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Cosmological gravitational waves in the limit of geometric optics

Thesis supervisor

Dr. Daniele Bertacca

Thesis co-supervisor

Prof. Sabino Matarrese

Candidate

Sujata Coccato

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Abstract

Detecting gravitational waves (GWs) propagating through cosmic structures can provide valuable information about the geometry and contents of our Universe, opening a completely new window for observational astrophysics. In order to carry out astrophysical and cosmological studies it is important to have a precise formalism for using GW observations. In this thesis we will consider GWs traveling through a perturbed FRW background and work with the geometric optics approximation. In particular, by observing the effect of cosmological perturbations on the GW waveform associated with a merging binary system, we calculate the correction due to the tensor contribution when estimating the luminosity distance anisotropies. Specifically, we compute the signatures left on the GW signal by primordial GWs and analytically derive their signature on the angular power spectrum associated with the relative correction to the luminosity distance.

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Chapter 1

Introduction

On 14 September 2015 for the first time gravitational waves were detected by the two Advanced LIGO detectors [1], a century after Albert Einstein predicted their existence. This first direct observation, named GW150914, and the following made by the LIGO-VIRGO collaborations opened a new chapter in astrophysics and cosmology.

Besides increasing our knowledge about the astrophysical sources which produced them and in general opening a new window for observational astrophysics (see e.g. [2–8]), GWs can be used to test cosmological models. For instance, on 17 August 2017, the joint observation of a gravitational wave signal [9], known as GW170817, and electromagnetic waves from the same astrophysical object, a binary neutron star merger, marked the beginning of a new era of multi-messenger astronomy. A detection of this type can be used to constrain the ratio between the GW velocity and the speed of light, providing a way to test general relativity. In the case of the GW170817 event and the associated gamma ray burst GRB170817A, the GW speed was fixed with high accuracy [10], $|\frac{v_{GW}-c}{c}| < 10^{-15}$, translating in the exclusion of many modified gravity theories. Other papers which show how to test general relativity and modified gravity theories through gravitational waves are given for example by [11–18].

Multi-messenger observations as GW170817 can be used as "standard sirens" [19] to measure the Hubble constant, which describes the expansion rate of the Universe. Standard sirens are the gravitational wave analog to standard candles: GW sources from which we can obtain a direct measurement of the luminosity distance. The GW170817 event provides an example of bright siren [20], which is a GW source that produces a detectable electromagnetic counterpart from which we can deduce the redshift. In the absence of an electromagnetic counterpart alternative methods can be employed in order to infer the source redshift, such as correlating galaxy catalogues with the inferred position of the GW source (see for example [21]). In this case GW sources are referred as dark sirens.

Besides GWs of astrophysical origin, gravitational waves produced in the early Universe are of great importance for cosmology [22–26]. All inflationary models predict a background of gravitational waves due to quantum tensor fluctuations. As a consequence primordial gravitational waves are considered a smoking-gun for inflation. Moreover, depending on the inflationary model which we consider, the features of the signal change, making possible to distinguish among different scenarios. There are models which, besides the quantum fluctuation of the gravitational field, predict additional mechanisms of primordial GW production, resulting in specific signatures.

In addition to the stochastic gravitational wave background of cosmic origin, it must be taken into account that the superposition of a large number of signals from unresolved astrophysical sources, too far or too faint to be detected separately, produces a stochastic gravitational wave background (ASGWB) [27–29]. From the detection of this background of astrophysical origin we can gain further information about the properties of the compact objects which generated it. Since the stochastic gravitational wave background is given by the combination of contributions of cosmic and astrophysical origin, it becomes necessary to develop techniques to disentangle these two contributions in order to extract precise information.

The number of gravitational wave detectors is increasing. Present detectors include the current ground-based interferometers, such as aLIGO/VIRGO/KAGRA collaboration, and the PTA collaboration (NANOGrav, EPTA/InPTA, PPTA, and CPTA), which recently detected a low frequency stochastic gravitational wave background (SGWB) [30–33]. Future detectors include both ground-based interferometers, such as Einstein Telescope (ET) [34–36] and Cosmic Explorer [37,38], and space-based detectors such as Laser Interferometer Space Antenna (LISA) [39], DECi-hertz Interferometer Gravitational Wave Observatory (DECIGO) [40] and Big Bang Observer (BBO) [41]. With the upcoming detectors the precision of the measurements will increase, opening an era of precise GW cosmology.

As a consequence it becomes necessary to develop precise formalisms in order to use GWs to carry out detailed astrophysical and cosmological studies. Early studies which started considering the effects of cosmological perturbations on the propagation of gravitational waves are [42] and [43], which analyzed the Integrated Sachs-Wolfe effect (ISW) respectively on the signal coming from supermassive black hole binaries and in the study of the anisotropies of the gravitational wave background. The effect of lensing magnification was treated in [44], while [45] considered the corrections on the GW signal due to environmental effects.

In this context, in this thesis we will drop the assumption of an unperturbed FRW universe and include cosmological perturbations. Precisely, we will calculate the corrections to the estimate of the luminosity distance of a merging binary system taking into consideration the effects of these cosmological perturbations on the propagation of the gravitational wave signal produced by the source. We will proceed as in [46]. With respect to it, in which the amplitude and the phase are calculated in the Poisson gauge, we will work in a general gauge. Moreover we will consider the perturbations at the observer. The original contribution of this thesis consists in evaluating, in addition to the scalar and vector contributions, the corrections to the luminosity distance due to tensor contributions. Actually the imprint left by the gravitational waves represented by these tensor perturbations can in principle provide information about them and therefore constitute a complementary probe of primordial gravitational waves. Thus we will calculate the analytical expression for the tensor contribution to the angular power spectrum associated with the correction to the luminosity distance and relate it to the primordial tensor power spectrum.

The thesis is organized as follows.

In this introductory chapter we will give a brief overview on cosmology, focusing the attention on the physical quantities and equations which will be used in the following chapters of the thesis. Furthermore we will briefly summarize the types of GW signals, concentrating on the GWs produced by a binary system made of two compact objects.

In Chapters 2 and 3 we will describe the Isaacson's geometric optics approximation for gravitational waves and the *Cosmic Rulers* formalism, which are used to study the propagation of gravitational waves over cosmological distances. By using the geometric optics approximation we assume that the metric can be written as $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$, where the small perturbation $h_{\mu\nu}$ represents the gravitational waves, which are characterized by a short wavelength and propagate over a curved background described by the metric $\tilde{g}_{\mu\nu}$ which varies on larger scales. The background metric $\tilde{g}_{\mu\nu}$ will be additionally split into the metric associated to a homogeneous and isotropic universe and first order perturbations which describe the large-scale structure (LSS) of the Universe. We will demonstrate that in the geometric optics limit gravitational waves travel on null geodesics of the background $\tilde{g}_{\mu\nu}$. Consequently the *Cosmic Rulers* formalism, initially introduced for the electromagnetic radiation, can be extended to gravitational waves. We will describe the *Redshift-GW frame* (RGW), used as reference system, and the $real frame$. Then we will see how to set a map between the two frames by decomposing each physical quantity of the real frame into a zero order contribution, given by the solution in the RGW frame, and a first order perturbation due to cosmic inhomogeneities. Chapter 2 will be devoted to the geometric optics approximation, while in Chapter 3 the attention will be focused on the Cosmic rulers formalism, in particular on the calculation of the wave-vector and geodesic perturbations in terms of the metric perturbations.

The effects of the LSS on the GW phase and amplitude will be analyzed in Chapter 4.

In Chapter 5 we will analytically derive in terms of scalar, vector and tensor perturbations the relative correction to the luminosity distance and calculate the angular power spectrum associated with it.

1.1 Sources and types of GW signals

In this section we provide a brief description of the different types of GW signals. Known GW sources span a frequency region of many orders of magnitude. As we can see in figure 1.1 different types of GW detectors are necessary to observe the entire GW spectrum: each instrument is designed to detect a specific frequency range. For example space-based detectors

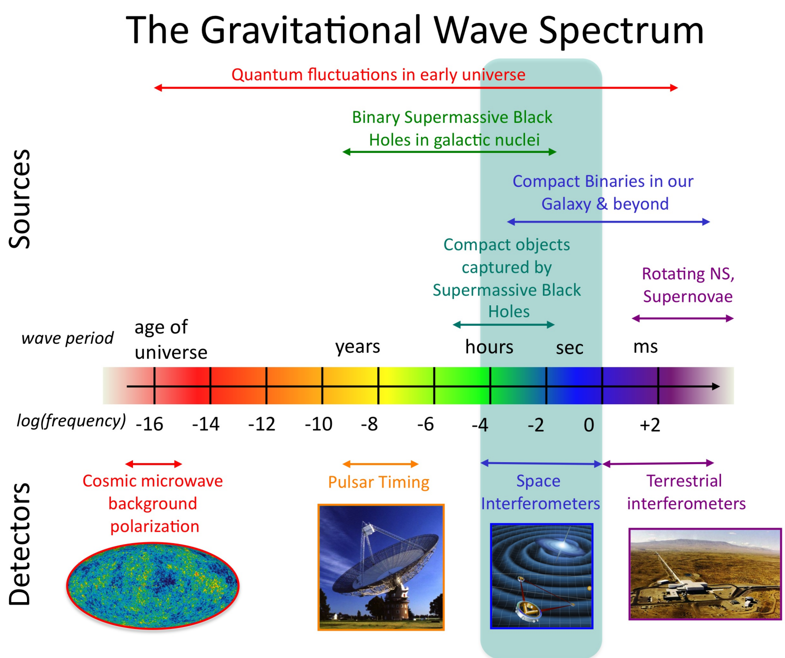

Figure 1.1: The GW spectrum and the associated sources and detectors. Figure credit: NASA Goddard Space Flight Center https://science.gsfc.nasa.gov/663/research/

like LISA are aimed at detecting gravitational waves with frequencies between 0.1 Hz and 1 Hz (the region highlighted in figure 1.1), quite lower in comparison with ground-based detectors. Gravitational waves observed by LISA could come from extreme mass ratio inspirals (EMRI), systems which consist of a stellar mass compact object orbiting around a massive black hole. GW sources that could be observed by LISA include also binary black holes in the early inspiral phase. Some of these events could become multi-band events if later detected by ground-based interferometers. Given that pulsar timing arrays operate at frequencies of the order of 10^{-9} Hz, they are sensitive to different types of sources with respect to the other GW detectors. A possible source of the low frequency stochastic gravitational wave background recently detected by the PTA collaboration is a population of supermassive black hole binaries which form in galaxy mergers and are distributed throughout the Universe. However there are alternative cosmological interpretations of the origin of the signal (see e.g. [47–49] for possible interpretations, both cosmological or astrophysical).

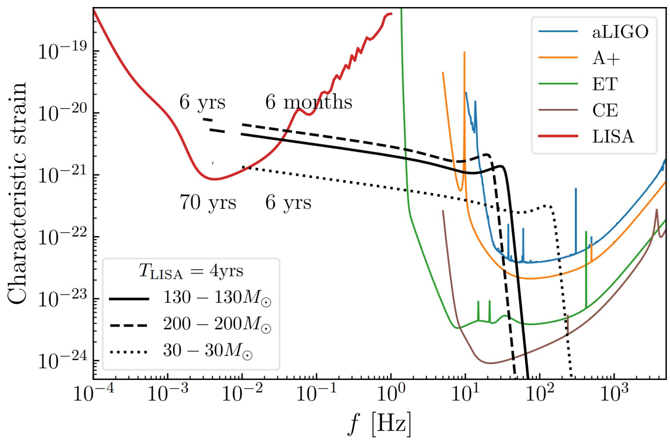

Figure 1.2: Examples of signals which could be observed by LISA and other ground-based detectors. The times indicate the time to merge. [50]

We proceed by classifying the different kinds of signals. The main distinction is between astrophysical gravitational waves, which, as the name suggests, are produced by astrophysical sources, and cosmological gravitational waves, whose sources are early Universe mechanisms.

1.1.1 Astrophysical sources

A classification of the astrophysical GW sources can be based on the type of signal they emit (see e.g. [51]). We can distinguish three kinds of signals: transient, continuous and stochastic.

Transient signal

Transient signals last for a relatively small amount of time in the detector bandwidth. This means that we are including both intrinsically short events and signals which can be observed only for a limited amount of time by the detector, given that it is sensitive only to a specific frequency range. As an example current ground-based interferometers cannot access the frequency region below 10 Hz because of seismic and Newtonian noise. This implies that they can observe only the final stages of binary inspirals.

Transient signals can be further divided in

- modelled signals, such as compact binaries close to coalescence; in this case we have a detailed knowledge of the shape of the signal in terms of a limited number of source parameters;

- GW bursts, which are not well modelled signals, such as supernova explosions; in this case we have no precise description of the shape of the signal, we are only able to make assumptions, imposing for instance constraints on the total duration of the signal, typically less than a second, and on the frequency band where the power is concentrated.

Continuous signal

Continuous signals refer to long-lasting signals that are present for the entire available time of observation, which can be of years. As the time of observation increases the signal-to-noise ratio increases. These signals can be emitted for example by non-axisymmetric spinning neutron stars, whose asymmetry could be due to imperfections in the spherical shape of the surface. This type of periodic source emits a quasi-monochromatic signal: intrinsic variation of the frequency of the source and modulation effects due to the motion of the Earth must be taken into account.

If we consider future space-based detectors another example of continuous GW signals can be provided by the early stages of binary inspirals.

Astrophysical stochastic background

This stochastic background is due to the superposition of a large number of signals from unresolved astrophysical sources, too far or too faint to be detected separately.

1.1.2 Cosmological GW

As already explained in the introduction, in this case we area talking about the stochastic gravitational wave background generated by processes active in the early Universe. Cosmological gravitational waves are predicted by any model of inflation. Besides it there are additional mechanisms which can produce gravitational waves in the early Universe, resulting in specific features of the signal.

1.2 Unperturbed FRW Universe

This section, in which we give some basic definitions used in cosmology, is mostly based on [52].

The real physical Universe has structures: we observe galaxies, filaments and walls, cluster and superclusters of galaxies, voids. These structures formed from initial small inhomogeneities in the energy density set at end of inflation, a period of accelerated expansion during the early universe. These initial perturbations grew by gravitational instability, leading to the large scale structures we observe today. Only on very large scales, above 100 Mpc, the Universe can be considered on average homogeneous and isotropic and therefore can be described by the Friedman-Robertson-Walker (FRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right], \tag{1.1}$$

where t is the cosmic time, $a(t)$ is the scale factor, r, θ and ϕ are the comoving spherical coordinates and k is the curvature parameter.

The evolution in time of the scale factor $a(t)$ is obtained by the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,. \tag{1.2}$$

If we consider a perfect fluid the stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{1.3}$$

where $g_{\mu\nu}$ is the metric tensor, ρ the density, P the pressure and u_μ is the 4-velocity of the fluid elements. Inserting (1.1) and (1.3) in the Einstein's equations we obtain the Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{1.4a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)\,, \tag{1.4b}$$

where the dot indicates the derivative with respect to the cosmic time t and $H = \dot{a}/a$ is the Hubble parameter.

1.2.1 The Hubble parameter

If we assume a ΛCDM cosmology, the Hubble parameter is a function of the matter (ρ_m), radiation (ρ_r) and dark energy (ρ_Λ) content of the Universe:

$$H^2(a) = H_0^2\left[\Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k,0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right], \tag{1.5}$$

where $\rho_{crit,0} = 3H_0^2/8\pi G$ is the critical density today, $\Omega_{i,0} = \rho_{i,0}/\rho_{crit,0}$ and $\Omega_{k,0} = -k/(a_0H_0)^2$ is the curvature density parameter. The last expression can be obtained from the first Friedmann equation (1.4a).

Taking into account that we can neglect radiation at present time we have

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z^2) + \Omega_\lambda}\,, \tag{1.6}$$

where z is the redshift and, as in literature, we dropped the lower index 0.

1.2.2 Redshift and comoving distance

The redshift of a luminous source is given by

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}\,, \tag{1.7}$$

where λ_e is the wavelength of radiation at emission time t_e and λ_0 is the wavelength of the light received at time t_0. The difference between λ_0 and λ_e is due to the expansion of the Universe.

The relation between the redshift z and the scale factor a is given by

$$1 + z = \frac{a_0}{a}\,, \tag{1.8}$$

where $a = a(t_e)$ and $a_0 = a(t_0) = 1$. The comoving distance from the light source to the observer is therefore given by

$$\chi = \int_{t_e}^{t_0}\frac{\mathrm{d}t}{a(t)} = \int_{a_e}^{1}\frac{\mathrm{d}a}{a^2H(a)} = \int_0^{z_e}\frac{\mathrm{d}z}{H(z)}\,. \tag{1.9}$$

1.2.3 Luminosity distance

The luminosity distance $\mathcal{D}_L$ is defined by

$$F = \frac{L}{4\pi\mathcal{D}_L^2}\,, \tag{1.10}$$

where F is the observed flux (the power received per unit area by the observer) at t_0 and L is the luminosity of the source (the energy emitted per second).

In a FRW Universe, if the source is located at the comoving distance χ, the flux is given by

$$F = \frac{L}{4\pi a_0^2\chi^2}\left(\frac{a}{a_0}\right)^2. \tag{1.11}$$

The last expression can be explained in the following way. The area of a sphere which at time t_0 is centred on the source and passes through the Earth is given by $4\pi a_0^2\chi^2$. The photons which are received are redshifted by a factor a/a_0 because of the expansion of the Universe. Moreover the photons emitted in an interval δt arrive at the observer in an interval $\delta t_0 = (a_0/a)\delta t$. Therefore, comparing (1.11) with (1.10) we obtain

$$\mathcal{D}_L = \frac{a_0^2}{a}\chi\,. \tag{1.12}$$

Consequently the luminosity distance in a FRW Universe is found to be

$$\mathcal{D}_L = (1+z)\int_0^z \frac{\mathrm{d}z'}{H(z')}\,, \tag{1.13}$$

where H is the Hubble parameter (1.5).
For $z \ll 1$ the last expression reduces to the Hubble law

$$z \simeq H_0 \mathcal{D}_L\,. \tag{1.14}$$

The Hubble parameter today, H_0, is called the Hubble constant. From the measurements of H_0 at early and late cosmological times emerged two sets values. This discrepancy goes under the name of the Hubble tension (see e.g. [53]). Using bright or dark sirens provides a third independent way to measure the Hubble constant. From the first bright siren observed by the LIGO-VIRGO collaboration, GW170817, it was inferred a value of $70^{+12}_{-8}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [20], but it was not precise enough given the larger error bars with respect to the other measurements.

If we consider higher redshifts, we can see from (1.13) and (1.5) that the luminosity distance encodes information about cosmic expansion at early epochs.

1.3 Coalescing compact binaries as standard sirens

In this section we see how compact binaries can be considered as standard sirens: GW sources from which we can obtain a direct measurement of the luminosity distance. We will calculate the GW waveform of the signal emitted by these objects and see how it depends on the redshifted chirp mass and the luminosity distance. This part is mostly based on [51].

1.3.1 Quadrupole radiation

We start by briefly summarizing the formulas necessary for describing the emission of gravitational waves by a binary system made of two compact objects. If we consider the quadrupole approximation, the expression for the emission of gravitational waves in the TT gauge is given by

$$\left[h_{ij}^{TT}(t,\mathbf{x})\right]_{quad} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}\left(t-\frac{r}{c}\right), \tag{1.15}$$

where

$$Q^{ij} = I^{ij} - \frac{1}{3}\delta^{ij} I_{kk} = \int \mathrm{d}^3x\,\rho(t,\mathbf{x})\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right) \tag{1.16}$$

is the reduced quadruple moment,

$$I^{ij} = \frac{1}{c^2}\int \mathrm{d}^3x\,T^{00}(t,\mathbf{x})x^i x^j\,, \tag{1.17}$$

is the second moment of the mass distribution, I_{kk} is the trace of I_{ij} and r is the distance from the source.

In order to analyze the evolution of the system due to the emission of gravitational waves we will need the total radiated power, which in the quadrupole approximation is given by

$$\frac{\mathrm{d}E_{gw}}{\mathrm{d}t} = \frac{G}{5c^2}\left\langle \dddot{I}_{ij}\dddot{I}_{ij} - \frac{1}{3}(\dddot{I}_{kk})^2\right\rangle, \tag{1.18}$$

where $\langle ... \rangle$ denotes an average over many periods of the GW and $\dddot{I}_{ij}$ are evaluated at the retarded time $t - r/c$.

1.3.2 Spiralling of a compact binary

We consider a binary system made of two compact objects. We use the Newtonian approximation to describe its dynamics. The compact objects are treated as point-like and their masses are denoted by m_1 and m_2. We assume they move on a circular Keplerian orbit. The orbital angular velocity ω_s is given by Kepler's law

$$\omega_s^2 = \frac{GM}{R^3}\,, \tag{1.19}$$

where R is the orbital separation and $M = m_1 + m_2$ the total mass.

Neglecting, for the moment, the back-reaction on the binary system due to the emission of gravitational waves, we calculate h_+ and $h_\times$. The second moment of the mass distribution for the binary system taken into account is given by

$$I_{jk} = m_1 x_1^j x_1^k + m_2 x_2^j x_2^k\,. \tag{1.20}$$

Choosing a reference frame (x, y, z) so that the two particles are in the xy plane and inserting their positions in (1.20), we obtain

$$I_{xx} = \mu R^2 \sin^2(\omega_s t)\,, \qquad I_{yy} = \mu R^2 \cos^2(\omega_s t)\,, \qquad I_{xy} = I_{yx} = \mu R^2 \cos(\omega_s t)\sin(\omega_s t)\,, \tag{1.21}$$

where $\mu = (m_1 m_2)/(m_1 + m_2)$ is the reduced mass.

Given the generic direction of propagation

$$\hat{\mathbf{n}} = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta) \tag{1.22}$$

we obtain

$$h_+(t) = \frac{4}{r}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{gw}}{c}\right)^{\frac{2}{3}}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{gw} t_{ret} + 2\phi)\,, \tag{1.23a}$$

$$h_\times(t) = \frac{4}{r}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{gw}}{c}\right)^{\frac{2}{3}}\cos\theta\sin(2\pi f_{gw} t_{ret} + 2\phi)\,, \tag{1.23b}$$

where $f_{gw} = 2\omega_s/(2\pi)$, r is the distance from the source, t_{ret} is the retarded time $t - r/c$ and

$$\mathcal{M}_c = \mu^{\frac{3}{5}} M^{\frac{2}{5}} = \frac{(m_1 m_2)^{\frac{3}{5}}}{(m_1 + m_2)^{\frac{1}{5}}} \tag{1.24}$$

is the chirp mass, a key quantity to describe the evolution of the system, as we will see now.

The next step consists in considering the evolution of the binary system due to the emission of gravitational waves. In order to find the radiated power we insert (1.21) in (1.18). We obtain

$$\frac{\mathrm{d}E_{gw}}{\mathrm{d}t} = \frac{32}{5}\frac{c^5}{G}\left(\frac{G\mathcal{M}_c\,\omega_{gw}}{2c^3}\right)^{\frac{10}{3}}\,. \tag{1.25}$$

Given that for a circular keplerian orbit the energy is given by

$$E = -G\frac{m_1 m_2}{2R}\,, \tag{1.26}$$

it can be easily seen that

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{2R^2}{G m_1 m_2}\frac{\mathrm{d}E_{gw}}{\mathrm{d}t}\,. \tag{1.27}$$

We are considering the regime of quasi-circular motion. In other words we are working under the assumption of a circular orbit with a slowly varying orbital radius. This approximation is valid as long as $\dot{\omega}_s \ll \omega_s^2$. Because of the emission of gravitational waves the energy of the system decreases, the orbital separation R decreases and, according to Kepler's law, the orbital frequency increases. From (1.25) we deduce that the radiated power increases, accelerating the process.

Combining (1.25) and (1.27), using (1.19) and $\omega_{gw} = 2\omega_s$ we obtain the equation for the frequency evolution:

$$\dot{f}_{gw} = \frac{96}{5}\pi^{\frac{8}{3}}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{\frac{5}{3}} f_{gw}^{\frac{11}{3}} . \tag{1.28}$$

Equation (1.28) shows that we can obtain the chirp mass measuring $\dot{f}_{gw}$ in correspondence to f_{gw}.

We are finally able to see how the back-reaction on the binary system due to the emission of gravitational waves has an impact on the gravitational wave signal itself. We have

$$h_+ = h_c \frac{1+\cos^2\theta}{2}\cos[\Phi(t_{ret})] , \tag{1.29a}$$

$$h_\times = h_c \cos\theta \sin[\Phi(t_{ret})] , \tag{1.29b}$$

where

$$\Phi(t) = \int_{t_0}^{t} \mathrm{d}t' \, \omega_{gw}(t') \tag{1.30}$$

and

$$h_c = \frac{4}{r}\left[\frac{G\mathcal{M}_c}{c^2}\right]^{\frac{5}{3}}\left[\frac{\pi f_{gw}(t_{ret})}{c}\right]^{\frac{2}{3}} . \tag{1.31}$$

Given that we are in the regime of quasi-circular motion, the time derivative of $R(t)$ and $\omega(t)$ were neglected.

While the gravitational wave is described by a tensor h_{ij}, the input of the detector is a scalar quantity and is given by

$$h(t) = F_+ h_+(t) + F_\times h_\times(t) , \tag{1.32}$$

where F_+ and $F_\times$ depend on the direction of propagation of the wave and on the geometry and orientation of the detector.

1.3.3 Sources at cosmological distance

Up to now we neglected the fact that the Universe is expanding. However if we consider sources at cosmological distances the expansion of the Universe has to be included. If we consider an unperturbed FRW Universe, the gravitational wave amplitude after propagation from the source to the observer is given by

$$h_c(t^{ret}) = \frac{4}{a(t_0)\chi}\left[\frac{G\mathcal{M}_c}{c^2}\right]^{\frac{5}{3}}\left[\frac{\pi f_{gw}^s(t_s^{ret})}{c}\right]^{\frac{2}{3}} \tag{1.33}$$

The next step consists in expressing the amplitude in terms of the quantities measured by the observer. Given that $f_{gw}^s = (1+z) f_{gw}^{obs}$ and using (1.12), (1.33) can be rewritten as

$$\begin{aligned} h_c(t^{ret}) &= \frac{4}{\mathcal{D}_L(z)}(1+z)\left[\frac{G\mathcal{M}_c}{c^2}\right]^{\frac{5}{3}}\left[\frac{\pi(1+z) f_{gw}^{obs}(t_{obs}^{ret})}{c}\right]^{\frac{2}{3}} \\ &= \frac{4}{\mathcal{D}_L(z)}\left[\frac{(1+z)G\mathcal{M}_c}{c^2}\right]^{\frac{5}{3}}\left[\frac{\pi f_{gw}^{obs}(t_{obs}^{ret})}{c}\right]^{\frac{2}{3}} \\ &= \frac{4}{\mathcal{D}_L(z)}\left[\frac{G\mathcal{M}_r}{c^2}\right]^{\frac{5}{3}}\left[\frac{\pi f_{gw}^{obs}(t_{obs}^{ret})}{c}\right]^{\frac{2}{3}} , \end{aligned} \tag{1.34}$$

where

$$\mathcal{M}_r = (1+z)\mathcal{M}_c \tag{1.35}$$

is the redshifted chirp mass. We see that we cannot obtain information about the redshift of the GW source from the gravitational wave itself: GW observations are sensitive only to the redshifted chirp mass.

1.4 Perturbed flat FRW Universe

Up to know we have considered a homogeneous and isotropic Universe. The next step consists in accounting for cosmological perturbations. As a consequence in this section we show how to write the perturbations of the FRW metric and the stress-energy tensor. This part is based on [54] and [22]. Finally we will focus the attention on the evolution of the primordial perturbations and the description of the primordial tensor power spectrum.

Perturbations of the metric tensor

The components of the perturbed spatially flat FRW metric are given by

$$g_{00} = -a^2(\eta)\left[1 + 2\sum_{r=1}^{+\infty}\frac{1}{r!}A^{(r)}\right], \tag{1.36a}$$

$$g_{0i} = g_{0i} = -a^2(\eta)\sum_{r=1}^{+\infty}\frac{1}{r!}B_i^{(r)}, \tag{1.36b}$$

$$g_{ij} = a^2(\eta)\left\{\left[1 - 2\sum_{r=1}^{+\infty}\frac{1}{r!}D^{(r)}\right]\delta_{ij} + \sum_{r=1}^{+\infty}\frac{1}{r!}h_{ij}^{(r)}\right\}, \tag{1.36c}$$

where $h_{ij}^{(r)}$ is traceless, η is the conformal time, which is related to the cosmic time by $\mathrm{d}\eta = \mathrm{d}t/a$, and $A^{(r)}$, $B_i^{(r)}$, $D^{(r)}$, $h_{ij}^{(r)}$ represent the rth-order perturbations of the metric. In this thesis we will stop at linear order ($r = 1$) in the metric perturbations.

Perturbations of the stress-energy tensor

The stress-energy tensor for a fluid is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} + \Pi_{\mu\nu}, \tag{1.37}$$

where ρ is the energy density, p is the pressure, $\Pi_{\mu\nu}$ is the anisotropic stress-tensor and u^μ is the 4-velocity. The energy density ρ and the 4-velocity u^μ of matter can be expanded as

$$\rho = \rho_{(0)} + \sum_{r=1}^{+\infty}\frac{1}{r!}\delta\rho^{(r)} \tag{1.38}$$

and

$$u^\mu = \frac{1}{a}\left(\delta_0^\mu + \sum_{r=1}^{+\infty}\frac{1}{r!}v_{(r)}^\mu\right), \tag{1.39}$$

where

$$u_{(0)}^\mu = \frac{1}{a}\delta_0^\mu = \frac{\delta_0^\mu}{\sqrt{-g_{00}^{(0)}}} \tag{1.40}$$

is the 4-velocity in a FRW Universe. Taking into account the normalization condition $u^\mu u_\mu = -1$ the first order perturbation $v^0_{(1)}$ can be written in terms of the lapse function $A_{(r)}$. We find

$$v^0_{(1)} = -A_{(1)}\,. \tag{1.41}$$

As concerns the pressure perturbation, using the equation of state, we have

$$\delta p = \left.\frac{\partial p}{\partial \rho}\right|_S \delta\rho + \left.\frac{\partial p}{\partial S}\right|_\rho \delta S = c_s^2 \delta\rho + \delta p_{non\,adiabatic}\,, \tag{1.42}$$

where S is the entropy and c_s is the adiabatic speed of sound of the fluid.

1.4.1 Gauge problem for cosmological perturbations

In general relativity when we consider perturbations of fields we have to take into account perturbations in the geometry itself. Since the comparison between two tensors has to be done at the same point, when we consider a perturbation of a generic tensor field given by

$$\Delta T = T - T_0\,, \tag{1.43}$$

where T and T_0 are the values in the physical perturbed and FRW background space-times, we need a one-to-one map between the two varieties. The choice of such a map corresponds to a gauge choice and a gauge transformation is a change of the map.

At linear order in the perturbations the expression for a generic tensor T after a gauge transformation is given by

$$\tilde{T} = T - \mathcal{L}_\xi T_0\,, \tag{1.44}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ and in order to define the gauge transformation we considered the passive coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ on the background manifold (for the details see Appendix A of [54]).

The tensor perturbation ΔT is gauge dependent. The relation between the perturbations in two different gauges is obtained as follows. The tensor perturbation is given by $\Delta T = T - T_0$ in the first gauge and $\Delta\tilde{T} = \tilde{T} - T_0$ in the second one. Inserting them in (1.44) we find

$$T_0 + \Delta\tilde{T} = T_0 + \Delta T - \mathcal{L}_\xi T_0\,, \tag{1.45}$$

which implies

$$\Delta\tilde{T} = \Delta T - \mathcal{L}_\xi T_0. \tag{1.46}$$

1.4.2 Power spectrum

As regards the cosmological perturbations, a useful statistical tool is the power spectrum. In order to define it we consider a generic random field $g(\mathbf{x}, t)$ and expand it in Fourier space:

$$g(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} g_\mathbf{k}(t)\,. \tag{1.47}$$

The power spectrum $P_g(k)$ is defined by

$$\langle g_\mathbf{k}, g^*_{\mathbf{k}'}\rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') P_g(k)\,, \tag{1.48}$$

where $\langle ...\rangle$ denotes an ensemble average. From (1.48) we can deduce that the power spectrum is the Fourier transform of the two point correlation function:

$$\xi(r) = \langle g(\mathbf{x}, t) g(\mathbf{x} + \mathbf{r}, t)\rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} P_g(k)\,, \tag{1.49}$$

As a consequence the variance is given by

$$\sigma^2 = \langle [g(\mathbf{x},t)]^2 \rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} P_g(k) = \frac{1}{2\pi^2} \int \mathrm{d}k\, k^2 P_g(k) = \int \frac{\mathrm{d}k}{k} \Delta_g(k)\,. \tag{1.50}$$

where

$$\Delta_g(k) = \frac{k^3}{2\pi^2} P_g(k)\,. \tag{1.51}$$

A quantity which is used to describe the shape of the power spectrum is the spectral index $n_g(k)$, which is given by

$$n_g(k) - 1 \equiv \frac{\mathrm{d}\ln \Delta_g}{\mathrm{d}\ln k}. \tag{1.52}$$

1.4.3 Perturbation evolution

In the final chapter of the thesis we will relate the angular power spectrum $C_l^{\mathcal{D}_L}$ to the primordial power spectra $P_\Psi(k)$ and $P_{T0}(k)$.

As regards the power spectrum $P_\Psi(k)$, which is defined by

$$\langle \Psi_p^*(\mathbf{k}) \Psi_p(\mathbf{k}') \rangle = (2\pi)^3 \delta_D^3(\mathbf{k} - \mathbf{k}') P_\Psi(k)\,, \tag{1.53}$$

we will use the relations in Fourier space between the linear perturbations $\Psi(a,\mathbf{k})$, $\Phi(a,\mathbf{k})$, $v(a,\mathbf{k})$ and $\Psi_p(\mathbf{k})$, which is the primordial value of the potential Ψ set during the inflation epoch. As concerns this part we will follow Appendix B of [46] and Appendix E of [55]. General relations which do not specify the DE model are given by

$$\Psi(a,\mathbf{k}) = \frac{9}{10} T_m(k) \frac{\mathcal{G}_\Psi(a,k)}{a} \Psi_p(\mathbf{k})\,, \tag{1.54a}$$

$$\Phi(a,\mathbf{k}) = \frac{9}{10} T_m(k) \frac{\mathcal{G}_\Phi(a,k)}{a} \Psi_p(\mathbf{k})\,, \tag{1.54b}$$

$$v(a,\mathbf{k}) = -\frac{9}{10} \frac{T_m(k)}{k} \mathcal{G}_v(a,k) \Psi_p(\mathbf{k})\,, \tag{1.54c}$$

where $T_m(k)$ is the Eisenstein Hu transfer function [56] and in the $\mathcal{G}$ functions is encoded the dark energy model.

If we consider ΛCDM and Dark Energy + Dark matter models the $\mathcal{G}$ functions become

$$\mathcal{G}_\Phi = \mathcal{G}_\Psi = \frac{D(\eta)}{D_{in}} a_{in}\,, \tag{1.55a}$$

$$\mathcal{G}_v = f \frac{2}{3} \frac{k\mathcal{H}}{\Omega_{m0} H_0^2} \frac{D(\eta)}{D_{in}} a_{in}\,, \tag{1.55b}$$

where $D(\eta)$ is the growth mode and $f = \mathrm{d}\ln D/\mathrm{d}\ln a$ is referred as the growth factor. Therefore we have

$$\Psi(a,\mathbf{k}) = \Phi(a,\mathbf{k}) = \frac{9}{10} T_m(k) \frac{D(a)}{D_{in}} \frac{a_{in}}{a} \Psi_p(k)\,, \tag{1.56a}$$

$$v(a,\mathbf{k}) = -\frac{3}{5} T_m(k) f \frac{\mathcal{H}}{\Omega_{m0} H_0^2} \frac{D(\eta)}{D_{in}} a_{in} \Psi_p(k)\,. \tag{1.56b}$$

We proceed with the description of the power spectrum $P_{T0}(k)$. The decomposition of the tensor perturbations h_{ij}^{TT} (which represent the primordial gravitational waves) is given by

$$h_{ij}^{TT}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} [h^+(\mathbf{k},\eta) e_{ij}^+(\hat{\mathbf{k}}) + h^\times(\mathbf{k},\eta) e_{ij}^\times(\hat{\mathbf{k}})]\,, \tag{1.57}$$

and the primordial tensor power spectrum $P_{T0}(k)$ is defined by

$$\left\langle h^{\lambda}_{prim}(\mathbf{k})\, h^{\lambda'}_{prim}(\mathbf{k}')\right\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}\frac{1}{4}P_{T0}(k)\,, \tag{1.58}$$

where $\lambda = +, \times$ and $h^{\lambda}_{prim}(\mathbf{k})$ are the primordial gravitational wave modes. The expression for the primordial tensor power spectrum $P_{T0}(k)$ is given by

$$P_{T0}(k) = 2\pi^2 k^{-3}\left(\frac{k}{k_0}\right)^{n_T}\Delta_T(k_0)\,, \tag{1.59}$$

where $\Delta_T(k_0)$ is the amplitude at a given pivot scale k_0, $P_{T0}(k)$ is related to $\Delta_T(k)$ by

$$P_{T0}(k) = \frac{2\pi^2}{k^3}\Delta_{T0}(k)\,. \tag{1.60}$$

and

$$n_T \equiv \frac{\mathrm{d}\ln\Delta_{T0}}{\mathrm{d}\ln k} \tag{1.61}$$

is the tensor spectral index. Standard single-field models of inflation predict a negative tensor spectral index n_T (red-tilted GW spectrum) which satisfy the consistency relation [57]

$$r = -8n_T\,, \tag{1.62}$$

where r is the tensor-to-scalar ratio. Currently a tight bound on CMB scales is given by $r < 0.032$ [58]. As regards other models which predict a blue-tilted ($n_T > 0$) GW spectrum and/or the violation of the consistency relation see for example [22].

