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C^1 isometric embeddings

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Chapter 1

Introduction

The problem of abstraction is fundamental in mathematics: whenever a problem is to be solved, one must ask which is the correct level of generality to work with. Riemannian manifolds are supposed to model space and surfaces, quite concrete objects usually considered subject to intuition. And yet Riemannian manifolds are very abstract in definition, to the point that on first analysis it is not even clear if a Riemannian manifold can be isometrically embedded in Euclidean space. It is clear that in order to understand how to work with them, one needs to develop various tools. With this goal in mind we can try to see how Riemannian Manifolds behave under regular transformations, i.e. isometries. For example, if our model does not distinguish between two completely different objects, we might reconsider our intuition about these entities. Instances of regularity with respect to isometries are the oldest historically: in 1827 Gauss proved the Theorema Egregium showing the preservation of curvature in space under C^2 isometries, and in 1929 Cohn-Vossen proved that convex surfaces in $\mathbb{R}^3$ are rigid under C^2 isometries. It was not until many years later that a surprising result about strong freedom of C^1 isometries was found: in 1954 Nash proved that any embedding that shortens distances in codimension two can be approximated arbitrarily well by C^1 isometric embeddings, and a year later Kuiper improved the result to codimension one, directly opposing previous results. From then on, different contributions have been made, and now it is known that in $\mathbb{R}^3$ C^1 embeddings whose derivatives are Hölder-continuous can or cannot approximate arbitrary embeddings depending on the level of Hölder continuity.
The scope of this work is to prove some results concerning our discussion above: first, we will show Nash's Theorem on isometric embeddings, and then we will use it to prove that any Riemannian manifold can be C^1 isometrically embedded in the Euclidean space.

Chapter 2

Statement

This chapter will be an overview of the whole work. Here we state the results we will cover along with other facts that make up the context our theorems are inserted in. We begin with some definitions.

Definition 2.1 (Short Map). *Consider a Riemannian manifold* (Σ, g) *and let* $u : \Sigma \longrightarrow \mathbb{R}^N$ *be a* C^1 *immersion. We say that* u *is short if the pullback of the Euclidean metric* e *satisfies* $u^{\#}e \leq g$ *in the sense of quadratic forms, that is* $u^{\#}e_q(v) \leq g_q(v)$ *for all* $q \in \Sigma$, $v \in T_q\Sigma$. *We say that* u *is strictly short if the above inequality holds strictly for all* $v \neq 0$ *and for all* $q \in \Sigma$.

The notion of short map is quite intuitive, for example if we take any curve γ on Σ, then the length of γ in Σ is greater than the length of $u \circ \gamma$ in $\mathbb{R}^N$.

Definition 2.2 (Limit set). *Consider a smooth manifold* Σ *and* $u : \Sigma \to \mathbb{R}^N$. *Consider an exhaustion by compact sets* $\{\Gamma_k\}_k$, *that is a collection of compact subsets* $\Gamma_k \subset \Sigma$ *with* $\Gamma_k \subset \Gamma_{k+1}$ *such that* $\bigcup_k \Gamma_k = \Sigma$. *The limit set of* u *is the set of points that are the limit of a sequence of points* $(u(x_n))_n$ *with* $x_n \in \Sigma \setminus \Gamma_n$.

The next two theorems constitute the core of this work, and chapters 3 and 4 will be entirely dedicated to their proofs.

Theorem 2.3 (Nash). *Let* (Σ, g) *be a smooth* n*-dimensional Riemannian manifold and* $v : \Sigma \longrightarrow \mathbb{R}^N$ *a* C^∞ *short immersion, with* $N \geq n + 2$, *such that the limit set of* v *does not intersect the image of* v. *Then for any* $\varepsilon > 0$ *there exists a* C^1 *isometric immersion* $u : \Sigma \longrightarrow \mathbb{R}^N$ *such that* $\|u - v\|_{C^0} < \varepsilon$. *If* v *is an embedding,* u *can be chosen to be an embedding. If* v *is strictly short, we can also ask for* u *and* v *to have the same limit set.*

Theorem 2.4. *Let* (Σ, g) *be a Riemannian manifold of dimension* n, *Then there exist a* C^1 *isometric embedding* $u : \Sigma \longrightarrow \mathbb{R}^{2n+1}$.

The question one might ask after seeing Theorem 2.3, is whether improvements to this already somewhat unsettling result can be made. If one looks at weakening the already very loose hypotheses, the following result shows up:

Theorem 2.5 (Kuiper). *Let (Σ, g) be a smooth n-dimensional Riemannian manifold and $v : \Sigma \longrightarrow \mathbb{R}^N$ be a C^∞ short immersion, with $N \geq n+1$, such that the limit set of v does not intersect the image of v. Then for any $\varepsilon > 0$ there exists a C^1 isometric immersion $u : \Sigma \longrightarrow \mathbb{R}^n$ such that $\|u - v\|_{C^0} < \varepsilon$. If v is an embedding, u can be chosen to be an embedding.*
If z is strictly short, we can also ask for u and v to have the same limit set.

See Kuiper's original article 6 for a proof. In section 4 we will highlight what parts of Nash's argument can be modified to account for the lower codimension.
In a much different way, one asks if the regularity of the isometric immersion can go beyond C^1. On this road one encounters an obstacle, that in a way is an expression of the rigidity that one would expect from isometries. The following is a classical result, see 1, 4:

Theorem 2.6. *Let $\Sigma \subset \mathbb{R}^3$ be a compact convex surface. If $u : \Sigma \longrightarrow \Gamma \subset \mathbb{R}^3$ is a C^2 isometric diffeomorphism, then Σ and Γ are congruent, that is there exists an isometry A of $\mathbb{R}^3$ such that $A(\Sigma) = \Gamma$.*

This theorem tells us that there is only one way of isometrically C^2-embed a manifold in $\mathbb{R}^3$. Indeed if u, v are two such map, then $u \circ v^{-1} : v(\Sigma) \longrightarrow u(\Sigma)$ is a C^2 isometry, and therefore $u(\Sigma)$ and $v(\Sigma)$ differ by a rigidity.
Then, almost no (short) map can be approximated by C^2 isometries. Suppose one such u is given: as among all $\lambda \in]0, 1]$, at most one of the maps λu has arbitrarily good C^2 isometric approximations.
From this opposition the question of regularity becomes more precise. Consider $\alpha \in (0, 1)$ and let $C^{1,\alpha}$ be the space of differentiable maps with first derivatives that are α-Hölder continuous: for what α, given any smooth short map $u : \Sigma \to \mathbb{R}^{n+1}$, can we find arbitrarily good isometric $C^{1,\alpha}$ approximations of u?
There have been various results pushing in both directions: here we state some, an extensive and detailed discussion also containing proofs can be found in 2.
For α small, the properties of the Nash-Kuiper Theorem still hold, indeed we have:

Theorem 2.7. *Let (Σ, g) be a smooth, n-dimensional compact Riemannian manifold. Suppose $N \geq n+1$ and $\alpha < (1 + n(n+1)^2)^{-1}$. Given any C^1 short map $u : \Sigma \to \mathbb{R}^N$, for all $\varepsilon > 0$ there exists an isometric immersion*

$v \in C^{1,\alpha}(\Sigma, \mathbb{R}^N)$ such that $\|u - v\|_{C^0} < \varepsilon$.
If u is an embedding, v can be chosen to be an embedding.

On the other side of the spectrum one finds that for α too close to 1 rigidity is still too strong:

Theorem 2.8. *Let (Σ, g) be a smooth, compact surface (2-dimensional manifold) with positive Gaussian curvature, and let $u, v \in C^{1,\alpha}(\Sigma, \mathbb{R}^3)$ be isometric immersions with $\alpha > 2/3$. Then $u(\Sigma)$ and $v(\Sigma)$ are congruent.*

As noted before, this implies that almost no map can be approximated by $C^{1,\alpha}$ isometric embeddings. There also exist rigidity results for dimensions higher than 2, we chose not to include them here. To this day the question remains open.

