Università degli Studi di Padova

Department of Industrial Engineering
Master Degree in Aerospace Engineering

PRELIMINARY SIZING OF AN ELECTROSPRAY THRUSTER

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I want to hug my family who helped me at the beginning of my journey and to give me love, patience and funny moments whenever I was disoriented or sad.

Last but not least, I thank all my friends and special people I have known in these incredible years and that I want to meet again in life.

"Overhead the albatross hangs motionless upon the air
and deep beneath the rolling waves in labyrinths of coral caves.
The echo of a distant tide
comes willowing across the sand
and everything is green and submarine.

And no one showed us to the land
and no one knows the wheres or whys,
but something stirs and something tries
and starts to climb towards the light."

Pink Floyd: *Echoes*, 1971.

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Chapter 1

Abstract

Recently electrospray thrusters have raised a new interest in their simple adaptation to many spatial applications. As there isn't a precise model of how they work what I will try to do in this master thesis is to find a mathematical model that describes them in a simple way. After exposing some physical concepts I will study the physics of conductive liquids and then study the dynamics of the fluid during the operation with the purpose of calculating their propulsive parameters. In the end all the equations found will be grouped together to implement in Matlab a code that can calculate the performances of these thrusters with different initial conditions.

Chapter 2

Introduction

Electrospray thrusters are electrostatic accelerators of charged particles produced from electrified liquid surfaces. At the moment, there are three types of electrospray thruster technologies:

1. Colloid thrusters, which are accelerators of charged droplets or ions, using solvents such as doped glycerol and formamide [$HCONH_2$] as propellants;

2. Field Emission Electric Propulsion (FEEP), which makes use of liquid metals, typically cesium (Cs) and lanthanides (Ln), and produce positively charged metallic ions;

3. Ionic Liquid Ion Sources (ILIS), that use room-temperature molten salts, also known as ionic liquids, and produce salt ion beams, or mixtures of ions and droplets.

A simple conceptual scheme of an electrospray thruster is shown in figure 2.1. We can see the emitter, the electrode (d), the power supply that gives the voltage V and the steady state dynamic of the conductive liquid (Taylor's cone, cone-jet, droplets and fragmentation).

The first form of electrospray propulsion (which can be tracked back to the beginning of the 20th century) came in the form of colloid thrusters. They were intensively studied from 1960 to 1975 as an alternative to normal ion engines. Their appeal at that time rested with the large molecular mass of the droplets, which was known to increase the thrust density of an ion engine. This is because the accelerating voltage is:

$$V = \frac{mc^2}{2q}$$

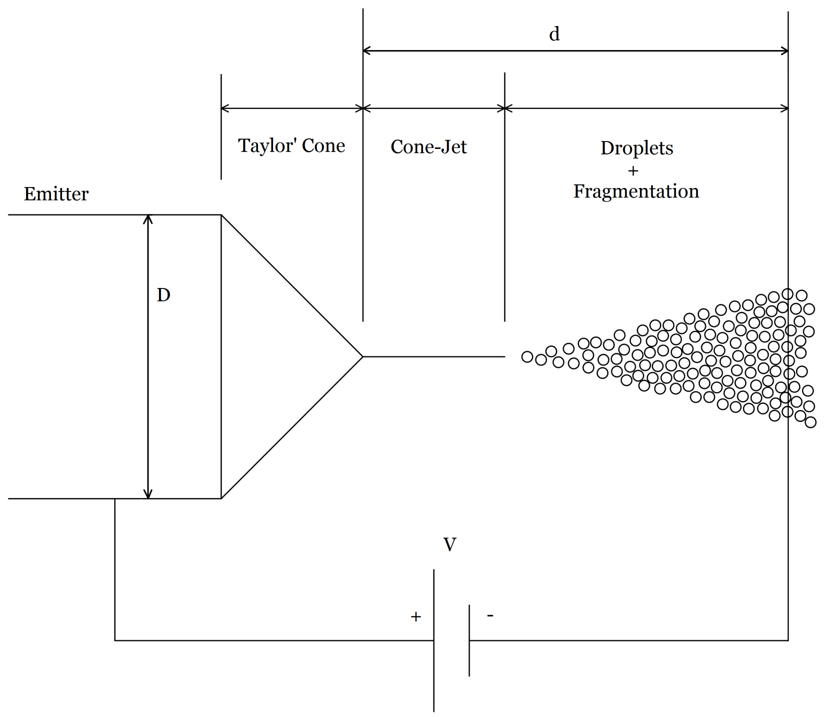

Figure 2.1: Electrospray thruster.

where:

- m: mass of a single droplet (or ion);
- q: electric charge of a single droplet (or ion);
- c: final speed reached by the droplets at the end of the applied electric field.

If c is mission requisite then the voltage V can be increased enhancing the ratio m/q. This also increases both the space charge limited current density (given by the Child-Langmuir's law):

$$j = \frac{4\varepsilon_0}{9}\sqrt{\frac{2q}{m}}\frac{V^{3/2}}{d^2}$$

and the thrust density, which is proportional to V^2 and therefore to $(m/q)^2$ (d is the grid spacing of the electrodes). In addition to the higher thrust density, the higher voltage also increases the efficiency since any cost-of-ion

voltage (V_{loss}) becomes then less significant.
Although everything seems to work well, high ratios of m/q create serious insulation and packaging problems, that made these devices unattractive, in spite of their demonstrated good performance ($I_{sp} \sim 1000$ s). In addition, the droplet generators were usually composed of arrays of a large number of individual liquid-dispensing capillaries, each providing a thrust of the order of 1 μN. This required fairly massive arrays, further discouraging implementation.

Nowadays, after years of lay-off on the subject, a strong interest on these devices is born again thanks to:

- the new emphasis on miniaturization of spacecraft. The very small thrust per emitter now becomes a positive feature, allowing designs with both fine controllability and high performance;

- the advances made by electrospray science in the following years. These have been motivated by other applications of electrospraying such as, in recent years, the extraction of charged biological macromolecules from liquid samples for very detailed mass spectroscopy. These advances now offer the potential for overcoming previous limitations on the specific charge q/m of droplets, and therefore may allow operation at more comfortable voltages (1-5 kV);

- the advances in micro-manufacturing technologies allow for efficient clustering of a large number of emitter tips on a small surface, potentially to the point of competing with plasma thrusters (ion or even Hall) in achievable current density.

One essential advantage of electrospray engines for space applications is the fact that no gas phase ionization is involved with their very small thrust levels. In these devices, as we will see, the charging mechanisms are variations of an electric field on the surface of a conductive liquid; small sizes naturally enhance local electric fields and facilitate this effect. In other electrical engines, attempts to miniaturize them (ion engines, Hall thrusters, arcjets) lead to the need to reduce the ionization mean free path $(\sigma_{ion} n_e)^{-1}$ by increasing n_e and the heat flux and energetic ion flux to walls as well; this leads inevitably to life reductions of the device.
As stated in many propulsion books, these thrusters are presently under development. [1]

From this overview, my supervisor and I got more keen on this subject and finally have decided to study the physics of these devices to develop a program for calculating their performance. The question we would like to answer is:

> *if I apply a given voltage to a simple electrospray thruster (with both the geometry and the fluid properties known), how much thrust will I get?*

So, starting from what we asked, the purpose of my thesis is to create a mathematical model to describe an electrospray thruster by studying the equations that are already present in scientific literature. During the work we dealt with problems of different nature: after searching for the relevant equations to our case we realized that the mathematical problem was undetermined (which is not surprising if one knows the embryonic state of art of this research field). Consequently, we had to find the last equations to create the model and implement it on a computer. My thesis, after a brief description of the basic physics involved, explains how we have found these equations and how they work together with the older and known physics laws.

Chapter 3

Basic physics

In colloid thrusters we have an interaction between two of the biggest phenomena ever: fluid mechanics and electrostatics. As I will describe in the next chapter, an ionic fluid is a conductive liquid, or rather, a fluid that can bring an electric current inside it. Therefore to model a conductive liquid we must put in relation the fundamental laws of both fluid mechanics and electrostatics.

3.1 Fluid mechanics

The most useful basic concepts of fluid mechanics to understand how an electrospray thruster works are:

- Continuity Equation (or mass conservation);
- the Laplace equation (for the surface tension).

We can also put the momentum equation but it's not interesting for what concerns the ionic fluid; it is the same that describes the thrust T in the basic propulsion physics and so it is always valid, also in our case:

$$T = \dot{m}c \tag{3.1}$$

The difference lies on how the thrust reacts: while in a rocket (or jet) engine the resulting pressure distribution, given by the momentum equation, reacts along the internal walls of the structure, in a colloid thruster this distribution works with capillary phenomena on the emitter's inner surface. So we mustn't worry to find another equation for the thrust, but rather to understand how to calculate the mass flow rate and the exhaust velocity with the geometry of the emitter, the fluid properties and the starting voltage V.

3.1.1 Continuity Equation

The principle of mass conservation (also known as the Continuity Equation) states that the mass $\mathcal{M}$ associated with a fluid portion that at the time t occupies the material volume $\mathcal{V}$ doesn't change with the motion of $\mathcal{V}$ and its variation depends only on the number of sources or wells inside $\mathcal{V}$.

Consider a finite volume $\mathcal{V}(\mathbf{x}, t)$ in a space $\{\mathbf{x}, t\}$, $\mathbf{x} \in \mathbb{R}^3$, in which the fluid density $\rho(\mathbf{x}, t)$ is defined:

$$\rho(\mathbf{x}, t) = \lim_{d\mathcal{V} \to 0} \frac{d\mathcal{M}}{d\mathcal{V}} \tag{3.2}$$

The mass inside the volume will be:

$$\mathcal{M} = \int_{\mathcal{V}} \rho(\mathbf{x}, t)\, d\mathcal{V} \tag{3.3}$$

If there are no sources or wells, we can translate what is stated in the principle above putting the total time derivative of mass $\mathcal{M}$ equal to zero:

$$\frac{d\mathcal{M}}{dt} = \frac{d}{dt} \int_{\mathcal{V}} \rho(\mathbf{x}, t)\, d\mathcal{V} = 0 \tag{3.4}$$

Now we meet a problem: we can't move the time derivative into the integral because of the time dependence of variable $\mathcal{V}$. But the Reynolds' transport theorem helps us.

Reynolds's transport theorem

Without showing the mathematical demonstration, the Reynolds' transport theorem permits to bring the time derivative into the integral also when the integrating volume $\mathcal{V}$ is time dependent. In other words, consider a generic function $\mathcal{F}(\mathbf{x}, t)$ inside the material volume $\mathcal{V}$, we can write the total time derivative of the integral of $\mathcal{F}$ on $\mathcal{V}$ in the following way:

$$\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F}(\mathbf{x},t)\, d\mathcal{V} \;=\; \int_{\mathcal{V}} \left(\frac{d\mathcal{F}}{dt} + \mathcal{F}\nabla\cdot\mathbf{v}\right) d\mathcal{V} \tag{3.5}$$

Now, recalling that the total derivative of a function $\mathcal{F}(\mathbf{x},t)$ of several variables can be written as:

$$\begin{aligned} \frac{d\mathcal{F}}{dt} &= \frac{\partial\mathcal{F}}{\partial t} + \nabla\mathcal{F}\cdot\frac{d\mathbf{x}}{dt} \\ &= \frac{\partial\mathcal{F}}{\partial t} + \nabla\mathcal{F}\cdot\mathbf{v} \end{aligned} \tag{3.6}$$

if we put equation (3.6) into (3.5) finally we have:

$$\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F}(\mathbf{x},t)\, d\mathcal{V} \;=\; \int_{\mathcal{V}} \left[\frac{\partial\mathcal{F}}{\partial t} + \nabla\cdot(\mathcal{F}\mathbf{v})\right] d\mathcal{V} \tag{3.7}$$

which is the most famous form of the Reynolds' transport theorem.

So we can use equation (3.7) putting $\mathcal{F} = \rho(\mathbf{x},t)$ to obtain the integral form of the continuity law:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho(\mathbf{x},t)\, d\mathcal{V} \;=\; \int_{\mathcal{V}} \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] d\mathcal{V} \tag{3.8}$$

If there are no sources or wells we can use equation (3.4) to find a local form of the principle:

$$\int_{\mathcal{V}} \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] d\mathcal{V} \;=\; 0$$

The integral must be null for each volume $\mathcal{V}$ taken into account. The only way to get this is that the integrating function is always equal to zero, so:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) \;=\; 0 \tag{3.9}$$

that represents the conservative local form of the Continuity Equation. If we suppose a steady state motion (that means all $\partial/\partial t\,(\dots) = 0$) and an incompressible fluid ($\rho = \text{UNIFORM}$) the Continuity Equation becomes [4]:

$$\boxed{\nabla \cdot \mathbf{v} = 0} \tag{3.10}$$

We can find also the non conservative local form of the Continuity Equation substituting (3.6) in (3.9), which is:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{3.11}$$

3.1.2 Surface tension

Each liquid is characterized by its surface tension γ that is the surface density of binding energy at the interface between a continuous body and a material of a different nature, for example a solid, a liquid or a gas. It derives from the one force reacting on a separation surface which presents curvatures. It is defined as a force per unit length and it depends on both the geometry of the surface and the material. It's related with the pressure difference Δp between the two faces of the surface by the Laplace equation:

$$\boxed{\Delta p = \tau = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).} \tag{3.12}$$

where $1/R_1$ and $1/R_2$ are two principal curvatures of the fluid. Equation (3.12) also states that each liquid has a specific breaking tension (τ) which, if it's exceeded, leads to the separation of the liquid particles. [4]

3.2 Electrostatics

With regard to electrostatics the fundamental concepts for my thesis are:

- electric current;
- Coulomb and Gauss theorems;
- Maxwell's equations;
- Continuity Equation;
- Child-Langmuir's law.

Let's go and analyze them.

3.2.1 Electric current

Operational definition

Consider a conductor of section $\mathcal{S}$ through which there is an orderly charge motion. Electric current is defined as the amount of electric charge Δq which in the time interval Δt crosses the surface $\mathcal{S}$:

$$I = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt} \tag{3.13}$$

Analytical definition

Always referring to a conductor of section $\mathcal{S}$ through which there is an orderly charge motion, consider the density of the number of charge carriers in a point of the section (n), each of them of charge q. Charge carriers move at an instantaneous velocity $\mathbf{v}$, called DRIFT SPEED, mediated on all carriers present at that point at that instant which is parallel or opposite to the direction of the electric field and of several orders of magnitude less than the thermal stirring speed of single particles. We define the electric charge density at that point as:

$$\sigma_{\mathcal{V}}(\mathbf{x}, t) = n(\mathbf{x}, t)q \tag{3.14}$$

The CURRENT DENSITY at a point $\mathbf{x}$ at time t is the vector given by the product of the electric charge density $\rho_{\mathcal{V}}(\mathbf{x},t)$ and the drift speed:

$$\mathbf{j}(\mathbf{x},t) \;=\; \rho_{\mathcal{V}}(\mathbf{x},t)\mathbf{v}(\mathbf{x},t) \;=\; n(\mathbf{x},t)q\,\mathbf{v}(\mathbf{x},t) \tag{3.15}$$

The current density is parallel to the drift speed but its direction depends on the charge of the carrier itself: it has the same direction of the drift speed in case of positive charge and vice versa. The electric current across the surface $\mathcal{S}$ is the flow through the surface of the electric current density:

$$I \;=\; \int_{\mathcal{S}} \mathbf{j} \cdot \hat{\mathbf{n}}\, d\mathcal{S} \tag{3.16}$$

in which $\hat{\mathbf{n}}$ is the normal to the surface $\mathcal{S}$ arbitrarily taken.