As concerns the expression of the transfer function $\mathcal{T}_T(k,\eta)$, which describes the sub-horizon evolution of gravitational waves when they enter the horizon after the phase of accelerated expansion and is defined by

$$h^{\lambda}(\mathbf{k},\eta) \equiv h^{\lambda}_{prim}(\mathbf{k})\mathcal{T}_T(k,\eta)\,, \tag{1.63}$$

the main reference is [59]. We denote by k_{eq} the wave-number of the modes which enter the horizon at the epoch of matter-radiation equality and by η_{eq} the conformal time corresponding to the matter-radiation equality. For $k > k_{eq}$ and $\eta < \eta_{eq}$ the transfer function is given by

$$\mathcal{T}_T(k,\eta) = j_0(k\eta)\,. \tag{1.64}$$

As regards the spherical Bessel functions $j_n(k\eta)$ see Appendix C. For $k > k_{eq}$ and $\eta > \eta_{eq}$ we have

$$\mathcal{T}_T(k,\eta) = \frac{\eta_{eq}}{\eta}\left[A(k)j_1(k\eta) + B(k)y_1(k\eta)\right]\,, \tag{1.65}$$

where $A(k)$ and $B(k)$ are obtained matching (1.64) and (1.65) (and their first derivatives) at matter-radiation equality [59]. Finally, for $k < k_{eq}$ and $\eta > \eta_{eq}$ the transfer function is given by

$$\mathcal{T}_T(k,\eta) = \frac{3j_1(k\eta)}{k\eta}\,. \tag{1.66}$$

Chapter 2

Geometric optics approximation

In this chapter we analyze the geometric optics approximation for gravitational waves which propagate through a curved space-time. This approach for the study of the propagation of gravitational waves was devised and developed by Richard Isaacson in [60, 61]. In addition to these two papers this chapter is mostly based on [51] and [62]. We will begin from the description of the geometric optics limit: gravitational waves, represented by the small perturbation $h_{\mu\nu}$ of the metric, vary on a length scale much smaller than the characteristic scale of variation of the background $\tilde{g}_{\mu\nu}$. Then we will expand the Einstein's equations in powers of $h_{\mu\nu}$, estimating the order of magnitude of each term. We will proceed by splitting the Einstein's equations in a low-frequency and high-frequency part. The second one will provide the propagation equation for the perturbation $h_{\mu\nu}$. The following step will consist in finding the explicit expression of the propagation equation in terms of $h_{\mu\nu}$. In order to simplify it we will change variable, substituting $h_{\mu\nu}$ with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}h$, and we will use the specific gauge condition $\tilde{\nabla}^{\nu}\bar{h}_{\mu\nu} = 0$. Finally we will obtain the evolution equations for the amplitude and the phase of the gravitational wave.

2.1 Hypothesis of high frequency

We assume that the metric $g_{\mu\nu}$ can be split into a slowly varying background $\tilde{g}_{\mu\nu}$ and a small amplitude perturbation $h_{\mu\nu}$ which is rapidly varying. Therefore we can write

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{2.1}$$

where

$$\tilde{g}_{\mu\nu} = \mathcal{O}(1)\,, \qquad h_{\mu\nu} \ll 1\,, \tag{2.2a}$$

$$\partial_\rho \tilde{g}_{\mu\nu} \sim \frac{1}{L_B}\,, \qquad \partial_\rho h_{\mu\nu} \sim \frac{h}{\lambda}\,, \qquad \frac{\lambda}{L_B} \ll 1\,. \tag{2.2b}$$

The perturbation varies on a scale λ much smaller than the scale of variation L_B of the background. The small parameters h and λ/L_B are linked: their relative strength will be deduced by looking at the Einstein equations.
The inverse metric to third order in h is given by

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\rho}h^{\rho\nu} - h^{\mu\rho}h^{\sigma}_{\rho}h^{\nu}_{\sigma} + o(h^3)\,, \tag{2.3}$$

where the indices of $h_{\mu\nu}$ are raised with the background metric $\tilde{g}^{\mu\nu}$.

The metric $g_{\mu\nu}$ satisfies the Einstein field equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\,, \tag{2.4}$$

where $T_{\mu\nu}$ is the stress energy tensor due to the presence of external matter, T is its trace and $R_{\mu\nu}$ is the Ricci tensor, whose expression is

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu} \,. \tag{2.5}$$

The Christoffel symbols which appear in (2.5) are given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\Big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\Big)\,. \tag{2.6}$$

The next step consists in inserting the decomposition (2.1) of the metric in the Einstein equations (2.4).

2.1.1 Expansion in powers of $h_{\mu\nu}$

Expansion of $R_{\mu\nu}$

We start with the expansion of the Ricci tensor to second order in the metric perturbation $h_{\mu\nu}$:

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(h^3)\,, \tag{2.7}$$

where

- $R^{(0)}_{\mu\nu}$ depends only on the background metric $\tilde{g}_{\mu\nu}$,
- $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$,
- $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$.

By looking at (2.2) we can see that the derivative ∂h is much higher than $\partial\tilde{g}$, $\tilde{g}$ and h. Each partial derivative applied to the perturbation $h_{\mu\nu}$ corresponds to a factor $1/\lambda$. It follows that the leading order terms of $R^{(1)}_{\mu\nu}$ come from the terms $\partial\Gamma$ which contain the second derivative $\partial^2 h$. A similar reasoning can be applied to $R^{(2)}_{\mu\nu}$. In this case the leading order terms come from $\partial^2 h$ and $(\partial h)^2$. Therefore, taking into account that the inverse metric to second order in h is $g^{-1} = \tilde{g}^{-1} - h\tilde{g}^{-2} + h^2\tilde{g}^{-3}$, the estimation of the order of magnitude of each term $R^{(n)}_{\mu\nu}$ is the following:

- $R^{(0)}_{\mu\nu} = \mathcal{O}\big(\frac{1}{L_B^2}\big)$,
- $R^{(1)}_{\mu\nu} \sim \tilde{g}^{-1}\partial^2 h = \mathcal{O}\big(\frac{h}{\lambda^2}\big)$,
- $R^{(2)}_{\mu\nu} \sim h\tilde{g}^{-2}\partial^2 h = \mathcal{O}\big(\frac{h^2}{\lambda^2}\big)$.

Expansion of $T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$

Since the stress-energy tensor $T_{\mu\nu}$ in general depends on the metric $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$, the same expansion to second order in the perturbation $h_{\mu\nu}$ is applied to the RHS of the Einstein's equations (2.4):

$$R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} = \frac{8\pi G}{c^2}\left[\Big(T^{(0)}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}T^{(0)}\Big) + \Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{(1)} + \Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{(2)}\right], \tag{2.8}$$

where

- $T^{(0)}_{\mu\nu}$ is constructed only with $\tilde{g}_{\mu\nu}$ and $T^{(0)} = \tilde{g}^{\rho\sigma}T^{(0)}_{\rho\sigma}$,

- $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(1)}$ and $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(2)}$ are respectively linear and quadratic in $h_{\mu\nu}$.

The explicit expression of the part of the stress-energy tensor linear in the perturbation $h_{\mu\nu}$ is the following:

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(1)} = T^{(1)}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\left(\tilde{g}^{\rho\sigma}T^{(1)}_{\rho\sigma}\right) - \frac{1}{2}h_{\mu\nu}\left(\tilde{g}^{\rho\sigma}T^{(0)}_{\rho\sigma}\right) + \frac{1}{2}\tilde{g}_{\mu\nu}\left(h^{\rho\sigma}T^{(0)}_{\rho\sigma}\right) \tag{2.9}$$

$$= T^{(1)}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}T^{(1)} - \frac{1}{2}h_{\mu\nu}T^{(0)} + \frac{1}{2}\tilde{g}_{\mu\nu}\left(h^{\rho\sigma}T^{(0)}_{\rho\sigma}\right), \tag{2.10}$$

where $T^{(1)}_{\mu\nu}$ is linear in h and $T^{(1)} = \tilde{g}^{\rho\sigma}T^{(1)}_{\rho\sigma}$.

As regards the estimation of the order of magnitude of the terms of the last expression, there are no derivatives of $h_{\mu\nu}$ in $(1/2)\,h_{\mu\nu}T^{(0)}$ and $(1/2)\,\tilde{g}_{\mu\nu}\left(h^{\rho\sigma}T^{(0)}_{\rho\sigma}\right)$. Furthermore, in our case the stress-energy tensor is given by a macroscopic distribution of matter, therefore also $T^{(1)}_{\mu\nu}$ and $(1/2)\,\tilde{g}_{\mu\nu}T^{(1)}$ do not contain derivatives of $h_{\mu\nu}$. It follows that $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(1)}$ does not contain terms of order h/λ^2 and h/λ.

2.1.2 Split in high-frequency and low-frequency parts

Having expanded the Einstein's equations in the small perturbation $h_{\mu\nu}$, the next step consists in splitting them in a high-frequency part and a low-frequency part. In order to do that it must be taken into account that:

- $R^{(0)}_{\mu\nu}$ does not contain high frequency modes since it depends only on the background metric $\tilde{g}_{\mu\nu}$,
- $R^{(1)}_{\mu\nu}$ contributes only to the high frequency part of the Einstein's equations since it is linear in $h_{\mu\nu}$,
- $R^{(2)}_{\mu\nu}$ contains both high and low frequency modes; this statement can be understood by thinking about the product $h_{\mu\nu}h_{\rho\sigma}$ of two metric perturbations where a high frequency mode of $h_{\mu\nu}$, characterized by a wave-vector $\mathbf{k_1}$, could combine with a high frequency mode $\mathbf{k_2} \simeq -\mathbf{k_1}$ coming from $h_{\rho\sigma}$, giving rise to a low frequency mode [51].

A similar reasoning can be applied to the term $T_{\mu\nu} - (1/2)\,g_{\mu\nu}T$. It is worth mentioning that we consider a stress-energy tensor due to a macroscopic distribution of matter, which is assumed to be smooth. It follows that the only high frequency components in $T_{\mu\nu}$ come from the fact that in general the stress-energy tensor depends on the metric $g_{\mu\nu}$ and consequently on the perturbation $h_{\mu\nu}$ [51]. The other high frequency components come from the fact that the trace of the stress-energy tensor is constructed with $g_{\mu\nu}$ and multiplied by it.

Therefore the high and low frequency parts of the Einstein's equations are:

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{high} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(1)} + \frac{8\pi G}{c^4}\left[\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(2)}\right]^{high} \tag{2.11a}$$

$$R^{(0)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{low} + \frac{8\pi G}{c^4}\left(T^{(0)}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}T^{(0)}\right) + \frac{8\pi G}{c^4}\left[\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{(2)}\right]^{low}. \tag{2.11b}$$

On the basis of equation (2.11b) the relative strength of the small parameters h and λ/L_B can be deduced. We can distinguish two following cases.

- If there is no external matter $T_{\mu\nu}=0$ and $R^{(0)}_{\mu\nu}$ is determined by $\left[R^{(2)}_{\mu\nu}\right]^{low}$. It follows that

$$\frac{1}{L_B^2}\sim\frac{h^2}{\lambda^2}\qquad\Longrightarrow\qquad h\sim\frac{\lambda}{L_B}\,. \tag{2.12}$$

- If the background curvature is determined by the stress-energy tensor $T_{\mu\nu}$ the contribution given by $\left[R^{(2)}_{\mu\nu}\right]^{low}$ is negligible. Therefore

$$\frac{1}{L_B^2}\sim\frac{h^2}{\lambda^2}+\text{matter contribution}\gg\frac{h^2}{\lambda^2}\qquad\Longrightarrow\qquad h\ll\frac{\lambda}{L_B}\,. \tag{2.13}$$

Averaging procedure

The split in the high and low frequency parts can be accomplished by averaging over a length scale $\bar{l}$ which is larger than λ and smaller compared to L_B:

$$\lambda\ll\bar{l}\ll L_B\,. \tag{2.14}$$

The averaging scheme is introduced by Isaacson in [61] under the the name of "Brill-Hartle averaging". Since the part which varies slowly remains constant on a length scale $\bar{l}$, the averaging procedure has no effect on it. For example $\left\langle R^{(0)}_{\mu\nu}\right\rangle_{\bar{l}}=R^{(0)}_{\mu\nu}$. On the other hand the part which rapidly oscillates averages to zero. For instance $\left\langle R^{(1)}_{\mu\nu}\right\rangle_{\bar{l}}=0$.

Therefore by averaging equation (2.8) the slowly varying part is extracted:

$$R^{(0)}_{\mu\nu}=-\left\langle R^{(2)}_{\mu\nu}\right\rangle_{\bar{l}}+\frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu}-\frac{1}{2}\tilde{g}_{\mu\nu}\bar{T}\right), \tag{2.15}$$

where $\langle ...\rangle_{\bar{l}}$ denotes an average over many wavelengths λ and

$$\bar{T}_{\mu\nu}-\frac{1}{2}\tilde{g}_{\mu\nu}\bar{T}=\left\langle T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right\rangle_{\bar{l}} \tag{2.16}$$

This is the "coarse-grained" part of the Einstein's equations. It shows a non-linear phenomenon: how the gravitational waves affect the background curvature. Indeed equation (2.15) can be rewritten as

$$R^{(0)}_{\mu\nu}-\frac{1}{2}\tilde{g}_{\mu\nu}\bar{R}=\frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu}+t_{\mu\nu}\right), \tag{2.17}$$

where

$$t_{\mu\nu}=-\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu}-\frac{1}{2}\tilde{g}_{\mu\nu}R^{(2)}\right\rangle_{\bar{l}} \tag{2.18}$$

is the effective stress-energy tensor associated to gravitational waves.

In order to find the fluctuating part we subtract from equation (2.8) the averaged part (2.15). We obtain

$$R^{(1)}_{\mu\nu}=-\left[R^{(2)}_{\mu\nu}\right]^{high}+\frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)^{(1)}+\frac{8\pi G}{c^4}\left[\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)^{(2)}\right]^{high}, \tag{2.19}$$

where $\left[R^{(2)}_{\mu\nu}\right]^{high}=R^{(2)}_{\mu\nu}-\left\langle R^{(2)}_{\mu\nu}\right\rangle_{\bar{l}}$.

Since, as will be explained later, we will consider only terms of order $(\lambda/L_B)^{-2}$ and $(\lambda/L_B)^{-1}$, the terms that contain the stress-energy tensor, which are of order $(\lambda/L_B)^0$, are neglected. We proceed by selecting the part which is linear in the amplitude h. We obtain the equation for the propagation of waves

$$R^{(1)}_{\mu\nu}(h) = 0\,. \tag{2.20}$$

The term $\left[R^{(2)}_{\mu\nu}\right]^{high}$ is responsible for non-linear correction $j_{\mu\nu}$ to $h_{\mu\nu}$:

$$R^{(1)}_{\mu\nu}(j) = -\left[R^{(2)}_{\mu\nu}(h)\right]^{high}\,. \tag{2.21}$$

These higher order corrections will not be investigated in this thesis.

2.1.3 Gauge transformations and invariance

Since it is useful to use a specific gauge in order to simplify the propagation equation (2.20), in this section we analyze how $R^{(1)}_{\mu\nu}$ changes under a gauge transformation. We consider a gauge transformation induced by a quadrivector ξ^μ of the same order of the metric perturbation.

The metric changes in the following way:

$$\tilde{g}_{\mu\nu} + h_{\mu\nu} \quad\longrightarrow\quad \tilde{g}_{\mu\nu} + h_{\mu\nu} - \tilde{\nabla}_\mu\xi_\nu - \tilde{\nabla}_\nu\xi_\mu\,. \tag{2.22}$$

In order to continue to consider $\tilde{g}_{\mu\nu}$ as the the background metric we demand that $(h_{\mu\nu} - \tilde{\nabla}_\mu\xi_\nu - \tilde{\nabla}_\nu\xi_\mu) \lesssim h$, which implies $\tilde{\nabla}_\mu\xi_\nu \lesssim h$. Given that $\tilde{\nabla}_\mu\xi_\nu = \partial_\mu\xi_\nu - \tilde{\Gamma}^\rho_{\mu\nu}\xi_\rho$, the previous condition corresponds to:

$$\partial\xi \lesssim h\,, \tag{2.23a}$$

$$\xi \lesssim hL_B\,. \tag{2.23b}$$

As concerns $R^{(1)}_{\mu\nu}$, after a gauge transformation it becomes

$$R^{(1)}_{\mu\nu} \quad\longrightarrow\quad R^{(1)}_{\mu\nu} - \mathcal{L}_\xi R^{(0)}_{\mu\nu}\,, \tag{2.24}$$

where the Lie derivative is given by

$$\mathcal{L}_\xi R^{(0)}_{\mu\nu} = \xi^\sigma\tilde{\nabla}_\sigma R^{(0)}_{\mu\nu} + R^{(0)}_{\mu\sigma}\tilde{\nabla}_\nu\xi^\sigma + R^{(0)}_{\sigma\nu}\tilde{\nabla}_\mu\xi^\sigma\,. \tag{2.25}$$

From

$$\tilde{\nabla}_\nu\xi^\sigma \lesssim h\,, \qquad \xi \lesssim hL_B \qquad \text{and} \qquad R^{(0)}_{\mu\nu} \sim \partial^2\tilde{g} \sim \frac{1}{L_B^2} \quad\Rightarrow\quad \tilde{\nabla}_\sigma R^{(0)}_{\mu\nu} \sim \partial^3\tilde{g}_{\mu\nu} \sim \frac{1}{L_B^3}$$

follows

$$\mathcal{L}_\xi R^{(0)}_{\mu\nu} \lesssim \frac{h}{L_B^2} = \frac{h}{\lambda^2}\frac{\lambda^2}{L_B^2}\,. \tag{2.26}$$

Since $R^{(1)}_{\mu\nu} \sim h/\lambda^2$,

$$\mathcal{L}_\xi R^{(0)}_{\mu\nu} \lesssim \left(\frac{\lambda}{L_B}\right)^2 R^{(1)}_{\mu\nu} \ll R^{(1)}_{\mu\nu}\,. \tag{2.27}$$

Therefore in the high frequency limit $\lambda/L_B \ll 1$ the perturbation $R^{(1)}_{\mu\nu}$ of the Ricci tensor is approximately gauge invariant.

$R^{(1)}_{\mu\nu}$ contains terms of order h/λ^2, h/λ and h, while $\mathcal{L}_\xi R^{(0)}_{\mu\nu}$ is of order h/L_B^2. Consequently the leading and next-to-leading order terms of $R^{(1)}_{\mu\nu}$ don't change under a gauge transformation. Therefore the terms which arise from a gauge transformation can be dropped by neglecting $(\lambda/L_B)^0$ contributions. Limiting to the leading and next-to-leading order in λ/L_B is exactly what will be done in the geometric optics approximation.

2.2 Propagation in a curved space-time

2.2.1 Calculation of $R^{(1)}_{\mu\nu}(h)$

The aim of this section is to find the expression for $R^{(1)}_{\mu\nu}(h)$, which is the part of the Ricci tensor linear in the perturbation $h_{\mu\nu}$. We follow the procedure suggested in [62]. The quantities denoted by a tilde are constructed with the background metric $\tilde{g}_{\mu\nu}$ only. We will write $R^{\rho}_{\sigma\mu\nu} - \tilde{R}^{\rho}_{\sigma\mu\nu}$ and $R_{\sigma\nu} - \tilde{R}_{\sigma\nu}$ in terms of the tensor $S^{\rho}_{\nu\sigma} = \Gamma^{\rho}_{\nu\sigma} - \tilde{\Gamma}^{\rho}_{\nu\sigma}$. Since in the calculations the expression for $S^{\rho}_{\nu\sigma}$ is never specified, the obtained results will be independent of the order at which we stop in the expansion in $h_{\mu\nu}$. Only later the calculations will be restricted to the linear case by not considering the terms quadratic in $S^{\rho}_{\nu\sigma}$ and neglecting in the expression for $S^{\rho}_{\nu\sigma}$ the quadratic terms in $h_{\mu\nu}$.

$R_{\mu\nu} - \tilde{R}_{\mu\nu}$ expressed in terms of $S^{\rho}_{\mu\nu}$

We begin by calculating $R^{\rho}_{\sigma\mu\nu} - \tilde{R}^{\rho}_{\sigma\mu\nu}$. Defining

$$S^{\rho}_{\ \nu\sigma} = \Gamma^{\rho}_{\nu\sigma} - \tilde{\Gamma}^{\rho}_{\ \nu\sigma} \tag{2.28}$$

and given that

$$R^{\rho}_{\ \sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\ \nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.29}$$

and

$$\tilde{R}^{\rho}_{\ \sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^{\rho}_{\nu\sigma} - \partial_\nu \tilde{\Gamma}^{\rho}_{\mu\sigma} + \tilde{\Gamma}^{\rho}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\ \nu\sigma} - \tilde{\Gamma}^{\rho}_{\nu\lambda}\tilde{\Gamma}^{\lambda}_{\mu\sigma}\,, \tag{2.30}$$

we can write

$$\begin{aligned} R^{\rho}_{\ \sigma\mu\nu} - \tilde{R}^{\rho}_{\ \sigma\mu\nu} &= \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\mu \tilde{\Gamma}^{\rho}_{\nu\sigma}\right) - \left(\partial_\nu \Gamma^{\rho}_{\mu\sigma} - \partial_\nu \tilde{\Gamma}^{\rho}_{\mu\sigma}\right) \\ &\quad + \left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \tilde{\Gamma}^{\rho}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\nu\sigma}\right) - \left(\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \tilde{\Gamma}^{\rho}_{\nu\lambda}\tilde{\Gamma}^{\lambda}_{\mu\sigma}\right) \\ &= \partial_\mu S^{\rho}_{\nu\sigma} - \partial_\nu S^{\rho}_{\mu\sigma} + \left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \tilde{\Gamma}^{\rho}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\nu\sigma}\right) - \left(\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \tilde{\Gamma}^{\rho}_{\nu\lambda}\tilde{\Gamma}^{\lambda}_{\mu\sigma}\right). \end{aligned} \tag{2.31}$$

Since

$$\nabla_\mu S^{\rho}_{\ \nu\sigma} = \partial_\mu S^{\rho}_{\ \nu\sigma} + \Gamma^{\rho}_{\mu\lambda} S^{\lambda}_{\ \nu\sigma} - \Gamma^{\lambda}_{\mu\nu} S^{\rho}_{\ \lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} S^{\rho}_{\ \lambda\nu} \tag{2.32}$$

the part which contains the partial derivatives of $S^{\rho}_{\nu\sigma}$ can be rewritten as

$$\begin{aligned} \partial_\mu S^{\rho}_{\ \nu\sigma} - \partial_\nu S^{\rho}_{\ \mu\sigma} &= \left(\nabla_\mu S^{\rho}_{\ \nu\sigma} - \Gamma^{\rho}_{\mu\lambda} S^{\lambda}_{\ \nu\sigma} + \Gamma^{\lambda}_{\mu\nu} S^{\rho}_{\ \lambda\sigma} + \Gamma^{\lambda}_{\mu\sigma} S^{\rho}_{\ \lambda\nu}\right) \\ &\quad - \left(\nabla_\nu S^{\rho}_{\ \mu\sigma} - \Gamma^{\rho}_{\nu\lambda} S^{\lambda}_{\ \mu\sigma} + \Gamma^{\lambda}_{\nu\mu} S^{\rho}_{\ \lambda\sigma} + \Gamma^{\lambda}_{\nu\sigma} S^{\rho}_{\ \lambda\mu}\right) \\ &= \nabla_\mu S^{\rho}_{\ \nu\sigma} - \nabla_\nu S^{\rho}_{\ \mu\sigma} - \Gamma^{\rho}_{\mu\lambda} S^{\lambda}_{\ \nu\sigma} + \Gamma^{\rho}_{\nu\lambda} S^{\lambda}_{\ \mu\sigma} + \Gamma^{\lambda}_{\mu\sigma} S^{\rho}_{\ \lambda\nu} - \Gamma^{\lambda}_{\nu\sigma} S^{\rho}_{\ \lambda\mu}\,. \end{aligned} \tag{2.33}$$

Moreover, using (2.28), we find

$$\tilde{\Gamma}^{\rho}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\ \nu\sigma} = \left(\Gamma^{\rho}_{\mu\lambda} - S^{\rho}_{\ \mu\lambda}\right)\left(\Gamma^{\lambda}_{\nu\sigma} - S^{\lambda}_{\ \nu\sigma}\right) = \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\mu\lambda} S^{\lambda}_{\ \nu\sigma} - \Gamma^{\lambda}_{\nu\sigma} S^{\rho}_{\ \mu\lambda} + S^{\rho}_{\ \mu\lambda} S^{\lambda}_{\ \nu\sigma}\,. \tag{2.34}$$

Finally, inserting (2.33) and (2.34) in (2.31), we obtain

$$\begin{aligned} R^{\rho}_{\ \sigma\mu\nu} - \tilde{R}^{\rho}_{\ \sigma\mu\nu} &= \nabla_\mu S^{\rho}_{\ \nu\sigma} - \nabla_\nu S^{\rho}_{\ \mu\sigma} - \Gamma^{\rho}_{\mu\lambda} S^{\lambda}_{\ \nu\sigma} + \Gamma^{\rho}_{\nu\lambda} S^{\lambda}_{\ \mu\sigma} + \Gamma^{\lambda}_{\mu\sigma} S^{\rho}_{\ \lambda\nu} - \Gamma^{\lambda}_{\nu\sigma} S^{\rho}_{\ \lambda\mu} \\ &\quad + \left(\Gamma^{\rho}_{\mu\lambda} S^{\lambda}_{\ \nu\sigma} + \Gamma^{\lambda}_{\nu\sigma} S^{\rho}_{\ \mu\lambda} - S^{\rho}_{\ \mu\lambda} S^{\lambda}_{\ \nu\sigma}\right) - \left(\Gamma^{\rho}_{\nu\lambda} S^{\lambda}_{\ \mu\sigma} + \Gamma^{\lambda}_{\mu\sigma} S^{\rho}_{\ \nu\lambda} - S^{\rho}_{\ \nu\lambda} S^{\lambda}_{\ \mu\sigma}\right) \\ &= \nabla_\mu S^{\rho}_{\ \nu\sigma} - \nabla_\nu S^{\rho}_{\ \mu\sigma} - S^{\rho}_{\ \mu\lambda} S^{\lambda}_{\ \nu\sigma} + S^{\rho}_{\ \nu\lambda} S^{\lambda}_{\ \mu\sigma}\,. \end{aligned} \tag{2.35}$$

As regards the Ricci tensor we find

$$R_{\sigma\nu} - \tilde{R}_{\sigma\nu} = R^{\rho}_{\ \sigma\rho\nu} - \tilde{R}^{\rho}_{\ \sigma\rho\nu} = \nabla_\rho S^{\rho}_{\ \nu\sigma} - \nabla_\nu S^{\rho}_{\ \rho\sigma} - S^{\rho}_{\ \rho\lambda} S^{\lambda}_{\ \nu\sigma} + S^{\rho}_{\ \nu\lambda} S^{\lambda}_{\ \rho\sigma}\,. \tag{2.36}$$

$S^{\rho}_{\mu\nu}$ expressed in terms of $h_{\mu\nu}$

Now we calculate $S^{\rho}_{\nu\sigma}$ at linear order in $h_{\mu\nu}$.
We begin by expanding in $h_{\mu\nu}$ the Christoffel symbols. Given that

$$\Gamma^{\rho}_{\nu\sigma} = \frac{1}{2} g^{\rho\lambda}\big(\partial_\nu g_{\lambda\sigma} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma}\big) \tag{2.37}$$

and

$$\tilde{\Gamma}^{\rho}_{\nu\sigma} = \frac{1}{2} \tilde{g}^{\rho\lambda}\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma}\big) \tag{2.38}$$

we obtain

$$\begin{aligned}
\Gamma^{\rho}_{\nu\sigma} &= \frac{1}{2}\big(\tilde{g}^{\rho\lambda} - h^{\rho\lambda}\big)\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\nu h_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma} - \partial_\lambda h_{\nu\sigma}\big) \\
&= \frac{1}{2}\tilde{g}^{\rho\lambda}\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma}\big) + \frac{1}{2}\tilde{g}^{\rho\lambda}\big(\partial_\nu h_{\lambda\sigma} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda h_{\nu\sigma}\big) \\
&\quad - \frac{1}{2}h^{\rho\lambda}\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma}\big) \\
&= \tilde{\Gamma}^{\rho}_{\nu\sigma} + \frac{1}{2}\tilde{g}^{\rho\lambda}\big(\partial_\nu h_{\lambda\sigma} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda h_{\nu\sigma}\big) - \frac{1}{2}h^{\rho\lambda}\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma}\big)\,,
\end{aligned} \tag{2.39}$$

where for the inverse metric $g^{\mu\nu}$ we used expression (2.3) neglecting terms quadratic in $h_{\mu\nu}$.
Since at linear order

$$\nabla_\nu h_{\lambda\sigma} = \tilde{\nabla}_\nu h_{\lambda\sigma} = \partial_\nu h_{\lambda\sigma} - \tilde{\Gamma}^{\rho}_{\nu\lambda} h_{\rho\sigma} - \tilde{\Gamma}^{\rho}_{\nu\sigma} h_{\lambda\rho}\,, \tag{2.40}$$

the sum of the three partial derivatives of $h_{\mu\nu}$ in the right-hand side of equation (2.39) can be rewritten as

$$\begin{aligned}
\partial_\nu h_{\lambda\sigma} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda h_{\nu\sigma} &= (\tilde{\nabla}_\nu h_{\lambda\sigma} + \tilde{\Gamma}^{\mu}_{\nu\lambda} h_{\mu\sigma} + \tilde{\Gamma}^{\mu}_{\nu\sigma} h_{\lambda\mu}) + (\tilde{\nabla}_\sigma h_{\nu\lambda} + \tilde{\Gamma}^{\mu}_{\sigma\nu} h_{\mu\lambda} + \tilde{\Gamma}^{\mu}_{\sigma\lambda} h_{\nu\mu}) \\
&\quad - (\tilde{\nabla}_\lambda h_{\nu\sigma} + \tilde{\Gamma}^{\mu}_{\lambda\nu} h_{\mu\sigma} + \tilde{\Gamma}^{\mu}_{\lambda\sigma} h_{\mu\nu}) \\
&= \tilde{\nabla}_\nu h_{\lambda\sigma} + \tilde{\nabla}_\sigma h_{\nu\lambda} - \tilde{\nabla}_\lambda h_{\nu\sigma} \;\; + 2\tilde{\Gamma}^{\mu}_{\nu\sigma} h_{\lambda\mu}\,.
\end{aligned} \tag{2.41}$$

As regards the last term in the right-hand side of equation (2.39) we can write

$$\frac{1}{2}h^{\rho\lambda}\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma}\big) = \frac{1}{2}h^{\rho}_{\mu}\,\tilde{g}^{\mu\lambda}\big(\partial_\nu \tilde{g}_{\lambda\sigma} + \partial_\sigma \tilde{g}_{\nu\lambda} - \partial_\lambda \tilde{g}_{\nu\sigma}\big) = h^{\rho}_{\mu}\,\tilde{\Gamma}^{\mu}_{\nu\sigma}\,. \tag{2.42}$$

Inserting (2.41) and (2.42) in (2.39), we find

$$\begin{aligned}
\Gamma^{\rho}_{\nu\sigma} &= \tilde{\Gamma}^{\rho}_{\nu\sigma} + \frac{1}{2}\tilde{g}^{\rho\lambda}(\tilde{\nabla}_\nu h_{\lambda\sigma} + \tilde{\nabla}_\sigma h_{\nu\lambda} - \tilde{\nabla}_\lambda h_{\nu\sigma}) + \frac{1}{2}\tilde{g}^{\rho\lambda}(2\tilde{\Gamma}^{\mu}_{\nu\sigma} h_{\lambda\mu}) - h^{\rho}_{\mu}\,\tilde{\Gamma}^{\mu}_{\nu\sigma} \\
&= \tilde{\Gamma}^{\rho}_{\nu\sigma} + \frac{1}{2}\tilde{g}^{\rho\lambda}(\nabla_\nu h_{\lambda\sigma} + \nabla_\sigma h_{\nu\lambda} - \nabla_\lambda h_{\nu\sigma})\,.
\end{aligned} \tag{2.43}$$

Therefore $S^{\rho}_{\nu\sigma}$ at linear order in $h_{\mu\nu}$ is given by

$$S^{\rho}{}_{\nu\sigma} = \frac{1}{2}\tilde{g}^{\rho\lambda}\big(\tilde{\nabla}_\nu h_{\lambda\sigma} + \tilde{\nabla}_\sigma h_{\nu\lambda} - \tilde{\nabla}_\lambda h_{\nu\sigma}\big)\,. \tag{2.44}$$

$R^{(1)}_{\mu\nu}$ expressed in terms of $h_{\mu\nu}$

As a consequence, given that we want to compute $R^{(1)}_{\nu\sigma}$, the terms $S^{\rho}{}_{\rho\lambda}S^{\lambda}{}_{\nu\sigma}$ and $S^{\rho}{}_{\nu\lambda}S^{\lambda}{}_{\rho\sigma}$, which are quadratic in $h_{\mu\nu}$, can be neglected in (2.36). Therefore we can write

$$R^{(1)}_{\mu\nu} = \tilde{\nabla}_\rho S^{\rho}{}_{\nu\mu} - \tilde{\nabla}_\nu S^{\rho}{}_{\rho\mu}\,, \tag{2.45}$$

where, since $S^{\rho}{}_{\nu\mu}$ does not contain zero order terms, we substituted ∇_ρ with $\tilde{\nabla}_\rho$.
Finally, in order to find the expression of $R^{(1)}_{\mu\nu}$ in terms of $h_{\mu\nu}$, we insert (2.44) in (2.45). We obtain

$$\begin{aligned}R^{(1)}_{\mu\nu} &= \tilde{\nabla}_\rho S^{\rho}{}_{\nu\mu} - \tilde{\nabla}_\nu S^{\rho}{}_{\rho\mu} \\ &= \frac{1}{2}\tilde{g}^{\rho\lambda}\left(\tilde{\nabla}_\rho\tilde{\nabla}_\nu h_{\lambda\mu} + \tilde{\nabla}_\rho\tilde{\nabla}_\mu h_{\nu\lambda} - \tilde{\nabla}_\rho\tilde{\nabla}_\lambda h_{\nu\mu}\right) - \frac{1}{2}\tilde{g}^{\rho\lambda}\left(\tilde{\nabla}_\nu\tilde{\nabla}_\rho h_{\lambda\mu} + \tilde{\nabla}_\nu\tilde{\nabla}_\mu h_{\rho\lambda} - \tilde{\nabla}_\nu\tilde{\nabla}_\lambda h_{\rho\mu}\right) \\ &= \frac{1}{2}\left(\tilde{\nabla}^\rho\tilde{\nabla}_\nu h_{\rho\mu} + \tilde{\nabla}^\rho\tilde{\nabla}_\mu h_{\nu\rho} - \tilde{\nabla}^\rho\tilde{\nabla}_\rho h_{\nu\mu} - \tilde{\nabla}_\nu\tilde{\nabla}^\rho h_{\rho\mu} - \tilde{\nabla}_\nu\tilde{\nabla}_\mu h + \tilde{\nabla}_\nu\tilde{\nabla}^\rho h_{\rho\mu}\right) \\ &= \frac{1}{2}\left(\tilde{\nabla}^\rho\tilde{\nabla}_\nu h_{\rho\mu} + \tilde{\nabla}^\rho\tilde{\nabla}_\mu h_{\rho\nu} - \tilde{\nabla}^\rho\tilde{\nabla}_\rho h_{\nu\mu} - \tilde{\nabla}_\nu\tilde{\nabla}_\mu h\right).\end{aligned} \tag{2.46}$$

2.2.2 Propagation equation

In this section the attention is focused on the propagation equation

$$R^{(1)}_{\mu\nu} = 0\,. \tag{2.47}$$

The expression $R^{(1)}_{\mu\nu}$ in terms of $h_{\mu\nu}$ was found in the previous section:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\tilde{\nabla}^\sigma\tilde{\nabla}_\nu h_{\mu\sigma} + \tilde{\nabla}^\sigma\tilde{\nabla}_\mu h_{\nu\sigma} - \tilde{\nabla}_\nu\tilde{\nabla}_\mu h - \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma h_{\mu\nu}\right). \tag{2.48}$$

$R^{(1)}_{\mu\nu}$ in terms of $\bar{h}_{\mu\nu}$

The next step consists in introducing the new variable

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}h \tag{2.49}$$

and rewriting $R^{(1)}_{\mu\nu}$ in terms of it. In order to do that $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\bar{h}$ is inserted in (2.48). We obtain

$$\begin{aligned}R^{(1)}_{\mu\nu} &= \frac{1}{2}\Big[\tilde{\nabla}^\sigma\tilde{\nabla}_\nu\bar{h}_{\mu\sigma} - \frac{1}{2}\tilde{g}_{\mu\sigma}\tilde{\nabla}^\sigma\tilde{\nabla}_\nu\bar{h} + \tilde{\nabla}^\sigma\tilde{\nabla}_\mu\bar{h}_{\nu\sigma} - \frac{1}{2}\tilde{g}_{\nu\sigma}\tilde{\nabla}^\sigma\tilde{\nabla}_\mu\bar{h} + \tilde{\nabla}_\nu\tilde{\nabla}_\mu\bar{h} \\ &\qquad - \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}\Big] \\ &= \frac{1}{2}\Big[\tilde{\nabla}^\sigma\tilde{\nabla}_\nu\bar{h}_{\mu\sigma} - \frac{1}{2}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\bar{h} + \tilde{\nabla}^\sigma\tilde{\nabla}_\mu\bar{h}_{\nu\sigma} - \frac{1}{2}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\bar{h} + \tilde{\nabla}_\nu\tilde{\nabla}_\mu\bar{h} - \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}\Big] \\ &= \frac{1}{2}\Big[\tilde{\nabla}^\sigma\tilde{\nabla}_\nu\bar{h}_{\mu\sigma} + \tilde{\nabla}^\sigma\tilde{\nabla}_\mu\bar{h}_{\nu\sigma} - \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}\Big].\end{aligned} \tag{2.50}$$

The last passage is due to the fact that $-\frac{1}{2}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\bar{h} - \frac{1}{2}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\bar{h} + \tilde{\nabla}_\nu\tilde{\nabla}_\mu\bar{h} = 0$, given that second covariant derivatives commute on scalars.

Since $2[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]A_{\lambda\rho} = R^{(0)}_{\lambda\sigma\mu\nu}A^{\sigma}{}_{\rho} - R^{(0)}_{\sigma\rho\mu\nu}A_{\lambda}{}^{\sigma}$ we can write

$$\begin{aligned}\tilde{\nabla}^\sigma\tilde{\nabla}_\nu\bar{h}_{\mu\sigma} &= \tilde{g}^{\sigma\tau}\tilde{\nabla}_\tau\tilde{\nabla}_\nu\bar{h}_{\mu\sigma} = \tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} + \tilde{g}^{\sigma\tau}\left(R^{(0)}_{\mu\lambda\tau\nu}\bar{h}^{\lambda}{}_{\sigma} - R^{(0)}_{\lambda\sigma\tau\nu}\bar{h}_{\mu}{}^{\lambda}\right) \\ &= \tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} + R^{(0)}_{\mu\lambda\sigma\nu}\bar{h}^{\lambda\sigma} + \tilde{g}^{\sigma\tau}R^{(0)}_{\sigma\lambda\tau\nu}\bar{h}_{\mu}{}^{\lambda} = \tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} - R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} + R^{(0)}_{\lambda\nu}\bar{h}_{\mu}{}^{\lambda}\,.\end{aligned} \tag{2.51}$$

Applying the same reasoning to $\tilde{\nabla}^\sigma\tilde{\nabla}_\mu\bar{h}_{\nu\sigma}$ we find

$$\begin{aligned}\tilde{\nabla}^\sigma\tilde{\nabla}_\nu\bar{h}_{\mu\sigma} + \tilde{\nabla}^\sigma\tilde{\nabla}_\mu\bar{h}_{\nu\sigma} &= \tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} + \tilde{\nabla}_\mu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma} - R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} - R^{(0)}_{\lambda\nu\sigma\mu}\bar{h}^{\lambda\sigma} + R^{(0)}_{\lambda\nu}\bar{h}_{\mu}{}^{\lambda} + R^{(0)}_{\lambda\mu}\bar{h}_{\nu}{}^{\lambda} \\ &= \tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} + \tilde{\nabla}_\mu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma} - 2R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} + R^{(0)}_{\lambda\nu}\bar{h}_{\mu}{}^{\lambda} + R^{(0)}_{\lambda\mu}\bar{h}_{\nu}{}^{\lambda}\,,\end{aligned} \tag{2.52}$$

where we used $R^{(0)}_{\lambda\nu\sigma\mu}\bar{h}^{\lambda\sigma} = R^{(0)}_{\sigma\mu\lambda\nu}\bar{h}^{\lambda\sigma} = R^{(0)}_{\sigma\mu\lambda\nu}\bar{h}^{\sigma\lambda}$.

