + \\
+ \int d^d k \frac{-2k \cdot p}{((k-p)^2 - m^2)\,(k^2 - m^2)^2}.
\end{aligned}
\tag{1.103}
$$

The numerators can be rewritten as:

- $2p^2 - 2k \cdot p = \big((k-p)^2 - m^2\big) + p^2 - \big(k^2 - m^2\big);$
- $-2k \cdot p = \big((k-p)^2 - m^2\big) - p^2 - \big(k^2 - m^2\big).$

Thus (1.103) reads:



(1.104)

1.8 Lorentz Invariance Identities (LIs)

Additional relations can be obtained recalling that integrals of the form (1.5) are *Lorentz Scalars*. So, by definitions, we are allowed to consider an *infinitesimal Lorentz transformation*:

$$
p_i^\mu \to p_i^\mu + \delta p_i^\mu = p_i^\mu + \omega^{\mu\nu} p_{i\,\nu}, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}, \quad \forall i = 1, \dots, n. \tag{1.105}
$$

Doing so, we obtain:

$$
\mathcal{I}^{(\ell,n)}(\mathbf{p}) = \mathcal{I}^{(\ell,n)}(\mathbf{p} + \delta\mathbf{p}), \quad \mathbf{p} = \{p_1, \dots, p_n\}. \tag{1.106}
$$

Thus we can expand the r.h.s. in (1.106) as:

$$
\mathcal{I}^{(\ell,n)}(\mathbf{p}) = \mathcal{I}^{(\ell,n)}(\mathbf{p}) + \omega^{\mu\nu} \sum_{j=1}^{n} p_{j\nu} \frac{\partial \mathcal{I}^{(\ell,n)}(\mathbf{p})}{\partial p_j^\mu}. \tag{1.107}
$$

Thus we obtain:

$$
0 = \sum_{j=1}^{n} \left(p_{j\nu} \frac{\partial}{\partial p_j^\mu} - p_{j\mu} \frac{\partial}{\partial p_j^\nu} \right) \mathcal{I}^{(\ell,n)}(\mathbf{p}). \tag{1.108}
$$

Finally, multiplying (1.108) by one of the $\frac{n(n-1)}{2}$ antisymmetric tensor of the form: $(p_{m,\mu}\, p_{n,\nu} - p_{m,\nu}\, p_{n\mu})$, other identities among the integrals are guaranteed, as stated above.

1.9 Symmetry Relations and Sector Symmetries

On top of that, we recall that an integr*al* is *invariant* under a *linear shift*:

$$
k_j^\mu \to (\mathbb{A})_{ji}\, k_i^\mu + (\mathbb{B})_{jm}\, p_m^\mu, \quad i, j = 1, \dots, l \;\; m = 1, \dots, n. \tag{1.109}
$$

On the other hand, (1.109) maps an integr*and* into a linear combination of different integr*ands*. Thus, restoring the integral sign, we obtain extra relation among integr*als* which might belong to different sectors.
Moreover, we could identify a *subset* of transformations (1.109), referred to as *Sector Symmetries*, SecSym, such that the original set of *denominators is mapped into itself.*

1.10 Master Integrals (MIs)

As pointed out for the first time by Laporta [2], given an Integral Family (1.5), we can generate identities (IBPs, LIs, SecSym) up to certain values of s and r, spanning the set of all the possible *sclar integrals* involved in the calculation. It is worth stressing that, on the one hand, raising r and s, the number of equations grows faster than the number of in tegrals involved; this would apparently lead to an *over constrained* system. On the other hand, it turns out that the equations we are dealing with are *not independent*, and in fact the (row) *rank* of the system is always smaller than the number of unknowns. Eventually, we can *"solve"* the system, namely we can express the *"hardest"* integrals in terms of the *"easiest'* ones, according to a careful *lexicographical ordering* established a priori. A proposal is sketched just below.

Lexicographical ordering
Given the set of indices which uniquely defines an integral:

$$\mathcal{I}^{(\ell,n)}[\{r_1,\ldots,r_t,s_1,\ldots,s_{n_{ISP}}\}], \quad r_i\in\mathbb{Z}, s_j\in\mathbb{N}_{\geq 0}, \quad i=1,\ldots,t,\ \ j=1,\ldots,n_{ISP}, \tag{1.110}$$

we can compute:

- $\rho=$ the number of *positive* (non null) indices in (1.110);
- $\sigma=$ the number of *negative* (non null) indices in (1.110);
- $r=$ the sum of *positive* indices in (1.110);
- $s=$ *minus* the sum of negative indices in (1.110).

Then, we define an auxiliary list of indices:

$$\mathcal{I}^{(\ell,n)}[\{r_1,\ldots,r_t,s_1,\ldots,s_{n_{ISP}}\}] \to \mathcal{I}^{(\ell,n)}[\{\rho,\sigma,r,s,r_1,\ldots,r_t,s_1,\ldots,s_{n_{ISP}}\}]. \tag{1.111}$$

Roughly speaking, we classify integrals according to the values of ρ: an higher value of ρ implies an higher complexity for the corresponding integral; being ρ the same for a subset of integrals we sort them (i.e.: integrals within this subset) according to the values of σ: an higher value of σ implies an higher complexity for the corresponding integral, as well, and so on. In `Mathematica` this ordering is accomplished by the built-in functions `Reverse` and `Sort`[6] Given a list of equations, we can solve the 1^{st} one w.r.t. to most complex integral, and we plug the solution in the whole system. Then we repeat the same for the 2^{nd} equation and so on. The set of integrals we are left with, called *Master Integrals*, MIs, forms a basis for the whole set of integrals: we might express every integral as a linear

[6] Starting from scratch, given a set of integral we detect:

$$n_{\max} = \max(r_{\max}+1, s_{\max}+1), \tag{1.112}$$

where $r_{\max}$, ($s_{\max}$) is the highest value of r (s) among the ones in the list. Then, let p_i be the i$^{\text{th}}$ element in the list $\{\rho,\sigma,r,s,r_1,\ldots,r_t,s_1,\ldots,s_{n_{ISP}}\}$, ($p_1\equiv s_{n_{ISP}}$), we define:

$$w = \sum_{i=1}^{4+t+n_{ISP}} p_i\, n_{\max}^{i-1}.$$

Finally w defines the complexity of the corresponding integral, namely, once again, an higher value of w implies an higher complexity for the integral.

combination of MIs performing a *back substitution* in our system. The coefficients in all these relations are rational valued functions in the kinematics invariants, masses and the space-time dimensions d.
The reduction in MIs dramatically simplifies the number of integrals needed for the Amplitude, tipically $\#\,\mathrm{MIs} \sim \mathcal{O}(10^2)$ while $\#\,\mathrm{Integrals} \sim \mathcal{O}(10^4)$ in a two-loop process. Not surprisingly, the reduction in MIs is one of the key points in multiloop calculations. On the one hand many public (and private) implementations of the Laporta Algotihm were developed over the years [52]-[59]; moreover, new ideas based on the Functional Reconstruction (over Finite Fields) [60]-[62] seem to be very promising in order to handle the tremendous complexity regarding algebraic manipulations (especially very large intermediate expressions) required by the reductions. On the other hand several criteria, such as the one proposed by Lee and Pomeransky [65], were suggested to determine at least the *Number* of MIs in a given Sector; this fact, which is quite interesting per se, could lead to new strategies in the whole IBPs reductions.

1.11 IBPs in Baikov's representation

As stated above, the integration variables within the Baikov's representation turn to be the t *Denominators* and the n_{ISP} *ISPs*; these variables are collectively denoted by $\{z_i\}_{i=1,\dots,n_{SP}}$, with $n_{SP} = t + n_{ISP}$. It seems natural to study how to implement the identities among various integrals, especially *IBPs*, (previously presented in momentum space), in this context. On the one hand, not surprisingly, *IBPs* arise from the the integral of a *total derivative*, (w.r.t. the current integration variables, namely $\{z_i\}_{i=1,\dots,n_{SP}}$, with $n_{SP} = t + n_{ISP}$), with vanishing boundary terms. A more formal, but equivalent in practice, discussion can be found in [10] and [11]. On the other hand, within this representation, concepts related to *Algebraic Geometry* naturally arise. In particular we will focus on *Sygyzy Polynomials* wich appeared for the first time in [9] in order to *avoid doubled denominators*, and revisited in [11] [12] in order to handle and reduce the huge number of equations generated by IBPs and related identities.
Let's consider consider, for the sake of simplicity, the following redefinitions (treating denominators and ISPs on an equal footing):

$$\{r_1, \dots, r_t, s_1, \dots, s_{n_{ISP}}\} \to \{a_1, \dots, a_t, a_{t+1}, \dots, a_{t+n_{ISP}}\} \equiv \mathbf{a}; \tag{1.113}$$

$$\{z_1, \dots, z_t, z_{t+1}, \dots, z_{t+n_{ISP}}\} \to \mathbf{z}; \tag{1.114}$$

$$t + n_{ISP} \to n_{SP}. \tag{1.115}$$

Thus, we deal with an integral of the form (avoiding overall factors, and keeping the notation as concise as possible):

$$\mathcal{I}[\mathbf{a}] = \int \prod_{i=1}^{n_{SP}} \frac{dz_i}{z_i^{a_i}} F(\mathbf{z})^{\gamma_d}, \quad \gamma_d = \frac{d-n-1-\ell}{2}. \tag{1.116}$$

We notice that γ_d caputeres the dependence on d, as well as on the number of the *independent* external legs, n, and the number of *loop*, ℓ.
Thus, in order to build up an *IBP* for *arbitrary indices*, we introduce a set of *Polynomials*, $\{v_j(\mathbf{z})\}_{j=1,\dots,n_{SP}}$, depending on the $\mathbf{z}$ variables (as well as on the kinematics involved), which is analogous to the vector v^μ, contained in (1.93). We have:

$$0 \equiv \int \prod_{i}^{n_{SP}} dz_i \sum_{j=1}^{n_{SP}} \partial_j \left(\frac{v_j(\mathbf{z}) F(\mathbf{z})^{\gamma_d}}{\prod_{i=1}^{n_{SP}} z_i^{a_i}} \right). \tag{1.117}$$

Performing the derivatives in (1.117), after a minor rearrangement, we obtain:

$$0 \equiv \int \prod_i^{n_{SP}} dz_i \frac{F(\mathbf{z})^{\gamma_d}}{\prod_i^{n_{SP}} z_i^{a_i}} \left(\sum_{j=1}^{n_{SP}} \frac{\gamma_d}{F(\mathbf{z})} v_j(\mathbf{z}) \partial_j F(\mathbf{z}) + \left(\sum_{j=1}^{n_{SP}} \partial_j v_j(\mathbf{z}) - \frac{a_j v_j(\mathbf{z})}{z_j} \right) \right). \quad (1.118)$$

Avoiding dimensional shift

Since the *exponent* of the Gram determinant, namely γ_d, is strictly related to the space-time dimensions, d, the term proportional to $F(\mathbf{z})^{-1}$ in (1.118), would involve lower dimensional integrals. Thus, it seem somewhat *desirable* looking for a set of polynomials:

$$\mathbf{v}(\mathbf{z}) = \{v_1(\mathbf{z}), \ldots, v_{n_{SP}}(\mathbf{z}), v_F(\mathbf{z})\} \quad (1.119)$$

such that:

$$\sum_{j=1}^{n_{SP}} v_j(\mathbf{z}) \partial_j F(\mathbf{z}) = v_F(\mathbf{z}) F(\mathbf{z}). \quad (1.120)$$

We left the discussions concerning these polynomials to the following subsections, we just notice here that once the latter are determined, then (1.118) reads:

$$0 \equiv \int \prod_i^{n_{SP}} dz_i \frac{F(\mathbf{z})^{\gamma_d}}{\prod_i^{n_{SP}} z_i^{a_i}} \left(\gamma_d v_F(\mathbf{z}) + \left(\sum_{j=1}^{n_{SP}} \partial_j v_j(\mathbf{z}) - \frac{a_j v_j(\mathbf{z})}{z_j} \right) \right). \quad (1.121)$$

Avoiding doubled denominators

Moreover, in this representation, the absence of *doubled denominators*, or better the absence of *denominators raised to a higher power*[7], originally advocated in [9], seems to be naturally implementable. In fact, it is sufficient looking for a new set of polynomials:

$$\mathbf{f}(\mathbf{z}) = \{f_1(\mathbf{z}), \ldots f_{n_{SP}}(\mathbf{z}), f_F(\mathbf{z}\}, \quad (1.122)$$

related to the previous ones by:

$$\begin{aligned} &v_j(\mathbf{z}) = z_j f_j(\mathbf{z}), \quad j = 1, \ldots, t, \\ &v_k(\mathbf{z}) \equiv f_k(\mathbf{z}), \quad k = t+1, \ldots, t+n_{ISP}, \\ &v_F(\mathbf{z}) \equiv f_F(\mathbf{z}). \end{aligned} \quad (1.123)$$

such that:

$$\sum_{j=1}^{t} z_j f_j(\mathbf{z}) \partial_j F(\mathbf{z}) + \sum_{k=t+1}^{t+n_{ISP}} f_k(\mathbf{z}) \partial_k F(\mathbf{z}) = f_F(\mathbf{z}) F(\mathbf{z}). \quad (1.124)$$

Sygyzies Equations

Finding one (or more than one) set of *polynomials*:

$$\mathbf{f}(\mathbf{z}) = \{f_1(\mathbf{z}), \ldots, f_{t+n_{ISP}}(\mathbf{z}), f_F(\mathbf{z})\}, \quad (1.125)$$

such that (1.123) holds, is a well-studied mathematical problem, known as *Sygyzies Equations*, or *Sygyzies Solutions* or simply *Sygyzies.* The set(s) of solutions can be provided by

[7]This constraint is relaxed for the ISPs, as it is clear from (1.123).

several computer algebra systems, like SINGULAR [13] and Macaulay2 [14].
Once the *Sygyzy Equation* (1.124) is solved, then (1.118) reads:

$$0 \equiv \int \prod_{i}^{n_{SP}} dz_i \frac{F(\mathbf{z})^{\gamma_d}}{\prod_i^{n_{SP}} z_i^{a_i}} \left(\gamma_d f_F(\mathbf{z}) + \left(\sum_{j=1}^{t} \partial_j (z_j f_j(\mathbf{z})) - a_j f_j(\mathbf{z}) \right) + \right. \\ \left. + \left(\sum_{k=t+1}^{t+n_{ISP}} \partial_k f_k(\mathbf{z}) - \frac{a_k f_k(\mathbf{z})}{z_k} \right) \right). \tag{1.126}$$

Let's conclude this section noting that, if we multiply a given set of solutions $\mathbf{f}(\mathbf{z})$, by an arbitrary *monomial* $m(\mathbf{z})$, we obtain a *new* set of solutions, $\tilde{\mathbf{f}}_m$, which satisfies (1.124) as well, but leads to a different *IBP*.

Sygyzies from Euler scaling

On the other hand, we might say that the strategy presented just above could be too demanding. In particular (1.124), especially for complicated 2-loops integrals, seems to be hardly solvable. Because of this, as a first step, we should relax the condition regarding the absence of doubled denominators, trying to avoid just integrals in different dimensions. Thus, let's recall the equation we are interested in:

$$\sum_{i=1}^{n_{SP}} v_i(\mathbf{z}) \frac{\partial F(\mathbf{z})}{\partial z_i} = v_F(\mathbf{z}) F(\mathbf{z}), \tag{1.127}$$

where:

$$\mathbf{v}(\mathbf{z}) = \{v_1(\mathbf{z}), \ldots, v(\mathbf{z})_{n_{SP}}, v_F(\mathbf{z})\}, \tag{1.128}$$

is the unknown set of polynomials. It is worth recalling that $F(\mathbf{z})$ is the *Gram determinant* of the whole set of *loop* and *external* momenta, expressed in terms of the $\mathbf{z} = \{z_1, \ldots, z_{n_{ISP}}\}$ variables. Despite its simplicity this fact turn to have important consequences. In fact, let's consider (1.62) and (1.64)

$$\begin{aligned} &G(\{k_1, \ldots, k_\ell, p_1, \ldots, p_n\}) = G(\{\lambda_1, \ldots, \lambda_\ell\})\, G(\{p_1, \ldots, p_n\}), \\ &F(\mathbf{z}) = G(\{k_1, \ldots, k_\ell, p_1, \ldots, p_n\})|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}. \end{aligned} \tag{1.129}$$

Then, $G(\{\lambda_1, \lambda_\ell\})$ turns to be a *homogeneous function of degree* ℓ in the scalar products $\{\lambda_{ij}\}_{1\le i\le j\le \ell}$.
Thus, we have (1.73):

$$\sum_{c\,\in\,\mathcal{C}} \lambda_c \frac{\partial\, G(\{\lambda_1, \ldots, \lambda_\ell\})}{\partial \lambda_c} = \ell\, G(\{\lambda_1, \ldots, \lambda_\ell\}), \quad \mathcal{C} = \{ij\}_{1\le i\le j\le \ell}. \tag{1.130}$$

Now, recalling that *denominators* can be rewritten in terms of $\{\lambda_{ij}\}_{1\le i\le j\le \ell}$, using the *chain rule* we find:

$$\frac{\partial G_\lambda}{\partial \lambda_c} = \sum_{i\,\in\,\mathcal{I}} \frac{\partial G_\lambda}{\partial z_i} \frac{\partial z_i}{\partial \lambda_c}, \tag{1.131}$$

where $\mathcal{I}$ is a carefully chosen set of indices, $\mathcal{I} \subseteq \{1, \ldots, n_{SP}\}$. Now, plugging (1.131) in (1.130), we obtain (expressing everything in terms of $\mathbf{z}$):

$$\sum_{i\,\in\,\mathcal{I}} \left(\sum_{c\,\in\mathcal{C}} \lambda_c \frac{\partial z_i}{\partial \lambda_c} \right) \frac{\partial G_\lambda}{\partial z_i} \Bigg|_{\lambda_c(\mathbf{z})} = \ell\, G_\lambda. \tag{1.132}$$

Finally, multiplying both sides in (1.132) by the *z-independent* factor $G(\{p_1, \ldots, p_n\})$, and using (1.64), we have:

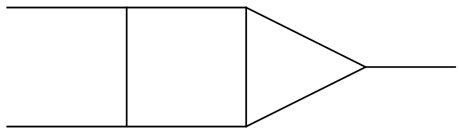
$$\sum_{i \in \mathcal{I}} w_i(\mathbf{z}) \frac{\partial F(\mathbf{z})}{\partial z_i} = \ell \, F(\mathbf{z}), \quad w_i(\mathbf{z}) = \sum_{c \in \mathcal{C}} \lambda_c \frac{\partial z_i}{\partial \lambda_c}, \tag{1.133}$$

which is exactly the kind of relations we are interested in. In many cases, $w_i(\mathbf{z})$, $i \in \mathcal{I}$, $\mathcal{I} \subseteq \{1, \ldots, n_{SP}\}$ turn to be *rational functions*, i.e.: there could be untrivial denominators; anyway this is not a big obstacle, in fact we can multiply the set $\{w_i(\mathbf{z})\}_{i=\mathcal{I}}$ by the *common denominator*, and recast (1.133) as the desired (1.127):

$$\sum_{i \in \mathcal{I}} v_i(\mathbf{z}) \frac{\partial F(\mathbf{z})}{\partial z_i} = v_F(\mathbf{z}) F(\mathbf{z}). \tag{1.134}$$

Sygyzies from Euler scaling, a 2-loop example

Let's cnsider a 2-loop example, namely:

$\mathcal{I}^{(2,2)} =$  , (1.135)

explicitly:

$$\mathcal{I}^{(2,2)} = \int d^d k_1 d^d k_2 \frac{(k_2 \cdot p_1)^{-s_1}}{(k_1^2)^{r_1}((k_1 - p_1)^2)^{r_2}((k_1 - p_1 - p_2)^2)^{r_3}((k_2 + p_1 + p_2)^2)^{r_4}(k_2^2)^{r_5}((k_1 + k_2)^2)^{r_6}},$$
(1.136)

with $p_1^2 = p_2^2 = 0$ and $p_1 \cdot p_2 = \frac{s}{2}$.

The *Baikov Polynomial*, namely $F(\mathbf{z})$, reads:

$$\begin{aligned} F(\mathbf{z}) = \frac{1}{16} \Big[& s^2 \left(z_2^2 - 2(z_5 + z_6 + 2z_7) z_2 + (z_5 - z_6 + 2z_7)^2 \right) + \\ & 2s \left(-(z_4 + z_5) z_2^2 + \left(z_5^2 - z_6 z_5 + 2 z_7 z_5 - 4 z_6 z_7 + 2 z_3 (z_5 + z_7) + \right.\right. \\ & \left.\left. z_4(-z_5 + z_6 + 2z_7) \right) z_2 - 2 z_3 z_7 (z_5 - z_6 + 2z_7) + \right. \\ & \left. z_1(-2z_3(z_5 + 2z_7) + z_2(z_4 + z_5 + 2z_7) + (z_4 - z_5 - 2z_7)(z_5 - z_6 + 2z_7)) \right) + \\ & (z_2(z_5 - z_4) + z_1(z_4 - z_5 - 2z_7) + 2 z_3 z_7)^2 \Big] \end{aligned} \tag{1.137}$$

On the other hand, in this 2-loop example we have:

$$\sum_{c \in \mathcal{C}} \lambda_c \frac{\partial G(\{\lambda_1, \lambda_2\})}{\partial \lambda_c} = 2 \, G(\{\lambda_1, \lambda_2\}), \quad \mathcal{C} = \{11, 12, 22\}. \tag{1.138}$$

We explicitly find:

$$\begin{aligned} \lambda_{11} &= \frac{\mathrm{s} z_2 - (z_1 - z_2)(z_2 - z_3)}{\mathrm{s}}, \\ \lambda_{12} &= \frac{-\mathrm{s}(z_2 + z_5 - z_6 + 2z_7) + z_2 z_4 - z_2 z_5 - 4 z_2 z_7 + 2 z_3 z_7 + z_1(-z_4 + z_5 + 2z_7)}{2\mathrm{s}}, \\ \lambda_{22} &= \frac{\mathrm{s}(z_5 + 2z_7) + 2z_7(-z_4 + z_5 + 2z_7)}{\mathrm{s}}. \end{aligned} \tag{1.139}$$

Then, solving the latter w.r.t. $\{z_3, z_4, z_6\}$, i.e. $\mathcal{I} = \{3, 4, 6\}$, we obtain:

$$\begin{aligned}
&z_3 = -\frac{-\lambda_{11}s + sz_2 + z_2^2 - z_1 z_2}{z_1 - z_2},\\
&z_4 = -\frac{\lambda_{22}s - 2sz_7 - sz_5 - 4z_7^2 - 2z_5 z_7}{2z_7},\\
&z_6 = \frac{1}{2(z_1 - z_2)z_7}\left(z_1^2(-\lambda_{22} + z_5 + 2z_7) + z_1(2z_7(2(\lambda_{12} + z_7) + z_5) - 2z_2(-\lambda_{22} + z_5 + z_7))\right.\\
&\left.-4\lambda_{11}z_7^2 - 2z_2 z_7(2\lambda_{12} + z_5) + z_2^2(z_5 - \lambda_{22})\right);
\end{aligned} \tag{1.140}$$

and then (1.133) gives:

$$\begin{aligned}
&w_3(\mathbf{z}) = \frac{sz_2 - (z_1 - z_2)(z_2 - z_3)}{z_1 - z_2},\\
&w_4(\mathbf{z}) = -\frac{s(z_5 + 2z_7) + 2z_7(-z_4 + z_5 + 2z_7)}{2z_7},\\
&w_6(\mathbf{z}) = -\frac{(z_5 + 2z_7)z_1^2 - 2((-z_5 + z_6 - 2z_7)z_7 + z_2(z_5 + z_7))z_1 + z_2(z_2 z_5 + 2(z_6 - z_5)z_7)}{2(z_1 - z_2)z_7}.
\end{aligned} \tag{1.141}$$

Thus, multiplying the latter by the *common denominator*, namely $2(z_1 - z_2)z_7$, we obtain the usual relation:

$$\sum_{i\in\mathcal{I}} v_i(\mathbf{z})\frac{\partial F(\mathbf{z})}{\partial z_i} = v_F(\mathbf{z})\,F(\mathbf{z}), \quad \mathcal{I} = \{3, 4, 6\}, \tag{1.142}$$

where:

$$\begin{aligned}
&v_3(\mathbf{z}) = 2\,z_7\left(sz_2 - (z_1 - z_2)(z_2 - z_3)\right),\\
&v_4(\mathbf{z}) = -(z_1 - z_2)\left(s(z_5 + 2z_7) + 2z_7(-z_4 + z_5 + 2z_7)\right),\\
&v_6(\mathbf{z}) = -(z_5 + 2z_7)z_1^2 - 2((-z_5 + z_6 - 2z_7)z_7 + z_2(z_5 + z_7))z_1 + z_2(z_2 z_5 + 2(z_6 - z_5)z_7),\\
&v_F(\mathbf{z}) = 4(z_1 - z_2)z_7.
\end{aligned} \tag{1.143}$$

Sygyzies from Laplace expansion

Let's consider once again the Gram determinant associated to the whole set of *loop* and *independent external* momenta, namely $G(\{k_1, \ldots, k_\ell, p_1, \ldots, p_n\})$. Being the Gram matrix symmetric, namely $s_{ij} = s_{ji}$, then (1.85) holds ($m \to \ell + n$):

$$\sum_{k=1}^{\ell+n}(1 + \delta_{jk})s_{ik}\frac{\partial G}{\partial s_{jk}} = 2\delta_{ij}G. \tag{1.144}$$

Then, taking $j \leq \ell$, i.e.: considering just the set of scalar products involving at least *one* loop momentum, collectively denoted by $\mathbf{s}$, and recalling the linear relations among $\mathbf{s}$ and $\mathbf{z} = \{z_1, \ldots, z_{n_{SP}}\}$, namely:

$$\mathbf{z} = \mathbb{A}\mathbf{s} + \mathbf{c}, \tag{1.145}$$

thanks to the *chain rule*, we infer:

$$\sum_{k=1}^{\ell+n}(1 + \delta_{jk})s_{ik}\sum_{a=1}^{n_{SP}}\frac{\partial G|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}}{\partial z_a}\frac{\partial z_a}{\partial s_{jk}} = 2\,\delta_{jk}G|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}. \tag{1.146}$$

Thanks to a minor rearrangement, and recalling the definition of the *Baikov Polynomial* $F(\mathbf{z})$, namely (1.70), we obtain again (1.127), namely:

$$\sum_{a=1}^{n_{SP}} v_a(\mathbf{z})\frac{\partial F(\mathbf{z})}{\partial z_a} = v_F(\mathbf{z})F(\mathbf{z}), \tag{1.147}$$

with:

$$v_a(\mathbf{z}) = \sum_{k=1}^{\ell+n} (1+\delta_{jk}) s_{ik} \frac{\partial z_a}{\partial s_{jk}}\bigg|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}, \quad v_F(\mathbf{z}) \equiv v_F = 2\,\delta_{ij}. \tag{1.148}$$

We stress that we are forced to consider: $1 \leq j \leq \ell$ and $1 \leq i \leq \ell + n$.
Furthermore, being $v_F(\mathbf{z})$ proportional to δ_{ij}, choosing $i \neq j$ the corresponding IBP (1.121) turns to be d-independent.

Sygyzies from Laplace expansion, a 2-loop example

As an example, let's consider once again the integral (1.136), which impose $\ell = n = 2$. Considering different choices compatible with $1 \leq j \leq 2$ and $1 \leq i \leq 4$, then (1.148) gives Sygyzies solutions:

$$\mathbf{v}(\mathbf{z}) = \{v_1(\mathbf{z}), v_2(\mathbf{z}), \ldots, v_{n_{SP}}(\mathbf{z}), v_F(\mathbf{z}\}. \tag{1.149}$$

For instance, among the possible choices, we present here:

- $i = 1$ and $j = 1$ which gives:
$$\mathbf{v}(\mathbf{z}) = \{2z_1, z_1 + z_2, -\mathrm{s} + z_1 + z_3, 0, 0, z_1 - z_6 + z_6, 0, 2\};$$
- $i = 1$ and $j = 2$ which gives:
$$\mathbf{v}(\mathbf{z}) = \left\{0, 0, 0, \mathrm{s} - z_3 - z_5 + z_6, -z_1 - z_5 + z_6, z_1 - z_5 + z_6, \frac{1}{2}(z_1 - z_2), 0\right\};$$
- $i = 2$ and $j = 1$ which gives:
$$\mathbf{v}(\mathbf{z}) = \{-z_1 - z_5 + z_6, -z_1 - z_5 + z_6 - 2z_7, \mathrm{s} - z_1 - z_4 + z_6, 0, 0, -z_1 + z_5 + z_6, 0, 0\}.$$

just to mention a few.

Dimension-shifted Integrals

Finally, we would like to exploit how relations between integrals in different dimensions naturally arise in Baikov representation, we refer the interested readers to [66] for a detailed discussion. To that end, we reintroduce the d-dependent prefactors, and let's consider:

$$\mathcal{I}^{d\,(\ell,n)} = C_d^\ell \left(G(p_1, \ldots, p_n)\right)^{\frac{n+1-d}{2}} \int \prod_{i=1}^{n_{SP}} \frac{dz_i}{z^{a_i}} (F(\mathbf{z}))^{\frac{d-n-\ell-1}{2}}. \tag{1.150}$$

We can factorize $F(\mathbf{z})$, and we consider the identity:

$$\begin{aligned}\mathcal{I}^{d\,(\ell,n)} =& C_d^\ell \left(G(p_1, \ldots, p_n)\right)^{\frac{n+1-d}{2}} \int \prod_{i=1}^{n_{SP}} \frac{dz_i}{z^{a_i}} (F(\mathbf{z}))^{\frac{d-n-\ell-1}{2}} \\ =& C_d^\ell \left(G(p_1, \ldots, p_n)\right)^{\frac{n+1-d}{2}} \int \prod_{i=1}^{n_{SP}} \frac{dz_i}{z^{a_i}}\, F(\mathbf{z}) (F(\mathbf{z}))^{\frac{d-2-n-\ell-1}{2}},\end{aligned} \tag{1.151}$$

thus, making explicit $F(\mathbf{z})$, and performing the simplifications against $\prod_{i=1}^{n_{SP}} z_i^{a_i}$, we can read the last line in (1.151) as a combination of integrlas in $d-2$ dimensions, provided the replacements:

$$\begin{aligned}&\left(G(p_1, \ldots, p_n)\right)^{\frac{n+1-d}{2}} = G(p_1, \ldots, p_n)^{-1} \left(G(p_1, \ldots, p_n)\right)^{\frac{n+1-(d-2)}{2}} \\ &C_d^\ell = C_{d-2}^\ell \left(\frac{C_d^\ell}{C_{d-2}^\ell}\right).\end{aligned} \tag{1.152}$$

1.12 Code Implementation

We illustrate here the main features of a `Mathematica` code written in order to generate IBPs in Baikov representation. Our implementation is an extension of a first version developed in [49].
Lets sketch here the main steps in our code.

