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STOCHASTIC FRONTIER ANALYSIS: A REVIEW OF ALTERNATIVE METHODS

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Ringrazio tutti quelli che mi hanno aiutato,

supportato, ma sopratutto sopportato...

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Chapter 1

Introduction

Abstract

This chapter will discuss, in general terms, *Stochastic Frontier Analysis*, but, above all, will introduce *Stochastic Frontier* modelling in an econometric contex. Moreover, it will contain one example of an estimation of the parameters that make it up, taken from a *William Green*'s paper. These results will then be compared with a new approch, which I propose, called *Fixed Effects Approch to the Estimation of a Stochastic Production Frontier Function with Battese-Coelli Time-Varying Inefficiency*, which uses the econometric theory of the *First Difference transformation of the data* to eliminate, unobserved heterogeneity. Furthermore, it will describe in detail the data which will be used later for the Estimation of a Stochastic Production Frontier Function in the various approaches.

1.1 Production Frontier Function

According to the concept of economic efficiency, a producer should maximize, in his working enviroment, the production function with the final goal of cost min-

imization, with an efficient allocation of the resources available to get maximum profit.
This maximization can be seen as an attainment of a *Production Frontier*, eventhough very frequently it is not reached, because of very many producer's inefficiency to manage the resources.
A *Production Frontier* can be considered either as the minimum input possible to produce many kinds of output, or the maximum output obtainable from various kinds of input, given an initial technology.
Hence, a good producer works on the basis of Production Frontiers. There are three types of production frontiers:

1) *Cost Frontier* : describes the minimum quantity of expenditure necessary to produce a given quantity of output.

2) *Revenue Frontier* : describes the maximum revenue possible from a given quantity of input.

3) *Profit Frontier* : describes the maximum profit attainable from a production, given the input costs, the output costs, with the available technology.

In the literature, there are very many definitions of production frontiers, some of them are:

- □ Production frontier is a model in which the estimation of the parameters is done by tying the non-positivity of the residuals, in order to obtain a consistent model with the economic theory of production function.

- □ Prodution function is a frontier which represents the maximum output obtainable from a given input factor, used as efficiently as possible, with a given technology available (*Richmond, 1976*).

- It is possible to observe points below the frontier, but not points above this. This position can reveal where the firm is placed below the production frontier, and this distance from the frontier can be considered a measure of its inefficiency (*Forsund, Lowell and Schmidt, 1980*).

According to the last definition, deviations (or errors, from here onwards) cannot have a symmetric distribution with mean zero when analysing a producer's behaviour.

In fact the errors, in the context of frontiers, can be considered *Composed Errors* formed by a symmetric component, called *random noise,* and a new one-side component that catches the producer's inefficiency. Therefore, their distribution must be skewed towards positive or negative values: it is negative if it is a profit or revenue frontier, positive for the costs frontier. Then, under this reformulation, the errors assume a stochastic form, through the random changes of operative enviroments and the deviations caused by very many kinds of inefficiency.

1.2 The Data: The World Health Organization (WHO) Data Set

The data set used in thesis was already used in Evans et al. (2000a,b). The entire dataset[1] is a panel of data observed for 191 member countries of WHO. The panel data was observed for 5 years, from 1993 to 1997. In this data a set of variables was observed which measure the level of public health cure in the countries presented and another set of country variables, which show exactly in which countries the public health cure needs to be improved.

The following outcome variables are observed:

[1] url: http://pages.stern.nyu.edu/∼wgreene/Econometrics/PanelDataSets.htm

- *DALE* : Disability Adjusted Life Expectancy
- *COMP* : Composite measure of success in 5 health goals, by year health, health distribution, responsiveness, responsiveness in distribution, fairness in financing. The components of this variable were constructed from survey data gathered by WHO for each country

The following country-specific information is collected:

- *HEXP* : Per procapita health expediture
- *HC3* : Educational attaintment
- *SMALL* : indicator for states, provinces, etc...

$$\begin{cases} SMALL > 0 & implies\ internal\ political\ unit \\ SMALL = 0 & implies\ country\ observation \end{cases}$$

- *GROUPTI* : Number of observation when $SMALL = 0$. Usually 5, some 1, one country 4

The variables that are indicators of cross country and timewise heterogenity are as follows:

- *GINI* : Gini coefficient, income inequality
- *VOICE* : World Bank measure of democratization and freedom of political unit
- *GEFF* : Measure of government effectiveness, World Bank measure
- *TROPICS* : Dummy variable for tropical location
- *POPDEN* : Population density, people per Km^2
- *PUBFIN* : Percentage of health care paid by government
- *GDPC* : Normalized per capita GDP

Finally, the time indicator is:

- *YEAR* : 1993, . . . ,1997

In what follows only, a subset of data will be used to obtain a dataset of balanced panel data with some of the country variables listed before. The final dataset consists of 700 observations obtained from 140 states, each followed for 5 years (140 x 5 = 700). Finally, all the variables chosen are expressed in the natural logarithmic form, in accordance with the Cobb-Douglass definition of production function.

Therefore, the actual variables chosen for the analysis are:

- *LDALE* $\}$ $outcome\ variable$
- *LHEXP*
- *LHC*
- *LHC2*

$\left.\begin{array}{l} LHEXP \\ LHC \\ LHC2 \end{array}\right\} regressor\ variables$

1.3 Article

Distinguishing between heterogeneity and inefficiency: Stochastic Frontier Analysis of the World Health Organization's panel data on national health care systems (William Green)

1.3.1 First model: Stochastic frontier with time invariant Inefficiency

The estimated frontier is:

$$\begin{aligned} LDALE &= \beta_0 + \beta_1 \times LHEXP + \beta_2 \times LHC + \beta_3 \times LHC2 + v_{it} - u_{it} \\ &= \beta_0 + \beta_1 \times LHEXP + \beta_2 \times LHC + \beta_3 \times LHC2 + \epsilon_{it} \end{aligned}$$

The parameters $\lambda, \sigma, \sigma_v$ and σ_u of a stochastic frontier were estimated with LIMDEP 9.0 Econometric Softwere created by William Green.

The results obtained are:

```
+--------------------------------------------+
| Limited Dependent Variable Model - FRONTIER |
| Dependent variable                LDALE     |
| Log likelihood function        501.4585     |
| Estimation based on N =    700, K =   6     |
| AIC =    -1.4156  Bayes IC =    -1.3766     |
| AICf.s. =    -1.4154  HQIC =    -1.4005     |
| Model estimated: May 08, 2009, 15:04:47     |
| Variances: Sigma-squared(v)=       .00131   |
|            Sigma-squared(u)=       .04305   |
|            Sigma(v)        =       .03623   |
|            Sigma(u)        =       .20749   |
| Sigma = Sqr[(s^2(u)+s^2(v)]=       .21063   |
| Stochastic Production Frontier, e=v-u.      |
```

```
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
+--------+Primary Index Equation for Model                            |
|Constant|    3.50709***       .03546       98.910   .0000            |
|LHEXP   |     .06636***       .00405       16.386   .0000     5.36533|
|LEDUC   |     .28811***       .03729        7.726   .0000     1.67163|
|LEDUC2  |    -.11018***       .02405       -4.582   .0000     1.55681|
+--------+Variance parameters for compound error                      |
|Lambda  |    5.72629***       .72268        7.924   .0000            |
|Sigma   |     .21063***       .00027      777.022   .0000            |
+--------+------------------------------------------------------------+
| Note: nnnnn.D-xx or D+xx => multiply by 10 to -xx or +xx.           |
| Note: ***, **, * = Significance at 1%, 5%, 10% level.               |
+---------------------------------------------------------------------+
```

The relative estimated efficiency per state is shown in table A.1 in the appendix.

1.3.2 Second model: Stochastic frontier with time-varying Inefficiency

The estimated frontier is:

$$\begin{aligned} LDALE &= \beta_0 + \beta_1 \times LHEXP + \beta_2 \times LHC + \beta_3 \times LHC2 + v_{it} - u_{it} \\ &= \beta_0 + \beta_1 \times LHEXP + \beta_2 \times LHC + \beta_3 \times LHC2 + v_{it} - \beta_t \times u_i \\ &= \beta_0 + \beta_1 \times LHEXP + \beta_2 \times LHC + \beta_3 \times LHC2 + \epsilon_{it} \end{aligned}$$

Where u_{it} varies over time according to the following analytical function proposed by *Battese-Coelli* (1992):

$$u_{it} = \underbrace{\exp\left\{[-\eta\,(t-T)]\right\}}_{\beta_t} \times u_i$$

The estimes of parameters λ, σ, σ_v and σ_u of a stochastic frontier, and η for β_t function obtained in this case (always with software LIMDEP 9.0) are:

```
+--------------------------------------------+
| Limited Dependent Variable Model - FRONTIER |
| Dependent variable               LDALE     |
| Log likelihood function        1718.675     |
| Estimation based on N =    700, K =   7     |
| AIC =    -4.8905  Bayes IC =    -4.8450     |
| AICf.s. =    -4.8903  HQIC =    -4.8729     |
| Model estimated: May 08, 2009, 16:03:13     |
+--------------------------------------------+
| Frontier model estimated with PANEL data.   |
| Estimation based on   140 individuals.      |
| Variances: Sigma-squared(v)=       .00011   |
|            Sigma-squared(u)=       .07338   |
|            Sigma(v)        =       .01063   |
|            Sigma(u)        =       .27089   |
| Sigma = Sqr[(s^2(u)+s^2(v)]=       .27110   |
| Stochastic Production Frontier, e=v-u.      |
| Time varying u(i,t)=exp[-eta(t-T)]*|U(i)|   |
+--------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
+--------+Primary Index Equation for Model                             |
|Constant|    3.87299***       .01383      280.004   .0000            |
|LHEXP   |     .01928***       .00195        9.887   .0000     5.36533|
|LEDUC   |     .19221***       .01655       11.612   .0000     1.67163|
|LEDUC2  |    -.05010***       .01424       -3.519   .0004     1.55681|
+--------+Variance parameters for compound error                       |
|Lambda  |    25.4801***       .00282     9026.173   .0000            |
|Sigma(u)|     .27089***       .00128      211.100   .0000            |
+--------+Eta parameter for time varying inefficiency                  |
|Eta     |    -.00772***       .00052      -14.973   .0000            |
+--------+-------------------------------------------------------------+
| Note: nnnnn.D-xx or D+xx => multiply by 10 to -xx or +xx.           |
| Note: ***, **, * = Significance at 1%, 5%, 10% level.               |
+---------------------------------------------------------------------+
```

The relative efficiency per state is shown in table A.2 in the Appendix.

Chapter 2

Inefficiency Estimation by means of a Stochastic Production Frontier Function

Abstract

As a first step, this chapter will give a definition of *Cobb-Douglass Production Function* and its relationship with a *Stochastic Production Frontier Function* (SFF, from here onwards). The second step is to provide a review of *SSF* estimation methods, already known in the literature, and in the thrid step the *inefficiency estimation* will be shown starting from a *SFF*. From here onwards, in terms of firms, the argument will be set (which it can be generalized to more cases, for example focused on states and so on). It will be focus throughout on the case of two input variables, through the approach described in what follows can be easily extended over the case of n input variables.

2.1 Production Frontier

From here onwards, uppercase and lowercase, respectively, to denote the levels and the logarithmic form of the variables, are used.

Given the input q_1 and q_2, output y and a given initial technology:

$$\begin{cases} q &= [q_1, q_2] \in \Re_{(+)} \times \Re_{(+)} = \Re^2_{(+)} \\ y &\in \Re_{(+)} \end{cases}$$

the goal unit becomes:

$$(q_1, q_2, y) = (q, y) \tag{2.1}$$

and the *production-set*, Ψ, is defined as:

$$\Psi \equiv \left\{(q, y) \in \Re^3_{(+)} \mid q \; can \; produce \; y\right\} \tag{2.2}$$

Therefore, for each $y \in \Psi$ there exist some *input-sets*, $C(y)$:

$$C(y) \equiv \left\{(q_1, q_2) \in \Re^2_{(+)} \mid \; (q_1, q_2, y) \in \Psi\right\} \tag{2.3}$$

$\Psi_m(q_0)$ is *the maximum expected production from* m *potential firms which use as maximum* $q_0 = (q_{01}, q_{02})$ *quantity as input*:

$$\Psi_m(q_0) \equiv E\left[\max_{1 \leq Y \leq m} (Y_i) \mid Q \leq q_0\right] \tag{2.4}$$

which represents the definition of a *production frontier*.
A graphical interpretation about (2.4) is shown in the Figure (2.1):

where $\Psi_m(q_0)$ represents the expected maximum level of output produced with

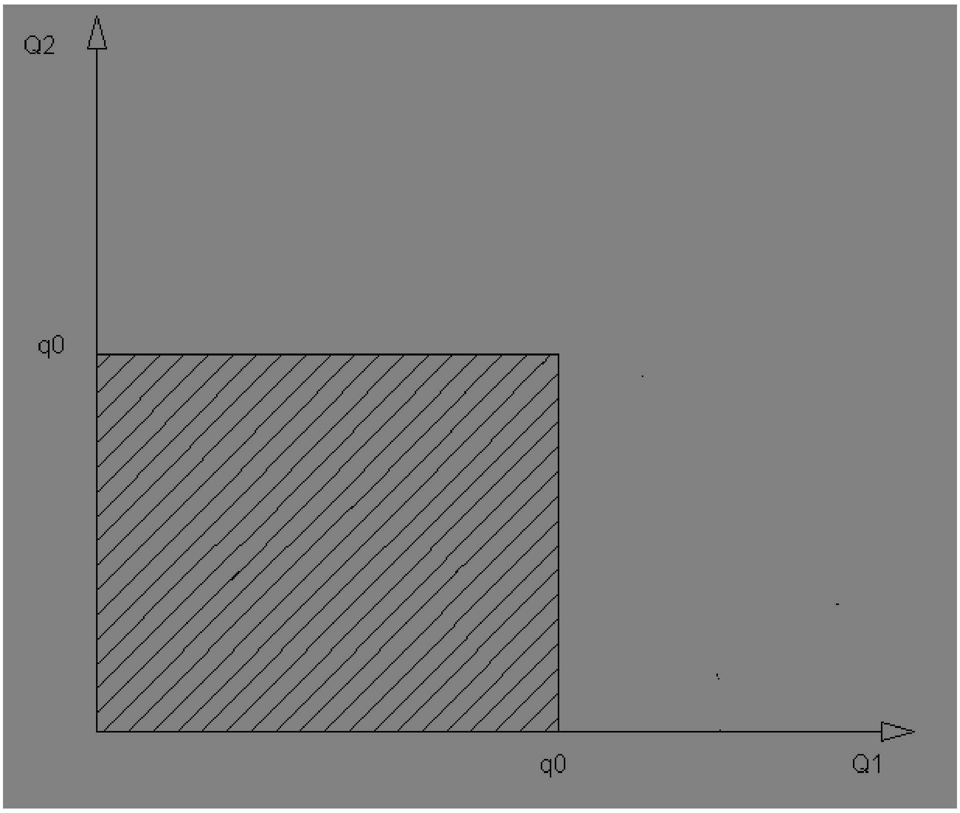

Figure 2.1: Production Function $\Psi_m(q_0)$

a combination of inputs (q_1, q_2), which lies inside the square, with a Cobb-Douglass specification to defines the technology adopted by the firm to produce its output. Thus, if with the same quantity q_0 in input, a firm obtains a lower output compared to the m firms represented by $\Psi_m(q_0)$, it means that it adds an inefficiency component in its production-cycle.

2.2 Cobb-Douglass Production Function

In the world of economy, a *Cobb-Douglass Production Function* (CDPF, from here onwards), is widely used to represent the *output-input* relationship.

The form of a *CDPF* is the following:

$$Y = A \times Q_1^{\beta_1} \times Q_2^{\beta_2} \tag{2.5}$$

and it is also called *log-linear* functions, because they become linear in the logarithmic form. If the logarithmic trasformation is applied the result is

$$y = \beta_0 + \beta_1 q_1 + \beta_2 q_2 \tag{2.6}$$

where:

- *y*: Total production
- *A*: Total Factor Productivity (TFP): a ratio between an output index and an input index, weighted average of the two input considered.
- β_0 : $\log(A)$
- β_1 and β_2 are two coefficients to estimate.

2.3 Stochastic Production Frontier Function

The first who introduced simultaneously *Stochastic Production Frontier* models were Aigner, Lovell, Schmidt (ALS) (1977) and Meeusen and van den Broeck (MB) (1977). These models allow for technical inefficiency, but they also assume that the random shocks outside the control of producers can effect output. The adjective *stochastic* is used precisely because these models presuppose distributional assumptions on their error component, consisting of the sum of two parts: a two-side symmetric noise component and a one-side skew nonnegative technical inefficiency component, usually assumed indipendent of each other and distributed indipendently of model regressors. This kind of error generally is called *composed error*.