Inserting (2.52) in (2.50), $R^{(1)}_{\mu\nu}$ becomes

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} + \tilde{\nabla}_\mu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma} - 2R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} + R^{(0)}_{\lambda\nu}\bar{h}_\mu{}^\lambda + R^{(0)}_{\lambda\mu}\bar{h}_\nu{}^\lambda - \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}\right]. \tag{2.53}$$

We contract the propagation equation $R^{(1)}_{\mu\nu} = 0$ with the background metric $\tilde{g}^{\mu\nu}$:

$$\begin{aligned} \tilde{g}^{\mu\nu}R^{(1)}_{\mu\nu} &= 2\tilde{\nabla}^\nu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma} - 2R^{(0)}_{\lambda\sigma}\bar{h}^{\lambda\sigma} + R^{(0)}_{\lambda\nu}\bar{h}^{\nu\lambda} + R^{(0)}_{\lambda\mu}\bar{h}^{\mu\lambda} - \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h} + 2\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h} \\ 0 &= 2\tilde{\nabla}^\nu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma} + \tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}\,. \end{aligned} \tag{2.54}$$

Therefore we obtain the following condition:

$$\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h} = -2\tilde{\nabla}^\nu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma}\,. \tag{2.55}$$

Inserting it back in $R^{(1)}_{\mu\nu} = 0$ we find

$$\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + \tilde{g}_{\mu\nu}\tilde{\nabla}^\lambda\tilde{\nabla}^\sigma\bar{h}_{\lambda\sigma} - \tilde{\nabla}_\nu\tilde{\nabla}^\sigma\bar{h}_{\mu\sigma} - \tilde{\nabla}_\mu\tilde{\nabla}^\sigma\bar{h}_{\nu\sigma} + 2R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} - R^{(0)}_{\lambda\nu}\bar{h}_\mu{}^\lambda - R^{(0)}_{\lambda\mu}\bar{h}_\nu{}^\lambda = 0\,. \tag{2.56}$$

Lorenz gauge

This equation can be simplified if $\bar{h}_{\mu\nu}$ satisfies the following condition:

$$\tilde{\nabla}^\nu\bar{h}_{\mu\nu} = 0\,. \tag{2.57}$$

In order to impose this condition a specific gauge must be chosen: ξ_μ has to be a solution of

$$\tilde{\nabla}^\nu\bar{h}_{\mu\nu} - R^{(0)}_{\mu\sigma}\xi^\sigma - \tilde{\nabla}^\nu\tilde{\nabla}_\nu\xi_\mu = 0\,. \tag{2.58}$$

The last equation was obtained by considering that under a gauge transformation

$$h \quad\rightarrow\quad \tilde{g}^{\rho\sigma}\Big(h_{\rho\sigma} - \tilde{\nabla}_\rho\xi_\sigma - \tilde{\nabla}_\sigma\xi_\rho\Big) = h - 2\tilde{\nabla}^\sigma\xi_\sigma \tag{2.59}$$

$$\bar{h}_{\mu\nu} \quad\rightarrow\quad \Big(h_{\mu\nu} - \tilde{\nabla}_\mu\xi_\nu - \tilde{\nabla}_\nu\xi_\mu\Big) - \frac{1}{2}\tilde{g}_{\mu\nu}\Big(h - 2\tilde{\nabla}^\sigma\xi_\sigma\Big) = \bar{h}_{\mu\nu} - \Big(\tilde{\nabla}_\mu\xi_\nu + \tilde{\nabla}_\nu\xi_\mu - \tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\xi_\sigma\Big)\,. \tag{2.60}$$

Given that $2[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\xi_\rho = R^{(0)}_{\rho\sigma\mu\nu}\xi^\sigma$,

$$\begin{aligned} \tilde{\nabla}^\nu\bar{h}_{\mu\nu} \quad\rightarrow\quad \tilde{\nabla}^\nu\bar{h}_{\mu\nu} - \tilde{\nabla}^\nu\tilde{\nabla}_\mu\xi_\nu - \tilde{\nabla}^\nu\tilde{\nabla}_\nu\xi_\mu + \tilde{g}_{\mu\nu}\tilde{\nabla}^\nu\tilde{\nabla}^\sigma\xi_\sigma &= \tilde{\nabla}^\nu\bar{h}_{\mu\nu} + 2[\tilde{\nabla}_\mu, \tilde{\nabla}^\nu]\xi_\nu - \tilde{\nabla}^\nu\tilde{\nabla}_\nu\xi_\mu \\ &= \tilde{\nabla}^\nu\bar{h}_{\mu\nu} - R^{(0)}_{\mu\sigma}\xi^\sigma - \tilde{\nabla}^\nu\tilde{\nabla}_\nu\xi_\mu\,. \end{aligned}$$

Propagation equation in Lorenz gauge

By inserting the Lorenz gauge condition (2.57) in (2.56) the propagation equation becomes

$$\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + 2R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} - R^{(0)}_{\lambda\nu}\bar{h}_\mu{}^\lambda - R^{(0)}_{\lambda\mu}\bar{h}_\nu{}^\lambda = 0\,. \tag{2.61}$$

The last two terms of equation (2.61) can be neglected. Indeed, in the vacuum case, as shown in (2.12), $R^{(0)}_{\mu\nu} \sim h^2/\lambda^2$. Therefore $R^{(0)}_{\lambda\mu}\bar{h}_\nu{}^\lambda \sim h^3/\lambda^2$ is of the same order of $R^{(3)}_{\mu\nu}$, which was neglected in the analysis. If instead the background curvature is determined by $T^{(0)}_{\mu\nu}$, we can still neglect the terms which contain the Ricci tensor because, as previously explained, $T^{(0)}_{\mu\nu}$ is of order $(\lambda/L_B)^0$. Indeed, as previously mentioned, given that we want to neglect terms which arise from a gauge transformation, we are not considering terms of order $(\lambda/L_B)^0$. Therefore at leading and next-to-leading order in λ/L_B the propagation equation becomes

$$\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\bar{h}_{\mu\nu} + 2R^{(0)}_{\lambda\mu\sigma\nu}\bar{h}^{\lambda\sigma} = 0\,. \tag{2.62}$$

2.2.3 Geometric optics Ansatz

Since the geometry of a curved background is locally flat we can consider the flat space-time solution $A_{\mu\nu}e^{ik_\rho x^\rho}$ as an approximate solution over scales of order L_B, which is the scale over which the background metric varies. Therefore we are searching for a solution with a slowly varying amplitude $A_{\mu\nu}$ and a rapidly varying phase $\varphi(x)$. The scale of variation of the amplitude $A_{\mu\nu}$ and of the wave-vector $\tilde{k}^\mu \equiv -\tilde{\nabla}_\mu\varphi$ is L_B, while the scale of variation of the phase is λ. Thus we are looking for a solution of this type:

$$\bar{h}_{\mu\nu} = A_{\mu\nu}e^{i\varphi/\epsilon} = e_{\mu\nu}\mathcal{A}e^{i\varphi/\epsilon}\,, \tag{2.63}$$

where $\mathcal{A} \equiv \left(A_{\mu\nu}A^{\mu\nu}\right)^{\frac{1}{2}}$ is the scalar amplitude, $e_{\mu\nu} = A_{\mu\nu}/\mathcal{A}$ is the polarization tensor and ϵ is a formal expansion parameter equal to unity used to keep in mind that a term multiplied by ϵ^n is proportional to $(\lambda/L_B)^n$ [62].

By inserting (2.63) in the propagation equation (2.62) we find

$$\begin{aligned}
\tilde{\nabla}^\rho\tilde{\nabla}_\rho\left[A_{\mu\nu}e^{i\varphi/\epsilon}\right] &= \tilde{\nabla}^\rho\left[e^{i\varphi/\epsilon}\tilde{\nabla}_\rho A_{\mu\nu} + \frac{i}{\epsilon}e^{i\varphi/\epsilon}A_{\mu\nu}\tilde{\nabla}_\rho\varphi\right] \\
&= e^{i\varphi/\epsilon}\bigg[\tilde{\nabla}^\rho\tilde{\nabla}_\rho A_{\mu\nu} + \frac{i}{\epsilon}(\tilde{\nabla}^\rho\varphi)(\tilde{\nabla}_\rho A_{\mu\nu}) + \frac{i}{\epsilon}(\tilde{\nabla}^\rho A_{\mu\nu})(\tilde{\nabla}_\rho\varphi) \\
&\qquad - \frac{1}{\epsilon^2}A_{\mu\nu}(\tilde{\nabla}^\rho\varphi)(\tilde{\nabla}_\rho\varphi) + \frac{i}{\epsilon}A_{\mu\nu}\tilde{\nabla}^\rho\tilde{\nabla}_\rho\varphi\bigg] \\
&= e^{i\varphi/\epsilon}\left[-\frac{1}{\epsilon^2}(A_{\mu\nu}\tilde{k}^\rho\tilde{k}_\rho) + \frac{i}{\epsilon}(2\tilde{k}^\rho\tilde{\nabla}_\rho A_{\mu\nu} + A_{\mu\nu}\tilde{\nabla}^\rho\tilde{k}_\rho) + \tilde{\nabla}^\rho\tilde{\nabla}_\rho A_{\mu\nu}\right].
\end{aligned} \tag{2.64}$$

Since we consider only terms of order $(\lambda/L_B)^{-2}$ and $(\lambda/L_B)^{-1}$, we neglected the second term of equation (2.62) and and we will not consider $\tilde{\nabla}^\rho\tilde{\nabla}_\rho A_{\mu\nu}$.

Equation for the wave-vector

At leading order we find that $\tilde{k}^\mu$ is a null vector:

$$A_{\mu\nu}\tilde{k}^\rho\tilde{k}_\rho = 0 \qquad \Rightarrow \qquad \tilde{k}^\rho\tilde{k}_\rho = 0\,. \tag{2.65}$$

Moreover the curves $x^\mu(l)$ defined by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}l} = \tilde{k}^\mu \tag{2.66}$$

are null geodesics. Indeed, by taking the covariant derivative of equation (2.65) and considering that covariant derivatives commute on the scalar φ we obtain

$$0 = \tilde{\nabla}_\sigma(\tilde{k}^\rho\tilde{k}_\rho) = 2\tilde{k}_\rho\tilde{\nabla}_\sigma\tilde{k}^\rho = -2\tilde{k}_\rho\tilde{\nabla}_\sigma\tilde{\nabla}^\rho\varphi = -2\tilde{k}_\rho\tilde{\nabla}^\rho\tilde{\nabla}_\sigma\varphi = -2\tilde{k}_\rho\tilde{\nabla}^\rho\tilde{k}_\sigma\,. \tag{2.67}$$

Equations for the amplitude and the polarization tensor

Moving to the next-to-leading order we find

$$\begin{aligned}
0 &= 2\tilde{k}^\rho\tilde{\nabla}_\rho A_{\mu\nu} + A_{\mu\nu}\tilde{\nabla}^\rho\tilde{k}_\rho \\
&= 2\mathcal{A}\tilde{k}^\rho\tilde{\nabla}_\rho e_{\mu\nu} + 2e_{\mu\nu}\tilde{k}^\rho\tilde{\nabla}_\rho\mathcal{A} + \mathcal{A}e_{\mu\nu}\tilde{\nabla}^\rho\tilde{k}_\rho \\
&= (2\tilde{k}^\rho\tilde{\nabla}_\rho\mathcal{A} + \mathcal{A}\tilde{\nabla}^\rho\tilde{k}_\rho)e_{\mu\nu} + 2\mathcal{A}\tilde{k}^\rho\tilde{\nabla}_\rho e_{\mu\nu}\,.
\end{aligned} \tag{2.68}$$

We can obtain two separate equations for the amplitude $\mathcal{A}$ and the polarization $e_{\mu\nu}$.
As regards the amplitude $\mathcal{A}$, by contracting with $e^{\mu\nu}$, we get

$$2\tilde{k}^\rho\tilde{\nabla}_\rho\mathcal{A} + \mathcal{A}\tilde{\nabla}^\rho\tilde{k}_\rho = 0\,, \tag{2.69}$$

where we used $e_{\mu\nu}e^{\mu\nu} = 1$ and $0 = \tilde{\nabla}_\rho(e_{\mu\nu}e^{\mu\nu}) = e^{\mu\nu}\tilde{\nabla}_\rho e_{\mu\nu} + e_{\mu\nu}\tilde{\nabla}_\rho e^{\mu\nu} = 2e^{\mu\nu}\tilde{\nabla}_\rho e_{\mu\nu}$. Equation (2.69) can be rewritten as

$$\tilde{k}^\rho \tilde{\nabla}_\rho \ln\mathcal{A} = -\frac{1}{2}\tilde{\nabla}^\rho \tilde{k}_\rho\,. \tag{2.70}$$

Since $\tilde{k}^\rho \tilde{\nabla}_\rho$ is applied to the scalar $\ln\mathcal{A}$, equation (2.70) can becomes

$$\frac{\mathrm{d}}{\mathrm{d}l}\ln\mathcal{A} = -\frac{1}{2}\tilde{\nabla}^\rho \tilde{k}_\rho\,. \tag{2.71}$$

By inserting (2.70) in (2.68) we find that the polarization tensor is parallel transported along the null geodesic $x^\mu(l)$:

$$\tilde{k}^\rho \tilde{\nabla}_\rho e_{\mu\nu} = 0\,. \tag{2.72}$$

Finally, inserting (2.63) in the gauge condition (2.57) we obtain

$$0 = \tilde{\nabla}^\nu [A_{\mu\nu} e^{i\varphi/\epsilon}] = e^{i\varphi/\epsilon}\left[\frac{i}{\epsilon}A_{\mu\nu}\tilde{\nabla}^\nu \varphi + \tilde{\nabla}^\nu A_{\mu\nu}\right]. \tag{2.73}$$

Given that $\tilde{\nabla}^\nu A_{\mu\nu}$ is order $(\lambda/L_B)^0$, we can neglect it. Therefore the polarization tensor is orthogonal to the rays:

$$\tilde{k}^\nu e_{\mu\nu} = 0\,. \tag{2.74}$$

Comoving metric

From now on using the comoving metric $\hat{g}_{\mu\nu} = \tilde{g}_{\mu\nu}/a^2$ will prove to be convenient. A change of metric of this kind is called conformal transformation. The details about conformal transformations are shown in Appendix A. Below, we just summarize how to change the quantities after a conformal transformation.

$$\tilde{g}_{\mu\nu} \quad \rightarrow \quad \hat{g}_{\mu\nu} = \frac{\tilde{g}_{\mu\nu}}{a^2} \tag{2.75a}$$

$$\tilde{\Gamma}^\mu_{\nu\rho} \quad \rightarrow \quad \hat{\Gamma}^\mu_{\nu\rho} - C^\mu_{\nu\rho} \tag{2.75b}$$

$$\tilde{k}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}l} \quad \rightarrow \quad \frac{1}{a^2}\hat{k}^\mu = \frac{1}{a^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\chi} \tag{2.75c}$$

where the relation between l and χ is given by

$$\frac{\mathrm{d}l}{\mathrm{d}\chi} = a^2 \tag{2.76}$$

and the expression for $C^\mu_{\nu\rho}$ is (A.6). In equations (2.75c) and (2.76) the proportionality constant $\mathbb{C}$ which appears in Appendix A is set equal to 1. The results obtained in this section do not change if we keep a constant of proportionality $\mathbb{C} \neq 1$.

Evolution equations after conformal transformation

As shown in appendix A, equations (2.65) and (2.67) are still valid if we substitute $\tilde{k}^\mu$ with $(1/a^2)\,\hat{k}^\mu$ and $\tilde{\nabla}_\sigma$ with $\hat{\nabla}_\sigma$. In other words null geodesics are left invariant under a conformal transformation. Condition (2.76) is imposed in order to have an affinely-parametrized geodesic equation after the conformal transformation.

As regards equation (2.70), in order to express it in terms of the comoving metric and the affine parameter χ, we proceed in the following way.
Using (2.76) and (2.75b) we can write

$$\begin{aligned}\frac{\mathrm{d}\chi}{\mathrm{d}l}\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\mathcal{A} &= -\frac{1}{2}\Big(\partial_\rho\tilde{k}^\rho + \tilde{\Gamma}^\rho_{\rho\lambda}\tilde{k}^\lambda\Big)\\ \frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\mathcal{A} &= -\frac{1}{2}\Big(\partial_\rho\tilde{k}^\rho + \hat{\Gamma}^\rho_{\rho\lambda}\tilde{k}^\lambda - C^\rho_{\rho\lambda}\tilde{k}^\lambda\Big) = -\frac{1}{2}\hat{\nabla}_\rho\tilde{k}^\rho + \frac{1}{2}C^\rho_{\rho\lambda}\tilde{k}^\lambda\,.\end{aligned} \tag{2.77}$$

Inserting the expression (A.6) for $C^\mu_{\nu\rho}$ calculated in Appendix A we obtain

$$\begin{aligned}\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\mathcal{A} &= -\frac{1}{2}\hat{\nabla}_\rho\tilde{k}^\rho + \frac{1}{2}a\Big[\delta^\rho_\rho\tilde{\nabla}_\lambda\Big(\frac{1}{a}\Big) + \delta^\rho_\lambda\tilde{\nabla}_\rho\Big(\frac{1}{a}\Big) - \tilde{g}^{\rho\sigma}\tilde{g}_{\rho\lambda}\tilde{\nabla}_\sigma\Big(\frac{1}{a}\Big)\Big]\tilde{k}^\lambda\\ &= -\frac{1}{2}\hat{\nabla}_\rho\tilde{k}^\rho - \frac{1}{2}\Big[4\tilde{\nabla}_\lambda(\ln a) + \tilde{\nabla}_\lambda(\ln a) - \tilde{\nabla}_\lambda(\ln a)\Big]\tilde{k}^\lambda\\ &= -\frac{1}{2}\hat{\nabla}_\rho\tilde{k}^\rho - 2\tilde{k}^\lambda\tilde{\nabla}_\lambda(\ln a)\,.\end{aligned} \tag{2.78}$$

By substituting $\tilde{k}^\mu$ with $(1/a^2)\,\hat{k}^\mu$ we get

$$\begin{aligned}\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\mathcal{A} &= -\frac{1}{2a^2}\hat{\nabla}_\rho\hat{k}^\rho - \frac{1}{2}\hat{k}^\rho\hat{\nabla}_\rho\Big(\frac{1}{a^2}\Big) - 2\frac{\mathrm{d}}{\mathrm{d}l}\ln a\\ &= -\frac{1}{2a^2}\hat{\nabla}_\rho\hat{k}^\rho - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\chi}\Big(\frac{1}{a^2}\Big) - 2\frac{1}{a^2}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\chi}\\ &= -\frac{1}{2a^2}\hat{\nabla}_\rho\hat{k}^\rho + \frac{1}{a^3}\frac{\mathrm{d}a}{\mathrm{d}\chi} - 2\frac{1}{a^3}\frac{\mathrm{d}a}{\mathrm{d}\chi}\,,\end{aligned} \tag{2.79}$$

where we used (2.76) and $\mathrm{d}/\mathrm{d}l = \tilde{k}^\lambda\tilde{\nabla}_\lambda$.
Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\mathcal{A} = -\frac{1}{2}\hat{\nabla}_\rho\hat{k}^\rho - \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\chi}\,, \tag{2.80}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\big(a\mathcal{A}\big) = -\frac{1}{2}\hat{\nabla}_\rho\hat{k}^\rho\,. \tag{2.81}$$

2.3 Space around a GW source

In this section we give a brief description of the different regions in which the space around a GW source can be divided.

- The *source* is characterized by a size L, which in the case of binary system corresponds to the orbital radius.
- The *near zone* is the region characterized by $r \ll \lambda$, where r is the comoving distance from the source. In this region retardation effects are negligible.
- The *wave zone* (or *far zone*) is the region described by $r \gg \lambda$. This is the region where we will apply the geometric optics approximation.

In the case of systems which contains compact objects, which are strong-field sources, we can further separate the near zone into two regions:

- the *strong-field near zone* is within a spherical region which has at the center the compact object and a radius of order a few times the Schwarzschild radius of the source [51];

- the *weak-field near zone* is the remaining part of the near zone.

We can call *wave generation region* the region which includes the source, the strong-field near zone and the weak-field near zone (it corresponds to $r \lesssim r_I$ in Figure 2.1).

Since we are interested in the propagation of the gravitational waves across cosmological distances it is useful to split the wave zone into two parts.

- The *local wave zone* is characterized by a comoving distance from the source sufficiently large so that the gravitational field displays the typical behaviour of waves and sufficiently small so that the effects of the background curvature of the universe can be neglected [51]. The background space can be considered asymptotically flat. The local wave zone acts as a matching region between the wave generation region and the wave propagation region [63].
- In the *distant wave zone* the propagation of the gravitational waves is perturbed by the effects which are due to the background curvature of the universe.

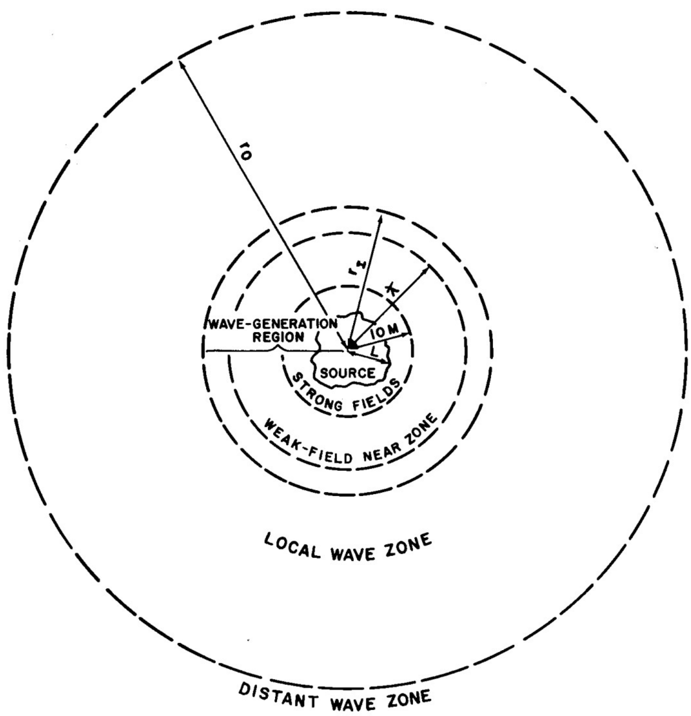

Figure 2.1: Regions in which the space around a GW source can be divided [63]. In this thesis the attention is focused on the distant wave zone, given that it is the region where the effects of the background curvature of the universe become important.

Chapter 3

Cosmic rulers formalism

In order to study the propagation of gravitational waves in a perturbed FRW Universe and the resulting effect on the GW waveforms, we follow the procedure used in [46]: we apply the *Cosmic Rulers* formalism, which was introduced for electromagnetic radiation in [64] and [65], to the gravitational radiation in the limit of geometric optics. This prescription provides a map between the observer's frame, also called Redshift-GW frame (RGW), which is considered as reference system, and the real frame. The definition of the real and RGW frames is the following.

- *Redshift-GW frame* (RGW)
  We assume we live in an unperturbed FRW Universe:

$$\mathrm{d}s^2 = a^2(\eta)[-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j] \,. \tag{3.1}$$

If the unit vector $\tilde{\mathbf{n}}$ is the observed direction of arrival of a gravitational wave and $\tilde{z}$ is the observed redshift of the electromagnetic counterpart, the inferred comoving position of the source at emission is

$$\bar{\eta} = \eta_0 - \bar{\chi}(\tilde{z}) \tag{3.2a}$$

$$\bar{\mathbf{x}} = \bar{\chi}(\tilde{z})\,\tilde{\mathbf{n}} \,, \tag{3.2b}$$

where $\bar{\chi}(z)$ is the distance-redshift relation in an unperturbed Universe, η_0 is the conformal time at observation and $\bar{\mathbf{x}}_o = (0,0,0)$ (the spatial origin corresponds to the location of the observer).
This position corresponds to the unique starting point of the null geodesic which arrives at the observer with direction $\tilde{\mathbf{n}}$ and is associated to a redshift $\tilde{z}$ [64]. Indeed, in the absence of perturbations, null geodesics are straight lines in conformal coordinates. In other words, null geodesics from the source to the observer, using the comoving distance $\bar{\chi}$ as affine parameter, can be written as:

$$\bar{x}^\mu = (\bar{\eta}, \bar{\mathbf{x}}) = (\eta_0 - \bar{\chi}, \bar{\chi}\,\mathbf{n}) \,. \tag{3.3}$$

The quantities in this frame are denoted with a bar.
The direction of arrival of the GW can be written as

$$n^i = \frac{\bar{x}^i}{\bar{\chi}} = \delta^{ij}\frac{\partial\bar{\chi}}{\partial\bar{x}^j} \,. \tag{3.4}$$

Indeed

$$\frac{\partial\bar{\chi}}{\partial\bar{x}^j} = \frac{\partial}{\partial\bar{x}^j}\sqrt{\bar{x}^i\bar{x}_i} = \frac{\partial}{\partial\bar{x}^j}\sqrt{\bar{x}^i\delta_{ik}\bar{x}^k} = \frac{1}{2\bar{\chi}}(\delta^i_j\delta_{ik}\bar{x}^k + \bar{x}^i\delta_{ik}\delta^k_j) = \frac{\bar{x}_j}{\bar{\chi}} = n_j \,. \tag{3.5}$$

In the RGW frame the wave-vector associated the null geodesic (3.3) is

$$\bar{k}^\mu = \frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}\bar{\chi}} = (-1, \mathbf{n}^i) . \tag{3.6}$$

Therefore the total derivative along the past GW-cone is

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} = \bar{k}^\mu \partial_\mu = -\frac{\partial}{\partial \bar{\eta}} + n^i \frac{\partial}{\partial \bar{x}^i} . \tag{3.7}$$

- *Real frame*
  This frame corresponds to a perturbed FRW Universe. The perturbed metric at linear order in a general gauge is

$$\mathrm{d}s^2 = a^2(\eta)[-(1+2A)\mathrm{d}\eta^2 - 2B_i \mathrm{d}\eta \mathrm{d}x^i + (\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j] , \tag{3.8}$$

  with

$$h_{ij} = -2D\delta_{ij} + h_{ij}^{TT} , \tag{3.9}$$

  where h_{ij}^{TT} is transverse and traceless.
  We denote by x^μ the actual comoving position of the source and by χ the comoving distance from the observer to the source. Given that the graviton path in a perturbed Universe is not straight, the inferred source's comoving position $\bar{x}^\mu$ found in the RGW frame does not coincide with the true spacetime point of emission x^μ (see Fig 3.1).

Below the subscript "e" will be used to denote quantities evaluated at the location where the gravitational waves are emitted, while "o" will stand for the position of the observer where the gravitational waves are received.

3.1 Map between real and RGW frames

Now that we have defined the real and the RGW frames, we can set up a map to relate them. Every quantity in the real frame will be decomposed into a zero order contribution, given by the solution in the RGW frame, which is the reference frame, plus a perturbation due to the cosmic inhomogeneities. Only first order perturbation will be considered.
We start from the comoving distance and define the first order perturbation

$$\delta\chi = \chi - \bar{\chi} . \tag{3.10}$$

Geodesic perturbation

As regards the graviton path, we want to define the map between $x^\mu(\chi)$, the actual comoving position located at a comoving distance χ from the observer, and the apparent position $\bar{x}^\mu(\bar{\chi})$, which we infer by assuming a homogeneous and isotropic Universe. In other words we want to find the expression for the perturbation $\Delta x^\mu(\bar{\chi}) = x^\mu(\chi) - \bar{x}^\mu(\bar{\chi})$. We proceed in the following way.
We start by introducing the perturbation of the null geodesic at fixed affine parameter:

$$x^\mu(\chi) = \bar{x}^\mu(\chi) + \delta x^\mu(\chi) . \tag{3.11}$$

We proceed by substituting χ with $\bar{\chi} + \delta\chi$ using (3.10) and Taylor expanding at linear order $\bar{x}^\mu$ around $\bar{\chi}$, while δx^μ is directly evaluated at $\bar{\chi}$ given that it is already a perturbation and we are

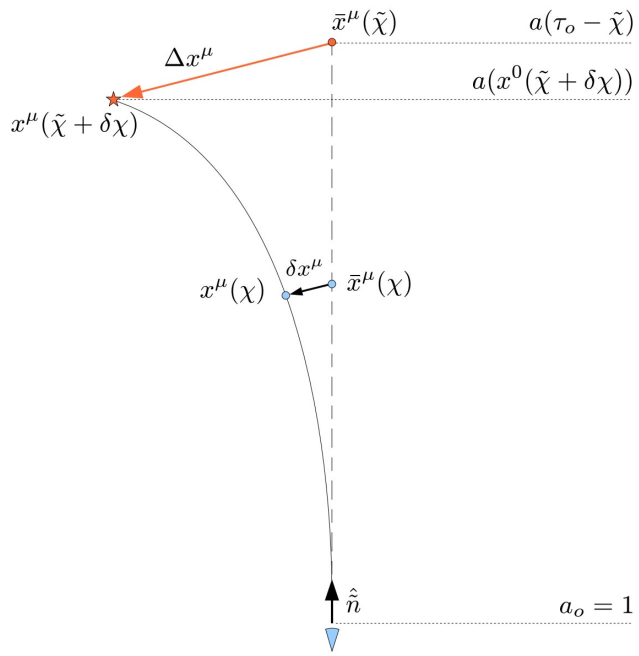

Figure 3.1: Comparison between the real frame and the Redshift-GW frame. The position of the GW source in the real frame is indicated by a star. The gravitational wave arrives with direction $\tilde{\mathbf{n}}$ at the observer located at the bottom, following the perturbed null geodesic represented by the solid line. The dashed line indicates the null geodesic which the graviton would follow in an unperturbed FRW Universe given the observed direction $\tilde{\mathbf{n}}$. This straight path traces back to the inferred GW source position indicated by the circle, which does not coincide with the real position. [64]

neglecting second order terms. Therefore we obtain

$$\begin{aligned} x^\mu(\chi) &= \bar{x}^\mu(\bar{\chi} + \delta\chi) + \delta x^\mu(\bar{\chi} + \delta\chi) \\ &= \bar{x}^\mu(\bar{\chi}) + \frac{\mathrm{d}\bar{\chi}}{\mathrm{d}\chi}\frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}\bar{\chi}}\delta\chi + \delta x^\mu(\bar{\chi}) \\ &= \bar{x}^\mu(\bar{\chi}) + \left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\chi}\right)\frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}\bar{\chi}}\delta\chi + \delta x^\mu(\bar{\chi}) \\ &= \bar{x}^\mu(\bar{\chi}) + \frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}\bar{\chi}}\delta\chi + \delta x^\mu(\bar{\chi}) \,, \end{aligned} \tag{3.12}$$

where in the last passage we neglected $\mathrm{d}\delta\chi/\mathrm{d}\chi$, given that it is multiplied by $\delta\chi$, which is already a first order perturbation. Finally we can write

$$x^\mu(\chi) = \bar{x}^\mu(\bar{\chi}) + \Delta x^\mu(\bar{\chi}) \,, \tag{3.13}$$

where

$$\Delta x^\mu(\bar{\chi}) = \delta x^\mu(\bar{\chi}) + \bar{k}^\mu \delta\chi \,. \tag{3.14}$$

We can notice two contributions to the perturbation $\Delta x^\mu(\bar{\chi})$: the first term comes from the perturbation of the GW geodesic at fixed $\bar{\chi}$, while the second one is proportional to the change $\delta\chi$ in the affine parameter.

Wave-vector perturbation

We can proceed with the perturbation of the wave-vector $\hat{k}^\mu = \mathrm{d}x^\mu/\mathrm{d}\chi$. From now on for simplicity we will write k^μ instead of $\hat{k}^\mu$ (which is the notation used in the previous chapter).

By using (3.14) we get

$$\begin{aligned} k^\mu(\chi) &= \frac{\mathrm{d}}{\mathrm{d}\chi}x^\mu(\chi) = \frac{\mathrm{d}}{\mathrm{d}\chi}\left[\bar{x}^\mu(\bar{\chi}) + \Delta x^\mu(\bar{\chi})\right] \\ &= \frac{\mathrm{d}\bar{\chi}}{\mathrm{d}\chi}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left[\bar{x}^\mu(\bar{\chi}) + \delta x^\mu(\bar{\chi}) + \bar{k}^\mu\delta\chi\right] \\ &= \left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\chi}\right)\left(\bar{k}^\mu + \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}\bar{k}^\mu}{\mathrm{d}\bar{\chi}}\delta\chi + \bar{k}^\mu\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right) \\ &= \bar{k}^\mu - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\chi}\bar{k}^\mu + \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}\bar{k}^\mu}{\mathrm{d}\bar{\chi}}\delta\chi + \bar{k}^\mu\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}} \end{aligned} \tag{3.15}$$

The difference between $\mathrm{d}\delta\chi/\mathrm{d}\bar{\chi}$ and $\mathrm{d}\delta\chi/\mathrm{d}\chi$ is second order. Indeed $\mathrm{d}\delta\chi/\mathrm{d}\chi = (\mathrm{d}\bar{\chi}/\mathrm{d}\chi)\,\mathrm{d}\delta\chi/\mathrm{d}\bar{\chi} = (1-\mathrm{d}\delta\chi/\mathrm{d}\chi)\,\mathrm{d}\delta\chi/\mathrm{d}\bar{\chi}$. Therefore at first order $-\bar{k}^\mu\,\mathrm{d}\delta\chi/\mathrm{d}\chi$ and $\bar{k}^\mu\,\mathrm{d}\delta\chi/\mathrm{d}\bar{\chi}$ cancel out. Moreover $\mathrm{d}\bar{k}^\mu/\mathrm{d}\bar{\chi} = 0$ since $\bar{k}^\mu$ satisfies the null geodesic equation $\bar{k}^\sigma\hat{\nabla}_\sigma\bar{k}^\mu = 0$. Thus

$$k^\mu(\chi) = \bar{k}^\mu + \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\bar{\chi}} = \bar{k}^\mu + \delta k^\mu\,. \tag{3.16}$$

Therefore $\Delta k^\mu(\bar{\chi}) = \delta k^\mu(\bar{\chi})$. In other words in this case there is no contribution proportional to $\delta\chi$. This statement is valid only at first order.
Then we can define $\delta\nu$ and δn^i so that

$$\delta k^\mu(\bar{\chi}) = (\delta\nu(\bar{\chi}), \delta n^i(\bar{\chi}))\,. \tag{3.17}$$

The next step consist in finding the expression for $\delta k^\mu(\bar{\chi})$ in terms of the metric perturbations. Before proceeding it is useful to define the parallel and perpendicular projection operators to the observed line-of-sight direction $\mathbf{n}$.

