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Symplectic Reflection Algebras

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Introduction

The present thesis studies a family of associative $\mathbb{C}$-algebras known as symplectic reflection algebras. These algebras, which were introduced by Etingof and Ginzburg [1], have a rich and varied theory which have applications in different mathematical contexts: they are encoded with a combinatorics which shows similarities with the one which arises from the classification of a family of irreducible representations of finite groups of Lie type [22], they are involved with the problem of the existence of symplectic resolutions for some symplectic quotient varieties [6], and they have many analogies with the universal enveloping algebra $U(\mathbf{g})$ of a semisimple Lie algebra $\mathbf{g}$.
In Chapter 1 we introduce some preliminary notions on Hopf algebras, graded and filtered algebras, symplectic $\mathbb{C}$-vector spaces, classical deformation theory and Poisson algebras.
In Chapter 2 we define the symplectic reflection algebras following [1] and we highlight some of their properties. In order to give their definition, Etingof and Ginzburg consider a larger class of algebras consisting of deformations of the smash product algebra $SV\#\Gamma$, where SV is the symmetric algebra of a finite dimensional $\mathbb{C}$-vector space V and Γ is a finite group of automorphisms of V.
In analagoy with Lie Thoery, the symplectic reflection algebras are the ones for which the PBW-Theorem holds (Theorem 2.1.2) and we denote them by $H_{t,c}$, where t, c are parameters.
Now, we study the substructures of a given symplectic reflection algebra $H_{t,c}$, in particular we focus on a subalgebra: the spherical subalgebra.
We describe the proof of [1, Theorem 1.6] in which the spherical subalgebra is interpreted as a deformation of the Poisson algebra $(SV)^\Gamma$ and where it is proved that for $t = 0$ it is commutative.
To conclude Chapter 2, we recall from [6] and [1] the results which allow us to relate the centre of the spherical subalgebra to the one of the corresponding symplectic reflection algebra and we restate the characterization of the centre of $H_{t,c}$ depending on the parameter $t \in \mathbb{C}$ that can be found in [6]:

- If $t \neq 0$, $Z(H_{t,c}) = \mathbb{C}$.

- If $t = 0$, $H_{0,c}$ is a finite module over $Z(H_{0,c})$.

In Chapters 3 we focus on a particular family of symplectic reflection algebras: the rational Cherednik algebras. Being a particular family of the main mathematical objects of this thesis, the results stated in the general case still hold. However, unlike general symplectic reflection algebras, rational Cherednik algebras are $\mathbb{Z}$-graded algebras and they admit a triangular decomposition as $\mathbb{C}$-vector spaces which is analogous to the one of the universal enveloping algebra $U(\mathbf{g})$ of a semisimple Lie algebra $\mathbf{g}$.
We move on setting the ground to study their representation theory. In particular, we focus on rational Cherednik algebras at $t = 0$ and we recall from [6] the definition of some quotients of algebras in the latter family: the restricted rational Cherednik algebras. They inherit a triangular decomposition as $\mathbb{C}$-vector spaces from the one of the corresponding rational Cherednik algebras which will play an important role in their representation theory.
In Chapter 4, in the spirit of the representation theory of $U(\mathbf{g})$, we report some results on the representation theory of a restricted rational Cherednik algebra from [14], obtained by exploiting its triangular decomposition. We present the construction given in [6] of a family of finitely generated left modules for this algebra called Baby Verma modules which are analogous to the Verma modules for the universal enveloping algebra $U(\mathbf{g})$ of a semisimple Lie algebra $\mathbf{g}$.
To conclude, we compute a concrete example for $\mathbb{C}^2$ and a cyclic group. In particular, we give an explicit description of the objects involved in the thesis: the associated rational Cherednik algebra and the restricted rational Cherednik algebra. Moreover, we compute the Baby Verma modules, writing the computations in full detail in the case of a cyclic group of order 2.

Contents

Chapter 1

Preliminaries

1.1 Hopf Algebras

1.1.1 Algebras and Coalgebras

Throughout this section F is a field and tensor products are over F. As in [3]:

Definition 1.1.1. A F-algebra (with unit) is a F-vector space A together with two F-linear maps, multiplication $m : A \otimes A \longrightarrow A$ and unit $u : F \longrightarrow A$, such that the following diagrams are commutative:

a) associativity:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ \downarrow{\scriptstyle id \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array}$$

b) unity:

$$\begin{array}{ccccc} & \overset{u \otimes id}{\nearrow} & A \otimes A & \overset{id \otimes u}{\nwarrow} & \\ F \otimes A & & \downarrow{\scriptstyle m} & & A \otimes F \\ & \searrow & A & \swarrow & \end{array}$$

Moreover, the two lower maps in b) are given by scalar multiplication, and the same diagram gives the usual identity element in A by setting $1_A = u(1_F)$.

Definition 1.1.2. For any F-vector spaces V and W, the *twist map* $\tau : V \otimes W \longrightarrow W \otimes V$ is given by $\tau(v \otimes w) = w \otimes v$.

Note that A is commutative if and only if $m \circ \tau = m$ on $A \otimes A$.

Definition 1.1.3. A F-coalgebra (with counit) is a F-vector space C together with two F-linear maps, comultiplication $\Delta : C \longrightarrow C \otimes C$ and counit $\varepsilon : C \longrightarrow F$ such that the following diagrams are commutative:

a) coassociativity:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes id} \\ C \otimes C & \xrightarrow{id\otimes\Delta} & C \otimes C \otimes C \end{array}$$

b) counity:

$$\begin{array}{ccccc} & \overset{1\otimes}{\swarrow} & C & \overset{\otimes 1}{\searrow} & \\ F \otimes C & & \downarrow{\scriptstyle\Delta} & & C \otimes F \\ & \underset{\varepsilon\otimes id}{\nwarrow} & C \otimes C & \underset{id\otimes\varepsilon}{\nearrow} & \end{array}$$

We say that C is cocommutative if and only if $\tau \circ \Delta = \Delta$ on C.

Remark 1.1.1.1. *Notice that given a F-algebra (B, m_B, u_B) then $(B \otimes B, \tilde{m}_{B\otimes B}, \tilde{u}_{B\otimes B})$ with $\tilde{m}_{B\otimes B} := (m_B \otimes m_B) \circ (id_B \otimes \tau \otimes id_B)$ and $\tilde{u}_{B\otimes B} := u_B \otimes u_B$, is again a F-algebra and similarly given a F-coalgebra $(C, \Delta_C, \varepsilon_C)$ then $(C \otimes C, \tilde{\Delta}_{C\otimes C}, \tilde{\varepsilon}_{C\otimes C})$, where $\tilde{\Delta}_{C\otimes C} := (id_C \otimes \tau \otimes id_C) \circ (\Delta_C \otimes \Delta_C)$ and $\tilde{\varepsilon}_{C\otimes C} := \varepsilon_C \otimes \varepsilon_C$, is a F-coalgebra.*

Definition 1.1.4. Let C and D be coalgebras, with comutiplications Δ_C and Δ_D, and counits ε_C and ε_D respectively.

a) A map $f : C \longrightarrow D$ is a *coalgebra morphism* if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and if $\varepsilon_C = \varepsilon_D \circ f$.

b) A subspace $I \subseteq C$ is a *coideal* if $\Delta I \subseteq I \otimes C + C \otimes I$ and if $\varepsilon(I) = 0$.

1.1.2 Bialgebras, Convolution, Summation Notation

Now we combine the notions of algebra and coalgebra;

Definition 1.1.5. A F-space B with maps m, u, Δ, ε is a *bialgebra* if (B, m, u) is an algebra, (B, Δ, ε) is a coalgebra, and either of the following (equivalent) conditions hold:

1) Δ and ε are algebra morphisms

2) m and u are coalgebra morphisms.

Definition 1.1.6. Let B, B' be bialgebras. A map $f : B \longrightarrow B'$ is a *bialgebra morphism* if it is both an algebra and a coalgebra morphism.

Definition 1.1.7. Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra and (A, m_A, u_A) be an algebra. Then $Hom_F(C, A)$ becomes an algebra under the *convolution product*:

$$(f * g)(c) = m_A \circ (f \otimes g)(\Delta_C)$$

for all $f \in Hom_F(C, A)$, $c \in C$. The unit element in $Hom_F(C, A)$ is $u_A \varepsilon_C$.

Notation 1.1.8. Let C be a coalgebra with comultiplication $\Delta : C \longrightarrow C \otimes C$. The *sigma notation* for Δ is given as follows: for any $c \in C$, we write:

$$\Delta c = \Sigma c_{(1)} \otimes c_{(2)}.$$

Remark 1.1.2.1. *When Δ must be applied more than once, using the coassociativity property in 1.1.3 a) we get:*
$\Sigma c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}} = \Sigma c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)}$*; this element is written as* $\Sigma c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = \Delta_2(c)$*, where $\Delta_2(c)$ is the element (necessarly unique) obtained applying coassociativity two times. Iterating this procedure we write:*

$$\Delta_{n-1}(c) = \Sigma c_{(1)} \otimes \cdots \otimes c_{(n)},$$

where $\Delta_{n-1}(c)$ is the element (necessarly unique) obtained applying coassociativity $(n-1)$-times. Moreover it follows that for all $c \in C$,

$$c = \Sigma \varepsilon(c_{(1)}) c_{(2)} = \Sigma \varepsilon(c_{(2)}) c_{(1)}$$

and the convolution product is given by:

$$(f * g)(c) = \Sigma f(c_{(1)}) g(c_{(2)}).$$

Now, we can give the definition of a *Hopf algebra*:

Definition 1.1.9. Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. Then H is a *Hopf algebra* if there exists an element $S \in Hom_F(H, H)$ which is an inverse of id_H under convolution $*$. The map S is called *antipode* for H.

Remark 1.1.2.2. *Note that in Σ-notation S satisfies:*

$$\Sigma(Sh_1)h_2 = \varepsilon(h)1_H = \Sigma h_1(Sh_2)$$

for all $h \in H$.

Definition 1.1.10. A map $f : H \longrightarrow K$ is a *Hopf algebra morphism* if it is a bialgebra morphism and if it holds:

$$f(S_H h) = S_K(f(h)), \text{ for any } h \in H,$$

where S_H and S_K are the antipodes for H and K, respectively.

Example 1.1.11. Let G be a finite group with unit e. The group algebra $H = FG$ is the space of all formal linear combinations $\Sigma_{g\in G} a_g g$, $a_g \in F$, with the multiplication:

$$(\Sigma_{g\in G} a'_g g)(\Sigma_{g\in G} a''_g g) = \Sigma_{g\in G}\Sigma_{uv=g} a'_u a''_v g,$$

and with unit e.
Then $H = FG$ becomes a Hopf algebra by defining $\Delta g = g \otimes g$, $\varepsilon(g) = 1$ and $Sg = g^{-1}$, for any $g \in G$.

Example 1.1.12. Let $\mathbf{g}$ be a Lie algebra and let $U(\mathbf{g})$ be its universal enveloping algebra. Then $U(\mathbf{g})$ becomes a Hopf algebra by defining
$\Delta x = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $Sx = -x$, for any $x \in \mathbf{g}$.

1.1.3 Smash Product of Hopf Algebras

In this subsection let H be a *F-Hopf algebra*.

Definition 1.1.13. Let H be a Hopf algebra and V, W be left *H-modules*. Then $V \otimes W$ is again a left *H-module* via:

$$h \cdot (v \otimes w) = \Sigma h_1 \cdot v \otimes h_2 \cdot w$$

for all $h \in H$, $v \in V$ and $w \in W$.
Rewriting the equality above in term of maps:

$$\Phi_{V\otimes W} = (\Phi_V \otimes \Phi_W) \circ (id_H \otimes \tau \otimes id_W) \circ (\Delta \otimes id_V \otimes id_W),$$

where $\Phi_V : H \otimes V \longrightarrow V$ and $\Phi_W : H \otimes W \longrightarrow W$ are the two given module actions, $\Phi_{V\otimes W} : H \otimes (V \otimes W) \longrightarrow V \otimes W$ is the module action on $V \otimes W$ and τ is the *twist map*: $H \otimes V \longrightarrow V \otimes H$.

Definition 1.1.14. An algebra A is a (left) *H-module algebra* if
for all $h \in H$, $a, b \in A$:

1) A is a left *H-module* via $h \otimes a \longmapsto h \cdot a$.

2) $h \cdot (ab) = \Sigma(h_1 \cdot a)(h_2 \cdot b)$.

3) $h \cdot 1_A = \varepsilon(h)1_A$.

Definition 1.1.15. Let A be a *left H-module algebra*. Then the *smash product algebra* $A\#H$ is defined as follows:

1) As F-vector spaces, $A\#H = A\otimes H$, and we write the element $a\#h = a\otimes h$, for $a \in A$ and $h \in H$.

2) Multiplication is given by:

$$(a\#h)(b\#k) = \Sigma a(h_1 \cdot b)\#h_2 k \ ,$$

for all a, $b \in A$, h, $k \in H$.

It follows from the definition that $A \simeq A \otimes 1$ and $H \simeq 1 \otimes H$, thus the element $a\#h$ is frequently abbreviated by ah.

Example 1.1.16. Let $H = FG$ and let A be an H-*module algebra*. Since $\Delta g = g \otimes g$ for any $g \in G$, by Definition 1.2.2 2), we have that
$g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all a, $b \in A$ and thus g acts as an endomorphism of A. Morevoer by Definition 1.2.2 1) g acts as an automorphism of A (since $gg^{-1} = 1$). As a consequence there is a group homomorphism $G \longrightarrow Aut_F(A)$ and also the converse holds: any such a map makes A into a FG-*module algebra*.
In this case $A\#FG = A * G$, the *skew group ring*, where multiplication is just:

$$(ag)(bh) = a(g \cdot b)gh, \text{ for all } a,\ b \in A,\ h,\ k \in G.$$

Example 1.1.17. Let H and G be groups such that G acts on H by automorphisms. Then FH is a FG-module algebra and the smash product $FH\#FG$ exists [7, Lemma 3.3.9]. Moreover by [7, Proposition 3.3.10], we have the following isomorphism of algebras:

$$FH\#FG \simeq F(H \rtimes G).$$

1.2 Symplectic Vector Spaces

Throughout this section fix $F = \mathbb{C}$.

1.2.1 Definitions and Basic Notions

Definition 1.2.1. Let V be a vector space over $\mathbb{C}$, and let
$\omega : V \times V \longrightarrow \mathbb{C}$ be a bilinear form which satisfies the following properties for any $x, y \in V$:

(i) $\omega(x, y) = -\omega(y, x)$ (skew-symmetric);

(*ii*) $\omega(x, x) = 0$ (totally isotropic);

(*iii*) If $\omega(y, x) = 0$ for all $y \in V$, then $x = 0$ (non-degenerate).

Then ω is a *symplectic form* on V and the pair (V, ω) is called *symplectic vector space.*
A subspace $W \subset V$ which satisfies $\omega(x, y) = 0$ for all $x, y \in W$ is said to be *isotropic.*

From now on V is a finite dimensional vector space over $\mathbb{C}$.

Corollary 1.2.2. *[8, Corollary 1.3] Let (V, ω) be a symplectic vector space. Then $dimV$ is even.*

Example 1.2.3. Let $V = \mathbb{C}^{2n}$. It is a symplectic vector space over $\mathbb{C}$ with symplectic form $\omega : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \longrightarrow \mathbb{C}$ such that its associated matrix with respect to the canonical basis of $\mathbb{C}^{2n}$ has the following form:

$$J = \begin{bmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{bmatrix} \tag{1.1}$$

Notice that the matrix associated to ω is skew-symmetric and non-singular.

Remark 1.2.1.1. *If (V, ω) is a symplectic vector space with $dimV = 2n$, then by [8, Theorem 1.4] there exists a basis of V such that the matrix associated to ω is as in Example* 1.2.3.

Definition 1.2.4. Let (V, ω) be a symplectic vector space and let $W \subset V$. W is said to be *Lagrangian* if it is a maximal isotropic subspace in V, that is:

(*i*) $\omega(x, y) = 0$ for all $x, y \in W$ (i.e. W is isotropic).

(*ii*) There is no $U \subset V$ such that (*i*) holds for U and $W \subset U$.

Definition 1.2.5. Let (V, ω) be a symplectic vector space and let $U \subset V$ be a subspace of V. We set $U^{\omega} = \{v \in V \mid \omega(v, w) = 0 \text{ for any } w \in U\}$. Then $U \subset V$ is said to be *symplectic* if $U \cap U^{\omega} = \{0\}$.
In particular it follows that:

$$dimU^{\omega} = dimV - dimU. \tag{1.2}$$

Thus, given a subspace $U \subset V$:

- U is isotropic if and only if $U \subseteq U^{\omega}$. Then by (1.2): $dimU \leq \frac{1}{2}dimV$.
- U is Lagrangian if and only if $U \subseteq U^{\omega}$ and $dimU = \frac{1}{2}dimV$, [8, Definition 1.6].

Remark 1.2.1.2. *In Example* 1.2.3 *we consider the subspaces* $U_j = Span\{e_1, \ldots, e_j\}$. *They are isotropic for any* $j \leq n$ *and Lagrangian if and only if* $j = n$.

Definition 1.2.6. Let (V, ω) be a symplectic vector space. The *symplectic group* $Sp(V) \leq GL(V)$ is the group of automorphisms of V which preserve the symplectic form ω:

$$\omega(g \cdot v, g \cdot u) = \omega(v, u), \text{ for any } g \in Sp(V), \text{ and } v, u \in V.$$

From now on through this section let (V, ω) be a *symplectic vector space*.

Definition 1.2.7. An element $s \in Sp(V)$ with $|s| < \infty$ is said to be a *symplectic reflection* if:

$$dim\{v \in V \mid s \cdot v = v\} = dimV - 2.$$

A group $\Gamma \subset Sp(V)$ which is generated by symplectic reflections is called a *symplectic reflection group.*

Remark 1.2.1.3. *Let* (V, ω) *be a symplectic vector space with* $dimV = 2n$ *and let* $\{v_1, \cdots, v_{2n}\}$ *be a basis of* V *such that the matrix associated to* ω *is as in Example* 1.2.3.
Then:

$$Sp(V) = \{X \in GL(2n, \mathbb{C}) \mid {}^tXJX = J\},$$

where J *is as in* (1.1).
Note that $Sp(V) \subseteq SL(V)$, *where* $SL(V) = \{X \in GL(2n, \mathbb{C}) \mid detX = 1\}$.
Indeed, for any $X \in Sp(V)$ *taking the determinants on both sides of the equation* ${}^tXJX = J$*:*

$$det({}^tXJX) = det^tXdetJdetX = (detX)^2detJ = detJ,$$

we get $detX = \pm 1$.
Then, viewing ω *as an element in* Λ^2V^* *we have* $\omega = v_1^* \wedge v_{n+1}^* + \cdots + v_n^* \wedge v_{2n}^*$, *where* $\{v_1^*, \cdots, v_{2n}^*\}$ *is the dual basis of the fixed basis in* V.
Consider $\omega^{\wedge n} := \underbrace{\omega \wedge \cdots \wedge \omega}_{n\text{-times}} \in \Lambda^{2n}V^*$. *By [18, Proposition 4.1]* $\Lambda^{2n}V^*$ *is one-dimensional, then* $\omega^{\wedge n} = \mu(v_1^* \wedge \cdots \wedge v_{2n}^*)$ *for some* $\mu \in \mathbb{C}^*$, *where* $v_1^* \wedge \cdots \wedge v_{2n}^*$ *is the volume form in* $\Lambda^{2n}V^*$. *More precisely, by [20]:*

$\omega^{\wedge n}(v_1, v_{n+1}, \cdots, v_n, v_{2n}) = \Sigma_{\sigma \in S_{n+n}} sgn(\sigma)\omega(v_{\sigma(1)}, v_{\sigma(n+1)}) \cdots \omega(v_{\sigma(n)}, v_{\sigma(n+n)})$.
Notice that since $\omega = \Sigma_{i=1}^n v_i^* \wedge v_{i+n}^*$, *the summands in the above equation are non zero if and only if* $\sigma(i+n) = \sigma(i) + n$, *for any* $i \in \{1, \cdots, n\}$. *In particular by [21, 2.7],* $\omega^{\wedge n}(v_1, v_{n+1}, \cdots, v_n, v_{2n}) = n!$, *i.e,* $0 \neq \omega^{\wedge n} \in \Lambda^{2n} V^*$.
By [18, Definition 7.1], $\omega^{\wedge n}(A) = \mu det(A)$, *for any* $A \in GL(2n, \mathbb{C})$.
Fix $X \in Sp(V)$. *Then* $\omega^{\wedge n}(XA) = \omega^{\wedge n}(A)$, *i.e,* $\mu det X det A = \mu det A$, *with* $\mu \in \mathbb{C}^*$. *Thus,* $det X = 1$ *and we conclude that* $Sp(V) \subseteq SL(V)$.

