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The heart of a t-structure in derived categories

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Introduction

Homological algebra studies chain complexes, meaning objects of the form

$$\cdots \xrightarrow{d^{-3}} A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots$$

where A^i are objects in an abelian category $\mathcal{A}$ and $d^{i+1} \circ d^i = 0$. A morphism of chain complexes $f : A^\bullet \to B^\bullet$ is a commutative diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}"] & A^{-1} \arrow[r, "d^{-1}"] \arrow[d, "f^{-1}"] & A^0 \arrow[r, "d^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "d^1"] \arrow[d, "f^1"] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & B^{-1} \arrow[r, "d^{-1}"] & B^0 \arrow[r, "d^0"] & B^1 \arrow[r, "d^1"] & \cdots
\end{tikzcd}$$

Chain complexes and chain maps form an abelian category $C(\mathcal{A})$ so one can talk about exact sequenceces

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0 \ .$$

A basic but fundamental result in homological algebra states that any such short exact sequence induces a long exact sequence

$$\cdots \longrightarrow H^iA^\bullet \longrightarrow H^iB^\bullet \longrightarrow H^iC^\bullet \longrightarrow H^{i+1}A^\bullet \longrightarrow \cdots$$

in $\mathcal{A}$ (see [5] for the proof). Two complexes in $C(\mathcal{A})$ are considered the same if and only if they are isomorphic. This notion of equivalence is very strict because we are more interested in the isomorphism classes of the cohomology objects rather than the isomorphism classes of the terms themselves. A more natural notion of equivalences between complexes is the notion of quasi-ismorphism: we want two complexes to be considered the same if and only if there exists a morphism between them that induces isomorphisms on all the cohomology objects. This leads to the definition of the derived category $\mathcal{D}(\mathcal{A})$. The goal of the first chapter is to construct the derived category as the localization of the homotopy category with respect to all quasi-isomorphisms. We will show that a morphism in the derived category can be represented as a roof

$$\begin{tikzcd}
 & C^\bullet & \\
A^\bullet \arrow[ur, "f"] & & B^\bullet \arrow[ul, "\phi"']
\end{tikzcd}$$

where ϕ is a quasi-isomorphism, modulo a certain equivalence relation. The derived category is always an additive category but in general it is not abelian so we can not talk about exact sequences in $\mathcal{D}(\mathcal{A})$. This problem is fixed by the triangulated structure in $\mathcal{D}(\mathcal{A})$. We will define the notion of triangulated category, prove that the homotopy category is triangulated and then prove that the localization functor induces a triangulated structure in $\mathcal{D}(\mathcal{A})$. Then we will show that the notion of distinguished triangle generalizes exact sequences. In the derived category $\mathcal{D}(\mathcal{A})$ one can consider the full subcategory $\mathcal{D}^{\leq 0}$ of complexes with zero cohomology in positive degree. Similarly one defines $\mathcal{D}^{\geq 0}$ as the full subcategory of complexes with zero cohomology in negative degree. The intersection $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is the full subcategory of complexes with non-zero cohomology only (at most) in degree zero which can be seen to be equivalent to $\mathcal{A}$. The idea of a t-structure is to replicate this situation in an arbitrary triangulated category by considering a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and imposing certain properties on them so that the heart $\mathcal{D}^{\heartsuit} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is abelian. This allows us to construct full abelian subcategories of triangulated categories hence also of $\mathcal{D}(\mathcal{A})$. In the second chapter we give the definition of a t-structure and prove that the heart is abelian. Then we show that t-structures induce special cohomological functors called cohomology functors that generalize the standard cohomology functor H^0 and explain the notion of t-exactness. Then we describe some of the most important constructions of t-structures: in particular the gluing technique used by Beilinson, Bernstein and Deligne in their paper [1] on perverse sheaves and the t-structure induced on $\mathcal{D}(\mathcal{A})$ by a torsion theory on $\mathcal{A}$. We conclude with a brief discussion on the concept of derived equivalence and its strong relation to t-structures.

Chapter 1

The derived category $\mathcal{D}(\mathcal{A})$ and its triangulated structure

1.1 Localization of categories

In this chapter I will introduce the notion of derived category of an abelian category and I will prove some basic properties. A derived category can be thought of as an ideal setting in which to do homological algebra. Given an abelian category $\mathcal{A}$ it is natural to consider the category $C(\mathcal{A})$ whose objects are chain complexes and the morphism are chain maps. However in this category complexes are studied up to isomorphism, which is a very strict notion of equivalence. It turns out that the most suitable notion for equivalence between two complexes is the notion of quasi-isomorphism.

Definition 1.1.1 (Quasi-isomorphism)**.** *A map $q : A^\bullet \to B^\bullet$ is a quasi-isomorphism if the induced map $H^n q$ is an isomorphism for all n.*

We want a category in which two complexes are considered the same if and only if they are quasi-isomorphic. That is to say we want a morphism to be invertible if and only if it is a quasi-isomorphism. The process of formally adjoining inverses to certain morphisms in a category is called localization. This process is very general and can be defined through a universal property in the following way:

Definition 1.1.2 (Localization)**.** *Let $\mathcal{C}$ be a category and S a class of morphisms. The localization of $\mathcal{C}$ with respect to S is a category $\mathcal{C}[S^{-1}]$ together with a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that the following two properties hold:*

(L1) Qf is an isomorphism for every $f \in S$.

(L2) Q is universal with respect to the previous condition. That is given a functor $F : \mathcal{C} \to \mathcal{D}$ such that Ff is an isomorphism for all $f \in S$ then there exists a functor $F[S^{-1}] : \mathcal{C}[S^{-1}] \to \mathcal{D}$ unique up to natural equivalence such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{C}[S^{-1}] \arrow[d, dashed, "F[S^{-1}]"] \\ & \mathcal{D} \end{tikzcd}$$

This construction is very general but it is not always well-behaved. Ideally we would want to represent morphisms in $\mathcal{C}[S^{-1}]$ as "fractions" of morphisms, where the numerator is a morphism in $\mathcal{C}$ and the denominator is in S. However this in general is not true because for example a morphism in $\mathcal{C}[S^{-1}]$ of the form $s_1^{-1} f s_2^{-2}$ can't apriori be written in the form $s_3^{-1} f'$ because there is no property of commutativity between morphisms. This observation leads to the following definition:

Definition 1.1.3. *A class of morphism S is a multiplicative system if the following properties hold:*

(MS1) the composition of two composable elements of S is in S. The arrow id_X is in S for every object X of $\mathcal{C}$.

(MS2) Given a map $\phi \in S$ and two maps $X \to X'', Y' \to Y$ there is always a commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "\phi'"] & X \arrow[r] \arrow[d, "\phi"] & X'' \arrow[d, "\phi''"] \\
Y' \arrow[r] & Y \arrow[r] & Y''
\end{tikzcd}$$

such that $\phi', \phi'' \in S$.

(MS3) Given $f, g : X \to Y$ maps in $\mathcal{C}$ there exists a map $\phi \in S$ such that $f \circ \phi = g \circ \phi$ if and only if there exists a map $\psi \in S$ such that $\psi \circ f = \psi \circ g$.

If S is a multiplicative system we get a nice represetation of the morphisms in $\mathcal{C}[S^{-1}]$: given two objects X, Y of $\mathcal{C}$ a morphism between them in $\mathcal{C}[S^{-1}]$ is an equivalence class of diagrams

$$\begin{tikzcd}
& Y' & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "\phi"']
\end{tikzcd}$$

such that $\phi \in S$, modulo the following equivalence relation: $(f_1, \phi_1) \sim (f_2, \phi_2)$ if there exists a commutative diagram

$$\begin{tikzcd}
& Y_1 \arrow[d, "f_3"] & \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "\phi_1"'] \arrow[dl, "\phi_2"] \\
& Y_2 \arrow[u, "\phi_3"'] &
\end{tikzcd}$$

such that $\phi_3 \in S$. In other words a morphism in the localized category is an equivalence class of fractions of morphisms such that the "denominator" is in S. Given morphisms $(f_1, \phi_1) : X \to Y$ and $(f_2, \phi_2) : Y \to Z$ in $\mathcal{C}[S^{-1}]$ the composition is the morphism $(f_3 \circ f_1, \phi_3 \circ \phi_2)$ where f_3 and ϕ_3 are any two maps such that $\phi_3 \in S$ and the following square commutes

$$\begin{tikzcd}
 & & C & & \\
 & A \arrow[ur, "f_3"] & & B \arrow[ul, "\phi_3"'] & \\
X \arrow[ur, "f_1"] & & Y \arrow[ul, "\phi_1"] \arrow[ur, "f_2"'] & & Z \arrow[ul, "\phi_2"']
\end{tikzcd}$$

This characterization is made precise in the following theorem.

Theorem 1.1.1. *Let $\mathcal{C}$ be a locally small category and S a multiplicative system. Then:*

(a) There exists a category $\mathcal{C}'$ such that $Obj(\mathcal{C}') = Obj(\mathcal{C})$ and the morphisms in $\mathcal{C}'$ are equivalence classes of diagrams as described above

(b) The category $\mathcal{C}'$ is the localization of $\mathcal{C}$ with respect to S

Proof. (a) First of all I need to prove that the relation I defined earlier is in fact an equivalence relation. The reflexive and symmetric properties are obvious. To prove the transitivity suppose $(f_1, \phi_1) \sim (f_2, \phi_2)$ and $(f_2, \phi_2) \sim (f_3, \phi_3)$ This means there exists a diagram

$$\begin{tikzcd}
 & & & Z & & & \\
 & & T_1 \arrow[ur, "z_1"] & & T_2 \arrow[ul, "z_2"'] & & \\
 & A \arrow[ur, "t_1"] & & B \arrow[ul, "t_2"'] \arrow[ur, "t_3"] & & C \arrow[ul, "t_4"'] & \\
X \arrow[ur, "f_1"] \arrow[urrr, "f_2"] \arrow[urrrrr, "f_3"'] & & & & & & Y, \arrow[ulllll, "\phi_1"] \arrow[ulll, "\phi_2"] \arrow[ul, "\phi_3"']
\end{tikzcd}$$

such that $t_1 \circ f_1 = t_2 \circ f_2$, $t_1 \circ \phi_1 = t_2 \circ \phi_2$, $t_3 \circ f_2 = t_4 \circ f_3, t_4 \circ \phi_3 = t_3 \circ \phi_2$ and $t_2, t_4 \in S$. Furthermore (MS2) implies that I can choose Z such that $z_2 \circ t_3 = z_1 \circ t_2$ and $z_2 \in S$. I need to prove that $z_1 \circ t_1 \circ f_1 = z_2 \circ t_4 \circ f_3$ and $z_1 \circ t_1 \circ \phi_1 = z_2 \circ t_4 \circ \phi_3$.

$$\begin{aligned} z_1 \circ t_1 \circ f_1 &= z_1 \circ t_2 \circ f_2 \\ &= z_2 \circ t_3 \circ f_2 \\ &= z_2 \circ t_4 \circ f_3 \end{aligned}$$

Similarly

$$\begin{aligned} z_2 \circ t_4 \circ \phi_3 &= z_2 \circ t_3 \circ \phi_2 \\ &= z_1 \circ t_2 \circ \phi_2 \\ &= z_1 \circ t_1 \circ \phi_1 \end{aligned}$$

And so $(f_1, \phi_1) \sim (f_3, \phi_3)$. This shows that $\sim$ is in fact an equivalence relation. Now I need to prove that the composition is well defined, always exists and is associative.

The existence is a direct consequence of (MS2). To prove that the composition is well defined consider the following diagram

$$\begin{tikzcd}
 & & T & & \\
 & C \arrow[ur, "t_1"] & & C' \arrow[ul, "t_2"'] & \\
 & A \arrow[u, "c_1"] \arrow[urr, "c_3"] & & B, \arrow[ull, "c_2"'] \arrow[u, "c_4"'] & \\
X \arrow[ur, "f_1"] & & Y \arrow[ul, "\phi_1"] \arrow[ur, "f_2"'] & & Z \arrow[ul, "\phi_2"']
\end{tikzcd}$$

in which $c_1 \circ \phi_1 = c_2 \circ f_2$, $c_3 \circ \phi_1 = c_4 \circ f_2$ and $c_2, c_4 \in S$.

As a consequence of (MS2) I can choose T such that $t_1 \circ c_2 = t_2 \circ c_4$. I need to prove that $t_1 \circ c_1 \circ f_1 = t_2 \circ c_3 \circ f_1$ and $t_2 \circ c_4 \circ \phi_2 = t_1 \circ c_2 \circ \phi_2$. The second condition obviously holds, as for the first:

$$\begin{aligned}
t_2 \circ c_4 &= t_1 \circ c_2 \\
t_2 \circ c_4 \circ f_2 &= t_1 \circ c_2 \circ f_2 \\
t_2 \circ c_3 \circ \phi_1 &= t_1 \circ c_1 \circ \phi_1 \\
\phi_1' \circ t_2 \circ c_3 &= \phi_1' \circ t_1 \circ c_1 \qquad (MS3) \\
\phi_1' \circ t_2 \circ c_3 \circ f_1 &= \phi_1' \circ t_1 \circ c_1 \circ f_1
\end{aligned}$$

So now by replacing T with T' and t_1, t_2 with $\phi_1' \circ t_1, \phi_1' \circ t_2$ we are done.

To prove that the composition is associative we use a similar argument. Consider the following diagram

$$\begin{tikzcd}
 & & & Z & & & \\
 & T_1 \arrow[urr, "z_1"] & & & & T_2 \arrow[ull, "z_2"'] & \\
 & & S_1 \arrow[urrr, "t_3"] & & S_2 \arrow[ulll, red, "t_2"'] & & \\
 & A \arrow[uu, "t_1"] \arrow[ur, "s_1"] & & B \arrow[ul, "s_2"'] \arrow[ur, "s_3"] & & C \arrow[ul, red, "s_4"'] \arrow[uu, red, "t_4"'] & \\
X_1 \arrow[ur, "f_1"] & & X_2 \arrow[ul, "\phi_1"] \arrow[ur, "f_2"'] & & X_3, \arrow[ul, "\phi_2"] \arrow[ur, "f_3"'] & & X_4 \arrow[ul, "\phi_3"']
\end{tikzcd}$$

in which $s_1 \circ \phi_1 = s_2 \circ f_2$, $s_3 \circ \phi_2 = s_4 \circ f_3$, $t_1 \circ \phi_1 = t_2 \circ s_3 \circ f_2$, $t_4 \circ f_3 = t_3 \circ s_2 \circ \phi_2$ and $s_2, s_4, t_2, t_4 \in S$.

Furthermore I can choose Z such that $z_1 \circ t_2 \circ s_4 = z_2 \circ t_4$ and $z_2 \in S$.

To see this simply apply (MS2) to the red diagram and notice that $t_2 \circ s_4 \in S$ as a consequence of (MS1). I need to prove that $z_1 \circ t_1 \circ f_1 = z_2 \circ t_3 \circ s_1 \circ f_1$.

$$\begin{aligned}
z_1 \circ t_2 \circ s_4 &= z_2 \circ t_4 \\
z_1 \circ t_2 \circ s_4 \circ f_3 &= z_2 \circ t_4 \circ f_3 \\
z_1 \circ t_2 \circ s_3 \circ \phi_2 &= z_2 \circ t_3 \circ s_2 \circ \phi_2 \\
\phi_2' \circ z_1 \circ t_2 \circ s_3 &= \phi_2' \circ z_2 \circ t_3 \circ s_2 && (MS3) \\
\phi_2' \circ z_1 \circ t_2 \circ s_3 \circ f_2 &= \phi_2' \circ z_2 \circ t_3 \circ s_2 \circ f_2 \\
\phi_2' \circ z_1 \circ t_1 \circ \phi_1 &= \phi_2' \circ z_2 \circ t_3 \circ s_1 \circ \phi_1 \\
\phi_1' \circ \phi_2' \circ z_1 \circ t_1 &= \phi_1' \circ \phi_2' \circ z_2 \circ t_3 \circ s_1 && (MS3) \\
\phi_1' \circ \phi_2' \circ z_1 \circ t_1 \circ f_1 &= \phi_1' \circ \phi_2' \circ z_2 \circ t_3 \circ s_1 \circ f_1
\end{aligned}$$

So by replacing Z with Z' and z_1, z_2 with $\phi_1' \circ \phi_2' \circ z_1, \phi_1' \circ \phi_2' \circ z_2$ we are done. To see that $(\mathrm{id}_X, \mathrm{id}_X)$ is the identity for any object X consider the following diagram:

$$\begin{tikzcd}
 & & Z' & & Z & & \\
 & Z' \arrow[ur, "\mathrm{id}"] & & X \arrow[ul, "\phi'"'] \arrow[ur, "f"] & & Z \arrow[ul, "\mathrm{id}"'] & \\
Y' \arrow[ur, "f'"] & & X \arrow[ul, "\phi'"'] \arrow[ur, "\mathrm{id}"] & & X \arrow[ul, "\mathrm{id}"'] \arrow[ur, "f"] & & Y \arrow[ul, "\phi"']
\end{tikzcd}$$

This shows that $\mathcal{C}'$ is in fact a category.

(b) First of all we need to define the localizing functor $Q : \mathcal{C} \to \mathcal{C}'$. We define Q such that it maps every object of $\mathcal{C}$ to itself and $Qf = (f, \mathrm{id})$. To prove that Q is a functor we need to check that it maps the identity to the identity and it respects the composition law. The fact that it maps the identity to the identity is obvious. The fact that it respects the composition is clear when looking at the following diagram:

$$\begin{tikzcd}
 & & Z & & \\
 & Y \arrow[ur, "f"] & & Z \arrow[ul, "Id"'] & \\
X \arrow[ur, "g"] & & Y \arrow[ul, "\mathrm{id}"'] \arrow[ur, "f"] & & Z \arrow[ul, "\mathrm{id}"']
\end{tikzcd} .$$

So Q is a functor. Now we want to show that it maps elements of S into isomorphisms. So let $s : X \to Y$ be an element of S. We know that $Qs = (s, \mathrm{id})$. An obvious candidate for the inverse is (id, s). The following diagram shows that this is the correct choice.

$$\begin{tikzcd}
 & & Y & & Y & & \\
 & Y \arrow[ur, "\mathrm{id}"] & & Y \arrow[ul, "\mathrm{id}"'] \arrow[ur, "\mathrm{id}"] & & Y \arrow[ul, "\mathrm{id}"'] & \\
Y \arrow[ur, "\mathrm{id}"] & & X \arrow[ul, "s"] \arrow[ur, "s"'] \arrow[dr, "\mathrm{id}"'] & & Y \arrow[from=uu, "\mathrm{id}"] \arrow[ul, "\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] & & X \arrow[ul, "s"'] \arrow[dl, "\mathrm{id}"] \\
 & & & & X \arrow[u, "s"] & &
\end{tikzcd}$$

Now let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that Fs is an isomorphism for all $s \in S$.

We need to prove that there exists a unique functor from $F[S^{-1}] : \mathcal{C}' \to \mathcal{D}$ such that $F[S^{-1}] \circ Q = F$. We define $F[S^{-1}]$ as follows: it maps an object X to FX and it maps the morphism (f, ϕ) to $(F\phi)^{-1} \circ Ff$. First of all we need to prove that it is indipendent from the choice of representatives. Consider two representations of the same morphism $(f_1, \phi_1) \sim (f_2, \phi_2)$. There exists a diagram that exhibits the equivalence:

$$\begin{tikzcd}
 & & T & & \\
 & Z \arrow[ur, "t_1"] & & Z' \arrow[ul, "t_2"'] & \\
X \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & Y \arrow[ul, "\phi_2"'] \arrow[ulll, "\phi_1"]
\end{tikzcd}$$

We know that

$$\begin{aligned}
t_1 \circ f_1 = t_2 \circ f_2 \qquad &\Longrightarrow \qquad Ft_1 \circ Ff_1 = Ft_2 \circ Ff_2 \\
t_1 \circ \phi_1 = t_2 \circ \phi_2 \qquad &\Longrightarrow \qquad Ft_1 \circ F\phi_1 = Ft_2 \circ F\phi_2
\end{aligned}$$

Now using the fact that $Ft_2, F\phi_1, F\phi_2$ are invertible we get:

$$Ff_2 = (Ft_2)^{-1} \circ Ft_1 \circ Ff_1 \quad Ft_1 = Ft_2 \circ F\phi_2 \circ (F\phi_1)^{-1}.$$

Substituting we get

$$\begin{aligned}
Ff_2 &= (Ft_2)^{-1} \circ Ft_2 \circ F\phi_2 \circ (F\phi_1)^{-1} \circ Ff_1 \\
&= F\phi_2 \circ (F\phi)^{-1} \circ Ff_1
\end{aligned}$$

Therefore

$$(F\phi_2)^{-1} \circ Ff_2 = (F\phi_1)^{-1} \circ Ff_1$$

It is obvious that $F[S^{-1}]$ maps the identity to the identity. To show that it respects the composition consider the following diagram:

$$\begin{tikzcd}
& & C & & \\
& A \arrow[ur, "c_1"] & & B \arrow[ul, "c_2"'] & \\
X \arrow[ur, "f_1"] & & Y \arrow[ul, "\phi_1"] \arrow[ur, "f_2"'] & & Z \arrow[ul, "\phi_2"']
\end{tikzcd}$$

The square is commutative and $c_2 \in S$ so we get

$$(F\phi_2)^{-1} \circ Ff_2 \circ (F\phi_1)^{-1} \circ Ff_1 = (F\phi_2)^{-1} \circ (Fc_2)^{-1} \circ Fc_1 \circ Ff_1$$

We have proved that $F[S^{-1}]$ is a well defined functor furthermore it is trivial to check that $F = F[S^{-1}] \circ Q$. The only thing left to prove is that $F[S^{-1}]$ is the only functor that makes the triangle commute. Every such functor must map (f, id) to Ff. But then for $\phi \in S$ it must map (id, ϕ) to $(F\phi)^{-1}$. Now notice that $(\mathrm{id}, \phi) \circ (f, \mathrm{id}) = (f, \phi)$ and so we are done.

