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New Methods for Scattering Amplitudes in Gauge Theories

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Abstract

In the present thesis, we combine the most recent theoretical recursive techniques for the efficient computation of scattering amplitudes in gauge theory at one-loop. The issue of computing one-loop amplitudes can be addressed in two stages: the reduction in terms of an integral basis and the evaluation of the elements of such a basis, called master integrals (MI's). The principle of a unitarity-based method is the extraction of the coefficients multiplying each MI by performing d-dimensional generalized cuts. The recently proposed Four dimensional formulation (FDF) of the d-dimensional regularization scheme, allows for a purely four-dimensional regularization of the amplitudes. As a consequence, an explicit four-dimensional representation of generalized states propagating around the loop can be formulated. Therefore, a straightforward implementation of d-dimensional generalized unitarity within exactly four space-time dimensions can be realized, avoiding any higher-dimensional extension of either the Dirac or the spinor algebra.

The method of differential equations is one of the most effective techniques for computing dimensionally regulated multi-loop integrals. Within the continuous dimensional regularization scheme, Feynman integrals fulfill identities that fall in the cathegory of the general class of integration-by-parts relations. Such relations can be exploited in order to identify a set of independent integrals (MI's), that can be used as a basis of functions for the contributions to scattering amplitudes. Afterward, convenient manipulations of the basis of MI's may be performed. Proper choices of MI's can simplify the form of the systems of differential equations, considering a form where the dependence on the dimensional parameter $\epsilon = (4-d)/2$ is factorized from the kinematic. The integration of a system in canonical form trivializes and can been addressed by using Magnus series expansion. In the thesis, we present the application of on-shell and unitarity-based techniques and of the differential equation methods via Magnus expansion to the evaluation of the one-loop scattering amplitudes contributing to $gg \to gH$ at Next-to-Leading-Order (NLO). Their analytic expressions were obtained with standard techniques, and our results are in full agreement with them.

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Contents

Introduction

Perturbation Theory (PT) is a powerful tool for describing the quantum behavior of particles. Feynman diagrams offer a compact formalism to encode the fundamental properties of particle scattering. At the leading order (LO), the scattering is depicted in terms of tree graphs, while higher accuracy is reached by including terms which, beyond LO, are represented by diagrams containing closed loops. The computation of scattering amplitudes requires the evaluation of highly non-trivial integrals which, in general diverge in four dimensions. Therefore, a regularization procedure needs to be introduced in order to accomplish the systematic subtraction of divergences, finally yielding the determination of finite results to be compared with the experimental measurements.

The study of scattering amplitudes is fundamental to our understanding of QFT. There are powerful new mathematical structures underlying the extraordinary properties of scattering amplitudes in gauge theories, and studying them bring us into direct contact with a very active area of current research in mathematics.

In the present thesis, we discuss modern recursive techniques for the efficient computation of scattering amplitudes in Yang-Mills theory, the general non-Abelian gauge theory embedding the whole Standard Model of Particle Physics. Due to the great success of the CERN experimental programmes of ATLAS and CMS [1, 2], that confirmed the existence of a boson, compatible with the one predicted by the Electroweak Symmetry Breaking mechanism, we study tree-level and one-loop scattering amplitudes involving the associated production of the Higgs boson and jets in gluon fusion, within the infinite top-mass approximation [3–5].
Our aim is to combine the most recent theoretical developments for obtaining the results of the relevant amplitudes in an efficient and elegant way, with the perspective of extending their application beyond one-loop, in order to contribute to the development of field of research in high energy theoretical physics which is extremely active and which is subject to a pletora of new developments.

In general, when a direct integration of Feynman integrals is prohibitive, the evaluation of scattering amplitudes beyond the leading order is addressed in two stages: *i)* the reduction in terms of an integral basis, and *ii)* the evaluation of the elements of such a basis, called master integrals (MI's).
At one-loop, the advantage of knowing apriori that the basis of MI's is formed by scalar one-loop functions, as well as the availability of their analytic expression, allowed the community to focus on the development of efficient algorithms for ex-

tracting the coefficients multiplying each MI's.
The unitarity of the S-matrix encodes the most profound property of a quantum system, namely the probability conservation. The optical theorem, that relates the difference between the transition amplitude and its complex conjugated one to their product, is the direct consequence of unitarity.
The unitarity-based methods [6–9] use two general properties of scattering amplitudes such as analyticity and unitarity. The former grants that the amplitudes can be reconstructed from their singularity structure while the latter grants that the residues at the singular points factorize into products of simpler amplitudes.
Integrand-reduction methods [10], instead, allow one to decompose the integrands of scattering amplitudes into multi-particle poles, where the multi-particle residues are expressed in terms of irreducible scalar products formed by the loop momenta and either external momenta or polarization vectors constructed out of them.
The principle of a unitarity-based method is the extraction of the coefficients multiplying each MI by matching the multiparticle cuts of the amplitude onto the corresponding cuts of the MI's. Cutting a propagating particle in an amplitude amounts to applying the on-shell condition and replacing its Feynman propagator by the corresponding δ-function, $(p^2-m^2+i0)^{-1} \to (2\pi i)\, \delta^{(+)}(p^2-m^2)$. As a result, the original function is substituted by a simpler one, easier to compute, which, nevertheless, still carries non-trivial information. In fact, the n-particle cut of an n-point one-loop master integral, I_n $(1 \leq n \leq 4)$, appears in the 0-trascendentality term (rational or irrational) of the corresponding cut-amplitude, multiplied by the same coefficient of I_n in the decomposition of the complete amplitude. Higher-transcendentality terms, such as logarithms, are associated to the cuts of higher-point MI's. In general, the fulfillment of multiple-cut conditions requires loop momenta with complex components. Since the loop momentum has four components, the effect of the cut-conditions is to fix some of them according to the number of the cuts. Any *quadruple*-cut [7] fixes the loop-momentum completely, yielding the algebraic determination of the coefficients of $I_n, (n \geq 4)$; the coefficient of 3-point functions, I_3, are extracted from *triple*-cut [11–15]; the evaluation of *double*-cut [12, 14–21] is necessary for extracting the coefficient of 2-point functions, I_2; and finally, in processes involving massive particles, the coefficients of 1-point functions, I_1, are detected by *single*-cut [14, 22, 23]. In cases where fewer than four denominators are cut, the loop momentum is not frozen: the free-components are left over as phase-space integration variables. Dimensionally-regulated amplitudes [24] are constituted by terms containing (poly)logarithms, also called cut-constructible terms, and rational terms. The former may be obtained by the discontinuity structure of integrals over the four-dimensional loop momentum. The latter ones, instead, escape any four-dimensional detectability and require to cope with integrations including also the $(d-4)$ components of the loop momentum. Within generalized-unitarity methods both terms can be in principle obtained by performing d-dimensional generalized cuts.
The recently proposed Four dimensional formulation (FDF) of the d-dimensional regularization scheme, allows for a purely four-dimensional regularization of the amplitudes [25]. Within FDF, the states in the loop are described as four dimensional

massive particles. The four-dimensional degrees of freedom of the gauge bosons are carried by massive vector bosons, and their $(d-4)$-dimensional ones by real scalar particles obeying a simple set of four-dimensional Feynman rules. A d-dimensional massive fermion is instead traded for a tachyonic Dirac field with an additional imaginary mass term proportional to γ^5. Moreover, the d dimensional algebraic manipulations are replaced by four dimensional ones complemented by a set of multiplicative selection rules. The latter are treated as an algebra describing internal symmetries.
Within unitarity-based techniques and integrand reduction methods, the FDF scheme allows for the simultaneous computation of both the cut-constructible and the rational terms at one, by relying on a purely four-dimensional formulation of the integrands. As a consequence, an explicit four-dimensional representation of generalized states propagating around the loop can be formulated. Therefore, a straightforward implementation of d-dimensional generalized unitarity within exactly four space-time dimensions can be realized, avoiding any higher-dimensional extension of either the Dirac or the spinor algebra.

The calculation of Master Integrals requires additional efforts. The MI's are functions of the kinematic invariants constructed from the external momenta and of the masses of the (internal and external) particles. For any given scattering process the set of MI's is not unique, and, in practice, their choice is rather arbitrary. Usually MI's are identified after applying the Laporta reduction algorithm [26],based on the solution of algebraic systems of equations obtained through integration-by-parts identities (IBP-id's) [27].
Remarkably, the aforementioned relations imply that the MI's obey linear systems of first-order differential equations (DE's) in the kinematic invariants, which can be used for the determination of their actual expression [28, 29]. Afterward, convenient manipulations of the basis of MI's may be performed. Proper choices of MI's can simplify the form of the systems of differential equations, hence, of their solution, although general criteria for determining such optimal sets are not available. In Ref. [30], Henn proposes to solve the systems of DE's forMI's with algebraic methods, by observing that with a *good* choice of MI's the system of DE's can be cast in a form - which we define *canonical* - where the dependence on the dimensional parameter $\epsilon = (4-d)/2$ is factorized from the kinematic. The integration of a system in canonical form trivializes and the analytic properties of its general solution are manifestly inherited from the matrix associated to the system, which is the kernel of the representation of the solutions in terms of repeated integrations. The integration can been addressed by using dedicated techniques for non-commutative systems of differential equations, such as Magnus series expansion [31]. The solution of the system, namely the MI's, is finally determined by imposing the boundary conditions at special values of the kinematic variables, properly chosen either in correspondence of configurations that reduce the MI's to simpler integrals or in correspondence of pseudo-thresholds. In this latter case, the boundary conditions are obtained by imposing the regularity of the MI's around unphysical singularities, ruling out

divergent behavior of the general solution of the systems.

Harmonic polylogarithms [32] and their properties are appropriated functions naturally appearing in the solutions of system of differential equations for MI's.

In the thesis, we present the application of on-shell and unitarity-based techniques and of the differential equation methods via Magnus expansion to the evaluation of the one-loop scattering amplitudes contributing to $gg \to gH$ at Next-to-Leading-Order (NLO). In particular, we address the calculation of the two independent (color ordered) helicity amplitudes $A(+,+,+,H)$ and $A(-,+,+,H)$ contributing to this process. Their analytic expressions were obtained in Ref. [33] with standard techniques, and our results are in full agreement with them. Our calculations have been carried out with the software *Mathematica*, using the packages S@M [34],for the spinor manipulation, and *Feyncalc* [35] ,for the tensor algebra, and the C++ code *Reduze2* [36], for the generation of the IBP-id's and of the differential equations.
The thesis is organized as follows. Chapter 1 contains a basic introduction to QCD, and its Feynman rules. In chapter 2 the Spinor-Helicity formalism is introduced, as a very useful tool to treat scattering amplitudes. Chapter 3 deals with on-shell methods to for the calculation of tree level amplitudes, as an alternative to Feynman diagrams. Chapter 4 contains the basics of the standard techniques used for one-loop amplitude, focusing on tensor reduction. In Chapter 5 the Unitarity-based methods are introduced , and the generalised unitarity strategy is defined, together with the definitions of the multiple-cuts . Chapter 6 deals with the formulation of FDF regularization scheme and the corresponding Feynman rules. Chapter 7 describes the idea s behind the Integrand Reduction, with the investigation of the polynomial structures appearing at the residues of the multiple-cuts. Chapter 8 contains the explicit calculation and the results of the one-loop scattering amplitudes for Higgs plus one jet, obtained from the generalised unitarity within the FDF scheme. In Chapter 9, we recall the basics methods for addressing the calculation of MI's, namely IBP-id's and differential equations for Feynman integrals. In Chapter 10, we show the recent ideas for improving the method of differential equations for MI's , using Magnus series expansion. Chapter 11 contains an introduction to HPL and 2dHPL, which are useful to write the solutions of DE for MI's. In Chapter 12, the differential equation method and Magnus expansion are used to obtain the analytic expressions of the MI's appearing in the amplitude decomposition for Higgs plus one jet production at one-loop.

Chapter 1

Perturbative QCD

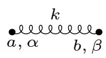
In the traditional approach to quantum field theory, one writes down a classical Lagrangian and can quantise the theory by defining the Feynman path integral. Perturbative physics can then be studied by drawing Feynman diagrams and using the Feynman rules generated by the path integral to calculate scattering amplitudes. For a non-Abelian gauge theory the classical theory is well-described by the Yang-Mills Lagrangian:

$$\begin{aligned}\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + gA^a_\mu\bar{\psi}\gamma^\mu T^a\psi - \\ - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{\mu c}A^{\nu d})\end{aligned} \tag{1.1}$$

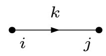
where ψ is a fermion field, A the gauge boson field and g is the coupling. Greek indices are associated with spacetime, while Roman indices describe the structure in gauge group space. This can then be used to construct the Feynman rules in the usual way. Although this construction is somewhat technical it is easy so see what these interactions will be from a heuristic standpoint. The first two terms in(1.1) will give the fermion and gauge boson propagators respectively. The third term involves two ψ and an A and thus represents a vertex where two fermions interact with a gauge boson. The fourth term involves 3 As and represents a 3-boson vertex while the fifth term gives a 4-boson vertex. If we work everything out properly then we find that, in Feynman gauge for example where we have set $\xi = 1$ in a more general gauge boson propagator of the form

$$\frac{-i}{p^2 + i\varepsilon}\big(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\big)\delta_{ab}, \tag{1.2}$$

the Feynman rules for an $SU(N_c)$ gauge theory are:

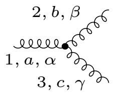


$$= -i\,\delta^{ab}\,\frac{g^{\alpha\beta}}{k^2 + i0} \quad \text{(gluon)}, \tag{1.3a}$$

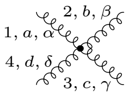


$$= i\,\delta^{ij}\,\frac{k\!\!\!/ + m}{k^2 - m^2 + i0}\,,$$

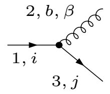
(fermion), (1.3b)



$$= -g\, f^{abc}\left[(k_1 - k_2)^\gamma g^{\alpha\beta} + (k_2 - k_3)^\alpha g^{\beta\gamma} + (k_3 - k_1)^\beta g^{\gamma\alpha}\right], \tag{1.3c}$$



$$\begin{aligned} = -ig^2\Big[& + f^{xad}\, f^{xbc}\left(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ & + f^{xac}\, f^{xbd}\left(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ & + f^{xab}\, f^{xdc}\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta}\right)\Big], \end{aligned} \tag{1.3d}$$



$$= -ig\,\left(t^b\right)_{ji}\gamma^\beta, \tag{1.3e}$$

(1.3f)

We have ignored the contributions due to ghost fields and will stick to these choices in what follows unless otherwise specified. Amplitudes for physical processes are obtained by drawing all the ways that the process can occur using the above rules and associating each of these with a specific mathematical expression. They are then evaluated and added up to produce the desired result. Classical results are obtained from diagrams without any closed loops while quantum corrections involve an increasing number of loops. Even though gauge theories present many technical challenges, the way to proceed (at least perturbatively) is in-principle well understood. In practice, however, the calculational complexity grows rapidly with the number of external particles (legs) and the number of loops. For example, even at tree-level where there are no loops to consider, the number of Feynman diagrams describing n-particle scattering of external gluons in QCD grows faster than factorially with n.