3.1.1 Projection operators and directional derivatives

For any spatial vector B^i and tensor A_{ij} we have:

$$\begin{aligned} B_\parallel &= n^i B_i\,, \\ A_\parallel &= n^i n^j A_{ij}\,, \\ B^i_\perp &= B^i - n^i B_\parallel = \left(\delta^{ij} - n^i n^j\right)B_j = \mathcal{P}^{ij}B_j\,, \end{aligned} \tag{3.18}$$

where

$$\mathcal{P}^{ij} = \delta^{ij} - n^i n^j\,. \tag{3.19}$$

As regards the perpendicular projection operator we calculate the following quantities:

$$\mathcal{P}^i_i = (\delta^i_i - n^i n_i) = 3 - 1 = 2 \tag{3.20}$$

and

$$\mathcal{P}^i_j\mathcal{P}^j_i = (\delta^i_j - n^i n_j)(\delta^j_i - n^j n_i) = \delta^i_j\delta^j_i - \delta^i_j n^j n_i - \delta^j_i n^i n_j + n^i n_j n^j n_i = \delta^i_i - 2n^i n_i + (n^i n_i)(n^j n_j) = 2\,. \tag{3.21}$$

The directional derivatives are defined in the following way:

$$\begin{aligned} \bar{\partial}_\parallel &= n^i\frac{\partial}{\partial\bar{x}^i}\,, \\ \bar{\partial}_{\perp i} &= \mathcal{P}^j_i\bar{\partial}_j = \left(\delta^j_i - n^j n_i\right)\bar{\partial}_j = \bar{\partial}_i - n_i\bar{\partial}_\parallel\,. \end{aligned} \tag{3.22}$$

Using (3.4) and (3.19) we find

$$\frac{\partial n^j}{\partial\bar{x}^i} = \frac{\partial}{\partial\bar{x}^i}\left(\frac{\bar{x}^j}{\bar{\chi}}\right) = \frac{\delta^j_i}{\bar{\chi}} - \frac{\bar{x}^j}{\bar{\chi}^2}\frac{\partial\bar{\chi}}{\partial\bar{x}^i} = \frac{1}{\bar{\chi}}\left(\delta^j_i - \frac{\bar{x}^j}{\bar{\chi}}\frac{\partial\bar{\chi}}{\partial\bar{x}^i}\right) = \frac{1}{\bar{\chi}}\left(\delta^j_i - n^j n_i\right) = \frac{1}{\bar{\chi}}\mathcal{P}^j_i\,. \tag{3.23}$$

Furthermore

$$\bar{\partial}_{\|} n^i = n^j \bar{\partial}_j n^i = n^j \frac{1}{\bar{\chi}} \left(\delta^i_j - n^i n_j \right) = \frac{1}{\bar{\chi}} \left(n^i - n^i n_j n^j \right) = 0 \tag{3.24}$$

and

$$\bar{\partial}_{\perp i} n^i = \bar{\partial}_i n^i - n_i \bar{\partial}_{\|} n^i = \bar{\partial}_i n^i = \frac{\mathcal{P}^i_i}{\bar{\chi}} = \frac{2}{\bar{\chi}} \, . \tag{3.25}$$

In addition $\mathrm{d}/\mathrm{d}\bar{\chi}$ and $\bar{\partial}_{\perp i}$ do not commute. In order to demonstrate it we proceed in the following way.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \bar{\partial}_{\perp i} &= \left(-\frac{\partial}{\partial \bar{\eta}} + n^j \frac{\partial}{\partial \bar{x}^j} \right) \left(\frac{\partial}{\partial \bar{x}^i} - n_i n^l \frac{\partial}{\partial \bar{x}^l} \right) \\
&= -\frac{\partial}{\partial \bar{\eta}} \frac{\partial}{\partial \bar{x}^i} + n_i n^l \frac{\partial}{\partial \bar{\eta}} \frac{\partial}{\partial \bar{x}^l} + n^j \frac{\partial}{\partial \bar{x}^j} \frac{\partial}{\partial \bar{x}^i} - n^j \frac{\partial}{\partial \bar{x}^j} \left(n_i n^l \frac{\partial}{\partial \bar{x}^l} \right) \\
&= -\frac{\partial}{\partial \bar{x}^i} \frac{\partial}{\partial \bar{\eta}} + n_i n^l \frac{\partial}{\partial \bar{x}^l} \frac{\partial}{\partial \bar{\eta}} + \left[\frac{\partial}{\partial \bar{x}^i} \left(n^j \frac{\partial}{\partial \bar{x}^j} \right) - \frac{\partial n^j}{\partial \bar{x}^i} \frac{\partial}{\partial \bar{x}^j} \right] \\
&\quad - \left[n_i n^j \frac{\partial}{\partial \bar{x}^j} \left(n^l \frac{\partial}{\partial \bar{x}^l} \right) + n^j \frac{\partial n_i}{\partial \bar{x}^j} n^l \frac{\partial}{\partial \bar{x}^l} \right] .
\end{aligned} \tag{3.26}$$

Given that $n^j \bar{\partial}_j n_i = 0$, as shown in (3.24), the last term is null. Finally, using (3.23), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \bar{\partial}_{\perp i} &= \left(\frac{\partial}{\partial \bar{x}^i} - n_i n^l \frac{\partial}{\partial \bar{x}^l} \right) \left(-\frac{\partial}{\partial \bar{\eta}} + n^j \frac{\partial}{\partial \bar{x}^j} \right) - \frac{\partial n^j}{\partial \bar{x}^i} \frac{\partial}{\partial \bar{x}^j} \\
&= \bar{\partial}_{\perp i} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} - \frac{\mathcal{P}^j_i}{\bar{\chi}} \frac{\partial}{\partial \bar{x}^j} = \bar{\partial}_{\perp i} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} - \frac{1}{\bar{\chi}} \bar{\partial}_{\perp i} \, .
\end{aligned} \tag{3.27}$$

Another useful relation is the following:

$$\begin{aligned}
\frac{\partial B^i}{\partial \bar{x}^j} &= \left(n_j \bar{\partial}_{\|} + \bar{\partial}_{\perp j} \right) \left(n^i B_{\|} + B^i_{\perp} \right) \\
&= n^i n_j \bar{\partial}_{\|} B_{\|} + n_j \bar{\partial}_{\|} B^i_{\perp} + B_{\|} \bar{\partial}_{\perp j} n^i + n^i \bar{\partial}_{\perp j} B_{\|} + \bar{\partial}_{\perp j} B^i_{\perp} \\
&= n^i n_j \bar{\partial}_{\|} B_{\|} + n_j \bar{\partial}_{\|} B^i_{\perp} + B_{\|} \frac{\mathcal{P}^i_j}{\bar{\chi}} + n^i \bar{\partial}_{\perp j} B_{\|} + \bar{\partial}_{\perp j} B^i_{\perp} \, ,
\end{aligned} \tag{3.28}$$

where in the second and third line we used respectively (3.24) and (3.23).
Moreover, using (3.23), (3.20) and (3.21), we find that

$$\begin{aligned}
\bar{\nabla}^2_{\perp} = \bar{\partial}_{\perp i} \bar{\partial}^i_{\perp} &= \left(\frac{\partial}{\partial \bar{x}^i} - n_i \bar{\partial}_{\|} \right) \left(\delta^{ij} \frac{\partial}{\partial \bar{x}^j} - n^i \bar{\partial}_{\|} \right) \\
&= \bar{\partial}_i \bar{\partial}^i - \bar{\partial}_i (n^i \bar{\partial}_{\|}) - n_i \bar{\partial}_{\|} \bar{\partial}^i + n_i \bar{\partial}_{\|} (n^i \bar{\partial}_{\|}) \\
&= \bar{\partial}_i \bar{\partial}^i - (\bar{\partial}_i n^i) \bar{\partial}_{\|} - n^i \bar{\partial}_i \bar{\partial}_{\|} - n_i \bar{\partial}_{\|} \bar{\partial}^i + n_i n^i \bar{\partial}^2_{\|} \\
&= \bar{\partial}_i \bar{\partial}^i - \frac{1}{\bar{\chi}} \mathcal{P}^i_i \bar{\partial}_{\|} - \bar{\partial}^2_{\|} - \bar{\partial}_{\|} (n_i \bar{\partial}^i) + \bar{\partial}^2_{\|} \\
&= \bar{\partial}_i \bar{\partial}^i - \frac{2}{\bar{\chi}} \bar{\partial}_{\|} - \bar{\partial}^2_{\|} \, .
\end{aligned} \tag{3.29}$$

3.1.2 Differential equation for δk^μ

In order to find the expression for $\delta k^\mu(\bar{\chi})$ in terms of the metric perturbations we need to know the differential equation satisfied by the wave-vector perturbation and then integrate it. Considering that $k^\mu = \bar{k}^\mu + \delta k^\mu$ and given that the geodesic equations satisfied by k^μ and $\bar{k}^\mu$

are known, we can proceed in the following way. We start from the geodesic equation satisfied by $k^\mu(\chi)$:

$$\frac{\mathrm{d}k^\mu(\chi)}{\mathrm{d}\chi} + \hat{\Gamma}^\mu_{\nu\rho}(x^\sigma)k^\nu(\chi)k^\rho(\chi) = 0\,, \tag{3.30}$$

where $\hat{\Gamma}^\mu_{\nu\rho}$ are the Christoffel symbols constructed with the comoving metric $\hat{g}_{\mu\nu} = \tilde{g}_{\mu\nu}/a^2$ and $\tilde{g}_{\mu\nu}$ is the metric associated to a perturbed FRW Universe.
Since $k^\mu(\chi) = \bar{k}^\mu(\bar{\chi}) + \delta k^\mu(\bar{\chi})$ and $\bar{k}^\mu(\bar{\chi})$ satisfies the geodesic equation $\mathrm{d}k^\mu/\mathrm{d}\bar{\chi} = 0$, the first term of equation (3.30) becomes, at linear order in the perturbations,

$$\frac{\mathrm{d}k^\mu}{\mathrm{d}\chi} = \left(\frac{\mathrm{d}\bar{\chi}}{\mathrm{d}\chi}\right)\left(\frac{\mathrm{d}\bar{k}^\mu}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}\delta k^\mu}{\mathrm{d}\bar{\chi}}\right) = \left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\chi}\right)\frac{\mathrm{d}\delta k^\mu}{\mathrm{d}\bar{\chi}} = \frac{\mathrm{d}\delta k^\mu}{\mathrm{d}\bar{\chi}}\,. \tag{3.31}$$

As concerns the second term of equation (3.30) we keep only the zero order component of the wave-vectors. This is due to the fact that the Christoffel symbols are already first order terms. Indeed the comoving metric of an unperturbed Universe is nothing else than the Minkowski metric and the associated Christoffel symbols are null. Finally, since

$$\hat{\Gamma}^\mu_{\nu\rho}(x^\sigma) = \delta\hat{\Gamma}^\mu_{\nu\rho}(x^\sigma) = \delta\hat{\Gamma}^\mu_{\nu\rho}(\bar{x}^\sigma) + \Delta x^\lambda \frac{\partial}{\partial \bar{x}^\lambda}\delta\hat{\Gamma}^\mu_{\nu\rho}(\bar{x}^\sigma)\,, \tag{3.32}$$

we obtain

$$\frac{\mathrm{d}\delta k^\mu}{\mathrm{d}\bar{\chi}}(\bar{\chi}) + \delta\hat{\Gamma}^\mu_{\nu\rho}(\bar{x}^\sigma)\bar{k}^\nu(\bar{\chi})\bar{k}^\rho(\bar{\chi}) = 0\,, \tag{3.33}$$

where the second order term $\Delta x^\lambda \bar{\partial}_\lambda \delta\hat{\Gamma}^\mu_{\nu\rho}(\bar{x}^\sigma)$ was neglected.
The $\mu = 0$ component gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}\delta\nu}{\mathrm{d}\bar{\chi}} + \delta\hat{\Gamma}^0_{00}\bar{k}^0\bar{k}^0 + 2\delta\hat{\Gamma}^0_{0i}\bar{k}^0\bar{k}^i + \delta\hat{\Gamma}^0_{ij}\bar{k}^i\bar{k}^j \\ &= \frac{\mathrm{d}\delta\nu}{\mathrm{d}\bar{\chi}} + A'(-1)(-1) + 2\bar{\partial}_i A(-1)(n^i) + \frac{1}{2}\left(\bar{\partial}_i B_j + \bar{\partial}_j B_i + h'_{ij}\right)n^i n^j \\ &= \frac{\mathrm{d}\delta\nu}{\mathrm{d}\bar{\chi}} + A' - 2n^i\bar{\partial}_i A + \frac{1}{2}n^i n^j\left(\bar{\partial}_i B_j + \bar{\partial}_j B_i + h'_{ij}\right)\,, \end{aligned} \tag{3.34}$$

where $' \equiv \partial/\partial\bar{\eta}$.
Since using (3.24) we get $n^j\bar{\partial}_\parallel B_j = \bar{\partial}_\parallel(n^j B_j)$, equation (3.34) becomes

$$0 = \frac{\mathrm{d}\delta\nu}{\mathrm{d}\bar{\chi}} + A' - 2\bar{\partial}_\parallel A + \bar{\partial}_\parallel B_\parallel + \frac{1}{2}h'_\parallel\,. \tag{3.35}$$

Considering that

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} = -\frac{\partial}{\partial\bar{\eta}} + n^i\frac{\partial}{\partial\bar{x}^i} = -\frac{\partial}{\partial\bar{\eta}} + \bar{\partial}_\parallel\,, \tag{3.36}$$

we obtain

$$\begin{aligned} \frac{\mathrm{d}\delta\nu}{\mathrm{d}\bar{\chi}} + 2\left(A' - \bar{\partial}_\parallel A\right) - B'_\parallel + \bar{\partial}_\parallel B_\parallel &= A' - B'_\parallel - \frac{1}{2}h'_\parallel \\ \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta\nu - 2A + B_\parallel\right) &= A' - B'_\parallel - \frac{1}{2}h'_\parallel\,. \end{aligned} \tag{3.37}$$

As regards the $\mu = i$ components we find

$$\begin{aligned}
0 &= \frac{\mathrm{d}\delta n^i}{\mathrm{d}\bar{\chi}} + \delta\hat{\Gamma}^i_{00}\bar{k}^0\bar{k}^0 + 2\delta\hat{\Gamma}^i_{0j}\bar{k}^0\bar{k}^j + \delta\hat{\Gamma}^i_{jl}\bar{k}^j\bar{k}^l \\
&= \frac{\mathrm{d}\delta n^i}{\mathrm{d}\bar{\chi}} + (-B^{i\prime} + \bar{\partial}^i A)(-1)(-1) + 2\left(-\frac{1}{2}\bar{\partial}_j B^i + \frac{1}{2}\bar{\partial}^i B_j + \frac{1}{2}h^{i\prime}_j\right)(-1)(n^j) \\
&\quad + \left(\frac{1}{2}\bar{\partial}_j h^i_l + \frac{1}{2}\bar{\partial}_l h^i_j - \frac{1}{2}\bar{\partial}^i h_{jl}\right)n^j n^l \\
&= \frac{\mathrm{d}\delta n^i}{\mathrm{d}\bar{\chi}} - B^{i\prime} + \bar{\partial}^i A + n^j\bar{\partial}_j B^i - n^j\bar{\partial}^i B_j - n^j h^{i\prime}_j + \frac{1}{2}n^l n^j\bar{\partial}_j h^i_l + \frac{1}{2}n^j n^l\bar{\partial}_l h^i_j - \frac{1}{2}n^j n^l\bar{\partial}^i h_{jl} \\
&= \frac{\mathrm{d}\delta n^i}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}B^i}{\mathrm{d}\bar{\chi}} + \bar{\partial}^i A - n^j\bar{\partial}^i B_j - n^j h^{i\prime}_j + n^j\partial_\parallel h^i_j - \frac{1}{2}n^j n^l\bar{\partial}^i h_{jl}\,.
\end{aligned} \tag{3.38}$$

Given that, using (3.23), $\bar{\partial}^i(n^j B_j) = n^j\bar{\partial}^i B_j + B_j\bar{\partial}^i n^j = n^j\bar{\partial}^i B_j + B_j\mathcal{P}^{ij}/\bar{\chi}$, we find

$$0 = \frac{\mathrm{d}\delta n^i}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}B^i}{\mathrm{d}\bar{\chi}} + \bar{\partial}^i A - \bar{\partial}^i(n^j B_j) + B_j\frac{\mathcal{P}^{ij}}{\bar{\chi}} + n^j\frac{\mathrm{d}h^i_j}{\mathrm{d}\bar{\chi}} - \frac{1}{2}n^j n^l\bar{\partial}^i h_{jl} \tag{3.39}$$

Finally, since $\mathrm{d}n^j/\mathrm{d}\bar{\chi} = 0$ and $\bar{\partial}^i h_\parallel = \bar{\partial}^i(n^j n^l h_{jl}) = n^j n^l\bar{\partial}^i h_{jl} + h_{jl}\bar{\partial}^i(n^j n^l)$, we obtain

$$0 = \frac{\mathrm{d}\delta n^i}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}B^i}{\mathrm{d}\bar{\chi}} + \bar{\partial}^i A - \bar{\partial}^i B_\parallel + \frac{B^i_\perp}{\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(n^j h^i_j) + \frac{1}{2}\bar{\partial}^i\left(n^j n^l\right)h_{jl} - \frac{1}{2}\bar{\partial}^i h_\parallel\,. \tag{3.40}$$

Therefore, using again (3.23), we get

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta n^i + B^i + n^j h^i_j\right) = -\bar{\partial}^i A + \bar{\partial}^i B_\parallel - \frac{B^i_\perp}{\bar{\chi}} - \frac{\mathcal{P}^{ji}}{\bar{\chi}}n^l h_{jl} + \frac{1}{2}\bar{\partial}^i h_\parallel\,. \tag{3.41}$$

The next step consists in integrating equations (3.37) and (3.41). In order to do that we need to know the boundary conditions for the wave-vector perturbations δk^μ at the observer's position $\bar{\chi} = 0$.

3.1.3 Boundary conditions at the observer for δk^μ

In order to determine the values of $\delta\nu$ and δn^i at $\bar{\chi} = 0$ we have to consider the graviton four-momentum measured by the observer. The observer's measurements are described in terms of the frame of reference built with an orthonormal tetrad $\Lambda^\mu_{\hat{\alpha}}$, which is defined through the following relations:

$$\tilde{g}^{\mu\nu}\Lambda^{\hat{\alpha}}_\mu\Lambda^{\hat{\beta}}_\nu = \eta^{\hat{\alpha}\hat{\beta}}\,, \qquad \eta_{\hat{\alpha}\hat{\beta}}\Lambda^{\hat{\alpha}}_\mu\Lambda^{\hat{\beta}}_\nu = \tilde{g}_{\mu\nu}\,, \qquad \tilde{g}^{\mu\nu}\Lambda^{\hat{\alpha}}_\nu = \Lambda^{\hat{\alpha}\mu}\,, \qquad \eta_{\hat{\alpha}\hat{\beta}}\Lambda^{\hat{\beta}}_\nu = \Lambda_{\hat{\alpha}\nu}\,, \tag{3.42}$$

where $\eta_{\hat{\alpha}\hat{\beta}}$ is the Minkowski metric, $\tilde{g}_{\mu\nu}$ is the metric associated to a perturbed FRW universe, $\hat{\alpha}$ and $\hat{\beta}$, which run from zero to three, are used as space-time indices of the tetrad $\Lambda_{\hat{\alpha}}$ and μ and ν denote its coordinate indices $(\Lambda_{\hat{\alpha}})^\mu$. Latin indices $\hat{a} = 1, 2, 3$ and $\hat{b} = 1, 2, 3$ will be used as space indices of the tetrad.

As regards the comoving tetrad, we define it through

$$\hat{g}^{\mu\nu}E^{\hat{\alpha}}_\mu E^{\hat{\beta}}_\nu = \eta^{\hat{\alpha}\hat{\beta}}\,, \qquad \eta_{\hat{\alpha}\hat{\beta}}E^{\hat{\alpha}}_\mu E^{\hat{\beta}}_\nu = \hat{g}_{\mu\nu}\,, \qquad \hat{g}^{\mu\nu}E^{\hat{\alpha}}_\nu = E^{\hat{\alpha}\mu}\,, \qquad \eta_{\hat{\alpha}\hat{\beta}}E^{\hat{\beta}}_\nu = E_{\hat{\alpha}\nu}\,, \tag{3.43}$$

where $\hat{g}_{\mu\nu} = \tilde{g}_{\mu\nu}/a^2$, $E^\mu_{\hat{\alpha}} = a\Lambda^\mu_{\hat{\alpha}}$ and $E_{\hat{\alpha}\mu} = (1/a)\Lambda_{\hat{\alpha}\mu}$.

In order find the expression at linear order for the tetrad $\Lambda^\mu_{\hat{\alpha}}$ in terms of the metric components $\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the metric of the background space-time, we consider the decomposition

$$\Lambda^\mu_{\hat{\alpha}} = \bar{\Lambda}^\mu_{\hat{\alpha}} + \delta\Lambda^\mu_{\hat{\alpha}}\,, \tag{3.44}$$

and we set $\bar{\Lambda}^{\mu}_{\hat{\alpha}} = \delta^{\mu}_{\hat{\alpha}}/a$. We proceed by using $\tilde{g}_{\mu\nu}\Lambda^{\mu}_{\hat{\alpha}}\Lambda^{\nu}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$ to obtain a system of three equations. For $\hat{\alpha} = \hat{\beta} = \hat{0}$ we find

$$-1 = \tilde{g}_{\mu\nu}\Lambda^{\mu}_{\hat{0}}\Lambda^{\nu}_{\hat{0}} = (\bar{g}_{00} + \delta g_{00})\big(\bar{\Lambda}^{0}_{\hat{0}} + \delta\Lambda^{0}_{\hat{0}}\big)^2 + 2\delta g_{0i}\big(\bar{\Lambda}^{0}_{\hat{0}} + \delta\Lambda^{0}_{\hat{0}}\big)\delta\Lambda^{i}_{\hat{0}} + (\bar{g}_{ij} + \delta g_{ij})\delta\Lambda^{i}_{\hat{0}}\delta\Lambda^{j}_{\hat{0}}\,. \quad (3.45)$$

Neglecting second order terms in the perturbations we get

$$-1 = (\bar{g}_{00} + \delta g_{00})\big(\bar{\Lambda}^{0}_{\hat{0}}\big)^2 + 2\bar{g}_{00}\bar{\Lambda}^{0}_{\hat{0}}\delta\Lambda^{0}_{\hat{0}} = -a^2(1+2A)\frac{1}{a^2} - 2a^2\frac{1}{a}\delta\Lambda^{0}_{\hat{0}}\,. \quad (3.46)$$

Therefore

$$\delta\Lambda^{0}_{\hat{0}} = -\frac{1}{a}A\,. \quad (3.47)$$

For $\hat{\alpha} = \hat{a}$ and $\hat{\beta} = \hat{b}$ we find, at first order in the perturbations,

$$\begin{aligned}\delta_{ab} = \tilde{g}_{\mu\nu}\Lambda^{\mu}_{\hat{a}}\Lambda^{\nu}_{\hat{b}} &= (\bar{g}_{ij} + \delta g_{ij})\big(\bar{\Lambda}^{i}_{\hat{a}} + \delta\Lambda^{i}_{\hat{a}}\big)\big(\bar{\Lambda}^{j}_{\hat{b}} + \delta\Lambda^{j}_{\hat{b}}\big)\\ &= a^2(\delta_{ij} + h_{ij})\left(\frac{1}{a}\delta^{i}_{a} + \delta\Lambda^{i}_{\hat{a}}\right)\left(\frac{1}{a}\delta^{j}_{b} + \delta\Lambda^{j}_{\hat{b}}\right)\\ &= \delta_{ab} + h_{ab} + a\delta_{aj}\delta\Lambda^{j}_{\hat{b}} + a\delta_{ib}\delta\Lambda^{i}_{\hat{a}}\,.\end{aligned} \quad (3.48)$$

We obtain

$$\delta\Lambda^{i}_{\hat{a}} = -\frac{1}{2a}h^{i}_{a}\,. \quad (3.49)$$

Finally, considering $\hat{\alpha} = \hat{0}$ and $\hat{\beta} = \hat{a}$, we have

$$\begin{aligned}0 = \tilde{g}_{\mu\nu}\Lambda^{\mu}_{\hat{0}}\Lambda^{\nu}_{\hat{a}} &= \bar{g}_{00}\bar{\Lambda}^{0}_{\hat{0}}\delta\Lambda^{0}_{\hat{a}} + \delta g_{0i}\bar{\Lambda}^{0}_{\hat{0}}\bar{\Lambda}^{i}_{\hat{a}} + \bar{g}_{ij}\delta\Lambda^{i}_{\hat{0}}\bar{\Lambda}^{j}_{\hat{a}}\\ &= -a^2\frac{1}{a}\delta\Lambda^{0}_{\hat{a}} - a^2 B_i\frac{1}{a}\frac{\delta^{i}_{a}}{a} + a^2\delta_{ij}\frac{\delta^{j}_{a}}{a}\delta\Lambda^{i}_{\hat{0}}\\ &= -a\delta\Lambda^{0}_{\hat{a}} - B_a + a\delta_{ia}\delta\Lambda^{i}_{\hat{0}}\,.\end{aligned} \quad (3.50)$$

In order to close the system we choose $\Lambda^{\mu}_{\hat{a}}$ to be orthogonal to the four-velocity u^{μ} of the observer. Since at linear order the four-velocity of the observer is given by $u^{\mu} = (1/a)(1-A, v^{i})$ and $u_{\mu} = a(-1-A, v_i - B_i)$, we find

$$0 = \Lambda^{\mu}_{\hat{a}}u_{\mu} = -(\Lambda^{0}_{\hat{a}})a(1+A) + \Lambda^{i}_{\hat{a}}a(v_i - B_i) = -a\delta\Lambda^{0}_{\hat{a}} + \delta^{i}_{a}(v_i - B_i)\,. \quad (3.51)$$

Therefore

$$\delta\Lambda^{0}_{\hat{a}} = \frac{1}{a}(v_a - B_a). \quad (3.52)$$

Thus equation (3.50) becomes

$$\begin{aligned}-v_a + B_a - B_a + a\delta_{ia}\delta\Lambda^{i}_{\hat{0}} &= 0\\ -\delta_{ia}v^{i} + a\delta_{ia}\delta\Lambda^{i}_{\hat{0}} &= 0\\ \delta\Lambda^{i}_{\hat{0}} &= \frac{1}{a}v^{i}\,.\end{aligned} \quad (3.53)$$

Summarizing:

$$\begin{aligned}\Lambda_{\hat{0}\mu} &= aE_{\hat{0}\mu} = a(-1-A, v_i - B_i) = u_{\mu}\,, & \Lambda_{\hat{a}\mu} &= aE_{\hat{a}\mu} = a\left(-v_a, \delta_{ai} + \frac{1}{2}h_{ai}\right),\\ \Lambda^{\mu}_{\hat{0}} &= \frac{E^{\mu}_{\hat{0}}}{a} = \frac{1}{a}(1-A, v^{i}) = u^{\mu}\,, & \Lambda^{\mu}_{\hat{a}} &= \frac{E^{\mu}_{\hat{a}}}{a} = \frac{1}{a}\left(v_a - B_a, \delta^{i}_{a} - \frac{1}{2}h^{i}_{a}\right).\end{aligned} \quad (3.54)$$

The components of the observed photon four-momentum $p^{\hat{\alpha}}\Lambda_{\hat{\alpha}} = p^{\mu}\partial_{\mu}$ with respect to the tetrad basis $\Lambda_{\hat{\alpha}}$ are

$$p^{\hat{\alpha}} = 2\pi f_o(1, -n^{\hat{a}}) , \tag{3.55}$$

where the minus sign is due to the fact that $n^{\hat{a}}$ points towards the source (see fig 3.1). The parametrization used for the four-momentum

$$p^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} = \frac{\mathrm{d}l}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}l} = \frac{\mathrm{d}l}{\mathrm{d}\lambda}\frac{1}{a^2}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\chi} \tag{3.56}$$

is given by

$$\mathrm{d}\lambda = -\frac{1}{2\pi f_o}\mathrm{d}l = -\frac{a^2}{2\pi f_o}\mathrm{d}\chi\,. \tag{3.57}$$

Indeed if we consider an unperturbed FRW Universe and use the affine parameter defined in (3.57) the four-momentum is given by $\bar{p}^{\mu} = -(2\pi f_o/a^2)(-1, n^i)$. If we take the projection of $\bar{p}^{\mu}$ on the tetrad basis $\bar{\Lambda}_{\hat{\alpha}}$ we obtain exactly the components (3.55):

$$p_{\hat{0}o} = (\bar{\Lambda}_{\hat{0}\mu}\bar{p}^{\mu})|_o = (a\bar{E}_{\hat{0}\mu})\left(-\frac{2\pi f_o}{\bar{a}^2}\bar{k}^{\mu}\right)\Big|_o = -\frac{2\pi f_o}{\bar{a}_o}(\bar{E}_{\hat{0}\mu}\bar{k}^{\mu})|_o = -2\pi f_o(-1)(-1) = -2\pi f_o \tag{3.58}$$

and

$$p_{\hat{a}o} = (\bar{\Lambda}_{\hat{a}\mu}\bar{p}^{\mu})|_o = -\frac{2\pi f_o}{\bar{a}_o}(\bar{E}_{\hat{a}\mu}\bar{k}^{\mu})|_o = -2\pi f_o[(\delta_{ai})(n^i)] = -2\pi f_o\, n_a\,, \tag{3.59}$$

where we used $\bar{a}_0 = 1$.

Now we are ready to calculate the boundary condition $\delta k^{\mu}_o = (\delta\nu_o, \delta n^i_o)$ in a perturbed FRW Universe. As regards the perturbation $\delta\nu_o$ we obtain

$$\begin{aligned} p^{GW}_{\hat{0}o} &= (\Lambda_{\hat{0}\mu}p^{\mu}_{GW})|_o = -\frac{2\pi f_o}{a_o}(E_{\hat{0}\mu}k^{\mu})|_o \\ &= -\frac{2\pi f_o}{1+\delta a_o}\left[(-1-A)(-1+\delta\nu) + (v_i - B_i)(n^i + \delta n^i)\right]\Big|_o \\ &= -2\pi f_o(1-\delta a_o)(1-\delta\nu_o + A + v_{\|o} - B_{\|o}) \\ &= -2\pi f_o(1 - \delta a_o - \delta\nu_o + A_o + v_{\|o} - B_{\|o})\,, \end{aligned} \tag{3.60}$$

where we used

$$a_o = a(\eta_o) = \bar{a}(\bar{\eta}_o) + \delta a_o = 1 + \delta a_o\,. \tag{3.61}$$

As concerns the perturbation δn^i we find

$$\begin{aligned} p^{GW}_{\hat{a}o} &= (\Lambda_{\hat{a}\mu}p^{\mu}_{GW})|_o = -\frac{2\pi f_o}{a_o}(E_{\hat{a}\mu}k^{\mu})|_o \\ &= -\frac{2\pi f_o}{1+\delta a_o}\left[(-v_a)(-1+\delta\nu) + \left(\delta_{ai} + \frac{1}{2}h_{ai}\right)(n^i + \delta n^i)\right]\Big|_o \\ &= -2\pi f_o(1-\delta a_o)\left[v_a + n_a + \delta n_a + \frac{1}{2}h_{ai}n^i\right]\Big|_o \\ &= -2\pi f_o\left[n_a - n_a\delta a_o + \delta n_{ao} + v_{ao} + \frac{1}{2}(h_{ai})_o n^i\right]. \end{aligned} \tag{3.62}$$

Therefore the initial conditions at the observer are:

$$\delta\nu_o = -\delta a_o + A_o + v_{\|o} - B_{\|o} \tag{3.63a}$$

$$\delta n_{ao} = n_a\delta a_o - v_{ao} - \frac{1}{2}(h_{ai})_o n^i. \tag{3.63b}$$

Now we have all the information necessary to integrate the differential equation satisfied by the wave-vector perturbation $\delta k^{\mu}(\bar{\chi})$.

3.1.4 Integration of the differential equation for δk^μ

The integration of equation (3.37) gives

$$\begin{aligned}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(\delta\nu - 2A + B_\parallel\right) &= \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right)\\ \delta\nu - 2A + B_\parallel &= \delta\nu_o - 2A_o + B_{\parallel o} + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right)\\ \delta\nu - 2A + B_\parallel &= \left(-\delta a_o + A_o + v_{\parallel o} - B_{\parallel o}\right) - 2A_o + B_{\parallel o} - 2I\\ \delta\nu &= -\delta a_o - A_o + v_{\parallel o} + 2A - B_\parallel - 2I\,,\end{aligned} \tag{3.64}$$

where we used (3.63a) and

$$I \equiv -\frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right) \tag{3.65}$$

is the Integrated Sachs-Wolfe contribution.

Finally we move to the integration of equation (3.41):

$$\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(\delta n^i + B^i + n^j h^i_j\right) = \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(-\tilde\partial^i A + \tilde\partial^i B_\parallel - \frac{B^i_\perp}{\tilde\chi} - \frac{\mathcal{P}^{ji}}{\tilde\chi}n^l h_{jl} + \frac{1}{2}\tilde\partial^i h_\parallel\right). \tag{3.66}$$

Using (3.63b) the left-hand side becomes

$$\begin{aligned}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(\delta n^i + B^i + n^j h^i_j\right) &= \delta n^i + B^i + n^j h^i_j - \delta n^i_o - B^i_o - n^j h^i_{j\,o}\\ &= \delta n^i + B^i + n^j h^i_j - n^i\delta a_o + v^i_o + \frac{1}{2}n^j h^i_{j\,o} - B^i_o - n^j h^i_{j\,o}\\ &= \delta n^i + B^i + n^j h^i_j - n^i\delta a_o + v^i_o - \frac{1}{2}n^j h^i_{j\,o} - B^i_o\,.\end{aligned} \tag{3.67}$$

As regards the right-hand side we can proceed in the following way.

$$\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(-\tilde\partial^i A+\tilde\partial^i B_\parallel-\frac{B^i_\perp}{\tilde\chi}-\frac{\mathcal{P}^{ji}}{\tilde\chi}n^l h_{jl}+\frac{1}{2}\tilde\partial^i h_\parallel\right) = \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[-\tilde\partial^i\left(A-B_\parallel-\frac{1}{2}h_\parallel\right)-\frac{1}{\tilde\chi}\left(B^i_\perp+n^l h_{jl}\mathcal{P}^{ji}\right)\right]. \tag{3.68}$$

Since

$$\bar\partial^i = \bar\partial^i_\perp + n^i\bar\partial_\parallel = \bar\partial^i_\perp + n^i\left(\frac{\mathrm{d}}{\mathrm{d}\bar\chi} + \frac{\partial}{\partial\bar\eta}\right) \tag{3.69}$$

we have

$$\begin{aligned}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\tilde\partial^i\left(A - B_\parallel - \frac{1}{2}h_\parallel\right) &= \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\tilde\partial^i_\perp\left(A - B_\parallel - \frac{1}{2}h_\parallel\right)\\ &\quad + n^i\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(A - B_\parallel - \frac{1}{2}h_\parallel\right) + n^i\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right)\\ &= \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\tilde\partial^i_\perp\left(A - B_\parallel - \frac{1}{2}h_\parallel\right)\\ &\quad + n^i\left(A - B_\parallel - \frac{1}{2}h_\parallel - A_o + B_{\parallel o} + \frac{1}{2}h_{\parallel o}\right) - 2n^i I\,.\end{aligned} \tag{3.70}$$

Therefore the right-hand side of (3.66) becomes

$$\begin{aligned}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(-\tilde\partial^i A + \tilde\partial^i B_\parallel - \frac{B^i_\perp}{\tilde\chi} - \frac{\mathcal{P}^{ji}}{\tilde\chi}n^l h_{jl} + \frac{1}{2}\tilde\partial^i h_\parallel\right) &= n^i\left(-A + B_\parallel + \frac{1}{2}h_\parallel + A_o - B_{\parallel o} - \frac{1}{2}h_{\parallel o}\right)\\ &\quad + 2n^i I + 2S^i_\perp\,,\end{aligned} \tag{3.71}$$

where

$$S^i_\perp = -\frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial^i_\perp\left(A - B_\| - \frac{1}{2}h_\|\right) + \frac{1}{\tilde\chi}\left(B^i_\perp + n^l h_{jl}\mathcal{P}^{ji}\right)\right] \tag{3.72}$$

is the perpendicular component of

$$S^i = -\frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial^i\left(A - B_\| - \frac{1}{2}h_\|\right) + \frac{1}{\tilde\chi}(B^i + n^k h^i_k)\right]. \tag{3.73}$$

Inserting (3.67) and (3.71) in (3.66) we obtain

$$\begin{aligned}\delta n^i &= -B^i - n^j h^i_j + n^i\delta a_o - v^i_o + \frac{1}{2}n^j h^i_{j\,o} + B^i_o + n^i\left(-A + B_\| + \frac{1}{2}h_\| + A_o - B_{\|o} - \frac{1}{2}h_{\|o}\right)\\ &\quad + 2n^i I + 2S^i_\perp\\ &= -B^i_\perp - \mathcal{P}^i_k n^j h^k_j - v^i_{\perp o} + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o} + B^i_{\perp o} + n^i\left(\delta a_o - v_{\|o} - A - \frac{1}{2}h_\| + A_o + 2I\right) + 2S^i_\perp\,,\end{aligned} \tag{3.74}$$

where we used $B^i = B^i_\perp + n^i B_\|$ and $n^j h^i_j - n^i h_\| = \delta^i_k n^j h^k_j - n^i n_k n^j h^k_j = (\delta^i_k - n^i n_k)n^j h^k_j = \mathcal{P}^i_k n^j h^k_j$. By defining

$$\delta n_\| = \delta a_o - v_{\|o} + A_o - A - \frac{1}{2}h_\| + 2I \tag{3.75a}$$

$$\delta n^i_\perp = -B^i_\perp + B^i_{\perp o} - v^i_{\perp o} - \mathcal{P}^i_k n^j h^k_j + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o} + 2S^i_\perp\,, \tag{3.75b}$$

we can rewrite (3.74) as

$$\delta n^i = n^i\delta n_\| + \delta n^i_\perp\,. \tag{3.76}$$

We are now ready to find the expression for the coordinate perturbations $\delta x^0(\bar\chi)$ and $\delta x^i(\bar\chi)$ by integrating (3.64) and (3.76). Before proceeding with the calculations we show how to derive the analytical expression for the coordinate perturbations at the observer. The derivation is based on [66] and [67].

3.1.5 Coordinate perturbations at the observer

The observer coordinates in the RGW frame and in the real frame do not coincide. This is due to the fact that the physical coordinate time t_o in an inhomogeneous universe does not correspond to the proper time $\mathcal{T}_o$ of the observer.

Conformal coordinate lapse δx^0_o

Considering that $\mathrm{d}t = a\,\mathrm{d}\eta = a\,\mathrm{d}x^0$, the physical coordinate time of the observer is calculated by integrating au^0 along the path of the observer. Therefore, given that in a perturbed universe the time component of the observer velocity is given by

$$u^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\mathcal{T}} = \frac{1-A}{a}\,, \tag{3.77}$$

we obtain

$$t_0 - t_{in} = \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} \frac{\mathrm{d}t}{\mathrm{d}\mathcal{T}}\mathrm{d}\mathcal{T} = \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} au^0\mathrm{d}\mathcal{T} = \mathcal{T}_0 - \mathcal{T}_{in} - \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} A[x^\mu(\mathcal{T})]\mathrm{d}\mathcal{T}\,. \tag{3.78}$$

Consequently in an unperturbed universe, since $A = 0$, the coordinate time of the observer coincides with the proper time:

$$\bar t_0 - t_{in} = \mathcal{T}_0 - \mathcal{T}_{in}\,. \tag{3.79}$$

On the other hand in a perturbed universe in general $A \neq 0$. This implies that the coordinate time of the observer is not synchronized with the proper time. Inserting (3.79) in (3.78) we find the following expression for the coordinate lapse:

$$\delta t_o = t_0 - \bar{t}_0 = -\int_{\mathcal{T}_{in}}^{\mathcal{T}_0} A[x^\mu(\mathcal{T})]\mathrm{d}\mathcal{T}\,. \tag{3.80}$$

Considering that, working at linear order, we can approximate at zeroth order the observer path $x^\mu(\mathcal{T})$ at which the perturbation A is evaluated, and using $\mathrm{d}\mathcal{T} = \bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}$, which comes from (3.77), to change variable, we obtain

$$\delta t_o = -\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} A(\bar{\eta}, \mathbf{0})\bar{a}\mathrm{d}\bar{\eta}\,. \tag{3.81}$$

At linear order the conformal coordinate lapse δx^0_o is equal to the coordinate lapse δt_o. Indeed

$$\delta t_o = a(\eta_0)\delta\eta_o = \bar{a}(\bar{\eta}_o)\delta\eta_o = \delta\eta_o = \delta x^0_o\,, \tag{3.82}$$

where the second equality is due to the fact that $\delta\eta_0$ is first order. Therefore

$$\delta x^0_o = -\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} A(\bar{\eta}, \mathbf{0})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}\,. \tag{3.83}$$

In other words the coordinate perturbation δx^0_o at the observer corresponds to the cumulative time delay which is due to the metric perturbation A along the trajectory of the observer [66].