- Given the set of Denominators and ISPs for the Integral Family under consideration, we generate the corresponding Baikov Polynomial, through the script `Baikov.m` [46];

- Then we consider the general form of IBPs in Baikov representation (1.126):

$$0 \equiv \int \prod_{i}^{n_{SP}} dz_i \frac{F(\mathbf{z})^{\gamma_d}}{\prod_i^{n_{SP}} z_i^{a_i}} \left[\gamma_d f_F(\mathbf{z}) + \left(\sum_{j=1}^{t} \partial_j (z_j f_j(\mathbf{z})) - a_j f_j(\mathbf{z}) \right) + \right. \\ \left. + \left(\sum_{k=t+1}^{t+n_{ISP}} \partial_k f_k(\mathbf{z}) - \frac{a_k f_k(\mathbf{z})}{z_k} \right) \right]; \tag{1.153}$$

  where: $\gamma_d = \frac{d-n-1-\ell}{2}$, ($\ell$ is the number of loops for the parent topology, n the *independent* external legs) and t the number of denominators, $n_{SP} = t + n_{ISP}$;

- We recall that $\mathbf{f}(\mathbf{z}) = \{f_1(\mathbf{z}), \ldots, f_t(\mathbf{z}), f_{t+1}(\mathbf{z}), \ldots f_{n_{ISP}}(\mathbf{z}), f_F(\mathbf{z})\}$ is a *minimal* set of Sygyzies; namely they fulfill the following relation:

$$\sum_{j=1}^{t} z_j f_j(\mathbf{z}) \partial_j F(\mathbf{z}) + \sum_{k=t+1}^{t+n_{ISP}} f_k(\mathbf{z}) \partial_k F(\mathbf{z}) = f_F(\mathbf{z}) F(\mathbf{z}). \tag{1.154}$$

  We stress that in (1.154), $F(\mathbf{z})$ and $\partial_k F(\mathbf{z})$ are considered known, and the set $\mathbf{f}(\mathbf{z})$ is unknown. In our code we obtain this set of polynomials thanks to SINGULAR [13];

- Multiplying the *minimal* set of Sygyzies by arbitrary Monomials, we obtain auxiliary Sygyzies and thus, thanks to (1.153) others IBPs.

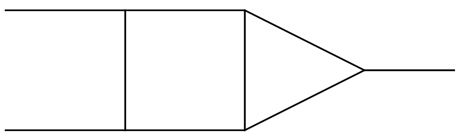
- Once the previous steps are accomplished we obtain a list of IBPs; in each of them (1.153) reduces to a linear combination of integrals with *arbitrary exponents*, with coefficients depending on d as well as on the *kinematics variables*. In the output, integrals are presented according to the string of their exponents, e.g.:

$$\int \prod_{i=1}^{n_{SP}} dz_i \, \frac{F(\mathbf{z})^{\gamma_d}}{z_1^{a_1} \, z_2^{a_2} \cdots z_{n_{SP}}^{a_{n_{SP}}}} \to \text{INT}[\{a_1, a_2, \ldots, a_{n_{SP}}\}]$$

- Finally we read integrals, and therefore IBPs, for different values of explicit indices: e.g.:

$$\{a_1, a_2, \ldots, a_{n_{SP}}\} \to \{1, 1, \ldots, 0\} \Rightarrow \\ \text{INT}[\{a_1, a_2, \ldots, a_{n_{SP}}\}] \to \text{INT}[1, 1, \ldots, 0\}];$$

An Explicit Example
In order to clarify the discussion we provide an explicit 2-loop example. Let's consider:

$$\mathcal{I}^{(2,2)} =$$  . (1.155)

We assume the *two* independent external momenta massless, namely $p_1^2 = p_2^2 = 0$, and $p_1 \cdot p_2 = \frac{s}{2}$.

Input
We initialize the input variables:

```
NLoops = 2;
NExtMom = 2;
Denominators = {k[1]^2,(k[1]-p[1])^2,(k[1]-p[1]-p[2])^2,(k[2]+p[1]+p[2])^2,
               k[2]^2,(k[1]+k[2])^2};
ISPs = {k[2] p[1]};
ExtRules = {ss[3,3]->0, ss[4,4]->0, ss[3,4]->S1/2};
gamma = (d-5)/2;
RankMonAux = {0, 1, 2};
```

where:

- `NLoops` is the number of loops;
- `NExtMom` is the number of independent external momenta:
- `Propagators` is the whole list of Denominators *and* ISPs (in momentum space);
- `ExtRules` consists in the definitions of external invariants. Notice that external momenta are labelled according to the redefinition: $p_1 \to p_{\ell+1}, \ldots, p_n \to p_{\ell+n}$, and so `ss[3, 3]` corresponds to $p_1 \cdot p_1$, and so on. Moreover the Mandelstam invariant s is identified by `S1` in our code.
- `gamma` was defined as γ_d;
- `RankMonAux` is needed in order generate auxiliary monomials to obtain auxiliary Sygyzies and IBPs, see below;

Baikov Polynomial Generation
We define the whole set of Propagators as: Denominators $\cup$ ISPs.

```
Propagators = Join[Denominators, ISPs];
```

Then, we generate the Baikov Polynomial $F(\mathbf{z})$; this step is accomplished thanks to the `Baikov.m` script [46].

```
Baikov[NLoops, NExtMom, Propagators, ExtRules];

F // Simplify
1/16((z[2](-z[4]+z[5])+z[1](z[4]-z[5]-2z[7])+2z[3]z[7])^2+
S1^2(z[2]^2+(z[5]-z[6]+2z[7])^2-2z[2](z[5]+z[6]+2z[7]))+
2S1(-z[2]^2(z[4]+z[5])-2z[3]z[7](z[5]-z[6]+2z[7])+
z[1](-2z[3](z[5]+2z[7])+z[2](z[4]+z[5]+2z[7])+
(z[4]-z[5]-2z[7])(z[5]-z[6]+2z[7]))+
z[2](z[5]^2-z[5]z[6]+2z[5]z[7]-
4z[6]z[7]+2z[3](z[5]+z[7])+z[4](-z[5]+z[6]+2z[7]))))
```

Monomials Generation
As a preliminary step, we (re)define the Denominators and ISPs in terms of $\mathbf{z}$ variables (notice the lower case):

```
rsp = Table[z[i], {i, Length[Denominators]}];
isp = Complement[Table[z[i], {i, Length[Propagators]}], rsp];
allz = Join[rsp, isp];
```

Then, thanks to the function MonomialListBuilder, we generate a list of monomials, auxilliartmonlist, in the allz variables, whose rank is compatible with the one (or with the ones) given in RankMonAux, defined in input. This list of monomials will be used in order to obtain auxiliary IBPs, see below.

```
(* RankMonAux = {0, 1, 2} *)

auxilliartmonlist
{1, z[7], z[6], z[5], z[4], z[3], z[2], z[1], z[7]^2, z[6] z[7],
 z[6]^2, z[5] z[7], z[5] z[6], z[5]^2, z[4] z[7], z[4] z[6],
 z[4] z[5], z[4]^2, z[3] z[7], z[3] z[6], z[3] z[5], z[3] z[4],
 z[3]^2, z[2] z[7], z[2] z[6], z[2] z[5], z[2] z[4], z[2] z[3],
 z[2]^2, z[1] z[7], z[1] z[6], z[1] z[5], z[1] z[4], z[1] z[3],
 z[1] z[2], z[1]^2}
```

IBPs Generation: Arbitrary indices
Then, given $F(\mathbf{z})$, and its derivatives, we can compute the *Sygyzies Solutions* for (1.154) by means of SINGULAR [13].
Moreover each list of Sygyzies solutions obtained by SINGULAR is multiplied by each auxiliary monomial. Doing so, we obtain others Sygyzies which lead to others IBPs. Then, plugging the *Sygyzies* in the square bracket in (1.153), we obtain the core expression for IBPs. The steps described above are accomplished by the function GenSyzIBP; a typycal output is presented below.

```
GenSyzIBP[F, gamma, rsp, isp, auxilliartmonlist]

{{5S1-5z[1]+5z[2]-a[3](-z[1]+z[2])-a[1](S1-z[1]+z[2])-
a[2](S1-z[1]+z[2])+10z[7]-2a[4]z[7]-a[5](S1+2z[7])-
a[6](S1-z[1]+z[2]+2z[7])+1/2(-5+d)(2S1-2z[1]+2z[2]+4z[7])-
(a[7](S1z[7]+2z[7]^2))/z[7], ...
```

Finally, thanks to the function RecasterLaurent, we *"convert"* or, better, we *"recast"* the square bracket into an IBP with arbitrary indices:

```
(-d+a[1]+a[2]+a[3]+a[6]) INT[{-1+a[1],a[2],a[3],a[4],a[5],a[6],a[7]}]
+(d-a[1]-a[2]-a[3]-a[6]) INT[{a[1],-1+a[2],a[3],a[4],a[5],a[6],a[7]}]
+2(d-a[4]-a[5]-a[6]-a[7]) INT[{a[1],a[2],a[3],a[4],a[5],a[6],-1+a[7]}]
+S1(d-a[1]-a[2]-a[5]-a[6]-a[7]) INT[{a[1],a[2],a[3],a[4],a[5],a[6],a[7]}], ...
```

IBPs Generation: Explicit Indices
Given a list of IBPs with arbitrary indices we have to fill the latter in order to obtain explicit identities among integrals. Moreover, the *same* relation with *arbitrary indices*, produces *different* relations with *explicit indices*, once the latter are chosen in different ways.
In a given set of indices, the *positive ones* correspond to (non trivial) *denominators*; the *negative* ones to (non trivial) *numerators* and *zeros* mean the absence of the corresponding variables. Furthermore the set of *positive* indices identifies the (sub)topology for the corresponding integral.
Thus, in order to obtain a set of exponents we have to specify:

- parentdeno: the number of denominators for the parent topology,
- parentisp: the number of ISPs for the parent topology,
- nsubdeno: the number of denominators for the considered subtopologies,

- `maxden`: the difference between the sum of powers of denominators, for the considered subtopologies, and the number of denominators for that subtopologies; e.g.: setting `maxden`= 0 each denominaotr in the considerd subtopology is raised to power *one*;

- `maxisp`: *minus* the sum of power of numerators for the considered subtopologies.

Then, the function `AssignSubTopo` generates lists of exponents compatible with its arguments:

```
parentdeno=6;
parentisp=1;
nsubdeno=6;
maxden=1;
maxisp=0;

AssignSubTopo[parentdeno, parenisp, nsubdeno, maxden, maxisp]
{{a[1]->1, a[2]->1, a[3]->1, a[4]->1, a[5]->1, a[6]->2, a[7]->0},
 {a[1]->1, a[2]->1, a[3]->1, a[4]->1, a[5]->2, a[6]->1, a[7]->0},
 {a[1]->1, a[2]->1, a[3]->1, a[4]->2, a[5]->1, a[6]->1, a[7]->0}, ...
```

Now, simply performing a substitution we get IBPs with explicit indices:

```
{-(-5+d)INT[{0,1,1,1,1,2,0}]+(-5+d)INT[{1,0,1,1,1,2,0}]+
2(-4+d)INT[{1,1,1,1,1,2,-1}]+(-5+d)S1 INT[{1,1,1,1,1,2,0}],
-(-4+d)INT[{0,1,1,1,2,1,0}]+(-4+d)INT[{1,0,1,1,2,1,0}]+
2(-4+d)INT[{1,1,1,1,2,1,-1}]+(-5+d)S1 INT[{1,1,1,1,2,1,0}],
-(-4+d)( INT[{0,1,1,2,1,1,0}]-INT[{1,0,1,2,1,1,0}]-
2INT[{1,1,1,2,1,1,-1}]-S1INT[{1,1,1,2,1,1,0}] ), ...
```

Zero Sectors
Moreover, we can trim the whole system imposing the vnishing of integrals whose ID number (1.1) is compatible with the Zero Sectors' IDs generated by the code `Reduze 2`, [58]; thanks to this step we get rid of integrals which are known to be vanishing in *Dimensional Regularization*, such as *scaleless* integrals.

Sector Symmetries
We might obtain extra identities among integrals thanks to redefinitions of loop momenta; an interesting class of relations, the so called *Sector Symmetries* (1.9), is generated shifting the loop momenta in such a way that the set of denominators in a given (sub)topology, is mapped into itself. Since these identities can be ontained with a minimal effort in *"momentum space"*, for example thanks to `Reduze 2`, and they greatly simplify the reduction in MIs, it is desirable to implement them within the Baikov representation.
Roughly speaking, since Baikov representation relies on the relation between scalar products and the set of denominators and ISPs, it is sufficient to shift loop momenta in *"momentum space* representation, and express the result in terms of $\{z_i\}_{i=1,\ldots,t+n_{ISP}}$. A given *Sector Symmetry* produces just a permutation of indices on denominators:

$$\{z_1, \ldots, z_t\} \to \{z_{\sigma(1)}, \ldots, z_{\sigma(t)}\}. \tag{1.156}$$

On the other hand, each ISP is mapped into a linear combination of the whole set of $\mathbf{z} = \{z_i\}_{i=1,\ldots,t+n_{ISP}}$ and a linear combination of kinematics invariants, collectively denoted by $\mathbf{s}$.

$$z_i \to \sum_{j=1}^{t+n_{ISP}} c_{ij}\, z_j + \mathbf{s}, \tag{1.157}$$

Finally the *Baikov polynomial*, $F(\mathbf{z})$, turns to be invariant under the transformation (1.156) and (1.157); this fact reflects the *shift invariance* property of Gram Determinants.
We stress that *Sector Symmetries* turn to be *d-independent* relations. On practical grounds, it turns to be convenient to consider the action of (1.156) and (1.157) on a set of denominaotrs with arbitrary indices and on a monomial in the corresponding ISPs with explicit indices, neglecting at all $F(\mathbf{z})$.

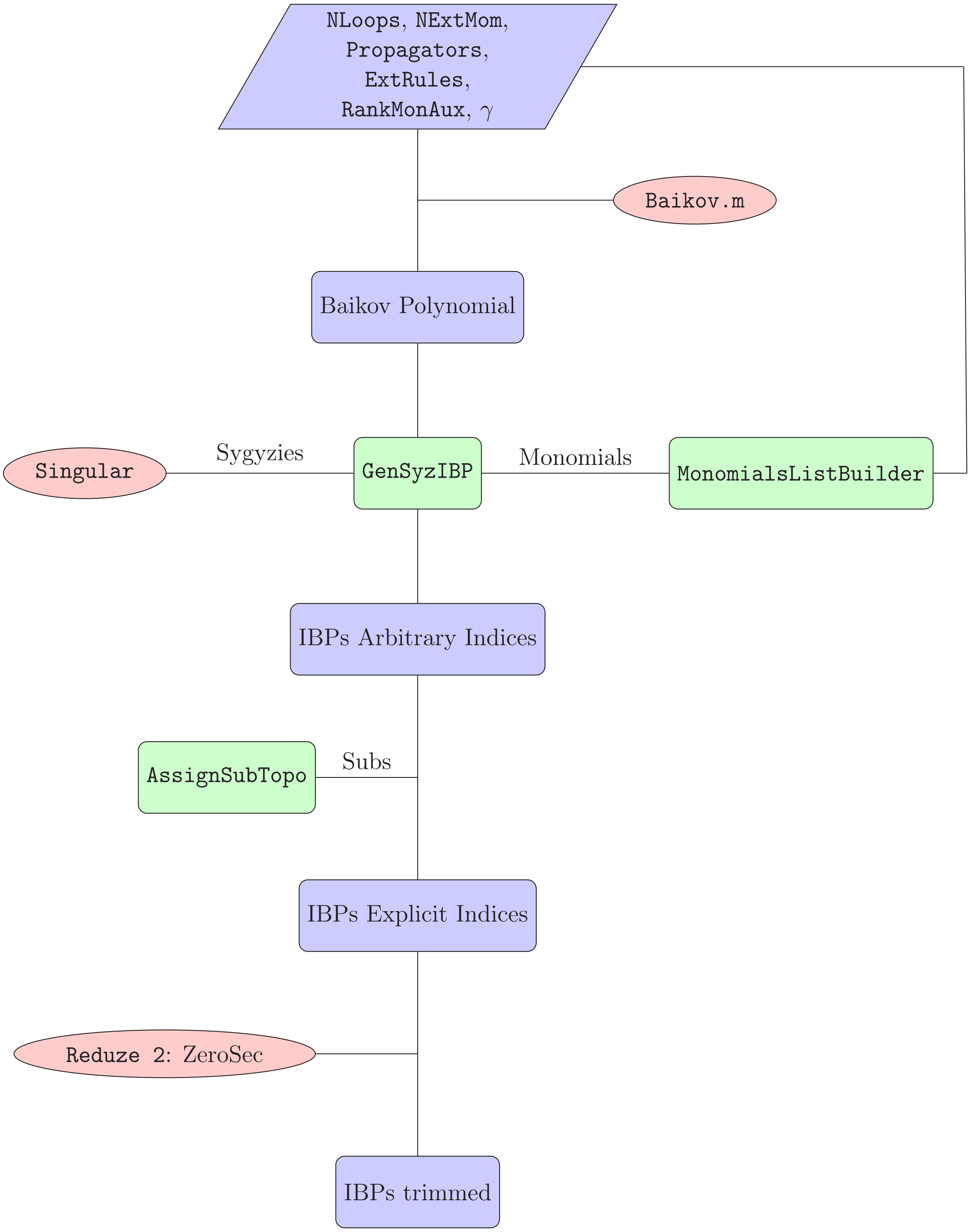


Figure 1.1: Flowchart for IBPs generation in Baikov representation. Red clouds represent external codes needed in the chain; green clouds represent implemented functions or chain of commands; blue blocks stand for key outputs in different steps.

System Solver
Once the system is built, the latter has to be solved. In simple cases, i.e. few integrals and few scales, the built-in `Mathematica` function `Solve`, is sufficient: provided the fact that the *number* of MIs is known, it is possible to solve the system w.r.t. *all* the integrals except for the ones chosen to be MIs. As a result, each integral in the system (excpet for the MIs) is writen as a linear combination of the chosen MIs.
On top of that, for the most complex cases, we plan to adopt an implementation of the Laporta algorithm [2] (or some of its variants).
It is worth stressing that the coefficients of the huge system are *rational functions* in d and in the kinematics, and the strategy advocated by Laporta heavily relies on Gaussian Elimination. Consequently, this type of approach suffers from the so-called *intermediate expression swell* i.e.: very large intermediate expressions appear. A very powerful idea to overcome this issue consists in reconstructing the analytic expression of the final result from multiple numerical evaluations. A detailed discussion of this topic, which goes beyond the scope of this Thesis can be found in [62] [63].
We present below some simple examples, in order to show the ideas behind these *Functional Reconstruction* techniques, taken from [64].

Univariate Polynomials
Let's consider a simple polynomial of degree *two*:

$$\mathcal{P}(x) = 5x^2 - 7x + 13 \tag{1.158}$$

Our goal is to guess (1.158), considering, as a starting point, its values at a given set of input values:

$$\begin{cases} \mathcal{P}(3) = 37 \\ \mathcal{P}(5) = 103 \\ \mathcal{P}(7) = 209 \\ \quad\vdots \end{cases} . \tag{1.159}$$

If, by any chance, the degree d of $\mathcal{P}$ is known in advance, we could simply build and solve a system of $d+1$ equations of the form:

$$\mathcal{P}(x) = n_0 + n_1 x + n_2 x^2 \Big|_{x=x_i}, \quad x_i = 3, 5, 7. \tag{1.160}$$

However, since in general one does not know *a priori* the degree d, another method must be used. The task at hand can be solved rewriting the polynomial in an alternative representation due to Newton:

$$\begin{aligned} \mathcal{P}(x) &= \sum_{i=0}^{R} a_i \prod_{j=0}^{i-1} (x - y_j) \\ &= a_0 + (x - y_0)\Big(a_1 + (x - y_1)\big(a_2 + (x - y_2)(\cdots + (x - y_{R-1})a_R)\big)\Big), \end{aligned} \tag{1.161}$$

where the a_i depend on the y_i, and the latter can be chosen arbitrarily. The values of the coefficients a_i can then be immediately extracted performing a smart sampling on x. Baring in mind the second line of (1.161), we have:

$$\text{choose } y_0 = 3 \Rightarrow \begin{cases} \mathcal{P}(3) = a_0 + (3-3)(\cdots) \\ \qquad\;\; = a_0 \\ \qquad\;\; \Downarrow \\ \quad a_0 = 37 \end{cases}$$

$$\text{choose } y_1 = 5 \Rightarrow \begin{cases} \mathcal{P}(5) = 37 + (5-3)(a_1 + (5-5)(\cdots)) \\ \qquad\;\; = 37 + (5-3)a_1 \\ \qquad\quad \Downarrow \\ a_1 = 33 \end{cases}$$

$$\text{choose } y_2 = 7 \Rightarrow \begin{cases} \mathcal{P}(7) = 37 + (7-3)(33 + (7-5)(a_2 + (7-7)(\cdots))) \\ \qquad\;\; = 37 + (7-3)(33 + (7-5)a_2) \\ \qquad\quad \Downarrow \\ a_2 = 5 \end{cases}$$

One can see that $a_i = 0$ for any y_i when $i > 2$, thus the final expression we find is

$$\mathcal{P}(x) = 37 + (x-3)(33 + (x-5)5) \tag{1.162}$$

which, after a minor rearrangement, coincides with (1.158).

Univariate rational functions

In a similar fashion, given the rational function:

$$\mathcal{R}(x) = \frac{n_0 + n_1 x + n_2 x^2 + \cdots + n_R x^R}{d_0 + d_1 x + d_2 x^2 + \cdots + d_{R'} x^{R'}} \tag{1.163}$$

we can use the so called *Thiele Interpolation Formula*, which is an alternative representation of (1.163):

$$\begin{aligned} \mathcal{R}(x) &= a_0 + \cfrac{x - y_0}{a_1 + \cfrac{x - y_1}{a_2 + \cfrac{x - y_2}{\cfrac{\vdots}{a_{N-1} + \cfrac{x - y_{N-1}}{a_N}}}}} \\ &= a_0 + (x - y_0)\left(a_1 + (x - y_1)\left((x - y_2)\left(\cdots + \frac{x - y_{N-1}}{a_N}\right)^{-1}\right)^{-1}\right)^{-1} \end{aligned} \tag{1.164}$$

Again, introducing a set of arbitrary constants y_i, the computation of the coefficients a_i is reduced to a systematic evaluation of $\mathcal{R}$ on the chosen y_i. The analytic expression of the a_i is computed recursively and is given by:

$$\begin{aligned} a_0 &= \mathcal{R}(y_0) \\ a_1 &= (\mathcal{R}(y_1) - a_0)^{-1}(y_1 - y_0) \\ &\vdots \\ a_r &= \left(\left((\mathcal{R}(y_N) - a_0)^{-1}(y_N - y_0) - a_1\right)^{-1}(y_N - y_1) - \cdots - a_{N-1}\right)^{-1}(y_N - y_{N-1}) \end{aligned} \tag{1.165}$$

IBP reconstruction: an explicit example

Finally we present a typical output of an IBP reduction[8] (namely an integral written as a combination of MIs), for an integral belonging to (1.136).

$$\begin{aligned} \text{INT}[\{1,1,1,1,1,2,-3\}] =& b_1 \text{INT}[\{0,1,0,1,1,1,0\}] + b_2 \text{INT}[\{1,0,1,1,1,0,0\}] + \\ & b_3 \text{INT}[\{1,0,0,1,0,1,0\}] \end{aligned} \tag{1.166}$$

[8] The identity is obtained by means of `Reduze 2`.

Each coefficient is interpreted as a *black box*, which associate to a given *input value* the corresponding output. We present the number of numerical evaluations needed to reconstruct the coefficients. The reconstructions are obtained by means of routines developed in [64], concerning the *Multivariate Reconstruction*, consisting in a further development of the ideas just sketched above.

Coefficient	Number of evaluations	Analytic expression
b_1	11	$-\frac{(d-5)(d-3)^2}{2(d-4)(d-2)}$
b_2	17	$\frac{(d-5)(d-3)d(d+2)}{8(d-6)(d-4)(d-2)(d-1)}$
b_3	37	$-\frac{3(3d-8)\left(3d^6-53d^5+373d^4-1349d^3+2678d^2-2764d+1080\right)}{8(d-6)^2(d-4)^2(d-2)(d-1)\,S1}$

(1.167)

Chapter 2

Differential Equations

In this Chapter we focus on the method of the Differential Equations for Feynman Integrals.
Given a set of unknown MIs, we can consider its derivatives w.r.t. the masses or the kinematics invariants and, thanks to IBPs, we obtain a suitable system of Differential Equations (DEQs). Solving the latter (and fixing the integration constants in the General Solution) we determine the analytic expression for the MIs, avoiding a direct integration over the loop momenta.
We recall the notion of *Canonical Basis*, a particular basis of Mis, in which ϵ (the regulator parameter) is completely factorized from the kinematics and the solution is naturally expressed in terms of iterated integrals over rational kernels. Moreover, we study an algorithm (purely *"algebraic"*), based on the *Magnus Exponential*, proposed to recast a DEQ system *linear* in ϵ in the *Canonical form*. We studied and implemented some of the properties fulfilled by the functions involved in the Solutions, namely Harmonic Polylogarithms and Goncharov Polylogarithms.
Thanks to these techniques, we propose the calculation of the MIs for the 1-loop 4-point function topology in the full massless case, and for the 1-loop 4 point function for the $\mu e \to \mu e$ scattering. Finally we obtain the Canonical System for a 2-loop *non planar* 3 point topology, which is needed in order to complete the calculation of the Mis for the $q\bar{q} \to t\bar{t}$ process, which are currently known just numerically.

2.1 Differential Equations (DEQs) for Master Integrals

Once the set of MIs has been identified, the latter has to be calculated. As proposed for the first time by Kotikov [6], then generalized by Remiddi [7] and fully exploited by Gehrmann and Remiddi [8], among others, a very powerful strategy consists in determining the unknown MI(s) as the *solution(s)* of a suitable *Differential Equations (System)* thus, avoiding the direct integration over the loop momenta. The procedure is sketched by the following example [4].

Derivative w.r.t. masses

Given a set of MIs, $\{\mathbf{F}_i\}_{i=1,\dots,n}$, for an Integral Family under consideration, let's consider the derivative of $\mathbf{F}_{\bar{i}}(m^2_{\bar{i}})$, $\bar{i} \in \{1,\dots,n\}$, w.r.t. $m^2_{\bar{i}}$, (assuming $(q^2_{\bar{i}} - m^2_{\bar{i}})^{-\bar{r}_i} \in \mathbf{F}_{\bar{i}}$, $\bar{r}_i > 0$, and considering, for the sake of simplicity, that $m^2_{\bar{i}}$ appears just in an unique Denominator

), then:

$$\begin{aligned}\frac{\partial}{\partial m_{\bar{i}}^2}\mathbf{F}_{\bar{i}}(m_{\bar{i}}^2) \approx \int \prod_{i=1}^{l} d^d k_i \dots \frac{\partial}{\partial m_{\bar{i}}^2}\left(\frac{1}{(q_{\bar{i}}^2 - m_{\bar{i}}^2)^{\bar{r}_i}}\right) \dots \approx \\ \approx \int \prod_{i=1}^{l} d^d k_i \dots \frac{\bar{r}_i}{(q_{\bar{i}}^2 - m_{\bar{i}}^2)^{\bar{r}_i+1}} \dots ;\end{aligned} \tag{2.1}$$

thus we notice that is possible to rearrange the r.h.s. through IBPs, in order to express the derivative of a MI in terms of *that particular* MI as well as *others* MIs chosen from the given set. Thus, iterating the procedure also for the other MIs, we end up with a *coupled system* of *Differential Equations*; roughly speaking, organizing the MIs in a vector with an increasing number of Denominators in each entry, $\mathbf{F}$, the DEQs System can be represented as:

$$\partial_{m^2}\mathbf{F} = \mathbb{A}\mathbf{F}. \tag{2.2}$$

We stress that the DEQs System is obtained using IBPs, and so the coefficients appearing in the System, turn to be rational functions in the kinematics invariants (i.e.: scalar products among external momenta, and masses) and in d, namely the space-time dimensions. Moreover, since IBPs relate integrals belonging to a given topology to integrals belonging to the *same topology* and its *sub-topologies*, the System turns to have a remarkable block triangular structure, where blocks are due to the fact that there could be more than one integral belonging to the same topology.

Derivative w.r.t. external kinematics invariants

In order to consider the derivative w.r.t. the external invariants, technical, simple but crucial details are needed [5], due to the fact that integrals do not depend explicitly on the external invarinats. Given $n+1$ external particles, we can identify only n *independent* external momenta, due to momentum conservation. Thus, we can build $\frac{n(n+1)}{2}$ external invariants, $\{s_{ij}\}$, $s_{ij} = p_i \cdot p_j$, and we can implement the chain rule:

$$\frac{\partial}{\partial p_i^{\mu}} = \sum_j \frac{\partial s_{ji}}{\partial p_i^{\mu}} \frac{\partial}{\partial s_{ij}}. \tag{2.3}$$

We can multiply the latter equation by a vector, let's say p_k^{μ}, obtaining a system of n^2 equations. The latter seems to be over determined, since we want to solve n^2 equations w.r.t. $\frac{n(n+1)}{2}$ derivative in the external invariants, thus dealing with $n^2 - \frac{n(n+1)}{2} = \frac{(n)(n-1)}{2}$ additional equations. But this is exactly the number of Lorentz Invariance identities (1.108): actually the different representations for a given differential operator, let's say $\frac{\partial}{\partial s_{ij}}$, turn to be related by LI symmetries.
As stated above, we can express the outcome of the derivative through IBPs; thus, we stress again that the derivative of a given MIs turns to be a combination of MIs in the same *topology* and its *subtopology.* Iterating the procedure for all the MIs, and for all the chosen kinematics invariants, we are able to build up a *Coupled System* of *(partial) Differential Equations* (if the kinematics invariants are more than one), which inherits from the IBPs a remarkable *block triangular* structre and *rational coefficients* in the kinematics invariants and d, namely the space-time dimensions. For the sake of simplicity in the rest of this section, we assume that MIs depends on a single variable, let's say x (often chosen

to be adimensional); thus, we deal with system in the form:

$$\frac{\partial \mathbf{F}(x,d)}{\partial x} = \mathbb{A}(x,d)\mathbf{F}(x,d); \tag{2.4}$$

We are free to redefine $d = d_c - 2\epsilon$, where, usually, $d_c = 4$.
Generally speaking, (2.4) gives, after integration, just the *general solution(s)*; in order to obtain the particular expression(s) for the unknown MI(s), let us anticipate that we have to adjust the constants coming from the integration, in order to match the *"physical"* result. This problem, which we referred to as *Fixing of the Boundary Conditions*, is particularly tricky; we will return on this delicate issue later on.
It is worth recalling that, on the one hand, the *number* of MIs is, somewhat, dictated by the problem, but, on the other hand, the *choice* of MIs is arbitrary. In fact we can imagine to express (2.4) in terms of a new basis of MIs, namely $\mathbf{I}$, related to the previous one by the relation $\mathbf{F}(x,\epsilon) = \mathbb{B}(x)\mathbf{I}(x,\epsilon)$. Thus, it is possible to recast (2.4) as:

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \left(\mathbb{B}^{-1}(x)\,\mathbb{A}(x,\epsilon)\,\mathbb{B}(x) - \mathbb{B}^{-1}(x)\frac{\partial \mathbb{B}(x)}{\partial x} \right) \mathbf{I}(x,\epsilon), \tag{2.5}$$

as we will see, this simple expression will play a crucial role in the proceeding. Even if, a priori, (2.4) could be expressed and solved w.r.t. any basis of MIs, a particularly brilliant and, in certain sense, *natural* basis, as it will be clear in a while, is the so called *canonical basis*, proposed by Henn, [15].