1.1 Color ordering

One prominent complication experienced by gauge theories is the extra structure inherent in their gauge invariance. This means that fields of the theory do not just carry spacetime indices but also indices relating to their transformation under the gauge group. In the standard model it has been found that $SU(N_c)$ groups are the most appropriate ones for describing the gauge symmetry and so unless otherwise specified we will consider gauge groups of this type. As is well-known, gluons carry an adjoint color index $a = 1, 2, \ldots, N_c^2 - 1$, while quarks and antiquarks carry fundamental (N_c) or anti-fundamental ($\bar{N}_c$) indices $i, j, = 1, 2, \ldots, N_c$. The $SU(N_c)$ generators in the fundamental representation are traceless Hermitian $N_c \times N_c$

matrices, $(T^a)_i^{\bar{j}}$ which we normalize to $\text{Tr}(T^aT^b) = \delta^{ab}$ The Lie algebra is defined by

$$[T^a, T^b] = if^{abc}T^c \tag{1.4}$$

where the structure constants f^{abc} satisfy the Jacobi Identity:

$$f^{ade}f^{bcd} + f^{bdc}f^{cad} + f^{cde}f^{abd} = 0 \tag{1.5}$$

Let us begin by considering a generic tree-level scattering amplitude. It is apparent from the Feynman rules that each quark-gluon vertex contributes a group theory factor of $(T^a)_i^{\bar{j}}$ and each three-boson vertex a factor of f^{abc}, while four-boson vertices contribute more complicated contractions involving pairs of structure constants such as $f^{abe}f^{cde}$. The quark and gluon propagators will then contract many of the indices together using their group theory factors of δ^{ab} and $\delta_i^{\bar{j}}$. We can now start to illuminate the general color structure of the amplitudes if we first use the definition of the Lie-algebra to re-write the structure constants as

$$f^{abc} = -i\text{Tr}\left(T^a[T^b, T^c]\right) \tag{1.6}$$

Doing this means that all color factors in the Feynman rules can be replaced by linear combinations of strings of T^as. In order to reduce the number of traces we make use of the identity

$$\sum_{a=1}^{N_c^2-1}(T^a)_i^{\bar{j}}(T^a)_k^{\bar{l}} = \delta_i^{\bar{l}}\delta_k^{\bar{j}} - \frac{1}{N_c}\delta_i^{\bar{j}}\delta_k^{\bar{l}} \tag{1.7}$$

which is just an algebraic statement of the fact that the generators T^a form a complete set of traceless Hermitian matrices. This in turn gives rise to simplifications such as

$$\begin{aligned}\sum_a \text{Tr}(T^{a_1}\dots T^{a_k}T^a)\text{Tr}(T^a\dots T^{a_{k+1}}T^{a_n}) =&\text{Tr}(T^{a_1}\dots T^{a_k}T^{a_{k+1}}\dots T^{a_n})-\\ &-\frac{1}{N_c}\text{Tr}(T^{a_1}\dots T^{a_k})\text{Tr}(T^{a_{k+1}}\dots T^{a_n})\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}\sum_a \text{Tr}(T^{a_1}\dots T^{a_k}T^a)(T^a\dots T^{a_{k+1}}T^{a_n})_i^{\bar{j}} =&(T^{a_1}\dots T^{a_k}T^{a_{k+1}}\dots T^{a_n})_i^{\bar{j}}-\\ &-\frac{1}{N_c}\text{Tr}(T^{a_1}\dots T^{a_k})\text{Tr}(T^{a_{k+1}}\dots T^{a_n})_i^{\bar{j}}\end{aligned} \tag{1.9}$$

In Eq. (1.7) the $1/Nc$ term corresponds to the subtraction of the trace of the $U(N_c)$ group in which $SU(N_c)$ is embedded and thus ensures tracelessness of the T^a. This trace couples directly only to quarks and commutes with $SU(N_c)$. As such the terms involving it disappear after one sums over all the permutations present - a fact which is easy to check directly. We can thus see that we are ultimately left with either sums of single traces of generators if we only have external gluons as in Eq. (1.8) or sums of strings of generators terminated by fundamental indices as in Eq. (1.9) if we also have external quarks. In most of what we do we will only be concerned with gluon scattering and can therefore write the color decomposition of amplitudes as

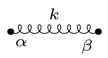
$$A_n^{tree} = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma_1}} T^{a_{\sigma_2}} \dots T^{a_{\sigma_n}}) A_n^{tree}(\sigma_1, \sigma_2, \dots, \sigma_n) \tag{1.10}$$

where S_n is the set of permutations of n objects and Z_n is the subset of cyclic permutations. g is the coupling constant of the theory. The A_n^{tree} sub-amplitudes are color-stripped and depend only on one ordering of the external particles. It is therefore sufficient to consider $A_n^{tree}(1, 2, \dots, n)$ - the 'reduced color-ordered amplitude' - and sum over all $(n-1)!$ non-cyclic permutations at the end. The primitive amplitudes, denoted here generically by $A(1, 2, ..., n)$, are by construction color independent and satisfy a number of important properties and relationships

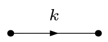
- $A(1, 2, ..., n)$ is gauge invariant. The proof follows the same lines of the QED Ward Identity, since after stripping the color factor, primitive amplitudes behave like in an Abelian theory.
- $A(1, 2, ..., n)$ is invariant under cyclic permutations of $1, 2, ..., n$.
- $A(n, n-1, ..., 2, 1) = (-1)^n A(1, 2, ..., n)$
- The dual Ward identity

$$A(1, 2, 3, ..., n) + A(2, 1, 3, ..., n) + A(2, 3, 1, ..., n) + ... + A(2, 3, ..., 1, n) = 0$$

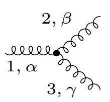
With the previous prescriptions we can write the color-ordered Feynman rules for QCD, they are readily obtained by the usual Feynman rules just by imposing a given ordering.



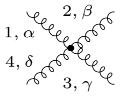
$$= -i \, \frac{g^{\alpha\beta}}{k^2 + i0} \, , \quad \text{(gluon)}, \tag{1.11a}$$



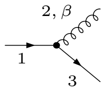
$$= i \, \frac{\not{k} + m}{k^2 - m^2 + i0} \, , \; \text{(fermion)}, \tag{1.11b}$$



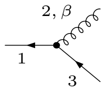
$$= \frac{i}{\sqrt{2}} \left[g_{\alpha\beta}(k_1 - k_2)_\gamma + g_{\beta\gamma}(k_2 - k_3)_\alpha g_{\gamma\alpha}(k_3 - k_1)_\beta \right] , \tag{1.11c}$$



$$= ig_{\alpha\gamma}g_{\beta\delta} - \frac{i}{2}\big(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\beta\gamma}\big)\,, \tag{1.11d}$$



$$= -\frac{i}{\sqrt{2}}\gamma^{\beta}\,, \tag{1.11e}$$



$$= \frac{i}{\sqrt{2}}\gamma^{\beta}\,, \tag{1.11f}$$

$$\tag{1.11g}$$

This means that there isn't a huge proliferation in the number of partial amplitudes that have to compute.

Chapter 2

Spinor-Helicity Formalism

In this section we review the spinor helicity formalism [37, 38]. This formalism is responsible for the existence of compact expressions for tree and loop amplitudes. It introduces a new set of kinematic objects, spinor products, which neatly capture the collinear behavior of these amplitudes.

2.1 Fermion wave-function

We start with the solutions of the massless Dirac equation in momentum space.

$$\not{p}\psi(p) = 0 \tag{2.1}$$

In this case, positive and negative energy solutions become identical up to normalization conventions. Considering a massless fermion, we have

$$\not{p}U(p) = 0 \tag{2.2}$$

Equation (2.2) has two solutions , the spinors for right-handed and left-handed fermions,

$$U_R(p) = \begin{pmatrix} 0 \\ u_R(p) \end{pmatrix}, \quad U_L(p) = \begin{pmatrix} u_L(p) \\ 0 \end{pmatrix} \tag{2.3}$$

where this solutions are eigenstates of helicity, satisfying

$$U_R(p) = RU(p), \quad U_L(p) = LU(p) \tag{2.4}$$

and R and L are projectors of helicity, defined as

$$R = \frac{1}{2}(1 + \gamma^5), \quad L = \frac{1}{2}(1 - \gamma^5) \tag{2.5}$$

The spinor $u_R(p)$ can be related to the spinor $u_L(p)$ by

$$u_R(p) = i\sigma^2 u_L^*(p) \tag{2.6}$$

To describe the antiparticles, we also need the solution $V(p)$ that describe their creation and destruction. However, for massless particles, $V(p)$ satisfies the same equation as $U(p)$. We can then use the same solutions $u_R(p)$ and $u_L(p)$, with $U_R(p) = V_L(p)$ and $U_L(p) = V_R(p)$. We will represent these spinors compactly as

$$U_R(p) = p\rangle, \quad U_L(p) = p], \quad \bar{U}_L(p) = \langle p, \quad \bar{U}_R(p) = [p \tag{2.7}$$

The Lorentz-invariant spinor products can be standardized as

$$\bar{U}_L(p)U_R(q) = \langle pq\rangle, \quad \bar{U}_R(p)U_L(q) = [pq] \tag{2.8}$$

The spinors are related to their lightlike vectors by identities

$$p\rangle[p = U_R(q)\bar{U}_L(p) = \not{p}\frac{1-\gamma^5}{2}, \quad p]\langle p = U_L(q)\bar{U}_R(p) = \not{p}\frac{1+\gamma^5}{2} \tag{2.9}$$

where the spinors are normalized according to

$$\frac{\langle p|\gamma^\mu|p]}{2} = p^\mu \tag{2.10}$$

From these formulas and using the properties of Dirac algebra we can prove a series of useful identities

$$\begin{gathered}\langle pq\rangle = [qp]^*,\\ \langle pq\rangle = -\langle qp\rangle, \quad [pq] = -[qp], \quad \langle pp\rangle = 0\\ \langle pq\rangle[qp] = 2p\cdot q\end{gathered} \tag{2.11}$$

Some further identities are useful in discussing the vector current built from spinors

$$u_L^\dagger(p)\bar{\sigma^\mu}u_L(p) = u_R^\dagger(p)\sigma^\mu u_R(p) \tag{2.12}$$

$$\langle p\gamma^\mu q] = [q\gamma^\mu p\rangle \tag{2.13}$$

The Fierz identity of sigma matrices,

$$(\bar{\sigma^\mu})_{ab}(\bar{\sigma_\mu})_{cd} = 2(i\sigma^2)_{ac}(i\sigma^2)_{bd}, \tag{2.14}$$

allows the simplification of contractions of spinor expressions, for instance

$$\langle p|\gamma^\mu|q]\langle k|\gamma_\mu|l] = 2\langle pk\rangle[lq], \quad \langle p|\gamma^\mu|q][k|\gamma_\mu|l\rangle = 2\langle pl\rangle[kq] \tag{2.15}$$

Finally the spinor products obey the Schouten identity

$$\langle pq\rangle\langle kl\rangle + \langle pk\rangle\langle lq\rangle + \langle pl\rangle\langle qk\rangle = 0, \tag{2.16}$$

$$[pq][kl] + [pk][lq] + [pl][qk] = 0 \tag{2.17}$$

This formalism hold only for massless particles, otherwise eigenstates of helicity and chirality are not the same. To treat a problem with massive fermions using spinor-helicity formalism we need an auxiliary on-shell momentum. In fact we suppose to have the Dirac equation

$$(\not{p} - m)U(p, n) = 0 \tag{2.18}$$

where $U(p,n)$ is an eigenstate of helicity and $n = (0, \frac{\vec{p}}{|p|})$ is the versor of direction of motion (in rest-frame) with properties $n^2 = -1$ and $p \cdot n = 0$. Defining the operator of helicity

$$P(n) = \frac{1 + \gamma^5 \gamma_\mu n^\mu}{2} \tag{2.19}$$

holds

$$P(n)U(p,n) = U(p,n), \qquad P(n)U(p,-n) = 0 \tag{2.20}$$

Then we can write the solution of (2.18) as

$$U(p,n) = A(\not{p} + m)U(q) \tag{2.21}$$

where $U(q)$ is a spinor of arbitrary momentum q that solve the Dirac massless equation $\not{q}U(q) = 0$. Therefore general expression of $U(p,n)$ is

$$U(p,n) = A\frac{1 + \gamma^5 \gamma_\mu n^\mu}{2}(\gamma_\mu p^\mu + m)U(q) \tag{2.22}$$

Introducing an auxiliary vector k such that $p = k + Bq$ and $k^2 = 0$ we have an explicit form for p and n

$$p = k + \frac{m^2}{2p \cdot q}q, \qquad n = -\frac{p}{m} + \frac{m}{p \cdot q}q \tag{2.23}$$

The massive spinor is written as

$$U(p,n) = A(\gamma_\mu p^\mu + m - \gamma^5 \gamma_\mu p^\mu \gamma_\nu p^\nu + m\gamma^5 \gamma_\nu n^\nu)U(q) \tag{2.24}$$

and using

$$\gamma_\nu n^\nu U(q) = -\frac{\gamma_\nu p^\nu}{m}U(q) \tag{2.25}$$

we obtain

$$U(p,n) = A(\gamma_\mu p^\mu + m)\frac{1 + \gamma^5}{2}U(q) = A(\gamma_\mu p^\mu + m)U_R(q) = A(\gamma_\mu p^\mu + m)|q\rangle \tag{2.26}$$

In order to compute the constant A use the normalization condition of spinors $\bar{U}(p,n)U(p,n) = 2m$ and obtain

$$|A|^2[q|\gamma_\mu p^\mu|q\rangle = 1, \quad A|^2[qk]\langle kq\rangle = 1 \tag{2.27}$$

From relations (2.1) we obtain $A = \frac{1}{\langle kq\rangle}$ or $A = \frac{1}{[qk]}$ and the massive spinors are written as

$$U(p,n) = \frac{1}{\langle kq\rangle}(\gamma_\mu p^\mu + m)|q\rangle = |k] + \frac{m}{\langle kq\rangle}|q\rangle, \tag{2.28}$$

$$U(p,-n) = \frac{1}{[qk]}(\gamma_\mu p^\mu + m)|q] = |k\rangle + \frac{m}{[qk]}|q] \tag{2.29}$$

and looking at spinors of negative energy

$$V(p,n) = \frac{1}{[qk]}(\gamma_\mu p^\mu - m)|q] = |k\rangle - \frac{m}{[qk]}|q], \tag{2.30}$$

$$V(p,-n) = \frac{1}{\langle kq\rangle}(\gamma_\mu p^\mu - m)|q\rangle = |k] - \frac{m}{\langle kq\rangle}|q\rangle \tag{2.31}$$

There is one subtlety that should be clarified to evaluate amplitudes by spinor-helicity formalism. In our calculations, could be useful to relate the brackets $|-p\rangle$ and $|-p]$ to $|p\rangle$ and $|p]$. It is consistent always to take [37]

$$|-p\rangle = i|p\rangle, \qquad |-p] = i|p] \tag{2.32}$$

The excuse of these relations come from (2.9), in fact to compensate the minus sign of momentum we need to add a $(-i)$ to each spinors

2.2 Vector boson wave function

We construct the massless polarization vectors by considering k to be the momentum of a photon (gluon), and p be another lightlike vector, chosen so that $p\cdot k \neq 0$. $u_R(p)$, $u_L(p)$ are the spinors of definite helicity for fermions with the light-like momentum p, defined according to previous conventions. The helicity one photon polarization vectors are

$$\varepsilon_+^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_+(k)\gamma^\mu u_+(p), \quad \varepsilon_-^\mu(k) = \frac{1}{\sqrt{4p\cdot q}}\bar{u}_-(k)\gamma^\mu u_-(p) \tag{2.33}$$

In the shorthand notation,

$$\varepsilon_+^\mu(k,q) = -\frac{\langle k|\gamma^\mu|q]}{\sqrt{2}[qk]} \qquad \varepsilon_-^\mu(k,q) = \frac{[k|\gamma^\mu|q\rangle}{\sqrt{2}\langle qk\rangle} \tag{2.34}$$

$$\varepsilon_+^{*\mu}(k,q) = \frac{[k|\gamma^\mu|q\rangle}{\sqrt{2}\langle qk\rangle}, \qquad \varepsilon_-^{*\mu}(k,q) = -\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}[qk]} \tag{2.35}$$

these polarization vectors are defined in terms of both the momentum vector k and a reference vector q. The gauge invariance of the scattering amplitudes of the spin-1 field manifests itself in the arbitrariness of the reference momentum q. The polarization vectors have the usual properties

$$\begin{gathered}(\varepsilon^\pm)^* = \varepsilon^\mp,\\ \varepsilon^\pm\cdot\varepsilon^\pm = 0,\\ \varepsilon^\pm\cdot\varepsilon^\mp = -1,\\ \sum_{\lambda=\pm}\varepsilon_\lambda^\mu(l,p)\varepsilon_\lambda^{*\nu}(k,q) = -g^{\mu\nu} + \frac{l^\mu k^\nu + k^\mu l^\nu}{l\cdot k}\end{gathered} \tag{2.36}$$

Chapter 3

On-Shell Methods at Tree-level

Calculation of tree-level scattering amplitudes is normally done by applying the Feynman rules; the expressions generated in this way are known not to be the most compact. Recursion relations have been utilised extensively in tree level matrix element calculations for many years. The main principle is to re-use calculations for lower multiplicity amplitudes to make up higher multiplicity amplitudes. The idea behind the derivation of the BCFW recursion relation [9, 39] is that tree-level amplitudes are analytic functions of the scattering momenta. Therefore, it should be possible to reconstruct amplitudes for generic scattering kinematics from their behavior in singular limiting kinematics. In these singular regions, amplitudes split, or factorize, into two causally disconnected amplitudes with fewer legs.