The model can be described as follows.
Given Q_1 and Q_2 as input, the Production Frontier Function is defined as:

$$Y = \Phi\left(Q_1, Q_2\right) \times V \tag{2.7}$$

where:

- $\Phi\left(Q_1, Q_2\right)$ is a function that depends on the inputs levels
- V is a stochastic shock (random noise)

Defining V as:

$$V = e^{-v} \tag{2.8}$$

and assuming a Cobb-Douglass relationship, $\Phi\left(Q_1, Q_2\right)$ can be given this form:

$$\Phi\left(Q_1, Q_2\right) : A \times Q_1^{\beta_1} \times Q_2^{\beta_2} \tag{2.9}$$

hence:

$$y = \beta_0 + \beta_1 q_1 + \beta_2 q_2 + v \tag{2.10}$$

which represents the ***Cobb-Douglass production frontier function***.
In this case $\Psi_m\left(q_0\right)$ can be modeling by $\beta_0 + \beta_1 q_1 + \beta_2 q_2$ and so

$$y = E[y|q_1, q_2] + v \tag{2.11}$$

where as before, $E[y|q_1, q_2]$ represent the expected maximum level of output given q_1 and q_2 as inputs. Thus, if $E[y|q_1, q_2]$ is the expected maximum, it is possible to observe only points below the function, but not points above this.

2.3.1 Inefficiency Component

Defining $U = e^{-u}$ as the inefficiency component which describes the deviations from the frontier (the points below this), using (2.10):

$$u \equiv y_i - \underbrace{\{\beta_0 + \beta_1 q_1 + \beta_2 q_2 + v\}}_{SF} \tag{2.12}$$

thus SF represent the frontier, and u can be only negative by definition ($u \leq 0$). Now, adding the component u to the equation (2.10):

$$\begin{aligned} y &= \beta_0 + \beta_1 q_1 + \beta_2 q_2 + v - u \\ &= \beta_0 + \beta_1 q_1 + \beta_2 q_2 + \epsilon \end{aligned} \tag{2.13}$$

the error component ϵ is formed by:

$$\epsilon = v - u \tag{2.14}$$

which gives the frontier a stochastic form. Hence, the equation (2.13) represent the ***Cobb-Douglas Stochastic Production Frontier function***.

2.3.2 Inefficiency Informers

In the literature, using a stochastic frontier, there are two important good informer about the firm inefficiency. They have the same interpretation, but a different formulation, though being made up on the same quantity.

Defining, in general terms, without assuming any distributional assumption on v and u error components, σ_v^2 and σ_u^2 their respective varinces

$$Var\left(\epsilon\right) = \sigma_v^2 + \sigma_u^2 = \sigma_\epsilon^2 \tag{2.15}$$

The most used in the literature inefficiency informer, is λ which it has the following definiton:

$$\lambda = \frac{\sigma_u}{\sigma_v} \tag{2.16}$$

where λ becomes the *goal parameter* which describes the inefficiency percentage that distinguishes one firm from the others. Its interpretation is outright: the more λ increases the more the inefficiency grows. The range of λ is : $\lambda \in (0, \infty)$.
Another inefficiency informer proposed by *Battese-Coelli*[1], is γ, which is defined as is follows:

$$\gamma = \frac{\sigma_u^2}{\sigma_\epsilon^2} \tag{2.17}$$

ie the proportion of variability due to u. It has the same interpretation of λ but its range is $\gamma \in [0, 1]$.

2.3.3 Composite Error Distribution

In the literature there are a lot of models with various kinds of distributional assumptions on v and u error components. In this thesis, a ***Normal-Half Normal model*** will be chosen as distributional error assumptions. Other types of distributional error assumptions are: *Normal-Truncated Normal model*, introduced by Stevenson (1980), a generalization of the model discussed here, and *Normal-Gamma* formulation introduced by Green (1980a,b) and Stevenson (1980), and extended by Green (1990). These distributional assumptions are used first for the maximum likelihood estimation method for the SF, and then to estimate the technical efficiency of each producer, the general final goal.

[1] see Battese-Coelli definition

Normal-Half Normal Model

The distributional assumptions of v and u are:

$$\begin{cases} (i) \quad v \quad \sim \quad N\left(0,\sigma_v^2\right) \equiv \dfrac{1}{\sigma_v}\phi\left(\dfrac{v}{\sigma_v}\right), \;\; v_i \; i.i.d. \\ \\ (ii) \;\; U \quad \sim \quad N\left(0,\sigma_u^2\right) \; so \; u \sim |U| = \dfrac{2}{\sigma_u}\phi\left(\dfrac{u}{\sigma_u}\right), with \; u \geq 0, \;\; u_i \; i.i.d. \\ \\ (iii) \;\; v_i \quad \perp \quad u_i \end{cases}$$

Where ϕ is a Normal distribution density function and $i = 1, \ldots, m$ (firms).

Under these distributional assumptions, the composed-error defined in (2.14) is distributed as follows:

$$f_\epsilon\left(\epsilon\right) = \frac{2}{\sigma}\phi\left(\frac{\epsilon}{\sigma}\right)\Phi\left(-\lambda\frac{\epsilon}{\sigma}\right) \tag{2.18}$$

which is a reparameterization of a *Skew-Normal*[2] distribution, where $\epsilon = v - u$, $\sigma = \sqrt{\sigma_v^2 + \sigma_u^2}$, ϕ is the Normal distribution density function and Φ is the cumulative Normal distribution density.
The first three moments of the distribution in (2.18) are[3]:

$$\begin{aligned} E\left[\epsilon\right] &= -\sqrt{\frac{2}{\pi}}\sigma_u \\ E\left[\epsilon^2\right] &= \sigma_v^2 + \sigma_u^2 \\ Var\left[\epsilon\right] &= \sigma_v^2 + \sigma_u^2\left[\frac{\pi-2}{\pi}\right] \\ E\left[\epsilon^3\right] &= \sqrt{\frac{2}{\pi}}\left[1-\frac{4}{\pi}\right]\sigma_u^3 \end{aligned}$$

[2]for the entire proof see the appendix
[3]see A. Azzalini (1985)

and the ratio of the variances of u and ϵ (a reparameterization of γ) is:

$$\frac{Var\left[u\right]}{Var\left[\epsilon\right]} = \frac{\sigma_u^2\left[\dfrac{\pi-2}{\pi}\right]}{\sigma_v^2+\sigma_u^2\left[\dfrac{\pi-2}{\pi}\right]} \tag{2.19}$$

which is a good informer about the inefficiency percentage: the closer the ratio to one the higher is inefficiency.

2.4 Cross Section data VS Panel Data: comparisons between the two types of data

Using *cross-sectional* data means to have available for each firm only one obsevation. Whilst, using *panel data*, where a firm is followed over time, there are more than one observation for each firm. Therefore using panel data means to have more informations about each firm, hence more reliable estimates and to have the way to consider an unobserved heterogeneity across the firms (see Chapter 3). The Panel can be:

- Balanced : each firm have the same number of observations ($T_i = T$, $i = 1, \dots m$)
- Unbalanced : each firm have a different number of observations ($T_i = T_i$)

However using a panel dataset, it will be introduced a correlation between the observations of the firm, but an indipendence across the firms.
In this thesis a balaced panel was chosen. Thus, the dataset is formed by $m \times T$ obeservations, where m is the number of firms, and T the number of period, or rather the number of observation for each firm.

2.5 Estimation by MLE of a Stochastic Production Frontier Function with Time-Invariant Technical Efficiency

Assuming a *Cobb Douglass* Stochastic Production Frontier function, the panel model of a SF that is going to be presented takes this form:

$$\begin{aligned} y_{it} &= \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + v_{it} - u_{it} \\ &= \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + \epsilon_{it} \end{aligned} \tag{2.20}$$

Where $i = 1, \ldots, m$ number of firms and $t = 1, \ldots, T$ periods number of panel data.

However, the use in this first panel model of the assumption on the composed error given in (§2.3.3), means a single record is treated as "new firm", i.e. the panel data set is used as a cross-sectional data set.
Making these hards assumptions implies to upset the structure of panel data, or rather to lose the dependence between the observations of a firm. Hence, a good question can be if these assumptions are reasonable. A possible answer is that: using this model can be allow to show how much a panel dataset brings more than a cross-sectional dataset to estimate the inefficiency, or rather if the estimates widely change using a model like this or a model which assumes a correlation between the inefficiency (see (§2.6)).
An other way can be to want to estimate a different inefficiency per time, or reather like starting every year from the *base-line* of *null-inefficiency*, assuming that the inefficiency at time $t-1$ does not affects the inefficiency at time t.
Therefore since the records are assumed mutually indipendent so are the residuals of the model after its estimation (consequently the inefficiency).

Thus, the estimes of parameters λ, σ, σ_v and σ_u of a SF are obtained maximizing the log-likelihood in (2.22) of the density in (2.18) respect σ_v, σ_u and β. The likelihood of $f_\epsilon(\epsilon_{it})$ is the following:

$$L(\epsilon_{it}|\sigma_v, \sigma_u, \beta) = \prod_{i=1}^{m\times T} f_\epsilon(\epsilon_{it}) \propto \sigma^{(mT)} \prod_{i=1}^{m\times T} \Phi\left(-\lambda\frac{\epsilon_{it}}{\sigma}\right) \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{m\times T}\epsilon_{it}\right\} \tag{2.21}$$

Hence the log-likelihood:

$$l(\epsilon_{it}|\sigma_v, \sigma_u, \beta) = \sum_{i=1}^{m\times T} \log\left(L(\epsilon_{it}|\lambda, \sigma)\right) \propto (mT)\log\sigma + \sum_{i=1}^{m\times T}\log\left[\Phi\left(-\lambda\frac{\epsilon_{it}}{\sigma}\right)\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{m\times T}\epsilon_{it} \tag{2.22}$$

where $\epsilon_{it} = y_{it} - \beta' x_{it}$,

The equation (2.22) can be maximized in only one step with a numerical algorithm, like Newton-Raphson.

2.5.1 Estimation of the Inefficiency Component

Now, with the estimates of the parameters σ_v and σ_u, obtained from the maximization of (2.18), it is possible to obtain an estimation of the producers' inefficiency, which is one of the main objectives of fitting the frontier models.

***Jondrow et al.*'s Inefficiency Estimator**

Standard estimator for the inefficiency of the firm, u_{it}, proposed by *Jondrow et al.* (1982), is $E[u|\epsilon]$.

Given the conditional distribution $f(u|\epsilon)$:

$$\begin{aligned} f(u|\epsilon) &= \frac{f(u,\epsilon)}{f(\epsilon)} \\ &= \frac{\dfrac{1}{\sqrt{2\pi}\sigma_*}\exp\left\{-\dfrac{(u-\mu_*)^2}{2\sigma_*^2}\right\}}{\left[1-\Phi\left(-\dfrac{\mu_*}{\sigma_*}\right)\right]} \end{aligned} \tag{2.23}$$

$f(u|\epsilon)$ is distributed as $N^+(\mu_*, \sigma_*^2)$, where $\mu_* = -\frac{\epsilon\,\sigma_u^2}{\sigma^2}$ and $\sigma_*^2 = \frac{\sigma_u^2\sigma_v^2}{\sigma^2}$.[4]

Then, $E[u|\epsilon]$, is defined as follows:

$$E[u_i|\epsilon_i] = \frac{\sigma\lambda}{1+\lambda^2}\left[\frac{\phi\left(\dfrac{\epsilon_i\lambda}{\sigma}\right)}{1-\Phi\left(\dfrac{\epsilon_i\lambda}{\sigma}\right)} - \left(\frac{\epsilon_i\lambda}{\sigma}\right)\right] \tag{2.24}$$

The equation (2.24) was obtained by a reparameterization of $\mu_{\star i}$ and $\sigma_\star$ respect λ and σ.[5]

2.6 Estimation by MLE of a Stochastic Production Frontier Function with Time - Varying Technical Inefficiency

Considering the dataset as a Panel, more information is introduced about each single firm. Therefore, it is reasonable to assume that the inefficiency of the firm changes over time, and, for this, Lee and Schmidt (1993) proposed an alternative

[4]for all calculations see the Appendix
[5]for all calculations see the Appendix

formulation of the model in (2.20), in which u_{it} is specified as:

$$u_{it} = \alpha_t \times u_i \tag{2.25}$$

where $t = 1, \ldots, T$ and $\alpha_1, \ldots, \alpha_t$ can be either specified as a set of time dummy variables same for all firms ($T-1$ additional parameters will be estimated, sosuitable for small Panel) or the values of a parametric function of time, $\alpha(t)$, which varies over time (*Kumbhakar*,1990 or *Battese-Coelli*,1992). This shows that the inefficiency varies over time through the α_t. Hence, in this case, the error assumes the following definition:

$$\epsilon_{it} = v_{it} - (\alpha_t \times u_i) \tag{2.26}$$

where $\epsilon_i = (\epsilon_{i,1}, \ldots, \epsilon_{i,T})'$.

Assuming a CD Stochastic Production Frontier function, the model has the following form:

$$y_{it} = \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + v_{it} - (\alpha_t \times u_i) \tag{2.27}$$

As it can seen the difference between the models (2.20) and (2.27) lies in the number of estimated inefficiencys for each firms. In fact, in the model (2.27) is estimated only one inefficiency per firm, whilst in the model (2.20) T inefficiency per firm are estimated.

Thus, the unique inefficiency estimated for the model (2.27) represent a sort of *sum of* T *effects* given from the inefficiency of each period. Indeed, inefficiency can be considered independent and identically distribuited across the firms but not over time (i.d.).

The distributional assumptions on the error for the model (2.27) are:

i) $v_{it} \sim N(0, \sigma_v^2) \quad with \quad v_i = (v_1, \ldots, v_T)'$

ii) $u_i \sim N^+(0, \sigma_u^2)$ half-normal distribution

Given these distributional assumptions, the density of error becomes:

$$\begin{aligned}
f_\epsilon(\epsilon_i) &= \int_0^\infty \prod_{t=1}^T f(\epsilon_{it} - \alpha_t u_i) f(u_i)\, du_i \\
&= \frac{2}{(2\pi)^{\frac{T+1}{2}} (\sigma_v^2)^T \sigma_u^2} \int_0^\infty \exp\left\{ -\frac{1}{2} \left[\sum_{t=1}^T \frac{(\epsilon_{it} - \alpha_t u_i)^2}{\sigma_v^2} + \frac{u_i^2}{\sigma_u^2} \right] \right\} du_i \\
&= \frac{2\sigma_\star \exp\left\{-\frac{1}{2} a_{\star i}\right\}}{(2\pi)^{\frac{T}{2}} (\sigma_v)^T \sigma_u} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_\star} \exp\left\{ -\frac{1}{2\sigma_\star^2} (u_i - \mu_{\star i})^2 \right\} du_i}_{1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)} \\
&= \frac{2\sigma_\star \exp\left\{-\frac{1}{2} a_{\star i}\right\}}{(2\pi)^{\frac{T}{2}} (\sigma_v)^T \sigma_u} \left[1 - \Phi\left(-\frac{\mu_{\star i}}{\sigma_\star} \right) \right]
\end{aligned} \tag{2.28}$$

where[6]:

$$\begin{cases}
\mu_{\star i} &= \dfrac{\sigma_u^2 \left(\sum_{t=1}^T \alpha_t \epsilon_{it} \right)}{\sigma_u^2 \sum_{t=1}^T \alpha_t^2 + \sigma_v^2} \\
\sigma_\star^2 &= \dfrac{\sigma_v^2 \sigma_u^2}{\sigma_u^2 \sum_{t=1}^T \alpha_t^2 + \sigma_v^2} \\
a_{\star i} &= \dfrac{1}{\sigma_v^2} \left[\sum_{t=1}^T \epsilon_{it}^2 - \dfrac{\sigma_u^2 \left(\sum_{t=1}^T \alpha_t \epsilon_{it} \right)^2}{\left(\sigma_u^2 \sum_{t=1}^T \alpha_t^2 + \sigma_v^2 \right)} \right] \\
\epsilon_{it} &= y_{it} - \beta' x_{it}
\end{cases}$$

[6] see the Appendix for all calculations

The likelihood function of the density given in (2.28) is:

$$L\left(\epsilon_{it}|\sigma_v,\sigma_u,\beta,\beta_t\right)\propto\left(\sigma_\star^2\right)^{\frac{m}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}a_{\star i}\right\}\left(\sigma_v^2\right)^{-\frac{mT}{2}}\left(\sigma_u^2\right)^{-\frac{m}{2}}\prod_{i=1}^{m}\left[1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right] \tag{2.29}$$

The log-likelihood of (2.29) is straight:

$$l\left(\epsilon_{it}|\sigma_v,\sigma_u,\beta,\beta_t\right)\propto\frac{m}{2}\log\sigma_\star^2-\frac{1}{2}\sum_{i=1}^{m}a_{\star i}-\frac{mT}{2}\log\sigma_v^2-\frac{m}{2}\log\sigma_u^2+\sum_{i=1}^{m}\log\left[1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right] \tag{2.30}$$

Where $i=1,\ldots,m$ number of firms, $t=1,\ldots,T$ period number of panel data.