Spatial coordinate shift δx^i_o

A similar procedure is used to calculate the spatial coordinate shift δx^i_o. Since in a perturbed universe the spatial component of the observer four-velocity is given by

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\mathcal{T}} = \frac{v^i}{a}\,, \tag{3.84}$$

we obtain

$$x^i_0 - x^i_{in} = \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} \frac{\mathrm{d}x^i}{\mathrm{d}\mathcal{T}}\mathrm{d}\mathcal{T} = \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} u^i\mathrm{d}\mathcal{T} = \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} \frac{v^i}{a}[x^\mu(\mathcal{T})]\mathrm{d}\mathcal{T}\,. \tag{3.85}$$

In a homogeneous Universe the path of the observer is static:

$$\bar{x}^i_0 = x^i_{in}\,. \tag{3.86}$$

In a perturbed FRW universe the spatial coordinate shift at linear order is

$$\delta x^i_o = \int_{\mathcal{T}_{in}}^{\mathcal{T}_0} \frac{v^i}{a}\mathrm{d}\mathcal{T} = \int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \frac{v^i}{\bar{a}}\bar{a}\mathrm{d}\bar{\eta} = \int_{\bar{\eta}_{in}}^{\bar{\eta}_0} v^i(\bar{\eta}, \mathbf{0})\mathrm{d}\bar{\eta}\,. \tag{3.87}$$

Scale factor perturbation δa_o

We are now able to find the analytical expression for the perturbation of the scale factor at the observer. Given that

$$a(\bar{\eta}_0 + \delta\eta_o) = \bar{a}(\bar{\eta}_0) + \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{\eta}}\Big|_{\bar{\eta}_0}\delta\eta_o = 1 + \mathcal{H}_0\delta\eta_o = 1 + \delta a_o\,, \tag{3.88}$$

we obtain

$$\delta a_o = \mathcal{H}_0\delta\eta_o = -\mathcal{H}_0\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} A(\bar{\eta}, \mathbf{0})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}\,. \tag{3.89}$$

3.1.6 Coordinate perturbations

In order to find the coordinate perturbations δx^μ we integrate the wave-vector perturbation δk^μ:

$$\delta x^\mu = \delta x^\mu_o + \int_0^{\bar\chi} \delta k^\mu \mathrm{d}\bar\chi \,. \tag{3.90}$$

As regards δx^0 we proceed in the following way.

$$\begin{aligned}\delta x^0 &= \delta x^0_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta\nu \\ &= \delta x^0_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(-\delta a_o - A_o + v_{\parallel o} + 2A - B_\parallel - 2I\right) \\ &= \delta x^0_o - \bar\chi \left(\delta a_o + A_o - v_{\parallel o}\right) + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(2A - B_\parallel\right) + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \int_0^{\tilde\chi} \mathrm{d}\tilde\chi' \left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right) ,\end{aligned} \tag{3.91}$$

where equation (3.64) has been used. As concerns the double integral, since the integrand is a function of $\tilde\chi'$, it is convenient to change the order of the integrations. This implies a change of the extremes of integration: at first we integrate in $\tilde\chi$ ranging from $\tilde\chi'$ to $\bar\chi$, then we integrate in $\tilde\chi'$ from 0 to $\bar\chi$. We obtain

$$\begin{aligned}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \int_0^{\tilde\chi} \mathrm{d}\tilde\chi' \left[A' - B'_\parallel - \frac{1}{2}h'_\parallel\right](\tilde\chi') &= \int_0^{\bar\chi} \mathrm{d}\tilde\chi' \int_{\tilde\chi'}^{\bar\chi} \mathrm{d}\tilde\chi \left[A' - B'_\parallel - \frac{1}{2}h'_\parallel\right](\tilde\chi') \\ &= \int_0^{\bar\chi} \mathrm{d}\tilde\chi' \left[A' - B'_\parallel - \frac{1}{2}h'_\parallel\right](\tilde\chi') \int_{\tilde\chi'}^{\bar\chi} \mathrm{d}\tilde\chi \\ &= \int_0^{\bar\chi} \mathrm{d}\tilde\chi' \left(\bar\chi - \tilde\chi'\right) \left[A' - B'_\parallel - \frac{1}{2}h'_\parallel\right](\tilde\chi') \,.\end{aligned} \tag{3.92}$$

Therefore the final expression for δx^0 is

$$\delta x^0 = \delta x^0_o - \bar\chi \left(\delta a_o + A_o - v_{\parallel o}\right) + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A - B_\parallel + (\bar\chi - \tilde\chi)\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right)\right] . \tag{3.93}$$

As regards the spatial coordinate perturbations δx^i, using the decomposition (3.76) we get

$$\begin{aligned}\delta x^i &= \delta x^i_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta n^i = \delta x^i_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(n^i \delta n_\parallel + \delta n^i_\perp\right) \\ &= n^i \delta x_{\parallel o} + n^i \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta n_\parallel + \delta x^i_{\perp o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta n^i_\perp \\ &= n^i \delta x_\parallel + \delta x^i_\perp \,,\end{aligned} \tag{3.94}$$

where $\delta x_\parallel = \delta x_{\parallel o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta n_\parallel$ and $\delta x^i_\perp = \delta x^i_{\perp o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta n^i_\perp$. Using (3.75a) the expression for the component parallel to the line of sight becomes

$$\begin{aligned}\delta x_\parallel &= \delta x_{\parallel o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \delta n_\parallel \\ &= \delta x_{\parallel o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\delta a_o - v_{\parallel o} + A_o - A - \frac{1}{2}h_\parallel + 2I\right] \\ &= \delta x_{\parallel o} + \bar\chi \left(\delta a_o - v_{\parallel o} + A_o\right) - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[A + \frac{1}{2}h_\parallel + (\bar\chi - \tilde\chi)\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right)\right] .\end{aligned} \tag{3.95}$$

Using (3.75b) we obtain the following expression for the perpendicular component:

$$
\begin{aligned}
\delta x^i_\perp &= \delta x^i_{\perp o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \, \delta n^i_\perp \\
&= \delta x^i_{\perp o} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[-B^i_\perp + B^i_{\perp o} - v^i_{\perp o} - \mathcal{P}^i_k n^j h^k_j + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o} + 2S^i_\perp \right] \\
&= \delta x^i_{\perp o} + \bar\chi \left(B^i_{\perp o} - v^i_{\perp o} + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o} \right) - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \, [B^i_\perp + \mathcal{P}^i_k n^j h^k_j] \\
&\quad - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \, (\bar\chi - \tilde\chi) \left[\tilde\partial^i_\perp \left(A - B_\parallel - \frac{1}{2} h_\parallel \right) + \frac{1}{\tilde\chi} \left(B^i_\perp + n^l h_{jl} \mathcal{P}^{ji} \right) \right] .
\end{aligned}
\tag{3.96}
$$

Moreover, using (3.93) and (3.95), we obtain

$$
\delta x^0 + \delta x_\parallel = \delta x^0_o + \delta x_{\parallel o} - T \, , \tag{3.97}
$$

where

$$
T = - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A - B_\parallel - \frac{1}{2} h_\parallel \right) \tag{3.98}
$$

is the Shapiro time delay.

3.1.7 Expressions for $\delta\chi$ and the components of Δx

We conclude by finding the expression for $\delta\chi$ in terms of the metric perturbations. We start from the following expansion of the scale factor in the real frame:

$$
\begin{aligned}
a[x^\mu(\chi)] &= a[\bar x^0(\bar\chi) + \Delta x^0(\bar\chi)] \\
&= \bar a[\bar x^0(\bar\chi)] + \Delta x^0(\bar\chi) \bar\partial_0 \bar a[\bar x^0(\bar\chi)] \\
&= \bar a[\bar x^0(\bar\chi)] \left(1 + \Delta x^0(\bar\chi) \frac{\bar\partial_0 \bar a}{\bar a} [\bar x^0(\bar\chi)] \right) \\
&= \bar a[\bar x^0(\bar\chi)] \Big(1 + \mathcal{H}[\bar x^0(\bar\chi)] \Delta x^0(\bar\chi) \Big) \, ,
\end{aligned}
\tag{3.99}
$$

where $\mathcal{H} = \bar a'/\bar a$. Defining

$$
\frac{a}{\bar a} = 1 + \Delta \ln a \, , \tag{3.100}
$$

we have, using equation (3.14),

$$
\Delta \ln a = \mathcal{H} \Delta x^0 = \mathcal{H} \big(-\delta\chi + \delta x^0 \big) \, . \tag{3.101}
$$

In order to compute explicitly $\Delta \ln a$ we consider the observed redshift, whose expression is given by

$$
1 + z = \frac{f_e}{f_o} = \frac{(u_\mu p^\mu)|_e}{(u_\mu p^\mu)|_o} = \frac{a_o}{a(\chi_e)} \frac{(E_{\hat 0 \mu} k^\mu)|_e}{(E_{\hat 0 \mu} k^\mu)|_o} = \frac{a_o}{a(\chi_e)} \frac{(E_{\hat 0 \mu} k^\mu)|_e}{a_o} = \frac{(E_{\hat 0 \mu} k^\mu)|_e}{a(\chi_e)} = \frac{1 + (E_{\hat 0 \mu} k^\mu)^{(1)}}{a} \, , \tag{3.102}
$$

where we used $(E_{\hat 0 \mu} k^\mu)|_o = a_o$ and $(E_{\hat 0 \mu} k^\mu)^{(0)} = E^{(0)}_{\hat 0 \mu} k^{\mu(0)} = (-1)(-1) = 1$. Using (3.54) and (3.64) and given that $\frac{1}{\bar a} = 1 + z$ we find

$$
\begin{aligned}
\Delta \ln a &= (E_{\hat 0 \mu} k^\mu)^{(1)} = E^{(1)}_{\hat 0 \mu} k^{\mu(0)} + E^{(0)}_{\hat 0 \mu} k^{\mu(1)} \\
&= (-A)(-1) + (v_i - B_i) n^i + (-1)(\delta\nu) \\
&= A + v_\parallel - B_\parallel - \delta\nu \\
&= A + v_\parallel - B_\parallel + (\delta a_o + A_o - v_{\parallel o} - 2A + B_\parallel + 2I) \\
&= -A + v_\parallel + \delta a_o + A_o - v_{\parallel o} + 2I \, .
\end{aligned}
\tag{3.103}
$$

Therefore, using equations (3.101), (3.103) and (3.93), we obtain

$$\begin{aligned}\delta\chi &= \delta x^0 - \frac{\Delta \ln a}{\mathcal{H}} \\ &= \delta x^0_o - \bar\chi\,(\delta a_o + A_o - v_{\|o}) + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A - B_\| + (\bar\chi - \tilde\chi)\left(A' - B'_\| - \frac{1}{2}h'_\|\right)\right] \\ &\quad - \frac{1}{\mathcal{H}}(-A + v_\| + \delta a_o + A_o - v_{\|o} + 2I) \\ &= \delta x^0_o - \left(\bar\chi + \frac{1}{\mathcal{H}}\right)(\delta a_o + A_o - v_{\|o}) + \frac{1}{\mathcal{H}}(A - v_\|) \\ &\quad + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A - B_\| + (\bar\chi - \tilde\chi)\left(A' - B'_\| - \frac{1}{2}h'_\|\right)\right] - \frac{2I}{\mathcal{H}}\,. \end{aligned} \tag{3.104}$$

Finally we can write the components of Δx^μ in terms of the metric perturbations. As regards $\mu = 0$, using (3.103), we have

$$\begin{aligned}\Delta x^0 &= \frac{\Delta \ln a}{\mathcal{H}} = \frac{1}{\mathcal{H}}(-A + v_\| + \delta a_o + A_o - v_{\|o} + 2I) \\ &= \frac{1}{\mathcal{H}}\left[(A_o - v_{\|o}) - A + v_\| + \delta a_o - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A' - B'_\| - \frac{1}{2}h'_\|\right)\right]. \end{aligned} \tag{3.105}$$

As concerns the component parallel to the line of sight, using (3.14), (3.97) and (3.105) we find

$$\begin{aligned}\Delta x_\| &= n_i \delta x^i + n_i \bar k^i \delta\chi = \delta x_\| + \delta\chi = \delta x_\| + \delta x^0 - \Delta x^0 = \delta x_{\|o} + \delta x^0_o - T - \Delta x^0 \\ &= \delta x_{\|o} + \delta x^0_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A - B_\| - \frac{1}{2}h_\|\right) \\ &\quad - \frac{1}{\mathcal{H}}\left[(A_o - v_{\|o}) - A + v_\| + \delta a_o - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A' - B'_\| - \frac{1}{2}h'_\|\right)\right]. \end{aligned} \tag{3.106}$$

Eventually, as regards the component perpendicular to the line of sight, using (3.96) we obtain

$$\begin{aligned}\Delta x^i_\perp &= \mathcal{P}^i_j(\delta x^j + \bar k^j \delta\chi) = \delta x^i_\perp \\ &= \delta x^i_{\perp o} + \bar\chi \left(B^i_{\perp o} - v^i_{\perp o} + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o}\right) - \int_0^{\bar\chi} \mathrm{d}\tilde\chi\,[B^i_\perp + \mathcal{P}^i_k n^j h^k_j] \\ &\quad - \int_0^{\bar\chi} \mathrm{d}\tilde\chi\,(\bar\chi - \tilde\chi)\left[\tilde\partial^i_\perp\left(A - B_\| - \frac{1}{2}h_\|\right) + \frac{1}{\tilde\chi}\left(B^i_\perp + n^l h_{jl}\mathcal{P}^{ji}\right)\right]. \end{aligned} \tag{3.107}$$

Chapter 4

Gravitational waves in the observed frame

In this chapter we calculate the effects of the large scale structure of the Universe on the gravitational waveforms. In the previous chapters we found that under the geometric optics approximation the GW waveform is given by

$$\bar{h}_{\mu\nu} = e_{\mu\nu}\mathcal{A}e^{i\varphi/\epsilon} = e_{\mu\nu}h\,. \tag{4.1}$$

and the evolution equations for the phase and amplitude are (2.65) and (2.81):

$$\hat{k}^{\mu}\hat{k}_{\mu} = 0\,, \tag{4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\ln\big(a\mathcal{A}\big) = -\frac{1}{2}\hat{\nabla}_{\rho}\hat{k}^{\rho}\,, \tag{4.2b}$$

where $\hat{k}_{\mu} = -\hat{\nabla}_{\mu}\varphi$.

The phase φ of a gravitational wave which propagates through a perturbed Universe will be described as the sum of a zero order contribution $\bar{\varphi}$, which corresponds to the solution in a homogeneous and isotropic Universe, and a correction $\Delta\varphi$ due to the cosmic inhomogeneities. The same procedure is applied to $\ln\mathcal{A}$. In order to find the explicit expressions for $\Delta\ln\mathcal{A}$ and $\Delta\varphi$ in terms of the metric perturbations we will insert their decompositions $\ln\bar{\mathcal{A}} + \Delta\ln\mathcal{A}$ and $\bar{\varphi} + \Delta\varphi$ in the evolution equations (2.65) and (2.81). Then, by subtracting the evolution equations for the background components we will be able to find the differential equations for the amplitude and phase perturbations.

Given that we will always work with the conformal metric, for simplicity from now on we will write k^{μ} and ∇_{μ} instead of $\hat{k}^{\mu}$ and $\hat{\nabla}_{\mu}$.

4.1 Phase

The evolution equation for the phase φ is given by (2.65). Indeed, given that $k_{\mu} = -\nabla_{\mu}\varphi$ and $\mathrm{d}/\mathrm{d}\chi = k^{\mu}\nabla_{\mu}$, we find

$$0 = k^{\mu}k_{\mu} = -k^{\mu}\nabla_{\mu}\varphi = -\frac{\mathrm{d}}{\mathrm{d}\chi}\varphi \quad\Longrightarrow\quad \frac{\mathrm{d}\varphi}{\mathrm{d}\chi} = 0\,. \tag{4.3}$$

We proceed with the following decomposition of the phase φ:

$$\begin{aligned}\varphi[x^{\mu}(\chi)] &= \varphi[\bar{x}^{\mu}(\bar{\chi}) + \Delta x^{\mu}(\bar{\chi})]\\ &= \varphi[\bar{x}^{\mu}(\bar{\chi})] + \Delta x^{\mu}(\bar{\chi})\bar{\nabla}_{\mu}\bar{\varphi}[\bar{x}^{\mu}(\bar{\chi})]\\ &= \bar{\varphi}[\bar{x}^{\mu}(\bar{\chi})] + \delta\varphi[\bar{x}^{\mu}(\bar{\chi})] + \Delta x^{\mu}(\bar{\chi})\bar{\nabla}_{\mu}\bar{\varphi}[\bar{x}^{\mu}(\bar{\chi})]\\ &= \bar{\varphi}[\bar{x}^{\mu}(\bar{\chi})] + \Delta\varphi(\bar{\chi})\,,\end{aligned} \tag{4.4}$$

where the zero order term $\bar{\varphi}[\bar{x}^\mu(\bar{\chi})]$ is the phase in the RGW frame and

$$\Delta\varphi(\bar{\chi}) = \delta\varphi[\bar{x}^\mu(\bar{\chi})] + \Delta x^\mu(\bar{\chi})\bar{\nabla}_\mu\bar{\varphi}[\bar{x}^\mu(\bar{\chi})] = \delta\varphi[\bar{x}^\mu(\bar{\chi})] - \Delta x^\mu(\bar{\chi})\bar{k}_\mu \tag{4.5}$$

is the total correction to the phase.

4.1.1 Evolution equation for $\delta\varphi$

We proceed by inserting the phase decomposition (4.4) in the evolution equation (4.3):

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\varphi[x^\mu(\chi)] = \frac{\mathrm{d}\bar{\chi}}{\mathrm{d}\chi}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}[\bar{\varphi} + \delta\varphi(\bar{x}^\mu) + \Delta x^\mu\bar{\nabla}_\mu\bar{\varphi}] = \left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right)\left(\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}\delta\varphi}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(\Delta x^\mu\bar{\nabla}_\mu\bar{\varphi})\right) = 0\,, \tag{4.6}$$

Given that

$$\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}\bar{\chi}} = \bar{k}^\mu\bar{\nabla}_\mu\bar{\varphi} = -\bar{k}^\mu\bar{k}_\mu = 0 \tag{4.7}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(\Delta x^\mu\bar{\nabla}_\mu\bar{\varphi}) &= \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}[(\delta x^\mu + \bar{k}^\mu\delta\chi)(-\bar{k}_\mu)] = \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(-\bar{k}_\mu\delta x^\mu - \bar{k}_\mu\bar{k}^\mu\delta\chi) = -\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(\bar{k}_\mu\delta x^\mu)\\ &= -\bar{k}_\mu\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta x^\mu = -\bar{k}_\mu\delta k^\mu\,,\end{aligned} \tag{4.8}$$

equation (4.6) becomes

$$0 = \left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right)\left(\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}\delta\varphi}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(\Delta x^\mu\bar{\nabla}_\mu\bar{\varphi})\right) = \left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\chi}\right)\left(\frac{\mathrm{d}\delta\varphi}{\mathrm{d}\bar{\chi}} - \bar{k}_\mu\delta k^\mu\right). \tag{4.9}$$

Therefore we obtain the evolution equations for the perturbation $\delta\varphi(\bar{\chi})$ at fixed comoving distance:

$$\frac{\mathrm{d}\delta\varphi}{\mathrm{d}\bar{\chi}}(\bar{x}^\mu) = \bar{k}_\mu\delta k^\mu(\bar{\chi})\,. \tag{4.10}$$

The same differential equation can be obtained as follows.

Alternative method

We start by perturbing directly $k^\mu = -\hat{g}^{\mu\nu}\nabla_\nu\varphi$:

$$k^\mu = -(\bar{g}^{\mu\nu}+\delta g^{\mu\nu})\nabla_\nu(\bar{\varphi}+\delta\varphi+\Delta x^\rho\bar{\nabla}_\rho\bar{\varphi}) = -\bar{g}^{\mu\nu}\nabla_\mu\bar{\varphi}-\delta g^{\mu\nu}\bar{\nabla}_\nu\bar{\varphi}-\bar{g}^{\mu\nu}\bar{\nabla}_\nu\delta\varphi-\bar{g}^{\mu\nu}\bar{\nabla}_\nu(\Delta x^\rho\bar{\nabla}_\rho\bar{\varphi})\,. \tag{4.11}$$

As concerns the first term on the right-hand side we get

$$\begin{aligned}-\bar{g}^{\mu\nu}\nabla_\mu\bar{\varphi} &= -\bar{g}^{\mu\nu}\left(\frac{\partial\bar{x}^\rho}{\partial x^\nu}\right)\frac{\partial\bar{\varphi}}{\partial\bar{x}^\rho} = \bar{g}^{\mu\nu}\frac{\partial(x^\rho - \Delta x^\rho)}{\partial x^\nu}\bar{k}_\rho = \bar{g}^{\mu\nu}\delta^\rho_\nu\bar{k}_\rho - \bar{g}^{\mu\nu}\frac{\partial\Delta x^\rho}{\partial\bar{x}^\nu}\bar{k}_\rho\\ &= \bar{k}^\mu - \bar{g}^{\mu\nu}\frac{\partial(\delta x^\rho + \bar{k}^\rho\delta\chi)}{\partial\bar{x}^\nu}\bar{k}_\rho = \bar{k}^\mu - \bar{g}^{\mu\nu}\bar{k}_\rho\frac{\partial\delta x^\rho}{\partial\bar{x}^\nu} - \bar{g}^{\mu\nu}\bar{k}_\rho\delta\chi\frac{\partial\bar{k}^\rho}{\partial\bar{x}^\nu}\\ &= \bar{k}^\mu - \bar{g}^{\mu\nu}\bar{k}_\rho\frac{\partial\delta x^\rho}{\partial\bar{x}^\nu}\,,\end{aligned} \tag{4.12}$$

where in the last passage we used

$$\bar{k}_\rho\frac{\partial\bar{k}^\rho}{\partial\bar{x}^\nu} = n_i\frac{\partial n^i}{\partial\bar{x}^\nu} = n_i\frac{\mathcal{P}^i_j}{\bar{\chi}} = 0\,. \tag{4.13}$$

As regards the last term on the right-hand side of (4.11) we obtain

$$\begin{aligned}-\bar{g}^{\mu\nu}\bar{\nabla}_\nu(\Delta x^\rho\bar{\nabla}_\rho\bar{\varphi}) &= \bar{g}^{\mu\nu}\bar{\nabla}_\nu(\delta x^\rho\bar{k}_\rho + \bar{k}^\rho\delta\chi\bar{k}_\rho) = \bar{g}^{\mu\nu}\bar{\nabla}_\nu(\delta x^\rho\bar{k}_\rho) = \bar{g}^{\mu\nu}\bar{k}_\rho\bar{\nabla}_\nu\delta x^\rho + \bar{g}^{\mu\nu}\delta x^\rho\bar{\nabla}_\nu\bar{k}_\rho \\ &= \bar{g}^{\mu\nu}\bar{k}_\rho\bar{\nabla}_\nu\delta x^\rho - \bar{g}^{\mu\nu}\delta x^\rho\bar{\nabla}_\nu\bar{\nabla}_\rho\bar{\varphi} = \bar{g}^{\mu\nu}\bar{k}_\rho\bar{\nabla}_\nu\delta x^\rho - \bar{g}^{\mu\nu}\delta x^\rho\bar{\nabla}_\rho\bar{\nabla}_\nu\bar{\varphi} \\ &= \bar{g}^{\mu\nu}\bar{k}_\rho\bar{\nabla}_\nu\delta x^\rho + \delta x^\rho\bar{\nabla}_\rho\bar{k}^\mu\,. \end{aligned} \tag{4.14}$$

As a consequence (4.11) becomes

$$\begin{aligned}k^\mu &= \bar{k}^\mu - \bar{g}^{\mu\nu}\bar{k}_\rho\frac{\partial\delta x^\rho}{\partial\bar{x}^\nu} - \delta g^{\mu\nu}\bar{\nabla}_\nu\bar{\varphi} - \bar{g}^{\mu\nu}\bar{\nabla}_\nu\delta\varphi + \bar{g}^{\mu\nu}\bar{k}_\rho\bar{\nabla}_\nu\delta x^\rho + \delta x^\rho\bar{\nabla}_\rho\bar{k}^\mu \\ &= \bar{k}^\mu - \delta g^{\mu\nu}\bar{\nabla}_\nu\bar{\varphi} - \bar{g}^{\mu\nu}\bar{\nabla}_\nu\delta\varphi + \delta x^\rho\bar{\nabla}_\rho\bar{k}^\mu\,.\end{aligned} \tag{4.15}$$

Therefore

$$\delta k^\mu(\bar{\chi}) = -\bar{g}^{\mu\nu}\bar{\nabla}_\nu\delta\varphi(\bar{x}^\mu) + \delta x^\rho\bar{\nabla}_\rho\bar{k}^\mu + \delta g^{\mu\nu}\bar{k}_\nu\,. \tag{4.16}$$

Starting from (4.16) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\varphi(\bar{x}^\mu) = -\bar{k}_\mu\delta k^\mu(\bar{\chi}) + \delta g^{\mu\nu}\bar{k}_\nu\bar{k}_\mu\,. \tag{4.17}$$

Indeed, multiplying (4.16) by $\bar{k}_\mu$ we get

$$\begin{aligned}\bar{k}_\mu\delta k^\mu(\bar{\chi}) &= -\bar{k}^\nu\bar{\nabla}_\nu\delta\varphi(\bar{x}^\mu) + \delta x^\rho\bar{k}_\mu\bar{\nabla}_\rho\bar{k}^\mu + \delta g^{\mu\nu}\bar{k}_\nu\bar{k}_\mu \\ &= -\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\varphi(\bar{x}^\mu) + \delta g^{\mu\nu}\bar{k}_\nu\bar{k}_\mu\,,\end{aligned} \tag{4.18}$$

where we used $\bar{k}_\mu\bar{\nabla}_\rho\bar{k}^\mu = 0$, which can be proved in the following way:

$$\bar{k}_\mu\bar{\nabla}_\rho\bar{k}^\mu = \bar{\nabla}_\rho(\bar{k}_\mu\bar{k}^\mu) - \bar{k}^\mu\bar{\nabla}_\rho\bar{k}_\mu = -\bar{k}^\mu\bar{\nabla}_\rho\bar{k}_\mu = -\bar{g}^{\mu\nu}\bar{k}_\nu\bar{\nabla}_\rho(\bar{g}_{\mu\lambda}\bar{k}^\lambda) = -\bar{k}_\nu\bar{\nabla}_\rho(\bar{g}^{\mu\nu}\bar{g}_{\mu\lambda}\bar{k}^\lambda) = -\bar{k}_\nu\bar{\nabla}_\rho\bar{k}^\nu\,. \tag{4.19}$$

In order to show that (4.17) is equivalent to (4.10) we proceed in the following way. Perturbing $k^\mu k_\mu = \hat{g}_{\mu\nu}k^\mu k^\nu = 0$ we obtain

$$\begin{aligned}0 &= \bar{g}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu + \bar{g}_{\mu\nu}\bar{k}^\mu\delta k^\nu + \bar{g}_{\mu\nu}\delta k^\mu\bar{k}^\nu + \delta g_{\mu\nu}\bar{k}^\mu\bar{k}^\nu \\ &= \bar{g}_{\mu\nu}\bar{k}^\mu\delta k^\nu + \bar{g}_{\nu\mu}\delta k^\mu\bar{k}^\nu + \delta g_{\mu\nu}\bar{g}^{\mu\rho}\bar{k}_\rho\bar{g}^{\nu\sigma}\bar{k}_\sigma \\ &= 2\bar{g}_{\mu\nu}\bar{k}^\mu\delta k^\nu - \delta g^{\rho\sigma}\bar{k}_\rho\bar{k}_\sigma\,,\end{aligned} \tag{4.20}$$

where we used $\bar{k}_\mu\bar{k}^\mu = 0$ and $\hat{g}^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu} = \bar{g}^{\mu\nu} - \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\delta g_{\rho\sigma}$.
Therefore

$$\delta g^{\rho\sigma}\bar{k}_\rho\bar{k}_\sigma = 2\bar{k}_\nu\delta k^\nu\,. \tag{4.21}$$

Inserting it in (4.17) we find

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\varphi(\bar{x}^\mu) = -\bar{k}_\mu\delta k^\mu(\bar{\chi}) + \delta g^{\mu\nu}\bar{k}_\nu\bar{k}_\mu = -\bar{k}_\mu\delta k^\mu(\bar{\chi}) + 2\bar{k}_\nu\delta k^\nu = \bar{k}_\mu\delta k^\mu\,, \tag{4.22}$$

which is exactly equation (4.10).

It can be noted that, defining $\varphi(\bar{x}^\mu) = \bar{\varphi}(\bar{x}^\mu) + \delta\varphi(\bar{x}^\mu)$, from (4.7) and (4.10) we get $\mathrm{d}\varphi(\bar{x}^\mu)/\mathrm{d}\bar{\chi} \neq 0$.

4.1.2 Total correction to the phase

In order to calculate $\delta\varphi$ in terms of the metric perturbations we integrate equation (4.10). Defining $\delta\varphi_o$ as the value of the perturbation $\delta\varphi$ at the position of the observer and using (3.97) we obtain

$$\begin{aligned}\delta\varphi &= \delta\varphi_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \frac{\mathrm{d}\delta\varphi}{\mathrm{d}\tilde\chi} = \delta\varphi_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \bar k_\mu \delta k^\mu(\tilde\chi) \\ &= \delta\varphi_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\delta x^0 + n_i \int_0^{\bar\chi} \mathrm{d}\tilde\chi \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\delta x^i \\ &= \delta\varphi_o + \delta x^0(\bar\chi) + \delta x_\parallel(\bar\chi) - \delta x^0_o - \delta x_{\parallel o} \\ &= \delta\varphi_o + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A - B_\parallel - \frac{1}{2}h_\parallel\right) \\ &= \delta\varphi_o - T\,.\end{aligned} \tag{4.23}$$

Given that we want to find the explicit expression for $\Delta\varphi = \delta\varphi + \Delta x^\mu \bar\nabla_\mu \bar\varphi$, which is the full correction to the phase, we proceed by calculating $\Delta x^\mu \bar\nabla_\mu \bar\varphi$. Using $\bar k_0 = \eta_{0\nu}\bar k^\nu = +1$ and (3.97) we obtain

$$\begin{aligned}\Delta x^\mu(\bar\chi)\bar\partial_\mu\bar\varphi[\bar x(\bar\chi)] &= -\Delta x^\mu \bar k_\mu = -\Delta x^0 - \Delta x^i n_i = -\Delta x^0 - \Delta x_\parallel \\ &= -(\bar k^0\delta\chi + \delta x^0 + n_i \bar k^i \delta\chi + \delta x_\parallel) = -(-\delta\chi + \delta x^0 + n_i n^i \delta\chi + \delta x_\parallel) \\ &= -\delta x^0 - \delta x_\parallel \\ &= T - \delta x^0_o - \delta x_{\parallel o}\,.\end{aligned} \tag{4.24}$$

Therefore the full correction to the phase is

$$\begin{aligned}\Delta\varphi(\bar\chi) &= \delta\varphi[\bar x^\mu(\bar\chi)] + \Delta x^\mu(\bar\chi)\bar\nabla_\mu\bar\varphi[\bar x^\mu(\bar\chi)] = \delta\varphi_o - T + T - \delta x^0_o - \delta x_{\parallel o} \\ &= \delta\varphi_o - \delta x^0_o - \delta x_{\parallel o}\,.\end{aligned} \tag{4.25}$$

4.2 Amplitude

Turning to the amplitude, we consider the following decomposition:

$$\begin{aligned}\ln\mathcal{A}\left[x^\mu(\chi)\right] &= \ln\mathcal{A}\left[\bar x^\mu(\bar\chi) + \Delta x^\mu(\bar\chi)\right] \\ &= \ln\bar{\mathcal{A}}\left[\bar x^\mu(\bar\chi)\right] + \delta\ln\mathcal{A}\left[\bar x^\mu(\bar\chi)\right] + \Delta x^\mu \bar\partial_\mu \ln\bar{\mathcal{A}}(\bar x^\mu) \\ &= \ln\bar{\mathcal{A}}[\bar x^\mu(\bar\chi)] + \Delta\ln\mathcal{A}(\bar\chi)\,.\end{aligned} \tag{4.26}$$

We perturb the evolution equation (2.81) up to linear order:

$$\left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar\chi}\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\ln\left[\mathcal{A}(\bar x^\mu + \Delta x^\mu)\,\bar a(1 + \Delta\ln a)\right] = -\frac{1}{2}\left[\frac{\partial\bar x^\sigma}{\partial x^\rho}\frac{\partial}{\partial\bar x^\sigma}(\bar k^\rho + \delta k^\rho) + \delta\hat\Gamma^\rho_{\rho\sigma}\bar k^\sigma\right], \tag{4.27}$$

where we used $\mathrm{d}\bar\chi/\mathrm{d}\chi = 1 - \mathrm{d}\delta\chi/\mathrm{d}\chi = 1 - \mathrm{d}\delta\chi/\mathrm{d}\bar\chi$. This is justified by the fact that the difference between $\mathrm{d}\delta\chi/\mathrm{d}\chi$ and $\mathrm{d}\delta\chi/\mathrm{d}\bar\chi$ is second order in the perturbations, as already explained for the calculation of the wave-vector perturbation.

Left-hand side of the evolution equation

We proceed by inserting (4.26) and (3.99) on the left-hand side of equation (4.27). We start by considering the term which contains the logarithm of the amplitude:

$$\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\ln\left[\mathcal{A}(\bar x^\mu + \Delta x^\mu)\right] = \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\ln\bar{\mathcal{A}} + \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\delta\ln\mathcal{A} + \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\Delta x^\mu\bar\partial_\mu\ln\bar{\mathcal{A}}\right]. \tag{4.28}$$

Considering (3.14) the third term on the right-hand side can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Big[\Delta x^\mu \bar{\partial}_\mu \ln \bar{\mathcal{A}}\Big] &= \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Big[\big(\bar{k}^\mu \delta\chi + \delta x^\mu\big)\bar{\partial}_\mu \ln \bar{\mathcal{A}}\Big] \\ &= \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Big[\delta\chi \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln \bar{\mathcal{A}}\Big] + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Big[\delta x^\mu \bar{\partial}_\mu \ln \bar{\mathcal{A}}\Big] \\ &= \left(\frac{\mathrm{d}^2}{\mathrm{d}\bar{\chi}^2}\ln \bar{\mathcal{A}}\right)\delta\chi + \left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln \bar{\mathcal{A}}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\chi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\bar{\partial}_\mu \ln \bar{\mathcal{A}}\right)\delta x^\mu + \big(\bar{\partial}_\mu \ln \bar{\mathcal{A}}\big)\delta k^\mu .\end{aligned} \tag{4.29}$$

Therefore (4.28) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\Big[\mathcal{A}(\bar{x}^\mu + \Delta x^\mu)\Big] = \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln \bar{\mathcal{A}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta \ln \mathcal{A} + \left(\frac{\mathrm{d}^2}{\mathrm{d}\bar{\chi}^2}\ln \bar{\mathcal{A}}\right)\delta\chi + \left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln \bar{\mathcal{A}}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\chi\right) \\ + \left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\bar{\partial}_\mu \ln \bar{\mathcal{A}}\right)\delta x^\mu + \big(\bar{\partial}_\mu \ln \bar{\mathcal{A}}\big)\delta k^\mu .\end{aligned} \tag{4.30}$$

As regards the term which depends on the scale factor, inserting (3.99) we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\big[\bar{a}(1+\Delta \ln a)\big] &= \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln \bar{a} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\left(1+\mathcal{H}\Delta x^0\right) \\ &= \frac{1}{\bar{a}}\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Big[\mathcal{H}(-\delta\chi + \delta x^0)\Big] \\ &= \frac{1}{\bar{a}}\left(-\frac{\partial}{\partial\bar{\eta}} + n^j\frac{\partial}{\partial\bar{x}^j}\right)\bar{a} + (-\delta\chi+\delta x^0)\left(-\frac{\partial}{\partial\bar{\eta}} + n^j\frac{\partial}{\partial\bar{x}^j}\right)\mathcal{H} \\ &\quad + \mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(-\delta\chi+\delta x^0) \\ &= -\frac{1}{\bar{a}}\frac{\partial\bar{a}}{\partial\bar{\eta}} - \mathcal{H}'(-\delta\chi+\delta x^0) + \mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}(-\delta\chi+\delta x^0) \\ &= -\mathcal{H} - \mathcal{H}'(-\delta\chi+\delta x^0) + \mathcal{H}\left(-\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}} + \delta k^0\right).\end{aligned} \tag{4.31}$$

Right-hand side of the evolution equation

As concerns the first term on the right-hand side of equation (4.27) we have

$$\begin{aligned}\frac{\partial\bar{x}^\sigma}{\partial x^\rho}\frac{\partial}{\partial\bar{x}^\sigma}(\bar{k}^\rho + \delta k^\rho) &= \left[\frac{\partial}{\partial x^\rho}(x^\sigma - \Delta x^\sigma)\right]\frac{\partial}{\partial\bar{x}^\sigma}(\bar{k}^\rho + \delta k^\rho) = \left[\delta^\sigma_\rho - \frac{\partial\Delta x^\sigma}{\partial\bar{x}^\rho}\right](\bar{\partial}_\sigma\bar{k}^\rho + \bar{\partial}_\sigma\delta k^\rho) \\ &= \bar{\partial}_\rho\bar{k}^\rho + \bar{\partial}_\rho\delta k^\rho - (\bar{\partial}_\rho\Delta x^\sigma)(\bar{\partial}_\sigma\bar{k}^\rho).\end{aligned} \tag{4.32}$$

Given that $\bar{\partial}_0\bar{k}^0 = 0$, the first term on the right-hand side becomes $\bar{\partial}_\rho\bar{k}^\rho = \bar{\partial}_i\bar{k}^i$. Moreover, given that $\bar{\partial}_\sigma\bar{k}^0 = 0$, we can set $\rho = i$ in the last term. As regards $\bar{\partial}_\rho\delta k^\rho$ we use (3.28). Therefore we obtain

$$\begin{aligned}\frac{\partial\bar{x}^\sigma}{\partial x^\rho}\frac{\partial}{\partial\bar{x}^\sigma}(\bar{k}^\rho + \delta k^\rho) = \bar{\partial}_i n^i + \bar{\partial}_0\delta k^0 + n^i n_i\bar{\partial}_\parallel\delta k_\parallel + n_i\bar{\partial}_\parallel\delta k^i_\perp + \delta k_\parallel\frac{\mathcal{P}^i_i}{\bar{\chi}} + n^i\bar{\partial}_{\perp i}\delta k_\parallel \\ + \bar{\partial}_{\perp i}\delta k^i_\perp - (\bar{\partial}_i\Delta x^\sigma)(\bar{\partial}_\sigma\bar{k}^i).\end{aligned} \tag{4.33}$$

Since

$$n_i\bar{\partial}_\parallel\delta k^i_\perp = \bar{\partial}_\parallel(n_i\delta k^i_\perp) = 0\,, \tag{4.34}$$

$$n^i\bar{\partial}_{\perp i}\delta k_\parallel = (n^i\bar{\partial}_i - n^i n_i\bar{\partial}_\parallel)\delta k_\parallel = (\bar{\partial}_\parallel - \bar{\partial}_\parallel)\delta k_\parallel = 0 \tag{4.35}$$

and $\bar{\partial}_0 n^i = 0$, we find

$$\frac{\partial \bar{x}^\sigma}{\partial x^\rho}\frac{\partial}{\partial \bar{x}^\sigma}(\bar{k}^\rho + \delta k^\rho) = \frac{\mathcal{P}^i_i}{\bar{\chi}} + \bar{\partial}_0 \delta k^0 + \bar{\partial}_\parallel \delta k_\parallel + \delta k_\parallel \frac{\mathcal{P}^i_i}{\bar{\chi}} + \bar{\partial}_{\perp i}\delta k^i_\perp - (\bar{\partial}_i \Delta x^j)(\bar{\partial}_j n^i)\,. \tag{4.36}$$

Finally, using (3.25), (3.14) and (3.21), we obtain

$$\begin{aligned}
\frac{\partial \bar{x}^\sigma}{\partial x^\rho}\frac{\partial}{\partial \bar{x}^\sigma}(\bar{k}^\rho + \delta k^\rho) &= \frac{2}{\bar{\chi}}(1+\delta k_\parallel) + \bar{\partial}_0 \delta k^0 + \bar{\partial}_\parallel \delta k_\parallel + \bar{\partial}_{\perp i}\delta k^i_\perp - \frac{\mathcal{P}^i_j}{\bar{\chi}}\bar{\partial}_i(\bar{k}^j \delta\chi + \delta x^j) \\
&= \frac{2}{\bar{\chi}}(1+\delta k_\parallel) + \bar{\partial}_0 \delta k^0 + \bar{\partial}_\parallel \delta k_\parallel + \bar{\partial}_{\perp i}\delta k^i_\perp - \frac{\mathcal{P}^i_j}{\bar{\chi}}\left(\frac{\mathcal{P}^j_i}{\bar{\chi}}\delta\chi + n^j \bar{\partial}_i \delta\chi + \bar{\partial}_i \delta x^j\right) \\
&= \frac{2}{\bar{\chi}}(1+\delta k_\parallel) + \bar{\partial}_0 \delta k^0 + \left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} + \bar{\partial}_0\right)\delta k_\parallel + \bar{\partial}_{\perp i}\delta k^i_\perp - \frac{2}{\bar{\chi}^2}\delta\chi \\
&\quad - \frac{1}{\bar{\chi}}\bar{\partial}_{\perp j}(n^j \delta x_\parallel + \delta x^j_\perp) \\
&= \frac{2}{\bar{\chi}}(1+\delta k_\parallel) + \bar{\partial}_0(\delta k^0 + \delta k_\parallel) + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta k_\parallel + \bar{\partial}_{\perp i}\delta k^i_\perp - \frac{2}{\bar{\chi}^2}\delta\chi - \frac{1}{\bar{\chi}}\bar{\partial}_{\perp j}\delta x^j_\perp \\
&\quad - \frac{1}{\bar{\chi}}\frac{\mathcal{P}^i_i}{\bar{\chi}}\delta x_\parallel \\
&= \frac{2}{\bar{\chi}}(1+\delta k_\parallel) + \bar{\partial}_0(\delta k^0 + \delta k_\parallel) + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta k_\parallel + \bar{\partial}_{\perp i}\delta k^i_\perp - \frac{2}{\bar{\chi}^2}(\delta\chi + \delta x_\parallel) - \frac{1}{\bar{\chi}}\bar{\partial}_{\perp j}\delta x^j_\perp\,.
\end{aligned} \tag{4.37}$$

Background solution

To the lowest order equation (4.27) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\bar{\mathcal{A}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\bar{a} &= -\frac{1}{\bar{\chi}} \\
\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln(\bar{a}\bar{\mathcal{A}}) &= -\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\bar{\chi}\,.
\end{aligned} \tag{4.38}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\left(\bar{a}\bar{\mathcal{A}}\bar{\chi}\right) = 0 \qquad \Longrightarrow \qquad \bar{\mathcal{A}}(\bar{x}^0, \bar{\chi}) = \frac{\mathcal{Q}}{\bar{a}(\bar{x}^0)\bar{\chi}} = \frac{\mathcal{Q}(1+z)}{\bar{\chi}}\,, \tag{4.39}$$

where $\mathcal{Q}$ is constant along the null geodesic. The value of $\mathcal{Q}$ is determined by the solution in the local wave zone. If we consider a gravitational wave produced by a compact binary inspiral, neglecting the post newtonian terms and considering the regime of "quasi-circular" motion, we have

$$\mathcal{Q} = \mathcal{M}_e(\pi f_e \mathcal{M}_e)^{\frac{2}{3}}\,, \tag{4.40}$$

where $\mathcal{M}_e$ is the chirp mass and f_e the frequency of the binary [68].