3.1 BCFW Recurrence Relation

Consider a color-ordered amplitude $iA(1,\ldots,n)$ of n gluons. Choose two legs i, j, and choose a value of z, a complex variable. Now define new spinors $\hat{i}$ and $\hat{j}$ shifting i and j spinors in this way

$$|i\rangle \to |\hat{i}\rangle = |i\rangle + z|j], \qquad |i] = |\hat{i}] \tag{3.1}$$

$$|j] \to |\hat{j}] = |j] - z|i\rangle, \qquad |j\rangle = |\hat{j}\rangle \tag{3.2}$$

Under this transformation, momenta shift as

$$k_i^\mu = \frac{1}{2}[i|\gamma^\mu|i\rangle \quad \to \quad \hat{k}_i^\mu(z) = k_i^\mu + \frac{z}{2}[i|\gamma^\mu|j\rangle \tag{3.3}$$

$$k_j^\mu = \frac{1}{2}[j|\gamma^\mu|j\rangle \quad \to \quad \hat{k}_j^\mu(z) = k_j^\mu - \frac{z}{2}[i|\gamma^\mu|j\rangle \tag{3.4}$$

The shift leave untouched the sum $k_i + k_j = \hat{k}_i + \hat{k}_j$ and other products $\hat{k}_{i,j}^2 = 0$, $\hat{k}_i \cdot \hat{k}_j = k_i \cdot k_j$. Now amplitude is a complex function of z, $A(z) = A(1,\ldots,\hat{i},\ldots,\hat{j},\ldots,n)$. If we consider tree-level amplitudes $A(z)$ is a rational function of z with some poles come from propagators. If we suppose $A(z)$ has no poles at infinity and multiply $A(z)$ times $\frac{1}{z}$ then this new function has a pole in $z = 0$ that corresponds to physical amplitude $iA(0)$. If we suppose that $\lim_{z\to+\infty} A(z) = 0$ and $C_R = (z : |z| = R)$ is a contour of very large R, then

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{C_R} dz \frac{iA(z)}{z} = 0 \tag{3.5}$$

but from the theorem of residues we knows that

$$\frac{1}{2\pi i} \oint_{C_R} dz \frac{iA(z)}{z} = iA(0) + \sum A^n_{i=1} Res_{z=z_i} \frac{iA(z)}{z} \tag{3.6}$$

Physical amplitude in terms of residues of poles in z

$$iA = iA(0) = -\sum_{i=1}^{n} Res_{z=z_i} \frac{iA(z)}{z} is \tag{3.7}$$

To determine the residues at each pole, we use the general factorization properties that any amplitude must satisfy at tree-level. In fact if we decide to shift k_i and k_j momenta amplitude is

$$iA(0) = A(z) = A(1, \dots, \hat{i}, \dots, l, \dots \hat{j}, \dots, n) \tag{3.8}$$

that we can represent as product of an amplitude $iA_l(z)$ with l external particles , a propagator $\frac{i}{Q(z)^2}$ and an amplitude $iA_{n-l}(z)$ with $n - l$ external particles. The global amplitude $iA(0)$ is then

$$iA(0) = iA_l(z) \frac{i}{Q(z)^2} iA_{n-l}(z) \tag{3.9}$$

with

$$\begin{aligned} Q(z)^2 &= \left(k_1^\mu + \dots + \hat{k}_i^\mu(z) + \dots + k_l^\mu(z)\right)^2 = \\ &= \left(k_1^\mu + \dots + k_i^\mu + \dots + k_l^\mu(z) + \frac{z}{2}[k_i|\gamma^\mu|k_j\rangle\right)^2 = \\ &= \left(Q + \frac{z}{2}[k_i|\gamma^\mu|k_j\rangle\right)^2 = Q^2 + z[k_i|\gamma_\mu Q^\mu|k_j\rangle \end{aligned} \tag{3.10}$$

In this case the pole is generated from $Q(z)^2 = 0$, that means

$$z_0 = -\frac{Q^2}{[k_i|\gamma_\mu Q^\mu|k_j\rangle} \tag{3.11}$$

Now we can evaluate residue of $\frac{iA(z)}{z}$, remembering equation (3.9)

$$\begin{aligned} &Res_{z\to z_0} \frac{iA(0)}{z} = \lim_{z\to z_0} (z - z_0) \frac{iA(z)}{z} = \\ &= \lim_{z\to z_0} \frac{Q^2 + z[k_i|\gamma_\mu Q^\mu|k_j\rangle}{[k_i|\gamma_\mu Q^\mu|k_j\rangle} \frac{iA_l(z)}{z} \frac{-i}{Q^2 + z[k_i|\gamma_\mu Q^\mu|k_j\rangle} iA_{n-l}(z) \\ &= iA_l(z_0) \frac{i}{Q^2} iA_{n-l}(z_0) \end{aligned} \tag{3.12}$$

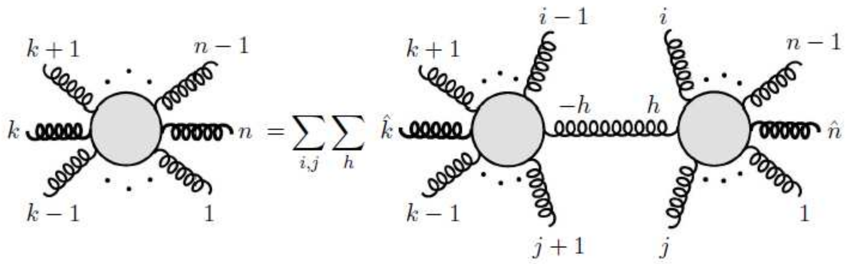


Figure 3.1: Representation of the BCFW recursion relation

The residue in (3.12) is the product of two simpler tree-level amplitudes (with less legs), analytically continued in the complex plane. We could obtain a more detailed expression considering an explicit form of shifted propagator, for example supposing to be a gluon. In this case we have

$$Res_{z_0} = iA_l^\mu(z_0)\frac{-ig_{\mu\nu}}{Q^2}iA_{n-l}^\nu(z_0) \tag{3.13}$$

and, making the polarization vectors to contract with gluon propagator explicit, we get

$$iA'_{l,\rho}(z_0)\epsilon^{\rho\mu}\frac{-ig_{\mu\nu}}{Q^2}\epsilon^{\nu\sigma}iA'_{n-l,\sigma}(z_0). \tag{3.14}$$

Using the completeness relation (2.2), the product of gluons with same helicity vanishes, so we obtain

$$iA(0) = \sum_{\alpha,h} iA_L^h(z_\alpha)\frac{-i}{Q_\alpha^2}iA_R^{-h}(z_\alpha) \tag{3.15}$$

where $iA_L(z_\alpha)$ and $iA_R(z_\alpha)$ are color ordered amplitude that are on left and right side respect to shifted propagator of z_α pole and h is helicty of intermediate gluon.

This is the BCFW recursion formula [9,39]. Recursion come from the fact to be possible to use (3.9) to rewrite amplitudes appearing in (3.15) in terms of amplitudes with a smaller number of legs; at the end we have our amplitudes of n gluons in terms of three external gluons amplitudes. It's useful notice that there is a prescription in order to respect $iA(z) \to 0$ when $z \to \infty$. Indeed to have this behavior we have to shift gluons with different helicity, in fact from (3.1) and (2.34) we have

$$\epsilon^\mu(q^+(z), a) = \frac{1}{\sqrt{2}(\langle aq\rangle + z\langle ak\rangle)}[q|\gamma^\mu|a\rangle \tag{3.16}$$

$$\epsilon^\mu(k^-(z), b) = \frac{1}{\sqrt{2}([bq] - z[bk])}[k|\gamma^\mu|b\rangle \tag{3.17}$$

that give a contribute of order $O \approx z^{-2}$ that ensure $\frac{iA(z)}{z}$ to go zero faster than $\frac{1}{z}$ at infinity in complex plane.

3.2 Example: 4-gluons Amplitude

To show how BCFW works we treat a 4-gluons amplitude $iA(1^-,2^-,3^+,4^+)$ and shift legs 2 and 3

$$k_2^\mu(z) = k_2^\mu - \frac{z}{2}[3|\gamma^\mu|2\rangle, \qquad k_3^\mu(z) = k_3^\mu + \frac{z}{2}[3|\gamma^\mu|2\rangle \tag{3.18}$$

so amplitudes of z is

$$iA(z) = iA(1^-,2^-,Q(z))\frac{-i}{Q^2(z)}iA_3(Q(z),3^+(z),4^+) \tag{3.19}$$

where $Q = k_1 + k_2$ is the momentum flows through propagator. Pole of this function come from $Q^2(z) = 0$, that means

$$Q^2(z_0) = Q^2 + z_0[3|\not{Q}|2\rangle = 0 \tag{3.20}$$

$$z_0 = -\frac{Q^2}{[3|\not{Q}|2\rangle} = -\frac{\langle 21\rangle[12]}{[3|(\not{1}+\not{2})|2\rangle} = -\frac{[21]}{[31]} \tag{3.21}$$

and using (3.15)

$$\begin{aligned} iA(0) =& iA(1^-,2^-,Q^+(z_0))\frac{-i}{Q^2}iA(Q^-(z_0),3^+(z),4^+)+ \\ &+ iA(1^-,2^-,Q^-(z_0))\frac{-i}{Q^2}iA(Q^+(z_0),3^+(z),4^+) \end{aligned} \tag{3.22}$$

but the second term of sum vanish because vanish each three point amplitude with identical helicity. Now using the expression for the three point amplitudes

$$iA(1^-,2^-,3^+) = ig\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \quad iA(1^+,2^+,3^-) = -ig\frac{[12]^4}{[12][23][31]}, \tag{3.23}$$

we reach this expression for $iA(0)$

$$iA(0) = -ig^2\frac{\langle 12(z_0)\rangle^4}{\langle 12(z_0)\rangle\langle 2(z_0)Q(z_0)\rangle\langle Q(z_0)1\rangle}\frac{1}{[34]\langle 43\rangle}\frac{[3(z_0)4]^4}{[3(z_0)4][4Q(z_0)][Q(z_0)3(z_0)]} \tag{3.24}$$

and semplifying

$$iA(0) = -ig^2\frac{\langle 12\rangle^3}{\langle 2Q(z_0)\rangle\langle Q(z_0)1\rangle}\frac{1}{\langle 43\rangle}\frac{[3(z_0)4]^2}{[34][4Q(z_0)][Q(z_0)3]} \tag{3.25}$$

using explicit form for $Q(z_0)$ we prove the relations

$$\begin{aligned} \langle 2Q(z_0)\rangle[Q(z_0)4] &= \langle 2|\gamma\cdot(3+4)|4] + \frac{1}{2}\Big(-\frac{[21]}{[31]}\Big)\langle 2|\gamma^\mu|4][3|\gamma_\mu|2\rangle \\ &= \langle 2|\not{3}|4] = \langle 23\rangle[34] \\ \langle 1Q(z_0)\rangle[Q(z_0)3] &= \langle 14\rangle[43] \end{aligned} \tag{3.26}$$

and the final compact expression for 4 gluon amplitude is

$$iA(0) = ig^2\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{3.27}$$

Chapter 4

Scattering Amplitudes at One-Loop

The tree-level amplitudes studied before do not give relevant information when we compare theory with experimentation, therefore is necessary to go to higher orders. In this chapter we are going to study how to compute one-loop amplitudes using analytic methods. As before, is important to establish a relation among kinematic and color information, for this, we consider the color decomposition to one-loop. To obtain kinematic information we review many ways to compute one-loop primite amplitudes as Passarino-Veltman decomposition, optical theorem and unitarity of the S-matrix. We focus in the unitarity of the S-matrix by studying the contributions that coming from box, triangle and bubble configurations, the tadpole configuration does not give any contributions because we only consider internal massless loop.

4.1 Color Ordering

In the section (1.1) we studied the color-ordered amplitudes at one-loop. Following the same procedure for the case of amplitudes at tree-level, we obtain [38]

$$\begin{aligned} A_n^{1-loop}(a_i) = & \\ & g^n\Big[\sum_{\sigma\in S_n/Z_n} N_c \mathrm{tr}(T^{a_{\sigma 1}}T^{a_{\sigma 2}}\dots T^{a_{\sigma n}})A_{n;1}^{1-loop}(\sigma 1,\sigma 2,\dots,\sigma n)+ \\ & + \sum_{c=2}^{\lfloor n/2\rfloor+1}\sum_{\sigma\in S_n/S_{n;c}} \mathrm{tr}(T^{a_{\sigma 1}}T^{a_{\sigma 2}}\dots T^{a_{\sigma(c-1)}})\mathrm{tr}(T^{a_{\sigma c}}\dots T^{a_{\sigma n}})\times \\ & \times A_{n;c}^{1-loop}(\sigma_1,\sigma_2,\dots,\sigma_n)\Big] \end{aligned} \tag{4.1}$$

where $A_{n;c}$ are the partial amplitudes that can be obtained from the primitive amplitudes $A_{n;1}$ by summing over all its permutations, Z_n and $S_{n;c}$ (previously defined) that leave the corresponding single and double trace structures invariant, and $\lfloor m\rfloor$is the greatest integer less than or equal to m. The primitive amplitudes $A_{n;1}$ can be computed using the color-ordered Feynman rules of section (1.1).

