



UNIVERSITA' DEGLI STUDI DI PADOVA

**DIPARTIMENTO DI SCIENZE ECONOMICHE ED AZIENDALI
"M. FANNO"**

DIPARTIMENTO DI MATEMATICA

CORSO DI LAUREA IN ECONOMIA E MANAGEMENT

PROVA FINALE

"Vertical Cooperative Advertising Models: A New Game Theoretic Approach"

RELATORE:

CH.MO PROF. BRUNO VISCOLANI

LAUREANDO: **FRANCESCO DI FONZO**

MATRICOLA N. **1065154**

ANNO ACCADEMICO 2015 – 2016

*Ai miei genitori. A mia sorella. Alla mia famiglia:
a chi è ancora qui e a chi purtroppo non c'è più.
Alla mia ragazza e ai miei amici.
A chi mi ha sostenuto e sopportato, giorno dopo giorno.*

Table of contents

SOMMARIO	3
ABSTRACT.....	3
1. INTRODUCTION.....	5
2. THEORETICAL FRAMEWORK	9
2.1. NASH EQUILIBRIUM.....	10
2.2. STACKELBERG EQUILIBRIUM	13
2.3. BERTRAND EQUILIBRIUM.....	15
2.4. COOPERATION	17
2.5. BARGAINING THEORY	18
2.5.1. <i>Symmetric Nash Bargaining Model.....</i>	<i>19</i>
2.5.2. <i>Asymmetric Nash Bargaining Model.....</i>	<i>20</i>
3. LITERATURE REVIEW.....	23
3.1. XIE AND NEYRET (2009)	23
3.1.1. <i>Model.....</i>	<i>23</i>
3.1.2. <i>Nash Game.....</i>	<i>27</i>
3.1.3. <i>Stackelberg Retailer and Manufacturer Game.....</i>	<i>28</i>
3.1.4. <i>Cooperation</i>	<i>31</i>
3.1.5. <i>Bargaining Model.....</i>	<i>34</i>
3.1.6. <i>Conclusions.....</i>	<i>34</i>
3.2. SEYEDSFAHANI, ET AL. (2011)	35
3.2.1. <i>Model.....</i>	<i>35</i>
3.2.2. <i>Nash Game.....</i>	<i>38</i>
3.2.3. <i>Stackelberg Retailer and Manufacturer Game.....</i>	<i>40</i>
3.2.4. <i>Cooperation</i>	<i>41</i>
3.2.5. <i>Bargaining Model.....</i>	<i>45</i>
3.2.6. <i>Conclusions.....</i>	<i>45</i>
3.3. AUST AND BUSER (2012)	48

3.3.1.	<i>Model</i>	48
3.3.2.	<i>Nash Game</i>	49
3.3.3.	<i>Stackelberg Retailer and Manufacturer Game</i>	50
3.3.4.	<i>Cooperation</i>	53
3.3.5.	<i>Bargaining Model</i>	55
3.3.6.	<i>A New Feasibility Condition</i>	57
3.3.7.	<i>Conclusions</i>	59
4.	CO-OP ADVERTISING IN SUPPLY CHAIN: A NEW MODEL	63
4.1.	MODEL.....	63
4.2.	NASH GAME.....	67
4.3.	STACKELBERG RETAILER AND MANUFACTURER GAME.....	70
5.	CONCLUSIONS	79
	BIBLIOGRAPHY	83
	APPENDIX: MATLAB SCRIPTS USED IN CHAPTER 4	87

Sommario

La cooperazione pubblicitaria verticale è una strategia di marketing secondo cui il produttore finanzia i dettaglianti, sobbarcandosi una porzione dei costi necessari per l'implementazione di campagne pubblicitarie locali, per incrementare le vendite dei suoi prodotti nei punti vendita.

Dopo aver introdotto l'argomento e presentato i concetti fondamentali di teoria dei giochi (Capitoli 1 e 2), nel Capitolo 3 approfondiamo criticamente tre lavori scientifici che propongono modelli diversi di cooperazione pubblicitaria verticale all'interno di una catena di distribuzione composta da un produttore e un dettagliante. In tutte le pubblicazioni si applica come strumento di indagine la teoria dei giochi, con i due giocatori -il produttore e il dettagliante- che fissano i prezzi e la somma da investire in pubblicità in quattro giochi diversi, di cui tre non cooperativi (Nash e Stackelberg -sia con il produttore che con il dettagliante *leader*-) e uno cooperativo.

Infine, nel Capitolo 4 proponiamo una nuova catena di distribuzione, composta da due dettaglianti e un produttore, presentando e analizzando gli equilibri di Nash e di Stackelberg ottenuti nel nuovo modello.

Abstract

Vertical cooperative advertising is a marketing strategy according to which manufacturers, in order to boost their sales, pay for a portion of retailer advertising costs to give an incentive to local advertising level.

Having presented the topic and introduced the basilar concepts of game theory (Chapter 1 and 2), in Chapter 3 we critically review three papers that propose different models to describe price along advertising decisions in a one-retailer-one-manufacturer supply chain by a game theoretic approach, which consists of four different games: Nash, Stackelberg Manufacturer, Stackelberg Retailer, and Cooperative game.

Lastly, in Chapter 4 we propose a two-retailers-one-manufacturer model, analyzing its Nash and Stackelberg equilibria.

1. Introduction

Nowadays, the global economy is characterized by the harshest competition ever. Technological advances have been one of the most crucial factors enhancing the competitiveness of the market. The creation of new distribution channels, such as the e-commerce, has dramatically increased the supply capacity of firms from all over the World, which are now able to enter and compete in new markets without huge investments. Thus, many Small and Medium Enterprises (SMEs) have decided to focus on their core business to compete efficiently. However, to rationalize firms' operations, more coordination with other channel parties is needed, to manage activities across all the levels of economic chain without wasting resources.

Competition interests not only manufacturers -implications of cheap work force have been already studied in depth-, but also retailers: today, Italian shirt retailers have to compete with German and Chinese ones, who could exploit their pronged network to expand through new markets.

Competition, pushing down good prices, has improved the overall welfare of customers, who have also started demanding for more multifaceted services and commodities. In this situation, factors as strategic positioning and operating efficiency have gained importance, since they represent fundamental chances to obtain competitive advantages.

A flourished literature about channel coordination already exists (Aust and Busher, 2013). However, some further researches regarding one particular aspect of the coordination, the cooperative advertising strategy, should be carried out.

Advertising constitutes an important part of many firms' marketing strategy. As shown in Fig. 1, in 2015 World's advertising expenditure was \$531 billion, and experts forecast a 4.6% increase for 2016 (Nielsen¹, 2015). Italian firms have spent \$6.3 billion on advertising last year (1.7% more than in 2014 if we consider digital advertisement expenditures too), recording the first increase since 2008, when the expenditure reached a peak at \$8.9 billion, and Italian biggest advertisers are Procter & Gamble (\$118 million), Volkswagen (\$101 million), and Barilla (\$91.8 million).

¹ Nielsen is a US global information and measurement company, which conducts market researches annually.

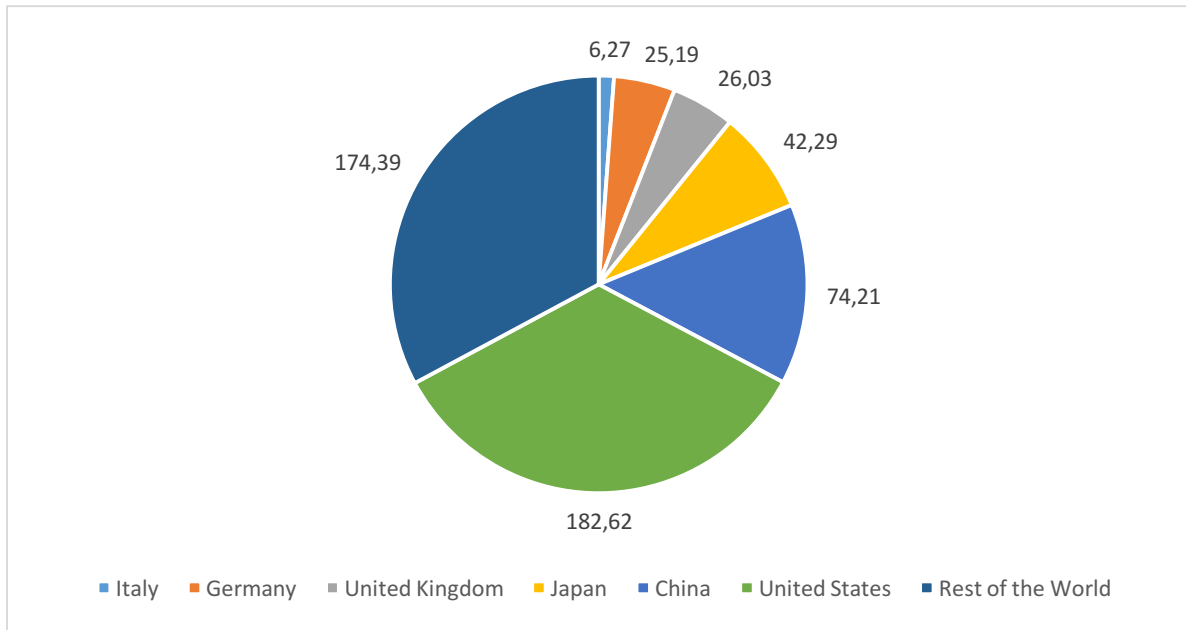


Fig. 1. Advertising expenditure in 2015 (data in billion)

Manufacturer's and retailer's advertising campaigns present several differences and have complementary effects. On the one hand, manufacturers' advertisement aims at influencing potential customers and raising brand awareness. It is long-term focused and has a national dimension, since manufacturers are more interested on firm's total sale than on results of a specific Point Of Sale (POS). Hence, this type of campaign wants to create and boost a specific brand image, influencing sales for several periods, but with an effect decreasing over time. On the other hand, retailers' advertisement tries to attract clients to particular POSs, offering discounts and promotion. It is often regional, short-term oriented, and focus on price as a key variable for product description.

Clearly, the two types of advertising offer balanced advantages and both parties could benefit from a strategic coordination. In particular, cooperation would be highly beneficial for the manufacturers, since their advertising campaigns enhance the brand image but, to grow as a firm, they need to increase their volume of sales. Thus, if the level of retailers' advertisements is not sufficient to satisfy the manufacturers' demand of local advertising, financial contribution of manufacturers to support retailers' advertising will be profitable, since manufacturers' additional revenues will exceed the higher costs implied. Another reason that can induce manufacturers to implement a cooperative advertising program is the competition of shelf spaces (Aust and Busher, 2013). According to this point, manufacturers can offer financial support for local advertising, in exchange for good space in retailers' POSs.

Anyhow, which is the most profitable manufacturers' cooperative advertising participation rate is still unknown. According to some studies about the US advertising market (Nagler, 2006), most of the manufacturers set their participation rate at 50% or 100%². This great difference and the absence of a clear theoretical background have suggested many researchers to analyse this topic, in order to find mathematical models able to identify the most efficient cooperative advertising strategy according to some specific variables (usually pricing and level/cost of advertising³).

Five different kinds of cooperative advertising have been considered in literature (Aust and Busher, 2013). Firstly, *Vertical Cooperative Advertising*, which is defined as a financial agreement where manufacturers offer to share a certain percentage of their retailers' advertising expenditures (Bergen and John, 1997). This type of cooperation is the most common in literature, and it is the one analysed in this work. Secondly, *Cooperative Advertising in Franchising*⁴, in which franchisors charge uniformly the cost of advertising campaigns between franchisees. Other authors consider the cooperation between competitors in the same supply chain (*Horizontal Cooperative Advertising*⁵). In this case, the aim of collaboration is to promote a general category of product and not, as usual, a specific brand. Another field of analysis is *Cooperative and Predatory Advertising*, where authors take into account the competitors' predatory advertising strategies and the positive influences of cooperation on the parties' demand⁶. Lastly, part of the literature is interested in *Joint Advertising Decisions*⁷, according to which cooperative advertising and collusive decision coincide, and parties can cooperate through either contracts or incentive strategies.

² Nagler (2006) collected data of 2,286 US firms, noting that 95% of them fixed a participation rate of 50% or 100%.

³ For more information regarding the consequences of considering either the cost or the level of advertising as a variable, see Aust and Busher (2013).

⁴ Related studies are Bhattacharyya and Lafontaine (1995), Sen (1995), Dant and Berger (1996), Desai (1997), Rao and Srinivasan (2001), Micheal (2002), Hempelmann (2006), and Sigué and Chintagunta (2009).

⁵ Related studies are Kinnucan (1997), Miles, White, and Munilla (1997), Krishnamurthy (2001 and 2000), Alston, Freebairn, and James (2001), Depken, Kamerschen, and Snow (2002), Bass, Krishnamoorthy, Chakravarti and Janiszewski (2004), Prasad, and Sethi (2005), Crespi and James (2007), and Lu, Thompson, and Tu (2007).

⁶ Related studies are Amrouche, Martin-Hérran, and Zaccour (2008), Depken and Snow (2008), Karray and Martin-Hérran (2008 and 2009), Erickson (2009), Viscolani and Zaccour (2009), Ma and Ulph (2012), and Viscolani (2012).

⁷ Related studies are El Ouardighi, Jorgensen, and Zaccour (2003), Buratto and Zaccour (2009), Simbanegavi (2009), and Karray (2011).

The aim of this work is to review critically three research contributions regarding mathematical modelling of *Vertical Cooperative Advertising* -from now on, we refer to this simply as cooperative advertising-, suggesting some hints and changes to improve the proposed models. We will apply a game-theoretic approach, whose basic concepts, required to understand the analysis, will be introduced in the next section.

2. Theoretical Framework

Myerson (Aust, 2014, p.1) defines game theory as “*the study of mathematical models of conflict and cooperation between intelligent rational decision-makers (...), which provides general mathematical techniques for analysing situations in which two or more individuals make decisions that will influence one another’s welfare*”. Therefore, a game theoretic approach, explicitly considering the interdependence between parties, is an appropriate tool to study cooperative advertising, allowing a characterization of players’ behaviours and bargaining power during decision-making process. In line with previous papers, we will present four different game-theoretic models based on one cooperative and three non-cooperative games (Nash, Stackelberg Manufacturer, and Stackelberg Retailer).

A game is defined as a “*situation in which more than one individual has to make a decision and these decisions are interdependent*” (Aust, 2014, p.12). It has three basic elements:

- Set of players;
- Set of actions or possible strategies for each player;
- Expected utility function over each possible action profile for each player.

A widely used notation indicates each player as p and the set of players as $N = \{1, \dots, n\}$, with $p \in N$. Players have to choose a strategy $s_p \in S_p$, where S_p represents the strategy set for every $p \in N$. Through a Cartesian product, we obtain all the possible strategy combinations of the game: $S = S_1 \times S_2 \times \dots \times S_n$. With $s \in S$ we indicate a single combination of strategies. Furthermore, it is required a utility function to assign players a specific utility, depending on their strategies. We denote this function by $u_p(s)$ and define $U = \{u_1(s), \dots, u_n(s)\}$. The utility levels obtained by players in each situation are called pay-offs. Defined these three aspects, we can refer to a specific game as $G = (N, S, U)$.

Chronology of decision-making process and time-period of games are other two crucial factors. Do the players play simultaneously? Or sequentially? Does the game last one period (static games) or more (dynamic games)? These characteristics are important to adopt the correct resolution method. Another interesting distinction among games is based on the players’ payoffs. There are both zero sum game -if the sum of the whole pay-offs achieved by players is 0-, and non-zero-sum game -if the sum is different-.

To analyse in depth the game solutions, we have to consider the information available to players. Games can be with complete information, in which all the knowledge is shared by the parties, and incomplete information (called Bayesian game), where players know game

conditions only partially. Cooperative advertising literature (and this work too) has used only complete information games to study cooperative advertising programs.

Lastly, it is important to notice that every game can be represented in different ways, according to its features. For instance, a static game with a finite number of strategies is usually summarized with its normal form, while a sequential game with its extensive form -a tree diagram-. Anyway, since the games studied in this work are static and have an infinite number of strategies, we will study them algebraically.

2.1. Nash Equilibrium

The concept of Nash Equilibrium (NE) was introduced by John Forbes Nash Jr. (1950), and it is used to solve a game in which power is equally distributed between players and all the decisions are taken simultaneously. Before applying this solution concept, it is useful to simplify the game. In fact, postulating complete information and rationality of players' behaviours and expectations, they will never use a Strictly Dominated Strategy (SDS) because such behaviours are inconsistent with their utility maximization. SDSs are, in fact, strategies that give to players a lower utility regardless of decisions taken by other players.

To be clearer, consider a game in which there are two players A and B , whose strategies are c and d and pay-offs are described in the normal form represented by Tab.1⁸. Formally, $N = \{A, B\}$ and $S_A = S_B = \{c, d\}$. In this case, strategy d is dominant for both player, as they always maximize their utilities playing d instead of c . Hence, we can simplify the game eliminating the row and column of strategy c , and the Nash Equilibrium of the game will be $NE = \{(d, d)\}$ ⁹.

⁸ By convention, the first number of the matrix refers to the utility of player A , whilst the second to the utility of player B . All different scenarios are obtained through the intersection of strategies on rows and columns.

⁹ By convention, with the first strategy we mean what player A will play and with the second what B will.

		Player B	
		c	d
Player A	c	-1; 5	3;6
	d	3; 4	4;5

Tab.1. Normal form of a game with two strictly dominated strategies

As seen above, the elimination of all the SDSs makes easier finding the Nash Equilibrium. But what is a Nash Equilibrium? A Nash Equilibrium is a strategy profile¹⁰ for which no player has incentive to deviate if he/she thinks that other players will play according to the equilibrium. In other words, a Nash equilibrium is a vector of strategies in which no player has an incentive to change unilaterally his/her strategy since he/she is already playing his/her best strategy given what the other players are doing. To apply this solution concept we have to assume complete information.

Formally, in a game with N players, $p \in N$, and $s_p \in S_p$, we use s_{-p} , with $s_{-p} \in S_{-p}$, to indicate the strategy profiles of other players. The function that maximizes the utility of player p given what is played by $-p$ is called Best Response function (BR), and it is denoted by $BR_p(s_{-p})$. If we define $s_p^* = BR_p(s_{-p}^*)$ and $s_{-p}^* = BR_{-p}(s_p^*)$, then the NE will be the combination of strategies $s^* = (s_p^*, s_{-p}^*)$.

A brief example shall illustrate how to use the NE concept in economic applications. Consider two firms ($N = \{1,2\}$), which produce quantities q_1 and q_2 respectively. Let the inverse demand function be

$$P(Q) = a - Q, \quad (2.1)$$

where a is a constant and $Q = q_1 + q_2$. The cost function for both firms is:

$$C_p(q_p) = c_p q_p \quad \text{with } p \in \{1,2\}. \quad (2.2)$$

¹⁰ If we consider mixed strategies, the number of Nash equilibria can be infinite (Tadelis, 2013).

Hence, considering identical constant marginal cost $c_1 = c_2 = c$, the players' payoff function (profits) will be:

$$u_p(q_1, q_2) = q_p(a - (q_1 + q_2) - c). \quad (2.3)$$

Now, to find a NE, we have to determine a pair of quantities (q_1^*, q_2^*) such that q_1^* is player 1's best reply against q_2^* , and vice versa¹¹. This means that q_1^* solves the following maximization problem:

$$\text{Max}_{q_1} u_1(q_1, q_2^*) = q_1(a - (q_1 + q_2^*) - c) \quad (2.4)$$

$$\text{s.t. } 0 \leq q_1 < +\infty.$$

In the same way, q_2^* is the solution of

$$\text{Max}_{q_2} u_2(q_1^*, q_2) = q_2(a - (q_1^* + q_2) - c) \quad (2.5)$$

$$\text{s.t. } 0 \leq q_2 < +\infty.$$

We obtain the player 1's *BR* applying the First Order Condition (FOC) at (2.3), that is:

$$\frac{\partial u_1(q_1, q_2^*)}{\partial q_1} = a - 2q_1 - q_2 - c = 0. \quad (2.6)$$

The same for player 2:

¹¹ This kind of game is known as *Cournot Duopoly*, outlined by Antoine Augustin Cournot in 1838.

$$\frac{\partial u_2(q_1^*, q_2)}{\partial q_2} = a - 2q_2 - q_1 - c = 0. \quad (2.7)$$

We can easily check the Second Order Condition (SOC), noting that both (2.6) and (2.7) present a negative second derivative. Therefore, we obtain the equilibrium quantities through the following system:

$$\begin{cases} q_1^* = \frac{a - q_2^* - c}{2} \\ q_2^* = \frac{a - q_1^* - c}{2} \end{cases} \quad (2.8)$$

with $q_p^* \leq a - c$, in order to respect the conditions of the problems (2.4) and (2.5). Thus, solving (2.8) we get $q_i^* = \frac{a-c}{3}$ and, therefore, the Nash Equilibrium will be: $NE = \left\{ q_1^* = \frac{a-c}{3}, q_2^* = \frac{a-c}{3} \right\}$.

