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Game theory

a microeconomic approach

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Purpose

During the past four years the word " economic crisis" has become sadly famous.

All around the world economists, politicians, state secretaries and university professors tried to give concrete advices in order to restore economic growth.

As a young engineering student, who doesn't have the knowledge of a professor, the competence of an economist or the skills of a politician, I would like to give my modest but passionate help by describing what, in my opinion, could help us building a better tomorrow, not only for us but for the future generations as well.

I think that the main way to change in better the unpleasant situation we are living is not only improving our technologic an scientific knowledge, but also creating more jobs.

To create new jobs and boost the innovation we need the birth of new kind of enterprises, we need to bring them in the global market and make them grow.

Obviously this is a very difficult quest, especially nowadays, but I personally saw in Game Theory one of the more useful lines that a young enterpreneur can follow to realize his ideas.

This is the reason that brings me to give in this thesis a description of this foudamental and ,in some ways, "different" mathematic theory and focus on the useful market applications that a hi-tech start-up founder can use in order to build a successful business which could create jobs and economic prosperity to the society.

1 Game theory

1.1 What is this?

First of all let's define what a game is. We consider a **game** a theoretical description of conflicts of interest. To be more clear, think about a political controversy between two countries, a business confict between two corporations in a specific sector, a football player shooting a penalty, a husband arguing with his wife about who should wash the dishes. These are few examples of what we can consider a game.

It is easy to see how wise is the range of this definition since we play several games during everyday life. But how can we mathematically describe such situations?

First of all we must consider all the decisions that could be made by players and their consequences. Based on the consequences we give to each player a payoff and then we analize the game. This explanation is rough and simple but we will deepen it later.

It is useful to describe all the possible forms a game can assume.

- **COOPERATIVE GAMES**

  In these games there are coalitions of players, and we specife only the payoff of each potential group although we can't say anything on how and why these groups decided to cooperate. A practical example could be the parliament or the senate during the discussion and the vote of a law.

- **NONCOOPERATIVE GAMES**

  Noncooperative games is concerned with the analysis of strategic conflicts where players act in their own interest only. This assuption does not exclude the possibility of a coalition, but this coalition's purpose is to maximise each own payoff.

- **EXTENSIVE GAMES**

  We will call extensive games that kind of games where there is no temporal content. Every decision is made simoultaneously without knowing the choices of other players.

- **NONEXTENSIVE GAMES**

  Nonextensive games, in opposition to the previous definition, consider the temporal content and players can be over time informed about the actions of others.

1.2   Terminology

We should give now a sort of glossary for the most common and important words used throughout this work.

- **Common knowledge**

  We define as common knowledge a fact that is noticed by all players, and e

- **Dominating strategy**

  We say that a strategy is dominating over an other strategy when it always gives a better payoff to the considered player regardless the actions of other players.

- **Extensive game**

  A game graphically described with a tree is called an extensive game.

- **Payoff**

  A payoff is a number that reflects the benefit of an outcome to a player.

- **Perfect information**

  We talk about perfect information when a player knows every move that has been made until then.

- **Rationality of players**
  This is a foundamental assumption. We say that the players are rational when they always play the strategy which maximises their own payoff.

- **Strategy**

  We call strategy an element of the set of all possible actions.

- **Mixed Strategy**

  A strategy is called mixed when a player considers probability in the decision-making process.

2 Dominating strategies

2.1 Dominance

We make the assumption that all players are rational. A rational player makes a choice that gives him the payoff he prefers most, considering what his opponents do.

In a limit case the player has two strategies, A and B, and he finds out that, regardless the combination of choices of other players, the outcome of A is always better than the one resulting from B. Then strategy A is said to be a **dominant strategy** over B.

Since every player is rational, nobody will choose to play a dominated strategy.

The following examples illustrate more clearly this ideas,

- **Corporations in a competitive market**

  Suppose we have two technology corporations, HR inc. and PINEAPPLE computers inc. (PA). These corporations must decide the volume of production of smartphones for the next year in order to take over the market and gain more profits. For now we assume that they are not forced to take their decision simultaneously and that both of them have perfect information on the decisions that have been made.
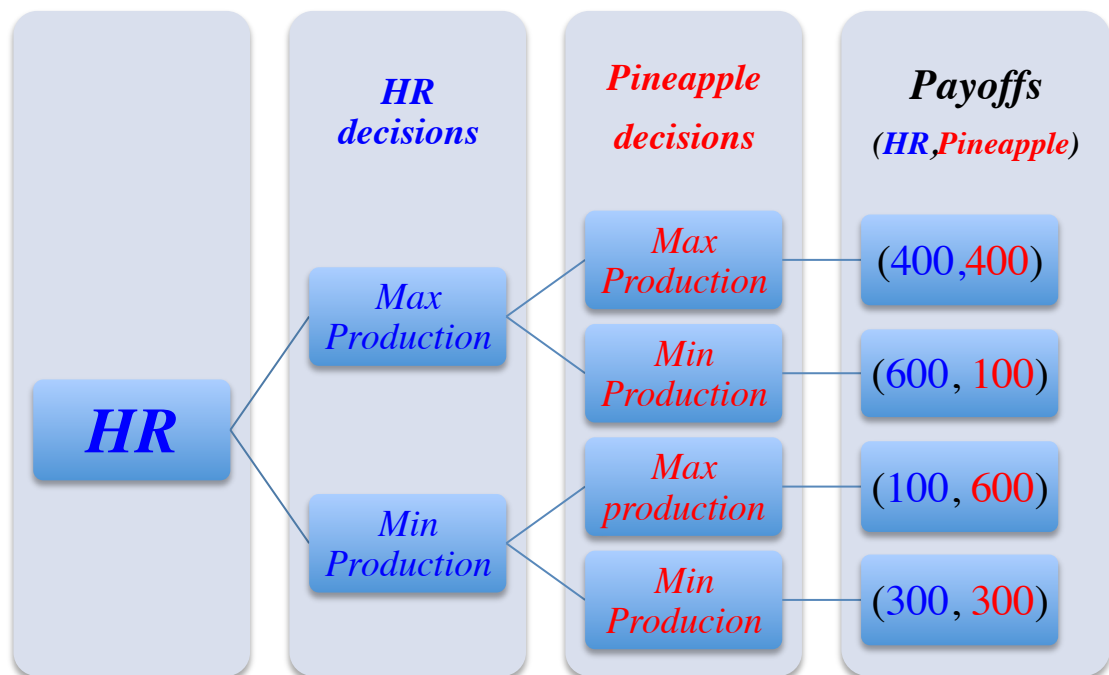

Figure 1.1: Tree representation of the game.

The graphic tree representation above gives a more clear view of the game.

