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Sharing nonlocality between sequential observers by
means of projective measurements

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Abstract

Weak measurements are considered fundamental for sharing the nonlocality of an entangled two-qubit state between several sequential observers. In this thesis work, we show that this is not necessarily true. Indeed it is possible to share the nonlocality using only standard projective measurements, without the need for any quantum ancilla. We will first show that two sequential observers can both violate the CHSH inequality when the initial state is a maximally entangled two-qubit one and all the observers are allowed to share classical randomness. Afterward, we will determine the optimal trade-off relation between the CHSH parameters in the same scenario. We also show that two sequential violations can be reached without the need of sharing classical randomness, but using only projective measurements and local randomness. Secondly, we will study what happens if the initial state is non-maximally entangled and we will show that not only it is always possible to have two sequential violations with a partially entangled state, but that in some cases these states make larger sequential violations. Lastly, we prove that it is also possible for three sequential observers to violate the CHSH inequality. These results show that standard, projective, measurements are a simple and useful resource for sharing quantum nonlocality between sequential observers.

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Preface

This thesis contains the main results obtained during my thesis internship at IQOQI and TU-Wien. The work has been carried out in the Huber group under the supervision of Prof. Marcus Huber and Dr. Armin Tavakoli. The results presented in this thesis has also been published in the article "Projective measurements are sufficient for recycling nonlocality" [1], recently accepted by Physical Review Letters.

Chapter 1

Introduction

Since the beginning of the 20th century Quantum Mechanics has changed the way we conceive the world revealing new interesting phenomena and leading to the research for new models and theories to describe them. Among the various new fields born thanks to the discovery of Quantum Mechanics we can acknowledge Quantum Information theory. Information theory is the field that studies how information can be quantified, stored, and processed. Its foundations can be traced back to 1948, when C. E. Shannon published "*A mathematical theory of communication*" [2] and has gained popularity ever since. The study of this field has deeply affected our lives allowing for the development of many tools we use daily. Quantum Information theory, on the other hand, proposes this study when the system considered is quantum mechanical. Indeed by considering a quantum system to carry information, researchers have introduced plenty of new protocols both for computing and for cryptography. While quantum computing is still at an early stage, quantum cryptography implementations are being achieved more and more often. Although there are still some technological limitations, such as the distance limit at which a quantum key can be efficiently broadcast with current technologies, the birth of several companies proposing quantum cryptography devices makes it reasonable that in the near future quantum cryptography will become more and more used.

In this thesis, we are going to study one of the most peculiar characteristics of quantum mechanics: non-locality. As we will see more in detail in the following sections, non-locality is indeed a useful resource to be used in quantum cryptography systems. When performing quantum cryptography protocols, it is important to consider the fact that the state and the devices used might be not reliable, i.e. they can be imperfect and behave differently from what the user wishes. To overcome this problem the idea of device-independent quantum cryptography has been born. In this scenario the user does not trust the used devices, seeing them as black boxes. This way the security proof of the quantum protocol will be independent of the device used. With this aim, in 1998 Mayers and Yao [3] proposed the concept of self-testing quantum apparatus. The idea is that the user does not assume the apparatus to be trusted in any way and that its properties are uniquely determined on the basis of their input-output statistics. A particularly useful test for self-testing the honesty of a device is the Bell test. To perform this kind of test, non-locality is

a fundamental resource.

In this scenario, we are going to study the possibility to share a non-local correlation between sequential observers sharing a two-qubit entangled state. In particular, we are going to propose a new way of performing this task in contrast with the standard method widely found in the literature. Indeed the standard procedure involves the use of sophisticated measurements, namely positive operator-valued measures (POVMs), whose implementation typically requires additional ancillary qubits. In our protocol instead, all the observers will be allowed to use only standard projective measurements which are much easier to implement. In this work we are first going to prove the validity of this protocol and study its limits in the case there are only three observers. Moreover, we are going to study what happens if the initial shared state is a general two-qubit pure state instead of a maximally entangled one. Finally, we are going to present an example showing the possibility to use it also when the observers are four. The importance of this work is firstly to point out that the common knowledge that POVMs are necessary to share non-locality between sequential observers is wrong. In addition to this conceptual statement, this thesis proposes a new easier way to perform this task.

In the following sections, we are going to introduce some useful mathematical tools used in quantum information as well as concepts like Bell inequalities and in particular the Clauser-Horne-ShimonyHolt (CHSH) inequality. Afterward, we are going to introduce the idea of sharing non-locality between sequential observers. Finally, in the following chapters, we are going to the main part of the thesis, in which we analyze and discuss what happens when the observers can only perform standard projective measurements.

1.1 Tools and concepts for Quantum Information

In this section, we are going to present some useful tools and concepts used when dealing with quantum information. In particular, we are introducing the qubit and the way to represent it, the idea of non-locality, Bell inequalities with a particular focus on the CHSH inequality, and finally quantum measurements.

This introductory section does not pretend to describe exhaustively all the topics presented, which lie outside the aim of this thesis work. We are indeed trying only to give an idea of the objects we are going to talk about and use.

1.1.1 Qubits

In classical information theory, the basic unit of information is the bit. It represents a logical state assuming one of two possible values, namely 0 and 1. Similarly, in quantum information theory the basic unit is the qubit (quantum bit). It represents the state of a system, such as a two-level system, belonging to the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$, having as basis vectors $|0\rangle$ and $|1\rangle$. The peculiar characteristic of the qubit is that it is not bound to be only in the basis states, it can be a linear combination of them. In this case, we say that the qubit is in a superposition. The qubit state $|\psi\rangle$ can be thus generally written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad (1.1)$$

where α and β are complex coefficients satisfying the normalization constraint $|\alpha|^2 + |\beta|^2 = 1$. Since quantum states are defined up to a global phase, it is possible to impose the condition $\alpha \in \mathbb{R}$ and rewrite the state as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \quad (1.2)$$

where θ and ϕ are real numbers. We can interpret these angles as spherical coordinates and identify the point $\vec{a} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ on the Bloch sphere. This is a unit sphere in \mathbb{R}^3 having the computational basis vectors $\{|0\rangle, |1\rangle\}$ as the z-axis, as x-axis the vectors $\{|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}$ and $\{|R\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}}, |L\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}\}$ for the y-axis. The vector \vec{a} is the Bloch vector associated with the state $|\psi\rangle$. In figure 1.1 we report a representation of the Bloch sphere.

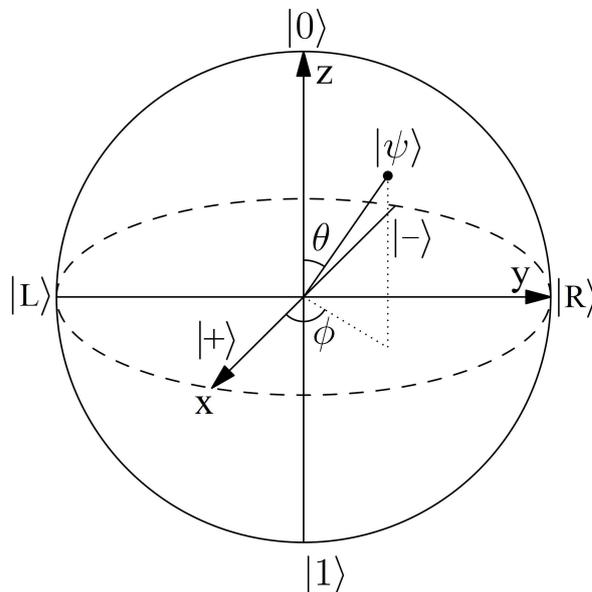


Figure 1.1: Bloch sphere representation.

We are also going to represent qubits with the matrix representation. In this representation, we associate the computational basis vectors with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.3)$$

which are the eigenvectors of the Pauli matrix σ_z .

The states we have used so far are only pure states. To talk about more general states, namely mixed states, it is necessary to introduce the density matrix formalism. In this formalism, the state can be written as

$$\rho = \frac{1}{2} (\mathbb{1} + \vec{r} \cdot \vec{\sigma}) = \frac{1}{2} (\mathbb{1} + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z) \quad (1.4)$$

where $\mathbb{1}$ is the identity matrix and σ_x , σ_y and σ_z are the Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.5)$$

The real vector $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector of the state and it represents the state in the Bloch sphere. Vectors with $\|\vec{r}\| = 1$ represent pure states. Mixed states are represented by vectors with $\|\vec{r}\| < 1$, thus inside the sphere.

1.1.2 Composite systems and entanglement

To introduce the representation of the state of a composite quantum system we can start again by comparing it with its classical counterpart. In the classical case, when we consider a system composed of n basic systems we simply obtain a n -bit string of 0s and 1s. In the quantum case, the state of a composite system belongs to the Hilbert space obtained as the tensor product of the sub-system Hilbert spaces. Considering n qubits each in a Hilbert space $\mathcal{H} \cong \mathbb{C}^2$, the state of the total system $|\Psi\rangle$ belongs to

$$\mathcal{H}_{\text{TOT}} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}}_{n \text{ times}} = \bigotimes_n \mathcal{H} \cong \mathbb{C}^{2^n} \quad (1.6)$$

Thus, the state of the composite system is represented by a $2n$ -component complex vector. If the total state can be written as the tensor product of the states of the sub-systems, the state is called *separable*. For example, if we consider a 2-qubit system, the total state $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, with $|\psi_1\rangle$ and $|\psi_2\rangle$ states of the two single qubits, is separable. When it is not possible to write the total state in this way, the state is not separable, it is *entangled*. For example, if we consider again a 2-qubit system, the state $\Psi = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is entangled.

Entanglement highlights the non-local nature of quantum theory. If for example we consider the 2-qubit entangled state $\Psi = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ and we measure the value of the first qubit, the outcome will be either 0 or 1 and this will also determine the value of the second qubit.

This quantum theory result does not satisfy the principle of locality stating that an object is directly influenced only by what happens in its immediate surroundings. For this reason, Einstein, Podolski and Rosen published the 1935 paper "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [4] considering the theory of quantum mechanics theory incomplete due to this violation of the locality principle. Indeed, they argued that, since information can not travel faster than light, it is impossible that an action taken on a particle, such as measuring the first qubit of Ψ , could affect instantaneously another particle, such as the second qubit of Ψ . They stated that "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.". From this, they stated that the second particle must already have a precise value before the measurement. The reason the observer does not know this value is due to his lack of knowledge of some "hidden variable" not

included in the theory. Theories trying to explain quantum mechanics results by including hidden variables are referred to as local hidden variable theories.

1.1.3 Bell theorem and CHSH inequality

In 1964, John Stewart Bell published "On the Einstein Podolsky Rosen paradox" [5] where he mathematically proved that the correlations between measurement outcomes predicted by quantum mechanics are incompatible with local hidden variable theories. Moreover, he stated that "If [a hidden-variable theory] is local it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local." [6], thus a non-local structure is an essential characteristic of any theory reproducing the predictions of quantum mechanics.

In particular, Bell considered the problem in which two independent observers perform measurements on two separated entangled particles and proved mathematically that, if the outcome of these measurements depends on some local hidden variables, the correlation between the outcomes is upper bounded. On the other hand, if we take into consideration quantum mechanics, the correlations can violate this inequality. This result is called Bell's theorem and the inequality is called Bell inequality. Beginning with this proof, several other versions of Bell inequality have been found. The first experiment showing a violation of a Bell inequality was accomplished by Freedman and Clauser [7]. Afterward, many other experiments has been carried out to test Bell's theorem, confirming that entangled states can violate Bell's inequalities. Between them we cite the experiment by Aspect, Dalibard and Roger [8].

In this thesis, we are going to consider the Clauser-Horne-Shimony-Holt (CHSH)[9] inequality. To introduce this inequality we consider the Bell experiment schematically reported in 1.2.

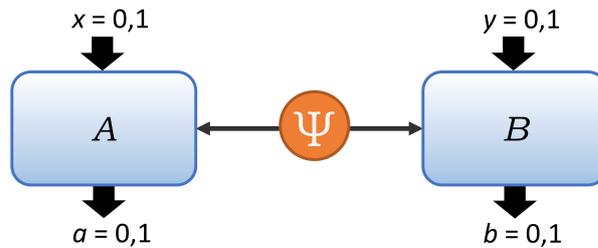


Figure 1.2: CHSH scenario - Alice (A) and Bob (B) share the two-qubit entangled state Ψ . Depending on their input value x and y , they obtain the outputs a and b , respectively.