2.2. Stackelberg Equilibrium

The Stackelberg equilibrium concept (established by von Stackelberg in 1934) can be applied in sequential games¹², and not in simultaneous as the NE. It allows us to incorporate a hierarchical structure in the game, modelling channel leadership (of retailer or manufacturer) in the supply chain relationships. The player who moves first is called leader, the other follower, and, as to find the Nash Equilibrium, we have to assume complete information. In game theory, this solution concept is defined as Sub-game Perfect Nash Equilibrium (SPNE), and it is a subset of the NE: all the SPNE are also NE, but the opposite is not true¹³.

¹² The model was thought to study and analyse oligopolistic situation, in which players compete in quantities. For more information, see *Marktform und Gleichgewicht* (von Stackelberg, 1934).

¹³ For more information, see Tadelis (2013).

Mathematically, to compute the set of SPNE, we have to proceed with backward induction: firstly, we have to find the best response function of the follower and, given the strategy chosen by him/her, we maximize the utility of the leader.

To exemplify, let us consider the same situation described in the previous section: there are two firms that have to choose quantities q_1 and q_2 . Their demand function is (2.1) and the utility function is (2.3). However, now players do not play simultaneously. Player 1 is the leader and player 2 the follower. Since it is a complete information game, player 1 can anticipate exactly what player 2 will play. Hence, proceeding by backward induction, we start maximizing for player 2. The maximization problem is the same as described in (2.5), and the solution will be obtained solving (2.7). Now, player 1 maximizes his/her profits considering player 2's *BR*. Thus, substituting $q_2 = \frac{a - q_1 - c}{2}$ in (2.4), the maximization problem for player 1 becomes:

$$\text{Max}_{q_1} u_1 \left(q_1, \frac{a - q_1 - c}{2} \right) = q_1 \left(a - q_1 - \frac{a - q_1 - c}{2} - c \right) \quad (2.9)$$

$$\text{s.t. } 0 \leq q_1 < +\infty.$$

Again, firstly we apply the FOC at (2.9) and, having checked the negativity of the second derivative, we solve the final system, finding the following solution:

$$q_1^* = \frac{a - c}{2}, \quad \text{with } a - c > 0 \quad (2.10)$$

$$q_2^* = \frac{a - c}{4}, \quad (2.11)$$

The Sub-game Perfect Nash Equilibrium will be $\text{SPNE} = \left\{ \left(\frac{a - c}{2}, \frac{a - c - q_1}{2} \right) \right\}$, with an equilibrium path -that is what players effectively produce- of $q_1 = \frac{a - c}{2}$ and $q_2 = \frac{a - c}{4}$.

2.3. Bertrand Equilibrium

The Bertrand equilibrium is completely different from the Stackelberg and Cournot ones, since we can apply these latter if and only if players' utility function is continuous, whilst Bertrand solution concept is used in case of discontinuous utility functions.

To exemplify, consider the same situation described in 2.1. and assume simultaneous price competition. Expression (2.1) indicates the inverse demand function. Here, we need the direct one. Hence, after few simple manipulations, we obtain:

$$Q(p) = a - p. \quad (2.12)$$

Having supposed homogenous goods, price sensitive consumers, and no capacity restrictions (both firms can produce any quantity at marginal cost), the two firms set a specific price. Consumers buy goods from the firm offering the lower price and, in the case of identical prices, we impose that consumers will equally split their purchases between the firms.

Therefore, calling p_1 and p_2 the prices set by the two firms, the quantity sold by firm i ¹⁴ will be:

$$q_i = \begin{cases} 0 & p_i > p_{-i} \\ \frac{Q(p)}{2} & p_i = p_{-i} \\ Q(p) & p_i < p_{-i} \end{cases} \quad (2.13)$$

with $i \in \{1,2\}$.

Denoting with c the marginal costs of the firms, the payoff functions will be:

¹⁴ In this section we denote the general firm with i , since we use the notation p to indicate the pricing variable.

$$u_i(p_i, p_{-i}) = \begin{cases} 0 & p_i > p_{-i} \\ \frac{1}{2}(p_i - c)(a - p_i) & p_i = p_{-i} \\ (p_i - c)(a - p_i) & p_i < p_{-i} \end{cases} \quad (2.14)$$

with $i \in \{1,2\}$.

Now, we can find the Bertrand Equilibrium. However, the process we have to apply is different from those described in the previous two sections. The utility function (2.14) is, in fact, discontinuous, and we cannot derive it; yet we can solve the problem finding the players' BRs. First of all, to simplify the resolution process, we assume that, if indifferent, player i will set his/her price at $p_i = c$. With regard to player 1, he/she has an incentive to lower his/her price and to seize the market, but only if the price set by the competitor is higher than player 1's marginal cost c . On the contrary, if competitors set a price equal to or lower than player 1's marginal cost, player 1's BR will be $p_i = c$. Formally¹⁵:

$$p_i^*(p_{-i}) = \begin{cases} c & p_{-i} \leq c \\ p_{-i} - \varepsilon & p_{-i} > c \end{cases}, \quad (2.15)$$

with $i \in \{1,2\}$.

Intersecting equations (2.15) -calculated for both the players-, we are able to derive the unique equilibrium of this Bertrand game¹⁶: $NE = \{p_1^* = c, p_2^* = c\}$.

It is interesting to note that, in the example above where players present identical marginal costs, equilibrium profits are 0 and retail price is equal to their marginal cost. This is called "*Bertrand's paradox*" since, through a price competition between two firms with identical marginal costs, we find the same equilibrium that we would have obtained with a perfect competition model, where there are an infinite number of competitors.

¹⁵ With the parameter ε we indicate an infinitesimal positive value.

¹⁶ Formally, the "*Bertrand Equilibrium*" is a Nash equilibrium. However, in this work we improperly refer to it as "*Bertrand Equilibrium*", but only to distinguish it from the equilibrium concept presented in section 2.1..

Nonetheless, this is not true for all the Bertrand games. The same process can be applied, with different results, to solve situations in which players present different cost function, as we will see in chapter 4.

2.4. Cooperation

The last solution concept we introduce is the cooperation one. Here, we consider all players as they were only one. They do not maximize their own utility singularly, but together. In the literature, inefficiency created by uncoordinated decisions has been already studied and, in this kind of game, we want to maximize the overall utility of the system. The profits obtained will be greater than (or at least equal to) those derived with NE or SPNE solution concepts. Algebraically, we have to sum all the utility functions of the players and maximize the function obtained.

Starting from the same situation described in the section 2.1., demand function will be (2.1), while the cost function (assuming that firms have the same marginal cost) will be:

$$C(Q) = cQ, \tag{2.16}$$

which is obtained summing the firms' cost functions (2.2). The maximization problem will be:

$$\text{Max}_Q u_1(Q) = Q(a - Q - c) \tag{2.17}$$

$$\text{s.t } 0 \leq Q < +\infty,$$

and the solution, calculated through the FOC, is:

$$Q = \frac{a-c}{2} \quad \text{with } a - c > 0. \tag{2.18}$$

The effective agreement about the levels of production and the amount of profits received by players is described in the following section, while Tab.2 represents a summary of the optimal production levels obtained in the solution concepts presented.

Tab.2. Optimal production in each game scenario.

Demand function	$P(Q) = a - Q = a - q_1 - q_2$
Cost function	$C(Q) = cQ$
Cournot duopoly production	$q_p = \frac{a - c}{3}$
Stackelberg duopoly production¹⁷	$q_1 = \frac{a - c}{2}$ $q_2 = \frac{a - c}{4}$
Cooperation production	$Q = \frac{a - c}{2}$

2.5. Bargaining Theory

Lastly, to fully comprehend the models that will be presented in chapter 3, it is necessary to introduce some concepts about the bargaining theory¹⁸. Bargaining models are commonly used to find a split of total profits gained in a cooperative game between two or more players. The results of the division depend on both the utility function of the players and the selected bargaining model. The most common model, widely used in the literature, is the Nash Bargaining one, which can be symmetrical or asymmetrical. In the former case, it considers only one parameter: the functional form of players' utility. In the latter, the parameters are two: the functional form and the bargaining power.

Formally, each player p has to fix his share y_p of total profits Y , with $0 \leq y_p \leq Y$ and $\sum_p y_p = Y$. Now, indicating with $v_p = u_p(y_p)$ each player's utility according to a specific split, we are able to present two different Nash Bargaining models: the symmetric and the asymmetric.

¹⁷ Where firm 1 is leader and firm 2 is follower.

¹⁸ For more information regarding to bargaining theory and its mathematical model, see Aust (2014).

2.5.1. Symmetric Nash Bargaining Model

Introduced by Nash in 1950, this model proposes a bargaining solution that maximizes the total utility v_N , where $v_N = \prod_p v_p$, $v_p = u_p(y_p)$, and y_p represents the share obtained by player p . Consider a situation where two players ($N = \{1,2\}$) bargain their shares y_1 and y_2 of a total amount Y of money. The maximization problem is given by:

$$\begin{aligned} \text{Max}_{y_1, y_2} v_{1+2} &= v_1 v_2 = u_1(y_1) u_2(y_2) & (2.19) \\ \text{s.t. } y_1 + y_2 &= Y, & 0 \leq y_1 \leq Y, & 0 \leq y_2 \leq Y. \end{aligned}$$

The utility function of players will be the following:

$$u_p(y_p) = y_p^{\lambda_p} \quad \text{with } p \in \{1, 2\}, \quad (2.20)$$

where λ_p depends on functional form of players' utilities: $\lambda_p > 1$ indicates convex function, $\lambda_p = 1$ linear, and $\lambda_p < 1$ concave. Replacing (2.20) in (2.19), the problem becomes:

$$\begin{aligned} \text{Max}_{y_1, y_2} v_{1+2} &= y_1^{\lambda_1} y_2^{\lambda_2} & (2.21) \\ \text{s.t. } y_1 + y_2 &= Y, & 0 \leq y_1 \leq Y, & 0 \leq y_2 \leq Y. \end{aligned}$$

Considering that $y_1 + y_2 = Y \Rightarrow y_2 = Y - y_1$, we can rewrite the problem (2.21) in a simpler univariable form:

$$\begin{aligned} \text{Max}_{y_1} v_{1+2} &= y_1^{\lambda_1} (Y - y_1)^{\lambda_2} & (2.22) \\ \text{s.t. } 0 &\leq y_1 \leq Y. \end{aligned}$$

Having set the FOCs of (2.22) and checked the SOC, the solution will be:

$$y_p = \frac{\lambda_p}{\lambda_1 + \lambda_2} Y \quad \text{with } p \in \{1,2\}. \quad (2.23)$$

2.5.2. Asymmetric Nash Bargaining Model

Harsanyi and Selten (1972) proposed a modification of the classic Nash Bargaining model, by integrating it with a new parameter, the bargaining power, denoted by μ_p , with $\sum_p \mu_p = 1$. The utility function proposed is the same of the previous model ($u_p(y_p) = y_p^{\lambda_p}$), but v_N , the overall utility of players, becomes $v_N = \prod_p v_p^{\mu_p}$. Adapting the situation described in (2.19), the maximization problem will be the following:

$$\begin{aligned} \text{Max}_{y_1, y_2} v_{1+2} &= y_1^{\mu_1 \lambda_1} y_2^{\mu_2 \lambda_2} & (2.24) \\ \text{s.t. } y_1 + y_2 &= Y, & 0 \leq y_1 \leq Y, & 0 \leq y_2 \leq Y. \end{aligned}$$

Again, since $y_1 + y_2 = Y \Rightarrow y_2 = Y - y_1$, problem (2.24) becomes:

$$\begin{aligned} \text{Max}_{y_1} v_{1+2} &= y_1^{\mu_1 \lambda_1} (Y - y_1)^{\mu_2 \lambda_2} & (2.25) \\ \text{s.t. } 0 &\leq y_1 \leq Y, \end{aligned}$$

and the solution of (2.25) leads to this share for both players:

$$y_p = \frac{\mu_p \lambda_p}{\mu_1 \lambda_1 + \mu_2 \lambda_2} Y \quad \text{with } p \in \{1,2\}. \quad (2.26)$$

Of course, this model is more complete and complex than the classical one, as it allows us to consider another players' feature -the parties' bargaining power- to determine the solution of the bargaining game.

3. Literature Review

Several papers have studied the co-op advertising strategy, according to which the manufacturer has some incentives to find a financial agreement with the retailer, supporting a certain fraction (participation rate) of the retailer's advertising expenditures in order to increase the overall sale volume. Yet three works in particular have dramatically contributed to the development of the co-op advertising mathematical static models. In this part of the work, we firstly analyse the model proposed by Xie and Neyret (2009). Then, we will present the model by SeyedEsfahani et al. (2011); and, lastly, the one by Aust and Busher (2012).

3.1. Xie and Neyret (2009)

Basing on two papers by Huang and Li (2001 and 2002), the authors want to develop a mathematical model that describes vertical co-op advertising. However, while Huang and Li considered only one variable -the advertising level-, Xie and Neyret deal with pricing as another decision variable, so as to describe static co-op advertising strategy. Other authors have already proposed models including the price variable, but always presupposing a dynamic environment. In Xie and Neyret (2009), the authors describe four game scenarios: (i) Nash game, (ii) Stackelberg Retailer game, (iii) Stackelberg Manufacturer game, and (iv) Cooperation game.

3.1.1. Model

The authors propose a single-manufacturer-single-retailer channel. Decision variables are the prices chosen by both the manufacturer -wholesale price p_M - and the retailer -market or retail price p_R -, the advertising levels, and the participation rate. Having called a and A the retailer's and the manufacturer's advertising investment, they consider the consumer demand function proposed by Jorgensen and Zaccour (1999):

$$V(a, A, p_R) = (\alpha - \beta p_R) \left(D - \frac{B}{a^\gamma A^\delta} \right), \quad (3.1)$$

where $\alpha, \beta, B, D, \gamma$, and δ are positive constants. It is important to note that, because demand function cannot be negative, $V(a, A, p_R)$ has to be positive. Hence $V(a, A, p_R) > 0 \Rightarrow p_R < \frac{\alpha}{\beta}$.

To simplify the algebra, the pricing variables have to be greater than 0 and less than 1.

The first part of the formula (25) - $(\alpha - \beta p_R)$ - is the demand-price curve, which decreases with p_R . The second - $(D - \frac{B}{a^\gamma A^\delta})$ - represents the effect of advertisements on sales, and it is the same as proposed by Huang and Li (2001). However, equation (3.1) does not take into consideration the *advertising saturation effect*: a diminishing marginal effect for increasing advertising expenditures (Aust, 2013). As we can see in the following two papers, a simple function that takes into consideration the *advertising saturation* can be obtained replacing a and A with their square roots¹⁹.

Another important variable is the participation rate t -fraction of the local advertising paid by the manufacturer-, whilst notable parameters are the manufacturer's unit production cost c (constant), and the retailer's unit cost d (constant). Tab.3 shows a brief summary of the model.

Profit functions are²⁰:

$$\Pi_M = (p_M - c)(\alpha - \beta p_R) \left(D - \frac{B}{a^\gamma A^\delta} \right) - ta - A, \quad (3.2)$$

$$\Pi_R = (p_R - p_M - d)(\alpha - \beta p_R) \left(D - \frac{B}{a^\gamma A^\delta} \right) - (1 - t)a, \quad (3.3)$$

$$\Pi_{M+R} = (p_R - c - d)(\alpha - \beta p_R) \left(D - \frac{B}{a^\gamma A^\delta} \right) - a - A. \quad (3.4)$$

Some conditions are required for the non-negativity of the manufacturer ($p_M > c$), retailer ($p_R > p_M + d$), and whole system ($p_R > c + d$) profits. Then, the authors, "*in order to handle*

¹⁹ The definition of the *saturation effect* given above is improper. In the following model, in fact, the marginal effect of an additional advertising unit is undoubtedly decreasing but, to have an actual saturation effect, the limit of the marginal effect, as advertising approaches infinity, has to be zero -and this is not the case-. However, according to the literature, we will refer to the decreasing effect of advertising as *advertising saturation effect*.

²⁰ Throughout this work, with subscripts M , R , and $M + R$ we mean the function regarding manufacturer, retailer, and both players together, respectively.

the problem in an equivalent but more convenient way” (Xie and Neyret, 2008, p.1378), apply some changes of variables²¹:

$$a' = \alpha - \beta(c + d),$$

$$p'_R = \frac{\beta}{a'}(p_R - (c + d)),$$

$$p'_M = \frac{\beta}{a'}(p_M - c),$$

$$B' = \frac{\alpha''}{\beta}B,$$

$$D' = \frac{\alpha''}{\beta}D,$$

$$A' = \frac{A}{\beta'^{\frac{1}{\gamma+\delta+1}}}B,$$

$$a' = \frac{a}{\beta'^{\frac{1}{\gamma+\delta+1}}},$$

$$\Pi' = \frac{\Pi}{\beta'^{\frac{1}{\gamma+\delta+1}}},$$

obtaining the following profit expressions:

$$\Pi'_M = p'_M(1 - p'_R) \left(\frac{A'}{B'^{\frac{1}{\gamma+\delta+1}}} - \frac{1}{a'^{\gamma}A'^{\delta}} \right) - ta' - A', \quad (3.5)$$

$$\Pi'_R = (p'_R - p'_M)(1 - p'_R) \left(\frac{A'}{B'^{\frac{1}{\gamma+\delta+1}}} - \frac{1}{a'^{\gamma}A'^{\delta}} \right) - (1 - t)a', \quad (3.6)$$

$$\Pi'_{M+R} = p'_R(1 - p'_R) \left(\frac{A'}{B'^{\frac{1}{\gamma+\delta+1}}} - \frac{1}{a'^{\gamma}A'^{\delta}} \right) - a' - A'. \quad (3.7)$$

²¹ See the appendix of Xie and Neyret (2009) for details.

Even though these changes of variables make computations indeed simpler, applying them we lose some important information about key parameters. With the new functions, for instance, we do not have a direct feedback regarding the influence of manufacturer's and retailer's cost on the optimal co-op advertising level. The price variable effect is not clear too, since we do not know how much manufacturer and retailer mark-ups influence profits. As we shall see, the model proposed in Aust and Busher (2012) is an evolution of that presented in Xie and Neyret (2009), as it operates without any substitutions of parameters, achieving more interesting and broad results.

Tab.3. Legend of Xie and Neyret's model

p_R	Retail price
p_M	Wholesale price
t	Participation rate
a	Retailer local advertising level
A	Manufacturer national advertising level
α	Intercept of the demand function
c	Manufacturer's production cost
d	Retailer's unit cost
β	Marginal effect of price on the demand
B	Effect of the advertising level on the demand (inversely proportional)
D	Sales saturate asymptote
γ	Effect of local advertising on the demand
δ	Effect of global advertising on the demand

3.1.2. Nash Game

As explained in section 2.1., the Nash Equilibrium concept is applied in situation in which players have the same decision power and play simultaneously, not cooperatively, to maximize their own profits. Formally, the manufacturer's decision problem is²²:

$$\text{Max}_{t,A,p_M} \Pi_M = p_M(1 - p_R) \left(\frac{A}{\frac{1}{B^{\gamma+\delta+1}}} - \frac{1}{a^\gamma A^\delta} \right) - ta - A \quad (3.8)$$

$$\text{s.t. } 0 \leq t \leq 1, \quad A \geq 0, \quad 0 \leq p_M \leq 1.$$

While, the retailer's problem is:

$$\text{Max}_{a,p_R} \Pi_R = (p_R - p_M)(1 - p_R) \left(\frac{A}{\frac{1}{B^{\gamma+\delta+1}}} - \frac{1}{a^\gamma A^\delta} \right) - (1 - t)a \quad (3.9)$$

$$\text{s.t. } a \geq 0, \quad 0 \leq p_R \leq 1.$$

Looking at (3.8), the optimal participation rate t is indeed zero: t negatively affects the profit function of the manufacturer and he/she will fix it at the lowest possible value. This is rational, as the game is played in one period only (static) and players' strategies are chosen at the same time. Therefore, there are no incentives for the manufacturer to participate in the local advertising expenditure. Then, manufacturer's profits increase with p_M . However, the optimal p_M cannot be equal to 1 (the maximum possible value), because in this case profits for both the players will be 0. In fact, we have that $0 \leq p_M \leq 1$, $0 \leq p_R \leq 1$, and $p_R \geq p_M$: thus if $p_M = 1 \Rightarrow p_R = 1 \Rightarrow \Pi_R = \Pi_M = 0$.

At this point, authors consider another assumption before solving (3.8) and (3.9), that is:

$$p_R - p_M = p_M, \quad (3.10)$$

²² For simplicity, hereafter we remove the superscript (') from (3.5), (3.6), and (3.7).

hence $p_M = \frac{p_R}{2}$. This condition imposes equal margins for the retailer and manufacturer. It is a very restrictive assumption valid only for specific values of the parameters. Without imposing (3.10) the authors would have been able to derive more interesting solutions (see Aust and Busher, 2012, section 3.3.).