Lemma 1.2.8. Let (V, ω) be a symplectic vector space with $dimV = 2n$ and let $\Gamma \subset Sp(V)$. Let S be the set of symplectic reflections in Γ. For each $s \in S$, the spaces $Im(1-s)$ and $Ker(1-s)$ are symplectic subspaces of V such that $V = Im(1-s) \oplus Ker(1-s)$ and $dimIm(1-s) = 2$.

Proof. For any $s \in S$, with $|s| = m < \infty$, s is diagonalizable over $\mathbb{C}$ with m-th roots of unity on the diagonal. Indeed $\mathbb{C}$ is an algebraically closed field with $char(\mathbb{C}) = 0$, thus it contains all the distinct roots (which are the m-th roots of unity) of its minimal polynomial. Since $|s| < \infty$, the Jordan blocks relative to the eigenvalues of s must be of finite order and since $char(\mathbb{C}) = 0$ they must be diagonal.
Moreover, for any $s \in S$, $dimV_1 = dimKer(1-s) = 2n-2$, where V_1 denotes the eigenspace relative to the eigenvalue 1. By Remark 1.2.1.3 $det(s) = 1$, then there are just two possibilities for the other eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ (counted with multiplicity):
(i) $\lambda_1 = \lambda_2 = -1$;
(ii) $\lambda_1, \lambda_2 = \lambda_1^{-1}$, with $\lambda_1 \neq \pm 1$.
Denote by $V_{\lambda_1}, V_{\lambda_2}$ the corresponding eigenspaces, then in (i) $V_{\lambda_1} = V_{\lambda_2} = V_{-1}$ and $dimV_{-1} = 2$, while in (ii) $V_{\lambda_1} \neq V_{\lambda_2}$ and $dimV_{\lambda_1} = dimV_{\lambda_2} = 1$.
Thus:

$$Im(1-s) = \begin{cases} V_{-1}, & \text{in (i)} \\ V_{\lambda_1} \oplus V_{\lambda_2}, & \text{in (ii)}. \end{cases}$$

Hence: $Ker(1-s) \cap Im(1-s) = \{0\}$ and $V = Ker(1-s) \oplus Im(1-s)$.
Now, it remains to prove that $Ker(1-s)^\omega = Im(1-s)$. Indeed, then we get that $Ker(1-s) \cap Ker(1-s)^\omega = \{0\}$ and we can conclude by Definition 1.2.5 that $Ker(1-s)$ and $Im(1-s)$ are symplectic subspaces of V.
Notice that $(1-s)_{|Im(1-s)}$ is an automorphism of $Im(1-s)$ and so any $v \in Im(1-s)$ is of the form $v = w - s \cdot w$, with $w \in V$. Moreover, for any $z \in Ker(1-s)$, we have that $s \cdot z = z$ and

$$\begin{aligned} \omega(v, z) = \omega(w - s \cdot w, z) = \omega(w, z) - \omega(s \cdot w, z) = \\ = \omega(w, z) - \omega(s \cdot w, s \cdot z) = 0, \end{aligned}$$

i.e,

$$Im(1-s) \subseteq Ker(1-s)^\omega.$$

The other inclusion follows by an argument on the dimensions of the subspaces of V we are considering.
Indeed, we know that $dimKer(1-s)^\omega = 2 = dimIm(1-s)$, hence it follows that

$$Im(1-s) = Ker(1-s)^\omega,$$

and we can conclude that $Ker(1-s)$ and $Im(1-s)$ are symplectic subspaces of V. □

From now on we denote by ω_s the 2-form on V whose restriction to $Im(1-s)$ is ω and whose restriction to $Ker(1-s)$ is zero.

Using the definition given in [1]:

Definition 1.2.9. The triple (V, ω, Γ), where $\Gamma \subset Sp(V)$, is said to be *indecomposable* if there is no ω-orthogonal direct sum decomposition:
$V = V_1 \oplus V_2$, where V_1 and V_2 are Γ-stable proper symplectic vector subspaces in V.

Definition 1.2.10. Let $\mathbf{h}$ be a finite dimensional vector space over $\mathbb{C}$ and let $W \subset GL(\mathbf{h})$ be a finite subgroup. A *pseudo-reflection* is an element $g \in W$ such that $rk(1-g) = 1$.
A finite subgroup $W \subset GL(\mathbf{h})$ is called a *complex reflection group* if it is generated by the *pseudo-reflections* that it contains.

Lemma 1.2.11. [6, Lemma 1.18] Let (V, ω, Γ) be an indecomposable triple.

1. Either V is a simple Γ-module or $V = U \oplus U^*$ with U a simple Γ-module and U, U^* Lagrangian with respect to ω.

2. If $V = U \oplus U^*$, then any symplectic reflection in Γ acts on U as a pseudo-reflection.

3. The space $(\Lambda^2 V^*)^\Gamma$ is one-dimensional.

Proof. Let $V = U_1 \oplus \cdots \oplus U_k$ be a decomposition of V into simple Γ-modules and assume that $V \neq U_1$. We claim that U_1 is an isotropic subspace of V.
First of all, the space U_1 cannot be a symplectic subspace of V because of the indecomposability of the triple.
Consequently the subspace $U_1 \cap U_1^\omega$ is a non-zero Γ-submodule of U_1. This implies that $U_1 = U_1 \cap U_1^\omega$, i.e, $U_1 \subseteq U_1^\omega$ and the claim is proved.

Let $0 \neq u \in U_1$. Now, ω is non-degenerate, so it is possible to find some $j \neq 1$ and $v \in U_j$ such that $\omega(u, v) \neq 0$. Hence $U_1 \oplus U_j$ is not isotropic.
We prove that $U_1 \oplus U_j$ is a symplectic subspace of V, i.e,
$(U_1 \oplus U_j) \cap (U_1 \oplus U_j)^\omega = \{0\}$.
Consider $U_1^\omega \cap U_j = \{w \in U_j \mid \omega(w, u) = 0$, for any $u \in U_1\} \subseteq U_j$, where U_j is a simple Γ-module. Let $w \in U_j \cap U_1^\omega$ and $g \in \Gamma$.
Then $\omega(g \cdot w, u) = \omega(w, g^{-1} \cdot u) = 0$ since U_1 is Γ-stable. Hence, we showed that $U_1^\omega \cap U_j$ is a Γ-submodule of U_j. Since U_j is simple and there exists $v \in U_j$ such that $\omega(v, u) \neq 0$ by construction, it follows that $U_1^\omega \cap U_j = \{0\}$.
Now, let $u_1 + u_j \in (U_1 \oplus U_j)^\omega \cap (U_1 \oplus U_j)$. By construction:
(i) $\omega(u_1 + u_j, u) = 0$ for any $u \in U_1$, and
(ii) $\omega(u_1 + u_j, w) = 0$ for any $w \in U_j$.
By (i) we have that $\omega(u_j, u) = 0$ for any $u \in U_1$ because U_1 is isotropic. Then, since $U_1^\omega \cap U_j = \{0\}$, it follows that $u_j = 0$. Analogously, by (ii) it follows that $u_1 = 0$. Then we get: $(U_1 \oplus U_j)^\omega \cap (U_1 \oplus U_j) = \{0\}$. Hence $U_1 \oplus U_j$ is a symplectic subspace. Therefore, by the indecomposability of the triple we get that $U_1 \oplus U_j$ cannot be a proper subspace of V, $V = U_1 \oplus U_j$. In this case, the symplectic form induces a Γ-module isomorphism $\omega : U_j \longrightarrow U_1^*$, $V = U_1 \oplus U_1^*$ and this proves 1).
Now we want to prove that a symplectic reflection in Γ must act on U_1 as a pseudo-reflection, that is: for all $g \in \Gamma \subset Sp(V)$, $g \neq 1$ such that
$dim\{v \in V \mid g \cdot v = v\} = dimV - 2$, then: $g_{|U_1} : U_1 \longrightarrow U_1$ is such that
$dim\{u \in U_1 \mid g \cdot u = u\} = dimU_1 - 1$.
This is true because $U_1 \simeq U_1^*$. Moreover, $dimU_1 = dimU_1^* = \frac{dimV}{2}$ because $U_1 \oplus U_1^* = V$. It also holds: $dimU_1^g = dim(U_1^*)^g$, for any $g \in \Gamma$. Indeed for any $u \in U_1$ such that $g \cdot u = u$, we have that $\phi : U_1 \longrightarrow U_1^*$, $u \longmapsto \omega(u, -)$ is such that: $g \cdot \phi(u) = \phi(g \cdot u) = \omega(g \cdot u, -) = \omega(u, -) = \phi(u)$. Thus, any symplectic reflection $s \in \Gamma$ acts on U_1 as a pseudo-reflection and 2) is proved.
Write from now on $U_1 = U$. Now we will show that the space $(\Lambda^2 V^*)^\Gamma$ is one-dimensional. If V is a simple Γ-module there is nothing to prove since $\omega \in (\Lambda^2 V^*)^\Gamma$ by hypothesis and by *Schur's Lemma* the dimension of the given space is at most equal to 1. If $V = U \oplus U^*$ we have that $\omega \in (\Lambda^2 V^*)^\Gamma$, hence the dimension of this space is at least 1.
So we have to prove that the dimension is exactly equal to 1 also in this case. Assume that there exists $0 \neq \nu \in (\Lambda^2 V^*)^\Gamma$, with $\nu \neq \lambda\omega$, for some $\lambda \in \mathbb{C}$. Then $Ker\nu$ is a Γ-module and we can decompose $V = Ker\nu \oplus V'$, where V' is some Γ-module. If V' is a proper submodule of V, then by 1)
$V = U \oplus U^*$ and it must be $V' \simeq U$. However, (V', ν) is a symplectic vector space, $\Gamma \subset Sp(V')$ and this is a contradiction by 2). Indeed, any symplectic reflection $s \in \Gamma$ must act on $V' \simeq U$ as a pseudo-reflection. Therefore ν must be non-degenerate and $V' = V$.

Take $s \in \Gamma$ to be a symplectic reflection and consider $H =< s >$.
By *Lemma* 1.2.8, it is possible to decompose $V = Im(1-s) \oplus Ker(1-s)$ into symplectic vector subspaces (with respect to both ω and ν since we proved that they are both non degenerate and by assumption
$\omega,\, \nu \in (\Lambda^2 V^*)^\Gamma$).
Let $V_0 := Im(1-s)$. Then it is a two-dimensional symplectic vector subspace of V and the restrictions ν_0 and ω_0 of ν and ω to V_0 are non-zero elements in the one-dimensional space $\Lambda^2 V_0^* = (\Lambda^2 V_0^*)^H$. Indeed, V_0 being a symplectic vector subspace of V is also a symplectic vector space in its own right, and requiring ω_0 and ν_0 being non-zero elements in $\Lambda^2 V_0^* = (\Lambda^2 V_0^*)^H$ is equivalent to requiring ω_0 and ν_0 being nondegenerate in $\Lambda^2 V_0^* = (\Lambda^2 V_0^*)^H$.
After rescaling ν if necessary, we may assume $\nu_0 = \omega_0$. If $\nu \neq \omega$, then $\nu - \omega$ is a non-zero but degenerate element in $(\Lambda^2 V^*)^\Gamma$, which is not possible since we have shown that any $0 \neq \nu \in (\Lambda^2 V^*)^\Gamma$ with $\nu \neq \lambda\omega$, $\lambda \in \mathbb{C}$ must be non-degenerate. Hence $\nu = \omega$. □

Example 1.2.12. Let $V = U \oplus U^*$ be a finite dimensional vector space over $\mathbb{C}$. The space $U \oplus U^*$ has a natural pairing: $(-,-) : U \times U^* \longrightarrow \mathbb{C}$, defined by $(x, y) := y(x)$, for any $x \in U$ and $y \in U^*$.
Then we define the standard symplectic form on V as follows:

$$\omega((x,\zeta),(y,\eta)) := (\eta, x) - (\zeta, y) = \eta(x) - \zeta(y),$$

for any $x, y \in U$ and $\zeta, \eta \in U^*$.

Now choose a basis $(e_1, \ldots, e_n)$ in U and let $(f_1, \ldots, f_n)$ be its dual basis in U^*, defined by: $f_j(e_i) = \delta_{ij}$.
The basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of V satisfies:

$$\omega(e_j, e_k) = \omega(f_j, f_k) = 0,\, \omega(f_j, e_k) = -\omega(e_j, f_k) = \delta_{jk}$$

and it is called a symplectic basis for V. Hence we have that (V, ω) is a symplectic vector space, since it follows from the definition of ω that it is a nondegenerate and skew-symmetric bilinear form of V.

If we write in coordinates with respect to the fixed basis we get:

$$\omega((x,\zeta),(y,\eta)) = \Sigma(\eta_j x_j - \zeta_j y_j),$$

with $x = \Sigma x_j e_j$, $y = \Sigma y_j e_j$, $\zeta = \Sigma \zeta_j f_j$ and $\eta = \Sigma \eta_j f_j$.

Thus, from the natural pairing $U \times U^* \longrightarrow \mathbb{C}$ it is possible to define a symplectic form on $V = U \oplus U^*$.

Moreover, if W is a complex reflection group acting on a finite dimensional complex vector space U, then W acts diagonally on $U \oplus U^*$. Explicitly, W acts on U^* by $(w \cdot x)(y) := x(w^{-1}y)$ for any $x \in U^*$, $y \in U$ and $w \in W$.
Then we have a W-action on $U \oplus U^*$: $w \cdot (x, y) = (w \cdot x, w \cdot y)$. In particular, since W is generated by pseudo-reflections, i.e, elements fixing hyperplanes in the action on U, then the same elements become symplectic reflections in the action on $U \oplus U^*$.
Furthermore, it follows from the diagonal action that the natural pairing is W-invariant. Hence, the symplectic form is also W-invariant and we can conclude that W acts on $V = U \oplus U^*$ as a symplectic reflection group.

1.3 Graded and Filtered Algebras

In this section we recall notions from [4]:

Definition 1.3.1. A $\mathbb{N}$-graded associative algebra is an algebra A such that $A = \oplus_{n\in\mathbb{N}} A_n$ as a vector space and $A_m A_n \subseteq A_{m+n}$ for all n, $m \in \mathbb{N}$.

Remark 1.3.0.1. *The tensor algebra TV is a $\mathbb{N}$-graded associative algebra: $TV = \oplus_{i\geq 0} T^i V$, where $T^i V := V^{\otimes i}$. The multiplication is the tensor product and the grading is the tensor degree: $(TV)_m := T^m V$.*

Definition 1.3.2. An *increasing filtration* on a vector space V is a sequence of subspaces $V_{\leq m} \subseteq V$ such that $V_{\leq m} \subseteq V_{\leq n}$ for all $m \leq n$ and $\cup_m V_{\leq m} = V$.

Definition 1.3.3. An *increasing filtration* on an associative algebra A is an increasing filtration $A_{\leq m}$ such that $A_{\leq m} A_{\leq n} \subseteq A_{\leq (m+n)}$, for all $m, n \in \mathbb{Z}$. An algebra equipped with such a filtration is called *filtered algebra*.

Definition 1.3.4. For a *filtered algebra* $A = \cup_{m\geq 0} A_{\leq m}$, the *associated graded algebra* is $grA := \oplus A_{\leq m}/A_{\leq (m-1)}$ where the product for any $a \in A_{\leq m}$ and $b \in A_{\leq n}$ is defined as follows:

$$(a + A_{\leq m-1})(b + A_{\leq n-1}) := ab + A_{\leq m+n-1},$$

and we set: $gr_m A = (grA)_m = A_{\leq m}/A_{\leq (m-1)}$.

Definition 1.3.5. A *filtered deformation* of a graded algebra B, is a filtered algebra A such that $grA \simeq B$ as graded algebras.

Example 1.3.6. Let $\mathbf{g}$ be a Lie algebra and let $U(\mathbf{g})$ its *universal enveloping algebra*. Then by the classical *PBW theorem*: $U(\mathbf{g})$ is a *filtered deformation* of $Sym(\mathbf{g})$, that is: $gr(U(\mathbf{g})) \simeq Sym(\mathbf{g})$.

1.3.1 Homogeneous and Nonhomogeneous Quadratic Algebras

Let V be a vector space over a field F and let TV be its tensor algebra over F. We recall some notions from [2]:

Definition 1.3.7. Fix a subspace $R \subseteq T^2V = V \otimes V$ and consider the two-sided ideal $J(R)$ in TV generated by R. A *homogenous quadratic algebra* is the quotient algebra $TV/J(R)$, which is denoted by $Q(V, R)$.

Now, we define their *filtered* analogous:

Definition 1.3.8. Let $T^{\leq i}(V) = \{\oplus T^j V | j \leq i\}$ of TV. Fix a subspace $P \subset T^{\leq 2}(V) = k \oplus V \oplus (V \otimes V)$ and consider the two-sided ideal $J(P)$ in TV generated by P. A *nonhomogeneouos quadratic algebra* is the quotient $TV/J(P)$ and it is denoted by $Q(V, P)$.

Remark 1.3.1.1. *Let $U = Q(V, P)$ be a nonhomogenous quadratic algebra. It inherits a filtration $U_{\leq 0} \subset U_{\leq 1} \subset ... \subset U_{\leq n}$ from TV with $U_{\leq i} = (T^{\leq i}(V) + J(P))/J(P)$ for any $i \in \mathbb{N}$.*
Consider now the natural projection $p : T^{\leq 2}(V) \longrightarrow V \otimes V$ on the homogenous component of degree 2, set $R = p(P)$ and consider the homogenous quadratic algebra $A = Q(V, R)$. Then there is a natural epimorphism (that we denote again by p): $p : A \longrightarrow grU$.
Explicitly, given an element $f \in TV$ denote by $LH(f)$ its leading homogeneous component, and for any $S \subset TV$ define $LH(S) := \{LH(f) \mid f \in S\}$. Recall from [19, 4] that $grU \simeq TV/J(LH(J(P)))$ and observe that $J(R) = J(p(P)) = J(LH(P))$.
Hence, there is a surjective map of graded algebras:

$$p : TV/J(R) \twoheadrightarrow TV/J(LH(J(P))),$$

$$x \longmapsto x + J(LH(J(P))),$$

where $J(R) = J(LH(P))$ is the two-sided ideal in TV generated by the projections of each element in $P \subset T^{\leq 2}(V)$ into its component of degree 2 (which coincides with $LH(P)$) and $J(LH(J(P)))$ is the two-sided ideal in TV generated by the highest homogeneous components of each element in $J(P)$.

Definition 1.3.9. A nonhomogenous quadratic algebra $U = Q(V, P)$ is of *PBW type* if the natural projection $p : A = Q(V, R) \longrightarrow grU$ is an isomorphism.

Example 1.3.10. Let $\mathbf{g}$ be a Lie algebra over a field F. Consider its tensor algebra $T(\mathbf{g})$ and the two-sided ideal $I \subset T^{\leq 2}(\mathbf{g}) = F \oplus \mathbf{g} \oplus (\mathbf{g} \otimes \mathbf{g})$ generated by elements of the form: $x \otimes y - y \otimes x - [x, y]$, for any $x, y \in \mathbf{g}$, where the brackets $[-,-]$ is the Lie bracket on $\mathbf{g}$. Notice that the universal enveloping algebra of $\mathbf{g}$, $U(\mathbf{g}) = T(\mathbf{g})/I$, is a *non-homogeneous* quadratic algebra. Consider now its symmetric algebra: $Sym(\mathbf{g}) = T(\mathbf{g})/ < x \otimes y - y \otimes x >_{x,y \in \mathbf{g}}$. Then by the classical PBW-theorem we have that the map $p : Sym(\mathbf{g}) \longrightarrow gr(U(\mathbf{g}))$ is an isomorphism, hence we can conclude that $U(\mathbf{g})$ is of PBW-type.

1.4 Deformations and Associated Poisson Structure

1.4.1 Classical Deformation Theory

We recall some notions from [10]:

Definition 1.4.1. Let (A, m_A, u_A) be a commutative associative F-algebra with unit e.
A homomorphism $\varepsilon : A \longrightarrow F$ is an *augmentation* of A if $\varepsilon u_A = id_F$.

The subspace $\bar{A} := ker\varepsilon$ is called the *augmentation ideal* of A.