At the end of the work, we will take a very specific surface, the Flat Torus, and put forward some calculations that if completed might gain insight into the possible regularity of isometric maps.

Chapter 3

Preliminary work

Among the tools we need to prove Theorem 2.3, an efficient way of handling sets of points on a manifold is important. The first reduction that we make is based on the fact that any smooth manifold is triangulable, and therefore we can work instead with simplicial complexes. These will serve as building blocks and we will split and glue them together to fix a regular enough geometric setting.

Definition 3.1 (Barycentric subdivision)**.** *We define the following operation inductively on the dimension of a simplex. Consider an n dimensional simplex Δ, spanned by points $p_1, ..., p_{n+1}$:*

1. *If $n = 0$, the barycentric subdivision of Δ is Δ itself.*
2. *If $n > 0$, define $b = (p_1 + ... + p_{n+1})/(n + 1)$. Δ has faces Δ_i for $i = 1, ..., n$ of dimension $n-1$. On each of the Δ_i the barycentric subdivision is defined, in particular Δ_i is covered by the $n-1$ dimensional simplices $\Delta_{i,1}, ..., \Delta_{i,m}$. Define $\Delta'_{i,k}$ the convex hull of $b \cup \Delta_{i,k}$ for $i = 1, ..., n$ and $k = 1, ..., m$. Finally, the barycentric subdivision of Δ is the simplex composed of the $\Delta'_{i,k}$'s.*

For a generic simplicial complex, the barycentric subdivision is the union of the subdivisions of its components.

Proposition 3.2. *Let Σ be an n-dimensional differentiable manifold and let $\{V_\lambda\}_\lambda$ be an open cover. Then there exists a covering $\{U_l\}_l$ such that:*

1. *For each l there exists λ such that $U_l \subset V_\lambda$;*
2. *The closure of each U_l is diffeomorphic to an n-dimensional closed ball;*
3. *Each U_l intersects finitely many U_m's;*

4. *Each point* $p \in \Sigma$ *has a neighbourhood intersecting at most* $n+1$ *different* U_l*'s;*

5. $\{U_l\}_l$ *can be split into* $n+1$ *classes* $\mathscr{C}_i$ *such that if* $U_l, U_m \in \mathscr{C}_i$ *and* $U_l \cap U_m \neq \varnothing$*, then* $l = m$*.*

Proof. Any differentiable manifold possesses a locally finite triangulation, so we can take one such triangulation and refine it so that for each point p, all the simplices touching p lie in the same V_λ[1]. We call S this triangulation, and enumerate as S_i^m its m dimensional components, for $m = 0, ..., n$. Let T be the barycentric subdivision of S, and for each S_i^m define $U_i^m = int(\bigcup \{\Delta : S_i^m \subset \Delta \in T\})$, where by *int* we mean the interior. That means we are taking all simplices in T containing S_i^m and calling U_i^m the interior of their union. Notice that if $i \neq j$ then $U_i^m \cap U_j^m = \varnothing$.
Define also $\mathscr{C}_m = \{U_i^m\}_i$ and $\mathscr{C} = \bigcup \mathscr{C}_m$. $\mathscr{C}$ is an open cover satisfying (1) and (5). Any simplex in T intersects finitely many elements S_i^m and thus intersects finitely many U_i^m, and since each U_l is contained in a finite union of simplices this gives (3). The closure of any of the U_i^m is homeomorphic to a n-dimensional ball but with sharp edges, so by choosing an appropriate smaller open set, we can ask it to be diffeomorphic to the n-dimensional closed ball, hence satisfying (2), while (1), (3), (5) are still satisfied.

To obtain (4), we can shrink once more all the U_l such that the closures of two elements in the same class $\mathscr{C}_i$ are disjoint, and this still satisfies all the properties above. Fix $p \in \Sigma$: if p is in the closure of no element of $\mathscr{C}_i$, then there is a neighborhood of p not touching $\mathscr{C}_i$, and if it is in the closure of an element of $\mathscr{C}_i$, there is a neighborhood of p not touching any other element in $\mathscr{C}_i$. The intersection of these neighborhoods satisfies (4). □

This result is quite easy but very important as it condenses most of the geometric regularity needed for our arguments, and we will use it on many occasions. In particular, the properties of local finiteness and separation make the treatments of limits and infinite series trivial on a local level and justify a great deal of decompositions that are to come. The condition on the U_l to be diffeomorphic to unit balls has its realization in Proposition 3.4.

Remark 3.3. *In the following proposition we will use some common machinery that we recall here:*

[1] Such refinement exists, indeed fixed local coordinates there exist positive r and R such that each point has a ball of radius r in some open set and each simplex has diameter at most R, therefore through a finite number of refinements we can decrease R to be lower that r and our requirement is satisfied.

1. *Gram-Schmidt algorithm. Suppose $u, v \in \mathbb{R}^d$ are two non-zero vectors. Then u and $w = v - \frac{u \cdot v}{\|u\|^2} u$ are orthogonal, and if u, v were linearly independent then $w \neq 0$. More in general, if $u, v : X \longrightarrow \mathbb{R}^d$ are two never vanishing functions, we can define $w(x) = v(x) - \frac{u(x) \cdot v(x)}{\|u(x)\|^2} u(x)$ and we get $u(x) \perp w(x)$ for all $x \in X$. Moreover, if u, v were always independent, w will be never vanishing. Notice that if z is orthogonal to u, v, it will be orthogonal also to w.*

2. *Smoothing by convolution. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ (the space of C^∞ functions with compact support), define $\phi_t(x) = t^{-d}\phi(x/t)$. Then, for any $f \in C_c(\mathbb{R}^d)$ we have $f * \phi \in C^\infty$ and $f * \phi_t(x) \to f(x)$ for $t \to 0$ uniformly in x. In particular after fixing ϕ and f we can choose t small so that $\|f * \phi_t - f\|_\infty < \varepsilon$ for $\varepsilon > 0$ small as we want. If instead $f = (f_1, ..., f_n) \in C_c(\mathbb{R}^d, \mathbb{R}^n)$, we can apply the above procedure to each of the f_i and choose t small enough to obtain an estimate on the whole function.*

Proposition 3.4. *Let $B \subset \mathbb{R}^n$ be diffeomorphic to the n-dimensional unit ball, and $w : B \longrightarrow \mathbb{R}^N$ a smooth immersion with $N \geq n + 2$. Then there exist smooth $\alpha, \beta : B \longrightarrow \mathbb{R}^N$ such that:*

1. *$\alpha(q) \perp \beta(q)$ for all $q \in B$*

2. *$\alpha(q), \beta(q)$ are orthogonal to T_qB for all $q \in B$.*

Proof. We can assume $B = B(0, 1) \subset \mathbb{R}^n$. We first prove there exist two such continuous α, β.
Consider the set $\mathscr{R}$ of positive radii ρ such that there are α, β on $B(0, \rho)$ satisfying our requirements. The set $\mathscr{R}$ is nonempty: indeed take $\alpha_0, \beta_0 \in \mathbb{R}^N$ orthogonal to $T_{w(0)}w(B)$ and to each other, and define $a(p) = \alpha_0, b(p) = \beta_0$ for all p in some neighbourhood of 0. Then, project a and b onto the normal bundle of $w(B)$, and if the neighborhood of 0 is small enough, the projected vectors will be independent and nonvanishing. Now make them orthogonal applying Gram-Schmidt, and finally normalize them. Notice that Gram-Schmidt keeps the vectors inside the normal bundle. The results are maps defined on some $B(0, \rho)$ that are continuous and satisfy the thesis, so $\mathscr{R} \neq \varnothing$.
We now show that $\sup \mathscr{R} \in \mathscr{R}$. Let $\rho \in \mathscr{R}$, with α, β defined in $B(0, \rho)$. For $\delta > 0$ define on $B(0, \rho + \delta)$ the maps α', β' extending α, β by:

$$\alpha'(x) = \alpha(\rho \frac{x}{|x|}) \qquad \beta'(x) = \beta(\rho \frac{x}{|x|}) \qquad \forall x \in B(0, \rho + \delta) \setminus B(0, \rho)$$