2.2 Solving DEQs, a General Strategy

As stated above, a key proprierty of a DEQs system for MIs is its *block triangular* structure, inherited from IBPs. Thus, it seems obvious to adopt a bottom-up solving strategy. Once a certain sector is solved, then the solution just determined is plugged in the inhomogeneous part of the next DEQ (i.e.: with an higher number of denominators). Let's consider, concretely, the case in which each sector contains just *one* MI. Then, the DEQ for the i^{th} MI, namely $F_i(x,\epsilon)$, reads:

$$\frac{\partial F_i(x,\epsilon)}{\partial x} = \mathbb{A}_{ii}(x,\epsilon)F_i(x,\epsilon) + S_i(x,\epsilon), \tag{2.6}$$

where $S_i(x,\epsilon)$ stands for the *subtopology*, considered known, for the given MI.
Then, we can solve (2.6) through *Euler's Variations of Constants*. Primarily, we solve the *homogeneous* DEQ associated to (2.6):

$$\frac{\partial F_{i,\,\text{hom}}(x,\epsilon)}{\partial x} = \mathbb{A}_{ii}(x,\epsilon)\,F_{i,\,\text{hom}}(x,\epsilon); \tag{2.7}$$

we have:

$$F_{i,\,\text{hom}}(x,\epsilon) = c_i(\epsilon)\,\exp\left(\int^x dt\,\mathbb{A}_{ii}(t,\epsilon)\right); \tag{2.8}$$

then, we look for a solution for (2.6) in the form of:

$$F_i^*(x,\epsilon) = \phi_i(x,\epsilon)\,F_{i,\text{hom}}(x,\epsilon). \tag{2.9}$$

Plugging the previous *ansatz* in (2.6), we obtain a DEQ for $\phi_i(x,\epsilon)$; solving the latter we can build-up (2.9), namely a particular solution for the *non-homogeneous* DEQ (2.6).
Then, a general solution for (2.6) reads:

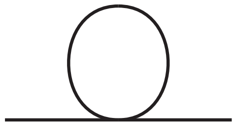
$$F_i(x,\epsilon) = F_{i,\text{hom}}(x,\epsilon) + F_i^*(x,\epsilon). \tag{2.10}$$

Bubble Integral, Closed Form

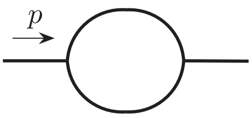
We consider the integral family given by:

$$\mathcal{B}^{(2,2)} = \int \widetilde{d^d k} \frac{1}{((k-p)^2 - m^2)^{r_1} (k^2 - m^2)^{r_2}}. \tag{2.11}$$

Thanks to `REDUZE2` we identify the following basis of MIs: $\mathbf{F} = \{F_1, F_2\}$ where:

$F_1 :=$  $= \int \widetilde{d^d k} \frac{1}{(k^2 - m^2)},$ (2.12)

and:

$F_2 :=$  $= \int \widetilde{d^d k} \frac{1}{((k-p)^2 - m^2)\,(k^2 - m^2)}.$ (2.13)

The DEQs system for $\mathbf{F}(s,\epsilon) = \{F_1(s,\epsilon), F_2(s,\epsilon)\}$ w.r.t. $p^2 = s$ reads:

$$\frac{\partial \mathbf{F}(s,\epsilon)}{\partial s} = \mathbb{A}(s,\epsilon)\mathbf{F}(s,\epsilon), \quad \mathbb{A}(s,\epsilon) = \begin{pmatrix} 0 & 0 \\ \frac{2(\epsilon-1)}{s(s-4m^2)} & \frac{2m^2 - s\epsilon}{s(s-4m^2)} \end{pmatrix} \tag{2.14}$$

We immediately notice that F_1 turns to be independent from the external momentum p; thus, the DEQ w.r.t. $s = p^2$, must be trivial:

$$\frac{\partial F_1(s,\epsilon)}{\partial s} = 0, \Rightarrow F_1(s,\epsilon) \equiv F_1(\epsilon) = \text{const.} \tag{2.15}$$

Moreover the analytic expression for $F_1(\epsilon)$ can be obtained from a direct integration[1] Then we can focus on the DEQ for F_2 considering its subtopology, namely the *non homogeneous* term in the DEQ completely known. We have:

$$\frac{\partial F_2(s,\epsilon)}{\partial s} = \frac{\left(2m^2 - \epsilon s\right)}{s\,(s-4m^2)} F_2(s,\epsilon) + \frac{2(\epsilon-1)}{s\,(s-4m^2)} F_1(\epsilon). \tag{2.16}$$

In order to simplify the DEQ, let's introduce the change of variable:

$$x = -\frac{s}{4m^2} \tag{2.17}$$

Thus, (2.16) reads:

$$\frac{\partial F_2(x,\epsilon)}{\partial x} = \left(\frac{1-2\epsilon}{2(x+1)} - \frac{1}{2x}\right) F_2(x,\epsilon) + \frac{2-2\epsilon}{4m^2}\left(\frac{1}{x} - \frac{1}{x+1}\right) F_1(\epsilon) \tag{2.18}$$

We stress again that F_1 has to be considered completely known, in this example: $F_1 =$ const.

Then, we can focus on the *homogeneous* DEQ in (2.18). Solving the latter we find:

$$F_{2,\text{hom}}(x,\epsilon) = c_2(\epsilon)\, x^{-\frac{1}{2}}\, (1+x)^{\frac{1}{2}-\epsilon}, \tag{2.19}$$

[1] The choice of the integration measure is not strictly necessary in this section.

where $c_2(\epsilon)$ is an arbitrary constant.
Then, we look for a solution in the form of:

$$F_2^*(x,\epsilon) = \phi_2(x)\, F_{2,\text{hom}}(x,\epsilon). \tag{2.20}$$

Plugging the latter in (2.18), we obtain a DEQ for $\phi_2(x,\epsilon)$ which reads:

$$\frac{\partial \phi_2(x,\epsilon)}{\partial x} = -\frac{(\epsilon-1)(x+1)^{\epsilon-\frac{3}{2}}}{2m^2\sqrt{x}}\, F_1(\epsilon). \tag{2.21}$$

Once again, F_1 is completely known, namely $F_1 = \text{const}$. Solving (2.21), we obtain:

$$\phi_2(x,\epsilon) = -\frac{\sqrt{x}(\epsilon-1)\, {}_2F_1\left(\frac{1}{2},\frac{3}{2}-\epsilon;\frac{3}{2};-x\right)}{m^2}\, F_1(\epsilon), \tag{2.22}$$

where ${}_2F_1$ is the *hypergeometric function*. Finally the general solution (2.10) reads:

$$F_2(x,\epsilon) = c_2(\epsilon)\, x^{-\frac{1}{2}}\,(1+x)^{\frac{1}{2}-\epsilon} - \frac{(\epsilon-1)(x+1)^{\frac{1}{2}-\epsilon}\, {}_2F_1\left(\frac{1}{2},\frac{3}{2}-\epsilon;\frac{3}{2};-x\right)}{m^2}\, F_1(\epsilon). \tag{2.23}$$

We stress that the general solution depends on an arbitrary constant, namely $c_2(\epsilon)$, which has to be fixed. We will discuss this issue in a dedicated section.

Solving DEQs, Laurent Expansion

Concretely, we are interested in determining the MIs in the $\epsilon \to 0$ limit (namely in the $d \to d_c$ limit). Thus the *closed* form (2.23), could be, somewhat, *unnecessary* and often too demanding.
Thus, we can consider the *Laurent Expansion* for the considered MIs:

$$F_i(x,\epsilon) = \sum_{k=\bar{k}}^{\infty} \epsilon^k\, F_i^{(k)}(x). \tag{2.24}$$

Plugging the previous expansion in (2.6) we obtain a *chained* DEQs system for the coefficients $F_i^{(k)}(x,\epsilon)$, which can be solved in a bottom-up approach, starting from the 1^{st} coefficient in the *Laurent Series*, namely $F_i^{\bar{k}}(x,\epsilon)$. Notice that this strategy, leads to a much more simple system, since, at least, the ϵ-dependence is factorized from the kinematics.

Bubble Integral, Laurent Expansion

Let's consider once again the DEQs system (2.14). As stated above, the 1^{st} MI, namely F_1 is a constant. We can adjust the integral measure, $\widetilde{d^dk}$, as:

$$\widetilde{d^dk} = \frac{d^dk}{i\pi^{\frac{d}{2}}}, \tag{2.25}$$

in such a way that:

$$F_1(\epsilon) = \frac{m^2}{\epsilon} - m^2\left(\log\left(\frac{m^2}{\mu^2}\right) + \gamma - 1\right) + \mathcal{O}(\epsilon). \tag{2.26}$$

Thus, F_1 and in particular its *Laurent Expansion* are completely known. Then let's consider the change of variable given by[2]

$$-\frac{s}{m^2} = \frac{(1-y)^2}{y}; \tag{2.27}$$

the DEQ for the second MI reads:

$$\frac{\partial F_2(y,\epsilon)}{\partial y} = -\frac{(y-1)^2\epsilon + 2y}{y(y^2-1)}\, F_1(y,\epsilon) - \frac{2(\epsilon-1)}{m^2\,(y^2-1)}\, F_1(\epsilon). \tag{2.28}$$

Then, let's suppose we are looking for [3] :

$$F_2(y,\epsilon) = \sum_{k=-2}^{\infty} \epsilon^k\, F_2^{(k)}(y). \tag{2.29}$$

Then, plugging the previous expansion in (2.28) we obtain:

$$\begin{aligned}\frac{\partial F_2^{(k)}(y)}{\partial y} =& -\frac{2}{y^2-1} F_2^{(k)}(y) - \frac{(y-1)^2}{y(y^2-1)}\, F_2^{(k-1)}(y) + \\ & \frac{2}{m^2\,(y^2-1)} F_1^{(k)} - \frac{2}{m^2\,(y^2-1)} F_1^{(k-1)}.\end{aligned} \tag{2.30}$$

Order ϵ^{-2}
Thus, the DEQ for $F_2^{(-2)}(y)$ reads (being $F_2^{(-2)}(y)$ the very first term in the Laurent Expansion, $F_2^{(-2-1)}(y)$ is vanishing; $F_1^{(-2)}$ and $F_1^{(-2-1)}$ are vanishing as well by construction):

$$\frac{\partial F_2^{(-2)}(y)}{\partial y} = -\frac{2}{y^2-1} F_2^{(-2)}(y); \tag{2.31}$$

we immediatly have:

$$F_2^{(-2)}(y) = -\frac{y+1}{y-1}\, c^{(-2)}. \tag{2.32}$$

We anticipate here that $F_2(y,\epsilon)$ turns to be finite in the $s \to 0 \Leftrightarrow y \to 1$ limit (see Section (2.6)), and so we are forced to set $c^{(-2)} \equiv 0$.

Order ϵ^{-1}
The DEQ for $F_2^{(-1)}(y)$ reads (taking into account: $F_2^{(-2)}(y) = 0$, $F_1^{(-1)} = m^2$ and $F_1^{(-1-1)} = 0$ by construction):

$$\frac{\partial F_2^{(-1)}(y)}{\partial y} = -\frac{2}{y^2-1} F_2^{(-1)}(y) + \frac{2}{y^2-1}. \tag{2.33}$$

The latter can be solved through the *Euler's variations of constants* described in the previous Subsections. The *homogeneous equation* is once again (2.31), thus:

$$F_{2,\,\mathrm{hom}}^{(-1)}(y) = -\frac{y+1}{y-1}\, c^{(-1)}; \tag{2.34}$$

[2]The usual redefinition: $-\frac{s}{m^2} = x$ would lead to more uncomfortable intermediate expression.
[3]It is well known (e.g.: by direct integration), that the *Laurent Expansion* for both the *Tadpole*, namely F_1 and the *Bubble*, namely F_2, are $\propto \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)$. Thus our guess is *"wrong"*. Anyway the *fixing of the Boundary Conditions* (see Section (2.6) will restore the correct *Laurent Expansion*. We believe that our assumption, namely considering an unnecessary term $\propto \frac{1}{\epsilon^2}$ in the *Laurent Expansion* makes the construction more transparent.

we look for a particular solution for (2.33) in the form of:

$$F_2^{*\,(-1)}(y) = -\frac{y+1}{y-1}\,\phi^{(-1)}(y); \tag{2.35}$$

plugging the latter in (2.33) we obtain a DEQ for $\phi^{(-1)}(y)$:

$$\frac{\partial \phi^{(-1)}(y)}{\partial y} = -\frac{2}{(y+1)^2}, \tag{2.36}$$

and a particular solution reads:

$$\phi^{(-1)}(y) = \frac{2}{y+1}. \tag{2.37}$$

Thus, a particular solution for (2.33) is:

$$F_2^{*\,(-1)}(y) = -\frac{2}{y-1}, \tag{2.38}$$

and so the general solution is:

$$F_2^{(-1)}(y) = F_{2,\,\mathrm{hom}}^{(-1)}(y) + F_2^{*\,(-1)}(y) = \frac{2+(1+y)c^{(-1)}}{1-y}. \tag{2.39}$$

The fixing of $c^{(-1)}$ is usually left as a final step in the calculation. However, as we anticipated above, the solution has to be finite in the $s \to 0 \Leftrightarrow y \to 1$ limit, and this is sufficient to set $c^{(-1)} = -1$.

Order ϵ^0

Then, we can solve the DEQ for $(F_2^{(0)}(y))$ which reads (taking into account the previous results):

$$\frac{\partial F_2^{(0)}(y)}{\partial y} = -\frac{2}{y^2-1}\,F_2^{(0)}(y) - \frac{2\left(\log\left(m^2\right)+\gamma-1\right)}{y^2-1} + \frac{1-y}{y(y+1)} - \frac{2}{y^2-1}; \tag{2.40}$$

Once again the *homogeneous solution* is known, namely:

$$F_{2,\,\mathrm{hom}}^{(0)}y = -\frac{y+1}{y-1}\,c^{(0)}, \tag{2.41}$$

where $c^{(0)}$ is an arbitrary constant. Then, we look for a particular solution for (2.40) in the form of:

$$F_2^{*(0)}(y) = -\frac{y+1}{y-1}\,\phi^{(0)}(y). \tag{2.42}$$

Plugging the latter in (2.40) we obtain a DEQ for $\phi^{(0)}(y)$ which reads:

$$\frac{\partial \phi^{(0)}(y)}{\partial y} = \frac{2y\log\left(m^2\right) + y^2 + 2(\gamma-1)y + 1}{y(y+1)^2}. \tag{2.43}$$

whose solution reads:

$$\phi^{(0)}(y) = \log(y) - \frac{2\left(\log\left(m^2\right)+\gamma-2\right)}{y+1}. \tag{2.44}$$

Thus we have:

$$F_2^{*(0)}(y) = -\frac{-2\log\left(m^2\right) + y\log(y) + \log(y) - 2\gamma + 4}{y-1}. \tag{2.45}$$

Finally the general solution for (2.40) is:

$$F_2^{(0)}(y) = F_{2,\,\text{hom}}^{(0)}(y) + F_2^{*(0)}(y) = -\frac{c^{(0)}y + c^{(0)} - 2\log\left(m^2\right) + y\log(y) + \log(y) - 2\gamma + 4}{y-1}. \tag{2.46}$$

Requiring that $F_2^{(2)}(y)$ has to be finite in the $s \to 0 \Leftrightarrow y \to 1$ limit, we fix $c^{(0)}$: $c^{(0)} = \log\left(m^2\right) + \gamma - 2$.

2.3 Canonical Basis

One variable case

In this section, we would try to delineate the principal features of the *Canonical Basis* [15], recalling (two of) the most remarkable properties:

- ϵ factorized form the kinematics, or explicitly the system in the form:

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \epsilon \mathbb{A}_c(x)\mathbf{I}(x,\epsilon), \tag{2.47}$$

- the matrix $\mathbb{A}_c(x)$ turns to have *simple poles.*

A few comments are mandatory; it is customary to rewrite the *canonical system* in a *dlog* form, namely:

$$d\,\mathbf{I}(\epsilon,x) = \epsilon d\,\mathbb{A}_c(x)\mathbf{I}(\epsilon,x), \quad \mathbb{A}_c = \sum_i^k \mathbb{M}_i d\,log\,\eta_i(x); \tag{2.48}$$

in the latter expression $\{\mathbb{M}_i\}_{i=1,\dots,k}$ are completely constant matrices, while $\{\eta_i(x)\}_{i=1,\dots,k}$ form the so-called *alphabet*, which constitutes the *kernel* for the integration.
Moreover, once the system is recast in the *canonical form* (modulo a rescaling by an appropriate power of ϵ for each MIs, in order to eliminate the ϵ-poles) the general solution can be obtained with a minimal effort, in terms of the *Dyson Series*, thus involving *Iterated Integrals*, up to an arbitrary order in the *Taylor expansion* in ϵ. More precisely, the bottom up approach can be avoided, and we can treat all the MIs on an equal footing, keeping them in a *vector*, $\mathbf{I}(x,\epsilon)$:

$$\mathbf{I}(x,\epsilon) = \left(\mathbb{1} + \epsilon\int^x dt_1\,\mathbb{A}_c(t) + \epsilon^2\int^x dt_1\int^{t_1} dt_2\,\mathbb{A}_c(t_1)\,\mathbb{A}_c(t_2) + \dots\right)\mathbf{I}_0(\epsilon). \tag{2.49}$$

We immediately notice that the solution at order k depends only on the solution at order $k-1$; the latter condition is encoded in:

$$\frac{\partial \mathbf{I}^{(k)}(x)}{\partial x} = \mathbb{A}_c(x)\,\mathbf{I}^{(k-1)}(x), \tag{2.50}$$

and:

$$\frac{\partial \mathbf{I}^{(0)}(x)}{\partial x} \equiv 0 \Leftrightarrow \mathbf{I}^{(0)}(x) \equiv \mathbf{I}_0(\epsilon), \tag{2.51}$$

namely the leading term in the ϵ expansion is a *constant.*

Two variables case
If the DEQs system is expressed in terms of *two variables*, let's say $\{x, y\}$, some additional effort is needed. More precisly, let's consider a partial DEQs system in the form:

$$\frac{\partial \mathbf{I}(x,y,\epsilon)}{\partial x} = \epsilon\, \mathbb{A}_x(x,y)\, \mathbf{I}(x,y,\epsilon), \qquad \frac{\partial \mathbf{I}(x,y,\epsilon)}{\partial y} = \epsilon\, \mathbb{A}_y(x,y)\mathbf{I}(x,y,\epsilon). \tag{2.52}$$

Then, we look for a solution:

$$\mathbf{I}(x,y,\epsilon) = \Big(\mathbb{1} + \epsilon \mathbb{B}^{(1)}(x,y) + \epsilon^2 \mathbb{B}^{(2)}(x,y) + \dots\Big)\, \mathbf{I}_0(\epsilon). \tag{2.53}$$

The matrices $\{\mathbb{B}^{(i)}(x,y)\}$ are computed according to the following steps.

- Given the general solution at order k, namely $\mathbb{B}^{(k)}(x,y,\epsilon)$, then we can compute:

  $$\int^x dt \mathbb{A}_x(t,y)\, \mathbb{B}^{(k)}(t,y).$$

  So the general solution at order $k+1$, will be:

  $$\mathbf{I}(x,y,\epsilon) = \left(\mathbb{1} + \dots + \epsilon^{(k)} \mathbb{B}^{(k)}(x,y) + \epsilon^{(k+1)}\left(\int^x dt \mathbb{A}_x(t,y)\, \mathbb{B}^{(k)}(t,y) + \mathbb{C}^{(k+1)}(y)\right)\right)\mathbf{I}_0(\epsilon);$$

  where $\mathbb{C}^{(k+1)}(y)$ is an unknown matrix depending on y.

- Plugging the latter in the DEQs system w.r.t. y, forgetting about the boundary vector $\mathbf{I}_0$, and collecting terms proportional to ϵ^{k+1}, we obtain:

  $$\frac{\partial \mathbb{C}^{(k+1)}(y)}{\partial y} = -\frac{\partial}{\partial y}\left(\int^x dt \mathbb{A}_x(t,y)\, \mathbb{B}^{(k)}(t,y)\right) + \mathbb{A}_y(x,y)\mathbb{B}^{(k)}(x,y).$$

  We stress that the r.h.s. must be x-independent; solving the latter DEQ, and thus obtaining $\mathbb{C}^{(k+1)}$, we determine the general solution at order $k+1$.

We also stress that, once the DEQs system is recast in the canonical form, the solutions turn to be *pure functions* of *uniform trascendentality*; more explicitly, following [15], we can define the *degree of trascendentaly* $\mathcal{T}_d(f)$, of a function f, as the number of integrations needed to obtain it, thus $\mathcal{T}_d(log) = \mathcal{T}_d(\pi) \equiv 1$, $\mathcal{T}_d(\zeta_n) \equiv n$ just to mention a couple of examples; and so, arbitrary imposing $\mathcal{T}_d(\epsilon) = -1$, each canonical MI is represented by a combination of functions wich degree of trascendentaly equals to zero. As a consequence of the previous discussion, investigating the nature of a certain function in terms of degree of trascendentality, and consequently manipulating the integrand, could offer a criterion to reach the canonical basis. Furthermore, it has been conjectured that MIs with *unit leading singularities* lead to pure functions of uniform trascendentality [15].
Besides its elegance, it should be noted that the general solution (2.49) can be obtained in an almost *"algebraic"* way, thus it's more than welcome in view of automation. Not surprisingly a lot of techniques, based on different approaches, have been proposed in the recent years in order to obtain the canonical basis of MIs, e.g.: [16]-[22]. Among the others, the method proposed for the first time in [23] based on the *Magnus Expansion* or, equivalently *Magnus Exponential*, requires, as a starting point, a system which is *linear in* ϵ, and had been used in presence of *several variables*; this method was successfully applied in many cases, as can be found in [23]-[27].

2.4 Magnus Exponential

In order to explain the method based on the *Magnus Exponential* it's worth reminding once again our starting assumption, namely, given a set of MIs $\mathbf{F}$, we consider a system *linear* in ϵ (just for the sake of simplicity, we assume that the MIs depends on a single variable, denoted by x):

$$\frac{\partial \mathbf{F}(x,\epsilon)}{\partial x} = \left(\mathbb{A}_0(x) + \epsilon \mathbb{A}_1(x)\right) \mathbf{F}(x,\epsilon). \tag{2.54}$$

Then, roughly speaking, the idea is to *"eat"*, or better, *"rotate away"* the ϵ-independent term, absorbing the latter via a change of basis for the MIs.
More concretely, we are free to redefine $\mathbf{F}(x,\epsilon) = \mathbb{B}(x)\mathbf{I}(x,\epsilon)$, where $\mathbb{B}(x)$ is an unknown matrix, and thus we recast (2.54) as:

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \mathbb{B}^{-1}(x)\left(-\frac{\partial \mathbb{B}(x)}{\partial x} + \mathbb{A}_0(x)\mathbb{B}(x) + \epsilon \mathbb{A}_1(x)\mathbb{B}(x)\right)\mathbf{I}(x,\epsilon). \tag{2.55}$$

We immediatly notice that, if $\mathbb{B}(x)$ is such that:

$$\frac{\partial \mathbb{B}(x)}{\partial x} = \mathbb{A}_0(x)\mathbb{B}(x), \tag{2.56}$$

then it is garanteed that (2.55) presents a manifest ϵ factorization:

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \epsilon \mathbb{B}^{-1}(x)\mathbb{A}_1(x)\mathbb{B}(x)\mathbf{I}(x,\epsilon) \equiv \epsilon \mathbb{A}_c(x)\mathbf{I}(x,\epsilon). \tag{2.57}$$

Thus, we are left with the problem of solving a DEQ (2.56), for the *linear operator* $\mathbb{B}(x)$. Following [29], a solution for (2.56), can be written in terms of the *exponential* of an, *a priori*, *infinite* series:

$$\mathbb{B}(x) = e^{\Omega[\mathbb{A}_0](x)}, \tag{2.58}$$

where $\Omega[\mathbb{A}_0](x)$ is given by:

$$\Omega[\mathbb{A}_0](x) = \sum_{k=1}^{\infty} \Omega_k[\mathbb{A}_0](x); \tag{2.59}$$

the various terms in the latter expression involve *iterated integrals* of *nested commutators* of the kernel $\mathbb{A}_0$. More explicitly the first terms read:

$$\begin{aligned}
\Omega_1[\mathbb{A}_0](x) &= \int^x dt_1 \mathbb{A}_0(t_1), \\
\Omega_2[\mathbb{A}_0](x) &= \int^x \int^{t_1} dt_1 dt_2 [\mathbb{A}_0(t_1), \mathbb{A}_0(t_2)].
\end{aligned} \tag{2.60}$$

Clearly, on a practical level, we have to furnish a closed solution to (2.56), since, after all, we are interested in finding out an explicit change of basis. Such an explicit transformation is guaranteed if the kernel $\mathbb{A}_0$ is a triangular matrix; the latter leads to a series (2.59) which contains just a finite number of terms, say $\bar{n}$, namely $\Omega_m[\mathbb{A}_0] \equiv 0 \ \forall m > \bar{n}$, as desired.
In order to prove the previous statement, we notice that we are free to split the *triangular* kernel $\mathbb{A}_0(x)$ as:

$$\mathbb{A}_0(x) = \mathbb{D}_0(x) + \mathbb{S}_0(x), \tag{2.61}$$

where $\mathbb{D}_0(x)$ is the *diagonal* part, and $\mathbb{S}_0(x)$ the *sub-triangular* one. Then, as a preliminary step, $\mathbb{D}_0(x)$ can be reabsorbed with a minimal effort (a *diagonal* matrix commutes with its integral, and so only the very first term in (2.60) is non-vanishing). On practical grounds, we introduce a new basis of MIs, $\mathbf{F}(x,\epsilon) = \mathbb{B}_1(x)\mathbf{F}^{[2]}(x,\epsilon)$, and we recast (2.54) as:

$$\frac{\partial \mathbf{F}^{[2]}(x,\epsilon)}{\partial x} = \mathbb{B}_1^{-1}(x)\left(-\frac{\partial \mathbb{B}_1(x)}{\partial x} + \mathbb{D}_0(x)\mathbb{B}_1(x) + \mathbb{S}_0(x)\mathbb{B}_1 + \mathcal{O}(\epsilon)\right)\mathbf{F}^{[2]}(x,\epsilon). \tag{2.62}$$

If the unknown matrix $\mathbb{B}_1(x)$ satisfies:

$$\partial_x \mathbb{B}_0(x) = \mathbb{D}_0(x)\mathbb{B}_0(x) \Rightarrow \mathbb{B}_0(x) = e^{\int^x dt\,\mathbb{D}_0(t)}; \tag{2.63}$$

then (2.62) reduces to:

$$\frac{\partial \mathbf{F}^{[2]}(x,\epsilon)}{\partial x} = \left(\underbrace{\mathbb{B}_1^{-1}(x)\mathbb{S}_0(x)\mathbb{B}_1(x)}_{=\mathbb{T}_0(x)} + \mathcal{O}(\epsilon)\right)\mathbf{F}^{[2]}(x,\epsilon); \tag{2.64}$$

furthermore the resulting matrix, $\mathbb{T}_0(x) = e^{-\int^x dt\,\mathbb{D}_0(t)}\ \mathbb{S}_0(x)\ e^{\int^x dt\,\mathbb{D}_0(t)}$, turns to be strictly triangular, being $\mathbb{S}_0(x)$ strictly triangular, and $e^{\pm\int^x dt\,\mathbb{D}_0(t)}$ diagonal ones. But now, we recall that the product of a set of m, (m,m) strictly triangular matrix is always vanishing. Thanks to this fact, given a (m,m) strictly triangular kernel, we argue that m *nested commutators* vanish as well (independently of iterated integrals), and so (2.59) has *at most* m terms.
On the other hand, it is worth stressing that even if the kernel $\mathbb{A}_0$ turns not to be triangular, then we could have a finite number of terms in (2.59) thanks to its *nilpotency* properties.

Bubble Integral, a 1-loop example

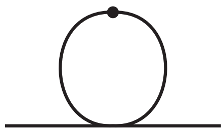
We show in this section an application of the algorithm described in the previous section. We consider the integral family given by:

$$\mathcal{B}^{(1,1)} = \int \widetilde{d^dk}\frac{1}{((k-p)^2-m^2)^{r_1}(k^2-m^2)^{r_2}}. \tag{2.65}$$

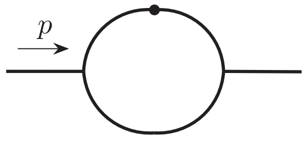
Thanks to `REDUZE2`, we identify a basis of MIs, $\mathbf{F} = \{F_1, F_2\}$[4] which fulfills a DEQs system linear in ϵ:

$$\frac{\partial \mathbf{F}(s,\epsilon)}{\partial s} = \mathbb{A}(s,\epsilon)\,\mathbf{F}(s,\epsilon), \quad \mathbb{A}(s,\epsilon) = \begin{pmatrix} 0 & 0 \\ \frac{\epsilon}{s(s-4m^2)} & \frac{2m^2-(\epsilon+1)s}{s(s-4m^2)} \end{pmatrix}, \tag{2.66}$$

where:

$$F_1(s,\epsilon) \equiv F_2(\epsilon) =$$  $$= \int \widetilde{d^dk}\frac{1}{(k^2-m^2)^2}, \tag{2.67}$$

and:

$$F_2(s,\epsilon) =$$ 

$$= \int \widetilde{d^dk}\frac{1}{(k^2-m^2)^2((k-p)^2-m^2)}. \tag{2.68}$$

[4]Strictly speaking we should multiply each MI by a factor ϵ in order to eliminate the ϵ-poles from the MIs. Anyway, this is not necessary here, and we postpone the issue to the following sections.

Then, splitting $\mathbb{A}(s,\epsilon)$ as: $\mathbb{A}(s,\epsilon) = \mathbb{A}_0(s) + \epsilon\mathbb{A}_1(s)$, being $\mathbb{A}_{0,1}(s)$ ϵ independent, the matrix $\mathbb{A}_0(s)$ reads:

$$\mathbb{A}_0(s) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{s-2m^2}{s(4m^2-s)} \end{pmatrix}. \tag{2.69}$$

Then, following the algorithm presented just above, we perform a change of basis: $\mathbf{F}(s,\epsilon) = \mathbb{B}(s)\mathbf{I}(s,\epsilon)$ and 2.66 can be recasted as:

$$\frac{\partial \mathbf{I}(s,\epsilon)}{\partial s} = \mathbb{B}^{-1}(s)\left(-\frac{\partial \mathbb{B}(s)}{\partial s} + \mathbb{A}_0(s)\,\mathbb{B}(s) + \epsilon\mathbb{A}_1(s)\mathbb{B}(s)\right)\mathbf{I}(s,\epsilon). \tag{2.70}$$

Being $\mathbb{A}_0(s)$ *diagonal* it commutes with its integral and the *Magnus Series* consists in the very first summand; we simply have:

$$\frac{\partial \mathbb{B}(s)}{\partial s} = \mathbb{A}_0(s)\mathbb{B}(s) \Rightarrow \mathbb{B}(s) = e^{\int ds \mathbb{A}_0(s)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{-s}\sqrt{4m^2-s}} \end{pmatrix}. \tag{2.71}$$

Then, the canonical MIs $\mathbf{I}(s,\epsilon) = \{I_1(\epsilon), I_2(s,\epsilon)\}$, reads:

$$I_1(\epsilon) = F_1(\epsilon), \quad I_2(s,\epsilon) = \sqrt{-s}\sqrt{4m^2-s}\,F_2(s,\epsilon). \tag{2.72}$$

Finally, performing the change of variable: $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$, (2.70) reads[5]

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \epsilon\mathbb{A}_c(x)\mathbf{I}(x,\epsilon), \quad \mathbb{A}_c = \begin{pmatrix} 0 & 0 \\ \frac{1}{x} & \frac{1}{x} - \frac{2}{x+1} \end{pmatrix}, \tag{2.73}$$

which has the desired properties.