Example 1.4.2. Let $F[[t]]$ be the unital ring of formal power series with coefficients in the field F. Then, $F[[t]]$ is *augmented* with *augmentation* $\varepsilon : F[[t]] \longrightarrow F$ given by: $\varepsilon(\Sigma_{i \in \mathbb{Z}_{\geq 0}} a_i t^i) := a_0$.

Example 1.4.3. Let $F[t]$ be the unital ring of polynomials with coefficients in F. Then $F[t]$ is *augmented* with *augmentation* $\varepsilon : F[t] \longrightarrow F$ given by: $\varepsilon(f) := f(0)$, for any $f \in F[t]$.

Example 1.4.4. Let G be a finite group with unit e and let $F[G]$ be its group algebra. Then, $F[G]$ is *augmented* by $\varepsilon : F[G] \longrightarrow F$ given by:

$$\varepsilon(\Sigma_{g \in G} a_g g) := \Sigma_{g \in G} a_g.$$

Example 1.4.5. An example of an associative and commutative F-algebra that does not admit an *augmentation* is given by any proper extension $\mathbf{F} \supset F$ of F.
Assume that an augmentation $\varepsilon : \mathbf{F} \longrightarrow F$ exists, then $Ker\varepsilon$ is an ideal in a field, i.e, it is trivial. Hence, ε is injective and it implies that $\mathbf{F} = F$, which is a contradiction with the assumption $\mathbf{F} \neq F$.

From now on in this subsection, (A, m_A, u_A) will be an associative, commutative, *augmented* and unital F-algebra with *augmentation* $\varepsilon : A \longrightarrow F$ and by a module we will understand a $left$-module.

Definition 1.4.6. Let W be an A-module. The *reduction* of W is the F-module $\overline{W} := F \otimes_A W$, with the F-action given by:

$$k'(k'' \otimes_A w) := k'k'' \otimes_A w, \text{ for any } k', k'' \in F \text{ and any } w \in W,$$

and tensor structure given by:

$$k' \otimes_A aw = \varepsilon(a)k' \otimes_A w, \text{ for any } a \in A,\ w \in W \text{ and } k' \in F.$$

Definition 1.4.7. Let now R be an associative F-algebra and A an augmented unital F-algebra. An A-*deformation* of R is a an associative A-algebra B together with a F-algebra isomorphism $\alpha : \overline{B} \longrightarrow R$.
In particular:

- If the A-module B is flat, i.e, if the functor $B \otimes_A -$ is left exact, then the deformation is said to be $flat$.
- If $A = F[[t]]$, i.e, the ring of formal power series, then the deformation is said to be $formal$.

Example 1.4.8. Let $A = F[t]$ with *augmentation* ε as in Example 1.4.3 and $R = F[x]$. Then the algebra $B = F[t, x]$ is an A-deformation of R since it is a $F[t]$-algebra and $\overline{B} = F \otimes_{F[t]} F[t, x] \simeq F[x]$.

1.4.2 Poisson Algebras

Definition 1.4.9. A *Poisson algebra* is a F-vector space V together with two operations: the multiplication: $\cdot : V \otimes V \longrightarrow V$ and the Poisson bracket: $\{-, -\} : V \otimes V \longrightarrow V$ such that:

(i) $(V, \cdot)$ is an associative algebra,

(ii) $(V, \{-, -\})$ is a Lie algebra, and

(iii) for any $v \in V$, the map $v \longmapsto \{u, v\}$ defines a derivation for any $u \in V$, i.e, $\{u, v \cdot w\} = \{u, v\} \cdot w + v \cdot \{u, w\}$, for any $w \in V$.

Then, the triple $(V, \cdot, \{-, -\})$ is a *Poisson algebra*.

Definition 1.4.10. Let $(A, \cdot, \{-,-\})$ be a Poisson algebra and assume that $A = \oplus_{i\in\mathbb{N}} A_i$ is $\mathbb{N}$-graded. The Poisson bracket is said to be *graded* if there exists some $l \in \mathbb{Z}$ such that $\{-,-\} : A_i \times A_j \longrightarrow A_{i+j+l}$, for all $i, j \in \mathbb{N}$. Then l is said to be the degree of the bracket.

Definition 1.4.11. Let $(A, \cdot_A, \{-,-\}_A)$, $(B, \cdot_B, \{-,-\}_B)$ be Poisson algebras. A map $f : A \longrightarrow B$ is a *Poisson algebras morphism* if it is an algebra morphism such that:

$$f(\{a_1, a_2\}_A) = \{f(a_1), f(a_2)\}_B,$$

for any $a_1, a_2 \in A$.

Example 1.4.12. Every Lie algebra $\mathbf{g}$ is a Poisson algebra with respect to the null associative product: $a \cdot b = 0$ and $\{-,-\}$ given by the Lie bracket. Every associative algebra A is a Poisson algebra with respect to the null Poisson bracket: $\{a, b\} = 0$, for $a, b \in A$.
Such an algebra is called *null Poisson algebra.*

Example 1.4.13. An associative algebra $(A, \cdot, u_A)$ is a Poisson algebra if we put $\{a, b\} = a \cdot b - b \cdot a$, for $a, b \in A$. Indeed, $(A, \{-,-\})$ is a Lie algebra and (iii) of Definition 1.5.8 follows from:

$$\begin{gathered}\{a, b \cdot c\} = a \cdot (b \cdot c) - (b \cdot c) \cdot a = \\ = (a \cdot b) \cdot c - b \cdot (c \cdot a) - (b \cdot a) \cdot c + b \cdot (a \cdot c) = \\ = \{a, b\} \cdot c + b \cdot \{a, c\}.\end{gathered}$$

for any $a, b, c \in A$.

Example 1.4.14. [11, Example 2.5] Let $\mathbf{g}$ be a Lie algebra with Lie bracket $[-,-]$.
By the classical PBW-theorem we know that: $gr(U(\mathbf{g})) \simeq Sym(\mathbf{g})$.
Hence, we can put on $gr(U(\mathbf{g}))$ a Poisson structure as follows.
Recall that $\mathbf{g}$ injects into $gr(U(\mathbf{g}))$ and that the latter algebra is filtered as:

$$\mathcal{U}_{\leq 0} \subset \mathcal{U}_{\leq 1} \subset \cdots \subset \mathcal{U}_{\leq k} \subset \cdots$$

where $\mathcal{U}_{\leq 0} = F$ and $\mathcal{U}_{\leq k}$ is generated by F and by elements of the form $x_1 \otimes \cdots \otimes x_h$, with $h \leq k$ and $x_i \in \mathbf{g}$.
Thus:

$$gr(U(\mathbf{g})) = \oplus_{k\geq 0} \mathcal{U}_{\leq k+1}/\mathcal{U}_{\leq k},$$

and an element x of degree k is an equivalence class, $[x]$, of tensor product of at most k elements of $\mathbf{g}$. Then we can define a skew-symmetric bilinear map $\{-,-\} : \mathcal{U}_{\leq k} \times \mathcal{U}_{\leq h} \longrightarrow \mathcal{U}_{\leq (k+h-1)}$ as:

$$\{[x],[y]\} := [xy - yx].$$

In this way, $gr(U(\mathbf{g}))$ becomes a Poisson algebra with respect to the bracket $\{-,-\}$ defined above.
Notice that the Leibniz identity follows by a computation, while Jacobi identity follows from the usual one in the Lie algebra $\mathbf{g}$ by an induction on the degree.
This algebra is called *Lie-Poisson* algebra over $\mathbf{g}$.

1.4.3 The Associated Poisson Structure

There is a connection between deformations and Poisson structures. More precisely, under the assumptions that we will state in this subsection, the construction in [1, 15] yields a Poisson structure on a commutative and associative algebra.

Let $\mathcal{A}$ be a flat formal deformation of a commutative $\mathbb{C}$-algebra A, i.e, let $\mathcal{A}$ be a flat $\mathbb{C}[[t]]$-algebra such that $\overline{\mathcal{A}} = \mathbb{C} \otimes_{\mathbb{C}[[t]]} \mathcal{A} = \mathcal{A}/t\mathcal{A} = A$.
There is a canonical Poisson bracket on A defined as follows.
First, given $\tilde{u}, \tilde{v} \in \mathcal{A}$, since A is commutative we have: $[\tilde{u}, \tilde{v}] \in t\mathcal{A}$, where $[\tilde{u}, \tilde{v}] := \tilde{u}\tilde{v} - \tilde{v}\tilde{u}$.
Let $m(\tilde{u}, \tilde{v}) \geq 1$ be the greatest integer (possibly ∞) such that $[\tilde{u}, \tilde{v}] \in t^{m(\tilde{u},\tilde{v})}\mathcal{A}$ and let $\mathbf{m} \geq 1$ be the minimum of the integers $m(\tilde{u}, \tilde{v})$ over all pairs $\tilde{u}, \tilde{v}$ as above. Then:

(i) If $\mathbf{m} = \infty$, we set the Poisson bracket on A to be zero,

(ii) If $\mathbf{m} < \infty$, given $u, v \in A$, choose $\tilde{u}, \tilde{v} \in \mathcal{A}$ so that:
$u = \tilde{u}\ mod(t\mathcal{A})$ and $v = \tilde{v}\ mod(t\mathcal{A})$, and put $\{u, v\} = (t^{-\mathbf{m}}[\tilde{u}, \tilde{v}])\ mod(t\mathcal{A})$.

The assignment: $u, v \longmapsto \{u, v\}$ gives rise to a well-defined Poisson bracket on A, which does not depend on the choices involved [1, 15].

Notice that $\mathbf{m} = \infty$ in the construction above implies that $[\tilde{u}, \tilde{v}] \in t^l\mathcal{A}$, for any $\tilde{u}, \tilde{v} \in \mathcal{A}$, and for any $l \geq 1$.
Since $\cap_{l\geq 1} t^l\mathcal{A} = 0$, we see that the vanishing of the Poisson bracket on A forces the whole deformation to be commutative ([1, Lemma 15.1]).

Chapter 2

Symplectic reflection algebras

2.1 Definition

Throughout this chapter V is a finite dimensional $\mathbb{C}$-vector space, TV is its tensor algebra and $\Gamma \subset GL(V)$ is a finite group.
Recall from Example 1.1.11 that $\mathbb{C}\Gamma$ is a *Hopf-algebra* and notice that TV is a (left) $\mathbb{C}\Gamma$- *module algebra*.
Hence it is possible to define the smash product $TV\#\Gamma$, as in Definition 1.1.15.

Following [1]:

Definition 2.1.1. Let $TV\#\Gamma$ be the *smash product* of the tensor algebra TV with $\mathbb{C}\Gamma$, the group algebra of Γ.
Given a skew-symmetric $\mathbb{C}$-bilinear paring $k : V \times V \longrightarrow \mathbb{C}\Gamma$, put

$$H_k := (TV\#\Gamma)/I < x \otimes y - y \otimes x - k(x,y) >_{x,y \in V},$$

where $I < .,. >$ stands for the two-sided ideal in $TV\#\Gamma$ generated by the indicated set. Thus H_k is an associative algebra.

Remark 2.1.0.1. *Notice that for $k \equiv 0$ all the generators $x \in V$ in H_k commute with each other. Hence the resulting algebra is isomorphic to the smash product algebra: $H_0 = SV\#\Gamma$.*
Moreover it results that in this particular case the algebra has a natural grading obtained by placing $\mathbb{C}\Gamma$ in degree zero and V in degree 1.
However, when $k \neq 0$ the resulting algebra H_k is no longer graded. Indeed the relations in the two-sided ideal I are no longer homogenous. Nevertheless, by assigning $\mathbb{C}\Gamma$ degree zero and V degree 1, H_k inherits an

$\mathbb{N}$*-increasing filtration* $(H_k)_{\leq\bullet}$ *and we can consider its associated graded algebra:* $gr(H_k) = \oplus(H_k)_{\leq i}/(H_k)_{\leq(i-1)}$.

Remark 2.1.0.2. *From the definition of* H_k *it follows that in the associated graded algebra* $gr(H_k)$*:*

$$x \otimes y - y \otimes x = 0 \text{ for any } x,\, y \in V \subset gr(H_k).$$

The reason is that $x \otimes y - y \otimes x = k(x, y) \in \mathbb{C}\Gamma$ *which has degree zero and the relation in* $gr(H_k)$ *becomes:* $x \otimes y - y \otimes x = 0$.
As in [1], the natural embedding $V \hookrightarrow gr(H_k)$ *extends to a well-defined and surjective graded algebra homomorphism* $H_{k=0} = SV\#\Gamma \longrightarrow gr(H_k)$.
We will say that the Poincaré-Birkhoff-Witt property (PBW for short) holds for H_k *if this homomorphism is also an isomorphism.*

2.1.1 The PBW Theorem for H_k

Now we focus on one of the main results in [1], the so-called *PBW Theorem* ([1, Theorem 1.3]).
Its proof is a generalization of a result in [2]. Using the notations in Definition 1.3.7 and Definition 1.3.8, Braveman and Gaitsgory found conditions on the subspace $P \subseteq T^{\leq 2}(V)$ under which a nonhomogenous quadratic algebra $U = Q(V, P)$ is of *PBW type* ([2, Theorem 4.1]). More precisely, they required $A = Q(V, R)$ to be a Koszul algebra and gave the following necessary and sufficient criteria [2, Lemma 3.3] on the subspace P:

$$(i)\ Im(\alpha \otimes id - id \otimes \alpha) \subset R \text{ (defined on } (R \otimes V) \cap (V \otimes R)); \quad (2.1)$$

$$(ii)\ \alpha \circ (\alpha \otimes id - id \otimes \alpha) = -(\beta \otimes id - id \otimes \beta); \quad (2.2)$$

$$(iii)\ \beta \circ (id \otimes \alpha - \alpha \otimes id) \equiv 0; \quad (2.3)$$

where $\alpha : R \longrightarrow V$ and $\beta : R \longrightarrow F$ are the two F-linear maps used in [2, 3.2] to give a description of $P \subseteq T^{\leq 2}(V)$ in term of maps, i.e,
$P = \{x - \alpha(x) - \beta(x) \mid x \in R\}$.

Remark 2.1.1.1. *Note that in [2], Braveman and Gaitsgory only consider algebras over a field, but as explained, e.g, in [13], everything works for quadratic algebras over any ground ring* R'*, provided* R' *is a finite dimensional semisimple* $\mathbb{C}$*-algebra. In our setting* $R' = \mathbb{C}\Gamma$ *and the result in [2] can be applied since* $\mathbb{C}\Gamma$ *is a finite dimensional* $\mathbb{C}$*-algebra and it is semisimple by Maschke's Theorem.*

Remark 2.1.1.2. *In order to apply Etingof and Ginzburg's generalization of [2, Theorem 4.1], we will work over $\mathbb{C}\Gamma$ rather than over a field F and we will need $SV\#\Gamma$ to be a $\mathbb{C}\Gamma$- Koszul algebra. This follows from the fact that SV is Koszul over $\mathbb{C}$ and it may be proved using an analogous of the following resolution computed in [5, Example 2.1]:*

$$\cdots \longrightarrow SV \otimes \Lambda^2 V \longrightarrow SV \otimes V \longrightarrow SV \longrightarrow F \longrightarrow 0,$$

where the differential is a map: $SV \otimes \Lambda^j V \longrightarrow SV \otimes \Lambda^{j-1} V$, $j \geq 1$ given by:

$f(x_1, ..., x_n) v_{i_1} \wedge ... \wedge v_{i_j} \longmapsto \Sigma_k (-1)^{1+k} f(x_1, ..., x_n) x_{i_k} v_{i_1} \wedge ... \wedge \widehat{v_{i_k}} \wedge ... \wedge v_{i_j}$, *where $f(x_1, \cdots, x_n) \in SV$ and $v_{i_1} \wedge ... \wedge v_{i_j} \in \Lambda^j V$.*

Theorem 2.1.2. *[1, Theorem 1.3] Assume that (V, ω, Γ) is an indecomposable triple. Then, the PBW-property holds for H_k if and only if there exists a constant $t \in \mathbb{C}$, and an $Ad\Gamma$-invariant function $c : S \longrightarrow \mathbb{C}$, $s \longmapsto c_s$, where S is the set of symplectic reflections in Γ; such that the pairing k has the form:*

$$k(x, y) = t\omega(x, y)1 + \Sigma c_s \omega_s(x, y)s, \text{ for all } x,\ y \in V.$$

Proof. The $\mathbb{C}$-bilinear form k has necessarily to be Γ-invariant (where Γ acts on itself by conjugation) otherwise the PBW fails already in degree two of the filtration; hence from now on we assume k to be Γ-invariant.
Set $K = \mathbb{C}\Gamma$, $E = V \otimes_{\mathbb{C}} \mathbb{C}\Gamma$ and write: $v \longmapsto v^g$ for the action of Γ on V by symplectomorphisms. The space E has a K-*bimodule* structure, with left and right Γ-actions given by:

$$g : v \otimes a \longmapsto v^g \otimes ga$$
$$g : v \otimes a \longmapsto v \otimes (ag)$$

for all $v \in V$, $a \in \mathbb{C}\Gamma$ and $g \in \Gamma$.
Consider the tensor algebra of the K-bimodule E, $T_K E = \oplus_i T_K^i E$. For any $i \geq 0$ there is a natural isomorphism: $T_K^i E \simeq (T_{\mathbb{C}}^i V) \otimes_{\mathbb{C}} \mathbb{C}\Gamma$. Thus there is a $\mathbb{N}$-graded algebra isomorphism: $T_K E \simeq (TV)\#\Gamma$. Hence we can write: $H_k = T_K E / I < P >$, where $I < P >$ is the two-sided ideal in $T_K E$ generated by the K-submodule $P \subset K \oplus (E \otimes E)$.
Following the proof in [1], write $E \wedge E$ for the K-submodule in $E \otimes_K E = V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} \mathbb{C}\Gamma$ spanned by elements of the form: $x \otimes y \otimes g - y \otimes x \otimes g$, for $x,\ y \in V$ and $g \in \Gamma$, and let k_K be the (unique) K-bimodule map: $E \otimes_K E \longrightarrow K$ that extends $k : V \otimes_{\mathbb{C}} V \longrightarrow K$. Then we have:

$$P = \{-k_K(p) + p \in K \oplus (E \otimes_K E) \mid p \in E \wedge E\}.$$

Using the notation introduced before, H_k is a *nonhomogenous quadratic K-algebra* and analogously $H_0 = SV\#\Gamma = T_K E/I < E \wedge E >$ is a *homogeneous quadratic K-algebra*. Hence to prove that H_k has the PBW property we can use a K-version of the criteria in [2] (we must verify $(2.1)-(2.2)-(2.3)$) and the thesis follows. Since $P \subset K \oplus (E \otimes_K E)$, then any element of P can be written as the sum of a non-zero element in $E \wedge E \subset E \otimes_K E = T^2_K E$ (of degree two) and a term in $K = T^0_K E$. Indeed $p \in E \wedge E$ and $k_K(p) \in K$. Hence, in this situation the two K-linear maps $\alpha : E \wedge E \longrightarrow E$ and $\beta : E \wedge E \longrightarrow \mathbb{C}\Gamma$ turn out to be $\alpha \equiv 0$ and $\beta \equiv k_K$.
Let the tensor products be over K. In our setting (2.1) and (2.3) become vacuous and following [1], condition (2.2) is equivalent to:

$k_K \otimes id_E - id_E \otimes k_K$ vanishes on $((E \wedge E) \otimes_K E) \cap (E \otimes_K (E \wedge E)) \subset T^3_K E$.