These satisfy (a). Notice that since w is smooth and the angle between α and the tangent space depends only on $\alpha \cdot D_i w$, due to compactness of B there is

some δ independent of ρ that keeps such angle greater than $\pi/2-\eta$ for some η small as we want. Then, by projecting onto the normal bundle, using Gram-Schmidt and normalizing, if we have chosen η small enough, we produce two maps $\bar{\alpha}, \bar{\beta}$ on $B(0, \rho+\delta)$ satisfying the thesis. Then if $\rho_0 = \sup \mathscr{R}$, we have $\rho_0 \geq \min\{1, \rho+\delta\}$ for all $\rho \in \mathscr{R}$, hence $\rho_0 = 1$ and we have found our global maps.
Now we smooth our functions by convolution: first extend α, β to be continuous with support on $B(0,2)$ (but without requirements of orthogonality outside of $B(0,1)$), then we take $\phi \in C_c^\infty(\mathbb{R}^n)$ and choose t small so that the projections of $\alpha * \phi_t, \beta * \phi_t$ onto the normal bundle are independent (since the projections of α and β are independent and B is compact, such a t exists). Now we use Gram-Schmidt and normalize to get our required functions. □

Remark 3.5. *In the above proposition we do not actually need $B \subset \mathbb{R}^n$. Indeed if B is an open subset of some smooth manifold and $\Phi : B \longrightarrow B(0,1) \subset \mathbb{R}^n$ is a diffeomorphism and $w : B \longrightarrow \mathbb{R}^N$ is the smooth immersion, we apply the proposition to $w \circ \Phi^{-1}$ to get $\bar{\alpha}, \bar{\beta}$, and $\alpha = \bar{\alpha} \circ \Phi, \beta = \bar{\beta} \circ \Phi$ satisfy the thesis.*

Notice how we used radial symmetry to progressively extend our function from a single point to the whole set: this would not have been possible if B was not diffeomorphic to the unit ball. Among all the preparatory propositions this is the only one to relate the manifold Σ to the Euclidean environment, and indeed we see a glimpse of the reason why we need codimension two: later on we will stretch our initial short map along these orthogonal directions in order to properly fit the abstract metric of the manifold inside of space.
The final element that we need to tackle the proof is some way to work with said metrics. In particular, the most efficient way to get into it is through decomposition into primitive metrics:

Definition 3.6 (Primitive metric)**.** *Let Σ be a smooth manifold. A primitive metric h on Σ is a smooth 2-tensor that can be written as $h(x) = a^2(x) d\psi(x) \otimes d\psi(x)$ for some $a, \psi \in C^\infty(\Sigma)$.*

The fact that such controlled objects span the whole set of Riemannian metrics is the object of the following proposition.

Proposition 3.7. *Let Σ be an n-dimensional differentiable manifold, h a Riemannian metric on Σ and $\{U_l\}_l$ an open cover. Then there exists a countable family $\{h_i\}_i$ of primitive metrics such that:*

(a) $h = \sum_{i \in \mathbb{N}} h_i$;

(b) Each h_i is supported in some U_l;

(c) Each point is in the support of at most $K(n) = \frac{n(n+1)^2}{2}$ different h_i's;

(d) The support of any h_i intersects the support of at most finitely many h_j's.

Proof. We will first write h as a sum of primitive metrics in a neighbourhood of $p \in M$ and then refine the family to fit the requirements. Fix a local coordinate map $\varphi : U \to W \subset \mathbb{R}^n$ around p.
Consider the space Sym_n of $n \times n$ symmetric matrices, which contains the convex cone of positive semidefinite (p.s.d.) quadratic forms on $T_{\varphi(p)}W \cong \mathbb{R}^n$, and let $m = n(n+1)/2 = \dim Sym_n$.
Consider $w_1, ..., w_m \in T_{\varphi(p)}W$ such that $\{w_k \otimes w_k\}_k$ generates Sym_n, and define $M = w_1 \otimes w_1 + ... + w_m \otimes w_m \in Sym_n$ which is positive definite. Recall that any p.s.d. matrix A can be written as $A = L_A^T L_A$ for some $L_A \in M_{n\times n}$, and if A is invertible so is L_A. Then if $\varphi_*(h(p)) = N$, we have $L_N^{-T} N L_N^{-1} = I = L_M^{-T} M L_M^{-1}$ since both M and N are positive definite and therefore invertible. Then $N = (L_M^{-1}L_N)^T M (L_M^{-1}L_N) = L^T M L$, and for each term in the decomposition of M we have $L^T(w_k \otimes w_k)L = (Lw_k) \otimes (Lw_k) = v_k \otimes v_k$, so $\varphi_*(h(p)) = N = v_1 \otimes v_1 + ... + v_m \otimes v_m$. The elements $v_i \otimes v_i$ generate all of Sym_n, and the maps $C_i : Sym_n \longrightarrow \mathbb{R}$ (for $i = 1, ..., m$) satisfying $A = \sum_i C_i(A) v_i \otimes v_i$ are smooth.
Define the maps $\psi_i : W \to \mathbb{R}$ as $\psi_i(x) = v_i \cdot x$, so that $d\psi_i(x) = v_i$.
Since for all $x \in W$ the space T_xW is canonically isomorphic to $T_{\varphi(p)}W$, the elements $d\psi_k(x) \otimes d\psi_k(x) = v_k \otimes v_k$ generate the space of positive semidefinite bilinear forms on T_xW. Therefore the maps $\alpha_k(x) = C_k(\varphi_* h(x))$ satisfy $\varphi_* h = \sum_k \alpha_k d\psi_k \otimes d\psi_k$ on W, and since h is smooth, all the α_k are smooth. Since $\alpha_k(\varphi(p)) = 1$ for all k, we can choose a neighbourhood $V_p \subset U$ so that on $\varphi(V_p)$ all the α_k are positive, so $\alpha_k(x) = a_k^2(x)$ for all $x \in \varphi(V_p)$ and the a_k are all smooth. We can finally define maps $\Psi_k = \psi_k \circ \varphi : V_p \to \mathbb{R}$ so that $h(q) = \sum_k a_k^2(\varphi(q)) d\Psi_k(q) \otimes d\Psi_k(q)$ for all $q \in V_p$.

The above method produces a covering $\{V_p\}_{p\in\Sigma}$ where on each V_p a local decomposition of primitive metrics is fixed. Refine the covering $\{V_p\}_p$ to $\{W_l\}_l$ as in Proposition 3.2, and take a partition of the unity $\{\beta_l\}_l$ subordinated to $\{W_l\}$. For each l, choose p such that $W_l \subset V_p$ and assign to W_l the primitive metrics $h_1^l, ..., h_m^l$ defined in V_p. Then if

$$\phi_l = \frac{\beta_l}{\sqrt{\Sigma_k \beta_k^2}}$$

the family of primitive metrics $\{\phi_l^2 h_i^l\}_{l,i}$ satisfies our conditions:
(a) and (b) are clear; since each point is in at most $n+1$ different W_i's, each associated with m primitive metrics, the point is in at most $(n+1)m = n(n+1)^2/2$ supports, so we have (c); in the same way, $\mathrm{supp}(h_i) \subset W_l$ for some l, and W_l intersects a finite number of different W_m's, each with finitely many metrics, so $\mathrm{supp}(h_i)$ intersects finitely many supports. □

Notice how we can once more retain properties of local finiteness, without which the problem of convergence of infinite series would be a serious issue.