In this experiment, a two-qubit entangled state is prepared. Each of the two qubits is sent to an independent observer. These observers, referred to as Alice and Bob, can choose one between two local measurements to perform on their half of the entangled state. In particular, they choose between two input values $x \in \{0, 1\}$ and $y \in \{0, 1\}$ for Alice and Bob respectively, and perform the corresponding measure. Thus, we are denoting as A_x and B_y Alice's and Bob's observables. The measurement outcome

are labelled as $a \in \{0, 1\}$ for Alice and $b \in \{0, 1\}$ for Bob. The CHSH parameter can be defined as

$$S = \sum_{x,y=0,1} (-1)^{xy} \langle A_x B_y \rangle = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \quad (1.7)$$

where with $\langle \cdot \rangle$ we indicate the expectation value.

If the system can be described by a local hidden variable theory, we can write the CHSH inequality $S \leq 2$. On the other hand, if we consider quantum mechanics the CHSH parameter can reach the Tsirelson's bound $S \leq 2\sqrt{2}$. This boundary can be reached when Alice measures along two orthonormal basis vectors and Bob along the diagonal directions. For example, for the state $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ Alice can measure $A_0 = \sigma_x$ and $A_1 = \sigma_z$, while Bob $B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}$ and $B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$. In the quantum case, we can rewrite the CHSH parameter as

$$S = \text{Tr}[\rho((A_0 + A_1) \otimes B_0)] + \text{Tr}[\rho((A_0 - A_1) \otimes B_1)] \quad (1.8)$$

where ρ is the state shared between Alice and Bob and Tr is the trace operator. These inequalities are fundamental for self-testing devices. Indeed when obtaining a CHSH inequality violation, we can deduce some properties of the state and the measurements that produced that violation. For example, if we reach the Tsirelson's bound we can affirm that the measurements are like the ones just reported, up to a global rotation.

1.1.4 Weak measurements

When an observer measures a quantum state it perturbs it. Weak measurements are a means for an observer to gain some partial information on a system while disturbing it only a bit. There will be a trade-off relation between how much information can be extracted from a system and how much the system is disturbed. On one hand, the weaker the measurement, the less information can be extracted, but the less the system is perturbed. The weakest measurement is the trivial measurement in which the observer simply does not perform any measurement, gaining no information, but leaving the state unperturbed. On the other hand, the stronger the measurement the more information can be obtained and the system is perturbed. The strongest measurement possible is the projective measurement, in which we obtain all the information of the state, but we project it on the measurement operator eigenspace. Practically, to perform this kind of measurement, it is necessary to weakly couple the system with the measurement device (usually called ancilla). The information will be obtained by measuring the ancilla.

Mathematically this kind of measurement can be described by a positive operator valued measure (POVM) $\{M_i\}$. If the dimension of the Hilbert space they are acting on is finite and the number of elements in the POVM is n , the POVM is simply a set of n positive semi-definite Hermitian matrices satisfying the completeness relation $\sum_{i=1}^n M_i = \mathbb{1}$. Each element M_i of the set is associated with the measurement outcome i . When performing a measurement on the state ρ , the probability of obtaining the outcome i is $P_i = \text{Tr}[\rho M_i]$.

In the case of two-outcome measurements, a POVM simply becomes an ordered pair of positive semi-definite matrices (M_0, M_1) such that $M_0 + M_1 = \mathbb{1}$.

Given an observable O having only two possible values (for simplicity we can refer to them as 0 and 1), we can write the POVM associated with it as $\{M_i\}_{i=0,1}$ such that $M_0 - M_1 = O$. This relation together with the completeness relation allows us to write $M_0 = \frac{\mathbb{1}+O}{2}$ and $M_1 = \frac{\mathbb{1}-O}{2}$, thus $M_i = \frac{\mathbb{1}+(-1)^i O}{2}$.

A more complete description of weak measurements can be found in [10].

1.1.5 Quantum channels and Kraus operators

A quantum channel is a completely positive, trace-preserving map that transforms a state (represented by its density matrix) into another state, $\Phi : \rho \rightarrow \rho'$. To describe the action of this map it is possible to use the Kraus operators K_i :

$$\rho' = \Phi(\rho) = \sum_i K_i \rho K_i^\dagger \quad (1.9)$$

For the channel to be trace-preserving, the Kraus operators must satisfy the condition $\sum_i K_i K_i^\dagger = \mathbb{1}$. If we consider the POVM $\{M_i\}$, the Kraus operators representing the quantum instrument are such that $M_i = K_i^\dagger K_i$. As long as the operators satisfy the previous relations, they do not have a univocal definition, thus we can use the Lüders rule and choose $K_i = \sqrt{M_i}$ [11].

In this thesis, we are going to consider projective measurements, which satisfy the additional property $M_i^2 = M_i$. In this case $K_i = \sqrt{M_i} = U_i M_i$ where U_i is an arbitrary unitary operator.

1.2 Sequential CHSH scenario

Up to this point, we only considered only two observers: Alice and Bob. In this section, we are going to introduce a more general problem that involves more observers on one of the two sides.

Since the work of Silva *et. al.*, [12], extensive research has been conducted on whether it is possible to re-use the post-measurement state of a Bell experiment to share nonlocality between several sequential observers. As in the standard scenario, the maximally entangled two-qubit state is shared between two initial observers Alice and Bob⁽¹⁾. They can both perform local measurements on their half of the state. Afterward, Bob⁽¹⁾ can relay his post-measurement state to another independent observer Bob⁽²⁾ who can also perform a local measurement, relay the post-measurement state to another observer, and so on. Each Bob has his own input $y_i \in \{0, 1\}$ and output $b_i \in \{0, 1\}$. This scenario is schematically reported in figure 1.3.

Lots of research has been carried out on whether it is possible for each of these Bob⁽ⁱ⁾ to violate the CHSH inequality together with Alice both in theory [13–23] and experiment [24–28]. A brilliant result recently obtained by Colbeck and Brown [29] is that an arbitrary number of Bob can achieve a CHSH violation with Alice.

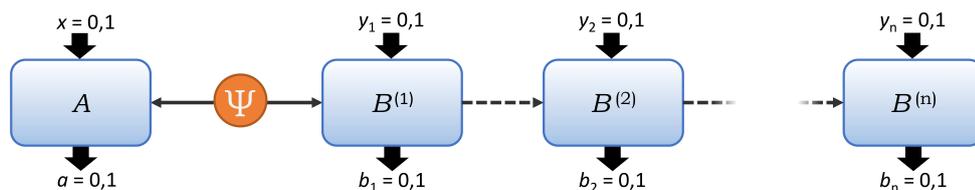


Figure 1.3: Sequential CHSH scenario - Alice (A) and a first Bob ($B^{(1)}$) share the two-qubit entangled state Ψ . They both measure their half of the state. After his measurement $B^{(1)}$, relays the post-measurement state to $B^{(2)}$, who measures it and so on.

To perform such a task, weak measurements are considered to be fundamental. Indeed a projective measurement leaves the state separable, making it impossible for other subsequent observers to violate the CHSH inequality. On the other hand, by interacting weakly with the state, it is possible to tune the amount of non-locality used to violate the CHSH inequality and the amount left for the other observers.

Using the notions introduced in 1.1, we are now going to write the CHSH parameter for each Bob.

Consider the set two-outcome observables $\{A_x, B_{y_1}^{(1)}, \dots, B_{y_n}^{(n)}\}$ with $x = \{0, 1\}$, $y_i = \{0, 1\}$ for $i = 1, \dots, n$. The CHSH parameter S_k between Alice and Bob $^{(k)}$ is

$$S_k = \text{Tr} \left[\rho^{(k)} \left((A_0 + A_1) \otimes B_0^{(k)} \right) \right] + \text{Tr} \left[\rho^{(k)} \left((A_0 - A_1) \otimes B_1^{(k)} \right) \right] \quad (1.10)$$

where $\rho^{(k)}$ is the state received by Bob $^{(k)}$.

To write this state we need to write the Kraus operators associated with the previous measurements. Each observable is associated with the measurement operators $A_{a|x}$ and $B_{b_k|y_k}^{(k)}$, for Alice and Bob $^{(k)}$ respectively. We denote as $K_{b_k|y_k}^{(k)}$ the Kraus operators that represent the instrument used by Bob $^{(k)}$ to realise the measurement $B_{b_k|y_k}^{(k)} = \left(K_{b_k|y_k}^{(k)} \right)^\dagger K_{b_k|y_k}^{(k)}$.

Since each Bob acts independently, Bob $^{(k+1)}$ is ignorant of the input and output values of Bob $^{(k)}$, y_k and b_k . Then state shared between Alice and Bob $^{(k+1)}$ is Bob $^{(k)}$'s post-measurement state averaged over y_k and b_k . The state shared by Alice and Bob $^{(k+1)}$ can be obtained by the recursive relation

$$\rho^{(k+1)} = \frac{1}{2} \sum_{b_k, y_k=0,1} \left(\mathbb{1} \otimes K_{b_k|y_k}^{(k)} \right) \rho^{(k)} \left(\mathbb{1} \otimes K_{b_k|y_k}^{(k)} \right)^\dagger \quad (1.11)$$

In chapters 2 and 3, we will use a slightly different notation for simplicity. Since we are going to consider only two observers other than Alice, we are going to call them Bob and Charlie. Bob's and Charlie's measurements, corresponding to the observables B_y and C_z with $y, z = \{0, 1\}$, will be noted simply as $B_{b|y}$ and $C_{c|z}$.

With this notation, we can rewrite equation (1.11) as

$$\rho_{AC} = \frac{1}{2} \sum_{b,y=0,1} (\mathbb{1} \otimes K_{b|y}) \rho_{AB} (\mathbb{1} \otimes K_{b|y})^\dagger \quad (1.12)$$

highlighting that ρ_{AB} is the state shared between Alice and Bob, while ρ_{AC} between Alice and Charlie.

As for the CHSH parameters, we can rewrite them as:

$$S_{AB} = \text{Tr}[\rho_{AB}((A_0 + A_1) \otimes B_0)] + \text{Tr}[\rho_{AB}((A_0 - A_1) \otimes B_1)] \quad (1.13)$$

$$S_{AC} = \text{Tr}[\rho_{AC}((A_0 + A_1) \otimes C_0)] + \text{Tr}[\rho_{AC}((A_0 - A_1) \otimes C_1)] \quad (1.14)$$

Chapter 2

Double violation

In this chapter, we will show that it is possible to have two sequential violations of the CHSH inequality between independent parties measuring one half of a two-qubit state using exclusively projective measurements.

First of all, we can observe that in the case of a single qubit, the only possible projective measurements are either basis measurements i.e. measurements in the direction of a certain Bloch vector $\{B_{0|y}, B_{1|y}\} = \{|\vec{v}\rangle, |-\vec{v}\rangle\}$, corresponding to a rank-1 projection, or trivial identity projections $\{B_{0|y}, B_{1|y}\} = \{\mathbb{1}, 0\}$, for which the measurement outcome does not depend on the state. When these measures are performed on one qubit of an entangled pair, the former measurement makes the whole post-measurement state separable, thus it disentangles it, while the latter leaves it unchanged.

Since both Alice and Charlie do not need to relay their post-measurement state to anyone they can simply perform basis projections and consume all the entanglement making the state separable. As for Bob, since he can perform only a combination of the measurements just described, he can use one of three different strategy types:

- (I) Both measurements are rank-1 projection, thus the state becomes separable. This way Bob can violate the CHSH inequality, but, being the state separable, it is not possible to have a second violation.
- (II) Both measurements are trivial i.e. identity measurements. In this case, it is not possible to obtain a first violation, while the second one is possible.
- (III) One measurement is trivial and the other is a basis projection. Since one measurement is the identity, one output is simply discarded. Thus a first violation will be impossible, but a second one is still possible.

As we have just observed, these strategies, individually, can not achieve more than one CHSH inequality violation. The idea is to overcome these unsuccesses by exploiting classical shared randomness between the parties. This way it is possible to stochastically combine these individually unsuccessful strategies to achieve the two sequential violations of the CHSH inequality.

Practically this means that before the beginning of the experiment, the parties need

to share a correlated string of classical data, such as a sequence of random numbers. These data will allow them to decide which strategy to choose for each run of the experiment. Let $\lambda = 1, 2, 3$ be the variable stating which strategy to use. It is subject to some probability distribution $\{p_\lambda\}_{\lambda=1}^3$. We can redefine the CHSH parameter as the expectation value

$$S_{AB} = \sum_{\lambda=1}^3 p_\lambda S_{AB}^{(\lambda)} \quad (2.1)$$

where $S_{AB}^{(\lambda)}$ is the CHSH parameter between Alice and Bob as defined in equation (1.13). Analogously we can define the CHSH parameter $S_{AC}^{(\lambda)}$ between Alice and Charlie.

This scenario is schematically represented in figure 2.1.