Anyhow, the unique Nash Equilibrium²³ will be:

$$p_M^N = \frac{1}{3}, \quad p_R^N = \frac{2}{3}, \quad t^N = 0, \quad (3.11)$$

$$a^N = \left[\frac{\left(\frac{\gamma}{\delta}\right)^\delta \gamma}{9} \right]^{\frac{1}{\gamma+\delta+1}}, \quad (3.12)$$

$$A^N = \frac{\delta}{\gamma} a^N. \quad (3.13)$$

3.1.3. Stackelberg Retailer and Manufacturer Game

The second and third situations studied aim at reproducing supply-chain scenarios in which one of the players has some advantages over the other parties. Interestingly, authors analyse two situations where either the manufacturer or the retailer is the leader (known as Manufacturer Stackelberg game and Retailer Stackelberg game, respectively).

The most prevalent Stackelberg game studied by co-op advertising literature is the Manufacturer leadership game. This is because scholars generally assume that this type of leadership is predominant, but it was true only until a few decades ago, when many manufacturers dominated retailers. Nowadays, the situation has changed, and even retail leadership relationships are common. Huang, Li, and Mahajan recognized this trend in 2002, citing as example Procter & Gamble (P&G) and Walmart: at the beginning, P&G

²³ Throughout this work, the superscript N indicates a Nash Equilibrium. For further information about algebra, see the appendix of Xie and Neyret (2009).

(manufacturer) dominated over Walmart (retailer), but then the relationship has evolved, and they have become partners.

Xie and Neyret (2009) start solving the Retailer leadership situation through backward induction. Firstly, they calculate the manufacturer best responses of t , A , and p_M . Again, as in the Nash game, they find that the optimal participation rate is $t = 0$, since the manufacturer plays second and does not have any incentive to fix different participation ratio. Equal margins are assumed too (3.10). Solving the maximization problem (3.8)²⁴, they obtain:

$$t = 0, \quad p_M = \frac{p_R}{2}, \quad A = \left[\frac{\delta a^{-\gamma} p_R (1 - p_R)}{2} \right]^{\frac{1}{\delta+1}}. \quad (3.14)$$

Replacing the manufacturer BRs in (3.9), we solve the maximization of retailer's profits:

$$\text{Max}_{a, p_R} \Pi_R = \frac{1}{2} p_R (1 - p_R) \left\{ \frac{A}{\frac{1}{B^{\gamma+\delta+1}}} - \frac{1}{a^{\frac{\gamma}{\delta+1} \delta^{\frac{\delta}{\delta+1}}} \left[\frac{1}{2} p_R (1 - p_R) \right]^{-\frac{\delta}{\delta+1}}} \right\} - a \quad (3.15)$$

$$\text{s.t. } a \geq 0, \quad 0 \leq p_R \leq 1.$$

Solutions²⁵ of (3.15) and the consequently simplifications of (3.14) allow us to calculate the Stackelberg Retailer Equilibrium:

$$p_M^{S_R} = \frac{1}{4}, \quad p_R^{S_R} = \frac{1}{2}, \quad t^{S_R} = 0, \quad (3.16)$$

$$a^{S_R} = \left[\frac{\left(\frac{\gamma}{\delta(\delta+1)} \right)^\delta \left(\frac{\gamma}{\delta+1} \right)}{8} \right]^{\frac{1}{\gamma+\delta+1}}, \quad (3.17)$$

$$A^{S_R} = \frac{\delta(\delta+1)}{\gamma} a^{S_R}. \quad (3.18)$$

²⁴ See the appendix of Xie and Neyret (2009) for details.

²⁵ Throughout this work, the superscript S_R indicates the Stackelberg Retailer Equilibrium.

With regard to the Stackleberg Manufacturer game, there are few differences with respect to the previous game. We start solving the maximization of the retailer profits to find his/her best responses:

$$p_R = \frac{1+p_M}{2}, \quad a = \left[\frac{\gamma(1-p_M)^2}{4(1-t)A^\delta} \right]^{\frac{1}{\gamma+1}}. \quad (3.19)$$

After that, taking into account expressions (3.19) and solving the maximization problem with regard to the manufacturer, they find the following value of t ²⁶:

$$t^{S_M} = \frac{(3 + \gamma)p_M - (\gamma + 1)}{(2 + \gamma)p_M - \gamma}. \quad (3.20)$$

Now, Xie and Neyret (2009) consider two situations. Since t cannot be negative, we have that either $p_M^{S_M} > \frac{1+\gamma}{3+\gamma} \Rightarrow t > 0$ or $p_M^{S_M} \leq \frac{1+\gamma}{3+\gamma} \Rightarrow t = 0$. However, because of the complexity of calculations, they do not solve analytically the problem, but through a numerical simulation. Using MATLAB, they derive the following diagrams²⁷ of the manufacturer price and participation ratio.

²⁶ Throughout this work, the superscript S_M indicates the Stackelberg Manufacturer Equilibrium.

²⁷ Source: *Co-op advertising and pricing models in manufacturer-retailer supply chains* by Xie and Neyret (2009).

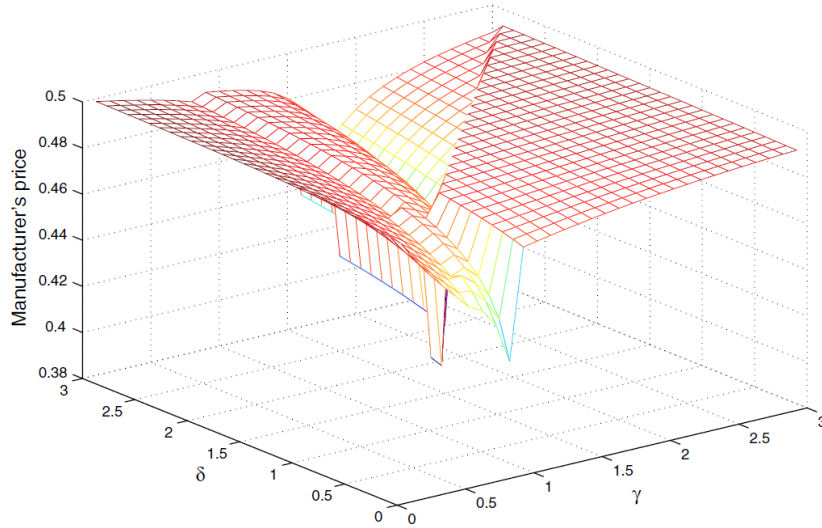


Fig.1. Manufacturer price p_M^{SM}

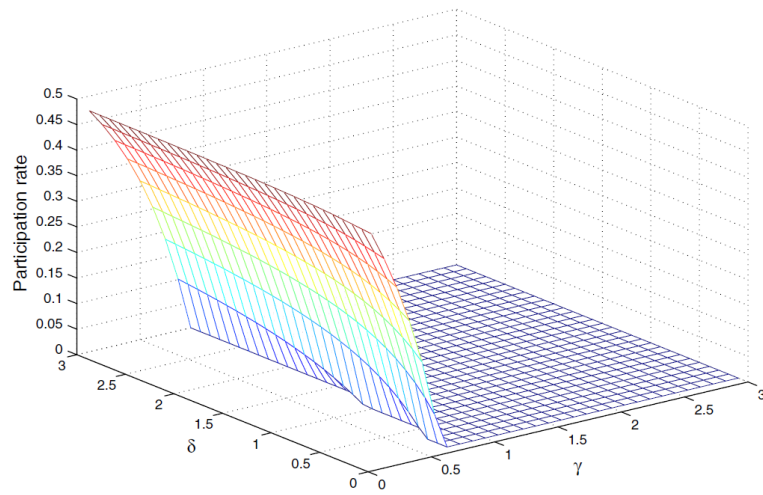


Fig.2. Participation ratio t^{SM}

3.1.4. Cooperation

The last game is the only cooperative among those studied in Xie and Neyret (2009). Bearing in mind the concepts introduced in section 2.4., the solution is obtained by this maximization problem:

$$\begin{aligned} \text{Max}_{p_R, a, A} \Pi_{M+R} &= p_R(1 - p_R) \left(\frac{A}{B^{\gamma+\delta+1}} - \frac{B}{a^\gamma A^\delta} \right) - a - A \\ \text{s.t. } a &\geq 0, \quad A \geq 0, \quad 0 \leq p_R \leq 1. \end{aligned} \quad (3.21)$$

The results²⁸ will be:

$$p_R^{co} = \frac{1}{2}, \quad A^{co} = \frac{\delta}{\gamma} a^{co}, \quad a^{co} = \left[\frac{\left(\frac{\gamma}{\delta}\right)^\delta \gamma}{4} \right]^{\frac{1}{\gamma+\delta+1}}. \quad (3.22)$$

Of course, players will accept to cooperate only if they obtain at least the same amount of profits received without cooperation. To find analytically the cases in which cooperation is feasible, they solve the following equation through a numerical simulation with MATLAB, assuming that the cooperation is feasible “*if and only if both the manufacturer and the retailer cannot get any higher profits in other games*” (Xie and Neyret, 2008, p.1378):

$$\Pi_{M+R}^{co} = \Pi_M^{co} + \Pi_R^{co} \geq \Pi_M^{\max} + \Pi_R^{\max} \quad (3.23)$$

With $\max(\Pi_M^{SM}, \Pi_M^{SR}, \Pi_M^N) = \Pi_M^{\max}$ and $\max(\Pi_R^{SM}, \Pi_R^{SR}, \Pi_R^N) = \Pi_R^{\max}$.

We do not report the analytical solutions of (3.23), since they do not give any useful insights for our analysis, but we summarize the results in the diagram below (Fig.3²⁹).

²⁸ See the appendix of Xie and Neyret for details. The superscript *co* indicates the Cooperative Equilibrium.

²⁹ Source: *Co-op advertising and pricing models in manufacturer-retailer supply chains* by Xie and Neyret (2009).

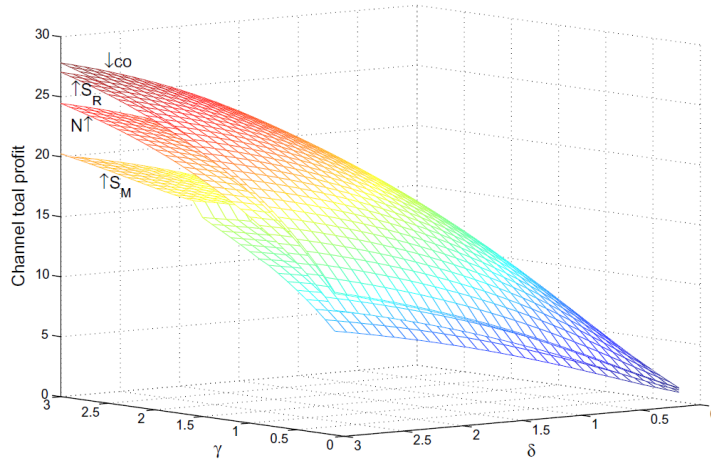


Fig.3. Whole channel's profits

Clearly, cooperation is always feasible, even though the criterion of cooperation presupposed by authors is restrictive. In fact, imposing condition (3.23), they assume that players can pass from one typology of the game to another, in whichever situation. On the contrary, the typology of the game is not a decision variable but a given situation. It is unjustified a comparison between total cooperative payoffs and the sum of the maximum payoffs available for retailer and manufacturer, since players have to participate at the same game and, consequently, they cannot be both leader or both follower. Thus, a condition imposing that cooperative payoffs have to be greater than the sum of manufacturer leader's and retailer leader's profits (which are usually the maximum profits available for the players), is clearly unwarranted. In other words, if the feasibility condition is bond with the overall payoffs received by players, cooperation should always be feasible because, to solve cooperative game, we have to maximize the total profits of the system. Therefore, condition (3.23) is, indeed, unnecessarily restrictive.

However, if we take into consideration the way in which players split the total cooperative profits, the results could be different. In this case, players will cooperate only if the amount of profits effectively received is greater than that obtained without cooperation. Thus, the feasibility will be strictly dependent on the bargaining results.

In section 3.3.6. we will propose a different feasibility condition that takes into accounts the factors aforementioned.

3.1.5. Bargaining Model

The approach used to split the cooperation profits is the Symmetric Nash bargaining model (see 2.4.1.), in which the bargaining outcome is obtained by maximizing the product of players' utilities.

The authors present three different situations, in which parameters λ_1 and λ_2 , which describe the functional form of players' utilities, assume different values: (i) $\lambda_1 = \lambda_2 = 1$, (ii) $\lambda_1 < 1$ and $\lambda_2 = 1$, and (iii) $\lambda_1 < 1$ and $\lambda_2 < 1$. Of course, if $\lambda_1 = \lambda_2$, players will equally split profits; but if, for instance, $\lambda_1 < \lambda_2$, the agreement will be beneficial for player 2 (see (2.23)).

Later, in Aust and Buser (2012), we will see a more developed bargaining model, the Asymmetric Nash one, which incorporates another parameter: the bargaining power.

3.1.6. Conclusions

The main results of the paper are about the optimal advertising and prices levels in the four different scenarios. Looking at the data, Xie and Neyret conclude that, firstly, in the coordinated situation players spend more in advertising but retail price is the lowest and, secondly, that leader, both in the Retailer and in the Manufacturer Stackelberg Game, invests less in advertising campaign than in the other cases, since he/she manages the follower to spend more. With regard to the profits, they notice that the manufacturer chooses to be leader only when $\delta > 0,5$. In the other cases, he/she prefers to play as follower (interestingly, the Nash equilibrium always provides him/her with the lowest profits, see Fig.4³⁰). Lastly, they find that coordination is always feasible, in spite of the condition stated in the equation (3.23), which is unnecessarily restrictive.

³⁰ Source: *Co-op advertising and pricing models in manufacturer-retailer supply chains* by Xie and Neyret (2009).

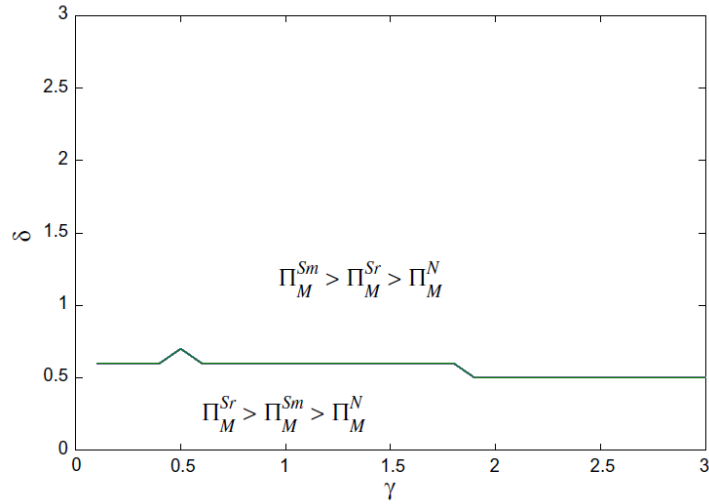


Fig.4. Manufacturer's profits in different scenarios

3.2. SeyedEsfahani, et al. (2011)

The second model analysed is an evolution of the previous one. Again, the authors want to derive the optimal level of co-op advertising in a supply-chain model, and they present the same four different games seen in Xie and Neyret (2009): three non-cooperative games (Nash, Stackelberg Retailer and Stackelberger Manufacturer) and one cooperative game. However, the model presents some significant differences.

3.2.1. Model

The model consists in a supply chain with one manufacturer and one retailer. The former supplies all his/her products to the same retailer at a price p_M , the latter sells the manufacturer's products only, at a price p_R . To simplify the solution process, authors impose that $p_M, p_R \in [0,1]$. National advertising expenditures are represented by the parameter A (manufacturer's advertising), while local advertising expenditures by the parameter a (retailer's advertising). Then, along the lines of standard literature, they propose the following demand function:

$$V(a, A, p_R) = V_0 g(p_R) h(a, A), \quad (3.24)$$

where V_0 is the base demand, whilst $g(p)$ and $h(a, A)$ are functions describing, respectively, the effect of retail price and of advertising levels on the demand. While Xie and Neyret assume a linear relationship between the demand and retail price, in this model the functional form of $g(p)$ is more general:

$$g(p_R) = (\alpha - \beta p_R)^{\frac{1}{v}}, \quad (3.25)$$

in which α, β and v are positive constants. In particular: α is the intercept of the demand function, β is the marginal effect of price on the demand, and v determines the functional form of $g(p)$. Values of $v < 1$, $v = 1$, and $v > 1$ correspond to convex, linear and concave demand-price curves, respectively. As suggested by Piana (2004), different types of society bring different shapes of demand-price curves: linear when the reserve prices are uniformly distributed, concave if a huge number of consumers have the same reserve price and only a few are rich or poor, and convex if there is a polarized distribution of reserve prices (heavy tail distribution).

The function describing the advertising effect - $h(a, A)$ - is another evolution of that proposed by Xie and Neyret (2009) -see equation (3.1)-, since it incorporates the *advertising saturation effect*:

$$h(a, A) = k_1 \sqrt{a} + k_2 \sqrt{A}. \quad (3.26)$$

Applying a square root to a and A , they obtain a function where additional advertising generates diminishing returns, reproducing the *saturation effect* already studied by past literature³¹. Parameters k_1 and k_2 reflect the effectiveness of local and national advertising. Replacing (3.26) and (3.25) in (3.24), we obtain the model demand function:

³¹ See for more information Simon and Arndt (1980), Kim and Staelin (1999) and Karray and Zaccour (2006).

$$V(a, A, p_R) = V_0(\alpha - \beta p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}). \quad (3.27)$$

The condition $p < \frac{\alpha}{\beta}$ is required to avoid negative demand. Now, calling c the manufacturer's production cost and d the retailer's unit cost, the profit functions are the following:

$$\Pi_M = V_0(p_M - c)(\alpha - \beta p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - ta - A, \quad (3.28)$$

$$\Pi_R = V_0(p_R - p_M - d)(\alpha - \beta p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - (1 - t)a, \quad (3.29)$$

$$\Pi_{M+R} = V_0(p_R - c - d)(\alpha - \beta p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - a - A. \quad (3.30)$$

Other conditions are necessary to avoid negative profits, thus: $p_M > c$ and $p_R > p_M + d$. Moreover, to simplify the analysis and calculations, the authors apply the following changes of variables:

$$\alpha' = \alpha - \beta(c + d),$$

$$p'_R = \frac{\beta}{\alpha'}(p_R - (c + d)),$$

$$p'_M = \frac{\beta}{\alpha'}(w - c),$$

$$k'_1 = \frac{V_0\alpha^{\frac{1}{v}+1}}{\beta} k_1,$$

$$k'_2 = \frac{V_0\alpha^{\frac{1}{v}+1}}{\beta} k_2.$$

Thus, the profits functions can be rewritten as follows:

$$\Pi'_M = p'_M(1 - p'_R)^{\frac{1}{v}}(k'_1\sqrt{a} + k'_2\sqrt{A}) - ta - A, \quad (3.31)$$

$$\Pi'_R = (p'_R - p'_M)(1 - p'_R)^{\frac{1}{v}}(k'_1\sqrt{a} + k'_2\sqrt{A}) - (1 - t)a, \quad (3.32)$$

$$\Pi'_{M+R} = p'_R(1 - p'_R)^{\frac{1}{v}}(k'_1\sqrt{a} + k'_2\sqrt{A}) - a - A. \quad (3.33)$$

For the sake of simplicity, since now we refer to (3.31), (3.32), and (3.33) without the superscript (').

Tab.4. Legend of SeyedEsfahani et al. (2011)

p_R	Retail price
p_M	Wholesale price
t	Participation rate
a	Retailer local advertising level
A	Manufacturer national advertising level
α	Intercept of the demand-price curve
c	Manufacturer's production cost
d	Retailer's unit cost
V_0	Base demand
β	Marginal effect of price on the demand
k_1	Marginal effect of local advertising (under square root)
k_2	Marginal effect of global advertising (under square root)
v	Parameter of the functional form of demand-price curve

3.2.2. Nash Game

In a Nash game players determine their strategies independently and simultaneously. To find the equilibrium, we have to solve the following maximization problems:

$$\text{Max}_{t,A,p_M} \Pi_M = p_M(1 - p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - ta - A \quad (3.34)$$

$$\text{s.t. } 0 \leq t \leq 1, \quad A \geq 0, \quad 0 \leq p_M \leq 1,$$

$$\text{Max}_{a,p_R} \Pi_R = (p_R - p_M)(1 - p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - (1 - t)a \quad (3.35)$$

$$\text{s.t. } a \geq 0, \quad p_M \leq p_R \leq 1.$$

As in the previous model, it is easy to prove that the optimal participation rate (t) is zero, since it has a negative impact on the objective function (3.34). Furthermore, function (3.34) is increasing with regard to p_M . However, $p_M = 1$ cannot be a solution, as $p_R \geq p_M$ and $p_M, p_R \in [0,1]$: if $p_M = 1 \Rightarrow p_R = 1 \Rightarrow \Pi_R = \Pi_M = 0$.

Again, the authors tackle the problem assuming a very restrictive constraint:

$$p_R - p_M = p_M, \quad (3.36)$$

hence $p_M = \frac{p_R}{2}$. In Aust and Buser (2012), authors will not use assumption (3.36) and obtain more interesting results. Anyway, now we can solve problems (3.34) and (3.35), finding the unique Nash Equilibrium³²:

$$p_M^N = \frac{v}{2v+1}, \quad p_R^N = \frac{2v}{2v+1}, \quad t^N = 0, \quad (3.37)$$

$$a^N = \frac{1}{4}k_1^2v^2 \left(\frac{1}{2v+1} \right)^{\frac{2}{v}+2}, \quad (3.38)$$

$$A^N = \frac{1}{4}k_2^2v^2 \left(\frac{1}{2v+1} \right)^{\frac{2}{v}+2}. \quad (3.39)$$

³² See the appendix of SeyedEsfahani et al. (2011) for further information.