□

1.2 Constructing the derived category $\mathcal{D}(\mathcal{A})$

Now we will apply this general construction to the case of the category $C(\mathcal{A})$ and the class of quasi-isomorphisms Q. We would like to simply consider $C(\mathcal{A})[Q^{-1}]$ but there is a problem, which is the fact that Q is not a multiplicative system in $C(\mathcal{A})$. So in order to get a calculus of fractions for the morphisms in the derived category we need an intermediate step.

Definition 1.2.1 (homotopy)**.** *Let $f, g : A^\bullet \to B^\bullet$ be chain maps. We say that f and g are homotopic if there exists a family of morphisms $k^n : A^n \to B^{n-1}$ such that $f - g = dk + kd$.*

Definition 1.2.2 (Homotopy category)**.** *Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$ is a category such that Obj($\mathcal{K}(\mathcal{A})$) = Obj($C(\mathcal{A})$) and the morphism in $\mathcal{K}(\mathcal{A})$ are chain maps modulo homotopy equivalence.*

The theorem that allows us to define $D(\mathcal{A})$ is the following.

Theorem 1.2.1. *The class of all quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$ is a multiplicative system*

To prove this we need some preliminary results. First of all I will make two observations that I will use throughout this and the following sections.

Remark. *Mitchell' s embedding theorem states that every small abelian category admits a fully faithful exact embedding in **R-Mod** for some ring R. Now for any diagram in an abelian category I can consider the full subcategory whose objects are in the diagram. This is obviously small and therefore can be embedded in **R-Mod**. This implies that I can "chase" elements in any diagram.*

Remark. *Consider a morphism $f : \oplus_{j=1}^{n} A_i \to \oplus_{k=1}^{m} B_j$ where A_i, B_j are objects of an abelian category. I can represent f as the matrix $\{f_{jk}\}$ where $f_{jk} : A_j \to B_k$ is given by $f_{jk} = p_j^B \circ f \circ i_j^A$ The properties of the canonical projections and inclusions imply that if I have two such functions the composition is given by the matrix product and if an element of $\oplus_{j=1}^{n} A_j$ is represented as a column vector $a = (a_1, \dots, a_n)^t$ then I can identify $f(a)$ with the usual matrix-vector product.*

Now we need some definitions.

Definition 1.2.3 (Shift functor)**.** *Let $A^\bullet$ be a complex we denote by $A[1]^\bullet$ the complex that satisfies $A[1]^n = A^{n+1}$ and $d_{A[1]} = -d_A$ This is clearly an endofunctor called the shift functor. We denote by $[n]$ the n-th iteration of $[1]$. This makes sense for all integers n since $[1]$ is invertible.*

Definition 1.2.4 (Cone)**.** *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. The cone of f is the complex*

$$Cone(f) = A[1]^\bullet \oplus B^\bullet.$$

with differentials given by $d = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}$. It is immediate to check that

$$d^2 = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}^2 = \begin{pmatrix} d_A^2 & 0 \\ -f d_A + d_B f & d_B^2 \end{pmatrix} = 0$$

and so $Cone(f)$ is a well defined complex.

Definition 1.2.5 (Cylinder)**.** *Let $f : A^\bullet \to B^\bullet$ be a morphims of complexes. The cylinder of f is the complex*

$$Cyl(f) = A^\bullet \oplus Cone(f) = A^\bullet \oplus A[1]^\bullet \oplus B^\bullet.$$

with differentials given by $d = \begin{pmatrix} d_A & -1 & 0 \\ 0 & -d_A & 0 \\ 0 & f & d_B \end{pmatrix}$. Once again it is easy to check that

$$d^2 = \begin{pmatrix} d_A & -1 & 0 \\ 0 & -d_A & 0 \\ 0 & f & d_B \end{pmatrix}^2 = \begin{pmatrix} d_A^2 & 0 & 0 \\ 0 & d_A^2 & 0 \\ 0 & -f d_A + d_B f & d_B^2 \end{pmatrix} = 0$$

The following lemma relates the notions of cone and quasi-isomorphism.

Lemma 1.2.1. *Let $f : A^\bullet \to B^\bullet$. Then:*

(1) There is a short exact sequence of complexes

$$0 \longrightarrow B^\bullet \xrightarrow{\ i\ } Cone(f) \xrightarrow{\ \pi\ } A[1]^\bullet \longrightarrow 0 \ .$$

(2) There is a long exact sequence of cohomology:

$$\dots \longrightarrow H^n(B) \xrightarrow{H^n i} H^n(Cone(f)) \xrightarrow{H^n \pi} H^{n+1}(A) \xrightarrow{H^{n+1} f} H^{n+1}(B) \longrightarrow \dots$$

(3) f is a quasi-isomorphism if and only if Cone(f) is acyclic.

Proof. (1) is obvious: one checks easily that the maps i, π are chain maps and the exactness of the sequence is trivial.
The short exact sequence on (1) induces a long exact sequence of cohomology. The only thing to prove is that the "Snake lemma" map is $H^{n+1}f$. To see this consider an element of $H^{n+1}(A)$ represented by $a^{n+1} \in A^{n+1}$. Lift it to $(a^{n+1}, 0) \in \text{Cone}(f)^n$, apply d to get $(d_A a^{n+1}, f a^{n+1})$ and lift it again to $f a^{n+1}$. This is a representative for the image of the snake lemma map which therefore coincides with Hf.
Now assume Cone(f) acyclic. This means that $H^n(\text{Cone}(f)) = 0$ for all n. Therefore the long exact sequence of cohomology breaks into the following exact pieces:

$$0 \longrightarrow H^n(A) \xrightarrow{H^n f} H^n(B) \longrightarrow 0\ .$$

But this means exactly that f is a quasi-isomorphism.
Conversely assume that f is a quasi-isomorphism. Then the exactness of the long exact sequence of cohomology at $H^n(A)$ and $H^n(B)$ for all n implies that Hi and $H\pi$ are zero. But now the exactness at $H^n(\text{Cone}(f))$ for all n implies that Cone(f) is acyclic. $\square$

The following lemma summarizes the main properties of the cone and the cylinder.

Lemma 1.2.2. *For any morphism $f : A^\bullet \to B^\bullet$ there exists a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[r] & B^\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & Cone(f) \arrow[r, "\pi"] \arrow[d, equal] & A[1]^\bullet \arrow[r] & 0 \\
0 \arrow[r] & A^\bullet \arrow[r, "i"] \arrow[d, equal] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & Cone(f) \arrow[r] & 0 & \\
 & A^\bullet \arrow[r, "f"] & B^\bullet & & &
\end{tikzcd}$$

such that the rows are exact, α and β are quasi-isomorphisms, $\beta\alpha = id_B$ and $\alpha\beta \sim id_{Cyl(f)}$.

Proof. The maps i and π are just the canonical inclusions and projections which can be readily checked to be chain maps. The exactness of the rows is obvious. Let α be the canonical inclusion and β be the morphism $(f, 0, 1)$. To see that β is a chain map:

$$\beta d_{\text{Cyl}(f)} = \begin{pmatrix} f & 0 & 1 \end{pmatrix} \begin{pmatrix} d_A & -1 & 0 \\ 0 & -d_A & 0 \\ 0 & f & d_B \end{pmatrix} = \begin{pmatrix} f d_A & 0 & d_B \end{pmatrix} = d_B \beta.$$

To see that the first square commutes:

$$\beta i = \begin{pmatrix} f & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f.$$

The commutativity of the second square is obvious. To see that $\beta\alpha = \text{id}_{B^\bullet}$:

$$\beta\alpha = \begin{pmatrix} f & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$

To see that $\alpha\beta \sim \mathrm{id}_{\mathrm{Cyl}(f)}$:

$$\begin{aligned}
\alpha\beta - Id &= \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}f & 0 & 1\end{pmatrix} - \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\\
&= \begin{pmatrix}-1&0&0\\0&-1&0\\f&0&0\end{pmatrix}\\
&= \begin{pmatrix}0&0&0\\d_A&-1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}-1&0&0\\-d_A&0&0\\f&0&0\end{pmatrix}\\
&= \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}d_A&-1&0\\0&-d_A&0\\0&f&d_B\end{pmatrix} + \begin{pmatrix}d_A&-1&0\\0&-d_A&0\\0&f&d_B\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\\
&= kd_{\mathrm{Cyl}(f)} + d_{\mathrm{Cyl}(f)}k
\end{aligned}$$

In particular α and β are invertible in $\mathcal{K}(\mathcal{A})$ and therefore they are quasi-isomorphisms. $\square$

Now we can prove that the class of quasi-isomorphisms forms a multiplicative system in $\mathcal{K}(\mathcal{A})$.

Proof of Theorem 2. We need to check three properties. The first one is obvious. To prove (MS2) suppose $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, $g : C^\bullet \to B^\bullet$ and $h : A^\bullet \to D^\bullet$ morphisms and consider the following diagram in $\mathcal{K}(\mathcal{A})$:

$$\begin{tikzcd}
 & \mathrm{Cone}(ig)[-1] \arrow[r, "\pi_*"] \arrow[d, "-\pi"] & C^\bullet \arrow[r, "ig"] \arrow[d, "g"] & \mathrm{Cone}(f) \arrow[d, equal] \\
\mathrm{Cone}(f)[-1] \arrow[r, "\pi"] \arrow[d, equal] & A^\bullet, \arrow[r, "f"] \arrow[d, "h"] & B^\bullet \arrow[r, "i"] \arrow[d, "-i"] & \mathrm{Cone}(f) \\
\mathrm{Cone}(f)[-1] \arrow[r, "h\pi"'] & D^\bullet \arrow[r, "i_*"'] & \mathrm{Cone}(h\pi) &
\end{tikzcd}$$

We need to prove that the two central squares commute up to homotopy equivalence and that π_* and i_* are quasi-isomorphisms. To prove the commutativity of the first square:

$$\begin{aligned}
g\pi_* + f\pi &= \begin{pmatrix}g & f & 0\end{pmatrix}\\
&= \begin{pmatrix}g & f & d_B\end{pmatrix} + \begin{pmatrix}0 & 0 & -d_B\end{pmatrix}\\
&= \begin{pmatrix}0 & 0 & -1\end{pmatrix}\begin{pmatrix}d_C&0&0\\0&d_A&0\\-g&-f&-d_B\end{pmatrix} + d_B\begin{pmatrix}0 & 0 & -1\end{pmatrix}\\
&= hd_{\mathrm{Cone}(ig)[-1]} + d_B h
\end{aligned}$$

It is easy to see that $h = \begin{pmatrix}0 & 0 & -1\end{pmatrix}$ is a family of morphisms from $\mathrm{Cone}(ig)[-1]^{n+1} = \mathrm{Cone}(ig)^n$ to B^n and therefore induces a well-defined homotopy between $g\pi_*$ and $-f\pi$.

Similarly to prove the commutativity of the second square:

$$\begin{aligned}
i_*h + if &= \begin{pmatrix} 0 \\ f \\ h \end{pmatrix} \\
&= \begin{pmatrix} -d_A \\ f \\ h \end{pmatrix} + \begin{pmatrix} d_A \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -d_A & 0 & 0 \\ f & d_B & 0 \\ h & 0 & d_D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} d_A \\
&= d_{\mathrm{Cone}(h\pi)} h' + h' d_A
\end{aligned}$$

and once again h' is a well-defined homotopy. To prove that π_* is a quasi-isomorphism notice that f quasi-isomorphism implies $\mathrm{Cone}(f)$ acyclic. Now consider the short exact sequence

$$0 \longrightarrow \mathrm{Cone}(f)[-1] \longrightarrow \mathrm{Cone}(ig) \xrightarrow{\pi_*} C[1]^\bullet \longrightarrow 0$$

The fact that $\mathrm{Cone}(f)$ is acyclic implies that the induced long exact sequence of cohomology breaks into the following exact pieces:

$$0 \longrightarrow H^n\mathrm{Cone}(ig) \xrightarrow{H^n\pi_*} H^nC^\bullet \longrightarrow 0 \qquad \forall n$$

But this shows that π_* is a quasi-isomorphism. The fact that i_* is a quasi-isomorphism can be seen in the same way. This concludes the proof of (MS2).
Now let $f : A^\bullet \to B^\bullet$ be a morphism in $\mathcal{K}(\mathcal{A})$ and $s : B^\bullet \to C^\bullet$ a quasi-isomorphism such that $sf = 0$. We need to show that there exists a quasi-isomorhism $t : D^\bullet \to A^\bullet$ such that $ft = 0$. To prove this let $h^n : A^n \to C^{n-1}$ be a homotopy between sf and the zero morphism and consider the following diagram:

$$\begin{tikzcd}
\mathrm{Cone}(s)[-1] \arrow[r, "\pi"] \arrow[d, equal] & B^\bullet \arrow[r, "s"] & C^\bullet \\
\mathrm{Cone}(s)[-1] & A^\bullet \arrow[l, "g"'] \arrow[u, "f"] & \mathrm{Cone}(g)[-1] \arrow[l, "t"']
\end{tikzcd}$$

Let g be the morphism $\begin{pmatrix} f \\ -h \end{pmatrix}$. We need to show that it is a chain map:

$$\begin{aligned}
gd_A - d_{\mathrm{Cone}(s)[-1]}g &= \begin{pmatrix} f \\ -h \end{pmatrix} d_A - \begin{pmatrix} d_B & 0 \\ -s & -d_C \end{pmatrix} \begin{pmatrix} f \\ -h \end{pmatrix} \\
&= \begin{pmatrix} fd_A - d_Bf \\ sf - hd_A - d_Ch \end{pmatrix} \\
&= 0
\end{aligned}$$

Now we need to show that ft and the zero morphism are homotopic. Clearly it is enough to show that gt is homotopic to the zero morphism.

$$\begin{aligned}
gt &= \begin{pmatrix} f \\ -h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f & 0 & 0 \\ -h & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f & d_B & 0 \\ -h & -s & -d_C \end{pmatrix} + \begin{pmatrix} 0 & -d_B & 0 \\ 0 & s & d_C \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} d_A & 0 & 0 \\ -f & -d_B & 0 \\ h & s & d_C \end{pmatrix} + \begin{pmatrix} d_B & 0 \\ -s & -d_C \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= h' d_{\mathrm{Cone}(g)[-1]} + d_{\mathrm{Cone}(s)[-1]} h'
\end{aligned}$$

Once again the fact that s is a quasi-isomorphism shows that $\mathrm{Cone}(s)[-1]$ is acyclic, the long exact sequence of cohomology breaks into pieces and shows that t is a quasi-isomorphism. The other part of (MS3) can be proved in the same way. $\square$

Now we can define the derived category of $\mathcal{A}$.

Definition 1.2.6 (Derived category)**.** *Let $\mathcal{A}$ be an abelian category, and S the class of all quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$. We define $D(\mathcal{A})$ as follows:*

$$D(\mathcal{A}) = \mathcal{K}(\mathcal{A})[S^{-1}].$$

We will conclude this section by proving that the derived category of an abelian category is always additive, altough not of course abelian in general.

Theorem 1.2.2. *Let $\mathcal{A}$ be an abelian category, then $\mathcal{D}(\mathcal{A})$ is an additive category.*

Proof. First of all we need to endow $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y)$ with an abelian group structure. Let $(f_1, \phi_1), (f_2, \phi_2) \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y)$ and consider the following diagram

$$\begin{tikzcd}
 & & Z_3 & & \\
 & Z_1 \arrow[ur, dotted, "\phi_4"] & & Z_2 \arrow[ul, dotted, "\phi_3"'] & \\
X \arrow[ur, "f_1"] & & Y \arrow[ul, "\phi_1"] \arrow[ur, "\phi_2"'] & & X \arrow[ul, "f_2"']
\end{tikzcd}$$

The property (MS2) implies that there exist dotted arrows that make the square commute and such that ϕ_3 is a quasi-isomorphism. But in a commutative square if three arrows are quasi-isomorphisms then so is the fourth. But now it is easy to see that $(f_1, \phi_1) \sim (\phi_4 \circ f_1, \phi_4 \circ \phi_1)$ and $(f_2, \phi_2) \sim (\phi_3 \circ f_2, \phi_4 \circ \phi_1)$. We define $(f_1, \phi_1) + (f_2, \phi_2) = (\phi_4 \circ f_1 + \phi_3 \circ f_2, \phi_4 \circ \phi_1)$. This definition is based on the idea of adding two fractions together for example in $\mathbb{Q}$: we simply find a common denominator and then add the numerators. To show that it is well defined consider the following diagram:

$$\begin{tikzcd}
& & R & & \\
& Z_3 \arrow[ur, "r_1"] & & Z_4 \arrow[ul, "r_2"'] & \\
& Z_1 \arrow[u, "\phi_4"] \arrow[urr, "\phi_6"'] & & Z_2 \arrow[u, "\phi_5"'] \arrow[ull, "\phi_3"] & \\
X \arrow[ur, "f_1"] & & Y \arrow[ul, "\phi_1"] \arrow[ur, "\phi_2"'] & & X \arrow[ul, "f_2"']
\end{tikzcd}$$

Repeating the argument we used to prove that the composition is well defined in the localized category we can find R such that $r_1 \circ \phi_4 \circ f_1 = r_2 \circ \phi_6 \circ f_1, \;\; r_1 \circ \phi_3 = r_2 \circ \phi_5$ and r_2 is a quasi-isomorphism. But then

$$\begin{aligned} r_2 \circ \phi_5 \circ \phi_2 &= r_1 \circ \phi_3 \circ \phi_2 \\ &= r_1 \circ \phi_4 \circ \phi_1. \end{aligned}$$

And so the diagram

$$\begin{tikzcd}
& & R & & \\
& Z_3 \arrow[ur, "r_1"] & & Z_4 \arrow[ul, "r_2"'] & \\
X \arrow[ur, "\phi_4\circ f_1+\phi_3\circ f_2"] \arrow[urrr, "\phi_6\circ f_1+\phi_5\circ f_2"'] & & & & Y \arrow[ul, "\phi_5\circ\phi_2"'] \arrow[ulll, "\phi_4\circ\phi_1"]
\end{tikzcd}$$

exhibits the equivalence $(\phi_4 \circ f_1 + \phi_3 \circ f_2, \phi_4 \circ \phi_1) \sim (\phi_6 \circ f_1 + \phi_5 \circ f_2, \phi_5 \circ \phi_2)$ as required. The commutativity follows at once from the definition. It is easy to check that $(0, \mathrm{id})$ is the neutral element and that the inverse of (f, ϕ) is $(-f, \phi)$.The associativity is also easy to see: given three morphisms in $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}$ I can always reduce them to a common denominator. But then the denominator of the sum is just the common denominator and the numertor of the sum is the sum of the numerators. Since the sum in $\mathcal{K}(\mathcal{A})$ is associative we are done. Next we need to prove that the composition is bilinear. To see this consider $(f_1, \phi_1), (f_2, \phi_2) \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B)$ and $(g, \psi) \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(B, C)$. We already observed that we can always reduce ourselves to the case where the denominators are the same so we can assume $\phi_1 = \phi_2$. But this implies that $(g, \psi) \circ (f_1, \phi_1) = (t_1 \circ f_1, t_2 \circ \psi)$ and $(g, \psi) \circ (f_2, \phi_1) = (t_1 \circ f_2, t_2 \circ \psi)$ with t_2 quasi-isomorphism. Therefore

$$\begin{aligned} (g, \psi) \circ (f_1, \phi_1) + (g, \psi) \circ (f_2, \phi_2) &= (t_1 \circ f_1, t_2 \circ \psi) + (t_1 \circ f_2, t_2 \circ \psi) \\ &= (t_1 \circ f_1 + t_1 \circ f_2, t_2 \circ \psi) \\ &= (t_1 \circ (f_1 + f_2), t_2 \circ \psi) \\ &= (g, \psi) \circ ((f_1, \phi_1) + (f_2, \phi_2)) \end{aligned}$$