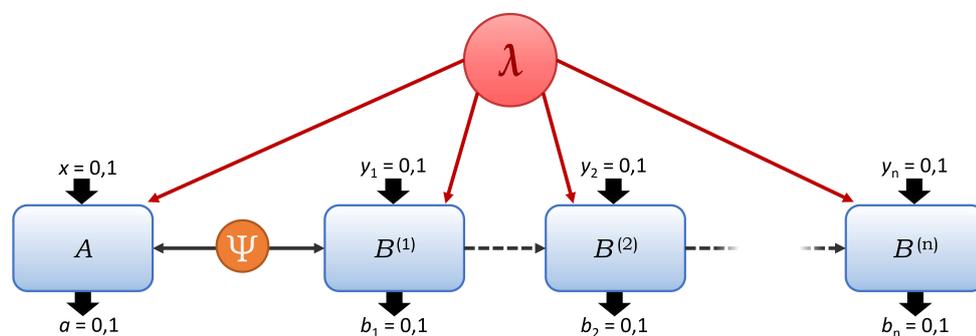


Figure 2.1: Sequential CHSH scenario - Alice (A) and a first Bob ($B^{(1)}$) share and measure the two-qubit entangled state Ψ . Afterwards, $B^{(1)}$ relays the post-measurement state to $B^{(2)}$, who measures it and so on. Before the beginning of the experiment, all the observers can share a string of classically correlated data λ .

In the following, we are going to consider as initial state shared between Alice and Bob $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, where $|0\rangle$ and $|1\rangle$ are the eigenstate of σ_z with eigenvalue 1 and -1 respectively.

Since all the states and measures we are going to consider lay on a 2D plane, we can, without loss of generality, restrain the problem to the XZ-disc of the Bloch sphere (the y component will be always 0).

2.1 Proof of principle

To begin with, we are proving that both Bob and Charlie can violate the CHSH inequality simply by using a strategy of type (I) to maximize $S_{AB}^{(1)}$, and (III) to maximize $S_{AC}^{(3)}$ and combining them afterward. Since we are not using strategy (II), p_2 is set to 0 and we can simply write $p_1 = p$ and $p_3 = 1 - p$. We are going to consider the simple case $S_{AB} = S_{AC}$. In general, we are not interested in this condition, but it simplifies the calculation and it is enough to show that both parameters can be greater than 2 at the same time. The strategies we are using present some

parameters, which we will use for the maximization process. Finally we are going to find p to combine the strategies together such that $S^{(1)} = S^{(2)}$.

Strategy type (I) ($\lambda = 1$): We start by choosing the observables. In particular Alice's observables are $A_0 = \frac{\sigma_x + \sigma_z}{\sqrt{2}}$ and $A_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$, Bob's are $B_0 = \cos \phi \sigma_x + \sin \phi \sigma_z$ and $B_1 = \sin \phi \sigma_x + \cos \phi \sigma_z$ and Charlie's are $C_0 = C_1 = \cos \phi \sigma_x + \sin \phi \sigma_z$ for a certain angle $\phi \in [0, 2\pi)$. As already mentioned in 1.1.5, Bob is allowed to perform a unitary operation which can depend on both Bob's input and output U_{yb} after the measurement. In this case, we choose them to be independent of the output b . When the input $y = 0$ we choose the identity operator, $U_0 = \mathbb{1}$, which means that no operation is performed on the measured state. As for the other input, $U_1 = e^{i(\phi - \frac{\pi}{4})\sigma_y}$, which, for a state having 0 y-component as in this case, is simply a rotation of $-\pi/4$ of the state in the XZ-plane.

We can use eq. (1.10) to compute the CHSH parameters for this strategy, obtaining

$$S_{AB}^{(1)} = 2\sqrt{2} \cos \phi \quad (2.2)$$

To compute the state shared between Alice and Charlie, ρ_{AC} we can use eq. (1.11). Since we are using projective measurements the Kraus operator can be simply written as $K_{b|y} = U_y B_{b|y}$, where $B_{b|y} = \frac{\mathbb{1} + (-1)^b B_y}{2}$ is the measurement operator associated with the observable B_y . From these calculations, we obtain the second CHSH parameter

$$S_{AC}^{(1)} = \sqrt{2} (\cos \phi + \sin \phi) \quad (2.3)$$

Now we can maximize $S_{AB}^{(1)}$, finding $S_{AB}^{(1)} = 2\sqrt{2}$ when $\phi = 0$. If we choose this measurement angle, the second CHSH parameter results in $S_{AC}^{(1)} = \sqrt{2}$.

Strategy type (III) ($\lambda = 3$): In this case we choose as observables $A_0 = \cos \theta \sigma_x + \sin \theta \sigma_z$ and $A_1 = \cos \theta \sigma_x - \sin \theta \sigma_z$ for a certain angle θ , $B_0 = \mathbb{1}$ and $B_1 = \sigma_z$, $C_0 = \sigma_x$ and $C_1 = \sigma_z$. In this case, Bob does not perform any unitary transformation of the state after the measurement. Following the steps performed for the first strategy we obtain the following CHSH parameters

$$S_{AB}^{(3)} = 2 \cos \theta \quad (2.4)$$

$$S_{AC}^{(3)} = 2 \sin \theta + \cos \theta \quad (2.5)$$

Maximizing $S_{AC}^{(3)}$, we find $S_{AC}^{(3)} = \sqrt{5}$ for $\theta = \arctan 2$ and $S_{AB}^{(3)} = \frac{4}{\sqrt{5}}$.

We want to use the two strategies just found to compute the final CHSH parameters as defined in equation (2.1). Being p and $1 - p$ the probabilities to use strategies (I) and (III) respectively, we obtain

$$S_{AB} = p S_{AB}^{(1)} + (1 - p) S_{AB}^{(3)} = 2\sqrt{2}p + \frac{4(1 - p)}{\sqrt{5}} \quad (2.6)$$

$$S_{AC} = p S_{AC}^{(1)} + (1 - p) S_{AC}^{(3)} = \sqrt{2}p + \sqrt{5}(1 - p) \quad (2.7)$$

To verify that it is possible for both S_{AB} and S_{AC} to exceed 2, it is sufficient to check whether their value when $S_{AB} = S_{AC}$ is greater than 2. Imposing this condition we find the value $p = \frac{\sqrt{5}}{5\sqrt{2}+\sqrt{5}} \approx 0.240$ which leads to $S_{AB} = S_{AC} = \frac{6\sqrt{10}}{5\sqrt{2}+\sqrt{5}} \approx 2.039$, which is indeed greater than 2.

This simple computation shows that it is indeed possible for two observers performing sequential measurements on one half of a two-qubit maximally entangled state $|\phi^+\rangle$ to both violate the CHSH inequality, when classical correlation between the measurement choices is present.

2.2 Optimal trade-off

Let's consider once again Alice and Bob sharing the maximally entangled state $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$.

Since the qubit is not relayed to anyone afterward, the optimal measurements for Alice are always rank-1 projective. This corresponds to Alice measuring the observables $A_0 = \vec{a}_0 \cdot \vec{\sigma}$ and $A_1 = \vec{a}_1 \cdot \vec{\sigma}$ with $\|\vec{a}_x\| = 1$, $x = 0, 1$.

Every rotation of the vectors \vec{a}_0 and \vec{a}_1 can be absorbed into a global rotation of Bob's measurements via the relation $O \otimes \mathbb{1} |\phi\rangle = \mathbb{1} \otimes O^T |\phi\rangle$. For this reason, we can simply choose $\vec{a}_0 = (\cos \theta, 0, \sin \theta)$ and $\vec{a}_1 = (\cos \theta, 0, -\sin \theta)$ and restrict the problem to the unit disk identified by the interception between the XZ-plane and the Bloch sphere.

To do so we introduce the unnormalized states remotely prepared by Alice on Bob's side $\rho_{a|x}$. These correspond to the eigenvectors of Alice's observables, namely $\rho_{a|x} = \frac{1}{4} (\mathbb{1} + (-1)^a \vec{a}_x \cdot \vec{\sigma})$. We observe that, since Alice's measurement operators are trace-one and that the local state is maximally mixed, $p(a|x) = \text{Tr}[\rho_{a|x}] = 1/2$.

We will now rewrite the equation for the CHSH parameter in a slightly different, more convenient, way. Firstly, from the definition of the unnormalized states remotely prepared by Alice on Bob's side $\rho_{a|x}$, we define $\rho_x = \rho_{0|x} - \rho_{1|x} = \frac{\vec{a}_x \cdot \vec{\sigma}}{2}$. Using this definition for ρ_x , the CHSH parameter between Alice and Bob becomes

$$S_{AB} = \sum_{x,y=0,1} (-1)^{xy} \text{Tr}[\rho_x B_y] \quad (2.8)$$

Considering that $C_z = 2C_0 - \mathbb{1}$ and remembering how the state ρ_{AC} can be obtained evolving ρ_{AB} from equation (1.12), we can rewrite S_{AC} as

$$S_{AC} = \frac{1}{2} \sum_{x,z=0,1} (-1)^{xz} \sum_{y,b} \text{Tr} \left(U_{by} \sqrt{B_{b|y}} \rho_x \sqrt{B_{b|y}} U_{by}^\dagger C_{0|z} \right) \quad (2.9)$$

Now we want to study the trade-off relation between S_{AB} and S_{AC} when Bob's measurements are projective, namely they satisfy $B_{b|y} B_{b'|y} = \delta_{b,b'} B_{b|y}$. As already discussed we can choose without loss of generality the Kraus operators in the form $K_{b|y} = U_{by} B_{b|y}$. Remembering also that the trace is cyclic the second CHSH

parameter becomes

$$S_{AC} = \cos \theta \sum_{y,b} \text{Tr} \left(B_{b|y} \sigma_X B_{b|y} U_{by}^\dagger C_{0|0} U_{by} \right) + \sin \theta \sum_{y,b} \text{Tr} \left(B_{b|y} \sigma_Z B_{b|y} U_{by}^\dagger C_{0|1} U_{by} \right) \quad (2.10)$$

Also in this case we chose Charlie's measurements to be projective since, as in Alice's case, he does not need to relay the qubit to anyone afterward.

In the following, we are examining the three strategies we discussed at the beginning of this chapter one by one.

2.2.1 Strategy type (I)

As already discussed, this strategy corresponds to Bob performing two projections on basis vectors. We also remember that since Bob's measurement breaks the entanglement, this kind of strategy allows for a first Bell inequality violation $S_{AB} > 2$, but not a second one $S_{AC} > 2$.

By defining the rank-one projectors $P_{zyb} = U_{by}^\dagger C_{0|z} U_{by}$ we can rewrite equation (2.10) as

$$S_{AC} = \cos \theta \sum_{y,b} \text{Tr} \left(B_{b|y} \sigma_X B_{b|y} P_{0yb} \right) + \sin \theta \sum_{y,b} \text{Tr} \left(B_{b|y} \sigma_Z B_{b|y} P_{1yb} \right) \quad (2.11)$$

We can now write an upper bound for the parameter S_{AC} by assuming that P_{0yb} is aligned with the eigenvector with the largest eigenvalue of the operator $B_{b|y} \sigma_X B_{b|y}$ and, analogously P_{1yb} is aligned with $B_{b|y} \sigma_Z B_{b|y}$. The upper bound on S_{AC} can be written as

$$S_{AC} \leq \cos \theta \sum_{y,b} \lambda_{\max} [B_{b|y} \sigma_X B_{b|y}] + \sin \theta \sum_{y,b} \lambda_{\max} [B_{b|y} \sigma_Z B_{b|y}] \quad (2.12)$$

where $\lambda_{\max} [B_{b|y} \sigma_X B_{b|y}]$ and $\lambda_{\max} [B_{b|y} \sigma_Z B_{b|y}]$ are the largest eigenvalues of $B_{b|y} \sigma_X B_{b|y}$ and $B_{b|y} \sigma_Z B_{b|y}$ respectively.

Observe that an operator defined as $P(\vec{u} \cdot \vec{\sigma})P$, with P rank-one projector, is a rank-one operator itself. Thus the operators $B_{b|y} \sigma_X B_{b|y}$ and $B_{b|y} \sigma_Z B_{b|y}$ are rank-one and their spectra have the form $(\lambda, 0)$. For this reason we can write their largest eigenvalue as $\lambda_{\max} [P(\vec{u} \cdot \vec{\sigma})P] = \max\{0, \text{Tr}[(\vec{u} \cdot \vec{\sigma})P]\}$.

Since $\text{Tr}[(\vec{u} \cdot \vec{\sigma})P]$ is the expectation value of the observable $\vec{u} \cdot \vec{\sigma}$ in the state P , we can expect that in the case $P = B_{b|y}$ this expectation value will be identical for $b = 0$ and $b = 1$, but with opposite signs, being one positive and one negative. For this reason, we expect that each of the two terms in equation (2.12) will have a contribution only from one value of b for each y value.