It is remarkable the relationship between the equilibrium levels of retail and manufacturer prices in (3.37) and those in (3.11). Solutions (3.37) are the general form of (3.11) because, to calculate the latter, authors have assumed the linearity of price-demand curve. On the contrary, in (3.37) the functional form is tied with the parameter ν . In fact, if we impose the condition $\nu = 1$ (linearity of price-curve demand), (3.11) and (3.37) will coincide.

Overall, the Nash Equilibrium of this model is more interesting than that found in Xie and Neyret (2009), as it provides some evidence about the influence of the functional form of the price-demand curve. Nonetheless, even this form is limited by condition (3.36).

3.2.3. Stackelberg Retailer and Manufacturer Game

We start solving the Stackelberg Retailer game. Firstly, we take manufacturer's best responses, which are the same as found in the Nash game:

$$t = 0, \quad p_M = \frac{p_R}{2}, \quad A = \frac{1}{4} k_2^2 \nu^2 \left(\frac{1}{2\nu+1} \right)^{\frac{2}{\nu}+2}. \quad (3.40)$$

Then, we replace equations (3.40) in the maximization problem (3.35). The S_R equilibrium obtained³³ is:

$$p_R^{S_R} = \frac{\nu}{\nu+1}, \quad p_M^{S_R} = \frac{\nu}{2(\nu+1)}, \quad t^{S_R} = 0, \quad (3.41)$$

$$a^{S_R} = \frac{k_1^2}{16} \nu^2 \left(\frac{1}{\nu+1} \right)^{\frac{2}{\nu}+2}, \quad (3.42)$$

$$A^{S_R} = \frac{k_2^2}{16} \nu^2 \left(\frac{1}{\nu+1} \right)^{\frac{2}{\nu}+2}. \quad (3.43)$$

³³ See the appendix of SeyedEsfahani et al. (2011) for further information.

Secondly, we calculate the equilibrium of the Stackelberg Manufacturer game. The BRs of the follower are again the same we have obtained in the Nash game:

$$a = \frac{1}{4} k_1^2 v^2 \left(\frac{1}{2v+1} \right)^{\frac{2}{v}+2}, \quad p_R = \frac{v+w}{v+1}. \quad (3.44)$$

Incorporating these value in problem (3.34), the S_M can be calculated³⁴:

$$p_R^{S_M} = \frac{v+p_M^{S_M}}{v+1}, \quad p_M^{S_M} = \frac{2v(v+1)k^2 + \sqrt{4v^2(v+1)^2 k^2 (k^2+1)v^2 + (v+2)^2}}{(v+2)^2 + 4k^2(v+1)^2} \quad (3.45)$$

$$\text{with } k = \frac{k_2}{k_1},$$

$$t^{S_M} = \frac{p_R^{S_M}(3v+2) - v}{p_R^{S_M}(v+2) + v}, \quad (3.46)$$

$$a^{S_R} = \left(\frac{v(1-p_M^{S_M})}{p_M^{S_M}(v+2) + v} k_1 v \left(\frac{1-p_M^{S_M}}{v+1} \right)^{\frac{1}{v}+1} \right)^2, \quad (3.47)$$

$$A^{S_M} = \left(\frac{p_M^{S_M} k_2}{2} \left(\frac{1-p_M^{S_M}}{v+1} \right)^{\frac{1}{v}} \right)^2. \quad (3.48)$$

From the expressions above, it is difficult to derive meaningful and intuitive results. However, in section 3.2.6., we will analyse the equilibrium values graphically.

3.2.4. Cooperation

Lastly we study the Cooperation game, in which both channel members agree to cooperate, maximizing their profits jointly. The maximization problem will be:

³⁴ See the appendix of SeyedEsfahani et al. (2011) for further information.

$$\text{Max}_{p_R, a, A} \Pi_{M+R} = p_R(1 - p_R)^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - a - A, \quad (3.49)$$

$$\text{s.t. } a \geq 0, \quad A \geq 0, \quad 0 \leq p_R \leq 1,$$

and the unique cooperative solution is³⁵:

$$p_R^{co} = \frac{v}{v+1}, \quad A^{co} = \left(\frac{1}{2} k_2 v \left(\frac{1}{v+1} \right)^{\frac{1}{v+1}} \right)^2, \quad a^{co} = \left(\frac{1}{2} k_1 v \left(\frac{1}{v+1} \right)^{\frac{1}{v+1}} \right)^2. \quad (3.50)$$

As in Xie and Neyret (2009), to determine when cooperation is actually feasible, the authors are not interested in p_M and t , and impose the following condition:

$$\Pi_{M+R}^{co} = \Pi_M^{co} + \Pi_R^{co} \geq \Pi_M^{\max} + \Pi_R^{\max}, \quad (3.51)$$

$$\text{with } \max(\Pi_M^{SM}, \Pi_M^{SR}, \Pi_M^N) = \Pi_M^{\max} \text{ and } \max(\Pi_R^{SM}, \Pi_R^{SR}, \Pi_R^N) = \Pi_R^{\max}.$$

Processing the profit functions of the four different games through MATLAB, they obtain Fig.5³⁶, in which there are five different regions. Tab.4 specifies the maximum players' profits in each of these regions.

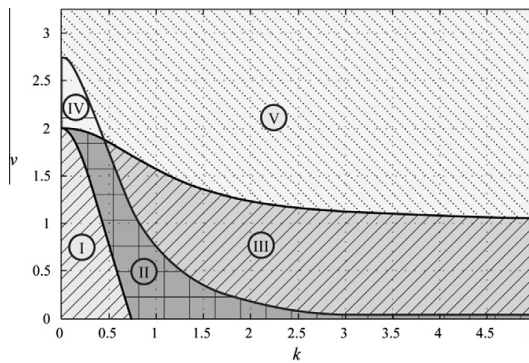


Fig.5. Five regions for the feasibility of the cooperation.

³⁵ See the appendix of SeyedEsfahani et al. (2011) for further information.

³⁶ Source: *A game theoretic approach to coordinate pricing and vertical co-op advertising in manufacturer-retailer supply chains* by SeyedEsfahani et al. (2011).

Tab.5. Maximum profits for players in the five situations of Fig.5.

Region	Π_M^{\max}	Π_R^{\max}
(i)	$\Pi_M^{S_R}$	$\Pi_R^{S_M}$
(ii) and (iii)	$\Pi_M^{S_R}$	$\Pi_R^{S_R}$
(iv) and (v)	$\Pi_M^{S_M}$	$\Pi_R^{S_R}$

Thanks to Fig.5 and Tab.5, we are now able to solve (3.51), since we can derive the value of Π_M^{\max} and Π_R^{\max} in the five different areas. Without further calculi, we can notice that cooperation is practicable in (ii) and (iii), because both players obtain maximum profits in S_R game and, by definition, the cooperation equilibrium maximizes the profits of the whole system in any situation. To check the other three areas, authors introduce the following equations:

$$\Delta_1 = \frac{\Pi_{M+R}^{co} - (\Pi_M^{S_R} + \Pi_R^{S_M})}{\Pi_{M+R}^{co}} \times 100, \quad (3.52)$$

$$\Delta_2 = \frac{\Pi_{M+R}^{co} - (\Pi_M^{S_M} + \Pi_R^{S_R})}{\Pi_{M+R}^{co}} \times 100. \quad (3.53)$$

Equations (3.52) and (3.53) represent the difference (in percentage) between cooperative profits and the maximum profits obtained by chain members in areas (i) and (iv)-(v), respectively.

As clearly represented by Fig.6³⁷ and Fig.7³⁸, the cooperation game is always feasible.

³⁷ Source: *A game theoretic approach to coordinate pricing and vertical co-op advertising in manufacturer-retailer supply chains* by SeyedEsfahani et al. (2011).

³⁸ Source: *A game theoretic approach to coordinate pricing and vertical co-op advertising in manufacturer-retailer supply chains* by SeyedEsfahani et al. (2011).

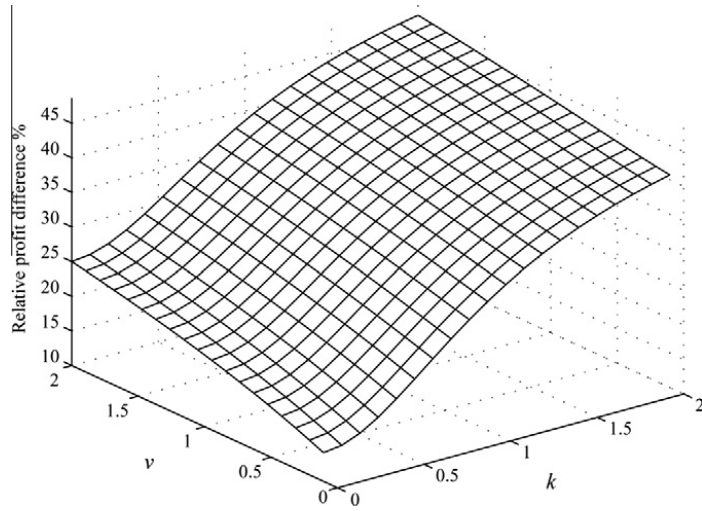


Fig.6. Graphic representation of (3.52).

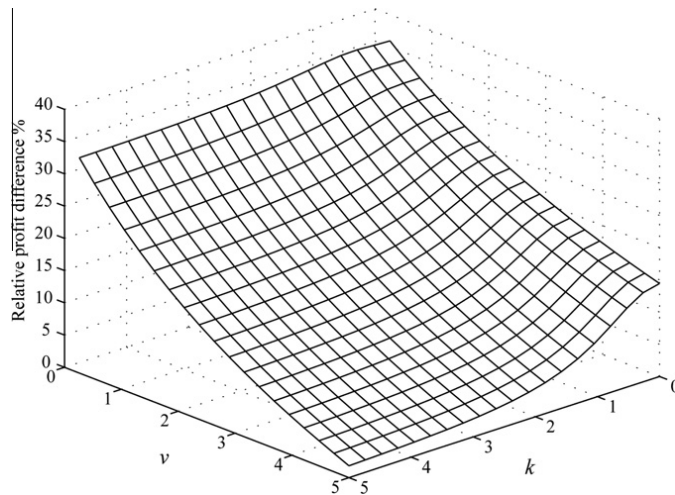


Fig.7. Graphic representation of (3.53).

Nevertheless, condition (3.51) is unnecessarily restrictive (see 3.1.4.), and we refer to section 3.3.6. to propose a different feasibility condition.

Moreover, until now, we have not considered other two variables: the wholesale price p_M and the participation rate t . Individual profits, calculated according to functions (3.31) and (3.32), are indeed dependent on these variables and they are analysed in the following section, where we will describe a model about internal agreement between cooperative players.

3.2.5. Bargaining Model

The bargaining model is the same as used by Xie and Neyret (2009) and takes into account only one parameter λ . According to the theoretical model explained in section 2.4.1., the utility functions of players are:

$$u_M = \Delta\Pi_M^{\lambda_M}, \quad (3.54)$$

$$u_R = \Delta\Pi_R^{\lambda_R}, \quad (3.55)$$

where λ indicates the form of the utility function, and $\Delta\Pi$ is the difference between the profits obtained through cooperation and the maximum profits received in other situations. Therefore, following the model proposed by Nash, the solution is derived by the following optimization problem:

$$\text{Max}_{\Delta\Pi_M, \Delta\Pi_R} u_M u_R = \Delta\Pi_M^{\lambda_M} \Delta\Pi_R^{\lambda_R} \quad (3.56)$$

$$\text{s.t. } \Delta\Pi_M + \Delta\Pi_R = \Delta\Pi, \quad \Delta\Pi_R \geq 0, \quad \Delta\Pi_M \geq 0.$$

Analytical solutions of (3.56) are not interesting for our analysis³⁹. It is remarkable only that, if the functional form is the same for both players, they will split the extra profits equally.

3.2.6. Conclusions

The proposed model, if compared to Xie and Neyret (2009), presents two significant improvements. Firstly, the relationship between price and demand is more general, including three different shapes of the demand-price function (linear, convex, and concave). Thus, we can observe the effects of the shape on the optimal values of players' decision variables and profits.

³⁹ For further information about the bargaining equilibrium, see the appendix of SeyedEsfahani et al. (2011).

Secondly, the model includes the *advertising saturation effect* in demand function, according to which each incremental amount of advertising causes a progressively lesser effect on demand increase.

Results, in contrast, are similar to those of past literature. Retail price is the lowest in the cooperative game, whilst advertising expenditures are higher in the non-cooperative ones. Again, the highest amount of profits is obtained in cooperative situation (see Fig.10⁴⁰). Lastly, manufacturers always prefer to be retailer’s follower rather than compete in a Nash game (Fig.8⁴¹), while this is not true for the retailer (Fig.9⁴²).

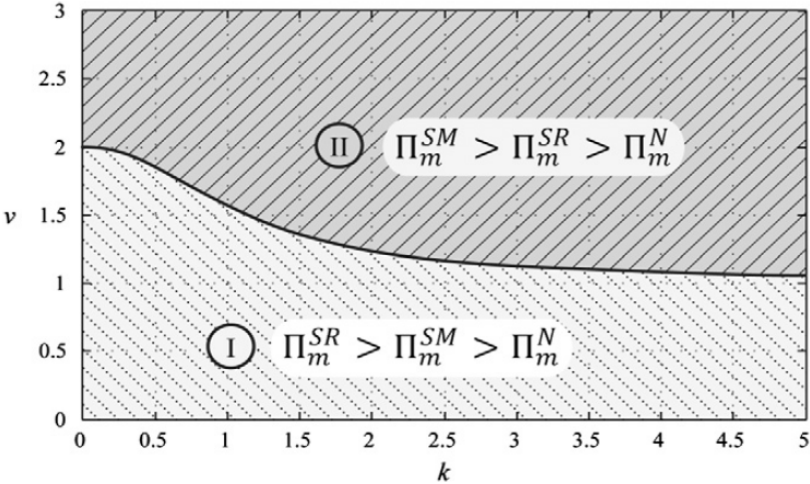


Fig.8. Manufacturer’s profits.

⁴⁰ Source: *A game theoretic approach to coordinate pricing and vertical co-op advertising in manufacturer-retailer supply chains* by SeyedEsfahani, et al. (2011).

⁴¹ Source: *A game theoretic approach to coordinate pricing and vertical co-op advertising in manufacturer-retailer supply chains* by SeyedEsfahani, et al. (2011).

⁴² Source: *A game theoretic approach to coordinate pricing and vertical co-op advertising in manufacturer-retailer supply chains* by SeyedEsfahani, et al. (2011).

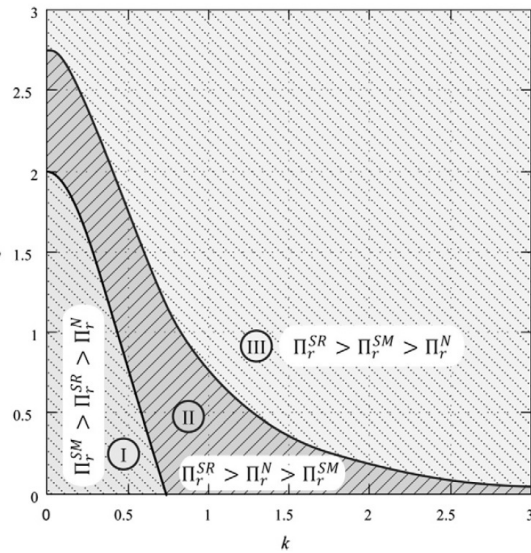


Fig.9. Retailer's profits

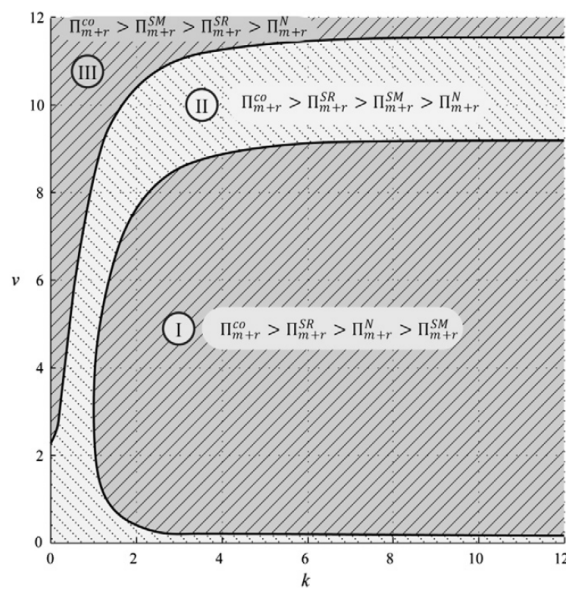


Fig.10. System's profits

With regard to the models flaws, two conditions are too restrictive: conditions (3.36) -that assumes equal margin for the two players-, and (3.51) -the feasibility condition-. In Aust and Busher (2012), condition (3.36) is not assumed and, about assumption (3.51), we will propose a modification in section 3.3.6..

3.3. Aust and Busher (2012)

Aust and Busher (2012) is an expansion of existing models about advertising and pricing decisions in a one-manufacturer-one-retailer supply chain. A game-theoretic approach is used to analyse the usual four types of game: Nash, Stackelberg Retailer, Stackelberg Manufacturer, and Cooperative.

3.3.1. Model

The model is very close to that proposed in SeyedEsfahani et al. (2011). The price and advertising demand functions are similar to (3.25) and (3.26) respectively, and there are few differences with regard to the variables and parameters considered. The base demand V_0 , the manufacturer's production cost c , and the retailer's unit cost d are not included; while the authors denote by p_M the manufacturer's margin and by p_R the retailer's one. Calling p the price of the good, we have:

$$p_M = p - p_R. \quad (3.57)$$

The rest of the model is summarized in Tab.4 (see section 3.2.1.). Manufacturer and retailer profit functions are the following:

$$\Pi_M = p_M(\alpha - \beta(p_M + p_R))^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - ta - A, \quad (3.58)$$

$$\Pi_R = p_R(\alpha - \beta(p_M + p_R))^{\frac{1}{v}}(k_1\sqrt{a} + k_2\sqrt{A}) - (1 - t)a. \quad (3.59)$$

The main difference between (3.58)-(3.59) and (3.28)-(3.29) is that, in Aust and Busher (2012), the retail price p is split into the wholesale price p_M and retailer's margin p_R . In this way, we do not have to introduce any condition assuming identical margins for both players, as in Xie and Neyret (2009) and SeyedEsfahani et al. (2011), in which this condition is required (see (3.10) and (3.36)) to avoid $\Pi_R = 0$ in equilibrium. Relaxing this assumption, we are able to get

further insights into pricing as a decision variable. Some conditions are necessary for the validity of the model: (i) all the parameters and variables have to be positive; (ii) $p_M + p_R < \frac{\alpha}{\beta}$ to avoid negative price-demand, and (iii) $0 \leq t < 1$ to facilitate calculations.

No further simplifications and change of variables are applied; therefore, we do not lose other important information about key parameters.

3.3.2. Nash Game

Having assumed a symmetrical distribution of power between the manufacturer and retailer, we can find the NE, where players take decisions simultaneously and not cooperatively. Players' decision problems are:

$$\begin{aligned} \text{Max}_{t,A,p_M} \Pi_M &= p_M(\alpha - \beta(p_M + p_R))^{\frac{1}{\nu}}(k_1\sqrt{a} + k_2\sqrt{A}) - ta - A & (3.60) \\ \text{s.t. } p_M &< \frac{\alpha}{\beta} - p_R, \quad A \geq 0, \quad 0 \leq t < 1, \end{aligned}$$

$$\begin{aligned} \text{Max}_{a,p_R} \Pi_R &= p_R(\alpha - \beta(p_M + p_R))^{\frac{1}{\nu}}(k_1\sqrt{a} + k_2\sqrt{A}) - (1 - t)a & (3.61) \\ \text{s.t. } p_R &< \frac{\alpha}{\beta} - p_M, \quad a \geq 0. \end{aligned}$$

Firstly, we have to impose the FOCs for both problems. The optimal value of t , as we noticed in previous models, is 0. The manufacturer does not have any incentive to fix a different level of participation rate, since t negatively affects his/her profit function. With regard to the other variables, we set $\frac{\partial \Pi_M}{\partial p_M}$, $\frac{\partial \Pi_M}{\partial A}$, $\frac{\partial \Pi_R}{\partial p_R}$, and $\frac{\partial \Pi_R}{\partial a}$ equal to 0. The solutions of the two maximization problems are the following⁴³:

⁴³ See the appendix of Aust and Busher (2012) for the proof.

$$p_R^N = p_M^N = \frac{\alpha v}{\beta(1+2v)}, \quad p^N = \frac{2\alpha v}{\beta(1+2v)}, \quad t^N = 0, \quad (3.62)$$

$$A^N = \frac{v^2 k_2^2}{4\beta^2} \left(\frac{\alpha}{1+2v} \right)^{\frac{2}{v}+2}, \quad (3.63)$$

$$a^N = \frac{v^2 k_1^2}{4\beta^2} \left(\frac{\alpha}{1+2v} \right)^{\frac{2}{v}+2}. \quad (3.64)$$

Interestingly, even without assuming equal margins for the players, the equilibrium leads to $p_R = p_M$, showing that assumptions (3.10) and (3.36), at least in this kind of game, are not restrictive. Moreover, both players will spend the same amount in advertisements if the effectiveness of local (k_1^2) and global (k_2^2) advertising coincide. Subdivision of advertising expenditure in the channel is influenced only by the effectiveness of the two kinds of advertising. Parameters α and β , on the contrary, cause only a decrease/increase of the overall level, without influencing advertising repartition between players.