To conclude that $\mathcal{D}(\mathcal{A})$ is additive it is enough to show that for every pair of objects in $A, B \in \mathcal{D}(\mathcal{A})$ the product $A \times B$ exists. We claim that for any A, B the object $A \times B$

(meaning the product of A and B in $\mathcal{K}(\mathcal{A})$) is the required product. Furthermore the canonical projections are given by (p_A, id) and (p_B, id). We have to show that $A \times B$ satisfies the universal property so let Z be equipped with morphism $(f_A, \phi_A) : Z \to A$ and $(f_B, \phi_B) : Z \to B$ and consider the following diagram

$$\begin{tikzcd}
 & A & & B & \\
A \arrow[ur, equal] \arrow[ddr, "\phi_A"'] & & A \times B \arrow[ul, "p_A"'] \arrow[ur, "p_B"] & & B \arrow[from=ul, equal] \arrow[ddl, "\phi_B"] \\
 & & Z \arrow[dl, "f_A"'] \arrow[dr, "f_B"] \arrow[dd, "f_A \oplus f_B"] & & \\
 & T_1 & & T_2 & \\
 & & T_1 \times T_2 \arrow[ul, "t_1"] \arrow[ur, "t_2"'] & &
\end{tikzcd}.$$

Now consider the map in $(\phi_A, \phi_B) : A \times B \to T_1 \times T_2$ in $\mathcal{K}(\mathcal{A})$. It is clear that it is a quasi-isomorphisms since ϕ_A and ϕ_B are quasi-isomorphisms. But then we have shown the existence of a morphism $(f_A \oplus f_B, \phi_A \oplus \phi_B) : Z \to A \times B$ in $\mathcal{D}(\mathcal{A})$. To see that this morphism satisfes the universal property consider the following diagram:

$$\begin{tikzcd}
 & & T_1 & & \\
 & A \arrow[ur, "\phi_A"] & & T_1 \times T_2 \arrow[ul, "p_1"'] & \\
A \arrow[ur, equal] \arrow[drr, "\phi_A"'] & & A \times B \arrow[ul, "p_A"] \arrow[ur, "\phi_A \oplus \phi_B"'] & & Z \arrow[ul, "f_A \oplus f_B"'] \arrow[dll, "f_A"] \\
 & & T_1 & &
\end{tikzcd}$$

This shows that $(f_A \oplus f_B, \phi_A \oplus \phi_B) \circ (p_A, \mathrm{Id}) = (f_A, \phi_A)$. The other condition can be checked in the same way. Now we need to show uniqueness. Suppose that there is another

morphism $\begin{tikzcd}
 & C & \\
Z \arrow[ur, "c_1"] & & A \times B \arrow[ul, "c_2"']
\end{tikzcd}$ that makes the diagram of the universal property commute. This means that we can construct the following commutative diagram:

$$\begin{tikzcd}
 & X_1 \arrow[d] & C \arrow[l] \arrow[r] & X_2 \arrow[d] & \\
A \arrow[ur] \arrow[dr, "\phi_A"'] & S_1 & Z \arrow[ul] \arrow[u, "c_1"] \arrow[ur] \arrow[dl, "f_A"] \arrow[dr, "f_B"'] & S_2 & B \arrow[ul] \arrow[dl, "\phi_B"] \\
 & T_1 \arrow[u] & & T_2 \arrow[u] &
\end{tikzcd}$$

But then the induced maps $C \to S_1 \times S_2$ and $T_1 \times T_2 \to S_1 \times S_2$ make the following diagram commute:

$$\begin{tikzcd}
 & T_1 \times T_2 \arrow[d] & \\
Z \arrow[ur, "f_A \oplus f_B"] \arrow[dr, "c_1"'] & S_1 \times S_2 & A \times B \arrow[ul, "\phi_A \oplus \phi_B"'] \arrow[dl, "c_2"] \\
 & C \arrow[u] &
\end{tikzcd}$$

Therefore the two morphism are actually the same and so we have proved uniqueness. $\square$

1.3 Triangulated categories

The goal of this section is to define triangulated categories and prove some basic properties.

Definition 1.3.1. *A triangulated category is a triple $(\mathcal{T}, T, \mathcal{F})$ wih $\mathcal{T}$ an additive category, T automorphism of $\mathcal{T}$ called the translation functor and $\mathcal{F}$ a family of distinguished triangles satisfying certain properties. I will write $X[n]$ for $T^n X$ and $f[n]$ for $T^n f$ whenever the translation functor is obvious from the context. A triangle is a diagram of the form*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

A morphism of triangles is a diagram of the form

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

The distinguished triangles are triangles that must satisfy the following properties:

(TR1) *(a)* $X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$ *is a distinguished triangle*

(b) Any triangle isomorphic to a distinguished triangle is distinguished.

(c) Any morphism $X \xrightarrow{u} Y$ *can be complete to a distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} X \xrightarrow{w} X[1]$.

(TR2) A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is distinguished if and only if the triangle* $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ *is distinguished.*

(TR3) Given two distinguished triangles and two morphisms $f : X \to X'$ and $g : Y \to Y'$ there exists $h : Z \to Z'$ such that the following diagram is a morphism of triangles.

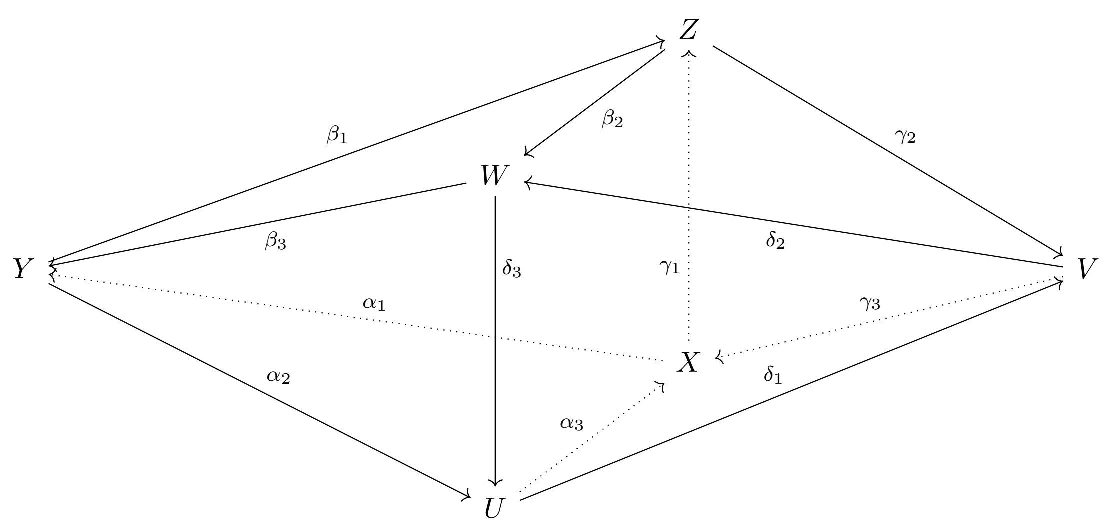
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

(TR4) Given 3 *distinguished triangles* $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3), (\gamma_1, \gamma_2, \gamma_3)$ *such that* $\beta_1 \circ \alpha_1 = \gamma_1$ *there exists a distinguished triangle* $(\delta_1, \delta_2, \delta_3)$ *such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, equal] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, dotted, "\delta_1"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\delta_3"] \arrow[d, dotted, "\delta_2"] & X[1] \arrow[d, "\alpha_1[1]"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, dotted, "\delta_3"] & Y[1] \\
 & Y[1] \arrow[r, "\alpha_2[1]"] & U[1] &
\end{tikzcd} .$$

A category satisfying only the first three axioms is said to be pre-triangulated.

The axiom (TR4) is usually called the octahedron axiom because of the following way of drawing the previous diagram:



Lemma 1.3.1. *Let $\mathcal{T}$ be a pre-triangulated category and let $\Delta = X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a distinguished triangle. Then for any object U of $\mathcal{T}$ the following sequences are exact :*

$$\dots \longrightarrow Hom(U, X[i]) \xrightarrow{u_*[i]} Hom(U, Y[i]) \xrightarrow{v_*[i]} Hom(U, Z[i])$$

$$\xrightarrow{w_*[i]} Hom(U, X[i+1]) \longrightarrow \dots ,$$

$$\dots \longrightarrow Hom(X[i+1], U) \xrightarrow{w^*[i]} Hom(Z[i], U) \xrightarrow{v^*[i]} Hom(Y[i], U)$$

$$\xrightarrow{u^*[i]} Hom(X[i], U) \longrightarrow \dots$$

Proof. I will prove the exactness of the first sequence, a similar argument applies for the second as well. First notice that it is enough to show exactness at the point $\mathrm{Hom}(U, Y[i])$ for all i. Indeed if this holds then by applying the result to Δ and the two distinguished triangles obtained from it applying (TR2) twice we are done. The first step of the proof consists in showing that the composition $vu = 0$. To do this consider the triangle $\Delta' = (X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow X[1])$ and complete id, u to a morphism of triangles $(\mathrm{id}, u, h) : \Delta' \to \Delta$.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dotted, "h"] & X[1] \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]
\end{tikzcd}$$

The map h must be the zero map buth then the commutativity of the middle square implies $vu = 0$. This also shows that $v_*[i] \circ u_*[i] = 0$ for all i. Now let $f \in \ker(v_*[i])$. This means $v[i] \circ f = 0$. Now consider the following diagram:

$$\begin{tikzcd}
U \arrow[r, equal] \arrow[d, dotted, "g"] & U \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & U[1] \arrow[d, "{g[1]}"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]
\end{tikzcd}$$

We can always find g that completes the diagram to a morphism of triangles. In fact I can rotate the diagram using the axiom (TR2) then use (TR3) to construct $g[1]$ and then simply recover $g = g[1][-1]$. But now the commutativity of the first square tells me that $f = ug$ which is equivalent to saying that $f \in \mathrm{Im}(u_*[i])$. □

Corollary. *Let $\mathcal{T}$ be pre-triangulated, suppose we are in the situation of axiom (TR3) and that f, g are isomorphisms. Then so is h. In particular this shows that the completion of a morphism to a triangle is unique up to isomorphism of triangles.*

Proof. The diagram in (TR3) induces the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(Z', X) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}(Z', Y) \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}(Z', Z) \arrow[r] \arrow[d, "h_*"] & \mathrm{Hom}(Z', X[1]) \arrow[r] \arrow[d, "{f[1]_*}"] & \mathrm{Hom}(Z', Y'[1]) \arrow[d, "{g[1]_*}"] \\
\mathrm{Hom}(Z', X') \arrow[r] & \mathrm{Hom}(Z', Y') \arrow[r] & \mathrm{Hom}(Z', Z') \arrow[r] & \mathrm{Hom}(Z', X'[1]) \arrow[r] & \mathrm{Hom}(Z', Y'[1])
\end{tikzcd}$$

Now the previous lemma implies that the rows are exact and we know that f and g are isomorphisms. This means we can apply the five lemma and get that h_* is also an isomorphism. In particular there exists $\phi : Z' \to Z$ such that $h\phi = \mathrm{Id}_{Z'}$. Applying the same reasoning to the other exact sequence we get a map $\psi : Z' \to Z$ such that $\psi h = \mathrm{Id}_Z$. This shows that h is an isomorphism as required. □

Now we will give an alternative description of axiom (TR4) and prove that it is equivalent to (TR4) modulo the first three axioms.

Lemma 1.3.2. *Let $\mathcal{T}$ be a triangulated category and consider the following:*

(TR4') Given morphisms $\alpha_1 : X \to Y$ and $\beta_1 : Y \to Z$ I can complete them to an octahedron diagram.

Then $\mathcal{T}$ satisfies (TR4) if and only if it satisfies (TR4').

Proof. It is clear that (TR4) implies (TR4'), in fact given α_1 and β_1 I can complete them and $\beta_1 \circ \alpha_1$ to three distinguished triangles and then apply (TR4). Now we need to show that (TR4') implies (TR4). Let $\Delta_1 = (\alpha_1, \alpha_2, \alpha_3), \Delta_2 = (\beta_1, \beta_2, \beta_3)$ and $\Delta_3 = (\gamma_1, \gamma_2, \gamma_3)$ be three distinguished triangles. Now use axiom (TR4') to complete α_1, β_1 to the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, equal] & Y \arrow[r, "\alpha'_2"] \arrow[d, "\beta_1"] & U' \arrow[r, "\alpha'_3"] \arrow[d, dotted, "\delta'_1"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma'_2"] \arrow[d, "\beta'_2"] & V' \arrow[r, "\gamma'_3"] \arrow[d, dotted, "\delta'_2"] & X[1] \arrow[d, "\alpha_1[1]"] \\
 & W' \arrow[r, equal] \arrow[d, "\beta'_3"] & W' \arrow[r, "\beta'_3"] \arrow[d, dotted, "\delta'_3"] & Y[1] \\
 & Y[1] \arrow[r, "\alpha_2[1]"] & U[1] &
\end{tikzcd}$$

If we set $\Delta'_1 = (\alpha_1, \alpha'_2, \alpha'_3), \Delta'_2 = (\beta_1, \beta'_2, \beta'_3)$ and $\Delta'_3 = (\gamma_1, \gamma_2, \gamma_3)$ then Δ_1 and Δ'_1 are completions of the same morphism to a distinguished triangle so they are isomorphic. This means that there exists an isomorphism $\phi_1 : U \to U'$ such that $\alpha'_2 = \phi_1 \circ \alpha_2$ and $\alpha_3 = \alpha'_3 \circ \phi_1$. Similarly there exist isomorphisms $\phi_2 : V \to V'$ and $\phi_3 : W \to W'$ such that $\beta'_2 = \phi_3 \circ \beta_2$, $\beta_3 = \beta'_3 \circ \phi_3$, $\gamma'_2 = \phi_2 \circ \gamma_2$ and $\gamma_3 = \gamma'_3 \circ \phi_2$. Now we set $\delta_1 = \phi_2^{-1} \circ \delta'_1 \circ \phi_1, \delta_2 = \phi_3^{-1} \circ \delta'_2 \circ \phi_2$ and $\delta_3 = \phi_1^{-1}[1] \circ \delta'_3 \circ \phi_3$ and we claim that this are the required maps. The triangle $(\delta_1, \delta_2, \delta_3)$ is obviously distinguished so we only need to show that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, equal] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] \arrow[dr, phantom, "\star"] & U \arrow[r, "\alpha_3"] \arrow[d, dotted, "\delta_1"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\delta_3"] \arrow[d, dotted, "\delta_2"] & X[1] \arrow[d, "\alpha_1[1]"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, dotted, "\delta_3"] & Y[1] \\
 & Y[1] \arrow[r, "\alpha_2[1]"] & U[1] &
\end{tikzcd}$$

For example to see the commutativity of $\star$:

$$\begin{aligned}
\delta_1 \circ \alpha_2 &= \phi_2^{-1} \circ \delta'_1 \circ \phi_1 \circ \alpha_2 \\
&= \phi_2^{-1} \circ \delta'_1 \circ \alpha'_2 \\
&= \phi_2^{-1} \circ \gamma'_2 \circ \beta_1 \\
&= \gamma_2 \circ \beta_1
\end{aligned}$$

The commutativity of the other squares can be proved in the same way. $\square$

1.4 The homotopy category $\mathcal{K}(\mathcal{A})$ is triangulated

Now we want to show that the homotopy category $\mathcal{K}(\mathcal{A})$ is triangulated. An obvious choice for the translation functor is the shift functor. We say that a triangle in $\mathcal{K}(\mathcal{A})$ is distinguished if it is isomorphic to a triangle of the form:

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{\pi} A[1]^\bullet$$

First we need a lemma.

Definition 1.4.1 (semi-split sequence). *A short exact sequence of complexes* $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ *is said to be semi-split if there exist maps* $h^n : C^n \to B^n$ *not necessarily forming a chain map such that* $g^n h^n = id_{C^n}$

Lemma 1.4.1. *Any distinguished triangle in* $\mathcal{K}(\mathcal{A})$ *is isomorphic to a triangle* $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]^\bullet$ *such that* $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ *is a semi-split short exact sequence. Conversely every semi-split short exact sequence in* $\mathcal{C}(\mathcal{A})$ *can be completed to a distinguished triangle in* $\mathcal{K}(\mathcal{A})$

Proof. For the first part consider the following diagram:

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r, "g"] \arrow[d, equal] & C^\bullet \arrow[r, "h"] \arrow[d, leftrightarrow, "\cong"'] & A[1] \arrow[d, equal] \\
A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & \mathrm{Cone}(f) \arrow[r, "\pi"] \arrow[d, equal] & A[1] \arrow[d, equal] \\
A^\bullet \arrow[r, "i"] & \mathrm{Cyl}(f) \arrow[r, "\pi"] & \mathrm{Cone}(f) \arrow[r, "\pi"] & A[1]
\end{tikzcd}$$

We already proved that α is invertible in $\mathcal{K}(\mathcal{A})$ with inverse β and that $\beta i = f$. This implies that the first and last triangles in the diagram are isomorphic but now simply notice that $0 \longrightarrow A^\bullet \longrightarrow \mathrm{Cyl}(f) \longrightarrow \mathrm{Cone}(f) \longrightarrow 0$ is a semi-split short exact sequence.