Finally, since when choosing Alice's measurement we decided to restrict the problem to the XZ-plane, the optimal choice for the Bloch vectors associated with Bob's measurements lay in the same plane, thus we can write $\vec{b}_y = (\cos \phi_y, 0, \sin \phi_y)$. With these choices, we can rewrite a simplified version for both the CHSH parameter

between Alice and Bob S_{AB} and equation (2.12)

$$S_{AB} = (\vec{a}_0 + \vec{a}_1) \cdot \vec{b}_0 + (\vec{a}_0 - \vec{a}_1) \cdot \vec{b}_1 = 2 \cos \theta \cos \phi_0 + 2 \sin \theta \sin \phi_1 \quad (2.13)$$

$$S_{AC} \leq \cos \theta (|\cos \phi_0| + |\cos \phi_1|) + \sin \theta (|\sin \phi_0| + |\sin \phi_1|) \quad (2.14)$$

These relations show that the pair (S_{AB}, S_{AC}) is fully characterized by the variables (θ, ϕ_0, ϕ_1) .

As a first thing, we want to recover the classical boundary. To do so we can simply choose $\phi_0 = \phi_1 = -\theta$, which leads to $S_{AB} = 2 \cos(2\theta)$ and $S_{AC} = 2$. While $\theta \in [0, \frac{\pi}{4}]$ we recover $0 \leq S_{AB} \leq 2$ and $S_{AC} = 2$. This choice is optimal because, being Bob's measurements both rank-1, the state relayed to Charlie is separable and can not violate the CHSH inequality, it can reach the value of 2 at the best.

Moving to the non-classical range $2 \leq S_{AB} \leq 2\sqrt{2}$, we are now going to look for the maximum value obtainable by S_{AC} for a certain S_{AB} . To do so, we are going to prove that for every pair (S_{AB}, S'_{AC}) obtained with (θ, ϕ_0, ϕ_1) , there exists another pair (S_{AB}, S_{AC}) , such that $S_{AC} \geq S'_{AC}$, obtained with $(\theta = \frac{\pi}{4}, \phi_0 = \phi, \phi_1 = \frac{\pi}{2} - \phi)$ for some $\phi \in [0, \frac{\pi}{2}]$, thus this triple is the optimal choice for (θ, ϕ_0, ϕ_1) .

Firstly, since the value for S_{AB} should not change with this choice of the parameters, ϕ satisfy

$$2 \cos \theta \cos \phi_0 + 2 \sin \theta \sin \phi_1 = 2\sqrt{2} \cos \phi \quad (2.15)$$

When $\phi_0 = 0$, $\phi_1 = \frac{\pi}{2}$ and $\pm\theta = \frac{\pi}{4}$ the left-hand-side of equation (2.15) is maximized/minimized and becomes $\pm 2\sqrt{2}$. Now we want to show that with this choice of ϕ we have $S_{AC} \geq S'_{AC}$. This last condition becomes

$$\cos \theta (|\cos \phi_0| + |\cos \phi_1|) + \sin \theta (|\sin \phi_0| + |\sin \phi_1|) \leq \sqrt{2} (\cos \phi + \sin \phi) \quad (2.16)$$

In the range of our interest, without loss of generality, we can simplify this relation and drop the absolute values by taking $\phi_0, \phi_1 \in [0, \frac{\pi}{2}]$.

After rearranging the disequality, squaring both sides and substituting ϕ via equation (2.15), we obtain

$$\cos^2 \theta (\cos^2 \phi_0 + \cos^2 \phi_1) + \sin^2 \theta (\sin^2 \phi_0^2 + \sin^2 \phi_1^2) + \sin(2\theta) \sin(\phi_0 + \phi_1) \leq 2 \quad (2.17)$$

We can finally differentiate the left-hand side with respect to ϕ_0 and ϕ_1 respectively and look for the maximum. We find that both the derivatives have two joint roots, one at $\theta = \frac{\pi}{4}$, $\phi_0 + \phi_1 = \frac{\pi}{2}$, and the other at $\theta = \phi_0 = \phi_1$. In both cases, the derivative with respect to θ vanishes. Both these solutions bring to a maximum value for the left-hand-side of equation (2.17) equal to 2, thus proving the inequality to hold.

Hence, we can choose parameters such that $\theta = \frac{\pi}{4}$, $\phi_0 = \phi$ and $\phi_1 = \frac{\pi}{2} - \phi$. This brings us to the relations

$$S_{AB} = 2\sqrt{2} \cos \phi \quad (2.18)$$

$$S_{AC} \leq \sqrt{2} (\cos \phi + \sin \phi) \quad (2.19)$$

To obtain the trade-off relation, we can simply substitute the former equation into the latter and obtain

$$S_{AC} \leq \frac{S_{AB}}{2} + \frac{1}{2}\sqrt{8 - (S_{AB})^2} \quad (2.20)$$

Since the inequality is tight, this trade-off relation is optimal. To prove the tightness of the inequality it is sufficient to observe that the strategy (I) proposed in section 2.1 has exactly this trade-off. We can also observe that the maximum of S_{AC} occurs at $S_{AB} = 2$ and gives $S_{AC} = 2$. This means that at the endpoint of its interval of validity, which is $2 \leq S_{AB} \leq 2\sqrt{2}$, this function meets the classical trade-off we already found in $0 \leq S_{AB} \leq 2$.

A plot of the optimal trade-off relation for this strategy type is reported in blue in figure 2.2. In the figure are also plotted two black dashed lines highlighting the classical bound $\{S_{AB}, S_{AC}\} = \{2, 2\}$ and two light green lines highlighting the Tsirelson's bound $\{S_{AB}, S_{AC}\} = \{2\sqrt{2}, 2\sqrt{2}\}$.

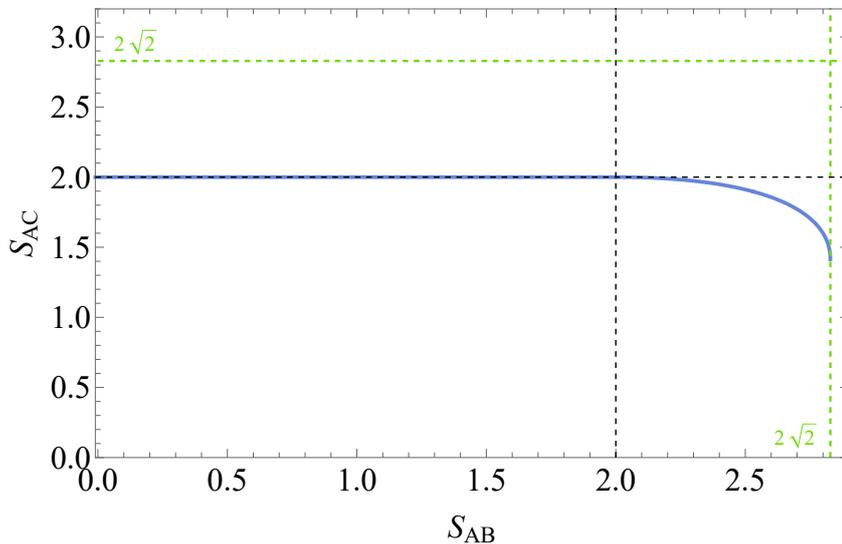


Figure 2.2: Trade-off relation between the CHSH parameters S_{AB} and S_{AC} for strategies of type (I) (blue). The dashed black and green lines represent the classical and Tsirelson's bounds respectively.

2.2.2 Strategy type (II)

We can now move to the second strategy, in which Bob performs only trivial projective measurements. With this strategy we do not expect any CHSH violation for Bob, only S_{AC} can be greater than 2. This corresponds to deterministically choosing a value for b based on y without taking into consideration the quantum state. The operators representing this kind of measurement are $(\mathbb{1}, 0)$ which gives always $b = 0$ as output, or $(0, \mathbb{1})$ giving always the output $b = 1$. From the expression for S_{AB} given in equation (2.9) it is immediate to find that $S_{AB} = 0$. In the case in which also Alice performs trivial (identity) measurements the value for S_{AB} could be increased to $S_{AB} = 2$, but this would imply that $S_{AC} \leq 2$. Even though this case exists, it is not interesting in our context since this does not lead to any CHSH violation at all.

We want to find an expression for S_{AC} in the case in which Alice performs rank-1 projective measurements and, thus $S_{AB} = 0$. Since Bob's instrument becomes only a unitary operator, the post-measurement state becomes $\rho' = \frac{1}{2} \sum_y (\mathbb{1} \otimes V_y) \phi^+ (\mathbb{1} \otimes V_y^\dagger)$ where V_y is the operator in $\{U_{0y}, U_{1y}\}$ associated to the only possible output (unit probability event) of Bob's measurements. It is trivial to see that an optimal choice can simply be $V_y = \mathbb{1}$, which leaves the state unperturbed, $\rho' = \phi^+$. If we consider Alice and Charlie now, we are in a simple Bell scenario and, thus Charlie can reach the Tsirelson bound $S_{AC} = 2\sqrt{2}$ by performing the measurements σ_X and σ_Z , while Alice chooses as measurement angle $\theta = \frac{\pi}{4}$. It is interesting to observe that contrary to the first and the third strategies, in which, as we will see, we can find a trade-off relation between S_{AB} and S_{AC} , this trivial strategy trade-off is simply the point: $(S_{AB}, S_{AC}) = (0, 2\sqrt{2})$.

2.2.3 Strategy type (III)

We recall that this kind of strategy consists in Bob performing one trivial (identity) measurement and one basis projection. With this strategy we expect only Charlie to be able to violate CHSH.

Firstly we observe that the CHSH parameter is invariant the following under coordinate permutations: $\{y \rightarrow \bar{y} \ \& \ a \rightarrow \bar{a} \text{ if } x = 1\}$ and $\{b \rightarrow \bar{b} \text{ if } y = 0 \ \& \ x \rightarrow \bar{x} \ \& \ a \rightarrow \bar{a}\}$ where with the bar we denote the bit-flip operation. For this reason, we can, without loss of generality, assign to the input $y = 0$ the single outcome $b = 0$, meaning that the first observable $B_0 = \mathbb{1}$, while the second measurement will be a basis projection corresponding to the observable $B_1 = \vec{b} \cdot \vec{\sigma}$, where \vec{b} is a unit Bloch vector. Thus the operators associated with the measurements can be written as $B_{0|0} = \mathbb{1}$, $B_{1|0} = \mathbb{1}$ and $B_{b|1} = \frac{1}{2} (\mathbb{1} + (-1)^b \vec{b} \cdot \vec{\sigma})$.

As for the unitaries implemented by Bob's measurements, thanks to the invariance under a global rotation of the unitaries U_{by} , we may fix a reference one, in this case, we choose $U_{00} = \mathbb{1}$. We can also notice that, since $B_{1|0} = 0$, the post-measurement state will not depend on the choice of U_{10} . Moreover, for $y = 1$ we can use some considerations pointed out when considering the first strategy. In particular, we remember that, given a unit vector \vec{u} , the eigenvector of $B_{b|y}(\vec{u} \cdot \vec{\sigma})B_{b|y}$ corresponding to its largest eigenvalue is identical for both $b = 0$ and $b = 1$, with one positive eigenvalue and the other zero.

Remembering from equation (2.10) that the unitaries aim to align the projectors $C_{0|0}$ and $C_{0|1}$ with the eigenvectors of $B_{b|y}(\vec{u} \cdot \vec{\sigma})B_{b|y}$. Since, as just discussed, these eigenvectors do not depend on b , we can optimally choose $U_{01} = U_{11} \equiv U_1$.