Chapter 4

Main proofs

Proposition 4.2 contains most of the real value of the proof. It is in particular Lemma 4.6 that makes the conceptual leap: there we see how we can bend the surface of Σ in order to precisely approximate the metric of Σ with the Euclidean metric; in doing so, we crucially use Proposition 3.4.

Definition 4.1. *On a Riemannian manifold (Σ, g), if $h : \Sigma \to M_{n\times n}$ and $U_l \subset \Sigma$ open, by $\|h\|_{0,U_l}$ we mean the supremum of the Hilbert-Schmidt norm of h over U_l, that is $\|h\|_{0,U_l} = \sup_{p\in U_l} \|h(p)\|$. We also define $\|h\|_0 = \sup_l \|h\|_{0,U_l}$.*

Proposition 4.2. *Let (Σ, g) be a smooth Riemannian manifold and $\{U_l\}_l$ an open cover, and let $w : \Sigma \longrightarrow \mathbb{R}^N$ be a strictly short immersion with $N \geq n+2$. Then, for any choice of positive numbers $\eta_l > 0$ and any $\delta > 0$, there exists a smooth immersion $z : \Sigma \longrightarrow \mathbb{R}^N$ such that :*

$$\|z - w\|_{0,U_l} < \eta_l \tag{4.3}$$

$$\|g - z^{\#}e\|_0 < \delta$$

$$\|Dz - Dw\|_0 < C\sqrt{\|g - w^{\#}e\|_0}$$

for some dimensional constant C. Also, if w is injective, z can be chosen to be injective.

Proof. We can assume $\{U_l\}_l$ to be as in Proposition 3.2, and we call $I(l) = \{j | U_j \cap U_l \neq \varnothing\}$ which is finite. Establish a partition of the unity $\{\varphi_l\}_l$ subordinated to $\{U_l\}_l$. Since each U_l is precompact and $I(l)$ finite, one can choose δ_l small enough so that $(1-\delta_l)g - w^{\#}e > 0$ on U_l and $\|\delta_l g\|_{0,U_j} < \delta/2$ for all $j \in I(l)$.
We set $\varphi = \sum_l \delta_l \varphi_l$ which is well defined and smooth since the sum is locally

finite. Call $h = (1-\varphi)g - w^{\#}e$, so that $g - (h + w^{\#}e) = \varphi g > 0$ and $\|g - (h + w^{\#}e)\|_0 < \delta/2$. Therefore, if we can find $z : \Sigma \longrightarrow \mathbb{R}^N$ such that

$$\|z^{\#}e - (h + w^{\#}e)\|_0 < \delta/2 \tag{4.4}$$

$$\|Dz - Dw\|^2_{0,U_l} < 2K(n)^2\|g - w^{\#}e\|_{0,U_l} \tag{4.5}$$

and (4.3) holds, the triangle inequality gives the thesis. Here $K(n) = n(n+1)^2/2$ is the maximum number of primitive metrics that don't vanish on any given point.
By Proposition 3.7, we can decompose h as a sum of primitive metrics: $h = \Sigma h_i$. We notice that $L(i) = \{j|\text{supp}(h_j) \cap \text{supp}(h_i)\}$ is finite.

Lemma 4.6. *There exists a family of "perturbation" maps $\{w_i^p\}_{i\in\mathbb{N}}$ such that supp(w_i^p) $\subset$ supp(h_i) and if we define $w_i = w + w_1^p + ... + w_i^p = w_{i-1} + w_i^p$ we have, for $i \geq 1$:*

$$\|w_i^p\|_{0,U_l} < \frac{\eta_l}{K(n)} \quad \forall l \in L(i) \tag{4.7}$$

$$\|Dw_i^p\|^2_{0,U_l} < 2\|h\|_{0,U_l} \quad \forall l \in L(i) \tag{4.8}$$

$$\|w_i^{\#}e - (w_{i-1}^{\#}e + h_i)\|_{0,U_l} < \frac{\delta_l}{2K(n)} \quad \forall l \in L(i) \tag{4.9}$$

Also, if w is injective, all w_i's are injective.

Proof. We proceed inductively. Take $w_0 = w$.
Choose $U_m \supset \text{supp}(h_i)$ and apply Proposition 3.4 to U_m with w_{i-1} in place of w as the immersion, to get α, β orthogonal to each other and to the tangent space of $w_{i-1}(U_m)$.
If $h_i(x) = a_i(x)^2 d\psi_i(x) \otimes d\psi_i(x)$ we define

$$w_i^p(x) = a_i(x)\frac{\alpha(x)}{\lambda}\cos\lambda\psi_i(x) + a_i(x)\frac{\beta(x)}{\lambda}\sin\lambda\psi_i(x)$$

for some parameter λ that we will choose later to be large enough. Since $\text{supp}a_i \subset U_m$, w_i^p can be defined as 0 outside of U_m. We now check the required properties:
Since $L(i)$ is finite, if λ is large enough, (4.7) holds.
We compute

$$\frac{d(w_i^p)_h}{dx_k}(x) = -a_i(x)\alpha_h(x)\sin\lambda\psi_i(x)\frac{d\psi_i(x)}{dx_k} + a_i(x)\beta_h(x)\cos\lambda\psi_i(x)\frac{d\psi_i(x)}{dx_k} + E_k(x)$$

Where $E_k(x)$ is proportional to λ^{-1}. Hence

$$Dw_i^p(x) = -a_i(x)\sin\lambda\psi_i(x)\alpha(x)\otimes d\psi_i(x) + a_i(x)\cos\lambda\psi_i(x)\beta(x)\otimes d\psi_i(x) + E(x)$$

where $|E(x)| \lesssim \lambda^{-1}$, by which we mean that $E(x)$ is smaller than λ^{-1} times a constant dependent on the smooth functions $\alpha, \beta, \psi_i, a_i$ but not on λ. We also call A and B the first two terms in Dw_i^p, respectively.
Then we can verify (4.8): if $l \in L(i)$ and $x \in U_l$ we have

$$|Dw_i^p(x)|^2 \leq a_i^2(x)|d\psi_i(x)|^2 + C\lambda^{-1} \leq \|h_i\|_{0,U_l}^2 + C\lambda^{-1} < 2\|h\|_{0,U_l}^2$$

and therefore

$$\|Dw_i^p\|_{0,U_l}^2 \leq 2\|h\|_{0,U_l}^2$$

Again by finiteness of $L(i)$, λ can be chosen large enough for the inequality to hold for all $l \in L(i)$.
We define $w_i = w + w_1^p + ... + w_i^p = w_{i-1} + w_i^p$ and call $\bar{h} = w_i^{\#}e - w_{i-1}^{\#}e$, which in terms of matrices identifies as

$$\bar{h} = Dw_i^T Dw_i - Dw_{i-1}^T Dw_{i-1}$$

Also $Dw_i = Dw_{i-1} + Dw_i^p = Dw_{i-1} + A + B + E$, and notice that, since $a, b \perp T_{w_{i-1}(p)}w_{i-1}(\Sigma)$ for all $p \in \text{supp}h_i$, and $a \perp b$:

$$0 = A^TB = B^TA = A^TDw_{i-1} = Dw_{i-1}^TA = B^TDw_{i-1} = Dw_{i-1}^TB$$

In order to find (4.9), we have to relate h_i to all these quantities:

$$\begin{aligned} A^TA + B^TB &= a(x)^2((\sin\lambda\psi_i(x))^2 + (\cos\lambda\psi_i(x))^2)d\psi_i(x) \otimes d\psi_i(x) = \\ &= a(x)^2 d\psi_i(x) \otimes d\psi_i(x) = h_i \end{aligned}$$