3.2.2 Coulomb and Gauss theorems and Maxwell's equations

The Gauss' theorem states that given a closed surface $\mathcal{S}$ containing any number of electric charges (positive or negative), the flux of the electric field $\mathbf{E}$ generated by the charges through this surface is equal to the ratio of the algebraic sum of the charges contained within the closed surface and the dielectric constant ε of the medium in which the charges are located (ε_0 in the vacuum):

$$\Phi_{\mathcal{S}}(\mathbf{E}) \;=\; \frac{\sum q_i}{\varepsilon} \tag{3.17}$$

If a space (or volumetric) electric charge density $\rho_{\mathcal{V}}(\mathbf{x},t)$ exists, the sum of the charges inside the volume $\mathcal{V}$ with his frontier $\partial\mathcal{V} = \mathcal{S}$ becomes:

$$\sum q_i \;=\; \int_{\mathcal{V}} \rho_{\mathcal{V}}\, d\mathcal{V} \tag{3.18}$$

The combination between (3.17) and (3.18) gives the integral form of the Gauss' theorem. As before we can find a local form of the theorem using the divergence theorem: with the mathematical definition of flux equation (3.17) (combined with (3.18)) becomes:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, d\mathcal{S} \;=\; \frac{1}{\varepsilon}\int_{\mathcal{V}} \rho_{\mathcal{V}}\, d\mathcal{V} \tag{3.19}$$

in which $\hat{\mathbf{n}}$ is the outward normal to the surface. So using the divergence theorem for the electric field and and then equalizing the two functions inside both integrals (which are the same) we have:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} \, d\mathcal{S} \;=\; \int_{\mathcal{V}} \nabla \cdot \mathbf{E} \, d\mathcal{V} \;=\; \frac{1}{\varepsilon} \int_{\mathcal{V}} \rho_{\mathcal{V}} \, d\mathcal{V} \tag{3.20}$$

that yields (assuming the independence of ε from $\mathcal{V}$) the local form of the Gauss' theorem:

$$\nabla \cdot \mathbf{E} \;=\; \frac{\rho_{\mathcal{V}}}{\varepsilon} \tag{3.21}$$

We can write the local form of the Gauss' theorem not considering the electric field vector but the electric displacement vector $\mathbf{D} = \varepsilon \mathbf{E}$ to find another helpful equation: in case of linear, homogeneous and isotropic material ($\varepsilon = \text{CONSTANT}$) we can write equation (3.21) in the following way:

$$\nabla \cdot (\varepsilon \mathbf{E}) \;=\; \nabla \cdot \mathbf{D} \;=\; \rho_{\mathcal{V}} \tag{3.22}$$

in which $\rho_{\mathcal{V}}$ doesn't take into account the polarization charges but only the free charges.

The second important theorem that will be useful is the Coulomb's theorem which affirms that, given a conductive body whose surface is characterized by a surface charge density $\sigma_{\mathcal{S}}$, the electric field produced near the surface is:

$$\mathbf{E} \;=\; \frac{\sigma_{\mathcal{S}}}{\varepsilon} \, \hat{\mathbf{n}} \tag{3.23}$$

where $\hat{\mathbf{n}}$ is always the normal to the body surface. In other words the theorem states that near a surface of a body in electrostatic equilibrium the electric field is orthogonal to the surface.

Maxwell's equations are a set of four equations (one of these is the Gauss' theorem already seen) that entirely describe the behavior of the electromagnetic phenomena. There are both the integral and the local form of these equations but for the purposes of this thesis I'm going to consider only the local form. Recalling that the electric displacement is $\mathbf{D} = \varepsilon \mathbf{E}$ and the magnetic induction is $\mathbf{B} = \mu \mathbf{H}$ the differential form of Maxwell's equations is:

$$\nabla \cdot \mathbf{D} = \rho_{\mathcal{V}} \tag{3.24a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.24b}$$

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{3.24c}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \tag{3.24d}$$

where $\mathbf{j}$ is the current density (previously defined) in the region in which $\mathbf{E}$ and $\mathbf{B}$ exist.

3.2.3 Continuity Equation

In the same way that we have defined a continuity equation for a fluid body we can describe a similar behavior for the electric charges which is known in literature as the Continuity Equation of the electric charge. This principle can be demonstrated in more than one way. I'll demonstrate it using the first and the last Maxwell's equations in the above list.

Consider equation (3.24d): with the tensorial calculus if we take the divergence of both sides of the equation we obtain:

$$\nabla \cdot \nabla \times \mathbf{H} = 0 \tag{3.25a}$$

$$\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} + \nabla \cdot \mathbf{j} = \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} + \nabla \cdot \mathbf{j} \tag{3.25b}$$

At this point we can substitute the first Maxwell's equation (3.24a) into equation (3.25b) and then rewrite the fourth Maxwell's equation applying the divergence operator; after some calculations we obtain the local form of the Continuity Equation for the electric charge, that is:

$$\frac{\partial \rho_{\mathcal{V}}}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{3.26}$$

As for the Continuity Equation for a fluid, considering a stationary state the Continuity Equation for the electric charge assumes the form:

$$\boxed{\nabla \cdot \mathbf{j} = 0} \tag{3.27}$$

3.2.4 Child-Langmuir's law

Let's now describe probably the most important concept for an electrospray thruster, the Child-Langmuir's law [5]. It explains the interrelation of the electrical and dynamical parameters in a one-dimensional model of an ion beam.

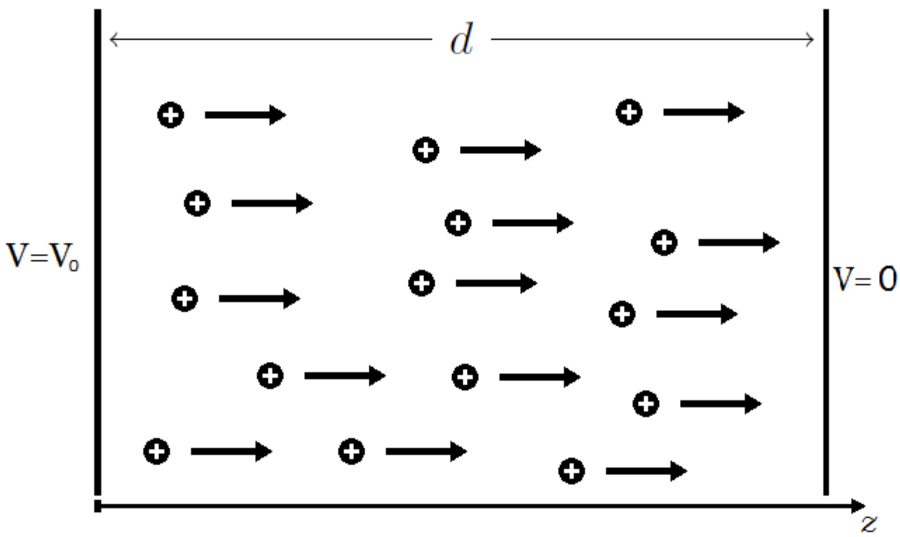

Figure 3.1: One-dimensional ion beam.

From figure 3.1 let z be the streamwise coordinate and $z = 0$ the position of the source at potential $V(0) = V_0$. The potential $V(z)$, the electric field $E(z) = -(dV/dz)$, the ion density $n(z)$ and the velocity $v(z)$ are all functions of z. In a steady state motion, equation (3.27) states that the current density is a constant along z:

$$
\begin{aligned}
&\nabla \cdot \mathbf{j} \;=\; 0 \quad\Longrightarrow\quad \frac{dj}{dz} \;=\; 0 \\
\Longrightarrow\quad & j(z) \;=\; n(z) q\, v(z) \;=\; \text{UNIFORM}
\end{aligned}
\tag{3.28}
$$

The ion velocity follows from the energy conservation: for ions of charge q and mass m emitted with negligible velocity at the source the principle gives:

$$
\begin{aligned}
q[V_0 - V(z)] \;&=\; \frac{1}{2} m v^2(z) \\
v(z) \;&=\; \left\{ \frac{2q\big[V_0 - V(z)\big]}{m} \right\}^{1/2}
\end{aligned}
\tag{3.29}
$$

The potential V is related to the charge density n by the Poisson's law, that is an extension of the first Maxwell's equations:

$$
\begin{aligned}
\nabla^2 V \;&=\; \frac{d^2 V}{dz^2} \;=\; -\frac{n(z) q}{\varepsilon_0} \\
&=\; -\frac{j}{\varepsilon_0 v(z)} \;=\; -\frac{j}{\varepsilon_0} \left\{ \frac{m}{2q[V_0 - V(z)]} \right\}^{1/2}
\end{aligned}
\tag{3.30}
$$

This may be integrated simply multiplying by $2(dV/dz)$ both sides of the equation; after calculating the primitive functions we obtain:

$$
\left(\frac{dV}{dz}\right)^2 - \left(\frac{dV}{dz}\right)^2_{z=0} \;=\; \frac{4j}{\varepsilon_0}\left[\frac{m(V_0 - V)}{2q}\right]^{1/2}
\tag{3.31}
$$

To simplify the problem the electric field at $z = 0$ is taken equal to 0. In this way we can calculate the highest possible current density in the beam, a limit which can't be overcome:

$$
\left(\frac{dV}{dz}\right)_{z=0} = 0 \quad\Longrightarrow\quad j \;=\; j_{\text{MAX}}.
\tag{3.32}
$$

After making these assumptions equation (3.31) together with the boundary conditions (3.32) leads to the following Cauchy's problem:

$$\begin{cases} \dfrac{dV}{dz} = -2\left(\dfrac{j}{\varepsilon_0}\right)^{1/2}\left\{\dfrac{m(V_0 - V)}{2q}\right\}^{1/4} \\ \\ V(0) = V_0 \end{cases} \tag{3.33}$$

in which the differential equation has separate variables and therefore can be easily solved. The final solution of (3.33) (which is unique) is:

$$V(z) = V_0 - \left[\frac{3}{2}\left(\frac{j}{\varepsilon_0}\right)^{1/2}\left(\frac{m}{2q}\right)^{1/4} z\right]^{4/3} \tag{3.34}$$

and if we put $z = d$ we can find the maximum value of the current density in the beam with a given applied voltage V_0 or rather the Child-Langmuir's law for a one-dimensional ion beam [5]:

$$\boxed{j_{\text{MAX}} = \frac{4\varepsilon_0}{9}\sqrt{\frac{2q}{m}}\frac{V^{3/2}}{d^2}} \tag{3.35}$$

3.3 Electric propulsion

In this section the fundamental relationships of electric propulsion will be shown. We start recalling the already known thrust equation (3.1):

$$T = \dot{m}c \tag{3.36}$$

in which the ions speed c can easily be calculated from the principle of energy conservation:

$$c = \sqrt{\frac{2qV}{m}} \tag{3.37}$$

Concerning the flow rate $\dot{m}$, we can proceed as follows: ideally an ion beam in electrospray propulsion is made by ions with different but constant ratios m/q: this means that their time derivative is equal to 0 which leads to the following relationship:

$$\begin{aligned} \frac{d}{dt}\left(\frac{m}{q}\right) &= \frac{\dot{m}q - m\dot{q}}{q^2} = 0 \\ \implies \quad \dot{m}q &= m\dot{q} \end{aligned} \tag{3.38}$$

and from equation (3.13) we obtain:

$$\boxed{\dot{m} = I \cdot \frac{m}{q}} \tag{3.39}$$

Now we are capable to calculate the thrust T. Consider a cylindrical beam of section $\mathcal{A}$: the electric current can be expressed, according to (3.16), by:

$$I = \mathcal{A}j \tag{3.40}$$

So putting together equations (3.40), (3.39), (3.37) into (3.36) the result is:

$$\boxed{T = j\mathcal{A}\sqrt{\frac{2mV}{q}}} \tag{3.41}$$

We can also define on the thrust density across $\mathcal{A}$:

$$\boxed{\frac{T}{\mathcal{A}} = j\sqrt{\frac{2mV}{q}}} \tag{3.42}$$

If we are in conditions to apply Child-Langmuir's law, it's possible to calculate the highest thrust (or thrust density) given by the ion beam by using equation (3.35). In this case the thrust density is:

$$\frac{T}{\mathcal{A}} = \frac{\varepsilon_0}{2}\left(\frac{4}{3}\frac{V}{d}\right)^2 \tag{3.43}$$

and we can see that $T/\mathcal{A}$ increases with V^2.

Finally we define the efficiency of the propulsion system as:

$$\eta = \frac{T^2}{2\dot{m}V_0 I} \tag{3.44}$$

Chapter 4

Physics of ionic liquids

4.1 Basic physics of ionic liquids

A conductive liquid is a liquid that deforms under the action of an electric field. This is due to the presence of charged free ions that can move freely in the volume of the liquid and therefore generate a current density which is related with the internal electric field by:

$$\mathbf{j} = K\mathbf{E} \tag{4.1}$$

in which K is the liquid conductivity.

Consider first a flat liquid surface subjected to a strong normal electric field $\mathbf{E}_0$: if the liquid is conductive an internal electric field $\mathbf{E}_i$ appears and free ions with an attractive polarity will be concentrated on it's surface as shown in figure 4.1. Suppose a perfectly conducting liquid ($\varepsilon_r \to +\infty$, $\mathbf{E}_i = 0$) and

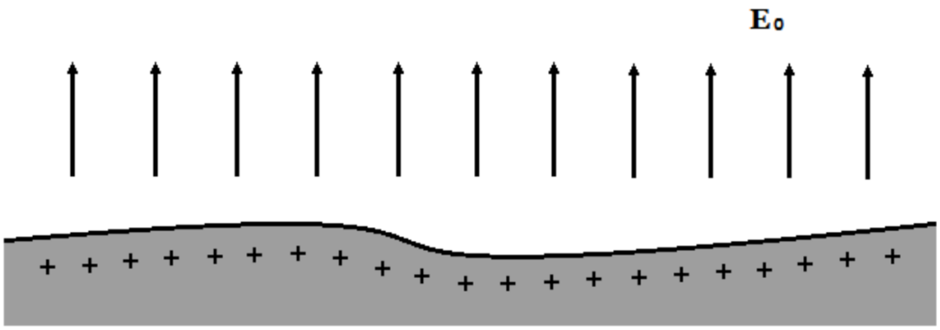

Figure 4.1: Conductive liquid.

a uniform volumetric charge density $\rho_\mathcal{V}$. We can determine its surface charge density $\sigma_\mathcal{S}$ by applying the first Maxwell's equation (3.24a) in the integral form to a control volume $\mathcal{V}$ as shown in the figure 4.2:

$$\int_\mathcal{V} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \int_\mathcal{V} \frac{\rho_\mathcal{V}}{\varepsilon_0}\, d\mathcal{V} \tag{4.2}$$

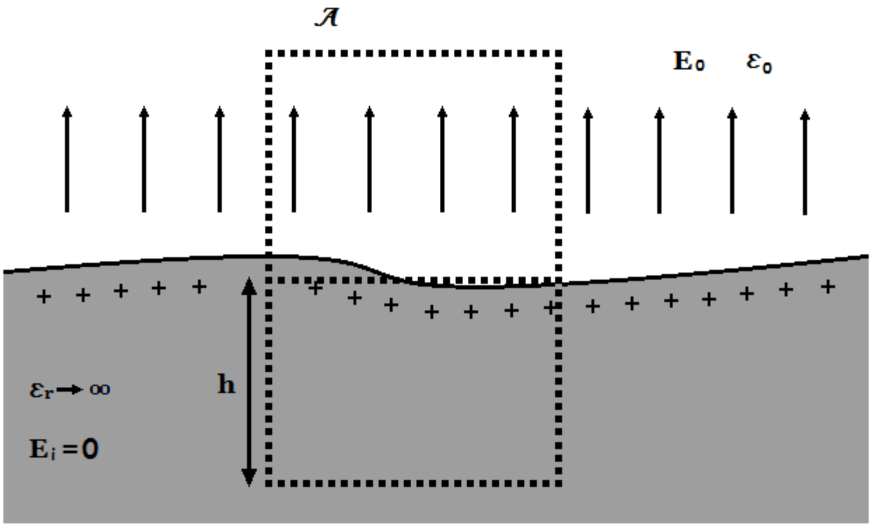

Figure 4.2: Control volume with a perfect conductor.