As discussed in the first strategy, since the states remotely prepared for Bob by Alice are in the XZ-plane, thus it is optimal for him to choose $\vec{b} = (\cos \phi, 0, \sin \phi)$. So the first CHSH parameter is $S_{AB} = 2 \sin \theta \sin \phi$. Taking into consideration the considerations just made on the unitaries, we remain with only one unitary. We can optimally take it as a rotation in the XZ-plane, $U_1 = e^{i\mu\sigma_y}$. Finally, writing Charlie's measurements in terms of Bloch vectors, always in the XZ-plane,

$\vec{c}_z = (\cos \phi_z, 0, \sin \phi_z)$, we can write the second CHSH parameter as

$$S_{AC} = \frac{1}{2} \cos \theta (\cos(2\mu + 2\phi - \phi_0) + \cos(2\mu - \phi_0) + 2 \cos \phi_0) + \frac{1}{2} \sin \theta (\sin(2\mu + 2\phi - \phi_1) - \sin(2\mu - \phi_1) + 2 \sin \phi_1) \quad (2.21)$$

Deriving this expression with respect to μ , ϕ_0 , ϕ_1 , ϕ , one finds out that when $\phi_0 = \mu = 0$ and $\phi = \phi_1 = \frac{\pi}{2}$ all the derivatives vanish and, considering the concavity of the manifold, we can deduce that this set of variables, which gives $S_{AC} = \cos \theta + 2 \sin \theta$, is optimal. We observe that $\phi = \frac{\pi}{2}$ is a maximum also for S_{AB} leading to $S_{AB} = 2 \sin \theta$. Comparing these two final expressions for the CHSH parameters we find the optimal trade-off relation which is

$$S_{AC} = S_{AB} + \sqrt{1 - \frac{(S_{AB})^2}{4}} \quad (2.22)$$

From this equation we see that when $S_{AB} = \frac{4}{\sqrt{5}}$, S_{AC} is maximized and we have $S_2 = \sqrt{5} > 2$. When $S_{AB} < \frac{4}{\sqrt{5}}$, which means $\theta < \arcsin \frac{2}{\sqrt{5}}$, this solution is not optimal. Anyway, $S_{AB} < \frac{4}{\sqrt{5}}$ the optimal trade-off is simply given by $S_{AC} = \sqrt{5}$ with no dependence on S_{AB} . Indeed this is the maximum value S_{AC} can assume when the only constraint on Bob's measurements is the rank, thus the strategy. Indeed, if we set $\phi_0 = 0$, $\phi = \phi_1 = \frac{\pi}{2}$ and $\mu = \frac{1}{2} \arccos \left(\frac{\sqrt{5} - \sin \theta}{\sin \theta + \cos \theta} \right)$, we obtain $S_{AC} = \sqrt{5}$ for every value of θ , while leaving S_{AB} free to change.

In figure 2.3 we report the plot of the optimal trade-off relation for this strategy type in yellow. In the figure are also plotted two dashed black and a green line highlighting the classical bound $\{S_{AB}, S_{AC}\} = \{2, 2\}$ and the Tsirelson's bound $S_{AC} = 2\sqrt{2}$ respectively.

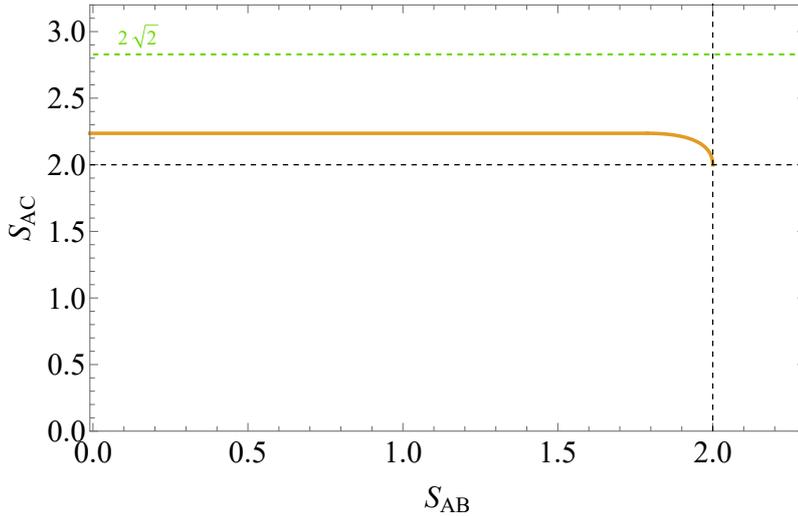


Figure 2.3: Trade-off relation between the CHSH parameters S_{AB} and S_{AC} for strategies of type (III) (yellow). The dashed black and green lines represent the classical and Tsirelson's bounds respectively.

2.2.4 Mixing via shared randomness

We are now going to combine stochastically these three projective strategies to find the optimal trade-off between S_{AB} and S_{AC} .

We start by observing that apart from the case (II) in which we found a single point in the plane, the other two trade-offs are concave functions.

The mixing of such functions via shared randomness is their linear combination with all non-negative coefficients which sum up to 1. This is justified if we remember that the combination coefficient associated with a function corresponds to the probability of using that strategy. Performing such a combination is equivalent to finding the convex hull of the set defined by the functions.

In the following, we are considering this mixing procedure case by case.

Mixing (II) and (III)

We start remembering that the optimal trade-off for strategy (II) and (III) are respectively the point $(0, 2\sqrt{2})$ and the function $S_{AC} = S_{AB} + \frac{1}{2}\sqrt{4 - S_{AB}^2}$.

For simplicity, in the following, we are going to consider the variable x and its function $f(x)$ instead of S_{AB} and S_{AC} respectively.

Thanks to the geometrical considerations we just made, we only need to look for the line tangent to the function $f(x) = x + \frac{1}{2}\sqrt{4 - x^2}$ and passing through the point $(0, 2\sqrt{2})$. To find the point of tangency we need to solve the equation

$$f(x_1) - 2\sqrt{2} = f'(x_1)(x_1 - 0) \quad (2.23)$$

where $f'(x)$ is the first derivative of f . Since $S_{AB} > 0$, we can consider only the positive solution obtaining the point $(x_1, f(x_1)) = (\sqrt{\frac{7}{2}}, \frac{1}{2\sqrt{2}} + \sqrt{\frac{7}{2}})$. It is important to notice that the point of tangency we found $x_1 = \sqrt{\frac{7}{2}}$ is greater than $\frac{4}{\sqrt{5}}$. Indeed, when we studied case (III) we discussed that equation (2.22) represents the trade-off only when $S_{AB} > \frac{4}{\sqrt{5}}$, while for $0 < S_{AB} < \frac{4}{\sqrt{5}}$ the trade-off is simply $S_{AC} = \sqrt{5}$.

Returning to the original notation, we write the tangent line as

$$S_{AC} = \left(1 - \frac{\sqrt{7}}{2}\right) S_{AB} + 2\sqrt{2} \quad (2.24)$$

which will be the boundary we are looking for in the interval $0 \leq x \leq \frac{\sqrt{7}}{2}$.

In figure 2.4 we report in yellow the trade-off between S_{AB} and S_{AC} for type III. The red line is its tangent line passing through the point defined by strategy type II, $(0, 2\sqrt{2})$. The black points highlight this point and the point of tangency. As in previous figures, we also reported with black and green dashed lines the classical and Tsirelson's bounds.

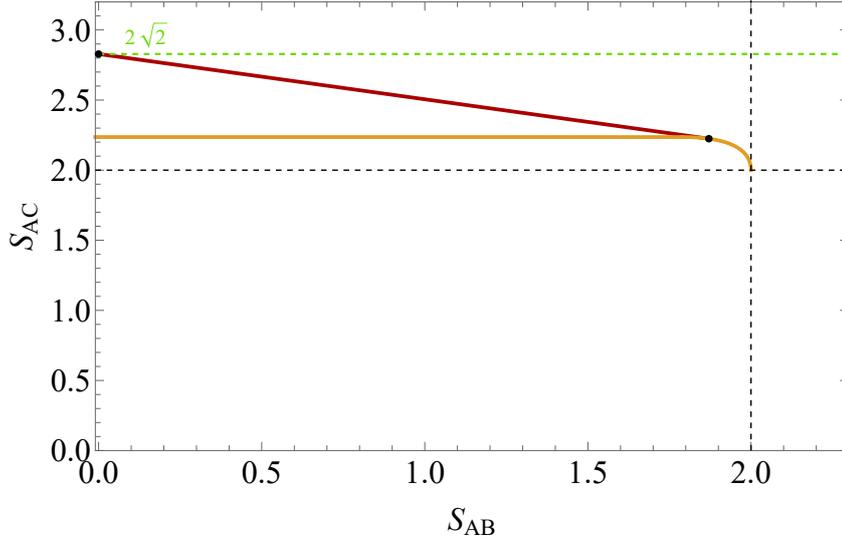


Figure 2.4: Tangent line (red) to the trade-off relation for strategies of type (III) (yellow) passing through the point $(0, 2\sqrt{2})$. The dashed black and green lines represent the classical and Tsirelson's bounds respectively.

Mixing (I) and (III)

Now we move to the mixing of the two non-trivial strategies. Thus we are looking for the line tangent to both the trade-off functions.

For simplicity, instead of equations (2.20) and (2.22) we are going to consider the functions $f(x) = x + \sqrt{1 - \frac{x^2}{4}}$ and $g(x) = \frac{x}{2} + \frac{1}{2}\sqrt{8 - x^2}$.

To find the points of tangency $(x_1, f(x_1))$ and $(x_2, g(x_2))$ we need to solve the two equations

$$f'(x_1) = g'(x_2) = \frac{f(x_1) - g(x_2)}{x_1 - x_2}, \quad (2.25)$$

where f' and g' are the first derivative of f and g respectively. The solution is

$$x_1 = 3\sqrt{\frac{2}{5}} \quad x_2 = 4\sqrt{\frac{2}{5}}. \quad (2.26)$$

Going back to the original notation we find the tangent

$$S_{AC} = \sqrt{10} - \frac{S_{AB}}{2} \quad (2.27)$$

In between the two points of tangency, namely $3\sqrt{\frac{2}{5}} \leq S_{AB} \leq 4\sqrt{\frac{2}{5}}$, this is another portion of the boundary.

In figure 2.5 we report in yellow and blue the optimal trade-off between S_{AB} and S_{AC} for type (III) and type (I) strategies respectively. The green line between them is their common tangent line and the black points highlight the points of tangency. As in previous figures, we also reported with black and green dashed lines the classical

and Tsirelson's bounds.

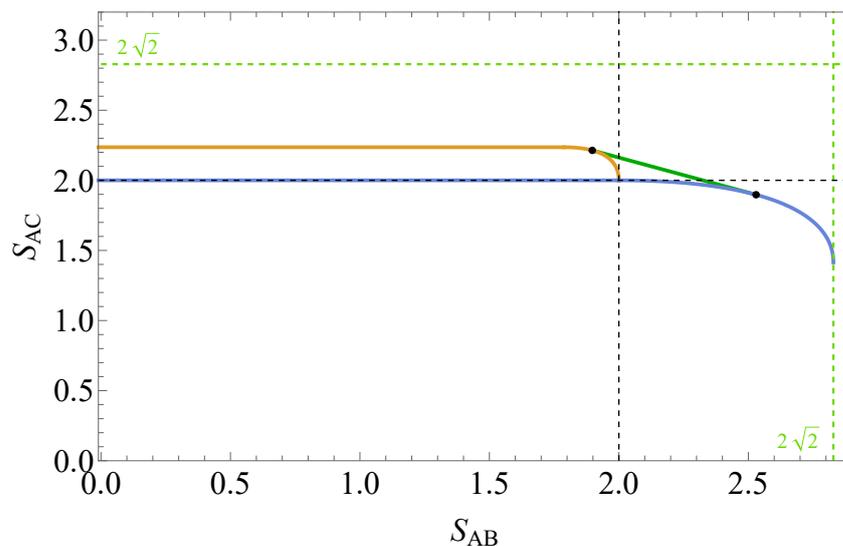


Figure 2.5: Common tangent line (green) to the trade-off curves for strategies of type (I) (blue) and (III) (yellow). The dashed black and green lines represent the classical and Tsirelson's bounds respectively.

Mixing (I) and (II)

We can now compute the tangent to the curve given by (2.20) through the point $(0, 2\sqrt{2})$ as already done previously.

Instead of writing the curve as in (2.20) we use $g(x) = \frac{x}{2} + \frac{1}{2}\sqrt{8-x^2}$ and look for the tangency point by solving

$$g(x_1) - 2\sqrt{2} = g'(x_1)(x_1 - 0) \quad (2.28)$$

and choosing the positive solution.

The point of tangency results to be $(\sqrt{6}, \frac{\sqrt{2}+\sqrt{6}}{2})$. And the tangent can be written as

$$S_{AC} = \frac{(1-\sqrt{3})S_{AB}}{2} + 2\sqrt{2} \quad (2.29)$$

As we can see in figure 2.6, this line (pink) is inside of the region described by the previously calculated curves (dashed red, yellow, and green lines), thus it is not going to be part of the boundary. This means that it is always possible to obtain a better trade-off by mixing other strategies, thus we are not going to use this case. In the plot, we highlighted points A, B, C, and D in which the function defining the boundary changes. In the subplot, we highlighted a region in which the optimal trade-off is simply given by choosing deterministically a strategy of type III. We are going to better discuss this region in the following.