4.2 Dimensional Regularization

Since one-loop calculations in quantum field theory lead to divergent expressions, we require regularization at intermediate stages of the calculation. Such regularization is accomplished by continuing momenta and polarization vectors of unobserved virtual particles to $D = 4-2\epsilon$ dimensions [24]. The divergences of one-loop amplitudes are regularized by the parameter ϵ. The final expression for any observable quantity should be have a well-defined limit as $D \to 4$. The application of Dimensional Regularization to different kinds of problems has led to the development of a variety of regularization schemes which share the dimensional regularization of momentum integrals but differ in their handling of observed states and spin degrees of freedom. We now seek to evaluate an one-loop integral, leaving the number of dimensions D unspecified. A typical example is

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+\Delta)^2} = \int \frac{d\Omega_D}{(2\pi)^D}\int_0^\infty d\ell \frac{\ell^{D-1}}{(\ell^2+\Delta)^2} \tag{4.2}$$

when we are considering integral over a D-dimensional Euclidean space. The first factor contain the area of unit sphere in D dimension, computing using trick

$$\begin{aligned}\sqrt{\pi}^D = \Big(\int dx e^{-x^2}\Big)^D = \int d^D x \exp\big(-\sum_{i=1}^{D} x_i^2\big) = \\ = \int d\Omega_D \int_0^\infty dx x^{D-1} e^{-x^2} = \Big(\int d\Omega_D\Big)\frac{1}{2}\Gamma[D/2]\end{aligned} \tag{4.3}$$

So the area of D-dimensional unit-sphere is

$$\int d\Omega_D = \frac{2\pi^{d/2}}{\Gamma[D/2]} \tag{4.4}$$

To complete the evaluation of (4.2), the radial integration must be performed.

$$\begin{aligned}\int_0^\infty d\ell \frac{\ell^{D-1}}{(\ell^2+\Delta)^2} = \frac{1}{2}\int_0^\infty d(\ell^2)\frac{\ell^{D/2-1}}{(\ell^2+\Delta)^2} = \\ = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-D/2}\int_0^1 dx x^{1-D/2}(1-x)^{D/2-1}\end{aligned} \tag{4.5}$$

Where we have substituted

$$x = \frac{\Delta}{(\ell^2+\Delta)} \tag{4.6}$$

and used definition of beta function

$$B[\alpha,\beta] = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma[\alpha]\Gamma[\beta]}{\Gamma[\alpha+\beta]} \tag{4.7}$$

The final result for D-dimensional integral is

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+\Delta)^2} = \frac{1}{(4\pi)^{D/2}}\frac{\Gamma[2-D/2]}{\Gamma[2]}\left(\frac{1}{\Delta}\right)^{2-D/2} \tag{4.8}$$

To find the behavior around $D = 4$ we use series expansion of the gamma function to $\epsilon \to 0$ and the result is

$$\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \mathcal{O}(\epsilon)\right) \tag{4.9}$$

where γ is Eulero-Mascheroni constant. Generalizing integral (4.2), we can easily verify more general expressions [40]

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+\Delta)^n} = \frac{1}{(4\pi)^{D/2}}\frac{\Gamma[n-D/2]}{\Gamma[n]}\left(\frac{1}{\Delta}\right)^{n-D/2} \tag{4.10}$$

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^2}{(\ell^2+\Delta)^n} = \frac{1}{(4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma[n-D/2-1]}{\Gamma[n]}\left(\frac{1}{\Delta}\right)^{n-D/2-1} \tag{4.11}$$

This result demonstrates the power of dimensional regularization. The only potentially divergent factor in this expression is the Gamma function, $\Gamma[n-D/2]$. Thus, if the number of propagators s is such that $n - D/2$ is a negative integer or zero in the physical value $D = 4$, the integral is divergent. It is common to express the dimensions as a deviation from the physical dimensions, $D = 4-2\epsilon$. When working in D dimensions, one must take special care of contractions and traces of Dirac gamma matrices which are abundant in loop calculations. First note that a trace of a space-time Kronecker delta is the number of space-time dimensions, $\delta^\mu_\mu = D$. This implies that contracting two metric tensors is also equal to the number of space-time dimensions,

$$g^{\mu\nu}g_{\mu\nu} = D \tag{4.12}$$

Keeping the normal definition of the gamma matrices,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \mathrm{Tr}[I] = 4 \tag{4.13}$$

let us see what happens when two matrices are contracted,

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}_{\mu\nu} = D \tag{4.14}$$

and in the case of more number of matrices,

$$\gamma^\mu\gamma^\nu\gamma_\mu = (2-D)\gamma^\nu \tag{4.15}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-D)\gamma^\nu\gamma^\rho \tag{4.16}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-D)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{4.17}$$

4.3 Tensor Reduction

The Passarino-Veltman [41] tensor reduction (PV) scheme has become a basic tool in the evaluation of one-loop integrals, making possible the calculation of countless amplitudes since its inception. The PV scheme is a much more efficient scheme allowing one to express any Feynman diagram as a sum of scalar integrals only, with each integral multiplied by some coefficient depending only on external kinematic quantities. When we do processes to one-loop that has n external particles, particle i has p_i external momenta (in $D = 4$ dimensions) and momentum conservation impose $\sum_{i=1}^{n} p_i = 0$,integrals appear as the following

$$I_n[f(k\ell)] = -i(4\pi)^{D/2} \int \frac{d^D\ell}{(2\pi)^D} \frac{f(\ell)}{D_1 D_2 \ldots D_n} \tag{4.18}$$

where the inverse scalar propagators are,

$$D_i = (\ell + q_i)^2 - m_i^2 \tag{4.19}$$

and

$$q_i = p_1 + p_2 + \cdots + p_i, \qquad q_n = 0 \tag{4.20}$$

$D = 4 - 2\epsilon$ is the number of dimensions in which we perform the loop integral in order to regularize either ultraviolet or infrared divergences. The function $f(\ell)$ contains all information from the loop momentum i.e. powers of loop momentum. If we consider $f(\ell) = 1$ we obtain the *scalar master integrals*

$$I_n = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 D_2 \ldots D_n} \tag{4.21}$$

Integral reduction [41,42] is a clearly defined procedure for expressing any one-loop Feynman integral as a linear combination of scalar boxes, scalar triangles, scalar bubbles, and scalar tadpoles, with rational coefficients:

$$A^{1-loop} = \sum_{n} \sum_{K_r} c_n[K] I_n[K] \tag{4.22}$$

In four dimensions, n ranges from 1 to 4. Additionally, in PV reduction , we work in $D = 4 - 2\epsilon$ dimensions and the coefficients of the loop integral functions depend on the dimensional regulator ϵ. Rational terms develop when ϵ-dependent pieces of the coefficients multiply poles in ϵ from the loop integral. The tadpole contributions with $n = 1$ arise only with internal masses. If we keep higher order contribution in ϵ, we find that the pentagons ($n = 5$) are independent as well. If we consider $f(\ell) = \ell^\mu$, one power of loop momentum in numerator

$$I_n[\ell^\mu] = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{D_1 D_2 \ldots D_n} \tag{4.23}$$

the result for this integral must be a function of the external momenta $p1, \ldots, p_{n-1}$ (by momentum conservation one momentum is not independent)

$$I_n[\ell^\mu] = \sum_{i=1}^{n-1} C_{n;i} p_i^\mu \tag{4.24}$$

Contracting both sides with p_j^μ,

$$I_n[\ell \cdot p_j] = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell \cdot p_j}{D_1 D_2 \ldots D_n} = \sum_{i=1}^{n-1} C_{n;i} \Delta^{ij} \tag{4.25}$$

where $\Delta^{ij} = p_i \cdot p_j$ is the Gram matrix. Since $p_j = q_j - q_{j-1}$ we can write the numerator of the integral as,

$$\ell \cdot p_j = \frac{1}{2}\Big[\big((\ell+q_j)^2 - m_j^2\big) - \big((\ell+q_{j-1})^2 - m_{j-1}^2\big) + m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2\Big] \tag{4.26}$$

this is the Passarino-Veltman reduction formula. Here the terms $\big((\ell+q_j)^2 - m_j^2\big)$ and $\big((\ell+q_{j-1})^2 - m_{j-1}^2\big)$ in the numerator can be used to cancel the D_j and D_{j-1} propagators respectively and so we end with a set of $n-1$ linear equations for the coefficients $C_{n;i}$.

$$\sum_{i=1}^{n-1} C_{n;i}\Delta^{ij} = \frac{1}{2}\Big(I_{n-1}^{(j)}[1] - I_{n-1}^{(j-1)}[1] + (m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2) I_n[1]\Big) \tag{4.27}$$

and

$$C_{n;i} = \frac{1}{2}\sum_j \Delta_{ij}^{-1}\Big(I_{n-1}^{(j)}[1] - I_{n-1}^{(j-1)}[1] + (m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2) I_n[1]\Big) \tag{4.28}$$

eq. (4.28) represent the set of linear equations. Now we consider $f(k) = \ell^\mu \ell^\nu$, two powers of loop momentum in numerator. The integral is a rank two tensor which can be formed out of the outer products of external momenta $p_i^\mu p_j^\nu$ and the metric tensor $g^{\mu\nu}$,

$$I_n[\ell^\mu \ell^\nu] = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{D_1 D_2 \ldots D_n} = C_{n;00} g^{\mu\nu} + \sum_{i=1}^{n-1} C_{n;i} p_i^\mu p_j^\nu \tag{4.29}$$

The first equation can be derived by contracting both sides with $g^{\mu\nu}$,

$$I_n[\ell^2] = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^2}{D_1 D_2 \ldots D_n} = C_{n;00} D + \sum_{i=1}^{n-1} C_{n;i} \Delta^{ij} \tag{4.30}$$

the other equations are obtained by contracting both sides with p_i, p_j and using eq. (4.28). For $f(k) = \ell^\mu \ell^\nu \ell^\rho$ and $f(k) = \ell^\mu \ell^\nu \ell^\rho \ell^\sigma$, more power of l we follow the same procedure,

$$I_n[\ell^\mu \ell^\nu \ell^\rho] = \sum_{i=1}^{4} C_{n;00i} g^{\{\mu\nu} p_i^{\rho\}} + \sum_{i,j,k=1}^{4} C_{n;ijk} p_i^{\{\mu} p_j^\nu p_\ell^{\rho\}} \tag{4.31}$$

to obtain a set of linear equations for the coefficients $C_{n;00i}$ or $C_{n;ijk}$ we need to contract with $g^{\mu\nu}p^{\rho}$ or with $p_r^{\mu}p_s^{\nu}p_t^{\rho}$. And, for four powers of loop momentum we have,

$$I_n[\ell^{\mu}\ell^{\nu}\ell^{\rho}\ell^{\sigma}] = C_{n;0000}g^{\{\mu\nu}g^{\rho\sigma\}} + \sum_{i,j=1}^{4} C_{n;00ij}g^{\{\mu\nu}p_i^{\rho}p_j^{\sigma\}} + \\ + \sum_{i,j,k,h=1}^{4} C_{n;ijkh}p_i^{\{\mu}p_j^{\nu}p_h^{\sigma}p_{\ell}^{\rho\}} \quad (4.32)$$

Here we need to contract with $g^{\mu\nu}g^{\rho\sigma}$, $g^{\mu\nu}p_r^{\rho}p_s^{\sigma}$, $p_r^{\mu}p_s^{\nu}p_t^{\sigma}p_u^{\rho}$ in order to project out the coefficients $C_{n;0000}$, $C_{n;00ij}$ and $C_{n;ijkh}$.

Chapter 5

Unitarity-based Methods for One-loop Amplitudes

Unitarity-based methods for loop calculations were suggested as an alternative to the Feynman-diagrammatic expansion long ago. It was argued that for gauge theories these methods lead to higher computational efficiency than traditional methods [6]. By means of the Cutkosky rules [43], we can calculate the imaginary or absorptive parts of one-loop amplitudes, directly as products of tree amplitudes. In this way, we cut the diagrams into two tree diagrams, while the loop integral is replaced by an integral over the phase space of the particles crossing the cut. This is much easier than a complete one-loop diagrammatic expansion, as tree-level amplitudes are easier to obtain and they have simple expressions that can be fed into the algorithm of the method. Unitarity cuts can be "generalized" in the sense of putting a different number of propagators on-shell. This operation selects different kinds of singularities of the amplitude; they are not physical momentum channels like ordinary cuts and do not have an interpretation relating to the unitarity of the S-matrix.

5.1 Optical Theorem

First, we investigate the structure of the S-matrix, which transforms in states to out states [44]. If particles do not interact, S is just the identity matrix. We isolate the interacting part of the S-matrix, by defining the T-matrix through the equation

$$S = 1 + iT \tag{5.1}$$

From the unitarity of the S-matrix, $S^\dagger S = 1$, we obtain the *optical theorem*

$$-i(T - T^\dagger) = T^\dagger T \tag{5.2}$$

In perturbation theory, T is a sum of Feynman diagrams, each one carrying a power of the coupling constant g, depending on the number of loops. The product of matrices $T^\dagger T$ implies a sum of contributions from all possible intermediate states f.

In terms of matrix elements $\mathcal{M}$ we have

$$-i[\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a)] = \sum_f \int d \prod \mathcal{M}^*(b \to f)\mathcal{M}(f \to a) \tag{5.3}$$

where, in addition to summing over the intermediate particle states f, we are also integrating over the complete phase space of these states. In practice we see that the imaginary part of the one-loop amplitude is related to a product of two tree amplitudes. Furthermore, the fact that we are dealing with amplitudes and not individual Feynman diagrams means that we do not have to consider the actual Feynman diagrams that make up the higher order term of the amplitude we are calculating. We simply need to identify the channel we are dealing with, i.e. the initial and final states, and then construct all the lower order amplitudes that could potentially contribute to the higher order term we seek to evaluate.

5.2 Cutkosky rules

The Cutkosky rules for computing the physical discontinuity of a specified diagram are given by the following algorithm [43]:

1. We cut the diagram so that the two propagators can simultaneously be put on-shell
2. For each cut propagator, we replace

   $$\frac{i}{p^2 - m^2 + i\epsilon} \to -1\pi i \delta^{(+)}(p^2 - m^2) \tag{5.4}$$

   here, the superscript (+) on the delta functions for the cut propagators denotes the choice of a positive-energy solution.
3. Then, perform the loop integrals
4. And finally, sum the contributions of all cuts

Using these rules "cutting rules", it is possible to prove the optical theorem to all orders in perturbation theory. The Cutkosky rules are expressed in the cut integral

$$\Delta A^{1-loop} \equiv \int d\mu A^{tree}_{left} A^{tree}_{right} \tag{5.5}$$

where A^{1-loop} is the color-ordered primitive amplitude and $d\mu$ the Lorentz-invariant phase space measure is defined by

$$d\mu = d^4p_1 d^4p_1 \delta^4(p_1 + p_2 - k)\delta^{(+)}(p_1^2)\delta^{(+)}(p_2^2) \tag{5.6}$$

with k loop momentum and p_1 and p_2. To compute the amplitude, we apply the cut Δ in various momentum channels where we get information about the coefficients of master integrals. If we apply a unitarity cut to the expansion (4.22) of an amplitude in master integrals, since the coefficients are rational functions, the branch cuts are located only in the master integrals. Thus we find that

$$\Delta A^{1-loop} = \sum_n \sum_{p_r} c_n(p_r)\Delta I_n(p_r) \tag{5.7}$$

Eq. (5.7) is the key to the unitarity method. It has two important features. First,

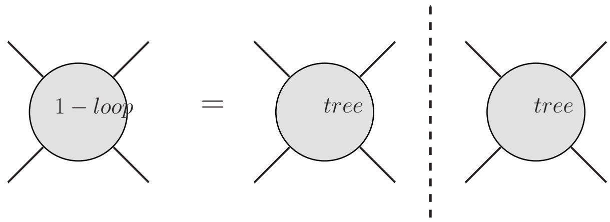


Figure 5.1: Diagram of Cutkowsky rule

we see from (5.6) that it is a relation involving tree-level quantities. Second, many of the terms on the right-hand side vanish, because only a subset of master integrals have a cut involving the given momentum K. The problem is to obtain the individual coefficients c_i. With generalized unitarity these coefficients are obtained easily.