The first member of equation (4.2), thanks the divergence theorem, yields:

$$\int_\mathcal{V} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \int_\mathcal{A} \mathbf{E} \cdot \hat{\mathbf{n}}\, d\mathcal{S} = E_0 \mathcal{A} \tag{4.3}$$

while the second member gives:

$$\int_\mathcal{V} \frac{\rho_\mathcal{V}}{\varepsilon_0}\, d\mathcal{V} = \frac{\rho_\mathcal{V}}{\varepsilon_0} \mathcal{A} h. \tag{4.4}$$

Now, substituting (4.3) and (4.4) in (4.2) finally we obtain the surface charge density [1]:

$$\sigma_\mathcal{S} = \rho_\mathcal{V} h = \varepsilon_0 E_0 \tag{4.5}$$

A similar effect occurs with dielectric liquids ($\varepsilon_r < +\infty$) even though there are no free charges; in this case an internal electric field $\mathbf{E}_i$ appears as shown in figure 4.3 and because there aren't free charges the first Maxwell's law gives:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{\mathcal{A}} \mathbf{D} \cdot \hat{\mathbf{n}}\, d\mathcal{S} = (\varepsilon_0 E_0 - \varepsilon_r \varepsilon_0 E_i)\mathcal{A} = 0$$

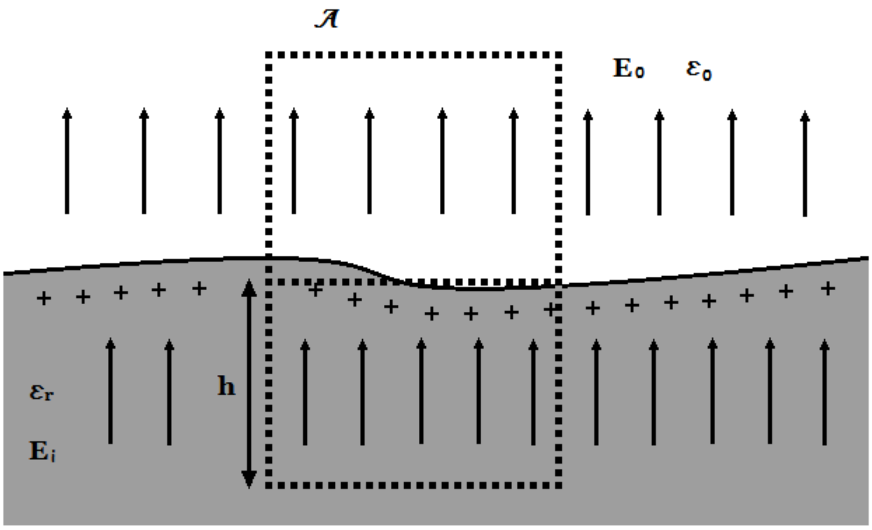

Figure 4.3: Control volume with a dielectric.

which yields the relationship between the internal electric field of the fluid and the applied external electric field in the absence of free charges:

$$\begin{aligned} \varepsilon_0 E_0 - \varepsilon_r \varepsilon_0 E_i &= 0 \\ \Longrightarrow E_i &= \frac{1}{\varepsilon_r} E_0 \end{aligned} \tag{4.6}$$

Now because there aren't free charges but there are polarized charges we can apply the Coulomb's theorem to the volume $\mathcal{V}$ considering a constant volumetric polarization charge density $\rho_{\mathcal{V}D}$ to get the surface charge density $\sigma_{\mathcal{S}D}$ for a dielectric [1]:

$$\begin{aligned} \int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} &= \int_{\mathcal{V}} \frac{\sigma_{\mathcal{S}D}}{\varepsilon_0}\, d\mathcal{V} \\ \sigma_{\mathcal{S}D} &= \rho_{\mathcal{V}D} h = \varepsilon_0 (E_0 - E_i) \end{aligned} \tag{4.7}$$

Now combining (4.6) and (4.7) we obtain the relationship between the surface charge density and the external electric field:

$$\sigma_{\mathcal{S}D} = \left(1 - \frac{1}{\varepsilon_r}\right) \varepsilon_0 E_0 \tag{4.8}$$

which if $\varepsilon_r \to +\infty$ is similar to (4.5).

First recalling that only in a perfect conductor the normal and tangential components E_{i_n} and E_{i_t} of the internal electric field are necessarily null, in a dielectric with $\varepsilon_r \gg 1$ the term:

$$E_{i_n} = \frac{1}{\varepsilon_r} E_0$$

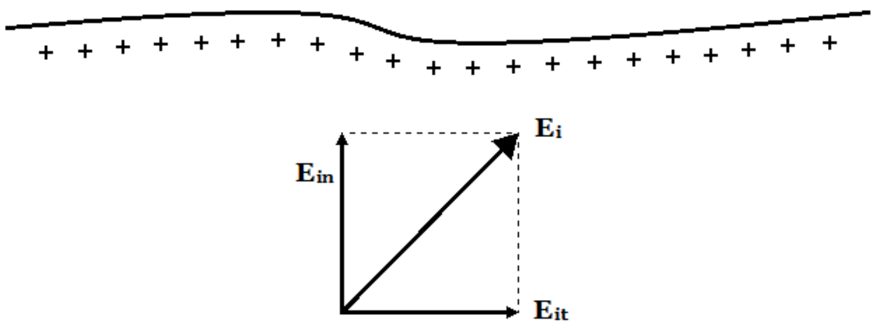

Figure 4.4: Normal and tangential components of $\mathbf{E}_i$.

can disappear but the tangential component, if exists, must be continuous through the interface of the liquid:

$$\frac{\partial E_{i_t}}{\partial y} \in C^0.$$

Consider a conductive liquid with a conductivity K, normally due to the motion of the ions of both polarities. If their concentration is $n^+ = n^- = n$ [m^{-3}] and their mobilities μ^+, μ^- [V/s] then:

$$K = ne(Z^+\mu^+ + Z^-\mu^-) \quad \left[\frac{\mathrm{Si}}{\mathrm{m}}\right]. \tag{4.9}$$

If there is a constant external electric field $\mathbf{E}_0$ applied on the gas side and initially the surface is uncharged, then the electric field draws positive ions to it (positive if $\mathbf{E}_0$ points away from the liquid) generating a variable surface free charge density σ_{SD} at a time rate [1]:

$$j = \frac{d\sigma_{SD}}{dt} = KE_i \tag{4.10}$$

So applying the first Maxwell's equation assuming a volumetric free charge density $\rho_{\mathcal{V}D}$ we find:

$$\varepsilon_0 E_0 - \varepsilon_r \varepsilon_0 E_i = \sigma_{SD} \tag{4.11}$$

in which eliminating E_i with equation (4.10) and set the initial conditions of uncharged surface at the time $t = 0$, $\sigma_{SD}(0) = 0$, we have to solve the following Cauchy's problem [1]:

$$\begin{cases} \dfrac{d\sigma_{SD}}{dt} + \dfrac{K}{\varepsilon_r\varepsilon_0}\sigma_{SD} = \dfrac{K}{\varepsilon_r}E_0 \\ \\ \sigma_{SD}(0) = 0 \end{cases} \tag{4.12}$$

whose solution is:

$$\sigma_{SD}(t) = \varepsilon_0 E_0 (1 - e^{-\frac{t}{\tau}}) \tag{4.13}$$

τ it's called 'charges relaxation time' that is the time after which the surface is charged at 66% of its final value:

$$\tau = \frac{\varepsilon_r \varepsilon_0}{K}. \tag{4.14}$$

4.2 Surface stability

4.2.1 General stress state

Consider an infinitesimal element of a continuous body subject to a force $\mathbf{F}$ and let's assume that this force generates an internal stress state $\boldsymbol{\tau}$ whose resultant is the force $\mathbf{F}$:

$$\mathbf{F} = \oint_{\mathcal{A}} \boldsymbol{\tau} \, d\mathcal{S} \tag{4.15}$$

We assume the Cauchy's Continuum Hypotheses that can be shortly described by:

$$\boldsymbol{\tau} = [\boldsymbol{\sigma}]\hat{\mathbf{n}} \tag{4.16}$$

where $[\boldsymbol{\sigma}]$ is a 3x3 symmetric tensor that describes for each point in the body volume the total stress state. $\hat{\mathbf{n}}$ is the normal to the surface $\mathcal{A}$. From the divergence theorem we have:

$$\mathbf{F} = \oint_{\mathcal{A}} [\boldsymbol{\sigma}]\hat{\mathbf{n}} \, d\mathcal{S} = \int_{\mathcal{V}} \nabla \cdot [\boldsymbol{\sigma}] \, d\mathcal{V} \tag{4.17}$$

The quantity inside the last integral is called force per unit volume:

$$\mathbf{f} = \nabla \cdot [\boldsymbol{\sigma}] \tag{4.18a}$$

$$f_j = \frac{\partial \sigma_{ij}}{\partial x_i}, \quad \text{i,j} = 1,2,3 \tag{4.18b}$$

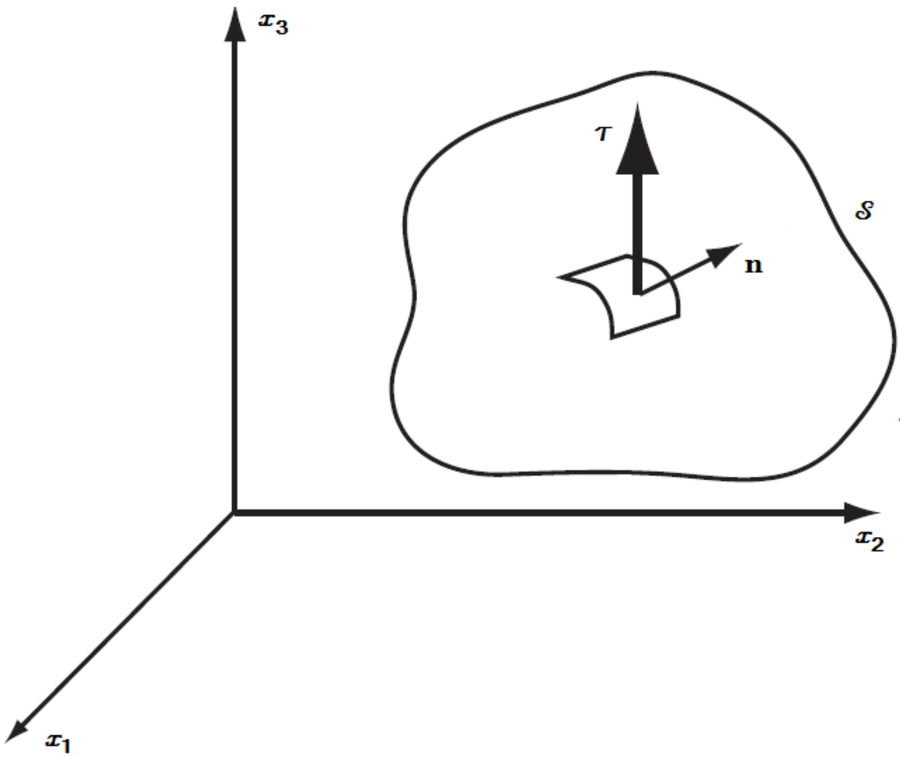

Figure 4.5: Infinitesimal element of a continuous body.

4.2.2 Maxwell's electric stress tensor

Due to the presence of charged particles in the ionic liquids, it's reasonable to consider, in addition to the forces per unit volume, also the electrical forces and the electric stress. This stress can be described in a similar way to the previous mechanical stress taking the following electrical forces per unit volume [3]:

$$\mathbf{f} = \rho_{\mathcal{V}}\mathbf{E} \qquad \text{(in the vacuum)} \tag{4.19a}$$

$$\mathbf{f} = \rho_{\mathcal{V}}\,\frac{\mathbf{E}}{\varepsilon_r} \qquad \text{(dielectric)} \tag{4.19b}$$

We saw that the volumetric charge density for a perfectly conducting liquid is given by equation (4.5) but in general if $\mathbf{E}$ is not uniform or we don't have a linear isotropic and homogeneous material we have:

$$\rho_{\mathcal{V}}(\mathbf{x}) = \frac{\partial(\varepsilon_0 E_i)}{\partial x_i}, \qquad \mathbf{x} \in \mathcal{V} \tag{4.20}$$

Substituting (4.20) in (4.19a) we obtain:

$$F_j = \left\{\frac{\partial(\varepsilon_0 E_i)}{\partial x_i}\right\} E_j = \frac{\partial}{\partial x_i}(\varepsilon_0 E_i E_j) - \varepsilon_0 E_i \frac{\partial E_j}{\partial x_i}. \tag{4.21}$$

Since the electric field is conservative then $\nabla \times \mathbf{E} = 0$, so:

$$\frac{\partial E_j}{\partial x_i} = \frac{\partial E_i}{\partial x_j}$$

and

$$\varepsilon_0 E_i \frac{\partial E_j}{\partial x_i} = \varepsilon_0 E_i \frac{\partial E_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{2}\varepsilon_0 E_k E_k\right) = \frac{\partial}{\partial x_i}\left(\frac{1}{2}\delta_{ij}\varepsilon_0 E^2\right).$$

Finally we obtain:

$$F_j = \frac{\partial}{\partial x_i}\left(\varepsilon_0 E_i E_j - \frac{1}{2}\delta_{ij}\varepsilon_0 E^2\right) \tag{4.22}$$

and, as we have defined the mechanical stress in (4.18b), we can define the electric stress as:

$$\boxed{\sigma_{ij}^{el} = \varepsilon_0 E_i E_j - \frac{1}{2}\delta_{ij}\varepsilon_0 E^2} \tag{4.23}$$

$$[\boldsymbol{\sigma}^{el}] = \begin{bmatrix} \varepsilon_0 E_1^2 - \frac{1}{2}\varepsilon_0 E^2 & \varepsilon_0 E_1 E_2 & \varepsilon_0 E_1 E_3 \\ \varepsilon_0 E_1 E_2 & \varepsilon_0 E_2^2 - \frac{1}{2}\varepsilon_0 E^2 & \varepsilon_0 E_2 E_3 \\ \varepsilon_0 E_1 E_3 & \varepsilon_0 E_2 E_3 & \varepsilon_0 E_3^2 - \frac{1}{2}\varepsilon_0 E^2 \end{bmatrix} \tag{4.24}$$

that is known as the Maxwell's tensor of electrical stress [3]. We see immediately that:

$$[\boldsymbol{\sigma}^{el}] = [\boldsymbol{\sigma}^{el}]^T$$

so there are three principle directions of stress. From Coulomb's theorem we know that the electric field near the surface of a conductor in electrostatic equilibrium is normal to the surface and null in the tangential directions, so if we refer the tensor to a surface reference system like the one shown in figure 4.6 we have:

$$E_1 = E_2 = 0$$

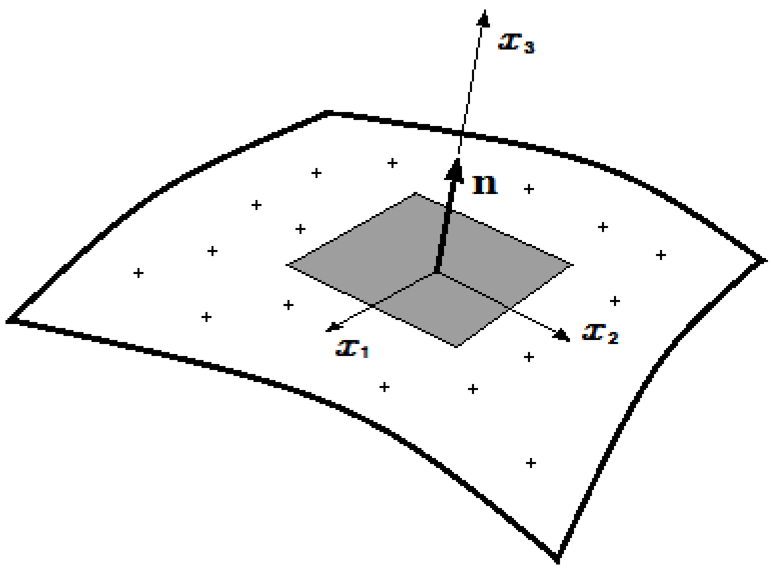

Figure 4.6: Surface reference system.