Hence, finally:

$$w_i^{\#}e - w_{i-1}^{\#}e - h_i = Dw_i^TDw_i - Dw_{i-1}^TDw_{i-1} - A^TA - B^TB = F^TE + E^TF + E^TE$$

where $F = Dw_{i-1} + A + B$ does not depend on λ, so for λ large enough we get (4.6):

$$\|w_i^{\#}e - (w_{i-1}^{\#}e + h_i)\| \lesssim \lambda^{-1} < \frac{\delta_l}{2K(n)}$$

We need to show that if w is injective, so is w_i, for λ large enough. We proceed by induction, as $w_0 = w$ is injective. need to show that w_i is also injective, Consider any $p, q \in \Sigma$, and let $\text{supp}(h_i) \subset U_m$. If p, q are both outside of $\text{supp}(h_i)$, then $w_i(p) = w_{i-1}(p) \neq w_{i-1}(q) = w_i(q)$.
If $p \in \text{supp}(h_j)$, and q is not in U_m: since $\text{supp}(h_i)$ is compact, there exists

$\gamma > 0$ such that $|w_{i-1}(x) - w_{i-1}(y)| > \gamma$ for all $x \in \text{supp}(h_j)$ and $y \notin U_m$. Then, if we take λ large so that $\|w_i^p\|_{0,U_l} < \gamma/2$, we have $|w_i(p) - w_i(q)| > |w_{i-1}(p) - w_{i-1}(q)| - |w_i^p(p)| > \gamma - \gamma/2 > 0$ so $w_i(p) \neq w_i(q)$.
The last case is $p \in \text{supp}(h_i)$ and $q \in U_l$. Since $\bar{U}_l$ is compact and w_{i-1} is injective, it is an embedding on $\bar{U}_l$. Then, for a small enough $\eta > 0$, there is a well defined orthogonal projection π from tubular neighbourhood T of $w_{i-1}(U_l)$ of thickness η to $w_{i-1}(U_l)$. For λ large enough $w_i(U_l) \subset T$, so $\pi(w_i(p)) = w_{i-1}(p) \neq w_{i-1}(q) = \pi(w_i(q))$, so $w_i(p) \neq w_i(q)$ and w_i is injective.

□

We can now finish the proof of Proposition 4.2: call $z = w + \Sigma_i w_i^p = \lim_i w_i$, which is well defined since the sum is locally finite, and we have:

$$\|z - w\|_{0,U_l} \leq \Sigma_i \|w_i^p\|_{0,U_l} < \eta_l$$

as at most $K(n)$ of the w_i^p are non-zero on U_l. Moreover

$$\|Dz - Dw\|_{0,U_l} \leq \Sigma_i \|Dw_i^p\|_{0,U_l} < \sqrt{2}K(n)\|h\|_{0,U_l} \leq \sqrt{2}K(n)\|g - w^{\#}e\|_{0,U_l}$$

where we used that $g - (h + w^{\#}e) > 0$, and

$$\begin{aligned} \|z^{\#}e - (w^{\#}e + h)\|_{0,U_l} &= \|\Sigma_i(w_i^{\#}e - w_{i-1}^{\#}e) - \Sigma_i h_i\|_{0,U_l} = \\ &= \|\Sigma_i(w_i^{\#}e - (w_{i-1}^{\#}e + h_i))\|_{0,U_l} < \delta_l/2 \end{aligned}$$

The injectivity is easy: for any $p, q \in \Sigma$, there is some $k \in \mathbb{N}$ such that $z(p) = w_k(p)$ and $z(q) = w_k(q)$ so $z(p) = w_k(p) \neq w_k(q) = z(q)$. □

We are now ready to prove Nash's theorem, which we state once more.

Theorem 2.3: *Let (Σ, g) be a smooth n-dimensional Riemannian manifold and $v : \Sigma \longrightarrow \mathbb{R}^N$ a C^∞ short immersion, with $N \geq n + 2$, such that the limit set of v does not intersect the image of v. Then for any $\varepsilon > 0$ there exists a C^1 isometric immersion $z : \Sigma \longrightarrow \mathbb{R}^n$ such that $\|z - v\|_{C^0} < \varepsilon$.
If v is an embedding, z can be chosen to be an embedding. If v is strictly short, we can also ask for the limit sets of z and v to be the same.*

Proof. We can assume v to be strictly short.
If v is just short, consider $\Phi \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ defined as

$$\Phi(x) = x - \varepsilon \frac{x}{|x|} e^{(-1/|x|)}$$

Notice that $|\nabla\Phi(x)| < 1$ for all $x \in \mathbb{R}^N$, so $\Phi \circ v$ is C^∞ and strictly short.

Also, $|x - \Phi(x)| = \varepsilon e^{-1/|x|} < \varepsilon$, so if we find z approximating $\Phi \circ v$ as in the thesis, we have $\|v - z\|_{C^0} \leq \|v - \Phi \circ v\|_{C^0} + \|\Phi \circ v - z\|_{C^0} < 2\varepsilon$, which gives us the theorem.

Consider a cover $\{U_l\}_l$ as in Proposition 3.2. Each U_l has positive distance d_l from the limit set. We now construct inductively a sequence of maps that will give us our isometry. Start defining $z_0 = v$. For $q \in \mathbb{N}$, set $\eta_l^q = 2^{-q-1} \min\{\varepsilon, d_l, 2^{-l}\}$ and $\delta_q = 4^{-q}$, and find z_q approximating z_{q-1} as in Proposition 4.2. We then have:

1. for $m < n$: $\|z_m - z_n\|_0 \leq \|z_m - z_{m+1}\|_0 + ... + \|z_{n-1} - z_n\|_0 \leq 2^{-m-2}\varepsilon + ... + 2^{-n-1}\varepsilon < 2^{-m}\varepsilon$

2. for $m < n$: $\|Dz_m - Dz_n\|_0 \leq C2^{-m} + ... + C2^{-n+1} < C2^{-m+1}$

Where C was the dimensional constant introduced in Proposition 4.2. Hence z_n is a Cauchy sequence in $C^1(\Sigma, \mathbb{R}^N)$ and converges to a map z. Since the differentials converge, we have $\|g - z^{\#}e\|_0 = \lim_n \|g - z_n^{\#}e\|_0 < \lim_n 4^{-n} = 0$ and so z is an isometry, in particular z has full rank and is therefore an immersion.
Also, $\|v - z\|_{0,U_l} \leq 2^{-l}\Sigma_{q\geq 1}2^{-q-1} = 2^{-l-1}$, so the limit set of z is the same as that of v, and at the same time $\|v - z\|_{0,U_l} \leq \beta_l \Sigma_{q\geq 1}2^{-q-1} = \beta_l/2$, so its image does not intersect said limit set.

Since the limit set of z does not intersect its image and z is already an immersion, it is an embedding if and only if it is injective. To prove that z can be taken injective if v is, we need to slightly adjust our choice of the η_l^q by taking them smaller, then we will repeat the argument above: everything we already found will therefore still be valid. For all $q \in \mathbb{N}$, consider $V_q = \bigcup_{l \leq q} U_l$, and define $2\gamma_i = \min\{|v_i(x) - v_i(y)| : d(x,y) \geq 2^{-i}, x, y \in V_i\}$, where d is the geodesic distance induced by the Riemannian metric g.
We now redefine $\bar{\eta}_l^q = \min\{\eta_l^q, 2^{-q-1}\gamma_1, 2^{-q-1}\gamma_2, ..., 2^{-q-1}\gamma_{q-1}\}$. At each step, Proposition 4.2 guarantees that z_n is injective.
Choose any $x, y \in \Sigma$, and take q such that $d(x,y) > 2^{-q}$ and $x, y \in V_q$. Then $|z(x) - z(y)| \geq |z_q(x) - z_q(y)| - \Sigma_{k\geq q}\|z_{k+1} - z_k\|_{0,V_q} \geq 2\gamma_q - \Sigma_{k\geq q}2^{-k-1}\gamma_q > \gamma_q > 0$. Hence $z(x) \neq z(y)$, this is true for any $x \neq y$ and so z is injective. $\square$

After seeing the proof, we shortly try to convey the idea behind the improvement in Theorem 2.5. Kuiper modified the argument so that Proposition 3.4 is no longer needed, and in particular the requirement of codimension 2 is loosened to codimension 1. Indeed, while Nash "pulls" the map along two directions orthogonal to the manifold, it is possible to move one of such directions to be tangent to Σ, so that on one side one bends the map, on the other side one "strains" the manifold like a rug, and the combined movement manages as in Nash's work to approximate the primitive metrics with the Euclidean one, though requiring much more careful treatment and many sub-steps.