A comment on a "Post-Canonical" scenario

We would like to mention the possibility to further iterate the ideas presented just above. Given the canonical system (2.73), we consider: $\mathbb{A}_c(x) = \mathbb{D}_c(x) + \mathbb{N}_c(x)$, being $D_c(x)$ the diagonal part of the Canonical Matrix. Then, we could perform an additional change of basis: $\mathbf{I}(x,\epsilon) = \mathbb{C}(x)\mathbf{H}(x,\epsilon)$, obtaining the usual:

$$\frac{\partial \mathbf{H}(x,\epsilon)}{\partial x} = \mathbb{C}^{-1}(x)\left(-\frac{\partial \mathbb{C}(x)}{\partial x} + \mathbb{D}_c(x)\mathbb{C}(x) + \mathbb{N}_c\mathbb{C}(x)\right)\mathbf{H}(x,\epsilon). \tag{2.74}$$

then we look for a matrix $\mathbb{C}(x)$ such that:

$$\frac{\partial \mathbb{C}(x)}{\partial x} = \epsilon\,\mathbb{D}_c(x)\,\mathbb{C}(x). \tag{2.75}$$

Being $\mathbb{D}_c(x)$ diagonal, the integration is straightforward; in this explicit example we obtain:

$$\mathbb{C}(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^{\epsilon}(1+x)^{-2\epsilon} \end{pmatrix}. \tag{2.76}$$

The new system reads:

$$\frac{\partial \mathbf{H}(x,\epsilon)}{\partial x} = \mathbb{A}_{pc}(x,\epsilon)\mathbf{H}(x,\epsilon), \quad \mathbb{A}_{pc}(x,\epsilon) = \epsilon\begin{pmatrix} 0 & 0 \\ x^{-(1+\epsilon)}(1+x)^{2\epsilon} & 0 \end{pmatrix}. \tag{2.77}$$

[5]We consistently assume: $0 < x < 1$.

Then, thnaks to its strictly triangular form, the Dyson Series for $\mathbb{A}_{ps}(x,\epsilon)$, has a finite number of terms [6]. In this explicit example we have:

$$\mathbf{H}(x,\epsilon) = \begin{pmatrix} 1 & 0 \\ -x^{-\epsilon}\,{}_2F_1(-2\epsilon,-\epsilon;1-\epsilon;-x) & 1 \end{pmatrix} \mathbf{H}_0(\epsilon). \tag{2.78}$$

We notice that, at least in principle, this decomposition could be applied to all the one-loop integrals, since, for these integrals, the Canonical System is triangular.

2.5 General Solution

Once the DEQs system is casted in the *canonical form* (2.48), and a suitable change of variable has been performed in order to obtain a *rational alphabet*[7], we are ready to determine the general solution in terms of a particular class of function, the so called *generalized polylogarithms* (GPLs) [30]-[**?**], thus exploiting the compactness and elegance, and so, finally, the *"naturalness"* offered by the *Canonical Basis*. In this context, the number of kinematics variables, i.e.: the ones w.r.t. we differentiate, considerably determines the richness of the mathematical structure involved; therefore we will follow here an *"historical"* approach, focusing , at the beginning, on the case with *one* variable present, and subsequently moving to the *two* variables case.

2.5.1 One variable case, HPLs

Definitions
In the *one* variable case, following [32] we assume that the *alphabet* is formed by:

$$\eta(x) = \{x, 1-x, 1+x\}. \tag{2.79}$$

Then, iterated integrals over the inverse of (2.79) lead to the so called *Harmonic Polylogarithms*: these functions are identified by a set of n indices, grouped into a n-dimensional vector $\vec{\omega}_n$, with entries chosen from the set: $\{0,1,-1\}$. Even if these functions are, nowadays, well known mathematical objects, we prefer to build them up, in a constructive and pedagogical way. As a first step, we can furnish a list of the rational functions, up to irrelevant constant factors, expected in the DEQ, which we referred to as *kernel*, namely:

$$\begin{aligned} \frac{1}{x} &= f(0,x), \\ \frac{1}{1-x} &= f(1,x), \\ \frac{1}{1+x} &= f(-1,x). \end{aligned} \tag{2.80}$$

Integrating (2.80) in the interval $[0,x]$ we find (define):

$$\begin{aligned} H(0,x) &\equiv \log(x), \\ H(1,x) &= \int_0^x \frac{dt}{1-t} = -\log(1-x), \\ H(-1,x) &= \int_0^x \frac{dt}{1+t} = \log(1+x). \end{aligned} \tag{2.81}$$

[6] Namely:

$$\begin{pmatrix} 0 & 0 \\ \clubsuit & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \spadesuit & 0 \end{pmatrix} = \mathbb{0}.$$

[7] A *non-rational* alphabet can be trated in terms of Chen's *iterated integrals* [34] .

Thus, roughly speaking and with a little abuse of notation, we can regard the *integration* as a *"replacement"*, at least in this very first step:

$$\int_0^x dt : f(\{\pm 1, 0\}, t) \to H(\{\pm 1, 0\}, x), \tag{2.82}$$

Trivially, the *derivative* brings us back to the fs, namely:

$$\frac{d}{dx} : H(\{\pm 1, 0\}, x) = f(\{\pm 1, 0\}, x). \tag{2.83}$$

Keeping in mind the structure given by the *Dyson Series*, we expect to face, at the n^{th} stpe, or, more properly at weight n, the *iterative structure* given by:

$$\begin{aligned} \int_0^x dt : f(\omega_n, t)\, H(\vec{\omega}_{n-1}, t) \to H(\vec{\omega}_n, x), \quad \vec{\omega}_n &= \{\omega_n, \omega_{n-1}, \ldots, \omega_1\} \\ &= \omega_{n,\, n-1,\, \ldots,\, 1}. \end{aligned} \tag{2.84}$$

accompanied by:

$$\frac{d}{dx} : H(\vec{\omega}_n, x) \to f(\omega_n, x)\, H(\vec{\omega}_{n-1}, x). \tag{2.85}$$

Through standard calculus, we can recognize the recursive relations:

$$\begin{aligned} H(\vec{1}_n, x) &= \frac{1}{n!}(-\log(1-x))^n, \\ H(-\vec{1}_n, x) &= \frac{1}{n!}\log^n(1+x), \end{aligned} \tag{2.86}$$

and, for consistency, we impose:

$$H(\vec{0}_n, x) = \frac{1}{n!}\log^n(x). \tag{2.87}$$

We immediately notice that all the HPLs, except from $H(\vec{0}_n, x)$, vanish in $x = 0$, due to the $\int_0^0 dt f(\omega_n, t)\, H(\vec{\omega}_{n-1}, t)$ integral.

Shuffle Algebra

Several of the properties fulfilled by HPLs can be proved by the usual integration by parts (ibp). Let's consider ($H(\{\omega_2, \omega_1\}, x) \equiv H(\omega_{2,1}, x)$):

$$\begin{aligned} H(\omega_{2,1}, x) = \int_0^x dt f(\omega_2, t)\, H(\omega_1, t) &= H(\omega_2, x) H(\omega_1, x) - \int_0^x dt H(\omega_2, t) f(\omega_1, t) \\ &= H(\omega_2, x) H(\omega_1, x) - H(\omega_{1,\,2}, x). \end{aligned} \tag{2.88}$$

The latter reads, after a minor rearrangement:

$$H(\omega_1, x) H(\omega_2, x) = H(\omega_{1,\,2}, x) + H(\omega_{2,\,1}, x). \tag{2.89}$$

Then, starting from (2.89) we can argue that HPLs obey the following relation:

$$\begin{aligned} H(\omega_n, x) H(\omega_{n-1,\, n-2,\, \ldots,\, 1}, x) = & H(\omega_{n,\, n-1,\, n-2,\, \ldots,\, 1}, x) \\ & + H(\omega_{n-1,\, n,\, n-2,\, \ldots,\, 1}, x) \\ & + \ldots \\ & + H(\omega_{n-1,\, n-2,\, \ldots,\, 1,\, n}, x). \end{aligned} \tag{2.90}$$

In fact, recalling the *"standard method"* presented in [31], we claim that (2.90) is guaranteed if and only if it holds in a point x_0 *and* it holds for its derivative, as well. We immediately notice that (2.90) holds in $x_0 = 0$, because all the HPLs are vanishing (it is sufficient to avoid the *"pathological case"* in which $\omega_i = 0\,\forall i$; in this case (2.90) is equivalent to (2.87)). So, if we consider the derivative w.r.t. x in (2.90) we obtain:

$$\begin{aligned}
&f(\omega_n, x)H(\omega_{n-1\,n-2\,\dots\,1}, x) + H(\omega_n, x)\,f(\omega_{n-1}, x)\,H(\omega_{n-2\,\dots\,1}, x) = \\
&f(\omega_n, x)H(\omega_{n-1\,n-2\,\dots\,1}, x) + \cdots + f(\omega_{n-1}, x)\,H(\omega_{n\,n-2\,\dots,\,1}, x) + \\
&f(\omega_{n-1}, x)\,H(\omega_{n-2\,\dots,\,1\,n}, x).
\end{aligned} \tag{2.91}$$

Thus, we notice that the first term in the first row cancels against the first term in the second row; moreover we can collect and remove an irrelevant $f(\omega_{n-1}, x)$ common factor. Then, we conclude that (2.91) holds, if and only if (2.90) is proven to be true at a lower weight; but, iterating the procedure, sooner or later, *lowering the weight*, we land on (2.89), wich holds by a direct calculation.
Moreover, (2.90) can be generalized to the product of two HPLs with arbitrary weights, namely $H(\vec{\omega}_p, x)\,H(\vec{\omega}_q, x)$, with p and q a priori both different form 1. In fact, thanks to the *"standard method"*, the so called *shuffle relation* holds:

$$H(\vec{\omega}_p, x)\,H(\vec{\omega}_q, x) = \sum_{r=p\uplus q} H(\vec{\omega}_r, x), \tag{2.92}$$

where $\vec{\omega}_{r=p\uplus q}$ represents all merges of $\vec{\omega}_p$ and $\vec{\omega}_q$ in which all the relative order of the elements in $\vec{\omega}_p$ and $\vec{\omega}_q$ is preserved.

2.5.2 Multiple variable case, GPLs

Definitions and properties
As promised, we are ready now to introduce a new and a bit more general, class of functions, namely the *Generalized Polylogarithms*. In this context, we have $\mathbf{x} = \{x_1, \dots x_m\}$ variables. Then, we can factor a generic letter in the alphabet, $\eta_k(\mathbf{x})$ w.r.t. each variable x_i, namely:

$$\eta_k(\mathbf{x}) = \prod_{i_k} (x_{i_k} - \omega_{i_k}), \tag{2.93}$$

where ω_{i_k} depends, a priori, on all the others variables.
Then, we define (notice that, again for *"historical reasons"*, the string of indices is reversed w.r.t. the previous case):

$$\begin{aligned}
G(\vec{\omega}_n, x_i) &= \int_0^{x_i} dt \frac{1}{t - \omega_1} G(\vec{\omega}_{n-1}, t), \\
G(\vec{0}_n, x_i) &= \frac{1}{n!} \log^n(x_i),
\end{aligned} \tag{2.94}$$

or, equivalently:

$$\frac{\partial}{\partial x_i} G(\vec{\omega}_n, x_i) = \frac{1}{x_i - \omega_1} G(\vec{\omega}_{n-1}, x_i). \tag{2.95}$$

Moreover, we can recover the classical polylogarithms:

$$G(\vec{\omega}_n, x_i) = \frac{1}{n!} \log^n(1 - \frac{x_i}{\omega}), \quad \vec{\omega} = \underbrace{\{\omega, \dots, \omega\}}_{n\ times}, \tag{2.96}$$

and the HPLs introduced in the previous subsections:

$$H(\vec{\omega}_n, x_i) = (-1)^p \, G(\vec{\omega}_n, x_i), \tag{2.97}$$

where p is the number of indices equal to $+1$ contained in $\vec{\omega}$. GPLs fulfill some interesting properties, often useful, or necessary, also during the calculations, such as the *shuffle algebra*, described in the previous section, a sort of *"rescale invariance"*, namely the fact that if the rightmost index is different from 0, $\omega_n \neq 0$, the $G(\vec{\omega}_n, x_i)$ is invariant under the rescaling of all its arguments by a common factor $z \in \mathbb{C}^*$:

$$G(\vec{\omega}_n, x_i) = G(z\,\vec{\omega}_n, z\,x), \quad \omega_n \neq 0,\ z \in \mathbb{C}^*. \tag{2.98}$$

Besides that, we can mention the *Holder convolution* [36], [35] considering $\omega_1 \neq 1$ and $\omega_n \neq 0$, we have:

$$G(\{\omega_1, \ldots, \omega_n\}, 1) = (-1)^n G(\{1-\omega_n, \ldots, 1-\omega_1\}, 1). \tag{2.99}$$

Derivative of GPLs w.r.t. their weights

On top of that, it is important to develop a strategy to compute the derivative of GPL w.r.t. their weights, or, better:

$$\frac{\partial}{\partial x_j} G(\{\omega_1, \ldots, \omega_k(x_j), \ldots, \omega_n\}, x_i) \equiv \frac{\partial G(\vec{\omega}(x_j), x_i)}{\partial x_j}, \quad x_j \neq x_i. \tag{2.100}$$

The latter can be computed recalling the very definition (2.94); then (2.100) trivially reads:

$$\begin{aligned}
\frac{\partial G(\vec{\omega}(x_j), x_i)}{\partial x_j} &= \int_0^{x_i} \frac{dt_1}{t_1-\omega_1} \cdots \int_0^{t_{k-1}} \frac{\partial}{\partial x_j}\left(\frac{dt_k}{t_k-\omega_k(x_j)}\right) \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n-\omega_n} = \\
&= \left(-\frac{\partial \omega_k(x_j)}{\partial x_j}\right) \int_0^{x_i} \frac{dt_1}{t_1-\omega_1} \cdots \int_0^{t_{k-1}} \frac{\partial}{\partial t_k}\left(\frac{dt_k}{t_k-\omega_k(x_j)}\right) \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n-\omega_n} = \\
&= \left(-\frac{\partial \omega_k(x_j)}{\partial x_j}\right) \int_0^{x_i} \frac{dt_1}{t_1-\omega_1} \cdots \int_0^{t_{k-1}} \frac{\partial}{\partial t_k}\left(\frac{dt_k}{t_k-\omega_k(x_j)}\right) G(\vec{\omega}_{n-k}, t_k).
\end{aligned} \tag{2.101}$$

Then, we can focus on the integral in dt_k in the last line above, and performing the standard integration by parts (dropping the x_j dependence):

$$\begin{aligned}
&\int_0^{t_{k-1}} \frac{\partial}{\partial t_k}\left(\frac{dt_k}{t_k-\omega_k}\right) G(\vec{\omega}_{n-k}, t_k) = \\
&= \frac{1}{t_{k-1}-\omega_k} G(\vec{\omega}_{n-k}, t_{k-1}) - \int_0^{t_{k-1}} \frac{dt_k}{t_k-\omega_k} \frac{\partial G(\vec{\omega}_{n-k}, t_k)}{\partial t_k} = \\
&= \frac{1}{t_{k-1}-\omega_k} G(\vec{\omega}_{n-k}, t_{k-1}) - \int_0^{t_{k-1}} \frac{dt_k}{(t_k-\omega_k)(t_k-\omega_{k+1})} G(\vec{\omega}_{n-k-1}, t_k).
\end{aligned} \tag{2.102}$$

Moreover, we can rewrite:

$$\frac{1}{(t_k-\omega_k)(t_k-\omega_{k+1})} = \frac{A}{t_k-\omega_k} + \frac{B}{t_k-\omega_{k+1}}, \tag{2.103}$$

where A, B are found to be:

$$A = -B = \frac{-1}{\omega_{k+1}-\omega_k}; \tag{2.104}$$

thus (2.102) results:

$$
\begin{aligned}
&\int_0^{t_{k-1}} \frac{\partial}{\partial t_k}\left(\frac{dt_k}{t_k-\omega_k(x_j)}\right) G(\vec{\omega}_{n-k},t_k) = \\
&= \frac{1}{t_{k-1}-\omega_k} G(\vec{\omega}_{n-k},t_{k-1}) - A\int_0^{t_{k-1}} \frac{dt_k}{t_k-\omega_k} G(\vec{\omega}_{n-k-1},t_k) + \\
&- B\int_0^{t_{k-1}} \frac{dt_k}{t_k-\omega_{k+1}} G(\vec{\omega}_{n-k-1},t_k).
\end{aligned}
\tag{2.105}
$$

Finally, we can plug the explicit result of the derivative, namely (2.105), in (2.101); again we have to perform the decomposition:

$$
\frac{1}{(t_{k-1}-\omega_{k-1})(t_{k-1}-\omega_k)} = \frac{C}{t_{k-1}-\omega_{k-1}} + \frac{D}{t_{k-1}-\omega_k}, \tag{2.106}
$$

where C, D reads:

$$
C = -D = \frac{-1}{\omega_k-\omega_{k-1}}. \tag{2.107}
$$

At the end, (2.101) results:

$$
\begin{aligned}
&\frac{\partial G(\vec{\omega}(x_j),x_i)}{\partial x_j} = \left(-\frac{\partial \omega_k(x_j)}{\partial x_j}\right) \times \\
&\times \left[\left(\frac{1}{\omega_k-\omega_{k-1}}\right)\left(G(\{\omega_1,\ldots,\not{\omega}_{k-1},\ldots,\omega_n\},x_i) - G(\{\omega_1,\ldots,\not{\omega}_k,\ldots,\omega_n\},x_i)\right) + \right. \\
&\left. + \left(\frac{1}{\omega_{k+1}-\omega_k}\right)\left(G(\{\omega_1,\ldots,\not{\omega}_k,\ldots,\omega_n\},x_i) - G(\{\omega_1,\ldots,\not{\omega}_{k+1},\ldots,\omega_n\},x_i)\right)\right].
\end{aligned}
\tag{2.108}
$$

The procedure exposed just above, can be extended with a minimal effort to the case in which there are *several* weights depending on x_j. In the *"unlucky"* configuration in which *two consecutive* weights are *equal*, ($\omega_k \equiv \omega_{k+1}$) the infinities appearing in a single derivative (2.108) turn to cancel out in the sum.

2.6 Boundary Conditions

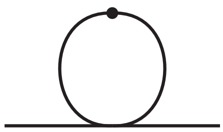
Once the general solution is determined, independently from the basis of MIs we are working with, we are left with the problem regarding the *fixing of the Boundary Conditions*, as anticipated above. We stress again that we are asked to choose the *integration constants* in such a way that the general solutions matches the *"physical result"*, namely the *"original"* values of the integrals. Generally speaking, this means that we have to obtain the value of the unknown integrals in a certain, preferable, kinematics points[8] through an independent method; then, evaluating our general solutions in those kineamtics limits, we have to adjust the free constants, in order to reproduce the desired results. Even if obtaining the value of an integral in a single point is, by far, less demanding than obtaining the full result, this step of the calculation is particularly tricky and not yet systematized. Among the various strategy often adopted, we could mention *Expansion by Regions* or the *Large Mass Expansion*. However it turns out that it is possible to infer *quantitative* informations regarding integrals from *qualitative* ones: for example, we could fix the boundary conditions in such a way that *unphysical* or *fictitious* singularities, are absent from the solution. On

[8]Boundary conditions for different MIs can be obtained, a priori, working in different kinematics points.

top of that, a careful inspection of the singularities, detecting *physical* and *unphysical* ones directly in the DEQ, and a subsequent manipulation on the latter, turns to be sufficient. Again, even if not strictly necessary, expressing the MIs in the *canonical basis*, makes this analysis particularly evident. In fact, the presence of a *pseudo-threshold* as a *simple pole* in the DEQ, let's say $(x - \bar{x}_0)^{-1}$, suggests that multiplying the latter by a factor $(x - \bar{x}_0)$, and then safely taking the limit $x \to \bar{x}_0$ in the expression, we obtain a relation between MIs in $x = \bar{x}_0$; and this can provide the fixing of BCs.
Moreover we notice that the *Magnus exponential* could introduce some *pseudo-threshold*, let's say again $(x - \bar{x}_0)$, directly in the *numerator* of the *canonical* MI. Thus, imposing the vanishing of *that* MI in the $x \to \bar{x}_0$ limit, we could fix the BCs.
Another possible strategy, suggested in [27], consists in solving the differential equation for a *massive* integral in a particular limit, and then exploiting the relation among the solution just obtained, the one associated to the whole differential equation (i.e.: no particular limit is assumed) and the value of the corresponding massless integral which is, generally speaking, much easier to compute and can be considered known. We will apply this proposal to an explicit example.

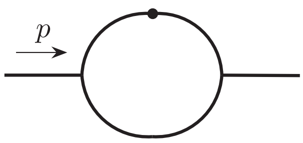
2.6.1 Massive BCs from Massless ones

Lets consider the DEQs system formed by:



$$= \int \widetilde{d^d k} \frac{1}{(k^2 - m^2)^2} =: \mathcal{T}_1, \tag{2.109}$$

and:



$$= \int \widetilde{d^d k} \frac{1}{(k^2 - m^2)^2((k-p)^2 - m^2)} =: \mathcal{T}_2, \tag{2.110}$$

with:

$$\widetilde{d^d k} = \frac{d^d k}{(2\pi)^d} \frac{(m^2)^\epsilon}{i\Gamma(1+\epsilon)(4\pi)^{\epsilon-2}}, \quad p^2 = s. \tag{2.111}$$

Then, introducing the *Landau variable*: $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$, and thanks to the *Magnus Exponential* (2.4), we identify the *canonical basis*, namely $\mathbf{I} = \{I_1, I_2\}$:

$$I_1 = \epsilon \mathcal{T}_1(\epsilon), \quad I_2(s,\epsilon) = \epsilon\sqrt{-s(4m^2 - s)}\mathcal{T}_2(s,\epsilon). \tag{2.112}$$

The DEQs system reads (x variable is assumed):

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \epsilon \mathbb{A}(x)\mathbf{I}(x,\epsilon), \quad \mathbb{A}(x) = \begin{pmatrix} 0 & & 0 \\ \frac{1}{x} & \frac{1}{x} & \frac{-2}{1+x} \end{pmatrix}. \tag{2.113}$$

The system can be solved, up to the Boundary Conditions (BCs), through the *Dyson Series*, in terms of *GPLs*. I_1 can be computed by direct integration; using (2.111) it turns to be:

$$I_1 = 1. \tag{2.114}$$

Following [27] we want to determine the BCs from the $x \to 0$, namely $m^2 \to 0$, limit of the DEQs system. In this limit (2.113) reduces to:

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} \overset{x\to 0}{\simeq} \frac{1}{x}\epsilon \mathbb{M}\mathbf{I}(x,\epsilon), \quad \mathbb{M} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2.115}$$

First of all we can identify a new basis of MIs, $\mathbf{H} = \{H_1, H_2\}$, related to the canonical one by: $\mathbf{H} = \mathbb{S}\mathbf{I}$, where $\mathbb{S}$ is such that $\mathbb{S}\mathbb{M}\mathbb{S}^{-1}$ reduces to the *Jordan Form*, $\mathbb{J}$.[9]

$$\mathbb{S}\mathbb{M}\mathbb{S}^{-1} = \mathbb{J}, \quad \mathbb{S} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.116}$$

In this new basis the system reads:

$$\frac{\partial \mathbf{H}(x,\epsilon)}{\partial x} = \epsilon \frac{1}{x}\mathbb{J}\mathbf{H}(x,\epsilon). \tag{2.117}$$

From the latter it follows that, integrating the second DEQ,:

$$H_2 = x^{\epsilon} H_{2,0} \Leftrightarrow H_{2,0} = x^{-\epsilon} H_2, \tag{2.118}$$

$H_{2,0}$ being a constant.
We can now express H_2 in terms of the canonical MIs $\{I_1, I_2\}$, through the relation $\mathbf{H} = \mathbb{S}\mathbf{I}$, and (2.118) becomes:

$$H_{2,0} = \lim_{x\to 0} x^{-\epsilon}(I_1(\epsilon) + I_2(x,\epsilon)); \tag{2.119}$$

in the latter expression only the BCs of I_2 are unknown.
The r.h.s in (2.119) can be also seen as a function of m^2, in the limit $m^2 \to 0$. In fact we can pull out the factor $(m^2)^{\epsilon}$ form the integral measure[10] (2.111) and, inverting $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$, we can re-write $\frac{x}{m^2}$ as a *Series Expansion* in m^2; in the $m^2 \to 0$ limit the latter reads:

$$\frac{x}{m^2} \overset{m^2\to 0}{=} -\frac{1}{s} \Rightarrow \left(\frac{x}{m^2}\right)^{-\epsilon} \overset{m^2\to 0}{=} \left(-\frac{1}{s}\right)^{-\epsilon} = (-s)^{\epsilon}. \tag{2.120}$$

We can now re-express $\{I_1, I_2\}$ in terms of $\{\mathcal{T}_1, \mathcal{T}_2\}$, and then take the limit $m^2 \to 0$ at the *integrand level* and in the *prefactor*. So we have:

$$\begin{aligned} H_{2,0} &= (-s)^{\epsilon}\,(I_1 + I_2)_{m^2\to 0} = (-s)^{\epsilon}\left(\epsilon\mathcal{T}_1 + \sqrt{-s(4m^2-s)}\epsilon\mathcal{T}_2\right)_{m^2\to 0} = \\ &= (-s)^{1+\epsilon}(\epsilon\mathcal{T}_2)_{m^2\to 0}; \end{aligned} \tag{2.121}$$

where in the last equality we use the fact that the *massless dotted Tadpole*, namely $(\mathcal{T}_1)_{m^2\to 0}$, is vanishing in *Dimensional Regularization.*
Now we can compute $(\epsilon\mathcal{T}_2)_{m^2\to 0}$ by direct integration:

$$(\epsilon\mathcal{T}_2)_{m^2\to 0} = (-s)^{-1-\epsilon}\mathcal{F}(\epsilon), \tag{2.122}$$

with:

$$\mathcal{F}(\epsilon) = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{2.123}$$

So it results:

$$H_{2,0} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{2.124}$$

[9] In this case $\mathbb{S} \equiv \mathbb{S}^{-1}$.
[10] After this, the integration measure is *m-independent*.

Imposing the equality of (2.124) and (2.119) order by order in ϵ we can determine the BCs for I_2. Denoting the latter by $I_{2,0} = \sum_{i=0}^{3} \epsilon^i I_{2,0}^{(i)}$, we find:

$$\begin{aligned} I_{2,0}^{(0)} &= 0, \\ I_{2,0}^{(1)} &= 0, \\ I_{2,0}^{(2)} &= -\frac{\pi^2}{6}, \\ I_{2,0}^{(3)} &= \texttt{PolyGamma[2, 1]} = -2\zeta_3 \quad {}^{11} \end{aligned} \tag{2.125}$$

The latter expressions are in fully agreement with the ones obtained through other standard methods, e.g.: Feynman Parametrization.

2.6.2 Massless BCs from Massive Ones

Furthermore we would like to mention the possibility to infer the BCs for *Massless Integrals* from *Massive* ones. This strategy is inspired by [87], where the Autors present a general method to solve *numerically* a DEQs system; roughly speaking it consists in introducing a common *fictitious mass* η to the whole set of MIs, determining the BCs in the limit $\eta \to \infty$, and finally reading the original value for the set of MIs at $\eta = 0$. The key point is that *all* the MIs in the $\eta \to \infty$ reduce to *Tadpoles*, which are much more easy to determine.
Let's consider $\mathbf{G}(m^2, \epsilon) = \{G_1(m^2, \epsilon), G_2(m^2, \epsilon)\}$, where

$$G_1(m^2, \epsilon) = \epsilon \mathcal{T}_1(m^2, \epsilon), \quad G_2(m^2, \epsilon) = \sqrt{-s}\sqrt{4m^2 - s}\, \epsilon \mathcal{T}_2(m^2, \epsilon), \tag{2.126}$$

where $\{\mathcal{T}_1(m^2, \epsilon), \mathcal{T}_2(m^2, \epsilon)\}$ are defined in (2.109) and (2.110) respectively, but assuming the *Integral Measure*:

$$\widetilde{d^d k} = \frac{d^d k}{(2\pi)^d} \frac{(-s)^\epsilon}{i\Gamma(1+\epsilon)(4\pi)^{\epsilon-2}}. \tag{2.127}$$

Then, they fulfill a DEQs system in the adimensional variable v, with: $\frac{v}{(1-v)^2} = -\frac{m^2}{s}$ (inherited from the one in m^2) wich reads:

$$\frac{\partial \mathbf{G}(v, \epsilon)}{\partial v} = \epsilon \mathbb{A}(v) \mathbf{G}(v, \epsilon), \quad \mathbb{A}(v) = \begin{pmatrix} \frac{2}{v-1} - \frac{1}{v} & 0 \\ \frac{1}{v} & \frac{2}{v-1} - \frac{2}{v+1} \end{pmatrix}. \tag{2.128}$$

The general solution for (2.128) can be obtained in terms of GPLs. Moreover, $G_1(v, \epsilon)$ can be obtained by direct integration, namely (taking into account (2.127)):

$$G_1(v, \epsilon) = \left(\frac{(1-v)^2}{v} \right)^\epsilon, \tag{2.129}$$

thus only the BCs for $G_2(v, \epsilon)$ are unknown in (2.128).
We can consider the $v \to 0$ limit in (2.128) i.e.:

$$\frac{\partial \mathbf{G}(v, \epsilon)}{\partial v} \overset{v \to 0}{\simeq} \frac{\epsilon}{v} \mathbb{M} \mathbf{G}(v, \epsilon), \quad \mathbb{M} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.130}$$

Following the discussion in the previous Subsection, we identify a new basis of MIs $\mathbf{H}(v, \epsilon)$, $\mathbf{H}(v, \epsilon) = \mathbb{S}\mathbf{G}(v, \epsilon)$, where $\mathbb{S}$ is such that: $\mathbb{S}\mathbb{M}\mathbb{S}^{-1}$ reduces to the *Jordan form* $\mathbb{J}$. We have:

$$\mathbb{S}\mathbb{M}\mathbb{S}^{-1} = \mathbb{J}, \quad \mathbb{S} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.131}$$

[11] `Mathematica: N[PolyGamma[2, 1]]=N[-2*Zeta[3]].`

In this new basis the system reads:

$$\frac{\partial \mathbf{H}(v,\epsilon)}{\partial v} = \frac{\epsilon}{v}\,\mathbb{J}\mathbf{H}(v,\epsilon), \tag{2.132}$$

thus we find:

$$H_2(v, \epsilon \equiv H_{2,0}(\epsilon). \tag{2.133}$$

Then, we can express the r.h.s. in terms of $\{G_1(v,\epsilon), G_2(v,\epsilon)\}$:

$$H_{2,0}(\epsilon) = \lim_{v\to 0} G_1(v,\epsilon) + G_2(v,\epsilon) \tag{2.134}$$

The r.h.s. can be seen as a function of $(-s, m^2)$ in the limit $m^2 \to 0$, namely:

$$H_{2,0}(\epsilon) = \lim_{m^2\to 0}\Big(\epsilon\mathcal{T}_1(m^2,\epsilon) + \sqrt{-s}\sqrt{4m^2-s}\,\epsilon\mathcal{T}_2(m^2,\epsilon)\Big) = (-s)\left(\epsilon\mathcal{T}_2\right)_{m^2\to 0}, \tag{2.135}$$

where in the last equality we use the fact that the *massless dotted Tadpole* is vanishing in *Dimensional Regularization.* An explicit calculations leads to:

$$H_{2,0}(\epsilon) = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{2.136}$$

and thus (2.134) could be used to fix the BCs for $G_2(v,\epsilon)$, as well as we did in the previous Subsection.
On the other hnad, we can look at (2.134) considering the r.h.s completely known: e.g. the BCs for $G_2(v,\epsilon)$ con be computed requiring that $G_2(v,\epsilon) \to 0$ in the $v \to 1$ limit (the latter corresponds to the $s \to 0$ limit, in which $\mathcal{T}_2$ is regular while the prefactor $\sqrt{-s}\sqrt{4m^2-s}$ is vanishing). Thus, $H_{2,0}(\epsilon)$ in (2.134) as well as in (2.135) can be considered known; finally we can use (2.135) to infer the (Boundary) value of $(-s)\left(\epsilon\mathcal{T}_2\right)_{m^2\to 0}$. We consistently find:

$$(-s)\left(\epsilon\mathcal{T}_2\right)_{m^2\to 0} = 1 - \epsilon^2\frac{\pi^2}{6} + \mathcal{O}(\epsilon^3), \tag{2.137}$$

in agreement with the Series Expansion of (2.136).