So the estimates of parameters λ, σ, σ_v and σ_u of a SF are obtained maximizing the log-likelihood in (2.30) of the density in (2.28) respect σ_v, σ_u, β and α_t.

2.6.1 Parametric Function of Time $\alpha(t)$

The $\alpha(t)$ is a parametric function of time that allows inefficiency to vary over time. In the literature there are many definitions of $\alpha(t)$; the two most important are those specified by *Kumbhakar* (1990) and *Battese-Coelli* (1992). The difference between the two lies in the different number of estimated parameters (two for *Kumbhakar* and one for *Battese-Coelli*), and in the range of the $\alpha(t)$.

Kumbhakar's $\alpha(t)$ Definition

Kumbhakar (1990) specified for $\alpha(t)$ the following parametric function:

$$\alpha(t)=\left[1+\exp\left\{\gamma t+\delta t^2\right\}\right]^{-1} \tag{2.31}$$

Kumbhakar's model contains two additional parameters to be estimated (in addition to those that need to be estimated in the model (2.27)), γ and δ, and t is a vector whose elements are the numbers from 1 to T number of periods of Panel. The $\alpha(t)$ function satisfies the following properties:

i) $0 \leq \alpha(t) \leq 1$

ii) $\alpha(t)$ can be monotonically increasing or decreasing, and concave or convex, depending on the signs and magnitudes of the two parameters γ and δ

If $\gamma = \delta = 0$, the inefficiency is time-invariant, and in which case $\alpha(t) = 1$. With this formulation, all the distributional assumptions made before on the components of the error v and u remain valid.

Battese-Coelli's $\alpha(t)$ Definition

Battese-Coelli (1992), instead, proposed for the $\alpha(t)$ the following definition:

$$\alpha(t) = \exp\left\{-\eta\left(t - T\right)\right\} \tag{2.32}$$

where t is a vector whose elements are the numbers from 1 to T number of periods of Panel. In this model there is only one additional parameter to estimate, η, and presents itself as a more parsimonious model than *Kumbhakar*'s model. The $\alpha(t)$ function satisfies the following properties:

i) $\alpha(t) \geq 0$

ii) $\alpha(t)$ decreases at an increasing rate if $\eta > 0$; increases at an increasing rate if $\eta < 0$; remains constant if $\eta = 0$.

In this case, the inefficiency is time-invariant if $\eta = 0$, in which case $\alpha(t) = 1$. According to the formulation for $\alpha(t)$ given in (2.32), the distributional assumpions change for the components of the error. In fact, v_{it} remains Normal, while u_i becomes a truncated normal[7].

2.6.2 Estimation of the Inefficiency Component

Before starting with this issue, it must be stated that the choice of the inefficiency estimator is influenced by the function $\alpha(t)$ choice. In this thesis the *Battese-Coelli*'s formulation for $\alpha(t)$ was chosen, however, maintaining a distribution for u_i Half-Normal because this case is contained in the Truncated Normal formulation.

Jondrow et al.'s Inefficiency Estimator

From the derivation of the log-likelihood function given in (2.30) it is easy to show that $u_i|\epsilon_i \sim N^+(\mu_{\star i}, \sigma_\star^2)$. An estimator for inefficiency has been proposed by *Jondrow et al.*, which is based on the mean of $u_i|\epsilon_i$, and it is given by:

$$\begin{aligned} E\left[u_{it}|\epsilon_{i1},\ldots,\epsilon_{iT}\right] &= \alpha(t)E\left[u_i|\epsilon_{i1},\ldots,\epsilon_{iT}\right] \\ &= \alpha(t)\left[\mu_{\star i} + \sigma_\star\left(\frac{\phi\left(\frac{\mu_{\star i}}{\sigma_\star}\right)}{\Phi\left(\frac{\mu_{\star i}}{\sigma_\star}\right)}\right)\right] \end{aligned} \tag{2.33}$$

where in this thesis $\alpha(t)$ function has the *Battese-Coelli*'s formulation.

[7] for all the proof see *Battese-Coelli* (1992) or (1995)

Chapter 3

Distinguishing Between Heterogeneity and Inefficiency: Mixture of Stochastic Frontier and Fixed Effects Analysis

Abstract

In this chapter a definition will be given of a new model for the Stochastic Frontier Analysis which allows to distinguish technical inefficiency from heterogeneity, using the *First Difference* trasformation of the Fixed Effects Analysis, and moreover it will speak about three models raised from three different definition of the inefficiency component, u_{it}.

3.1 Fixed Effects Models

In the literature, there are very many books and papers about Fixed Effects. In fact, they just deal with their utility to eliminate *management ability*, i.e to eliminate the problem of *omitted variables*, or rather characteristics typically not observable to the analyst but known to the firm. Usually, these characteristics are costant in time. In addiction there are very many methods which allow to eliminate this problem, some of them are:

- Estimating $m-1$ additional incidental parameters of firm (m if the intercept is not estimated), which catching the *management ability* of each firm.
- Using the *Within Group* estimator, which consisting in subtracting the mean of T period from each observation, or rather $\Delta w_{it} = w_{it} - \bar{w}$ where $\bar{w} = \frac{1}{T}\sum_{t=1}^{T} w_{it}$.
- Using the *First Differences* estimator (see §3.1.2)

A *first difference* estimator in this thesis will be chosen.

3.1.1 Could Heterogeneity to be seen as the Inefficiency?

The answer is outright: heterogeneity can be seen like an inefficiency, because it depend on the initial technology of the firm.
Therefore, it can be seen like the *starting inefficiency*, which is different for each the firm, and which sum itself with the inefficiency grown during production-cycle.
Thus, if it will want to obtain comparable inefficiencies across the firms, it must be remove the *starting inefficiency*, to compare, only the *true inefficiencies*, or rather the *production-cycle inefficiency*.
Finilly, the problem becomes: which is the inefficiency of each firm net of initial technology?

3.1.2 First Differences Estimator

Given a *Cobb-Douglass* production function (see defintion in (2.5)):

$$y_{it} = x_{it}\beta + \delta_i + \epsilon_{it} \tag{3.1}$$

Suppose that the main purpose of the analysis is to estimate β, or rather the marginal effect of x on the response variable y, based on a sample (y_i, x_i) and allowing *non-observable heterogeneity*, δ (costant in time), can correlate with the x regressors. But, a regression of y on x identifies the goal parameter β if and only if the *omitted variable*, δ:

1) is not correlated with x, that is $E(\eta|x) = 0$; or

2) does not determine y, that is $\beta = 0$; however clearly contradicts the initial specifiation of the model.

Hence, this problem can be solved by exploiting *longitudinal data* on the same units, i.e panel data. Suppose to have T period in the panel:
given Δ the operator *first difference*

$$\Delta w_{it} = w_{it} - w_{it-1} \tag{3.2}$$

Applying (3.2) on the T equations implied by (3.1), the following system will be otained of $(T-1)$ equations for *i-th* producer:

$$\begin{cases} \Delta y_{i2} &= \Delta x_{i2}\beta + \Delta u_{i2} \\ \quad\vdots & \quad\vdots \\ \Delta y_{iT} &= \Delta x_{iT}\beta + \Delta u_{iT} \end{cases}$$

In matrix form:

$$Dy_i = Dx_i\beta + Du_i \tag{3.3}$$

where D is an appropriate matrix $(T-1)\times T$ dimensional.

If $E(u_i|x_i,\eta_i)=0$, and so $E(Du_i|\eta_i)=0$, a OLS estimate of β from (3.3)

$$\hat{\beta}_{FE} = \left(\sum_{i=1}^{m}(Dx_i)'\,Dx_i\right)^{-1}\left(\sum_{i=1}^{m}(Dx_i)'\,Dy_i\right) \tag{3.4}$$

is correct and consistent if $N\rightarrow\infty$. Therefore the *First Differences Estimator* coincides with OLS estimator given by the regression of Δy_{it} on Δx_{it}. And the correctness and consistency of (3.4) derives from the fact that the model (3.3) does not depend on component δ.

Through this trasformation, all the parts of the model which did not vary over time have been eliminated, because, being constant, they cancel each other. So, the *starting inefficiency* has been eradicated.

However, (3.4) is not efficient, because a non-null correlation exists between the errors in first difference trasformation:

$$\begin{aligned}\Delta y_{i2} &= \Delta x_{i2}\beta + \boxed{u_{i2}} - u_{i1}\\ \Delta y_{i3} &= \Delta x_{i3}\beta + u_{i3} - \boxed{u_{i2}}\\ \vdots\quad &\quad \vdots\end{aligned}$$

and the errors in the first difference model are $MA(1)$. Anyhow, when this estimator is used with the Stochastic Frontier, it ia reasonable to think that the inefficiency, ie errors of the model, are correlated, becouse the inefficiency at time $t-1$ affects the inefficiency at time t.

3.2 Fixed Effects Models for Stochastic Frontier with Time-Varying Technical Inefficiency

These models of stochastic frontier arise from the need to estimate the inefficiency of firm net of management ability.
Starting from the first model defined before in (2.20) and adding on it the *non-observable heterogeneity*, δ, the initial model becomes:

$$y_{it} = \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + \delta_i + \epsilon_{it} \tag{3.5}$$

where, as usual, $\epsilon_{it} = v_{it} - u_{it}$. Applying the first difference operator (§3.1.2) on the equation above, the result is:

$$\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \Delta\epsilon_{it} \tag{3.6}$$

Focusing on the error component, $\Delta\epsilon_{it}$, of model (3.6), it will be obtain three particular cases, or rather three new sub-model:

1)

$$\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \underbrace{\Delta v_{it} - \Delta u_{it}}_{\Delta\epsilon_{it}} \tag{3.7}$$

2)

$$\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \underbrace{\Delta v_{it} - (\alpha_t - \alpha_{t-1}) u_i}_{\Delta\epsilon_{it}} \tag{3.8}$$

3)

$$\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \underbrace{\Delta v_{it}}_{\Delta\epsilon_{it}} \tag{3.9}$$

Where α_t is a parametric function of time like those described in the Chapter 2 (*Kumbhakar* (2.31) or *Battese-Coelli* (2.32)).
The following paragraphs will explain the three models listed above and the iterative method for the models estimation.

3.2.1 Iterative Estimation Procedure

The algorithm that is proposed here consisting of the iteration of D steps until the condition $\max||\hat{\theta}_k - \hat{\theta}_{k-1}|| < \zeta$ (where $\hat{\theta}_k$ is the vector of estimates at the itaration k and ζ is a quantity sufficiently small) is verified.
N.B: Given the underlying structure of dependence of the T errors of the firm, if it will want to maximize the log-likehood (or the function of the model), it must chosen two sets of indipendent differences (like for exsample, if $T = 5$, the differences sets are composed by differences of times $1-2$ and $4-5$ of each firm and so on) and maximize the function on them as described below.
Thus, to obtain the indipendence, a part of the data is lost, or rather all the sets of data which are dependent (like for exsample, if $T = 5$, the set of differeces of the times $3-4$ is rejected becouse is dependent either from the times $1-2$ and from times $4-5$). The two sets of differences can be seen, respectively, as a sort of the *training* and *validation* sets. If, it hold a large Panel ($T \to \infty$) a sort of *cross-validation* procedure, across the indipendent sets of differences, can be done, changing every time, after the D steps (one iteration), the two set of differences (for exsample, if $T = 10$, in the first iteration it could use the sets of differences from $1-2$ and $4-5$, in the second iteration il will use the $6-7$ and $9-10$ sets, and making this change of sets until the algorithm converges).
Defining *Tset*, the training set, and *Vset*, the validation set, the D steps of the estimation procedure are:

A) Maximize (3.26) respect β_1, β_2, σ_v, σ_u and η, on the *Tset*, obtaining: $\hat{\beta}_1^{(1)}$, $\hat{\beta}_2^{(1)}$, $\hat{\sigma}_v^{(1)}$, $\hat{\sigma}_u^{(1)}$, $\hat{\eta}^{(1)}$.

B) Fixing $\beta_1 = \hat{\beta}_1^{(1)}$, $\beta_2 = \hat{\beta}_2^{(1)}$, maximize (3.26) respect σ_v, σ_u and η on *Vset*, obteining: $\hat{\beta}_1^{(1)}$, $\hat{\beta}_2^{(1)}$, $\hat{\sigma}_v^{(2)}$, $\hat{\sigma}_u^{(2)}$ and $\hat{\eta}^{(2)}$.

C) Fixing $\beta_1 = \hat{\beta}_1^{(1)}$, $\beta_2 = \hat{\beta}_2^{(1)}$ and $\eta = \hat{\eta}^{(2)}$, maximize (3.26) respect σ_v and σ_u on *Tset*, obtaing: $\hat{\beta}_1^{(1)}$, $\hat{\beta}_2^{(1)}$, $\hat{\sigma}_v^{(3)}$, $\hat{\sigma}_u^{(3)}$ and $\hat{\eta}^{(2)}$.

D) Fixing $\beta_1 = \hat{\beta}_1^{(1)}$, $\beta_2 = \hat{\beta}_2^{(1)}$, $\eta = \hat{\eta}^{(2)}$ and $\sigma_v = \hat{\sigma}_v^{(3)}$, maximize (3.26) respect σ_u on *Vset*, obtaing the vector of final estimates,
$\hat{\theta}_k = [\hat{\beta}_1^{(1)}, \hat{\beta}_2^{(1)}, \hat{\eta}^{(2)}, \hat{\sigma}_v^{(3)}, \hat{\sigma}_u^{(4)}]$ with $k = 1, \ldots, K$ number of needed iterations.

3.2.2 Fixed Effect Stochastic Frontier Model with Time-Varying Tecnical Inefficiency

Starting from the equation model (3.7), the error components Δu_{it} end Δv_{it} is defined as:

$$\begin{cases} \Delta u_{it} = u_{it} - u_{it-1} \\ \Delta v_{it} = v_{it} - v_{it-1} \end{cases} \tag{3.10}$$

So the error is identically distribuited and not indipendent across the T observations of the firm but it becomes indipendent across the firms.

$$\begin{aligned} \Delta y_{i2} &= \beta_1 \Delta x_{1i2} + \beta_2 \Delta x_{2i2} + \left(\boxed{v_{i2}} - v_{i1} \right) - \left(\boxed{u_{i2}} - u_{i1} \right) \\ \Delta y_{i3} &= \beta_1 \Delta x_{1i3} + \beta_2 \Delta x_{2i3} + \left(v_{i3} - \boxed{v_{i2}} \right) - \left(u_{i3} - \boxed{u_{i2}} \right) \\ \vdots & \quad \vdots \end{aligned}$$

As usual, v_{it} and u_{it} are i.i.d. and also indipendent each other, consequently $\Delta v_{it} \perp \Delta u_{it}$. According to the last hypothesis on the errors components, them distributions become:

$$\begin{cases} \Delta u_{it} &= u_{it} - u_{it-1} \sim N^{+}(0,\sigma_u) - N^{+}(0,\sigma_u) = f_{\Delta u_{it}}(\Delta u_{it}) \\ \Delta v_{it} &= v_{it} - v_{it-1} \sim N(0,\sigma_v) - N(0,\sigma_v) = f_{\Delta v_{it}}(\Delta v_{it}) \sim N(0, 2\sigma_v^2) \end{cases} \tag{3.11}$$

Through a easy convolution, a closed form of the distribution of Δu_{it}, $f_{\Delta u_{it}}(\Delta u_{it})$, can be otained:

$$\begin{aligned} f_{\Delta u_{it}}(\Delta u_{it}) &= \int_0^\infty f_{u_{it}}\left(u_{it-1} + \Delta u_{it}\right) f_{u_{it-1}}\left(u_{it-1}\right) du_{it-1} \\ &= \frac{4}{2\pi\sigma_u^2}\int_0^\infty \exp\left\{\frac{\left(u_{it-1}+\Delta u_{it}\right)^2}{\sigma_u^2} + \frac{u_{it-1}^2}{\sigma_u^2}\right\} du_{it-1} \\ &= \frac{4\sqrt{2}}{\sqrt{2\pi}\sigma_u}\underbrace{\int_0^\infty \frac{1}{2\sqrt{\pi}\sigma_\star}\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_{it-1}+\mu_{\star it}\right)^2\right\} du_{it-1}}_{1-\Phi\left[\frac{\sqrt{2}}{2}\left(\frac{\Delta u_{it}}{\sigma_u}\right)\right]} \\ &= \frac{4}{\sqrt{\pi}\sigma_u}\Phi\left[-\frac{\sqrt{2}}{2}\left(\frac{\Delta u_{it}}{\sigma_u}\right)\right]. \end{aligned} \tag{3.12}$$

where[1]:

$$\begin{cases} \mu_{\star it} &= -\dfrac{\Delta u_{it}}{2} \\ \\ \sigma_\star^2 &= \dfrac{\sigma_u^2}{2} \end{cases} \tag{3.13}$$

[1] for all calculations see the Appendix

Now, with the distributions of Δv_{it} and Δu_{it}, it will obtain through another easy convolution, the distribution of the $\Delta\epsilon_{it} = \Delta v_{it} - \Delta u_{it}$:

$$\begin{aligned} f_{\Delta\epsilon_{it}}(\Delta\epsilon_{it}) &= \int_0^{\infty} f_{\Delta v_{it}}\left(\Delta\epsilon_{it} + \Delta u_{it}\right) f_{\Delta u_{it}}\left(\Delta u_{it}\right) d\Delta u_{it} \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_v}\phi\left(\frac{\Delta\epsilon_{it} + \Delta u_{it}}{\sqrt{2}\sigma_v}\right)\frac{4}{\sqrt{\pi}\sigma_u}\Phi\left[-\frac{\sqrt{2}}{2}\left(\frac{\Delta u_{it}}{\sigma_u}\right)\right] d\Delta u_{it} \end{aligned} \tag{3.14}$$

To find a cloused form of the distribution of $\Delta\epsilon_{it}$ solving the integral, it is not so easy. Thus, it will take an alternative route for the calculation of the integral, or rather the *Trapezoidal Method*.