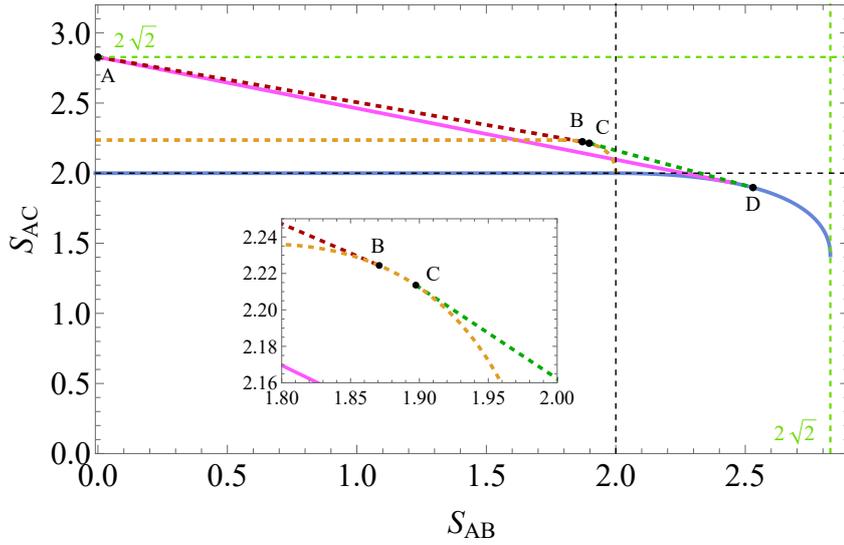


Figure 2.6: Plot - Tangent line (solid pink line) to the trade-off curve for strategies of type (I) (solid blue line) passing through the point $(0, 2\sqrt{2})$. This line is always under the curves obtained by mixing strategies of type (II) and (III) (dashed red line between A and B), (II) and (III) (dashed green line between C and D) or deterministically using type (III) (dashed yellow between B and C). The dashed black and green lines represent the classical and Tsirelson's bounds respectively. Subplot - Zoom on the region between points B and C.

Intermediate regions

Up to now, we found the optimal trade-off in the intervals $0 \leq S_{AB} \leq \sqrt{\frac{7}{2}}$ and $3\sqrt{\frac{2}{5}} \leq S_{AB} \leq 4\sqrt{\frac{2}{5}}$. To cover the complete interval $0 \leq S_{AB} \leq 2\sqrt{2}$ we need to determine the optimal trade-off in

$$\sqrt{\frac{7}{2}} < S_{AB} < 3\sqrt{\frac{2}{5}} \quad (2.30)$$

$$4\sqrt{\frac{2}{5}} < S_{AB} \leq 2\sqrt{2} \quad (2.31)$$

Since these intervals are not covered by the mixture of different strategies, the boundary in those regions is simply a deterministic strategy. In particular, in the interval (2.30) the deterministic strategy (III) is optimal, thus in that interval, the boundary is represented by equation (2.22). As for the other interval, the deterministic the optimal deterministic strategy is (I), thus the boundary is given by (2.20). The complete boundary of the set (S_{AB}, S_{AC}) reachable by means of projective measurements and with shared randomness is given by the four-part

piecewise function

$$S_{AC} = \begin{cases} \left(1 - \frac{\sqrt{7}}{2}\right) S_{AB} + 2\sqrt{2} & \text{if } 0 \leq S_{AB} \leq \sqrt{\frac{7}{2}} \\ S_{AB} + \frac{1}{2}\sqrt{4 - (S_{AB})^2} & \text{if } \sqrt{\frac{7}{2}} < S_{AB} < 3\sqrt{\frac{2}{5}} \\ \sqrt{10} - \frac{S_{AB}}{2} & \text{if } 3\sqrt{\frac{2}{5}} \leq S_{AB} \leq 4\sqrt{\frac{2}{5}} \\ \frac{S_{AB}}{2} + \frac{1}{2}\sqrt{8 - (S_{AB})^2} & \text{if } 4\sqrt{\frac{2}{5}} < S_{AB} \leq 2\sqrt{2} \end{cases} \quad (2.32)$$

This function is represented in figure 2.7. In the plot, we report as dashed lines the trade-off relations found for strategies of type (I) and (III) when they are not part of the boundary. The solid lines form the boundary, while the black points highlight the passage from one definition interval to another one. The colors of the lines are the same used in the previous plots, namely yellow for type III, blue for type I, red for a combination of (II) and III, and green for the combination of (I) and III. In the subplot, we highlight the region where the boundary is reached by deterministically choosing type (III) strategies.

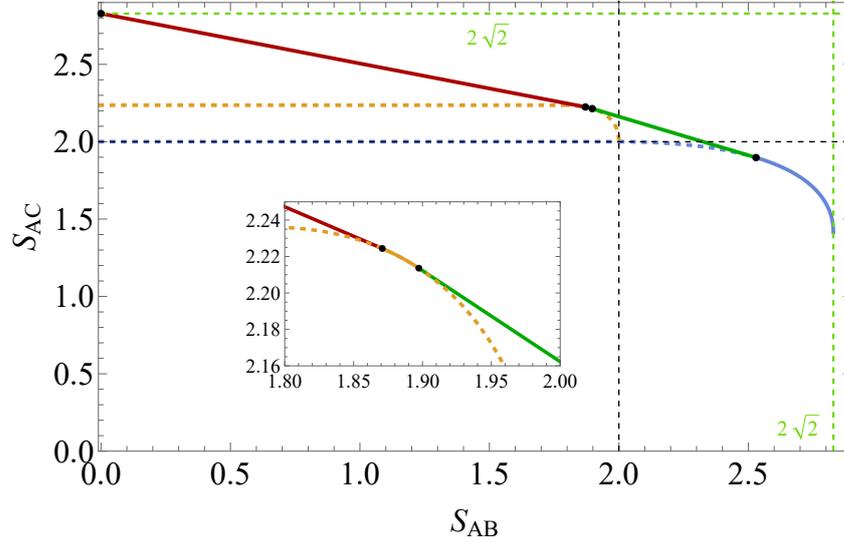


Figure 2.7: Plot - Boundary of the set (S_{AB}, S_{AC}) reachable under projective measurements and shared randomness (solid lines). The color of the lines corresponds to the strategy needed to reach it: yellow for type III, blue for type I, red for a combination of (II) and III, and green for the combination of (I) and III. The dashed lines are the relations obtained in cases (I) (blue) and (III) (yellow) when they are not part of the boundary. The black points highlight the change in the definition of the function. Subplot - Zoom on the region where strategy (III) is part of the boundary.

2.3 Only local randomness

In the above protocols, we have exploited some classical shared randomness. We are now going to show, with an example that this useful resource is not strictly necessary. Indeed even classically independent parties, thus with no shared randomness, can reach at least two sequential CHSH violations using projective measurements. In this case, the observers are only allowed to generate classical randomness locally. This corresponds to replacing the collective variable λ with a triple of variables associated with Alice, Bob, and Charlie respectively $(\lambda_A, \lambda_B, \lambda_C)$. The independence of the observers can be expressed by the factorization of the probability, namely $p(\lambda) = p(\lambda_A)p(\lambda_B)p(\lambda_C)$.

In this quantum strategy, we are allowing only Bob to use local randomness, while Alice and Charlie will not use it.

Alice's and Charlie's observables are $A_0 = \frac{\sqrt{3}}{2}\sigma_X + \frac{1}{2}\sigma_Z$, $A_1 = \cos(2)\sigma_X - \sin(2)\sigma_Z$, $C_0 = \cos\left(\frac{2\pi}{3e}\right)\sigma_X - \sin\left(\frac{2\pi}{3e}\right)\sigma_Z$, $C_1 = \cos\left(\frac{1}{3}\right)\sigma_X + \sin\left(\frac{1}{3}\right)\sigma_Z$. Bob, instead, can randomly choose between two strategies, labeled $\lambda_B \in \{0, 1\}$. The probability for him to use the strategy $\lambda_B = 0$ is $q = p(\lambda_B = 0)$. When $\lambda_B = 0$, Bob measures $B_0^{(0)} = \cos\left(\frac{2}{17}\right)\sigma_X + \sin\left(\frac{2}{17}\right)\sigma_Z$ and $B_1^{(0)} = \frac{\sigma_X + \sigma_Z}{\sqrt{2}}$. Then he applies the unitary operators $U_{00}^{(0)} = U_{10}^{(0)} = \mathbb{1}$ and $U_{01}^{(0)} = U_{11}^{(0)} = e^{-\frac{2\pi}{27}i\sigma_Y}$. When $\lambda_B = 1$, Bob measures $B_0^{(1)} = \mathbb{1}$ and $B_1^{(1)} = \frac{\sigma_X + \sigma_Z}{\sqrt{2}}$. Then he applies $U_{00}^{(1)} = e^{-\frac{5\pi}{81}i\sigma_Y}$, $U_{10}^{(1)} = \mathbb{1}$ and $U_{01}^{(1)} = U_{11}^{(1)} = e^{-\frac{2\pi}{27}i\sigma_Y}$.

If we look for the condition $S_{AB} = S_{AC}$, we find the solution $q \approx 0.358$, which gives $S_{AB} = S_{AC} \approx 2.046 > 2$, thus a double violation of the inequality.

It is interesting to point out that, since in this case Alice and Charlie do not rely on any shared randomness, from their point of view this experiment is identical to the standard CHSH scenario. The remarkable difference is that Bob can decide which set of observables to measure, independently from Alice and Charlie.

Chapter 3

Partially entangled state

In the standard CHSH scenario maximally entangled states are necessary to reach the largest CHSH parameter. We are now going to show that in the sequential scenario when using projective measurements, this is not true anymore. Thus we are going to study the same scenario as in chapter 2 if, instead of using the initial state $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, we use the pure state $|\psi_\varphi\rangle = \cos \varphi |00\rangle + \sin \varphi |11\rangle$ with $\varphi \in [0, \frac{\pi}{4}]$. As in the previous case, we can restrict Alice's and Charlie's measurement to the XZ-plane without loss of generality. As for Bob's measurements, we can restrict them on that plane too when they are represented by rank-1 operators.

Firstly, we are going to describe some analytical strategies producing couples of CHSH parameters outside the boundary described by equation (2.32). These strategies will show that, in certain cases, with partially entangled states it is possible to outperform maximally entangled ones. Afterwards, we are proving that every pure state $|\psi_\varphi\rangle$ can produce a couple $(S_{AB}, S_{AC}) > (2, 2)$.

3.1 Outperforming maximally entangled states

3.1.1 Strategy type (I)

To start we remember that in this strategy type Bob performs two rank-1 projective measurements.

In the following, we are going to present a particular strategy. We were not able to numerically obtain any type (I) strategy better than this, for any value of the angle φ .

Alice's observables of this strategy are $A_0 = \sigma_X$ and $A_1 = \sigma_Z$, while Bob's ones are $B_0 = \cos \phi \sigma_X + \sin \phi \sigma_Z$ and $B_1 = \cos \phi \sigma_X - \sin \phi \sigma_Z$. Finally Charlie's observables are $C_0 = \cos \phi \sigma_X + \sin \phi \sigma_Z$ and $C_1 = -\cos \phi \sigma_X - \sin \phi \sigma_Z$. The unitaries used by Bob after the measurements are $U_{by} = U_y$ with $U_0 = \mathbb{1}$ and $U_1 = e^{i(\phi - \frac{\pi}{2})\sigma_Y}$. The

CHSH parameters obtained from these measurements are

$$S_{AB} = 2(\cos \phi \sin(2\phi) + \sin \phi) \quad (3.1)$$

$$S_{AC} = 2 \sin \phi \quad (3.2)$$

Finding $\sin \phi$ from the first equation and substituting it into the second one we find

$$S_{AC} = \frac{1}{1 + \sin(2\phi)^2} \left(S_{AB} + \sin(2\phi) \sqrt{4(1 + \sin(2\phi)^2) - (S_{AB})^2} \right) \quad (3.3)$$

As expected, if we choose $\phi = \frac{\pi}{4}$ we recover (2.20). In this case, we could not find any evidence that partially entangled states could produce larger violations than maximally entangled ones. In figure 3.1, we report this trade-off relation for some ϕ values.

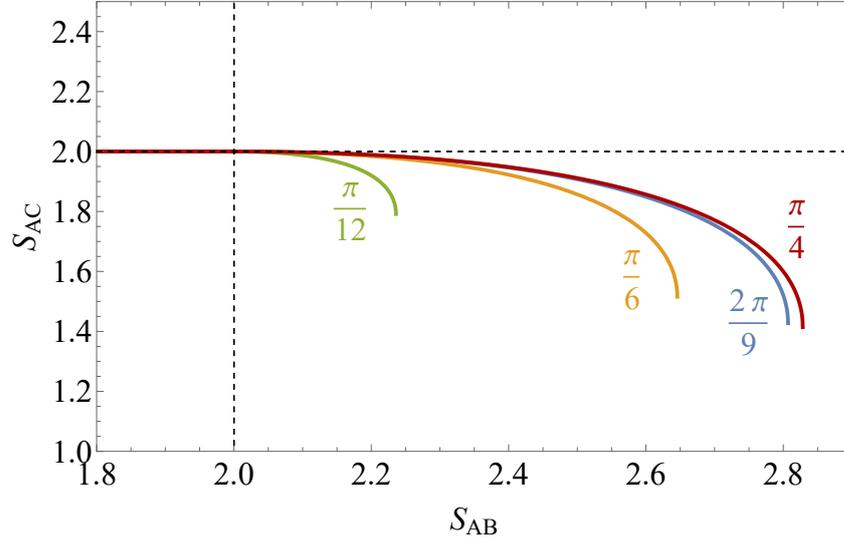


Figure 3.1: Type (I) strategy trade-off for partially entangled states for some values of ϕ .