2.7 One-loop massless 4-point topology

Integral Family

We consider in this section the 1-loop massless 4-point function; we assume all the external momenta incoming and massless, namely:

$$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0, \quad p_i^2 = 0, \,\forall i = 1,\dots,4. \tag{2.138}$$

Moreover we can define the Mandelstam invariants as:

$$(p_1+p_2)^2 = s, \quad (p_1+p_3)^2 = t, \quad (p_1+p_4)^2 = u, \tag{2.139}$$

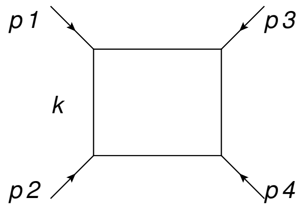
related by the usual constraint:

$$s + t + u = 0. \tag{2.140}$$

The denominators read:

$$D_1 = (k-p_1)^2, \quad D_2 = k^2, \quad D_3 = (k+p_2)^2, \quad D_4 = (k+p_2+p_4)^2. \tag{2.141}$$

and the integral family is:

$$\mathcal{B}^{(1,3)} = \quad$$ 

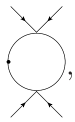
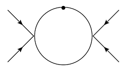
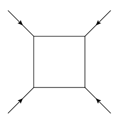
$$= \int \widetilde{d^d k} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}}, \tag{2.142}$$

where the definition of $\widetilde{d^d k}$ is:

$$\widetilde{d^d k} = \frac{d^d k}{(2\pi)^d} \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)} \tag{2.143}$$

Magnus Exponential and Canonical Form

Thanks to `Reduze 2`, we identify *three* MIs, $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$, which fulfill a DEQs system in (s,t) linear in ϵ:

$\mathcal{T}_1(t,\epsilon) =$ , $\quad \mathcal{T}_2(s,\epsilon) =$ , $\quad \mathcal{T}_3(s,t,\epsilon) =$  (2.144)

The DEQs systems read:

$$\frac{\partial \mathcal{T}(s,t,\epsilon)}{\partial s} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon+1}{s} & 0 \\ \frac{2}{s(s+t)} & -\frac{2}{s(s+t)} & -\frac{s+\epsilon t+t}{s(s+t)} \end{pmatrix} \mathcal{T}(s,t,\epsilon), \tag{2.145a}$$

$$\frac{\partial \mathcal{T}(s,t,\epsilon)}{\partial t} = \begin{pmatrix} -\frac{\epsilon+1}{t} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{t(s+t)} & \frac{2}{t(s+t)} & -\frac{\epsilon s+s+t}{t(s+t)} \end{pmatrix} \mathcal{T}(s,t,\epsilon). \tag{2.145b}$$

A change of variables which greatly simplifies the problem is:

$$s = s, \quad t = sz. \tag{2.146}$$

In fact, thanks to the chain rule, we can write:

$$\frac{\partial \mathcal{T}(s,z,\epsilon)}{\partial s} \equiv \frac{\partial \mathcal{T}(s,t=sz,\epsilon)}{\partial s} = \left[\frac{\partial \mathcal{T}}{\partial s} + \frac{\partial \mathcal{T}}{\partial t} z\right](s,z,\epsilon), \tag{2.147}$$

and plugging in the latter (2.145a) and (2.145b) we have:

$$\frac{\partial \mathcal{T}(s,z,\epsilon)}{\partial s} = \begin{pmatrix} -\frac{\epsilon+1}{s} & 0 & 0 \\ 0 & -\frac{\epsilon+1}{s} & 0 \\ 0 & 0 & -\frac{\epsilon+2}{s} \end{pmatrix} \mathcal{T}(s,z,\epsilon); \tag{2.148}$$

Similarly we have:

$$\frac{\partial \mathcal{T}(s,z,\epsilon)}{\partial z} = \frac{\partial \mathcal{T}(s,t=sz,\epsilon)}{\partial z} = \frac{\partial \mathcal{T}(s,z,\epsilon)}{\partial t} z; \tag{2.149}$$

finally, thanks to (2.145b) the latter reads:

$$\frac{\partial \mathcal{T}(s,z,\epsilon)}{\partial z} = \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{sz(z+1)} & \frac{2}{sz(z+1)} & -\frac{\epsilon+z+1}{z(z+1)} \end{pmatrix} \mathcal{T}(s,z,\epsilon). \tag{2.150}$$

We immediately notice that the DEQs system in s, (2.148), is absolutely trivial: solving just *three decoupled homogeneous* DEQs we completely determine the s-dependent part of the MIs. Moreover, we can recast $\mathcal{T}$, as:

$$\mathcal{T}(s,z,\epsilon) = \mathbb{S}(s)\widetilde{\mathbf{F}}(z,\epsilon), \tag{2.151}$$

with:

$$\mathbb{S}(s) = \begin{pmatrix} (-s)^{-\epsilon-1} & 0 & 0 \\ 0 & (-s)^{-\epsilon-1} & 0 \\ 0 & 0 & (-s)^{-\epsilon-2} \end{pmatrix}, \tag{2.152}$$

being $\widetilde{\mathbf{F}}(z,\epsilon)$ the unknown (part of the) MIs; the DEQs system for $\widetilde{\mathbf{F}}(z,\epsilon)$ trivially reads:

$$\begin{aligned} \frac{\partial \widetilde{\mathbf{F}}(z,\epsilon)}{\partial z} &= \mathbb{S}^{-1}(s) \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{sz(z+1)} & \frac{2}{sz(z+1)} & -\frac{\epsilon+z+1}{z(z+1)} \end{pmatrix} \mathbb{S}(s)\, \widetilde{\mathbf{F}}(z,\epsilon) \\ &= \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{z(z+1)} & -\frac{2}{z(z+1)} & -\frac{\epsilon+z+1}{z(z+1)} \end{pmatrix} \widetilde{\mathbf{F}}(z,\epsilon). \end{aligned} \tag{2.153}$$

We can now go on, trying to remove the ϵ-poles from the various MIs, i.e.: looking for a *Taylor expansion* in ϵ, rather than a *Laurent Series.* It's well known that $\widetilde{F}_{1,2}$ have simple poles in ϵ, and so we introduce *two* new MIs which are *free* from ϵ *poles*:

$$F_i(z,\epsilon) = \epsilon \widetilde{F}_i(z,\epsilon), \quad i = 1,2. \tag{2.154}$$

Moreover, in order to facilitate the convergence of the Magnus Series, it is desirable to have as many entries as possible in the system proportional to ϵ . Thus, we consider:

$$F_3(z,\epsilon) = \epsilon^2 \widetilde{F}_3(z,\epsilon); \tag{2.155}$$

So, the DEQs system we are interested in, is:

$$\frac{\partial \mathbf{F}(z,\epsilon)}{\partial z} = \mathbb{A}(z,\epsilon)\mathbf{F}(z,\epsilon), \quad \mathbf{F}(z,\epsilon) = \mathrm{Diag}(\epsilon,\epsilon,\epsilon^2)\widetilde{\mathbf{F}}(z); \tag{2.156}$$

with:

$$\mathbb{A}(z,\epsilon) = \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\epsilon}{z(z+1)} & -\frac{2\epsilon}{z(z+1)} & -\frac{\epsilon+z+1}{z(z+1)} \end{pmatrix}. \tag{2.157}$$

Thus, in order to recast the DEQs system in the *Canonical Form*, we apply the algorithm based on the *Magnus Exponential*, described in (2.4). In virtue of that, we split $\mathbb{A}(x,\epsilon)$ as:

$$\mathbb{A}(z,\epsilon) = \mathbb{A}_0(z) + \epsilon \mathbb{A}_1(z), \tag{2.158}$$

with:

$$\mathbb{A}_0(z) = \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{z} \end{pmatrix} \tag{2.159a}$$

$$\mathbb{A}_1(z) = \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{z} - \frac{2}{z+1} & \frac{2}{z+1} - \frac{2}{z} & \frac{1}{z+1} - \frac{1}{z} \end{pmatrix}. \tag{2.159b}$$

We notice that, in this simple example, the DEQ which should be solved thanks to the *Magnus Exponential* is:

$$\frac{\partial \mathbb{B}(z)}{\partial z} = \mathbb{A}_0(z)\mathbb{B}(z). \tag{2.160}$$

But, thanks to the remarkable *Diagonal Structure*, the *Series* involved in the *Magnus Exponential* which has $\mathbb{A}_0(x)$ as a kernel, consists in the very first term, since a *Diagonal Matrix* always commutes with its integral, thus:

$$\mathbb{B}(z) = e^{\int^z dt\, \mathbb{A}_0(t)} = \text{diag}\left(\frac{1}{z}, 1, \frac{1}{z}\right). \tag{2.161}$$

As stated in (2.4), the *canonical* MIs, namely $\mathbf{I}(z,\epsilon)$, are related to the previous ones by the relation:

$$\mathbf{F}(z,\epsilon) = \mathbb{B}(z)\,\mathbf{I}(z,\epsilon). \tag{2.162}$$

Then, the DEQs system for the *Canonical MIs* reads:

$$\frac{\partial \mathbf{I}(z,\epsilon)}{\partial z} = \epsilon \mathbb{A}_c(z)\mathbf{I}(z,\epsilon), \quad \mathbb{A}_c(z) = \mathbb{B}^{-1}(z)\mathbb{A}_1(z)\mathbb{B}(z), \tag{2.163a}$$

$$\mathbb{A}_c(z) = \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{z} - \frac{2}{z+1} & -\frac{2}{z+1} & \frac{1}{z+1} - \frac{1}{z} \end{pmatrix}. \tag{2.163b}$$

General Solution

Then, the *General Solution* can be written in terms of *Iterated Integrals* and, more precisely, it involves just HPLs (2.5.1). Let's consider the *General Solution* in the form of:

$$\mathbf{I}(z,\epsilon) = \left(\mathbb{1} + \epsilon\mathbb{B}^{(1)}(z) + \epsilon^2\mathbb{B}^{(2)}(z) + \epsilon^3\mathbb{B}^{(3)}(z) + \mathcal{O}(\epsilon^4)\right)\mathbf{I}(\epsilon). \tag{2.164}$$

The entries of $\mathbb{B}^{(1)}$ reads:

$$\begin{aligned}
&\mathbb{B}^{(1)}_{11} = -H[\{0\},z], \quad \mathbb{B}^{(1)}_{12} = \mathbb{B}^{(1)}_{13} = 0;\\
&\mathbb{B}^{(1)}_{21} = \mathbb{B}^{(1)}_{22} = \mathbb{B}^{(1)}_{23} = 0;\\
&\mathbb{B}^{(1)}_{31} = -2H[\{-1\},z] + 2H[\{0\},z], \quad \mathbb{B}^{(1)}_{32} = -2H[\{-1\},z],\\
&\mathbb{B}^{(1)}_{33} = H[\{-1\},z] - H[\{0\},z].
\end{aligned} \tag{2.165}$$

The entries of $\mathbb{B}^{(2)}$ are:

$$\begin{aligned}
&\mathbb{B}^{(2)}_{11} = H[\{0,0\},z], \quad \mathbb{B}^{(2)}_{12} = \mathbb{B}^{(2)}_{13}) = 0;\\
&\mathbb{B}^{(2)}_{21} = \mathbb{B}^{(2)}_{22} = \mathbb{B}^{(2)}_{23} = 0;\\
&\mathbb{B}^{(2)}_{31} = -2H[\{-1,-1\},z] + 4H[\{-1,0\},z] + 2H[\{0,-1\},z] - 4H[\{0,0\},z],\\
&\mathbb{B}^{(2)}_{32} = -2H[\{-1,-1\},z] + 2H[\{0,-1\},z],\\
&\mathbb{B}^{(2)}_{33} = H[\{-1,-1\},z] - H[\{-1,0\},z] - H[\{0,-1\},z] + H[\{0,0\},z].
\end{aligned} \tag{2.166}$$

Finally the entries of $\mathbb{B}^{(3)}$ read:

$$
\begin{aligned}
\mathbb{B}^{(3)}_{11} =& - H[\{0,0,0\},z], \quad \mathbb{B}^{(3)}_{12} = \mathbb{B}^{(3)}_{13} = 0; \\
\mathbb{B}^{(3)}_{21} =& \mathbb{B}^{(3)}_{22} = \mathbb{B}^{(3)}_{23} = 0; \\
\mathbb{B}^{(3)}_{31} =& - 2H[\{-1,-1,-1\},z] + 4H[\{-1,-1,0\},z] + 2H[\{-1,0,-1\},z]+ \\
& - 6H[\{-1,0,0\},z] + 2H[\{0,-1,-1\},z] - 4H[\{0,-1,0\},z]+ \\
& - 2H[\{0,0,-1\},z] + 6H[\{0,0,0\},z], \\
\mathbb{B}^{(3)}_{32} =& - 2H[\{-1,-1,-1\},z] + 2H[\{-1,0,-1\},z] + 2H[\{0,-1,-1\},z]+ \\
& - 2H[\{0,0,-1\},z], \\
\mathbb{B}^{(3)}_{33} =& H[\{-1,-1,-1\},z] - H[\{-1,-1,0\},z] - H[\{-1,0,-1\},z]+ \\
& + H[\{-1,0,0\},z] - H[\{0,-1,-1\},z] + H[\{0,-1,0\},z]+ \\
& + H[\{0,0,-1\},z] - H[\{0,0,0\},z]
\end{aligned}
\tag{2.167}
$$

Boundary conditios

We notice that I_1 (and I_2) can be obtained in a closed form, by direct integration, with a minimal effort, see e.g. [42]. Thus, their expansion in terms of *HPLs* is not necessary. So the only unknown for this problem can be considered I_3.
A careful inspection shows that this MI is *regular* in te $z \to -1$ limit, since *planar integrals* are known to be *regular* in the $u \to 0 \Leftrightarrow z \to -1$ limit. Thus, multiplying the DEQ for the 3rd MI by a factor $(z+1)$ and considering the $z \to -1$ limit, we obtain a relation between the *three* MIs at $z = -1$ (which holds *order by order* in ϵ), namely:

$$
I_3(-1) = 2I_1(-1) + 2I_2(-1). \tag{2.168}
$$

Thanks to the analytic expression for $I_{2,3}$ mentioned just above and (2.143), then the latter equality gives:

$$
I_1(-1) = 1 - i\pi\epsilon - \frac{\pi^2\epsilon^2}{2} + \frac{i\pi^3\epsilon^3}{6} + O\left(\epsilon^4\right), \quad I_2(-1) = 1. \tag{2.169}
$$

Being *all* the MIs known at $z = -1$ we can fix the BCs *order by order* in ϵ in the *General Solution* in order to match (2.169). Doing so, we obtain:

$$
I_3(z,\epsilon) = \sum_{i=0}^{3} \epsilon^i I_3^{(i)}(z) + O(\epsilon^4), \tag{2.170}
$$

with:

$$
\begin{aligned}
I_3^{(0)}(z) =& 4, \\
I_3^{(1)}(z) =& - 2H[\{0\},z], \\
I_3^{(2)}(z) =& - \pi^2, \\
I_3^{(3)}(z) =& - 2(H[\{0,0,-1\},z] + \zeta(3)) + \frac{1}{3}H[\{0\},z]^3 + \\
& + \left(2H[\{0,-1\},z] + \pi^2\right) H[\{0\},z] - H[\{-1\},z]\left(H[\{0\},z]^2 + \pi^2\right).
\end{aligned}
\tag{2.171}
$$

2.8 One-loop QED 4-point topology μe scattering

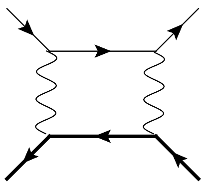
In this Section we discuss the calculation of the MIs needed the one-loop four-point function for the $\mu e \to \mu e$ scattering, as presented in [27].
Currently the study of QED corrections to $\mu e \to \mu e$, and in particular the NNLO QED corrections which goes beyond the scope of this work [27], [28] , are crucial in order to interpret the high-precision data of future experiments like MUonE, recently proposed at CERN, dedicated to the study of the differential cross section of high energy muons on atomic electrons as a function of the spacelike squared momentum transfer [67], [68]. These measurements will lead to the knowledge of the running of the electromagnetic coupling in the spacelike region , allowing a new and independent determination of the *hadronic contributions* to g-2. The success of the program requires to measure the differential cross section with statistical and systematic uncertainties of the order 10 ppm, and thus the same accuracy is required in the theoretical predictions.
Moreover, these higher order QED corrections could be useful for the crossing related process: $e^+e^- \to \mu^+\mu^-$, which is planned to be studied at low energy e^+e^- experiments, like Belle-II and VEPP-2000.

Integral Family

The process we are considering is:

$$\mu^+(p_1) + e^-(p_2) \to e^-(p_3) + \mu^+(p_4), \tag{2.172}$$

where, due to momentum conservation, $p_1 + p_2 = p_3 + p_4$; in particular we focus on the Feynman Diagram:



where thick lines represent *muons*.
We define the Mandelstam invariants as:

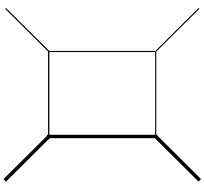
$$s = (p_1 + p_2)^2, \quad t = (p_2 - p_3)^2, \quad u = (p_1 - p_3)^2. \tag{2.173}$$

The previous variables are related by the well-known relation $s + t + u = 2m^2$, where we assumed the electron massless. Thanks to this assumption, the denominators can be written as:

$$D_1 = (k^2 - m^2), \quad D_2 = (k + p_1)^2, \quad D_3 = (k + p_1 + p_2)^2, \quad D_4 = (k + p_4)^2, \tag{2.174}$$

and the Integral Family which we are referring to, reads:

$\mathcal{B}^{(1,3)}_{\mu e} =$



$$= \int \widetilde{d^d k} \frac{1}{D_1^{r_1} D_2^{r_2} D_3^{r_3} D_4^{r_4}} \quad r_i \geq 0; \tag{2.175}$$

where we have introduced, for later convenience[12]

$$\widetilde{d^d k} = \frac{d^d k}{(2\pi)^d} \left(\frac{i S_\epsilon}{16\pi^2}\right)^{-1} \left(\frac{m^2}{\mu^2}\right)^{\epsilon}, \quad S_\epsilon = (4\pi)^\epsilon \Gamma(1 + \epsilon). \tag{2.176}$$

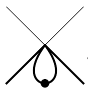
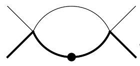
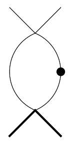
[12]Being μ the 't Hooft parameter.

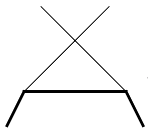
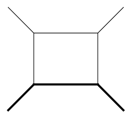
Magnus Exponential and Canonical Form

We identify the following set of MIs, which obeys a coupled system of Differential Equations *linear* in ϵ, in the variables s and t:

$$\begin{aligned} &F_1(\epsilon) = \epsilon \mathcal{T}_1(\epsilon), \quad F_2(s,\epsilon) = \epsilon \mathcal{T}_2(s,\epsilon), \quad F_3(t,\epsilon) = \epsilon \mathcal{T}_3(t,\epsilon), \\ &F_4(t,\epsilon) = \epsilon^2 \mathcal{T}_4(t,\epsilon), \quad F_5(s,t,\epsilon) = \epsilon^2 \mathcal{T}_5(s,t,\epsilon), \end{aligned} \tag{2.177}$$

where:

$\mathcal{T}_1(\epsilon) =$ , $\mathcal{T}_2(s,\epsilon) =$ , $\mathcal{T}_3(t,\epsilon) =$ ,

$\mathcal{T}_4(t,\epsilon) =$ , $\mathcal{T}_5(s,t,\epsilon) =$  (2.178)

Thanks to the *Magnus Exponential* we can rotate away the ϵ-*independent* part, and recast the system of Differential Equation in the *Canonical Form*. The new set of MIs are related to the previous one by the following relations:

$$\begin{aligned} &I_1(\epsilon) = F_1(\epsilon), \quad I_2(s,\epsilon) = -sF_2(s,\epsilon), \quad I_3(t,\epsilon) = -tF_3(t,\epsilon), \\ &I_4(t,\epsilon) = \sqrt{-t}\sqrt{4m^2-t}F_4(t,\epsilon), \quad I_5(s,t,\epsilon) = -t(m^2-s)F_5(s,t,\epsilon). \end{aligned} \tag{2.179}$$

Finally we introduce a couple of *adimensional variables*, namely x and y, related to s and t by:

$$-\frac{s}{m^2} = x, \quad -\frac{t}{m^2} = \frac{(1-y)^2}{y}. \tag{2.180}$$

So the unknown vector of MIs $\mathbf{I} = \{I_i\}_{i=1,\ldots,5}$ satisfies the following Differential Equations:

$$\frac{\partial \mathbf{I}(x,y,\epsilon)}{\partial x} = \epsilon \mathbb{A}_x(x,y)\mathbf{I}(x,y,\epsilon), \quad \frac{\partial \mathbf{I}(x,y,\epsilon)}{\partial y} = \epsilon \mathbb{A}_y(x,y)\mathbf{I}(x,y,\epsilon); \tag{2.181}$$

the matrices $\mathbb{A}_x$ and $\mathbb{A}_y$ are presented below:

$$\mathbb{A}_x(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{x+1} & \frac{1}{x}-\frac{2}{x+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{y}{xy+1}+\frac{2}{x+1}-\frac{1}{x+y} & 2\left(\frac{2}{x+1}-\frac{1}{x}\right)+2\left(\frac{1}{x}-\frac{y}{xy+1}\right)-\frac{2}{x+y} & -\frac{y}{xy+1}-\frac{1}{x+y} & \frac{y}{xy+1}-\frac{1}{x+y} & \frac{y}{xy+1}-\frac{2}{x+1}+\frac{1}{x+y} \end{pmatrix}, \tag{2.182a}$$

$$\mathbb{A}_y(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{y}-\frac{2}{y-1} & 0 & 0 \\ \frac{1}{y} & 0 & -\frac{1}{y} & \frac{2}{y+1}-\frac{2}{y-1} & 0 \\ -\frac{x}{xy+1}+\frac{1}{y}-\frac{1}{x+y} & -\frac{2x}{xy+1}+\frac{2}{y}-\frac{2}{x+y} & -\frac{x}{xy+1}+\frac{2}{y-1}-\frac{1}{x+y} & \frac{x}{xy+1}-\frac{1}{x+y} & \frac{x}{xy+1}-\frac{2}{y-1}+\frac{1}{x+y} \end{pmatrix}. \tag{2.182b}$$

General Solution

Along the lines presented in (2.3), the general solutions for the various MIs, up to order ϵ^2 read:

$$\mathbf{I}(x,y,\epsilon) = \Big(\mathbb{1} + \epsilon\,\mathbb{B}^{(1)}(x,y) + \epsilon\,\mathbb{B}^{(2)}(x,y)\Big)\,\mathbf{I}_0(\epsilon), \tag{2.183}$$

where the entries of $\mathbb{B}^{(1)}$ read:

$$\begin{aligned}
&\mathbb{B}^{(1)}_{11} = \mathbb{B}^{(1)}_{12} = \mathbb{B}^{(1)}_{13} = \mathbb{B}^{(1)}_{14} = \mathbb{B}^{(1)}_{15} = 0;\\
&\mathbb{B}^{(1)}_{21} = -G[\{-1\},x], \quad \mathbb{B}^{(1)}_{22} = -2G[\{-1\},x] + G[\{0\},x], \quad \mathbb{B}^{(1)}_{23} = \mathbb{B}^{(1)}_{24} = \mathbb{B}^{(1)}_{25} = 0;\\
&\mathbb{B}^{(1)}_{33} = G[\{0\},y] - 2G[\{1\},y], \quad \mathbb{B}^{(1)}_{31} = \mathbb{B}^{(1)}_{32} = \mathbb{B}^{(1)}_{34} = \mathbb{B}^{(1)}_{35} = 0;\\
&\mathbb{B}^{(1)}_{41} = G[\{0\},y], \quad \mathbb{B}^{(1)}_{43} = -G[\{0\},y], \quad \mathbb{B}^{(1)}_{44} = 2G[\{-1\},y] - 2G[\{1\},y], \quad \mathbb{B}^{(1)}_{42} = \mathbb{B}^{(1)}_{45} = 0;\\
&\mathbb{B}^{(1)}_{51} = 2G[\{-1\},x] - G[\{-(1/y)\},x] - G[\{-y\},x],\\
&\mathbb{B}^{(1)}_{52} = 4G[\{-1\},x] - 2G[\{-(1/y)\},x] - 2G[\{-y\},x],\\
&\mathbb{B}^{(1)}_{53} = -G[\{0\},y] + 2G[\{1\},y] - G[\{-(1/y)\},x] - G[\{-y\},x],\\
&\mathbb{B}^{(1)}_{54} = -G[\{0\},y] + G[\{-(1/y)\},x] - G[\{-y\},x],\\
&\mathbb{B}^{(1)}_{55} = -2G[\{-1\},x] + G[\{0\},y] - 2G[\{1\},y] + G[\{-(1/y)\},x] + G[\{-y\},x].
\end{aligned} \tag{2.184}$$

and the ones for $\mathbb{B}^{(2)}$ are:

$$
\begin{aligned}
&\mathbb{B}^{(2)}_{11} = \mathbb{B}^{(2)}_{12} = \mathbb{B}^{(2)}_{13} = \mathbb{B}^{(2)}_{14} = \mathbb{B}^{(2)}_{15} = 0;\\
&\mathbb{B}^{(2)}_{21} = 2G[\{-1,-1\},x] - G[\{0,-1\},x],\\
&\mathbb{B}^{(2)}_{22} = 4G[\{-1,-1\},x] - 2G[\{-1,0\},x] - 2G[\{0,-1\},x] + G[\{0,0\},x],\\
&\mathbb{B}^{(2)}_{23} = \mathbb{B}^{(2)}_{24} = \mathbb{B}^{(2)}_{25} = 0;\\
&\mathbb{B}^{(2)}_{33} = G[\{0,0\},y] - 2G[\{0,1\},y] - 2G[\{1,0\},y] + 4G[\{1,1\},y],\\
&\mathbb{B}^{(2)}_{31} = \mathbb{B}^{(2)}_{32} = \mathbb{B}^{(2)}_{34} = \mathbb{B}^{(2)}_{35} = 0;\\
&\mathbb{B}^{(2)}_{41} = 2G[\{-1,0\},y] - 2G[\{1,0\},y],\\
&\mathbb{B}^{(2)}_{43} = -2G[\{-1,0\},y] - G[\{0,0\},y] + 2G[\{0,1\},y] + 2G[\{1,0\},y],\\
&\mathbb{B}^{(2)}_{44} = 4G[\{-1,-1\},y] - 4G[\{-1,1\},y] - 4G[\{1,-1\},y] + 4G[\{1,1\},y],\\
&\mathbb{B}^{(2)}_{42} = \mathbb{B}^{(2)}_{45} = 0;\\
&\mathbb{B}^{(2)}_{51} = G[\{0\},y]G[\{-(1/y)\},x] - G[\{0\},y]G[\{-y\},x] - 8G[\{-1,-1\},x] +\\
&\quad 2G[\{-1,-(1/y)\},x] + 2G[\{-1,-y\},x] - G[\{0,0\},y] + 4G[\{-(1/y),-1\},x] +\\
&\quad - G[\{-(1/y),-(1/y)\},x] - G[\{-(1/y),-y\},x] + 4G[\{-y,-1\},x] - G[\{-y,-(1/y)\},x] +\\
&\quad - G[\{-y,-y\},x],\\
&\mathbb{B}^{(2)}_{52} = -16G[\{-1,-1\},x] + 4G[\{-1,0\},x] + 4G[\{-1,-(1/y)\},x] + 4G[\{-1,-y\},x] +\\
&\quad 8G[\{-(1/y),-1\},x] - 2G[\{-(1/y),0\},x] - 2G[\{-(1/y),-(1/y)\},x] - 2G[\{-(1/y),-y\},x]\\
&\quad + 8G[\{-y,-1\},x] - 2G[\{-y,0\},x] - 2G[\{-y,-(1/y)\},x] - 2G[\{-y,-y\},x],\\
&\mathbb{B}^{(2)}_{53} = 2G[\{-1\},x]G[\{0\},y] - 4G[\{-1\},x]G[\{1\},y] - 3G[\{0\},y]G[\{-(1/y)\},x] + 4G[\{1\},y]G[\{-(1/y)\},x\\
&\quad - G[\{0\},y]G[\{-y\},x] + 4G[\{1\},y]G[\{-y\},x] + 2G[\{-1,-(1/y)\},x] + 2G[\{-1,-y\},x] +\\
&\quad - G[\{0,0\},y] + 4G[\{0,1\},y] + 4G[\{1,0\},y] - 8G[\{1,1\},y] - G[\{-(1/y),-(1/y)\},x] +\\
&\quad - G[\{-(1/y),-y\},x] - G[\{-y,-(1/y)\},x] - G[\{-y,-y\},x],\\
&\mathbb{B}^{(2)}_{53} = 2G[\{-1\},x]G[\{0\},y] + 2G[\{-1\},y]G[\{-(1/y)\},x] - G[\{0\},y]G[\{-(1/y)\},x] +\\
&\quad - 2G[\{1\},y]G[\{-(1/y)\},x] - 2G[\{-1\},y]G[\{-y\},x] - G[\{0\},y]G[\{-y\},x] +\\
&\quad 2G[\{1\},y]G[\{-y\},x] - 2G[\{-1,-(1/y)\},x] + 2G[\{-1,-y\},x] - 2G[\{0,-1\},y] +\\
&\quad - G[\{0,0\},y] + 2G[\{0,1\},y] + 2G[\{1,0\},y] + G[\{-(1/y),-(1/y)\},x] +\\
&\quad - G[\{-(1/y),-y\},x] + G[\{-y,-(1/y)\},x] - G[\{-y,-y\},x].
\end{aligned}
\tag{2.185}
$$

Boundary Conditions

We are left with the problem regarding the *fixing of the Boundary Conditions*. Let us notice that, thanks to the normalization chosen in (1.9), I_1 is totally trivialized:

$$I_1 \equiv 1. \tag{2.186}$$

Moreover, $I_3(y,\epsilon)$ can be computed through the standard Feynman parametrization with a minimal effort, obtaining (taking into account (2.176):

$$I_3(y,\epsilon) = \left(\frac{(1-y)^2}{y}\right)^{-\epsilon} \left(1 - \epsilon^2\zeta_2 + \mathcal{O}(\epsilon^3)\right; \tag{2.187}$$

and so the *BCs* are, simply, given by the second factor in the latter. On the contrary the remaining MIs requires a careful examination.
We notice that $\mathcal{T}_2(s,\epsilon)$ is regular in the $s \to 0$, i.e. $x \to 0$ limit; then recalling (2.177) and (2.179), we have:

$$I_2(s,\epsilon) = \epsilon(-s)\mathcal{T}_2(s,\epsilon), \tag{2.188}$$

and so we immediately argue that $I_2(s,\epsilon)$ vanishes in the $s \to 0 \Leftrightarrow x \to 0$ limit, thanks to the prefator $(-s)$ and the regular behaviour of $\mathcal{T}_2(s,\epsilon)$ in this limit. We immediately stress that the latter condition holds *order by order* in ϵ, and so the *Boundary Conditions* for $I_2(x,\epsilon)$, namely $I_{0,2}(\epsilon)$ read:

$$\begin{aligned} I_{0,2} &= \sum_{i=0}^{2} \epsilon^i I_{0,2}^{(i)}, \\ I_{0,2}^{(0)} &= I_{0,2}^{(1)} = I_{0,2}^{(2)} = 0. \end{aligned} \tag{2.189}$$

We can focus now on the 4$^{\text{th}}$ MI. $\mathcal{T}_4(t,\epsilon)$ turns to be regulat at $t \to 4m^2$ (i.e.: $y \to -1$), and so the corresponding MI, namely $I_4(t,\epsilon)$ is vanishing in this limit due to the prefactor $\sqrt{4m^2 - t}$ in the numerator; therefore we have:

$$\lim_{y \to -1} I_4(y,\epsilon) = 0, \tag{2.190}$$

and the latter information, which holds order by order in ϵ, is sufficient in order to fix the BCs. We obtain:

$$\begin{aligned} I_{0,4} &= \sum_{i=0}^{2} \epsilon^i I_{0,4}^{(i)}, \\ I_{0,4}^{(0)} &= I_{0,4}^{(1)} = 0, \\ I_{0,4}^{(2)} &= G[\{0,0\},-1] - 2G[\{0,1\},-1]; \end{aligned} \tag{2.191}$$