This method is a used to approximate the area under a curve. This is done by inscribing or circumscribing n number of trapezoids under a curve. The areas of the trapezoids are then summed. In fact, the integral (3.14) represent an area under a curve that is its argument. Obviously, using the *Trapezoidal Method*, an approximation error, is introduced, because it will trying to approximate a curve with a lot of straight line.

The general rule of the methods is:

$$A = \frac{b-a}{2m}\left[f(x_0) + 2f(x_1) + 2f(x_3) + \ldots + 2f(x_{m-1}) + f(x_m)\right] \tag{3.15}$$

Thus, to abtain $n = m - 1$ trapezoids under the function, in m points $(x_1, \ldots, x_m)$ $f(x)$ it must be evalutated (as usual, m is the number of the firms). To use this method, it must choose a reasonable interval for Δu_{it} (which extremes are a and b) to be divided into $m - 1$ sub-intervals of equal width, in which extremes the functions, $f(x)$ (and $f_{\Delta u_{it}}(x)$), is evalutated.

In this case $f(x)$ assumes this forms:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_v}\phi\left(\frac{\Delta\epsilon_{it} + \Delta u_{it}}{\sqrt{2}\sigma_v}\right)\frac{4}{\sqrt{\pi}\sigma_u}\Phi\left[-\frac{\sqrt{2}}{2}\left(\frac{\Delta u_{it}}{\sigma_u}\right)\right] \tag{3.16}$$

Where $\Delta\epsilon_{it} = \Delta y_{it} - \beta_1\Delta x_{1it} - \beta_2\Delta x_{2it}$, and Δu_{it} is the value of $f_{\Delta u_{it}}$ evaluated in the point x.

Hence, according to the definition on Δu_{it} and $\Delta\epsilon_{it}$ just made, (3.15) represent an approximation of the density $f_{\Delta\epsilon_{it}}$. Thus, according to the definition, it will can build on it the log likelihood:

$$\ell\left(\Delta\epsilon_{it}|\beta_1, \beta_2, \sigma_v, \sigma_u\right) = \sum_i \log(f_{\Delta\epsilon_{it}}(\Delta\epsilon_{it})) \tag{3.17}$$

and maximizing it respect to β_1, β_2, σ_v and σ_u, using the iterative algorithm shown in (§3.2.1).

Finally, it will obtain with this method, and approximation of the value of λ or alternatively of γ.

A graphic representation of the method is shown in Figure below.

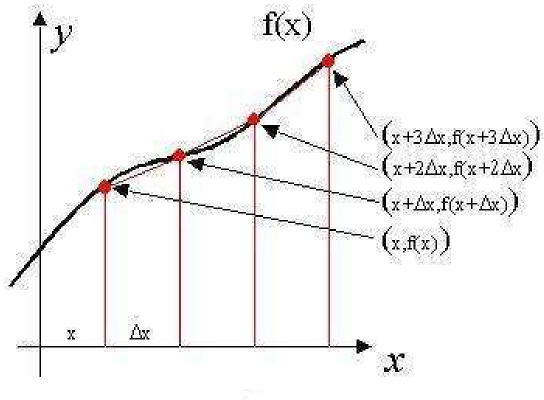

Figure 3.1: Trapezoidal Method: graphic representation

3.2.3 Fixed Effects Model for Stochastic Frontier with Time-Varying Technical Inefficiency by means of α_t Parametric Function

Starting from the equation model (3.8), the error components Δu_{it} end Δv_{it} is defined as:

$$\begin{cases} \Delta u_{it} & = \underbrace{(\alpha_t - \alpha_{t-1})}_{\Delta\alpha_t} u_i \\ \\ \Delta v_{it} & = v_{it} - v_{it-1} \end{cases} \tag{3.18}$$

So the error is identically distribuited and not indipendent across the T observations of the firm but it becomes indipendent across the firms.

$$\begin{aligned} \Delta y_{i2} &= \beta_1 \Delta x_{1i2} + \beta_2 \Delta x_{2i2} + \left(\boxed{v_{i2}} - v_{i1} \right) - \left(\boxed{\alpha_2} - \alpha_1 \right) u_i \\ \Delta y_{i3} &= \beta_1 \Delta x_{1i3} + \beta_2 \Delta x_{2i3} + \left(v_{i3} - \boxed{v_{i2}} \right) - \left(\alpha_3 - \boxed{\alpha_2} \right) u_i \\ \vdots & \quad \vdots \end{aligned}$$

Dividing by $\Delta\alpha_t$ the equation (3.8):

$$\frac{1}{\Delta\alpha_t} \Delta y_{it} = \frac{1}{\Delta\alpha_t} \left[\Delta x_{it} \beta + \Delta v_{it} \right] + u_i \tag{3.19}$$

Where:

- $\Delta v_{it} \sim N \left(0, 2\sigma_v^2 \right)$
- $u_i \sim N^+ \left(0, \sigma_u^2 \right)$
- $\Delta\alpha_t = \alpha_{t-1} - \alpha_t$

The result is

$$\frac{1}{\Delta\alpha_t} \left[\Delta x_{it} \beta + \Delta v_{it} \right] \sim N \left(\frac{\Delta x_{it} \beta}{\Delta\alpha_t}, \frac{2\sigma_v^2}{\Delta\alpha_t^2} \right) \tag{3.20}$$

Therefore the distribution of $f \left(\frac{1}{\Delta\alpha_t} \Delta y_{it} | \Delta x_{it} \right)$ is the following:

$$f \left(\frac{1}{\Delta\alpha_t} \Delta y_{it} | \Delta x_{it} \right) \sim N + N^+ \tag{3.21}$$

According to *Battese-Coelli*'s definition (§2.6.1)

- $\alpha_t = \exp \left\{ -\eta \left(t - T \right) \right\}$
- $\alpha_{t-1} = \exp \left\{ -\eta \left(t - 1 - T \right) \right\}$

and so[2]

$$\begin{aligned} \Delta\alpha_t &= \alpha_{t-1} - \alpha_t \\ &= \exp\left\{-\eta\left(t-T\right)\right\}\left(e^{\eta}-1\right) \end{aligned} \tag{3.22}$$

consequently

$$\Delta\alpha_t^2 = \exp\left\{-2\eta\left(t-T\right)\right\}\left(e^{\eta}-1\right)^2 \tag{3.23}$$

Defining:

$$\begin{cases} \Delta y_{it\eta} &= \frac{1}{\Delta\alpha_t}\Delta y_{it} \\ \Delta_F &= \frac{1}{\Delta\alpha_t}\left[\Delta x_{it}\beta + \Delta v_{it}\right] \\ \mu_F &= \dfrac{\Delta x_{it}\beta}{\Delta\alpha_t} \\ \sigma_F^2 &= \dfrac{2\sigma_v^2}{\Delta\alpha_t^2} \end{cases}$$

the distribution (3.21) is obtained through a convolution:

$$f_{\Delta y_{it\eta}}\left(\Delta y_{it\eta}\right) = \int_0^{\infty}\prod_{t=1}^{T-1}\left(\Delta y_{it\eta} - \Delta\alpha_t u_i\right) f_U\left(u\right) du_i =$$

[2]see all calulations in the Appendix

$$
\begin{aligned}
&= \frac{2}{(2\pi)^{\frac{2T-1}{2}} (\sigma_F)^{T-1} \sigma_u} \int_0^{\infty} \exp\left\{ -\frac{1}{2} \left[\frac{\sum_{t=1}^{T-1} (\Delta y_{it\eta} - \Delta\alpha_t u_i - \mu_F)^2}{\sigma_F^2} + \frac{u_i^2}{\sigma_u^2} \right] \right\} du_i \\
&= \frac{2\sigma_\star \exp\left\{-\frac{1}{2} a_{\star i}\right\}}{(2\pi)^{T-1} (\sigma_F)^{T-1} \sigma_u} \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\star} \exp\left\{ -\frac{1}{2\sigma_\star^2} (u_i - \mu_{\star i})^2 \right\} du_i \\
&= \frac{2\sigma_\star \exp\left\{-\frac{1}{2} a_{\star i}\right\}}{(2\pi)^{T-1} (\sigma_F)^{T-1} \sigma_u} \left[1 - \Phi\left(-\frac{\mu_{\star i}}{\sigma_\star} \right) \right]
\end{aligned}
\tag{3.24}
$$

where[3]:

$$
\begin{cases}
\mu_{\star i} = \dfrac{\sigma_u^2 \sum_{t=1}^{T-1} \left[\Delta\alpha_t \left(\Delta y_{it\eta} - \mu_F \right) \right]}{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2 + \sigma_F^2} \\
\sigma_\star^2 = \dfrac{\sigma_F^2 \sigma_u^2}{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2 + \sigma_F^2} \\
a_{\star i} = \dfrac{1}{\sigma_F^2} \left[\sum_{t=1}^{T-1} (\Delta y_{it\eta} - \mu_F)^2 - \dfrac{\sigma_u^2 \left(\sum_{t=1}^{T-1} \left[\Delta\alpha_t \left(\Delta y_{it\eta} - \mu_F \right) \right] \right)^2}{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2 + \sigma_F^2} \right]
\end{cases}
$$

As can be seen, using the Fixed effect, the first observation of the firm is lost, by definition of first differences. Hence, in this case, for each firm there are $T-1$ observations.

[3]for all calculations see the Appendix

The likelihood of (3.24) is the following:

$$L\left(\Delta y_{it\eta}|\sigma_v,\sigma_u,\beta\right)=\prod_{i=1}^{m} f_{\Delta y_{it\eta}}\left(\Delta y_{it\eta}\right)$$

$$\propto\left(\sigma_\star^2\right)^{\frac{m}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}a_{\star i}\right\}\left(\sigma_F^2\right)^{-\frac{(T-1)m}{2}}\left(\sigma_u^2\right)^{-\frac{m}{2}}\prod_{i=1}^{m}\left[1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right] \tag{3.25}$$

So, the log-likelihood is:

$$\ell\left(\Delta y_{it\eta}|\sigma_v,\sigma_u,\beta\right)=\sum_{i=1}^{m}\log\left(L\left(\Delta y_{it\eta}|\sigma_v,\sigma_u,\beta\right)\right)$$

$$\propto\frac{m}{2}\log\sigma_\star^2-\frac{1}{2}\sum_{i=1}^{m}a_{\star i}-\frac{(T-1)m}{2}\log\sigma_F^2-\frac{m}{2}\log\sigma_u^2+\sum_{i=1}^{m}\log\left[1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right] \tag{3.26}$$

To estimate the parameters and the quantities of the (3.26) via MLE, it can be used the procedure presented in (§3.2.1).

Estimation of Inefficiency Component

Knowing that $f(u,\Delta y_{it\eta})$ is:

$$\begin{aligned} f\left(u,\Delta y_{it\eta}\right) &= \frac{2}{(2\pi)^{\frac{2T-1}{2}}\left(\sigma_F\right)^{T-1}\sigma_u}\exp\left\{-\frac{1}{2}\left[\frac{\sum_{t=1}^{T-1}\left(\Delta y_{it\eta}-\Delta\alpha_t u_i-\mu_F\right)^2}{\sigma_F^2}+\frac{u_i^2}{\sigma_u^2}\right]\right\} \\ &\vdots \\ &= \frac{2\sigma_\star\exp\left\{-\frac{1}{2}a_{\star i}\right\}}{(2\pi)^{T-1}\left(\sigma_F\right)^{T-1}\sigma_u}\left[\frac{1}{\sqrt{2\pi}\sigma_\star}\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}\right] \end{aligned} \tag{3.27}$$

Then, with easy calculations, the distribution of $f(u|\Delta y_{it\eta})$ can be derived[4]:

$$f\left(u_i|\Delta y_{it\eta}\right) = \frac{f\left(u, \Delta y_{it\eta}\right)}{f\left(\Delta y_{it\eta}\right)}$$

$$= \frac{\left[\frac{1}{\sqrt{2\pi}\sigma_\star} \exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_i - \mu_{\star i}\right)^2\right\}\right]}{\left[1 - \Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right]} \tag{3.28}$$

As can be seen, the distribution of $u_i|\Delta y_{it\eta}$ is a truncated normal, or rather $N^+(\mu_{\star i}, \sigma_\star^2)$. So an estimator for u_i can be obtained from the mean of $u|\Delta y_{it\eta}$, which is given by:

$$E\left(u_i|\Delta y_{it\eta}\right) = \mu_{\star i} + \sigma_\star \left[\frac{\phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)}{1 - \Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)}\right]. \tag{3.29}$$

3.2.4 Degenerate Fixed Effetcs Model of Stochastic Frontier

Analyzing the models (3.7) and (3.8), the model (3.9) can be derivated as a particulr case of them. Or rather:

- From di model (3.7): If $\Delta u_{it} = 0$ then $u_{it} = u_{it-1} \Rightarrow u_{it} = u_i$ and when the Δ operator is applied they cancel each other.
- From model (3.8): $\Delta u_{it} = 0$ if $(\alpha_t - \alpha_{t-1}) = 0$.

If it will want to estimate this model, via OLS or by MLE, the problem rises in the end of the estimation: the unobserved heterogeneity and the true inefficiency can not be distinguished becouse of they are mixed in the residuals of the model.

[4] for all calculation see the Appendix

Thus, the inefficiency can not be correctly estimated: this explains why the model is called degenerate.