5.3 Generalized Unitarity and Dimensional Regularization

In this section we discuss a consequence of using internal lines in $(4-2\epsilon)$-dimensions [14, 15, 45] One consequence was obtaining of an effective mass μ^2 en 4 dimensions. We begin by writing a general 1-loop amplitude in terms of a D-dimensional n-point function,

$$A_n^{(1)} = \int \frac{d^D\ell}{(4\pi)^{D/2}} \frac{\mathcal{N}(\{p_i\}, l)}{(\ell^2 - m_1^2)((\ell - p_1)^2 - m_2^2)\dots((\ell + p_n)^2 - m_n^2)}. \tag{5.8}$$

The numerator function $\mathcal{N}$ contains all information from external polarization states and wave functions and tensor structures from the loop momenta. Using D-dimensional Passarino-Veltman reduction techniques on (5.8) allows us to reduce to a basis of scalar integral functions with rational, but D-dimensional, coefficients [15, 46],

$$\begin{aligned} A_n^{(1),D} &= \sum_{p_5} \tilde{\mathcal{C}}_{5;p_5}(D) I_{5;p_5}^D + \sum_{p_4} \tilde{\mathcal{C}}_{4;p_4}(D) I_{4;p_4}^D \\ &+ \sum_{p_3} \mathcal{C}_{3;p_3}(D) I_{3;p_3}^D + \sum_{p_2} \mathcal{C}_{2;p_2}(D) I_{2;p_2}^D + \mathcal{C}_1(D) I_1^D, \end{aligned} \tag{5.9}$$

where we define the sets of external momenta, p_r, as the set of all ordered partitions of the n external particles into r distinct groups (the ordering is defined by that of the full amplitude $A_n^{(1)}$).

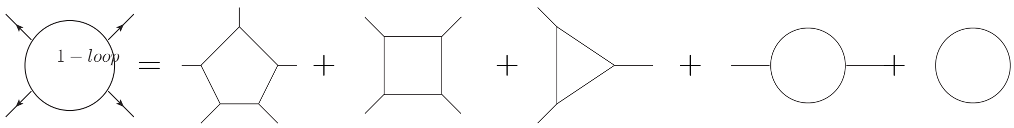


Figure 5.2: Passarino-Veltman decomposition

Since we are concerned with computing the four dimensional limit it is useful to decompose the loop momenta as,

$$\bar{\ell}^\nu = \ell^\nu + \tilde{\ell}^\nu, \tag{5.10}$$
$$\bar{\ell}^2 = \ell^2 + \tilde{\ell}^2 \equiv \ell^2 - \mu^2. \tag{5.11}$$

where $\bar{\ell}$ contains the four-dimensional components and $\tilde{\ell}$ contains the remaining $D - 4 = -2\epsilon$ dimensional components.

We see then that any dimensional dependence of the numerators arises only through dependence on μ^2. In QCD, the maximum number of power of loop momentum appearing in the numerator of an n-point tensor integral is n, so the boxes can have at most a μ^4 while the triangles and bubbles can have up to a μ^2 . The pentagon integral is an independent function in D dimensions since we can find poles in the $D - 4$ dimensional sub-space, then the coefficient of this function in $D = 4 - 2\epsilon$, residue around the extra dimensional poles, can have no dependence on ϵ. With this prescription, the master integrals in $D = 4 - 2\epsilon$-dimensions would take the form,

$$\int \frac{d^D \ell_1}{(2\pi)^D} = \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}} \int \frac{d^4 l_1}{(2\pi)^4}. \tag{5.12}$$

It is fairly straightforward to write the four new integrals in terms of higher-dimensional scalar integrals using [47],

$$I_n^D[\mu^{2r}] = \frac{1}{2^r} I_n^{D+2r}[1] \prod_{k=0}^{r-1} (D - 4 + k). \tag{5.13}$$

This procedure changes the dimension of the integral and that D -dependence appears in the coefficients of the master integrals. Writing A_{1-loop} n in D-dimension in terms of $(\mu^2)^k$, $k = 0, 1, 2$,

$$\begin{aligned} A_n^{(1),D} &= \sum_{p_5} \tilde{C}_{5;p_5} I^D_{5;p_5} \\ &+ \sum_{p_4} C^{[0]}_{4;p_4} I^D_{4;p_4}[1] + \sum_{p_4} C^{[2]}_{4;p_4} I^D_{4;p_4}[\mu^2] + \sum_{p_4} C^{[4]}_{4;p_4} I^D_{4;p_4}[\mu^4] + \sum_{p_3} C_{3;p_3} I^D_{3;p_3}[1] \\ &+ \sum_{p_3} C^{[2]}_{3;p_3} I^D_{3;p_3}[\mu^2] + \sum_{p_2} C_{2;p_2} I^D_{2;p_2}[1] + \sum_{p_2} C^{[2]}_{2;p_2} I^D_{2;p_2}[\mu^2] + C_1 I_1^D. \end{aligned} \tag{5.14}$$

We also use the dimensional shift identity [48] to decompose the pentagon integrals

$$I_5^D[1] = \frac{(D-4)}{2} I_5^{D+2}[1] \left(\sum_{i,j} S_{ij}^{-1} \right) + \frac{1}{2} \sum_{i=1}^{5} \sum_{j} S_{ij}^{-1} I_{4;p_5^{(i)}}, \tag{5.15}$$

$$S_{ij} = \frac{1}{2}\left(m_i^2 + m_j^2 - p_{ij}^2\right). \tag{5.16}$$

In the above $p_5^{(i)}$ is one of the five sets of four partitions obtained cyclically merging two adjacent partitions of a given pentagon configuration p_5. From identity (3-43) we obtain:

$$I_{4;p_4}^D[\mu^2] = \frac{D-4}{2} I_{4;p_4}^{D+2}[1] \tag{5.17}$$

$$I_{4;p_4}^D[\mu^4] = \frac{(D-4)(D-2)}{4} I_{4;p_4}^{D+4}[1] \tag{5.18}$$

$$I_{3;p_3}^D[\mu^2] = \frac{D-4}{2} I_{3;p_3}^{D+2}[1] \tag{5.19}$$

$$I_{2;p_2}^D[\mu^2] = \frac{D-4}{2} I_{2;p_2}^{D+2}[1] \tag{5.20}$$

And the full amplitude:

$$\begin{aligned} A_n^{(1),D} &= \frac{D-4}{2} \sum_{p_5} C_{5;p_5} I_{5;p_5}^{D+2} \\ &+ \sum_{p_4} C_{4;p_4} I_{4;p_4}^D + \frac{D-4}{2} \sum_{p_4} C_{4;p_4}^{[2]} I_{4;p_4}^{D+2} + \frac{(D-4)(D-2)}{4} \sum_{p_4} C_{4;p_4}^{[4]} I_{4;p_4}^{D+4} + \sum_{p_3} C_{3;p_3} I_{3;p_3}^D \\ &+ \frac{D-4}{2} \sum_{p_3} C_{3;p_3}^{[2]} I_{3;p_3}^{D+2} + \sum_{p_2} C_{2;p_2} I_{2;p_2}^D + \frac{D-4}{2} \sum_{p_2} C_{2;p_2}^{[2]} I_{2;p_2}^{D+2} + C_1 I_1^D, \end{aligned} \tag{5.21}$$

where

$$C_{4;p_4} = C_{4;p_4}^{[0]} + \sum_{i=1}^{5} \sum_{j} S_{ij}^{-1} \tilde{C}_{5;p_5^{(i)}}. \tag{5.22}$$

$$C_{5;p_5} = \tilde{C}_{5;p_5} \sum_{i,j} S_{ij}^{-1}. \tag{5.23}$$

Now, we take the 4 -dimensional limit $D = 4 - 2\epsilon$ around $\epsilon \to 0$:

$$A_n^{1-loop} = \text{Cut-Constructible} + \text{Rational Terms} \tag{5.24}$$

The cut-constructible amplitude can be obtained just by studying our amplitude in $D = 4$ dimensions and is given by,

$$\text{Cut-Constructible} = \sum_{p_4} C_{4;p_4} I_{4;p_4}^{4-2\epsilon} + \sum_{p_3} C_{3;p_3} I_{3;p_3}^{4-2\epsilon} + \sum_{p_2} C_{2;p_2} I_{2;p_2}^{4-2\epsilon} + C_1 I_1^{4-2\epsilon} \tag{5.25}$$

Rational terms, Rn, arise in $D = 4 - 2\epsilon$ dimensions,

$$R_n = \frac{D-4}{2} \sum_{p_5} C_{5;p_5} I_{5;p_5}^{D+2} + \frac{D-4}{2} \sum_{p_4} C_{4;p_4}^{[2]} I_{4;p_4}^{D+2} + \frac{(D-4)(D-2)}{4} \sum_{p_4} C_{4;p_4}^{[4]} I_{4;p_4}^{D+4}$$
$$+ \frac{D-4}{2} \sum_{p_3} C_{3;p_3}^{[2]} I_{3;p_3}^{D+2} + \frac{D-4}{2} \sum_{p_2} C_{2;p_2}^{[2]} I_{2;p_2}^{D+2} \qquad (5.26)$$

5.4 Coefficients Projections

Then to extract the integral coefficient using generalized unitarity [12, 14, 15, 49, 50] we need to solve the constraints which put the various propagators on-shell. Moreover in $D = 4 - 2\epsilon$ we need to extract the μ dependence of the coefficients. By studying internal lines in $D = 4 - 2\epsilon$ we obtain an effective mass term, therefore it is possible to construct the full amplitude from tree amplitudes where the internal lines have an uniform mass,

$$\bar{\ell}_i^2 = \ell_i^2 - \mu^2 \quad \Rightarrow \quad \ell_i^2 = \mu^2 \qquad (5.27)$$

where ℓ_i is in 4 dimension. For each cut, we decompose ℓ, namely the 4-dimensional part of $\bar{\ell}$, into a specific basis of four massless vectors e_i,

$$\ell = -p_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 \qquad (5.28)$$

such that

$$e_i^2 = 0\ , \quad e_1 \cdot e_3 = e_1 \cdot e_4 = 0 \quad e_2 \cdot e_3 = e_2 \cdot e_4 = 0 \quad e_1 \cdot e_2 = -e_3 \cdot e_4 \qquad (5.29)$$

and where e_1 and e_2 are real vectors, while e_3 and e_4 are complex.
The massless vectors e_1 and e_2 can be written as a linear combination of the two external legs at the edges of the propagator carrying momentum $\bar{\ell} + p_0$, say p_1 and p_2,

$$e_1^\nu = \frac{1}{\beta}\left(p_1^\nu - \frac{p_1^2}{\gamma} p_2^\nu\right), \qquad e_2^\nu = \frac{1}{\beta}\left(p_2^\nu - \frac{p_2^2}{\gamma} p_1^\nu\right), \qquad (5.30)$$

with

$$\beta = 1 - \frac{p_1^2 p_2^2}{\gamma^2}\ , \qquad \gamma = p_1 \cdot p_2 \pm \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}\ . \qquad (5.31)$$

The massless vectors e_3 and e_4 can be then obtained as,

$$e_3^\nu = \frac{\langle e_1|\gamma^\nu|e_2]}{2}, \qquad e_4^\nu = \frac{\langle e_2|\gamma^\nu|e_1]}{2}. \qquad (5.32)$$

Now solving the system of on-shell constraints can be fixed coefficients of master integrals.

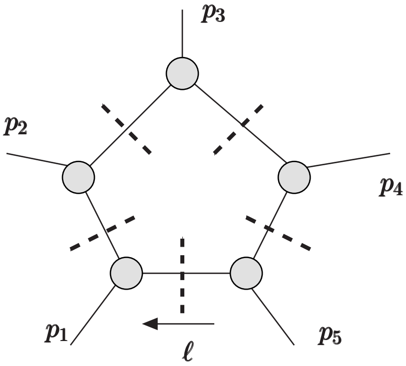


Figure 5.3: Quintuple-cut

5.4.1 Pentagon Coefficient

To compute the coefficient of pentagon integral there are five constraints,

$$D_0(\ell) = \ell^2 = \mu^2 \tag{5.33}$$
$$D_1(\ell) = (\ell + p_1)^2 = \mu^2 \tag{5.34}$$
$$D_2(\ell) = (\ell + p_1 + p_2)^2 = \mu^2 \tag{5.35}$$
$$D_3(\ell) = (\ell + p_1 + p_2 + p_3)^2 = \mu^2 \tag{5.36}$$
$$D_4(\ell) = (\ell - p_5)^2 = \mu^2 \tag{5.37}$$

therefore the quintuple-cut has one solution in the variables x_1, x_2, x_3, x_4 and μ^2, which can be found as follow. Using $D_0 = \mu^2$, the subsystem $D_1 = D_2 = D_3 = D_4 = \mu^2$ will impose four linear constraints on ℓ, which fix the four components x_1, x_2, x_3, x_4 in terms of external kinematic variables. Finally, equation $D_0 = \mu^2$,

$$x_1 x_2 - x_3 x_4 = \frac{\mu^2}{e_1 \cdot e_2} \tag{5.38}$$

freezes the value of μ^2.

5.4.2 Box Coefficient

Without loss of generality we can consider, in this section and in the next, a process with 4 external legs. We can use momenta p_1 and p_4 (see figure) to build the basis vectors (5.30) For the quadrupole cut [fig] in $4 - 2\epsilon$-dimensions, the on-shell cut conditions are

$$D_0(\bar{\ell}) = D_1(\bar{\ell}) = D_2(\bar{\ell}) = D_3(\bar{\ell}) = 0 \tag{5.39}$$

or equivalently with q in 4 dimension

$$D_0(\ell) = D_1(\ell) = D_2(\ell) = D_3(\ell) = \mu^2 \tag{5.40}$$

where μ^2 represents an effective mass term that comes from the (-2ϵ)-dimensional components Using momentum conservation and writing all loop momenta in terms

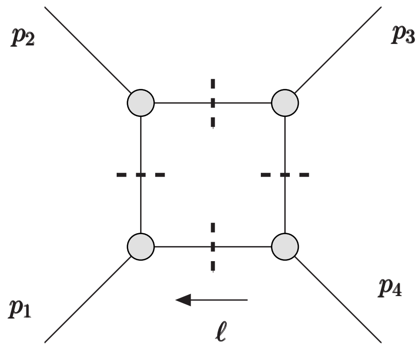


Figure 5.4: Quadruple-cut

of ℓ,

$$D_0(\ell) = \ell^2 = \mu^2 \quad (5.41)$$
$$D_1(\ell) = (\ell + p_1)^2 = \mu^2 \quad (5.42)$$
$$D_2(\ell) = (\ell + p_1 + p_2)^2 = \mu^2 \quad (5.43)$$
$$D_3(\ell) = (\ell - p_4)^2 = \mu^2 \quad (5.44)$$

from first equation x_4 takes the form

$$x_4 = \frac{\gamma x_1 x_2 - \mu^2}{\gamma x_3} \quad (5.45)$$

Solving the other on-shell conditions, we find

$$x_1 = \frac{p_1^2(p_4^2 + \gamma)}{\gamma^2 - p_1^2 p_4^2}, \quad x_2 = -\frac{p_4^2(p_1^2 + \gamma)}{\gamma^2 - p_1^2 p_4^2}, \quad x_3^{(\pm)} = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0 c_2}}{2c_2} \quad (5.46)$$

with

$$c_1 = x_1 e_2 \cdot p_2 + x_2 e_1 \cdot p_2 - p_2^2 \quad (5.47)$$
$$c_2 = e_4 p_2 \quad (5.48)$$
$$c_0 = (x_1 x_2 - \frac{\mu^2}{\gamma}) e_1 \cdot p_2 \quad (5.49)$$

There are two solutions corresponding two values $c_\pm$(we could think there are four solutions because plus and minus of γ but we have to choose one basis, so this ambiguity vanish) how expected because we have equations of second order. To determine the full box coefficient, we must average over these solutions.

$$C_4^{[0]} = \frac{i}{2} \sum_\sigma A_1 A_2 A_3 A_4(q^{(\sigma)}) \quad (5.50)$$

$$C_4^{[4]} = \frac{i}{2} \mathrm{Inf}_{\mu^2} [A_1 A_2 A_3 A_4(q^{(\sigma)})]_{\mu^4} \quad (5.51)$$