3.3.3. Stackelberg Retailer and Manufacturer Game

In this part, we study the model presupposing an asymmetrical distribution of power between players. In the first case, we assume an advantage for the manufacturer (Stackelberg Manufacturer game), who is aware of the retailer's best response prior to maximize his/her utility. Formally, we start solving the retailer's maximization problem, which is the same as described in (3.61) and leads to these best response functions:

$$p_R = \frac{v(\alpha - \beta p_M)}{\beta(1+v)}, \quad (3.65)$$

$$a = \frac{k_1^2 p_R^2 (\alpha - \beta(p_M + p_R))^{\frac{2}{v}}}{4(1-t)^2}. \quad (3.66)$$

Replacing (3.65) and (3.66) in (3.60), the manufacturer's maximization problem becomes:

$$\begin{aligned} \text{Max}_{t,A,p_M} &= p_M \left(\alpha - \beta \left(p_M + \frac{v(\alpha - \beta p_M)}{\beta(1+v)} \right) \right)^{\frac{1}{v}} \left(k_1 \sqrt{\frac{k_1^2 p_R^2 (\alpha - \beta(p_M + p_R))^{\frac{2}{v}}}{4(1-t)^2}} + k_2 \sqrt{A} \right) \\ &\quad - t \frac{k_1^2 p_R^2 (\alpha - \beta(p_M + p_R))^{\frac{2}{v}}}{4(1-t)^2} - A \\ \text{s.t. } p_M &< \frac{\alpha}{\beta} - p_R, \quad A \geq 0, \quad 0 \leq t < 1. \end{aligned} \quad (3.67)$$

After some simplifications, the Stackelberg Manufacturer equilibrium is the following (note that, for the sake of simplicity, we have maintained in the solution the variable p_M ⁴⁴):

$$\begin{aligned} p_R^{SM} &= \frac{v(\alpha - \beta p_M)}{\beta(1+v)} \\ p_M^{SM} &= \frac{2\alpha v k^2 (v+1) + \alpha v \sqrt{4k^2(1+v)^2(k^2+1) + (v+2)^2}}{\beta((v+2)^2 + 4k^2(v+1)^2)} \quad \text{with } k = \frac{k_2}{k_1}, \end{aligned} \quad (3.68)$$

$$p^{SM} = \frac{\alpha v + \beta p_M}{\beta(1+v)}, \quad t^{SM} = \frac{\beta p_M(2+3v) - \alpha v}{\beta p_M(2+v) + \alpha v},$$

$$A^{SM} = \frac{k_2^2 p_M^2}{4} \left(\frac{\alpha - \beta p_M}{1+v} \right)^{\frac{2}{v}}, \quad (3.69)$$

$$a^{SM} = \frac{k_1^2 (\beta v p_M + 2\beta p_M + \alpha v)^2}{16\beta^2 (v+1)^2} \left(\frac{\alpha - \beta p_M}{1+v} \right)^{\frac{2}{v}}. \quad (3.70)$$

In the second case, the retailer plays as a leader (Stackelberg Retailer game), and we follow the same steps described above to deduce the solutions. This time, we start deriving the manufacturer's best responses of problem (3.60), finding that:

$$t = 0, \quad p_M = \frac{\alpha - \beta p_R}{2\beta}, \quad A = \frac{k_2^2 p_M^2 (\alpha - \beta(p_M + p_R))^2}{4}. \quad (3.71)$$

Substituting (3.71) in (3.61), the maximization problem becomes:

⁴⁴ For further details, see the appendix of Aust and Buser (2012).

$$\begin{aligned}
\text{Max}_{a,p_R} \Pi_R &= p_R \left(\alpha - \beta \left(\frac{\alpha - \beta p_R}{2\beta} + p_R \right) \right)^{\frac{1}{v}} \times \\
&\times (k_1 \sqrt{a} + k_2 \sqrt{\frac{k_2^2 p_M^2 (\alpha - \beta (p_M + p_R))^2}{4}}) - a \\
\text{s.t. } p_R &< \frac{\alpha}{\beta} - p_M, \quad a > 0,
\end{aligned} \tag{3.72}$$

whose solution is

$$p_R^{S_R} = \frac{-\alpha v(1+v-k^2(2+3v))+\sqrt{(1+v)^2-2k^2v(1+v)+k^4(v+2)^2}}{2\beta(v+1)(2vk^2-v-1)}, \quad \text{with } k = \frac{k_2}{k_1}, \tag{3.73}$$

$$p_M^{S_R} = \frac{v(\alpha-\beta p_R)}{\beta(1+v)}, \quad p^{S_R} = \frac{\alpha v+\beta p_R}{\beta(1+v)}, \quad t^{S_R} = 0,$$

$$A^{S_R} = \frac{k_2^2 v^2}{4\beta^2} \left(\frac{\alpha - \beta p_M}{1+v} \right)^{\frac{2}{v}+2}, \tag{3.74}$$

$$a^{S_R} = \frac{k_1^2 p_R^2}{4} \left(\frac{\alpha - \beta p_R}{1+v} \right)^{\frac{2}{v}}. \tag{3.75}$$

The existence condition of (3.73) is $v \neq (2k^2 - 1)^{-1}$.

Looking at (3.68) and (3.73), it is clear that, this time, the assumption of identical margins seen in Xie and Neyret (2009) and SeyedEsfahani et al. (2011) is restrictive. In fact, considering that $p_M + p_R = p$, $p_M = p_R$ implies that $\frac{p_M}{p} = \frac{1}{2}$ and, as shown in Fig.11⁴⁵, this condition holds only for specific values of parameters v and k .

⁴⁵ Source: *Vertical cooperative advertising and pricing decision in a manufacturer-retailer supply chain: A game-theoretic approach* by Aust and Buscher (2012).

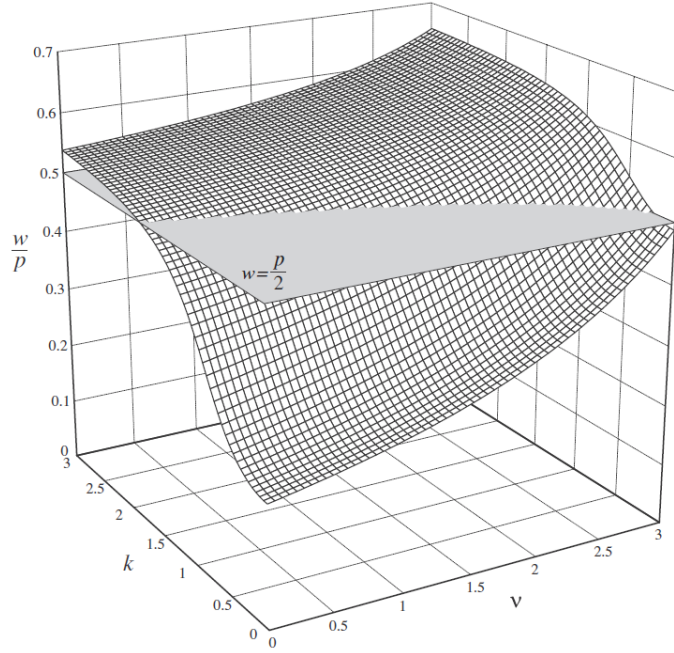


Fig.11. Ratio of wholesale price and price in a Stackelberg Retailer equilibrium.

3.3.4. Cooperation

Finally, we analyse the same cooperative game studied before, adapted to this new model. The decision problem is:

$$\begin{aligned} \text{Max}_{p,A,a} \Pi_{M+R} &= p(\alpha - \beta p)^{\frac{1}{\nu}}(k_1\sqrt{a} + k_2\sqrt{A}) - A - a & (3.76) \\ \text{s.t. } p &< \frac{\alpha}{\beta}, \quad A \geq 0, \quad a \geq 0. \end{aligned}$$

In (3.76), we no longer consider variables p_M and p_R , since they have no influence on the total profits, and we maximize it with respect to p , a , and A . The optimal solution obtained is the following⁴⁶:

⁴⁶ See the appendix of Aust and Busher (2012) for further details.

$$p^{co} = \frac{\alpha v}{\beta(v+1)}, \quad A^{co} = \frac{v^2 k_2^2}{4\beta^2} \left(\frac{\alpha}{1+v} \right)^{\frac{2}{v}+2}, \quad a^{co} = \frac{v^2 k_1^2}{4\beta^2} \left(\frac{\alpha}{1+v} \right)^{\frac{2}{v}+2}. \quad (3.77)$$

As in the Nash game, the optimal levels of advertising (a and A) will differ only if players' publicity effectiveness (k_1 and k_2) takes different values.

With regard to the feasibility condition, in Aust and Buscher (2012) it is the same as assumed in Xie and Neyret (2009) and SeyedEsfahani et al. (2011) -see (3.23) and (3.51)-. The authors impose that players will agree to cooperate only if they receive a higher amount profits than in any other game. As noticed in section 3.1.4., this condition is restrictive without reason, since the authors consider the game played a choice and not an exogenous condition.

Anyhow, the solution method is the same as seen in SeyedEsfahani et al. (2011), but conclusions are significantly different. In fact, if in SeyedEsfahani et al. (2011) cooperation is always feasible, here it is not. Using a numerical computation with MATLAB, the authors prove that for high value of v the solution could be unfeasible, as it is clearly shown in Tab.6⁴⁷ (where other parameters are set equal to $\alpha = 10, \beta = 1, v = 8, k_M = 2$, and $k_R = 1$), since $\Pi_M^{S_M} + \Pi_R^{S_R} > \Pi_{M+R}^{co}$ ⁴⁸.

It is important to note that, to have a complete cooperative solution, we need an effective division of cooperative profits, which can be obtained through the bargaining game described in the following section. Furthermore, in section 3.3.6., we will present a different feasibility condition, which takes into account the actual profits received by players -the split of cooperative profits in the bargaining game- to determine whether a game is feasible.

⁴⁷ Source: *Vertical cooperative advertising and pricing decision in a manufacturer-retailer supply chain: A game-theoretic approach* by Aust and Buscher (2012).

⁴⁸ Take into consideration the bold numbers in Tab.6. to check that $\Pi_M^{S_M} + \Pi_R^{S_R} > \Pi_{M+R}^{co}$.

Tab.6. Numerical computation with MATLAB.

	p_M	p_R	p	A	a	t	Π_M	Π_R	Π_{M+R}
N	4.71	4.71	9.41	19.39	4.85	0	29.09	43.64	72.73
S_M	8.77	1.09	9.86	46.77	13.19	0.88	59.96	13.15	73.11
S_R	4.45	4.99	9.44	17.13	5.38	0	26.72	43.77	70.49
co	-	-	8.89	81.12	20.28	-	-	-	101.40

3.3.5. Bargaining Model

In Xie and Neyret (2009) and SeyedEsfahani et al. (2011) the bargaining model used is the Symmetrical one, in which the only parameter taken into account is λ , which describes the players' utility functional form.

On the contrary, in Aust and Busher (2012), the authors introduce the Nash Asymmetrical model (see section 2.4.2.) that, firstly presented in Harsanyi and Selten (1972), incorporates a new parameter, the bargaining power μ . According to section 2.4.2. and calling $\Delta\Pi_p$ the difference between the profits obtained by player p in a cooperative game and the maximum profits received by him/her among the other games, the manufacturer's and retailer's utility functions remain the same as described in SeyedEsfahani et al (2011) -see (3.54) and (3.55)-. The formulation of the bargaining model, instead, is different and includes the bargaining power too (with $\mu_M + \mu_R = 1$). Hence, the objective function v_{M+R} that we have to maximize is given by:

$$v_{M+R} = \Delta\Pi_M^{\lambda_M\mu_M} \Delta\Pi_R^{\lambda_R\mu_R}. \quad (3.78)$$

The maximization problem will be the following:

$$\text{Max}_{\Delta\Pi_M, \Delta\Pi_R} v_{M+R} = \Delta\Pi_M^{\lambda_M \mu_M} \Delta\Pi_R^{\lambda_R \mu_R} \quad (3.79)$$

$$\text{s.t. } \Delta\Pi = \Delta\Pi_M + \Delta\Pi_R, \quad \Delta\Pi_M \geq 0, \quad \Delta\Pi_R \geq 0$$

and the solution⁴⁹ is:

$$\Delta\Pi_M = \frac{\lambda_M \mu_M}{\lambda_M \mu_M + \lambda_R \mu_R} \Delta\Pi, \quad (3.80)$$

$$\Delta\Pi_R = \frac{\lambda_R \mu_R}{\lambda_M \mu_M + \lambda_R \mu_R} \Delta\Pi. \quad (3.81)$$

Observing (3.80) and (3.81), we can note that identical utility functional form (λ) and bargaining power (μ) between players lead to an equal split of extra profits. With regard to the bargaining power, the authors have studied its effect setting $\lambda_R = \lambda_M = c$, with c a generic constant. They conclude, as it can be intuitively deduced from (3.80) and (3.81), that “*the player with the higher bargaining power will be able to get the bigger share of profits*” (Aust and Busher, 2012, p.477).

In addition, focusing on (3.79), we can reformulate the problem in a simpler way, through a new parameterization. Firstly, since we are interested only in the fraction of the extra profits received by players and not in the absolute value of their earnings, we can impose, without losing significant information, $\Delta\Pi = 1$. Therefore, the situation becomes a distribution problem of two positive quantities ($\Delta\Pi_M$ and $\Delta\Pi_R$) whose sum is one. Then, to generalize the problem, we replace the variables $\Delta\Pi_M$ and $\Delta\Pi_R$ with σ_M and σ_R , and the exogenous parameters $\lambda_M \mu_M$ and $\lambda_R \mu_R$ with ω_M and ω_R . The problem becomes:

$$\text{Max}_{\sigma_M, \sigma_R} \sigma_M^{\omega_M} \sigma_R^{\omega_R} \quad (3.82)$$

$$\text{s.t. } \sigma_M + \sigma_R = 1, \quad \sigma_M \geq 0, \quad \sigma_R \geq 0.$$

⁴⁹ For further details, see the appendix of Aust and Busher (2012).

And its solution will be :

$$\sigma_M^* = \frac{\omega_M}{\omega_M + \omega_R}, \quad (3.83)$$

$$\sigma_R^* = \frac{\omega_R}{\omega_M + \omega_R}, \quad (3.84)$$

with $\sigma_M^* * 100$ and $\sigma_R^* * 100$ that represent the percentage of the extra profits received by the manufacturer and the retailer, respectively.

3.3.6. A New Feasibility Condition

As already stressed above, the feasibility condition imposed in the previous three models is inadequate and restrictive. In fact, coordinated games should always be feasible if we consider the overall payoffs obtained the key variable to determine whether cooperation is practicable, since, by definition, cooperating players maximize system's profits. In Xie and Neyret (2009) and SeyedEsfahani et al. (2011), the authors, in spite of assuming conditions (3.23) and (3.51), arrive at the same conclusion: for each value of the parameters, players have incentive to cooperate. In Aust and Busher (2012), instead, the results are different because of the unwarranted feasibility condition (3.51).

Anyway, starting from the previous section we can propose a different feasibility condition, focusing on the amount of profits received by players, individually. To present formally this condition, we have to modify slightly the bargaining model solved in the section above. In Aust and Busher (2012) players split only the extra profits they acquire thanks to cooperation ($\Delta\Pi_{M+R}$), while our bargaining game is interested in the overall coordinated profits. Denoting with Π_M and Π_R the share they get of the overall cooperative profits Π_{M+R} , with $\Pi_M + \Pi_R = \Pi_{M+R}$, the objective function (3.78) will become:

$$v_{M+R} = \Pi_M^{\lambda_M \mu_M} \Pi_R^{\lambda_R \mu_R}, \quad (3.85)$$

with λ and μ that indicate the players' utility functional form and the bargaining power. Thus, the maximization problem will be:

$$\text{Max}_{\Pi_M, \Pi_R} v_{M+R} = \Pi_M^{\lambda_M \mu_M} \Pi_R^{\lambda_R \mu_R} \quad (3.86)$$

$$\text{s.t. } \Pi_{M+R} = \Pi_M + \Pi_R, \quad \Pi_M \geq 0, \quad \Pi_R \geq 0$$

Solutions of (3.86) will correspond to those of the previous section, see (3.80) and (3.81):

$$\Pi_M = \frac{\lambda_M \mu_M}{\lambda_M \mu_M + \lambda_R \mu_R} \Pi_{M+R}, \quad (3.87)$$

$$\Pi_R = \frac{\lambda_R \mu_R}{\lambda_M \mu_M + \lambda_R \mu_R} \Pi_{M+R}. \quad (3.88)$$

If we call Π_M^G and Π_R^G the profits received by manufacturer and retailer in a specific game G , with $G \in \{N, S_R, S_M\}$, cooperation will be feasible if and only if:

$$\begin{cases} \Pi_M = \frac{\lambda_M \mu_M}{\lambda_M \mu_M + \lambda_R \mu_R} \Pi_{M+R} \geq \Pi_M^G \\ \Pi_R = \frac{\lambda_R \mu_R}{\lambda_M \mu_M + \lambda_R \mu_R} \Pi_{M+R} \geq \Pi_R^G \end{cases}. \quad (3.89)$$

Expression (3.89) offers a more realistic condition associated with the individual profits received by players. According to (3.89), they decide to cooperate taking into account several factors: (i) the non-cooperative game they could play, (ii) the cooperative profits, (iii) their utility functional form, and (iv) bargaining power. Replacing conditions (3.23) and (3.51) with (3.89), we could surely derive more precisely when cooperation is actually feasible.

3.3.7. Conclusions

One of the main contributions of Aust and Buser (2012) is that, without the assumptions (3.10) and (3.36) that presuppose identical margins, the authors derive unrestricted and broader results. Thus, we obtain the three diagrams below about the wholesale (Fig.12⁵⁰), retailer (Fig.13⁵¹), and retail price (Fig.14⁵²). Note that no manufacturer costs are assumed. Hence, wholesale and retailer prices coincide with the manufacturer's and retailer's margin.

Focusing on Fig.14, we can see that consumers usually benefit more from a non-cooperative Nash game than from a Stackelberg Retailer/Manufacturer game, since the retail price is generally lower. However, even if not represented in the picture, the lowest retail price results from supply chain members' cooperation, which ensures the maximum customer welfare, due to the lowest price.

Looking at Fig.11, we can see how restrictive is the condition regarding identical margins between retailer and manufacturer assumed by previous works. This condition is valid only for particular values of the parameters, while, for instance, for small values of k and v the retailer obtains a higher margin. However, this paper presents a couple of flaws, since authors remove two parameters to simplify the calculations: the manufacturer's production and retailer's unit cost (denoted in this work by c and d , respectively). An analysis of a model that includes both players' costs (as in SeyedEsfahani et al., 2011) and splits the price in manufacturer's and retailer's margin (as in Aust and Buser, 2012) would be surely interesting, as it would allow us to observe how much costs influence the optimal value of decision variables, without losing information about players' margin.

⁵⁰ Source: *Vertical cooperative advertising and pricing decision in a manufacturer-retailer supply chain: A game-theoretic approach* by Aust and Buscher (2012).

⁵¹ Source: *Vertical cooperative advertising and pricing decision in a manufacturer-retailer supply chain: A game-theoretic approach* by Aust and Buscher (2012).

⁵² Source: *Vertical cooperative advertising and pricing decision in a manufacturer-retailer supply chain: A game-theoretic approach* by Aust and Buscher (2012).

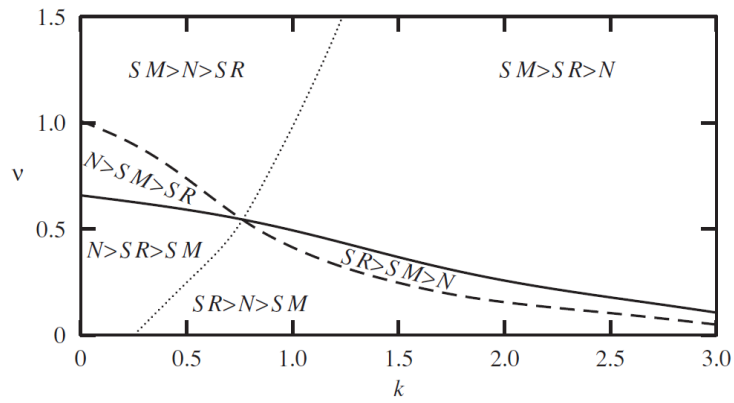


Fig.12. Manufacturer's price p_M .