Now let $0 \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0$ be a semi-split short exact sequence. Since it is semi-split we can assume $B^n = A^n \oplus C^n$, the maps $A^n \to B^n$ and $B^n \to C^n$ the canonical inclusions and projections. The differential d_B can be written as $\begin{pmatrix} d_A & -f \\ g & d_C \end{pmatrix}$.The fact that u is a chain map implies:

$$\begin{aligned}
0 &= d_B u - u d_A \\
&= \begin{pmatrix} d_A & -f \\ g & d_C \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} d_A \\
&= \begin{pmatrix} 0 \\ g \end{pmatrix}.
\end{aligned}$$

This implies $g = 0$. Furthermore the fact that d_B is a chain map implies:

$$\begin{aligned}
0 &= d_B^2 \\
&= \begin{pmatrix} d_A & -f \\ 0 & d_C \end{pmatrix}^2 \\
&= \begin{pmatrix} d_A^2 & -d_A f - f d_C \\ 0 & d_C^2 \end{pmatrix}
\end{aligned}$$

This implies that $f : C^\bullet \to A[1]^\bullet$ is a chain map. The map f completes the semi-split short exact sequence to a triangle. Now we need to show that it is distinguished. To see this consider the following diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "1"] & (A \oplus C)^\bullet \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "2"] & C^\bullet \arrow[r, "f"] \arrow[d, "h"] \arrow[dr, phantom, "3"] & A[1]^\bullet \arrow[d, equal] \\
A^\bullet \arrow[r, "u"'] & (A \oplus C)^\bullet \arrow[r, "i"'] & \mathrm{Cone}(u) \arrow[r, "\pi"'] & A[1]^\bullet
\end{tikzcd}$$

where $h = \begin{pmatrix} f \\ 0 \\ 1 \end{pmatrix}$. To see that h is a chain map:

$$\begin{aligned}
h d_C - d_{\mathrm{Cone}(u)} h &= \begin{pmatrix} f \\ 0 \\ 1 \end{pmatrix} d_C - \begin{pmatrix} -d_A & 0 & 0 \\ 1 & d_A & -f \\ 0 & 0 & d_C \end{pmatrix} \begin{pmatrix} f \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} f d_C \\ 0 \\ d_C \end{pmatrix} - \begin{pmatrix} -d_A f \\ f - f \\ d_C \end{pmatrix} \\
&= 0
\end{aligned}$$

The commutativity of squares 1 and 3 are obvious. To see the commutativiy of 2 :

$$\begin{aligned}
hv - i &= \begin{pmatrix} f \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & f \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -d_A & f \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d_A & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_A & -f \\ 0 & d_C \end{pmatrix} + \begin{pmatrix} -d_A & 0 & 0 \\ 1 & d_A & -f \\ 0 & 0 & d_C \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= k d_{A \oplus C} + d_{\mathrm{Cone}(u)} k
\end{aligned}$$

To see that h is an isomorphism in $\mathcal{K}(\mathcal{A})$ let $\pi : \mathrm{Cone}(u) \to C^\bullet$ be the canonical projection. Clearly $\pi h = \mathrm{id}_{C^\bullet}$. To see that $h\pi \sim \mathrm{id}$:

$$\begin{aligned}
\mathrm{Id} - h\pi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} f \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & d_A & -f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -d_A & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d_A & 0 & 0 \\ 1 & d_A & -f \\ 0 & 0 & d_C \end{pmatrix} + \begin{pmatrix} -d_A & 0 & 0 \\ 1 & d_A & -f \\ 0 & 0 & d_C \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= k' d_{\mathrm{Cone}(u)} + d_{\mathrm{Cone}(u)} k'
\end{aligned}$$

□

Theorem 1.4.1. *The homotopy category $\mathcal{K}(\mathcal{A})$ with the shift functor and the distinguished triangles defined above is triangulated.*

Proof. We need to check the four axioms of a triangulated category:

(TR1) The conditions (b) and (c) follow at once from the definition of distinguished triangle in $\mathcal{K}(\mathcal{A})$. To prove (a) consider the following diagram

$$\begin{tikzcd}
X^\bullet \arrow[r, "\mathrm{id}"] \arrow[d, equal] & X^\bullet \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & X[1]^\bullet \arrow[d, equal] \\
X^\bullet \arrow[r, "\mathrm{id}"] & X^\bullet \arrow[r, "i"] & \mathrm{Cone}(\mathrm{id}) \arrow[r, "\pi"] & X[1]^\bullet
\end{tikzcd}$$

We claim that this is an isomorphism of triangles. The only non trivial things are the fact that the middle square commutes and that the map $0 \to \mathrm{Cone}(\mathrm{id})$ is invertible in $\mathcal{K}(\mathcal{A})$. To see that the middle square commutes:

$$\begin{aligned}
i - 0 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -d_X \\ 1 \end{pmatrix} + \begin{pmatrix} d_X \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -d_X & 0 \\ 1 & d_X \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} d_X \\
&= d_{\mathrm{Cone}(\mathrm{id})} h + h d_X
\end{aligned}$$

To see that the map $0 \to \operatorname{Cone}(\mathrm{id})$ is an isomorphism it is enough to check that $0_{\operatorname{Cone}(\mathrm{id})} \sim \mathrm{Id}_{\operatorname{Cone}(\mathrm{id})}$. To see this:

$$\begin{aligned}
\mathrm{id}_{\operatorname{Cone}(\mathrm{id})} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -d_X \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & d_X \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -d_X & 0 \\ 1 & d_X \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_x & 0 \\ 1 & d_X \end{pmatrix} \\
&= d_{\operatorname{Cone}(\mathrm{id})} h + h d_{\operatorname{Cone}(\mathrm{id})}
\end{aligned}$$

(TR2) Suppose $A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]^\bullet$ is distinguished and consider the following diagram:

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r, "g"] \arrow[d, equal] & C^\bullet \arrow[r, "h"] \arrow[d, leftrightarrow, "\cong"] & A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, equal] & B[1]^\bullet \arrow[d, equal] \\
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "i_*"] \arrow[d, equal] & \operatorname{Cone}(f) \arrow[r, "\pi"] \arrow[d, equal] \arrow[dr, phantom, "1"] & A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, dotted, "\theta"] \arrow[dr, phantom, "2"] & B[1]^\bullet \arrow[d, equal] \\
 & B^\bullet \arrow[r, "i_*"] & \operatorname{Cone}(f) \arrow[r, "i"] & \operatorname{Cone}(i_*) \arrow[r, "\pi"] & B[1]^\bullet
\end{tikzcd}$$

Let θ be the morphism $\begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix}$. To see that it is a chain map:

$$\begin{aligned}
d_{\operatorname{Cone}(i_*)}\theta - \theta d_{A[1]} &= \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ 1 & f & d_B \end{pmatrix} \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix} d_A \\
&= \begin{pmatrix} d_B f \\ -d_A \\ -f + f \end{pmatrix} + \begin{pmatrix} -f d_A \\ d_A \\ 0 \end{pmatrix} = 0
\end{aligned}$$

To see that the square 1 commutes:

$$\begin{aligned}
i - \theta\pi &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} f & d_B \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -d_B \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} + \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ 1 & f & d_B \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= h d_{\operatorname{Cone}(f)} + d_{\operatorname{Cone}(i_*)} h
\end{aligned}$$

The commutativity of square 2 is obvious. Now let $\pi_* : \mathrm{Cone}(i_*) \to A[1]^\bullet$ be the map $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. It is clear that $\pi_*\theta = \mathrm{id}_{A[1]^\bullet}$. If we show that $\theta\pi_* \sim \mathrm{id}_{\mathrm{Cone}(i_*)}$ we are done. To see this:

$$
\begin{aligned}
\mathrm{id} - \theta\pi_* &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & f & d_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -d_B \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ 1 & f & d_B \end{pmatrix} + \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ 1 & f & d_B \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= h d_{\mathrm{Cone}(i_*)} + d_{\mathrm{Cone}(i_*)} h
\end{aligned}
$$

Conversely assume $B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet$ is distinguished.

By applying twice what we just proved we get that $A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet \xrightarrow{-g[1]} C[1]^\bullet \xrightarrow{-h[1]} A[2]$ is distinguished.

But since the shift functor reflects isomorphisms and we have $\mathrm{Cone}(-f[1]) \cong \mathrm{Cone}(f)[1]$ the triangle $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1]^\bullet$ is also distinguished.

(TR3) We can assume we are in the following situation:

$$
\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d, "f"] & B^\bullet \arrow[r, "i"] \arrow[d, "g"] & \mathrm{Cone}(u) \arrow[r, "\pi"] \arrow[d, dotted, "h"] & A[1]^\bullet \arrow[d, "f[1]"] \\
A'^\bullet \arrow[r, "u'"'] & B'^\bullet \arrow[r, "i"'] & \mathrm{Cone}(u') \arrow[r, "\pi"'] & A'[1]^\bullet
\end{tikzcd}
$$

We need to find the dotted arrow h so that the diagram is a morphism of triangles. It is easy to see that $h = f[1] \oplus g$ is the correct choice.

(TR4) To prove the octahedral axiom we will rely heavily on Lemma 1.4.1. We can assume that the triangles $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3), (\gamma_1, \gamma_2, \gamma_3)$ are semi-split so that the maps $\alpha_1, \beta_1, \gamma_1$ are canonical inclusions, the maps $\alpha_2, \beta_2, \gamma_2$ are canonical projections.

Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, equal] & X \oplus U \arrow[r, "\pi"] \arrow[d, "i"] & U \arrow[r, "\alpha_3"] \arrow[d, dotted, "\delta_1"'] & X[1] \arrow[d, equal] \\
X \arrow[r, "i"] & X \oplus U \oplus W \arrow[r, "\pi"] \arrow[d, "\pi"'] & U \oplus W \arrow[r, "\gamma_3"] \arrow[d, dotted, "\delta_2"'] \arrow[dr, phantom, "\star"] & X[1] \arrow[d, "i"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"'] & W \arrow[r, "\beta_3"'] \arrow[d, dotted, "\delta_3"'] & X[1] \oplus U[1] \\
 & X[1] \oplus U[1] \arrow[r, "\pi"'] & U[1] &
\end{tikzcd}$$

In Lemma 1.4.1 we proved that there is a relation between the differential of the second object and the third map in a semi-split triangle. If $d_{X \oplus U} = \begin{pmatrix} d_X & -f \\ 0 & d_U \end{pmatrix}$ then $\alpha_3 = f$. Furthermore if we set $\gamma_3 = \begin{pmatrix} f & g' \end{pmatrix}$ and $\beta_3 = \begin{pmatrix} g'' \\ h \end{pmatrix}$ then since the second object in both triangles is the same we get $g' = g'' = g$. Now we need to construct the dotted arrows, prove that the diagram commutes and that $(\delta_1, \delta_2, \delta_3)$ is distinguished. Let δ_1, δ_2 be the canonical inclusion and projection. By applying the relation between the third map and the differential to the triangle (i, π, β_3) we get :

$$d_{X \oplus U \oplus W} = \begin{pmatrix} d_X & -f & -g \\ 0 & d_U & -h \\ 0 & 0 & d_W \end{pmatrix}.$$

But then since (i, π, γ_3) is also semi-split we get

$$d_{U \oplus W} = \begin{pmatrix} d_U & -h \\ 0 & d_W \end{pmatrix}.$$

This means that if we set $\delta_3 = h$ the triangle $(\delta_1, \delta_2, \delta_3)$ is automatically distinguished so the only thing left to prove is commutativity. All the squares commute trivially except for $\star$. To see that this commutes:

$$\begin{aligned}
i\gamma_3 - \beta_3\delta_2 &= \begin{pmatrix} f & g \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & g \\ 0 & h \end{pmatrix} \\
&= \begin{pmatrix} f & 0 \\ 0 & -h \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ d_U & -h \end{pmatrix} - \begin{pmatrix} -f & 0 \\ d_U & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_U & -h \\ 0 & d_W \end{pmatrix} - \begin{pmatrix} d_X & -f \\ 0 & d_U \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= k d_{U \oplus W} + d_{(X \oplus U)[1]} k
\end{aligned}$$

□

1.5 The derived category $\mathcal{D}(\mathcal{A})$ is triangulated

Now we want to prove that the derived category is triangulated. Instead of proving it directly we will show that the localization of a triangulated category with respect to a class of morphisms S is triangulated provided that S satisfies certain properties.

Definition 1.5.1. *Let $\mathcal{T}$ be a triangulated category with translation functor T and S a multiplicative system of morphism. Then S is said to be compatible with the triangulation if the following properties hold:*

(a) $s \in S$ if and only if $T(s) \in S$.

(b) In the diagram of axiom (TR3) if $f, g \in S$ then h can be required to be in S.

Theorem 1.5.1. *Let $\mathcal{T}$ be a triangulated category and S a multiplicative system of morphisms compatible with the triangulation. Let $\mathcal{T}_S := \mathcal{T}[S^{-1}]$ and define $T_S : \mathcal{T}_S \to \mathcal{T}_S$ in the natural way: $T_S = T$ on objects and $T(f, \phi) = (Tf, T\phi)$. A triangle in $\mathcal{T}_S$ is distinguished if and only if it is ismorphic to the image of a distinguished triangle in $\mathcal{T}$ under the localization functor. Then $\mathcal{T}_S$ with the translation functor T_S and the distinguished described above is triangulated.*

Proof. We need to check that the four axioms of a triangulated category hold.

(TR1) Let $(f, \phi) : X \to Y$ be a morphism in $\mathcal{T}_S$. Complete f to a distinguished triangle $X \xrightarrow{f} Z \xrightarrow{u} U \xrightarrow{w} X[1]$ in $\mathcal{T}$. Now consider the following diagram

$$\begin{tikzcd}
 & Z & & U \arrow[dr, equal] & & X[1] \arrow[dr, equal] & \\
X \arrow[ur, "f"] \arrow[dd, equal] & & Y \arrow[ul, "\phi"'] \arrow[ur, "u\circ\phi"] \arrow[dd, "\phi"] & & U \arrow[ur, "w"] \arrow[dd, equal] & & X[1] \arrow[dd, equal] \\
 & Z \arrow[dr, equal] & & U \arrow[dr, equal] & & X[1] \arrow[dr, equal] & \\
X \arrow[ur, "f"] & & Z \arrow[ur, "u"] & & U \arrow[ur, "w"] & & X[1]
\end{tikzcd}$$

This is clearly an isomorphism of triangles. The bottom one is distinguished and so the top one must be distinguished. This proves that any morphism in $\mathcal{T}_S$ can be completed to a distinguished triangle so we are done.

(TR2) This is an obvious consequence of the fact that the rotation of a triangle commutes with the localization functor.

(TR3) We can assume that the given distinguished triangles are the image of distinguished triangles in $\mathcal{T}$.

$$\begin{tikzcd}
 & X'' \arrow[rr, dotted, "u''"] & & Y'' \arrow[rr, dotted, "v''"] & & Z'' \arrow[rr, dotted, "w''"] & & X[1] \\
X \arrow[ur, "\phi_1"] \arrow[rr, "u"] & & Y \arrow[ur, "\phi_2"] \arrow[rr, "v"] & & Z \arrow[ur, dotted, "\phi_3"] \arrow[rr, "w"] & & X[1] \arrow[ur, "\phi_1[1]"] & \\
X' \arrow[uur, "f_1"'] \arrow[rr, "u'"] & & Y' \arrow[uur, "f_2"'] \arrow[rr, "v'"] & & Z' \arrow[uur, dotted, "f_3"'] \arrow[rr, "w'"] & & X'[1] \arrow[uur, "f_1[1]"'] &
\end{tikzcd}$$

In this diagram the bottom two rows are the given distinguished triangles and for example by u I mean the morphism (u, id) in $\mathcal{T}_S$. We are given morphisms $(f_1, \phi_1), (f_2, \phi_2)$ and we want to find a morphism (f_3, ϕ_3) that completes them to a morphism of triangles. The maps u'', v'', w'' are auxiliary maps that I will now construct. The first part of the proof consists in showing that up to replacing the representation of the morphism (f_2, ϕ_2) it is always possible to find u'' such that the squares $XX''Y''Y$ and $X'X''Y''Y'$ commute. To see this first consider the following diagram:

$$\begin{tikzcd}
X'' \arrow[r, dotted, "t"] & Y''_1 \\
X' \arrow[u, "\phi"] \arrow[r, "\phi_2 \circ u'"'] & Y'' \arrow[u, dotted, "s"']
\end{tikzcd}$$

The property (MS2) of a multiplicative system implies the existence of the dotted arrows that make the previous diagram commute and such that $s \in S$. It is obvious that (f_2, ϕ_2) and $(s \circ f_2, s \circ \phi_2)$ so I can replace Y'' with Y''_1 and set $u'' = t$. This makes the square $XX''Y''Y$ commutative. The square $XX'Y'Y$ may not commute in $\mathcal{T}$ but we know that it commutes in $\mathcal{T}_S$. Now notice that $(u'' \circ f_1, \phi_2)$ represents $u' \circ (f_1, \phi_1)$ and $(f_2 \circ u, \phi_2)$ represents $(f_2, \phi_2) \circ u$. The fact that the square commutes means that there exists Y''_2 and maps $t_1, s_1 : Y'' \to Y''_2$ such that $s_1 \in S$ and the following diagram commutes:

$$\begin{tikzcd}
 & Y'' \arrow[d, "t_1"] & \\
X \arrow[ur, "u'' \circ f_1"] \arrow[dr, "f_2 \circ u"'] & Y''_2 & Y' \arrow[ul, "\phi_2"'] \arrow[dl, "\phi_2"] \\
 & Y'' \arrow[u, "s_1"'] &
\end{tikzcd}$$

The commutativity of the right triangle implies that $t_1 \circ \phi_1 = s_1 \circ \phi_2$. But then by (MS3) there exists Y''_3 and a map $\phi'_2 : Y''_2 \to Y''_3$ such that $\phi'_2 \circ t_1 = \phi'_2 \circ s_1$. Now we replace Y'' with Y''_3, ϕ_2 with $\phi'_2 \circ s_1 \circ \phi_2$, f_2 with $\phi'_2 \circ s_1 \circ f_2$ and u'' with $\phi'_2 \circ s_1 \circ u''$. Again it is easy to see that the morphism (f_2, ϕ_2) does not change, but now the commutativity of the left triangle implies that the square $XX'Y'Y$ commutes in $\mathcal{T}$. Now the rest is easy: complete u'' to a distinguished triangle $(u'', v'', w'') = \Delta''$ in $\mathcal{T}$. Set $(u, v, w) = \Delta$ and $(u', v', w') = \Delta'$ for simplicity. Complete f_1, f_2 to a morphism of triangles $(f_1, f_2, f_3) : \Delta' \to \Delta''$ and ϕ_1, ϕ_2 to a morphism of triangles $(\phi_1, \phi_2, \phi_3) : \Delta \to \Delta''$. We can do this because of TR3 and we can also require $\phi_3 \in S$ because S is compatible with the triangulation. But now it is easy to see that (f_3, ϕ_3) completes $(f_1, \phi_1), (f_2, \phi_2)$ to a morphism of triangles and so we are done.

(TR4) Let $f : X \to Y, g : Y \to Z$ be morphisms in $\mathcal{T}_S$. Up to changing the representation for the morphism g we can assume that $f = (p, t), g = (qt, s)$ and $gf = (qp, s)$ coming from the following diagram

$$\begin{tikzcd}
 & & Z' & & \\
 & Y' \arrow[ur, "q"] & & & \\
X \arrow[ur, "p"] & & Y \arrow[ul, "t"'] & & Z \arrow[uull, "s"']
\end{tikzcd}$$

Now using (TR4) in $\mathcal{T}$ we complete p, q to the following octahedron diagram:

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, equal] & Y' \arrow[r, "\alpha_2"] \arrow[d, "qt"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1"] & X[1] \arrow[d, equal] \\
X \arrow[r, "qp"] & Z' \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\delta_3"] \arrow[d, "\delta_2"] & X[1] \arrow[d, "\alpha_1[1]"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, "\delta_3"] & Y'[1] \\
 & Y'[1] \arrow[r, "\alpha_2[1]"] & U[1] &
\end{tikzcd}$$

We claim that the following diagram is the required octahedron diagram in $\mathcal{T}_S$ with the convention that a morphism $A \xrightarrow{h} B$ means the morphism (h, id) in $\mathcal{T}_S$.

$$\begin{tikzcd}
X \arrow[r, "{(p,t)}"] \arrow[d, equal] & Y \arrow[r, "\alpha_2 \circ t"] \arrow[d, "{(qt,s)}"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1"] & X[1] \arrow[d, equal] \\
X \arrow[r, "{(qp,s)}"] & Z \arrow[r, "\gamma_2 \circ s"] \arrow[d, "\beta_2 \circ s"] & V \arrow[r, "\delta_3"] \arrow[d, "\delta_2"] & X[1] \arrow[d, "{(p[1],t[1])}"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, "\delta_3"] & Y[1] \\
 & Y[1] \arrow[r, "(\alpha_2 \circ t)[1]"] & U[1] &
\end{tikzcd}$$

The commutativity is an obvious consequence of the commutativity of the octahedron in $\mathcal{T}$ so we only need to show that the four triangles are distinguished. For example to check that the first row is distinguished:

$$\begin{tikzcd}
X \arrow[r, "{(p,t)}"] \arrow[d, equal] & Y \arrow[r, "\alpha_2 \circ t"] \arrow[d, "t"] & U \arrow[r, "\alpha_3"] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r, "p"'] & Y' \arrow[r, "\alpha_2"'] & U \arrow[r, "\alpha_3"'] & X[1]
\end{tikzcd}$$

This diagram commutes and so it is a morphism of triangles, but t is an isomorphism in $\mathcal{T}_S$ so it is actually an isomorphism of triangles. This implies that the first row is distinguished. Similarly the other triangles can be shown to be distinguished so we are done.

□

Corollary. *Derived categories are triangulated*

Proof. Clearly it is enough to show that the class of quasi-isomorphism in $\mathcal{K}(\mathcal{A})$ is compatible with the triangulation. The first property is obvious: if q is a quasi-isomorphism then $\mathcal{H}^n q$ is an isomorphism for all n but then $\mathcal{H}^n q[1] = \mathcal{H}^{n+1} q$ is also an isomorphism for all n and so $q[1]$ is a quasi-isomorphism. To prove the second property recall that given an exact triangle in $\mathcal{K}(\mathcal{A})$ there is an associated long exact sequence of cohomology. Now consider the following situation:

$$\begin{tikzcd}
A^\bullet \arrow[r, "\alpha_1"] \arrow[d, "f"] & B^\bullet \arrow[r, "\alpha_2"] \arrow[d, "g"] & C^\bullet \arrow[r, "\alpha_3"] \arrow[d, dotted, "h"] & A[1]^\bullet \arrow[d, "f[1]"] \\
A'^\bullet \arrow[r, "\beta_1"] & B'^\bullet \arrow[r, "\beta_2"] & C'^\bullet \arrow[r, "\beta_3"] & A'[1]^\bullet
\end{tikzcd}$$

The two rows are distinguished, f, g are quasi-isomorphism, the first square commutes and h is any morphism that completes the diagram to a morphism of triangles (which exists because of (TR3)). This induces the following diagram for all n:

$$\begin{tikzcd}
H^n(A^\bullet) \arrow[r] \arrow[d, "H^n f"] & H^n(B^\bullet) \arrow[r] \arrow[d, "H^n g"] & H^n(C^\bullet) \arrow[r] \arrow[d, "H^n h"] & H^{n+1}(A^\bullet) \arrow[r] \arrow[d, "H^{n+1} f"] & H^{n+1}(B^\bullet) \arrow[d, "H^{n+1} g"] \\
H^n(A'^\bullet) \arrow[r] & H^n(B'^\bullet) \arrow[r] & H^n(C'^\bullet) \arrow[r] & H^{n+1}(A'^\bullet) \arrow[r] & H^{n+1}(B'^\bullet)
\end{tikzcd}$$

Applying the five-lemma to the previous diagram we get that h is also a quasi-isomorphism.