4.2.1 Evolution equation for $\delta \ln \mathcal{A}$

With the background solution (4.39) equation (4.27) can be simplified: we use (4.39) to calculate the derivatives $\frac{\mathrm{d}^2}{\mathrm{d}\bar{\chi}^2}\ln\bar{\mathcal{A}}$ and $\bar{\partial}_\mu \ln\bar{\mathcal{A}}$ which appear in (4.30). Using (3.7) and (3.5) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\bar{\mathcal{A}} = -\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\bar{a} - \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\bar{\chi} = \frac{\bar{\partial}_0\bar{a}}{\bar{a}} - n^j\frac{\bar{\partial}_j\bar{\chi}}{\bar{\chi}} = \mathcal{H} - n^j\frac{n_j}{\bar{\chi}} = \mathcal{H} - \frac{1}{\bar{\chi}}\,, \tag{4.41a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\bar{\chi}^2}\ln\bar{\mathcal{A}} = -\mathcal{H}' + \frac{1}{\bar{\chi}^2} \tag{4.41b}$$

and

$$\bar{\partial}_\mu \ln \bar{\mathcal{A}} = \bar{\partial}_\mu \ln\left(\frac{\mathcal{Q}}{\bar{a}(\bar{x}^0)\bar{\chi}}\right) = -\bar{\partial}_\mu \ln \bar{a} - \bar{\partial}_\mu \ln \bar{\chi} = -\frac{\bar{\partial}_\mu \bar{a}}{\bar{a}} - \frac{\bar{\partial}_\mu \bar{\chi}}{\bar{\chi}} \,, \tag{4.42}$$

which correpsonds to

$$\bar{\partial}_0 \ln \bar{\mathcal{A}} = -\mathcal{H} \,, \tag{4.43a}$$

$$\bar{\partial}_i \ln \bar{\mathcal{A}} = -\frac{n_i}{\bar{\chi}} \,. \tag{4.43b}$$

Consequently

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\bar{\partial}_\mu \ln \bar{\mathcal{A}}\right)\delta x^\mu + \left(\bar{\partial}_\mu \ln \bar{\mathcal{A}}\right)\delta k^\mu &= -\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\bar{\chi}}\delta x^0 - \left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\frac{n_i}{\bar{\chi}}\right)\delta x^i - \mathcal{H}\delta k^0 - \frac{n_i}{\bar{\chi}}\delta k^i \\ &= \mathcal{H}'\delta x^0 + \frac{1}{\bar{\chi}^2}\delta x_\parallel - \mathcal{H}\delta k^0 - \frac{1}{\bar{\chi}}\delta k_\parallel \,,\end{aligned} \tag{4.44}$$

where we used

$$\left(\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\frac{n_i}{\bar{\chi}}\right) = (-\bar{\partial}_0 + n^j\bar{\partial}_j)\frac{n_i}{\bar{\chi}} = n^j\frac{\mathcal{P}_{ij}}{\bar{\chi}^2} - \frac{n^j n_i}{\bar{\chi}^2}\bar{\partial}_j\bar{\chi} = -\frac{n^j n_i}{\bar{\chi}^2}\frac{\bar{x}_j}{\bar{\chi}} = -\frac{n_i}{\bar{\chi}^2} \,. \tag{4.45}$$

Therefore (4.30) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\left[\mathcal{A}(\bar{x}^\mu + \Delta x^\mu)\right] = \mathcal{H} - \frac{1}{\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta \ln \mathcal{A} + \left(-\mathcal{H}' + \frac{1}{\bar{\chi}^2}\right)\delta\chi + \left(\mathcal{H} - \frac{1}{\bar{\chi}}\right)\left(\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right) \\ +\mathcal{H}'\delta x^0 + \frac{1}{\bar{\chi}^2}\delta x_\parallel - \mathcal{H}\delta k^0 - \frac{1}{\bar{\chi}}\delta k_\parallel \,.\end{aligned} \tag{4.46}$$

Using (4.46) and (4.31) the left-hand side of equation (4.27) becomes

$$\begin{aligned}&\left(1 - \frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right)\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\ln\left[\mathcal{A}(\bar{x}^\mu + \Delta x^\mu)\,\bar{a}(1 + \Delta \ln a)\right] = \\ &\quad = \mathcal{H} - \frac{1}{\bar{\chi}} - \left(\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right)\left(\mathcal{H} - \frac{1}{\bar{\chi}}\right) + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta \ln \mathcal{A} + \left(-\mathcal{H}' + \frac{1}{\bar{\chi}^2}\right)\delta\chi \\ &\qquad + \left(\mathcal{H} - \frac{1}{\bar{\chi}}\right)\left(\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}}\right) + \mathcal{H}'\delta x^0 + \frac{1}{\bar{\chi}^2}\delta x_\parallel - \mathcal{H}\delta k^0 - \frac{1}{\bar{\chi}}\delta k_\parallel \\ &\qquad - \mathcal{H} + \mathcal{H}\frac{\mathrm{d}\delta\bar{\chi}}{\mathrm{d}\bar{\chi}} - \mathcal{H}'(-\delta\chi + \delta x^0) + \mathcal{H}\left(-\frac{\mathrm{d}\delta\chi}{\mathrm{d}\bar{\chi}} + \delta k^0\right) \\ &\quad = -\frac{1}{\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta \ln \mathcal{A} + \left(\frac{1}{\bar{\chi}^2}\right)\delta\chi + \frac{1}{\bar{\chi}^2}\delta x_\parallel - \frac{1}{\bar{\chi}}\delta k_\parallel \,.\end{aligned} \tag{4.47}$$

Inserting (4.47) and (4.37) in equation (4.27) we obtain

$$\begin{aligned}-\frac{1}{\bar{\chi}} + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta \ln \mathcal{A} + \left(\frac{1}{\bar{\chi}^2}\right)\delta\chi + \frac{1}{\bar{\chi}^2}\delta x_\parallel - \frac{1}{\bar{\chi}}\delta k_\parallel = -\frac{1}{\bar{\chi}}(1 + \delta k_\parallel) - \frac{1}{2}\bar{\partial}_0(\delta k^0 + \delta k_\parallel) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta k_\parallel \\ - \frac{1}{2}\bar{\partial}_{\perp i}\delta k^i_\perp + \frac{1}{\bar{\chi}^2}(\delta\chi + \delta x_\parallel) + \frac{1}{2}\frac{1}{\bar{\chi}}\bar{\partial}_{\perp j}\delta x^j_\perp - \frac{1}{2}\hat{\Gamma}^\rho_{\rho\sigma}\bar{k}^\sigma \,,\end{aligned} \tag{4.48}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta \ln \mathcal{A} = -\frac{1}{2}\bar{\partial}_0(\delta k^0 + \delta k_\parallel) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta k_\parallel - \frac{1}{2}\bar{\partial}_{\perp i}\delta k^i_\perp + \frac{1}{2}\frac{1}{\bar{\chi}}\bar{\partial}_{\perp j}\delta x^j_\perp - \frac{1}{2}\hat{\Gamma}^\rho_{\rho\sigma}\bar{k}^\sigma \,. \tag{4.49}$$

Since, using (3.27), the third term of the right-hand side of equation (4.49) can be rewritten as

$$\bar{\partial}_{\perp i}\delta k^i_\perp = \bar{\partial}_{\perp i}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta x^i_\perp = \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\bar{\partial}_{\perp i}\delta x^i_\perp + \frac{1}{\bar{\chi}}\bar{\partial}_{\perp i}\delta x^i_\perp\,, \tag{4.50}$$

and given that

$$\bar{\partial}_{\perp i}\Delta x^i_\perp = \bar{\partial}_{\perp i}(\mathcal{P}^{ij}n_j\delta\chi + \delta x^i_\perp) = \bar{\partial}_{\perp i}\delta x^i_\perp\,, \tag{4.51}$$

equation (4.49) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\ln\mathcal{A} = -\frac{1}{2}\left[\bar{\partial}_0(\delta k^0 + \delta k_\parallel) + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta k_\parallel - 2\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\kappa + \hat{\Gamma}^\rho_{\rho\sigma}\bar{k}^\sigma\right]\,, \tag{4.52}$$

where

$$\kappa = -\frac{1}{2}\bar{\partial}_{\perp i}\Delta x^i_\perp \tag{4.53}$$

is the weak lensing convergence term.

As concerns the first term of the right-hand side of equation (4.52), by using (3.64) and (3.75a) we obtain

$$\bar{\partial}_0(\delta k^0{+}\delta k_\parallel) = \bar{\partial}_0(\delta\nu{+}\delta n_\parallel) = \bar{\partial}_0\left(2A{-}B_\parallel{-}2I{-}A{-}\frac{1}{2}h_\parallel{+}2I\right) = \bar{\partial}_0\left(A{-}B_\parallel{-}\frac{1}{2}h_\parallel\right) = A'{-}B'_\parallel{-}\frac{1}{2}h'_\parallel\,. \tag{4.54}$$

Moving to the last term of equation (4.52)

$$\begin{aligned}\delta\hat{\Gamma}^\rho_{\rho\sigma}\bar{k}^\sigma &= \delta\hat{\Gamma}^\rho_{\rho 0}(-1) + \delta\hat{\Gamma}^\rho_{\rho i}n^i = -\delta\hat{\Gamma}^0_{00} - \delta\hat{\Gamma}^k_{k0} + \delta\hat{\Gamma}^0_{0i}n^i + \delta\hat{\Gamma}^k_{ki}n^i\\ &= -A' - \frac{1}{2}\left(\bar{\partial}^k B_k - \bar{\partial}_k B^k + h^{k\prime}_k\right) + n^i\bar{\partial}_i A + \frac{1}{2}n^i(\bar{\partial}_i h^k_k + \bar{\partial}_k h^k_i - \bar{\partial}^k h_{ki})\\ &= -A' - \frac{1}{2}h^{k\prime}_k + \bar{\partial}_\parallel A + \frac{1}{2}\bar{\partial}_\parallel h^k_k = \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A + \frac{1}{2}h^k_k\right)\,.\end{aligned} \tag{4.55}$$

Therefore, inserting (4.54) and (4.55) in (4.52), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta\ln\mathcal{A} = -\frac{1}{2}\left[A' - B'_\parallel - \frac{1}{2}h'_\parallel + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\delta n_\parallel - 2\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\kappa + \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A + \frac{1}{2}h^k_k\right)\right]\,. \tag{4.56}$$

4.2.2 Total correction to the amplitude

The integration of the last equation yields

$$\delta\ln\mathcal{A} = \delta\ln\mathcal{A}_o - \frac{1}{2}\left[\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left(A' - B'_\parallel - \frac{1}{2}h'_\parallel\right) + \delta n_\parallel - \delta n_{\parallel o} - 2\kappa + 2\kappa_o + A + \frac{1}{2}h^k_k - A_o - \frac{1}{2}(h^k_k)_o\right]\,. \tag{4.57}$$

Setting $\kappa_o = 0$ (being an integrated effect) and using (3.65) and (3.75a), it becomes

$$\begin{aligned}\delta\ln\mathcal{A} &= \delta\ln\mathcal{A}_o - \frac{1}{2}\left[-2I - A + A_o - \frac{1}{2}h_\parallel + \frac{1}{2}h_{\parallel o} + 2I - 2\kappa + A + \frac{1}{2}h^k_k - A_o - \frac{1}{2}(h^k_k)_o\right]\\ &= \delta\ln\mathcal{A}_o - \frac{1}{2}\left[-\frac{1}{2}h_\parallel + \frac{1}{2}h_{\parallel o} - 2\kappa + \frac{1}{2}h^k_k - \frac{1}{2}(h^k_k)_o\right]\,.\end{aligned} \tag{4.58}$$

Finally, given that $h^k_k = \delta^{kj}h_{jk} = (\delta^{kj} - n^k n^j)h_{jk} + n^k n^j h_{jk} = \mathcal{P}^{kj}h_{jk} + h_\parallel$, we obtain

$$\begin{aligned}\delta\ln\mathcal{A} &= \delta\ln\mathcal{A}_o - \frac{1}{2}\left[-2\kappa + \frac{1}{2}\mathcal{P}^{kj}h_{jk} - \frac{1}{2}\mathcal{P}^{kj}(h_{jk})_o\right]\\ &= \delta\ln\mathcal{A}_o + \kappa - \frac{1}{4}\mathcal{P}^{ij}[h_{ij} - (h_{ij})_o]\,.\end{aligned} \tag{4.59}$$

Given that, for (4.59) to be consistent, $\delta \ln \mathcal{A}_o = -\frac{1}{4}\mathcal{P}^{ij}(h_{ij})_o$, we have

$$\delta \ln \mathcal{A} = \kappa - \frac{1}{4}\mathcal{P}^{ij} h_{ij} \,. \tag{4.60}$$

Since the aim is to write the full correction to the amplitude in terms of the metric perturbations, given that $\Delta \ln \mathcal{A} = \delta \ln \mathcal{A} + \Delta x^\mu \bar{\partial}_\mu \ln \bar{\mathcal{A}}$ and we have already calculated $\delta \ln \mathcal{A}$, the next step consists in finding the expression for $\Delta x^\mu \bar{\partial}_\mu \ln \bar{\mathcal{A}}$. Using (4.43) and (4.24) we obtain

$$\begin{aligned} \Delta x^\mu \bar{\partial}_\mu \ln \bar{\mathcal{A}} &= \Delta x^0 \bar{\partial}_0 \ln \left(\frac{\mathcal{Q}}{\bar{a}\bar{\chi}} \right) + \Delta x^i \bar{\partial}_i \ln \left(\frac{\mathcal{Q}}{\bar{a}\bar{\chi}} \right) \\ &= -\mathcal{H}\Delta x^0 - \frac{n_i}{\bar{\chi}}\Delta x^i \\ &= -\Delta \ln a - \frac{\Delta x_\parallel}{\bar{\chi}} - \frac{\Delta x^0}{\bar{\chi}} + \frac{\Delta x^0}{\bar{\chi}} \\ &= -\Delta \ln a \left(1 - \frac{1}{\mathcal{H}\bar{\chi}} \right) + \frac{1}{\bar{\chi}}(T - \delta x^0_o - \delta x_{\parallel o}) \,. \end{aligned} \tag{4.61}$$

Consequently the full correction to the amplitude is given by

$$\Delta \ln \mathcal{A} = \kappa - \frac{1}{4}\mathcal{P}^{ij} h_{ij} - \Delta \ln a \left(1 - \frac{1}{\mathcal{H}\bar{\chi}} \right) + \frac{1}{\bar{\chi}}(T - \delta x^0_o - \delta x_{\parallel o}) \,. \tag{4.62}$$

Chapter 5

Correction to the luminosity distance

We start by calculating the analytical expression for the correction to the luminosity distance due to the cosmological perturbations. We will see that $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$ is related to $\Delta \ln \mathcal{A}$. After finding the explicit expression of $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$ in terms of the metric perturbations we will calculate the angular power spectrum, taking into account the scalar contribution due to the metric perturbations Φ and Ψ, the vector contribution due to the velocity term $v_\parallel$ and the tensor contributions h_{ij}^{TT} due to primordial gravitational waves.

5.1 Luminosity distance in terms of the metric perturbations

We start from the full description of the perturbed gravitational wave in the geometric optics limit:

$$\begin{aligned} h(\eta_e, \mathbf{x}_e) = \mathcal{A}(\eta_e, \mathbf{x}_e) e^{i\varphi(\eta_e, \mathbf{x}_e)} &= \bar{\mathcal{A}}\,(1 + \Delta \ln \mathcal{A}) e^{i(\bar{\varphi} + \Delta\varphi)} = \frac{\mathcal{Q}}{\bar{a}\bar{\chi}}(1 + \Delta \ln \mathcal{A}) e^{i(\bar{\varphi} + \Delta\varphi)} \\ &= \frac{\mathcal{Q}(1+z)}{\bar{\chi}}(1 + \Delta \ln \mathcal{A}) e^{i(\bar{\varphi} + \Delta\varphi)} = \frac{\mathcal{Q}(1+z)^2}{\bar{\mathcal{D}}_L}(1 + \Delta \ln \mathcal{A}) e^{i(\bar{\varphi} + \Delta\varphi)} \,, \end{aligned} \tag{5.1}$$

where $\Delta\varphi$ and $\Delta \ln \mathcal{A}$ are given by (4.25) and (4.62), the background amplitude is (4.39), $\bar{a}(\eta_e) = 1/(1+z)$ and $\bar{\mathcal{D}}_L = (1+z)\bar{\chi}$.
Given that

$$h_e = \frac{\mathcal{Q}(1+z)^2}{\mathcal{D}_L} e^{i\varphi} \,, \tag{5.2}$$

the luminosity distance is given by

$$\mathcal{D}_L = \frac{\bar{\mathcal{D}}_L}{1 + \Delta \ln \mathcal{A}} = \bar{\mathcal{D}}_L (1 - \Delta \ln \mathcal{A}) \,. \tag{5.3}$$

In other words the relative correction to the luminosity distance can be expressed in terms of the full correction to the gravitational wave amplitude:

$$\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L} = \frac{\mathcal{D}_L - \bar{\mathcal{D}}_L}{\bar{\mathcal{D}}_L} = \frac{\bar{\mathcal{D}}_L(1 - \Delta \ln \mathcal{A} - 1)}{\bar{\mathcal{D}}_L} = -\Delta \ln \mathcal{A} \,. \tag{5.4}$$

Therefore from (4.62) we obtain

$$\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L} = -\kappa + \frac{1}{4}\mathcal{P}^{ij} h_{ij} + \Delta \ln a \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right) - \frac{1}{\bar{\chi}}(T - \delta x_o^0 - \delta x_{\parallel o}) \,. \tag{5.5}$$

The next step consists in writing (5.5) in terms of the metric perturbations.

5.1.1 Explicit expression of κ

We start by finding the explicit expression for the weak lensing convergence term. By inserting (3.107) in (4.53) we obtain

$$\kappa = -\frac{1}{2}\partial_{\perp i}\Delta x^i_\perp = \kappa_1 + \kappa_2 + \kappa_3\,, \tag{5.6}$$

where

$$\kappa_1 = \frac{1}{2}\bar{\partial}_{\perp i}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\tilde{\partial}^i_\perp\left(A - B_\parallel - \frac{1}{2}h_\parallel\right), \tag{5.7a}$$

$$\kappa_2 = \frac{1}{2}\bar{\partial}_{\perp i}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\frac{\bar{\chi}}{\tilde{\chi}}\left(B^i_\perp + \mathcal{P}^i_k n^j h^k_j\right), \tag{5.7b}$$

$$\kappa_3 = -\frac{1}{2}\bar{\partial}_{\perp i}\delta x^i_{\perp o} - \frac{1}{2}\bar{\partial}_{\perp i}\bar{\chi}\left(B^i_{\perp o} - v^i_{\perp o} + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o}\right). \tag{5.7c}$$

In order to move the perpendicular derivative $\bar{\partial}_{\perp j}$ inside the integral we have to take into account that $\bar{\chi}$ can be different from $\tilde{\chi}$ (and consequently $\bar{x}^i = \bar{\chi}n^i$ is different from $\tilde{x}^i = \tilde{\chi}n^i$). Hence when the perpendicular derivative is moved inside the integral we obtain an extra factor $\tilde{\chi}/\bar{\chi}$. In order to prove it we proceed as in [69]. The perpendicular derivative can be written as

$$\bar{\partial}_{\perp j} = \mathcal{P}^k_j\frac{\partial}{\partial\bar{x}^k} = \mathcal{P}^k_j\frac{\partial\tilde{x}^i(\tilde{\chi})}{\partial\bar{x}^k}\frac{\partial}{\partial\tilde{x}^i}\,, \tag{5.8}$$

where

$$\frac{\partial}{\partial\bar{x}^k}\tilde{x}^i(\tilde{\chi}) = \frac{\partial}{\partial\bar{x}^k}\tilde{\chi}n^i = n^i\frac{\partial\tilde{\chi}}{\partial\bar{x}^k} + \tilde{\chi}\frac{\partial n^i}{\partial\bar{x}^k} = n^i\frac{\partial\tilde{\chi}}{\partial\bar{x}^k} + \tilde{\chi}\frac{\mathcal{P}^i_k}{\bar{\chi}}\,, \tag{5.9}$$

which corresponds to

$$\begin{aligned}\frac{\partial}{\partial\bar{x}^k}\tilde{x}^i(\tilde{\chi}) = n^i n_k + \mathcal{P}^i_k = n^i n_k + \delta^i_k - n^i n_k = \delta^i_k \qquad &\text{if}\quad \tilde{\chi} = \bar{\chi}\,,\\ \frac{\partial}{\partial\bar{x}^k}\bar{x}^i(\tilde{\chi}) = \frac{\tilde{\chi}}{\bar{\chi}}\mathcal{P}^i_k \qquad &\text{if}\quad \tilde{\chi} \neq \bar{\chi}\,.\end{aligned} \tag{5.10}$$

Consequently, for $\tilde{\chi} \neq \bar{\chi}$, (5.8) becomes

$$\bar{\partial}_{\perp j} = \mathcal{P}^k_j\frac{\tilde{\chi}}{\bar{\chi}}\mathcal{P}^i_k\frac{\partial}{\partial\tilde{x}^i} = \frac{\tilde{\chi}}{\bar{\chi}}\mathcal{P}^i_j\frac{\partial}{\partial\tilde{x}^i} = \frac{\tilde{\chi}}{\bar{\chi}}\tilde{\partial}_{\perp j}\,. \tag{5.11}$$

Therefore, using (5.11) and given that

$$\bar{\partial}_{\perp i}\bar{\chi} = \mathcal{P}^j_i\bar{\partial}_j\bar{\chi} = \mathcal{P}^j_i n_j = 0\,, \tag{5.12}$$

$$\tilde{\partial}_{\perp i}\mathcal{P}^i_j = \mathcal{P}^k_i\tilde{\partial}_k(\delta^i_j - n^i n_j) = -\mathcal{P}^k_i\frac{1}{\tilde{\chi}}(\mathcal{P}^i_k n_j + n^i\mathcal{P}_{jk}) = -\frac{2n_j}{\tilde{\chi}} \tag{5.13}$$

and

$$\bar{\partial}_{\perp i}v^i_{\perp o} = \bar{\partial}_{\perp i}(v^i_o - n^i n_j v^j_o) = -v_{\parallel o}\bar{\partial}_{\perp i}n^i - v^j_o n^i\bar{\partial}_{\perp i}n_j = -v_{\parallel o}\frac{\mathcal{P}^i_i}{\bar{\chi}} - v^j_o\mathcal{P}_{ij}n^i = -\frac{2}{\bar{\chi}}v_{\parallel o}\,, \tag{5.14}$$

we find

$$\kappa_1 = \frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}^2_\perp\left(A - B_\parallel - \frac{1}{2}h_\parallel\right), \tag{5.15a}$$

$$\begin{aligned}
\kappa_2 &= \frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial_{\perp i}B^i_\perp + (\tilde\partial_{\perp i}\mathcal{P}^i_k)n^j h^k_j + \mathcal{P}^i_k(\tilde\partial_{\perp i}n^j)h^k_j + \mathcal{P}^i_k n^j(\tilde\partial_{\perp i}h^k_j)\right] \\
&= \frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial_{\perp i}B^i_\perp - \frac{2n_k}{\tilde\chi}n^j h^k_j + \mathcal{P}^i_k \frac{\mathcal{P}^j_i}{\tilde\chi}h^k_j + \mathcal{P}^i_k n^j \mathcal{P}^l_i \tilde\partial_l h^k_j\right] \\
&= \frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial_{\perp i}B^i_\perp - \frac{2h_\parallel}{\tilde\chi} + \frac{\mathcal{P}^j_k}{\tilde\chi}h^k_j + n^j\mathcal{P}^i_k\tilde\partial_i h^k_j\right] \\
&= \frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial_{\perp i}B^i_\perp - \frac{3h_\parallel}{\tilde\chi} + \frac{h^i_i}{\tilde\chi} + n^j\mathcal{P}^i_k\tilde\partial_i h^k_j\right],
\end{aligned} \tag{5.15b}$$

$$\begin{aligned}
\kappa_3 &= -\frac{1}{2}\bar\partial_{\perp i}\delta x^i_{\perp o} - \frac{1}{2}\bar\partial_{\perp i}\bar\chi\left(B^i_{\perp o} - v^i_{\perp o} + \frac{1}{2}\mathcal{P}^i_k n^j h^k_{j\,o}\right) \\
&= \frac{1}{\bar\chi}\delta x_{\parallel o} + (B_{\parallel o} - v_{\parallel o}) - \frac{1}{4}\bar\chi h^j_{k\,o}(\bar\partial_{\perp i}n^k)\mathcal{P}^i_j - \frac{1}{4}\bar\chi h^j_{k\,o}n^k(\bar\partial_{\perp i}\mathcal{P}^i_j) \\
&= \frac{1}{\bar\chi}\delta x_{\parallel o} + (B_{\parallel o} - v_{\parallel o}) - \frac{1}{4}\bar\chi h^j_{k\,o}\left(\frac{\mathcal{P}^k_i}{\bar\chi}\right)\mathcal{P}^i_j - \frac{1}{4}\bar\chi h^j_{k\,o}n^k\left(-\frac{2n_j}{\bar\chi}\right) \\
&= \frac{1}{\bar\chi}\delta x_{\parallel o} + (B_{\parallel o} - v_{\parallel o}) - \frac{1}{4}(h^i_{i\,o} - h_{\parallel o}) + \frac{1}{2}h_{\parallel o} \\
&= \frac{1}{\bar\chi}\delta x_{\parallel o} + (B_{\parallel o} - v_{\parallel o}) - \frac{1}{4}(h^i_{i\,o} - 3h_{\parallel o}).
\end{aligned} \tag{5.15c}$$

5.1.2 Explicit expression of $\Delta\mathcal{D}_l/\bar{\mathcal{D}}_L$

We proceed by finding the explicit expression of $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$ in terms of the metric perturbations. We will consider the metric

$$\mathrm{d}s^2 = a(\eta)^2[-(1+2\Phi)\mathrm{d}\eta^2 + (1-2\Psi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + h^{TT}_{ij}\mathrm{d}x^i\mathrm{d}x^j], \tag{5.16}$$

which, compared to (3.8), corresponds to the following choice:

$$A = \Phi, \qquad B_i = 0, \qquad h_{ij} = -2\Psi\delta_{ij} + h^{TT}_{ij}, \tag{5.17}$$

where h^{TT}_{ij} is traceless ($h^{TT\,i}_{\ \ i} = 0$) and transverse ($\partial^i h^{TT}_{ij} = 0$). We report here all the quantities which contain the spatial part of the metric making explicit the components Ψ and h^{TT}_{ij}. As regards the projection operators applied to h_{ij} we find

$$\mathcal{P}^{ij}h_{ij} = (\delta^{ij} - n^i n^j)(-2\Psi\delta_{ij} + h^{TT}_{ij}) = -6\Psi + 2\Psi - n^i n^j h^{TT}_{ij} = -4\Psi - h^{TT}_\parallel \tag{5.18}$$

and

$$h_\parallel = -2\Psi\delta_{ij}n^i n^j + h^{TT}_\parallel = -2\Psi + h^{TT}_\parallel. \tag{5.19}$$

The trace becomes

$$h^i_i = -2\Psi\delta_{ij}\delta^{ij} = -6\Psi. \tag{5.20}$$

As concerns the Shapiro time delay and the integrated Sachs-Wolfe contribution, using the metric (5.16) we obtain respectively

$$T = -\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(\Phi + \Psi - \frac{1}{2}h^{TT}_\parallel\right) \tag{5.21}$$

and

$$I = -\frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(\Phi' + \Psi' - \frac{1}{2}h^{TT\,\prime}_\parallel\right). \tag{5.22}$$

As a consequence the relative correction to the luminosity distance written in terms of the metric perturbations becomes

$$\begin{aligned}\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L} &= -\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_\perp^2\left(\Phi+\Psi-\frac{1}{2}h_\parallel^{TT}\right)\\ &\quad-\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\left[\frac{6\Psi}{\tilde{\chi}}-\frac{3h_\parallel^{TT}}{\tilde{\chi}}-\frac{6\Psi}{\tilde{\chi}}-2n^j\mathcal{P}^i_j\tilde{\partial}_i\Psi+n^j\mathcal{P}^i_k\tilde{\partial}_i h^{TTk}{}_j\right]\\ &\quad-\frac{1}{\bar{\chi}}\delta x_{\parallel o}+v_{\parallel o}+\frac{1}{4}(-6\Psi_o+6\Psi_o-3h_{\parallel o}^{TT})-\Psi-\frac{1}{4}h_\parallel^{TT}\\ &\quad+\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\left[-\Phi+v_\parallel+\delta a_o+\Phi_o-v_{\parallel o}-\int_0^{\bar{\chi}} d\tilde{\chi}\left(\Phi'+\Psi'-\frac{1}{2}h_\parallel^{TT\prime}\right)\right]\\ &\quad+\frac{1}{\bar{\chi}}\int_0^{\bar{\chi}} d\tilde{\chi}\left(\Phi+\Psi-\frac{1}{2}h_\parallel^{TT}\right)+\frac{1}{\bar{\chi}}(\delta x_o^0+\delta x_{\parallel o})\,.\end{aligned} \tag{5.23}$$

Finally, since

$$\begin{aligned}-\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,n^j\mathcal{P}^i_k\tilde{\partial}_i h^{TTk}{}_j &= -\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,n^j\tilde{\partial}_{\perp k}h^{TTk}{}_j = -\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,(n^j\tilde{\partial}_k h^{TTk}{}_j - n^j n_k\tilde{\partial}_\parallel h^{TTk}{}_j)\\ &= \frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,\tilde{\partial}_\parallel h_\parallel^{TT} = \frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\left(\frac{d}{d\tilde{\chi}}+\frac{\partial}{\partial\tilde{\eta}}\right)h_\parallel^{TT}\\ &= \frac{1}{2}h_\parallel^{TT}-\frac{1}{2}h_{\parallel o}^{TT}+\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,h_\parallel^{TT\prime}\,,\end{aligned} \tag{5.24}$$

we obtain

$$\begin{aligned}\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L} &= -\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_\perp^2\left(\Phi+\Psi-\frac{1}{2}h_\parallel^{TT}\right)+\frac{1}{2}h_\parallel^{TT}-\frac{1}{2}h_{\parallel o}^{TT}-\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\left[-\frac{3h_\parallel^{TT}}{\tilde{\chi}}-h_\parallel^{TT\prime}\right]\\ &\quad+v_{\parallel o}-\frac{3}{4}h_{\parallel o}^{TT}-\Psi-\frac{1}{4}h_\parallel^{TT}\\ &\quad+\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\left[-\Phi+v_\parallel+\delta a_o+\Phi_o-v_{\parallel o}-\int_0^{\bar{\chi}} d\tilde{\chi}\left(\Phi'+\Psi'-\frac{1}{2}h_\parallel^{TT\prime}\right)\right]\\ &\quad+\frac{1}{\bar{\chi}}\int_0^{\bar{\chi}} d\tilde{\chi}\left(\Phi+\Psi-\frac{1}{2}h_\parallel^{TT}\right)+\frac{1}{\bar{\chi}}\delta x_o^0\,.\end{aligned} \tag{5.25}$$

For the following calculations it is useful to separate the contributions from scalar, vector and tensor perturbations. We denote them respectively by the indices (S), (V) and (T). We have

$$\begin{aligned}\left[\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}\right]^{(S)} &= -\frac{1}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_\perp^2(\Phi+\Psi)-\Psi+\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\left[-\Phi-\int_0^{\bar{\chi}} d\tilde{\chi}\,(\Phi'+\Psi')\right]\\ &\quad+\frac{1}{\bar{\chi}}\int_0^{\bar{\chi}} d\tilde{\chi}\,(\Phi+\Psi)+\frac{1}{\bar{\chi}}\delta x_o^0+\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)(\Phi_o+\delta a_o)\,,\end{aligned} \tag{5.26}$$

$$\left[\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}\right]^{(V)} = \left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)v_\parallel+\frac{1}{\mathcal{H}\bar{\chi}}v_{\parallel o} \tag{5.27}$$

and

$$\begin{aligned}\left[\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}\right]^{(T)} &= \frac{1}{4}h_\parallel^{TT}+\frac{3}{2}\int_0^{\bar{\chi}} d\tilde{\chi}\,\frac{1}{\tilde{\chi}}h_\parallel^{TT}-\frac{1}{2}\frac{1}{\bar{\chi}}\int_0^{\bar{\chi}} d\tilde{\chi}\,h_\parallel^{TT}+\left(1-\frac{1}{2}\frac{1}{\mathcal{H}\bar{\chi}}\right)\int_0^{\bar{\chi}} d\tilde{\chi}\,h_\parallel^{TT\prime}\\ &\quad+\frac{1}{4}\int_0^{\bar{\chi}} d\tilde{\chi}(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_\perp^2 h_\parallel^{TT}-\frac{5}{4}h_{\parallel o}^{TT}\,.\end{aligned} \tag{5.28}$$

5.2 Angular power spectrum

We proceed by calculating the angular power spectrum associated with $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$. In order to define the angular power spectrum we start from the expansion of a scalar function A in terms of the spherical harmonics $Y_{lm}(\mathbf{n})$:

$$A(\mathbf{n}) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\mathbf{n}) \tag{5.29}$$

where, using Y^*_{lm} to denote the complex conjugate of Y_{lm},

$$\alpha_{lm} = \int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) A(\mathbf{n})\,. \tag{5.30}$$

The last expression can be easily proved by inserting (5.29) in (5.30) and using

$$\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) Y_{l'm'}(\mathbf{n}) = \delta_{ll'}\delta_{mm'}\,. \tag{5.31}$$

Indeed we obtain

$$\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) \sum_{l'm'} \alpha_{l'm'} Y_{l'm'}(\mathbf{n}) = \sum_{l'm'} \alpha_{l'm'} \int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) Y_{l'm'}(\mathbf{n}) = \sum_{l'm'} \alpha_{l'm'}\delta_{ll'}\delta_{mm'} = \alpha_{lm}\,. \tag{5.32}$$

The angular power spectrum C_l is defined as follows:

$$C_l = \frac{1}{2l+1}\sum_{m=-l}^{+l} \langle \alpha^*_{lm}\alpha_{lm}\rangle\,, \tag{5.33}$$

where $\langle ...\rangle$ is an ensemble average.