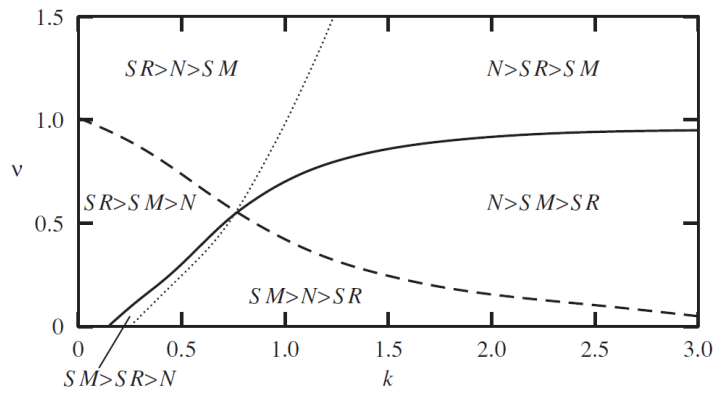


Fig.13. Retailer margin p_R .

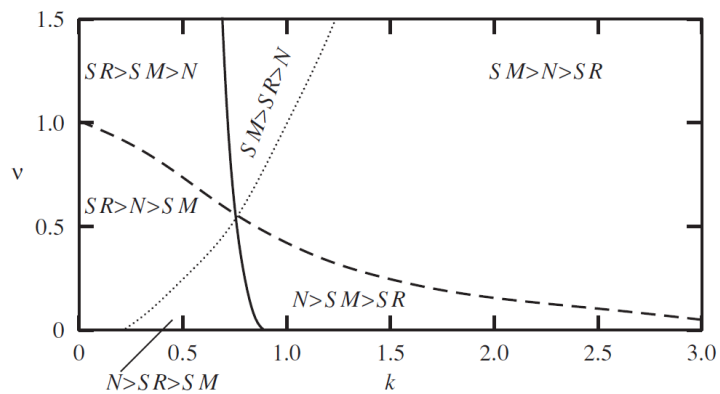


Fig.14. Retail price p .

With regard to the bargaining model, here the authors apply the Nash Asymmetrical one, in order to derive a bargaining solution that incorporates both players' utility functional form and the bargaining power. Therefore, results are more complete, as they take into account a new

variable. However, the feasibility condition implied is restrictive -see 3.1.4.-. According to (3.51), in fact, players could reject cooperation even though there are no possibility to achieve higher profits through non-cooperative behaviours, given the game structure. In section 3.3.6., we propose a different feasibility condition (3.89.), which states that players will cooperate if and only if they receive, after the bargaining game, a higher amount of profits than through non-cooperative behaviours: this feasibility condition allows us to derive more realistic and meaningful results.

Above all, Aust and Buser (2012) presents the most complete and interesting model of co-op advertising in supply-chain. However, there are some key points that need improvements. In the final chapter of this work, we propose a different co-op advertising model, adding another supply chain member (a retailer), and including manufacturer's and retailer's costs.

4. Co-op advertising in supply chain: A new model

In the last part of this work, we present a new game-theoretical analysis of co-op advertising in supply chain management. The aim of this new model is to improve and develop the ideas presented in past researches, with some significant changes. We propose a different game structure with three players⁵³ (one manufacturer and two retailers) where, firstly, the two retailers compete in a Bertrand game and, secondly, the winner of the first game plays with the manufacturer in a Nash, Stackleberg Retailer, or Stackelberg Manufacturer game (see Fig.15).

4.1. Model

We introduce a single-manufacturer-double-retailer channel, in which the overall game presents two steps: (i) firstly, the two retailers compete in a Bertrand game (see section 2.3. for further details); (ii) secondly, the retailer with the lower costs will play against the manufacturer in one of the three alternative scenarios: Nash, Stackleberg Retailer, and Stackelberg Manufacturer game.

Our aim is to reproduce a situation in which the manufacturer and one of the two retailers have to sign an exclusive partnership agreement, according to which both parties can buy/sell products of the counterparty only. Assuming that retailers can compete only by price (without considering other possible differentiations, such as networking, clients, reputation, and so forth), the manufacturer will deal with the winner of the Bertrand game, who is able to sell products at the lowest price (increasing the manufacturer's sales and profits).

Denoting with the subscripts M , R_1 , and R_2 the variables regarding to the manufacturer and the two retailers, we call p_M , p_{R_1} , and p_{R_2} the players' margin (with p_{R_*} that indicates the margin of the retailer who wins the Bertrand competition⁵⁴). We obtain the following retail price p :

$$p = p_M + p_*, \quad (4.1)$$

⁵³ Another two-retailers-one-manufacturer model is studied in Aust (2014).

⁵⁴ Throughout this part, with the subscript $*$ we indicate all the variables with respect to the retailer who has won the first-step Bertrand game.

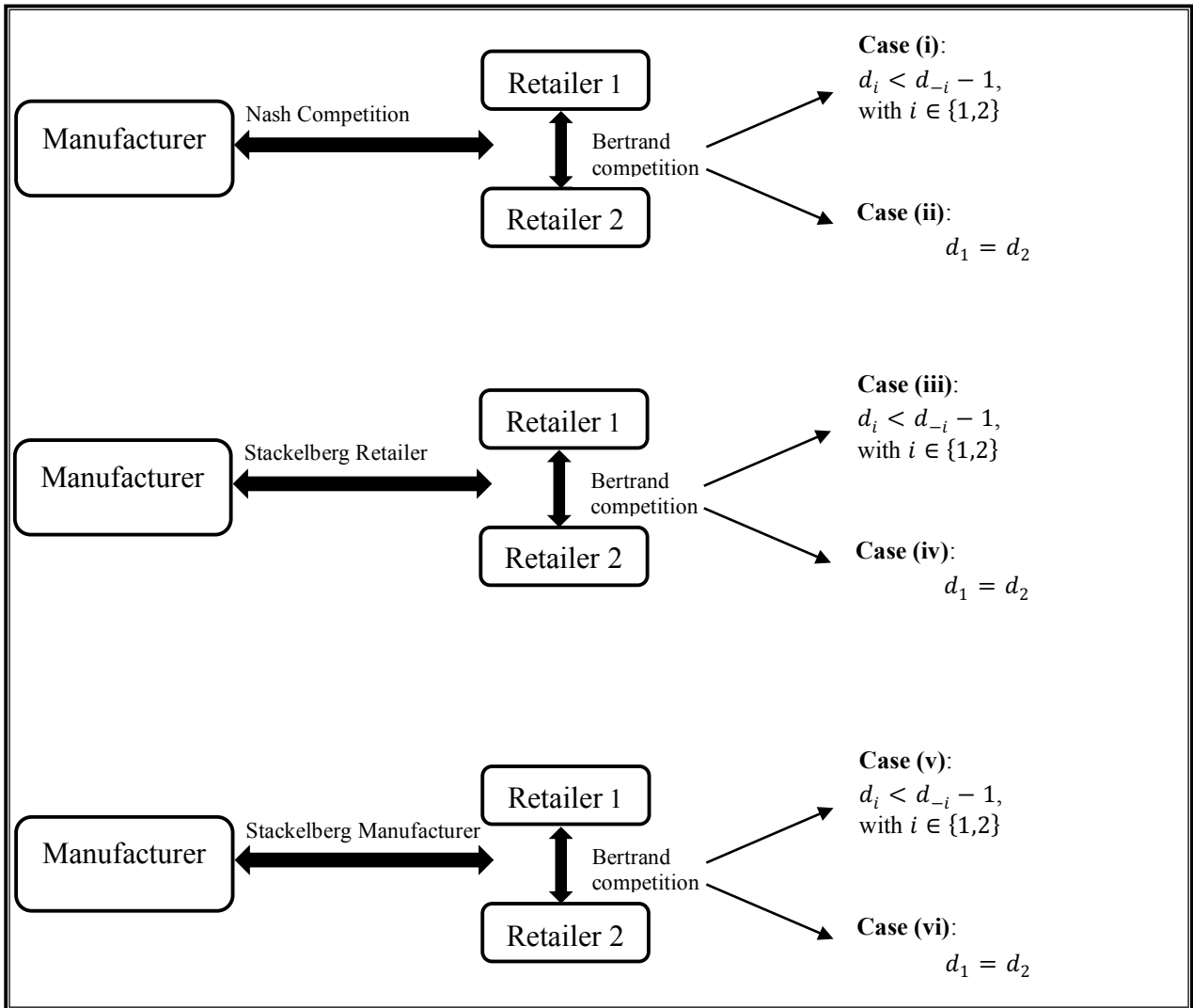


Fig.15. Summary of the model.

With regard to the consumer demand function, it is the same as used in Aust and Busher (2012)⁵⁵:

$$V(a_*, A, p) = (\alpha - \beta p)^{\frac{1}{\nu}} (k_{R^*} \sqrt{a_*} + k_M \sqrt{A}). \quad (4.2)$$

⁵⁵ For further details with regard to the nomenclature used, see Tab.7.

As in past literature⁵⁶, the demand price curve $(\alpha - \beta p)^{\frac{1}{v}}$ decreases with p , and its functional form depends on the parameter v : $v > 1$, $v < 1$, and $v = 1$ correspond to concave, convex, and linear function, respectively. The function describing the advertising effect $(k_{1*}\sqrt{a_*} + k_M\sqrt{A})$ takes into account the saturation effect, as the partial derivatives with respect to a_* and A are negative. Unlike Aust and Buser (2012), here we consider the retailers' (d_1 and d_2) and manufacturer's (c) marginal costs too.

Profit functions are:

$$\Pi_M = (p_M - c)(\alpha - \beta p)^{\frac{1}{v}}(k_{R_*}\sqrt{a_*} + k_M\sqrt{A}) - t_*a_* - A, \quad (4.3)$$

$$\Pi_{R_*} = (p_{R_*} - d_*)(\alpha - \beta p)^{\frac{1}{v}}(k_{R_*}\sqrt{a_*} + k_M\sqrt{A}) - (1 - t_*)a_*, \quad (4.4)$$

$$\Pi_{M+R_*} = (p - c - d_*)(\alpha - \beta p)^{\frac{1}{v}}(k_{R_*}\sqrt{a_*} + k_M\sqrt{A}) - a_* - A. \quad (4.5)$$

Lastly, we have to assume several conditions. Parameters d_1 , d_2 , c , k_{R_1} , k_{R_2} , k_M , α , and β must be positive and, similarly, variables a_1 , a_2 , and A have to be ≥ 0 . The demand function has to be positive too. Hence, looking at (4.2):

$$\alpha - \beta p > 0, \quad (4.6)$$

and, consequently, $p < \frac{\alpha}{\beta}$. With regard to the participation rates, $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$.

⁵⁶ See SeyedEsfahani et al. (2011) and Aust and Buser (2012).

Tab.7. Legend

p_{R_1}	Retailer 1 margin
p_{R_2}	Retailer 2 margin
p_{R_*}	Margin of the Bertrand game's winner
p_M	Manufacturer margin
t_1	Participation rate for player 1 advertising
t_2	Participation rate for player 2 advertising
t_*	Participation rate for Bertrand game winner
a_1	Retailer 1 local advertising level
a_2	Retailer 2 local advertising level
a_*	Bertrand game winner local advertising level
A	Manufacturer national advertising level
α	Intercept of the demand-price curve
β	Marginal effect of price on the demand
c	Manufacturer's production cost
d_1	Retailer 1's unit cost
d_2	Retailer 2's unit cost
d_*	Bertrand game winner's unit cost
k_{R_1}	Marginal effect of retailer 1's local advertising (under square root)
k_{R_2}	Marginal effect of retailer 2's local advertising (under square root)
k_{R_*}	Marginal effect of the Bertrand game winner's advertising (under square root)
k_M	Marginal effect of global advertising (under square root)
v	Parameter of the functional form of demand-price curve

4.2. Nash Game

Before we apply the classic NE concept, we have to solve the Bertrand game where players' strategic variable is the price.

An essential assumption to facilitate calculations is to set the minimum quantity for which retailers can undercut their price at 1⁵⁷, and not ε . In addition, we have to impose the other conditions seen in section 2.1.3. too: homogenous goods, price sensitive consumers, and no capacity restrictions for the firms. Furthermore, we hypothesize that, in case of identical prices set by retailers, the consumers will equally split their purchases between the firms. As a result, both retailers have an incentive to lower their price to seize the market, taking into account that their profits have to be positive.

We can distinguish two cases: in the first one, retailers present different marginal cost ($d_1 \neq d_2$); in the second one, retailers' marginal costs are identical ($d_1 = d_2$). We are not interested in the latter situation since, as we have seen in 2.3., if players have the same marginal costs the Bertrand competition will lead to zero profits. Therefore, players do not have any incentive to compete. With regard to the former case, marginal costs can be either $d_1 < d_2$ or $d_1 > d_2$. These scenarios are symmetrical, so we can focus on the circumstance in which $d_1 < d_2 + 1$ - remember that costs can decrease/increase by at least one- and generalize the results.

Hence, imposing the condition:

$$d_1 < d_2 + 1, \quad (4.7)$$

we obtain the following players' BRs:

$$p_1^*(p_2) = \begin{cases} d_1, & p_2 \leq d_1 \\ p_2, & p_2 = d_1 + 1, \\ p_2 - 1, & p_2 > d_1 + 1 \end{cases} \quad (4.8)$$

$$p_2^*(p_1) = \begin{cases} d_2, & p_1 \leq d_2 \\ p_1, & p_1 = d_2 + 1, \\ p_1 - 1, & p_1 > d_2 + 1 \end{cases} \quad (4.9)$$

⁵⁷ The minimum increase and decrease for price variables is 1.

Intersecting (4.8) and (4.9), we derive the optimal retailers' margins⁵⁸:

$$p_1^{BE} = d_2 - 1, \quad (4.10)$$

$$p_2^{BE} = d_2. \quad (4.11)$$

The equilibrium will be: $NE = \{p_1^* = d_2 - 1, p_2^* = d_2\}$. Retailer 1, being more efficient than retailer 2, has incentive to lower his/her price at $d_2 - 1$, so as to supply all the demands on his/her own (note that assumption (4.6) ensures positive profits). At the same time, player 2 is indifferent to offer any price $p_2 \geq d_2$ but, for the sake of simplicity, we impose that he/she sets price at his/her marginal cost.

At the end, the exclusive partnership agreement will be signed between the manufacturer and retailer 1, who now have to compete in a Nash game. Referring to functions (4.3) and (4.4), profits are:

$$\Pi_M = (p_M - c)(\alpha - \beta(p_M + d_2 - 1))^{\frac{1}{\nu}}(k_{R_1}\sqrt{a_1} + k_M\sqrt{A}) - t_1 a_1 - A, \quad (4.12)$$

$$\Pi_{R_1} = (d_2 - d_1 - 1)(\alpha - \beta(p_M + d_2 - 1))^{\frac{1}{\nu}}(k_{R_1}\sqrt{a_1} + k_M\sqrt{A}) - (1 - t_1)a_1, \quad (4.13)$$

in which we have replaced p with $p_M + p_R$ and p_R with $d_2 - 1$. Consequently, the maximization problems will be:

$$\text{Max}_{t_1, A, p_M} \Pi_M = (p_M - c)(\alpha - \beta(p_M + d_2 - 1))^{\frac{1}{\nu}}(k_{R_1}\sqrt{a_1} + k_M\sqrt{A}) - t_1 a_1 - A \quad (4.14)$$

$$\text{s.t. } 0 \leq t_1 \leq 1, \quad A \geq 0, \quad 0 \leq p_M \leq 1,$$

⁵⁸ Throughout this work, the superscript *BE* indicates a Bertrand Equilibrium.

$$\text{Max}_a \Pi_{R_1} = (d_2 - d_1 - 1)(\alpha - \beta(p_M + d_2 - 1))^{\frac{1}{v}}(k_{R_1}\sqrt{a_1} + k_M\sqrt{A}) - (1 - t_1)a_1 \quad (4.15)$$

s.t. $a \geq 0$.

With regard to t_1 , it is easy to see that its optimal value is 0, since it negatively affects the manufacturer's profit function. Then, deriving Π_M with respect to A and p_M , and Π_R with respect to a_1 , we obtain:

$$\frac{\partial \Pi_M}{\partial p_M} = \frac{(\sqrt{A}k_M + \sqrt{a_1}k_{R_1})(\alpha + \beta - \beta d_2 - \beta p_M)^{\frac{1}{v}-1}(\beta c - \beta p_M + \alpha v + \beta v - \beta d_2 v - \beta p_M v)}{v}, \quad (4.16)$$

$$\frac{\partial \Pi_M}{\partial A} = \frac{k_M(p_M - c)(\alpha - \beta(d_2 + p_M - 1))^{\frac{1}{v}}}{2\sqrt{A}} - 1, \quad (4.17)$$

$$\frac{\partial \Pi_{R_1}}{\partial a_1} = \frac{k_{R_1}(d_2 - d_1 - 1)(\alpha - \beta(d_2 + p_M - 1))^{\frac{1}{v}}}{2\sqrt{a_1}} - 1, \quad (4.18)$$

where $v \neq 0$. Applying the FOCs to (4.17) and (4.18) -that are $\frac{\partial \Pi_M}{\partial A} = 0$ and $\frac{\partial \Pi_{R_1}}{\partial a_1} = 0$ - we obtain the following values for A and a_1 :

$$A = \frac{k_M^2(p_M - c)^2(\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}}}{4}, \quad (4.19)$$

$$a_1 = \frac{k_{R_1}^2(d_2 - d_1 - 1)^2(\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}}}{4}. \quad (4.20)$$

Replacing (4.19) and (4.20) in (4.16), the expression becomes:

$$\frac{\partial \Pi_M}{\partial p_M} = \frac{\left(k_M \sqrt{\frac{k_M^2 (p_M - c)^2 (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}}}{4}} + k_{R_1} \sqrt{\frac{k_{R_1}^2 (d_1 - d_2 + 1)^2 (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}}}{4}} \right)}{v} \times \quad (4.21)$$

$$\times \frac{(\alpha + \beta - \beta d_2 - \beta p_M)^{\frac{1}{v}-1} (\beta c - \beta p_M + \alpha v + \beta v - \beta d_2 v - \beta p_M v)}{v}$$

Imposing $\frac{\partial \Pi_M}{\partial p_M} = 0$ (FOC), we find the optimal value of p_M ⁵⁹. Expression (4.21) is equal to 0 when the numerator is 0, which is true if and only if $\beta c - \beta p_M + \alpha v + \beta v - \beta d_2 v - \beta p_M v = 0$ is null, since the other two factors are strictly positive: the former, because it is a sum of two positive factors; the latter, because we have already imposed that the demand function has to be positive. Substituting the optimal value of p_M in (4.17) and (4.18), we calculate⁶⁰ all the equilibrium values of the decision variables:

$$p_M^N = \frac{\beta c + \alpha v + \beta v - \beta d_2 v}{\beta + \beta v}, \quad t^N = 0, \quad (4.22)$$

$$A^N = \frac{k_M^2 v^2 \left(\frac{\alpha + \beta - \beta c - \beta d_2}{v + 1} \right)^{\frac{2}{v}} (\alpha + \beta - \beta c - \beta d_2)^2}{4 \beta^2 (v + 1)^2}, \quad (4.23)$$

$$a_1^N = \frac{k_{R_1}^2 \left(\frac{\alpha + \beta - \beta c - \beta d_2}{v + 1} \right)^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4}. \quad (4.24)$$

4.3. Stackelberg Retailer and Manufacturer Game

The next two games analysed are the Stackelberg ones, where it is assumed an asymmetrical distribution of power. Firstly, we study the Stackelberg Retailer situation. With regard to the first step, the Bertrand game, the reasoning is the same proposed before. Again,

⁵⁹ In this entire chapter, we assume a profit curve progress similar to those derived in papers studied in chapter 3, where the Hessian matrix is always negative definite. Thus, through the FOCs, we find the optimal values of the decision variables.

⁶⁰ The calculations have been performed through a MATLAB computation, whose script is shown in the appendix.

we are interested only in the situation in which $d_1 \neq d_2$, because if marginal costs are identical the retailers' profits will be 0. The same conditions assumed before must be true and the solutions of the game are (4.10) and (4.11), with retailer 1 who signs the exclusive agreement with the manufacturer.

Formally, we start solving the Stackleberg game maximizing the follower's profits. The maximization problem is (4.14) and derivatives with respect to p_M and A are (4.16) and (4.17), respectively. Applying the FOC to (4.17), we derive (4.19) and, replacing (4.19) in (4.16), we obtain the following equation:

$$\frac{\partial \Pi_M}{\partial p_M} = \frac{\left(k_M \sqrt{\frac{k_M^2 (p_M - c)^2 (\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}}}{4}} + k_{R_1} \sqrt{a_1} \right)}{v} \times \frac{(\alpha + \beta - \beta d_2 - \beta p_M)^{\frac{1}{v}-1} (\beta c - \beta p_M + \alpha v + \beta v - \beta d_2 v - \beta p_M v)}{v} \quad (4.25)$$

To find manufacturer's BRs, we need the value of p_M for which $\frac{\partial \Pi_M}{\partial p_M} = 0$. As equation (4.21), expression (4.25) is equal to 0 if and only if $\beta c - \beta p_M + \alpha v + \beta v - \beta d_2 v - \beta p_M v = 0$, since the other two factors at numerator cannot zero -the first one, because it is a square root times a positive parameter; the second one because in section 4.1. we have imposed a positive demand function-.