Therefore (4.24) becomes:

$$[\boldsymbol{\sigma}^{el}] = \frac{1}{2}\varepsilon_0 E^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.25}$$

where $E_3 = E$. So the stress along x_3 is:

$$\boxed{\sigma^{el} = \frac{1}{2}\varepsilon_0 E^2} \tag{4.26}$$

4.2.3 Instability

If the breaking tension is exceeded instability occurs [1]. Consider a deformed liquid surface: its shape will depend on the field shape and on the initial shape and stress state that, if the fluid is quiet, is only pressure. The charged particles will concentrate on the external protuberances, so intensifying locally the electric field.

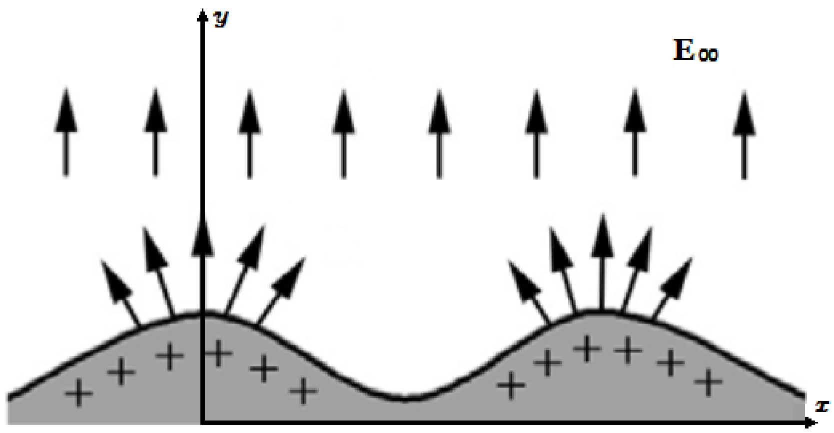

Figure 4.7: Deformed surface of a conductive liquid.

To determine the minimum electric field necessary to generate instability, we start by assuming a sinusoidal shape of the deformed surface:

$$y = A\cos(\alpha x).$$

with $\alpha = (2\pi/\lambda)$. If $\alpha \ll 1$ the external potential ϕ that satisfies the Laplace equation $\nabla^2\phi = 0$ with $\phi = 0$ on the surface can be described by the sum of a constant electric field E_0 and a small perturbation (fig.4.7):

$$\phi(x,y) \approx -E_\infty y + \phi_1 e^{-\alpha y}\cos(\alpha x). \tag{4.27}$$

For $\phi = 0$ we obtain the deformed surface:

$$y \approx \frac{\phi_1}{E_\infty}\cos(\alpha x) \tag{4.28}$$

which presents a curvature equal to:

$$\frac{1}{R_c} = \left|\frac{d^2y}{dx^2}\right| = \alpha^2\frac{\phi_1}{E_\infty}\cos(\alpha x)$$

whose maximum is reached for $\cos(\alpha x) = 1$. For these values we obtain the minimum radius of curvature:

$$R_{c,\mathrm{MIN}} = \frac{E_\infty}{\alpha^2\phi_1} \tag{4.29}$$

and the maximum breaking tension (or recalling tension):

$$\tau_b = \frac{\gamma}{R_{c,\mathrm{MIN}}} = \gamma\frac{\alpha^2\phi_1}{E_\infty} \tag{4.30}$$

The electric field on the tips is:

$$E_y = -\left(\frac{\partial\phi}{\partial y}\right)_{\cos(\alpha x)=1} = E_\infty + \alpha\phi_1 e^{-\alpha y}$$

and considering $\alpha y \ll 1$ it becomes:

$$E_y = E_\infty + \alpha\phi_1 \tag{4.31}$$

The electric perturbation is therefore:

$$\begin{aligned} \delta\mathcal{P} &= \frac{1}{2}\varepsilon_0(E_y^2 - E_\infty^2) \\ &= \frac{1}{2}\varepsilon_0(E_\infty^2 + \alpha^2\phi_1^2 + 2E_\infty\alpha\phi_1 - E_\infty^2) \\ &\sim \varepsilon_0 E_\infty\alpha\phi_1. \end{aligned} \tag{4.32}$$

Instability occurs if $\delta\mathcal{P} > \tau_b$:

$$\begin{aligned} \varepsilon_0 E_\infty \alpha \phi_1 &> \gamma \frac{\alpha^2 \phi_1}{E_\infty} \\ E_\infty &> \sqrt{\frac{\gamma\alpha}{\varepsilon_0}} \end{aligned} \tag{4.33}$$

Thus, the higher is the distance between the peaks, the less is the electric field required to get instability. We are interested to extract the liquid from small capillaries of diameter D so we have [1]:

$$\begin{aligned} \lambda_{\text{MAX}} &= 2D \\ \alpha_{\text{MIN}} &= \frac{\pi}{D} \\ E_\infty &> \sqrt{\frac{\pi\gamma}{\varepsilon_0 D}} \end{aligned} \tag{4.34}$$

For example considering formamide [$HCONH_2$] and an emitter with the following geometry:

$$\begin{aligned} \lambda &= 0,05\,\mathrm{N/m} \\ D &= 0,1\,\mathrm{mm} \end{aligned}$$

the minimum required electric field is:

$$E_\infty = 1,33 \cdot 10^7\,\mathrm{V/m}.$$

Chapter 5

Electrospray propulsion

5.1 Conceptual scheme

A colloid thruster is basically composed by:

- a small emitter that presents the conductive liquid inside it;
- an external electrode (with a power supply) to generate the electric voltage V.

We will consider only cylindrical emitters and cylindrical jets to simplify the treatment (in particular for the Taylor's Cone phenomenon). The hypothesis of one-dimensional flow will still be taken being z the one-dimensional coordinate. A simple geometry of an electrospray thruster is given in figure 5.1 in which:

- V: applied voltage;
- D: emitter's diameter;
- d: grid spacing of the electrodes.

The figure also shows the dynamic of the conductive liquid when the thruster is working which can be modeled with the following sequence of events:

- Taylor's Cone: a cone that appears only when the device is operating;
- Cone-Jet: a little cylindrical jet;
- Droplets and Fragmentation: at an unknown point the jet breaks and droplets start to form. Together with this phenomenon there is also the fragmentation of the droplets due to collisions of these with each other.

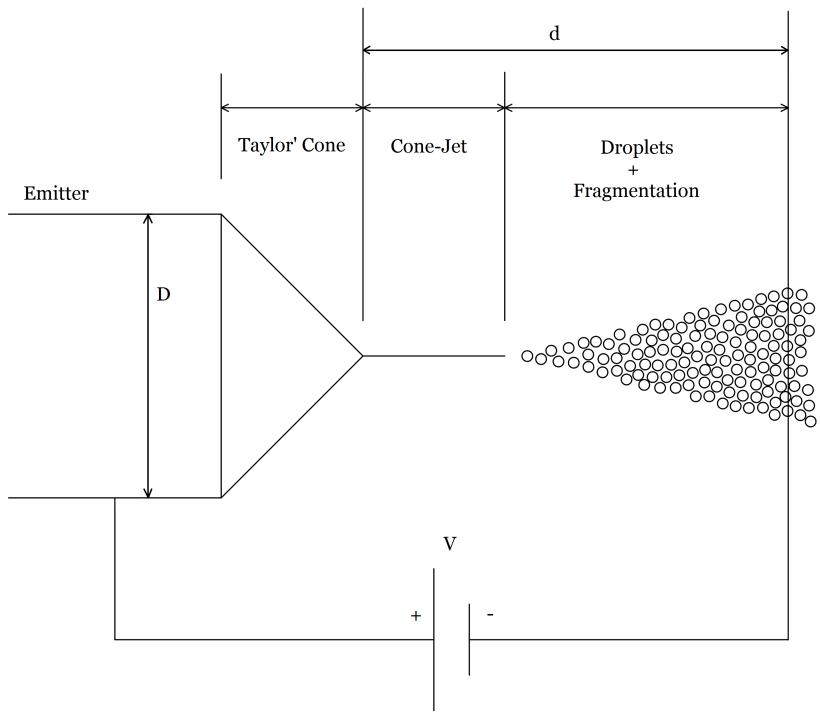

Figure 5.1: Conceptual scheme.

5.2 The Taylor's Cone

From early experimental observations it was known that, when a strong field is applied to the liquid coming out from the end of a tube, the liquid surface adopts a conical shape with a very thin fast-moving jet emitted from it apex. In 1965, G.I. Taylor explained analytically (and verified experimentally) this behavior [1][8]. The basic idea is that the surface traction due to the electric field must be balanced everywhere on the conical surface by the pull of the surface tension:

$$\sigma_{el} = \tau \tag{5.1}$$

The latter is given by equation (3.12):

$$\tau = \gamma\left(\frac{1}{R_{c1}} + \frac{1}{R_{c2}}\right) \tag{5.2}$$

where $1/R_{c1}$ and $1/R_{c2}$ are the principal curvatures of the surface.

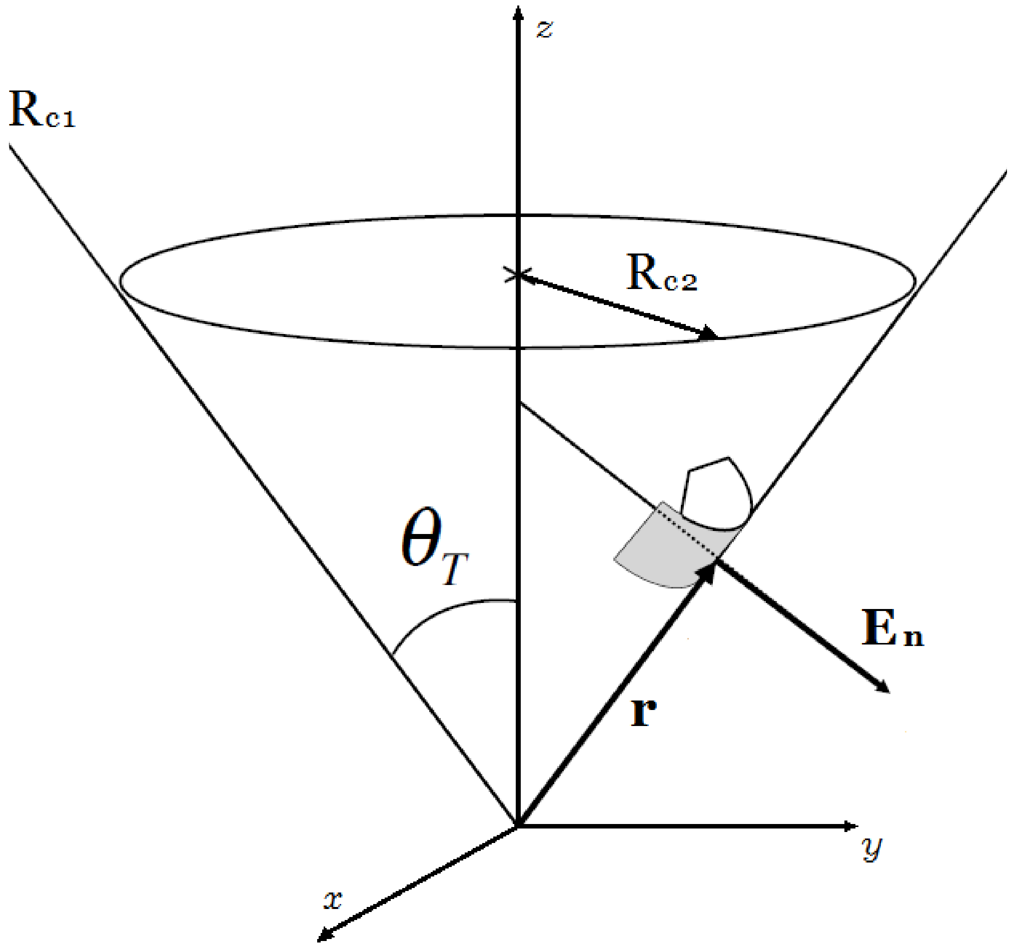

Figure 5.2: Taylor's Cone.

From differential geometry (Meusnier's theorem) we have:

$$\frac{1}{R_{c1}} = 0$$

along the generator, while for sections normal to the rotation axis we have:

$$\frac{1}{R_{c2}} = \frac{\cos(\theta)}{R} = \frac{\cos(\theta)}{r\sin(\theta)} = \frac{\cot(\theta)}{r}. \tag{5.3}$$

So from (5.2), (5.3) and (4.26) we obtain:

$$\begin{aligned}\frac{1}{2}\varepsilon_0 E_n^2 &= \gamma\frac{\cot(\theta)}{r} \\ E_n &= \sqrt{\frac{2\gamma\cot(\theta)}{\varepsilon_0 r}}\end{aligned} \tag{5.4}$$

The question is to find an external electrostatic field such that the cone is an equipotential surface with the normal field varying as in equation (5.4). Adopting a cylindrical coordinate system and owing to the cylindrical geometry of the cone $(\partial\phi/\partial\psi) = 0$, the Laplace equation becomes:

$$\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\phi}{\partial\theta}\right) = 0 \tag{5.5}$$

and admits solutions of the type:

$$\phi = A\mathcal{P}_\nu\big(\cos(\theta)\big)r^\nu + \phi_0 \tag{5.6a}$$

$$\phi = A\mathcal{Q}_\nu\big(\cos(\theta)\big)r^\nu + \phi_0 \tag{5.6b}$$

where $\mathcal{P}_\nu(\bullet)$ and $\mathcal{Q}_\nu(\bullet)$ are Legendre's functions of the first and the second type. $\mathcal{P}_\nu(\bullet)$ presents a singularity for $\theta = \pi$ while $\mathcal{Q}_\nu(\bullet)$ for $\theta = 0$ so only the last solution is acceptable because we are interested to find solutions external to the cone. So the normal field is:

$$\begin{aligned}\mathbf{E}_n = \mathbf{E}_\theta &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{u}_r \\ &= A\frac{d\mathcal{Q}_\nu}{d\cos(\theta)}\sin(\theta)r^{\nu-1}\mathbf{u}_r\end{aligned} \tag{5.7}$$

in which $\mathbf{E}_n$ is referred to a cone's reference system and $\mathbf{E}_\theta$ is referred to the spherical coordinates.

To find a relation like $E_n \propto 1/\sqrt{r}$ we have to set $\nu = 1/2$, from which:

$$\phi \;=\; A\mathcal{Q}_{1/2}\big(\cos(\theta)\big)r^{1/2} + \phi_0 \tag{5.8}$$

The function $\mathcal{Q}_{1/2}\big(\cos(\theta)\big)$ admits one single zero for:

$$\boxed{\theta_T \;=\; 49,29^\circ} \tag{5.9}$$

which can then be taken as an equipotential surface $\phi = \phi_0$.

This value is universal and independent of both fluid properties and applied voltage [1]. Taylor (and others) have verified experimentally this value, as long as no strong space charge effects are present, no flow, and as long as the electrode geometry is "reasonably similar" to what is implied in equation (5.8). The experimental fact that stable Taylor's cones do form even when the electrodes applying the voltage are substantially different from the shape given by (5.8) apparently indicates that the external potential distribution near the cone is dictated by the equilibrium condition (5.4), and that the transition to some other potential distribution capable of matching the real electrode shape takes place far enough from the liquid to be of little consequence. We should expect however that the Taylor's cone solution will be disturbed by non-ideal conditions and could eventually disappear. In one respect at least, the Taylor's cone can't be an exact solution: in the tip of the cone we have:

$$\lim_{r\to 0} E_n \;=\; \lim_{r\to 0}\sqrt{\frac{2\gamma\cot(\theta_T)}{\varepsilon_0 r}} \;=\; +\infty$$

that physically doesn't make sense.

5.3 The Cone Jet

We have seen that when the electrical traction overcomes the surface tension a Taylor's cone appears. The Taylor's cone is a mathematical idealization in which a liquid flow is not present on the tip, which instead happens in reality. The idealization requires also an infinite electric field on the tip and a perfect electric relaxation along r. Actually this relaxation can't be continuous if there is a flow rate Q (typically $\sim 10^{-13}\ \mathrm{m}^3/\mathrm{s}$) starting from the cone and forming a really thin jet ($20 \div 50\,\mathrm{nm}$). Because the liquid is conductive this implies the existence of a current I (typically $\sim 10^{-9}\,\mathrm{A}$).