It results that $((E \wedge E) \otimes_K E) \cap (E \otimes_K (E \wedge E)) = \Lambda^3 V \otimes_{\mathbb{C}} \mathbb{C}\Gamma$, where $\Lambda^3 V \subset T^3 V$ denotes the space of totally skew-symmetric tensors.
Let $Alt : TV \longrightarrow TV$ be the antisymmetrization on the tensor algebra TV defined as follows. Let $T^r V$ be the space of homogeneous tensors of degree r, which is spanned by decomposable tensors: $v_1 \otimes \cdots \otimes v_r$, $v_i \in V$. Then the antisymmetrization of a decomposable tensor is defined in the following way:

$$Alt(v_1 \otimes \cdots \otimes v_r) = \tfrac{1}{r!}\Sigma_{\sigma \in S_r}(-1)^{sgn(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)},$$

and it can be extended by linearity and homogeneity on the tensor algebra TV. Moreover $Alt(TV) \simeq \Lambda V$, where ΛV is the exterior algebra of V.
Hence, since $((E \wedge E) \otimes_K E) \cap (E \otimes_K (E \wedge E)) = \Lambda^3 V \otimes_{\mathbb{C}} \mathbb{C}\Gamma$ condition (ii) is equivalent to $k \otimes id_V - id_V \otimes k$ on $\Lambda^3 V$. Furthermore, since $k : V \otimes V \longrightarrow K$ in H_k is such that $k(x, y) = x \otimes y - y \otimes x$, by a direct computation it follows that (ii) is equivalent also to the identity:
$Alt([x, y]z - x[y, z]) = 0$, where $[x, y] := x \otimes y - y \otimes x$. Moreover, the latter identity is precisely the standard Jacobi identity:

$$[z, [x, y]] = [[z, x], y] + [x, [z, y]]. \tag{2.4}$$

Thus, we find that (2.4) is equivalent to:

$$[z, k(x, y)] = [k(z, x), y] + [x, k(z, y)] \text{ in } H_k,$$

for all x,y,$z \in V$.
Now, write $k(x, y) = \Sigma_{g \in \Gamma} b(g, x, y)g$, where $b(g, x, y) \in \mathbb{C}$. We claim that $b(g, x, y) = 0$ unless $g = 1$ or $g \in S$. This would imply the statement.
Using the following identity in E by omitting the tensor products:
$[z, g] = zg - z^g g = (z - z^g)g$, we obtain:

$[z,[x,y]] = \Sigma_{g\in\Gamma} b(g,x,y)[z,g] = \Sigma_{g\in\Gamma} b(g,x,y)(z-z^g)g$, and writing similar expressions for the other terms: $[[z,x],y]$ and $[x,[z,y]]$, (2.4) becomes:

$$\Sigma_{g\in\Gamma} b(g,x,y)(z-z^g)g = \Sigma_{g\in\Gamma} b(g,z,x)(y-y^g)g + \Sigma_{g\in\Gamma} b(g,z,y)(x-x^g)g.$$

Hence for any $g \in \Gamma$ we must have:

$$b(g,x,y)(z-z^g) = b(g,z,x)(y-y^g) + b(g,z,y)(x-x^g), \qquad (2.5)$$

for all x,y,$z \in V$.
Fix now $g \neq 1$ and assume $b(g,x,y)$ is not identically zero. Then we choose x and y such that $b(g,x,y) \neq 0$ and from (2.5) we observe that $(y-y^g)$ and $(x-x^g)$ span $Im(1-g) = \{v \in V \mid \exists\ w \in V\ ;\ w - w^g = v\}$, so $dim Im(1-g) \leq 2$.
Moreover by Remark 1.2.1.3, $det g = 1$ for any $g \in \Gamma$, and if $g \neq 1$ then $dim Im(1-g) \geq 2$ (since $g \in \Gamma$ is diagonalizable over $\mathbb{C}$ and $dim Ker(1-g) \leq 2n-2$, with $dim V = 2n$). Hence we get $dim Im(1-g) = 2$, and by Definition 1.2.7 it holds that $g \in S$.
Fix $s \in S$. The assignment $x,y \longmapsto b(s,x,y)$ gives a skew-symmetric bilinear form: $V \times V \longrightarrow \mathbb{C}$. Assuming this form is non zero, we find $x,y \in V$ such that $b(s,x,y) \neq 0$. Thus, for $g = s$ and $z \in Ker(1-s)$, the above equation yelds: $b(s,x,z) = 0 = b(s,y,z)$, hence $Ker(1-s)$ is contained in the radical of the bilinear form: $x,y \longmapsto b(s,x,y)$ and it must also be proportional to ω_s, since any alternating 2-form must be proportional on the two-dimensional vector space $Im(1-s)$.
Thus Γ-equivariance of k implies the existence of an $Ad\Gamma$-invariant function $c: S \longrightarrow \mathbb{C}$ and an $Ad\Gamma$-invariant skew-symmetric bilinear form $f: V \times V \longrightarrow \mathbb{C}$ such that:

$$k(x,y) = f(x,y)1 + \Sigma_{s\in S} c_s \omega_s(x,y)s, \text{ for any } x,y \in V.$$

The form f belongs to $(\Lambda^2 V^*)^\Gamma$ and by Lemma 1.2.11 it must be proportional to ω, and we get the thesis. □

Let C denote the space of $Ad\Gamma$-invariant functions on S.

Remark 2.1.1.3. *From now on we will always assume k to have the form described in Theorem 2.1.2, for some $t \in \mathbb{C}$ and $c \in C$. The corresponding algebra $H_{t,c} := H_k$ will be referred to as symplectic $reflection$ algebra. By Theorem 2.1.2:*

$$gr(H_{t,c}) \simeq SV\#\Gamma \text{ and } H_{0,0} = SV\#\Gamma.$$

In this case, we will say that $H_{t,c}$ is a PBW-deformation of $SV\#\Gamma$.

Remark 2.1.1.4. *Notice that for any* $r \in \mathbb{C}^*$ *there is an algebra isomorphism between* $H_{t,c}$ *and* $H_{rt,rc}$ *induced by the map:* $TV\#\Gamma \longrightarrow TV\#\Gamma$, $v \longmapsto \frac{1}{\sqrt{r}}v$, $g \longmapsto g$. *Assume* k *to be as in the statement of Theorem* 2.1.2 *and consider the following map:*

$$F : TV\#\Gamma \twoheadrightarrow H_{rt,rc} = TV\#\Gamma/I < x \otimes y - y \otimes x - rk(x, y) >_{x,y \in V},$$

$$v \longmapsto \frac{1}{\sqrt{r}}v, \quad g \longmapsto g.$$

Then, $KerF = \{Y \in TV\#\Gamma \mid F(Y) \in I < x \otimes y - y \otimes x - rk(x, y) >_{x,y\in V}\}$ *and it is an ideal in* $TV\#\Gamma$.
Note that $F(x \otimes y - y \otimes x - k(x, y)) = \frac{1}{r}(x \otimes y - y \otimes x - rk(x, y)) = 0$ *in* $TV\#\Gamma/I < x \otimes y - y \otimes x - rk(x, y) >_{x,y\in V}$.
Then $I < x \otimes y - y \otimes x - k(x, y) >_{x,y\in V} \subseteq KerF$, *and by the fundamental homomorphism theorem:*

$$TV\#\Gamma/I < x \otimes y - y \otimes x - k(x, y) >_{x,y\in V} = H_{t,c} \twoheadrightarrow H_{rt,rc}$$

is again a surjective algebra morphism.
Recall from Remark 2.1.0.1 *that* $H_{t,c}$ *and* $H_{rt,rc}$ *are* $\mathbb{N}$*-filtered algebras, i.e,* $\cup_{l\in\mathbb{N}}(H_{t,c})_{\leq l} = H_{t,c}$ *as a* $\mathbb{C}$*-vector space,* $(H_{t,c})_{\leq n}(H_{t,c})_{\leq m} \subseteq (H_{t,c})_{n+m}$ *and analogously for* $H_{rt,rc}$.
Moreover, the algebra morphism F *respects the* $\mathbb{N}$*-filtrations, i.e, for any* $l \in \mathbb{N}$ *its restriction to the* l*-filtered components* $(H_{t,c})_{\leq l}$ *and* $(H_{rt,rc})_{\leq l}$ *induces a surjective* $\mathbb{C}$*-linear map:*

$$F_l : (H_{t,c})_{\leq l} \twoheadrightarrow (H_{rt,rc})_{\leq l}.$$

Notice that for any $l \in \mathbb{N}$, $(H_{t,c})_{\leq l}$ *and* $(H_{rt,rc})_{\leq}$ *are finite dimensional* $\mathbb{C}$*-vector spaces such that* $dim((H_{t,c})_{\leq l}) = dim((H_{rt,rc})_{\leq l})$.
Indeed, let $\{v_1, \cdots, v_n\}$ *be a basis of* V, *by Theorem* 2.1.2 *it is possible to find for both of them a basis made of ordered monomials in* $\{v_i\}_i, g$ *of degree* $\leq l$ *with* $\{v_i\}_i \in V$ *and* $g \in \Gamma$.
Since, F_l *is a surjective map of finite dimensional* $\mathbb{C}$*-vector spaces of the same dimension, it is also injective. Then,* F_l *is a bijection for any* $l \in \mathbb{N}$.
Thus, F *is an isomoprhism of algebras and* $H_{t,c} \simeq H_{rt,rc}$, *for any* $r \in \mathbb{C}^*$.
Hence, we can conclude that the family of algebras $\{H_{t,c}\}_{(t,c)\in((\mathbb{C}\oplus C)\smallsetminus(0,0))}$ *is parametrized by the projective space* $\overline{C} := (\mathbb{C} \oplus C)/\mathbb{C}^*$ *of dimension* $dimC$.

2.2 The Spherical Subalgebra

From now on we will always assume k to be as in the statement of Theorem 2.1.2 and we will write $H_k = H_{t,c}$.

Definition 2.2.1. Let $\mathbf{e} = \frac{1}{|\Gamma|}\Sigma_{g\in\Gamma} g$ denote the trivial idempotent in $\mathbb{C}\Gamma$. The subalgebra $\mathbf{e}H_{t,c}\mathbf{e} \subset H_{t,c}$ is called the *spherical subalgebra* of $H_{t,c}$.

Remark 2.2.0.1. *It is shown in* [1, 2.15] *that there is a canonical algebra isomorphism:* $A^\Gamma \longrightarrow \boldsymbol{e}(A\#\Gamma)\boldsymbol{e}, \quad a \longmapsto a\cdot\boldsymbol{e} = \boldsymbol{e}\cdot a$, *where* A *is an associative algebra and the group* Γ *acts on* A *by algebra automorphisms.*

Remark 2.2.0.2. *The spherical subalgebra, being a subalgebra of* $H_{t,c}$ *inherits a filtration from* $H_{t,c}$.
In particular, let $(H_{t,c})_{\leq\bullet}$ *be the* $\mathbb{N}$*-increasing filtration in Remark* 2.1.0.1.
Then, $(\boldsymbol{e}H_{t,c}\boldsymbol{e})_{\leq\bullet} := (H_{t,c})_{\leq\bullet} \cap \boldsymbol{e}H_{t,c}\boldsymbol{e}$ *is an* $\mathbb{N}$*-increasing filtration of* $\boldsymbol{e}H_{t,c}\boldsymbol{e}$.
Moreover, by Remark 2.1.0.1 *there holds* $deg\boldsymbol{e} = 0$ *since* $\boldsymbol{e} \in \mathbb{C}\Gamma$.
Thus, $(\boldsymbol{e}H_{t,c}\boldsymbol{e})_{\leq\bullet} = \boldsymbol{e}(H_{t,c})_{\leq\bullet}\boldsymbol{e}$.
Hence, combining Remark 2.1.1.3 *and Remark* 2.2.0.1*: for* $t = 0$ *and* $c = 0$ *we have:* $\boldsymbol{e}(H_{0,0})\boldsymbol{e} = \boldsymbol{e}(SV\#\Gamma)\boldsymbol{e} \simeq (SV)^\Gamma$*, and:*

$$gr(\mathbf{e}H_{t,c}\mathbf{e}) \simeq SV^\Gamma, \textit{ for any } (t,c) \in \mathbb{C}\oplus C,$$

where SV^Γ *is precisely the centre of* $SV\#\Gamma$.
Moreover, it is shown in the proof of [1, Theorem 1.8] that for $c = 0$ *and any* $t \neq 0$ *it holds:* $H_{t,0} = A_t\#\Gamma$*, where* $A_t = TV/I < x\otimes y - y\otimes x - t\omega(x,y) >_{x,y\in V}$ *is the Weyl algebra of the symplectic vector space* $(V, t\omega)$.
Hence, by Remark 2.2.0.1 *the spherical subalgebra* $\boldsymbol{e}H_{t,0}\boldsymbol{e}$ *is isomorphic to* A_t^Γ.

Example 2.2.2. Let $(\mathbb{C}^2, \omega)$ be a symplectic vector space where ω is the canonical symplectic form on $\mathbb{C}^2$ defined in Example 1.2.12 and let
$G \leq Sp(2,\mathbb{C}) = SL(2,\mathbb{C})$ be the finite group generated by the symplectic reflection $s = -1 \in Sp(2,\mathbb{C})$, with $s^2 = 1$, so $\omega_s = \omega$.
Let $\{x, y\}$ be a basis of $\mathbb{C}^2$ such that $\omega_s(x,y) = \omega(x,y) = 1$.
Then, the group $G \simeq C_2$ acts on $\mathbb{C}^2$ as follows: $s\cdot x = -x$, $s\cdot y = -y$, i.e, $sx = -xs$ and $sy = -ys$.
Thus, the symplectic reflection algebra associated with the triple $(\mathbb{C}^2, \omega, G)$ is:

$$H_{t,c} = \mathbb{C} < x, y > \rtimes G/ < xy - yx - t - c(s)s >_{x,y\in V}.$$

By *Remark* 2.1.1.3 we can find a basis of $H_{t,c}$ as a $\mathbb{C}$-vector space made of ordered monomials in x, y, s, i.e, $B = \{x^iy^j,\ x^iy^js \mid i,j \geq 0\}$.
Furthermore, given two elements $x^iy^j, x^ay^b \in H_{t,c}$ we would be able to write their product $x^iy^jx^ay^b$ with respect to the basis B. To this aim, notice that the element y^jx^a can be rewritten using the relation $xy - yx = t + c(s)s$ in $H_{t,c}$. Indeed, by induction on the power of y, we get:

$$y^n x = \begin{cases} xy^n - nty^{n-1} - c(s)y^{n-1}s, & \text{if } n \neq 2l, l \in \mathbb{N} \\ xy^n - nty^{n-1}, & \text{if } n = 2l, l \in \mathbb{N}, \end{cases} \tag{2.6}$$

Then, by induction on the power of x in (2.6) it is possible to find an expression for $y^j x^a$, with $j, a \in \mathbb{N}$.
Let $\mathbf{e} = \frac{1}{2}(1+s)$ be the trivial idempotent in $\mathbb{C}G$ and consider the spherical subalgebra $\mathbf{e}H_{t,c}\mathbf{e}$ of $H_{t,c}$.
Since by Remark 2.2.0.2 $gr(\mathbf{e}H_{t,c}\mathbf{e}) \simeq \mathbf{e}gr(H_{t,c})\mathbf{e} \simeq \mathbf{e}(SV\#G)\mathbf{e} \simeq (SV)^G$, we can find a basis of the spherical subalgebra as a $\mathbb{C}$-vector space made of ordered monomials in x, y, s after multiplying them on both sides by $\mathbf{e}$.
In particular, given the basis B of $H_{t,c}$, we can find a basis of $\mathbf{e}H_{t,c}\mathbf{e}$ as follows.

Note that since $s\mathbf{e} = \mathbf{e}$ in $\mathbb{C}G$, then $\mathbf{e}x^iy^js\mathbf{e} = \mathbf{e}x^iy^j\mathbf{e}$, for any $i, j \geq 0$. Thus we consider only the elements of the form: $\mathbf{e}x^iy^j\mathbf{e}$.
Since $sx = -xs$ and $sy = -ys$, it holds: $sx^iy^j = (-1)^{i+j}x^iy^js$, i.e, x^iy^j commutes with s (and then with $\mathbf{e}$) if and only if $i+j$ is even.
Explicity:

$$\mathbf{e}x^iy^j\mathbf{e} = \tfrac{1}{2}(x^iy^j + (-1)^{i+j}x^iy^js)\mathbf{e} =$$

$$= \frac{1}{2}(x^iy^j\mathbf{e} + (-1)^{i+j}x^iy^j\mathbf{e}) = \begin{cases} 0, & \text{if } i+j \neq 2l, l \in \mathbb{N} \\ x^iy^j\mathbf{e}, & \text{if } i+j = 2l, l \in \mathbb{N}. \end{cases}$$

Observe that the commutation relation $[x, y] = t + c(s)s$ in $H_{t,c}$ becomes $xy\mathbf{e} - yx\mathbf{e} = (t + c(s))\mathbf{e}$ in $\mathbf{e}H_{t,c}\mathbf{e}$. This, together with the ones obtained from it by induction on the power of x, y, are the only relations among the ordered monomials in $\mathbf{e}H_{t,c}\mathbf{e}$. Hence, the desired basis as a $\mathbb{C}$-vector space of the spherical subalgebra is:

$$B' = \{x^iy^j\mathbf{e} \mid i, j \geq 0,\ i+j = 2l,\ l \in \mathbb{N}\}.$$

Note that computing the product of two non-zero elements $\mathbf{e}x^iy^j\mathbf{e}$, $\mathbf{e}x^ay^b\mathbf{e}$ of the spherical subalgebra $\mathbf{e}H_{t,c}\mathbf{e}$ is equivalent to computing the product of two elements x^iy^j, x^ay^b of the symplectic reflection algebra $H_{t,c}$. Indeed:

$$\mathbf{e}x^iy^j\mathbf{e}\mathbf{e}x^ay^b\mathbf{e} = \mathbf{e}x^iy^j\mathbf{e}x^ay^b\mathbf{e} = \mathbf{e}x^iy^jx^ay^b\mathbf{e}.$$

By induction on the power of x in (2.6), we can compute the product above with respect to the basis B'.
We choose as generators of $\mathbf{e}H_{t,c}\mathbf{e}$ as a $\mathbb{C}$-algebra: $xy\mathbf{e}$, $x^2\mathbf{e}$ and $y^2\mathbf{e}$.
Indeed any element can be obtained by adding and multilplying their linear combinations.
Then, by induction on the power of x in (2.6), the relations among the chosen generators are:

$$[xy\mathbf{e}, x^2\mathbf{e}] = xyx^2\mathbf{e} - x^2xy\mathbf{e} = x(x^2y - 2tx)\mathbf{e} - x^3y\mathbf{e} =$$
$$= x^3y\mathbf{e} - 2tx^2\mathbf{e} - x^3y\mathbf{e} = -2tx^2\mathbf{e},$$

$$[xy\mathbf{e}, y^2\mathbf{e}] = xyy^2\mathbf{e} - y^2xy\mathbf{e} = xy^3\mathbf{e} - y^2xy\mathbf{e} = xy^3\mathbf{e} - (xy^2 - 2ty)y\mathbf{e} = 2ty^2\mathbf{e},$$

$$[x^2\mathbf{e}, y^2\mathbf{e}] = x^2y^2\mathbf{e} - y^2x^2\mathbf{e} = x^2y^2\mathbf{e} - (y^2x)x\mathbf{e} =$$
$$= x^2y^2\mathbf{e} - (xy^2 - 2ty)x\mathbf{e} = x^2y^2\mathbf{e} - x(xy^2 - 2ty)\mathbf{e} + 2tyx\mathbf{e} =$$
$$= 2txy\mathbf{e} + 2tyx\mathbf{e} = 2t(xy + yx)\mathbf{e} = 4txy\mathbf{e} - 2t^2\mathbf{e} - 2tc\mathbf{e}.$$

2.2.1 The Spherical Subalgebra for $t = 0$

In this subsection we will state a result on the commutativity of the spherical subalgebra [1, Theorem 1.6].
Its proof is based on the construction in [1, 15] and we can notice that Etingof and Ginzburg used the geometric structure of the spherical subalgebra in order to get informations on its centre.

Remark 2.2.1.1. *Note that the choice of a basis $\{v_1, \cdots, v_n\}$ of V identifies SV with $\mathbb{C}[V]$, i.e, with the polynomial algebra $\mathbb{C}[v_1, \cdots, v_n]$ and $(SV)^\Gamma = \mathbb{C}[V]^\Gamma$. Then by [6, Lemma 1.17] it is possible to endow $(SV)^\Gamma$ with a Poisson bracket $\{-,-\}_\omega$ obtained by restricting the Poisson bracket on $\mathbb{C}[V]$ induced by the symplectic structure on V:*

$$\{f, g\}_\omega := \omega(f, g), \textit{ for any } f, g \in V,$$

and by extending it as a derivation.
Moreover, from the standard grading in $SV^\Gamma = \oplus_{i\geq 0}(S^iV)^\Gamma$, it follows:

$$\{-,-\}_\omega : (S^iV)^\Gamma \times (S^jV)^\Gamma \longrightarrow (S^{i+j-2}V)^\Gamma.$$

Thus the Poisson bracket $\{-,-\}_\omega$ on SV^Γ is of degree -2.