□

Remark. *A complex $A^\bullet$ is said to be bounded above if there exists $n \in \mathbb{Z}$ such that $A^i = 0$ for $i > n$. A complex satisfying the dual condition is said to be bounded below and a complex is bounded if it both bounded above and below. One can consider the full subcategory $C(\mathcal{A})^+ \subset C(\mathcal{A})$ of bounded above complexes. Similarly one can consider the full subcategory of bounded below complexes $C(\mathcal{A})^-$ and the full subcategory of bounded complexes $C(\mathcal{A})^b$. By considering morphisms modulo homotopy one gets $\mathcal{K}(\mathcal{A})^*$ for $* = +, -, b$. Then if S denotes the class of quasi-ismorphisms then one can define*

$$\mathcal{D}^*(\mathcal{A}) = \mathcal{K}^*(\mathcal{A})[S^{-1}]$$

for $* = +, -, b$. By repeating all the proofs in this chapter one finds that a morphism in $\mathcal{D}^*(\mathcal{A})$ is represented by a roof (f, ϕ) modulo an equivalence relation and that $\mathcal{D}^*(\mathcal{A})$ has a triangulated structure induced by $\mathcal{K}^*(\mathcal{A})$. This works because taking the cone or cylinder of a morphisms between bounded above (resp. below) complexes yields a bounded above (resp, below) complex.

1.6 Distinguished triangles as generalized exact sequences

Previously we proved that distinguished triangles in the homotopy category $\mathcal{K}(\mathcal{A})$ are in correspondence with semi-split exact sequences in $C(\mathcal{A})$. This shows that the triangulated structure in $\mathcal{K}(\mathcal{A})$ is inadequate to describe all exact sequences in $C(\mathcal{A})$. The derived category can also be seen as the extension of $\mathcal{K}(\mathcal{A})$ that fixes this problem. In fact the following holds:

Proposition 1.6.1. *Every short exact sequence* $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ *in $C(\mathcal{A})$ can be completed to a distinguished triangle* $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$ *in $\mathcal{D}(\mathcal{A})$. Conversely every distinguished triangle in $\mathcal{D}(\mathcal{A})$ is isomorphic to one obtained in this way.*

Proof. Let $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ be a short exact sequence in $C(\mathcal{A})$. Then there is a morphism $\gamma : \mathrm{Cone}(f) \to C^\bullet$ given by $f = \begin{pmatrix} 0 & g \end{pmatrix}$. We claim that this map is a quasi-isomorphism. To see this first we notice that g is an epimorphism and therefore so is γ. Now we consider the comples $\ker\gamma$ and we want to prove that it is acyclic. Indeed let $\begin{pmatrix} a \\ b \end{pmatrix}$ be a cocycle. Since it belongs to a term of $\ker\gamma$ we know that $g(b) = 0$ which means $b = fa'$ by the exactness of the sequence. Furthermore the fact that it is a cocycle means that $-d_A a = 0$ and $fa + d_B b = 0$. The last condition together with the fact that f is a monomorphism implies $a = -d_A a'$. But then we have

$$\begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \begin{pmatrix} a' \\ 0 \end{pmatrix} = \begin{pmatrix} -d_A a' \\ fa' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

which shows that $\begin{pmatrix} a \\ b \end{pmatrix}$ is a coboundary and so $\ker\gamma$ is acyclic. Now consider the obvious short exact sequence

$$0 \longrightarrow \ker\gamma \longrightarrow \mathrm{Cone}(f) \xrightarrow{\gamma} C^\bullet \longrightarrow 0 \ .$$

This induces a long exact sequence of cohomology and applying the fact that the first complex is acyclic we get precisely that γ is a quasi-isomorphism. Now consider the following diagram

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r, "g"] \arrow[d, equal] & C^\bullet \arrow[r, dotted] & A[1] \arrow[d, equal] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "i"] & \mathrm{Cone}(f) \arrow[r, "\pi"] \arrow[u, "\gamma"] & A[1] \end{tikzcd}$$

It is trivial to check that the first two squares commute. Furthermore since γ is a quasi-isomorphism it is invertible in $\mathcal{D}(\mathcal{A})$ and so we can find the dotted arrow h that makes the third square commute. This map completes the diagram to an isomorphism of triangles and since the bottom triangle is distinguished so is the top one. This shows why this works only in $\mathcal{D}(\mathcal{A})$ and not in the homotopy category: we need γ to be invertible. The converse is very easy: we alredy proved this for the homotopy category and it remains true when passing to the derived category. □

This shows that the distinguished triangles in the derived category generalize the notion of exact sequences. Thinking of distinguished triangles as generalized exact sequences leads to the following interpretation of the octahedron axiom:

Remark. *A prototype for exact sequences in an abelian category is given by*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook] & Y \arrow[r, two heads] & \frac{Y}{X} \arrow[r] & 0 \end{tikzcd}$$

Now consider the chain of inclusions $X \hookrightarrow Y \hookrightarrow Z$ *. We can form the following diagram*

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, equal] & Y \arrow[r, two heads] \arrow[d, hook] & \frac{Y}{X} \arrow[r] & X[1] \\
X \arrow[r, hook] & Z \arrow[r, two heads] \arrow[d, two heads] & \frac{Z}{X} \arrow[r] & X[1] \\
 & \frac{Z}{Y} \arrow[d] & & \\
 & Y[1] & &
\end{tikzcd}$$

where all the triangles involved are distinguished and the square is obviously commutative. By applying the octahedron axiom we get

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, equal] & Y \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\star"] & \frac{Y}{X} \arrow[r] \arrow[d, hook] & X[1] \arrow[d, equal] \\
X \arrow[r, hook] & Z \arrow[r, two heads] \arrow[d, two heads] & \frac{Z}{X} \arrow[r] \arrow[d, two heads] & X[1] \arrow[d] \\
 & \frac{Z}{Y} \arrow[r, equal] \arrow[d] & \frac{Z}{Y} \arrow[r] \arrow[d] & Y[1] \\
 & Y[1] \arrow[r] & \frac{Y}{X}[1] &
\end{tikzcd}$$

This yields a distinguished triangle

$$\frac{Y}{X} \hookrightarrow \frac{Z}{X} \twoheadrightarrow \frac{Z}{Y} \longrightarrow \frac{Y}{X}[1]$$

and the commutativity of the marked square implies that the map $\frac{Y}{X} \hookrightarrow \frac{Z}{X}$ *is the canonical inclusion. But then since all completions of a morphism to a distinguished triangle are isomorphic we get an isomorphism*

$$\frac{Z}{Y} \simeq \frac{\frac{Z}{X}}{\frac{Y}{X}}$$

In other words the octahedron axiom can be thought of as a version of the third isomorphism theorem.

The previous remark justifies the following definition:

Definition 1.6.1. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{D} \subset \mathcal{T}$ a subcategory. The category $\mathcal{D}$ is said to be closed by extensions if for any distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{T}$ such that $X, Z \in \mathcal{D}$ then $Y \in \mathcal{D}$.

The fact that distinguished triangles generalize exact sequences suggests that the triangulated structure is the right setting to define cohomological functors.

Definition 1.6.2. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. A functor $F : \mathcal{T} \to \mathcal{A}$ is called cohomological if for every distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{T}$ the sequence

$$\cdots \xrightarrow{Fh[-1]} FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{Fh} FX[1] \xrightarrow{Ff[1]} \cdots$$

is exact.

Example 1.6.1. *Let $\mathcal{T}$ be any triangulated category and $X \in \mathcal{T}$ an object. Then the functors $Hom(X, -)$ and $Hom(-, X)$ are cohomological.*

Example 1.6.2. *The functor $H^n : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ is cohomological.*

Chapter 2

t-structures

2.1 t-structures and truncation functors

In this chapter I will define t-structures and prove that the heart of a t-sructure is an abelian category. As I mentioned in the previous chapter the derived category $\mathcal{D}(\mathcal{A})$ is additive but not abelian. It can be very important to identify full abelian subcategories of derived categories and then ask what information this subcategories provide about $\mathcal{D}(\mathcal{A})$ or even $\mathcal{A}$ itself. The tool used to construct full abelian subcategories of triangulated categories (hence also of derived categories) is precisely the formalism of t-structures. Before giving the formal definition we will start with a motivating example. Suppose $\mathcal{D} = \mathcal{D}(\mathcal{A})$ is the derived category of an abelian category $\mathcal{A}$. Denote by $\mathcal{D}^{\geq 0}$ the full subcategory formed by all the complexes $A^\bullet$ such that $H^i(A^\bullet) = 0$ for all $i < n$. Similarly denote by $\mathcal{D}^{\leq 0}$ the full subcategory formed by all the complexes $A^\bullet$ such that $H^i(A^\bullet) = 0$ for all $i > n$ and set $\mathcal{D}^\heartsuit = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Proposition 2.1.1. *The obvious functor $F : \mathcal{A} \to \mathcal{D}^\heartsuit$ is an equivalence of categories.*

In particular this shows that any abelian category can be embedded as a full subcategory of its derived category.

Proof. We will show that the functor is fully faithful and essentially surjective. The essential surjectiviy is easy to see: every object $A^\bullet \in \mathcal{D}^\heartsuit$ is isomorphic to the complex with $H^0(\mathcal{A}^\bullet)$ in degree zero and 0 everywhere else, but this is exactly $FH^0(A)$. Now we need to show that the map

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\Phi} \mathrm{Hom}_{\mathcal{D}^\heartsuit}(FA, FB)$$

is an isomorphism for all A, B in $\mathcal{A}$. This map sends f to the morphism $\Phi(f)$ such that $\Phi(f)^0 = f = (f, \mathrm{id})$ and $\Phi(f)^n = 0$ for all n different from zero. Now we define a map

$$\mathrm{Hom}_{\mathcal{D}^\heartsuit}(FA, FB) \xrightarrow{\Psi} \mathrm{Hom}_{\mathcal{A}}(A, B)$$

defined as follows: $\Psi((f, s)) = (H^0 s)^{-1} \circ H^0 f$. It is obvious that $\Psi \circ \Phi = \mathrm{id}$ so we only need to check that $\Phi \circ \Psi = \mathrm{id}$. To see this consider a generic element of $\mathrm{Hom}_{\mathcal{D}^\heartsuit}(A, B)$

represented by the roof

$$\begin{tikzcd} & C & \\ FA \arrow[ur, "f"] & & FB \arrow[ul, "s"'] \end{tikzcd} .$$

We need to show that $\Phi \circ \Psi((f,s)) = (\Psi((f,s)), \mathrm{id})$ is equivalent to (f,s). To do this consider the complex V defined as follows:

$$V^n = \begin{cases} 0 & \text{if } n < 0 \\ B & \text{if } n = 0 \\ C^n & \text{if } n > 0 \end{cases}$$

where the only non obvious differential d^0 is simply the zero map. It is quite easy to check that V is in $\mathcal{D}^{\heartsuit}$ and the map $g : FB \to V$ which is the identity in degree zero and 0 everywhere else is a quasi-isomorphism. Furthermore there is a map $h : C \to V$ which is 0 in negative degree, the identity in positive degree and $(H^0 s)^{-1}$ in degree zero. If we show that the following diagram is commutative we are done:

$$\begin{tikzcd} & & V & & \\ & C \arrow[ur, "h"] & & FB \arrow[ul, "g"'] & \\ FA \arrow[ur, "f"] \arrow[urrr, "\Psi((f,s))"'] & & & & FB \arrow[ulll, "s"'] \arrow[ul, equal] \end{tikzcd}$$

But since all the non trivial maps are in degree zero it is enough to show that the following diagram commutes:

$$\begin{tikzcd} & & B & & \\ & C^0 \arrow[ur, "(H^0 s)^{-1}"] & & B \arrow[ul, equal] & \\ A \arrow[ur, "H^0 f"] \arrow[urrr, "(H^0 s)^{-1} \circ H^0 f"'] & & & & B \arrow[ulll, "H^0 s"'] \arrow[ul, equal] \end{tikzcd}$$

□

The idea of trying to imitate this structure in a general triangulated category led to this definition:

Definition 2.1.1. *A t-structure on a triangulated category $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ such that the conditions $a) - c)$ below are satisisfied. Let $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.*

(TS1) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ *and* $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.

(TS2) $Hom(X,Y) = 0$ *for any* $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$

(TS3) For any $X \in D$ there exists a distinguished triangle $A \to X \to B \to A[1]$ with $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$

The core of the t-structure is the category $\mathcal{D}^{\heartsuit} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$

Proposition 2.1.2. *If $\mathcal{D} = \mathcal{D}(\mathcal{A})$ then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ as defined in the example above is a t-structure.*

Proof. The property (TS1) is trivial. For any complex $A^\bullet$ we can consider the complex $\tau_{\leq 0}A^\bullet$ defined as follows:

$$\tau_{\leq 0}A^n = \begin{cases} A^n & n < 0 \\ \ker d^0 & n = 0 \\ 0 & n > 0 \end{cases} \qquad\qquad d^n_{\tau_{\leq 0}} = \begin{cases} d^n_A & n > 0 \\ 0 & n \geq 0 \end{cases}$$

Similarly we can consider the complex $\tau_{\geq 1}A^\bullet$ defined as follows:

$$\tau_{\geq 1}A^n = \begin{cases} 0 & n < 0 \\ \mathrm{Im} d^0 & n = 0 \\ A^n & n > 0 \end{cases} \qquad\qquad d^n_{\tau_{\geq 1}} = \begin{cases} 0 & n < 0 \\ i & n = 0 \\ d^n_A & n > 0 \end{cases}$$

Now we notice that $A^\bullet \in \mathcal{D}^{\leq 0}$ if and only if $A^\bullet$ is quasi-isomorphic to $\tau_{\leq 0}A^\bullet$. Similarly $B^\bullet \in \mathcal{D}^{\geq 1}$ if and only if $B^\bullet$ is quasi-isomorphic to $\tau_{\geq 1}B^\bullet$. This implies that for $A^\bullet \in \mathcal{D}^{\leq 0}, B^\bullet \in \mathcal{D}^{\geq 1}$ the following holds:

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\tau_{\leq 0}A^\bullet, \tau_{\geq 1}B^\bullet).$$

Now consider a generic morphism in $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\tau_{\leq 0}A^\bullet, \tau_{\geq 0}B^\bullet)$ represented by the diagram

$$\begin{tikzcd} & C^\bullet & \\ \tau_{\leq 0}A^\bullet \arrow[ur, "f"] & & \tau_{\geq 1}B^\bullet \arrow[ul, "\phi"'] \end{tikzcd}.$$

The existance of the morphism ϕ implies that $C^\bullet \in \mathcal{D}^{\geq 1}$. But then up to changing the representation of the morphism we can assume we are in this situation:

$$\begin{tikzcd} & \tau_{\geq 1}C^\bullet & \\ \tau_{\leq 0}A^\bullet \arrow[ur, "f"] & & \tau_{\geq 1}B^\bullet \arrow[ul, "\phi"'] \end{tikzcd}$$

To prove (TS2) it is enough to show that $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\tau_{\leq 0}A^\bullet, \tau_{\geq 1}B^\bullet) = 0$. Indeed let f be any such map:

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "i"] & C^1 \arrow[r] & C^2 \arrow[r] & \cdots \\ \cdots \arrow[r] & A^{-2} \arrow[r] \arrow[u] & A^{-1} \arrow[r] \arrow[u] & X^0 \arrow[r] \arrow[u, "f^0"] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

Clearly $f^n = 0$ for all n different from zero. Furthermore by commutativity we have $i \circ f^0 = 0$. But i is an inclusion so it is is monomorphism and this implies $f^0 = 0$. But then f is the zero morphism and so we are done. To prove (TS3) it is enough to notice that for all $A^\bullet$ there is a short exact sequence in $\mathcal{K}(\mathcal{A})$:

$$0 \longrightarrow \tau_{\leq 0}A^\bullet \longrightarrow A^\bullet \longrightarrow \tau_{\geq 1}A^\bullet \longrightarrow 0$$

which induces a distinguished triangle in $\mathcal{D}(\mathcal{A})$

$$\tau_{\leq 0}A^\bullet \longrightarrow A^\bullet \longrightarrow \tau_{\geq 1}A^\bullet \longrightarrow (\tau_{\leq 0}A^\bullet)[1] \ .$$

□

This proof relies on the existence of $\tau_{\leq 0}A^\bullet$ and $\tau_{\geq 1}A^\bullet$ for all $A^\bullet \in \mathcal{D}$. It is easy to check that $\tau_{\leq 0}, \tau_{\geq 1}$ are actually functors called the truncation functors. The term t-structure is an abbreviation for truncation structure. In fact the axioms for a t-structure were chosen so that they would imply the existance of functors $\tau_{\leq n}, \tau_{\geq n}$ satisfying the properties listed in the following lemma that are trivial to check in the case of $\mathcal{D} = \mathcal{D}(\mathcal{A})$.

Lemma 2.1.1. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ a t-structure. Then:*

(a) There exist functors $\tau_{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$ (resp. $\tau_{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$) that are right (resp, left) adjoint to the correponding embedding functors.

(b) For any $X \in \mathcal{D}$ and $n \in \mathbb{Z}$ there exists a distinguished triangle of the form

$$\tau_{\leq n}X \longrightarrow X \longrightarrow \tau_{>n}X \longrightarrow (\tau_{\leq n}X)[1] \ .$$

and any two distinguished triangles $A \to X \to B \to A[1]$ with $A \in \mathcal{D}^{\leq n}, B \in \mathcal{D}^{>n}$ are canonically isomorphic.

(c) For any $x \in \mathcal{D}$ the following are equivalent:

(1) $\tau_{\leq n}X = 0$

(2) the morphism $X \to \tau_{>n}X$ is an isomorphism.

(3) $X \in \mathcal{D}^{>n}$

(4) For all $Z \in \mathcal{D}^{\leq n}$ $Hom(Z, X) = 0$

(d) For $m \leq n$ there exist natural isomorphisms

$$\tau_{\leq m} \cong \tau_{\leq m}\tau_{\leq n} \cong \tau_{\leq n}\tau_{\leq m} \quad \text{and} \quad \tau_{\geq n} \cong \tau_{\geq m}\tau_{\geq n} \cong \tau_{\geq n}\tau_{\geq m}.$$

(e) The categories $\mathcal{D}^{\leq m}$ and $\mathcal{D}^{\geq m}$ are closed by extensions.

(f) For any $m, n \in \mathbb{Z}$ there exists a natural isomorphism

$$\tau_{\geq m}\tau_{\leq n} \cong \tau_{\leq n}\tau_{\geq m}.$$

Proof. (a) We will first show this for $\tau_{\leq 0}$ and $\tau_{\geq 1}$. For any X we can choose a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{D}^{\leq 0}$, $B \in \mathcal{D}^{\geq 1}$ and set $\tau_{\leq 0}X = A$, $\tau_{\geq 1}X = B$. Of course this depends on the choice of the distinguished triangle which is not necessarily unique but we will show that all such choices yield isomorphic functors. Now for any morphism $X \xrightarrow{f} Y$ we can consider the distinguished triangles

$$\begin{tikzcd}
A \arrow[r, "a"] & X \arrow[r] \arrow[d, "f"] & B \arrow[r] & A[1] \\
A' \arrow[r, "a'"] & Y \arrow[r] & B' \arrow[r] & A'[1]
\end{tikzcd}$$

We know that for any exact triangle there is an associated long exact sequence obtained applying the Hom-functor:

$$\cdots \longrightarrow \mathrm{Hom}(A, B'[-1]) \longrightarrow \mathrm{Hom}(A, A') \xrightarrow{a' \circ -} \mathrm{Hom}(A, Y) \longrightarrow \mathrm{Hom}(A, B') \longrightarrow \cdots$$

However $A \in \mathcal{D}^{\leq 0}, B' \in \mathcal{D}^{\geq 1}$ and $B'[-1] \in \mathcal{D}^{\geq 1}[-1] = \mathcal{D}^{\geq 2} \subset \mathcal{D}^{\geq 1}$ by (TS1). Then (TS2) implies that $\mathrm{Hom}(A, B') = \mathrm{Hom}(A, B'[-1]) = 0$ and so by exactness we get an isomorphism

$$\mathrm{Hom}(A, A') \xrightarrow[a' \circ -]{\cong} \mathrm{Hom}(A, Y)$$

This can be rephrased by saying that any morphism $A \to Y$ factors uniquely through h. Now the morphism $f \circ a$ induces a morphism $f^* : A \to A'$ and I simply set $\tau_{\leq 0} f = f^*$. It is easy to see that this maps the identity to the identity so I only need to show that it respects composition. To see this consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[dr, "\tau_{\leq 0} f"'] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
& A' \arrow[ur, "a'"'] \arrow[dr, "\tau_{\leq 0} g"'] & & \\
& & A'' \arrow[uur, "a''"'] &
\end{tikzcd}$$

The morphism $\tau_{\leq 0}(g \circ f)$ is the unique morphism that composed with a'' gives $g \circ f \circ a$. But we have

$$a'' \circ \tau_{\leq 0} g \circ \tau_{\leq 0} f = g \circ a' \circ \tau_{\leq 0} f = g \circ f \circ a$$

and so $\tau_{\leq 0}(g \circ f) = \tau_{\leq 0} g \circ \tau_{\leq 0} f$. Now for any $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}$ we have

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X, \tau_{\leq 0} Y).$$

It is also easy to check the naturality in both arguments so $\tau_{\leq 0}$ is right adjoint to the embedding functor. A similar argument shows that $\tau_{\geq 1}$ is a functor and is the left adjoint of the correponding embedding. Now we define $\tau_{\leq n} = T^{-n} \circ \tau_{\leq 0} \circ T^n$

and $\tau_{\geq n} = T^{1-n} \circ \tau_{\geq 1} \circ T^{n-1}$. To see that $\tau_{\leq n}$ is right adjoint to the embedding:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^{\leq n}}(X, \tau_{\leq n} Y) &= \mathrm{Hom}_{\mathcal{D}^{\leq n}}(X, T^{-n} \circ \tau_{\leq 0} \circ T^n Y) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\leq 0}}(T^n X, \tau_{\leq 0} \circ T^n Y) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(T^n X, T^n Y) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\leq n}}(X, Y) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(X, Y)
\end{aligned}$$

The naturality of this isomorphism is a direct consequence of the naturality in the case $n = 0$. A similar argument shows that $\tau_{\geq n}$ is the left adjoint to the corresponding embedding.