HR has the first move and must decide before Pinapple the volume of production of the product. If HR chooses to produce smartphones at the maximum capacity of its facilities, in response Pineapple can decide to produce at the top of its resources or at the minimum. If Pineapple chooses the first option they will gain a profit of 400 millions $ both, while, if the second option is preferred, HR will recieve profits for 600 millions $ and Pineapple only for 100 millions $. But these decisions are simple actions, they are not strategies. We define **strategy** a set of decions that a player (in this case the two corporations) can take in contrast to the situations which can develop during the game. In the tree described above, a strategy put in display what the player will do for every decisional node of the tree. When, like in this case, the decisions of the playes are not at the same time, we can define a **decision rule** which specify what decision the player who moves for second should take in response to other players choices. For example a decision rule for Pineapple could be: " if HR opts for the max production, we will opt for max production too".

We conclude this paragraph showing another useful way to represent graphically a game, the so called **matrix representation.**

HR Pineapple	Max Production	Min production
Max production	400 400	100 600
Min production	600 100	300 300

This matrix, which is very simple in this case, shows clearly the various options and their payoff for every player and can be used in a large variety of games. But while the tree

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[1] Usually the names of the players are written in the first cell of the table, while in the other cells of the first column and row are written, respectively, the options of the first and of the second player. In the other cells usually are reported the payoffs of the player, higlighted with a different color in this case.

representation gives us the possibility to describe decisions taken in different times, the matrix drawed above suits more for cuncurrent selections of the action to play. That is the reason why we will use the table representation for *extensive games* while we will prefer the tree representation for *nonextensive games*.

After having introduced in a general way the notion of dominance, we give some formal mathematic definitions. We will always refer to vectors by writing them in bold letters.

Definition: Payoff

For an N-person game we say that the function $e_i(x_1, x_2,\ldots, x_i,\ldots, x_N)$ is the **payoff** to player i if players $1, 2, 3,\ldots i-1, i, i+1,.., N$ play the strategies $x_1, x_2,\ldots x_{i-1}, x_i, x_{i+1},\ldots, x_N$.

Definition: Dominance

For a *two person* game we say that a couple of solutions (x_2, y_2) **dominates** (x_1, y_1) if $e_1(x_2, y_2) \geq e_1(x_1, y_1)$ with strict inequality in at least one case. If a game is between more than two players we can extend the above definition using vectors of solutions $(x_1, y_1, a_1, b_1,\ldots, z_1)$.

Definition: Pareto Optimality

A pair of strategies (x, y) is **Pareto[2] optimal** if it is not dominated.

Definition: Strictly sense solution

A game have a solution in the **strictly sense** if:

1) There is an equilibrium pairs among Pareto optimal pairs.
2) All Pareto optimal equilibrium are **interchangeable** and have the same payoffs. The solution is the set of Pareto optimal equilibrium pairs.

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[2] Named after Vilfredo Pareto (1848, 1923), Italian mathematician and economist (Enciclopedia Treccani).

2.2 Equilibrium in dominating strategies

After having introduced the notion of dominance, we move toward a first defininition of quilibrium by using the previous example. Suppose HR's Board of directors decides to produce the new smartphone using minimum production capabilities and you are the CEO of Pineapple. Using the matrix or the tree illustrated in the previous pages, we can see that the best solution for Pineapple is to produce the maximum possible number of smartphones and gain profits for 600 millions, which are far more appetizing than the 100 millions of the other choice. In the same way, if Hr chooses to conquer the market using all the available productive potential, Pineapple's CEO will prefer to compete directly by maximizing the production since his payoff will be of 400 millions, which is better than the other payoff. The same line of reasoning can be follow by HR's CEO when deciding the strategy his company should follow. From the graphs above it is clear that the best option for HR is to produce the higher number possible of goods in order to recieve the highest payoff without depending on Pineapple's decisions. Therefore a strategy that works well at least as any other and doesn't depend on other players decision is defined **dominant.** There is no reason a player should play a non-dominant strategy if he has got a dominant strategy. As a consequence of this attitude every player during the game will choose his favourite dominating strategy. Hence we can conclude that *when each player can play a dominating strategy, we have the only reasonable equilibrium solution when every player follow his own dominating strategy*. The set of dominating strategies and the payoffs of the resulting game are the foudamentals of the **equilibrium in dominating strategies.**

2.2.1 The Nash and reliability conditions

Choosing the dominating strategy is not the only necessary condition to reach equilibrium. In fact there are other two foundamental conditions we have never mentioned before but that must be introduced, the **Nash condition**[3] and the **reliability condition**.

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[3] John Forbes Nash, Jr (born June 13, 1928), Nobel Laureate 1994. He discussed the propreties of the equilibrium which later will take his name in 1950 during his Phd dissertation under the doctoral advice of Albert.W. Tucker in Princeton University, New Jersey, United States of America (A beautiful mind: a biography of John Forbes Nash ,jr. , winner of the nobel prize in economics).

The first condition, named after the name of the famous mathematician John Forbes Nash, states that none player can change unilaterally his decision during the game, in other words every player should have determined an optimum response to the other players decisions. Because every dominating strategy is an optimum response to every kind of strategy, an equilibrium of dominating strategies satisfies for sure the Nash condition. Now it is useful to give the formal mathematical definition of this condition.

Definition: Nash equilibrium

Let (S, f) be a game with n players, where S_i is the strategy set for player i, $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of strategy profiles and $f = (f_1(x),\ldots,f_n(x))$ the payoff function for $x \in S$. Let x_i be a strategy profile for player i and x_{-1} be a strategy profile for every player except player i. When each player $i \in \{1,2,3,\ldots.N\}$ chooses strategy x_i resulting in strategy profile $x =(x_1, x_2,\ldots,x_n)$ then player I obtains a payoff $f(i)$.

The payoff depends on the strategy profile chosen, on the strategy chosen by player i as well on the strategies chosen by other players. This strategy $x^* \in \boldsymbol{S}$ profile is a *Nash equilibrium* if no unilateral deviation in strategy by any single player is profitable for that player, that is

$$\forall\, i,\, x_i \in S,\, x_i \neq x_i^* : f_i(x_i^*,\, x_{-i}) \geq f_i(x_i^*,\, x_{-i}^*)$$

In addition when the above inequality holds strictly (with $>$ instead of $\geq$) for all players and all practicable strategies we talk about **strict Nash equilibrium.** If instead, for some player, there is an exact equality between x_i^* and another strategy in the set $\boldsymbol{S},$ we call the equilibrium a **weak Nash equilibrium**.

The **reliability condition,** instead, request that during the selection process of an action, the player has an effective interest in choosing the action contained in his strategy.

Reporting these considerations in the previous example, we see that HR's strategy satisfies the reliability condition without any doubts, since this strategy contemplate only one action and the Nash condition is enough to guarantee that is in HR's interest adopting the

dominating strategy. We now demonstrate that Pineapple's strategy too satisfies these two conditions even if a little bit more difficult.