3.1.2 Strategy type (II)

In this case, Bob can only perform trivial measurements. The strategy we are going to describe is the optimal one for this kind. Since Bob observables are $B_0 = B_1 = \mathbb{1}$, the first CHSH parameter is $S_{AB} = 2\langle A_0 \otimes \mathbb{1} \rangle = 2 \text{Tr} [A_0 (\cos^2 \varphi |0\rangle\langle 0| + \sin^2 \varphi |1\rangle\langle 1|)]$. When Alice measures $A_0 = \sigma_z$, the parameter becomes $S_{AB} = 2 \cos(2\varphi)$, which is optimal because 0 and 1 are eigenvectors of sigma z. To choose Charlie's measurements we consider that the optimal CHSH parameter value for any state $|\psi_\varphi\rangle$ is $2\sqrt{1 + \sin(2\varphi)^2}$ [30]. This can be reached if we let Charlie measure $C_0 = \cos \phi \sigma_x + \sin \phi \sigma_z$ and $C_1 = -\cos \phi \sigma_x + \sin \phi \sigma_z$ with $\phi = \arctan(\sin(2\varphi))$. From this, we can deduce that the strategy is optimal. We can conclude that optimal

type (II) strategies produce

$$S_{AB} = 2 \cos(2\varphi) \quad (3.4)$$

$$S_{AC} = 2\sqrt{1 + \sin(2\varphi)^2} \quad (3.5)$$

Inverting the first equation and substituting $\varphi = \frac{1}{2} \arccos\left(\frac{S_{AB}}{2}\right)$ in the second one we find the trade-off relation

$$S_{AC} = \sqrt{8 - (S_{AB})^2} \quad (3.6)$$

Even if, being deterministic, this strategy cannot produce double violations, we can observe that it can outperform what is attainable with maximally entangled states. Indeed, comparing this function with the boundary we found for maximally entangled states in equation (2.32), we find the former exceeds the latter when $0 < S_{AB} < h$ where

$$h = \frac{8\sqrt{2}}{113} (7\sqrt{7} - 2) \approx 1.65 \quad (3.7)$$

In figure 3.2 we report the plot of this trade-off together with the boundary function.

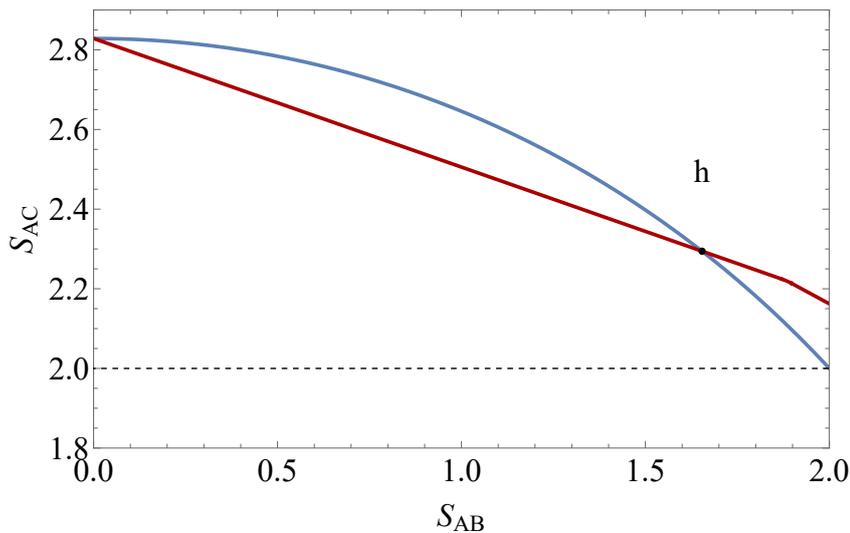


Figure 3.2: Type (II) strategy trade-off for partially entangled states φ (blue) compared with the boundary function (red).

3.1.3 Strategy type (III)

In this case, Bob has one trivial and one rank-1 projective measurement. Also, we are going to show that there exist strategies producing couples (S_{AB}, S_{AC}) laying outside the boundary for the maximally entangled state. We are doing so by presenting an example strategy. Again these points will not be double violations since we are considering a deterministic strategy.

The measurements are $A_x = (-1)^x \cos \theta \sigma_X + \sin \theta \sigma_Z$ for Alice, $B_0 = \mathbb{1}$ and $B_1 = \sigma_X$

for Bob, $C_0 = \sigma_Z$ and $C_1 = \sigma_X$ for Charlie. Bob is not required to perform any unitary operation after his measurement ($U_{by} = \mathbb{1}$). The expressions for the CHSH parameters in this case are

$$S_{AB} = 2 \sin(\theta + 2\varphi) \quad (3.8)$$

$$S_{AC} = \sin \theta + 2 \cos \theta \sin(2\varphi) \quad (3.9)$$

Inverting the first one we find $\theta = \pi - 2\varphi - \arcsin\left(\frac{S_{AB}}{2}\right)$. Substituting this relation in the second one we obtain

$$S_{AC} = \sin(2\varphi) \sqrt{1 - \left(\frac{S_{AB}}{2}\right)^2} (1 - 2 \cos(2\varphi)) + \frac{S_{AB}}{2} (2 \sin(2\varphi)^2 + \cos(2\varphi)) \quad (3.10)$$

Maximizing this last equation over φ we find an expression for the optimal angle φ

$$\varphi = \arccos \left(\frac{1}{4} \sqrt{9 - \sqrt{g(S_{AB})} + \sqrt{33 - g(S_{AB}) + \frac{8S_{AB}^2}{\sqrt{g(S_{AB})}}}} \right) \quad (3.11)$$

where

$$\begin{aligned} g(x) &= 11 + h(x) + (121 - 24x^2)/h(x) \\ h(x) &= \left(8x^4 - 396x^2 + 8x^2\sqrt{x^4 + 117x^2 - 484} + 1331\right)^{1/3} \end{aligned} \quad (3.12)$$

The optimal trade-off for this strategy, which is not, in general, the best for all strategies of type (III), can be recovered by substituting this expression for φ in equations (3.12).

To show that this strategy can outperform what is obtainable by the maximally entangled state we can simply compare the trade-off obtained with the boundary in equation (2.32). As shown in figure 3.3 this is the case between points A and B, hence while $1.84 \lesssim S_{AB} \lesssim 1.99$. Observing how the angle φ changes with S_{AB} , we notice that the entanglement becomes weaker when S_{AB} increases and reaches $\varphi \approx 0.686$ for $S_{AB} \approx 1.99$.

We studied this strategy because it brings an analytic expression for the trade-off. We also numerically searched over general quantum strategies of this kind. As a result, we found that for any value of S_{AB} , the improvement in S_{AC} is at most 2×10^{-3} .

3.1.4 Double violation

Now we report an immediate way to see that with partially entangled states it is possible to outperform the result found with maximally entangled states. We can simply consider the three strategies presented in the previous sections and use shared randomness to combine them. Using equations (3.1), (3.2), (3.4), (3.5), (3.8) and

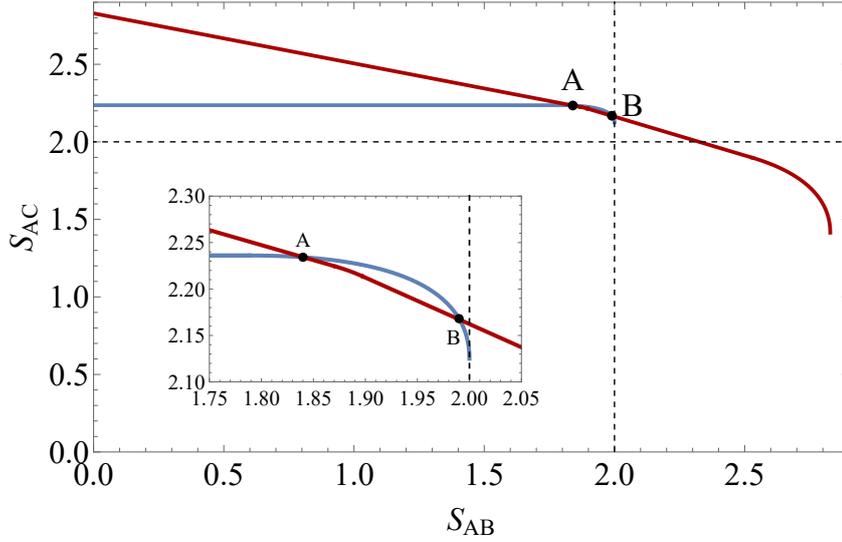


Figure 3.3: Plot - Type (III) strategy trade-off for partially entangled states φ (blue) compared with the boundary function (red). Subplot - Zoom of the region where a partially entangled state can outperform a maximally entangled one.

(3.9) obtain the following CHSH parameters

$$S_{AB} = 2(p_1 \sin(2\varphi) \cos \phi + p_1 \sin \phi + p_2 \cos(2\varphi) + p_3 \sin(\theta + 2\varphi)) \quad (3.13)$$

$$S_{AC} = 2p_1 \sin \phi + 2p_2 \cos(2\varphi) + p_3 \sin \theta (2 \sin(2\varphi) + 1) \quad (3.14)$$

where p_1 , p_2 , and p_3 are the probabilities to use the strategies of type (I), (II), and (III) respectively.

We can consider the simple case in which $\phi = 2\varphi$, $\theta = 2\varphi$, and $p_2 = 0$. With these simplifying assumptions and remembering that $p_1 + p_2 + p_3 = 1$, the relations for S_{AB} and S_{AC} become

$$S_{AB} = 2p_1 \sin(2\varphi) + (2 - p_1) \sin(4\varphi) \quad (3.15)$$

$$S_{AC} = \sin(2\varphi)(1 + p_1 + 2(1 - p_1) \sin(2\varphi)) \quad (3.16)$$

If we now consider the angle $\varphi = \frac{7\pi}{36}$ and impose the condition $S_{AB} = S_{AC}$, we find that for $p_1 \approx 0.644$ $S_{AB} = S_{AC} \approx 2.136$. Even if these choices for the strategy parameters are sub-optimal, this CHSH value is larger than what it is possible to obtain with maximally entangled states. Indeed if we impose the same $S_{AB} = S_{AC}$ constrain to equation (2.32), we find $S_{AB} = S_{AC} = 2.108 < 2.136$.

To better see how certain states can outperform maximally entangled ones, we considered generic measurements for Alice, Bob, and Charlie, unitaries for Bob as well as the distribution $\{p_\lambda\}_{\lambda=1}^3$. Then we numerically maximized S_{AC} for a given value of S_{AB} .

In figure 3.4 we illustrate the results of this maximization for some fixed values of φ together with the boundary given by equation (2.32). Notice that for both $\varphi = \frac{\pi}{6}$

and $\varphi = \frac{2\pi}{9}$ we found many points going beyond the boundary. In particular, as shown in the right plot of figure 3.4 the angle $\varphi = \frac{2\pi}{9}$ presents an improvement also in the double violation region.

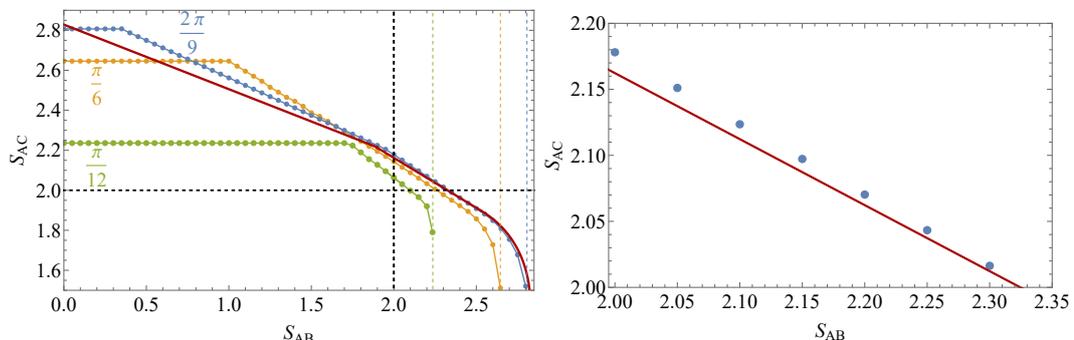


Figure 3.4: Left - Numeric optimization of a generic strategy for some values φ together with the boundary function (red). Right - Focus on the double violation region for $\varphi = \frac{2\pi}{9}$.