Conclusions

In this thesis the main subject are the *Stochastic Frontiers*: starting from their definition to arrive at their estimate, and consequently to the estimates of the inefficiencies. This thesis is divided into three chapters. The first is an introduction of the world of the *Stochastic Frontiers*, where it was given a general definition about them, or rather like a maximun level of reachable output, given a known quantity of inputs, with the available tecnology. Thus, a good firm use them as an indicator of their ineffeciency, whereas the frontier like a upper border, which should be close to be efficient. The distance from the frontier is a measure about the inefficiency of the firm. In the second part of this chapter it was shown the Panel dataset (WHO), where the two models for the frontier, already present in the literature, were estimated (with LIMDEP 9.0), and the results obtained. In the second chapter with greater statistical rigor the frontier was defined, ussuming a Cobb-Douglass formulation, or rather, $\underbrace{y = \beta_0 + \beta_1 q_1 + \beta_2 q_2 + v}_{Frontier} - u$, where u is the inefficiency component. In the sencond part of the chapter the focus shifts to the error component of frontier, $\epsilon = v - u$, on the distributions of v and u, for which was chosen, respectively, $N(0, \sigma_v^2)$ and $N^+(0, \sigma_u^2)$, called *Normal-Half Normal* model. In addiction it was reported here all the inference on the first two models already present in literature, or rather:

1) Estimation by MLE of a Stochastic Production Frontier Function with Time-

Invariant Technical Efficiency

- $y_{it} = \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + v_{it} - u_{it}$

2) Estimation by MLE of a Stochastic Production Frontier Function with Time-Varying Technical Inefficiency:

- $y_{it} = \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + v_{it} - (\alpha_t u_i)$

In the third and final chapter they were exposed three new stochastic frontier models to reach the target of distinguishing heterogeneity from inefficiency, the true final goal of the thesis. For eliminating the heterogeneity, or rather the technology available, different for each firm, which makes the inefficiencies not comparable, it was used the *First Difference* estimator used in the *Fixed Effect* models. Applying the FD operator on the two models listed before, its was obtained three new models:

3) Fixed Effect Stochastic FrontierModel with Time-Varying Tecnical Inefficiency:

- $\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \underbrace{\Delta v_{it} - \Delta u_{it}}_{\Delta \epsilon_{it}}$

4) Fixed Effects Model for Stochastic Frontier with Time-Varying Technical Inefficiency by means of α_t Parametric Function:

- $\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \underbrace{\Delta v_{it} - (\alpha_t - \alpha_{t-1}) u_i}_{\Delta \epsilon_{it}}$

5) Degenerate Fixed Effetcs Model of Stochastic Frontier:

- $\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \underbrace{\Delta v_{it}}_{\Delta \epsilon_{it}}$

In this Thesis it was showed only the theory and the estimation procedure of each model, becouse of maximize their log-likelihoods is not so easy to manage becouse them shape is not regular. Hence, this would require further analysis, that

for lack of time were not made.
Thus, all the models that were reported here can be expanded and analyzed in more detail, but morover they can be implemented, and their results compared with other similar works on it.
For example in the model (§3.2.2), it can be found an estimation method for the inefficiency and so on.
Whilst, if it want instead to concentrate on the concept of frontier as a maximum, it can try to express the frontiers in terms of *Extreme Values* where the error componet assumes this form: $\epsilon_{it} = v_{it} + (\xi_{it} - \max_i \xi_{it})$. Or rather, if it assumes v_{it} and ξ_{it} are two random variables independent, ϵ_{it} measure the sum of a random noise, v_{it}, and a component that measure how mauch the efficiency of the firm, ξ_{it}, departs from the maximum efficiency in the market, $\max_i \xi_{it}$.
An other possible extension to other models of stochastic frontier is to consider the error defined in this way: $\epsilon_{it} = v_{it} - u_{it}$ where $u_{it} = \rho u_{it-1} + \zeta_{it}$. Or rather, assuming that the inefficiency is a persistent process over the time and hence observations of u belonging to the same firm are not independet.

Appendix A

Calculations of formulas listed before

Model 1: Estimation of a stochastic production frontier function with Time-Invariant Technical Efficiency

PROOF: The distribution of ϵ is a Shew Normal.

Knowing that $X = a|Z_1| + bZ_2$
where:

- $Z_1 \sim |Z|$ with $Z \sim N(0,1)$ and $Z_2 \sim N(0,1)$
- $a^2 + b^2 = 1$ with a = δ and b = $\sqrt{1-\delta^2}$
- $\delta = \dfrac{\alpha}{\sqrt{1+\alpha}}$
- $\alpha = \dfrac{a}{b}$

Then X has a Shew-Normal distribution, that is $X \sim SN(\alpha)$[1].

In this particular $\alpha = \dfrac{\sigma_u}{\sigma_v}$ and $\sigma^2 = \sigma_v^2 + \sigma_u^2$. Hence, knowing that $\epsilon = v - u$, where v and u are, respectively, representations of V and U, random variables ($V \perp U$), respectively distribuited as $N(0, \sigma_v^2)$ and $|Z|$ with $Z \sim N(0, \sigma_u^2)$ (Half-Normal distribution), then:

multiplying both members of ϵ by -1:

$-\epsilon = -v + u$

And now recalling the well-known property of Normal distribution:

- if $Y \sim N(\mu, \eta^2)$ then $cY \sim N(c\mu, c^2\eta^2)$, with c costant

In this case:

$(-1)V \sim N(0, \sigma_v^2)$

Therefore $-V \equiv V$. Thus, it follows that:

$-\epsilon = v + u$ so $-\epsilon \sim SN(\alpha)$ and for the proprety of Shew Normal distribution $\epsilon \sim SN(-\alpha)$.

□

Derivation of $-\epsilon$ distribution: ($-\epsilon = v + u$)

Set $z = -\epsilon$:

$$\begin{aligned}
F_Z[z] &= Pr\{V + U \le z\} = Pr\{-U \le z - V\} = \\
&= Pr\{U \ge V - z\} = 1 - Pr\{U \le V - z\} = \\
&= E_V[1 - Pr\{U \le v - z | V = v\}] = \qquad (V \perp U) \\
&= 1 - E_V[Pr\{U \le v - z\}] = \\
&= 1 - E_V[F_U(v - z)] = \\
&= 1 - \int_0^{\infty} F_U(v - z) f_V(v)\, dv
\end{aligned}$$

[1] see A.Azzalini and Dalla Valle 2006

Now with the obtained cumulative probability function of Z, $F_z[z]$, its own density probability function can be derived by means of derivation respect z, $F_z[z]$, and we get:

$$f_Z(z) = \int_0^{\infty} f_U(v-z) f_V(v)\ dv$$

<u>Developing the calculations:</u>

Set $a = \sigma_u^2$ and $b = \sigma_v^2$ then:[2]

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \frac{2}{2\pi\sigma_v\sigma_u} \exp\left\{-\frac{1}{2}\left[\frac{(v-z)^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2}\right]\right\} dv \\
&= \int_{-\infty}^{z} \frac{2}{2\pi\sigma_v\sigma_u} \exp\left\{-\frac{1}{2}\left[\frac{(\sigma_v^2 z^2 - 2\sigma_v^2 zv + \sigma_v^2 v^2 + \sigma_u^2 v^2)}{\sigma_v^2\sigma_u^2}\right]\right\} dv \\
&= \int_{-\infty}^{z} \frac{2}{2\pi\sigma_v\sigma_u} \exp\left\{-\frac{1}{2}\left[\frac{\left(z^2\sigma_u^2\sigma_v^2 - z^2\sigma_u^2\sigma_v^2 + \sigma_v^2 z^2 - 2\sigma_v^2 zv + v^2\underbrace{\left(\sigma_u^2+\sigma_v^2\right)}_{1}\right)}{\sigma_v^2\sigma_u^2}\right]\right\} dv \\
&= \int_{-\infty}^{z} \frac{2}{2\pi\sigma_v\sigma_u} \exp\left\{-\frac{1}{2}\left[\frac{\left(z^2\sigma_u^2\sigma_v^2 + z^2\sigma_v^2\underbrace{\left(1-\sigma_u^2\right)}_{\sigma_v^2} - 2\sigma_v^2 zv + v^2\right)}{\sigma_v^2\sigma_u^2}\right]\right\} dv \\
&= \int_{-\infty}^{z} \frac{2}{2\pi\sigma_v\sigma_u} \exp\left\{-\frac{1}{2}\left[\frac{(z^2\sigma_u^2\sigma_v^2 + z^2\sigma_v^4 - 2\sigma_v^2 zv + v^2)}{\sigma_v^2\sigma_u^2}\right]\right\} dv
\end{aligned}$$

[2] see above for the property about a and b in the SN

$$= \int_{-\infty}^{z} \frac{2}{2\pi\sigma_v\sigma_u} \exp\left\{-\frac{1}{2}\left[z^2 + \frac{\left(v - \sigma_v^2 z\right)^2}{\sigma_v^2\sigma_u^2}\right]\right\} dv$$

$$= \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\sigma_v\sigma_u} \exp\left\{-\frac{1}{2\sigma_v^2\sigma_u^2}\left(v - \sigma_v^2 z\right)^2\right\} dv$$

$$= 2\,\phi(z)\,\Phi\left(\frac{z - \sigma_v^2 z}{\sigma_v\sigma_u}\right) = 2\,\phi(z)\,\Phi\left(\frac{z\overbrace{\left(1 - \sigma_v^2\right)}^{\sigma_u^2}}{\sigma_v\sigma_u}\right)$$

$$= 2\,\phi(z)\,\Phi\left(\frac{\sigma_u}{\sigma_v}z\right) = 2\,\phi(z)\,\Phi(\lambda z)$$

<u>Now adding to the distribution, $f_Z(z)$, scale (ω) and location (ξ) parameters:</u>

Considering the following linear trasformation:

$$Y = \xi + \omega Z$$

The distribution of Y becomes:

$$Y \sim SN(\xi, \omega^2, \alpha)$$

<u>PROOF:</u>

$$z = y\frac{1}{\omega} - \frac{1}{\omega}\xi$$

$$\frac{\partial z}{\partial y} = \frac{1}{\omega} \Longrightarrow Jacobiano\,(J)$$

$$\begin{aligned} f_Y(y) &= f_Z(y) \times |J| \\ &= f_Y\left(y\frac{1}{\omega} - \frac{1}{\omega}\xi\right)\frac{1}{\omega} \\ &= \frac{2}{\omega}\phi\left(y\frac{1}{\omega} - \frac{1}{\omega}\xi\right)\Phi\left(\alpha\left(y\frac{1}{\omega} - \frac{1}{\omega}\xi\right)\right) \end{aligned}$$

In this case $\xi = 0$, $\omega = \sigma$ and $\alpha = \lambda$, so:

$$f_Z(z) = \frac{2}{\sigma}\phi\left(\frac{z}{\sigma}\right)\Phi\left(\lambda\frac{z}{\sigma}\right)$$

It follows that the distribution of ϵ is[3]:

$$f_\epsilon(\epsilon) = \frac{2}{\sigma}\phi\left(\frac{\epsilon}{\sigma}\right)\Phi\left(-\lambda\frac{\epsilon}{\sigma}\right)$$

□

Derivation of $\mu_\star$ and $\sigma_\star$:

$$\begin{aligned} F_\epsilon[\epsilon] &= Pr\{V - U \le \epsilon\} = Pr\{V \le \epsilon + U\} = \\ &= E_U[Pr\{V \le \epsilon + u | U = u\}] = \qquad (V \perp U) \\ &= E_U[Pr\{V \le \epsilon + u\}] = E_U[F_V(\epsilon + u)] = \\ &= \int_0^\infty F_V(\epsilon + u) f_U(u) du \end{aligned}$$

Now with the obtained cumulative probability function of Z, $F_z[z]$, its own density probability function can be derived by mean of derivation respect z, $F_z[z]$:

$$f_Z(z) = \int_0^\infty f_V(\epsilon + u) f_U(u)\, du$$

[3]see above the proprety of a $shew - normal$ distribution

Developing the calculations:

$$f_Z(z) = \frac{2}{2\pi\sigma_v\sigma_u}\int_0^{\infty}\exp\left\{-\frac{1}{2}\left[\frac{(\epsilon+u)^2}{\sigma_u^2}+\frac{u^2}{\sigma_v^2}\right]\right\}du$$

$$= \int_0^{\infty}\frac{2}{2\pi\sigma_v\sigma_u}\exp\left\{-\frac{1}{2}\left[\frac{\sigma_v^2u^2}{\sigma_v^2\sigma_u^2}+\frac{\sigma_u^2\epsilon^2}{\sigma_v^2\sigma_u^2}+\frac{2\sigma_u^2\epsilon u}{\sigma_v^2\sigma_u^2}+\frac{\sigma_u^2u^2}{\sigma_v^2\sigma_u^2}\right]\right\}du$$

$$= \int_0^{\infty}\frac{2}{2\pi\sigma_v\sigma_u}\exp\left\{-\frac{1}{2}\left[\frac{\sigma_u^2\epsilon^2}{\sigma_v^2\sigma_u^2}+u^2\left(\frac{\sigma_v^2}{\sigma_v^2\sigma_u^2}+\frac{\sigma_u^2}{\sigma_v^2\sigma_u^2}\right)-2u\left(-\frac{2\sigma_u^2\epsilon}{\sigma_v^2\sigma_u^2}\right)\right]\right\}du$$

knowing that:

$$ax^2+bx = a\left(x^2+\frac{b}{a}x\right)$$

$$a\left[x^2+\left(\frac{b}{a}x\right)+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right] \quad \text{(A.1)}$$

$$\boxed{a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}}$$

Hence, in this case:

$$\begin{cases} a = \dfrac{\sigma_v^2+\sigma_u^2}{\sigma_v^2\sigma_u^2} \\ b = -2\left(-\dfrac{2\sigma_u^2\epsilon}{\sigma_v^2\sigma_u^2}\right) \\ x = u \end{cases}$$

Developing the calculations:

$$\frac{\sigma_v^2+\sigma_u^2}{\sigma_v^2\sigma_u^2}\left(x+\frac{\sigma_u^2\epsilon}{\cancel{\sigma_v^2\sigma_u^2}}\frac{\cancel{\sigma_v^2\sigma_u^2}}{\sigma_v^2+\sigma_u^2}\right)^2-\left[\cancel{4}\left(\frac{\cancel{\sigma_u^2}\epsilon}{\sigma_v^2\cancel{\sigma_u^2}}\right)^2\frac{1}{\cancel{4}}\frac{\cancel{\sigma_v^2}\sigma_u^2}{\sigma_v^2+\sigma_u^2}\right]=$$

$$=\underbrace{\frac{\sigma_v^2+\sigma_u^2}{\sigma_v^2\sigma_u^2}}_{\frac{1}{\sigma_\star^2}}\left(x+\underbrace{\frac{\sigma_u^2\epsilon}{\sigma_v^2+\sigma_u^2}}_{-\mu_\star}\right)^2-\frac{\sigma_u^2\epsilon^2}{\sigma_v^2\left(\underbrace{\sigma_v^2+\sigma_u^2}_{\sigma^2}\right)}$$

Going back to the exponential argument:

$$\exp\left\{-\frac{1}{2}\left[\frac{\sigma_u^2\epsilon^2}{\sigma_v^2\sigma_u^2}+\frac{1}{\sigma_\star^2}\left(u+\mu_{\star i}\right)^2-\frac{\sigma_u^2\epsilon^2}{\sigma_v^2\sigma^2}\right]\right\}=$$

$$=\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u+\mu_{\star i}\right)^2\right\}\exp\left\{-\frac{1}{2}\frac{\sigma^2\sigma_u^2\epsilon^2-\left(\sigma_u^2\right)^2\epsilon^2}{\sigma_v^2\sigma_u^2\sigma^2}\right\}=$$

$$=\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u+\mu_{\star i}\right)^2\right\}\exp\left\{-\frac{1}{2}\frac{\epsilon^2}{\sigma^2}\right\}$$

Putting the expression just derived to the integral:

$$\frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\epsilon^2}{\sigma^2}\right\} \int_0^\infty \frac{1}{\sqrt{2\pi}\frac{\sigma_v\sigma_u\sigma}{\sigma}} \exp\left\{-\frac{1}{2\sigma_\star^2}(u+\mu_\star)^2\right\} =$$

$$= \frac{2}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\frac{\epsilon^2}{\sigma^2}\right\} \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_\star} \exp\left\{-\frac{1}{2\sigma_\star^2}(u+\mu_\star)^2\right\} =$$

$$= \frac{2}{\sigma}\phi\left(\frac{\epsilon}{\sigma}\right)\left[1-\Phi\left(0+\frac{\epsilon\sigma_u^2}{\sigma^2}\frac{\sigma}{\sigma_v\sigma_u}\right)\right] =$$

$$= \frac{2}{\sigma}\phi\left(\frac{\epsilon}{\sigma}\right)\left[1-\Phi\left(\frac{\lambda\epsilon}{\sigma}\right)\right] = \frac{2}{\sigma}\phi\left(\frac{\epsilon}{\sigma}\right)\Phi\left(-\frac{\lambda\epsilon}{\sigma}\right)$$

Where:

$$\begin{cases} \mu_\star &= -\frac{\sigma_u^2\epsilon}{\sigma^2} \\ \sigma_\star^2 &= \frac{\sigma_v^2\sigma_u^2}{\sigma^2} \\ \sigma &= \sqrt{\sigma_v^2+\sigma_u^2} \\ \lambda &= \frac{\sigma_u}{\sigma_v} \end{cases}$$

Reparameterization $\mu_\star$, $\sigma_\star^2$, σ_v^2 and σ_u^2, respect to λ and σ, fondamental parameter of the SF, it will obtain:

$$\begin{cases} \mu_\star &= \frac{-\epsilon\lambda^2}{1+\lambda^2} \\ \sigma_\star^2 &= \frac{\lambda\sigma}{1+\lambda^2} \\ \sigma_v^2 &= \frac{\sigma^2}{1+\lambda^2} \\ \sigma_u^2 &= \frac{\lambda^2\sigma^2}{1+\lambda^2} \end{cases}$$