This steady state is called Cone Jet and both Q and I are constant. This regime can be obtained with all conductive liquids, in particular with electrolytic solutions. In a good highly polar solvent ($\varepsilon_r \gg 1$) the salt in solution is highly dissociated (at least in low concentrations) for example lithium chloride [LiCl] in solution with formamide [$HCONH_2$]. For higher concentrations instead the degree of dissociation decreases.

Since the electrical conductivity K of the electrolytic solution is a finite value, this requires the existence of a tangential electric field $\mathbf{E}_r \neq 0$: this contradicts the assumption of equipotential surface especially near the tip where the current density must be stronger. Fortunately $\mathbf{E}_r \ll \mathbf{E}_\theta$ in most of the cone.

Figure 5.3: Taylor's Cone and Cone-Jet.

5.3.1 Transition region

Let's assume to have a conical structure in which because of the flow the charges are moving slowly everywhere except near the tip of the cone and suppose that this transition region is located at a distance r^* from the tip. The liquid surface ceases to be an equipotential surface when [2]:

$$\frac{r^{*3}}{Q} \sim \tau = \frac{\varepsilon_r \varepsilon_0}{K}$$

namely when the time passage of the fluid becomes of the order of the charge relaxation time τ given by equation (4.14). We obtain the characteristic dimension of the transition region:

$$r^* = \left(\frac{\varepsilon_r \varepsilon_0 Q}{K} \right)^{1/3} \tag{5.10}$$

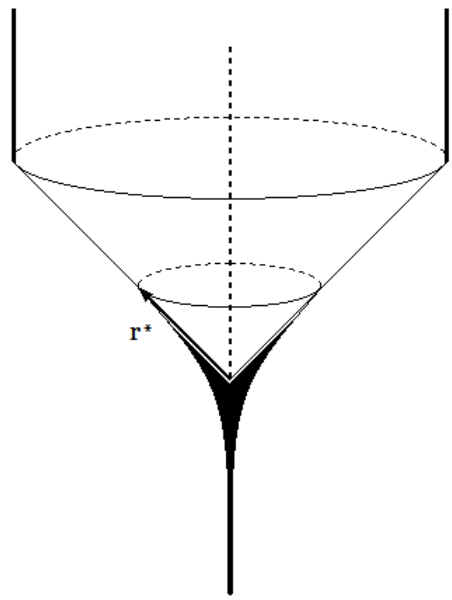

Figure 5.4: r^* and the beginning of the cone-jet.

r^* plays a fundamental role on the scaling and understanding of electrospray thrusters. In this region we assume that most of the surface transport will be convected ($I = I_{\mathcal{S}}$) but still most of the surface is relaxed $\sigma_{\mathcal{S}} \sim \varepsilon_0 E_n$. The surface current I_S is associated with the fluid velocity $\mathbf{v}$ and is characterized by a charge density dependent only on r; from (5.10) the flow rate Q will be:

$$\boxed{Q = \frac{K r^{*3}}{\varepsilon_r \varepsilon_0}} \tag{5.11}$$

5.3.2 Current

Fernandez de la Mora verified experimentally that the current transported by the cone-jet is given by [2]:

$$\boxed{I = f(\varepsilon_r)\sqrt{\frac{\gamma K Q}{\varepsilon_r}}} \tag{5.12}$$

in which $f(\varepsilon_r) \sim 18$ for $\varepsilon_r > 40$. Equation (5.12) is remarkable in several respects:

- Current is independent of applied voltage;
- Current is independent of electrode shape;
- Current is independent of fluid viscosity even though some of the fluids tested are very viscous.

From equation (3.39) we see that the maximum specific charge is obtained with the minimum flow:

$$\frac{q}{m} = \frac{I}{\rho Q} = \frac{f(\varepsilon_r)}{\rho}\sqrt{\frac{\gamma K}{\varepsilon_r Q}}. \tag{5.13}$$

5.3.3 Current density of the jet

For the current density we can apply the model of conducting liquids from equation (4.10). From the continuity law (3.26) and still assuming a one-dimensional flow in a steady state we know that j is uniform inside the cone-jet and therefore also the electric field is constant too:

$$j_{jet}(z) = \text{UNIFORM} \quad \Longrightarrow \quad E_{jet}(z) = \text{UNIFORM} \tag{5.14}$$

The current density remains uniform even after the cone-jet when the droplets are formed ($\nabla \cdot \mathbf{j} = 0$ everywhere in the space grid in a steady state):

$$j_{jet} = j_D = j = \text{UNIFORM} \tag{5.15}$$

Assuming a cylindrical jet of length $\mathcal{L}_{jet}$ and cross section $\mathcal{A}_{jet} = \pi\mathcal{R}_{jet}^2$, the current density and the current are related by:

$$I = \pi\mathcal{R}_{jet}^2 j \tag{5.16}$$

5.3.4 The length of the jet

Between the Taylor's cone and the first drop we have a cylindrical jet of radius $\mathcal{R}_{jet}$ and length $\mathcal{L}_{jet}$. Several experiments show that if the applied voltage increases then $\mathcal{L}_{jet}$ decreases. We can calculate the potential V_D at which the droplets begin to form in this way:

$$\begin{aligned} V_D &= V_0 - E_{jet}\mathcal{L}_{jet} \\ E_{jet} &= \frac{V_0 - V_D}{\mathcal{L}_{jet}} \end{aligned} \tag{5.17}$$

Assuming a semi-spherical end of the jet (still of radius $\mathcal{R}_{jet}$) the surface tension is given by equation (3.12):

$$\tau = \frac{2\gamma}{\mathcal{R}_{jet}} \tag{5.18}$$

If the electric perturbation exceeds the surface tension:

$$\delta\mathcal{P} \geq \tau \tag{5.19}$$

then the cone-jet break-up occurs. The electric perturbation is the same calculated in the previous chapter and in this case its amount is:

$$\delta\mathcal{P} \;=\; \frac{1}{2}\varepsilon_0 E_{jet}^2 - \frac{1}{2}\varepsilon_0 E_\infty^2 \tag{5.20}$$

So (5.19) together with (5.17),(5.18) and (5.20) leads to:

$$V_D^2 - 2V_0V_D + V_0^2 - \left(\frac{4\gamma}{\varepsilon_0\mathcal{R}_{jet}}\right)\mathcal{L}_{jet}^2 \;=\; 0 \tag{5.21}$$

whose solution is (we must take the solution with the minus):

$$V_D \;=\; V_0 - \sqrt{\frac{4\gamma}{\varepsilon_0\mathcal{R}_{jet}} + E_\infty^2} \cdot \mathcal{L}_{jet} \tag{5.22}$$

Comparing equation (5.22) with (5.17) and using the approximation (4.6) wee see that:

$$E_{jet} \;\sim\; \sqrt{\frac{4\gamma}{(1-\varepsilon_r)\varepsilon_0\mathcal{R}_{jet}}} \tag{5.23}$$

5.3.5 Flow rate, speed and energy

The jet carries a flow rate Q which can be expressed by:

$$Q \;=\; \mathcal{A}_{jet}v \;=\; \pi\mathcal{R}_{jet}^2 v \tag{5.24}$$

where v is the particle speed inside the jet. From the continuity law of fluid mechanics we know that in a steady state (and noting that $\rho = $ UNIFORM inside jet) $\nabla \cdot \mathbf{v} = 0$ and therefore, as for the current density:

$$v_{jet}(z) \;=\; \text{UNIFORM}$$

Unlike what happens for the current density, particles speed after the cone-jet doesn't remain constant. In fact, in a steady state the continuity law states that:

$$\nabla \cdot (\rho \mathbf{v}) = \nabla \cdot \mathbf{Q} = 0$$

that is the flow rate necessarily remains constant but not the speed; what happens is that $\mathbf{v}$ increases while ρ decreases. Let's call c the final speed that the particles have when they reach the grid and v_D the particle speed in a generic point between the cone-jet break-up and the grid: the energy conservation yields:

$$\frac{1}{2}mc^2 = qV_D + \frac{1}{2}mv^2 = qV + \frac{1}{2}mv_D^2 \tag{5.25}$$

or rearranging the equation:

$$\frac{1}{2}\frac{m}{q}c^2 = V_D + \frac{1}{2}\frac{m}{q}v^2 = V_D - V + \frac{1}{2}\frac{m}{q}v_D^2 \tag{5.26}$$

We can consider the last term of equation (5.26) a specific kinetic energy of the particle that we call K_D:

$$\mathrm{K}_D = \frac{1}{2}\frac{m}{q}v_D^2 \tag{5.27}$$

5.4 Droplet size and charge

From the nature of the Taylor's cone, while the liquid is moving towards the tip jet maintains an equilibrium on its surface between electrostatic and surface tension forces. This equilibrium is disturbed near the tip but it is reasonable to conjecture that something close to it will be sustained into the jet and even after jet break-up into the droplets which result. If we postulate this for a droplet of radius $\mathcal{R}_D$ and charge q the equilibrium condition becomes [2]:

$$\begin{cases} \frac{1}{2}\varepsilon_0 E_n^2 = \frac{2\gamma}{\mathcal{R}_D} \\ \\ E_n = \frac{q}{4\pi\epsilon_0 \mathcal{R}_D^2} \end{cases} \tag{5.28}$$

whence:

$$\boxed{q_R = 8\pi\sqrt{\varepsilon_0\gamma}\,\mathcal{R}_D^{3/2}} \tag{5.29}$$

which is known as the Rayleigh's Limit and represents the maximum charge that a droplet can hold (above which a coulombian explosion is expected). It derives that the maximum ratio q/m is:

$$\begin{aligned} \left(\frac{q}{m}\right)_{\text{MAX}} &= \left(\frac{I}{\rho Q}\right)_{\text{MAX}} = \frac{q_R}{\frac{4}{3}\pi\mathcal{R}_D^3} \\ \left(\frac{I}{\rho Q}\right)_{\text{MAX}} &= \frac{6\sqrt{\varepsilon_0\gamma}}{\rho\mathcal{R}_D^{3/2}} \end{aligned} \tag{5.30}$$

However in practice a small departure from the full spherical shape will trigger the instability when close to this limit. The outcome of a Coulombic explosion is fragmentation into small spherical droplets. It can be proved that N daughter droplets, if fragmenting symmetrically from a droplet at the Rayleigh limit, will be charged to about $100\,N^{-1/2}\,\%$ of their corresponding limit and therefore will be stable (neglecting solvent evaporation).

Experiments have shown that the electrospray technology produces streams of droplets charged to about 1/2 of their Rayleigh's Limit. This can be explained taking the total energy of a droplet [2]:

$$\mathrm{E} = \frac{1}{2}q\phi + 4\pi\mathcal{R}_D^2\gamma$$

in which the first term is the electrostatic energy where:

$$\phi = \frac{q}{4\pi\varepsilon_0\mathcal{R}_D}$$

is the scalar potential of a charge q, while the second term is the energy associated with the surface tension.

We know that for N droplets with radius $\mathcal{R}_D$ deriving from one droplet mother of mass m_i and charge q_i the ratio q/m doesn't change:

$$N = \frac{m_i}{m_f}$$

$$q_f = \frac{q_i}{N} = \frac{q_i}{m_i} m_f$$

$$\left(\frac{q}{m}\right)_f = \left(\frac{q}{m}\right)_i .$$

So we can find N and after we can calculate the total energy of droplet mother summing the energies of the N droplets without considering any kind of losses:

$$\begin{aligned} N &= \frac{3m_i}{4\rho\pi\mathcal{R}_D^3} \\ \mathrm{E} &= N\left(\frac{1}{2}\frac{q_f^2}{4\pi\varepsilon_0\mathcal{R}_D} + 4\pi\mathcal{R}_D^2\gamma + C\right) \end{aligned} \tag{5.31}$$

Because both the droplets and the droplet mother are in equilibrium the total energy E must have a minimum for $\mathcal{R}_D$:

$$\frac{d\mathrm{E}}{d\mathcal{R}_D} = 0$$

Therefore the final results are:

$$\mathcal{R}_D = \left[\frac{9}{\rho^2}\left(\frac{m_i}{q_i}\right)^2 \varepsilon_0\gamma\right]^{1/3} \tag{5.32}$$

and:

$$\left(\frac{q}{m}\right)_{\text{MINE}} = \frac{3\sqrt{\varepsilon_0\gamma}}{\rho\mathcal{R}_D^{3/2}} \tag{5.33}$$

If the droplet size $\mathcal{R}_D$ is assumed to be known we can deduce the radius $\mathcal{R}_{jet}$ of the jet from whose breakdown they originate. Several experiments confirm that this jet break-up conforms closely to the classical Rayleigh-Taylor stability theory for uncharged jets [2][7], which predicts a ratio:

$$\frac{\mathcal{R}_D}{\mathcal{R}_{jet}} \sim 1.89 \tag{5.34}$$

from which using (5.30) and (5.13) we can express the radius of the jet as a function of the flow and fluid properties:

$$\boxed{\mathcal{R}_{jet} = \frac{1}{1.89}\left[\frac{6}{f(\varepsilon_r)}\right]^{2/3} r^*} \tag{5.35}$$

This value is in the range of the data published in the literature, so it strongly supports the validity of the arguments used. It can be also observed that $f(\varepsilon_r)$ is known to fall for less than about 40 and (5.35) constitutes a prediction for a corresponding increase in the jet diameter. No direct data appear to be available on this point.

As mentioned before a high specific charge is important to reduce the holding voltage V_0 for a given specific impulse I_{sp}:

$$V_0 = \frac{c^2}{2}\left(\frac{q}{m}\right)^{-1}$$

Note that from equation (5.13) we find:

$$\begin{aligned} Q &\propto C\frac{V^2}{c^4} \\ T &= \rho Q c \propto C\frac{V^2}{c^3}. \end{aligned} \tag{5.36}$$

So, the requirement in flow rate is more sensitive than the requirement on the thrust. For small Δv missions, where a high specific impulse is not imperative, the design can be facilitated by both reducing V and increasing Q.

5.5 Fragmentation

It's difficult for this thesis to take into account the effects of fragmentation, the last phenomenon of the entire jet. We can only state that the possible results of this thesis will be the maximum limits which can't be overcome because fragmentation is essentially a dissipative phenomenon in which a part of energy is lost [6]. A simple equation that we can write is:

$$\boxed{q(V_0 - V_D) + \frac{1}{2}mv^2 = \frac{1}{2}mc^2 + \Delta\mathcal{E}_{\text{FRAG}}} \tag{5.37}$$

From the literature we can see that fragmentation occurs at specific potentials during the path depending on the mass of the ions involved:

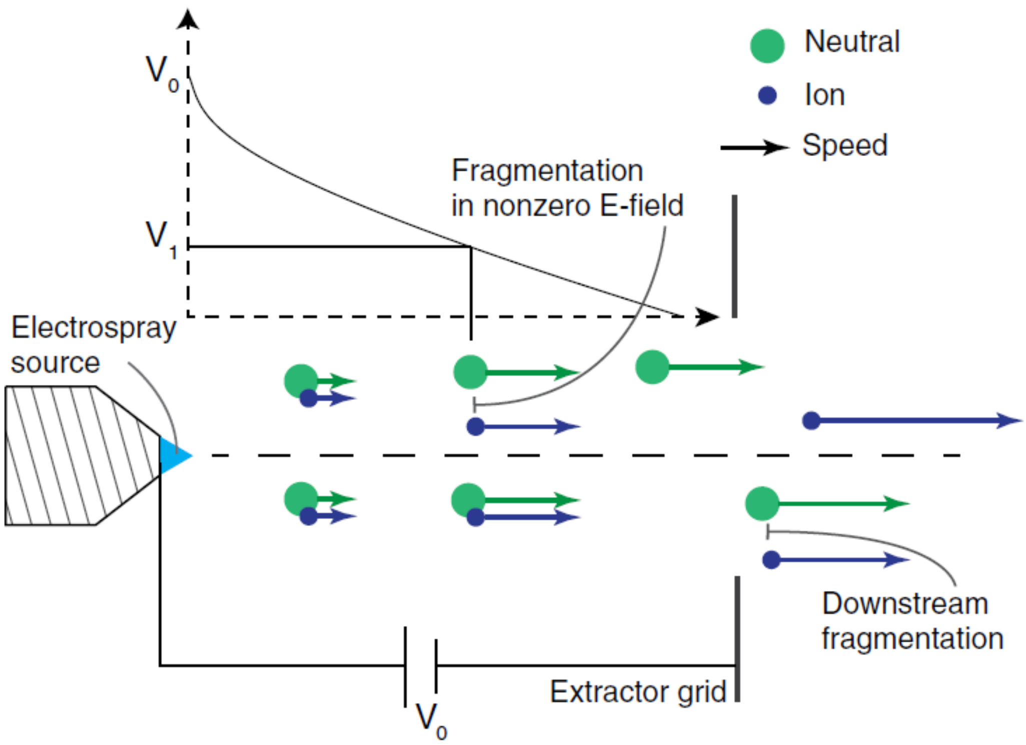

Figure 5.5: Fragmentation of the droplets.