I_5 turns to be finite in the $s = -t \to 4m^2$ limit; we notice that the latter condition is equivalent to $x = -y \to -\frac{1}{2}$. Thus let's consider the DEQ w.r.t. x for I_5, multiply the latter by a factor $(x+y)$ and then consider the limit $x = -y \to -\frac{1}{2}$. Being all the MIs regular in this limit, as well as $\partial_x I_5$, we select a linear combination of MIs involving I_5, evaluated in $(x,y) = (-\frac{1}{2},\frac{1}{2})$, which was exactly the one proportional to $\frac{1}{(x+y)}$ in the canonical DEQ, namely:

$$0 = -I_1 - 2I_2 - I_3 - I_4 + I_5. \tag{2.192}$$

We stress again that this relation holds oreder by order in ϵ, and fully determine the BCs for I_5, which are, at this level, the only unknowns we are left with. We find:

$$\begin{aligned} I_{0,5} =& \sum_{i=0}^{2} \epsilon^i I_{0,5}^{(i)}, \quad I_{0,5}^{(0)} = 2, \quad I_{0,5}^{(1)} = 0, \\ I_{0,5}^{(2)} =& \frac{1}{6}\left(-\pi^2 + 12G[\{-1\},-(1/2)]G[\{0\},1/2] + \right. \\ & - 24G[\{-1\},-(1/2)]G[\{1\},1/2] + 24G[\{-1,-1\},-(1/2)] + \\ & - 12G[\{0,-1\},-(1/2)] + 6G[\{0,0\},-1] - 12G[\{0,1\},-1] + \\ & \left. -12G[\{1,0\},1/2] + 24G[\{1,1\},1/2]\right). \end{aligned} \tag{2.193}$$

Final Result and GPLs manipulations

Finally, taking into accout the matrices (2.184) and (2.185), and the MIs directly computed, namely (2.186), (2.187), and the ones explicitly worked out in (2.189), (2.191) and (2.193), then (2.183) gives all the MIs; the final expressions are:

$$I_i^{(k)}(x, y, \epsilon) = \epsilon^k \sum_{k=0}^{2} I_i^{(k)}(x, y) + \mathcal{O}(\epsilon^3), \quad i = 2, 4, 5; \tag{2.194}$$

with:

$$\begin{aligned} I_2^{(0)} &= 0, \\ I_2^{(1)}(x) &= -G[\{-1\}, x], \\ I_2^{(2)}(x) &= 2G[\{-1, -1\}, x] - G[\{0, -1\}, x] \end{aligned} \tag{2.195}$$

$$\begin{aligned} I_4^{(0)} &= 0, \\ I_4^{(1)}(y) &= 0, \\ I_4^{(2)}(y) &= G[\{0, 0\}, -1] - 2G[\{0, 1\}, -1] - G[\{0, 0\}, y] + 2G[\{0, 1\}, y]; \end{aligned} \tag{2.196}$$

$$\begin{aligned} I_5^{(0)} &= 2, \\ I_5^{(1)}(x, y) &= -2G[\{-1\}, x] + G[\{0\}, y] - 2G[\{1\}, y], \\ I_5^{(2)}(x, y) &= -2G[\{-1\}, x]G[\{0\}, y] + 4G[\{-1\}, x]G[\{1\}, y] + \\ &\frac{1}{6}(-\pi^2 + 12G[\{-1\}, -(1/2)]G[\{0\}, 1/2] - 24G[\{-1\}, -(1/2)]G[\{1\}, 1/2] + \\ &24G[\{-1, -1\}, -(1/2)] - 12G[\{0, -1\}, -(1/2)] + \\ &6G[\{0, 0\}, -1] - 12G[\{0, 1\}, -1] - 12G[\{1, 0\}, 1/2] + 24G[\{1, 1\}, 1/2]). \end{aligned} \tag{2.197}$$

As an exercise we could recast (2.196) and (2.197) in a much more compact form in particular as concerns the constants, thanks to GPLs properties described in (2.5.2). In fact, thanks to *Shuffle Algebra*, namely (2.92) we have:

$$G[\{0, 0\}, -1] = \frac{1}{2}\left(G[\{0\}, -1]\right)^2 = \frac{1}{2}\log^2(-1) = -\frac{\pi^2}{2}. \tag{2.198}$$

Moreover, by the very definition, we have:

$$G[\{0, 1\}, -1] = -H[\{0, 1\}, -1] = -\mathrm{Li}_2(-1) = \frac{\pi^2}{12}, \tag{2.199}$$

where in the first equality we used (2.97); so we have

$$\begin{aligned} I_4^{(2)}(y) &= G[\{0, 0\}, -1] - 2G[\{0, 1\}, -1] - G[\{0, 0\}, y] + 2G[\{0, 1\}, y] = \\ &= -\frac{2\pi^2}{3} - G[\{0, 0\}, y] + 2G[\{0, 1\}, y] = \\ &= -4\zeta_2 - G[\{0, 0\}, y] + 2G[\{0, 1\}, y], \end{aligned} \tag{2.200}$$

in perfect agreement with [27].
Unfortunately $I_5^{(2)}(x,y,\epsilon)$ is more tedious. Let's consider the constant term in $I_5^{(2)}(x,y,\epsilon)$, namely:

$$\begin{aligned}\mathcal{K}_{I_5^{(2)}} =& \frac{1}{6}(-\pi^2 + 12G[\{-1\},-(1/2)]G[\{0\},1/2] - 24G[\{-1\},-(1/2)]G[\{1\},1/2]+ \\ &+ 24G[\{-1,-1\},-(1/2)] - 12G[\{0,-1\},-(1/2)]+ \\ &+ 6G[\{0,0\},-1] - 12G[\{0,1\},-1] - 12G[\{1,0\},1/2] + 24G[\{1,1\},1/2]);\end{aligned} \tag{2.201}$$

now, thanks to the *Scaling Property* (2.98), and assuming we have:

$$\begin{aligned}&G[\{-1\},-(1/2)] = G[\{1\},1/2], \\ &G[\{-1,-1\},-(1/2)] = G[\{1,1\},1/2], \\ &G[\{0,-1\},-1/2] = G[\{0,1\},1/2].\end{aligned} \tag{2.202}$$

Thus, we can rewrite (2.201) as:

$$\begin{aligned}\mathcal{K}_{I_5^{(2)}} =& -\frac{\pi^2}{6} + 2G[\{0\},1/2]G[\{1\},1/2] - 4G[\{1\},1/2]^2 + G[\{0,0\},-1]+ \\ &- 2G[\{0,1\},-1] - 2G[\{0,1\},1/2] - 2G[\{1,0\},1/2] + 8G[\{1,1\},1/2];\end{aligned} \tag{2.203}$$

now, we can re-use the *Shuffle Relation*, namely (2.92):

$$\begin{aligned}&G[\{0\},1/2]\,G[\{1\},1/2] = G[\{0,1\},1/2] + G[\{1,0\},1/2], \\ &(G[\{1\},1/2])^2 = 2\,G[\{1,1\},1/2],\end{aligned} \tag{2.204}$$

which leads to:

$$\mathcal{K}_{I_5^{(2)}} = -\frac{\pi^2}{6} + G[\{0,0\},-1] - 2G[\{0,1\},-1]. \tag{2.205}$$

Finally, thanks to (2.198) and (2.199) we have:

$$\mathcal{K}_{I_5^{(2)}} = -\frac{5\pi^2}{6} = -5\zeta_2. \tag{2.206}$$

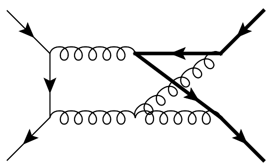
Thus, we obtain:

$$I_5^{(2)}(x,y) = -5\zeta_2 + -2G[\{-1\},x]G[\{0\},y] + 4G[\{-1\},x]G[\{1\},y], \tag{2.207}$$

which is in fully agreement with [27].

2.9 Two-loop non planar Vertex

We consider here a 3-point 2-loop non planar Integral Family. The MIs considered here are a subset of those, currently unknown, needed for the 2-loops non planar box Feynman Diagram:



which contribute to the QCD corrections to the $t\bar{t}$ production at hadron colliders (thick lines represent *top quarks*, thin lines *lighter quarks*). The complete 2-loops QCD corrections to $pp \to t\bar{t}$ are, in fact, known only numerically [75]-[79].
The analytic evaluation of the MIs concerning the leading-colour corrections to $pp \to t\bar{t}$, due to planar diagrams only, were considered in [80]-[83]. The set of available functions for considering also sub-leading color contributions were extended in [27] [28]. The analytic calculations for the planar double box, with a closed top loop was presented in [84]; moreover another non planar 3-point subgraph for the non planar double box with a closed top loop, was published in [85]

Integral Family

Let's consider the set of denominaotrs:

$$\begin{gathered} D_1 = k_1^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_2 - p_3)^2, \quad D_4 = (k_1 - p_3 - p_4)^2, \\ D_5 = (k_1 - k_2)^2 - m^2, \quad D_6 = (k_1 - k_2 - p_4)^2, \quad D_7 = (k_2 + p_4)^2. \end{gathered} \tag{2.208}$$

We assume:

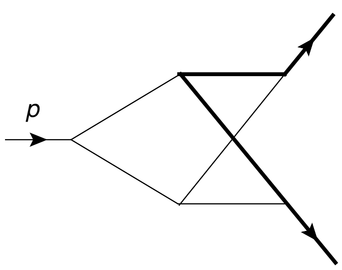
$$p^2 = s = (p_3 + p_4)^2 \tag{2.209}$$

being p the incoming momentum, and $p_{3,4}$ the outgoing ones; furthermore we have:

$$p_3^2 = p_4^2 = m^2. \tag{2.210}$$

Then, the Integral Family reads:

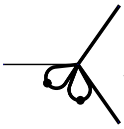
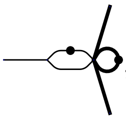
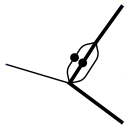
$\mathcal{T}^{(2,2)} =$

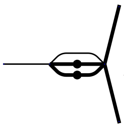
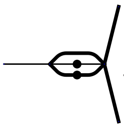
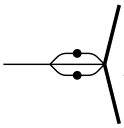


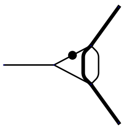
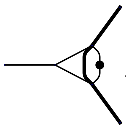
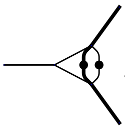
$$= \int \widetilde{d^d k_1} \widetilde{d^d k_2} \frac{D_7^{-s_1}}{D_1^{r_1} D_2^{r_2} D_3^{r_3} D_4^{r_4} D_5^{r_5} D_6^{r_6}}. \tag{2.211}$$

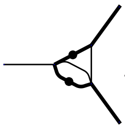
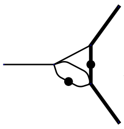
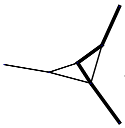
MIs linear in ϵ

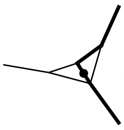
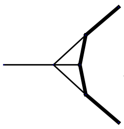
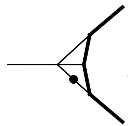
Thanks to Reduze2, we identify a basis of MIs, $\mathbf{F}(s,\epsilon) = \{F_i(s,\epsilon)\}_{i=1,\ldots,16}$ which fulfills a DEQs system w.r.t. s, *linear* ϵ:

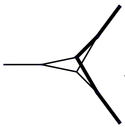
$F_1(s,\epsilon) = \epsilon^2$  , $F_2(s,\epsilon) = \epsilon^2$  , $F_3(s,\epsilon) = \epsilon^2$  ,

$F_4(s,\epsilon) = \epsilon^2$  , $F_5(s,\epsilon) = \epsilon^2$  , $F_6(s,\epsilon) = \epsilon^2$  ,

$F_7(s,\epsilon) = (-1+2\epsilon)\epsilon^2$  , $F_8(s,\epsilon) = \epsilon^3$  , $F_9(s,\epsilon) = \epsilon^2$  ,

$F_{10}(s,\epsilon) = \epsilon^2$  , $F_{11}(s,\epsilon) = \epsilon^2$  , $F_{12}(s,\epsilon) = \epsilon^3$  ,

$F_{13}(s,\epsilon) = \epsilon^2$  , $F_{14}(s,\epsilon) = \epsilon^4$  , $F_{15}(s,\epsilon) = (1+2\epsilon)\epsilon^2$  ,

$F_{16}(s,\epsilon) = \epsilon^4$  .

(2.212)

We point out that the overall factor ϵ^2 in (2.212) is required in order to have a basis of MIs *free* from ϵ poles. In particular, a common factor to the whole set of MIs does not modify at all the DEQs system and so the residual factors, depending on ϵ, are the source of the *linear dependence* on ϵ, as well as the choice of squared denominators in the basis of MIs.

Magnus Exponential

The DEQs sysetm for the basis $\mathbf{F}(s,\epsilon)$ reads:

$$\frac{\partial \mathbf{F}(s,\epsilon)}{\partial s} = \left(\mathbb{A}_0(s) + \epsilon \mathbb{A}_1(s)\right) \mathbf{F}(s,\epsilon). \tag{2.213}$$

The matrices $\mathbb{A}_0(s)$ and $\mathbb{A}_1(s)$ are presented in Appendix A. In order to obtain an ϵ-factorized form, following (2.4), we proceed in *two* steps. As a first step we split the matrix $\mathbb{A}_0(s)$ as:

$$\mathbb{A}_0(s) = \mathbb{D}_0(s) + \mathbb{N}_0(s), \tag{2.214}$$

where $\mathbb{D}_0(s)$ is the *diagonal part* of $\mathbb{A}_0(s)$. Then we switch to a new basis of MIs, $\mathbf{F}(s,\epsilon) = \mathbb{B}_1(s)\mathbf{F}^{[2]}(s,\epsilon)$:

$$\frac{\partial \mathbf{F}^{[2]}(s,\epsilon)}{\partial s} = \mathbb{B}_1^{-1}(s)\left(-\frac{\partial \mathbb{B}_1(s)}{\partial s} + \mathbb{D}_0(s)\mathbb{B}_1(s) + \mathbb{N}_0(s)\mathbb{B}_1(s) + \epsilon\mathbb{A}_1(s)\mathbb{B}_1(s)\right)\mathbf{F}^{[2]}(s,\epsilon). \tag{2.215}$$

We impose that $\mathbb{B}_1(s)$ is such that:

$$\frac{\partial \mathbb{B}_1(s)}{\partial s} = \mathbb{D}_0(s)\mathbb{B}_1(s), \quad \mathbb{B}_1(s) = e^{\int ds \mathbb{D}_0(s)}. \tag{2.216}$$

We stress that the latter can be trivially solved, being $D_0(s)$ a *diagonal matrix*; therefore resulting matrix $\mathbb{B}_1(s)$ is *diagonal* as well. The matrix $\mathbb{B}_1(s)$ is presented in Appendix A. Then, (2.215) results:

$$\frac{\partial \mathbf{F}^{[2]}(s,\epsilon)}{\partial s} = \underbrace{\mathbb{B}_1^{-1}(s)\left(\mathbb{N}_0(s) + \epsilon \mathbb{A}_1(s)\right)\mathbb{B}_1(s)}_{=\mathbb{A}^{[2]}(s,\epsilon)} \mathbf{F}^{[2]}(s,\epsilon) = \mathbb{A}^{[2]}(s,\epsilon)\mathbf{F}^{[2]}(s,\epsilon). \tag{2.217}$$

The matrix $\mathbb{A}^{[2]}(s,\epsilon)$ is, by construction, *linear* in ϵ, namely $\mathbb{A}^{[2]}(s,\epsilon) = \mathbb{A}_0^{[2]}(s) + \epsilon\mathbb{A}_1^{[2]}(s)$; moreover the *diagonal elements* are null. We introduce once again a new basis of MIs, $\mathbf{F}^{[3]}(s,\epsilon)$, where: $\mathbf{F}^{[2]}(s,\epsilon) = \mathbb{B}_2(s)\mathbf{F}^{[3]}(s,\epsilon)$, and so:

$$\frac{\partial \mathbf{F}^{[3]}(s,\epsilon)}{\partial s} = \mathbb{B}_2^{-1}(s)\left(-\frac{\partial \mathbb{B}_2(s)}{\partial s} + \mathbb{A}_0^{[2]}(s)\mathbb{B}_2(s) + \epsilon\mathbb{A}_1^{[2]}(s)\mathbb{B}_2(s)\right)\mathbf{F}^{[3]}(s,\epsilon). \tag{2.218}$$

We immediatly notice that the desired ϵ-factorized form is achieved when:

$$\frac{\partial \mathbb{B}_2(s)}{\partial s} = \mathbb{A}_0^{[2]}(s)\mathbb{B}_2(s). \tag{2.219}$$

The latter can be solved by means of the *Magnus Exponential* (2.4), and the solution is presented in Appndix A.
The DEQs system now reads:

$$\frac{\partial \mathbf{F}^{[3]}(s,\epsilon)}{\partial s} = \epsilon \underbrace{\mathbb{B}_2^{-1}(s)\mathbb{A}_1^{[2]}(s)\mathbb{B}_2(s)}_{=\mathbb{A}^{[3]}(s)} \mathbf{F}^{[3]}(s,\epsilon) = \epsilon\mathbb{A}^{[3]}(s)\mathbf{F}^{[3]}(s,\epsilon). \tag{2.220}$$

Finally we can express the DEQs system in the *Landau Variable*[13] x, $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$, and we introduce a new basis of MIs, in order to *"clean-up"* the DEQs system from the residual m^2 factor: $\mathbf{F}^{[3]}(x,\epsilon) = \mathbb{B}_3(m^2)\mathbf{I}(x,\epsilon)$ (the matrix $\mathbb{B}_3(m^2)$ is presented in Appendix A):

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \epsilon\mathbb{A}_c(x)\mathbf{I}(x,\epsilon). \tag{2.221}$$

[13]We assumed $0 < x < 1$.

Canonical Basis

The basis $\mathbf{I}(s,\epsilon)$ are related to $\mathbf{F}(s,\epsilon)$ by the following relation:

$$
\begin{aligned}
I_1(s,\epsilon) =& F_1(s,\epsilon), \quad I_2(s,\epsilon) = sF_2(s,\epsilon), \\
I_3(s,\epsilon) =& m^2 F_3(s,\epsilon), \quad I_4(s,\epsilon) = sF_4(s,\epsilon), \\
I_5(s,\epsilon) =& \sqrt{s}\sqrt{s-4m^2}\left(\frac{1}{2}F_4(s,\epsilon) + F_5(s,\epsilon)\right), \quad I_6(s,\epsilon) = sF_6(s,\epsilon), \\
I_7(s,\epsilon) =& \sqrt{s}\sqrt{s-4m^2}\left(F_2(s,\epsilon) + F_7(s,\epsilon)\right), \quad I_8(s,\epsilon) = \sqrt{s}\sqrt{s-4m^2}F_8(s,\epsilon), \\
I_9(s,\epsilon) =& s\left(m^2 F_9(s,\epsilon) - \frac{F_2(s,\epsilon)}{2} - F_7(s,\epsilon) + F_8(s,\epsilon)\right), \\
I_{10}(s,\epsilon) =& -\frac{\sqrt{s}\sqrt{s-4m^2}\left(4m^4 F_{10}(s,\epsilon) + 6m^2 F_3(s,\epsilon) - 3sF_4(s,\epsilon) + F_1(s,\epsilon)\right)}{8m^2-4s}, \\
I_{11}(s,\epsilon) =& \frac{1}{4}\sqrt{s}\sqrt{s-4m^2}\left(4m^2 F_{11}(s,\epsilon) + 3F_6(s,\epsilon)\right), \\
I_{12}(s,\epsilon) =& \sqrt{s}\sqrt{s-4m^2}F_{12}(s,\epsilon), \\
I_{13}(s,\epsilon) =& \frac{m^2\left(12m^2F_3(s,\epsilon) + s(6F_3(s,\epsilon) - 7F_4(s,\epsilon) - 4(F_5(s,\epsilon) + 3F_8(s,\epsilon))) + 2F_1(s,\epsilon)\right)}{12m^2-6s} + \\
& - \frac{s(5sF_4(s,\epsilon)s - 4sF_5(s,\epsilon) - 12sF_8(s,\epsilon)s + 24sF_{12}(s,\epsilon) - 2F_1(s,\epsilon))}{12\left(2m^2-s\right)} + m^2 sF_{13}(s,\epsilon) \\
& \frac{4m^2 s\left(m^2F_{10}(s,\epsilon) + 3F_{12}(s,\epsilon)\right)}{3\left(2m^2-s\right)}, \\
I_{14}(s,\epsilon) =& \sqrt{s}\sqrt{s-4m^2}F_{14}(s,\epsilon), \\
I_{15}(s,\epsilon) =& -\frac{1}{24}s\left(8(F_{11}(s,\epsilon) - 3F_{15}(s,\epsilon)m^2 + 3F_6(s,\epsilon) + 48F_{14}(s,\epsilon)\right), \\
I_{16}(s,\epsilon) =& s^{3/2}\sqrt{s-4m^2}F_{16}(s,\epsilon)
\end{aligned}
\tag{2.222}
$$

The matrix $\mathbb{A}_c(x)$ in (2.221), turns to have a rational alphabet, with letters $\{x, 1+x, 1-x\}$:

$$
\mathbb{A}_c(x) = \frac{1}{x}\mathbb{M}_1 + \frac{1}{1+x}\mathbb{M}_2 + \frac{1}{1-x}\mathbb{M}_3. \tag{2.223}
$$

Thus, the general solution can be obtained in terms of HPLs with a minimal effort. We present here the explicit expression for: $\{\mathbb{M}_i\}_{i=1,2,3}$:

$$
\mathbb{M}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 3 & -\frac{3}{4} & -\frac{3}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & 0 & -\frac{1}{2} & -\frac{1}{24} & \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{6} & 0 & -1 & \frac{5}{6} & \frac{7}{6} & 0 & 2 & 1 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & -2 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{3} & 0 & 0 & 4 & 2 & 0 \\
-\frac{1}{3} & -1 & -2 & \frac{5}{6} & 0 & \frac{5}{8} & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 2
\end{pmatrix}, \tag{2.224}
$$

$$
\mathbb{M}_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}, \tag{2.225}
$$

$$
\mathbb{M}_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 1 & 1 & 0 & \frac{4}{3} & 0 & 2 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & -2 & \frac{5}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}. \tag{2.226}
$$

Chapter 3

Differential Equation and Homogeneous solution

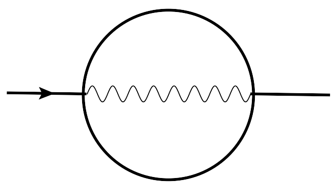
In this Chapter we reconsider the algorithm based on the Magnus Exponential, and we apply it to the so-called *"QED Sunrise"* Integral Family. In particular we emphasize its connection with the solution of the DEQs system at $\epsilon = 0$: obtaining different solutions for the system at $\epsilon = 0$ we build a matrix, which is similar, and, on practical grounds, fully equivalent to the one obtained through the Magnus algorithm. Then we consider a different basis of MIs, and we show how Cut Integrals, and their natural implementation in Baikov Representation, provide the *whole* set of solutions for the *homogeneous* part of an higher order DEQ.
Finally, thanks to the full set of homogeneous solutions obtained through Cuts and IBPs, we obtain another matrix for the original basis of MIs, similar but fully equivalent on practical grounds, to the one obtained through the Magnus algorithm.

3.1 Magnus Exponential

3.1.1 Magnus Exponential, application to the QED Sunrise

Let's consider:

$\mathcal{S}^{(2,1)} =$

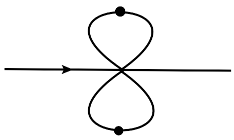


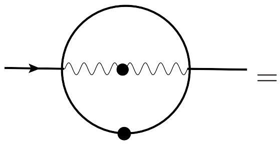
$=$

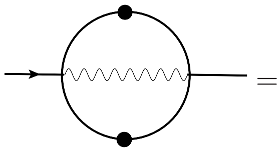
$$= \int \widetilde{d^d k_1} \widetilde{d^d k_2} \frac{(k_1+p)^{-s_1}(k_2+p)^{-s_2}}{(k_1^2-m^2)^{r_1}((k_1+k_2+p)^2)^{r_2}(k_2^2-m^2)^{r_3}}, \quad s_i \leq 0, \quad i = 1, 2. \tag{3.1}$$

In this Section we will apply the algorithm based on the *Magnus Exponential* (2.4) to the Integral Family (3.1). Thanks to `Reduze 2` we identify the following basis of MIs given by,

$\mathbf{F} = \{F_1, F_2, F_3\}$:

$F_1 =$  $= \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{1}{(k_2^2-m^2)^2(k_1^2-m^2)^2},$

$F_2 =$  $= m^2 \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{1}{(k_2^2-m^2)^2((k_1+k_2+p)^2)^2(k_1^2-m^2)},$ (3.2)

$F_3 =$  $= m^2 \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{1}{(k_2^2-m^2)^2(k_1+k_2+p)^2(k_1^2-m^2)^2}.$

They fulfill a DEQs system *linear* in ϵ, expressed w.r.t. the *adimensional* variable x, where $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$:

$$\frac{\partial \mathbf{F}(x,\epsilon)}{\partial x} = \mathbb{A}(x,\epsilon)\mathbf{F}(x,\epsilon),$$
$$\mathbb{A}(x,\epsilon) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\epsilon}{2(x-1)} - \frac{\epsilon}{2(x+1)} & \frac{-6\epsilon-1}{x+1} - \frac{1}{x-1} + \frac{3\epsilon+1}{x} & \frac{1-2\epsilon}{2(x-1)} - \frac{\epsilon}{x} + \frac{6\epsilon-1}{2(x+1)} \\ 0 & \frac{2\epsilon}{x} - \frac{4\epsilon}{x-1} & \frac{1}{x} - \frac{2}{x-1} \end{pmatrix}. \tag{3.3}$$

We can now split $\mathbb{A}(x,\epsilon)$ as: $\mathbb{A}(x,\epsilon) = \mathbb{A}_0(x) + \epsilon\mathbb{A}_1(x)$, i.e. into a part *linear in* ϵ, namely $\epsilon\mathbb{A}_1(x)$ and another one which is ϵ*-independent*, namely $\mathbb{A}_0(x)$, which reads:

$$\mathbb{A}_0(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x-1} & \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \\ 0 & 0 & \frac{1}{x} - \frac{2}{x-1} \end{pmatrix} \tag{3.4}$$

The latter, namely $\mathbb{A}_0(x)$, can be further decomposed as $\mathbb{A}_0(x) = \mathbb{D}_0(x) + \mathbb{N}_0(x)$ (being $\mathbb{D}_0$(x) the *diagonal part*):

$$\mathbb{A}_0(x) = \mathbb{D}_0(x) + \mathbb{N}_0(x), \quad \mathbb{D}_0(x) = \mathrm{diag}\left(0, \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x-1}, \frac{1}{x} - \frac{2}{x-1}\right). \tag{3.5}$$

We can now obtain a DEQs system for a new basis of MIs, $\mathbf{G}(x,\epsilon)$, related to the previous one by the relation: $\mathbf{F}(x,\epsilon) = \mathbb{B}_1(x)\mathbf{G}(x,\epsilon)$; we trivially find:

$$\frac{\partial \mathbf{G}(x,\epsilon)}{\partial x} = \mathbb{B}_1^{-1}(x)\left(-\frac{\partial \mathbb{B}_1(x)}{\partial x} + \mathbb{D}_0(x)\mathbb{B}_1(x) + \mathbb{N}_0(x)\mathbb{B}_1(x) + \epsilon\mathbb{A}_1(x)\mathbb{B}_1(x)\right)\mathbf{G}(x,\epsilon). \tag{3.6}$$

We impose now that $\mathbb{B}_1(x)$ is such that:

$$\frac{\partial \mathbb{B}_1(x)}{\partial x} = \mathbb{D}_0(x)\mathbb{B}_1(x), \quad \Rightarrow \mathbb{B}_1(x) = e^{\int dx\, \mathbb{D}_0(x)} =$$
$$\mathbb{B}_1(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{1-x^2} & 0 \\ 0 & 0 & \frac{x}{(1-x)^2} \end{pmatrix}, \tag{3.7}$$

and so (3.6) can be recast as:

$$\frac{\partial \mathbf{G}(x,\epsilon)}{\partial x} = \underbrace{\mathbb{B}_1^{-1}(x)(\mathbb{N}_0(x) + \epsilon\mathbb{A}_1(x))\mathbb{B}_1(x)}_{=:\mathbb{A}^{[2]}(x,\epsilon)}\,\mathbf{G}(x,\epsilon) = \mathbb{A}^{[2]}(x,\epsilon)\mathbf{G}(x,\epsilon). \tag{3.8}$$

We can now consider $A^{[2]}(x,\epsilon) = A_0^{[2]}(x) + \epsilon\, A_1^{[2]}(x)$, as well. It results:

$$\mathbb{A}_0^{[2]}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{(x-1)^2} \\ 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

We immediately perform another change of basis, namely $\mathbf{G}(x,\epsilon) = \mathbb{B}_2(x)\,\mathbf{I}(x,\epsilon)$, and, thanks to the latter, we have:

$$\frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} = \mathbb{B}_2^{-1}(x)\left(-\frac{\partial \mathbb{B}_2(x)}{\partial x} + \mathbb{A}_0^{[2]}(x)\,\mathbb{B}_2(x) + \epsilon\mathbb{A}_1^{[2]}(x)\mathbb{B}_2(x)\right)\mathbf{I}(x,\epsilon); \tag{3.10}$$

we notice that we are close to the desired ϵ-*factorized from*; the latter can be achieved provided the fact that we are able to furnish an explicit expression for the matrix $\mathbb{B}_2(x)$, with $\mathbb{B}_2(x)$ such that:

$$\frac{\partial \mathbb{B}_2(x)}{\partial x} = \mathbb{A}_0^{[2]}(x)\mathbb{B}_2(x). \tag{3.11}$$

This very last matrix DEQ (3.11), can be solved through the *Magnus Exponential* (2.59), (2.60); as stated above, we have:

$$\mathbb{B}_2(x) = e^{\Omega[\mathbb{A}^{[2]}{}_0](x)}. \tag{3.12}$$

Being $\mathbb{A}_0^{[2]}(x)$ a strictly upper $(3,3)$ triangular matrix $\Omega[\mathbb{A}^{[2]}{}_0](x)$ is expected to have a finite number of summands; an explicit calculation show that $\Omega[\mathbb{A}^{[2]}{}_0](x)$ consists in the very first term of the series:

$$\Omega[\mathbb{A}_0^{[2]}](x) = \int dx\,\mathbb{A}_0^{[2]}(x), \tag{3.13}$$

since $\mathbb{A}_0^{[2]}$ commutes with its integral; and so it results:

$$\mathbb{B}_2(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{x-1} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

Moreover the *Canonical System* reads:

$$\begin{aligned} \frac{\partial \mathbf{I}(x,\epsilon)}{\partial x} &= \epsilon\mathbb{A}_c(x)\mathbf{I}(x,\epsilon), \\ \mathbb{A}_c(x) &= \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{x} & \frac{5}{x} - \frac{6}{x+1} - \frac{2}{x-1} & -\frac{6}{x} + \frac{3}{x+1} + \frac{2}{x-1} \\ 0 & \frac{2}{x} & \frac{2}{x-1} - \frac{2}{x} \end{pmatrix}, \\ \mathbf{F}(x,\epsilon) &= \mathbb{B}_1(x)\mathbf{G}(x,\epsilon) = \mathbb{B}_1(x)\,\mathbb{B}_2(x)\mathbf{I}(x,\epsilon). \end{aligned} \tag{3.15}$$

In this worked out example, the algorithm based on the *Magnus Exponential*, produces the *"rotation"* matrix:

$$\mathbb{R}_{\text{Magnus}}(x) = \mathbb{B}_1(x)\mathbb{B}_2(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{1-x^2} & \frac{x}{(x-1)(1-x^2)} \\ 0 & 0 & \frac{x}{(1-x)^2} \end{pmatrix}. \tag{3.16}$$

3.1.2 A first look beyond Magnus Exponential

Moreover, we could notice that the *matrix solution* given by the *Magnus Exponential* is deeply related to the solution of the DEQs system itself, at $\epsilon = 0$, i.e.: $d = d_c$. More in detail, we can consider the DEQs system, at $\epsilon = 0$:

$$\frac{\partial \mathbf{F}^0(x)}{\partial x} = \mathbb{A}_0(x)\mathbf{F}^0(x). \tag{3.17}$$

A standard result in the general theory of Differential Equations, states that we can consider $\mathbf{F}^0_{[1]}(x), \ldots, \mathbf{F}^0_{[m]}(x)$ different solutions for (3.17), and consequently build the matrix[1]:

$$\Phi(x) = \left(\mathbf{F}^0_{[1]}(x), \cdots, \mathbf{F}^0_{[m]}(x)\right); \tag{3.18}$$

then, essentially simply preforming the row-column multiplication, we can argue that $\mathbf{\Phi}$ is such that:

$$\frac{\partial \Phi(x)}{\partial x} = \mathbb{A}_0(x)\Phi(x), \tag{3.19}$$

which is exactly the DEQs system (2.56) solved in terms of the *Magnus Exponential*. Furthermore, considering a constant matrix $\mathbb{C}$, then:

$$\Psi_C(x) = \Phi(x)\,\mathbb{C} \tag{3.20}$$

satisfies the same DEQs system, namely (2.56).