□

Jondrow et al.'s Inefficiency estimator:

$$
\begin{aligned}
E\left[u_i|\epsilon_i\right] &= \mu_{*i} + \sigma_* \left[\frac{\phi\left(-\dfrac{\mu_{*i}}{\sigma_*}\right)}{1-\Phi\left(-\dfrac{\mu_{*i}}{\sigma_*}\right)}\right] \\
&= \sigma_* \left[\frac{\phi\left(-\dfrac{\mu_{*i}}{\sigma_*}\right)}{1-\Phi\left(-\dfrac{\mu_{*i}}{\sigma_*}\right)} + \frac{\mu_{*i}}{\sigma_*}\right] \qquad \text{(A.2)} \\
&= \frac{\sigma\lambda}{1+\lambda^2} \left[\frac{\phi\left(\dfrac{\epsilon_i\lambda}{\sigma}\right)}{1-\Phi\left(\dfrac{\epsilon_i\lambda}{\sigma}\right)} - \left(\frac{\epsilon_i\lambda}{\sigma}\right)\right]
\end{aligned}
$$

Model 2: Estimation of a stochastic production frontier function with Time - Varying Technical Inefficiency that depends on time through the α_ts parameters

$$f_\epsilon(\epsilon_i) = \int_0^\infty \prod_{t=1}^{T} f(\epsilon_{it} - \alpha_t u_i) f(u_i)\, du_i$$

$$= \frac{2}{(2\pi)^{\frac{T+1}{2}}(\sigma_v^2)^T \sigma_u^2} \int_0^\infty \exp\left\{-\frac{1}{2}\left[\sum_{t=1}^{T} \frac{(\epsilon_{it} - \alpha_t u_i)^2}{\sigma_v^2} + \frac{u_i^2}{\sigma_u^2}\right]\right\} du_i$$

$$= \frac{2}{(2\pi)^{\frac{T+1}{2}}(\sigma_v^2)^T \sigma_u^2} \int_0^\infty \exp\left\{-\frac{1}{2}\left[\frac{\sigma_u^2 \sum_{t=1}^{T} \epsilon_{it}^2}{\sigma_v^2\sigma_u^2} - \frac{2\sigma_u^2 u_i \sum_{t=1}^{T} \alpha_t \epsilon_{it}}{\sigma_v^2\sigma_u^2} + \frac{\sigma_u^2 \sum_{t=1}^{T} \alpha_t^2}{\sigma_v^2\sigma_u^2} + \frac{\sigma_v^2 u_i^2}{\sigma_v^2\sigma_u^2}\right]\right\} du_i$$

$$= \frac{2}{(2\pi)^{\frac{T+1}{2}}(\sigma_v^2)^T \sigma_u^2} \int_0^\infty \exp\left\{-\frac{1}{2}\left[\frac{\sigma_u^2 \sum_{t=1}^{T} \epsilon_{it}^2}{\sigma_v^2\sigma_u^2} + u_i^2\left(\frac{\sigma_u^2 \sum_{t=1}^{T} \alpha_t^2 + \sigma_v^2}{\sigma_v^2\sigma_u^2}\right) - 2u_i\left(\frac{\sigma_u^2 \sum_{t=1}^{T} \alpha_t \epsilon_{it}}{\sigma_v^2\sigma_u^2}\right)\right]\right\} du_i$$

In this case, recalling the (A.1), it will obtain:

$$\begin{cases} a = \dfrac{\sigma_u^2 \sum_{t=1}^{T} \alpha_t^2 + \sigma_v^2}{\sigma_v^2\sigma_u^2} \\ b = -2\left(\dfrac{\sigma_u^2 \sum_{t=1}^{T} \alpha_t \epsilon_{it}}{\sigma_v^2\sigma_u^2}\right) \\ x = u_i \end{cases}$$

Developing the calculations:

$$\frac{\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2}{\sigma_v^2\sigma_u^2}\left(u_i-\frac{\sigma_u^2\sum_{t=1}^T\alpha_t\epsilon_{it}}{\cancel{\sigma_v^2}\cancel{\sigma_u^2}}\frac{\cancel{\sigma_v^2}\cancel{\sigma_u^2}}{\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2}\right)^2$$

$$-\left(\cancel{4}\left(\frac{\cancel{\sigma_u^2}\sum_{t=1}^T\alpha_t\epsilon_{it}}{\sigma_v^2\cancel{\sigma_u^2}}\right)^2\frac{1}{\cancel{4}}\frac{\cancel{\sigma_v^2}\sigma_u^2}{\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2}\right)=$$

$$=\underbrace{\frac{\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2}{\sigma_v^2\sigma_u^2}}_{\frac{1}{\sigma_\star^2}}\left(u_i-\underbrace{\frac{\sigma_u^2\sum_{t=1}^T\alpha_t\epsilon_{it}}{\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2}}_{\mu_{\star i}}\right)^2-\left(\frac{\sigma_u^2\left(\sum_{t=1}^T\alpha_t\epsilon_{it}\right)^2}{\sigma_v^2\left(\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2\right)}\right)$$

Going back to the exponential argument:

$$\exp\left\{-\frac{1}{2}\left[\frac{\sigma_u^2\sum_{t=1}^T\epsilon_{it}^2}{\sigma_v^2\sigma_u^2}+\frac{1}{\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2-\frac{\sigma_u^2\left(\sum_{t=1}^T\alpha_t\epsilon_{it}\right)^2}{\sigma_v^2\left(\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2\right)}\right]\right\}=$$

$$=\exp\left\{-\frac{1}{2}\underbrace{\frac{1}{\sigma_v^2}\left[\sum_{t=1}^T\epsilon_{it}^2-\frac{\sigma_u^2\left(\sum_{t=1}^T\alpha_t\epsilon_{it}\right)^2}{\left(\sigma_u^2\sum_{t=1}^T\alpha_t^2+\sigma_v^2\right)}\right]}_{a_{\star i}}\right\}\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}$$

Putting the expression just derived to the integral:

$$\frac{2\sigma_\star\exp\left\{-\frac{1}{2}a_{\star i}\right\}}{(2\pi)^{\frac{T}{2}}(\sigma_v)^T\sigma_u}\underbrace{\int_0^\infty\frac{1}{\sqrt{2\pi}\sigma_\star}\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}du_i}_{1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)}=$$

Hence:

$$f_\epsilon(\epsilon_i) = \frac{2\sigma_\star \exp\left\{-\frac{1}{2}a_{\star i}\right\}}{(2\pi)^{\frac{T}{2}}(\sigma_v)^T\sigma_u}\left[1 - \Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right]$$

Where:

$$\begin{cases} \mu_{\star i} = \dfrac{\sigma_u^2\left(\sum_{t=1}^T \alpha_t\epsilon_{it}\right)}{\sigma_u^2\sum_{t=1}^T \alpha_t^2 + \sigma_v^2} \\[2ex] \sigma_\star^2 = \dfrac{\sigma_v^2\sigma_u^2}{\sigma_u^2\sum_{t=1}^T \alpha_t^2 + \sigma_v^2} \\[2ex] a_{\star i} = \dfrac{1}{\sigma_v^2}\left[\displaystyle\sum_{t=1}^T \epsilon_{it}^2 - \frac{\sigma_u^2\left(\sum_{t=1}^T \alpha_t\epsilon_{it}\right)^2}{\left(\sigma_u^2\sum_{t=1}^T \alpha_t^2 + \sigma_v^2\right)}\right] \end{cases}$$

□

Model 1 FE: Fixed Effect Stochastic Frontier Model with Time-Varying Tecnical Inefficiency

Derivation of the distribution of Δu_{it}:

$$\Delta u_{it} = u_{it} - u_{it-1}, \quad u_{it} \perp u_{it-1}$$

$$\begin{aligned}
F_{\Delta} u_{it} \left[\Delta u_{it}\right] &= Pr\left\{u_{it} - u_{it-1} \leq \Delta u_{it}\right\} = Pr\left\{u_{it} \leq \Delta u_{it} + u_{it-1}\right\} = \\
&= E_{u_{it-1}}\left[Pr\left\{u_{it} \leq u_{it} + u_{it-1} | U = u_{it-1}\right\}\right] = \\
&= E_{u_{it-1}}\left[Pr\left\{u_{it} \leq \Delta u_{it} + u_{it-1}\right\}\right] = E_{u_{it-1}}\left[F_{u_{it}}(\Delta u_{it} + u_{it-1})\right] = \\
&= \int_0^\infty F_{u_{it}}(\Delta u_{it} + u_{it-1}) f_{u_{it-1}}(u_{it-1})\, du_{it-1}
\end{aligned}$$

$$\begin{aligned}
f_{\Delta u_{it}}(\Delta u_{it}) &= \int_0^\infty f_{u_{it}}\left(u_{it-1} + \Delta u_{it}\right) f_{u_{it-1}}\left(u_{it-1}\right) du_{it-1} \\
&= \frac{4}{2\pi\sigma_u^2} \int_0^\infty \exp\left\{\frac{\left(u_{it-1} + \Delta u_{it}\right)^2}{\sigma_u^2} + \frac{u_{it-1}^2}{\sigma_u^2}\right\} du_{it-1} \\
&= \frac{4}{2\pi\sigma_u^2} \int_0^\infty \exp\left\{\frac{u_{it-1}^2 + 2u_{it-1}\Delta u_{it} + \Delta u_{it}^2}{\sigma_u^2} + \frac{u_{it-1}^2}{\sigma_u^2}\right\} du_{it-1} \\
&= \frac{4}{2\pi\sigma_u^2} \int_0^\infty \exp\left\{\frac{2u_{it-1}^2}{\sigma_u^2} + \frac{2u_{it-1}\Delta u_{it}}{\sigma_u^2} + \frac{\Delta u_{it}^2}{\sigma_u^2}\right\} du_{it-1} \\
&= \frac{4}{2\pi\sigma_u^2} \int_0^\infty \exp\left\{u_{it-1}^2\left(\frac{2}{\sigma_u^2}\right) - 2u_{it-1}\left(\frac{-\Delta u_{it}}{\sigma_u^2}\right) + \frac{\Delta u_{it}^2}{\sigma_u^2}\right\} du_{it-1}
\end{aligned}$$

In this case, recalling the (A.1), it will obtain:

$$\begin{cases} a = \dfrac{2}{\sigma_u^2} \\ \\ b = -2\left(\dfrac{\Delta u_{it}}{\sigma_u^2}\right) \\ \\ x = u_i \end{cases}$$

Developing the calculations:

$$= \underbrace{\frac{2}{\sigma_u^2}}_{\frac{1}{\sigma_\star^2}} \left(u_{it-1} + \underbrace{\frac{\Delta u_{it}}{2}}_{-\mu_{\star_{it}}} \right)^2 - \left(4 \frac{\Delta u_{it}^2}{(\sigma_u^2)^2} \frac{1}{4} \sigma_u^2 \right) =$$

$$= \frac{1}{\sigma_\star^2} (u_{it-1} + \mu_{\star_{it}})^2 - \left(\frac{\Delta u_{it}^2}{\sigma_u^2} \right)$$

Going back to the exponential argument:

$$\exp\left\{ -\frac{1}{2} \left[\frac{\Delta u_{it}^2}{\sigma_u^2} - \frac{\Delta u_{it}^2}{\sigma_u^2} + \frac{1}{\sigma_\star^2} (u_{it-1} + \mu_{\star_{it}})^2 \right] \right\}$$

Putting the expression just derived to the integral:

$$= \frac{4\sqrt{2}}{\sqrt{2\pi}\sigma_u} \int_0^\infty \frac{1}{2\sqrt{\pi}\frac{\sigma_u}{\sqrt{2}}} \exp\left\{ -\frac{1}{2\sigma_\star^2} (u_{it-1} + \mu_{\star it})^2 \right\} du_{it-1}$$

$$= \frac{4\sqrt{2}}{\sqrt{2\pi}\sigma_u} \left[1 - \Phi\left(0 + \frac{\Delta u_{it}}{2} \frac{\sqrt{2}}{\sigma_u} \right) \right]$$

$$= \frac{4}{\sqrt{\pi}\sigma_u}\Phi\left[1 - \Phi\left(\frac{\sqrt{2}\Delta u_{it}}{2\sigma_u}\right)\right]$$

$$= \frac{4}{\sqrt{\pi}\sigma_u}\Phi\left[-\frac{\sqrt{2}}{2}\left(\frac{\Delta u_{it}}{\sigma_u}\right)\right].$$

Where:

$$\begin{cases} \mu_{\star it} & = & -\dfrac{\Delta u_{it}}{2} \\ \\ \sigma_\star^2 & = & \dfrac{\sigma_u^2}{2} \end{cases}$$

□

Derivation of the distribution of $\Delta\epsilon_{it}$:

$$\Delta\epsilon_{it} = \Delta v_{it} - \Delta u_{it},\ \Delta v_{it} \perp \Delta u_{it}$$

$$\begin{aligned} F_{\Delta\epsilon_{it}}\left[\Delta\epsilon_{it}\right] &= Pr\left\{\Delta v_{it} - \Delta u_{it} \leq \Delta\epsilon_{it}\right\} = Pr\left\{\Delta v_{it} \leq \Delta u_{it} + \Delta\epsilon_{it}\right\} = \\ &= E_{\Delta u_{it}}\left[Pr\left\{\Delta v_{it} \leq \Delta u_{it} + \Delta\epsilon_{it} | U = \Delta u_{it}\right\}\right] = \\ &= E_{\Delta u_{it}}\left[Pr\left\{\Delta v_{it} \leq \Delta u_{it} + \Delta\epsilon_{it}\right\}\right] = E_{\Delta u_{it}}\left[F_{\Delta v_{it}}(\Delta u_{it} + \Delta\epsilon_{it})\right] = \\ &= \int_0^\infty F_{\Delta v_{it}}(\Delta u_{it} + \Delta\epsilon_{it}) f_{\Delta u_{it}}(\Delta u_{it})\, d\Delta u_{it} \end{aligned}$$

$$\begin{aligned} f_{\Delta\epsilon_{it}}(\Delta\epsilon_{it}) &= \int_0^\infty f_{\Delta v_{it}}(\Delta u_{it} + \Delta\epsilon_{it}) f_{\Delta u_{it}}(\Delta u_{it})\, d\Delta u_{it} \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_v}\phi\left(\frac{\Delta\epsilon_{it} + \Delta u_{it}}{\sqrt{2}\sigma_v}\right)\frac{4}{\sqrt{\pi}\sigma_u}\Phi\left[-\frac{\sqrt{2}}{2}\left(\frac{\Delta u_{it}}{\sigma_u}\right)\right] d\Delta u_{it} \end{aligned}$$

Model 2 FE: Fixed Effects Model for Stochastic Frontier with Time-Varying Technical Inefficiency by means of α_t Parametric Function

$$f_{\Delta y_{it\eta}}\left(\Delta y_{it\eta}\right) = \int_0^\infty \prod_{t=1}^{T-1} \left(\Delta y_{it\eta} - \Delta\alpha_t u_i\right) f_U\left(u\right) du_i$$

$$= \underbrace{\frac{2}{\left(2\pi\right)^{\frac{2T-1}{2}} \left(\sigma_F\right)^{T-1} \sigma_u}}_{K} \int_0^\infty \exp\left\{ -\frac{1}{2} \left[\frac{\sum_{t=1}^{T-1} \left(\Delta y_{it\eta} - \Delta\alpha_t u_i - \mu_F\right)^2}{\sigma_F^2} + \frac{u_i^2}{\sigma_u^2} \right] \right\} du_i$$

$$= K \int_0^\infty \exp\left\{ -\frac{1}{2} \left[\frac{\sigma_u^2 \sum_{t=1}^{T-1} \Delta y_{it\eta}^2}{\sigma_F^2\sigma_u^2} + \frac{\sigma_u^2 u_i^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2}{\sigma_F^2\sigma_u^2} + \frac{\sigma_u^2 (T-1)\mu_F^2}{\sigma_F^2\sigma_u^2} - \frac{2\sigma_u^2\mu_F \sum_{t=1}^{T-1} \Delta y_{it\eta}}{\sigma_F^2\sigma_u^2} - \frac{2\sigma_u^2 u_i \sum_{t=1}^{T-1} \left(\Delta y_{it\eta}\Delta\alpha_t\right)}{\sigma_F^2\sigma_u^2} + \frac{2\sigma_u^2\mu_F u_i \sum_{t=1}^{T-1} \Delta\alpha_t}{\sigma_F^2\sigma_u^2} + \frac{\sigma_F^2 u_i^2}{\sigma_F^2\sigma_u^2} \right] \right\} du_i$$

$$= K \int_0^\infty \exp\left\{ -\frac{1}{2} \left[\frac{\sigma_u^2 \sum_{t=1}^{T-1} \Delta y_{it\eta}^2}{\sigma_F^2\sigma_u^2} - \frac{2\sigma_u^2\mu_F \sum_{t=1}^{T-1} \Delta y_{it\eta}}{\sigma_F^2\sigma_u^2} + \frac{\sigma_u^2 (T-1)\mu_F^2}{\sigma_F^2\sigma_u^2} + u_i^2 \left(\frac{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2}{\sigma_F^2\sigma_u^2} + \frac{\sigma_F^2}{\sigma_F^2\sigma_u^2} \right) - 2u_i \left(\frac{2\sigma_u^2 \sum_{t=1}^{T-1} \left(\Delta y_{it\eta}\Delta\alpha_t\right)}{\sigma_F^2\sigma_u^2} - \frac{2\sigma_u^2\mu_F \sum_{t=1}^{T-1} \Delta\alpha_t}{\sigma_F^2\sigma_u^2} \right) \right] \right\} du_i$$

$$= K\int_0^\infty \exp\left\{-\frac{1}{2}\left[\frac{\sigma_u^2\sum_{t=1}^{T-1}\left(\Delta y_{it\eta}-\mu_F\right)^2}{\sigma_F^2\sigma_u^2}+u_i^2\left(\frac{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}{\sigma_F^2\sigma_u^2}\right)\right.\right.$$
$$\left.\left.-2u_i\left(\frac{\sigma_u^2\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]}{\sigma_F^2\sigma_u^2}\right)\right]\right\}du_i$$