Our aim is to calculate $C_l^{\mathcal{D}_L}$, which is the angular power spectrum associated with $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$. In the final part of the previous section we saw that there are scalar (S), vector (V) and tensor (T) contributions to $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$. In the end we will obtain an expression of this type:

$$C_l^{\mathcal{D}_L} = C_l^{(S+V)} + C_l^{(T)}\,. \tag{5.34}$$

In order to calculate $C_l^{\mathcal{D}_L}$ we need to find the explicit expression of

$$\alpha_{lm}^{\mathcal{D}_L} = \int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) \frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}\,. \tag{5.35}$$

We start by calculating the α_{lm}'s associated with the terms which contain the scalar perturbations Ψ and Φ.

5.2.1 Contributions from Φ and Ψ

The first term which we consider is $[\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L]^{(S1)} = -\Psi - [1 - 1/(\mathcal{H}\bar{\chi})]\Phi$. We use the generic term A to indicate $-\Psi$ or $-[1 - 1/(\mathcal{H}\bar{\chi})]\Phi$. We proceed by inserting in (5.30) the Fourier decomposition

$$A(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} A(\mathbf{k},\eta)\,. \tag{5.36}$$

Using the plane wave expansion in spherical harmonics

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(kx) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{x}})\,, \tag{5.37}$$

we find

$$\begin{aligned}\alpha_{lm} &= \int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\bar{\chi}\mathbf{n}} A(\mathbf{k},\eta) \\ &= \int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, 4\pi \sum_{l'm'} i^{l'} j_{l'}(k\bar{\chi}) Y^*_{l'm'}(\hat{\mathbf{k}}) Y_{l'm'}(\mathbf{n}) A(\mathbf{k},\eta) \\ &= 4\pi \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \sum_{l'm'} i^{l'} j_{l'}(k\bar{\chi}) Y^*_{l'm'}(\hat{\mathbf{k}}) A(\mathbf{k},\eta) \int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) Y_{l'm'}(\mathbf{n})\,,\end{aligned} \tag{5.38}$$

where j_l are the spherical Bessel functions (see Appendix C). Proceeding with the calculations, using (5.31), we obtain

$$\begin{aligned}\alpha_{lm} &= 4\pi \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \sum_{l'm'} i^{l'} j_{l'}(k\bar{\chi}) Y^*_{l'm'}(\hat{\mathbf{k}}) A(\mathbf{k},\eta) \delta_{ll'}\delta_{mm'} \\ &= 4\pi \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, i^l j_l(k\bar{\chi}) Y^*_{lm}(\hat{\mathbf{k}}) A(\mathbf{k},\eta) \\ &= 4\pi \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, i^l j_l(k\bar{\chi}) Y^*_{lm}(\hat{\mathbf{k}}) \mathcal{T}_A(k,\eta) \Psi_p(\mathbf{k})\,.\end{aligned} \tag{5.39}$$

In the last step we wrote $A(\mathbf{k},\eta)$ in terms of the primordial value $\Psi_p(\mathbf{k})$ set during inflation:

$$A(\mathbf{k},\eta) = \mathcal{T}_A(k,\eta)\Psi_p(\mathbf{k})\,, \tag{5.40}$$

where $\mathcal{T}_A(k,\eta)$ is the transfer function. In our case we have

$$\mathcal{T}_\Psi = \frac{9}{10} T_m(k) \frac{\mathcal{G}_\Psi(a,k)}{a}\,, \tag{5.41a}$$

$$\mathcal{T}_\Phi = \frac{9}{10} T_m(k) \frac{\mathcal{G}_\Phi(a,k)}{a}\,. \tag{5.41b}$$

For the details see the introductory chapter, section 1.4.3. Therefore we obtain

$$\alpha^{(S1)}_{lm} = -4\pi \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} Y^*_{lm}(\hat{\mathbf{k}}) i^l j_l(k\bar{\chi}) \frac{9}{10} T_m(k) \left[\frac{\mathcal{G}_\Psi(\bar{a},k)}{\bar{a}} + \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right) \frac{\mathcal{G}_\Phi(\bar{a},k)}{\bar{a}} \right] \Psi_p(\mathbf{k})\,. \tag{5.42}$$

We proceed by considering the term

$$\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}^{(S2)} = \frac{1}{\bar{\chi}} \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, (\Phi + \Psi)\,. \tag{5.43}$$

We find

$$\alpha^{(S2)}_{lm} = 4\pi \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} Y^*_{lm}(\hat{\mathbf{k}}) \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, \frac{1}{\bar{\chi}} i^l j_l(k\tilde{\chi}) \frac{9}{10} T_m(k) \left[\frac{\mathcal{G}_\Phi(\tilde{a},k) + \mathcal{G}_\Psi(\tilde{a},k)}{\tilde{a}} \right] \Psi_p(\mathbf{k})\,. \tag{5.44}$$

As concerns the term

$$\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}^{(S3)} = -\left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right) \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, (\Phi' + \Psi') \tag{5.45}$$

we obtain

$$\begin{aligned}\alpha^{(S3)}_{lm} &= -4\pi \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right) \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} Y^*_{lm}(\hat{\mathbf{k}}) \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, i^l j_l(k\tilde{\chi}) [\mathcal{T}'_\Phi(k,\tilde{\eta}) + \mathcal{T}'_\Psi(k,\tilde{\eta})] \Psi_p(\mathbf{k}) \\ &= -4\pi \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right) \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} Y^*_{lm}(\hat{\mathbf{k}}) \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, i^l j_l(k\tilde{\chi}) \frac{9}{10} T_m(k) \tilde{a}\mathcal{H}(\tilde{a}) \\ &\qquad \times \frac{\mathrm{d}}{\mathrm{d}\tilde{a}} \left[\frac{\mathcal{G}_\Phi(\tilde{a},k) + \mathcal{G}_\Psi(\tilde{a},k)}{\tilde{a}} \right] \Psi_p(\mathbf{k})\,.\end{aligned} \tag{5.46}$$

Finally, as regards the term

$$\frac{\Delta\mathcal{D}_l}{\bar{\mathcal{D}}_l}^{(S4)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_\perp^2(\Phi+\Psi)\,, \tag{5.47}$$

we obtain

$$\begin{aligned}
\alpha_{lm}^{(S4)} &= -\frac{1}{2}\int \mathrm{d}\Omega\, Y_{lm}^*(\mathbf{n})\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_\perp^2(\Phi+\Psi) \\
&= -\frac{1}{2}\int \mathrm{d}\Omega\, Y_{lm}^*(\mathbf{n})\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\frac{\tilde{\chi}}{\bar{\chi}}\frac{\Delta_\Omega}{\tilde{\chi}^2}4\pi\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\sum_{l'm'} i^{l'} j_{l'}(k\tilde{\chi})Y_{l'm'}^*(\hat{\mathbf{k}})Y_{l'm'}(\mathbf{n}) \\
&\qquad\times[\Phi(\mathbf{k},\tilde{\eta})+\Psi(\mathbf{k},\tilde{\eta})] \\
&= -4\pi\int \mathrm{d}\Omega\, Y_{lm}^*(\mathbf{n})\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\frac{(\bar{\chi}-\tilde{\chi})}{2\bar{\chi}\tilde{\chi}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\sum_{l'm'} i^{l'} j_{l'}(k\tilde{\chi})Y_{l'm'}^*(\hat{\mathbf{k}})\Delta_\Omega Y_{l'm'}(\mathbf{n}) \\
&\qquad\times[\Phi(\mathbf{k},\tilde{\eta})+\Psi(\mathbf{k},\tilde{\eta})]\,.
\end{aligned} \tag{5.48}$$

In the last equations we used (3.29) and

$$\Delta_\Omega \equiv \tilde{\chi}^2\tilde{\nabla}_\perp^2 = \tilde{\chi}^2\left(\tilde{\nabla}^2 - \frac{2}{\tilde{\chi}}\tilde{\partial}_\parallel - \tilde{\partial}_\parallel^2\right) = \partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\,, \tag{5.49}$$

which is the angular part of the Laplacian. Given that

$$\Delta_\Omega Y_{lm} = -l(l+1)Y_{lm} \tag{5.50}$$

we get

$$\begin{aligned}
\alpha_{lm}^{(S4)} &= 4\pi\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\frac{(\bar{\chi}-\tilde{\chi})}{2\bar{\chi}\tilde{\chi}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\sum_{l'm'} i^{l'} j_{l'}(k\tilde{\chi})Y_{l'm'}^*(\hat{\mathbf{k}})[\Phi(\mathbf{k},\eta)+\Psi(\mathbf{k},\eta)] \\
&\qquad\times\int \mathrm{d}\Omega\, l'(l'+1)Y_{lm}^*(\mathbf{n})Y_{l'm'}(\mathbf{n})\,.
\end{aligned} \tag{5.51}$$

Using (5.31) and (5.41), we obtain

$$\alpha_{lm}^{(S4)} = l(l+1)4\pi\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}Y_{lm}^*(\hat{\mathbf{k}})\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\frac{(\bar{\chi}-\tilde{\chi})}{2\bar{\chi}\tilde{\chi}} i^l j_l(k\tilde{\chi})\frac{9}{10}T_m(k)\left[\frac{\mathcal{G}_\Phi(\tilde{a},k)+\mathcal{G}_\Psi(\tilde{a},k)}{\tilde{a}}\right]\Psi_p(\mathbf{k})\,. \tag{5.52}$$

Each contribution $\alpha_{ml}^{(Sr)}$ calculated up to now can be written as

$$\alpha_{ml}^{(Sr)} = 4\pi i^l\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}Y_{lm}^*(\hat{\mathbf{k}})\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, j_l(k\tilde{\chi})W_{(Sr)}\left[\bar{\chi},\tilde{\chi},\bar{\eta},\tilde{\eta},\frac{\partial}{\partial\tilde{\eta}},\delta_D(\bar{\chi}-\tilde{\chi})\right]\mathcal{T}_{(Sr)}(k,\eta_0-\tilde{\chi})\Psi_p(\mathbf{k})\,. \tag{5.53}$$

For example, as concerns $\alpha_{ml}^{(S4)}$,

$$W_{(S4)} = l(l+1)\frac{\bar{\chi}-\tilde{\chi}}{2\bar{\chi}\tilde{\chi}} \tag{5.54}$$

and

$$\mathcal{T}_{(S4)} = \frac{9}{10}T_m(k)\left[\frac{\mathcal{G}_\Phi(\tilde{a},k)+\mathcal{G}_\Psi(\tilde{a},k)}{\tilde{a}}\right]\,. \tag{5.55}$$

Therefore, in order to find the contributions to $C_l^{\mathcal{D}_L}$ from scalar perturbations we have to take into account terms of this type:

$$\begin{aligned}\langle \alpha_{lm}^{(Sr)*}\alpha_{lm}^{(Sr')}\rangle &= \Big\langle (4\pi)^2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}Y_{lm}(\hat{\mathbf{k}})\int_0^{\bar\chi}\mathrm{d}\tilde\chi\, j_l(k\tilde\chi)W_{(Sr)}(\tilde\chi)\mathcal{T}_{(Sr)}(k,\eta_0-\tilde\chi)\Psi_p^*(\mathbf{k})\\ &\quad\times\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}Y^*_{lm}(\hat{\mathbf{k}}')\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\, j_l(k'\tilde\chi')W_{(Sr')}(\tilde\chi')\mathcal{T}_{(Sr')}(k',\eta_0-\tilde\chi')\Psi_p(\mathbf{k}')\Big\rangle\\ &= (4\pi)^2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\, j_l(k\tilde\chi)W_{(Sr)}\mathcal{T}_{(Sr)}\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\, j_l(k'\tilde\chi')W_{(Sr')}\mathcal{T}_{(Sr')}\\ &\quad\times Y_{lm}(\mathbf{k})Y^*_{lm}(\mathbf{k}')\langle\Psi_p^*(\mathbf{k})\Psi_p(\mathbf{k}')\rangle\end{aligned}\tag{5.56}$$

Since

$$\langle\Psi_p^*(\mathbf{k})\Psi_p(\mathbf{k}')\rangle = (2\pi)^3\delta_D^3(\mathbf{k}-\mathbf{k}')P_\Psi(k)\tag{5.57}$$

we find

$$\langle \alpha_{lm}^{(Sr)*}\alpha_{lm}^{(Sr')}\rangle = (4\pi)^2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\, j_l(k\tilde\chi)W_{(Sr)}\mathcal{T}_{(Sr)}\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\, j_l(k\tilde\chi')W_{(Sr')}\mathcal{T}_{(Sr')}|Y_{lm}(\hat{\mathbf{k}})|^2P_\Psi(k)\tag{5.58}$$

Given that, using (5.31), the angular part of the integral, $\int\mathrm{d}\Omega\,|Y_{lm}(\hat{\mathbf{k}})|^2$, is equal to 1, we obtain

$$\begin{aligned}\langle \alpha_{lm}^{(Sr)*}\alpha_{lm}^{(Sr')}\rangle &= \frac{2}{\pi}\int_0^\infty\mathrm{d}k\,k^2P_\Psi(k)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\, j_l(k\tilde\chi)W_{(Sr)}\left[\bar\chi,\tilde\chi,\bar\eta,\tilde\eta,\frac{\partial}{\partial\tilde\eta},\delta_D(\bar\chi-\tilde\chi)\right]\mathcal{T}_{(Sr)}(k,\eta_0-\tilde\chi)\\ &\quad\times\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\, j_l(k\tilde\chi')W_{(Sr')}\left[\bar\chi,\tilde\chi',\bar\eta,\tilde\eta',\frac{\partial}{\partial\tilde\eta'},\delta_D(\bar\chi-\tilde\chi')\right]\mathcal{T}_{(Sr')}(k,\eta_0-\tilde\chi')\\ &= \frac{2}{\pi}\int_0^\infty\mathrm{d}k\,k^2P_\Psi(k)F_l^{(Sr)}(k)F_l^{(Sr')}(k)\,,\end{aligned}\tag{5.59}$$

where

$$F_l^{(Sr)}(k)\equiv\int_0^{\bar\chi}\mathrm{d}\tilde\chi\, j_l(k\tilde\chi)W_{(Sr)}\left[\bar\chi,\tilde\chi,\bar\eta,\tilde\eta,\frac{\partial}{\partial\tilde\eta},\delta_D(\bar\chi-\tilde\chi)\right]\mathcal{T}_{(Sr)}(k,\eta_0-\tilde\chi)\,.\tag{5.60}$$

Observer terms

We proceed by considering the scalar contributions due the terms evaluated at the observer. We start from $[1-1/(\mathcal{H}\bar\chi)]\Phi_o$. We find that its contribution to $\alpha_{lm}^{\mathcal{D}_l}$ is given by

$$\alpha_{lm}^{\Phi_o} = \left(1-\frac{1}{\mathcal{H}\bar\chi}\right)\lim_{\bar\chi\to0}4\pi\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}i^lj_l(k\bar\chi)Y^*_{lm}(\hat{\mathbf{k}})\mathcal{T}_\Phi(k,\eta_0-\bar\chi)\Psi_p(\mathbf{k})\,.\tag{5.61}$$

Since the only non-zero contribution comes from $l=0$ and given that

$$\lim_{\bar\chi\to0}j_0(k\bar\chi)=1\,,\tag{5.62}$$

we obtain

$$\alpha_{lm}^{\Phi_o} = \delta_{l0}\left(1-\frac{1}{\mathcal{H}\bar\chi}\right)4\pi\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}Y^*_{0m}(\hat{\mathbf{k}})\mathcal{T}_\Phi(k,\eta_0)\Psi_p(\mathbf{k})\,.\tag{5.63}$$

Terefore in this case we have

$$F_l^{\Phi_o}(k)\equiv\delta_{l0}\left(1-\frac{1}{\mathcal{H}\bar\chi}\right)\mathcal{T}_\Phi(k,\eta_0)\,.\tag{5.64}$$

As regards the observer term $(1/\bar{\chi})\delta x^0_o$ we proceed as follows. In section 3.1.5 we found the expression for the coordinate time lapse (3.83), which is given by

$$\delta x^0_o = -\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \Phi(\bar{\eta}, \mathbf{0})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}\,. \tag{5.65}$$

Therefore

$$\begin{aligned}
\alpha_{lm}^{\delta x^0_o} &= -\frac{1}{\bar{\chi}}\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n})\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \Phi(\bar{\eta}, \mathbf{0})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta} \\
&= -\frac{1}{\bar{\chi}}\lim_{\bar{\chi}\to 0}\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n})\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\bar{\chi}\mathbf{n}}\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \mathcal{T}_\Phi(k,\bar{\eta})\Psi_p(\mathbf{k})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta} \\
&= -\frac{1}{\bar{\chi}}4\pi\lim_{\bar{\chi}\to 0}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}i^l j_l(k\bar{\chi})Y^*_{lm}(\hat{\mathbf{k}})\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \mathcal{T}_\Phi(k,\bar{\eta})\Psi_p(\mathbf{k})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta} \\
&= -\delta_{l0}\frac{1}{\bar{\chi}}4\pi\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}Y^*_{0m}(\hat{\mathbf{k}})\Psi_p(\mathbf{k})\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \mathcal{T}_\Phi(k,\bar{\eta})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}\,.
\end{aligned} \tag{5.66}$$

An analogous reasoning can be applied to the observer term $[1 - 1/(\mathcal{H}\bar{\chi})]\delta a_o$. We report here (3.89), which is the explicit expression for δa_o. We have

$$\delta a_o = \mathcal{H}_0\delta\eta_o = -\mathcal{H}_0\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \Phi(\bar{\eta}, \mathbf{0})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}\,. \tag{5.67}$$

Therefore we obtain

$$\alpha_{lm}^{\delta a_o} = -\delta_{l0}\mathcal{H}_0\left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right)4\pi\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}Y^*_{0m}(\hat{\mathbf{k}})\Psi_p(\mathbf{k})\int_{\bar{\eta}_{in}}^{\bar{\eta}_0} \mathcal{T}_\Phi(k,\bar{\eta})\bar{a}(\bar{\eta})\mathrm{d}\bar{\eta}\,. \tag{5.68}$$

5.2.2 Contribution from $\boldsymbol{v}_\parallel$

We proceed calculating the contribution to $\alpha_{lm}^{\mathcal{D}_L}$ due to the term

$$\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}^{(V)} = \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right)v_\parallel\,. \tag{5.69}$$

Assuming an irrotational velocity field, which implies $v_j(\mathbf{k},\eta) = ik_j v(\mathbf{k},\eta)$, we have

$$\begin{aligned}
\alpha_{lm}^{(V)} &= \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right)\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n})\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\bar{\chi}\mathbf{n}}n^j v_j(\mathbf{k},\eta) \\
&= \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right)\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n})\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\bar{\chi}\mathbf{n}}in^j k_j v(\mathbf{k},\eta)\,,
\end{aligned} \tag{5.70}$$

where $v(\mathbf{k},\eta)$ is the velocity potential. Given that

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}e^{i\mathbf{k}\cdot\bar{\chi}\mathbf{n}} = in^j k_j\,, \tag{5.71}$$

we find

$$\alpha_{lm}^{(V)} = \left(1 - \frac{1}{\mathcal{H}\bar{\chi}}\right)\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n})\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}e^{i\mathbf{k}\cdot\bar{\chi}\mathbf{n}}\right]v(\mathbf{k},\eta)\,. \tag{5.72}$$

Inserting (5.37) we obtain

$$\begin{aligned}
\alpha_{lm}^{(V)} &= \left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\int \mathrm{d}\Omega\, Y_{lm}^*(\mathbf{n})\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}v(\mathbf{k},\eta)\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}4\pi\sum_{l'm'} i^{l'} j_{l'}(k\bar{\chi})Y_{l'm'}^*(\hat{\mathbf{k}})Y_{l'm'}(\hat{\mathbf{n}}) \\
&= \left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}v(\mathbf{k},\eta)4\pi k\sum_{l'm'} i^{l'}\left[\frac{\mathrm{d}}{\mathrm{d}(k\bar{\chi})}j_{l'}(k\bar{\chi})\right]Y_{l'm'}^*(\hat{\mathbf{k}})\int \mathrm{d}\Omega\, Y_{lm}^*(\mathbf{n})Y_{l'm'}(\hat{\mathbf{n}}) \\
&= 4\pi\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}k i^l\left[\frac{\mathrm{d}}{\mathrm{d}(k\bar{\chi})}j_l(k\bar{\chi})\right]Y_{lm}^*(\hat{\mathbf{k}})v(\mathbf{k},\eta)\,.
\end{aligned} \tag{5.73}$$

We can substitute the derivative of the Bessel functions with

$$\partial_x j_l = -j_{l+1} + \frac{l}{x}j_l \tag{5.74}$$

which is obtained from the identities

$$\partial_x j_l = j_{l-1} - \frac{l+1}{x}j_l \tag{5.75}$$

and

$$j_{l-1} + j_{l+1} = \frac{2l+1}{x}j_l\,. \tag{5.76}$$

Therefore (5.73) becomes

$$\alpha_{lm}^{(V)} = 4\pi\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}k i^l\left[\frac{l}{k\bar{\chi}}j_l(k\bar{\chi}) - j_{l+1}(k\bar{\chi})\right]Y_{lm}^*(\hat{\mathbf{k}})\mathcal{T}_v(k,\eta)\Psi_p(\mathbf{k})\,. \tag{5.77}$$

In this case the transfer function is given by

$$\mathcal{T}_v(a,k) = -\frac{9}{10}\frac{T_m(k)}{k}\mathcal{G}_v(a,k)\,. \tag{5.78}$$

As concerns the contributions $\langle\alpha_{lm}^{(V)*}\alpha_{lm}^{(V)}\rangle$ and $\langle\alpha_{lm}^{(Sr)*}\alpha_{lm}^{(V)}\rangle$ we obtain a result similar to (5.59), the only difference being the expression of $F_l^{(V)}(k)$, which is given by

$$F_l^{(V)}(k) \equiv k\left[\frac{l}{k\bar{\chi}}j_l(k\bar{\chi}) - j_{l+1}(k\bar{\chi})\right]\mathcal{T}_v(k,\eta_0-\bar{\chi})\,. \tag{5.79}$$

Observer terms

In order to find the contribution from the observer term $[1/(\mathcal{H}\bar{\chi})]v_{\|o}$ we proceed in the following way.

$$\alpha_{lm}^{v_{\|o}} = 4\pi\frac{1}{\mathcal{H}\bar{\chi}}\lim_{\bar{\chi}\to 0}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}k i^l\left[\frac{l}{k\bar{\chi}}j_l(k\bar{\chi}) - j_{l+1}(k\bar{\chi})\right]Y_{lm}^*(\hat{\mathbf{k}})\mathcal{T}_v(k,\eta)\Psi_p(\mathbf{k})\,. \tag{5.80}$$

Given that the only non-zero contribution comes from $l=1$ and

$$\lim_{\bar{\chi}\to 0}\frac{j_1(k\bar{\chi})}{k\bar{\chi}} = \frac{1}{3}\,, \tag{5.81}$$

we obtain

$$\alpha_{lm}^{v_{\|o}} = \delta_{l1}\frac{1}{3}4\pi\frac{1}{\mathcal{H}\bar{\chi}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}kY_{1m}^*(\hat{\mathbf{k}})\mathcal{T}_v(k,\eta)\Psi_p(\mathbf{k})\,. \tag{5.82}$$

We are now ready to calculate $C_l^{(S+V)}$, which is the contribution to $C_l^{\mathcal{D}_L}$ from scalar and vector perturbations.

5.2.3 Contribution to $C_l^{\mathcal{D}_L}$ from scalar and vector perturbations

The scalar and vector contributions to $C_l^{\mathcal{D}_L}$ are given by

$$C_l^{(S+V)} = \frac{2}{\pi}\int \mathrm{d}k\, k^2 \Big[I_l^{(S+V)}(k)\Big]^2 P_\Psi(k)\,, \tag{5.83}$$

where [1]

$$\begin{aligned} I_l^{(S+V)} = \frac{9}{10}T_m(k)\Bigg\{ & -j_l(k\bar{\chi})\left[\frac{\mathcal{G}_\Psi(\bar{a},k)}{\bar{a}} + \left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\frac{\mathcal{G}_\Phi(\bar{a},k)}{\bar{a}}\right] \\ & - \left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\left[\frac{l}{k\bar{\chi}}j_l(k\bar{\chi}) - j_{l+1}(k\bar{\chi})\right]\mathcal{G}_v(\bar{a},k) \\ & + \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, j_l(k\tilde{\chi})\frac{1}{\bar{\chi}}\left(\frac{\mathcal{G}_\Psi(\tilde{a},k)+\mathcal{G}_\Phi(\tilde{a},k)}{\tilde{a}}\right) \\ & - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, j_l(k\tilde{\chi})\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\tilde{a}\mathcal{H}(\tilde{a})\frac{\mathrm{d}}{\mathrm{d}\tilde{a}}\left(\frac{\mathcal{G}_\Psi(\tilde{a},k)+\mathcal{G}_\Phi(\tilde{a},k)}{\tilde{a}}\right) \\ & + \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\, j_l(k\tilde{\chi})\frac{l(l+1)(\bar{\chi}-\tilde{\chi})}{2\bar{\chi}\tilde{\chi}}\left(\frac{\mathcal{G}_\Psi(\tilde{a},k)+\mathcal{G}_\Phi(\tilde{a},k)}{\tilde{a}}\right) \\ & + \delta_{l0}\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\frac{\mathcal{G}_\Phi(a_0,k)}{a_0} - \delta_{l0}\left[\frac{1}{\bar{\chi}} + \mathcal{H}_0\left(1-\frac{1}{\mathcal{H}\bar{\chi}}\right)\right]\int_{\bar{\eta}_{in}}^{\bar{\eta}_0}\mathcal{G}_\Phi(k,\bar{a})\mathrm{d}\bar{\eta} \\ & - \delta_{l1}\frac{1}{3}\frac{1}{\mathcal{H}\bar{\chi}}\mathcal{G}_v(a_0,k)\Bigg\}\,. \end{aligned} \tag{5.84}$$

5.2.4 Contributions from $h_{\parallel}^{TT}$

The final step consists in calculating the tensor contributions to the angular power spectrum associated with the relative correction to the luminosity distance. We follow a procedure similar to what has been done in [70] for the calculation of the effects of the stochastic GW background on the large scale-structure observables.

We start from the decomposition of the tensor perturbation h_{ij}^{TT} into plane waves of the two polarization tensors. We have

$$\begin{aligned} h_{ij}^{TT}(\mathbf{x},\eta) &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} h_{ij}^{TT}(\mathbf{k},\eta) \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}[h^+(\mathbf{k},\eta)e_{ij}^+(\hat{\mathbf{k}}) + h^\times(\mathbf{k},\eta)e_{ij}^\times(\hat{\mathbf{k}})]\,, \end{aligned} \tag{5.85}$$

where the polarization tensors $e_{ij}^\lambda(\hat{\mathbf{k}})$, which are denoted as $\lambda = +, \times$, are transverse and traceless and normalized through $e_{ij}^\lambda e^{\lambda' ij} = 2\delta^{\lambda\lambda'}$. The solution $h^\lambda(\mathbf{k},\eta)$ at a generic time η can be written as

$$h^\lambda(\mathbf{k},\eta) \equiv h_{prim}^\lambda(\mathbf{k})\mathcal{T}_T(k,\eta)\,, \tag{5.86}$$

where

- $h_{prim}^\lambda(\mathbf{k})$ is the primordial gravitational wave mode which remains constant on super-horizon scales;
- $\mathcal{T}_T(k,\eta)$ describes the sub-horizon evolution of the gravitational wave modes when they enter the horizon after the phase of accelerated expansion (see section 1.4.3).

[1]For numerical estimations, which are not considered in this thesis, we should multiply the right-hand side of (5.84) by $\bar{\chi}^2\mathcal{W}_x$, where $\mathcal{W}_x$ is the normalized object selection function, and integrate.

The power spectra of the polarizations are given by

$$\Big\langle h_\lambda(\mathbf{k},\eta)\, h_{\lambda'}(\mathbf{k}',\eta')\Big\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}\frac{1}{4}P_T(k,\eta,\eta')\,, \tag{5.87}$$

where

$$P_T(k,\eta,\eta') = \mathcal{T}_T(k,\eta)\mathcal{T}_T(k,\eta')P_{T0}(k) \tag{5.88}$$

and P_{T0} is the primordial tensor power spectrum (see section 1.4.3).

We report here (5.28), which contains all the tensor contributions to the relative correction to the luminosity distance.

$$\begin{aligned}\left[\frac{\Delta\mathcal{D}_L}{\bar{\mathcal{D}}_L}\right]^{(T)} &= \frac{1}{4}h_\parallel^{TT} + \frac{3}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{1}{\tilde\chi}h_\parallel^{TT} - \frac{1}{2}\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,h_\parallel^{TT} + \left(1-\frac{1}{2}\frac{1}{\mathcal{H}\bar\chi}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,h_\parallel^{TT\,\prime}\\ &+\frac{1}{4}\int_0^{\bar\chi}\mathrm{d}\tilde\chi(\bar\chi-\tilde\chi)\frac{\tilde\chi}{\bar\chi}\tilde\nabla_\perp^2 h_\parallel^{TT} - \frac{5}{4}h_{\parallel o}^{TT}\,.\end{aligned} \tag{5.89}$$

Each term in (5.89) can be written in the generic form

$$\begin{aligned}A(\mathbf{n},\bar\chi) &= \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_A\left[\tilde\chi,\bar\chi,\frac{\partial}{\partial\tilde\eta},\delta_D(\bar\chi-\tilde\chi)\right]h_\parallel^{TT}(\tilde\chi\mathbf{n},\eta_0-\tilde\chi)\\ &= \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_A\left[\tilde\chi,\bar\chi,\frac{\partial}{\partial\tilde\eta},\delta_D(\bar\chi-\tilde\chi)\right]\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{n}\tilde\chi}n^in^jh_{ij}^{TT}(\mathbf{k},\eta_0-\tilde\chi)\,.\end{aligned} \tag{5.90}$$

We proceed considering the contribution of a plane wave tensor perturbation with the wave-vector $\mathbf{k}$ oriented along the z-axis. We denote this contribution to $A(\mathbf{n},\bar\chi)$ as $A(\mathbf{n},\mathbf{k},\bar\chi)$. In this case

$$n^in^jh_{ij}^{TT}(\mathbf{k},\eta) = \sin^2\theta\,[\cos 2\phi\, h^+(\mathbf{k},\eta) + \sin 2\phi\, h^\times(\mathbf{k},\eta)] = \sin^2\theta\,[e^{i2\phi}h_1 + e^{-i2\phi}h_2]\,, \tag{5.91}$$

where

$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{5.92}$$

and

$$h_{1,2} \equiv \frac{1}{2}(h^+ \mp ih^\times)\,. \tag{5.93}$$

are the circular polarizations. Therefore

$$A(\mathbf{n},\mathbf{k},\bar\chi) = \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_A(\tilde\chi)e^{ik\tilde\chi\mu}(1-\mu^2)\big[e^{i2\phi}h_1(\mathbf{k},\eta_0-\tilde\chi) + e^{-i2\phi}h_2(\mathbf{k},\eta_0-\tilde\chi)\big]\,, \tag{5.94}$$

where $\mu = \cos\theta$ is the cosine of the angle between between the direction of observation $\mathbf{n}$ and the wave-vector $\mathbf{k}$. Comparing it with the scalar case, we see that the main difference is given by the factors $e^{\pm i2\phi}$.

The multipole coefficients of A can be written as

$$\begin{aligned}\alpha_{lm}^A &= \int\mathrm{d}\Omega\,Y_{lm}^*(\mathbf{n})A(\mathbf{n},\bar\chi) = \int\mathrm{d}\Omega\,Y_{lm}^*(\mathbf{n})\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}A(\mathbf{n},\mathbf{k},\bar\chi)\\ &= \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_A(\tilde\chi)\int\mathrm{d}\Omega\,Y_{lm}^*(\mu,\phi)e^{ik\tilde\chi\mu}(1-\mu^2)\big[e^{i2\phi}h_1(\mathbf{k},\eta_0-\tilde\chi)+e^{-i2\phi}h_2(\mathbf{k},\eta_0-\tilde\chi)\big]\\ &= \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\alpha_{lm}^A(\mathbf{k})\,,\end{aligned} \tag{5.95}$$

where

$$\alpha_{lm}^A(\mathbf{k}) = \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_A(\tilde\chi)\int\mathrm{d}\Omega\,Y_{lm}^*(\mu,\phi)e^{ik\tilde\chi\mu}(1-\mu^2)\big[e^{i2\phi}h_1(\mathbf{k},\eta_0-\tilde\chi)+e^{-i2\phi}h_2(\mathbf{k},\eta_0-\tilde\chi)\big]\,. \tag{5.96}$$

Given that

$$\int \mathrm{d}\Omega\, Y^*_{lm}(1-\mu^2)e^{\pm i2\phi}e^{ix\mu} = -\sqrt{4\pi(2l+1)}\sqrt{\frac{(l+2)!}{(l-2)!}}\,i^l\frac{j_l(x)}{x^2}\delta_{m\pm2}\,, \tag{5.97}$$

we obtain

$$\alpha^A_{lm}(\mathbf{k}) = -\sqrt{\frac{2l+1}{4\pi}\frac{(l+2)!}{(l-2)!}}4\pi i^l\int_0^{\bar\chi}\mathrm{d}\tilde\chi\, W_A(\tilde\chi)\big[h_1(\mathbf{k},\eta_0-\tilde\chi)\delta_{m2}+h_2(\mathbf{k},\eta_0-\tilde\chi)\delta_{m-2}\big]\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2} \tag{5.98}$$

The proof of (5.97) is given in Appendix C. Before proceeding we calculate the power spectra of h_1 and h_2. We find

$$\begin{aligned}\Big\langle h_1^*(\mathbf{k},\eta)\,h_1(\mathbf{k}',\eta)\Big\rangle &= \frac{1}{4}\Big\langle\big[h^+(\mathbf{k},\eta)+ih^\times(\mathbf{k},\eta)\big]\big[h^+(\mathbf{k}',\eta)-ih^\times(\mathbf{k}',\eta)\big]\Big\rangle\\ &= \frac{1}{4}\Big\langle h^+(\mathbf{k},\eta)\,h^+(\mathbf{k}',\eta)\Big\rangle+\frac{1}{4}\Big\langle h^\times(\mathbf{k},\eta)\,h^\times(\mathbf{k}',\eta)\Big\rangle\\ &= \frac{1}{8}(2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P_T(k)\,,\end{aligned} \tag{5.99}$$

where we used (5.87). With an analogous reasoning we obtain

$$\Big\langle h_2^*(\mathbf{k},\eta)\,h_2(\mathbf{k}',\eta)\Big\rangle = \frac{1}{8}(2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P_T(k)\,, \tag{5.100a}$$

$$\Big\langle h_2^*(\mathbf{k},\eta)\,h_1(\mathbf{k}',\eta)\Big\rangle = \Big\langle h_1^*(\mathbf{k},\eta)\,h_2(\mathbf{k}',\eta)\Big\rangle = 0\,. \tag{5.100b}$$

We are now ready to calculate the tensor contribution $C_l^{(T)}$ to the angular power spectrum $C_l^{\mathcal{D}_L}$. Each term of $C_l^{(T)}$ will be of this type:

$$\frac{1}{2l+1}\sum_m\Big\langle\alpha^{(Tr)*}_{lm}\alpha^{(Tr')}_{lm}\Big\rangle\,. \tag{5.101}$$

Inserting (5.98) in it we obtain

$$\begin{aligned}\frac{1}{2l+1}\sum_m\Big\langle\alpha^{(Tr)*}_{lm}\alpha^{(Tr')}_{lm}\Big\rangle &=\\ &= \frac{(l+2)!}{(l-2)!}4\pi\sum_m\Big\langle\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_{(Tr)}(\tilde\chi)\big[h_1^*(\mathbf{k},\eta_0-\tilde\chi)\delta_{m2}+h_2^*(\mathbf{k},\eta_0-\tilde\chi)\delta_{m-2}\big]\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\\ &\quad\times\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\,W_{(Tr')}(\tilde\chi')\big[h_1(\mathbf{k}',\eta_0-\tilde\chi')\delta_{m2}+h_2(\mathbf{k},'\eta_0-\tilde\chi')\delta_{m-2}\big]\frac{j_l(k'\tilde\chi')}{(k'\tilde\chi')^2}\Big\rangle\\ &= \frac{(l+2)!}{(l-2)!}4\pi\sum_m\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_{(Tr)}(\tilde\chi)\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\,W_{(Tr')}(\tilde\chi')\frac{j_l(k'\tilde\chi')}{(k'\tilde\chi')^2}\\ &\quad\times\big[\langle h_1^*(\mathbf{k},\eta_0-\tilde\chi)h_1(\mathbf{k}',\eta_0-\tilde\chi')\rangle(\delta_{m2})^2+\langle h_2^*(\mathbf{k},\eta_0-\tilde\chi)h_2(\mathbf{k}',\eta_0-\tilde\chi')\rangle(\delta_{m-2})^2\big]\\ &= \frac{(l+2)!}{(l-2)!}4\pi\sum_m\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_{(Tr)}(\tilde\chi)\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\,W_{(Tr')}(\tilde\chi')\frac{j_l(k'\tilde\chi')}{(k'\tilde\chi')^2}\\ &\quad\times\left[(2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\frac{P_T}{8}(\delta_{m2})^2+(2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\frac{P_T}{8}(\delta_{m-2})^2\right],\end{aligned} \tag{5.102}$$

where we used (5.99) and (5.100). Proceeding with the calculations we find

$$\begin{aligned}\frac{1}{2l+1}\sum_m\left\langle\alpha_{lm}^{(Tr)*}\alpha_{lm}^{(Tr')}\right\rangle &= \frac{(l+2)!}{(l-2)!}4\pi\sum_m\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_{(Tr)}(\tilde\chi)\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\,W_{(Tr')}(\tilde\chi')\frac{j_l(k\tilde\chi')}{(k\tilde\chi')^2}\\ &\quad\times\frac{P_T}{8}\left[(\delta_{m2})^2+(\delta_{m-2})^2)\right]\\ &= \frac{(l+2)!}{(l-2)!}4\pi\int\frac{4\pi k^2}{(2\pi)^3}\mathrm{d}k\,\frac{2}{8}P_{T0}(k)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_{(Tr)}(\tilde\chi)\mathcal{T}_T(k,\eta_0-\tilde\chi)\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\\ &\quad\times\int_0^{\bar\chi}\mathrm{d}\tilde\chi'\,W_{(Tr')}(\tilde\chi')\mathcal{T}_T(k,\eta_0-\tilde\chi')\frac{j_l(k\tilde\chi')}{(k\tilde\chi')^2}\,.\end{aligned} \tag{5.103}$$

Finally, defining

$$F_l^X(k)\equiv\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,W_x\left[\tilde\chi,\bar\chi,\frac{\partial}{\partial\tilde\eta},\delta_D(\bar\chi-\tilde\chi)\right]\mathcal{T}_T(k,\eta_0-\tilde\chi)\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\,, \tag{5.104}$$

we obtain

$$\frac{1}{2l+1}\sum_m\left\langle\alpha_{lm}^{(Tr)*}\alpha_{lm}^{(Tr')}\right\rangle=\frac{1}{2\pi}\frac{(l+2)!}{(l-2)!}\int k^2\mathrm{d}k\,P_{T0}(k)F_l^{(Tr)}(k)F_l^{(Tr')}(k)\,. \tag{5.105}$$

Observer term

We proceed by calculating the contribution due to the observer term $-(5/4)h_{\|o}$. Considering that the only non-zero term comes from $l=2$ and given that

$$\lim_{\bar\chi\to0}\frac{j_2(k\bar\chi)}{(k\bar\chi)^2}=\frac{1}{15}\,, \tag{5.106}$$

we obtain

$$\begin{aligned}F_l^{h_{\|o}}(k) &= -\frac{5}{4}\lim_{\bar\chi\to0}\mathcal{T}_T(k,\eta_0-\bar\chi)\frac{j_l(k\bar\chi)}{(k\bar\chi)^2}\\ &= -\delta_{l2}\frac{5}{4}\frac{1}{15}\mathcal{T}_T(k,\eta_0)\,.\end{aligned} \tag{5.107}$$

5.2.5 Contribution to $C_l^{\mathcal{D}_L}$ form tensor perturbations

The tensor contribution to $C_l^{\mathcal{D}_L}$ is given by

$$C_l^{(T)}=\frac{1}{2\pi}\frac{(l+2)!}{(l-2)!}\int k^2\mathrm{d}k\left[I_l^{(T)}(k)\right]^2P_{T0}(k)\,, \tag{5.108}$$

where

$$\begin{aligned}I_l^{(T)}(k) &= \frac{1}{4}\frac{j_l(k\bar\chi)}{(k\bar\chi)^2}\mathcal{T}_T(k,\eta_0-\bar\chi)-\frac{1}{2\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\mathcal{T}_T(k,\eta_0-\tilde\chi)\\ &\quad+\frac{3}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\frac{1}{\tilde\chi}\mathcal{T}_T(k,\eta_0-\tilde\chi)+\left(1-\frac{1}{2}\frac{1}{\mathcal{H}\bar\chi}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\frac{\mathrm{d}}{\mathrm{d}\tilde\eta}\mathcal{T}_T(k,\eta_0-\tilde\chi)\\ &\quad-\frac{l(l+1)}{4}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{\bar\chi-\tilde\chi}{\bar\chi\tilde\chi}\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\mathcal{T}_T(k,\eta_0-\tilde\chi)-\frac{\delta_{l2}}{12}\mathcal{T}_T(k,\eta_0)\,.\end{aligned} \tag{5.109}$$

It can be seen that the sum of the two terms which contain $1/\tilde\chi$, which is given by

$$\left[\frac{3}{2}-\frac{l(l+1)}{4}\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{j_l(k\tilde\chi)}{(k\tilde\chi)^2}\frac{1}{\tilde\chi}\mathcal{T}_T(k,\eta_0-\tilde\chi)\,, \tag{5.110}$$

is regular for $l\geq3$ and null for $l=2$. Indeed $\lim_{\tilde\chi\to0}j_l(k\tilde\chi)/(k\tilde\chi)^2=0$ for $l\geq3$ and $3/2-l(l+1)/4=0$ for $l=2$.