DEQs system at $\epsilon = 0$, the QED Sunrise

Let's consider the DEQ (3.3) at $d = 4 \Leftrightarrow \epsilon = 0$, namely [2]:

$$\frac{\partial \mathbf{F}^0(x)}{\partial x} = \mathbb{A}_0(x)\mathbf{F}^0(x), \quad \mathbb{A}_0(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x-1} & \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \\ 0 & 0 & \frac{1}{x} - \frac{2}{x-1} \end{pmatrix}, \tag{3.21}$$

where $\mathbf{F}^0(x) = \{F^0_1(x), F^0_2(x), F^0_3(x)\}$. The solution for $F_{1,0}(x)$ is absolutely trivial:

$$F^0_1(x) \equiv F^0_1 = c_1, \tag{3.22}$$

where c_1 is a constant. The solution for $F^0_3(x)$ is straightforward as well, namely:

$$F^0_3(x) = c_3\,\frac{x}{(1-x)^2}. \tag{3.23}$$

Then, once the solution for $F^0_3(x)$ is known, the DEQ for $F^0_2(x)$ is a 1^{st} *order non homogeneous DEQ*, and can be solved with standard techniques (i.e.: *Euler's variation of constants method*). We find:

$$F^0_2(x) = c_3\,\frac{x}{(x-1)(1-x^2)} + c_2\,\frac{x}{1-x^2}. \tag{3.24}$$

[1] I.e.: the i^{th}-column of $\mathbf{\Phi}$ is given by $F_i(x)$.

[2] We notice that the system is *"homogeneous"*, in the sense that each DEQ involves Mis belonging to the same topology, i.e. with the same denominators.

Then, for example, choosing the set of constants $\{c_1, c_2, c_3\}$ as:

$$\begin{aligned}
&\{c_1, c_2, c_3\} = \{1, 0, 0\} \Rightarrow \mathbf{F}^{0,T}_{[1]} = (1, 0, 0) \\
&\{c_1, c_2, c_3\} = \{0, 0, -1\} \Rightarrow \mathbf{F}^{0,T}_{[2]}(x) = \left(0, \frac{x}{(x-1)^2(x+1)}, -\frac{x}{(1-x)^2}\right) \\
&\{c_1, c_2, c_3\} = \{0, -1, -1\} \Rightarrow \mathbf{F}^{0.T}_{[3]}(x) = \left(0, \frac{x^2}{(x-1)^2(x+1)}, -\frac{x}{(1-x)^2}\right).
\end{aligned} \tag{3.25}$$

We can build the matrix:

$$\Phi(x) = \left(\mathbf{F}^0_{[1]}(x), \mathbf{F}^0_{[2]}(x), \mathbf{F}^0_{[3]}(x)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{(x-1)^2(x+1)} & \frac{x^2}{(x-1)^2(x+1)} \\ 0 & -\frac{x}{(1-x)^2} & -\frac{x}{(1-x)^2} \end{pmatrix}, \tag{3.26}$$

and we can verify that:

$$\frac{\partial \Phi(x)}{\partial x} = \mathbb{A}_0(x)\, \Phi(x), \tag{3.27}$$

and in fact $\mathbb{R}_{\text{Magnus}}(x)$ and $\Phi(x)$ are related by the relation:

$$\mathbb{R}_{\text{Magnus}}(x) = \Phi(x)\mathbb{C}_\Phi, \quad \mathbb{C}_\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.28}$$

After all, thanks to the Magnus Exponential, we solve a 1^{st}-order differential equation, and an arbitrariness in choosing the constants should be expected.

2^{nd}-order DEQ, the QED Sunrise

Finally we would like to mention the possibility to recast a DEQs system into a unique *higher order DEQ* for one of the unknown. This observation will play an important role especially in the next section; even if it's not strictly necessary, we apply this method to (3.21) . As a first step, let's consider[3]:

$$\begin{cases} \frac{\partial F^0_2(x)}{\partial x} = a_{22}(x)\, F^0_2(x) + a_{23}(x)\, F^0_3(x) \\ \frac{\partial F^0_3(x)}{\partial x} = a_{32}(x)\, F^0_2(x) + a_{33}(x)\, F^0_3(x) \end{cases} \tag{3.29}$$

We can express F^0_3 in terms of F^0_2 and $\partial_x F^0_2$, namely:

$$F^0_3(x) = \frac{1}{a_{23}(x)} \left(\partial_x F^0_2(x) - a_{22}(x) F^0_2(x)\right). \tag{3.30}$$

Then, considering the derivative in (3.29) we have[4]

$$\partial^2_x F^0_2(x) = \partial_x a_{22}(x)\, F^0_2(x) + a_{23}(x)\, \partial_x F^0_2(x) + \partial_x a_{23}(x)\, F^0_3(x) + a_{23}(x)\, \partial_x F^0_3(x), \tag{3.31}$$

but thanks to (3.30) and the 2^{nd} DEQ in (3.29) we have:

$$\begin{aligned}
\partial^2_x F^0_2(x) = &\left(a_{22}(x) + a_{33}(x) + \frac{\partial_x a_{23}(x)}{a_{23}(x)}\right) \partial_x F^0_2(x) + \\
&+ \left(\partial_x a_{22}(x) + a_{23}(x)\, a_{32}(x) - \frac{a_{22}\,(a_{23}\, a_{33} + \partial_x a_{23})}{a_{23}}\right) F^0_2(x).
\end{aligned} \tag{3.32}$$

[3] We consider here a DEQs system with *two* unknowns. In our example the 1^{st} MI is just a *subtopolgy*, thus it does not play any role. Moreover, in our explicit example the *two* DEQs for $F_{0,2}(x)$ and $F_{0,3}(x)$ are decoupled; thus even if it is possible, it is not necessary to recast the *two* DEQs into an *unique* DEQ, as stated above.

[4] We assume the standard notation: $\frac{\partial^k}{\partial x^k} = \partial^k_x$.

We immediately notice that, once $F_2^0(x)$ is solved then $F_3^0(x)$ can be determined, thanks to (3.30). In the explicit example (3.21) we have:

$$\partial_x^2 F_2^0(x) = \frac{2\left(2x^2+x+1\right)}{x-x^3}\partial_x F_2^0(x) - \frac{2\left(x^2-x+1\right)}{(x-1)^2x^2}F_2^0(x). \tag{3.33}$$

Solving the latter we find:

$$F_2^0(x) = k_1\,\frac{x^2}{(x-1)^2(x+1)} + k_2\frac{x}{(x-1)^2(x+1)}, \tag{3.34}$$

being $\{k_1, k_2\}$ arbitrary constants.
Choosing $\{k_1, k_2\} = \{0, 1\}$, and expressing $F_3^0(x)$ in terms of $\{F_2^0(x), \partial_x F_2^0(x)\}$, through (3.30) we find a solution for (3.29), namely:

$$\widetilde{\mathbf{F}}_{[2]}^{0,T}(x) = \left(\frac{x}{(x-1)^2(x+1)}, -\frac{x}{(1-x)^2}\right). \tag{3.35}$$

On the other hand, repeating the same steps with $\{k_1, k_2\} = \{1, 0\}$ the solution reads:

$$\widetilde{\mathbf{F}}_{[3]}^{0,T}(x) = \left(\frac{x^2}{(x-1)^2(x+1)}, -\frac{x}{(1-x)^2}\right) \tag{3.36}$$

Being (3.32) solved we build-up *two* solution for (3.21):

$$\mathbf{F}_{[2]}^{0,T}(x) = \left(0, \widetilde{\mathbf{F}}_{[2]}^{0,T}(x)\right), \quad \mathbf{F}_{[3]}^{0,T}(x) = \left(0, \widetilde{\mathbf{F}}_{[3]}^{0,T}(x)\right). \tag{3.37}$$

Finally, considering:

$$\mathbf{F}_{[1]}^{0,T} = (1, 0, 0)\,, \tag{3.38}$$

we reobtain (3.26), namely:

$$\Phi(x) = \left(\mathbf{F}_{[1]}^0, \mathbf{F}_{[2]}^0(x), \mathbf{F}_{[3]}^0(x)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{(x-1)^2(x+1)} & \frac{x^2}{(x-1)^2(x+1)} \\ 0 & -\frac{x}{(1-x)^2} & -\frac{x}{(1-x)^2} \end{pmatrix}. \tag{3.39}$$

On top of that we can recast (3.32) as:

$$\partial_x \begin{pmatrix} F_2^0(x) \\ \partial_x F_2^0(x) \end{pmatrix} = \mathbb{A}_{\text{eq}}(x) \begin{pmatrix} F_2^0(x) \\ \partial_x F_2^0(x) \end{pmatrix}, \tag{3.40}$$

with:

$$\mathbb{A}_{\text{eq}} = \begin{pmatrix} 0 & 1 \\ -\frac{2(x^2-x+1)}{(x-1)^2x^2} & \frac{2(2x^2+x+1)}{x-x^3} \end{pmatrix}. \tag{3.41}$$

We can build the *Wronski matrix*, starting from (3.34):

$$\begin{aligned} \mathbb{W}(x) &= \begin{pmatrix} F_{0,2}^{k_1=1,k_2=0}(x) & F_{0,2}^{k_1=0,k_2=1}(x) \\ \partial_x F_{0,2}^{k_1=1,k_2=0}(x) & \partial_x F_{0,2}^{k_1=0,k_2=1}(x) \end{pmatrix} = \\ &= \begin{pmatrix} \frac{x^2}{(x-1)^2(x+1)} & \frac{x}{(x-1)^2(x+1)} \\ -\frac{x\left(x^2+x+2\right)}{(x-1)^3(x+1)^2} & -\frac{2x^2+x+1}{(x-1)^3(x+1)^2} \end{pmatrix}. \end{aligned} \tag{3.42}$$

We can verify that the following relation holds:

$$\partial_x \mathbb{W}(x) = \mathbb{A}_{\text{eq}}(x)\mathbb{W}(x). \tag{3.43}$$

3.2 Homogeneous solution, a general strategy

In this Section we will study Cut Integrals in the context of DEQs. Cut Integrals turn to be an unrivaled tool, since, as it was intensively shown by many autohrs [43]-[47], they provide the *homogeneous solution* for DEQ. Things are even more interesting looking at an *higher order DEQ*, i.e.: when there is more than one MI for a given topology, since, taking care about the integration contours, they provide the *whole set* of homogeneous solutions. This fact seems to be extremely important: no general strategy to compute the homogeneous solution of an higher order DEQ is known, and this *"physical input"* seems essential. Moreover, once the homogeneous solution is known, the whole solution could be recovered with a standard technique, namely *Euler variations of constants.* Even if it goes beyond the scope of this thesis, we would mention the possibility to use these techniques in the study of Feynman Integral, and the associated DEQs, beyond Polylogarithms [3]. Finally we would reveal the connection between the (set of) homogeneous solutions an the Magnus Exponential (2.4).

3.2.1 Feynman Cut Integrals

We review here some basic definitions and properties about Cut Integrals, closely following [3].
Roughly speaking, a Feynman Cut Integral is an integral in which we impose one, or even more, virtual particles to be on-shell. On practical grounds this means that we substitute one, or more denominators, with the corresponding δ function under the integral sign, according to the prescription:

$$\int d^d k \, \frac{1}{D_k(k)} \to \int d^d k \, \delta(D_k(k)). \tag{3.44}$$

Moreover, we can furnish an operative definition for a Cut Integral involving a denominator raised to a power greater than one, namely $D_k^{r_k}(k)$, with $r_k > 1$. For the sake of simplicity, let's consider $r_k = 2$ (the generalization to $a_k > 2$ can be obtained with a minimal effort) and modify the mass in the corresponding denominator according to: $m_k^2 \to \hat{m}_k^2$; thus we have:

$$I^{(\ell,n)}_{1\ldots2\ldots1} = - \lim_{\hat{m}_k^2 \to m_k^2} \partial_{\hat{m}_k^2} I^{(\ell,n)}_{1\ldots1\ldots1}. \tag{3.45}$$

Finally, we can apply the usual operative definition for a Cut Integrals (3.44), assuming the *Cut Operation* is insensible (roughly speaking it *"commutes"*) to $\partial_{\hat{m}_k^2}$:

$$\text{Cut}[I^{(\ell,n)}_{1\ldots2\ldots1}] = - \lim_{\hat{m}_k^2 \to m_k^2} \partial_{\hat{m}_k^2} \text{Cut}[I^{(\ell,n)}_{1\ldots1\ldots1}]. \tag{3.46}$$

3.2.2 Cut Integrals in Baikov representation

Not surprisingly the *Baikov representation* is particularly well suited for *Cut Integrals*; in fact, in this representation a cut over the k^{th} denominator, reads:

$$\text{Cut}[\mathcal{I}^{(\ell,n)}] = (G(\{p_1, \ldots, p_n\}))^{(n+1-d)/2} \int \prod_{i=1,\, i\neq k}^{n_{SP}} \frac{dz_i}{z_i^{a_i}} \oint_{z_k=0} dz_k \frac{(F(\mathbf{z}))^{\gamma_d}}{z_k^{a_k}}, \tag{3.47}$$

where n is the number of *independent* external momenta, and $\gamma_d = d - n - 1 - \ell$, see (1.4). We immediatly notice that (3.47) can be computed by means of *Residue Theorem.*

Furthermore, in the case in which $r_i = 1$, we recover the standard prescription:

$$\frac{1}{z_i} \to 2\pi i \delta(z_i). \tag{3.48}$$

Then, considering an Integral with the whole set of t denominators raised to power 1, the corresponding *Maximal Cut*, i.e.: the Integral in which the whole set of denominator is *cut*, reads:

$$\text{MCut}[\mathcal{I}^{(\ell,n)}] = \mathcal{I}^{(\ell,n)}_{\text{M.C.}} = (2\pi i)^t \left(G(\{p_1,\dots,p_n\})\right)^{(n+1-d)/2} \int \prod_{i=t+1}^{n_{SP}} dz_i\, z^{-a_i} \left(F(\mathbf{z})\right)^{\gamma_d}\Big|_{z_1=\dots=z_t=0}. \tag{3.49}$$

Then, in order to exploit (3.49) and its consequences some comments are mandatory:

- at *one loop*, i.e.: $\ell = 1$, the number of integration variables always matches the number of denominators. Thus, a Maximal Cut Integral is *completely localized*, i.e.: for $\ell = 1$ there is *no integration* left in (3.49);
- since IBPs rely on the vanishing of the integr*and* on the integration boundaries, imposing that IBPs hold for (3.49), beyond *one loop*, it is natural to identify an *Integration Domain*, Γ, such that the *Baikov Polynomial on the Cut*, $\text{MCut}[F(\mathbf{z})] = F(\mathbf{z})\Big|_{z_1=\dots=z_t=0}$, is vanishing on $\partial\Gamma$:

  $$\text{MCut}[F(\mathbf{z})]\Big|_{\partial\Gamma} = 0. \tag{3.50}$$

  A remarkable fact is that we can identify *more than one region*, namely $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$ such that:

  $$\text{MCut}[F(\mathbf{z})]\Big|_{\partial\Gamma_i} = 0, \quad \forall i = 1,\dots n. \tag{3.51}$$

- Given an IBP, we can apply to both sides of the equation our operative definitions of *"Cut"*. In a t^*-*Cut*, Integrals with a number of denominators lower than t^* turn to be vanishing, since they do not support the t^*-*Cut*. Furthermore, if we apply a *Maximal Cut* compatible with the *"highest topology"*, i.e.: the topology with the *highest* number of denominators, in a given relation, we select the homogeneous part of the relation. By virtue of this, since a DEQ is obtained by a chained series of IBPs, the *Cut Integrals* turn to satisfy the *homogeneous* DEQ; moreover, considering an higher order DEQ, each representation of maximal Cut Integrals (3.51) should satisfies the DEQ itself.
  Moreover, thanks to IBPs it is always possible to express an Integral with t propagators, each of them raised to powers *greater or equal to one*, as a linear combination of Integrals of the same topology with denominators strictly raised to power *one* and subtopologies:

  $$\begin{aligned}\mathcal{I}^{(\ell,n)}[\{a_1,\dots,a_t,a_t,\dots a_{n_{SP}}\}] = &\sum_i c_i \mathcal{I}^{(\ell,n)}[\{1,\dots,1,a^i_{t+1},\dots,a^i_{n_{SP}}\}]+\\ &+\text{subtopologies}.\end{aligned} \tag{3.52}$$

  Then, applying a Maximal Cut, we have:

  $$\text{MCut}[\mathcal{I}^{(\ell,n)}[\{a_1,\dots,a_t,a_{t+1},\dots,a_{n_{SP}}\}]] = \sum_i c_i \text{MCut}[\mathcal{I}^{(\ell,n)}[\{1,\dots,1,a^i_{t+1},\dots,a^i_{n_{SP}}\}]]; \tag{3.53}$$

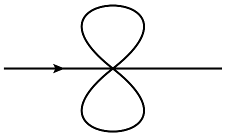
  the latter can be seen as an alternative definition for (Maximal) Cut Integrals, involving denominators raised to powers greater than one.

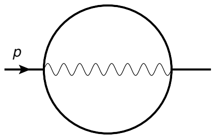
3.3 Homogeneous DEQ for the QED Sunrise, Laporta Basis

Let's consider once again the Integral Family (3.1):

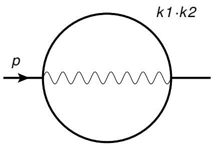
$$\mathcal{S}^{(2,1)} = \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{(k_1\cdot k_2)^{-a_4}(k_1\cdot p)^{-a_5}}{(k_1^2-m^2)^{a_1}((k_1+k_2+p)^2)^{a_2}(k_2^2-m^2)^{a_3}}, \quad a_i \leq 0,\ i=4,5. \tag{3.54}$$

Thanks to `Reduze 2` we identify the following *"minimal"* basis of MIs, $\mathbf{L} = \{L_1, L_2, L_3\}$:

$L_1 =$  $= \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{1}{(k_1^2-m^2)(k_2^2-m^2)},$

$L_2 = m^2$ 

$= m^2\int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{1}{(k_1^2-m^2)(k_1+k_2+p)^2(k_2^2-m^2)},$ (3.55)

$L_3 =$ 

$= \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{(k_1\cdot k_2)}{(k_1^2-m^2)(k_1+k_2+p)^2(k_2^2-m^2)}.$

The latter fulfills a DEQs system in the *adimensional variable* x, $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$, which reads:

$$\frac{\partial \mathbf{L}(x,d)}{\partial x} = \mathbb{A}(x,d)\mathbf{L}(x,d), \quad \mathbb{A}(x,d) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{d-2}{2(x^2-1)} & \frac{d(x^2-4x+1)-4(x-1)^2}{2x(x^2-1)} & -\frac{3(d-2)}{x^2-1} \\ \frac{d-2}{2(x^2-1)} & \frac{-dx^2+2x^2+4dx-12x-d+2}{2x-2x^3} & \frac{(d-2)(x^2-x+1)}{x(x^2-1)} \end{pmatrix}.$$

(3.56)

Then, starting from (3.56), we can obtain a 2^{nd}-order *homogeneous differential equation* ($L_1 = 0$) for L_2 in $d = 4-2\epsilon$ dimensions:

$$\partial_x^2 L_{2,\text{h}}^{\epsilon} + \frac{3(x-1)^2\epsilon+6x-2}{x\,(x^2-1)}\partial_x L_{2,\text{h}}^{\epsilon} + \frac{(2\epsilon-1)\left(\left(x^2-4x+1\right)\epsilon+2x\right)}{(x-1)^2x^2}L_{2,\text{h}}^{\epsilon} = 0. \tag{3.57}$$

Moreover at $\epsilon = 0$, we have:

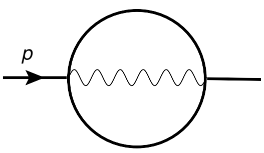
$$\partial_x^2\, L_{2,\text{h}}^{\epsilon=0} + \frac{2-6x}{x-x^3}\,\partial_x\, L_{2,\text{h}}^{\epsilon=0} - \frac{2}{(x-1)^2x}\, L_{2,\text{h}}^{\epsilon=0} = 0. \tag{3.58}$$

Solving the latter we find 2-independent solutions (as expected for a 2^{nd} order DEQ):

$$\begin{aligned} h_1^0(x) &= \frac{x^2-x+1}{(x-1)^2}, \\ h_2^0(x) &= \left(\frac{x^2-x+1}{(x-1)^2}\right)\left(-\frac{3(x-2)}{x^2-x+1}+x-\frac{1}{x}+4\log(x)\right). \end{aligned} \tag{3.59}$$

3.4 Baikov on the Maximal Cut

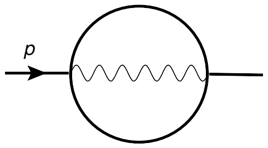
On the other hand, let's consider the integral:

$\frac{L_2}{m^2} =$ 

$= \int \widetilde{d^dk_1}\widetilde{d^dk_2}\frac{1}{(k_1^2-m^2)(k_1+k_2+p)^2(k_2^2-m^2)};$ (3.60)

$\frac{L}{m^2}$ can be expressed in terms of the *Baikov representation*[5] (we assume $z_4 = k_1 \cdot p$, $z_5 = k_2 \cdot p$)

$$\frac{L_2}{m^2} =$$



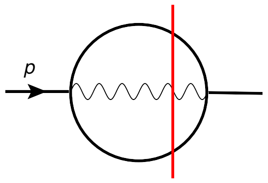
$$= (G(p))^{\frac{2-d}{2}} \int \prod_{i=1}^{5} dz_i F(\mathbf{z})^{\frac{d-4}{2}} \frac{1}{z_1 z_2 z_3}. \tag{3.61}$$

Where:

$$G(p) = p^2 = s. \tag{3.62}$$

Then, (3.61) on the Maximal Cut, reads:

$$\text{MCut}[\frac{L_2}{m^2}] =$$



$$= (G(p))^{\frac{2-d}{2}} \int dz_4 dz_5 \left(F(\mathbf{z})\right)^{\frac{d-4}{2}} \Big|_{z_1=z_2=z_3=0}, \tag{3.63}$$

where, $F(\mathbf{z})$ on the *Maximal Cut*, i.e.: $F(\mathbf{z})\Big|_{\text{M.C.}} = F(\mathbf{z})\Big|_{z_1=z_2=z_3=0}$, reads:

$$F(\mathbf{z})\Big|_{\text{M.C.}} = -m^2(s + z_4 + z_5)^2 - \frac{1}{4}(s + 2z_4)(s + 2z_5)(s + 2(z_4 + z_5)). \tag{3.64}$$

A comment on the Lee-Pomeransky criterium

We would like to mention the possibility to apply the Lee-Pomeransky criterium [65], already cited in Section (1.10), within the Baikov representation on the *Maximal Cut*. The criterium states that the number of *Proper Critical Points* $\{\bar{\mathbf{z}}_\alpha\}$, where:

$$\{\bar{\mathbf{z}}_\alpha\} = \{\mathbf{z} \mid \quad \nabla F(\mathbf{z})\Big|_{\text{M.C.}} = 0, \quad F(\mathbf{z})\Big|_{\text{M.C.}} \neq 0\} \tag{3.65}$$

is equal to the number of MIs for the corresponding topology, *modulo symmetry relations*. For what concerns the DEQs, thanks to this analysis we can determine *a priori* (i.e.: without performing explicitly the IBPs reduction) the *order* of the homogeneous DEQ for (one of) the MI for the considered topology.
Working on (3.64), we identify *three* Proper Critical Points according to (3.65):

$$\begin{aligned} &\bar{\mathbf{z}}_1 = -\frac{1}{3}(2m^2 + s, 2m^2 + s), \quad \bar{\mathbf{z}}_2 = -\frac{1}{4}(s + \sqrt{s}\sqrt{8m^2 + s}, s - \sqrt{s}\sqrt{8m^2 + s}), \\ &\bar{\mathbf{z}}_3 = -\frac{1}{4}(s - \sqrt{s}\sqrt{8m^2 + s}, s + \sqrt{s}\sqrt{8m^2 + s}). \end{aligned} \tag{3.66}$$

Actually, $\bar{\mathbf{z}}_2$ and $\bar{\mathbf{z}}_3$ turn to be related by the symmetry: $\{z_4 \to z_5, z_5 \to z_4\}$, as we can see simply looking at (3.64), which corresponds to the symmetry: $\{k_1 \to k_2, k_2 \to k_1\}$ in momentum space. Thus, we have just *two* independent Proper Critical Points and thus there are *two* MIs for the corresponding topology, as confirmed by the explicit IBPs reduction.

Since multiloop calculations heavily rely on integration by parts relations, the integration domain in (3.63) has to be chosen consistently in such a way that the total derivative under the integral sign is vanishing. This means that the integration boundaries in (3.63)

[5] We are forgetting about irrelevant over-all constants.

should be identified among the zeros of the integrand. Moreover, generally speaking, dealing with a 2-fold integration, such as (3.63), is quite uncomfortable. Thus, we will perform some change of variables in order to decouple the two integration variables. More precisely, our goal is to obtain an "irrelevant" variable , namely one of the two residual variables in (3.63) should give just an overall numerical factor and on practical grounds the associated integration in (3.63) should depend on s and m^2 neither at the integrand level, nor in the boundaries and so it turns to be negligible in the study of the homogeneous solution.

3.4.1 Change of Variables

We can consider the following redefinitions:

$$(z_4, z_5) \to \left(\frac{s}{2} \left(\frac{w(v + l(1+v)^2)}{v(1+v)} - 1 \right), \frac{s}{2} \left(\frac{w(v + l(1+v)^2)}{(1+v)} - 1 \right) \right), \tag{3.67}$$

where:

$$l = -\frac{m^2}{s}. \tag{3.68}$$

Thanks to (3.67), the *Baikov Polynomial* on the *Maximal Cut* reads:

$$F(v, w)\Big|_{\text{M.C.}} = -\frac{s^3(w-1)w^2 \left(l(v+1)^2 + v\right)^3}{4v^2(v+1)^2}, \quad l = -\frac{m^2}{s}. \tag{3.69}$$

For the sake of simplicity we will consider: $l > 0 \Leftrightarrow s < 0$.

We notice that in (3.69) the v-dependent term and the w-dependent one are completely factorized. In order to build-up the whole integrand we have to compute the jacobian associated to (3.67); since in each redefinition the new variables are always factorized, the same holds also for the corresponding jacobian:

$$\text{jac}(v, w) = \frac{s^2}{4} \frac{w(v + l(1+v)^2)^2}{(v(v+1))^2}. \tag{3.70}$$

So, our starting point for the integration is:

$$\text{MCut}[\frac{L_2}{m^2}] = s^{d-3} \int_\Gamma dv\, dw\; v^{2-d}\; ((v{+}1)^2)^{\frac{2-d}{2}}\; (l(v{+}1)^2{+}v)^{\frac{3d}{2}-4}\; w^{d-3}\; (1{-}w)^{\frac{d-4}{2}}, \quad l > 0. \tag{3.71}$$

We notice that we can get rid of the w-dependent term, as promised. Moreover the quadratic form $(l(v+1)^2 + v)$ can be naturally recast as:

$$\begin{aligned} (l(v+1)^2 + v) &= l \left(v - \frac{-2l - \sqrt{4l+1} - 1}{2l} \right) \left(v - \frac{-2l + \sqrt{4l+1} - 1}{2l} \right) \\ &= l\,(v - \bar{v}_-)(v - \bar{v}_+). \end{aligned} \tag{3.72}$$

Thus, (3.71) seems to impose 5 integration regions:

$$\Gamma = \bigcup_{i=1}^{5} \Gamma_i \tag{3.73}$$

with:

$$\begin{aligned}
\Gamma_1 &= \left(-\infty, \frac{-2l-\sqrt{4l+1}-1}{2l}\right); \\
\Gamma_2 &= \left(\frac{-2l-\sqrt{4l+1}-1}{2l}, -1\right); \\
\Gamma_3 &= \left(-1, \frac{-2l+\sqrt{4l+1}-1}{2l}\right); \\
\Gamma_4 &= \left(\frac{-2l+\sqrt{4l+1}-1}{2l}, 0\right); \\
\Gamma_5 &= (0, +\infty)\,.
\end{aligned} \tag{3.74}$$

3.4.2 Integration in $d = 4$ dimension

We can consider the $d = 4$ limit in (3.71) both at the integrand level and in the prefactor[6]:

$$\text{MCut}[L_2]\Big|_{\Gamma}^{d=4} = l^{-1} \int_{\Gamma} dv \frac{\left(l\left(v+\frac{1}{v}+2\right)+1\right)^2}{(v+1)^2}. \tag{3.75}$$

Unfortunately the integral is divergent at: $\{\pm\infty, -1, 0\}$. Nevertheless, we can try to integrate over: $\Gamma_2 \cup \Gamma_3$ using *integration by parts* to regularize the integral. Let's consider:

$$\text{MCut}[L_2]\Big|_{\Gamma_2\cup\Gamma_3}^{d=4} = l^{-1} \int_{\frac{-2l-\sqrt{4l+1}-1}{2l}}^{\frac{-2l+\sqrt{4l+1}-1}{2l}} dv \frac{\left(l\left(v+\frac{1}{v}+2\right)+1\right)^2}{(v+1)^2}, \tag{3.76}$$

which is divergent at $v = -1$. As stated above, we can overcome this issue using the standard *integration by parts* [7]; as a first step we notice that the *numerator* in (3.76), is vanishing at the boundaries; by virtue of this fact we can safely neglect the *surface term* obtaining:

$$\text{MCut}[L_2]\Big|_{\Gamma_1\cup\Gamma_2}^{d=4} = l^{-1} \int_{\frac{-2l-\sqrt{4l+1}-1}{2l}}^{\frac{-2l+\sqrt{4l+1}-1}{2l}} dv \frac{2l(v-1)\left(l(v+1)^2+v\right)}{v^3}. \tag{3.77}$$

Then, *no singular point* is present on the integration path. A direct integration gives:

$$\text{MCut}[L_2]\Big|_{\Gamma_2\cup\Gamma_3}^{d=4} = 2\left(\frac{l}{2v^2} + lv + \frac{l+1}{v} + (l+1)\log(v)\right)\Big|_{\frac{-2l-\sqrt{4l+1}-1}{2l}}^{\frac{-2l+\sqrt{4l+1}-1}{2l}}, \tag{3.78}$$

which is equivalent:

$$\text{MCut}[L_2]\Big|_{\Gamma_2\cup\Gamma_3}^{d=4} = l^{-1}\left(\sqrt{4l+1}(1-2l) + 2l(l+1)\log\left(\frac{2l+\sqrt{4l+1}+1}{2l-\sqrt{4l+1}+1}\right)\right). \tag{3.79}$$

As a final step we can express l in terms of x, namely: $l = -\frac{m^2}{s} = \frac{x}{(1-x)^2}$, and we obtain *one* solution for the *homogeneous* DEQ

$$\text{MCut}[L_2]\Big|_{\Gamma_2\cup\Gamma_3}^{d=4}(x) = \mathscr{H}_1^0(x) = \frac{x^4-4x^3+4\left(x^2-x+1\right)x\log(x)+4x-1}{(x-1)^2x}. \tag{3.80}$$

[6]We considered $s \to -m^2\, l^{-1}$, and we dropped the factor proportional to m^2.