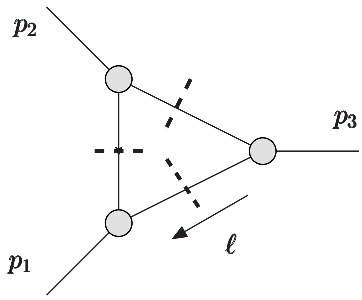


Figure 5.5: Triple-cut

5.4.3 Triangle coefficient

We defining e_1 and e_2 using momenta p_1 and p_3, analogously to see before The three delta function constraints imposed by the cuts:

$$D_0(\ell) = \ell^2 = \mu^2 \tag{5.52}$$
$$D_1(\ell) = (\ell + p_1)^2 = \mu^2 \tag{5.53}$$
$$D_2(\ell) = (\ell - p_3)^2 = \mu^2 \tag{5.54}$$

Using the same parametrization of the box and the on-shell conditions, we find two family of solutions

$$q = x_1 e_1 + x_2 e_2 + t e_3 + \frac{\gamma x_1 x_2 - \mu^2}{\gamma t} e_4 \tag{5.55}$$
$$q^* = x_1 e_1 + x_2 e_2 + t e_4 + \frac{\gamma x_1 x_2 - \mu^2}{\gamma t} e_3 \tag{5.56}$$

where

$$x_1 = \frac{p_1^2(p_3^2 + \gamma)}{\gamma^2 - p_1^2 p_3^2}, \tag{5.57}$$
$$x_2 = -\frac{p_3^2(p_1^2 + \gamma)}{\gamma^2 - p_1^2 p_3^2}, \tag{5.58}$$
$$\gamma = p_1 \cdot p_3 \pm \sqrt{(p_1 \cdot p_3)^2 - p_1^2 p_3^2} \tag{5.59}$$

and the complex parameter t is free. The box integrals also contain triple cuts, so we must extract the triangle coefficients using the limiting behavior of the integrand. The coefficients therefore contain an Inf term that is a polynomial expansion in t, but only the order t^0 is retained.

$$C_3^{[0]} = -\frac{1}{2} \sum_\sigma [\mathrm{Inf}_t A_1 A_2 A_3(q^{(\sigma)})](t)|_{t^0} \tag{5.60}$$
$$C_3^{[0]} = -\frac{1}{2} \sum_\sigma \mathrm{Inf}_{\mu^2} [Inf_t A_1 A_2 A_3(q^{(\sigma)})](t)|_{\mu^2, t^0} \tag{5.61}$$

The sum is over the solutions, including the conjugate momentum solution. In $C_3^{[0]}$, μ^2 is set to zero, while the expansion in $C_3^{[2]}$ is restricted to the coefficients of the μ^2 term.

5.4.4 Bubble Coefficient

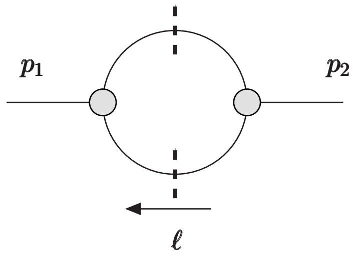


Figure 5.6: Double-cut

To extract the coefficients of bubble integrals, we impose the cuts that define the bubble topology (see figure)

$$D_0(\ell) = \ell^2 = \mu^2 \tag{5.62}$$

$$D_1(\ell) = (\ell + p_1)^2 = \mu^2 \tag{5.63}$$

Only one bubble configuration will satisfy these cuts, but multiple triangle and box configurations will do so. Since we only have one external momentum, p_1, in a bubble configuration, we can choose an arbitrary massless momentum χ^μ to define our parametrization: using the on-shell conditions, we have only one solution for the bubble cut contribution,

$$\ell = ye_1 + \frac{p_1^2}{\gamma}e_2 + te_3 + \frac{y(1-y)p_1^2 - \mu^2}{\gamma t}e_4 \tag{5.64}$$

where ℓ has two free complex parameters y and t. By studying the triangle contribution to the bubble coefficient, we fix the parameter y by using another on-shell condition,

$$(\ell + p_3)^2 = \mu^2 \tag{5.65}$$

the solutions for y is

$$y_\pm = \frac{B_1 \pm \sqrt{B_1^2 + 4B_0B_2}}{2B_2}, \tag{5.66}$$

where

$$B_2 = p_1^2 e_4 \cdot p_3, \tag{5.67}$$

$$B_1 = \gamma te_1 \cdot p_3 - p_1^2 te_2 \cdot p_3 + p_1^2 e_4 \cdot p_3, \tag{5.68}$$

$$B_0 = \gamma t^2 e_3 \cdot p_3 - \mu^2 e_4 \cdot p_3 + \gamma tp_3^2 + tp_1^2 e_2 \cdot p_3 \tag{5.69}$$

We then calculate the triple cut integrand $A1A2A3$ for all triple cuts that share two cuts with the original double cut. The bubble coefficients are given by

$$C_2^{[0]} = -i\mathrm{Inf}_t(\mathrm{Inf}_y)[A_1A_2(q(y,t))]|_{t^0,y^i\to Y_i} - \frac{1}{2}\sum_{C_tri}\sum_{\sigma_y}\mathrm{Inf}_t[A_1A_2A_3(q(t))]|_{t_i\to T_i} \tag{5.70}$$

$$C_2^{[2]} = -i\mathrm{Inf}_{\mu^2}(\mathrm{Inf}_t(\mathrm{Inf}_y)[A_1A_2(q(y,t))])|_{\mu^2,t^0,y^i\to Y_i} - \\ -\frac{1}{2}\sum_{C_tri}\sum_{\sigma_y}\mathrm{Inf}_{\mu^2}\mathrm{Inf}_t[A_1A_2A_3(q(t))]|_{\mu^2,t_i\to T_i} \tag{5.71}$$

the functions T_i and Y_i have been computed in Ref. [14] for arbitrary kinematics. Explicitly with an uniform mass we have,

$$Y_0 = 1 \quad Y_1 = \frac{1}{2} \quad Y_2 = \frac{1}{3}\left(1 - \frac{\mu^2}{p_1^2}\right). \tag{5.72}$$

and

$$T_1 = -\frac{p_1^2 e_4\cdot p_3}{\gamma\Delta}, \tag{5.73}$$

$$T_2 = -\frac{3p_1^2(e_4\cdot p_3)^2}{2\gamma^2\Delta^2}\left(p_1^2p_3^2 + p_1\cdot p_3 p_1^2\right), \tag{5.74}$$

$$T_3 = -\frac{(e_4\cdot p_3)^3}{4\gamma^3\Delta^3}\Big(15(p_1^2p_3^2)^3 + 30p_1\cdot p_3 {p_1^2}^3 p_3^2 + 11(p_1\cdot p_3)^2 {p_1}^3 \\ + 4(p_1^2)^4p_3^2 + 16\mu^2(p_1^2)^2\Delta\Big), \tag{5.75}$$

where $\Delta = (p_1\cdot p_3)^2 - p_1^2p_3^2$.

5.4.5 Tadpole Coefficient

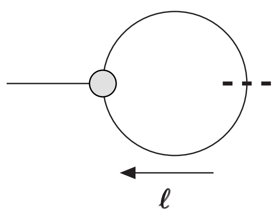


Figure 5.7: Single-cut

The tadpole coefficients can be extracted by an extension of the same procedure. The only constraint that the loop momentum must satisfy is that

$$\ell^2 = \mu^2 \tag{5.76}$$

Satisfying this constraint leaves three free parameters in ℓ

$$\ell = ye_1 + we_2 + te_3 + \frac{1}{t}\left(wy - \frac{\mu^2}{\gamma}\right) e_4 \tag{5.77}$$

The complete tadpole coefficient is found by combining the pure tadpole, bubble and triangle contributions.

$$\begin{aligned}
a_m = & \left[\text{Inf}_w \left[\text{Inf}_y \left[\text{Inf}_t A^{(m)}\right]\right]\right](t, y, w)\Big|_{t\to 0,(wy)^n \to D(n)} \\
& - i \sum_{\text{bubbles}} \left[\left[\text{Inf}_y \left[\text{Inf}_t \tilde{A}_1^{(m)} \tilde{A}_2^{(m)}\right]\Big|_{w=w_0}\right]\Big|_{y^k t^m \to E_w(k,m)} \right. \\
& + \left. \left[\text{Inf}_w \left[\text{Inf}_t \tilde{A}_1^{(m)} \tilde{A}_2^{(m)}\right]\Big|_{y=y_0}\right]\Big|_{w^k t^m \to E_y(k,m)}\right] \\
& - \frac{1}{2} \sum_{\text{triangles}} \sum (w, y) = w_1, y_1{}^{(w_2,y_2)} \left[\text{Inf}_t \hat{A}_1^{(m)} \hat{A}_2^{(m)} \hat{A}_3^{(m)}\right](t)\Big|_{t^n \to F(n)}
\end{aligned} \tag{5.78}$$

where the $\tilde{A}_n^{(m)}$ are the amplitudes formed by cutting a propagator in $A^{(m)}$, adding a second condition $(\ell + p_1)^= \mu^2$and $\hat{A}_n^{(m)}$ are the amplitudes formed by cutting two propagators in $A^{(m)}$, adding a third condition $(\ell + p_2)^2 = \mu^2$. Integral value $E(k, m)$ and $F(n)$ are tabulated in [14]

Chapter 6

Four-Dimensional Formulation of Dimensional Regularization

In this section we show a pure four-dimensional formulation (FDF) of the d-dimensional regularization of one-loop scattering amplitudes following [25, 51]. Within FDF, the states in the loop are described as four dimensional massive particles. The four-dimensional degrees of freedom of the gauge bosons are carried by massive vector bosons of mass μ and their $(d-4)$-dimensional ones by real scalar particles obeying a simple set of four-dimensional Feynman rules. A d-dimensional fermion of mass m is instead traded for a tachyonic Dirac field with mass $m + i\mu\gamma5$. The d dimensional algebraic manipulations are replaced by four-dimensional ones complemented by a set of multiplicative selection rules. The latter are treated as an algebra describing internal symmetries. The FDH scheme [47, 52, 53] defines a d-dimensional vector space embedded in a larger d_s-dimensional space, $d_s \equiv (4-2\epsilon) > d > 4$. The scheme is determined by the following rules

- The loop momenta are considered to be d-dimensional. All observed external states are considered as four-dimensional. All unobserved internal states, *i.e.* virtual states in loops and intermediate states in trees, are treated as d_s-dimensional.

- Since $d_s > d > 4$, the scalar product of any d- or d_s-dimensional vector with a four-dimensional vector is a four-dimensional scalar product. Moreover any dot product between a d_s-dimensional tensor and a d-dimensional one is a d-dimensional dot product.

- The Lorentz and the Clifford algebra are performed in d_s dimensions, which has to be kept distinct from d. The matrix γ^5 is treated using the 't Hooft-Veltman prescription, *i.e.* γ^5 commutes with the Dirac matrices carrying -2ϵ indices.

- After the γ-matrix algebra has been performed, the limit $d_s \to 4$ has to be performed, keeping d fixed. The limit $d \to 4$ is taken at the very end.

6.1 Feynman Rules

In the following d_s-dimensional quantities are denoted by a bar. One can split the d_s-dimensional metric tensor as follows

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}\,, \tag{6.1}$$

in terms of a four-dimensional tensor g and a -2ϵ-dimensional one, $\tilde{g}$, such that

$$\tilde{g}^{\mu\rho}\, g_{\rho\nu} = 0\,, \qquad \tilde{g}^{\mu}{}_{\mu} = -2\epsilon \underset{d_s\to 4}{\longrightarrow} 0\,, \qquad g^{\mu}{}_{\mu} = 4\,, \tag{6.2}$$

The tensors g and $\tilde{g}$ project a d_s-dimensional vector $\bar{q}$ into the four-dimensional and the -2ϵ-dimensional subspaces respectively,

$$q^{\mu} \equiv g^{\mu}{}_{\nu}\, \bar{q}^{\nu}\,, \qquad\qquad \tilde{q}^{\mu} \equiv \tilde{g}^{\mu}{}_{\nu}\, \bar{q}^{\nu}\,. \tag{6.3}$$

At one loop the only d-dimensional object is the loop momentum $\bar{\ell}$. The square of its -2ϵ dimensional component is defined as:

$$\tilde{\ell}^2 = \tilde{g}^{\mu\nu}\, \bar{\ell}_\mu\, \bar{\ell}_\nu \equiv -\mu^2\,. \tag{6.4}$$

The properties of the matrices $\tilde{\gamma}^{\mu} = \tilde{g}^{\mu}{}_{\nu}\, \bar{\gamma}^{\nu}$ can be obtained from Eq.(6.2)

$$[\tilde{\gamma}^{\alpha}, \gamma^5] = 0\,, \qquad\qquad \{\tilde{\gamma}^{\alpha}, \gamma^{\mu}\} = 0\,, \tag{6.5a}$$

$$\{\tilde{\gamma}^{\alpha}, \tilde{\gamma}^{\beta}\} = 2\,\tilde{g}^{\alpha\beta}\,. \tag{6.5b}$$

We remark that the -2ϵ tensors can not have a four-dimensional representation. Indeed the metric tensor $\tilde{g}$ is a tripotent matrix

$$\tilde{g}^{\mu\rho}\tilde{g}_{\rho\nu}\tilde{g}^{\nu\sigma} = \tilde{g}^{\mu\sigma}\,, \tag{6.6}$$

and its square is traceless

$$\tilde{g}^{\mu\rho}\tilde{g}_{\rho\mu} = \tilde{g}^{\mu}{}_{\mu} \underset{d_s\to 4}{\longrightarrow} 0\,, \tag{6.7}$$

but in any integer-dimension space the square of any non-null tripotent matrix has an integer, positive trace. Moreover the component $\tilde{\ell}$ of the loop momentum vanishes when contracted with the metric tensor g,

$$\tilde{\ell}^{\mu}\, g_{\mu\nu} = \bar{\ell}_{\rho}\, \tilde{g}^{\rho\mu}\, g_{\mu\nu} = 0\,, \tag{6.8}$$

and in four dimensions the only four vector fulfilling (6.8) is the null one. Finally in four dimensions the only non-null matrices fulfilling the conditions (6.5a) are proportional to γ^5, hence $\tilde{\gamma} \sim \gamma^5$. However the matrices $\tilde{\gamma}$ fulfill the Clifford algebra (6.5b), thus

$$\tilde{\gamma}^{\mu}\, \tilde{\gamma}_{\mu} \underset{d_s\to 4}{\longrightarrow} 0\,, \qquad \text{while} \qquad \gamma^5\gamma^5 = \mathbb{I}\,. \tag{6.9}$$

These arguments exclude any four-dimensional representation of the -2ϵ subspace. It is possible, however, to find such a representation by introducing additional rules, called in the following -2ϵ *selection rules*, (-2ϵ)-SRs. Indeed the Clifford algebra (6.5b) is equivalent to

$$\cdots\tilde{\gamma}^{\alpha}\cdots\cdots\tilde{\gamma}_{\alpha}\cdots = 0, \qquad \tilde{\ell}\tilde{\ell} = -\mu^2\,. \tag{6.10}$$

Therefore any regularization scheme which is equivalent of FDH has to fulfill the conditions (6.2) – (6.5a), and (6.10). The orthogonality conditions (6.2) and (6.3) are fulfilled by splitting a d_s-dimensional gluon onto a four-dimensional one and a colored scalar, s_g, while the other conditions are fulfilled by performing the substitutions:

$$\tilde{g}^{\alpha\beta} \to G^{AB}, \qquad \tilde{\ell}^{\alpha} \to i\,\mu\,Q^{A}\,, \qquad \tilde{\gamma}^{\alpha} \to \gamma^5\,\Gamma^{A}\,. \tag{6.11}$$

The -2ϵ-dimensional vectorial indices are thus traded for (-2ϵ)-SRs such that

$$G^{AB}G^{BC} = G^{AC}, \qquad G^{AA} = 0, \qquad G^{AB} = G^{BA}, \tag{6.12}$$

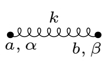
$$\Gamma^{A}G^{AB} = \Gamma^{B}, \qquad \Gamma^{A}\Gamma^{A} = 0, \qquad Q^{A}\Gamma^{A} = 1, \tag{6.13}$$

$$Q^{A}G^{AB} = Q^{B}, \qquad Q^{A}Q^{A} = 1. \tag{6.14}$$

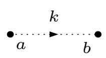
The exclusion of the terms containing odd powers of μ completely defines the FDF, and allows one to build integrands which, upon integration, yield to the same result as in the FDH scheme.