Consequently, we find that:

$$p_M = \frac{\beta c + \alpha v + \beta v - \beta d_2 v}{\beta + \beta v}, \quad (4.26)$$

which is exactly the same value of equation (4.22). Of course, replacing (4.26) in (4.19), we derive player 1's best response with regard to variable A , that will be equal to (4.23):

$$A = \frac{k_M^2 v^2 \left(\frac{\alpha + \beta - \beta c - \beta d_2}{v + 1} \right)^{\frac{2}{v}} (\alpha + \beta - \beta c - \beta d_2)^2}{4\beta^2 (v + 1)^2}. \quad (4.27)$$

Looking at (4.18) and considering that the optimal values of p_M and A are the same of the Nash game, we can conclude that even the equilibrium value of a_1 will be identical to (4.24). However, for the sake of clarity, we solve the problem in the analytical way, replacing (4.26) and (4.27) in (4.13), that becomes:

$$\Pi_{R_1} = \frac{(d_2 - d_1 - 1)(\alpha + \beta - \beta c - \beta d_2)^{\frac{1}{v}}}{(v + 1)^{\frac{1}{v}}} \left(k_{R_1} \sqrt{a_1} + \frac{k_M^2 v \sqrt{\frac{(\alpha + \beta - \beta c - \beta d_2)^{\frac{2(v+1)}{v}}}{2}}}{2(v + 1)^{\frac{(v+1)}{v}}} \sqrt{A} \right) - a_1. \quad (4.28)$$

Deriving with respect to a_1 , we find

$$\frac{\partial \Pi_{R_1}}{\partial a_1} = \frac{k_{R_1} (d_2 - d_1 - 1)(\alpha + \beta - \beta c - \beta d_2)^{\frac{1}{v}}}{2\sqrt{a_1} (v + 1)^{\frac{1}{v}}} - 1, \quad (4.29)$$

which is equal to 0 for the same value of a_1 expressed in (4.24). Summing up, the equilibrium values will be:

$$p_M^{S_R} = \frac{\beta c + \alpha v + \beta v - \beta d_2 v}{\beta + \beta v}, \quad t^{S_R} = 0, \quad (4.30)$$

$$A^{S_R} = \frac{k_M^2 v^2 \left(\frac{\alpha + \beta - \beta c - \beta d_2}{v + 1} \right)^{\frac{2}{v}} (\alpha + \beta - \beta c - \beta d_2)^2}{4\beta^2 (v + 1)^2}, \quad (4.31)$$

$$a_1^{S_R} = \frac{k_{R_1}^2 \left(\frac{\alpha + \beta - \beta c - \beta d_2}{v + 1} \right)^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4}. \quad (4.32)$$

But why, in this model, do the Nash and the Stackelberg Retailer equilibrium coincide?

The reason is that, introducing a first-step Bertrand game, we remove one decision variable, the retailer price, which is set equal to $d_2 - 1$. Thus, the maximization of the objective function (4.12) is completely independent of the retailer's decision variable a_1 (look at (4.16) and (4.17)). Therefore, solving (4.12) and (4.13) simultaneously or before the (4.12) and successively the (4.13) will obviously lead to the same results.

Let me investigate further on the Nash and Stackelberg Retailer's solutions found above, focusing on the the optimal values for decision variables describing the local (a_1) and global (A) advertising levels. Setting $\alpha = 10$, $\beta = 1$, $c = 1$, and $d_1 = 4$, we plot the following three graphs, considering on the x and y axis the retailer 2's marginal costs d_2 and the retailer 1 local advertising efficiency k_{R_1} , respectively. Some constraints are required. On the x axis, d_2 has to be greater than 5 since the initial condition states that $d_1 < d_2 - 1$ and, at the same time, it has to be smaller than 10, because for every value of v the optimal level of a_1 must be positive.

In Fig.16, in which we have set $v = 1$, we can see that the optimal value of a_1 is positively influenced by k_{R_1} . This is rational, because advertising expenditures will be higher the more is the advertising effectiveness. The effect of d_2 , instead, is positive at the beginning, whilst is negative for higher values of d_2 . The explanation is intuitive: at the beginning, an increase of d_2 affects positively retailer 1's margin ($p_R = d_2 - d_1 - 1$) yet, for higher values, the effect is compensated by the negative influence that a higher margin has on the consumer demand.

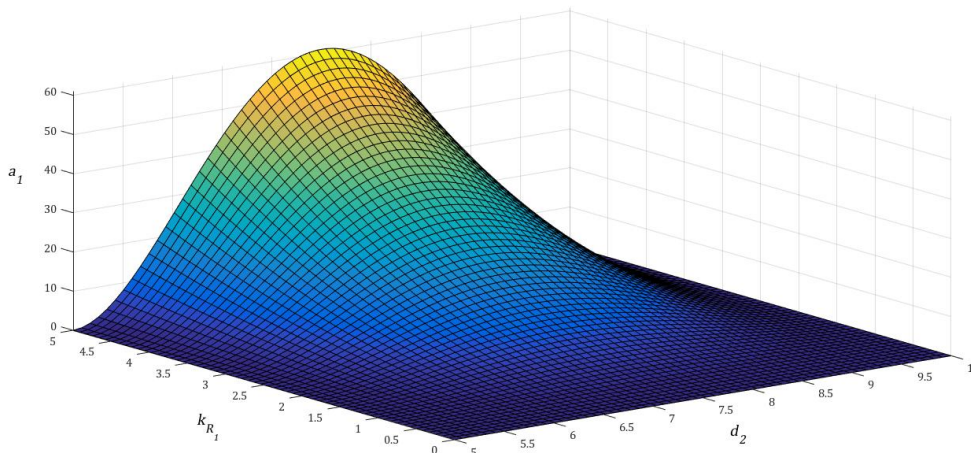


Fig.16. Local advertising level $a_1^N = a_1^{SR}$, with $v = 1$

Moreover, the demand functional form has a significant impact on the advertising level. In Fig.17 and Fig.18, we represent the same situation described before, changing the value of ν , which is equal to 2 and $\frac{1}{2}$ respectively. As clearly shown, a concave demand function (Fig.17, where $\nu > 1$) leads to lower values of a_1 , while a convex one (Fig.18, where $\nu < 1$) to higher.

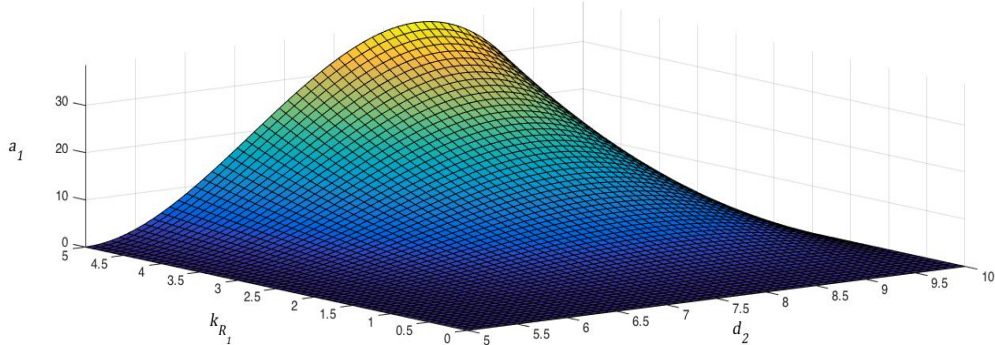


Fig.17. Local advertising level $a_1^N = a_1^{SR}$, with $\nu = 2$

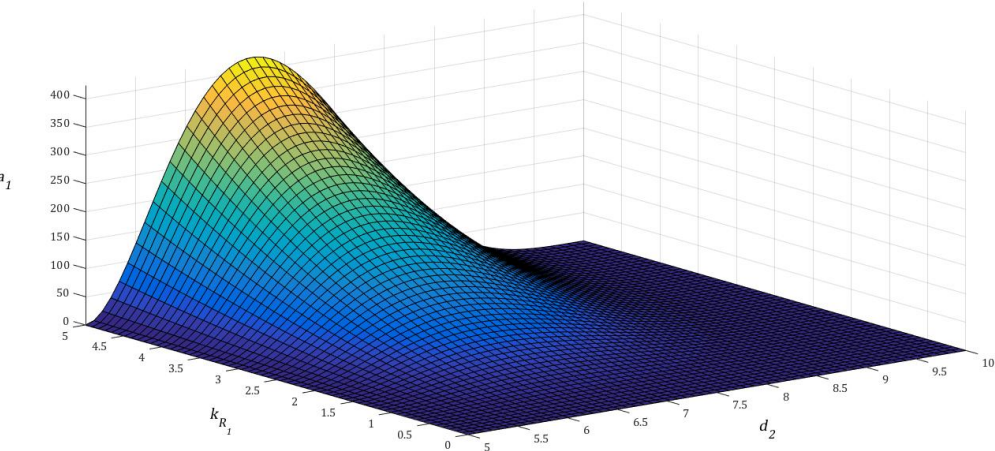


Fig.18. Local advertising level $a_1^N = a_1^{SR}$, with $\nu = \frac{1}{2}$

Overall, the lower is the value of ν , the higher is the retailer 1’s incentive to increase his/her expenditures in advertising.

With regard to the global advertising level A , the results are quite different. Looking at Fig.19, the effectiveness of global advertising is, again, positively related with variable A but, differently from what seen in Fig.16, the optimal value decreases with respect to d_2 . The

manufacturer, in fact, prefers a retailer margin as low as possible, to maximize the volume of sales.

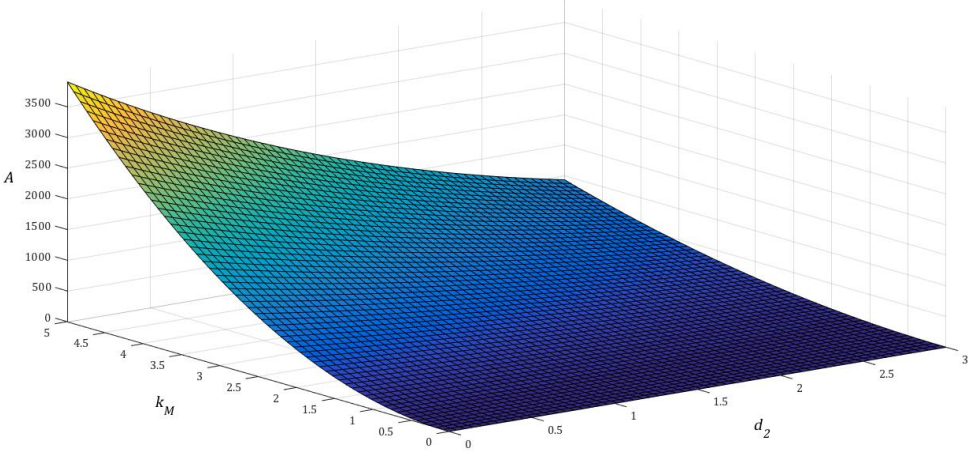


Fig.19. Global advertising level $A^N = A^{S_R}$, with $v = 1$

The effect of the demand functional form and of player’s marginal costs are the same as appreciated for retailer: small values of v and c increase the global advertising level (Fig.20⁶¹).

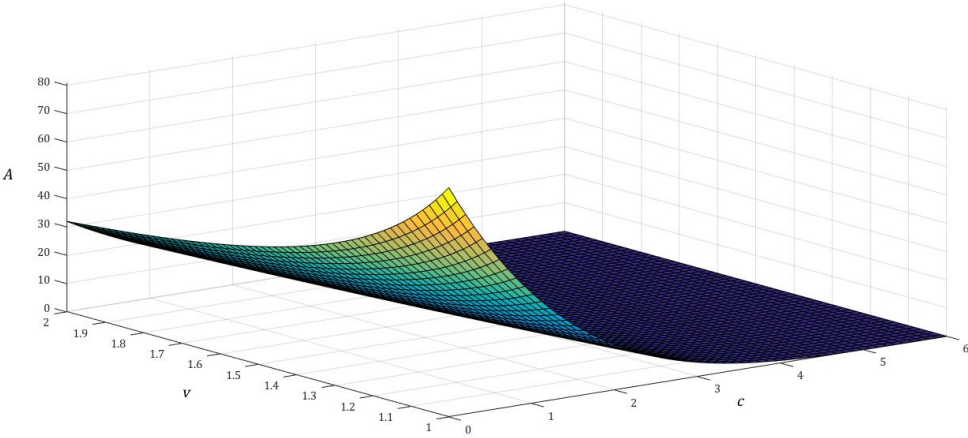


Fig.20. Global advertising level $A^N = A^{S_R}$

⁶¹ To plot this graph we have to set other two parameters. We impose $k_M = 2$ and $d_2 = 5$.

The last situation we analyse is the Stackelberg Manufacturer game. The main difference with the previous scenario is that we cannot set ex ante the optimal participation rate equal to 0. Objective functions are the same -(4.12) and (4.13)-, but now we start deriving (4.13) with regard to a_1 , finding (4.18). Applying the FOC to (4.18):

$$a_1 = \frac{k_{R_1}^2 (\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2}. \quad (4.33)$$

Substituting (4.33) in (4.12), manufacturer's profits becomes:

$$\begin{aligned} \Pi_M = (p_M - c)(\alpha - \beta(p_M + d_2 - 1))^{\frac{1}{v}} & \left(K_{R_1} \sqrt{\frac{k_{R_1}^2 (\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2} + k_M \sqrt{A}} \right) - \\ & - t_1 \frac{k_{R_1}^2 (\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2} - A. \end{aligned} \quad (4.34)$$

Taking the derivative of (4.34) with respect to A , p_M , and t_1 , we find:

$$\frac{\partial \Pi_M}{\partial A} = \frac{k_M (p_M - c)(\alpha - \beta(d_2 + p_M - 1))^{\frac{1}{v}}}{2\sqrt{A}} - 1, \quad (4.35)$$

$$\begin{aligned} \frac{\partial \Pi_M}{\partial t_1} = & \frac{k_{R_1}^2 (\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{2(t_1 - 1)^2} \times \\ & \times \left(\frac{t_1}{t_1 - 1} + \frac{k_{R_1}^2 (c - p_M)(\alpha - \beta(d_2 + p_M - 1))^{\frac{1}{v}}}{2(t_1 - 1) \sqrt{\frac{k_{R_1}^2 (\alpha - \beta(d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2}}} - \frac{1}{2} \right), \end{aligned} \quad (4.36)$$

$$\begin{aligned}
\frac{\partial \Pi_M}{\partial p_M} &= \left(\sqrt{A} k_M + \sqrt{\frac{k_{R_1}^2 (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2}} k_{R_1} \right) (\alpha - \beta (d_2 + p_M - 1))^{\frac{1}{v}} + \\
&+ \frac{\beta \left(\sqrt{A} k_M + \sqrt{\frac{k_{R_1}^2 (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2}} k_{R_1} \right) (\alpha - \beta (d_2 + p_M - 1))^{\frac{1}{v}-1} (c - p_M)}{v} + \\
&+ \frac{\beta k_{R_1}^2 t_1 (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}-1} (d_2 - 1 - d_1)^2}{2v(t_1 - 1)^2} + \\
&+ \frac{\beta k_{R_1}^3 (c - p_M) (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}-1} (\alpha - \beta (d_2 + p_M - 1))^{\frac{1}{v}} (d_2 - 1 - d_1)^2}{4v(t_1 - 1)^2 \sqrt{\frac{k_{R_1}^2 (\alpha - \beta (d_2 + p_M - 1))^{\frac{2}{v}} (d_2 - d_1 - 1)^2}{4(t_1 - 1)^2}}}.
\end{aligned} \tag{4.37}$$

The first step is to find the manufacturer's best response with regard to A . Being the derivative (4.35) exactly the same of (4.17), the BR will be the same as found in (4.19). Now, imposing the FOC on (4.36) $-\frac{\partial \Pi_M}{\partial t_1} = 0$, we derive two values for t :

$$t_1^1 = \frac{2c + d_1 - d_2 - 2p_M + 1}{d_1 - 2c - d_2 + 2p_M + 1}, \quad t_1^2 = \frac{d_1 - 2c - d_2 + 2p_M + 1}{2c + d_1 - d_2 - 2p_M + 1}. \tag{4.38}$$

Bearing in mind that the participation rate has to be $0 \leq t_1 \leq 1$ by definition, solutions in (4.38) can be valid only for specific values of parameters and p_M . To solve the problem we have to replace (4.38) and (4.19) in (4.37), in order to find the optimal value of p_M . Unfortunately, because of the complexity of the expressions, we are not able to conclude the calculations⁶² and we interrupt here our analysis.

⁶² The calculations have been performed through a MATLAB computation, whose script is shown in the appendix.

5. Conclusions

In this work, we have considered several aspects of vertical cooperative advertising strategies, analysing three similar works. All of them presuppose a one-retailer-one-manufacturer supply chain model, in which manufacturers have some incentives to participate in retailers' advertising expenditure, in order to increase the level of local-oriented advertising campaign. This support is known, in literature, as participation rate and its optimal value depends on several crucial factors.

In the models proposed, it is applied a game-theoretic approach, since this tool allows us to incorporate the interdependencies between players. The situation is indeed simplified and only few parameters are taken into account, such as the effectiveness of advertising, the price elasticity of the demand, and players' marginal costs. The effects of other factors, like the competition between multiple manufacturers/retailers, or the influence of complementary and substitutive products, deserve further research.

In chapter 2, we have introduced the essential notions related to game theory, which are required to understand the analysis, focused on four game scenarios. In the Nash game, players do not cooperate and play simultaneously. No one has any informational advantage and participation rate is always zero, since manufacturers have no incentive to set the variable at a different level. On the contrary, in the Stackelberg Retailer and Stackelberg Manufacturer game, one of the players has the leadership: he/she plays before of the other party and, bearing in mind his/her preference and utility function, he/she maximizes his/her profit function. These two kinds of game aim at reproducing a situation in which either the manufacturer or the retailer obtains the channel leadership. The last game is the cooperative one, where authors maximize the overall system profits. However, with regard to the feasibility condition for this game, the one used in the model presented is too restrictive. In previous papers, in fact, the admissibility condition has been always related to the overall cooperative profits, which are compared with the sum of the maximum payoffs achievable by players in the other games. In section 3.3.6., we propose an alternative, according to which we should consider the effective division of cooperative profits between players (through the bargaining model) to determine whether an equilibrium is feasible.

In chapter three, we start our literature review with Xie and Neyret (2009). The authors present an interesting demand function -(3.1)-, finding results for all the scenarios. Yet, so as to derive the objective functions, they assume a very limiting restriction, according to which

manufacturer and retailer set the same price, both in Nash and Stackelberg games. Furthermore, using the Symmetric Nash bargaining model, authors overlook the bargaining power effect.

In SeyedEsfahani et al. (2011), a more general demand function is assumed, which involves a shape parameter (v) and the *saturation advertising effect*. In any case, because of the same assumption about players' prices aforementioned and of the bargaining model used (Symmetric Nash), even these results are not too indicative.

Lastly, we have presented the work by Aust and Busher (2012), which is the most complete and interesting among those studied. The demand function is the same as used in SeyedEsfahani et al. (2011), even though the players' marginal costs are not considered anymore; however, removing the condition imposing identical players' margins, the results obtained are noteworthy. Interestingly, without assumptions (3.10) or (3.36), Nash game's solution leads to equal players' margins but, looking at Fig.11., it is clear that, in Stackelberg games, the identical margin condition is reasonable only for specific values of parameters. Then, applying the Asymmetric Nash bargaining model, authors derive broader bargaining solutions too.

At this point, there are several possible directions for further investigation. First of all, a model involving more decision makers (and not only one retailer and one manufacturer) could be closer to reality. In the same way, a different demand function, a more complex bargaining game, and a new feasibility condition (as that theorised in section 3.3.6.) could enrich the results. We have tried to propose an evolution in chapter 4, where the model becomes a one-manufacturer-two-retailers supply chain. Adding a new player, we have been able to reproduce horizontal price competition between retailers, simulating a situation in which the manufacturer wants to sign an exclusive partnership agreement with the retailer offering the lowest margin, in order to increase the volume of sales. In this case, since the players' functions are discontinuous, we cannot apply the classic solution procedure, but the Bertrand game (see 2.3.). Having solved the first game between retailers, we apply three of the game concepts seen before: the Nash, Stackelberg Manufacturer, and Stackelberg Retailer games.

With regard to the Nash and Stackelberg Retailer games, the results are exactly the same since the retailer's maximization problem is independent of the optimal value of manufacturer's decision variables. Hence, if we solve all the FOCs simultaneously or with a specific order, the results do not change. Looking at Fig.16., Fig.17., Fig.18., and Fig.19., we can see how the retailer 2's marginal costs influence the local (positively at the beginning, negative for higher values of d_2) and global (negatively) advertising level. This result is interesting, since it gives us some insights on the influence that other retailers' marginal costs have on Nash and

Stackelberg equilibria. Lastly, we have tried to solve the Stackelberg Manufacturer game. Here, the calculation process is considerably more complex, because we cannot set the optimal participation rate automatically equal to 0. Unfortunately, even with the support of software such as MATLAB, we have not been able to derive a complete solution for our problem. With regard to this point, next studies could be helpful.