Theorem 2.2.3. *[1, Theorem 1.6] For any $c \in C$, the algebra $\mathbf{e}H_{t,c}\mathbf{e}$ is commutative if and only if $t = 0$.*

Proof. Consider the family of algebras $\{H_{rt,rc}\}_{r\in\mathbb{C}\smallsetminus\{0\}}$.
By Remark 2.1.1.4, these algerbas are isomorphic to each other for any $r \in \mathbb{C} \smallsetminus \{0\}$. Then, we treat the family $\{H_{rt,rc}\}_{r\in\mathbb{C}\smallsetminus\{0\}}$ as a single algebra.
Explicity, consider $\hbar$ as an auxiliary variable and let $TV[\hbar] = TV \otimes \mathbb{C}[\hbar]$.
We regard $TV[\hbar]$ as a graded algebra with $deg\hbar = 2$ and we assume that the group Γ acts trivially on $\hbar$. Then:

$$\tilde{H} := (TV[\hbar]\#\Gamma)/I < x \otimes y - y \otimes x - k(x, y)\hbar \in T^2V \oplus \mathbb{C}\Gamma[\hbar] >_{x,y\in V},$$

is a flat $\mathbb{C}[\hbar]$-algebra.
Let $\mathcal{A} := \mathbf{e}\tilde{H}\mathbf{e}$ be its spherical subalgebra. It is a flat $\mathbb{C}[\hbar]$-algebra and it holds $\overline{\mathcal{A}} = \mathbb{C} \otimes_{\mathbb{C}[\hbar]} \mathcal{A} = \mathcal{A}/\hbar\mathcal{A}$. Notice that $\mathcal{A}/\hbar\mathcal{A} \simeq gr(\mathbf{e}H_{t,c}\mathbf{e})$, since by Remark 2.2.0.2 $\mathcal{A}/\hbar\mathcal{A} = \mathbf{e}H_{0,0}\mathbf{e} \simeq gr(\mathbf{e}H_{t,c}\mathbf{e})$ for any $(t,c) \in \mathbb{C} \oplus C$, i.e,
$\mathcal{A}$ is a flat $\mathbb{C}[\hbar]$-deformation of $gr(\mathbf{e}H_{t,c}\mathbf{e})$.
Then we can see any member of the family $\{\mathbf{e}H_{rt,rc}\mathbf{e}\}_{r\in\mathbb{C}\setminus\{0\}}$ is the specialization at $\hbar = r$ of the $\mathbb{C}[\hbar]$-algebra $\mathcal{A}$.
Since $\mathcal{A}/(\hbar-1)\mathcal{A} = \mathbf{e}H_{t,c}\mathbf{e}$, we can view the $\mathbb{C}$-algebra $\mathbf{e}H_{t,c}\mathbf{e}$ as a flat $\mathbb{C}[\hbar]$-deformation of the commutative algebra $gr(\mathbf{e}H_{t,c}\mathbf{e}) \simeq (SV)^\Gamma$.
Thus, for any $(t,c) \in \mathbb{C} \oplus C$, by the general construction in [1, 15], this deformation gives rise to a well-defined Poisson bracket $B_{t,c}$ on $(SV)^\Gamma$.
Let $\mathbf{m}_{t,c}$ be the integer involved in the construction of the Poisson bracket. Then:

(i) There are only two alternatives: either $\mathbf{m}_{t,c} = 1$, or $\mathbf{m}_{t,c} = \infty$. The algebra $\mathbf{e}H_{t,c}\mathbf{e}$ is non-commutative if $\mathbf{m}_{t,c} = 1$ and commutative if $\mathbf{m}_{t,c} = \infty$.
Indeed since $\mathcal{A}/\hbar\mathcal{A} = gr(\mathbf{e}H_{t,c}\mathbf{e})$, with $deg\hbar = 2$, the Poisson bracket $B_{t,c}$ on $gr(\mathbf{e}H_{t,c}\mathbf{e}) = (SV)^\Gamma$ has degree $(-2\mathbf{m}_{t,c})$.
By [1, Lemma 2.23 (ii)], in order for $B_{t,c}$ to be non-zero, we must have $\mathbf{m}_{t,c} = 1$. However, by [1, Lemma 15.1], vanishing of $B_{t,c}$ implies commutativity of the algebra $\mathcal{A}$ and (i) follows.

(ii) If $\mathbf{m}_{t,c} = 1$, then $B_{t,c} = f(t,c)\{-,-\}_\omega$, where $f : \mathbb{C} \oplus C \longrightarrow \mathbb{C}$ is a non-zero linear function.
Indeed we regard t, c as variables with $degt = degc = 2$, while we set $deg\hbar = 0$. Thus, $\mathcal{A}$ becomes a $\mathbb{C}[\mathbb{C} \oplus C]$-algebra, depending on the parameter $\hbar$.
Applying the Poisson bracket construction of [1, 15] we get, for
$\mathbf{m}_{t,c} = \mathbf{m} = 1$ (since t, c now are variables and not parameters), a bracket B on $(SV)^\Gamma \otimes \mathbb{C}[\mathbb{C} \oplus C]$ of degree (-2).
By [1, Lemma 2.23 (i)], we have: $B = f(t,c)\{-,-\}_\omega$, for some $f(t,c) \in \mathbb{C}$.
Since the relations in $H_{t,c}$ become homogeneous in our new grading, i.e, $degt = degc = 2$ and $deg\hbar = 0$, it follows that $\{H_{rt,rc}\}_{r\in\mathbb{C}^*}$ and $\mathcal{A}$ are graded algebras. Thus: $(t,c) \longmapsto f(t,c)$ is a linear function on $\mathbb{C} \oplus C$.
Moreover, for any $(t,c) \in \mathbb{C}\oplus C$ such that $B_{t,c} \neq 0$, the Poisson bracket $B_{t,c}$ is clearly a specialization of B at the point (t,c), that is: $B_{t,c} = f(t,c)\{-,-\}_\omega$ and (ii) is proved.

By (i), the algebra $\mathbf{e}H_{t,c}\mathbf{e}$ is commutative if and only if $f(t,c) = 0$.
Hence, the paramters (t,c) such that $\mathbf{e}H_{t,c}\mathbf{e}$ is commutative form an hyperplane in $\mathbb{C} \oplus C$ given by the equation $f(t,c) = 0$.

In order to complete the proof it suffices to show that $f(t,c) = \lambda t$ for some $\lambda \in \mathbb{C}^*$, i.e, that the above hyperplane is the one given by the equation $t = 0$. To prove this, assume that for some (t,c) there holds $f(t,c) = 0$, and hence the algebra $\mathbf{e}H_{t,c}\mathbf{e}$ is commutative. In this case, choose a generic character $\chi : \mathbf{e}H_{t,c}\mathbf{e} \longrightarrow \mathbb{C}$, let $T_\chi = H_{t,c} \otimes_{\mathbf{e}H_{t,c}\mathbf{e}} \chi$ be the induced $H_{t,c}$-module and denote by ρ the corresponding structure morphism. By [1, Lemma 2.24], the $H_{t,c}$-module T_χ is isomorphic, as a Γ-module, to the regular representation of Γ.
We know that in the regular Γ-representation, for any $g \in \Gamma$ such that $g \neq 1$, it holds $tr(\rho(g)) = 0$.
Now take the traces in T_χ on both sides of the relation: $x \otimes y - y \otimes x = k(x,y)$, for any $x, y, \in V$. Then:

$$tr(\rho(x \otimes y - y \otimes x)) = tr(t\omega(x,y)\rho(1_\Gamma) + \Sigma_{s \in S} c(s)\omega_s(x,y)\rho(s)).$$

Since $tr(\rho(s)) = 0$ for any $s \in S$, the equation above is equal to:

$$tr(\rho(x \otimes y)) - tr(\rho(y \otimes x)) = tr(t\omega(x,y)\rho(1_\Gamma)).$$

Notice that $tr(\rho(x \otimes y)) = tr(\rho(y \otimes x))$ for any $x, y \in V$ by a property of the trace of two endomorphisms. Then, using the non-degeneracy of ω we get: $0 = |\Gamma|t$. Hence $f(t,c) = \lambda t$, $\lambda \in \mathbb{C}^*$, i.e, the hyperplanes $t = 0$ and $f(t,c) = 0$ coincide and the statement is proved. □

Remark 2.2.1.2. *The family $\{\mathbf{e}H_{t,c}\mathbf{e}\}_{t \in \mathbb{C}}$ is a flat deformation of $\mathbf{e}H_{0,c}\mathbf{e}$ to be interpreted as in the proof of Theorem* 2.2.3*. Hence, by the construction in [1, 15], this deformation induces a Poisson bracket on $\mathbf{e}H_{0,c}\mathbf{e}$ to be denoted $\{-,-\}$.*

2.3 The Centre of the Symplectic Reflection Algebra

In this section we will state some results which allow us to describe the centre of the symplectic reflection algebra $H_{t,c}$.
In particular, in order to get our aim, the commutativity of the spherical subalgebra will play an important role.

Theorem 2.2.3 together with [1, Theorem 1.5 (iv)], [1, Theorem 3.1] and a result by Brown and Gordon in [9, 7.2], imply that the symplectic reflection algebra is a finite module over its centre if and only if $t = 0$.

Now we state the so-called *Satake isomorphism*:

Theorem 2.3.1. *[1, Theorem 3.1] For any $c \in C$ the map $Z(H_{0,c}) \longrightarrow \boldsymbol{e}H_{0,c}\boldsymbol{e}$ such that $z \longmapsto z\boldsymbol{e}$ is a Poisson algebra isomorphism.*

Remark 2.3.0.1. *For any $z \in Z(H_{0,c})$, $\boldsymbol{e}z\boldsymbol{e} \in Z(\boldsymbol{e}H_{0,c}\boldsymbol{e}) = \boldsymbol{e}H_{0,c}\boldsymbol{e}$. Moreover, z commutes with $\boldsymbol{e} \in \mathbb{C}\Gamma$ and so $\boldsymbol{e}z\boldsymbol{e} = z\boldsymbol{e}^2 = z\boldsymbol{e} \in \boldsymbol{e}H_{0,c}\boldsymbol{e}$. Note that we stated Theorem* 2.3.1 *just for $t = 0$, but there is an algebra isomorphism for all $t \in \mathbb{C}$: $Z(H_{t,c}) \xrightarrow{\sim} Z(\boldsymbol{e}H_{t,c}\boldsymbol{e})$, [6, Theorem 2.5].*

Example 2.3.2. Let $H_{t,c}$ be the symplectic reflection algebra in Example 2.2.2 and $\mathbf{e}H_{t,c}\mathbf{e}$ be its spherical subalgebra.
By Remark 2.1.1.3:

$$gr(H_{t,c}) \simeq \mathbb{C}[x,y] \ltimes G = H_{0,0},$$

and by Remark 2.2.0.2:

$$gr(\mathbf{e}H_{t,c}\mathbf{e}) \simeq \mathbb{C}[x,y]^G = \mathbf{e}H_{0,0}\mathbf{e}.$$

Fix $t = 0$. By the defining relations of the spherical subalgebra in Example 2.2.2 we get that $Z(\mathbf{e}H_{0,c}\mathbf{e}) = \mathbf{e}H_{0,c}\mathbf{e}$.
Choosing $xy\mathbf{e}$, $x^2\mathbf{e}$ and $y^2\mathbf{e}$ as generators of $\mathbf{e}H_{0,c}\mathbf{e}$ as a $\mathbb{C}$-algebra, we get: $Z(\mathbf{e}H_{0,c}\mathbf{e}) = \mathbb{C}[x^2\mathbf{e}, y^2\mathbf{e}, xy\mathbf{e}]$.
Thus, by the *Satake isomorphism* we get: $Z(H_{0,c}) \simeq Z(\mathbf{e}H_{0,c}\mathbf{e})$ and then: $Z(H_{0,c}) = \mathbb{C}[x^2, y^2, xy]$.

Hence, the *Satake isomoprhism* allows us to relate the centre of the symplectic reflection algebra to the spherical subalgebra. Then making use also of [1, Theorem 1.5 (iv)] we can state the following characterization of the centre of the symplectic reflection algebra:

Theorem 2.3.3. *[6, Theorem 2.6] The centre of the symplectic reflection algebra $H_{t,c}$ is described as follows:*

1. *If $t = 0$ then the Satake isomorphism identifies $Z_c := Z(H_{t,c}) \xrightarrow{\sim} \boldsymbol{e}H_{0,c}\boldsymbol{e}$ and $H_{0,c}$ is a finite module over Z_c.*
2. *If $t \neq 0$ then $Z(H_{t,c}) = \mathbb{C}$.*

Proof. Statement 1. follows from [1, Theorem 3.1] and [1, Theorem 1.5 (iv)], while it is possibile to find a proof of 2. in [9, Proposition 7.2]. □

Chapter 3

Rational Cherednik Algebras

3.1 A Particular Family of Symplectic Reflection Algebras

Let (V, ω) be a finite dimensional symplectic vector space over $\mathbb{C}$,
let $\Gamma \subset Sp(V)$ be a finite group and let $< S > \subset \Gamma$ be the subgroup generated by the set S of symplectic reflections in Γ.
Notice that it is a normal subgroup in Γ, indeed for any $s \in S$, $rk(1-s) = 2$ by Definition 1.2.6, and:

$$rk(1 - gsg^{-1}) = rk(g(1-s)g^{-1}) = rk(1-s) = 2, \text{ for any } g \in \Gamma.$$

Thus, $gsg^{-1} \in S$.
Moreover, as noted in [15, 4.2], the defining relations of the symplectic reflection algebra $H_{t,c}(\Gamma)$ associated to the indecomposable triple (V, ω, Γ), show that:

$$H_{t,c}(\Gamma) \simeq H_{t,c}(< S >) \rtimes (\Gamma / < S >).$$

Therefore, the symplectic reflection algebras associated to an indecoposable triple (V, ω, Γ), where Γ is a symplectic reflection group are particularly important.
As we showed in Example 1.2.10, a standard way to construct symplectic reflection groups is creating them out of complex reflection groups and we will adopt this strategy in the sequel.

Remark 3.1.0.1. *Let (V, ω, Γ) be an indecomposable triple. By Lemma 1.2.9 there are two cases:*

- *V is a simple Γ-module such that $(\Lambda^2 V^*)^\Gamma$ is one-dimensional.*

- *$V = \boldsymbol{h} \oplus \boldsymbol{h}^*$ such that $(\Lambda^2 V^*)^\Gamma$ is one-dimensional, $\boldsymbol{h}$ is an irreducible Γ-module and it is a Lagrangian subspace.*

In the latter case, any symplectic reflection $g \in \Gamma$ acts on $\boldsymbol{h}$ as a pseudo-reflection.
Thus, saying that Γ is generated by symplectic reflections amounts to saying that Γ is a finite complex reflection group in a vector space $\boldsymbol{h}$ and that it acts on $V = \boldsymbol{h} \oplus \boldsymbol{h}^*$ by induced symplectic automoprhisms.
Indeed any pseudo-reflection $s \in \Gamma$ on $\boldsymbol{h}$ induces the corresponding symplectic reflection of $V = \boldsymbol{h} \oplus \boldsymbol{h}^*$.

Thus, by *Remark* 3.1.0.1 we will be interested in studying the symplectic reflection algebras associated to indecomposable triples of the form $(\mathbf{h} \oplus \mathbf{h}^*, \omega, W)$, where W is a complex reflection group acting diagonally on $V = \mathbf{h} \oplus \mathbf{h}^*$ and ω is as in Example 1.2.12.

3.1.1 Root Systems and Weyl Groups

Before giving the definition of rational Cherednik algebras, we need some further definitions:

Definition 3.1.1. Let $\mathbb{E}$ be a Euclidean vector space with a positive-definite inner product $(-,-)$. A *reduced root system* is a finite collection R of nonzero vectors in $\mathbb{E}$ such that:
(i) R spans $\mathbb{E}$ as a vector space;

(ii) $\mathbb{R}\alpha \cap R = \{\alpha, -\alpha\}$, for any $\alpha \in R$;

(iii) $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in R$, for any $\alpha, \beta \in R$;

(iv) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, for any $\alpha, \beta \in R$.

If we omit condition (ii) then R is said to be a root system.

Definition 3.1.2. Let R be a *root system* in a Euclidean vector space $(\mathbb{E}, (-,-))$. For any $\alpha \in R$ the *coroot* $\alpha^\vee$ is defined as:

$$\alpha^\vee := 2\frac{\alpha}{(\alpha,\alpha)}.$$

The set of coroots also forms a root system in $\mathbb{E}$ called the *dual root system.*

Definition 3.1.3. Let R be a root system in a Euclidean vector space $(\mathbb{E}, (-,-))$. The *Weyl group* corresponding to the root system R is the subgroup $W \subseteq GL(\mathbb{E})$ such that $W :=< s_\alpha \mid \alpha \in R > \subseteq GL(\mathbb{E})$, where s_α for any $\alpha \in R$ is given by:

$$s_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha, \text{ for any } v \in \mathbb{E}.$$

Remark 3.1.1.1. *Note that* $W \subseteq O(\mathbb{E})$. *Indeed:*

$$\begin{aligned}(s_\alpha(v), s_\alpha(w)) = (v - 2\tfrac{(\alpha,v)}{(\alpha,\alpha)}\alpha, w - 2\tfrac{(\alpha,w)}{(\alpha,\alpha)}\alpha) = \\ = (v,w) - 2\tfrac{(\alpha,w)}{(\alpha,\alpha)}(v,\alpha) - 2\tfrac{(\alpha,v)(\alpha,w)}{(\alpha,\alpha)} + 4\tfrac{(\alpha,v)(\alpha,w)}{(\alpha,\alpha)} = \\ = (v,w) - 4\tfrac{(\alpha,w)(\alpha,v)}{(\alpha,\alpha)} + 4\tfrac{(\alpha,v)(\alpha,w)}{(\alpha,\alpha)} = (v,w),\end{aligned}$$

for any $\alpha \in R$ *and* $v, w \in \mathbb{E}$.

Example 3.1.4. Let $(\mathbb{E}, (-,-))$ be the Euclidean vector space $\mathbb{E} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \Sigma x_i = 0\}\}$, with the standard scalar product in $\mathbb{R}^n$. Let $\mathbf{A_{n\text{-}1}} := \{\varepsilon_i - \varepsilon_j \mid i \neq j,\ i,j \in \{1, \cdots, n\}\}$, where $(\varepsilon_i)_{i \in \{1,\cdots,n\}}$ is the canonical orthonormal basis of $\mathbb{R}^n$.
Notice that $\mathbf{A_{n\text{-}1}}$ is a finite reduced root system in $(\mathbb{E}, (-,-))$.
Let $W =< s_{\varepsilon_i - \varepsilon_j} \mid i \neq j,\ i,j \in \{1, \cdots, n\} >$ be the Weyl group associated to $\mathbf{A_{n\text{-}1}}$. Any reflection $s_{\varepsilon_i - \varepsilon_j} \in W$ permutes the coordinates i and j of any element $(x_1, \cdots, x_n) \in \mathbb{E}$.
Thus, the reflection $s_{\varepsilon_i - \varepsilon_j}$ corresponds to the transposition $(i,j) \in S_n$.
Since $\{(i,j)\}_{i \neq j, i,j \in \{1,\cdots,n\}}$ generate S_n, we get that W acts on $\mathbb{E}$ as the symmetric group S_n.

Remark 3.1.1.2. *Let* W *be a complex reflection group acting on a finite dimensional complex vector space* $\boldsymbol{h}$. *We say that* $\boldsymbol{h}$ *is a representation of real type if there exists a real vector subspace* $\boldsymbol{h}^{re} \subset \boldsymbol{h}$ *such that* W *acts on* $\boldsymbol{h}^{re}$ *as a real reflection group and* $\boldsymbol{h} = \boldsymbol{h}^{re} \otimes \mathbb{C}$.
Moreover, assume $\boldsymbol{h}$ *to be a simple* W*-module. Then there exists a unique (up to scalars)* W*-invariant inner product* $(-,-)_{re}$, *i.e,* $(wu, wv)_{re} = (u,v)_{re}$ *for any* $w \in W$ *and* $u, v \in \boldsymbol{h}^{re}$. *It can be extended by linearity to a non degenerate, positive definite* W*-ivariant* $\mathbb{C}$*-bilinear form* $(-,-)$ *on* $\boldsymbol{h}$.

Remark 3.1.1.3. *Notice that if* W *is the Weyl group associated to a finite reduced root system in* $\mathbb{E}$ *then* $\boldsymbol{h} = \mathbb{E} \otimes_{\mathbb{R}} \mathbb{C}$ *is a representation of real type.*

3.1.2 Different Defining Relations and Triangular Decomposition

As in [1, 4], we will focus on the special case of a symplectic reflection algebra associated to an indecomposable triple (V, ω, W), where $V = \mathbf{h} \oplus \mathbf{h}^*$, ω is as in Example 1.2.12 and W is the *Weyl group* associated to a reduced root system $R \subset \mathbf{h}^*$.