In this case, recalling the (A.1), it will obtain:

$$\begin{cases} a = \dfrac{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}{\sigma_F^2\sigma_u^2} \\ b = -2\left(\dfrac{\sigma_u^2\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]}{\sigma_F^2\sigma_u^2}\right) \\ x = u_i \end{cases}$$

Developing the calculations:

$$\underbrace{\frac{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}{\sigma_F^2\sigma_u^2}}_{\frac{1}{\sigma_\star^2}}\left(u_i-\frac{\sigma_u^2\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]}{\sigma_F^2\sigma_u^2}\frac{\sigma_F^2\sigma_u^2}{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}\right)^2+$$
$$-4\left(\frac{\sigma_u^2\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]}{\sigma_F^2\sigma_u^2}\right)^2\frac{1}{4}\frac{\sigma_F^2\sigma_u^2}{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}=$$

$$=\frac{1}{\sigma_\star^2}\left(u_i-\underbrace{\frac{\sigma_u^2\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]}{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}}_{\mu_{\star i}}\right)^2-\frac{\sigma_u^2\left(\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]\right)^2}{\sigma_F^2\left(\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2\right)}$$

Going back to the exponential argument:

$$\exp\left\{-\frac{1}{2}\left[\frac{\sum_{t=1}^{T-1}\left(\Delta y_{it\eta}-\mu_F\right)^2}{\sigma_F^2}-\frac{\sigma_u^2\left(\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]\right)^2}{\sigma_F^2\left(\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2\right)}\right.\right.$$
$$\left.\left.+\frac{1}{\sigma_{\star i}}\left(u_i-\mu_{\star i}\right)^2\right]\right\}=$$

$$=\exp\left\{\underbrace{-\frac{1}{2}\frac{1}{\sigma_F^2}\left[\sum_{t=1}^{T-1}\left(\Delta y_{it\eta}-\mu_F\right)^2-\frac{\sigma_u^2\left(\sum_{t=1}^{T-1}\left[\Delta\alpha_t\left(\Delta y_{it\eta}-\mu_F\right)\right]\right)^2}{\sigma_u^2\sum_{t=1}^{T-1}\Delta\alpha_t^2+\sigma_F^2}\right]}_{a_{\star i}}\right\}\times$$
$$\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}$$

Putting the expression just derived to the integral:

$$\frac{2\sigma_\star\exp\left\{-\frac{1}{2}a_{\star i}\right\}}{(2\pi)^{T-1}\left(\sigma_F\right)^{T-1}\sigma_u}\int_0^\infty\frac{1}{\sqrt{2\pi}\sigma_\star}\exp\left\{-\frac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}du_i$$

$$f_{\Delta}y_{it\eta}\left(\Delta y_{it\eta}\right)=\frac{2\sigma_\star\exp\left\{-\frac{1}{2}a_{\star i}\right\}}{(2\pi)^{T-1}\left(\sigma_F\right)^{T-1}\sigma_u}\left[1-\Phi\left(-\frac{\mu_{\star i}}{\sigma_\star}\right)\right]$$

Where:

$$\begin{cases} \mu_{\star i} = \dfrac{\sigma_u^2 \sum_{t=1}^{T-1} \left[\Delta\alpha_t \left(\Delta y_{it\eta} - \mu_F\right)\right]}{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2 + \sigma_F^2} \\ \\ \sigma_\star^2 = \dfrac{\sigma_F^2 \sigma_u^2}{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2 + \sigma_F^2} \\ \\ a_{\star i} = \dfrac{1}{\sigma_F^2} \left[\displaystyle\sum_{t=1}^{T-1} \left(\Delta y_{it\eta} - \mu_F\right)^2 - \dfrac{\sigma_u^2 \left(\sum_{t=1}^{T-1} \left[\Delta\alpha_t \left(\Delta y_{it\eta} - \mu_F\right)\right] \right)^2}{\sigma_u^2 \sum_{t=1}^{T-1} \Delta\alpha_t^2 + \sigma_F^2} \right] \end{cases}$$

□

Calculations of $\Delta\alpha_t$:

$$\begin{aligned} \Delta\alpha_t &= \alpha_{t-1} - \alpha_t \\ &= \exp\left\{-\eta\left(t-1-T\right)\right\} - \exp\left\{-\eta\left(t-T\right)\right\} \\ &= \exp\left\{-\eta\left(t-T\right)+\eta\right\} - \exp\left\{-\eta\left(t-T\right)\right\} \\ &= \exp\left\{-\eta\left(t-T\right)\right\} e^{\eta} - \exp\left\{-\eta\left(t-T\right)\right\} \\ &= \exp\left\{-\eta\left(t-T\right)\right\} \left(e^{\eta}-1\right) \end{aligned} \tag{A.3}$$

Estimation of Inefficiency Component:

$$
\begin{aligned}
f\left(u_i|\Delta y_{it\eta}\right) &= \frac{f\left(u,\Delta y_{it\eta}\right)}{f\left(\Delta y_{it\eta}\right)} \\
&= \frac{\dfrac{2\sigma_\star \exp\left\{-\frac{1}{2}a_{\star i}\right\}}{\left(2\pi\right)^{T-1}\left(\sigma_F\right)^{T-1}\sigma_u}\left[\dfrac{1}{\sqrt{2\pi}\sigma_\star}\exp\left\{-\dfrac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}\right]}{\dfrac{2\sigma_\star \exp\left\{-\frac{1}{2}a_{\star i}\right\}}{\left(2\pi\right)^{T-1}\left(\sigma_F\right)^{T-1}\sigma_u}\left[1-\Phi\left(-\dfrac{\mu_{\star i}}{\sigma_\star}\right)\right]} \\
&= \frac{\left[\dfrac{1}{\sqrt{2\pi}\sigma_\star}\exp\left\{-\dfrac{1}{2\sigma_\star^2}\left(u_i-\mu_{\star i}\right)^2\right\}\right]}{\left[1-\Phi\left(-\dfrac{\mu_{\star i}}{\sigma_\star}\right)\right]}
\end{aligned}
$$

Tables listed before in the chapters

Table A.1: Time Invariant efficiency per state

	stati	eff93	rank93	eff94	rank94	eff95	rank95	eff96	rank96	eff97	rank97
1	Malta	0.9081	63	0.9098	59	0.9110	61	0.9114	61	0.9145	59
2	Oman	0.8828	93	0.8831	92	0.8900	84	0.8874	89	0.8882	89
3	Singapore	0.9723	6	0.9854	2	0.9630	9	0.9644	10	0.9677	8
4	Italy	0.9075	65	0.9087	62	0.9099	62	0.9110	63	0.9122	63
5	Jamaica	0.9022	74	0.9036	71	0.9051	70	0.9064	68	0.9074	69
6	SaudiArabia	0.7352	127	0.7211	127	0.7189	126	0.7120	126	0.7072	126
7	Japan	0.9035	72	0.9044	70	0.9061	68	0.9068	67	0.9083	67
8	Morocco	0.9174	52	0.8862	87	0.8853	92	0.8801	94	0.8807	93
9	France	0.7863	116	0.7706	117	0.7593	118	0.7483	119	0.7339	122
10	Spain	0.8912	83	0.8951	81	0.9006	80	0.9007	76	0.8970	78
11	Greece	0.9327	41	0.9358	33	0.9330	38	0.9438	26	0.9386	30
12	Portugal	0.9056	67	0.9077	63	0.9122	60	0.9137	59	0.9138	61
13	Bahrain	0.7722	118	0.7741	115	0.7775	115	0.7758	116	0.7754	116
14	Netherlands	0.8931	81	0.9071	65	0.9126	58	0.9002	77	0.8914	86
15	Chinag	0.8197	108	0.8224	109	0.8298	109	0.8228	109	0.8360	108
16	ElSalvador	0.8895	85	0.8935	83	0.8893	87	0.8918	87	0.8933	82
17	Colombia	0.8988	77	0.8991	78	0.9013	78	0.8973	79	0.8915	85
18	UnitedKingdom	0.6019	134	0.5798	136	0.5593	139	0.5407	140	0.5208	139
19	CostaRica	0.7873	115	0.7618	120	0.7430	122	0.7328	124	0.7186	124
20	Austria	0.8955	79	0.9014	74	0.9029	75	0.9061	69	0.9085	66
21	Belgium	0.8870	88	0.8883	85	0.8899	85	0.8925	85	0.8957	79
22	Cyprus	0.9488	19	0.9489	23	0.9471	23	0.9427	27	0.9415	26
23	Venezuela	0.9708	8	0.9697	9	0.9671	8	0.9644	9	0.9614	11
24	Turkey	0.8585	103	0.8517	104	0.8389	108	0.8257	108	0.8114	111
25	Switzerland	0.7423	123	0.7457	123	0.7354	124	0.7400	123	0.7268	123
26	DominicanRepubl	0.7443	122	0.7501	122	0.7410	123	0.7401	122	0.7368	121

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27	Mexico	0.9456	23	0.9347	35	0.9270	41	0.9184	57	0.9123	62
28	UnitedArabEmir	0.8146	112	0.8275	108	0.8400	107	0.8525	105	0.8622	100
29	Canada	0.9810	3	0.9704	8	0.9726	6	0.9733	6	0.9742	5
30	Indonesia	0.9471	22	0.9528	20	0.9548	16	0.9531	16	0.9531	16
31	CapeVerde	0.9436	26	0.9419	28	0.9412	28	0.9365	31	0.9375	31
32	Luxembourg	0.9010	75	0.9055	68	0.9075	66	0.9145	58	0.9187	56
33	IranIslamicRe	0.8730	99	0.8735	98	0.8732	97	0.8751	98	0.8774	95
34	Chile	0.8747	98	0.8762	96	0.8769	96	0.8754	97	0.8741	96
35	Australia	0.9558	15	0.9570	13	0.9594	12	0.9581	12	0.9566	15
36	Yemen	0.9166	54	0.9187	49	0.9230	45	0.9285	41	0.9288	41
37	Sweden	0.9427	27	0.9449	25	0.9477	22	0.9490	20	0.9502	19
38	Armenia	0.9452	24	0.9442	26	0.9452	27	0.9440	25	0.9460	22
39	Norway	0.8876	87	0.8815	94	0.8781	95	0.8826	91	0.8935	81
40	Israel	0.7526	121	0.7391	125	0.7275	125	0.7157	125	0.7043	127
41	SriLanka	0.8936	80	0.8986	79	0.9014	76	0.9023	73	0.9035	72
42	Brazil	0.8846	91	0.8858	88	0.8897	86	0.8897	88	0.8901	88
43	Ireland	0.9212	49	0.9224	45	0.9223	47	0.9235	47	0.9234	50
44	Germany	0.9222	48	0.9227	44	0.9237	44	0.9275	42	0.9291	40
45	Honduras	0.9620	9	0.9779	5	0.9764	4	0.9744	5	0.9737	6
46	Egypt	0.7670	119	0.7712	116	0.7750	116	0.7779	115	0.7973	114
47	Argentina	0.8835	92	0.9013	75	0.9058	69	0.9118	60	0.9235	49
48	TrinidadandTob	0.7762	117	0.7694	118	0.7487	120	0.7515	117	0.7650	117
49	Croatia	0.7369	125	0.7622	119	0.7632	117	0.7427	121	0.7370	120
50	UnitedStatesof	0.9582	11	0.9558	14	0.9543	18	0.9496	19	0.9494	21
51	Iceland	0.9033	73	0.9099	58	0.9181	53	0.9316	37	0.9297	38
52	Paraguay	0.9569	14	0.9556	15	0.9560	14	0.9442	24	0.9440	24
53	Georgia	0.9472	21	0.9549	16	0.9551	15	0.9566	14	0.9577	13

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54	Tunisia	0.9427	28	0.9426	27	0.9352	33	0.9329	33	0.9337	35
55	Slovenia	0.8067	113	0.8117	111	0.8026	114	0.8045	114	0.8051	112
56	Barbados	0.8888	86	0.8853	91	0.8886	89	0.8943	84	0.9006	73
57	Finland	0.9597	10	0.9610	11	0.9603	11	0.9635	11	0.9671	9
58	Uruguay	0.8403	107	0.8411	106	0.8440	105	0.8468	106	0.8507	104
59	Denmark	0.9137	58	0.9151	54	0.9125	59	0.9191	55	0.9090	65
60	CzechRepublic	0.9134	59	0.9112	57	0.9162	57	0.9226	50	0.9201	54
61	Panama	0.9139	57	0.9089	61	0.9030	74	0.8970	80	0.8830	92
62	Qatar	0.9052	69	0.9060	67	0.9061	67	0.9048	72	0.9042	70
63	Poland	0.9054	68	0.9007	76	0.9014	77	0.8969	81	0.8910	87
64	Nicaragua	0.9269	43	0.9289	41	0.9331	37	0.9233	48	0.9251	45
65	Slovakia	0.9803	4	0.9807	4	0.9818	3	0.9806	2	0.9798	3
66	Kuwait	0.9079	64	0.9066	66	0.9077	65	0.8988	78	0.8986	75
67	Malaysia	0.9389	31	0.9391	30	0.9383	31	0.9384	30	0.9395	28
68	Bulgaria	0.8897	84	0.8776	95	0.8670	101	0.8634	102	0.8588	103
69	Iraq	0.7380	124	0.7243	126	0.7087	128	0.6940	129	0.6812	129
70	Lithuania	0.8670	100	0.8696	99	0.8695	99	0.8643	101	0.8649	99
71	Ukraine	0.8851	90	0.8856	89	0.8860	90	0.8864	90	0.8695	97
72	Thailand	0.9240	46	0.9142	56	0.9081	64	0.9105	65	0.8384	107
73	Pakistan	0.9718	7	0.9713	7	0.9715	7	0.9718	7	0.9736	7
74	Lebanon	0.6292	132	0.6229	133	0.6198	132	0.6119	132	0.6122	132
75	NewZealand	0.9285	42	0.9184	50	0.9204	49	0.9220	51	0.9275	43
76	RepublicofMold	0.8999	76	0.9019	73	0.9008	79	0.9011	75	0.8952	80
77	Ecuador	0.9084	61	0.8935	84	0.8859	91	0.8823	92	0.8982	76
78	Guatemala	0.9733	5	0.9723	6	0.9747	5	0.9744	4	0.9767	4
79	Peru	0.9374	32	0.9295	40	0.9270	40	0.9199	53	0.9243	47
80	RepublicofKore	0.8147	111	0.8161	110	0.8150	110	0.8128	111	0.8163	110

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81	Tonga	0.9178	50	0.9183	51	0.9262	42	0.9305	39	0.9285	42
82	Romania	0.8176	110	0.8059	114	0.8092	113	0.8084	112	0.7970	115
83	Hungary	0.9537	16	0.9541	19	0.9542	19	0.9525	17	0.9524	17
84	Bangladesh	0.9370	33	0.9181	52	0.9186	51	0.9236	46	0.9154	58
85	RussianFederati	0.8197	109	0.8091	113	0.8100	112	0.8054	113	0.8005	113
86	Guyana	0.7361	126	0.7394	124	0.7465	121	0.7448	120	0.7511	118
87	Uzbekistan	0.8635	101	0.8643	100	0.8659	102	0.8575	103	0.8677	98
88	Nepal	0.5725	139	0.5654	140	0.5662	138	0.5549	138	0.5452	138
89	Mauritius	0.9335	38	0.9366	31	0.9349	34	0.9310	38	0.9295	39
90	SyrianArabRepu	0.5734	138	0.5690	138	0.5680	137	0.5656	136	0.5609	136
91	Jordan	0.7216	129	0.7102	129	0.7163	127	0.7006	127	0.7101	125
92	Myanmar	0.6737	131	0.6851	131	0.6906	131	0.6977	128	0.6922	128
93	Belarus	0.9269	44	0.9323	39	0.9356	32	0.9324	35	0.9394	29
94	Philippines	0.9082	62	0.9096	60	0.9096	63	0.9113	62	0.9081	68
95	India	0.9039	70	0.9046	69	0.9046	71	0.9059	70	0.9156	57
96	Estonia	0.9227	47	0.9195	48	0.9178	54	0.9322	36	0.9321	36
97	Latvia	0.8860	89	0.8872	86	0.8886	88	0.8924	86	0.8930	83
98	Gambia	0.9835	2	0.9817	3	0.9822	2	0.9803	3	0.9803	2
99	Kazakhstan	0.8929	82	0.8951	80	0.8993	81	0.9051	71	0.9094	64
100	VietNam	0.9332	40	0.9364	32	0.9326	39	0.9327	34	0.9357	33
101	Samoa	0.8755	97	0.8816	93	0.8833	94	0.8807	93	0.8839	90
102	Fiji	0.9172	53	0.9198	47	0.9207	48	0.9227	49	0.9265	44
103	Maldives	0.9244	45	0.9249	42	0.9242	43	0.9211	52	0.9225	52
104	Bolivia	0.9345	36	0.9338	38	0.9343	36	0.9245	45	0.9237	48
105	Tajikistan	0.9479	20	0.9458	24	0.9459	25	0.9427	28	0.9424	25
106	Sudan	0.8521	104	0.8633	101	0.8675	100	0.8649	99	0.8596	102