First of all the Pineapple's strategy to produce a lot of smartphones if HR do the same satisfies the reliability condition; in reality this is what the corporation does in an equilibrium condition. The most critical aspect to verify is if the *threat* of Pineapple to use all her production capabilities in case HR produce a low quantity of goods is trustworthy or not. But if we look at the game tree we see that it is: in fact if HR chooses the minimum production Pineapple would have a payoff of 600 millions producing at the top of her capabilities and a payoff of 300 millions in the other case.

It is not surprising that Pineapple's strategy is trustworthy since is dominating and so, whatever HR decides to do, works at least as well as any other available strategy.

We finally found out that the equilibrium that we find satisfies both Nash and reliability conditions.

2.2.2 Perfect equilibrium

The previous game was pretty simple because there was an equilibrium made of dominating strategies but unfortunately this isn't the most common case.

Suppose we have the same previous example but with different payoffs like illustrated in the tree below.

In this case there isn't an equilibrium of dominating strategies. In fact while Pineapple can produce a lot if HR decides to produce less and vice versa, which is a dominating strategy, HR hasn't got this possibility. As we can see from the tree, HR has the handicap to have the first move and so the Board of directors of this company should foresee the decisions of Pineapple's management. From the considerations made in the preceding paragraphs, it would be logically correct to expect that Pineapple will choose its dominating strategy but, at the same time, one of the main axioms of game theory states that every player is conscious of other players rationality in the game. As a result of that HR should expect that its rival will choose one of its dominating strategy because every other decision would be irrational. Before proceeding with our reasoning, it is very useful to introduce a method of analysis of the game tree that we will call *induction*. This method can be described in the following steps referred to the game tree:

1) Find the last decision that one of the players must make before the payoffs are revealed.

2) For every of these decisional nodes, detect the ones that maximizes the payoff of the player.

3) Develop a strategy assuming that every player chooses the strategy that maximizes his payoff for every decisional node.

4) Use this strategy in order to direct the actions of the player that must move for first.

It is simple to see that this line of reasoning forces HR to ignore every type of menace or agreement with Pineapple. This example will help us to understand that the Nash condition only is not sufficient to provide equilibrium. In fact if we search the equilibrium by using only the Nash condition we would have two possible conditions:

- HR produces the maximum quantity of products, Pineapple produces the minimal amount

- HR produces the minimal amount, Pineapple produces the maximum quantity.

However the arguments behind the second equilibrium point isn't that convincing. In fact HR would choose the strategy to produce less in order to avoid that Pineapple's decision could damage both corporations profits. But to be trustworthy a menace must give an effective bonus to who can realize it. Let's use this concept in the formulation of a strategy. It is extremely important that, when some player must decide, that it is in his own interest to take in *that very moment* the decision which suits best to his equilibrium strategy. Suppose that Pineapple menaced to invade the market if HR's strategy include a massive production of goods but let's assume also that this menace isn't considered trustworthy by HR director who decides to follow the more aggressive strategy. What reaction should he wait from Pineapple's manager? Unfortunately for Pineapple's investors the menace didn't worked in the hoped way, so the manager they have appointed must "surrender" and produce less than HR in order to receive more profits. Finally we arrived at the equilibrium point where HR's optimum response to Pineapple's dominating strategy is producing the maximum possible number of smartphones and the dominating strategy of Pineapple is the best reply to HR's strategy. We showed that these strategies satisfy both the Nash and reliability condition and we call ***perfect equilibrium*** the equilibrium founded in the previous lines.
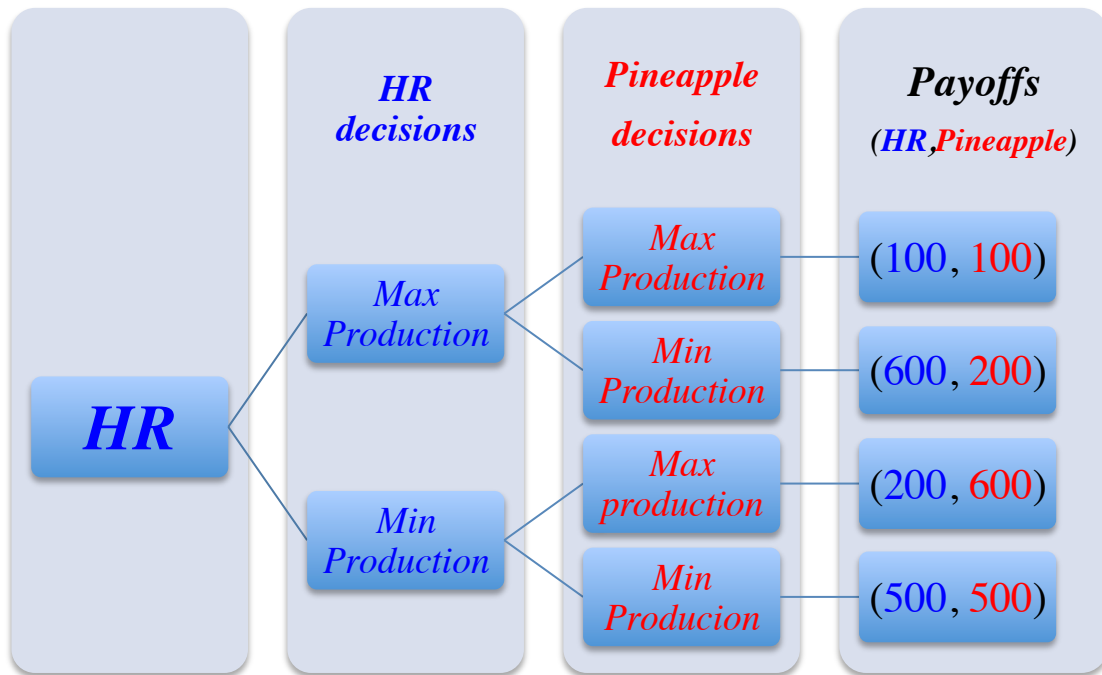
Figure 2.1: *Tree representation of the game with the new payoffs.*

3 Games with imperfect information

3.1 Definition

In the previous paragraphs we analized only games where players make their decisions consequentially and every player is perfectly aware about every move of his rivals when he must take a decision. Unfortunately the reality of facts isn't that simple. It is very likely that a player, at the time he has to make a choice, doesn't know which strategic line his opponent decided to follow. That can occur because players decide simoultaneously, so we are in the case of a game in strategic form or *nonextensive game*, or because they simply hide their actions. Real life by the way is plenty of examples. Think about the Cold War when U.S.A. and U.R.S.S. were in strict comptetition to reach the space for first. Their both main concern was to hide to the opponent their technology status although they both was very intrested to find out their rival's scientific achievements. We can find the same line of reasoning when talking about the competition between two or more hi-tech companies which are going to launch a new product in the market. They both would be very pleased if their competitors give them a draft with the progress of their technology, their business plan and their intentions for the future. But this will remain nothing but a dream and that's the reason there are patents and corporations usually hire a lot of lawyers, to protect their own technlgy and informations.