By Remark 3.1.1.2 and Remark 3.1.1.3, $\mathbf{h}$ has a non degenerete W-invariant inner product $(-,-)$, and W is a finite group of orthogonal transformations of $\mathbf{h}$.
Let R be a finite reduced root system in $\mathbf{h}^*$ and let S' be the set of reflections in the associated *Weyl group* W.
For $\alpha \in R \subset \mathbf{h}^*$, we let s_α denote the corresponding reflection in W, and $\alpha^\vee := 2\frac{\alpha}{(\alpha,\alpha)} \in \mathbf{h}^*$ the coroot corresponding to $\alpha \in R$.
Notice that $\alpha^\vee$ can be regarded as a $\mathbb{C}$-linear function $R \longrightarrow \mathbb{C}$, i.e,
$\alpha^\vee = 2\frac{(\alpha,-)}{(\alpha,\alpha)} \in (\mathbf{h}^*)^* = \mathbf{h}$.
From now on we fix a W-invariant linear function $c : R \longrightarrow \mathbb{C}$, $\alpha \longmapsto c_\alpha$ and $t \in \mathbb{C}$.

Let now $H_{t,c}$ be the symplectic reflection algebra associated to the indecomposable triple of the form $(\mathbf{h} \oplus \mathbf{h}^*, \omega, W)$, a complex number $t \in \mathbb{C}$ and a W-invariant function $c : S' \longrightarrow \mathbb{C}$ as above.
According to Definition 2.1.1 the algebra $H_{t,c}$ is generated by $\mathbf{h}$, $\mathbf{h}^*$ and by the group W, subject to the following defining relations:

$$wxw^{-1} = w \cdot x, \text{ for any } x \in \mathbf{h} \text{ and any } w \in W,$$

$$wyw^{-1} = w \cdot y, \text{ for any } y \in \mathbf{h}^* \text{ and any } w \in W,$$

$$[x_1, x_2] = 0 \text{ in } H_{t,c}, \text{ for any } x_1, x_2 \in \mathbf{h},$$

(3.1)

$$[y_1, y_2] = 0 \text{ in } H_{t,c}, \text{ for any } y_1, y_2 \in \mathbf{h}^*,$$

$$[x, y] = t(x, y) - \tfrac{1}{2}\Sigma_{\alpha \in R} c_\alpha (x, \alpha)(\alpha^\vee, y) s_\alpha \text{ in } H_{t,c},$$

$$\text{for any } x \in \mathbf{h} \text{ and } y \in \mathbf{h}^*.$$

Indeed by Theorem 2.1.2: $k(x, y) = t\omega(x, y) + \Sigma_{s \in S} c(s)\omega_s(x, y)s$, for any $x, y \in V$. Thus we can rewrite k in the following way:

$$k(x, y) = t(x, y) - \tfrac{1}{2}\Sigma_{\alpha \in R} c_\alpha (\alpha^\vee, y)(x, \alpha) s_\alpha,$$

where $(x, y) := y(x)$ for any $x \in \mathbf{h}$, $y \in \mathbf{h}^*$ and where ω_α is written in coordinates with respect to the basis $\{\alpha, \alpha^\vee\}$ of the two-dimensional vector space $Im(1 - s_\alpha)_{|\mathbf{h} \oplus \mathbf{h}^*}$ (as in Example 1.2.12). In particular α is a basis of the one dimensional vector space $Im(1 - s_\alpha)_{|\mathbf{h}^*}$ and $\alpha^\vee$ is a basis of the one dimensional vector space $Im(1 - s_\alpha)_{|\mathbf{h}}$.
Moreover, since α and $-\alpha$ give rise to the same reflection, we put on the

right hand side $-\frac{1}{2}$ and the other defining relations follow from the definition of the symplectic form on $\mathbf{h} \oplus \mathbf{h}^*$.
Indeed we have that $[x_1, x_2] = 0$ for any $x_1, x_2 \in \mathbf{h}$ since:
$k(x_1, x_2) = \omega_s((x_1, 0), (x_2, 0))_{|Im(1-s)_{|\mathbf{h}\oplus\mathbf{h}^*}} = \omega((x_1, 0), (x_2, 0)) = 0$ (and similarly it follows that $[y_1, y_2] = 0$, for any $y_1, y_2 \in \mathbf{h}^*$).

Now, we can give the definition of a rational Cherednik algebra as in [1]:

Definition 3.1.5. A rational Cherednik algebra with parameters (t, c) is the symplectic reflection algebra $H_{t,c}$ corresponding to the W-diagonal action on $\mathbf{h} \oplus \mathbf{h}^*$. Hence it is an associative $\mathbb{C}$-algebra $H_{t,c}$ generated by the spaces $\mathbf{h}$, $\mathbf{h}^*$ and by the group W with defining relations (3.1).

Remark 3.1.2.1. *Note that if the W-action is irreducible, the set S' of reflections in W forms either one or two conjugacy classes depending on whether all the roots have the same length or not [1].*
Thus, giving a W-invariant function $c : S' \longrightarrow R$ amounts to giving a map:

$$\alpha \longmapsto c_\alpha \in \mathbb{C},$$

where c_α depends only on the length of $\alpha \in R$.

Remark 3.1.2.2. *Recall from Remark* 2.1.1.4 *that $H_{t,c} \simeq H_{rt,rc}$, for any $r \in \mathbb{C} \setminus \{0\}$.*
In the sequel we will focus in particular on the case $t = 0$, therefore we are free to rescale c by r whenever it is convenient.
Moreover, unlike a general symplectic reflection algebra, one can see from the relations above that setting $deg(\boldsymbol{h}^*) = -1$, $deg(\boldsymbol{h}) = 1$ and $deg(W) = 0$, makes the Rational Cherednik algebra $H_{t,c}$ into a $\mathbb{Z}$-graded algebra (as in [6]).

Example 3.1.6. [1, 4 Example] Let $\mathbf{h} = \mathbb{C}^n$, $R = \mathbf{A_{n\text{-}1}}$ and $W = S_n$ as in Example 3.1.4.
We write down the defining relations of the Rational Cherednik algebra associated with the indecomposable triple: $(\mathbf{h} \oplus \mathbf{h}^*, \omega, W)$, where ω is the canonical symplectic form of $\mathbf{h} \oplus \mathbf{h}^*$ defined as in the Example 1.2.12.
We use the standard coordinates on $\mathbb{C}^n$ to write:

$$\mathbf{h} = \{(x_1, \cdots, x_n) \in \mathbb{C}^n \mid \Sigma x_i = 0\} \text{ and}$$

$$\mathbf{h}^* = \{(y_1, \cdots, y_n) \in (\mathbb{C}^n)^* \mid \Sigma y_i = 0\}.$$

Moreover, since $\mathbf{h} \hookrightarrow \mathbb{C}^n$, then there is a surjective map $F : \mathbb{C}[\mathbb{C}^n] \twoheadrightarrow \mathbb{C}[\mathbf{h}]$, where $\mathbb{C}[x_1, \cdots, x_n] \simeq \mathbb{C}[\mathbb{C}^n]$.
Note that $KerF = (x_1 + \cdots + x_n)$, thus $\mathbb{C}[\mathbf{h}] \simeq \mathbb{C}[x_1, \cdots, x_n]/(x_1 + \cdots + x_n)$,

and analogously $\mathbb{C}[\mathbf{h}^*] \simeq \mathbb{C}[x_1^*, \cdots, x_n^*]/(x_1^* + \cdots + x_n^*)$.
Recall that in the $\mathbf{A_{n\text{-}1}}$ case all the roots are W-conjugate and so the linear function $c : R \longrightarrow \mathbb{C}$ is a constant function and we choose $c = 1$. Then we view the parameter $k = (t, 1)$ as a point of $\mathbb{P}^1$. Thus the case $c = 0$ corresponds to the point $k = \infty$.

Write now s_{ij} for the transposition: $i \longleftrightarrow j$. The algebra $H_t = H_{t,1}(S_n)$ is generated by $\mathbf{h}$, $\mathbf{h}^*$ and by the group S_n, with the following defining relations (which are a specialization of the ones in (3.1)):

$$s_{ij}x_i = x_j s_{ij}, \quad s_{ij}y_i = y_j s_{ij}, \quad \text{for any } i, j \in \{1, 2, \cdots, n\},\ i \neq j;$$

$$[y_i, x_j] = s_{ij}, \quad [x_i, x_j] = 0 = [y_i, y_j], \quad \text{for any } i, j \in \{1, 2, \cdots, n\},\ i \neq j;$$

$$[y_k, x_k] = t - \Sigma_{i \neq k} s_{ik}.$$

Let $H_{t,c}$ be the rational Cherednik algebra associated to the triple $(\mathbf{h} \oplus \mathbf{h}^*, \omega, W)$. Then, as a consequence of Remark 2.1.1.3, we get:

Corollary 3.1.7. *[1, Corollary 4.4] For any $k \in \overline{C} := (\mathbb{C} \oplus C)/\mathbb{C}^*$, multiplication in H_k induces a vector spaces isomoprhism:*

$$\mathbb{C}[\boldsymbol{h}] \otimes \mathbb{C}[\boldsymbol{h}^*] \otimes \mathbb{C}W \xrightarrow{\sim} H_k.$$

Remark 3.1.2.3. *[12] Let $\boldsymbol{g}$ be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$. Then:*

$$\boldsymbol{g} = \boldsymbol{n}_- \oplus \boldsymbol{h} \oplus \boldsymbol{n}_+,$$

where $\boldsymbol{h}$ is a Cartan subalgerba and $\boldsymbol{n}_+$ and $\boldsymbol{n}_-$ are the spans of the positive and negative root spaces in $\boldsymbol{g}$.
Moreover, let $U(\boldsymbol{g})$ be the universal enveloping algebra of the Lie algebra we are considering. Then, by the classical PBW-theorem, we get the following triangular decomposition:

$$U(\boldsymbol{g}) = U(\boldsymbol{n}_-) \otimes U(\boldsymbol{h}) \otimes U(\boldsymbol{n}_+).$$

Remark 3.1.2.4. *Note that by Corollary* 3.1.7 *there exists a decomposition of the rational Cherednik algebra $H_{t,c}$ which is analogous to the one of a finite dimensional semi-simple Lie algebra $\boldsymbol{g}$, as in Remark* 3.1.2.3.

3.2 The Centre of the Rational Cherednik Algebra for $t = 0$

In this section we we will describe the centre of the rational Cherednik algebra $H_{t,c}$ for $t = 0$.
Note that since it is a symplectic reflection algebra, we already have a description of its centre (Theorem 2.3.3) but we will state some more properties.

Example 3.2.1. Let $H_{t,c}$ be the symplectic reflection algebra associated to the triple $(\mathbb{C}^2, \omega, G)$ as in Example 2.2.2.
Choose a basis $\{x, y\}$ of $V = \mathbb{C}^2$ such that: $V = \mathbf{h} \oplus \mathbf{h}^* = \mathbb{C}x \oplus \mathbb{C}y$, where $\mathbf{h} = Span\{x\} = \mathbb{C}x$ and $\mathbf{h}^* = Span\{y\} = \mathbb{C}y$ and $y(x) = 1$.
Let ω be the canonical symplectic form on V as in Example 1.2.12 with $\omega(x, y) = 1$ and let $W =< s' >\subset GL(\mathbf{h})$, with $s' = -1 \in \mathbb{C}$ be the *Weyl group* associated to $\mathbf{h}^*$ acting diagonally on V.
Then $H_{t,c}$ in Example 2.2.2 is precisely the rational Cherednik algebra associated to the triple $(\mathbb{C}x \oplus \mathbb{C}y, \omega, W)$, a complex number $t \in \mathbb{C}$, a constant function $c : \mathbf{h}^* \longrightarrow \mathbb{C}$, with defining relations (which are a specialization of the ones in (3.1)):

$$s'x = -xs',\ s'y = -ys',$$

$$[x, y] = t + cs'.$$

Fix $t = 0$. By Theorem 2.3.1, it holds: $Z(H_{0,c}) \simeq Z(\mathbf{e}H_{0,c}\mathbf{e}) = \mathbf{e}H_{0,c}\mathbf{e}$ and by Corollary 3.1.7 and Remark 2.1.1.3 we get:

$$\mathbb{C}[\mathbf{h} \oplus \mathbf{h}^*]^W \xrightarrow{\sim} \mathbf{e}H_{0,c}\mathbf{e}.$$

Then, $Z(H_{0,c}) \simeq \mathbf{e}H_{0,c}\mathbf{e} \simeq \mathbb{C}[\mathbf{h} \oplus \mathbf{h}^*]^W$.
Consider the following invariant subalgebras of $Z(H_{0,c})$: $\mathbb{C}[\mathbf{h}]^W = \mathbb{C}[x^2]$ and $\mathbb{C}[\mathbf{h}^*]^W = \mathbb{C}[y^2]$.
Recall from Example 2.3.2 that $Z(H_{0,c}) = \mathbb{C}[x^2, y^2, xy]$. Then, notice that the two invariant subalgebras above are both contained in the centre of $H_{t,c}$ and thus $\mathbb{C}[x^2] \otimes \mathbb{C}[y^2] \subset Z(H_{0,c}) = \mathbb{C}[x^2, y^2, xy]$.
Furthermre, observe that $Z(H_{0,c}) = \mathbb{C}[\mathbf{h} \oplus \mathbf{h}^*]^W$ is a free module over its subalgebra $\mathbb{C}[\mathbf{h}]^W \otimes \mathbb{C}[\mathbf{h}^*]^W = \mathbb{C}[x^2] \otimes \mathbb{C}[y^2]$ by Chevalley's Theorem. Moreover the rank of the free module $Z(H_{0,c})$ over $\mathbb{C}[\mathbf{h}]^W \otimes \mathbb{C}[\mathbf{h}^*]^W$ is $|W| = 2$. Indeed we can choose $\{1, xy\}$ as a basis.

In the previous example we showed some properties of the centre of the rational Cherednik algebra $H_{0,c}$ we were considering. However, it is not the only situation in which they hold.
In general we have:

Proposition 3.2.2. *[6, Proposition 2.7] Let $H_{0,c}$ be the rational Cherednik algebra associated to the complex reflection group W, then:*

- *The subalgebras $\mathbb{C}[\boldsymbol{h}]^W$ and $\mathbb{C}[\boldsymbol{h}^*]^W$ are contained in $Z_{0,c} := Z(H_{0,c})$.*
- *The algebra $Z_{0,c}$ is a free $\mathbb{C}[\boldsymbol{h}]^W \otimes \mathbb{C}[\boldsymbol{h}^*]^W$-module of rank $|W|$.*

3.2.1 Restricted Rational Cherednik Algebras

Recall from Proposition 3.2.2 that given a rational Cherednik algebra $H_{t,c}$ associated to an indecomposable triple $(\mathbf{h} \oplus \mathbf{h}^*, \omega, W)$, at parameter $t = 0$ there is an inclusion of algebras:

$$A := \mathbb{C}[\mathbf{h}]^W \otimes \mathbb{C}[\mathbf{h}^*]^W \hookrightarrow Z(H_{0,c}).$$

Thus, we are allowed to give the following definition:

Definition 3.2.3. [6] Let $H_{0,c}$ be a rational Cherednik algebra at parameter $t = 0$. Then we define the *restricted rational Cherednik algebra* associated to it as:

$$\overline{H_c} = H_{0,c}/A_+H_{0,c},$$

where, A_+ denotes the ideal in A of elements with zero constant term.

It is shown in [14] that $\overline{H}_c$ is a $\mathbb{Z}$-graded algebra where $deg(x) = 1$, $deg(y) = -1$ and $deg(w) = 0$, for any $x \in \mathbf{h}$, $y \in \mathbf{h}^*$ and $w \in W$.

Definition 3.2.4. Let $\mathbf{h}$ be a finite dimensional $\mathbb{C}$-vector space and let W be any finite group of $GL(\mathbf{h})$. We denote by $\mathbb{C}[\mathbf{h}]_+$ the ideal of all functions with constant term equal to zero.
The ring of co-invariants for W is defined to be the finite dimensional quotient algebra:

$$\mathbb{C}[\mathbf{h}]^{CoW} := \mathbb{C}[\mathbf{h}]/ < \mathbb{C}[\mathbf{h}]_+^W >.$$

In general the ring of coinvariants is not easy to describe.
However if W is a complex reflection group, Chevalley gave a description of $\mathbb{C}[\mathbf{h}]^{CoW}$ as a W-module in [16]:

Proposition 3.2.5. *[16] Let W be a complex reflection group. Then as a W-module the ring of coinvariants $\mathbb{C}[\boldsymbol{h}]^{CoW}$ is isomoprhic to the regular representation. In particular: $dim\mathbb{C}[\boldsymbol{h}]^{CoW} = |W|$.*

Remark 3.2.1.1. *By Corollary* 3.1.7 *and by Definitions* 3.2.3 *and* 3.2.4 *it follows that* $\overline{H_c}$ *has the following triangular decomposition as a vector space:*

$$\overline{H_c} \xrightarrow{\sim} \mathbb{C}[\boldsymbol{h}]^{CoW} \otimes \mathbb{C}[\boldsymbol{h}^*]^{CoW} \otimes \mathbb{C}W.$$

Moreover, by Proposition 3.2.5 *we have:*

$$dim(\overline{H_c}) = |W|^3.$$

Example 3.2.6. Let $H_{t,c}$ be the rational Cherednik algebra in Example 3.2.1. Fix $t = 0$ and consider the corresponding restricted rational Cherednik algebra $\overline{H_c}$.
By Remark 3.2.1.1 it follows that:

$$\overline{H_c} \xrightarrow{\sim} \mathbb{C}[\mathbf{h}]^{CoW} \otimes \mathbb{C}[\mathbf{h}^*]^{CoW} \otimes \mathbb{C}W,$$

with

$$\mathbb{C}[\mathbf{h}]^{CoW} = \mathbb{C}[x]/ < \mathbb{C}[x]_+^W >= \mathbb{C}[x]/(x^2),$$

and

$$\mathbb{C}[\mathbf{h}^*]^{CoW} = \mathbb{C}[y]/ < \mathbb{C}[y]_+^W >= \mathbb{C}[y]/(y^2).$$

Then, by Definitions 3.1.5 and 3.2.3, the restricted rational Cherednik algebra $\overline{H_c}$ is the $\mathbb{C}$-algebra generated by $\mathbf{h} = \mathbb{C}x$, $\mathbf{h}^* = \mathbb{C}y$, the Weyl group $W =< s' >\subset GL(\mathbb{C})$, with $s' = -1 \in \mathbb{C}$ associated to $\mathbf{h}^*$ and a constant function $c : \mathbf{h}^* \longrightarrow \mathbb{C}$, subject to the following defining relations:

$$s'x = -xs',\ s'y = -ys',$$

$$x^2 = 0,\ y^2 = 0,$$

$$[x, y] = cs'.$$

Moreover, by Example 2.2.2 a basis of $\overline{H_c}$ as a $\mathbb{C}$-vector space is given by:

$$\{1, x, y, xy, s', xs', ys', xys'\}.$$

Thus: $dim\overline{H_c} = |W|^3 = 8$.

Chapter 4

On the representation theory of the restricted rational Cherednik algebras

Recall from Remark 3.1.2.4 that a rational Cherednik algebra at $t = 0$ admits a triangular decomposition which is analogous to the one of the universal enveloping algebra $U(\mathbf{g})$ of a semisimple Lie algebra $\mathbf{g}$.
Moreover, recall from Remark 3.2.1.1 that a restricted rational Cherednik algebra inherits a triangular decomposition from the one of the corresponding rational Cherednik algebra.
Throughout this chapter, in the spirit of the representation theory of $U(\mathbf{g})$, we state some results on the representation theory of a restricted rational Cherednik algebra $\overline{H_c}$ by exploiting its triangular decomposition.

4.1 Baby Verma Modules

In this section we introduce *baby Verma modules* and we state the *Brauer-type reciprocity* ([17]).

Let $\overline{H_c}$ be a restricted rational Cherednik algebra as in Definition 3.2.3. Recall from Remark 3.2.1.1 that it admits the following triangular decomposition as a $\mathbb{C}$-vector space:

$$\overline{H_c} \xrightarrow{\sim} \mathbb{C}[\mathbf{h}]^{CoW} \otimes \mathbb{C}[\mathbf{h}^*]^{CoW} \otimes \mathbb{C}W,$$

and consider the subalgebra $\mathbb{C}[\mathbf{h}^*]^{CoW} \# \mathbb{C}W$ of $\overline{H_c}$.