Chapter 6

Mathematical models

In this section we will develop simplified models to calculate the performance of an electrospray thruster without taking into account fragmentation. I will explain two mathematical models, the first to give a simple idea of how my supervisor and I had operated although it isn't a real model, the second to try to find the right performance of the thruster.

6.1 First model

The first model we're going to develop doesn't consider the ion's speed inside the jet and their kinetic energy ($v = 0$ and K = 0), so the system we have to solve is:

$$\begin{cases} \dfrac{q}{m} = \dfrac{6(\varepsilon_0\gamma)^{1/2}}{\rho} \cdot \dfrac{1}{(1.89\mathcal{R}_{jet})^{3/2}} \\ \\ \mathcal{R}_{jet} = \dfrac{1}{1.89}\left[\dfrac{6}{f(\varepsilon_r)}\right]^{2/3}\left(\dfrac{\varepsilon_r\varepsilon_0 Q}{K}\right)^{1/3} \\ \\ I = \pi\mathcal{R}_{jet}^2 j = f(\varepsilon_r)\sqrt{\dfrac{\gamma K Q}{\varepsilon_r}} \\ \\ j = KE_{jet} \end{cases} \tag{6.1}$$

that presents 4 equations and 6 unknowns (q/m, $\mathcal{R}_{jet}$, I, j, E, Q), so we need two more equations.

6.1.1 Modified Child-Langmuir's Law

To find the other two equations we start saying that the maximum charge density in the electrostatic accelerator is given by the Child-Langmuir's law to a flow of ions accelerated by a potential V_0 with a distance d between the source and the electrostatic grid (as stated in chapter 3):

$$j = \frac{4\varepsilon_0}{9}\sqrt{\frac{2q}{m}}\frac{V_0^{3/2}}{d^2} \tag{6.2}$$

This law is true only if both the electric field on the source and the ion's speed are null, but that is not true in this case because of the existence of both an electric field E_{jet} (to simplify the demonstration I will call it just E) and an initial speed v_D of the ion. In this first model we start assuming both $\mathcal{L}_{jet} \sim 0 \implies V_D \sim V_0$ and $v = 0$ but considering a flow rate not null. This is only to give an idea of how the second model will be developed. So we have to place in equation (3.31) the following relationship:

$$\left(\frac{dV}{dz}\right)_0 = -E \tag{6.3}$$

So the equation that has to be solved is:

$$\left(\frac{dV}{dz}\right)^2 - E^2 = \frac{4j}{\varepsilon_0}\left[\frac{m(V_0 - V)}{2q}\right]^{1/2} \tag{6.4}$$

Let's solve this equation. First we have to separate the variables as follows (the sign minus is because the potential decreases along z):

$$\frac{dV}{dz} = -\left\{\frac{4j}{\varepsilon_0}\left[\frac{m(V_0 - V)}{2q}\right]^{1/2} + E^2\right\}^{1/2}$$

Placing:

$$\begin{cases} \theta = V/V_0 & dV = V_0\, d\theta \\ \xi = x/d & dx = d\, d\xi \end{cases} \tag{6.5}$$

we obtain a simpler non-dimensional form of the previous equation:

$$-\left[1+\frac{4j}{\varepsilon_0 E^2}\left(\frac{mV_0}{2q}\right)^{1/2}\sqrt{1-\theta}\right]^{-1/2} d\theta = \frac{Ed}{V_0}\, d\xi\,.$$

Now substituting the following variables:

$$a = \frac{4j}{\varepsilon_0 E^2}\left(\frac{mV_0}{2q}\right)^{1/2} \tag{6.6a}$$

$$b = \frac{Ed}{V_0} \tag{6.6b}$$

which are constants, we finally obtain the differential equation that we must integrate:

$$-\frac{d\theta}{\sqrt{1+a\sqrt{1-\theta}}} = b\,dx \tag{6.7}$$

A primitive function of equation (6.7) is:

$$\int \frac{d\theta}{\sqrt{1+a\sqrt{1-\theta}}} = -\frac{4(a\sqrt{1-\theta}-2)\sqrt{1+a\sqrt{1-\theta}}}{3a^2} + C$$

$$= \frac{\Theta_1(\theta)}{a^2} + C.$$

So taking the complete integration along the z axis with the following boundary conditions:

- for $z = 0$, $\xi = 0$, $V = V_0$, $\theta = 1$;
- for $z = d$, $\xi = 1$, $V = 0$, $\theta = 0$.

the final equation that we were searching is:

$$\boxed{\begin{aligned} a^2 b &= \Theta_1(1) - \Theta_1(0) \\ &= \frac{4}{3}\left[2 + (a-2)\sqrt{1+a}\right] \end{aligned}} \tag{6.8}$$

that is independent from the previous equations. Thus we now lack an equation to close the entire problem.

6.1.2 Dimensionless and Operating Parameter

Now we have 5 equations and 6 unknowns. We know that the only physical quantity that we can change during the steady state is the voltage V_0 so we have to find a quantity, the Operating Parameter $P.O.$, as a function only of V_0:

$$P.O. = f(V_0, \text{Fluid}, \text{Geometry})$$

To find this unknown function we start recalling Buckingham's theorem (see appendix B): we have 6 variables that depend on 4 fundamental physics quantities: [m] [s] [kg] [C]. So the system can be described with 2 non-dimensional groups as follows:

- substituting the relation for $\mathcal{R}_{jet}$ in the equation of q/m we obtain the first dimensionless number π_1:

$$\boxed{\pi_1 = \frac{q}{m}\frac{Q^{1/2}\rho}{\gamma^{1/2}K^{1/2}} = \frac{f(\varepsilon_r)}{\varepsilon_r^{1/2}}} \tag{6.9}$$

- substituting for $\mathcal{R}_{jet}$ in the equation of j we obtain the second dimensionless number π_2:

$$\boxed{\pi_2 \; = \; \frac{j Q^{1/6} \varepsilon_0^{2/3}}{\gamma^{1/2} K^{7/6}} \; = \; \frac{1.89^2}{\pi \, 6^{4/3}} \left[\frac{f(\varepsilon_r)}{\varepsilon_r^{1/2}} \right]^{7/3}} \tag{6.10}$$

We can rewrite the above system in a simpler form:

$$\begin{cases} \dfrac{q}{m} \; = \; \pi_1 \, \dfrac{\gamma^{1/2} K^{1/2}}{\rho} \, Q^{-1/2} \\[2ex] j \; = \; \pi_2 \, \dfrac{\gamma^{1/2} K^{7/6}}{\varepsilon_0^{2/3}} \, Q^{-1/6} \end{cases} \tag{6.11}$$

We see that for a specific fluid:

$$\begin{bmatrix} \dfrac{q}{m} \\[2ex] j \end{bmatrix} \; = \; \mathbf{\Lambda_1}(Q)$$

or in other words that the quantities are functions only of the flow rate.
So from equations (6.6a), (6.6b) we can rewrite a and b as functions of Q and V_0:

$$a \; = \; a(Q, V_0) \tag{6.12a}$$

$$b \; = \; b(Q, V_0) \tag{6.12b}$$

Let's find the first of these relations. Substituting (4.1) in (6.6a) and subsequently substituting equations (6.11) we find:

$$a \; = \; \frac{2^{3/2}}{\pi_1^{1/2} \pi_2} \, \frac{\rho^{1/2} K^{7/12}}{\varepsilon_0^{1/3} \gamma^{3/4}} \, Q^{5/12} \, V_0^{1/2} \tag{6.13}$$

In a similar way from equations (6.6b) and (4.1) we obtain:

$$b = \pi_2 \frac{\gamma^{1/2} K^{1/6} d}{\varepsilon_0^{2/3}} \frac{1}{Q^{1/6} V_0} \tag{6.14}$$

Now taking the product $ab^{5/2}$ we delete Q and we obtain a function depending only on V_0:

$$ab^{5/2} = \frac{2^{3/2}\pi_2^{3/2}}{\pi_1^{1/2}} \frac{K\rho^{1/2}\gamma^{1/2}d^{5/2}}{\varepsilon_0^2} \frac{1}{V_0^2} \tag{6.15}$$

that is the equation of the Operating Parameter, namely:

$$\boxed{P.O. = ab^{5/2}} \tag{6.16}$$

This is the last equation that we were searching for and now the system is balanced: 6 equations and 6 unknowns. Unfortunately this is a non linear system so we can't state that it has a unique solution; from the algebra we can only say that the set of the solutions has measure 0.

6.1.3 Closure of the problem

We are now able to write the complete system of equations that governs an electrospray thruster:

$$\begin{cases} \dfrac{q}{m} = \dfrac{q}{m}(Q) \\ j = j(Q) \\ a = a(Q, V_0) \\ b = b(Q, V_0) \\ a^2 b = \Theta_1(1) - \Theta_1(0) \\ ab^{5/2} = P.O. \end{cases} \tag{6.17}$$

with $\Theta_1(1) - \Theta_1(0) = g(a,b) = h(Q,V_0)$ and $P.O. = f(Q,V_0)$. So the solution comes from the last two equations:

$$\begin{cases} a^2 b = \Theta_1(1) - \Theta_1(0) \\ \\ ab^{5/2} = P.O. \end{cases} \tag{6.18}$$

and the problem is solved.

6.2 Second model

Considering also the initial speed v of the ions inside the jet and their kinetic energy we can write the following equations to describe the steady state operation:

$$\begin{cases} \dfrac{q}{m} = \dfrac{6(\varepsilon_0\gamma)^{1/2}}{\rho} \cdot \dfrac{1}{(1.89\mathcal{R}_{jet})^{3/2}} \\ \\ \mathcal{R}_{jet} = \dfrac{1}{1.89}\left[\dfrac{6}{f(\varepsilon_r)}\right]^{2/3}\left(\dfrac{\varepsilon_r\varepsilon_0 Q}{K}\right)^{1/3} \\ \\ I = f(\varepsilon_r)\sqrt{\dfrac{\gamma K Q}{\varepsilon_r}} \\ \\ j = KE \\ \\ Q = \pi\mathcal{R}_{jet}^2 v \\ \\ \mathrm{K}_D = \dfrac{1}{2}\dfrac{m}{q}v^2 \end{cases} \tag{6.19}$$

This is a system of 6 equations with 8 unknowns $(q/m, \mathcal{R}_{jet}, I, j, E, Q, v_D, \mathrm{K}_D)$, so we need two more equations.

6.2.1 Modified Child-Langmuir's law

Like before, we start taking $\mathcal{L}_{jet} \sim 0 \implies V_D \sim V_0$ and changing equation (3.29) with the following equation: if we take the the energy conservation of the motion of the droplets we have:

$$\begin{aligned} qV_0 + \frac{1}{2}mv^2 &= qV + \frac{1}{2}mv_D^2 \\ v_D(z) &= \left[\frac{2q[V_0 - V(z)]}{m} + v^2\right]^{1/2} \end{aligned} \tag{6.20}$$

Substituting equation (6.20) in Poisson's equation we find:

$$\frac{d^2V}{dz^2} = -\frac{j}{\varepsilon_0}\left[\frac{m}{2q(V_0 - V) + mv^2}\right]^{1/2}. \tag{6.21}$$

that can be easily integrated placing:

$$\begin{cases} \psi = V_0 - V + \dfrac{1}{2}\dfrac{m}{q}v^2 = V_0 - V + \mathrm{K}_D \\ \\ d\psi = -dV \end{cases} \tag{6.22}$$

whence

$$\frac{d^2\psi}{dz^2} = \frac{j}{\varepsilon_0}\left(\frac{m}{2q\psi}\right)^{1/2}. \tag{6.23}$$

In the same way as we have seen before, the first integration yields:

$$\left(\frac{d\psi}{dz}\right)^2 - \left(\frac{d\psi}{dz}\right)_0^2 = \frac{4j}{\varepsilon_0}\left(\frac{m\psi}{2q}\right)^{1/2} \tag{6.24}$$

and placing

$$\left(\frac{d\psi}{dz}\right)_{z=0} = -E$$

we obtain another Cauchy's problem:

$$\begin{cases} \left(\dfrac{d\psi}{dz}\right)^2 - E^2 = \dfrac{4j}{\varepsilon_0}\left(\dfrac{m\psi}{2q}\right)^{1/2} \\ \\ \psi(0) = \mathrm{K}_D \\ \\ \psi(d) = V_0 + \mathrm{K}_D \end{cases} \tag{6.25}$$

The equation has separate variables so:

$$\left(\frac{4j}{\varepsilon_0}\sqrt{\frac{m}{2q}}\psi^{1/2} + E^2\right)^{-1/2} d\psi = dz \tag{6.26}$$

With the usual substitutions as before we take:

$$a = \frac{4j}{\varepsilon_0 E^2}\left(\frac{mV_0}{2q}\right)^{1/2} \tag{6.27a}$$

$$b = \frac{Ed}{V_0} \tag{6.27b}$$

and:

$$\begin{cases} \theta = \dfrac{a^2}{V_0}\psi, \qquad \theta = \dfrac{a^2}{V_0}d\psi \\ \\ \xi = dz, \qquad \xi = d\,dz \end{cases} \tag{6.28}$$

we obtain:

$$\frac{d\theta}{\sqrt{\theta^{1/2}+1}} = a^2 b\, d\xi \tag{6.29}$$

A primitive function of (6.29) is:

$$\Theta_2(\theta) = \int \frac{d\theta}{\sqrt{\theta^{1/2}+1}} = \frac{4}{3}\left(\theta^{1/2}-2\right)\sqrt{\theta^{1/2}+1} + C$$

So integrating from $z = 0$ to $z = d$ with the following boundary conditions:

$$\begin{cases} V(0) = V_0,\ \psi(0) = \mathrm{K}_D,\ \theta(0) = \dfrac{a^2\mathrm{K}_D}{V_0},\ \xi(0) = 0 \\ \\ V(d) = 0,\ \psi(d) = V_0 + \mathrm{K}_D,\ \theta(d) = a^2\left(1 + \dfrac{K_D}{V_0}\right),\ \xi(d) = 1 \end{cases} \tag{6.30}$$

finally we obtain:

$$\boxed{\Theta_2\left[a^2\left(1 + \frac{\mathrm{K}_D}{V_0}\right)\right] - \Theta_2\left[\frac{a^2\mathrm{K}_D}{V_0}\right] \;=\; a^2 b} \tag{6.31}$$

which is independent from the first six equations and so it can be added to the initial system.