(b) It is enough to consider the case $n = 0$ since the other cases can be obtained applying the translation functor. The existance of the triangle is an obvious consequence of the definition of the truncation functors. Now consider two distinguished triangles

$$\begin{tikzcd}
\Delta = (A \arrow[r, "a"] & X \arrow[r, "b"] \arrow[d, equal] & B \arrow[r] & A[1]) \\
\Delta' = (A' \arrow[r, "a'"] & X \arrow[r, "b'"] & B' \arrow[r] & A'[1])
\end{tikzcd}.$$

The morphism a factors uniquely through a' yielding a morphism $f : A \to A'$ and a' factors uniquely through a yielding $f' : A' \to A$. Similarly we get $g : B \to B'$ and $g' : B' \to B$. We claim that f and f' are inverses. To see this notice that $a' \circ f \circ f' = a'$. Now a' factors through a' also in the trivial way $a' = a' \circ \mathrm{id}$. By uniqueness we get precisely $f \circ f' = \mathrm{id}$. Similarly $a \circ f' \circ f = a$ and since a factors also in the trivial way $a = a \circ \mathrm{id}$ we get $f' \circ f = \mathrm{id}$. A similar argument shows that g and g' are inverses. Now the pair (f, id) can be completed to isomorphism of triangles $(f, \mathrm{id}, g_*) : \Delta \to \Delta'$. By comparing factorizations of b' through b we get that $g_* = g$ so that the isomorphism is canonical.

(c) We will first show that (1) is equivalent to (2). Indeed for any $X \in \mathcal{D}$ we have:

$$\begin{aligned}
\tau_{\leq n} X = 0 \quad &\Leftrightarrow \quad T^{-n} \circ \tau_{\leq 0} \circ T^n X = 0 \\
&\Leftrightarrow \quad \tau_{\leq 0} T^n X = 0 \\
&\Leftrightarrow \quad T^n X \to \tau_{\geq 1} T^n X \quad \text{is an isomorphism} \\
&\Leftrightarrow \quad X \to T^{-n} \circ \tau_{\geq 1} \circ T^n X \quad \text{is an isomorphism} \\
&\Leftrightarrow \quad X \to \tau_{>n} X \quad \text{is an isomorphism}
\end{aligned}$$

Next we will show that (2) $\implies$ (3) $\implies$ (4) $\implies$ (2). Suppose $X \to \tau_{>n} X$ is an isomorphism. We know that $\tau_{>n} X \in \mathcal{D}^{>n}$ which is stable under isomorphisms and so $X \in \mathcal{D}^{>n}$. Now assume $X \in \mathcal{D}^{>n}$ and let $Z \in \mathcal{D}^{\leq n}$. This means $X = X_0[-n], Z = Z_0[-n]$ with $Z \in \mathcal{D}^{\leq 0}, X \in \mathcal{D}^{\geq 1}$. But then

$$\mathrm{Hom}(Z, X) = \mathrm{Hom}(Z_0[-n], X_0[-n]) \cong \mathrm{Hom}(Z_0, X_0) = 0.$$

Now suppose $\mathrm{Hom}(Z, X) = 0$ for all $Z \in \mathcal{D}^{\leq n}$. In particular this is true for $Z = \tau_{\leq n} X$. But now I consider the distinguished triangle

$$\tau_{\leq n} X \longrightarrow X \longrightarrow \tau_{>n} X \longrightarrow (\tau_{\leq n} X)[1]$$

The first morphism is zero so the second must be an isomorphism.

(d) The isomorphisms $\tau_{\leq m} \cong \tau_{\leq n}\tau_{\leq m}$ and $\tau_{\geq n} \cong \tau_{\geq m}\tau_{\geq n}$ are an obvious consequence of part (c). Now for any $X \in \mathcal{D}, Y \in \mathcal{D}^{\leq m}$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^{\leq m}}(Y, \tau_{\leq m}\tau_{\leq n} X) &\cong \mathrm{Hom}_{\mathcal{D}^{\leq n}}(Y, \tau_{\leq n} X) \cong \mathrm{Hom}_{\mathcal{D}}(Y, \tau_{\leq n} X) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(Y, X) \cong \mathrm{Hom}_{\mathcal{D}^{\leq m}}(Y, \tau_{\leq m} X)\end{aligned}$$

which shows $\tau_{\leq m} \cong \tau_{\leq n}\tau_{\leq m}$. The other isomorphism can be checked in the same way.

(e) I will prove that the categories $\mathcal{D}^{\leq n}$ are closed by extensions for all n. The other case can be proved in a similar way. Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

be a distinguished triangle with $X, Z \in \mathcal{D}^{\leq n}$. By applying the functor $\mathrm{Hom}(-, T)$ we get the exact sequence

$$\mathrm{Hom}(Z, T) \longrightarrow \mathrm{Hom}(Y, T) \longrightarrow \mathrm{Hom}(X, T)$$

The first and third terms are zero which implies that $\mathrm{Hom}(Y, T) = 0$ for all $T \in \mathcal{D}^{>n}$ which is equivalent to $Y \in \mathcal{D}^{\leq n}$.

(f) For $m > n$ both functors are identically zero as a consequence of (c) so we can assume $m \leq n$. Consider the following distinguished triangle

$$\tau_{\leq n}\tau_{\geq m} X \longrightarrow \tau_{\geq m} X \longrightarrow \tau_{>n}\tau_{\geq m} X \longrightarrow (\tau_{\leq n}\tau_{\geq m} X)[1] \; .$$

We know from part (d) that $\tau_{>n}\tau_{\geq m} X = \tau_{>n} X$ so that the triangle becomes

$$\tau_{\leq n}\tau_{\geq m} X \longrightarrow \tau_{\geq m} X \longrightarrow \tau_{>n} X \longrightarrow (\tau_{\leq n}\tau_{\geq m} X)[1] \; .$$

Now we notice that $\tau_{\geq m} X, (\tau_{>n} X)[-1] \in \mathcal{D}^{\geq m}$ so by rotating the previous triangle and applying the fact that the category $\mathcal{D}^{\geq m}$ is closed by extensions we get that $\tau_{\leq n}\tau_{\geq m} X \in \mathcal{D}^{\geq m}$. Similarly there is a distinguished triangle

$$\tau_{>m} X \longrightarrow \tau_{\leq n} X \longrightarrow \tau_{\geq m}\tau_{\leq n} X \longrightarrow (\tau_{>m} X)[1]$$

which shows that $\tau_{\geq m}\tau_{\leq n} X \in \mathcal{D}^{\leq n}$. We will first define a morphism $\phi_X : \tau_{\geq m}\tau_{\leq n} X \to \tau_{\leq n}\tau_{\geq m} X$ and then prove that it is an isomorphism. Consider the following diagram

$$\begin{tikzcd} \tau_{\geq m}\tau_{\leq n} X \arrow[d] \arrow[dr, dotted, "\exists!"] & \\ \tau_{\geq m} X & \tau_{\leq n}\tau_{\geq m} X \arrow[l] \end{tikzcd}$$

The horizontal arrow is the canonical morphism and the vertical arrow is the image under $\tau_{\leq m}$ of the canonical morphism $\tau_{\leq n}X \to X$. Since $\tau_{\geq m}\tau_{\leq n}X \in \mathcal{D}^{\leq n}$ we can use the universal property of the adjunction and say that there is a unique dotted arrow ϕ that makes the triangle commute. To prove that it is an isomorphism we will need to use the octahedron axiom. Indeed consider the following diagram

$$\begin{tikzcd}
\tau_{<m}X \arrow[r] \arrow[d, equal] & \tau_{\leq n}X \arrow[r] \arrow[d] & \tau_{\geq m}\tau_{\leq n}X \arrow[r] & (\tau_{<m}X)[1] \\
\tau_{<m}X \arrow[r] & X \arrow[r] \arrow[d] & \tau_{\geq m}X \arrow[r] & (\tau_{<m}X)[1] \\
 & \tau_{>n}X \arrow[d] & & \\
 & (\tau_{\leq n}X)[1] & &
\end{tikzcd}$$

It is clear that the three triangles are distinguished and the square commutes trivially since the canonical morphism $\tau_{<m}X \to \tau_{\leq n}X$ is the unique dotted arrow that makes the following diagram commute:

$$\begin{tikzcd}
\tau_{\leq n}X \arrow[r] & X \\
\tau_{<m}X \arrow[u, dotted] \arrow[ur] &
\end{tikzcd}$$

We can apply the octahedron axiom and get the following diagram

$$\begin{tikzcd}
\tau_{<m}X \arrow[r] \arrow[d, equal] & \tau_{\leq n}X \arrow[r] \arrow[d] & \tau_{\geq m}\tau_{\leq n}X \arrow[r] \arrow[d] & (\tau_{<m}X)[1] \arrow[d] \\
\tau_{<m}X \arrow[r] & X \arrow[r] \arrow[d] & \tau_{\geq m}X \arrow[r] \arrow[d] & (\tau_{<m}X)[1] \\
 & \tau_{>n}X \arrow[r, equal] \arrow[d] & \tau_{>n}X \arrow[d] & \\
 & (\tau_{\leq n}X)[1] \arrow[r] & (\tau_{\geq m}\tau_{\leq n}X)[1] &
\end{tikzcd}$$

This implies the existence of a distinguished triangle

$$\tau_{\geq m}\tau_{\leq n}X \longrightarrow \tau_{\geq m}X \longrightarrow \tau_{>n}X \longrightarrow (\tau_{\geq m}\tau_{\leq n}X)[1]$$

We already proved that $\tau_{\geq m}\tau_{\leq n}X \in \mathcal{D}^{\leq n}$ and so (b) implies the existence of a unique isomorphism of triangles

$$\begin{tikzcd}
\tau_{\geq m}\tau_{\leq n}X \arrow[r] \arrow[d, "\phi_X"] & \tau_{\geq m}X \arrow[r] \arrow[d, equal] & \tau_{>n}X \arrow[r] \arrow[d, equal] & (\tau_{\geq m}\tau_{\leq n}X)[1] \arrow[d, "\phi_X[1]"] \\
\tau_{\leq n}\tau_{\geq m}X \arrow[r] & \tau_{\geq m}X \arrow[r] & \tau_{>n}X \arrow[r] & (\tau_{\leq n}\tau_{\geq m}X)[1]
\end{tikzcd}$$

To conclude we will show that the collection of morphisms ϕ_X is a natural transformation. To see this recall that for any X we have the commutative diagram

$$\begin{tikzcd} \tau_{\geq m}\tau_{\leq n}X \arrow[d, "g_X"'] \arrow[dr, "\phi_X"] & \\ \tau_{\geq m}X & \tau_{\leq n}\tau_{\geq m}X \arrow[l, "h_X"] \end{tikzcd}$$

and g_X, h_X are natural transformations. For any morphism $f : X \to Y$ in $\mathcal{D}$ we have following diagram

$$\begin{tikzcd} & & & \tau_{\geq m}\tau_{\leq n}Y \arrow[dl, "g_Y"'] \arrow[dr, "\phi_Y"] & \\ & \tau_{\geq m}\tau_{\leq n}X \arrow[urr, "\tau_{\geq m}\tau_{\leq n}f"] \arrow[dl, "g_X"'] \arrow[dr, "\phi_X"] & \tau_{\geq m}Y & & \tau_{\leq n}\tau_{\geq m}Y \arrow[ll, "h_Y"] \\ \tau_{\geq m}X \arrow[urr, "\tau_{\geq m}f"'] & & \tau_{\leq n}\tau_{\geq m}X \arrow[ll, "h_X"] \arrow[urr, "\tau_{\leq n}\tau_{\geq m}f"'] & & \end{tikzcd}$$

We need to show that $\phi_Y \circ \tau_{\geq m}\tau_{\leq n}f = \tau_{\leq n}\tau_{\geq m}f \circ \phi_X$. Indeed we have

$$\begin{aligned} h_Y \circ \phi_Y \circ \tau_{\geq m}\tau_{\leq n}f &= g_Y \circ \tau_{\geq m}\tau_{\leq n}f \\ &= \tau_{\geq m}f \circ g_X \\ &= \tau_{\geq m}f \circ h_X \circ \phi_X \\ &= h_Y \circ \tau_{\leq n}\tau_{\geq m}f \circ \phi_X \end{aligned}$$

But now we simply recall that $\tau_{\geq m}\tau_{\leq n}X \in \mathcal{D}^{\leq n}$ which implies that the morphism $h_Y \circ \phi_Y \circ \tau_{\geq m}\tau_{\leq n}f$ factors uniquely through h_Y and so we are done.

□

The functor $\tau_{\geq m}\tau_{\leq n} = \tau_{\leq n}\tau_{\geq m}$ is usually denoted by $\tau_{[n,m]}$.

2.2 The heart $\mathcal{D}^{\heartsuit}$ is abelian

In this section we will prove the main result of this thesis.

Lemma 2.2.1. *Let $\mathcal{D}$ be a triangulated category and $\Delta_1 = (f, g, h)$, $\Delta_2 = (f', g', h')$ two distinguished triangles. Then $\Delta_1 \oplus \Delta_2 = (f \oplus f', g \oplus g', h \oplus h')$ is distinguished. Furthermore for any two objects $X, Z \in \mathcal{D}$ there is a distinguished triangle*

$$X \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow X[1]$$

Proof. Given two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$$

we can consider the morphism $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y'$ and complete it to a distinguished triangle

$$X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \longrightarrow Q \longrightarrow (X \oplus X')[1]$$

We can use axiom (TR3) to complete the canonical projections to a morphism of distinguished triangles

$$\begin{tikzcd}
X \oplus X' \arrow[r, "f \oplus f'"] \arrow[d, "\pi_X"] & Y \oplus Y' \arrow[r] \arrow[d, "\pi_Y"] & Q \arrow[r] \arrow[d, "\phi_Z"] & (X \oplus X')[1] \arrow[d, "\pi_X[1]"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

obtaining $\phi_Z : Q \to Z$. In a similar way we construct $\phi'_Z : Q \to Z'$ and therefore we get $\phi = (\phi_Z, \phi'_Z) : Q \to Z \oplus Z'$. The way we constructed ϕ implies that the diagram

$$\begin{tikzcd}
X \oplus X' \arrow[r] \arrow[d, equal] & Y \oplus Y' \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d, "\phi"'] & (X \oplus X')[1] \arrow[d, equal] \\
X \oplus X' \arrow[r] & Y \oplus Y' \arrow[r] & Z \oplus Z' \arrow[r] & (X \oplus X')[1]
\end{tikzcd}$$

is a morphism of triangles. The first triangle is distinguished so if we prove that ϕ is an isomorphism we are done. Applying the functor $\mathrm{Hom}(A, -)$ we get the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(A, X \oplus X') \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A, Y \oplus Y') \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A, Q) \arrow[r] \arrow[d, "\phi\circ-"'] & \mathrm{Hom}(A, (X \oplus X')[1]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A, (Y \oplus Y')[1]) \arrow[d, equal] \\
\mathrm{Hom}(A, X \oplus X') \arrow[r] & \mathrm{Hom}(A, Y \oplus Y') \arrow[r] & \mathrm{Hom}(A, Z \oplus Z') \arrow[r] & \mathrm{Hom}(A, (X \oplus X')[1]) \arrow[r] & \mathrm{Hom}(A, (Y \oplus Y')[1])
\end{tikzcd}$$

The first row is obviously exact. As for the second we notice that the Hom functor commutes with direct sums so that the second row is the direct sum of two exact sequences and so it is exact. Applying the five lemma we get that $\phi \circ - : \mathrm{Hom}(A, Z \oplus Z') \to \mathrm{Hom}(A, Q)$ is an isomorphism for all A and so by Yoneda lemma we get that $\phi : Z \oplus Z' \to X$ is an isomorphism. This proves the first part of the lemma. For the second part simply notice that for any X, Z there are distinguished triangles

$$X = X \longrightarrow 0 \longrightarrow X[1]$$

$$0 \longrightarrow Z = Z \longrightarrow 0$$

But then using the first part of the lemma we get that their direct sum is also distinguished and so we are done. □

Now we are ready to prove the main result of this chapter.

Theorem 2.2.1. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t-structure. Then*

(i) The heart $\mathcal{D}^{\heartsuit}$ is an abelian category.

(ii) *An exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{D}^\heartsuit$ *induces a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

in $\mathcal{D}$.