The developments of the literature regarding vertical cooperative advertising strategies can be various. Future research could apply first-step Bertrand competition to different situations, introducing new supply chain members or changing the second-step game (i.e. a cooperative game). Moreover, presupposing a horizontal competition among manufacturers or a different demand function, related to new parameters such as player reputation or market expansion, could be another interesting direction for further investigation⁶³.

⁶³ Number of words: 14,407.

Bibliography

- Alston, J. M., Freebairn, J. W., and James, J. S., 2001, Beggar-Thy-Neighbor Advertising: Theory and Application to Generic Commodity Promotion Programs. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Amrouche, N., Martin-Hérran, G., and Zaccour, G., 2008. Pricing and Advertising of Private and National Brands in a Dynamic Marketing Channel. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Aust, G., 2014. *Vertical Cooperative Advertising in Supply Chain Management: A Game-Theoretic Analysis*, 1st ed. London: Springer.
- Aust, G., and Busher, U., 2012. Vertical Cooperative Advertising and Pricing Decisions in a Manufacturer-Retailer Supply Chain: A Game-Theoretic Approach, *European Journal of Operational Research, 223 (2012): 473-482.*
- Aust, G., and Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, *European Journal of Operational Research, 234 (2013): 1-14.*
- Bass, F. M., et al., 2005. Generic and Brand Advertising Strategies in a Dynamic Duopoly. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Bergen, M., and John, G., 1997. Understanding Cooperative Advertising Participation Rates in Conventional Channels. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Bhattacharyya, S., and Lafontaine, F., 1995, Double Sided Moral Hazard and the Nature of Share Contracts. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Buratto, A., Zaccour, G., 2009. Coordination of Advertising Strategies in a Fashion Licensing Contract. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Chakravarti, A., and Janiszewski, C., 2004. The Influence of Generic Advertising on Brand Preferences. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Crespi, J. M., and James, J. S., 2007. Bargaining Rationale for Cooperative Generic Advertising. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Dant, R. P., and Berger, P. D., 1996. Modeling Cooperative Advertising Decisions in Franchising. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Depken, C. A., Kamerschen, D. R., and Snow, A., 2002, Generic Advertising of Intermediate Goods: Theory and Evidence of Free Riding. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Depken, C. A., and Snow, A., 2008. The Strategic Nature of Advertising in Segmented Markets. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Desai, P. S., 1997. Advertising Fee in Business-Format Franchising, Management Science. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- El Ouardighi, F., Jorgensen, S., and Pasin, F., 2008. A Dynamic Game of Operations and Marketing Management in as Supply Chain. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*
- Erickson, G. M., 2009. An Oligopoly Model of Dynamic Advertising Competition. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research, 234 (2013), pp. 1-3.*

- Harsanyi, J. C., and Selten, R., 1972. A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information. *In: Aust, G., 2014, Vertical Cooperative Advertising in Supply Chain Management: A Game-Theoretic Analysis*, 1st ed. London: Springer, pp.25-26.
- Hempelmann, B., 2006, Optimal Franchise Contracts with Private Cost Information. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Huang, Z., and Li, S. X., 2001. Co-op Advertising Models in Manufacturer-Retailer Supply Chains: A Game Theory Approach. *In: Xie, J., and Neyret, A., 2009. Co-op Advertising and Pricing Models in a Manufacturer-Retailer Supply Chains, Computer and Industrial Engineering*, 56(2009), pp.1375-1376.
- Huang, Z., Li, S. X., and Mahajan, V., 2002. An Analysis of Manufacturer-Retailer Supply Chain. *In: Xie, J., and Neyret, A., 2009. Co-op Advertising and Pricing Models in a Manufacturer-Retailer Supply Chains, Computer and Industrial Engineering*, 56(2009), pp.1375-1376.
- Jorgensen, S., and Zaccour, G., 1999. Equilibrium Pricing and Advertising Strategies in a Marketing Channel. *In: SeyedEsfahani, M. M., et al., 2011. A Game Theoretic Approach to Coordinate Pricing and Vertical Co-op Advertising in Manufacturer-Retailer Supply Chains, European Journal of Operational Research*, 211(2010), pp.264-265.
- Karray, S., 2011. Effectiveness of Retail Joint Promotions Under Different Channel Structures. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Karray, S., and Martin-Hèrran, G., 2008. Investigating the Relationship between Advertising and Pricing in a Channel with Private Label Offering: A Theoretic Model. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Karray, S., and Martin-Hèrran, G., 2009. A Dynamic Model for Advertising and Pricing Competition between National and Store Brands. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Karray, S., and Zaccour, G., 2006. Could Co-op Advertising Be a Manufacturer's Counterstrategy to Store Brands?. *In: SeyedEsfahani, M. M., et al., 2011. A Game Theoretic Approach to Coordinate Pricing and Vertical Co-op Advertising in Manufacturer-Retailer Supply Chains, European Journal of Operational Research*, 211(2010), pp.264-265.
- Kim, S., and Staelin, R., 1999. Manufacturer allowances and retailer pass-through rates in a competitive environment. *In: SeyedEsfahani, M. M., et al., 2011. A Game Theoretic Approach to Coordinate Pricing and Vertical Co-op Advertising in Manufacturer-Retailer Supply Chains, European Journal of Operational Research*, 211(2010), pp.264-265.
- Kinnucan, H. W., 1997. Middlemen Behaviour and Generic Advertising Rents in Competitive Interrelated Industries. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Krishnamurthy, S., 2000. Enlarging the Pie vs Increasing One's Slice: An Analysis of the Relationship Between Generic and Brand Advertising. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Krishnamurthy, S., 2001. The Effect of Provision Points on Generic Advertising Funding. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Lu, M., Thompson, S., and Tu, Y., 2007. An Alternative Strategy for Building Sales of Computers: Generic Advertising. *Journal of Business and Economics Research*, *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Ma, J., and Ulph, A. M., 2012. Advertising Subsidy and International Oligopolistic Competition. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Micheal, S. C., 2002, Can a Franchise Chain Coordinate?. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.

- Miles, M. P., White, J. B., and Munilla, L. S., 1997. Advertising Budgeting Practice in Agribusiness: The Case of Farmer Cooperatives. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Myerson, R. B., 1997. Game Theory: Analysis of Conflict. *In: Aust, G., 2014. Vertical Cooperative Advertising in Supply Chain Management: A Game-Theoretic Analysis*, 1st ed. London: Springer, p.1.
- Nagler, M., 2006. An Explanatory Analysis of the Determinants of Cooperative Advertising Participation Rates, *Marketing Letters*, 17 (2006): 91-102.
- Nash, J. F., 1950. The Bargaining Model, *Econometrica*, 18(1950): 155-162.
- Nielsen, 2015. *Il Mercato Pubblicitario in Italia*, <<http://www.nielsen.com/it/it/insights/news/2016/il-mercato-pubblicitario-in-italia-nel-2015.html>>, visited on 28th March, 2016.
- Nielsen, 2015. *The Total Audience Report*, <<http://www.nielsen.com/us/en/insights/reports/2015.html>>, visited on 22nd April, 2016.
- Piana, V., 2004. Consumer Decision Rules for Agent-Based Models. *In: SeyedEsfahani, M. M., et al., 2011. A Game Theoretic Approach to Coordinante Pricing and Vertical Co-op Advertising in Manufacturer-Retailer Supply Chains, European Journal of Operational Research*, 211(2010), pp.264-265.
- Rao R. C., and Srinivasan, S., 2001. An Analysis of Advertisement Payments in Franchise Contracts. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- SeyedEsfahani, M. M., et al., 2011. A Game Theoretic Approach to Coordinante Pricing and Vertical Co-op Advertising in Manufacturer-Retailer Supply Chains, *European Journal of Operational Research*, 211(2010): 263-273.
- Sen, K. C., 1995, Advertising Fees in the Franchised Channel. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Sigué, S. P., and Chintagunta, P., 2009, Advertising Strategies in a Franchise System. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Simbanegavi, W., 2009., Informative Advertising. Competition or Cooperation?. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Simon, J., Arndt, J., 1980. The shape of the advertising response function. *In: SeyedEsfahani, M. M., et al., 2011. A Game Theoretic Approach to Coordinante Pricing and Vertical Co-op Advertising in Manufacturer-Retailer Supply Chains, European Journal of Operational Research*, 211(2010), pp.264-265.
- Tadelis, S., 2013. *Game Theory: An Introduction*, (s.l.): Princeton University Press.
- Viscolani, B., 2012. Pure Strategy Nash Equilibria in an Advertising Game with Interference. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Viscolani, B., and Zaccour, G., 2009. Advertising Strategies in a Differential Game with Negative Competitors Interference. *In: Aust, G., Busher, U., 2013. Cooperative Advertising Model in Supply Chain Management: A Review, European Journal of Operational Research*, 234 (2013), pp. 1-3.
- Von Stackelberg, H., 1934, *Marktform und Gleichgewicht*, 1st ed. Berlin: Springer.
- Xie, J., and Neyret, A., 2009. Co-op Advertising and Pricing Models in a Manufacturer-Retailer Supply Chains, *Computer and Industrial Engineering*, 56(2009): 1375-1385.

Appendix: MATLAB scripts⁶⁴ used in chapter 4

A.1 Nash Game (Section 4.2.)

```
clear; % remove all variables from the current workspace
clc; % clear all input and output from the Command Window display, giving
a "clean screen"

% Parameters
syms alpha beta v km c d1 d2 kr1 kr2 km;
% Variables of interest
syms pm pr1 pr2 t1 t2 a1 a2 A;

% General conditions
% -----
% 1. Costs must be positive
assume(d1>0);
assume(d2>0);
assume(c>0);

% 2. % Positive advertising effects
assume(kr1>0);
assume(kr2>0);
assume(km>0);

% 3. % Positive advertising level
assume(a1>0);
assume(a2>0);
assume(A>0);

% 4. Positive functional parameter v
assume(v>0);

% 5. Participation rates in [0,1]
assume(t1<=1 & t1>=0);
assume(t2<=1 & t2>=0);

% Nash game. Case 1: d1 < d2 - 1
assume(d1<d2-1);
assume(pm<(alpha/beta-d2+1)); % To avoid negative demand function

t1 = 0; % Optimal value of t1 for Pim (Manufacturer Profit function)

% Pim: Manufacturer Profit function
Pim = (pm - c)*(alpha - beta*(pm + d2 - 1))^(1/v) * (kr1*sqrt(a1) +
km*sqrt(A)) - t1*a1 - A;

% Pir1: Retailer 1 Profit function
Pir1 = (d2 - 1 - d1)*(alpha - beta*(pm + d2 - 1))^(1/v) * (kr1*sqrt(a1) +
km*sqrt(A)) - (1 - t1)*a1;

dPim_pm = diff(Pim, 'pm');
dPim_pm = simplify(dPim_pm);
pretty(dPim_pm);
```

⁶⁴ MATLAB Symbolic Toolbox is needed to run the scripts.

```

dPim_A = diff(Pim,'A');
dPim_A = simplify(dPim_A); % pretty(dPim_A);
% Solution to dPim_A = 0 for A: it depends only on pm
A_sol = solve(dPim_A,A,'ReturnConditions',true);

dPir1_a1 = diff(Pir1,'a1');
dPir1_a1 = simplify(dPir1_a1); % pretty(dPir1_a1);
% Solution to dPir1_a1 = 0 for a1: it depends only on pm
a1_sol = solve(dPir1_a1,a1,'ReturnConditions',true);

dPim_pm_Aalsol = subs(dPim_pm,[A,a1],[A_sol.A,a1_sol.a1]);
dPim_pm_Aalsol = simplify(dPim_pm_Aalsol);

syms Z;
Z = (beta*c - beta*pm + alpha*v + beta*v - beta*d2*v - beta*pm*v);
pmstar = solve(Z,pm,'ReturnConditions',true);
Astar = subs(A_sol.A,pm,pmstar.pm); Astar = simplify(Astar);
alstar = subs(a1_sol.a1,pm,pmstar.pm); alstar = simplify(alstar);

```

A.2 Stackelberg Retailer Game (Section 4.3.)

```

clear; % removes all variables from the current workspace
clc; % clears all input and output from the Command Window display,
giving a "clean screen"

% Parameters
syms alpha beta v km c d1 d2 kr1 kr2 km;
% Variables of interest
syms pm pr1 pr2 t1 t2 a1 a2 A;

% General conditions
% -----
% 1. Costs must be positive
assume(d1>0);
assume(d2>0);
assume(c>0);

% 2. % Positive advertising effects
assume(kr1>0);
assume(kr2>0);
assume(km>0);

% 3. % Positive advertising level
assume(a1>0);
assume(a2>0);
assume(A>0);

% 4. Positive functional parameter v
assume(v>0);

% 5. Participation rates in [0,1]
assume(t1<=1 & t1>=0);
assume(t2<=1 & t2>=0);

% Stackelberg Retailer Game. Case 1:  $d1 < d2 - 1$ 
assume(d1<d2-1);
assume(pm<(alpha/beta-d2+1)); % To avoid negative demand function

```

```

t1 = 0; % Optimal value of t1 for Pim (Manufacturer Profit function)

% Pim: Manufacturer Profit function
Pim = (pm - c)*(alpha - beta*(pm + d2 - 1))^(1/v) * (kr1*sqrt(a1) +
km*sqrt(A)) - t1*a1 - A;

% Pirl: Retailer 1 Profit function
Pirl = (d2 - 1 - d1)*(alpha - beta*(pm + d2 - 1))^(1/v) * (kr1*sqrt(a1) +
km*sqrt(A)) - (1 - t1)*a1;

dPim_pm = diff(Pim, 'pm');
dPim_pm = simplify(dPim_pm); % pretty(dPim_pm);

dPim_A = diff(Pim, 'A');
dPim_A = simplify(dPim_A); % pretty(dPim_A);
% Solution to dPim_A = 0 for A: it depends only on pm
A_sol = solve(dPim_A,A, 'ReturnConditions',true);

dPim_pm_Asol = subs(dPim_pm,A,A_sol.A);
dPim_pm_Asol = simplify(dPim_pm_Asol);

syms Z;
Z = (beta*c - beta*pm + alpha*v + beta*v - beta*d2*v - beta*pm*v);
pmstar = solve(Z,pm, 'ReturnConditions',true);
Astar = subs(A_sol.A,pm,pmstar.pm); Astar = simplify(Astar);

%Replace pmstar and Astar in Pirl
Pirl_pmstar_Astar = subs(Pirl,[pm,A],[pmstar.pm,Astar]);
Pirl_pmstar_Astar = simplify(Pirl_pmstar_Astar);

dPirl_pmstar_Astar_a1 = diff(Pirl_pmstar_Astar, 'a1');
dPirl_pmstar_Astar_a1 = simplify(dPirl_pmstar_Astar_a1);

% Solution to dPirl_pmstar_Astar_a1 = 0
alstar = solve(dPirl_pmstar_Astar_a1,a1, 'ReturnConditions',true); alstar.a1
= simplify(alstar.a1);

```

A.3 Stackelberg Manufacturer Game (Section 4.3.)

```

clear; % removes all variables from the current workspace
clc; % clears all input and output from the Command Window display,
giving a "clean screen"

% Parameters
syms alpha beta v km c d1 d2 kr1 kr2 km;
% Variables of interest
syms pm pr1 pr2 t1 t2 a1 a2 A;

% General conditions
% -----
% 1. Costs must be positive
assume(d1>0);
assume(d2>0);
assume(c>0);

```

```

% 2. % Positive advertising effects
assume(kr1>0);
assume(kr2>0);
assume(km>0);

% 3. % Positive advertising level
assume(a1>0);
assume(a2>0);
assume(A>0);

% 4. Positive functional parameter v
assume(v>0);

% 5. Participation rates in [0,1]
assume(t1<=1 & t1>=0);
assume(t2<=1 & t2>=0);

% Stackelberg Manufacturer Game. Case 1: d1 < d2 - 1
assume(d1<d2-1);
assume(pm<(alpha/beta-d2+1)); % To avoid negative demand function

% Pim: Manufacturer Profit function
Pim = (pm - c)*(alpha - beta*(pm + d2 - 1))^(1/v) * (kr1*sqrt(a1) +
km*sqrt(A)) - t1*a1 - A;

% Pir1: Retailer 1 Profit function
Pir1 = (d2 - 1 - d1)*(alpha - beta*(pm + d2 - 1))^(1/v) * (kr1*sqrt(a1) +
km*sqrt(A)) - (1 - t1)*a1;

dPir1_a1 = diff(Pir1,'a1');
dPir1_a1 = simplify(dPir1_a1);

% Solution to dPir1_a1 = 0;
dPir1_alsol = solve(dPir1_a1,a1,'ReturnConditions',true);
dPir1_alsol.a1 = simplify(dPir1_alsol.a1);

% Replace the optimal value of a1 in Pim;
Pim_a1 = subs(Pim,a1,dPir1_alsol.a1);
Pim_a1 = simplify(Pim_a1);

dPim_a1_pm = diff(Pim_a1,'pm'); dPim_a1_pm = simplify(dPim_a1_pm);
dPim_a1_A = diff(Pim_a1,'A'); dPim_a1_A = simplify(dPim_a1_A);
dPim_a1_t1 = diff(Pim_a1,'t1'); dPim_a1_t1 = simplify(dPim_a1_t1);

% Solution to dPim_a1_A = 0;
dPim_a1_Asol = solve(dPim_a1_A, A, 'ReturnConditions',true);
dPim_a1_Asol = simplify(dPim_a1_Asol.A);

% Solution to dPim_a1_t1 = 0;
dPim_a1_t1sol = solve(dPim_a1_t1,t1, 'ReturnConditions',true);
dPim_a1_t1sol = simplify(dPim_a1_t1sol.t1);

% Two solutions found
dPim_a1_t1sol1 = dPim_a1_t1sol(1); dPim_a1_t1sol2 = dPim_a1_t1sol(2);

% Substitution of Asol and t1sol1
dPim_a1_pm_Asol_t1sol1 =
subs(dPim_a1_pm,[A,t1],[dPim_a1_Asol,dPim_a1_t1sol1]);
dPim_a1_pm_Asol_t1sol1 = simplify(dPim_a1_pm_Asol_t1sol1);

% Solve to get pmstar1. Warning: Cannot find explicit solution.

```

```

pmstar1 = solve(dPim_a1_pm_Asol_t1sol1,pm,'ReturnConditions',true);

% Substitution of Asol and t1sol2
dPim_a1_pm_Asol_t1sol2 =
subs(dPim_a1_pm,[A,t1],[dPim_a1_Asol,dPim_a1_t1sol2]);
dPim_a1_pm_Asol_t1sol2 = simplify(dPim_a1_pm_Asol_t1sol2);

% Solve to get pmstar2. Warning: Cannot find explicit solution.
pmstar2 = solve(dPim_a1_pm_Asol_t1sol2,pm,'ReturnConditions',true);

% Solution to dPim_a1_pm = 0; Warning: Cannot find explicit solution.
%dPim_a1_pmsol = solve(dPim_a1_pm, pm, 'ReturnConditions',true);

```

A.4 Fig.16, 17, and 18

```

clear;
clc;

% Plot of a1* with a linear demand function (v=1). Fig.16.
v=1;
alpha=10;
beta=1;
c=1;
d1=4;

syms a1(kr1,d2);

a1(kr1,d2) = (kr1^2*((alpha+beta-beta*c-beta*d2)/(v+1))^(2/v)*(d2-d1-1)^2)/4;
ezsurf(a1,[5,10,0,5]);

pause;

clear;

% Plot of a1* with a concave demand function (v>1). Fig.17.
v=2;
alpha=10;
beta=1;
c=1;
d1=4;

syms a1(kr1,d2);

a1(kr1,d2) = (kr1^2*((alpha+beta-beta*c-beta*d2)/(v+1))^(2/v)*(d2-d1-1)^2)/4;
ezsurf(a1,[5,10,0,5]);

pause;

clear;

% Plot of a1* with a convex demand function (v<1). Fig.18.
v=0.5;

```

```

alpha=10;
beta=1;
c=1;
d1=4;

syms a1(kr1,d2);

a1(kr1,d2) = (kr1^2*((alpha+beta-beta*c-beta*d2)/(v+1))^(2/v)*(d2-d1-
1)^2)/4;
ezsurf(a1,[5,10,0,5])

```

A.5 Fig.19

```

clear;

% Plot of A* with a linear demand function (v=1). Fig.19.
v=1;
alpha=10;
beta=1;
c=1;

syms A(km,d2);

A(km,d2) = (km^2*v^2*((alpha+beta-beta*c-beta*d2)/(v+1))^(2/v)*(alpha+beta-
beta*c-beta*d2)^2)/(4*beta^2*(v+1)^2);
ezsurf(A,[0,3,0,5]);

```

A.6 Fig.20

```

% Plot of A* with v[1,2]. Fig.20.
km=2
alpha=10;
beta=1;
d2=5;

syms A(v,c);

A(v,c) = (km^2*v^2*((alpha+beta-beta*c-beta*d2)/(v+1))^(2/v)*(alpha+beta-
beta*c-beta*d2)^2)/(4*beta^2*(v+1)^2);
ezsurf(A,[0,6,1,2]);

```