Remark 4.1.0.1. *The subalgebra $\mathbb{C}[\boldsymbol{h}^*]^{CoW} \# \mathbb{C}W$ should be seen as an analogue of the subalgebra $U(\boldsymbol{h} \oplus \boldsymbol{n})$ of the universal enveloping algebra $U(\boldsymbol{g})$ of* *a semisimple Lie algebra* $\mathfrak{g}$*.*

The algebra map:

$$\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W \longrightarrow \mathbb{C}W,\ p\#w \longmapsto p(0)w,$$

for any $p \in \mathbb{C}[\mathbf{h}^*]^{CoW}$ and $w \in W$, makes any W-module into a $\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W$-module.
Moreover, $\overline{H_c}$ is a right $\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W$-module since $\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W$ is a subalgebra of $\overline{H_c}$.
Let $Irr(W)$ denote a set of complete, non-isomorphic, simple W-modules.
Following [6]:

Definition 4.1.1. Let $\lambda \in Irr(W)$. The *baby Verma module* of $\overline{H_c}$ associated to λ is:

$$\Delta(\lambda) := \overline{H_c} \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} \lambda,$$

where $\mathbb{C}[\mathbf{h}^*]^{CoW}_{+}$ acts on λ as zero.
Moreover the $\overline{H_c}$-action is given by:

$$h' \cdot (h \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} v) = (h'h \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} v),$$

for any $h, h' \in \overline{H_c}$ and $v \in \lambda$.

Remark 4.1.0.2. *Let A be a $\mathbb{C}$-algebra and let M be a left A-module. Then M^* becomes a right A-module, where the action of A on M^* is defined to be:*

$$(f \cdot a)(m) := f(a \cdot m), \text{ for any } f \in M^*,\ m \in M \text{ and } a \in A.$$

Analogously, let M be a right A-module. Then M^* is a left A-module, where the A-action on M^* is defined to be:

$$(a \cdot f)(m) := f(m \cdot a), \text{ for any } f \in M^*,\ m \in M \text{ and } a \in A.$$

Now consider the subalgebra $\mathbb{C}[\mathbf{h}]\#\mathbb{C}W$ of $\overline{H_c}$. The algebra map:

$$\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W \longrightarrow \mathbb{C}W,\ q\#w \longmapsto q(0)w,$$

for any $q \in \mathbb{C}[\mathbf{h}]^{CoW}$ and $w \in W$, makes any W-module into a $\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W$-module. Moreover, by Remark 4.1.0.2 given a left W-module (which is also a left $\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W$-module), λ^* is a right W-module (and a right $\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W$-module) with W-action given by:

$$(f \cdot w)(v) := f(w \cdot v),$$

for any $f \in \lambda^*$, $v \in \lambda$ and $w \in W$.
Furthermore, $\overline{H_c}$ is a left $\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W$-module since $\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W$ is a subalgebra of $\overline{H_c}$.

Definition 4.1.2. Let $\lambda \in Irr(W)$. The *dual baby Verma module* of $\overline{H_c}$ associated to λ is:

$$\nabla(\lambda) := (\lambda^* \otimes_{\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W} \overline{H_c})^*,$$

where $\mathbb{C}[\mathbf{h}]^{CoW}_+$ acts on λ^* as zero.
Moreover, the $\overline{H_c}$-action is given by:

$$(h \cdot \phi)(f \otimes_{\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W} h') := \phi(f \otimes_{\mathbb{C}[\mathbf{h}]^{CoW}\#\mathbb{C}W} h'h),$$

for any $\phi \in \nabla(\lambda)$, $h, h' \in \overline{H_c}$ and $f \in \lambda^*$.

Remark 4.1.0.3. *By Remark* 3.2.1.1 *it follows that as a* $\mathbb{C}$*-vector space:*

$$dim(\Delta(\lambda)) = dim(\nabla(\lambda)) = |W| dim\lambda, \text{ for any } \lambda \in Irr(W).$$

Remark 4.1.0.4. *Recall from Chapter* 3 *that any restricted rational Cherendik algebra* $\overline{H_c}$ *is a* $\mathbb{Z}$*-graded algebra by assigning* $deg(x) = 1$, $deg(y) = -1$ *and* $deg(w) = 0$*, for any* $x \in \boldsymbol{h}$*,* $y \in \boldsymbol{h}^*$ *and* $w \in W$*.*
Indeed by Remark 3.1.1.2 *the associated rational Cherendik algebra* $H_{0,c}$ *is a* $\mathbb{Z}$*-graded algebra. Moreover the ideal* $A_+H_{0,c}$ *is a* $\mathbb{Z}$*-graded ideal in* $H_{0,c}$*, thus* $\overline{H_c} = H_{0,c}/A_+H_{0,c}$ *is a* $\mathbb{Z}$*-graded algebra.*

Following [6, Subsection 2.5], denote by $\overline{H_c} - mod_{\mathbb{Z}}$ the category of finitely generated, $\mathbb{Z}$-graded (left) $\overline{H_c}$-modules.
Given $M \in Ob(\overline{H_c} - mod_{\mathbb{Z}})$, it has a decomposition (as an abelian group) $M = \oplus_{n\in\mathbb{Z}} M_n$ where each M_n is a $(\overline{H_c})_0$-module and $(\overline{H_c})_m M_n \subseteq M_{m+n}$, for any $m, n \in \mathbb{Z}$.
The morphisms in $\overline{H_c} - mod_{\mathbb{Z}}$ are graded morphisms of degree zero, i.e, any $f \in Hom_{\overline{H_c}-mod_{\mathbb{Z}}}(M, N)$ is such that: $f : M \longrightarrow N$ is a morphism of underlying modules that respects grading: $f(M_n) \subset N_n$, for any $n \in \mathbb{Z}$.
By a submodule N of a module $M \in Ob(\overline{H_c} - mod_{\mathbb{Z}})$ we understand $N \in Ob(\overline{H_c} - mod_{\mathbb{Z}})$, i.e, $N = \oplus_{n\in\mathbb{N}} N_n$, and for any $n \in \mathbb{N}$ there holds $i(N_n) \subset M_n$ where $i : N \longrightarrow M$ is the set-theoretic inclusion, i.e, $i \in Hom_{\overline{H_c}-mod_{\mathbb{Z}}}(N, M)$.
Moreover, if $M \in Ob(\overline{H_c} - mod_{\mathbb{Z}})$, then $M[i]$ will denote the $\mathbb{Z}$-graded $\overline{H_c}$-module with same underlying abelian group as M, same module structure, and with grading $M[i]_j = M_{j-i}$, with $i, j \in \mathbb{Z}$.
We denote by F the forgetful functor:

$$F : \overline{H_c} - mod_{\mathbb{Z}} \longrightarrow \overline{H_c}\text{-}mod,$$

where $\overline{H_c} - mod$ denotes the category of finitely generated (left) $\overline{H_c}$-modules.

Remark 4.1.0.5 ([17])**.** *Let A be a $\mathbb{Z}$-graded $\mathbb{C}$-algebra, let B be a $\mathbb{Z}$-graded subalgebra of A and let N be a $\mathbb{Z}$-graded B-module.*
Then $A \otimes_B N$ ia a $\mathbb{Z}$-graded A-module and for any $i \in \mathbb{Z}$, its i-th homogeneous component is defined as follows:

$$(A \otimes_B N)_i = Span_{\mathbb{C}}\{a \otimes_B n \mid a \in A_j, n \in N_{i-j}\}.$$

Remark 4.1.0.6. *[17] Let A be a $\mathbb{Z}$-graded $\mathbb{C}$-algebra, let M be a (left) $\mathbb{Z}$-graded A-module and let $M^* = Hom_{\mathbb{C}}(M, \mathbb{C})$.*
Then M^* is a (right) $\mathbb{Z}$-graded A-module with grading:
$(M^*)_i := \{f \in M^* \mid f(M_j) = 0$ for any $j \neq -i\}$, for any $i, j \in \mathbb{Z}$.

Remark 4.1.0.7. *By Definitions 4.1.1, 4.1.2 and by Remarks 4.1.0.5, 4.1.0.6 it follows that for any $\lambda \in Irr(W)$ the Baby Verma module $\Delta(\lambda)$ and the dual Baby Verma module $\nabla(\lambda)$ can be regarded also as finitely generated $\mathbb{Z}$-graded $\overline{H_c}$-modules, i.e, as objects in $\overline{H_c} - mod_{\mathbb{Z}}$.*
Indeed, by assigning $degv = 0$ for any $v \in \lambda \in Irr(W)$ and $degf = 0$ for any $f \in \lambda^*$, we get: $(\Delta(\lambda))_i = Span_{\mathbb{C}}\{a \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \# \mathbb{C}W} v \mid a \in (\overline{H_c})_i, v \in \lambda\}$, for any $i \in \mathbb{Z}$. Analogously also $\nabla(\lambda)$ inherits a $\mathbb{Z}$-grading from $\overline{H_c}$. Indeed:
$(\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \# \mathbb{C}W} \overline{H_c})_i = Span_{\mathbb{C}}\{f \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \# \mathbb{C}W} h \mid f \in \lambda^*, h \in (\overline{H_c})_i\}$. Then, consider $(\nabla(\lambda)) = (\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \# \mathbb{C}W} \overline{H_c})^* = Hom_{\mathbb{C}}(\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \# \mathbb{C}W} \overline{H_c}, \mathbb{C})$, by Remark 4.1.0.6 it holds:
$(\nabla(\lambda))_i = Span_{\mathbb{C}}\{f \in \nabla(\lambda) \mid f((\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \# \mathbb{C}W} \overline{H_c})_j) = 0$ for any $j \neq -i\}$.

Before stating the main result of this section we need some further definitions. Let $\mathcal{C}$ be either $\overline{H_c} - mod$ or $\overline{H_c} - mod_{\mathbb{Z}}$.

Definition 4.1.3. Let $M \in Ob(\mathcal{C})$ and let N be a submodule of M. Then N is said to be a superfluous submodule of M if for any submodule P of M such that $P + N = M$, then $P = M$.

Definition 4.1.4. Let $M \in Ob(\mathcal{C})$. Then M is said to be projective in $\mathcal{C}$ if for every epimorphism $f \in Hom_{\mathcal{C}}(M, N)$ and every morphism $g \in Hom_{\mathcal{C}}(P, N)$, there exists a morphism $h \in Hom_{\mathcal{C}}(P, M)$ such that $f \circ h = g$.

Definition 4.1.5. Let $M \in Ob(\mathcal{C})$. A projective cover of M is a pair (X, f) where $X \in Ob(\mathcal{C})$ is projective and $f : X \longrightarrow M$ is a superfluous epimorphism in $Hom_{\mathcal{C}}(X, M)$, i.e, $Kerf$ is a superfluous submodule of X.

Definition 4.1.6. Let $M \in Ob(\mathcal{C})$. A submodule N of M is called maximal if M/N is a simple module. The *radical* of M is the intersection of all maximal submodules of M, i.e,

$$Rad(M) = \cap_{N \text{ is a maximal submodule of } M} N.$$

Now, we summarize the results of [17] applied to this situation:

Proposition 4.1.7. *[6, Proposition 2.11] Let $\lambda, \mu \in Irr(W)$.*

(i) The baby Verma module $\Delta(\lambda)$ has a simple head, i.e, $L(\lambda) := \Delta(\lambda)/Rad(\Delta(\lambda))$ is a simple module in $\overline{H_c} - mod_{\mathbb{Z}}$ and $\Delta(\lambda)$ is indecomposable.

(ii) $\Delta(\lambda)$ is isomorphic to $\Delta(\mu)[i]$ if and only if $\lambda \simeq \mu$ and $i = 0$.

(iii) The set $\{L(\lambda)[i] \mid \lambda \in Irr(W),\ i \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic, simple $\mathbb{Z}$-graded $\overline{H_c}$-modules.

(iv) $F(L(\lambda))$ is a simple $\overline{H_c}$-module and $\{F(L(\lambda)) \mid \lambda \in Irr(W)\}$ is a complete set of pairwise non-isomorphic simple $\overline{H_c}$-modules.

(v) If $P(\lambda)$ is the projective cover of $L(\lambda)$ in $\overline{H_c} - mod_{\mathbb{Z}}$, then $F(P(\lambda))$ is the projective cover of $F(L(\lambda))$ in $\overline{H_c}$-mod.

Example 4.1.8. Let $\overline{H_c}$ be the restricted rational Cherednik algebra in Example 3.2.6.
We compute the baby Verma modules of $\overline{H_c}$ for any $\lambda \in Irr(W)$, where $W =< s' >\subset GL(\mathbb{C})$, with $s' = -1 \in \mathbb{C}$.
We know that $Irr(W) = \{\mathbb{C}_+, \mathbb{C}_-\}$, where $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the trivial and the sign representation respectively.
Then:

$$\Delta(\mathbb{C}_\pm) = \overline{H_c} \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} \mathbb{C}_\pm.$$

By Remark 4.1.0.3 they are both two dimensional as $\mathbb{C}$-vector spaces.
Indeed, $\{x \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1, 1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1\}$ is a basis of both $\Delta(\mathbb{C}_+)$ and $\Delta(\mathbb{C}_-)$. The $\overline{H_c}$-action is given by:

$$y \cdot (1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = (y \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = (1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} y \cdot 1) = 0 \text{ and}$$
$$y \cdot (x \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = (xy - cs' \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) =$$
$$= (xy \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) - (cs' \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) =$$
$$= (x \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} y \cdot 1) - (cs' \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) =$$
$$= \mp c \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1,$$

for any $y \in \mathbf{h}^*$, since $s' \cdot 1 = \pm 1$, while

$$x \cdot (1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = (x \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) \text{ and}$$
$$x \cdot (x \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = (x^2 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = 0,$$

for any $x \in \mathbf{h}$, since $x^2 = 0 \in \overline{H_c}$, and

$$s' \cdot (1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \pm 1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1 \text{ and}$$
$$s' \cdot (x \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = (-xs' \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \mp x \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1,$$

for $s' \in W$.
Note that $\Delta(\mathbb{C}_\pm)$ are cyclic $\overline{H_c}$-modules. Indeed $\{1 \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1\}$ is a generator for them as $\overline{H_c}$-modules. Moreover, by the relation $[x, y] = cs'$, we distinguish two cases depending on $c \in \mathbb{C}$.

- If $c \neq 0$, $\Delta(\mathbb{C}_\pm)$ are simple modules.

  Indeed, let $(a + bx + gy + dxy + lyx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1)$ with $a, b, g, d, l \in \mathbb{C}$, be a non-zero element of $\Delta(\mathbb{C}_\pm)$. Note that:

$$\begin{gathered}(a + bx + gy + dxy + lyx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \\ = (a + bx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) + (g \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} y \cdot 1) + \\ + (dx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} y \cdot 1) + (lyx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \\ = (a + bx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) + (lxy - lcs' \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \\ = (a + bx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) + (lx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} y \cdot 1) - (lcs' \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \\ = (a + bx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) + (\mp lc \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1) = \\ = (a \mp lc + bx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1).\end{gathered}$$

  Then, without loss of generality let $a + bx \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1$ be a non-zero element of $\Delta(\mathbb{C}_\pm)$, i.e, with $a, b \in \mathbb{C}$ such that $(a, b) \neq (0, 0)$. Note that it must hold $b \neq 0$. Otherwise, if $b = 0$, $a \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1$ is a generator of $\Delta(\mathbb{C}_\pm)$ for any $a \neq 0$, while if $a = 0$ it is the zero element in $\Delta(\mathbb{C}_\pm)$. Moreover, by the $\overline{H_c}$-action we get that $a + bx \otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W} 1$ generates $\Delta(\mathbb{C}_\pm)$, for any $a \in \mathbb{C}$, $b \in \mathbb{C}^*$. Indeed:

$$\begin{gathered}y \cdot (a + bx \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1) = \\ = (ya \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1) + (byx \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1) = \\ = (a \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} y \cdot 1) + (\mp bc \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1) = \mp bc \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1,\end{gathered}$$

  for any $y \in \mathbf{h}^*$.
  Since for any $b \in \mathbb{C}^*$, $\mp bc \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1$ generates the cyclic $\overline{H_c}$-modules $\Delta(\mathbb{C}_\pm)$, we get that any non zero element of $\Delta(\mathbb{C}_\pm)$ is a generator. Then, there are no proper submodules of $\Delta(\mathbb{C}_\pm)$, except for the zero submodule.

- If $c = 0$, then $\mathbb{C}(x \otimes_{\mathbb{C}[y]/(y^m)\#\mathbb{C}W} 1)$ is the only proper submodule of $\Delta(\mathbb{C}_\pm)$, so it is a maximal submodule.

Indeed, let $a+bx\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}1$ be an element of $\Delta(\mathbb{C}_{\pm})$, with $a,b\in\mathbb{C}$ such that $(a,b)\neq(0,0)$. Then, $\mathbb{C}(a+bx\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}1)$ is not a submodule of $\Delta(\mathbb{C}_{\pm})$ for any $a\in\mathbb{C}^*$, $b\in\mathbb{C}$. Indeed:

$$x\cdot(a+bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)=(ax\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1),$$

for any $x\in\mathbf{h}$, and $(ax\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)\notin\mathbb{C}(a+bx\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}1)$.
If we consider the submodule generated by: $ax\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1$, $a+bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1$, we get that it is not proper and it is precisely $\Delta(\mathbb{C}_{\pm})$.
Fix $a=0$ and consider $\mathbb{C}(bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)$, for any $b\in\mathbb{C}^*$.
Note that it is a proper submodule:

$$x\cdot(bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)=0,$$

for any $x\in\mathbf{h}$, since $x^2=0\in\overline{H_c}$,

$$s'\cdot(bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)=\mp bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1,$$

for any $s'\in W$, and

$$\begin{aligned}y\cdot(bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)&=(bxy\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W})-(bcs'\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}1)=\\&=(bx\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}y\cdot1)+(\mp bc\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)=\\&=\mp bc\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}1=0,\end{aligned}$$

for any $y\in\mathbf{h}^*$, since $c=0$.
Hence, $\mathbb{C}(bx\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)=\mathbb{C}(x\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)$ is the only proper submodule of $\Delta(\mathbb{C}_{\pm})$ for any $b\neq0$, so it is maximal and $Rad(\Delta(\mathbb{C}_{\pm}))=\mathbb{C}(x\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)$.
Then: $L(\mathbb{C}_+)=\Delta(\mathbb{C}_+)/Rad(\Delta(\mathbb{C}_+))$ and $L(\mathbb{C}_-)=\Delta(\mathbb{C}_-)/Rad(\Delta(\mathbb{C}_-))$ are their simple heads. More precisely, as $\mathbb{C}$-vector space $L(\mathbb{C}_+)$ and $L(\mathbb{C}_-)$ are 1-dimensional and a basis for them is given by: $\{\overline{1\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1}\}$. As $\overline{H_c}$-modules they are generated by the same element and the $\overline{H_c}$-action is as follows:

$$y\cdot\overline{(1\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)}=\overline{(1\otimes_{\mathbb{C}[y]/(y^2)\#\mathbb{C}W}y\cdot1)}=0,$$

for any $y\in\mathbf{h}^*$,

$$x\cdot\overline{(1\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1)}=\overline{x\otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W}1}=0,$$

for any $x \in \mathbf{h}$, since $x \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1 \in Rad(\Delta(\mathbb{C}_\pm))$ and

$$s' \cdot \overline{(1 \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1)} = \overline{\pm 1 \otimes_{\mathbb{C}[\mathbf{h}^*]^{CoW}\#\mathbb{C}W} 1},$$

with $s' \in W$.

4.1.1 Brauer-Type Reciprocity

Before stating the *Brauer-type reciprocity* we recall some notions on module theory:

Definition 4.1.9. Let R be a ring and let M be a R-module. A composition series for M is a finite chain of submodules of M

$$\{0\} = J_0 \subset \cdots \subset J_n = M,$$

where all inclusions are strict and J_k is a maximal submodule of J_{k+1} for any k.

Remark 4.1.1.1. *An R-module M has a composition series if and only if it is both Artinian and Noetherian.*
In particular if R is Artinian and M is finitely generated over R, then by Hopkins' Theorem it admits a composition series.
If an R-module M has a composition series, then any finite strictly increasing series of submodules of M may be refined to a composition series, and any two composition series for M are equivalent.
In this case, the simple quotient modules J_{k+1}/J_k for each k, are known as the composition factors *of M.*
By Jordan-Holder's Theorem the number of occurrences of each isomorphism type of simple R-module, i.e, $|\{i;\ J_{i+1}/J_i \simeq J_{k+1}/J_k\}|$ for any fixed k, as a composition factor does not depend on the choice of composition series.