[7]We use: $\int dvF(v)g(v) = F(v)G(v) - \int dvf(v)G(v)$, with $F(v) = \left(l\left(v+\frac{1}{v}+2\right)+1\right)^2$ and $g(v) = \frac{1}{(v+1)^2}$.

Moreover, once a solution for a 2nd order *homogeneous* DEQ is known, the other one can be recovered thanks to *Euler's variation of constants method.* More in detail we look for a solution:

$$\mathscr{H}_2^0(x) = \mathscr{H}_1^0(x)\,\mathscr{F}(x). \tag{3.81}$$

Plugging the latter in (3.58) we obtain a 1^{st} order DEQ for $f(x)$, being $f(x) = \frac{\partial \mathscr{F}(x)}{\partial x}$. The latter reads (after minor rearrangements):

$$\partial_x f(x) + f(x)\frac{2\left(x^5+5x^4-12x^3+4x^2+4\left(2x^3-6x^2+3x-1\right)\log(x)+11x-9\right)}{(x^2-1)\left(x^4-4x^3+4\left(x^2-x+1\right)x\log(x)+4x-1\right)} = 0, \tag{3.82}$$

and $f(x)$ is trivially determined [8]:

$$f(x) = -\frac{(1-x)^2(x+1)^4}{\left(x^4-4x^3+(4x^3-4x^2+4x)\log(x)+4x-1\right)^2}. \tag{3.83}$$

Despite its complexity, a primitive for $f(x)$ can be explicitly found:

$$\mathscr{F}(x) = \int dx f(x) = \frac{x\left(x^2-x+1\right)}{x^4-4x^3+4\left(x^2-x+1\right)x\log(x)+4x-1}, \tag{3.84}$$

and so, quite surprisingly, the other solution is:

$$\mathscr{H}_2^0(x) = \mathscr{H}_1^0(x)\mathscr{F}(x) = \frac{\left(x^2-x+1\right)}{(x-1)^2}. \tag{3.85}$$

3.4.3 Integration in d dimensions

Some additional effort is needed in order to explicitly perform the integration, in particular to recognize a primitive for (3.71) in d dimensions.
First of all let's recast (3.71) as[6]:

$$\text{MCut}[L_2]\Big|_\Gamma = l^{3-d}\int_\Gamma dv\,\frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v}, \tag{3.86}$$

and introduce the auxiliary variable p defined as:

$$p(v) = \frac{(1+v)^2}{v}. \tag{3.87}$$

The function $p(v)$ is not injective over $\mathbb{R}_{\neq 0}$; however it turns to admit an inverse function at least on: $\mathcal{I}_1 = (-\infty,-1)$, $\mathcal{I}_2 = (-1,0)$, $\mathcal{I}_3 = (0,+1)$ and $\mathcal{I}_4 = (+1,+\infty)$, being the inverse function respectively: $v_-(p)$, $v_+(p)$, $v_-(p)$ and $v_+(p)$, with:

$$v_+(p) = \frac{(p-2)+\sqrt{p(p-4)}}{2}, \quad v_-(p) = \frac{(p-2)-\sqrt{p(p-4)}}{2}. \tag{3.88}$$

Finally, (3.86) can be computed recalling the relation:

$$\frac{dv}{v} = \frac{1}{v_\pm(p)}\frac{\partial v_\pm(p)}{\partial p}dp = \frac{\partial \log(v_\pm(p))}{\partial p}dp, \tag{3.89}$$

and it results:

$$\text{MCut}[L_2]\Big|_\Gamma = l^{3-d}\int_\Gamma dp\frac{\partial \log(v_\pm(p))}{\partial p}p^{\frac{2-d}{2}}\left(l\,p+1\right)^{\frac{3d}{2}-4}, \tag{3.90}$$

where $v_\pm(p) = \{v_+(p), v_-(p)\}$ should be chosen according to the integration region. In this explicit example, the choice among $v_\pm(p) = \{v_+(p), v_-(p)\}$ produces just an overall *different sign*, which is negligible in the study of the *homogeneous differential equation.*

[8] Forgetting about a trivial integration constant.

3.4.4 Integration over Γ_1

The integration over Γ_1 reads:

$$\begin{aligned}\text{MCut}[L_2]\Big|_{\Gamma_1} &= l^{3-d}\int_{-\infty}^{\frac{-2l-\sqrt{4l+1}-1}{2l}} dv\,\frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v} = \\ &= l^{3-d}\int_{-\infty}^{-\frac{1}{l}} dp\,\frac{\partial\log(v_-(p))}{\partial p}p^{\frac{2-d}{2}}(l\,p+1)^{\frac{3d}{2}-4} = \\ &= -2^{2-d}l^{3-d}\left(\sqrt{p-4}(4l+1)^{\frac{3d}{2}-4}F_1\left(\frac{1}{2};\frac{d-1}{2},4-\frac{3d}{2};\frac{3}{2};1-\frac{p}{4},-\frac{l(p-4)}{4l+1}\right)\right|_{-\infty}^{-\frac{1}{l}};\end{aligned} \tag{3.91}$$

where F_1 is the *Appel function.* Then, assuming:

$$\lim_{p\to-\infty} l^{3-d}\sqrt{p-4}(4l+1)^{\frac{3d}{2}-4}F_1\left(\frac{1}{2};\frac{d-1}{2},4-\frac{3d}{2};\frac{3}{2};1-\frac{p}{4},-\frac{l(p-4)}{4l+1}\right)=0, \tag{3.92}$$

which holds for some value of d, we have:

$$\text{MCut}[L_2]\Big|_{\Gamma_1} = \frac{-2^{2-d}i\sqrt{\pi}l^{\frac{5}{2}-d}(4l+1)^{\frac{3d}{2}-\frac{7}{2}}\Gamma\left(\frac{3d}{2}-3\right)\,{}_2F_1\left(\frac{1}{2},\frac{d-1}{2};\frac{3d}{2}-\frac{5}{2};1+\frac{1}{4l}\right)}{2\,\Gamma\left(\frac{3d}{2}-\frac{5}{2}\right)}. \tag{3.93}$$

3.4.5 Integration over Γ_2

The integration over Γ_2 is:

$$\begin{aligned}\text{MCut}[L_2]\Big|_{\Gamma_2} &= l^{3-d}\int_{\frac{-2l-\sqrt{4l+1}-1}{2l}}^{-1} dv\,\frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v} = \\ &= l^{3-d}\int_{-\frac{1}{l}}^{0} dp\,\frac{\partial\log(v_-(p))}{\partial p}p^{\frac{2-d}{2}}(l\,p+1)^{\frac{3d}{2}-4} = \\ &= -2^{2-d}l^{3-d}\left(\sqrt{p-4}(4l+1)^{\frac{3d}{2}-4}F_1\left(\frac{1}{2};\frac{d-1}{2},4-\frac{3d}{2};\frac{3}{2};1-\frac{p}{4},-\frac{l(p-4)}{4l+1}\right)\right|_{-\frac{1}{l}}^{0}.\end{aligned} \tag{3.94}$$

The latter is equivalent to:

$$\begin{aligned}\text{MCut}[L_2]\Big|_{\Gamma_2} =& -2^{2-d}i\sqrt{\pi}\left(\frac{l^{3-d}(4l+1)^{\frac{3d}{2}-4}\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)\,{}_2F_1\left(\frac{1}{2},4-\frac{3d}{2};\frac{1-d}{2}+\frac{3}{2};\frac{4l}{4l+1}\right)}{\Gamma\left(2-\frac{d}{2}\right)}+\right.\\ &\left.-\frac{l^{\frac{5}{2}-d}(4l+1)^{\frac{3d}{2}-\frac{7}{2}}\Gamma\left(\frac{3d}{2}-3\right)\,{}_2F_1\left(\frac{1}{2},\frac{d-1}{2};\frac{3d}{2}-\frac{5}{2};1+\frac{1}{4l}\right)}{2\,\Gamma\left(\frac{3d}{2}-\frac{5}{2}\right)}\right).\end{aligned} \tag{3.95}$$

3.4.6 Integration over Γ_3

The integration over Γ_3 reads:

$$
\begin{aligned}
\mathrm{MCut}[L_2]\Big|_{\Gamma_3} &= l^{3-d}\int_{-1}^{\frac{-2l+\sqrt{4l+1}-1}{2l}} dv\, \frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v} = \\
&= l^{3-d}\int_{0}^{-\frac{1}{l}} dp\, \frac{\partial \log(v_+(p))}{\partial p} p^{\frac{2-d}{2}} (l\,p+1)^{\frac{3d}{2}-4} = \\
&= 2^{2-d} l^{3-d}\left(\sqrt{p-4}(4l+1)^{\frac{3d}{2}-4} F_1\left(\frac{1}{2}; \frac{d-1}{2}, 4-\frac{3d}{2}; \frac{3}{2}; 1-\frac{p}{4}, -\frac{l(p-4)}{4l+1}\right)\right|_{0}^{-\frac{1}{l}},
\end{aligned}
\tag{3.96}
$$

and so we argue that:

$$
\mathrm{MCut}[L_2]\Big|_{\Gamma_2} = \mathrm{MCut}[L_2]\Big|_{\Gamma_3} \tag{3.97}
$$

3.4.7 Integration over Γ_4

The integration over Γ_4 is:

$$
\begin{aligned}
\mathrm{MCut}[L_2]\Big|_{\Gamma_4} &= l^{3-d}\int_{\frac{-2l+\sqrt{4l+1}-1}{2l}}^{0} dv\, \frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v} = \\
&= l^{3-d}\int_{-\frac{1}{l}}^{-\infty} dp\, \frac{\partial \log(v_+(p))}{\partial p} p^{\frac{2-d}{2}} (l\,p+1)^{\frac{3d}{2}-4} = \\
&= -2^{2-d} l^{3-d}\left(\sqrt{p-4}(4l+1)^{\frac{3d}{2}-4} F_1\left(\frac{1}{2}; \frac{d-1}{2}, 4-\frac{3d}{2}; \frac{3}{2}; 1-\frac{p}{4}, -\frac{l(p-4)}{4l+1}\right)\right|_{-\infty}^{-\frac{1}{l}},
\end{aligned}
\tag{3.98}
$$

and so:

$$
\mathrm{MCut}[L_2]\Big|_{\Gamma_4} = \mathrm{MCut}[L_2]\Big|_{\Gamma_1}. \tag{3.99}
$$

3.4.8 Integration over Γ_5

The integration over Γ_5 reads:

$$
\begin{aligned}
\mathrm{MCut}[L_2]\Big|_{\Gamma_5} &= l^{3-d}\int_{0}^{+\infty} dv\, \frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v} = \\
&= l^{3-d}\left(\int_{0}^{+1} dv + \int_{+1}^{+\infty} dv\right) \frac{\left(\frac{(v+1)^2}{v}\right)^{\frac{2-d}{2}}\left(\frac{l(v+1)^2}{v}+1\right)^{\frac{3d}{2}-4}}{v} = \\
&= l^{3-d}\left(\int_{+\infty}^{4} dp \frac{\partial \log(v_-(p))}{\partial p} p^{\frac{2-d}{2}} (lp+1)^{\frac{3d}{2}-4} + \int_{4}^{+\infty} dp \frac{\partial \log(v_+(p))}{\partial p} p^{\frac{2-d}{2}} (lp+1)^{\frac{3d}{2}-4}\right) = \\
&= l^{3-d}\int_{4}^{+\infty} dp \frac{\partial}{\partial p} \log\left(\frac{v_+(p)}{v_-(p)}\right) p^{\frac{2-d}{2}} (lp+1)^{\frac{3d}{2}-4} = \\
&= 2^{3-d}\, l^{3-d}\left(\sqrt{p-4}(4l+1)^{\frac{3d}{2}-4} F_1\left(\frac{1}{2}; \frac{d-1}{2}, 4-\frac{3d}{2}; \frac{3}{2}; 1-\frac{p}{4}, -\frac{l(p-4)}{4l+1}\right)\right|_{4}^{+\infty}.
\end{aligned}
\tag{3.100}
$$

In the latter expression the $p \to 4$ limit is vanishing as well as the $p \to +\infty$ limit, which is vanishing for some value of d, as stated above. So we have:

$$\mathrm{MCut}[L_2]\Big|_{\Gamma_5} = 0. \tag{3.101}$$

We can verify that each term $\mathrm{MCut}[L_2]\Big|_{\Gamma_i}$, $1 \leq i \leq 5$ is a solution for (3.57). Moreover, we notice that the 2^{nd} term in $\mathrm{MCut}[L_2]\Big|_{\Gamma_2}$ precisely corresponds to $-\mathrm{MCut}[L_2]\Big|_{\Gamma_1}$; thus, we argue that each term in (3.95) is a solution for (3.57). Therefore we are free to identify ($l \to \frac{x}{(1-x)^2}$ is assumed):

$$\begin{aligned} L^{(1)\,d}_{2,\,\mathrm{h}} &= \mathrm{MCut}[L_2]\Big|_{\Gamma_1} = \\ &= -\frac{i\sqrt{\pi}2^{1-d}\left(\frac{x}{(x-1)^2}\right)^{\frac{5}{2}-d}\left(\frac{(x+1)^2}{(x-1)^2}\right)^{\frac{1}{2}(3d-7)}\Gamma\left(\frac{3d}{2}-3\right)\,{}_2F_1\left(\frac{1}{2},\frac{d-1}{2};\frac{1}{2}(3d-5);\frac{(x+1)^2}{4x}\right)}{\Gamma\left(\frac{3d}{2}-\frac{5}{2}\right)}, \end{aligned} \tag{3.102}$$

and:

$$\begin{aligned} L^{(2)\,d}_{2,\,\mathrm{h}} &= \mathrm{MCut}[L_2]\Big|_{\Gamma_1} + \mathrm{MCut}[L_2]\Big|_{\Gamma_2} = \\ &= -\frac{i\sqrt{\pi}2^{2-d}x^4\left(\frac{x}{(x-1)^2}\right)^{-d-1}\left(\frac{(x+1)^2}{(x-1)^2}\right)^{3d/2}\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)\,{}_2F_1\left(\frac{1}{2},4-\frac{3d}{2};2-\frac{d}{2};\frac{4x}{(x+1)^2}\right)}{(x+1)^8\Gamma\left(2-\frac{d}{2}\right)}. \end{aligned} \tag{3.103}$$

The latter are *two* independent solutions for (3.57), in d dimensions.

3.4.9 Limit at $d = 4$

Using the `Mathematica` package `HypExp 2` [51], we can consider the limit $d \to 4$:

$$\begin{aligned} L^{(1)\,d=4}_{2,\,\mathrm{h}} = \lim_{d\to 4} L^{(1)\,d}_{2,\,\mathrm{h}} &= -\frac{4\left(x^2-x+1\right)x\,\mathrm{HPL}\left(\{\mathrm{plus}\},\frac{x+1}{x-1}\right)+x^4-4x^3+4x-1}{2(x-1)^2x} = \\ &\overset{9}{=} -\frac{x^4-4x^3+4\left(x^2-x+1\right)x\log(-x)+4x-1}{2(x-1)^2x} = \\ &= -\frac{2i\pi\left(x^2-x+1\right)}{(x-1)^2} - \frac{x^4-4x^3+4\left(x^2-x+1\right)x\log(x)+4x-1}{2(x-1)^2x}. \end{aligned} \tag{3.104}$$

On the other hand, we have:

$$L^{(2)\,d=4}_{2,\,\mathrm{h}} = \lim_{d\to 4} L^{2,\,d}_{(2)\,\mathrm{h}} = -\frac{2i\pi\left(x^2-x+1\right)}{(x-1)^2}. \tag{3.105}$$

We can verify that (3.104) and (3.105) are (independent) solutions for 3.58).

[9] Using the `Mathematica` package `HPL` [50], and the function `HPLConvertToKnownFunction` implemented therein.

3.5 Magnus Matrix from Cut Integrals

In this Section we will build the matrix $\mathbb{T}(x)$:

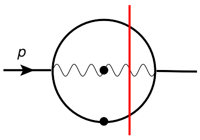
$$\mathbb{T}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F^{(1)\,\epsilon=0}_{2,\,\mathrm{h}} & F^{(2)\,\epsilon=0}_{2,\,\mathrm{h}} \\ 0 & F^{(1)\,\epsilon=0}_{3,\,\mathrm{h}} & F^{(2)\,\epsilon=0}_{3,\,\mathrm{h}} \end{pmatrix}, \tag{3.106}$$

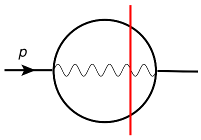
where each column in (3.106) is a solution for the system (3.3) at $\epsilon = 0$. The matrix $\mathbb{T}(x)$ is such that:

$$\frac{\partial \mathbb{T}(x)}{\partial x} = \mathbb{A}_0(x)\mathbb{T}(x), \tag{3.107}$$

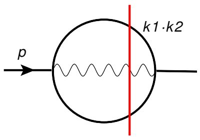
which is exactly the DEQ solved by the *Magnus Exponential.*

First of all, we can read $F^{\epsilon}_{2,\mathrm{h}}$ through IBPs *"on the Cut"*, namely:

$F^{\epsilon}_{2,\,\mathrm{h}} := m^2$ 

$= A(x)\, m^2$ 

$+$

$+\, B(x)$ 

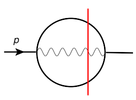
$,$ (3.108)

where:

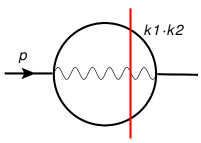
$$A(x) = x(2\epsilon - 1)\left[\frac{(1 + 12x - 105x^2 + 200x^3 - 105x^4 + 12x^5 + x^6)\epsilon}{2(x-1)^2(x+1)^6} + \right.$$
$$\left. - \frac{\left(x^6 + 4x^5 - 89x^4 + 176x^3 - 89x^2 + 4x + 1\right)}{2(x-1)^2(x+1)^6} - \frac{12(x-1)^2x^2}{2(x-1)^2(x+1)^6\epsilon}\right], \tag{3.109}$$

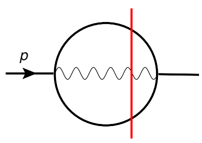
and:

$$B(x) = 3x^2(2\epsilon - 1)\left[\frac{\left(x^4 - 20x^3 + 30x^2 - 20x + 1\right)\epsilon}{(x-1)^2(x+1)^6} - \frac{\left(x^4 - 24x^3 + 34x^2 - 24x + 1\right)}{(x-1)^2(x+1)^6} + \right.$$
$$\left. - \frac{4x(1 - x + x^2)}{(x-1)^2(x+1)^6\epsilon}\right]. \tag{3.110}$$

On the one hand, *two* solutions for $L^{\epsilon}_{2,\mathrm{h}} = m^2$ 

, were computed in the previous Section (3.102) and (3.103), namely $L^{(1)\,\epsilon}_{2,\,\mathrm{h}}$ and $L^{(2)\,\epsilon}_{2,\,\mathrm{h}}$; then, the corresponding *homogeneous solutions* for L_3, namely $L^{(i)\,\epsilon}_{3,\,\mathrm{h}}$, $i = 1, 2$ can be computed inverting the *homogeneous* DEQ for L_2:

$L^{\epsilon}_{3,\,\mathrm{h}} =$ 

$= \left[\frac{\left(x^2 - 4x + 1\right)\epsilon + 4x}{6x(\epsilon - 1)} + \frac{x^2 - 1}{6(\epsilon - 1)}\partial_x\right] m^2$ 

(3.111)

Thus, given *two* pairs of solutions: $\{L_{2,\,\mathrm{h}}^{(i)\,\epsilon}, L_{3,\,\mathrm{h}}^{(i)\,\epsilon}\}$, $i = 1, 2$, by means of (3.108) we obtain $F_{2,\,\mathrm{h}}^{(i)\,\epsilon}$, $i = 1, 2$.
Considering the leading coefficient in the Laurent Expansion for each of them, we obtain *two* solutions for the *homogeneous* DEQ (3.33), namely $F_{2,\,\mathrm{h}}^{(i)\,\epsilon=0}$, $i = 1, 2$. We explicitly find:

$$F_{2,\,\mathrm{h}}^{(1)\,\epsilon=0} = \frac{x}{x^2-1}, \quad F_{2,\,\mathrm{h}}^{(2)\,\epsilon=0} = \frac{x(2x-1)}{(x-1)^2(x+1)}. \tag{3.112}$$

Finally, simply inverting the DEQ for F_2 at $\epsilon = 0$, and given $F_{2,\,\mathrm{h}}^{(i)\,\epsilon=0}$, $i = 1, 2$, we obtain the corresponding solutions $F_{3,\,\mathrm{h}}^{(i)\,\epsilon=0}$, $i = 1, 2$:

$$F_{3,\,\mathrm{h}}^{(i)\,\epsilon} = \left[\frac{x^2+1}{x} + (x^2-1)\partial_x\right] F_{2,\,\mathrm{h}}^{(i)\,\epsilon}, \quad i = 1, 2. \tag{3.113}$$

Thus, we explicitly reads:

$$F_{3,\,\mathrm{h}}^{(1)\,\epsilon=0} = 0, \quad F_{3,\,\mathrm{h}}^{(2)\,\epsilon=0} = -\frac{x}{(x-1)^2}, \tag{3.114}$$

and (3.106) reads:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{x^2-1} & \frac{x(2x-1)}{(x-1)^2(x+1)} \\ 0 & 0 & -\frac{x}{(x-1)^2} \end{pmatrix} \tag{3.115}$$

In conclusion, the matrix $\mathbb{T}(x)$ defined just above, and the one obtained through the algorithm based on the Magnus Exponential (3.16), namely $\mathbb{R}_{\mathrm{Magnus}}(x)$, are related by:

$$\mathbb{R}_{\mathrm{Magnus}}(x) = \mathbb{T}(x)\,\mathbb{D}, \quad \mathbb{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.116}$$

Conclusions

In this Thesis we discussed and applied some of the novel ideas concerning Feynman Integrals. In the first part of this work, we focused on the main source of relations among integrals, namely Lorentz invariance identities, Sector Symmetries and especially Integration by Parts Identities. Studying the latter in *Baikov representation*, we are naturally led to consider *highly non trivial* concepts regarding *Polynomials* and their manipulations, especially coming from *Algebraic Geometry*, such as *Sygyzies Equations.* Inspired by this, we developed a `Mathematica` code devoted to IBPs generation in Baikov representation. Furthermore we independently derive and apply algorithm for Sygyzies involving determinants, and we present a new method to address this issue based on Eulers theorem on *homogeneous functions.*
In the second part we studied the method of Differential Equations for Feynman Integrals; the latter fulfill suitable System of Differential Equations and thus Feynman Integrals are obtained solving this system, thus avoiding a direct integration over loop momenta. We focused on the *Canonical Form* for Differential Equations, and the algorithm based on the Magnus Exponential to obtain the latter. Following this strategy, Algebra naturally emerges: the Magnus Exponential involves nested commutators, and the algorithm produces a set of Matrix similarity transformations. Moreover, once the system is recast in the Canonical Form, the solution can obtained via the *Dyson Series*, keeping the unknown Integrals tight together in a single vector. This approach remarkably simplifies the solution, both from a conceptual and a practical point of view. We applied these techniques to a few known examples, one of them concerning the 1-loop box topology; moreover we obtained the Canonical Form for a *non-planar* 2-loops 3-points Integral Family whose integrals are a subset of the ones, currently unknown, needed for a *non-planar* 2-loops graph which contributes to $q\bar{q} \to t\bar{t}$ process.
In the last part we exploit Cut Integrals, and their role as solutions for *homogeneous* Differential Equations, working on an explicit example, namely the so-called *QED Sunrise* Integral Family. Remarkably the Baikov representation turns to be a very powerful tool: on the one hand Cut Integrals are naturally implementable within this representation, and, on the other hand, it allows to identify *different*, IBP-compatible, integration regions, which provide the *whole* set of solutions for a higher order Differential Equation. Moreover we showed the relation between the solution given by the Magnus algorithm and the homogeneous solutions obtained by means of Cut Integrals.
Concluding, we would like to mention some possible developments and future prospects related to the work presented in this Thesis:

- a robust implementation of a system solver for the IBPs system. As we sketched in the First Chapter solving such a huge system is very challenging from a computational point of view; in this sense, approaches based on the *functional reconstruction*, namely the reconstruction of the analytic shape of a function from multiple numerical evaluations, seem to be very promising in order to avoid the large intermediate expressions. Moreover other strategies, which rely on the *a priori* knowledge of the *number* of Mis, as advocated in [73], could simplify the task;

- the evaluation of the MIs identified in Section (2.9); the general solution is really at hand. We plan to fix the *Boundary Conditions* along the lines suggested in Subsection (2.6.1);
- for the time being, the algorithm based on the Magnus Exponential produces the desired transformation matrix only if the series has a finite number of terms (and this was the case in many many examples); however a counterexample is given by the *Massive Sunrise* [74], which evaluates to elliptic integrals; it would be interesting to investigate how to obtain a *Resummation* for such an infinite series and, the relations with Cut Integrals goes in this direction.

The methods presented in this Thesis are important, or even fundamental, in several branches of Physics. These and related topics are mandatory, in order to study processes involving massive particles, like heavy quarks, electroweak bosons and Higgs Boson. Beyond Phenomenology, they can be used to reveal new unexpected relations concerning the underlying framework of Scattering Amplitudes. On top of that, we believe that they can contribute to develop new ideas in other Fields, namely Algebraic Geometry, Number Theory and Computing, just to mention a few, triggering a virtuous cycle between them.

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Appendix A

Matrices for 2-loop non planar vertex

We present here the matrices obtained along the lines described in (2.9), namely the matrices $\mathbb{A}_0(s)$ and $\mathbb{A}_1(s)$ which form the DEQs system *linear* in ϵ: $\mathbb{A}(s,\epsilon) = \mathbb{A}_0(s) + \epsilon\mathbb{A}_1(s)$. Moreover we present the matrices: $\mathbb{B}_1(s), \mathbb{B}_2(s)$ and $\mathbb{B}_3(m^2)$ needed to obtain the Canonical Form.

$$
\mathbb{A}_0(s) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{m^2}{4m^2 s-s^2} & \frac{s-2m^2}{s(4m^2-s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2m^2}{4m^2 s-s^2} & 0 & 0 & 0 & 0 & \frac{s-2m^2}{s(4m^2-s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s-2m^2}{s(4m^2-s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{4m^2 s-s^2} & 0 & 0 & 0 & 0 & \frac{2}{4m^2 s-s^2} & -\frac{2}{4m^2 s-s^2} & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{8m^4 s-6m^2 s^2+s^3} & 0 & -\frac{6m^2}{8m^4 s-6m^2 s^2+s^3} & \frac{3}{8m^4-6m^2 s+s^2} & 0 & 0 & 0 & 0 & 0 & -\frac{4m^4}{8m^4 s-6m^2 s^2+s^3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{8m^2 s-2s^2} & 0 & 0 & 0 & 0 & \frac{s-2m^2}{s(4m^2-s)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s-2m^2}{s(4m^2-s)} & 0 & 0 & 0 & 0 \\
-\frac{2}{3s(8m^4-6m^2 s+s^2)} & 0 & -\frac{4m^2}{8m^4 s-6m^2 s^2+s^3} & \frac{2m^2+5s}{24m^4 s-18m^2 s^2+3s^3} & \frac{2}{12m^2 s-3s^2} & 0 & 0 & \frac{2}{4m^2 s-s^2} & 0 & -\frac{8m^4}{3s(8m^4-6m^2 s+s^2)} & 0 & \frac{4}{s(s-4m^2)} & -\frac{1}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s-2m^2}{s(4m^2-s)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{8m^2 s-2s^2} & 0 & 0 & 0 & 0 & \frac{2m^2}{12m^2 s-3s^2} & 0 & 0 & \frac{4}{4m^2 s-s^2} & -\frac{1}{s} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\left(s-3m^2\right)}{s(s-4m^2)}
\end{pmatrix}. \tag{A.1}
$$

$$
\mathbb{A}_1(s) = \left(\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4m^2s-s^2} & 0 & 0 & \frac{2m^2+s}{s\left(s-4m^2\right)} & -\frac{3}{s-4m^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{s} & \frac{2\left(s-2m^2\right)}{(2\mathrm{ep}-1)s\left(s-4m^2\right)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{s} & -\frac{4m^2}{4m^2s-s^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{4}{s\left(s-4m^2\right)} & \frac{6}{4m^2s-s^2} & 0 & 0 & 0 & \frac{8m^2-s}{4m^4s-m^2s^2} & -\frac{2}{4m^2s-s^2} & \frac{s-8m^2}{s\left(4m^2-s\right)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{s-2m^2}{m^2s\left(4m^2-s\right)} & -\frac{4m^2+s}{4m^4s-m^2s^2} & 0 & 0 & 0 & 0 & \frac{2\left(s-2m^2\right)}{s\left(s-4m^2\right)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{4m^2s-s^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s-8m^2}{s\left(4m^2-s\right)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 \\
\frac{1}{4m^4s-m^2s^2} & \frac{1}{4m^4-m^2s} & 0 & -\frac{2m^2+s}{4m^4s-m^2s^2} & -\frac{1}{m^2s} & 0 & 0 & \frac{2}{4m^2s-s^2} & 0 & 0 & 0 & 0 & -\frac{4m^2}{4m^2s-s^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4m^2}{4m^2s-s^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8m^2}{4m^2s-s^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{4m^2s-s^2} & 0 & \frac{2}{s\left(s-4m^2\right)} & 0 & -\frac{2\left(s-6m^2\right)}{s\left(s-4m^2\right)}
\end{array}\right)
\tag{A.2}
$$

$$
\mathbb{B}_1(s)\mathbb{B}_2(s) = \left(\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2s} & \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2m^2 s} & 0 & 0 & 0 & 0 & \frac{1}{m^2\sqrt{s}\sqrt{s-4m^2}} & -\frac{1}{m^2\sqrt{s}\sqrt{s-4m^2}} & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{s-2m^2}{8m^6-4m^4 s} & 0 & \frac{3\left(s-2m^2\right)}{4m^4-2m^2 s} & \frac{3}{4m^4} & 0 & 0 & 0 & 0 & 0 & \frac{s-2m^2}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3}{4m^2 s} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & 0 & 0 \\
-\frac{1}{6m^2 s} & 0 & -\frac{1}{s} & \frac{5}{12m^2 s} & \frac{1}{3m^2\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & \frac{1}{m^2\sqrt{s}\sqrt{s-4m^2}} & 0 & \frac{4m^2}{3\sqrt{s}\sqrt{s-4m^2}} & 0 & -\frac{2}{m^2\sqrt{s}\sqrt{s-4m^2}} & \frac{1}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{8m^2 s} & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{s}\sqrt{s-4m^2}} & 0 & 0 & \frac{2}{m^2\sqrt{s}\sqrt{s-4m^2}} & \frac{1}{s} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s^{3/2}\sqrt{s-4m^2}}
\end{array}\right)
\tag{A.3}
$$

$$\mathbb{B}_3(m^2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{m^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m^4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{A.4}$$

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