6.2.2 Dimensionless and Operating Parameter

Now we have 7 equations and 8 unknowns. As before, we search for a function of the type:

$$P.O. \;=\; f(V_0, \textsc{Fluid}, \textsc{Geometry})$$

so similarly to the first model we start to develop the previous equations in dimensionless form. It's easy to prove that π_1 and π_2 are the same as before. There's also another dimensionless number due to the kinetic energy K_D (N.B: the non dimensional numbers that really describe the system remain 2. This third number it's only useful but it's not necessary to describe the system): substituting for $\mathcal{R}_{jet}$ and $v = v(Q)$ in the equation of K_D we obtain the third dimensionless number π_3:

$$\boxed{\pi_3 \;=\; \frac{1.89^4}{2\pi^2 6^{8/3}}\left[\frac{f(\varepsilon_r)}{\varepsilon_r^{1/2}}\right]^{5/3}} \tag{6.32}$$

We can rewrite the above equations in a simpler form:

$$\begin{cases} \dfrac{q}{m} = \pi_1 \dfrac{\gamma^{1/2}K^{1/2}}{\rho} Q^{-1/2} \\ \\ j = \pi_2 \frac{\gamma^{1/2}K^{7/6}}{\varepsilon_0^{2/3}} Q^{-1/6} \\ \\ \mathcal{K}_D = \pi_3 \dfrac{\rho K^{5/6}}{\varepsilon_0^{4/3}\gamma^{1/2}} Q^{7/6} \end{cases} \tag{6.33}$$

We can still see that for a specific fluid all the three quantities are functions only of the flow rate:

$$\begin{bmatrix} \dfrac{q}{m} \\ \\ j \\ \\ \mathrm{K}_D \end{bmatrix} = \mathbf{\Lambda_2}(Q)$$

π_3 isn't necessary to find the equation of the Operating Parameter, but below it will be helpful; it's easy to prove that still:

$$P.O. = ab^{5/2} \tag{6.34}$$

and the problem is mathematically closed again.

6.2.3 Closure of the problem

The system becomes:

$$\begin{cases} \dfrac{q}{m} \;=\; \dfrac{q}{m}(Q) \\[2ex] j \;=\; j(Q) \\[2ex] \mathrm{K}_D \;=\; \mathrm{K}_D(Q) \\[2ex] a \;=\; a(Q, V_0) \\[2ex] b \;=\; b(Q, V_0) \\[2ex] a^2 b \;=\; \Theta_2 \left[a^2 \left(1 + \dfrac{\mathrm{K}_D}{V_0} \right) \right] - \Theta_2 \left[\dfrac{a^2 \mathrm{K}_D}{V_0} \right] \\[2ex] a b^{5/2} \;=\; P.O. \end{cases} \tag{6.35}$$

If we find a relation of the type:

$$\frac{\mathrm{K}_D}{V_0} \;=\; \frac{\mathrm{K}_D}{V_0}(a, b)$$

the solution is obtained by solving together the last two equations. Let's begin to find another relation for the voltage starting from (6.6b):

$$\begin{aligned} V_0 \;&=\; \frac{Ed}{b} \;=\; \frac{jd}{K\,b} \\ &=\; \pi_2 \, \frac{\gamma^{1/2} K^{1/6}\, d}{\varepsilon_0^{2/3} b} \, Q^{1/6} \end{aligned} \tag{6.36}$$

from which combining it with the third equation of system (6.33) we get:

$$\frac{\mathrm{K}_D}{V_0} \;=\; \frac{\pi_3}{\pi_2} \frac{\rho K^{2/3} b}{\varepsilon_0^{2/3} \gamma d} \, Q^{4/3} \tag{6.37}$$

Substituting equation (6.36) in (6.13) we obtain:

$$a = \frac{2^{3/2}}{\pi_1^{1/2}\pi_2^{1/2}} \frac{\rho^{1/2} K^{2/3} d^{1/2}}{\varepsilon_0^{2/3} \gamma^{1/2} b^{1/2}} Q^{1/3} \tag{6.38}$$

that leads to:

$$Q^{4/3} = \frac{\pi_1^2 \pi_2^2}{64} \frac{a^4 b^2 \varepsilon_0^{8/3} \gamma^2}{\rho^2 K^{8/3} d^2} \tag{6.39}$$

Now substituting the new relationship for Q in equation (6.37) we obtain:

$$\begin{aligned} \frac{\mathrm{K}_D}{V_0} &= \frac{\pi_1^2 \pi_2 \pi_3}{64} \frac{\varepsilon_0^2 \gamma}{\rho K^2 d^3} a^4 b^3 \\ &= \pi^* a^4 b^3 \end{aligned} \tag{6.40}$$

in which π^* is another dimensionless number characterizing the ratio K_D/V_0. We can rewrite equation (6.31) as:

$$\Theta_2\left[a^2(1 + \pi^* a^4 b^3)\right] - \Theta_2\left[\pi^* a^6 b^3\right] = a^2 b. \tag{6.41}$$

and placing:

$$\begin{cases} m = a^6 b^3 \\ a^2 b = m^{1/3} \end{cases} \tag{6.42}$$

equation (6.41) becomes finally:

$$\boxed{\Theta_2(a^2 + \pi^* m) - \Theta_2(\pi^* m) = m^{1/3}} \tag{6.43}$$

Always from the equation (6.42) we can rewrite the Operating Parameter as follows:

$$\boxed{m^{5/6} = a^4 P.O.} \tag{6.44}$$

So the final parametric system is:

$$\begin{cases} \Theta_2(a^2 + \pi^* m) - \Theta_2(\pi^* m) \; = \; m^{1/3} \\ \\ m^{5/6} \; = \; a^4 P.O. \end{cases} \tag{6.45}$$

which presents different solutions for different values of $P.O.$.

Chapter 7

Matlab programs

7.1 First implementation

In this case we are going to solve the final system of the first model with Matlab's function `fsolve`. To avoid numerical errors due to the quadratic function `sqrt` we write the system (6.18) in the following way:

$$\begin{cases} 9a^2b^2 - 48b - 16a + 48 = 0 \\ \\ a^2b^5 - P.O.^2 = 0 \end{cases} \tag{7.1}$$

The results are reported on Table 7.1, 7.2 and 7.3 and in figure 7.1. Formamide was used as the conductive liquid and a geometry of the emitter like the one used in the example in the end of chapter 3 was adopted.

```
% MAIN

% project requirements

% voltage [V]
V_0 = 5000:100:7000;

% electrode distance [mm]
d = 5;

% emitter's diameter [mm]
D = 0.1;

% input(1) = d
% input(2) = D

double input;
input = [d*10^(-3) D*10^(-3)];

fid = fopen('project_requirements.txt','w+');
fprintf(fid,'%10.4f\n',input);
fclose(fid);

% fluid properties

% density [kg/m^3]
rho = 1130;

% surface tension [N/m]
gamma = 0.059;

% conductivity [Si/m]
K = 1;

% current parameter
f = 18;

% relative dielectric constant
e = 100;

% fluid(1) = rho
% fluid(2) = gamma
% fluid(3) = K
```

```
% fluid(4) = f
% fluid(5) = e

double fluid;
fluid = [rho gamma K f e];

fid = fopen('fluid_properties.txt','w+');
fprintf(fid,'%10.4f\n',fluid);
fclose(fid);

% physics constants

% dielectric constant [F/m]
e_0 = 8.85418781762*10^(-12);

% gravity acceleration [m/s^2]
g = 9.80665;

% taylor cone's angle [deg]
theta_t = 49.29;

% cost(1) = e_0
% cost(2) = g
% cost(3) = teta_t

double cost;
cost = [e_0 g theta_t*pi/180];

fid = fopen('physics_constants.txt','w+');
fprintf(fid,'%10.25f\n',cost);
fclose(fid);

% dimensionless numbers
pi = dimensionless_numbers(fluid);

% operating parameter
PO = operating_parameter(V_0,input,fluid,cost,pi);

fid = fopen('operating_parameter.txt','w+');
fprintf(fid,'%10.4f\n',PO);
fclose(fid);

% initial point of the iterative method
```

```
x_0 = [600 500];

for i = 1:((7000 - 5000)/100 + 1)

    fid = fopen('operating_parameter.txt','w+');
    fprintf(fid,'%10.4f\n',P0(i));
    fclose(fid);

    fun = @final_system;
    options = optimoptions('fsolve','MaxFunEval',3000);
    x = fsolve(fun,x_0,options);

    a(i) = x(1);
    b(i) = x(2);

    x_0 = x;

end

% flow rate
Q = flow_rate(a,V_0,pi,fluid,cost);

% steady state operation
[E_ct, j, q_m] = steady_state_operation(Q,pi,fluid,cost);

% propulsive parameters
[I, m_dot, v, T, I_sp, eta] = propulsive_parameters(q_m,Q,V_0,
                                fluid,cost);
```

```
% DIMENSIONLESS NUMBERS

function pi = dimensionless_numbers(fluid)

W = fluid(4)/sqrt(fluid(5));

% pi(1) = dimensionless number of q/m

pi(1) = W;

% pi(2) = dimensionless number of j

pi(2) = W^(7/3)*0.1043;
```

```
% OPERATING PARAMETER
```

```
function PO = operating_parameter(V_0,input,fluid,cost,pi)

PO = sqrt(8*pi(2)^3)/sqrt(pi(1))*(fluid(3)*sqrt(fluid(1)*fluid(2))
    *input(1)^(5/2))./(cost(1)^2.*V_0.^2);
```

```
% FINAL SYSTEM

function F = final_system(x)

fid = fopen('operating_parameter.txt');
Par_Op = fscanf(fid,'%g',[1,inf]);
fclose(fid);

F(1) = 9*x(1)^2*x(2)^2-48*x(2)-16*x(1)+48;
F(2) = x(1)^2*x(2)^5 - Par_Op^2;
```

```
% FLOW RATE

function Q = flow_rate(a,V_0,pi,fluid,cost)

Q = (a*cost(1)^(1/3)*fluid(2)^(3/4)*pi(2)*sqrt(pi(1)))./(2^(3/2)
   *fluid(3)^(7/12)*sqrt(fluid(1)).*V_0.^(1/2)).^(12/5);
```

```
% STEADY STATE OPERATION

function [E_ct, j, q_m] = steady_state_operation(Q,pi,fluid,cost)

q_m = pi(1)*sqrt(fluid(2)*fluid(3))/fluid(1).*Q.^(-1/2);

j = pi(2)*sqrt(fluid(2))/cost(1)^(2/3)*(fluid(3))^(7/6)*Q.^(-1/6);

E_ct = fluid(3)*j;
```

```
% PROPULSIVE PARAMETERS

function [I, m_dot, v, T, I_sp, eta] = propulsive_parameters(q_m,
                                       Q,V_0,fluid,cost)

I = fluid(4)*sqrt(fluid(2)*fluid(3)/fluid(5))*Q.^(0.5);

m_dot = fluid(1)*Q;

v = sqrt(2*q_m.*V_0);
```

```

T = m_dot.*v;

I_sp = v/cost(2);

eta = m_dot.*v.^2./(2*V_0.*I);
```

V_0 [V]	a	b	$P.O. \cdot 10^9$	Q [pm^3/s]
5000	562,35	555,62	4,09	5,00
5100	545,10	553,74	3,93	4,74
5200	527,73	552,32	3,78	4,48
5300	510,26	551,33	3,64	4,23
5400	492,71	550,81	3,51	4,00
5500	475,11	550,74	3,38	3,77
5600	457,47	551,13	3,26	3,55
5700	439,82	552,00	3,15	3,34
5800	422,18	553,37	3,04	3,14
5900	404,19	555,44	2,94	2,95
6000	386,25	558,06	2,84	2,76
6100	368,39	561,26	2,75	2,58
6200	353,92	562,96	2,66	2,43
6300	339,51	565,12	2,58	2,29
6400	329,11	565,03	2,50	2,18
6500	316,61	566,78	2,42	2,06
6600	305,95	567,62	2,35	1,95
6700	292,28	571,18	2,28	1,83
6800	278,77	575,24	2,21	1,71
6900	265,44	579,81	2,15	1,60
7000	252,33	584,91	2,09	1,50

Table 7.1: Dimensionless, fluid [$HCONH_2$].

V_0 [V]	q/m [C/kg]	j [A/m^2] $\cdot 10^8$	E [V/m] $\cdot 10^8$
5000	172,98	1,78	1,78
5100	177,79	1,80	1,80
5200	182,81	1,82	1,82
5300	188,05	1,83	1,83
5400	193,53	1,85	1,85
5500	199,27	1,87	1,87
5600	205,28	1,89	1,89
5700	211,59	1,91	1,91
5800	218,24	1,93	1,93
5900	225,34	1,95	1,95
6000	232,85	1,97	1,97
6100	240,80	1,99	1,99
6200	248,08	2,01	2,01
6300	255,74	2,03	2,03
6400	262,21	2,05	2,05
6500	269,84	2,07	2,07
6600	277,02	2,09	2,09
6700	286,00	2,11	2,11
6800	295,46	2,13	2,13
6900	305,45	2,16	2,16
7000	316,00	2,18	2,18

Table 7.2: Steady state operation, fluid [$HCONH_2$].

V_0 [V]	I [μA]	$\dot{m}$ [nkg/s]	c [km/s]	T [μN]	I_{sp} [s]	η
5000	97,8	5,65	1,32	7,44	134,11	1.00
5100	95,1	5,35	1,35	7,21	137,32	1.00
5200	92,5	5,06	1,38	6,98	140,61	1.00
5300	90,0	4,78	1,41	6,75	143,97	1.00
5400	87,4	4,52	1,45	6,53	147,42	1.00
5500	84,9	4,26	1,48	6,31	150,97	1.00
5600	82,4	4,01	1,52	6,09	154,62	1.00
5700	79,9	3,78	1,55	5,87	158,37	1.00
5800	77,5	3,55	1,59	5,65	162,24	1.00
5900	75,1	3,33	1,63	5,43	166,28	1.00
6000	72,7	3,12	1,67	5,22	170,45	1.00
6100	70,3	2,92	1,71	5,00	174,78	1.00
6200	68,2	2,75	1,75	4,82	178,85	1.00
6300	66,1	2,59	1,80	4,64	183,05	1.00
6400	64,5	2,46	1,83	4,51	186,81	1.00
6500	62,7	2,32	1,87	4,35	190,99	1.00
6600	61,1	2,20	1,91	4,22	195,00	1.00
6700	59,2	2,07	1,96	4,05	199,62	1.00
6800	57,3	1,94	2,00	3,88	204,41	1.00
6900	55,4	1,81	2,05	3,72	209,36	1.00
7000	53,5	1,69	2,10	3,56	214,48	1.00

Table 7.3: Propulsive parameters, fluid [$HCONH_2$].

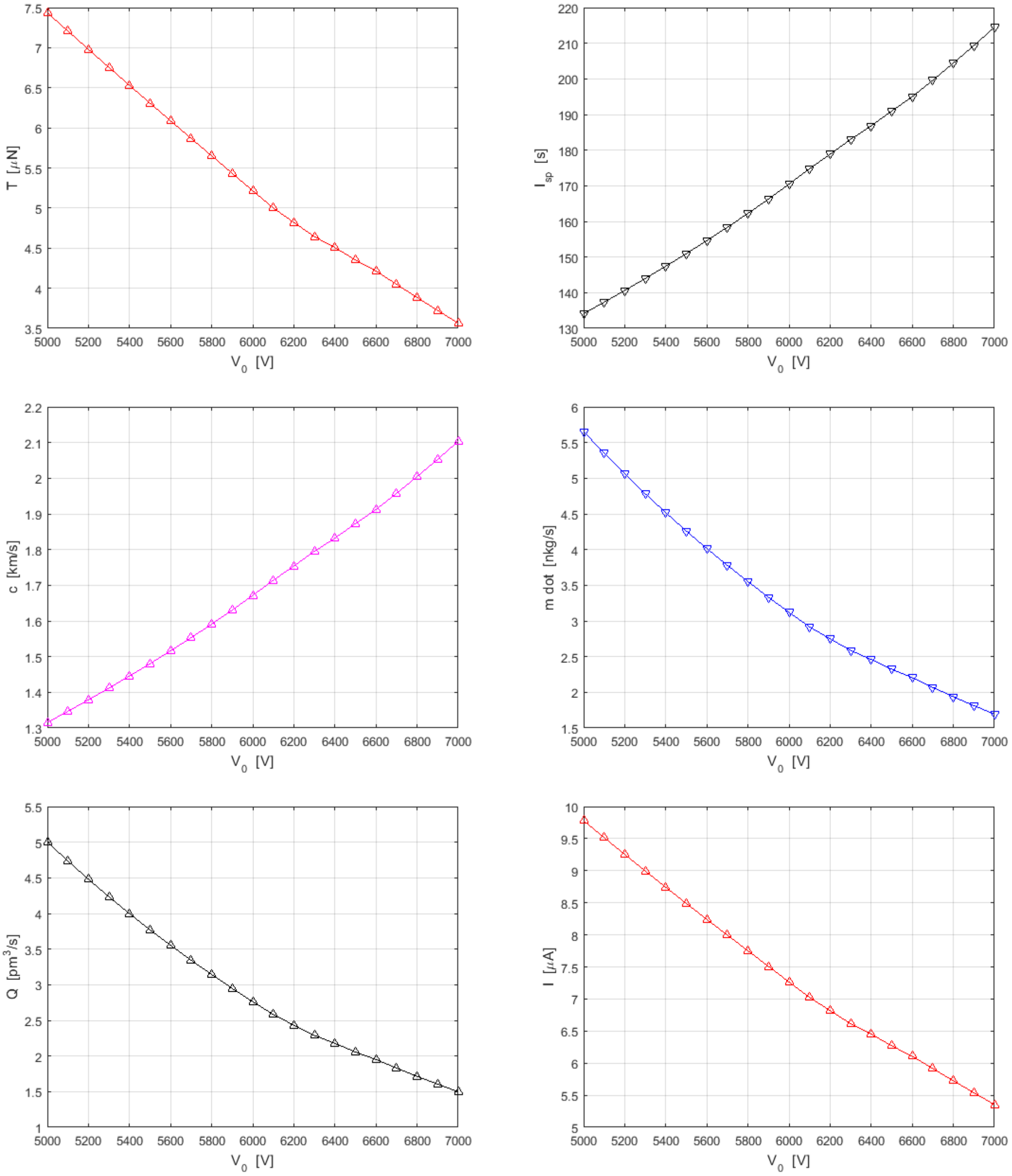

Figure 7.1: Propulsive parameters, fluid [$HCONH_2$].