When a player must choose an action without knowing any decisions taken before or simultaneously by other players we talk about an ***imperfect information game.***

For these reasons, when a player must make up his mind, he doesnt' know at which node of the game tree he is although we can define the set that contains player's actions: it is the set of the nodes in which a player could be find because of his opponents' moves.

3.2 The prisoner's dilemma

Here we describe one of the most famous example of a imperfect information game.

Suppose you and your best friend robbed a bank but, unfortunately, the police collected enough proofs to be sure that you both are the culprits and, as a result, closed you in two

separate and remote cells. The chief officer of the police station, who must question you about the fact, however, gives you the possibility of being freed immediately if you testify against your friend and ,at the same time, his assistant make the same proposal to your friend. What you don't know is this:

- If you testify one against the other (confess), you both will recieve 20 years of prison.
- If you testify but your friend doesn't, you will be immediately freed and your friend will recieve 40 years of prison,
- If you don't testify but your friend isn't the kind of friend you thought and testifies against you, you will recieve 40 years of prison and he will be freed immediately.
- If you both decide to be loyal and don't confess you will recieve 10 years of prison each.

Let's make this situation clearer by drawing the matrix of this game.

Your friend **You**	**Confess**	**Doesn't confess**
Confess	**- 20** **- 20**	**- 40** **0**
Don't confess	**0** **- 40**	**- 10** **- 10**

Figure 3.1: *the prisoner dilemma game. Payoffs shows the respective years of prison for each player and are negative because are token away from players' lives.*

Because no one knows the decision of the other, it is quite simple to determine the equilibrium condition. For you and your best friend one dominating strategy could be *confess*. We have the only equilibrium solution when you both confess, although it would be better if you both don't confess at all. More generally we will define as ***prisoner's dilemma*** every game in which the simultaneous decisions of dominating strategies by all players gives an output situation in which every player recieve a worse payoff than the one

recieved if he had choose a different strategy. Let's return to the example of HR and Pineapple to prove it.

HR Pineapple	Max Production	Min production
Max Production	400 400	300 700
Min production	700 300	700 700

Figure 3.2: the prisoner dilemma game applied to the corporations game.

As we can see from the matrix, HR an Pineapple would recieve the maximum payoff if they would sign smultaneously an agreement to enter the market in a more "soft" way. Unfortunately this type of collaboration ,in theory, isn't that realistic and very difficult to obtain in real life. In fact the equilibrium point will be reached only if Pineapple and HR decide to enter the market in an aggressive way by producing all the smartphones they can. But as we saw before this way of reasoning will in some ways damage both the corporations' profits. This problem is very usual in the real world, especially when the players are represented by oligopolist[4] corporations.

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[4] In economic theory we define an oligopoly a market form in which a market of a specific good or industry is dominated by a small number of sellers which are called oligopolist.

4 Mixed strategies

4.1 Definition

A game in a strategic form does not always have a Nash equilibrium in which each player chooses one of his strategies. There are cases in which players can decide to play a game by selecting *randomly* among the available pure strategies. This kind of behaviour in which a player randomizes his own choice is defined ***mixed strategy game***. Nash proved in 1951 that, if mixed strategy are allowed, any finite strategic-form game has an equilibrium. It is quite clear that in this case we can't no longer consider payoffs in the way we did in previous chapters since the total payoff could vary during the game according to the chosen strategies and their probabilities. Therefore we will define the ***average payoff***, which is the ***expected*** payoff that must be considered since the result of the game may be random.

4.2 Practical example

Nothing is more useful in these cases than an example to explain how mixed strategy works. In order to mantain a useful link with other paragraphs we will focus this example on the technology industry, so suppose that, as players, we have Pineapple and a young company that we call YoungCo.
As said above, YoungCo is a developing and growing firm and hasn't got any time or capitals to make itself a develop software necessary to program the brand new operative system for its PCs. This is the reason that forced the managment to purchase a limited use licence from Pineapple for its develop software package. But by purchasing this package, YoungCo signed an agreement on some very restrictive rules that could damage the productivity of its software division, as a result YoungCo has a lot of incentives to violate the agreement. Pineapple would like to verify that its client is respecting the agreement, this can be done through some or several inspections which are expensive although the corporation can require a large penalty payment in case of noncompliance. Let's schematize the situation with the game matrix.

YoungCo / Pineapple	Comply	Cheat
Don't inspect	**0** **0**	**20** **- 20**
Inspect	**0** **- 5**	**- 100** **-15**

Figure 4.1: The matrix representation for the game discussed above. Notice that for Pineapple inspection gives a negative payoff since it is costly. Numbers represents thousands of dollars.

The previous figure gives a more explicit view of the situation. It is clear that we have four possible outcomes:

- (**Don't inspect**, **Comply**). This is the situation where YoungCo is totally honest and Pineapple trusts YoungCo's loyalty. In this case they both recieve a payoff of 0 since Pineapple doesn't spend any money on the inspection and YoungCo doesn't anything illegal.

- (**Inspect**, **Comply**). In this case YoungCo honors the agreement but Pineapple prefers not to trust it and to make an inspection. That's the reason why Pineapple loses 5000 $ while YoungCo loses nothing.

- (**Don't inspect**,**Cheat**). That's the situation in which Pineapple is too trustful of YoungCo and doesn't make any inspection even if it should because YoungCo is saving a lot of money (20000 $) by cheating its software. The more money YoungCo earns, the more money Pineapple loses, so it goes under of 20000$.

- **(Inspect,Cheat).** YoungCo tried to fool Pineapple by using its software without the appropriate licence but Pineapple made an inspection to prevent it.

  Consequences are tough for YoungCo which now must pay a 100000$ dollars bill to Pineapple which, in contrast, can cover the losses caused by the unlicensed use of its program although it losses as well.

The game seems pretty difficult but let's analize it rationally. In all cases Pineapple would strongly prefer that YoungCo decides to comply, but unfortunately it is outside its control. However if Pineapple always decides to don't inspect, this would be a dominating strategy and this would be part of an unique equilibrium point where YoungCo cheats. It is easy to see that this game has no equilibrium in pure strategies.

In fact if any of the players set on a deterministic choice, the best response of the other player would be unique. That is the case, for example, of Pineapple which chooses to don't inspect and YoungCo that, as a consequence, decides to cheat. Remember that the strategies in a Nash equilibrium must be best reponses to each other, so in this game this fails to hold for any pure strategy combination.