Remark 4.1.1.2. *By Remark* 4.1.1.1*, for any $\lambda \in Irr(W)$ $\Delta(\lambda)[i]$ and $\nabla(\lambda)[i]$ for $i \in \mathbb{Z}$ have composition series. Indeed they are finitely generated, $\mathbb{Z}$-graded $\overline{H_c}$-modules, $\overline{H_c}$ is a finitely generated $\mathbb{C}$-algebra and $\mathbb{C}$ is Artinian.*

Definition 4.1.10 ([17])**.** Let $M \in Ob(\overline{H_c} - mod_{\mathbb{Z}})$. Then M is said to have a Δ-filtration if it has a filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

by submodules such that for each $0 \leq j \leq n$, $M_j/M_{j-1} \simeq \Delta(\lambda)[i]$, for some $\lambda \in Irr(W)$ and $i \in \mathbb{Z}$.
Note that for a given Δ-filtration of M by [17, Corollary 4.3], for any $i \in \mathbb{Z}$ the numbers:

$$[M : \Delta(\lambda)[i]] = |\{j \; |M_j/M_{j-1} \simeq \Delta(\lambda)[i]\}|,$$

are independent of the filtration used.

As an application of [17, Theorem 4.5], we have the following Brauer-type reciprocity result:

Theorem 4.1.11. *[14, 4.6] Any projective object in $\overline{H_c} - mod_{\mathbb{Z}}$ has a Δ-filtration. In particular for $\lambda, \mu \in Irr(W)$ and $i \in \mathbb{Z}$ the projective cover $P(\lambda)$ of $L(\lambda)$ has a Δ-filtration and*

$$[P(\lambda) : \Delta(\mu)[i]] = (\nabla(\mu)[i] : L(\lambda)),$$

where $(\nabla(\mu)[i] : L(\lambda))$ denotes the multiplicity of $L(\lambda)$ as a composition factor of $\nabla(\mu)[i]$.

4.2 Example: the Cyclic Group

4.2.1 The Symplectic Reflection Algebra and Its Centre at $t = 0$

Let $(\mathbb{C}^2, \omega)$ be a symplectic vector space where ω is the canonical symplectic form as in Example 1.2.12. Let $\{x, y\}$ be a basis of $\mathbb{C}^2$ such that $\omega(x, y) = 1$ and let $G \subseteq Sp(2, \mathbb{C}) = SL(2, \mathbb{C})$ be the finite group generated by:

$$\varepsilon = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix},$$

where ξ is a fixed primitive *m-th* root of unity.
Notice that it is a symplectic reflection. Indeed, $\varepsilon \neq 1$ and by Remark 1.2.1.3 $det\varepsilon = 1$, so it cannot have 1 as eigenvalue, i.e, $dimKer(1 - \varepsilon) = 0$. By Definition 1.2.7 we conclude that ε is a symplectic reflection and analogously ε^i is a symplectic reflection for any $i \in \{1, \cdots, m-1\}$.
Observe that by choosing such a basis it holds:

$$\omega_\varepsilon(x, y) = \cdots = \omega_{\varepsilon^{m-1}}(x, y) = 1.$$

The group $G \simeq C_m$ acts on $\mathbb{C}^2$ as follows:

$$\varepsilon \cdot x = \xi x,\ \varepsilon \cdot y = \xi^{-1} y,$$

so in $\mathbb{C} < x, y > \ltimes G$ we have the relations $\varepsilon x = \xi x \varepsilon$ and $\varepsilon y = \xi^{-1} y \varepsilon$, and the symplectic reflection algebra associated with the triple $(\mathbb{C}^2, \omega, G)$ is:

$$H_{t,c} = \mathbb{C} < x, y > \ltimes G / I < xy - yx - t - \Sigma_{i=1}^{m-1} c_i \varepsilon^i >.$$

By Remark 2.1.1.3 we can find a basis of $H_{t,c}$ as a $\mathbb{C}$-vector space made of ordered monomials in x, y, ε^l, for $l \in \{0, \cdots, m-1\}$, i.e,

$$B = \{x^i y^j \varepsilon^l \mid i, j \in \mathbb{N},\ l \in \{0, \cdots, m-1\}\}.$$

As in Example 2.2.2, given two elements $x^i y^j, x^a y^b$ in $H_{t,c}$ we would be able to write their product $x^i y^j x^a y^b$ with respect to the basis B. Notice that we can rewrite the element $y^j x^a$ by using the relation $[x, y] = t + \Sigma_{i=1}^{m-1} c_i \varepsilon^i$ in $H_{t,c}$. Indeed by induction on the power of y, we get:

$$(4.1)\quad y^n x = x y^n - n t y^{n-1} - \Sigma_{i=1}^{m-1} c_i (1 + \xi^{-i} + \cdots + \xi^{-(n-1)i}) y^{n-1} \varepsilon^i,$$

for any $n \in \mathbb{N}_{\geq 2}$.
Observe that if $n \equiv m$ (mod m), then $\Sigma_{i=1}^{m-1} c_i (1 + \xi^{-i} + \cdots + \xi^{-(n-1)i}) = 0$ since each summand is the sum of roots of unity and:

$$1 + \xi^{-i} + \cdots + \xi^{-i(n-1)} = \frac{\xi^{-in} - 1}{\xi^{-1} - 1} = 0,$$

for any $i \in \{1, \cdots, m-1\}$.
Then by induction on the power of x in (4.1) it is possible to find an expression for $y^j x^a$, $j, a \in \mathbb{N}$.
Let $\mathbf{e} = \frac{1}{m} \Sigma_{i=0}^{m-1} \varepsilon^i$ be the trivial idempotent in $\mathbb{C}G$ and consider the spherical subalgebra $\mathbf{e} H_{t,c} \mathbf{e}$ of $H_{t,c}$.
As it is noticed in Example 2.2.2, it is possible to find a generating set of $\mathbf{e} H_{t,c} \mathbf{e}$ made of ordered monomials in x, y, ε^l, with $l \in \{0, \cdots, m-1\}$ after multypling them on both sides by $\mathbf{e}$.
Note that $\varepsilon^l \mathbf{e} = \mathbf{e}$, for any $l \in \{0, \cdots, m-1\}$. Indeed:

$$\varepsilon^l \frac{1}{m} \Sigma_{i=0}^{m-1} \varepsilon^i = \frac{1}{m} \Sigma_{i=0}^{m-1} \varepsilon^{i+l} = \mathbf{e},$$

after rescaling the index i in the sum.
Then we only consider elements of the form: $\mathbf{e} x^i y^j \mathbf{e}$, for $i, j \in \mathbb{N}$.
Since $\varepsilon x = \xi x \varepsilon$ and $\varepsilon y = \xi^{-1} y \varepsilon$, it holds: $\varepsilon x^i y^j = \xi^i \xi^{-j} x^i y^j \varepsilon$, i.e, $x^i y^j$ commutes with ε if and only if $i - j = mh$ with $h \in \mathbb{N}$, i.e, if and only if $i \equiv j$ (mod m). Analogously, it follows that $\varepsilon^l x^i y^j = (\xi^i)^l (\xi^j)^{-l} x^i y^j \varepsilon^l$, i.e, $x^i y^j$ commutes with ε^l for any $l \in \{0, \cdots, m-1\}$ (and then with $\mathbf{e}$) if and only if $il - jl = mh$, with $h \in \mathbb{N}$, i.e, if and only if $il \equiv jl$ (mod m). Explicity:

$$\mathbf{e}x^ay^b\mathbf{e} = \frac{1}{m}\Sigma_{i=0}^{m-1}\varepsilon^i x^a y^b\mathbf{e} = \frac{1}{m}\Sigma_{i=0}^{m-1}\xi^{ia}\xi^{-ib}x^ay^b\mathbf{e} =$$

$$= \frac{1}{m}\Sigma_{i=0}^{m-1}\xi^{ia-ib}x^ay^b\mathbf{e} = \begin{cases} 0, & \text{if } a \not\equiv b \text{ (mod m)} \\ x^ay^b\mathbf{e}, & \text{if } a \equiv b \text{ (mod m)}. \end{cases}$$

Indeed, let $a \not\equiv b \pmod m$. Then there are two cases:
(i) there exists $d \in \mathbb{N}$ such that $a \equiv b \pmod d$ with $d|m$, i.e, $a - b = dc_1$, for some $c_1 \in \mathbb{N}$, and $a \not\equiv b \pmod r$ with $r = \frac{m}{d}$.
In this case, $\frac{1}{m}\Sigma_{i=0}^{m-1}\xi^{ia-ib}x^ay^b\mathbf{e} = 0$. Indeed,

$$\Sigma_{i=0}^{m-1}\xi^{i(a-b)} = \Sigma_{i=0}^{m-1}\xi^{idc_1} = \underbrace{\Sigma_{i=0}^{r-1}\xi^{idc_1} + \cdots + \Sigma_{i=m-1-(r-1)}^{m-1}\xi^{idc_1}}_{d\text{-summands}} = 0,$$

and each summand is the sum of r-th roots of unity which is equal to zero.
(ii) $a \not\equiv b \pmod d$ for any $d \in \mathbb{N}$ such that $d|m$.
Then, $\frac{1}{m}\Sigma_{i=0}^{m-1}\xi^{ia-ib}x^ay^b\mathbf{e} = 0$ since $\Sigma_{i=0}^{m-1}\xi^{ia-ib}$ is the sum of m-th roots of unity and it is equal to zero.
Notice that the commutation relation $[x,y] = t + \Sigma_{i=1}^{m-1}c_i\varepsilon^i$ in $\mathbf{e}H_{t,c}\mathbf{e}$ gives: $xy\mathbf{e} - yx\mathbf{e} = t\mathbf{e} + \Sigma_{i=1}^{m-1}c_i\mathbf{e}$. This, together with the ones obtained by induction on the power of x and y, are the non-zero relations among the ordered monomials in $\mathbf{e}H_{t,c}\mathbf{e}$. Then a basis made of ordered monomials in x, y, ε^l with $l \in \{0, \cdots, m-1\}$ of $\mathbf{e}H_{t,c}\mathbf{e}$ as a $\mathbb{C}$-vector space is:

$$B' = \{x^iy^j\mathbf{e} \mid i \equiv j \text{ (mod m)}, \, i,j \in \mathbb{N}\}.$$

Note that, as it is shown in Example 2.2.2, computing the product of two non zero elements $\mathbf{e}x^iy^j\mathbf{e}, \mathbf{e}x^ay^b\mathbf{e}$ in $\mathbf{e}H_{t,c}\mathbf{e}$ is equivalent to computing the product of x^iy^j, x^ay^b in $H_{t,c}$.
We choose as generators of $\mathbf{e}H_{t,c}\mathbf{e}$ as a $\mathbb{C}$-algebra: $x^m\mathbf{e}, y^m\mathbf{e}$ and $xy\mathbf{e}$.
Indeed, let $x^{lm+a}y^{l'm+a}\mathbf{e}$ be a non-zero element of $\mathbf{e}H_{t,c}\mathbf{e}$. We prove by induction on the power of $xy\mathbf{e}$ that:

$$(4.2) \qquad x^ay^a\mathbf{e} = (xy\mathbf{e})^a + \Sigma_{j=0}^{a-1}p_j^{(a)}(xy\mathbf{e})^j, \text{ for some } p_j^{(a)} \in \mathbb{C},$$

for any $a \in \mathbb{N}_{\geq 2}$.
Note that by (4.1):

$$(xy\mathbf{e})^2 = (xy\mathbf{e})(xy\mathbf{e}) = xyxy\mathbf{e} = x^2y^2\mathbf{e} - txy\mathbf{e} - \Sigma_{i=1}^{m-1}c_i\xi^{-i}xy\mathbf{e},$$

i.e, (4.2) holds for $a = 2$:

$$x^2y^2\mathbf{e} = (xy\mathbf{e})^2 + t(xy\mathbf{e}) + \Sigma_{i=1}^{m-1}c_i\xi^{-i}xy\mathbf{e}.$$

Assume that (4.2) holds for $a - 1$, with $a \in \mathbb{N}$:

$$(4.3) \qquad x^{a-1}y^{a-1}\mathbf{e} = (xy\mathbf{e})^{a-1} + \Sigma_{j=1}^{a-2} p_j^{(a-1)}(xy\mathbf{e})^j, \text{ for some } p_j^{(a-1)} \in \mathbb{C}$$

then we prove that (4.2) holds also for $a \in \mathbb{N}$. Indeed, by (4.3):

$$(x^{a-1}y^{a-1}\mathbf{e})(xy\mathbf{e}) = (xy\mathbf{e})^a + \Sigma_{j=1}^{a-2} p_j^{(a-1)}(xy\mathbf{e})^{j+1}, \text{ for some } p_j^{(a-1)} \in \mathbb{C},$$

and, by (4.1) there holds:

$$(x^{a-1}y^{a-1}\mathbf{e})(xy\mathbf{e}) = x^{a-1}y^{a-1}xy\mathbf{e} =$$
$$= x^{a-1}(xy^{a-1} - (a-1)ty^{a-2} - \Sigma_{i=1}^{m-1} c_i(1+\xi^{-1}+\cdots+\xi^{-i(a-2)})y^{a-2}\varepsilon^i)y\mathbf{e} =$$
$$= x^a y^a \mathbf{e} - (a-1)tx^{a-1}y^{a-1}\mathbf{e} - \Sigma_{i=1}^{m-1} c_i(1+\xi^{-1}+\cdots+\xi^{-i(a-2)})x^{a-1}y^{a-1}\mathbf{e}.$$

Thus, by combining the last two equations we get:

$$x^a y^a \mathbf{e} = (xy\mathbf{e})^a + \Sigma_{j=1}^{a-1} p_j^{(a)}(xy\mathbf{e})^j, \text{ for some } p_j^{(a)} \in \mathbb{C},$$

with $a \in \mathbb{N}$, i.e, (4.2) holds for any $a \in \mathbb{N}_{\geq 2}$.
Hence, we can rewrite $x^{lm+a}y^{l'm+a}\mathbf{e} = (x^m)^l x^a y^a (y^m)^{l'}\mathbf{e} = (x^m\mathbf{e})^l(x^a y^a\mathbf{e})(y^m\mathbf{e})^l$, where $x^a y^a \mathbf{e}$ can be written with respect to the chosen generators. Thus, any element can be obtained by adding and multypling linear combinations of $x^m\mathbf{e}$, $y^m\mathbf{e}$ and $xy\mathbf{e}$.
Moreover, by induction on the power of x in (4.1) we can compute $[x^m\mathbf{e}, y^m\mathbf{e}]$, $[x^m\mathbf{e}, xy\mathbf{e}]$ and $[y^m\mathbf{e}, xy\mathbf{e}]$.
Now let $t = 0$. By Theorem 2.3.3, $Z(H_{0,c}) \simeq Z(\mathbf{e}H_{0,c}\mathbf{e})$ and by Theorem 2.2.3, $Z(\mathbf{e}H_{0,c}\mathbf{e}) = \mathbf{e}H_{0,c}\mathbf{e}$.
Then the commutation relations $[x^m\mathbf{e}, y^m\mathbf{e}]$, $[x^m\mathbf{e}, xy\mathbf{e}]$ and $[y^m\mathbf{e}, xy\mathbf{e}]$ are all equal to zero for $t = 0$ and we get that $\mathbf{e}H_{0,c}\mathbf{e} = \mathbb{C}[x^m\mathbf{e}, y^m\mathbf{e}, xy\mathbf{e}]$.
Hence, by the Satake isomorphism in Theorem 2.3.1 it holds
$Z(H_{0,c}) = \mathbb{C}[x^m, y^m, xy]$.

4.2.2 The Restricted Rational Cherednik Algebra

Note that the symplectic reflection algebra $H_{t,c}$ we are considering is a rational Cherednik algebra by choosing a basis $\{x, y\}$ such that
$\mathbb{C}^2 = \mathbf{h} \oplus \mathbf{h}^* = \mathbb{C}x \oplus \mathbb{C}y$, where $\mathbf{h} = Span\{x\}$, $\mathbf{h}^* = Span\{y\}$ and $y(x) = 1$.
Let ω be as above, i.e, as in Example 1.2.12 and let $W =< s' >\subset GL(\mathbb{C})$, with $s' = \xi \in \mathbb{C}$ such that $\xi^m = 1$, be the Weyl group associated to $\mathbf{h}^*$ acting diagonally on $\mathbb{C}x \oplus \mathbb{C}y$. Then, $H_{t,c}$ is the rational Cherednik algebra associated with the triple $(\mathbb{C}x\oplus\mathbb{C}y, \omega, W)$, a complex number t and a function $c : S' \longrightarrow \mathbb{C}$ as in Definition 3.1.5.
Thus we can rewrite the defining relation of $H_{t,c}$ in the following way:

$$s'x = \xi x s', \; s'y = \xi^{-1}ys', \; [x, y] = t + \Sigma_{i=1}^{m-1} c_i \xi^i.$$

From now on we fix $t = 0$. Let $A = \mathbb{C}[\mathbf{h}]^W \otimes \mathbb{C}[\mathbf{h}^*]^W = \mathbb{C}[x^m] \otimes \mathbb{C}[y^m]$ be a subalgebra of $Z(H_{0,c})$ as in Subsection 3.2.1. By Proposition 3.2.2, $Z(H_{0,c}) = \mathbb{C}[x^m, y^m, xy]$ is a free module over A of rank $|W| = m$. Indeed, a basis of $Z(H_{0,c})$ as a module over $\mathbb{C}[x^m] \otimes \mathbb{C}[y^m]$ is given by:

$$\{1, xy, \cdots, x^{m-1}y^{m-1}\}.$$

Consider the ideal A_+ in A of elements with zero constant term. The restricted rational Cherednik algebra associated to $H_{0,c}$ is $\overline{H_c} = H_{0,c}/A_+H_{0,c}$, and it is the $\mathbb{C}$-algebra generated by $\mathbf{h}$, $\mathbf{h}^*$, W and a function c as above, with defining relations:

$$s'x = \xi x s',\ s'y = \xi^{-1} y s',\ [x, y] = \Sigma_{i=1}^{m-1} c_i \xi^i \text{ and}$$

$$x^m = 0,\ y^m = 0.$$

Moreover, by Remark 3.2.1.1 it follows that as a $\mathbb{C}$-vector space $dim\overline{H_c} = |W|^3 = m^3$.

4.2.3 On the Representation Theory: the Baby Verma Modules

Let $W =< \xi >\subset GL(\mathbb{C})$ as above, then $W \simeq C_m$ and $|W| = m$.
Let $Irr(W)$ be a complete set of non-isomorphic simple W-modules. Since W is abelian, all its irreducible representations are one dimensional.
Moreover, any irreducible representation V of C_m is specified by the requirement that the fixed generator ξ of C_m must act on V through the multiplication by a *m-th* root of unity. Then $Irr(W) = \{\mathbb{C}_1, \mathbb{C}_\xi, \cdots, \mathbb{C}_{\xi^{m-1}}\}$, where $\mathbb{C}_{\xi^i}$ is the 1-dimensional $\mathbb{C}$-vector space on which ξ acts through multiplication by ξ^i, for any $i \in \{0, \cdots, m-1\}$.
Then, the baby Verma modules for $\overline{H_c}$ are as follows:

$$\Delta(\mathbb{C}_{\xi^i}) = (\overline{H_c} \otimes_{\mathbb{C}[y]/(y^m)\#\mathbb{C}W} \mathbb{C}_{\xi^i}),$$

for any $i \in \{0, \cdots, m-1\}$.
Recall from Remark 4.1.0.3 that they are m-dimensional as $\mathbb{C}$-vector spaces. Indeed, for any $i \in \{0, \cdots, m-1\}$ a basis for $\Delta(\mathbb{C}_{\xi^i})$ is given by:
$\{1 \otimes_{\mathbb{C}[y]/(y^m)\#\mathbb{C}W} 1, x \otimes_{\mathbb{C}[y]/(y^m)\#\mathbb{C}W} 1, \cdots, x^{m-1} \otimes_{\mathbb{C}[y]/(y^m)\#\mathbb{C}W} 1\}$.
Then, as in Example 4.1.8, by the relation: $[x, y] = \Sigma_{i=1}^{m-1} c_i \xi^i$ and by induction on the power of x and y, we can compute the action of $\overline{H_c}$ on the elements of the basis.

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