The rules (6.14) constitute an abstract algebra which is similar to the algebras implementing internal symmetries. For instance, in a Feynman diagrammatic approach the (-2ϵ)-SRs can be handled as the color algebra and performed for each diagram once and for all. In each diagram, the indices of the (-2ϵ)-SRs are fully contracted and the outcome of their manipulation is either 0 or ± 1. It is worth to remark that the replacement of $\tilde{\gamma}^{\alpha}$ with γ^5 takes care of the d_s-dimensional Clifford algebra automatically, thus we do not need to introduce any additional scalar particle for each fermion flavor. These particles and their interactions have been instead introduced in Ref. [54], where a method for the reconstruction of the μ^2-dependent part of the numerator has been proposed.

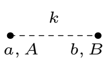
To summarize, the QCD d-dimensional Feynman rules in the 't Hooft-Feynman gauge, collected in Ref. [55], may have the following four-dimensional formulation:



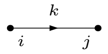
$$= -i\,\delta^{ab}\,\frac{g^{\alpha\beta}}{k^2-\mu^2+i0} \quad \text{(gluon)}, \tag{6.15a}$$



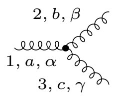
$$= i\,\delta^{ab}\,\frac{1}{k^2-\mu^2+i0} \quad \text{(ghost)}, \tag{6.15b}$$



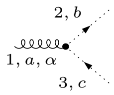
$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k^2-\mu^2+i0}\,, \quad \text{(scalar)}, \tag{6.15c}$$



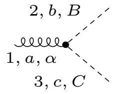
$$= i\,\delta^{ij}\,\frac{\not k + i\mu\gamma^5 + m}{k^2 - m^2 - \mu^2 + i0} \quad \text{(fermion)}, \tag{6.15d}$$



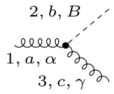
$$\begin{aligned} = -g\, f^{abc} \big[& (k_1 - k_2)^\gamma g^{\alpha\beta} \\ & + (k_2 - k_3)^\alpha g^{\beta\gamma} \\ & + (k_3 - k_1)^\beta g^{\gamma\alpha} \big] , \end{aligned} \tag{6.15e}$$



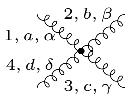
$$= -g\, f^{abc}\, k_2^\alpha , \tag{6.15f}$$



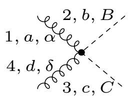
$$= -g\, f^{abc}\, (k_2 - k_3)^\alpha\, G^{BC} , \tag{6.15g}$$



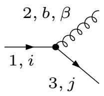
$$\begin{aligned} &= \mp g\, f^{abc}\, (i\mu)\, g^{\gamma\alpha}\, Q^B , \\ &\quad (\tilde{k}_1 = 0, \quad \tilde{k}_3 = \pm \tilde{\ell}) \end{aligned} \tag{6.15h}$$



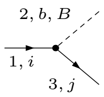
$$\begin{aligned} = -ig^2 \big[& \\ + f^{xad}\, f^{xbc}\, & \left(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\beta\delta} \right) \\ + f^{xac}\, f^{xbd}\, & \left(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\delta} g^{\beta\gamma} \right) \\ + f^{xab}\, f^{xdc}\, & \left(g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta} \right) \big] , \end{aligned} \tag{6.15i}$$



$$= 2ig^2\, g^{\alpha\delta}\, \left(f^{xab}\, f^{xcd} + f^{xac}\, f^{xbd} \right) G^{BC} , \tag{6.15j}$$



$$= -ig\, \left(t^b \right)_{ji} \gamma^\beta , \tag{6.15k}$$



$$= -ig\, \left(t^b \right)_{ji} \gamma^5\, \Gamma^B . \tag{6.15l}$$

In the Feynman rules (6.15) all the momenta are incoming and the scalar particle s_g can circulate in the loop only. The terms μ^2 appearing in the the propagators (6.15a)–(6.15e) enter only if the corresponding momentum k is d-dimensional, *i.e.* only if the corresponding particle circulates in the loop. In the vertex (6.15i) the momentum k_1 is four-dimensional while the other two are d-dimensional. The possible combinations of the -2ϵ components of the momenta involved are

$$\{\tilde{k}_1\,, \tilde{k}_2\,, \tilde{k}_3\,\} = \{0\,, \mp\tilde{\ell}\,, \pm\tilde{\ell}\,\} . \tag{6.16}$$

The overall sign of the Feynman rule (6.15i) depends on which of the combinations (6.16) is present in the vertex.

6.2 Generalized Unitarity

Generalized-unitarity methods in d dimensions require an explicit representation of the polarization vectors and the spinors of d-dimensional particles. The latter ones are essential ingredients for the construction of the tree-level amplitudes that are sewn along the generalized cuts. In this respect, the FDF scheme is suitable for the four-dimensional formulation of d-dimensional generalized unitarity. The main advantage of the FDF is that the four-dimensional expression of the propagators of the particles in the loop admits an explicit representation in terms of generalized spinors and polarization expressions, whose expression is collected below. In the following discussion we will decompose a d-dimensional momentum $\bar{\ell}$ as follows

$$\bar{\ell} = \ell + \tilde{\ell}\,, \qquad \bar{\ell}^2 = \ell^2 - \mu^2 = m^2\,, \tag{6.17}$$

while its four-dimensional component ℓ will be expressed as

$$\ell = \ell^\flat + \hat{q}_\ell\,, \qquad \hat{q}_\ell \equiv \frac{m^2+\mu^2}{2\,\ell\cdot q_\ell} q_\ell\,, \tag{6.18}$$

in terms of the two massless momenta $\ell^\flat$ and q_ℓ. The spinors of a d-dimensional fermion have to fulfill a completeness relation which reconstructs the numerator of the cut propagator,

$$\sum_{\lambda=1}^{2^{(d_s-2)/2}} u_{\lambda,(d)}\left(\bar{\ell}\right)\bar{u}_{\lambda,(d)}\left(\bar{\ell}\right) = \bar{\not{\ell}} + m\,, \tag{6.19}$$

$$\sum_{\lambda=1}^{2^{(d_s-2)/2}} v_{\lambda,(d)}\left(\bar{\ell}\right)\bar{v}_{\lambda,(d)}\left(\bar{\ell}\right) = \bar{\not{\ell}} - m\,. \tag{6.20}$$

The substitutions (6.11) allow one to express Eq. (6.20) as follows:

$$\sum_{\lambda=\pm} u_\lambda\left(\ell\right)\bar{u}_\lambda\left(\ell\right) = \not{\ell} + i\mu\gamma^5 + m\,, \tag{6.21}$$

$$\sum_{\lambda=\pm} v_\lambda\left(\ell\right)\bar{v}_\lambda\left(\ell\right) = \not{\ell} + i\mu\gamma^5 - m\,. \tag{6.22}$$

The generalized massive spinors

$$u_+\left(\ell\right) = \left|\ell^\flat\right\rangle + \frac{(m-i\mu)}{[\ell^\flat\, q_\ell]}\left|q_\ell\right]\,, \quad u_-\left(\ell\right) = \left|\ell^\flat\right] + \frac{(m+i\mu)}{\langle\ell^\flat\, q_\ell\rangle}\left|q_\ell\right\rangle\,, \tag{6.23a}$$

$$v_-\left(\ell\right) = \left|\ell^\flat\right\rangle - \frac{(m-i\mu)}{[\ell^\flat\, q_\ell]}\left|q_\ell\right]\,, \quad v_+\left(\ell\right) = \left|\ell^\flat\right] - \frac{(m+i\mu)}{\langle\ell^\flat\, q_\ell\rangle}\left|q_\ell\right\rangle\,, \tag{6.23b}$$

$$\bar{u}_+\left(\ell\right) = \left[\ell^\flat\right| + \frac{(m+i\mu)}{\langle q_\ell\,\ell^\flat\rangle}\left\langle q_\ell\right|\,, \quad \bar{u}_-\left(\ell\right) = \left\langle\ell^\flat\right| + \frac{(m-i\mu)}{[q_\ell\,\ell^\flat]}\left[q_\ell\right|\,, \tag{6.23c}$$

$$\bar{v}_-\left(\ell\right) = \left[\ell^\flat\right| - \frac{(m+i\mu)}{\langle q_\ell\,\ell^\flat\rangle}\left\langle q_\ell\right|\,, \quad \bar{v}_+\left(\ell\right) = \left\langle\ell^\flat\right| - \frac{(m-i\mu)}{[q_\ell\,\ell^\flat]}\left[q_\ell\right|\,, \tag{6.23d}$$

fulfill the completeness relation (6.22). The spinors (6.23b) are solutions of the tachyonic Dirac equations

$$\left(\not\ell + i\mu\gamma^5 + m\right)\, u_\lambda\left(\ell\right) = 0\,, \tag{6.24}$$
$$\left(\not\ell + i\mu\gamma^5 - m\right)\, v_\lambda\left(\ell\right) = 0\,, \tag{6.25}$$

which leads to a Hermitian Hamiltonian. It is worth to notice that the spinors (6.23) fulfill the Gordon's identities

$$\frac{\bar{u}_\lambda\left(\ell\right)\;\gamma^\nu\; u_\lambda\left(\ell\right)}{2} = \frac{\bar{v}_\lambda\left(\ell\right)\;\gamma^\nu\; v_\lambda\left(\ell\right)}{2} = \ell^\nu\,. \tag{6.26}$$

The d-dimensional polarization vectors of a spin-1 particle fulfill the following relation

$$\sum_{i=1}^{d-2} \varepsilon^\mu_{i\,(d)}\left(\bar\ell,\bar\eta\right)\varepsilon^{*\nu}_{i\,(d)}\left(\bar\ell,\bar\eta\right) = -\bar g^{\mu\nu} + \frac{\bar\ell^\mu\,\bar\eta^\nu + \bar\ell^\nu\,\bar\eta^\mu}{\bar\ell\cdot\bar\eta}\,, \tag{6.27}$$

where $\bar\eta$ is an arbitrary d-dimensional massless momentum such that $\bar\ell\cdot\bar\eta \neq 0$. Gauge invariance in d dimensions guarantees that the cut is independent of $\bar\eta$. In particular the choice

$$\bar\eta^\mu = \ell^\mu - \tilde\ell^\mu\,, \tag{6.28}$$

with ℓ, $\tilde\ell$ defined in Eq. (6.17), allows one to disentangle the four-dimensional contribution form the d-dimensional one:

$$\sum_{i=1}^{d-2} \varepsilon^\mu_{i,(d)}\left(\bar\ell,\bar\eta\right)\varepsilon^{*\nu}_{i,(d)}\left(\bar\ell,\bar\eta\right) = \left(-g^{\mu\nu} + \frac{\ell^\mu\ell^\nu}{\mu^2}\right) - \left(\tilde g^{\mu\nu} + \frac{\tilde\ell^\mu\tilde\ell^\nu}{\mu^2}\right)\,.$$

The first term is related to the cut propagator of a massive gluon and can be expressed as follows

$$-g^{\mu\nu} + \frac{\ell^\mu\ell^\nu}{\mu^2} = \sum_{\lambda=\pm,0} \varepsilon^\mu_\lambda(\ell)\,\varepsilon^{*\nu}_\lambda(\ell)\,, \tag{6.29}$$

in terms of the polarization vectors of a vector boson of mass μ [56],

$$\varepsilon^\mu_+\left(\ell\right) = -\frac{\left[\ell^\flat\left|\gamma^\mu\right|\hat q_\ell\right\rangle}{\sqrt2\mu}\,,\quad \varepsilon^\mu_-\left(\ell\right) = -\frac{\left\langle\ell^\flat\left|\gamma^\mu\right|\hat q_\ell\right]}{\sqrt2\mu}\,,\quad \varepsilon^\mu_0\left(\ell\right) = \frac{\ell^{\flat\mu} - \hat q^\mu_\ell}{\mu}\,. \tag{6.30}$$

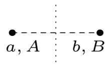
The latter fulfill the well-known relations

$$\varepsilon^2_\pm(\ell) = \;\;0\,, \qquad \varepsilon_\pm(\ell)\cdot\varepsilon_\mp(\ell) = -1\,, \tag{6.31}$$
$$\varepsilon^2_0(\ell) = -1\,, \qquad \varepsilon_\pm(\ell)\cdot\varepsilon_0(\ell) = \;\;0\,, \tag{6.32}$$
$$\varepsilon_\lambda(\ell)\cdot\ell = \;\;0\,. \tag{6.33}$$

The second term of the r.h.s. of Eq. (6.29) is related to the numerator of cut propagator of the scalar s_g and can be expressed in terms of the (-2ϵ)-SRs as:

$$\tilde g^{\mu\nu} + \frac{\tilde\ell^\mu\tilde\ell^\nu}{\mu^2} \quad\to\quad \hat G^{AB} \equiv G^{AB} - Q^A Q^B\,. \tag{6.34}$$

The factor $\hat{G}^{AB}$ can be easily accounted for by defining the cut propagator as



$$= \hat{G}^{AB}\,\delta^{ab}\,. \tag{6.35}$$

The generalized four-dimensional spinors and polarization vectors defined above can be used for constructing tree-level amplitudes with full μ-dependence. Therefore, in the context of on-shell and unitarity-based methods, they are a simple alternative to approaches introducing explicit higher-dimensional extension of either the Dirac or the spinor algebra [46, 57].

Chapter 7

Integrand Reduction

How showed in section 4.3, any one-loop amplitude can be expressed as a linear combination of a limited set of Master Integrals. Therefore, the evaluation of one-loop corrections reduces to evaluating the coefficients that multiply each MI. A very useful way to do this is *Integrand Reduction Method*, originally proposed in a four-dimensional framework by Ossola, Papadopoulos and Pittau (OPP) [10, 58]. The principle of an integrand-reduction method is the underlying multi-particle pole expansion for the integrand of any scattering amplitude, or, equivalently, a representation where the numerator of each Feynman integral is expressed as a combination of products of the corresponding denominators, with polynomial coefficients. Each residue is a (multivariate) polynomial in the *irreducible scalar products* (ISP's) formed by the loop momenta and either external momenta or polarization vectors constructed out of them. We do a short brief of this method, supposing to have a process with 4 particle and using shorthand notation. Recalling the PV decomposition (4.25) in four dimension, we know that

$$\int d^4\ell A^{1-loop}(\ell) = c_4 \int \frac{d^4\ell}{D_3D_2D_1D_0} + c_3 \int \frac{d^4\ell}{D_2D_1D_0} + c_2 \int \frac{d^4\ell}{D_1D_0} + c_1 \int \frac{d^4\ell}{D_0} \quad (7.1)$$

Clearly at the integrand level we have

$$A(\ell) \neq \frac{c_4}{D_3D_2D_1D_0} + \frac{c_3}{D_2D_1D_0} + \frac{c_2}{D_1D_0} + \frac{c_1}{D_0} \quad (7.2)$$

but, introducing certain functions of q we could say that

$$A(\ell) \equiv \frac{\Delta_{0123}(\ell)}{D_3D_2D_1D_0} + \frac{\Delta_{012}(\ell)}{D_2D_1D_0} + \frac{\Delta_{01}(\ell)}{D_1D_0} + \frac{\Delta_0(\ell)}{D_0} \quad (7.3)$$

Now we can write $A(\ell)$ in terms of numerators $N(\ell)$ and denominators $D_i(\ell)$ the come from Feynman diagrams, and using cuts rules, obtain the Δ's in a recursive way. Particularly the coefficient c_i will be the constant term of Δ's functions. Retreat these naive passages in a more rigorous and general form. Any one-loop n-point amplitude can be written as

$$A_n = \int d^D\bar{\ell}\ \mathcal{A}(\bar{\ell}, \epsilon)\ , \tag{7.4}$$

$$\mathcal{A}(\bar{\ell}, \epsilon) = \frac{N(\bar{\ell}, \epsilon)}{D_0 D_1 \cdots D_{n-1}}\ , \tag{7.5}$$

$$D_i = (\bar{\ell} + p_i)^2 - m_i^2 = (\ell + p_i)^2 - m_i^2 - \mu^2, \qquad (p_0 \neq 0)\,. \tag{7.6}$$

By using the Passarino-Veltman decomposition (4.25) the multi-pole nature of the integrand of any one-loop n-point amplitude becomes exposed,

$$\begin{aligned} A(\ell, \mu^2) = &\sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\ell, \mu^2)}{D_i D_j D_k D_\ell D_m} + \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\ell, \mu^2)}{D_i D_j D_k D_\ell} + \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\ell, \mu^2)}{D_i D_j D_k} \\ &+ \sum_{i<j}^{n-1} \frac{\Delta_{ij}(\ell, \mu^2)}{D_i D_j} + \sum_{i}^{n-1} \frac{\Delta_{i}(\ell, \mu^2)}{D_i}\ . \end{aligned} \tag{7.7}$$

where $i << m$ is the lexicographic ordering $i < j < k < \ell < m$. The functions $\Delta(\ell, \mu^2)$ are polynomials in the components of q and in μ^2.