4.3 Mixed equilibrium

How should Pineapple and YoungCo behave in a game like that? One solution is that they both prepare for the worst. A strategy whose main target is maximizing the player's worst payoff is called a ***maximum-minimum strategy***[5]. A max-min strategy for Pineapple is to ***inspect*** and for YoungCo is to ***comply***, however this strategy does not give a Nash equilibrium since YoungCo could switch his strategy and decide to cheat in order to recieve an higher payoff. So what can be a mixed strategy? The answer is pretty simple. For example a good mixed strategy for Pineapple is to inspect YoungCo only witha a certain frequency and probability. Randomizing the inspections is also an approach that reduces costs and discourage YoungCo from using the software illegaly since even a low probability of being caught can "scare" its management. But how many times Pineapple should inspect and with which probability? Let's answer to this question evaluating the

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[5] In game theory we define a *maximum-minimum strategy* a sequence of choices that aim to maximizes a player payoff against all possible choices of the opponent.

possible payoffs of YoungCo. If, for example, the probability of inspection is very low, say 3%, YoungCo recieves a payoff of 0 for fulfill the agreement and a payoff of **0.97*20 + 0.03*(-100)** which gives a **16.4** payoff for cheat, which is bigger than what the company could recieve if it doesn't cheat. So it is useful to see what happens if Pineapple raises the inspection probabilities. Suppose that the new probability is much higher, say 0.3, according to the previous calculation we have **0.7*20+0.3*(-100)** whose result is **-16**, which is a negative payoff. In this case it is clear that is far more better for YoungCo to comply since the expected payoff isn't worth the risk. From the previous considerations we see that if the probability of being caught is too high or too low, YoungCo has only one best response and, as shown above, this pure strategy can not be a part of a pure equilibrium. The only case where YoungCo could randomize between its strategies is if *both strategies* give the same payoff. Logically is never optimal for a player to assing a positive and high probability to a strategy that is inferior given the choices of other players. Let's find out the probability that make YoungCo indifferent. By solving a simple equation we find out that the value of this probability is around 0,83, in fact with this value we have **0.83*20+0.83(-100)=0**. With this mixed strategy of Pineapple, YoungCo is indifferent between its strategies. As a result it can mixed them without losing profits. There is only one case where, in turn, the original mixed strategy is a best response and that is if Pineapple is *indifferent*. According to the payoffs given above this condition requires YoungCo to comply and cheat with a probability of 0.5. Then the expected payoffs of Pineapple are then **0.5*0+0.5*(-20)=-10** and **0.5*(-5)+0.5*(-15)=-10** and Pineapple is indifferent and its mixed strategy is the best response for the strategy of YoungCo. This case defines the only Nash equilibrium of the game. Since it uses mixed strategies it is called ***mixed equilibrium.***

Definition: Mixed equilibrium

In an n-person non-cooperative game, the n-tuple of strategies $x^*_1, x^*_2,....., x^*_n$, where player *i* plays the mixed strategy x^*_i , is an equilibrium n-tuple if for all other strategies $y_1, y_2, ...y_n$:

$$e_i(x^*_1, x^*_2,.,x^*_i, ..., x^*_n) \geq e_i(x^*_1, x^*_2,.,y_i, ..., x^*_n) \qquad 1 \leq i \leq n$$

Theorem: Any finite n-person non-cooperative has at least one equilibrium couple.

4.4   Interpretation of mixed strategy equilibrium

Let's sum up some conclusions from the previous considerations. As we saw above, mixed strategies are in some ways "odd" if compared to other games since payoffs of this kind of games can express much more than the mere monetary profits. In fact from the payoff recieved by a player we can recognize his attitude in some situations since we can see if he is a risk lover or is more judicious and prefers not to gamble against probabilities in order to recieve an higher payoff. Furthermore the reward of the game can represent other things less tangible like the satisfaction or the delusion of a player consequent to the win or the loss. All these parametres which are non-directly quantifiable represent the so called *political features* of game theory but this analysis exceeds the purpose of this research. Another feature of mixed strategies which deserves some attention is the that tha fact of mixing could seem paradoxical when the player is indifferent in equilibrium. Why, for example, should YoungCo gamble if it can equally comply or cheat? In fact YoungCo could be safer by always complying and, by doing so, recieving a payoff of zero. But the fact is that there is no incentive to prefer a strategy over an other, so the player can mix without any problems and reach the equilibrium. The last aspect of the mixed equilibrium, which is the least clear too, is that the probabilies depend on the payoff of the *opponent*, and not on player's own. In fact it would be very reasonable to expect that increasing the penalty for cheating decreases the probabilty of Pineapple being defrauded in the final equilibrium. But it does not, the only thing that changes is the probability of inspection, which is gradually decreased until the consumer is indifferent.

5 Market Games

5.1 Edgeworth Market Games

In the previous paragraphs we have analized market conflict situation in a very specific way, now it's the time to generalize what we saw and give a more global view of the subject. One of the earliest application of game theory was in mathematical economics to describe some rules in trading and commerce. The most direct mathematical model for this kind of situations was given by Francis Ysidro Edgeworth[6]. In the economic model which takes his name, Edgeworth supposed that there were only two commoditites to be traded. Since electronic goods didn't exist in XIXth century, he used apples and bread for his example. So let's identify apples with A and bread with B in order to simplify things. Assume there are M apple traders an N bread traders. Assume in the same way that an apple trader starts with a_i apples and that a bread trader strarts with b_i bread. In the following lines we will refer to an apple or a bread trader as, respectively, *(a$_i$, 0) i=1, 2, …., N* and *(0, b$_i$) i=1, 2, …..M* . The meaning of the overhanging two-tuples is that, at the beginning of the game, the apple trader has a_i apples and 0 pieces of bread and vice versa. We call the *utility* of trader *i* the amount of apples and bread he or she has, and we write it *u$_i$(a, b)*.

5.2 [1, 1] Market games

Since there is only one trader for each type of commodity, this is the most simple condition, where the A trader strats with *(a, 0)* apples and the B trader starts with *(0, b)* pieces of bread. Hedgeworth developed a graph that represents the results of the trading between the two players which is called, not surprisingly, the *Edgeworth box*.