Proof. (i) The category $\mathcal{D}^\heartsuit$ is obviously pre-additive because it is the intersection of two full subcategories of an additive category. To show that it is additive we need to prove that given $X, Y \in \mathcal{D}^\heartsuit$ the direct sum $X \oplus Y$ is in $\mathcal{D}^\heartsuit$ but this is clear: the previous lemma implies that the direct sum of X and Y is an extension and since $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$ are stable under extensions so is their intersection $\mathcal{D}^\heartsuit$. Now we need to define kernels and cokernels in $\mathcal{D}^\heartsuit$. Let $f : X \to Y$ be a morphism in $\mathcal{D}^\heartsuit$. We can complete it to a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

The stability under extensions implies that $Z \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$ and $Z[-1] \in \mathcal{D}^{\leq 1} \cap \mathcal{D}^{\geq 0}$. Now we define $K = \tau_{\leq 0} Z[1]$ and $C = \tau_{\geq 0} Z$. Clearly $K, C \in \mathcal{D}^\heartsuit$ and there are canonical morphisms $K \to Z[1] \to X$ and $Y \to Z \to C$. We claim that K and C are respectively the kernel and cokernel of f and the morphisms we just defined are the canonical inclusion and projection associated to the kernel and cokernel. For any $W \in \mathcal{D}^\heartsuit$ we can apply the Hom functor to the distinguished triangle and get long exact sequences

$$\mathrm{Hom}(X[1], W) \longrightarrow \mathrm{Hom}(Z, W) \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W)$$

$$\mathrm{Hom}(W, Y[-1]) \longrightarrow \mathrm{Hom}(W, Z[-1]) \longrightarrow \mathrm{Hom}(W, X) \longrightarrow \mathrm{Hom}(W, Y)$$

First of all we notice that $\mathrm{Hom}(X[1], W) = \mathrm{Hom}(W, Y[-1]) = 0$ by (TS2). Furthermore since $W \in \mathcal{D}^\heartsuit$ we can use the adjunctions between $\tau_{\leq 0}$ and $\tau_{\geq}$ and the correspondent embeddings and obtain

$$\mathrm{Hom}(Z, W) \cong \mathrm{Hom}(\tau_{\geq 0} Z, W) = \mathrm{Hom}(C, W)$$
$$\mathrm{Hom}(W, Z[-1]) \cong \mathrm{Hom}(W, \tau_{\leq 0} Z[-1]) = \mathrm{Hom}(W, K)$$

This means that the long exact sequences can be rewritten as

$$0 \longrightarrow \mathrm{Hom}(C, W) \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W)$$

$$0 \longrightarrow \mathrm{Hom}(W, K) \longrightarrow \mathrm{Hom}(W, X) \longrightarrow \mathrm{Hom}(W, Y)$$

The fact that this sequences are exact for all $W \in \mathcal{D}^{\heartsuit}$ means exactly that K and C are the kernel and cokernel of f. To conclude the proof we need to show that the coimage and the image of f are isomorphic. We start by embedding the canonical morphism $Y \to C$ in a distinguished triangle

$$I \longrightarrow Y \longrightarrow C \longrightarrow I[1]$$

In particular we notice that $I \in \mathcal{D}^{\geq 0}$. Now we consider the distinguished triangle

$$\tau_{<0}Z \longrightarrow Z \longrightarrow \tau_{\geq 0}Z \longrightarrow (\tau_{<0}Z)[1]$$

and we notice that $\tau_{\geq 0}Z = C$ and $(\tau_{<0}Z)[1] = K[2]$. Now consider the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X[1] \arrow[r] & Y[1] \\
Y \arrow[r] & C \arrow[r] \arrow[d] & I[1] \arrow[r] & Y[1] \\
 & K[2] \arrow[d] & & \\
 & Z[1] & &
\end{tikzcd}$$

The three triangles are distinguished and the square commutes by definition of the map $Y \to C$. This means that we can apply the octahedron axiom and get the following diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & Y[1] \arrow[d, equal] \\
Y \arrow[r] & C \arrow[r] \arrow[d] & I[1] \arrow[r] \arrow[d] & Y[1] \\
 & K[2] \arrow[r, equal] \arrow[d] & K[2] \arrow[d] & \\
 & Z[1] \arrow[r] & X[2] &
\end{tikzcd}$$

which means that the triangle

$$X[1] \longrightarrow I[1] \longrightarrow K[2] \longrightarrow X[2]$$

is distinguished. This shows that $I \in \mathcal{D}^{\leq 0}$ and so $I \in \mathcal{D}^{\heartsuit}$. Furthermore we have distinguished triangles

$$I \longrightarrow Y \longrightarrow C \longrightarrow I[1]$$

$$K \longrightarrow X \longrightarrow I \longrightarrow K[1]$$

and the commutativity of the bottom square of the octahedron diagram implies that the morphism $K \to X$ is exactly the canonical morphism of the kernel. Now by repeating the same argument as before (applying the Hom functors and noticing that the first term of the exact sequence vanishes) we get that

$$\mathrm{Im} f = \ker(Y \to \mathrm{coker} f) \cong I \cong \mathrm{coker}(\ker f \to X) = \mathrm{coIm} f$$

(ii) Let $0 \longrightarrow X \xrightarrow{f} Y \longrightarrow Z \longrightarrow 0$ be an exact sequence in $\mathcal{D}^\heartsuit$. We can embed f in a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow W \longrightarrow X[1]$$

As usual the stability under extensions implies $W \in \mathcal{D}^{\leq 0}$. The exactness of the short exact sequence tells us that $\ker f = 0$ and $\mathrm{coker} f = Z$. In the previous proof we showed how to construct kernels and cokernels in $\mathcal{D}^\heartsuit$. In particular we have $0 = \ker f = (\tau_{\leq 0}) W[1]$ which means that $W[1] \in \mathcal{D}^{>0} \implies W \in \mathcal{D}^{\geq 0} \implies W \in \mathcal{D}^\heartsuit$. Furthermore we have $Z = \mathrm{coker} f \cong \tau_{\geq 0} W \cong W$ and so we are done. □

2.3 Cohomological functors induced by t-structures

Previously we defined the functors $\tau_{[n,m]} = \tau_{\leq 0}\tau_{\geq 0} = \tau_{\geq 0}\tau_{\leq 0}$. In order to understand the behaviour of the functor $\tau_{[0,0]} : \mathcal{D} \to \mathcal{D}^\heartsuit$ we will first consider the special case $\mathcal{D} = \mathcal{D}(\mathcal{A})$ with the standard t-structure. Let $A^\bullet \in \mathcal{D}(\mathcal{A})$ be the complex

$$\cdots \longrightarrow A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots .$$

Then we have

$$\begin{aligned}\tau_{[0,0]} A^\bullet &= \tau_{\leq 0}(\ldots \to 0 \to \mathrm{im} d^{-1} \to A^0 \to A^1 \to \ldots) \\ &= (\cdots \to 0 \to \mathrm{im} d^{-1} \to \ker d^0 \to 0 \to \ldots)\end{aligned}$$

This complex is quasi-isomorphic to the complex $H^0 A^\bullet$. This means that composing the functor $\tau_{[0,0]}$ with the equivalence $\mathcal{D}^\heartsuit \to \mathcal{A}$ yields a functor isomorphic to the 0-cohomology functor. This justifies the notation H^0 for $\tau_{[0,0]}$, furthermore we will also define $H^n = H^0 \circ T^n = T^n \circ \tau_{[n,n]}$. This functors are called the n-cohomology functors induced by the $t$structure and they turn out to be cohomological functors also in the general case.

Theorem 2.3.1. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t-structure with heart $\mathcal{D}^\heartsuit$. Then the functors $H^n : \mathcal{D} \to \mathcal{D}^\heartsuit$ are cohomological.*

Proof. Let $\Delta = \; X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ be a distinguished triangle. This proof is divided in three steps.

(a) First we will show that if $X, Y, Z \in \mathcal{D}^{\geq 0}$ the sequence

$$0 \longrightarrow H^0 X \longrightarrow H^0 Y \longrightarrow H^0 Z$$

is exact. To see this consider an object $W \in \mathcal{D}^{\heartsuit}$, applying the cohomological functor $\mathrm{Hom}(W,-)$ to Δ we get the following exact sequence:

$$\mathrm{Hom}(W,Z[-1]) \longrightarrow \mathrm{Hom}(W,X) \longrightarrow \mathrm{Hom}(W,Y) \longrightarrow \mathrm{Hom}(W,Z) \ .$$

Now we notice that $\mathrm{Hom}(W,Z[-1]) = 0$, $\mathrm{Hom}(W,X) \cong \mathrm{Hom}(W,\tau_{\leq 0}X) \cong \mathrm{Hom}(W,H^0X)$ and similarly for Y and Z so that the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}^{\heartsuit}}(W,H^0X) \longrightarrow \mathrm{Hom}_{\mathcal{D}^{\heartsuit}}(W,H^0Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}^{\heartsuit}}(W,H^0Z)$$

is exact. Since W is an arbitrary object in $\mathcal{D}^{\heartsuit}$ by the left-exactness of the Hom-functor we get that the sequence

$$0 \longrightarrow H^0X \longrightarrow H^0Y \longrightarrow H^0Z \tag{2.1}$$

is exact. Dually we get that if $X,Y,Z \in \mathcal{D}^{\leq 0}$ then the sequence

$$H^0X \longrightarrow H^0Y \longrightarrow H^0Z \longrightarrow 0 \tag{2.2}$$

is exact.

(b) Now we only assume $Z \in \mathcal{D}^{\geq 0}$ and let $W \in \mathcal{D}^{<0}$. Arguing as before and noticing that $\mathrm{Hom}(W,Z) = \mathrm{Hom}(W,Z[-1]) = 0$ we get $\mathrm{Hom}(W,X) \cong \mathrm{Hom}(W,Y)$ we get $\tau_{<0}X \cong \tau_{<0}Y$ and so the canonical morphism $\tau_{<0}X \to \tau_{<0}Y$ is an isomorphism. Now consider the diagram

$$\begin{tikzcd}
\tau_{<0}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau_{\geq 0}X \arrow[r] & (\tau_{<0}X)[1] \\
\tau_{<0}X \arrow[r] & Y \arrow[r] \arrow[d] & \tau_{\geq 0}Y \arrow[r] & (\tau_{<0}X)[1] \\
 & Z \arrow[d] & & \\
 & X[1] & &
\end{tikzcd}$$

The triangles are distinguished and the square commutes so by applying the octahedron axiome we get a distinguished triangle

$$\tau_{\geq 0}X \longrightarrow \tau_{\geq 0}Y \longrightarrow Z \longrightarrow (\tau_{\geq 0}X)[1]$$

By applying (a) we get that the sequence (2.1) is exact under the assumption $Z \in \mathcal{D}^{\geq 0}$. Dually we get that the sequence (2.2) is exact provided that $X \in \mathcal{D}^{\leq 0}$.

(c) Now we will consider the general case. We can consider the composition $\tau_{\leq 0}X \to X \to Y$ and embed it into a distinguished triangle

$$\tau_{\leq 0}X \longrightarrow Y \longrightarrow W \longrightarrow (\tau_{\leq 0}X)[1] \ .$$

Applying (b) to this triangle we get that

$$H^0X \longrightarrow H^0Y \longrightarrow H^0Z$$

is exact. Now consider the following diagram

$$\begin{tikzcd}
\tau_{\leq 0}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau_{>0}X \arrow[r] & (\tau_{\leq 0}X)[1] \\
\tau_{\leq 0}X \arrow[r] & Y \arrow[r] \arrow[d] & W \arrow[r] & (\tau_{\leq 0})[1] \\
& Z \arrow[d] & & \\
& X[1] & &
\end{tikzcd}$$

The square commutes by definition and the triangles are distinguished so by applying the octahedron axiom we get

$$\begin{tikzcd}
\tau_{\leq 0}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau_{>0}X \arrow[r] \arrow[d] & (\tau_{\leq 0}X)[1] \\
\tau_{\leq 0}X \arrow[r] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\star"] & W \arrow[r] \arrow[d] & (\tau_{\leq 0})[1] \\
& Z \arrow[r, equal] \arrow[d] & Z \arrow[d] & \\
& X[1] \arrow[r] & (\tau_{>0}X)[1] &
\end{tikzcd} .$$

In particular by rotating we get that the triangle

$$W \longrightarrow Z \longrightarrow (\tau_{>0}X)[1] \longrightarrow W[1]$$

is distinguished. We notice that $\tau_{>0}[1] \in \mathcal{D}^{\geq 0}$ so by applying (b) again we get an exact sequence

$$0 \longrightarrow H^0W \longrightarrow H^0Z$$

By gluing the two exact sequences we get an exact sequence

$$H^0X \longrightarrow H^0Y \longrightarrow H^0Z$$

and the commutativity of the marked square in the octahedron diagram implies that the composition $H^0Y \to H^0W \to H^0Z$ is precisely $H^0(Y \to Z)$ and so we are done.

$\square$

If we consider $\mathcal{D} = \mathcal{D}(\mathcal{A})$ with the standard t- structure then a complex $A^\bullet$ is zero if and only if it is quasi-isomorphic to the zero complex and this is equivalent to $H^n A^\bullet = 0$ for all integers. This result does not generalize to arbitrary t- structures but it works if we impose the following condition.

Definition 2.3.1. *Let $\mathcal{D}$ be a triangulated category. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is non-degenerate if*

$$\bigcap_{n\in\mathbb{Z}} \mathcal{D}^{\leq n} = \bigcap_{n\in\mathbb{Z}} \mathcal{D}^{\geq 0} = \{0\}.$$

Proposition 2.3.1. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a non-degenerate t-structure. Then*

(a) A morphism $f : X \to Y$ is an isomorphism in $\mathcal{D}$ if and only if $H^i f$ are isomorphisms in $\mathcal{D}^\heartsuit$ for all $i \in \mathbb{Z}$.

(b) $X \in \mathcal{D}^{\leq n}$ if and only if $H^i X = 0$ for all $i > n$. Dually $X \in \mathcal{D}^{\geq n}$ if and only if $H^i X = 0$ for all $i < n$.

Proof. (a) If f is an isomorphism then clearly $H^i f$ is an isomorphism for all $i \in \mathbb{Z}$. To prove the converse we will first show that $H^i X = 0$ for all i implies $X = 0$. If we add the condition $X \in \mathcal{D}^{\geq 0}$ then $H^0 X = \tau_{\leq 0} X = 0$ so that $X \in \mathcal{D}^{\geq 1}$ and by iterating this reasoning we get $X \in \bigcup_{n\in\mathbb{Z}} \mathcal{D}^{\geq n} = \{0\}$. A similar argument shows that if we add the condition $X \in \mathcal{D}^{\leq 0}$ then $X = 0$. The general case follows from the equalities $H^i \tau_{\leq 0} X = \tau_{\leq 0} H^i X = 0, H^i \tau_{\geq 1} X = \tau_{\geq 1} H^i X = 0$ and the distinguished triangle

$$\tau_{\leq 0} X \longrightarrow X \longrightarrow \tau_{\geq 1} X \longrightarrow (\tau_{\leq 0} X)[1] \ .$$

Now let f be a morphism sucht that $H^i f$ is an isomorphism for all i. We can embed f in a distiguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1] \ .$$

By applyling the cohomological functor H^0 we get the exact sequence

$$\cdots \longrightarrow H^0 X \xrightarrow{H^0 f} H^0 Y \longrightarrow H^0 Z \longrightarrow H^1 X \xrightarrow{H^1 f} \cdots \ .$$

Since the maps $H^i f$ are isomorphism we get that $H^i Z = 0$ for all i. This implies $Z = 0$ but then f is an isomorphism as required.

(b) It is enough to show this for $n = 0$ since the other cases can be obtained by shifting. Suppose $H^i X = 0$ for all $i > 0$. Then the identity $H^i \tau_{\geq 1} X = \tau_{\geq 1} H^i X$ implies $H^i \tau_{\geq 1} = 0$ for all i. To see this simply notice $H^i X$ is zero for $i > n$ and belongs to $\mathcal{D}^{\leq 0}$ for $i \leq 0$. But then applyig (a) we get that $\tau_{\geq 1} X = 0$ which implies $X \in \mathcal{D}^{\leq 0}$. Conversely if $X \in \mathcal{D}^{\leq 0}$ then $\tau_{\geq 1} X = 0$ and so $H^i \tau_{\geq 1} X = 0$ for all i. For $i > 0$ we have $H^i X \in \mathcal{D}^{\geq 1}$ and so

$$0 = H^i \tau_{\geq 1} X = \tau_{\geq 1} H^i X = H^i X$$

The other case follows from the dual argument so we are done.

$\square$

2.4 t-exact functors

A functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between two triangulated categories is said to be exact if it commutes with the shift functors and preserves distinguished triangles. This notion of exactness if fairly weak in the sense that we are only requiring the functor to preserve the triangulated structure. A more refined notion of exactness can be defined if we also have t-structures $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$ and $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$ on $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

Definition 2.4.1. *Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be an exact functor between triagulated categories and $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ t-structures on $\mathcal{D}_i$ for $i = 1, 2$. We say that F is left t-exact if $F(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$. Dually F is t-right exact if $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$ and F is t-exact if it is both left and right t-exact.*

The interest in t-exact functors lies in the fact that they induce exact functors (in the traditional sense) between the hearts.

Definition 2.4.2. *In the same setting as the previous definition we define*

$$^pF = H^0 \circ F \circ \epsilon_1 : \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_2^{\heartsuit}$$

where ϵ_1 is the embedding $\mathcal{D}_1^{\heartsuit} \to \mathcal{D}_1$ and H_0 is the cohomology functor induced by the t-strcture on $\mathcal{D}_2$.

Proposition 2.4.1. *In the same setting as above if we assume F to be left t-exact then*

(a) For any $X \in \mathcal{D}_1$ we have $\tau_{\leq 0} \circ F \circ \tau_{\leq 0} X \cong \tau_{\leq 0} \circ FX$. In particular for $X \in \mathcal{D}_1^{\geq 0}$ there exists an isomorphism $^pF \circ H^0 X \cong H^0 \circ FX$ in $\mathcal{D}^{\heartsuit}$.

(b) $^pF : \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_2^{\heartsuit}$ is a left exact functor between abelian categories.

Proof. (a) There is a canonical morphism $\tau_{\leq 0} \circ F \circ \tau_{\leq 0} X \to \tau_{\leq 0} \circ FX$ obtained by applying the functor $\tau_{\leq 0} \circ F$ to the canonical morphism $\tau_{\leq 0} X \to X$. This morphism induces a canonical morphism

$$\mathrm{Hom}_{\mathcal{D}_2^{\leq 0}}(W, \tau_{\leq 0} \circ F \circ \tau_{\leq 0} X) \to \mathrm{Hom}_{\mathcal{D}_2^{\leq 0}}(W, \tau_{\leq 0} \circ FX)$$

and by the Yoneda lemma if we show that this morphism is an isomorphism for all $W \in \mathcal{D}_2^{\leq 0}$ then we are done. This is equivalent to proving that the canonical morphism

$$\mathrm{Hom}_{\mathcal{D}_2^{\leq 0}}(W, F \circ \tau_{\leq 0} X) \to \mathrm{Hom}_{\mathcal{D}_2^{\leq 0}}(W, FX)$$

is an isomorphism for all $W \in \mathcal{D}_2^{\leq 0}$. Now consider the triangle

$$F \circ \tau_{\leq 0} X \longrightarrow FX \longrightarrow F \circ \tau_{\geq 1} X \longrightarrow (F \circ \tau_{\leq 0} X)[1] \ .$$

This is distinguished because it is obtained by applying the exact functor F to a distinguished triangle. We can apply to it the cohomological functor H^0 and obtain the exact sequence

$$\mathrm{Hom}_{\mathcal{D}_2}(W, (F \circ \tau_{\geq 1} X)[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}_2}(W, F \circ \tau_{\leq 0} X) \longrightarrow \mathrm{Hom}_{\mathcal{D}_2}(W, FX)$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{D}_2}(W, F \circ \tau_{\geq 1} X)$$

Now we notice that the left t-exactness of F implies $(F \circ \tau_{\geq 1} X)[1]$ and $F \circ \tau_{\geq 1} X$ are objects in $\mathcal{D}_2^{\geq 1}$ and so the first and fourth term in the sequence are zero by (TS2). This means that the canonical morphism

$$\mathrm{Hom}_{\mathcal{D}_2^{\leq 0}}(W, F \circ \tau_{\leq 0} X) \to \mathrm{Hom}_{\mathcal{D}_2}(W, FX)$$

is an isomorphism. In particular for $X \in \mathcal{D}_1^{\geq 0}$ we have the chain of isomorphisms

$$\begin{aligned} {}^pF \circ H^0 X &\cong H^0 \circ F \circ H^0 X \\ &\cong H^0 \circ F \circ \tau_{\leq 0} \circ \tau_{\geq 0} X \\ &\cong H^0 \circ F \circ \tau_{\leq 0} X \\ &\cong H^0 \circ FX \end{aligned}$$

(b) For an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{D}_1^{\heartsuit}$ we can always consider a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{D}_1$. By applying the exact functor F we get the distinguished triangle

$$FX \longrightarrow FY \longrightarrow FZ \longrightarrow FX[1]$$

and applying to it the cohomological functor H^0 we get the exact sequence

$$H^{-1} \circ FZ \longrightarrow H^0 \circ FX \longrightarrow H^0 \circ FY \longrightarrow H^0 \circ FZ \ .$$

Since F is left t-exact we have $FZ \in \mathcal{D}_2^{\geq 0}$ and so

$$H^{-1} \circ FZ = (\tau_{\geq -1} \circ \tau_{\leq -1} \circ FZ)[-1] = 0.$$

This means that the exact sequence can be rewritten as

$$0 \longrightarrow {}^pFX \longrightarrow {}^pFY \longrightarrow {}^pFZ$$

which shows that ${}^pF$ is left exact. Dually if F is right t-exact then ${}^pF$ is right exact and if F is t-exact then F is an exact functor.

□

2.5 Constructing t-structures

In this section we will descrive some methods and techniques to construct t-structures. A triangulated category has many different t-structures but it may not be easy to construct them. Furthermore most techniques to construct a t-structure rely on other t-structures so they may not be easy to apply. We will start with some trivial constructions.

Example 2.5.1. *If $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$ then*

(a) $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is a t-sructure on $\mathcal{D}$ for all n.

(b) $((\mathcal{D}^{\geq 0})^{op}, (\mathcal{D}^{\leq 0})^{op})$ is a t-structure on $\mathcal{D}^{op}$.