We close this chapter with a direct application that we had anticipated.

Theorem 2.4: *Let (Σ, g) be a Riemannian manifold of dimension n, Then there exist a C^1 isometric embedding $u : \Sigma \longrightarrow \mathbb{R}^{2n+1}$.*

Proof. Our strategy is to first construct a short embedding of Σ into $\mathbb{R}^N$ for some large N, then we will show we can lower N up to $N = 2n + 1$ and finally use Theorem 2.3 to complete the proof. We consider separately the cases of Σ compact and noncompact.

Assume Σ is compact. Consider an atlas $\{(U_i, \phi_i)\}_{i=1,\dots,m}$ such that $B(0, 2) \subset \phi_i(U_i) \subset \mathbb{R}^n$ for all i and the sets $\phi_i^{-1}(B(0, 1))$ cover Σ. Consider $\lambda \in C_c^\infty(\mathbb{R}^n)$ such that $\lambda = 1$ on $B(0, 1)$ and $\mathrm{supp}\lambda \subset B(0, 2)$. Now define the maps $\lambda_i \in C_c^\infty(M)$ as

$$\lambda_i(p) = \lambda \circ \phi_i(p) \quad \forall p \in U_i$$

and $\phi_i(p) = 0$ otherwise. Clearly $\phi_i^{-1}(B(0, 1)) \subset \lambda_i^{-1}(1) = B_i$ so the B_i cover Σ. Then define maps $f_i : \Sigma \to \mathbb{R}^n$ as:

$$f_i(p) = \lambda_i(p)\phi_i(p) \quad \forall p \in U_i$$

and $f_i(p) = 0$ otherwise. Finally set

$$F_i = (f_i, \lambda_i) : \Sigma \to \mathbb{R}^{n+1}$$

and

$$F = (F_i)_{i=1,\dots,m} : \Sigma \to \mathbb{R}^{m(n+1)}$$

Since the sets $B_i = \lambda_i^{-1}(1)$ cover Σ, for each point we can find $U_k \supset B_k \ni p$ so that $f_k|_{B_k} = \phi_k|_{B_k}$, therefore F has injective differential at p. Consider $x, y \in \Sigma$ and $x \neq y$. Then either there exists k such that $x, y \in B_k$, so that $F_k(x) = (\phi_k(x), 1) \neq (\phi_k(y), 1) = F_k(y)$, or there exists k such that $x \in B_k, y \notin B_k$, in which case $(F_k(x))_{n+1} = 1 \neq (F_k(y))_{n+1}$. Therefore, F is injective. Since Σ is compact, that is enough to make F an embedding.
Since the metric g is positive definite and Σ is compact, there exists $\eta > 0$ such that $u^\# e < \eta g$ in the sense of quadratic forms. By defining $z = F/\eta : \Sigma \to \mathbb{R}^{m(n+1)}$ we obtain our short embedding.

Suppose now that Σ is not compact. We prove there exists a short embedding $z : \Sigma \to \mathbb{R}^N$, for $N = (n+1)(n+2)$, with limit set $\{0\} \notin z(\Sigma)$.
Consider an atlas $\{(U_l, \Phi_l)\}_l$ such that $\{U_l\}_l$ is as in Proposition 3.2, in particular there exist disjoint classes $\mathscr{C}_i$ such that $\bigcup_i \mathscr{C}_i = \{U_l\}_l$. Multiplying by constants, we can assume that $|\Phi_l(x)| \leq 1$ for all $x \in U_l$ and for all l. Consider also a collection of C^∞ maps $\{\phi_l\}_l$ with $\mathrm{supp}\phi_l \subset U_l$ and $0 \leq \phi_l \leq 1$, and for each $p \in \Sigma$ there exists l such that $\phi_l = 1$ in a neighbourhood of p. Let $\{\varepsilon_l\}_l$ be a strictly decreasing sequence of positive numbers, that we will choose later. We now define the short immersion z by components as follows: for $i = 1, ..., n+1$

$$z_{(i-1)(n+2)+j}(p) = \sum_{U_l \in \mathscr{C}_i} \varepsilon_l \phi_l(p)(\Phi_l(p))_j \quad j = 1, ..., n$$

$$z_{(i-1)(n+2)+n+1}(p) = \sum_{U_l \in \mathscr{C}_i} \varepsilon_l \phi_l(p)$$

$$z_{i(n+2)} = \sum_{U_l \in \mathscr{C}_i} \varepsilon_l^2 \phi_l(p)$$

All of the sums are locally finite because $\mathrm{supp}(\phi_l) \subset U_l$, so z is well defined. To be more explicit, for $i = 1, ..., n+1$: if p is in no element in $\mathscr{C}_i$, then $z_{(i-1)(n+2)+j}(p) = 0$ for $j = 1, ..., n+2$; if there exists $U_l \in \mathscr{C}_i$ with $p \in U_l$, it must be unique and:

$$z_{(i-1)(n+2)+j}(p) = \varepsilon_l \phi_l(p)(\Phi_l(p))_j \quad j = 1, ..., n$$

$$z_{(i-1)(n+2)+n+1}(p) = \varepsilon_l \phi_l(p)$$

$$z_{i(n+2)} = \varepsilon_l^2 \phi_l(p)$$

We will choose $\varepsilon_l \to 0$, so 0 is the only element in the limit set. Since there exists l such that $\phi_l(x) = 1$ in a neighbourhood of p, and $U_l \in \mathscr{C}_i$, then:

$$\frac{dz_{(i-1)(n+2)+h}}{dx_k}(p) = \varepsilon_l \frac{d(\Phi_l)_h}{dx_k}(p) \quad h, k = 1, ..., n$$

And therefore $D\Phi_l(p)$ has full rank. Also, $z_{(i-1)(n+2)+n+1}(p) = 1$ so $0 \notin z(\Sigma)$. Finally, we check that z is injective. Take any $p, q \in \Sigma$, and $U_l \in \mathscr{C}_i$ such that $\phi_l = 1$ in a neighbourhood of p. Suppose $q \in U_l$, if $\phi_l(q) \neq 1$ then $z_{(i-1)(n+2)+n+1}(p) \neq z_{(i-1)(n+2)+n+1}(q)$, if $\phi_l(q) = 1$ then $\Phi_l(p) \neq \Phi_l(q)$ and so $z_{(i-1)(n+2)+1}(p) \neq z_{(i-1)(n+2)+1}(q)$. If $q \notin U_l$, then either q is in no element in $\mathscr{C}_i$, in which case $0 = z_{i(n+2)}(q) \neq z_{i(n+2)}(p)$, or $q \in U_m \in \mathscr{C}_i$, and we can assume that $m > l$, and so $z_{i(n+2)}(q) = \varepsilon_m^2 \phi_m(q) \leq \varepsilon_m^2 < \varepsilon_l^2 = \varepsilon_l^2 \phi_l(p) =$

$z_{i(n+2)}(p)$.
Since the covering is locally finite and the U_l are precompact, we can choose at each step ε_l such that z is strictly short.