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[6] Named after Francis Ysidro Edgeworth (1845-1926). Edgeworth was an Irish philosopher who made significant contributions to economics and statistics during the period between IX and X cebtury (Enciclopedia Treccani).
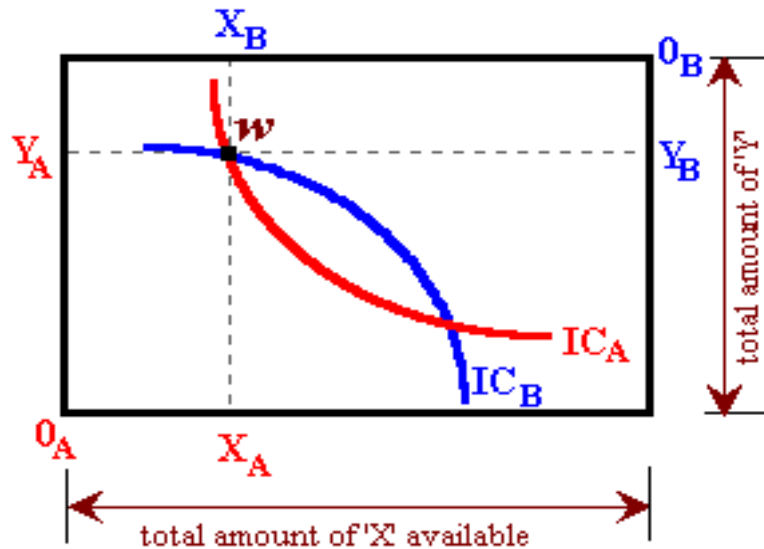
Figure 5.1: *Example of Edgeworth Box. In red is drawn the indifference curve that represents the amount of commodities of trader A while the blue one represents the goods of trader B.*

In the Edgeworth box each point represents a possible outcome of the trade where A has *(x, y)* and B has *(a-x, b-y)* commodities. From the box we can see also that trader A will only consider points to the north-east of the red line where his utility function gives at least $u_1(a, 0)$ which is the utility he gets if he doesn' t trade. Analogously trader B will only consider points to the south-west of the blue line, which is his indifference curve of utility values equivalent to $u_2(0, b)$. What is the solution of this type of trading? Edgeworth in his work suggested that the equilibrium in the trade could be find in the curve that joins the intersections of the two indifference curves, which he called the *contract curve*. It is pretty symple to consider this model as a two-person game, and, by doing that, it results that the contract curve is in fact part of the set of optimal solutions for the players.

5.3 [M, N] Market game

Another Edgeworth market game which is worth to touch on since it is the most common in real life is the [M, N] market game. In this game there are M-type traders and N-type traders as an M+N person game. To make the description simpler, we will make the same assumption of the previous paragraph, and we will consider M traders A and N traders B. In the same way we will assume that all the traders have the same utilty function *u(x, y)* and each A-trader starts with *a* of A an each B-trader starts with *b* of B. In order to find out the charateristic function, let's focus on a subset U of the M+N players with u_1 A-traders and u_2 B-traders. We can see that the best condition U can ensure itself is the highest sum of the utilities of its members is obtained when they trade each other. In opposition the worst case for U is when the traders don't belonging to U decide to don't trade with players inside of U, with a sort of protectionististic policy. The characteristic function formulation requires some type of side-payment, in fact we should not forget that A and B traders trade respectively apples and slices of bread and the best way to standardize the price of a good is to "transfer" its value into another divisible commodity like money in order to make a succesfull trade. However at the end of the bargaing we will consider only the quantity of apples and bread of each player. After these considerations, we are ready to write

$$v(U) = max_{x1,......,x(s1+s2), y1,.....,y(s1+s2)} \sum_{i=1}^{u1+u2} u(xi, yi)$$

where

$$\sum_{i=1}^{u1+u2} xi = u1a \quad \text{and} \quad \sum_{i=1}^{u1+u2} xi = u2b$$

This formula gives the charachteristic function for subset U of traders. As we can see it depends on the utility fuction, in particular on the maximum of this function which in turn is detemined by the number of commodities A and B.

5.4 Duopoly and oligopoly

The previous two paragraphs were about a very simple game model which, however, was in some ways linked to bartering since the two type of traders swapped goods one another. This model can be easy to understand but does not fit well in a real market and, since the various applications game theory has in economic sciences dealt essentialy with the study of how firms compete with each other, it is useful to focus on how these general models can be helpful in reality. To do this we must first of all give some definitions. A market condition where there are only two firms which are producing very similar product is called a *duopoly,* if there are more than two but still a very restricted number we talk about an *oligopoly*. These firms can decide the price of their products by their own and the amount of production, these two factors will determine the demand for the product. As we saw in the examples about Pineapple and HR, firms in these cases are considered like players in a game where payoffs are the profits they make. Now we can explain how the duo and oligopoly models fit in a [M, N] market game. Essentialy duopoly and oligopoly are respectively **[2, ∞]** and **[M, ∞]** market games, where firms are the first type of traders and they want to sell their products to an infinite (∞) population of potential buyers who exchange the product for their money (this is a mere mathematical assumption to semplify the explanation since there isn't an infinite number of humans in the world). However the number of buyers is still high enough to be a good reason to consider their reqirements by one utility function $u(p_1, p_2,.....,p_m, q_1, q_2,.....,q_m)$ where p_i is the price decided by the *i-th* producing firm and q_j is the amount of that good bought by consumers. We make the assumption that buyers know the prices and then choose the quantities in order to maximizes their utility function. This consideration gives us the permission to develop an equation which connect the demand q_i for *i'*s firm product with the prices $(p_1, p_2,.....,p_m)$ setted by the firms, that gives

$$q_i = f_i(p_1, p_2,.....,p_m)$$

With this equation we can now try to understand producers' profits,

$$e_i(p_1, p_2,.....,p_M) = p_i q_i - c_i(q_i) = p_i(f_i(p_1, p_2,.....,p_M)) - c_i(f_i(p_1, p_2,.....,p_M))$$

In this equation appears the production cost function c_i for i.

This equation is a very valid, even though simplicistic, model to describe the price of a product in a oligopolistic or duopolistic situation. It is worth noting that oligopoly is between monopoly and perfect competition. In fact in a monopoly theory there is only one producer who selects the price in order to maximize his profit while in a perfect competition situation due to the surplus of producers the demand is unlimited and the price will be fixed around a constant value. However these two theories lie outside game theory for the reason that there is only an available strategy.

Since this moment we haven't talked about any form of equilibrium for the oligopoly and duopoly situations, and it's now the time to give at least a brief explanation of this topic. It is in some ways curious that the equilibrium formula for these market conditions have been developed more time before the actual study of market games by a famous French mathematician named Antoine Augustin Cournot. This kind of equilibrium, which surprisingly took his name, says:

Definition: A ***Cournot equilibrium*** is a vector of prices $p^c = (p^c_1, p^c_2, ..., p^c_M)$ so that for all the firms $i = 1, ..., M$ holds,

$$e_i(p^c_1, p^c_2, .., p^c_i, ., p^c_M) = max_{pi} \; e_i(p^c_1, p^c_2, .., p^c_i, ., p^c_M)$$

Cournot discovered that, if the others firms prices are fixed, the Cournot equilibrium for i-th firm set the price which maximises its profits, and this holds for every firm involved in the oligopoly. Despite the fact that this definition corrisponds to the idea of an equilibrium n-tuple in a n-person game, these games have an infinite number of pure strategies for each player which is the price they choose, and as a result we can't appeal to Nash's equilibrium. However there is an implicit upper bound for the price which is made by customers, in fact if the price is too high and it becomes literally *out of market* noone would buy it, forcing the firm to low it. This helps to stabilize not only prices but also the prediction of the demand and offer dynamic.