(c) Let $\mathcal{D}'$ be a triangulated subcategory of $\mathcal{D}$ meaning a strictly full subcategory closed with respect to the shift functor and such that if the first two objects in a distinguished triangle are in $\mathcal{D}'$ then so is the third. Then

$$((\mathcal{D}')^{\leq 0}, (\mathcal{D}')^{\geq 0}) = (\mathcal{D}' \cap \mathcal{D}^{\leq 0}, \mathcal{D}' \cap \mathcal{D}^{\geq 0})$$

is a t-structure on $\mathcal{D}'$ provided that $\mathcal{D}'$ is stable under $\tau_{\leq 0}$. To check the firs axiom:

$$(\mathcal{D}')^{\leq 0} = \mathcal{D}' \cap \mathcal{D}^{\leq 0} \subset \mathcal{D}' \cap \mathcal{D}^{\leq 1} = (\mathcal{D}')[-1] \cap \mathcal{D}^{\leq 1} = (\mathcal{D}' \cap \mathcal{D}^{\leq 0})[-1] = (\mathcal{D}')^{\leq 1}$$

and

$$(\mathcal{D}')^{\geq 1} = (\mathcal{D}' \cap \mathcal{D}^{\geq 0})[-1] = \mathcal{D}'[-1] \cap \mathcal{D}^{\geq 1} = \mathcal{D}' \cap \mathcal{D}^{\geq 1} \subset \mathcal{D}' \cap \mathcal{D}^{\geq 0} = (\mathcal{D}')^{\geq 0}.$$

The second axiom for $((\mathcal{D}')^{\leq 0}, (\mathcal{D}')^{\geq 0})$follows directly from the second axiom applied to $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. To check the third axiom let $X \in \mathcal{D}'$ and consider the distinguished triangle

$$\tau_{\leq 0} X \longrightarrow X \longrightarrow \tau_{\geq 1} \longrightarrow (\tau_{\leq 0} X)[1]$$

in $\mathcal{D}$. Since $X \in \mathcal{D}'$ then $\tau_{\leq 0} X \in \mathcal{D}'$ because of the stability under the truncation functor but then since $\mathcal{D}'$ is a triangulated subcategory we have that $\tau_{\geq 1} X, (\tau_{\leq 0} X)[1] \in \mathcal{D}'$ and the previous triangle is distinguished in $\mathcal{D}'$. To conclude we simply observe that $\tau_{\leq 0} X \in \mathcal{D}' \cap \mathcal{D}^{\leq 0} = (\mathcal{D}')^{\leq 0}$ and $\tau_{\geq 1} X \in \mathcal{D}' \cap \mathcal{D}^{\geq 1} = (\mathcal{D}')^{\geq 1}$. The t-structure $((\mathcal{D}')^{\leq 0}, (\mathcal{D}')^{\geq 0}$ is called the induced t-structure. It is obvious that $(\mathcal{D}')^{\heartsuit} = \mathcal{D}' \cap \mathcal{D}^{\heartsuit}$, furthermore it follows from the proof of the third axiom that the truncation and cohomology functors of the induced t-structure are simply the restriction of the truncation and cohomology functors to $\mathcal{D}'$.

Now we will present three important ways to construct a t-structure.

Gluing

Let $\mathcal{D}_F, \mathcal{D}_U, \mathcal{D}$ be triangulated categories and

$$\begin{aligned} i_* &: \mathcal{D}_F \to \mathcal{D} \\ i^*, i^! &: \mathcal{D} \to \mathcal{D}_F \\ j_!, j_* &: \mathcal{D}_U \to \mathcal{D} \\ j^* &: \mathcal{D} \to \mathcal{D}_U \end{aligned}$$

exact functors satisfying the following properties:

(a) The two sequences $(j_!, j^*, j_*)$ and $(i^*, i_*, i^!)$ are triples of adjoint functors.

(b) The functors $i_*, j_*, j_!$ are fully faithful and they satisfy $j^* i_* = 0$.

(c) For every $X \in \mathcal{D}$ there exist unique maps $i_*i^*X \to j_!j^*X[1]$ and $j_*j^*X \to i_*i^!X[1]$ that complete the unit and counit of the adjunctions descrived in (a) to distinguished triangles

$$j_!j^*X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow j_!j^*X[1]$$

$$i_*i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow i_*i^!X[1]$$

A collection of three triangulated categories and six exact functors satisfying this properties is colled gluing data. Now suppose further that $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ and $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ are t-structures on $\mathcal{D}_F$ and $\mathcal{D}_U$ respectively and we set

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} : \ i^*X \in \mathcal{D}_F^{\leq 0}, \ j^*X \in \mathcal{D}_U^{\leq 0}\}$$
$$\mathcal{D}^{\geq 0} = \{X \in \mathcal{D} : \ i^!X \in \mathcal{D}_F^{\geq 0}, \ j^*X \in \mathcal{D}_U^{\geq 0}\}.$$

Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$.

Proof. We need to check the three axioms of the definition of t-structure. Axiom (TS1) for $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ follows directly from axiom (TS1) applied to $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ and $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$. To check axiom (TS2) let $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$. Applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ to the first triangle in (c) we get an exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(i_*i^*X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(j_!j^*X, Y) \ .$$

Now from the fact that $(i_*, i^!)$ is an adjoint pair and axiom (TS2) applied to $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ we get

$$\mathrm{Hom}_{\mathcal{D}}(i_*i^*X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(i^*X, i^!Y) = 0$$

since $i^*X \in \mathcal{D}_F^{\leq 0}$ and $i^!Y \in \mathcal{D}_F^{\geq 1}$. A similar argument shows

$$\mathrm{Hom}_{\mathcal{D}}(j_!j^*X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(j^*X, j^*Y) = 0.$$

But then by exactness we get $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ as required. Now we need to check axiom (TS3) so let X be an object in $\mathcal{D}$. There is a canonical morphism $X \to j_*\tau_{\geq 1}j^*X$ obtained by composing the unit of the adjunction (j^*, j_*) and j_* applied to the canonical morphism $j^*X \to \tau_{\geq 1}^U j^*X$. We can embed this morphism in a distinguished triangle

$$\Delta_1 = (Y \longrightarrow X \longrightarrow j_*\tau_{\geq 1}^U j^*X \longrightarrow Y[1])$$

Similarly there is a canonical morphism $Y \to i_*\tau_{\geq 1}^F i^*Y$ which can be embedded in a distinguished triangle

$$\Delta_2 = (A \longrightarrow Y \longrightarrow i_*\tau_{\geq 1}^F i^*Y \longrightarrow A[1]) \ .$$

Now we can consider the morphism $A \to X$ obtained simply by composing $A \to Y \to X$ and embed it in a distinguished triangle

$$\Delta_3 = (A \longrightarrow X \longrightarrow B \longrightarrow A[1]) \ .$$

We claim that Δ_3 is the triangle required in axiom (TS3). In order to prove this we have to show that $j^*A \in \mathcal{D}_U^{\leq 0}, i^*A \in \mathcal{D}_F^{\leq 0}, j^*B \in \mathcal{D}_U^{\geq 1}$ and $i^!B \in \mathcal{D}_F^{\geq 1}$. The three triangles we just constructed fit in the following commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & i_*\tau^F_{\geq 1}i^*Y \arrow[r] & A[1] \\
A \arrow[r] & X \arrow[r] \arrow[d] & B \arrow[r] & A[1] \\
& j_*\tau^U_{\geq 1}j^*X \arrow[d] & & \\
& Y[1] & &
\end{tikzcd}$$

so we can use the octahedron axiom and get the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & i_*\tau^F_{\geq 1}i^*Y \arrow[r] \arrow[d] & A[1] \arrow[d, equal] \\
A \arrow[r] & X \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A[1] \arrow[d] \\
& j_*\tau^U_{\geq 1}j^*X \arrow[r, equal] \arrow[d] & j_*\tau^U_{\geq 1}j^*X \arrow[r] \arrow[d] & Y[1] \\
& Y[1] \arrow[r] & i_*\tau^F_{\geq 1}i^*Y[1] &
\end{tikzcd} .$$

In paricular the triangle

$$\Delta_4 = (i_*\tau^F_{\geq 1}i^*Y \longrightarrow B \longrightarrow j_*\tau^U_{\geq 1}j^*X \longrightarrow i_*\tau^F_{\geq 1}i^*Y[1])$$

is distinguished. Now we apply the exact functor j^* to the distinguished triangle Δ_4 and obtain the distinguished triangle

$$j^*\Delta_4 = (j^*i_*\tau^F_{\geq 1}i^*Y \longrightarrow j^*B \longrightarrow j^*j_*\tau^U_{\geq 1}j^*X \longrightarrow j^*i_*\tau^F_{\geq 1}i^*X[1]) \ .$$

Since $j^*i_* = 0$ the first term of the triangle is zero. Furthermore since (j^*, j_*) is an adjoint pair and j_* is fully faithful the counit is an isomorphism which implies $j^*j_* \cong \mathrm{id}$. Putting everything together we proved that $j^*B \cong \tau^U_{\geq 1}j^*X$. In particular $j^*B \in \mathcal{D}_U^{\geq 1}$. Now we apply the exact functor j^* to the distinguished triangle Δ_3 and obtain the distiguished triangle

$$j^*\Delta_3 = (j^*A \longrightarrow j^*X \longrightarrow j^*B \longrightarrow j^*A[1]) \ .$$

We notice the the third object is isomorphic to $\tau^U_{\geq 1}j^*X$ and so by uniqueness of the canonical triangles induced by truncation we have $j^*A \cong \tau^U_{\leq 0}j^*X$ which shows $j^*A \in \mathcal{D}_U^{\leq 0}$.

By applying the exact functor i^* to the distinguished triangle Δ_2 we get the distinguished triangle

$$i^*\Delta_2 = (i^*A \longrightarrow i^*Y \longrightarrow i^*i_*\tau^F_{\geq 1}i^*Y \longrightarrow i^*A[1]) \ .$$

A similar argument as before shows that $i^*i_* \cong \mathrm{id}$ which implies $i^*A \cong \tau^F_{\leq 0}i^*Y$ and so $i^*A \in \mathcal{D}^{\leq 0}_F$. Finally applying the exact functor $i^!$ to the distinguished triangle Δ_1 we get the distinguished triangle

$$i^!\Delta_1 = (i^!i_*\tau^F_{\geq 1}i^*Y \longrightarrow i^!B \longrightarrow i^!j_*\tau^U_{\geq 0}j^*X \longrightarrow i^!i_*\tau^F_{\geq 1}i^*Y[1]) \ .$$

As before we have $i^!i_*\tau^F_{\geq 1}Y \cong \tau^F_{\geq 1}Y$. Furthemore for $X \in \mathcal{D}_U$ and $Y \in \mathcal{D}_F$ we have

$$\mathrm{Hom}_{\mathcal{D}_F}(Y, i^!j_*X) \cong \mathrm{Hom}_{\mathcal{D}}(i_*Y, j_*X) \cong \mathrm{Hom}_{\mathcal{D}_U}(j^*i_*Y, X) \cong \mathrm{Hom}_{\mathcal{D}_U}(0, X) = 0.$$

This shows that the third object in the distinguished triangle is zero so the first two must be isomorphic. This implies $i^!B \cong \tau^F_{\geq 1}Y$ and so in particular $i^!B \in \mathcal{D}^{\geq 1}_F$. We proved that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$ which shows that Δ_3 is the required triangle. □

This construction was used by Beilinson Bernstein and Deligne in their paper on perverse sheaves [1]. Their goal was to show that a certain subcategory of the bounded derived category of constructible sheaves over a topological space X was abelian. The word constructible means some kind of compatibility with a stratification of the space X (meaning a filtration satisfying certain conditions). The technique of gluing t-structures allowed them to prove by induction on the number of strata that the category of perverse sheaves was the heart of the so called "perverse t-structure" and so it was an abelian category.

t-structures induced by torsion theories

Let $\mathcal{A}$ be an abelian category. A torsion theory in $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ thath satisfy the following properties:

(a) $\mathcal{T} = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(X, Y) = 0 \ \forall \ Y \in \mathcal{F}\}$

(b) $\mathcal{F} = \{Y \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(X, Y) = 0 \ \forall \ X \in \mathcal{T}\}$

(c) For all $X \in \mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory then $\mathcal{T}$ is called the torsion class and $\mathcal{F}$ is calledd the torsion free class. An object $T \in \mathcal{T}$ is called a torsion object while $F \in \mathcal{F}$ is called torsion free object. Finally if

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

is a short exact sequence with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ then T is called the torsion part of X and F is the torsion free part of X. It can be shown that the torsion and torsion free parts of an object X are unique up to isomorphism by showing that they correspond to the adjoint functors to the embeddings $\mathcal{T} \to \mathcal{D}$ and $\mathcal{F} \to \mathcal{D}$. Now we will show that a torsion theory on $\mathcal{A}$ induces a t-structure on $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$. Indeed if we set

$$\begin{aligned}\mathcal{D}^{\leq 0} &= \{A^\bullet \in \mathcal{D} \mid H^i A^\bullet = 0 \text{ for } i > 0, H^0 A^\bullet \in \mathcal{T}\}\\ \mathcal{D}^{\geq 0} &= \{A^\bullet \in \mathcal{D} \mid H^i A^\bullet = 0 \text{ for } i < -1, H^{-1} A^\bullet \in \mathcal{F}\}\end{aligned}$$

then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$.

Proof. The axiom (TS1) is clear. To check (TS2) let $A^\bullet \in \mathcal{D}^{\leq 0}$, $B^\bullet \in \mathcal{D}^{\geq 1}$ and $g \in \mathrm{Hom}_{\mathcal{D}}(A^\bullet, B^\bullet)$ represented by the roof

$$\begin{tikzcd} & C^\bullet & \\ A^\bullet \arrow[ur, "f"] & & B^\bullet \arrow[ul, "s"'] \end{tikzcd} .$$

In particualr $f \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, C^\bullet)$ and $C^\bullet \cong A^\bullet$ in $\mathcal{D}$ so that $C^\bullet \in \mathcal{D}^{\geq 1}$. Now we consider the standard t-structure on $\mathcal{D}^b(\mathcal{A})$ and we will denote it by $(\mathcal{D}_s^{\leq 0}, \mathcal{D}_s^{\geq 0})$. We notice that $\mathcal{D}^{\leq 0} \subset \mathcal{D}_s^{\leq 0}$ so that $\tau_{\leq 0}^s A^\bullet \cong A^\bullet$. Furthermore $\mathcal{D}^{\geq 1} \subset \mathcal{D}_s^{\geq 0}$ and so $\tau_{\leq 0} C^\bullet \cong H^0 C^\bullet[0] \cong F[0]$ where $F[0]$ denotes the complex with F in degree zero and 0 everywhere else and $F \in \mathcal{F}$. Now consider the commutative diagram

$$\begin{tikzcd} \tau_{\leq 0}^s A^\bullet \arrow[r, "\sim"] \arrow[d, "\tau_{\leq 0}^s f"'] & A^\bullet \arrow[d, "f"] \\ \tau_{\leq 0}^s C^\bullet \arrow[r] & C^\bullet \end{tikzcd} .$$

If we show that $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\tau_{\leq 0}^s A^\bullet, F[0]) \cong 0$ we are done because this would imply $\tau_{\leq 0}^s f = 0 \implies f = 0 \implies g = 0$. Let $h \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\tau_{\leq 0} A^\bullet, F[0])$. We notice that the maps in degree different than zero are necessarily zero. Now consider the following diagram

$$\begin{tikzcd} A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d] & \ker d_A^0 \arrow[r] \arrow[d, "h^0"'] & \operatorname{coker} d_A^{-1} = H^0 A^\bullet = T \arrow[dl, dotted] \\ 0 \arrow[r] & F & \end{tikzcd} .$$

The commutativity of the square shows that $h^0 \circ d_A^{-1} = 0$ and so by the universal property of the cokernel h^0 factors uniquely through T. But any map $T \to F$ is necessarily zero and so $h^0 = 0$. To prove (TS3) let $A^\bullet \in \mathcal{D}$ and consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Im} d^{-1} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \mathrm{Im} d^{-1} \arrow[r] & \ker d^0 \arrow[r] \arrow[d] & H^0 A^\bullet \arrow[r] \arrow[d] & 0 \\
 & & F \arrow[r, equal] \arrow[d] & F \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}.$$

To construct it we start with the middle horizontal sequence and the right vertical sequence which is exact by definition of torsion theory. Then we construct the horizontal sequence on the top by pullback along μ and we define the map $\ker d^0 \to F$ as the only map that makes the square commute. A simple diagram chasing shows that the middle vertical sequence is also exact. Now we define a complex $A'^\bullet$ as follows: $A'^i = A^i$ for $i \leq -1, A'^0 = E$ and $A'^i = 0$ for $i > 0$. The only non obvious differential $d_{A'}^{-1}$ is given by the compositon $A^{-1} \to \mathrm{Im} d^{-1} \to E$. Then we have $H^0 A'^\bullet = T$ so that $A'^\bullet \in \mathcal{D}^{\leq 0}$. Finally if we set $A''^\bullet = \dfrac{A^\bullet}{A'^\bullet}$ then we have $H^0 A''^\bullet = F$ so that $A''^\bullet \in \mathcal{D}^{\geq 1}$. The exact sequence

$$0 \longrightarrow A'^\bullet \longrightarrow A^\bullet \longrightarrow A''^\bullet \longrightarrow 0$$

in $C(\mathcal{A})$ induces a distinguished triangle

$$A'^\bullet \longrightarrow A^\bullet \longrightarrow A''^\bullet \longrightarrow A'^\bullet[1]$$

which is the required triangle. □

An easy example of this construction if the following: consider $\mathcal{A} = \mathrm{Ab}$ the category of abelian groups. Then there is a torsion theory in $\mathcal{A}$ given by torsion and torsion free groups. In particular this shows that the category

$$\mathcal{D}^\heartsuit = \{A^\bullet \in \mathcal{D}^b(\mathrm{Ab}) \mid H^0 A^\bullet \text{ is a torsion group, } H^{-1} A^\bullet \text{ is torsion free, } H^i A^\bullet = 0 \text{ for } i \neq 0, -1\}$$

is a full abelian subcategory of $\mathcal{D}^b(\mathrm{Ab})$.

Derived equivalences

Whenever mathematicians study a certain class of objects they do so up to a certain equivalence relation. If this equivalence is very strict then they are studying this objects in detail but it will be harder to prove general results. Conversely if the equivalence is very weak then it is generally easier to study but a lot of information is lost. It is always an important challenge in mathematics to find an equivalence relation between objects that is strict enough to capture relevant information but also weak enough that it makes the study manageable. One of the reasons why category theory is so prevalent in modern mathematics is that it allows the study of categories not up to isomorphism but up to equivalence of categories. In the case of abelain categories there is a weaker notion of equivalence between them that is still interesting.

Definition 2.5.1. *Two abelian categories $\mathcal{A}, \mathcal{B}$ are said to be derived equivalent if their bounded derived categories are equivalent as triangulated categories, meaning there is an equivalence of categories that is also an exact functor.*

There many examples in mathematics were two interesting abelian categories are not equivalent but are derived equivalent. One way of thinking about this is the following: an equivalence between abelian categories must be exact. Passing to the bounded derived categories fixes the lack of exactness so that if this was the only obstruction to constructing an equivalence we obtain a derived equivalence. We will now show that the notion of derived equivalence is strictly related to the concept of t-structures. Indeed suppose $\mathcal{A}, \mathcal{B}$ are derived equivalent but not equivalent and let

$$F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$$

be an exact equivalence. We know that the heart of the standard t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{A}}, \mathcal{D}^{\geq 0}_{\mathcal{A}})$ is $\mathcal{A}$ and similarly the heart of the standard t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{B}}, \mathcal{D}^{\geq 0}_{\mathcal{B}})$ is $\mathcal{B}$. It is immediate to check that an exact equivalence maps t-structures to t-structures and so we get a t-structure $(F\mathcal{D}^{\leq 0}_{\mathcal{A}}, F\mathcal{D}^{\geq 0}_{\mathcal{A}})$ in $\mathcal{B}$ with heart $\mathcal{A}$ and a t-structure $(F^{-1}\mathcal{D}^{\leq 0}_{\mathcal{B}}, F^{-1}\mathcal{D}^{\geq 0}_{\mathcal{B}})$ in $\mathcal{A}$ with heart $\mathcal{B}$. This shows that if $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent then $\mathcal{A}$ is a full subcategory of $\mathcal{D}^b(\mathcal{B})$ and $\mathcal{B}$ is a full subcategory of $\mathcal{D}^b(\mathcal{A})$. Another way of saying this is that if $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent then there is an abstract triangulated category $\mathcal{T}$ that is equivalent to the bounded derived categories of $\mathcal{A}$ and $\mathcal{B}$ and interepreting $\mathcal{T}$ as the derived category of $\mathcal{A}$ is equivalent to declaring that the t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{A}}, \mathcal{D}^{\geq 0}_{\mathcal{A}})$ is the standard t-structure and the same holds for $\mathcal{B}$. However it should be mentioned that not all t-structures in $\mathcal{D}^b(\mathcal{A})$ induce derived equivalences and in fact if $\mathcal{B}$ is the heart of a t-structure in $\mathcal{D}^b(\mathcal{A})$ there is not even a canonical functor from $\mathcal{D}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{A})$.

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