The results are plotted as functions of the voltage. From a first inspection it can be seen that if T decreases then the flow rate Q decreases and the specific impulse I_{sp} increases, which is compatible with the current literature. It can be also seen that both I and Q have the same increasing and decreasing properties with the voltage, that is another result in agreement with the scientific literature. However the magnitude of the quantities is not really compatible with the experimental data; this can be explained by our assumption of null speed at the beginning of the jet that doesn't take into account a part of energy.

7.2 Second implementation

The second implementation doesn't resolve the entire system with `fsolve` but solves the resulting equation (7.1) of variable b. Substituting (6.16) into equation (6.8) we obtain:

$$48b^4 - 48b^3 + 16P.O.b^{1/2} - 9P.O.^2 = 0 \tag{7.2}$$

that is a parametric equation with parameter $P.O.$ and variable b. The program plots the function:

$$P.O. = f(b) \tag{7.3}$$

To plot (7.3) we must fix the value of b and solve for $P.O.$: we take b has a vector $[1] * [N]$ with N chosen at will.

$$9P.O.^2 - 16\sqrt{b}P.O. + 48b^3 - 48b^4 = 0 \tag{7.4}$$

The discriminant $\Delta/4$ is equal to:

$$\frac{\Delta}{4} = 64b - 432b^3 + 432b^4. \tag{7.5}$$

Once again, not to make numerical errors due to a small discriminant (actually we don't know the size of $\Delta/4$ but we're going to implement the same algorithm as a precaution) we have to write the solution of a generic quadratic equation $Ax^2 + Bx + C = 0$ in the following way:

$$\begin{aligned} q &= -\left[\frac{B}{2} + \sqrt{\frac{\Delta}{4}}\right] \\ P.O. &= \frac{q}{A} \end{aligned} \tag{7.6}$$

Domain of definition

The domain of definition of equation (7.4) is $\Delta/4 \geq 0$, that's equal to:

$$b(64 - 432b^2 + 432b^3) \geq 0 \tag{7.7}$$

Setting $b > 0$ the domain is given by:

$$b^3 - b^2 + \frac{4}{27} \geq 0 \tag{7.8}$$

that is an equation of the type $Ax^3 + Bx^2 + Cx + D = 0$. To calculate the solutions of this equation first we have to calculate the following numbers:

$$\begin{aligned} Q &= \frac{A^2 - 3B}{9} \\ R &= \frac{2A^3 - 9AB + 27C}{54} \end{aligned} \tag{7.9}$$

and then verify the condition:

$$Q^3 - R^2 \geq 0. \tag{7.10}$$

If (7.10) is true we have to calculate another quantity:

$$\alpha = \arccos\left(\frac{R}{\sqrt{Q^3}}\right) \tag{7.11}$$

The solutions will be:

$$\begin{aligned} x_1 &= -2\sqrt{Q}\cos\left(\frac{\alpha}{3}\right) - \frac{A}{3} \\ x_2 &= -2\sqrt{Q}\cos\left(\frac{\alpha + 2\pi}{3}\right) - \frac{A}{3} \\ x_3 &= -2\sqrt{Q}\cos\left(\frac{\alpha + 4\pi}{3}\right) - \frac{A}{3} \end{aligned} \tag{7.12}$$

So let's go find the solutions of (7.8): from (7.9) we have:

$$Q = \frac{1}{9}, \qquad R = \frac{1}{27} \tag{7.13}$$

and it's easy to see that (7.10) is true and equal to 0. So equation (7.8) admit solutions that are:

$$\alpha = 0, \qquad b_1 = -\frac{1}{3}, \qquad b_2 = b_3 = \frac{2}{3} \tag{7.14}$$

The domain can be written as:

$$b\left(b + \frac{1}{3}\right)\left(b - \frac{2}{3}\right)^2 \geq 0 \tag{7.15}$$

whose solution is:

$$b \leq -\frac{1}{3} \quad b \geq 0. \tag{7.16}$$

From the physics we know that b is a positive quantity so we can plot without worries for $b \geq 0$.

The second implementation of the method is shown below; there are differences in the main program but the functions of the dimensionless numbers, operating parameter, flow rate, steady state operation and propulsive parameters are the same as before.

```
% MAIN 2

% project requirements

% voltage [V]
V_0 = 4001:100:7000;

% electrode distance [mm]
d = 5;

% emitter's diameter [mm]
D = 0.1;

% input(1) = d
% input(2) = D

double input;
input = [d*10^(-3) D*10^(-3)];

fid = fopen('project_requirements.txt','w+');
fprintf(fid,'%10.4f\n',input);
fclose(fid);

% fluid properties

% density [kg/m^3]
rho = 1130;

% surface tension [N/m]
gamma = 0.059;

% conductivity [Si/m]
K = 1;

% current parameter
f = 18;

% relative dielectric constant
e = 100;

% fluid(1) = rho
% fluid(2) = gamma
% fluid(3) = K
```

```
% fluid(4) = f
% fluid(5) = e

double fluid;
fluid = [rho gamma K f e];

fid = fopen('fluid_properties.txt','w+');
fprintf(fid,'%10.4f\n',fluid);
fclose(fid);

% physics constants

% dielectric constant [F/m]
e_0 = 8.85418781762*10^(-12);

% gravity acceleration [m/s^2]
g = 9.80665;

% taylor cone's angle [deg]
theta_t = 49.29;

% cost(1) = e_0
% cost(2) = g
% cost(3) = teta_t

double cost;
cost = [e_0 g theta_t*pi/180];

fid = fopen('physics_constants.txt','w+');
fprintf(fid,'%10.25f\n',cost);
fclose(fid);

% dimensionless numbers
pi = dimensionless_numbers(fluid);

% second method

b = 1:100:3000;

delta_quarti = 64*b -432*b.^3 + 432*b.^4;

q = - (-0.5*16*b.^0.5 - delta_quarti.^0.5);

```

```
PO = q/9;

a = PO.*b.^(-2.5);

% flow rate
Q = flow_rate(a,V_0,pi,fluid,cost);

%steady state operation
[E_ct, j, q_m] = steady_state_operation(Q,pi,fluid,cost);

% propulsive parameters
[I, m_dot, v, T, I_sp, eta] = propulsive_parameters(q_m,Q,V_0,
                              fluid,cost);

% plot
plot(b,PO); xlabel('b'); ylabel('P.O.');
```

```
% DIMENSIONLESS NUMBERS

function pi = dimensionless_numbers(fluid)

W = fluid(4)/sqrt(fluid(5));

% pi(1) = dimensionless number of q/m

pi(1) = W;

% pi(2) = dimensionless number of j

pi(2) = W^(7/3)*0.1043;
```

```
% OPERATING PARAMETER

function PO = operating_parameter(V_0,input,fluid,cost,pi)

PO = sqrt(8*pi(2)^3)/sqrt(pi(1))*(fluid(3)*sqrt(fluid(1)*fluid(2))
    *input(1)^(5/2))./(cost(1)^2.*V_0.^2);
```

```
% FLOW RATE

function Q = flow_rate(a,V_0,pi,fluid,cost)

Q = (a*cost(1)^(1/3)*fluid(2)^(3/4)*pi(2)*sqrt(pi(1)))./(2^(3/2)
   *fluid(3)^(7/12)*sqrt(fluid(1)).*V_0.^(1/2)).^(12/5);
```

```
% STEADY STATE OPERATION

function [E_ct, j, q_m] = steady_state_operation(Q,pi,fluid,cost)

q_m = pi(1)*sqrt(fluid(2)*fluid(3))/fluid(1).*Q.^(-1/2);

j = pi(2)*sqrt(fluid(2))/cost(1)^(2/3)*(fluid(3))^(7/6)*Q.^(-1/6);

E_ct = fluid(3)*j;
```

```
% PROPULSIVE PARAMETERS

function [I, m_dot, v, T, I_sp, eta] = propulsive_parameters(q_m,
                                        Q,V_0,fluid,cost)

I = fluid(4)*sqrt(fluid(2)*fluid(3)/fluid(5))*Q.^(0.5);

m_dot = fluid(1)*Q;

v = sqrt(2*q_m.*V_0);

T = m_dot.*v;

I_sp = v/cost(2);

eta = m_dot.*v.^2./(2*V_0.*I);
```

.

The plot of function (7.3) is shown in figure 7.2: it can be seen the direct proportionality of $P.O.$ with b and so we can say, according to (6.6b), that:

$$P.O. \propto \frac{1}{V_0} \tag{7.17}$$

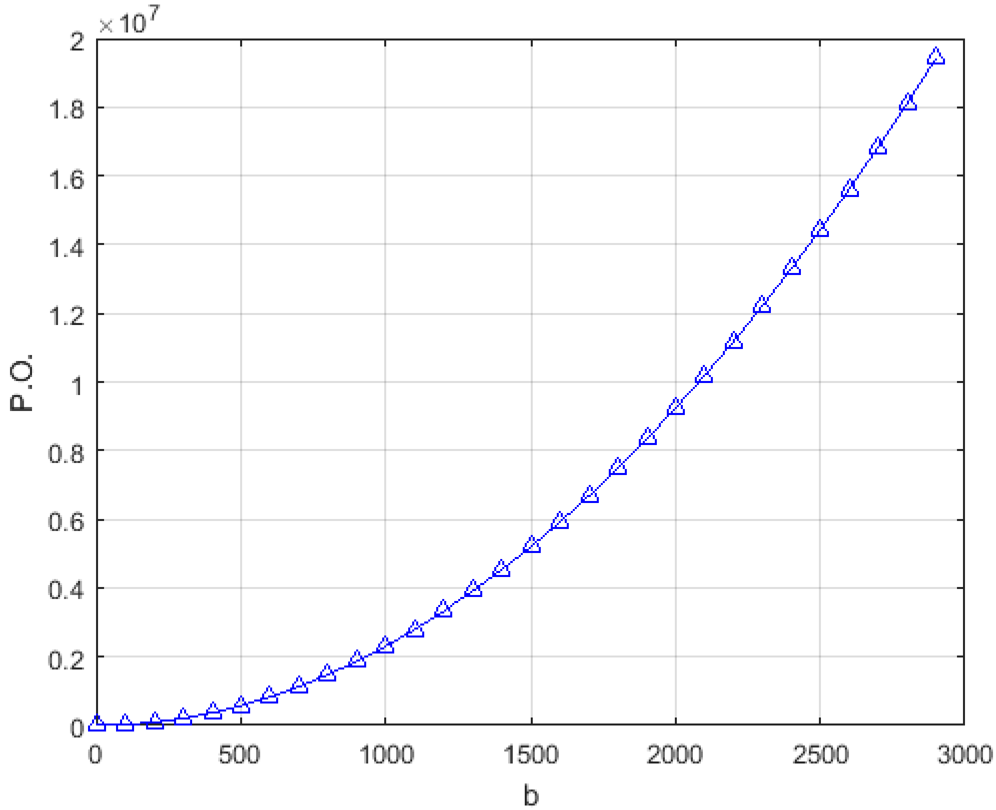

Figure 7.2: Function $P.O. = f(b)$.

Chapter 8

Conclusions

The models work differently and give different results. With a first inspection we can see that the first model works better than the second and shows results that are of the same order of magnitude of the data in the literature of electrospray engines. We also note that the dependence of the thrust with the voltage it's not correct because from equations (5.36) and (3.37):

$$\left.\begin{aligned} T &\propto \frac{V^2}{c^3} \\ c &\propto \sqrt{V} \end{aligned}\right\} \quad \Longrightarrow \quad T \propto \sqrt{V} \tag{8.1}$$

and it's clear from the graphics in figure 7.1 that the thrust doesn't have a trend like (8.1). The deviation from the actual data can be explained by our assumptions of absence of fragmentation and of the initial speed of the jet: actually the jet has a speed not null and so the first model doesn't take into account a part of energy present in the phenomenon. However, with a large range of initial voltages V_0 and without taking into account the losses of fragmentation it can be seen that the calculated efficiencies for each voltage is exactly 1 so it can be said that the physics principles present are respected and also that the program is correctly implemented.
Now, if the speed of the jet is taken into account the program shows an unexpected behavior yielding us complex solutions, that don't make sense. An hypothesis after seeing step by step the program can be searched in the complexity of the solving system: it's possible that for some values of the Operating Parameter the system doesn't have real solutions, so the subsequent iterations are all complex numbers. Another possibility could be the sensibility of function `fzero`, for which with a specific initial point gives complex solutions: for a large range of initial points the system always gives complex

solutions so this last supposition seems to be the weakest. Unfortunately we don't have actual data of the Operating Parameter with which we can compare the results, so our calculation cannot be continued without other assumptions.

Appendix A

Meusnier's theorem

In differential geometry Meusnier's theorem states that the radius of curvature R_α of an oblique flat section whose normal forms with the normal surface an angle θ is equal to the radius of curvature R_n of the normal section having the same tangent multiplied by the cosine of θ:

$$R_n \cos(\theta) = R_\alpha \tag{A.1}$$

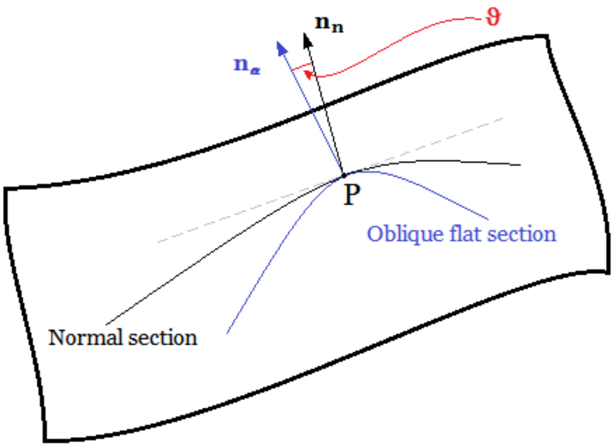

Figure A.1: Meusnier's theorem.

Appendix B

Buckingham's theorem

The Buckingham's theorem (or π-theorem) states that in mathematical terms, if we have a physically meaningful equation such as:

$$f(q_1, q_2, \dots, q_N) = 0 \tag{B.1}$$

where the q_j are the N physical variables, and they are expressed in terms of K independent physical units, then the above equation can be restated as:

$$F(\pi_1, \pi_2, \dots, \pi_P) = 0 \tag{B.2}$$

where the π_j are dimensionless parameters constructed from the q_j by $P = N - K$ dimensionless equations the so called Π groups, of the form:

$$\pi_i = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdot \dots \cdot q_N^{\alpha_N} \tag{B.3}$$

where the exponents α_j are rational numbers (they can always be taken to be integers by redefining π_i as being raised to a power that clears all denominators).

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