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107	Benin	0.9424	29	0.9495	22	0.9458	26	0.9298	40	0.9298	37
108	Comoros	0.8815	95	0.8331	107	0.8615	103	0.8643	100	0.8779	94
109	Bahamas	0.6151	133	0.6233	132	0.6138	133	0.6091	133	0.6077	133
110	Senegal	0.9529	18	0.9546	17	0.9567	13	0.9572	13	0.9571	14
111	Turkmenistan	0.8506	105	0.8552	103	0.8497	104	0.8562	104	0.8613	101
112	CotedIvoire	0.8026	114	0.8095	112	0.8120	111	0.8165	110	0.8198	109
113	Haiti	0.9531	17	0.9542	18	0.9514	21	0.9500	18	0.9507	18
114	Ghana	0.9581	12	0.9595	12	0.9545	17	0.9540	15	0.9593	12
115	Mali	0.9355	35	0.9343	36	0.9183	52	0.9097	66	0.9036	71
116	Mauritania	0.8956	78	0.8948	82	0.8901	83	0.8962	82	0.8995	74
117	Mozambique	0.9125	60	0.9150	55	0.9176	56	0.9188	56	0.9203	53
118	Cameroon	0.5904	136	0.5940	134	0.5991	134	0.6015	134	0.6074	134
119	GuineaBissau	0.9059	66	0.9072	64	0.9041	73	0.9105	64	0.9139	60
120	Togo	0.7645	120	0.7562	121	0.7590	119	0.7491	118	0.7450	119
121	Congo	0.9037	71	0.8995	77	0.8976	82	0.8950	83	0.8925	84
122	BurkinaFaso	0.8600	102	0.8563	102	0.8721	98	0.8755	96	0.8833	91
123	EquatorialGuine	0.8776	96	0.8744	97	0.8840	93	0.8779	95	0.8462	105
124	Ethiopia	0.9358	34	0.9199	46	0.9225	46	0.9245	44	0.9233	51
125	CentralAfrican	0.9342	37	0.9404	29	0.9403	29	0.9391	29	0.9371	32
126	Burundi	0.9149	56	0.9181	53	0.9177	55	0.9196	54	0.9195	55
127	Kenya	0.9452	25	0.9502	21	0.9531	20	0.9488	21	0.9495	20
128	Niger	0.7087	130	0.7000	130	0.6927	130	0.6840	130	0.6746	130
129	Uganda	0.7293	128	0.7117	128	0.6956	129	0.6710	131	0.6538	131
130	UnitedRepublic	0.9423	30	0.9358	34	0.9398	30	0.9471	22	0.9407	27
131	Nigeria	0.9160	55	0.9024	72	0.9042	72	0.9016	74	0.8979	77
132	SouthAfrica	0.8433	106	0.8416	105	0.8424	106	0.8429	107	0.8437	106

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133	Lesotho	0.9178	51	0.9230	43	0.9197	50	0.9270	43	0.9249	46
134	Swaziland	0.9579	13	0.9614	10	0.9614	10	0.9675	8	0.9621	10
135	Rwanda	0.9333	39	0.9342	37	0.9345	35	0.9345	32	0.9351	34
136	Namibia	0.8827	94	0.8853	90	0.9469	24	0.9460	23	0.9449	23
137	Zimbabwe	0.9841	1	0.9855	1	0.9852	1	0.9854	1	0.9834	1
138	Botswana	0.5589	140	0.5765	137	0.5802	135	0.5835	135	0.5872	135
139	Malawi	0.5812	137	0.5659	139	0.5552	140	0.5443	139	0.5156	140
140	Zambia	0.5940	135	0.5832	135	0.5719	136	0.5652	137	0.5505	137

Table A.2: Time - Varying efficiency per state

	stati	eff93	eff94	eff95	eff96	eff97	rank
1	Malta	0.9972	0.9971	0.9971	0.9971	0.9971	1
2	Oman	0.9969	0.9969	0.9969	0.9968	0.9968	2
3	Singapore	0.9947	0.9946	0.9945	0.9945	0.9944	3
4	Italy	0.9935	0.9934	0.9933	0.9933	0.9932	4
5	Jamaica	0.9928	0.9928	0.9927	0.9926	0.9925	5
6	SaudiArabia	0.9884	0.9883	0.9882	0.9881	0.9879	6
7	Japan	0.9672	0.9669	0.9665	0.9662	0.9659	7
8	Morocco	0.9649	0.9646	0.9642	0.9639	0.9636	8
9	France	0.9639	0.9636	0.9633	0.9629	0.9626	9
10	Spain	0.9635	0.9632	0.9629	0.9625	0.9622	10
11	Greece	0.9528	0.9524	0.9519	0.9515	0.9510	11
12	Portugal	0.9458	0.9453	0.9448	0.9442	0.9437	12
13	Bahrain	0.9398	0.9393	0.9387	0.9382	0.9376	13
14	Netherlands	0.9392	0.9386	0.9381	0.9375	0.9369	14
15	Chinag	0.9385	0.9379	0.9373	0.9368	0.9362	15
16	ElSalvador	0.9374	0.9368	0.9362	0.9356	0.9350	16
17	Colombia	0.9365	0.9360	0.9354	0.9348	0.9342	17
18	UnitedKingdom	0.9346	0.9340	0.9334	0.9328	0.9322	18
19	CostaRica	0.9344	0.9338	0.9332	0.9325	0.9319	19
20	Austria	0.9343	0.9337	0.9331	0.9325	0.9318	20
21	Belgium	0.9308	0.9302	0.9295	0.9289	0.9282	21
22	Cyprus	0.9276	0.9269	0.9262	0.9256	0.9249	22
23	Venezuela	0.9254	0.9248	0.9241	0.9234	0.9227	23
24	Turkey	0.9236	0.9229	0.9222	0.9215	0.9207	24
25	Switzerland	0.9227	0.9220	0.9213	0.9206	0.9198	25
26	DominicanRepubl	0.9213	0.9206	0.9199	0.9192	0.9184	26
27	Mexico	0.9193	0.9185	0.9178	0.9170	0.9163	27
28	UnitedArabEmir	0.9192	0.9185	0.9177	0.9170	0.9162	28
29	Canada	0.9178	0.9171	0.9163	0.9156	0.9148	29
30	Indonesia	0.9173	0.9165	0.9158	0.9150	0.9142	30
31	CapeVerde	0.9172	0.9165	0.9157	0.9149	0.9142	31
32	Luxembourg	0.9171	0.9164	0.9156	0.9148	0.9141	32
33	IranIslamicRe	0.9168	0.9160	0.9152	0.9145	0.9137	33
34	Chile	0.9161	0.9153	0.9145	0.9137	0.9130	34
35	Australia	0.9145	0.9137	0.9129	0.9122	0.9113	35
36	Yemen	0.9144	0.9136	0.9128	0.9120	0.9112	36
37	Sweden	0.9127	0.9120	0.9111	0.9103	0.9095	37
38	Armenia	0.9115	0.9107	0.9099	0.9091	0.9083	38

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39	Norway	0.9105	0.9097	0.9089	0.9080	0.9072	39
40	Israel	0.9100	0.9092	0.9084	0.9076	0.9067	40
41	SriLanka	0.9090	0.9082	0.9074	0.9065	0.9057	41
42	Brazil	0.9057	0.9048	0.9040	0.9031	0.9022	42
43	Ireland	0.9045	0.9037	0.9028	0.9019	0.9010	43
44	Germany	0.9035	0.9026	0.9017	0.9008	0.8999	44
45	Honduras	0.8988	0.8979	0.8970	0.8960	0.8951	45
46	Egypt	0.8958	0.8949	0.8939	0.8930	0.8920	46
47	Argentina	0.8949	0.8940	0.8930	0.8921	0.8911	47
48	TrinidadandTob	0.8932	0.8922	0.8912	0.8902	0.8892	48
49	Croatia	0.8930	0.8920	0.8910	0.8901	0.8891	49
50	UnitedStatesof	0.8928	0.8918	0.8908	0.8899	0.8889	50
51	Iceland	0.8888	0.8877	0.8867	0.8857	0.8847	51
52	Paraguay	0.8867	0.8857	0.8847	0.8837	0.8826	52
53	Georgia	0.8863	0.8853	0.8843	0.8832	0.8822	53
54	Tunisia	0.8841	0.8830	0.8820	0.8809	0.8798	54
55	Slovenia	0.8839	0.8828	0.8818	0.8807	0.8796	55
56	Barbados	0.8830	0.8820	0.8809	0.8799	0.8788	56
57	Finland	0.8820	0.8809	0.8798	0.8788	0.8777	57
58	Uruguay	0.8801	0.8790	0.8780	0.8769	0.8758	58
59	Denmark	0.8767	0.8756	0.8744	0.8733	0.8722	59
60	CzechRepublic	0.8755	0.8744	0.8732	0.8721	0.8710	60
61	Panama	0.8754	0.8743	0.8732	0.8720	0.8709	61
62	Qatar	0.8710	0.8699	0.8687	0.8675	0.8664	62
63	Poland	0.8685	0.8673	0.8661	0.8649	0.8637	63
64	Nicaragua	0.8684	0.8673	0.8661	0.8649	0.8637	64
65	Slovakia	0.8665	0.8653	0.8641	0.8629	0.8617	65
66	Kuwait	0.8645	0.8633	0.8620	0.8608	0.8596	66
67	Malaysia	0.8644	0.8632	0.8620	0.8608	0.8596	67
68	Bulgaria	0.8621	0.8609	0.8597	0.8584	0.8572	68
69	Iraq	0.8594	0.8582	0.8569	0.8557	0.8544	69
70	Lithuania	0.8564	0.8552	0.8539	0.8526	0.8513	70
71	Ukraine	0.8562	0.8549	0.8536	0.8523	0.8510	71
72	Thailand	0.8553	0.8540	0.8528	0.8515	0.8501	72
73	Pakistan	0.8533	0.8520	0.8507	0.8494	0.8481	73
74	Lebanon	0.8532	0.8519	0.8506	0.8493	0.8479	74
75	NewZealand	0.8525	0.8512	0.8499	0.8486	0.8473	75
76	RepublicofMold	0.8516	0.8502	0.8489	0.8476	0.8463	76

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77	Ecuador	0.8508	0.8495	0.8481	0.8468	0.8455	77
78	Guatemala	0.8486	0.8473	0.8459	0.8446	0.8432	78
79	Peru	0.8457	0.8444	0.8430	0.8416	0.8403	79
80	RepublicofKore	0.8454	0.8441	0.8427	0.8413	0.8399	80
81	Tonga	0.8450	0.8437	0.8423	0.8409	0.8395	81
82	Romania	0.8429	0.8415	0.8401	0.8387	0.8373	82
83	Hungary	0.8428	0.8415	0.8401	0.8387	0.8373	83
84	Bangladesh	0.8422	0.8408	0.8395	0.8381	0.8366	84
85	RussianFederati	0.8421	0.8407	0.8393	0.8379	0.8365	85
86	Guyana	0.8405	0.8391	0.8377	0.8362	0.8348	86
87	Uzbekistan	0.8397	0.8383	0.8369	0.8355	0.8341	87
88	Nepal	0.8397	0.8383	0.8369	0.8354	0.8340	88
89	Mauritius	0.8385	0.8371	0.8357	0.8342	0.8328	89
90	SyrianArabRepu	0.8380	0.8366	0.8351	0.8337	0.8322	90
91	Jordan	0.8372	0.8358	0.8344	0.8329	0.8315	91
92	Myanmar	0.8321	0.8306	0.8292	0.8277	0.8262	92
93	Belarus	0.8297	0.8282	0.8267	0.8252	0.8237	93
94	Philippines	0.8278	0.8263	0.8248	0.8233	0.8218	94
95	India	0.8262	0.8247	0.8231	0.8216	0.8201	95
96	Estonia	0.8236	0.8220	0.8205	0.8190	0.8174	96
97	Latvia	0.8187	0.8171	0.8155	0.8139	0.8123	97
98	Gambia	0.8175	0.8160	0.8144	0.8128	0.8112	98
99	Kazakhstan	0.8165	0.8149	0.8133	0.8117	0.8101	99
100	VietNam	0.8162	0.8146	0.8130	0.8114	0.8098	100
101	Samoa	0.8037	0.8020	0.8003	0.7986	0.7969	101
102	Fiji	0.8017	0.8000	0.7983	0.7966	0.7949	102
103	Maldives	0.7976	0.7959	0.7941	0.7924	0.7906	103
104	Bolivia	0.7799	0.7781	0.7762	0.7743	0.7725	104
105	Tajikistan	0.7729	0.7710	0.7691	0.7672	0.7652	105
106	Sudan	0.7703	0.7684	0.7665	0.7645	0.7626	106
107	Benin	0.7689	0.7670	0.7651	0.7631	0.7611	107
108	Comoros	0.7612	0.7593	0.7573	0.7553	0.7532	108
109	Bahamas	0.7606	0.7586	0.7566	0.7546	0.7526	109
110	Senegal	0.7487	0.7466	0.7445	0.7424	0.7403	110
111	Turkmenistan	0.7468	0.7447	0.7426	0.7405	0.7384	111
112	CotedIvoire	0.7461	0.7440	0.7419	0.7398	0.7377	112
113	Haiti	0.7328	0.7307	0.7285	0.7263	0.7241	113
114	Ghana	0.6935	0.6911	0.6887	0.6862	0.6838	114

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115	Mali	0.6850	0.6825	0.6800	0.6775	0.6750	115
116	Mauritania	0.6812	0.6787	0.6762	0.6736	0.6711	116
117	Mozambique	0.6768	0.6743	0.6717	0.6692	0.6666	117
118	Cameroon	0.6725	0.6699	0.6674	0.6648	0.6622	118
119	GuineaBissau	0.6648	0.6622	0.6596	0.6569	0.6543	119
120	Togo	0.6634	0.6608	0.6582	0.6556	0.6529	120
121	Congo	0.6573	0.6546	0.6520	0.6493	0.6467	121
122	BurkinaFaso	0.6540	0.6514	0.6487	0.6460	0.6433	122
123	EquatorialGuine	0.6531	0.6505	0.6478	0.6451	0.6424	123
124	Ethiopia	0.6439	0.6412	0.6384	0.6357	0.6330	124
125	CentralAfrican	0.6305	0.6278	0.6250	0.6222	0.6194	125
126	Burundi	0.6293	0.6265	0.6237	0.6209	0.6181	126
127	Kenya	0.6277	0.6249	0.6221	0.6192	0.6164	127
128	Niger	0.6067	0.6038	0.6009	0.5980	0.5950	128
129	Uganda	0.6065	0.6036	0.6007	0.5978	0.5949	129
130	UnitedRepublic	0.6015	0.5986	0.5957	0.5928	0.5898	130
131	Nigeria	0.5978	0.5949	0.5919	0.5889	0.5860	131
132	SouthAfrica	0.5859	0.5829	0.5799	0.5769	0.5739	132
133	Lesotho	0.5757	0.5726	0.5696	0.5665	0.5634	133
134	Swaziland	0.5526	0.5495	0.5463	0.5432	0.5400	134
135	Rwanda	0.5445	0.5413	0.5382	0.5350	0.5318	135
136	Namibia	0.5245	0.5213	0.5180	0.5148	0.5115	136
137	Zimbabwe	0.5194	0.5161	0.5129	0.5096	0.5063	137
138	Botswana	0.5068	0.5035	0.5002	0.4969	0.4936	138
139	Malawi	0.4969	0.4936	0.4903	0.4870	0.4836	139
140	Zambia	0.4723	0.4689	0.4655	0.4621	0.4587	140

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