We now need the following Proposition to complete the argument:

Proposition 4.10. *Suppose $M \subset \mathbb{R}^N$ is an n-dimensional submanifold. If $N > 2n+1$, then M can be shortly embedded into $\mathbb{R}^{N-1}$.*

Proof. $\mathbb{R}^{N-1}$ can be identified with any hyperplane $v^\perp = \{w \in \mathbb{R}^N : v \cdot w = 0\}$, therefore if we can find $v \in \mathbb{S}^{N-1}$ such that the projection $f_v : \mathbb{R}^N \to v^\perp = \mathbb{R}^{N-1}$ restricted to M is an embedding the proof will be complete. Since projections are open maps, we only need to check for injectivity of f_v and of Df_v on M.
The map f_v is injective if and only if M has no secant line parallel to v, that is:

$$v \neq \frac{x-y}{|x-y|} \quad \forall x, y \in M \;\; x \neq y \tag{4.11}$$

And for Df_v to be injective, it must be that no tangent vector to M is parallel to v, that is:

$$v \neq \frac{w}{|w|} \quad \forall w \in TM \tag{4.12}$$

We will now define two maps σ and π that encode these requirements.
First consider $\Delta = \{(x,x) \in M^2 : x \in M\}$: Δ is closed in $M \times M$ and therefore $M_2 = M \times M \setminus \Delta \neq \varnothing$ is a $2n$-dimensional smooth submanifold. Then we can define

$$\sigma : M_2 \to \mathbb{S}^{N-1}$$

$$\sigma(x,y) = \frac{x-y}{|x-y|}$$

A vector $v \in \mathbb{S}^{N-1}$ satisfies (4.11) if v is not in the image of σ.
Define $T_1M = \{(x,w) \in TM : |w| = 1\}$. The map

$$s : TM \to \mathbb{R}$$

$$(x,w) \mapsto |w|^2$$

has 1 as a regular value, so $T_1M = s^{-1}(1)$ is a $(2n-1)$-dimensional submanifold of TM. Define

$$\pi : T_1M \to \mathbb{S}^{N-1}$$

$$(x,w) \mapsto w$$

Again, a vector $v \in \mathbb{S}^{N-1}$ satisfies (4.12) if v is not in the image of π.
We now use the following, see chapter 3 of 5 for a proof:

Lemma 4.13. *Let P, Q be two smooth manifolds, and let $u : P \to Q$ be a C^1 map. If dim(P)<dim(Q), then the complement of the image of u is dense in Q.*

First consider $P = M_2$, $Q = \mathbb{S}^{N-1}$ and $u = \sigma$: since $\dim(M_2) = 2n < N-1 = \dim(\mathbb{S}^{N-1})$ we get that $\sigma(M_2)^c$ is dense in $\mathbb{S}^{N-1}$. In the same way, since $\dim(T_1M) = 2n-1 < N-1 = \dim(\mathbb{S}^{N-1})$, we have that $\pi(T_1M)^c$ is dense in $\mathbb{S}^{N-1}$.
Consider now $\{\Gamma_k^1\}_k$ an exhaustion by compact sets of M_2. Since σ is continuous $G_k^1 = \sigma(\Gamma_k^1)$ is closed in $\mathbb{S}^{N-1}$ for all k, and since $G_k \subset \sigma(M_2)$, $(G_k^1)^c$ is dense in $\mathbb{S}^{N-1}$ for all k. Identically, consider $\{\Gamma_k^2\}_k$ an exhaustion by compact sets of T_1M: then $G_k^2 = \pi(\Gamma_k^2)$ is closed and its complement is dense in $\mathbb{S}^{N-1}$ for all k. But then by Baire's Category Theorem, since $\{G_k^i\}_{i=1,2;k\in\mathbb{N}}$ is a countable family of closed sets with dense complements, and since $\mathbb{S}^{N-1}$ is a complete metric space, we have that $\bigcup_{i,k} G_k^i$ has dense complement in $\mathbb{S}^{N-1}$, and in particular $\mathbb{S}^{N-1} \setminus \bigcup_{i,k} G_k^i$ is nonempty. But $\sigma(M_2) = \bigcup_k G_k^1$ and $\pi(T_1M) = \bigcup_k G_k^2$ and therefore $\sigma(M_2) \cup \pi(T_1M)$ has non empty complement. Then if we take any $v \in \mathbb{S}^{N-1} \setminus \sigma(M_2) \cup \pi(T_1M)$ our requirements are satisfied. □

We still need a modification for our situation:

Corollary 4.14. *Suppose $M \subset B_N(0,R) \subset \mathbb{R}^N$ is an n-dimensional submanifold such that the limit set of M is $\{a\} \subset \mathbb{R}^N \setminus M$. If $N > 2n+1$, there exists a short embedding $u : M \to B_{N-1}(0,R) \subset \mathbb{R}^{N-1}$ with limit set a single point $u(a) \notin u(M)$.*

Proof. The have the same requirements as Proposition 4.10, we just added another condition:

$$v \neq \frac{x-a}{|x-a|} \qquad \forall x \in M$$

indeed if $y_n = u(x_n)$ is a sequence of points in $u(M)$ converging to some point in the limit set of $u(M)$, then $(x_n)_n \subset B_N(0,R)$ is bounded and therefore there exists a convergent subsequence $x_{n_k} \to b$. But then b is in the limit set of M, hence $a = b$ and $u(x_n) \to u(a)$ so the limit set of $u(M)$ is just $u(a)$, and the only condition is, therefore, $u(a) \notin u(M)$.
We can define the map $\rho : M \to \mathbb{S}^{N-1}$ as

$$\rho(x) = \frac{x-a}{|x-a|} \qquad \forall x \in M.$$

Since the map $\rho : M \to \mathbb{S}^{N-1}$ is differentiable on M and $\dim(M) = n < N-1 = \dim(\mathbb{S}^{N-1})$ we can apply Lemma 4.13 and see that $\rho(M)^c$ is dense in $\mathbb{S}^{N-1}$.

The argument then repeats as before: fixed exhaustions by compact sets of M_2, T_1M, M their images are three countable collections of closed nowhere dense subsets of $\mathbb{S}^{N-1}$ so by Baire's Category theorem their union is nowhere dense and thus there exists some $v \in \mathbb{S}^{-1}$ satisfying all three conditions. Also, since $f_v(B_N(0,R)) = B_{N-1}(0,R)$ we retain the boundedness of $f_v(M)$. $\square$

We go back to the Theorem 2.4: we have already built a short embedding z of Σ in $\mathbb{R}^N$ for some large N. Notice that in the non-compact case, since all the components of z are bounded by $\varepsilon_l < 1$, we have that $z(\Sigma) \subset B(0,N)$. In the compact case, as long as $N > 2n+1$, by Proposition 4.10 we can find a short embedding f of $z(\Sigma)$ onto $\mathbb{R}^{N-1}$, so that $f \circ z : \Sigma \to \mathbb{R}^{N-1}$ is a short embedding. Iterating we finally find a short embedding $\bar{z} : \Sigma \to \mathbb{R}^{2n+1}$. In the non compact case, by Corollary 4.14 we can find a short embedding f pf $z(\Sigma)$ into $\mathbb{R}^{N-1}$ such that the hypotheses for Corollary 4.14 are satisfied also for $f \circ z(\Sigma)$, again by iteration we find again a short embedding $\bar{z} : \Sigma \to \mathbb{R}^{2n+1}$. By Theorem 2.3, there exists a C^1 isometric embedding $u : \Sigma \to \mathbb{R}^{2n+1}$, and the proof is complete. $\square$

Chapter 5

The flat Torus

Consider the equivalence relation in $\mathbb{R}^2$: $x \sim y$ if and only if $x - y \in 2\pi\mathbb{Z}^2$.

Definition 5.1. *The Flat torus is the* $2D$*-surface defined as*

$$\mathbb{T} = \mathbb{R}^2 / \sim$$

equipped with the induced metric from $\mathbb{R}^2$.