6 Game theory application in business

6.1 Introduction

After having discussed the foundamentals of game theory it is time to use our new knowledge to solve a possible real life problem. Suppose you had a great idea for a new kind of electronic device and you patented it. Since your idea is really brilliant, you have been able to raise the necessary capitals from angel investors in order to give birth to your own start-up, but the hi-tech market is competitive an mercyless so you have to program every single step with accuracy. So you meet with your board of directors in order to decide a succesful business plan, and to do this you use game theory. The hi-tech market can be considered as an oligpolistic situation, since there are few big and famous firms. But this is a different kind of oligopoly since you are free to access the market in every moment if you have a competitive idea, the problem is how to do it and this is exactly what you and your board are trying to fix.
So no more words and lets get down to business.

6.2 Access the oligopoly

As said before, the hi-tech market can be seen as an oligopoly which gives the possibility to other firms to enter the matket if they are " skilled" enough. To simplify the situation, suppose that the specific field of your firm that from now we will call Start-up is dominated by another firm only, Pineapple, which produces a device which is similar and comparable to the product of Start-Up. This market condition can induce someone to think this as a monopoly instead of an oligopoly, but in fact this situation has nothing to do with monopoly. A monopolistic firm doesn't have any type of competitors and doesn't have even the risk that someone else could access to its market, as a result this firm does not take any decision based on other players strategies. Start-Up now must decide if it is more convenient to enter the field of the market dominated by Pinapple or to try in another field

by targeting another type of customers with its new product. To do this, Start-Up must make some affordable predictions of the possible reactions of Pineapple and their consequences. If Start-Up predicts that the game it is going to play with Pineapple will reach a profitable equilibrium, that is an equilibrium where it has a positive payoff, then it should face directly the other firm. At the opposite, if the predicted equilibrium gives an unsatisfactory payoff, then Start-Up should change its product application in order to enter successfully other markets. Recalling paragraph 4, if this was a perfect competition condition it would have been possible to predict accurately Start-Up's profits, since its entrance in the market would not have changed the equilibrium significantly. On the other hand Pineapple has all the interest in discouraging every possible competitor, and it can reach this target by "scaring" all the firms that express some interest in its specific production sector. There are lots of possible strategies Pineapple can adopt to do this, from illegal acts like industrial espionage to legal actions like buying the rights on all possible cuncurrent products or like increasing the production of the good which is menaced by other firms. Suppose that Pineapple, after having become acknowleged about Start-Up's intentions decides to make an offer for new patent, what Start-Up should do? A tree graph can come in our help.
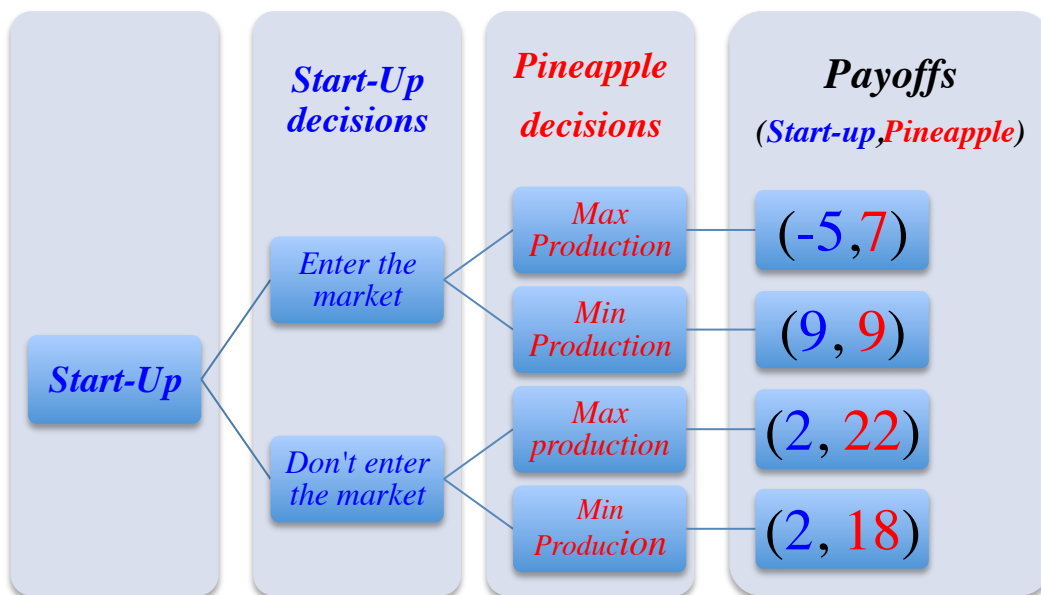

Figure 6.1: Tree graph representing the game discussed above. As usual we write Start-up moves in blue and Pineapple ones in red. Payoff are expressed in millions $.

Since decisions are not simoultaneous but are taken in different times, the tree representation suits best for this game. We made the assumption that Start-Up has the first

move because it must decide to sell its patent or to enter the market against Pineapple. From the tree we can see what payoffs are expected for both firms, in particular we see that if Start-Up decides to stay out of the market, it recieves a 2 millions dollar payoff since this is the price for the rights of production of its patent, otherwise Start-Up enters the market and faces Pineapple. Since Pineapple takes its decision after Start-Up, its stategy shows how how to reply properly to its possible future competitors, then a strategy for Pineapple could be the following: *if Start-up decides to enter, then we will start the production at the top of our facilities, instead we won't do this if Strat-Up keeps itself out from the market*. In this game there are two strategies which satisfies Nash condition:

- (Don't enter the market, Max production): In this strategy Start-Up prefers to stay out and sell its patent to Pineapple while Pineapple decides to produce at the maximum of its capabilities whatever decision Start-up take.

- (Enter, Min Production): In this strategy Start-Up decides to enter the market while Pineapple makes the following consideration: if Start-Up enters, we will produce the minimum we can, instead if Start-Up stays out, we will produce the maximum.

Let's consider the first case. The couple of decisions (Don't enter the market, Max production) gives a Nash equilibrium. To verify this we must control that each firm chooses an optimal response to the stategy adopted by the other. One or both the two firms could increase its own profit if the strategy of the other stays the same? Let' take a look to the profits. With the first strategy Start-up would recieve a payoff of 2 millions while Pineapple would recieve profits for 22 millions. Instead if Start-Up decides to enter the market, its losses would be massive since Pineapple doesn't change its strategy. In the same way we can see that Pineapple maximize its payoff producing the maximum possible number of devices in every case, as a result Pineapple hasn't got any interest in change its strategy and we just proved that this couple of strategies gives a Nash equilibrium.