Chapter 6

Conclusions

In this work we studied the propagation of gravitational waves through a perturbed FRW Universe in the limit of geometric optics and considered the luminosity distance inferred from the GW waveform associated with a merging binary system. In particular we calculated the tensor contribution to the correction to the estimate of the luminosity distance. Indeed the GW signal coming from a merging binary system should be affected by the primordial GWs present in the spacetime through which the signal propagates and therefore can encode informations about them. Therefore we calculated the imprint of the tensor contribution on the angular power spectrum $C_l^{\mathcal{D}_L}$ associated with the relative correction to the luminosity distance $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$.

Chapter 2 was entirely focused on the description of the geometric optics approximation [60, 61], which consists in separating the background metric $\tilde{g}_{\mu\nu}$ from the metric perturbation $h_{\mu\nu}$ relying on their different scales of variation: the perturbation $h_{\mu\nu}$ varies on a length scale λ smaller than the scale of variation L_B of the background. The propagation equation for the gravitational waves was obtained by subtracting from the Einstein's equations the low frequency part by means of an averaging procedure on an intermediated scale $\lambda \ll \bar{l} \ll L_B$ (section 2.1.2). We took into account the leading and next-to-leading orders in λ/L_B, neglecting terms of order $(\lambda/L_B)^0$. In this way the propagation equation becomes gauge invariant and its expression can be simplified choosing the Lorenz gauge (2.57). The final part of the chapter consisted in inserting the ansazt (2.63) in the propagation equation, thus obtaining the evolution equations for the amplitude and the phase of the gravitational waves and finding that gravitational waves in the limit of geometric optics propagate on null geodesics of the background. Finally the equations were written in terms of the comoving metric $\hat{g}_{\mu\nu} = \tilde{g}_{\mu\nu}/a^2$, which proves to be convenient in the following chapters.

Since in the thesis we assumed a perturbed FRW Universe, the background metric $\hat{g}_{\mu\nu} = \tilde{g}_{\mu\nu}/a^2$ was additionally split in the metric $\bar{g}_{\mu\nu}$ associated with a homogeneous and isotropic FRW Universe and the perturbation $\delta g_{\mu\nu}$ due to cosmological perturbations. In Chapter 3 we introduced the *Cosmic Rulers* formalism [64, 65] and provided a map between the observer's frame, which is characterized by the assumption that we live in an unperturbed FRW Universe described by the metric $\bar{g}_{\mu\nu}$, and the real frame, where we take into account the effects of the cosmological perturbations and therefore we consider the perturbed metric $\bar{g}_{\mu\nu} + \delta g_{\mu\nu}$. From the integration of the differential equation for the wave-vector perturbation δk^μ we obtained the perturbations δx^μ at fixed comoving distance in terms of the metric perturbations. With respect to [46] we took into account the perturbations of the observer's coordinate (section 3.1.5) and worked in a general gauge. Finally, in section 3.1.7 we calculated the total correction $\Delta x^\mu = \delta x^\mu + \bar{k}^\mu \delta\chi$ finding the expressions for $\delta\chi$, Δx^0, $\Delta x_\parallel$ and $\Delta x^i_\perp$ in terms of the metric perturbations.

Chapter 4 was entirely devoted to the calculation of the effects of the cosmological perturbations on the GW waveform. We inserted $\bar{\varphi}[\bar{x}^\mu(\bar{\chi})] + \Delta\varphi(\bar{\chi})$ and $\ln\bar{\mathcal{A}}[\bar{x}^\mu(\bar{\chi})] + \Delta\ln\mathcal{A}(\bar{\chi})$ in the evolution equations (2.65) and (2.81) for the phase and the amplitude calculated at the end of Chapter 2, thus obtaining the corrections $\Delta\varphi(\bar{\chi})$ and $\Delta \ln \mathcal{A}(\bar{\chi})$ in terms of the metric perturbations (equations (4.25) and (4.62)).

In Chapter 5, given that the total correction to the amplitude is related to the relative correction to the luminosity distance by (5.4), we calculated the expression for $\Delta\mathcal{D}_L/\bar{\mathcal{D}}_L$, which is given by (5.25). The final step consisted in the calculation of the angular power spectrum $C_l^{\mathcal{D}_L}$ associated with the relative correction to the luminosity distance. The final expression $C_l^{\mathcal{D}_L} = C_l^{(S+V)} + C_l^{(T)}$ is given by two contributions. The first one is due to the scalar and vector corrections which contain the metric perturbations Φ, Ψ and the velocity term $v_\parallel = \mathbf{n}\cdot\mathbf{v}$, the second one to the tensor corrections coming from the primordial gravitational waves h_{ij}^{TT}. We see from (5.108) that $C_l^{(T)}$ encodes information about primordial gravitational waves through the presence of the primordial tensor power spectrum $P_{T0}(k)$. Future work requires to include a numerical evaluation of the tensor contributions to $C_l^{\mathcal{D}_L}$ in order to estimate the impact of these relativistic corrections.

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Appendix A

Conformal transformations

The following part is based on Appendix G of [71] and Appendix D of [72].
A metric $\hat{g}_{\mu\nu}$ arises from $\tilde{g}_{\mu\nu}$ via a conformal transformation if

$$\hat{g}_{\mu\nu} = \omega^2(x)\tilde{g}_{\mu\nu}\,, \tag{A.1}$$

where $\omega^2(x)$ is a nonvanishing function.
It can be easily verified that the inverse conformal transformation is

$$\hat{g}^{\mu\nu} = \frac{1}{\omega^2(x)}\tilde{g}^{\mu\nu}\,. \tag{A.2}$$

Indeed

$$\hat{g}^{\mu\rho}\hat{g}_{\rho\nu} = \frac{1}{\omega^2(x)}\tilde{g}^{\mu\rho}\omega^2(x)\tilde{g}_{\rho\nu} = \tilde{g}^{\mu\rho}\tilde{g}_{\rho\nu} = \delta^\mu_\nu\,. \tag{A.3}$$

In our case, since $\hat{g}_{\mu\nu} = \frac{\tilde{g}_{\mu\nu}}{a^2}$,

$$\omega = \frac{1}{a}\,. \tag{A.4}$$

The covariant derivatives associated with $\tilde{g}_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ are denoted respectively as $\tilde{\nabla}_\mu$ and $\hat{\nabla}_\mu$. The Christoffel symbols $\hat{\Gamma}^\mu_{\nu\rho}$ associated with $\hat{g}_{\mu\nu}$ can be written as $\hat{\Gamma}^\mu_{\nu\rho} = \tilde{\Gamma}^\mu_{\nu\rho} + C^\mu_{\nu\rho}$, where $C^\mu_{\nu\rho}$ is a tensor, given that it is the difference of two connections. In order to find the expression of $C^\mu_{\nu\rho}$ in terms of a and $\tilde{g}_{\mu\nu}$ we proceed in the following way:

$$\begin{aligned}
\hat{\Gamma}^\mu_{\nu\rho} &= \frac{1}{2}\hat{g}^{\mu\sigma}\Big[\partial_\nu\hat{g}_{\rho\sigma} + \partial_\rho\hat{g}_{\nu\sigma} - \partial_\sigma\hat{g}_{\nu\rho}\Big] \\
&= \frac{1}{2\omega^2}\tilde{g}^{\mu\sigma}\Big[\partial_\nu(\omega^2\tilde{g}_{\rho\sigma}) + \partial_\rho(\omega^2\tilde{g}_{\nu\sigma}) - \partial_\sigma(\omega^2\tilde{g}_{\nu\rho})\Big] \\
&= \tilde{\Gamma}^\mu_{\nu\rho} + \frac{1}{2\omega^2}\tilde{g}^{\mu\sigma}2\omega\Big[\tilde{g}_{\rho\sigma}\partial_\nu\omega + \tilde{g}_{\nu\sigma}\partial_\rho\omega - \tilde{g}_{\nu\rho}\partial_\sigma\omega\Big] \\
&= \tilde{\Gamma}^\mu_{\nu\rho} + \frac{1}{\omega}\Big[\delta^\mu_\rho\tilde{\nabla}_\nu\omega + \delta^\mu_\nu\tilde{\nabla}_\rho\omega - \tilde{g}^{\mu\sigma}\tilde{g}_{\nu\rho}\tilde{\nabla}_\sigma\omega\Big] \\
&= \tilde{\Gamma}^\mu_{\nu\rho} + C^\mu_{\nu\rho}\,,
\end{aligned} \tag{A.5}$$

where

$$C^\mu_{\nu\rho} \equiv \frac{1}{\omega}\left(\delta^\mu_\rho\tilde{\nabla}_\nu\omega + \delta^\mu_\nu\tilde{\nabla}_\rho\omega - \tilde{g}^{\mu\sigma}\tilde{g}_{\nu\rho}\tilde{\nabla}_\sigma\omega\right). \tag{A.6}$$

In the following part we will demonstrate that conformal transformations leave null geodesics invariant.
We begin by showing that null curves are left null under a conformal transformation. Indeed

the tangent vector of a curve $x^\mu(l)$ which is null with respect to $\tilde{g}_{\mu\nu}$ is also null with respect to $\hat{g}_{\mu\nu}$:

$$\hat{g}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}l}\frac{\mathrm{d}x^\nu}{\mathrm{d}l} = \frac{\tilde{g}_{\mu\nu}}{a^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}l}\frac{\mathrm{d}x^\nu}{\mathrm{d}l} = 0\,. \tag{A.7}$$

The next step consists in showing that a null geodesic with respect to $\tilde{\nabla}_\sigma$ is also a null geodesic with respect to $\hat{\nabla}_\sigma$.

If $x^\rho(l)$ is a geodesic with respect to $\tilde{\nabla}_\sigma$ and l is an affine parameter, the geodesic equation takes the form

$$\tilde{k}^\rho\tilde{\nabla}_\rho\tilde{k}^\sigma = 0\,, \tag{A.8}$$

where $\tilde{k}^\rho = \frac{\mathrm{d}x^\rho}{\mathrm{d}l}$.

In order to see if $\tilde{k}^\rho$ satisfies the geodesic equation with respect to $\hat{\nabla}_\sigma$, we analyze $\tilde{k}^\rho\hat{\nabla}_\rho\tilde{k}^\sigma$. By using (A.5) and (A.8) we obtain

$$\begin{aligned}
\tilde{k}^\rho\hat{\nabla}_\rho\tilde{k}^\sigma &= \tilde{k}^\rho\partial_\rho\tilde{k}^\sigma + \tilde{k}^\rho\hat{\Gamma}^\sigma_{\rho\lambda}\tilde{k}^\lambda \\
&= \tilde{k}^\rho\partial_\rho\tilde{k}^\sigma + \tilde{k}^\rho\tilde{\Gamma}^\sigma_{\rho\lambda}\tilde{k}^\lambda + \tilde{k}^\rho C^\sigma_{\rho\lambda}\tilde{k}^\lambda \\
&= \tilde{k}^\rho\tilde{\nabla}_\rho\tilde{k}^\sigma + a\tilde{k}^\rho\left[\delta^\sigma_\rho\tilde{\nabla}_\lambda\left(\frac{1}{a}\right) + \delta^\sigma_\lambda\tilde{\nabla}_\rho\left(\frac{1}{a}\right) - \tilde{g}^{\sigma\tau}\tilde{g}_{\rho\lambda}\tilde{\nabla}_\tau\left(\frac{1}{a}\right)\right]\tilde{k}^\lambda \\
&= -\tilde{k}^\sigma\tilde{k}^\lambda\tilde{\nabla}_\lambda(\ln a) - \tilde{k}^\sigma\tilde{k}^\rho\tilde{\nabla}_\rho(\ln a) + \tilde{k}^\rho\tilde{k}_\rho\tilde{g}^{\sigma\tau}\tilde{\nabla}_\tau(\ln a)\,.
\end{aligned} \tag{A.9}$$

Therefore in general a geodesic with respect to $\tilde{\nabla}_\rho$ is not a geodesic with respect to $\hat{\nabla}_\rho$. However, if we consider the specific case of null geodesics ($\tilde{k}^\rho\tilde{k}_\rho = 0$), the previous equation becomes

$$\tilde{k}^\rho\hat{\nabla}_\rho\tilde{k}^\sigma = -2\tilde{k}^\sigma\tilde{k}^\rho\tilde{\nabla}_\rho(\ln a)\,, \tag{A.10}$$

which is a non-affinely parametrized geodesic equation. Indeed, if $x^\mu(\chi)$ is a geodesic with respect to $\hat{\nabla}_\sigma$ and χ is an affine parameter, the geodesic equation written in terms of a generic parameter $l(\chi)$ becomes

$$\begin{aligned}
\frac{\mathrm{d}l}{\mathrm{d}\chi}\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\frac{\mathrm{d}x^\mu}{\mathrm{d}l}\right) + \hat{\Gamma}^\mu_{\nu\rho}\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\right)^2\frac{\mathrm{d}x^\nu}{\mathrm{d}l}\frac{\mathrm{d}x^\rho}{\mathrm{d}l} &= 0 \\
\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\right)^2\frac{\mathrm{d}^2x^\mu}{\mathrm{d}l^2} + \frac{\mathrm{d}^2l}{\mathrm{d}\chi^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}l} + \hat{\Gamma}^\mu_{\nu\rho}\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\right)^2\frac{\mathrm{d}x^\nu}{\mathrm{d}l}\frac{\mathrm{d}x^\rho}{\mathrm{d}l} &= 0
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\frac{\mathrm{d}^2x^\mu}{\mathrm{d}l^2} + \hat{\Gamma}^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}l}\frac{\mathrm{d}x^\rho}{\mathrm{d}l} &= -\frac{\frac{\mathrm{d}^2l}{\mathrm{d}\chi^2}}{\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\right)^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}l} \\
\tilde{k}^\rho\hat{\nabla}_\rho\tilde{k}^\mu &= -\frac{\frac{\mathrm{d}^2l}{\mathrm{d}\chi^2}}{\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\right)^2}\tilde{k}^\mu\,.
\end{aligned} \tag{A.12}$$

Comparing it to equation (A.9) we get

$$\frac{\frac{\mathrm{d}^2l}{\mathrm{d}\chi^2}}{\left(\frac{\mathrm{d}l}{\mathrm{d}\chi}\right)^2} = 2\frac{\mathrm{d}\chi}{\mathrm{d}l}\frac{\mathrm{d}}{\mathrm{d}\chi}\ln a\,. \tag{A.13}$$

Thus the relation between the two parameters l and χ is given by

$$\frac{\mathrm{d}l}{\mathrm{d}\chi} = \mathbb{C}a^2\,, \tag{A.14}$$

where $\mathbb{C}$ is a constant of proportionality.
Therefore

$$\tilde{k}^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}l} = \frac{\mathrm{d}\chi}{\mathrm{d}l}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\chi} = \frac{1}{\mathbb{C}}\frac{1}{a^2}\hat{k}^{\mu}\,, \tag{A.15}$$

where $\hat{k}^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\chi}$.
Summarizing, if $x^{\mu}(l)$ is a null geodesic with respect to $\tilde{\nabla}_{\sigma}$ and l is an affine parameter, $x^{\mu}(\chi) = x^{\mu}[l(\chi)]$ is a null geodesic with respect to $\hat{\nabla}_{\sigma}$ (and χ is an affine parameter if $\mathrm{d}l \propto a^2\mathrm{d}\chi$). In other words, if the null vector $\tilde{k}^{\mu}$ satisfies $\tilde{k}^{\rho}\tilde{\nabla}_{\rho}\tilde{k}^{\sigma} = 0$ and $\mathrm{d}l \propto a^2\mathrm{d}\chi$, $\hat{k}^{\mu} \propto a^2\tilde{k}^{\mu}$ satisfies $\hat{k}^{\rho}\hat{\nabla}_{\rho}\hat{k}^{\sigma} = 0$.

Appendix B

Connection coefficients in a general gauge

The components of $\tilde{g}_{\mu\nu} = a^2 \hat{g}_{\mu\nu}$ and $\tilde{g}^{\mu\nu} = (1/a^2)\hat{g}^{\mu\nu}$ are given by

$$
\begin{aligned}
\tilde{g}_{00} &= a^2 \hat{g}_{00} = -a^2(1+2A)\,, & \tilde{g}^{00} &= \frac{1}{a^2}\hat{g}^{00} = -\frac{1}{a^2}(1-2A)\,, \\
\tilde{g}_{0i} &= a^2 \hat{g}_{0i} = -a^2 B_i\,, & \tilde{g}^{0i} &= \frac{1}{a^2}\hat{g}^{0i} = -\frac{1}{a^2}B^i\,, \\
\tilde{g}_{ij} &= a^2 \hat{g}_{ij} = a^2(\delta_{ij}+h_{ij})\,, & \tilde{g}^{ij} &= \frac{1}{a^2}\hat{g}^{ij} = -\frac{1}{a^2}(\delta^{ij}-h^{ij})\,.
\end{aligned}
\tag{B.1}
$$

We calculate, at linear order in the metric perturbations, the Christoffel symbols $\hat{\Gamma}^{\mu}_{\nu\rho}$ associated to the comoving metric $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. Since

$$
\hat{\Gamma}^{\mu}_{\nu\rho} = \delta\hat{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2}\bar{g}^{\mu\sigma}(\partial_\rho \delta g_{\nu\sigma} + \partial_\nu \delta g_{\rho\sigma} - \partial_\sigma \delta g_{\nu\rho})\,,
\tag{B.2}
$$

we find

$$
\begin{aligned}
\hat{\Gamma}^{0}_{00} = \delta\hat{\Gamma}^{0}_{00} &= \frac{1}{2}\bar{g}^{0\sigma}(\partial_0\delta g_{0\sigma} + \partial_0\delta g_{0\sigma} - \partial_\sigma\delta g_{00}) = \frac{1}{2}\bar{g}^{00}\partial_0\delta g_{00} = -\frac{1}{2}\partial_0(-2A) \\
&= \partial_0 A\,, \\
\hat{\Gamma}^{0}_{0i} = \delta\hat{\Gamma}^{0}_{0i} &= \frac{1}{2}\bar{g}^{0\sigma}(\partial_i\delta g_{0\sigma} + \partial_0\delta g_{i\sigma} - \partial_\sigma\delta g_{0i}) = \frac{1}{2}\bar{g}^{00}(\partial_i\delta g_{00} + \partial_0\delta g_{i0} - \partial_0\delta g_{0i}) \\
&= \partial_i A\,, \\
\hat{\Gamma}^{0}_{ij} = \delta\hat{\Gamma}^{0}_{ij} &= \frac{1}{2}\bar{g}^{0\sigma}(\partial_j\delta g_{i\sigma} + \partial_i\delta g_{j\sigma} - \partial_\sigma\delta g_{ij}) = \frac{1}{2}\bar{g}^{00}(\partial_j\delta g_{i0} + \partial_i\delta g_{j0} - \partial_0\delta g_{ij}) \\
&= -\frac{1}{2}(-\partial_j B_i - \partial_i B_j - \partial_0 h_{ij}) = \frac{1}{2}\partial_j B_i + \frac{1}{2}\partial_i B_j + \frac{1}{2}\partial_0 h_{ij}\,, \\
\hat{\Gamma}^{i}_{00} = \delta\hat{\Gamma}^{i}_{00} &= \frac{1}{2}\bar{g}^{i\sigma}(\partial_0\delta g_{0\sigma} + \partial_0\delta g_{0\sigma} - \partial_\sigma\delta g_{00}) = \frac{1}{2}\delta^{ij}(\partial_0\delta g_{0j} + \partial_0\delta g_{0j} - \partial_j\delta g_{00}) \\
&= \frac{1}{2}(-\partial_0 B^i - \partial_0 B^i + 2\partial^i A) = -\partial_0 B^i + \partial^i A\,, \\
\hat{\Gamma}^{i}_{0j} = \delta\hat{\Gamma}^{i}_{0j} &= \frac{1}{2}\bar{g}^{i\sigma}(\partial_j\delta g_{0\sigma} + \partial_0\delta g_{j\sigma} - \partial_\sigma\delta g_{0j}) = \frac{1}{2}\delta^{ik}(\partial_j\delta g_{0k} + \partial_0\delta g_{jk} - \partial_k\delta g_{0j}) \\
&= \frac{1}{2}(-\partial_j B^i + \partial_0 h^i_j + \partial^i B_j) = -\frac{1}{2}\partial_j B^i + \frac{1}{2}\partial^i B_j + \frac{1}{2}\partial_0 h^i_j\,, \\
\hat{\Gamma}^{i}_{jk} = \delta\hat{\Gamma}^{i}_{jk} &= \frac{1}{2}\bar{g}^{i\sigma}(\partial_k\delta g_{j\sigma} + \partial_j\delta g_{k\sigma} - \partial_\sigma\delta g_{jk}) = \frac{1}{2}\delta^{il}(\partial_k\delta g_{jl} + \partial_j\delta g_{kl} - \partial_l\delta g_{jk}) \\
&= \frac{1}{2}\partial_k h^i_j + \frac{1}{2}\partial_j h^i_k - \frac{1}{2}\partial^i h_{jk}\,.
\end{aligned}
\tag{B.3}
$$

For completeness we report the Christoffel symbols $\tilde{\Gamma}^{\mu}_{\nu\rho}$ associated to the metric $\tilde{g}_{\mu\nu}$. Given that we already calculated $\hat{\Gamma}^{\mu}_{\nu\rho}$, in order to find the explicit expression for $\tilde{\Gamma}^{\mu}_{\nu\rho}$ we can use (A.5) and (A.6), which we rewrite here:

$$\tilde{\Gamma}^{\mu}_{\nu\rho} = \hat{\Gamma}^{\mu}_{\nu\rho} - C^{\mu}_{\nu\rho} = \hat{\Gamma}^{\mu}_{\nu\rho} + \frac{1}{a}\big(\delta^{\mu}_{\rho}\tilde{\nabla}_{\nu}a + \delta^{\mu}_{\nu}\tilde{\nabla}_{\rho}a - \tilde{g}^{\mu\sigma}\tilde{g}_{\nu\rho}\tilde{\nabla}_{\sigma}a\big)\,. \tag{B.4}$$

Therefore, neglecting second order terms in the metric perturbations, we obtain

$$\begin{aligned}
\tilde{\Gamma}^{0}_{00} &= \hat{\Gamma}^{0}_{00} + \frac{1}{a}\big(2\delta^{0}_{0}\partial_0 a - \tilde{g}^{0\sigma}\tilde{g}_{00}\partial_\sigma a\big) = \partial_0 A + \frac{1}{a}\big(2\partial_0 a - \tilde{g}^{00}\tilde{g}_{00}\partial_0 a\big)\\
&= \partial_0 A + \frac{\partial_0 a}{a}\big[2 - (1-2A)(1+2A)\big] = \partial_0 A + \mathcal{H}\,,\\
\tilde{\Gamma}^{0}_{0i} &= \hat{\Gamma}^{0}_{0i} + \frac{1}{a}\big(\delta^{0}_{0}\partial_i a + \delta^{0}_{i}\partial_0 a - \tilde{g}^{0\sigma}\tilde{g}_{0i}\partial_\sigma a\big) = \partial_i A + \frac{1}{a}\big(-\tilde{g}^{00}\tilde{g}_{0i}\partial_0 a\big)\\
&= \partial_i A + \frac{\partial_0 a}{a}\big[-(1-2A)B_i\big] = \partial_i A - \mathcal{H}B_i\,,\\
\tilde{\Gamma}^{0}_{ij} &= \hat{\Gamma}^{0}_{ij} + \frac{1}{a}\big(\delta^{0}_{j}\partial_i a + \delta^{0}_{i}\partial_j a - \tilde{g}^{0\sigma}\tilde{g}_{ij}\partial_\sigma a\big) = \hat{\Gamma}^{0}_{ij} + \frac{1}{a}\big(-\tilde{g}^{00}\tilde{g}_{ij}\partial_0 a\big)\\
&= \hat{\Gamma}^{0}_{ij} + \frac{\partial_0 a}{a}(1-2A)(\delta_{ij}+h_{ij}) = \frac{1}{2}\partial_j B_i + \frac{1}{2}\partial_i B_j + \frac{1}{2}\partial_0 h_{ij} + \mathcal{H}(\delta_{ij} + h_{ij} - 2A\delta_{ij})\,,\\
\tilde{\Gamma}^{i}_{00} &= \hat{\Gamma}^{i}_{00} + \frac{1}{a}\big(\delta^{i}_{0}\partial_0 a + \delta^{i}_{0}\partial_0 a - \tilde{g}^{i\sigma}\tilde{g}_{00}\partial_\sigma a\big) = -\partial_0 B^i + \partial^i A + \frac{1}{a}\big(-\tilde{g}^{i0}\tilde{g}_{00}\partial_0 a\big)\\
&= -\partial_0 B^i + \partial^i A + \frac{\partial_0 a}{a}\big[-B^i(1+2A)\big] = -\partial_0 B^i + \partial^i A - \mathcal{H}B^i\,,\\
\tilde{\Gamma}^{i}_{0j} &= \hat{\Gamma}^{i}_{0j} + \frac{1}{a}\big(\delta^{i}_{j}\partial_0 a + \delta^{i}_{0}\partial_j a - \tilde{g}^{i\sigma}\tilde{g}_{0j}\partial_\sigma a\big) = \hat{\Gamma}^{i}_{0j} + \frac{1}{a}\big(\delta^{i}_{j}\partial_0 a - \tilde{g}^{i0}\tilde{g}_{0j}\partial_0 a\big)\\
&= \hat{\Gamma}^{i}_{0j} + \frac{\partial_0 a}{a}(\delta^{i}_{j} - B^i B_j) = -\frac{1}{2}\partial_j B^i + \frac{1}{2}\partial^i B_j + \frac{1}{2}\partial_0 h^{i}_{j} + \mathcal{H}\delta^{i}_{j}\,,\\
\tilde{\Gamma}^{i}_{jk} &= \hat{\Gamma}^{i}_{jk} + \frac{1}{a}\big(\delta^{i}_{k}\partial_j a + \delta^{i}_{k}\partial_j a - \tilde{g}^{i\sigma}\tilde{g}_{jk}\partial_\sigma a\big) = \hat{\Gamma}^{i}_{jk} + \frac{1}{a}\big(-\tilde{g}^{i0}\tilde{g}_{jk}\partial_0 a\big) = \hat{\Gamma}^{i}_{jk} + \frac{\partial_0 a}{a}B^i(\delta_{jk} + h_{jk})\\
&= \frac{1}{2}\partial_k h^{i}_{j} + \frac{1}{2}\partial_j h^{i}_{k} - \frac{1}{2}\partial^i h_{jk} + \mathcal{H}B^i\delta_{jk}\,.
\end{aligned} \tag{B.5}$$

Appendix C

Special functions and some useful properties

Spherical Bessel functions

Spherical Bessel functions satisfy the differential equation

$$\partial_x^2 j_l = -2\frac{\partial_x j_l}{x} + \left[\frac{l(l+1)}{x^2} - 1\right] j_l \, . \tag{C.1}$$

Other important relations used in this thesis are

$$\partial_x j_l = j_{l-1} - \frac{l+1}{x} j_l \tag{C.2}$$

and

$$j_{l-1} + j_{l+1} = \frac{2l+1}{x} j_l \, . \tag{C.3}$$

We report here the lowest spherical Bessel functions:

$$j_0(x) = \frac{\sin x}{x} \, , \tag{C.4a}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \, , \tag{C.4b}$$

$$j_2(x) = \frac{(3 - x^2)\sin x}{x^3} - \frac{3\cos x}{x^2} \, . \tag{C.4c}$$

Spherical harmonics

In order to prove the useful relation

$$\int \mathrm{d}\Omega \, Y^*_{lm} (1-\mu^2)^{\frac{|r|}{2}} e^{ir\phi} e^{ix\mu} = \sqrt{4\pi(2l+1)} \sqrt{\frac{(l+|r|)!}{(l-|r|)!}} i^r i^l \frac{j_l(x)}{x^{|r|}} \delta_{mr} \, . \tag{C.5}$$

we proceed as in App. A2 in [65]. Relation (C.5) for $r = 2$ is equivalent to (5.97), which is used in the thesis for the calculation of the tensor contributions to the angular power spectrum associated with the relative variation of the luminosity distance. Following the notation and convention of [65], the spherical harmonics which appear in (C.5) are given by

$$Y_{lm}(\theta, \phi) = \epsilon_m \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\phi} \tag{C.6}$$

where P_l^m are the associated Legendre polynomials, $\epsilon_m = 1$ if $m > 0$ and $\epsilon_m = (-1)^m$ if $m \leqslant 0$. The Legendre polynomials satisfy the differential equation

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x) \qquad (m \geq 0)\,. \tag{C.7}$$

Spherical harmonics are orthonormal, with normalization

$$\int \mathrm{d}\Omega\, Y^*_{lm}(\mathbf{n}) Y_{l'm'}(\mathbf{n}) = \delta_{ll'}\delta_{mm'}\,. \tag{C.8}$$

In order to prove (C.5) we will also use the plane wave expansion in spherical harmonics

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(kx) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{x}})\,. \tag{C.9}$$

By using the definition (C.6) of spherical harmonics the left-hand side of (C.5) becomes

$$\begin{aligned}
&\int \mathrm{d}\Omega\, Y^*_{lm} (1-\mu^2)^{\frac{|r|}{2}} e^{ir\phi} e^{ix\mu} \\
&= \epsilon_m \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}} \int_0^{2\pi} \mathrm{d}\phi\, e^{i\phi(r-m)} \int_0^{\pi} \mathrm{d}\theta\, \sin\theta (1-\cos^2\theta)^{\frac{|r|}{2}} P_l^{|m|}(\cos\theta) e^{ix\cos\theta} \\
&= \epsilon_r \sqrt{\frac{2l+1}{4\pi}\frac{(l-|r|)!}{(l+|r|)!}} 2\pi\delta_{rm} \int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)^{\frac{|r|}{2}} P_l^{|r|}(\mu) e^{ix\mu} \\
&= \epsilon_r \sqrt{4\pi(2l+1)} \sqrt{\frac{(l-|r|)!}{(l+|r|)!}} \delta_{rm} \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)^{\frac{|r|}{2}} P_l^{|r|}(\mu) e^{ix\mu} \\
&= \epsilon_r \sqrt{4\pi(2l+1)} \sqrt{\frac{(l-|r|)!}{(l+|r|)!}} \delta_{rm} I_l^{|r|}(x)\,,
\end{aligned} \tag{C.10}$$

where $\mu = \cos\theta$ and

$$I_l^{|r|}(x) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)^{\frac{|r|}{2}} P_l^{|r|}(\mu) e^{ix\mu}\,. \tag{C.11}$$

Comparing (C.10) and (C.5) and since $(-1)^r i^{|r|} = i^r$ for $r < 0$, we deduce that we have to prove the following relation:

$$I_l^{|r|}(x) = \frac{(l+|r|)!}{(l-|r|)!} i^{|r|+l} \frac{j_l(x)}{x^{|r|}}\,. \tag{C.12}$$

In order to prove it we proceed by induction: we show that if (C.12) holds for a generic $r > 0$, the it holds for $r+1$. We start by proving that (C.5) holds for $r = 0$. Therefore we have to prove that

$$\int \mathrm{d}\Omega\, Y^*_{lm} e^{ix\mu} = \sqrt{4\pi(2l+1)}\, i^l j_l(x) \delta_{m0}\,. \tag{C.13}$$

Using (C.8) and given that

$$e^{ix\cos\theta} = 4\pi \sum_{lm} i^l j_l(x) Y^*_{lm}(\hat{\mathbf{z}}) Y_{lm}(\mathbf{n})\,, \tag{C.14}$$

we find

$$\begin{aligned}\int \mathrm{d}\Omega\, Y_{lm}^{*} e^{ix\mu} &= \int \mathrm{d}\Omega\, Y_{lm}^{*}(\mathbf{n}) 4\pi \sum_{l'm'} i^{l} j_{l}(x) Y_{l'm'}^{*}(\hat{\mathbf{z}}) Y_{l'm'}(\mathbf{n}) \\ &= 4\pi \sum_{l'm'} i^{l} j_{l}(x) Y_{l'm'}^{*}(\hat{\mathbf{z}}) \int \mathrm{d}\Omega\, Y_{lm}^{*}(\mathbf{n}) Y_{l'm'}(\mathbf{n}) \\ &= 4\pi \sum_{l'm'} i^{l} j_{l}(x) Y_{l'm'}^{*}(\hat{\mathbf{z}}) \delta_{ll'} \delta_{mm'} \\ &= 4\pi i^{l} j_{l}(x) Y_{lm}^{*}(\hat{\mathbf{z}}) \\ &= 4\pi i^{l} j_{l}(x) \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}\,,\end{aligned} \tag{C.15}$$

which is exactly (C.13).

Having proved that (C.5) holds for $r = 0$, we proceed by showing that, if it holds for $r > 0$, then it holds for $r + 1$. By inserting (C.7) in (C.11) we obtain

$$\begin{aligned} I_{l}^{r+1}(x) &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)^{\frac{r+1}{2}} P_{l}^{r+1}(\mu) e^{ix\mu} \\ &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)^{\frac{r+1}{2}} (-1)^{(r+1)} (1-\mu^2)^{\frac{r+1}{2}} \frac{\mathrm{d}^{r+1}}{\mathrm{d}\mu^{r+1}} P_{l}(\mu) e^{ix\mu} \\ &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, (-1)^{r} \left[\frac{\mathrm{d}^{r}}{\mathrm{d}\mu^{r}} P_{l}(\mu)\right] \frac{\mathrm{d}}{\mathrm{d}\mu} \left[(1-\mu^2)^{(r+1)} e^{ix\mu}\right], \end{aligned} \tag{C.16}$$

where in the last passage we integrated by parts. Proceeding with the calculations we find

$$\begin{aligned} I_{l}^{r+1}(x) &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, (-1)^{r} \left[\frac{\mathrm{d}^{r}}{\mathrm{d}\mu^{r}} P_{l}(\mu)\right] (1-\mu^2)^{r} [-2\mu(r+1) + ix(1-\mu^2)] e^{ix\mu} \\ &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)^{-\frac{r}{2}} P_{l}^{r}(\mu) (1-\mu^2)^{r} [2(r+1)\partial_x + x(1+\partial_x^2)] i e^{ix\mu} \\ &= [2(r+1)\partial_x + x(1+\partial_x^2)] i I_{l}^{r}(x)\,, \end{aligned} \tag{C.17}$$

where in the second and last passages we used respectively (C.7) used (C.11). Assuming that (C.5) holds for r, we insert (C.12) in (C.17). We obtain

$$I_{l}^{r+1}(x) = \frac{(l+r)!}{(l-r)!} i^{r+l+1} [2(r+1)\partial_x + x(1+\partial_x^2)] \frac{j_l(x)}{x^r}\,. \tag{C.18}$$

Using the differential equation (C.1) we find

$$\begin{aligned} [2(r+1)\partial_x + x(1+\partial_x^2)] \frac{j_l(x)}{x^r} &= -2r(r+1)\frac{j_l(x)}{x^{r+1}} + 2(r+1)\frac{\partial_x j_l(x)}{x^r} + \frac{j_l(x)}{x^{r-1}} \\ &\quad + \frac{\partial_x^2 j_l(x)}{x^{r-1}} + r(r+1)\frac{j_l(x)}{x^{r+1}} - 2r\frac{\partial_x j_l(x)}{x^r} \\ &= -r(r+1)\frac{j_l(x)}{x^{r+1}} + l(l+1)\frac{j_l(x)}{(x^{r+1})} \\ &= (l+r+1)(l-r)\frac{j_l(x)}{x^{r+1}}\,. \end{aligned} \tag{C.19}$$

Therefore (C.18) becomes

$$\begin{aligned} I_{l}^{r+1}(x) &= \frac{(l+r)!}{(l-r)!} i^{r+l+1} (l+r+1)(l-r) \frac{j_l(x)}{x^{r+1}} \\ &= \frac{(l+r+1)!}{(l-r-1)!} i^{r+l+1} \frac{j_l(x)}{x^{r+1}}\,, \end{aligned} \tag{C.20}$$

which proves the hypothesis (C.12).

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