Remark 5.2. *The flat torus has Gaussian curvature* 0 *in any point. Indeed, consider the submanifold* Σ *obtained by removing the boundary of* $[0, 2\pi]^2$ *with the induced metric. Then* $\Sigma = (-\pi, \pi)^2$ *and it smoothly and isometrically embeds in* $\mathbb{R}^2$. *Clearly the curvature, in the* $\mathbb{T}$ *metric, of any point in* $\mathbb{T} \cap \Sigma$ *is the same as the curvature in the* Σ *metric, which is* 0 *due to the Theorema Egregium. By continuity, the Gaussian curvature is* 0 *in all points of the flat torus.*

Lemma 5.3. *Let* $\Sigma \subset \mathbb{R}^3$ *be a* 2*-dimensional* C^2 *compact submanifold. Then* Σ *has a point with positive Gaussian curvature.*

Proof. Since Σ is compact, there exists $p \in \Sigma$ such that $|p| = \max_{q\in\Sigma}|q|$. By rotating the coordinate axes, we can assume that $p = -Me_3$, in particular $\Sigma \subset \{q \in \mathbb{R}^3 : |q| \leq M\} = B(0, M)$ and p is a contact point. Also, $T_p\Sigma = \{z = -M\}$, indeed the function $\phi(q) = -q \cdot e_3$ has a maximum in p, as $|\phi(q)| \leq |q| \leq |p| = \phi(p)$Thefore $T_p\Sigma \perp \nabla\phi(p) = -e_3$.
Since $T_p\Sigma \perp e_3$, we can find $V \ni p$ open in Σ small enough so that V is the graph of a function of (x, y), that is: there exist $U \subset \mathbb{R}^2$ open and $\Phi : U \longrightarrow \mathbb{R}$ such that $q = (x, y, z) = (x, y, \Phi(x, y))$ for all $q \in V$. In particular, close to p, we have

$$\Phi(x, y) = \Phi(0, 0) + \langle \nabla\Phi(0, 0)|(x, y)\rangle + 1/2 \langle H\Phi(0, 0)(x, y)|(x, y)\rangle + o(|(x, y)|^2) =$$

$$= -M + 1/2 \langle H\Phi(0,0)(x,y)| \, , |(x,y)\rangle + o(|(x,y)|^2)$$

But notice that since $V \subset \Sigma \subset B(0,M)$, we have

$$\Phi(x,y) > -\sqrt{M^2 - (x^2+y^2)} = -M\sqrt{1-(x^2+y^2)/M^2} =$$

$$= -M + 1/M \langle \mathbf{I}(x,y)|(x,y)\rangle + o(|(x,y)|)^2$$

. Therefore $1/2 \langle H\Phi(0,0)(x,y)|(x,y)\rangle \geq 1/M \langle \mathbf{I}(x,y)|(x,y)\rangle$ for all (x,y) small and so in particular $\det(\Phi(0,0)) > 0$. But apart from positive normalization factors, $\det(H\Phi(0,0))$ is the Gaussian curvature of Σ at p, which is therefore positive. □

This applies directly to our case:

Corollary 5.4. *There can be no C^2 isometric embedding of the Flat torus into $\mathbb{R}^3$.*

Proof. Suppose there exists $u : \mathbb{T} \longrightarrow \mathbb{R}^3$ isometric embedding and $u \in C^2$. Then $u(\Sigma) \subset \mathbb{R}^3$ is a 2-dimensional C^2 submanifold, and therefore there exists $x = u(p) \in u(\Sigma)$ such that the gaussian curvature of $u(\Sigma)$ at x is positive. But since u is a C^2 isometry, by the Theorema Egregium, the Gaussian curvature of $u(\Sigma)$ at x is the same as that of Σ at p, which is therefore positive. But the flat torus has vanishing curvature in all of its points, so this is impossible, and such u cannot exist. □

Let's try some formal computation:
Since $\mathbb{T} = \mathbb{R}^2/\sim$, by using the induced coordinates, any map $f \in C^1(\mathbb{T}, \mathbb{R}^3)$ has a Fourier series expansion:

$$f(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny}$$

where $a_{n,m} \in \mathbb{C}^n$ are the Fourier coefficients of f.
At the same time, the directional derivatives satisfy $D_x f, D_y f \in C^0(\Sigma, \mathbb{R}^3) \subset (L^2(\mathbb{T}))^3$, they also have their Fourier series:

$$D_x f(x,y) = \sum_{m,n\in\mathbb{Z}} m a_{m,n} e^{imx} e^{iny}$$

$$D_y f(x,y) = \sum_{m,n\in\mathbb{Z}} n a_{m,n} e^{imx} e^{iny}$$

We now impose that f is an isometry.
Since f is real valued , we get that $a_{m,n} = \bar{a}_{-m,-n}$.

The first fundamental form of $\mathbb{T}$ is $\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$, so f is an isometry if and only if:

$$|D_x f(x,y)| = |D_y f(x,y)| = 1 \qquad \langle D_x f(x,y) | D_y f(x,y) \rangle = 0 \qquad \forall x, y \in \mathbb{T}$$

We can expand each condition:

$$\begin{aligned}
1 &= \langle D_x f(x,y) | D_x f(x,y) \rangle \\
&= \left\langle \sum_{n,m \in \mathbb{Z}} n a_{n,m} e^{inx} e^{imy} \middle| \sum_{\mu,\nu \in \mathbb{Z}} \mu a_{\mu,\nu} e^{i\mu x} e^{i\nu y} \right\rangle \\
&= \sum_{m,n,\mu,\nu \in \mathbb{Z}} m\mu a_{m,n} \cdot \bar{a}_{\mu,\nu} e^{i(m-\mu)x} e^{i(n-\nu)y} \\
&= \sum_{m,n,\mu,\nu \in \mathbb{Z}} m\mu a_{m,n} \cdot a_{-\mu,-\nu} e^{i(m-\mu)x} e^{i(n-\nu)y} \\
&= \sum_{m,n,M,N \in \mathbb{Z}} m(m-M) a_{m,n} \cdot a_{M-m,N-n} e^{iMx} e^{iNy}
\end{aligned}$$

and since this has to be identically true, we can decompose into the equations:

$$\begin{aligned}
\sum_{m,n \in \mathbb{Z}} a_{m,n} \cdot a_{-m,-n} m^2 &= -1 \\
\sum_{m,n \in \mathbb{Z}} a_{m,n} \cdot a_{M-m,N-n} m(M-m) &= 0 \quad \forall (M,N) \in \mathbb{Z}^2 \setminus \{(0,0)\}.
\end{aligned}$$

Repeating identical computation with the other conditions yields:

$$\begin{aligned}
\sum_{m,n \in \mathbb{Z}} a_{m,n} \cdot a_{-m,-n} n^2 &= -1 \\
\sum_{m,n \in \mathbb{Z}} a_{m,n} \cdot a_{M-m,N-n} n(N-n) &= 0 \quad \forall (M,N) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\
\sum_{m,n \in \mathbb{Z}} a_{m,n} \cdot a_{M-m,N-n} m(N-n) &= 0 \quad \forall (M,N) \in \mathbb{Z}^2.
\end{aligned}$$

We possess some insight about the solutions of this set of equations: Theorem 2.5 guarantees that a solution exists, as one might start from the classical Torus in $\mathbb{R}^3$ and construct a C^1 isometric embedding; at the same time Corollary 5.4 shows that no C^2 solutions exist. These translate into different conditions for the decay of the coefficients $(a_{m,n})_{(m,n)}$, but most importantly we are interested in finding a solution such that $a_{m,n}$ decays as $(|m| + |n|)^{-(2+\alpha+\varepsilon)}$, for some $\alpha \in (0,1)$ and $\varepsilon > 0$, as that would guarantee the existence of a $C^{1,\alpha}$ isometric embedding, putting a lower bound on rigidity theorems in general.

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