What can we say about the strategic couple (Enter, Min Production)? In this case Start-Up decides to enter the market while Pineapple adopts the following strategy:" *If Start-up decides to enter we will produce less, otherwise we will produce all the devices we*

can". If Start-up enters the market, Pineapple has no interest in producing more products since its payoff would be only 7 millions against the 9 of the other choiche. In the same way if Start-Up accepts the offer of Pineapple and doesn't enter the market Pineapple itself would recieve a payoff of 22 millions by producing all the devices it can instead of the 18 millions given by the other choice. We just proven that this couple too forms a Nash equilibrium. Naturally, Pineapple would strongly prefer to maintain its momentary "monopoly" while entering the market is the preferred strategy of Start-Up. We can predict what will be the effective result of the game by applying the relieability condition. In fact let us consider the first equilibrium condition. As we repeated before Start-Up decides to sell its patent to Pineapple because its management is scared about the possible strong response of their competitor but suppose for a moment that Start-Up decides to enter the market despite Pineapple's menaces. By doing this Start-Up forces Pineapple to change its strategy and produce less than planned, since it will recieve an higher payoff. As a result of that the menace made by Pineapple to take over the market with massive production is not trustworthy and so the reliability condition can't be satisfied. Therefore the only couple of strategies that satisfies both Nash and Reliability conditions and gives a *perfect equilibrium* is the second. In fact Pineapple's strategy is fully trustworthy because it bring an higher payoff than every other when adopted and the same can be said for Start-Up decisions.

Finally it is very intresting to see that, in a more general situation, if the "older" firm can sign a cooperative agreement with the new one and fix the volumes of the production, this can harm its profits. In fact the new firm which aspires to enter the market will consider the possibility of a collusive treaty with the competitor and the advantages it can take when choosing its strategy. In conclusion this kind of behaviour will give a strong stimulus for the new firm to enter the market and change the previous situation of oligopoly, which is always better desirable for the firm or firms which were in the market already. This is the main reason why it is challenging, but not impossible, for a new firm to enter a market slice dominated by other companies.

7 Other applications of game theory

Game theory can be used in a very wide range of applications following, basically, the same approach we described in the previous chapters and, since these applications are very important, it is useful to describe them briefly.

7.1 Biology

In biology, for example, we can study and make previsions about the possible outputs of the hunter-prey dynamic between animals, or, as well, game theory can be used to foresee the result of a the so called *evolution-game,* where the dominance of a specie over another is considered, as an example think about the competitive game played by Homo Sapiens and Homo Neanderthalensis. It is useful to describe a simple example of the hunter-prey dynamic in order to see clearly the correlation between the two applications. Suppose for example you are a zoologist and you have to make a study on the hunting dynamic of the hawk. To simplify things we can consider both the hunter , in this case a hawk, and the prey, say a dove, as the two players. It is clear that payoffs are no more profits but we can assign a number to a certain event to simulate a "natural payoff". Suppose we are in a situation where a hawk have spotted a dove. Since both the hawk and the dove act by instinct, they both can represent two different instincts that can be considered as fixed strategies: the hunter instict and the prey instinct. The hunter instinct forces the player taken into account to attack the prey without considering the circumstances, while the prey instict always forces the player to escape when in danger. In the game matrix we consider the opposite strategies too since there can be some circumstances that can change both hunter's and prey's mind.

Dove Hawk	Escape	Keep fliying
Attack	3 3	0 10
Stand still	10 0	1 1

From the matrix above we can analize the dynamic hunter-prey.

- (Escape, Attack) : in this case the hawk decides to hunt the dove and the dove tries to escape. The hawk can injury and kill the dove but, at the same time, can be injured because it can misst he target and hit the ground, while the dove can successfully escape or be catched by the hawk. That is the reason why they both have the same payoff.

- (Keep flying, Attack) : this is the case where the hawk attacks the dove, but the dove can not escape because it doesn't spotted its hunter. The dove has no chances against the hawk, while the hawk can easily kill the dove.

- (Escape, Stand still) : in this situation the dove becomes aware of the danger and escapes but the hawk decides to keep flying in order to catch another prey. The hawk has no loss and the dove is safe.

- (Keep flying, Stand still) : this is the case where they don't spot each other.

This pretty simple and trivial example gives us a clear but incomplete view of the biological applications of Game theory.

7.2 Social sciences

Another and maybe a more natural extension of the microeconomic application is the field concernig social sciences. In this case the microeconomic approach is extended from firms to people, nations and their interactions. Think about the competition between two or more nations in order to achieve economic and commercial supremacy and the decisions they should take. It is not difficult to understand how Game Theory can be helpful and useful in the decision making process and can, in some cases, make the difference between good and bad decisions that affect lives of many. Social sciences are tightly linked to microeconomics, as a result reporting an example would be redundant, since we can simply generalize the cases treated in previous chapters by replacing firms with nations or populations.

7.3 Computer science

Anther field in which Game Theory plays an increasingly important role is computer science. Not surprisingly computer scientists understood the enourmous potential of this theory in the modelling of algorithms which regulates interactive computations between computers and multi agent systems. But the most common computer science application can be seen in everyday life, when we are surfing the internet. In fact the online algorithms which manage requests and answers between personal computers and servers are based on games between a number k of servers where a great variety of variables and payoffs are considered, such as response time, distance, energetic cost, speed and quality of the communication. Though computer science shares the foundamentals of game theory with the other sciences, there are a lot of differences between its application in this field and in others. In order to give a brief example of a game which is often used to mathematically model distributed computing, such as k servers interaction, it is useful to describe the so

called *Byzantyne agreement problem*. Suppose there are *n* soldiers and that between these *n* soldiers there are up to *t* possible faulty soldiers (the *t* stands for traitor), *n* and *t* are assumed to be knowledge. Each soldier start with an initial preference toward *attack* or *retreat*. We want to develop a protocol where

- All *nonfaulty soldiers* reach the same decision.
- If all the soldiers are nonfaulty and their initial preferences are the same, then the decision is coherent with their initial preference.

It is pretty simple to see how this example can relate to a computer science enviroment, and this is a typical problem usually solved by game theory methods.

7.4 Philosophy

Just to give an idea of how broadly this theory is applied in a lot of sides of the human knowledge, it is interesting to mention its use in philosophy, especially in the field related to psychology and social interaction. Thanks to Game Theory philosophers have been able to study with a mathematic instrument relations between social behaviour, ethic, morality, uses and costumes.

From these last considerations it is pretty simple to see that Game theory plays a foundamental role in several aspects of human knowledge. Thanks to its development scientists, economists, philosophers, biologists, engineers and managers are able to approach problems in a very effective way finding brilliant solutions which can change in better the lives of many.

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Abstract

This thesis aims to give an introduction of game theory with a particular focus on microeconomics topics. In detail all foundametals of game theory are given in order to make a final analysis on how this mathematical instrument can be used to study the best ways for a new Start up to enter the technology market.