3.2 Double violation with generic pure entangled states

Now we are going to show that every pure entangled state $|\psi_\varphi\rangle$ can produce a double CHSH violation. We are proving this by considering only combinations of strategies of type (I) and (II) for simplicity. In particular, we are going to consider the strategies described in section 3.1.1 and 3.1.2. We are looking for the tangent line to the trade-off equations (3.3) passing through the point described by (3.4) and (3.5). Given this tangent, we can find its intersections with the lines $S_{AB} = 2$ and $S_{AC} = 2$. Finally, we can check whether the non-fixed coordinate of these points is greater than 2. If this is the case for both the points, we can conclude that there exists a region in the (S_{AB}, S_{AC}) -plane inside which we find both CHSH parameters violate the Bell inequality.

More in detail, to find the tangent we start by looking for the point of tangency $(x_1, f(x_1))$ by solving the following equation

$$f(x_1) - y_P = f'(x_1)(x_1 - x_P) \quad (3.17)$$

where the function $f(x)$ is simply equation (3.3) with the substitution $S_{AB} \rightarrow x$, x_P and y_P are the coordinates of the point given by equations (3.4) and (3.5).

$$f(x) = \frac{1}{1 + \sin^2(2\varphi)} \left(x + \sin(2\varphi) \sqrt{4 \left(1 + \sin^2(2\varphi) \right) - x^2} \right) \quad (3.18)$$

$$(x_P, y_P) = \left(2 \cos(2\varphi), 2\sqrt{1 + \sin^2(2\varphi)} \right) \quad (3.19)$$

Once obtained the point x_1 , the tangent line is simply $t(x) = f'(x_1)(x - x_1) + f(x_1)$. Finally we can find the coordinates of the intersection points $S_{AC}(S_{AB}=2) = (2, t(2))$ and $S_{AB}(S_{AC}=2) = (t^{-1}(2), 2)$ where $t^{-1}(x)$ is the inverse function of $t(x)$.

Unfortunately, both the expression for the tangent and these points are cumbersome, so we do not report it. On the other hand, we are going to plot the y-coordinate of $S_{AC}(S_{AB}=2)$ and the x-coordinate of $S_{AB}(S_{AC}=2)$ as a function of φ in figure 3.5. From the plot it is already possible to observe that these curves are greater than 2 for $\varphi \neq 0$, thus when $\varphi \neq 0$ both S_{AB} and S_{AC} exceed the local bound.

To be more quantitative we can expand the expression for these curves for $\varphi \approx 0$. We find

$$S_{AB}(S_{AC}=2) = 2 + 4(\sqrt{2} - 1)\varphi^2 + O(\varphi^3) \quad (3.20)$$

$$S_{AC}(S_{AB}=2) = 2 + 2(2 - \sqrt{2})\varphi^2 + O(\varphi^3) \quad (3.21)$$

This proves that both interception points are above the local bound, thus that for $\varphi \neq 0$ the tangent passes through the double violation region, namely $(S_{AB}, S_{AC}) > 2$, in the (S_{AB}, S_{AC}) -plane.

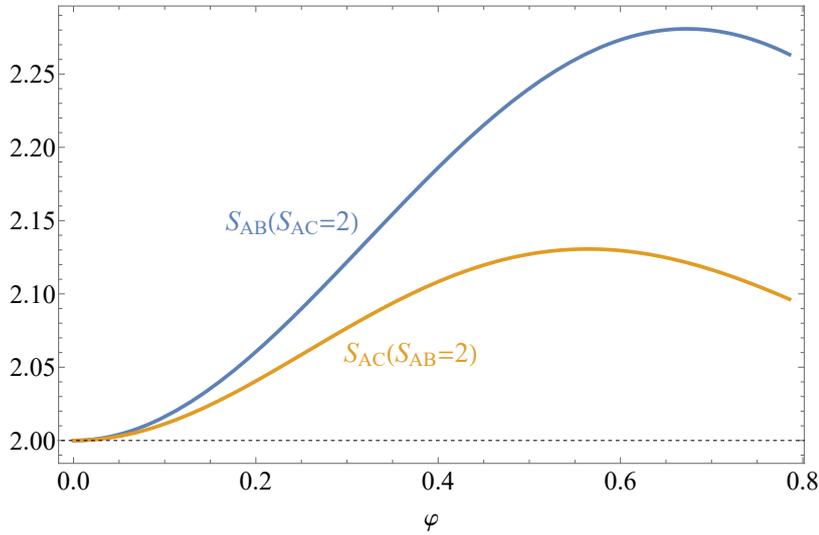


Figure 3.5: Dependence of the points $S_{AC}(S_{AB}=2)$ and $S_{AB}(S_{AC}=2)$ as a function of φ for the tangent line between the curve (3.18) and the point (3.19). For $\varphi \neq 0$ both the curves are always greater than 2, thus the tangent passes through the double violation region $(S_{AB}, S_{AC}) > 2$.

Chapter 4

Triple violation

Now we are going to present evidence that it is possible to reach 3 CHSH violations in the scenario in which, apart from Alice, we have three sequential observers on Bob's side. The initial state we are going to consider is again maximally entangled, specifically $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. For this mean, we are switching back to the more general formalism used in section 1.2. This way we are going to call the observers Alice, Bob⁽¹⁾, Bob⁽²⁾ and Bob⁽³⁾, their input x, y_1, y_2 and y_3 and their output a, b_1, b_2 and b_3 , each in $\{0, 1\}$, respectively. With this formalism their observables are $A_x, B_{y_1}^{(1)}, B_{y_2}^{(2)}, B_{y_3}^{(3)}$ and the measurements operators $A_{a|x}, B_{b_1|y_1}^{(1)}, B_{b_2|y_2}^{(2)}$ and $B_{b_3|y_3}^{(3)}$. Moreover, we are going to write the CHSH parameters between Alice and Bob⁽¹⁾, Bob⁽²⁾ and Bob⁽³⁾ as $S_1^{(\lambda)}, S_2^{(\lambda)}$ and $S_3^{(\lambda)}$ respectively and for each value of λ .

We can repeat an argument similar to the one already made at the beginning of chapter 2. In this case, though, while Alice is still allowed to perform two basis projections, i.e. rank-1 measurements, and make the state separable, the second Bob is not. Indeed after the measurement, the second Bob will relay his post-measurement state to the third Bob. Bob⁽³⁾ on the other hand will not relay his post-measurement state to anyone, thus he does not need to save any entanglement. For this reason, he can perform two basis projections too. As for Bob⁽¹⁾ and Bob⁽²⁾, they can perform a combination of trivial measurements and basis projections generalizing what is discussed in chapter 2. Afterward, they are allowed to apply unitary operators before sending the state to the following Bob. In the following, we are not going to give a complete description of the problem. We only intend to show a simple strategy producing 3 violations. Since in this example we are going to use only three kinds of strategies, we are not going to present all the possible combinations.

In particular, we are going to consider three cases:

- $\lambda = 1$ is the analogous of the first case considered in chapter 2. In this case, all the observers perform two basis projections. We can expect a Bell violation for the first CHSH parameter S_1 , but not for the following ones.
- $\lambda = 2$ provides for the second violation, $S_2 > 2$. Indeed in this case Bob⁽¹⁾ performs a trivial and a rank-1 measurement allowing a violation for S_2 , but

not for S_1 . Afterwards, Bob⁽²⁾ and Bob⁽³⁾ performs two rank-1 measurements. Since all the state arriving to Bob⁽³⁾ is separable, S_3 can not violate the Bell inequality.

- $\lambda = 3$ allows the third violation, $S_3 > 2$. In this case both Bob⁽¹⁾ and Bob⁽²⁾ perform one trivial and one rank-1 measurement. Finally only Bob⁽³⁾ performs two basis projections, thus S_1 and S_2 show a violation, but S_3 can.

4.1 Example strategy

We are now proposing a strategy showing that it is possible to reach three sequential violations. We built this strategy using the cases just described such that each case provides for a CHSH violation, while the other two parameters only nearly fail for this purpose.

When $\lambda = 1$, Alice's observables are $A_x = \frac{\sigma_X + (-1)^x \sigma_Z}{\sqrt{2}}$ and Bob⁽¹⁾'s ones are $B_0^{(1)} = \cos \phi \sigma_X + \sin \phi \sigma_Z$, $B_1^{(1)} = \sin \phi \sigma_X + \cos \phi \sigma_Z$. Moreover he applies the unitaries $U_0^{(1)} = \mathbb{1}$ and $U_1^{(1)} = e^{i(\phi - \frac{\pi}{4})\sigma_Y}$ with $U_{b_1 y_1}^{(1)} = U_{y_1}^{(1)}$. The other two Bob both measure $B_y^{(2)} = B_y^{(3)} = \cos \phi \sigma_X + \sin \phi \sigma_Z$ independently of y and do not use unitaries. The three CHSH parameters obtained are $S_1^{(1)} = 2\sqrt{2} \cos \phi$ and $S_2^{(1)} = S_3^{(1)} = \sqrt{2} (\cos \phi + \sin \phi)$.

When $\lambda = 2$ we can choose the following observables $A_x = \cos \hat{\phi} \sigma_X + (-1)^x \sin \hat{\phi} \sigma_Z$, $B_0^{(1)} = \mathbb{1}$ and $B_1^{(1)} = \sigma_Z$. Bob⁽²⁾ and Bob⁽³⁾ perform the same measurements, namely $B_0^{(2)} = B_0^{(3)} = \sigma_X$ and $B_1^{(2)} = B_1^{(3)} = \sigma_Z$. In this case, none of the observers performs any unitary ($U = \mathbb{1}$). With these choices we obtain the parameters $S_1^{(2)} = 2 \sin \hat{\phi}$, $S_2^{(2)} = \cos \hat{\phi} + 2 \sin \hat{\phi}$ and $S_3^{(2)} = \frac{\cos \hat{\phi}}{2} + \sin \hat{\phi}$.

Finally, if $\lambda = 3$ Alice measures $A_x = \cos \tilde{\phi} \sigma_X + (-1)^x \sin \tilde{\phi} \sigma_Z$. Bob⁽¹⁾ and Bob⁽²⁾ perform the same measurements $B_0^{(1)} = B_0^{(2)} = \mathbb{1}$ and $B_1^{(1)} = B_1^{(2)} = \sigma_Z$, and Bob⁽²⁾ measure $B_0^{(3)} = \sigma_X$ and $B_1^{(3)} = \sigma_Z$. As in the previous case, no unitaries are used. The CHSH parameters obtained with these measurements are $S_1^{(3)} = 2 \sin \tilde{\phi}$, $S_2^{(3)} = 2 \sin \tilde{\phi}$ and $S_3^{(3)} = \frac{1}{2} (\cos \tilde{\phi} + 4 \sin \tilde{\phi})$.

The final CHSH parameter is obtained by combining the parameters obtained for each value of λ as

$$S_i = p_1 S_i^{(1)} + p_2 S_i^{(2)} + p_3 S_i^{(3)} \quad (4.1)$$

where S_i with $i = \{1, 2, 3\}$ is the CHSH parameter between Alice and Bob⁽ⁱ⁾, p_1 , p_2 and p_3 are the probabilities associated with $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$ respectively.

There are several choices for the angles and probabilities to reach three sequential violations.

For example, we can choose as measurement angles $(\phi, \hat{\phi}, \tilde{\phi}) = (\frac{31\pi}{132}, \frac{88\pi}{245}, \frac{16\pi}{33})$ and impose the condition $S_1 = S_2 = S_3 \equiv S$. From the condition we get the values for the probabilities $p_1 \approx 0.086$, $p_2 \approx 0.019$ and $p_3 \approx 0.895$ and for the CHSH parameter

$S \approx 2.0023$. Although this is not a large violation, it still shows that three sequential violations are possible.

Chapter 5

Conclusions

We have proven the incorrectness of the common knowledge that sees weak measurements as necessary to produce multiple sequential violations of the CHSH inequality. Indeed we have shown that by allowing the observers to use some shared classical randomness it is possible to use projective measurements to obtain at least three sequential violations. Notice that the shared randomness does not modify the fundamental structure of CHSH, indeed this only affects the strategy type the observers are going to use. Once decided the type, for each strategy the observers can choose independently which measurements to perform. We have also studied in detail the optimal trade-off obtainable between the CHSH parameters in the case that there are only two sequential observers. Lastly, we have considered what happens if the party shares a generic two-qubit pure entangled state, instead of a maximally entangled one. In this case, we have shown that not only it is always possible to reach a double CHSH violation, but a non-maximally entangled state can outperform maximally entangled ones. These results are relevant not only from a conceptual point of view but can also be applied in the self-testing scenario. Indeed, it is possible to certify whether measurement devices implement weak measurements by comparing the CHSH parameters with the results of this thesis [31].

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