Then the numerator $N(\ell, \mu^2)$ can be expressed in terms of denominators D_i, as follows

$$\begin{aligned} N(\ell, \mu^2) = &\sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\ell, \mu^2) \prod_{h\neq i,j,k,\ell,m}^{n-1} D_h + \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\ell, \mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} D_h \\ &+ \sum_{i<<k}^{n-1} \Delta_{ijk}(\ell, \mu^2) \prod_{h\neq i,j,k}^{n-1} D_h + \sum_{i<j}^{n-1} \Delta_{ij}(\ell, \mu^2) \prod_{h\neq i,j}^{n-1} D_h \\ &+ \sum_{i}^{n-1} \Delta_{i}(\ell, \mu^2) \prod_{h\neq i}^{n-1} D_h \end{aligned} \tag{7.8}$$

Now we are able to find the expression for Δ functions, and then for c_i, by using multiple-cuts in a recursive method [OPP] [58]. Proceding in a top-down precess we can obtain all coefficients.

Quintuple-cut

The algorithm starts from the quintuple-cut. Imposing

$$D_i = D_j = D_k = D_l = D_m = 0 \tag{7.9}$$

remain only the first term of right side of (7.8) and we have

$$N(\ell, \mu^2) = \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\ell, \mu^2) \prod_{h\neq i,j,k,\ell,m}^{n-1} D_h \tag{7.10}$$

The evaluetion of the left hand side of (7.10) in the solution of the quintuple cut give us the value of $\Delta_{ijk\ell m}$.

Quadruple-cut

The next step is imposing the quadruple cut constraints

$$D_i = D_j = D_k = D_l = 0 \tag{7.11}$$

The solutions of (7.11) do survive only the pentagons and the boxes coefficients in the right side of (7.8). Because we know $\Delta_{ijk\ell m}$s thanks previous step, $\Delta_{ijk\ell}$s are given by

$$N(\ell,\mu^2) - \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell,m}^{n-1} D_h = \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} D_h \tag{7.12}$$

Triple-cut

Performing this recursive method, we can do the triple-cut imposing

$$D_i = D_j = D_k = 0 \tag{7.13}$$

and we obtain the expression

$$\begin{aligned} N(\ell,\mu^2) - \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell,m}^{n-1} D_h - \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} D_h = \\ = \sum_{i<<k}^{n-1} \Delta_{ijk}(\ell,\mu^2) \prod_{h\neq i,j,k}^{n-1} D_h \end{aligned} \tag{7.14}$$

Double-cut

Similarly we can do for the double-cut

$$\begin{aligned} N(\ell,\mu^2) - \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell,m}^{n-1} D_h - \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} D_h - \\ - \sum_{i<<k}^{n-1} \Delta_{ijk}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} = \sum_{i<j}^{n-1} \Delta_{ij}(\ell,\mu^2) \prod_{h\neq i,j}^{n-1} D_h \end{aligned} \tag{7.15}$$

Single-cut

At the end we perform the single cut

$$\begin{aligned} N(\ell,\mu^2) - \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell,m}^{n-1} D_h - \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} D_h - \\ - \sum_{i<<k}^{n-1} \Delta_{ijk}(\ell,\mu^2) \prod_{h\neq i,j,k,\ell}^{n-1} - \sum_{i<j}^{n-1} \Delta_{ij}(\ell,\mu^2) \prod_{h\neq i,j}^{n-1} D_h = \sum_{i}^{n-1} \Delta_{i}(\ell,\mu^2) \prod_{h\neq i}^{n-1} D_h \end{aligned} \tag{7.16}$$

The functions Δs are polynomial in the loop momentum variable ℓ [49,50,59,60]. Using decomposition of ℓ seen in section (5.4) we can also define the vectors v and $v_\perp$

$$v^\mu = (e_4 \cdot K_3)\, e_3^\mu + (e_3 \cdot K_3)\, e_4^\mu, \qquad v_\perp^\mu = (e_4 \cdot K_3)\, e_3^\mu - (e_3 \cdot K_3)\, e_4^\mu, \tag{7.17}$$

where K_3 is the third leg of the 4-point function associated to the considered quadruple-cut, and the vector ℓ is written as

$$q^\nu = -p_{i_1}^\nu + \frac{1}{e_1 \cdot e_2}\left(x_1 e_1^\nu + x_2 e_2^\nu\right) + \frac{1}{v^2}\left(x_{3,v}\, v^\nu - x_{4,v}\; v_\perp^\nu\right). \tag{7.18}$$

The universal parametric form of the residues $\Delta_{i_1 \cdots i_k}$ is then

$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0\, \mu^2 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_{4,v} + \mu^2 \left(c_2 + c_3 x_{4,v} + \mu^2 c_4\right) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_3 + c_5 x_3^2 + c_6 x_3^3 + \mu^2 \left(c_7 + c_8 x_4 + c_9 x_3\right) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_1 + c_2 x_2^2 + c_3 x_4 + c_4 x_4^2 + c_5 x_3 + c_6 x_3^2 + c_7 x_1 x_4 + c_8 x_1 x_3 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_2 + c_2 x_1 + c_3 x_4 + c_4 x_3.
\end{aligned} \tag{7.19}$$

where we understand that the unknown coefficients c_j depend on the indexes of the residue (e.g. $c_j = c_j^{(i_1 \cdots i_k)}$), while the scalar products x_i and $x_{i,v}$ depend on the both the indexes of the residue and the loop momentum ℓ.

Chapter 8

Higgs plus jet production in gluon-fusion at One-loop

Higgs production in the gluon-gluon fusion mechanism is mediated by triangular loops of heavy quarks. In the SM, only the top quark and, to a lesser extent, the bottom quark will contribute to the amplitude. The decreasing Hgg form factor with rising loop mass is counterbalanced by the linear growth of the Higgs coupling with the quark mass.
In this section we discuss the calculation of $gg \to gH$ process at one-loop. The result of the helicity amplitude had been computed by Schmidt in ref. [33]. We rederive it in a very efficient way using generalized-unitarity methods discussed in the previous sections.

8.1 Effective Vertex

From the standard model Higgs boson does not couple to massless particles at tree-level. This suggests us that the process $\gamma\gamma \to H$ to lower order must be treated to one-loop, this loop has to be fermion due to the Higgs/photon couples to fermions [61]. Moreover, to developing one-loop diagram calculation it becomes complicated, for this reason we need to take certain approaches such as: H momentum is small (i.e. $M_H \ll M_{loop}$), it implies top mass going to infinite ($m_t \to \infty$). In addition, this approach is correct because the processes at the LHC are given by 95% when there is a top quark loop and 5% with a bottom quark [62, 63]

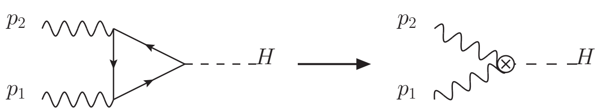


Figure 8.1: Effective vertex for the process $\gamma\gamma \to H$

To compute this amplitude we need the photon self energy due to the bubble configuration [51]. Here we consider the approximation when the transfered momentum in the top loop is much larger than the Higgs momentum, the first triangle configuration can be studied as the derivative of a bubble. Computing the amplitude for the bubble configuration and calculating the derivative of the fermion photon self-energy,

$$\prod_{\mu\nu}^{\gamma\gamma} = -iN_c e^2 e_f^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\{\gamma_\mu(\not{k}+m)\gamma_\nu(\not{p}+\not{k})+m\}}{[(p+k)^2-m^2](k^2-m^2)} \tag{8.1}$$

with $N_c = 3(1)$ for quarks (leptons) and e_f the electric charge for the fermions in the loop. Applying rules for traces of gamma matrices and writing the denominator with Feynman parameters as usually [37] we obtain

$$\prod_{\mu\nu}^{\gamma\gamma} = \frac{N_c e^2 e_f^2}{4\pi^2}(g_{\mu\nu - p_\mu p_\nu}) \int_0^1 dx \int_0^\infty dyy \frac{2x(1-x)}{[y+m^2-p^2x(1-x)]^2} \tag{8.2}$$

We now compute the $\gamma\gamma \to H$ vertex. It is important to see that external photons are on-shell and $p_{1,2} \neq p$ (and must be symmetrize, i.e. $A_{\mu\nu}^{H\gamma\gamma} \to 2A_{\mu\nu}^{H\gamma\gamma}$) but $p^2 = p_1 \cdot p_2 = \frac{1}{2}M_H^2$. We write down our amplitude

$$\begin{aligned} A_{\mu\nu}^{H\gamma\gamma} &= -2\frac{m}{v}\prod_{\mu\nu}^{\gamma\gamma}(p_1,p_2) = \\ &= -\frac{2m^2}{v}\frac{N_c e^2 e_f^2}{\pi^2}(g_{\mu\nu}p_1\cdot p_2 - p_{1\mu}p_{2\nu}) \int_0^1 dx \int_0^\infty dyy \frac{-2x(1-x)}{[y+m^2-p^2x(1-x)]^3} \end{aligned}$$

As we mentioned before, we are studying $m_t \to \infty$, this implies $m^2 \gg p^2 M_H^2$. Taking into account this prescription inside the integral and integrate out over x and y,

$$\int x(1-x)dx = \frac{1}{6}, \qquad \int \frac{ydy}{(y+m^2)^3} = \frac{1}{2}m^2 \tag{8.3}$$

Finally,

$$A_{\mu\nu}^{H\gamma\gamma} = \frac{2}{3v}N_c e_f^2 \frac{\alpha}{\pi}(g_{\mu\nu}p_1\cdot p_2 - p_{1\mu}p_{2\nu}) \tag{8.4}$$

The amplitude obtained is finite and there is not tree level contribution, then the approximation $m_f \gg M_H$ is in practice good up to $M_H \approx 2m_f$. By the way, only top quarks contribute, other fermions have negligible Yukawa coupling. The calculation made for photon can be used also for gluons if we make the changes:

$$Q_e \to g_s T^a, \qquad \alpha \to \alpha_s \qquad N_c \to \mathrm{Tr}\{T^a T^b\} \tag{8.5}$$

Taking into account the result obtained in eq. (8.4) we can construct an effective Lagrangian for infinitely heavy quarks. Writing the effective $H\gamma\gamma$ Lagrangian [5]

$$\mathcal{L}(H\gamma\gamma) = \frac{1}{4}(\sqrt{2}G_F)^{1/2}e_q^2\beta'(1+\delta)HF_{\mu\nu}F^{\mu}\nu \tag{8.6}$$

with $\beta' = 2(\alpha/\pi)(1 + \alpha_s/\pi)$ and the Higgs-quark vertex correction $\delta = 2\alpha_s/\pi$ the $H\gamma\gamma$ coupling can be readily derived:

$$\mathcal{L}(H\gamma\gamma) = \frac{1}{2\pi}(\sqrt{2}G_F)^{1/2}\alpha e_q^2(1+\delta)HF_{\mu\nu}F^{\mu}\nu \tag{8.7}$$

The generalization to the Hgg coupling follow from $\beta' = \frac{1}{3}(\alpha/\pi)(1 + \frac{10}{3}\alpha_s/\pi)$ and $\delta = 2\alpha_s/\pi$ that

$$\mathcal{L}(H\gamma\gamma) = \frac{1}{12\pi}(\sqrt{2}G_F)^{1/2}\alpha(1 + \frac{11}{4}\alpha_s/\pi)HF_{\mu\nu}F^{\mu}\nu \tag{8.8}$$

8.1.1 Feynman Rules in FDF scheme

The color-ordered Feynman rules describing the interaction among an external Higgs boson and gluons in the infinite top-mass limit can be derived by previous relations. In particular we are interesting to the interactions with the internal gluons in order to do the one-loop calculations. Considering the kinematic term of amplitude (8.4), where gluons are in $D = 4 - 2\epsilon$ and using FDF scheme,

$$\bar{g}^{\mu\nu}\bar{p}_1 \cdot \bar{p}_2 - \bar{p}_1^{\mu}\bar{p}_2^{\nu} \tag{8.9}$$

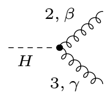
we can use prescriptions seen in chapter (6) about FDF to explicit (8.9) as

$$\left(g^{\mu\nu} + \tilde{g}^{\mu\nu}\right)\left(p_1 \cdot p_2 + \mu^2\right) - \left(p_1^{\mu} + \tilde{p}_1^{\mu}\right)\left(p_2^{\nu} + \tilde{p}_2^{\nu}\right) \tag{8.10}$$

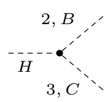
and, using (12.58),

$$\left(g^{\mu\nu} + G^{AB}\right)\left(p_1 \cdot p_2 + \mu^2\right) - \left(p_1^{\mu} + \mu Q^A p_1\right)\left(p_2^{\nu} + \mu Q^B p_2\right) \tag{8.11}$$

Combining the terms of (8.11) we can obtain the Feynman rules for two gluons and Higgs vertex and two scalars and Higgs vertex [25]

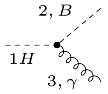


$$= -2i[k_3^{\beta}k_2^{\gamma} - g^{\beta\gamma}(k_2 \cdot k_3 + \mu^2)]\,, \tag{8.12a}$$

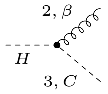


$$= -2i\left[\mu^2\, Q^B Q^C - G^{BC}(k_2 \cdot k_3 + \mu^2)\right] \tag{8.12b}$$

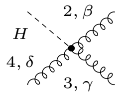
where $2i$ factor take into account the normalization terms of the structure constants of the color algebra. In the same way we can extract the others Feynman rules



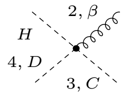
$$= \pm 2\, k_2^{\gamma}\, \mu\, Q^B \qquad (\tilde{k}_3 = \pm\tilde{\ell})\,, \tag{8.13a}$$



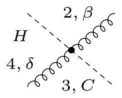
$$= \pm 2\, k_3^{\beta}\, \mu\, Q^C \qquad (\tilde{k}_2 = \pm\tilde{\ell})\,, \tag{8.13b}$$



$$= i\sqrt{2}\left[g_{\beta\gamma}(k_2 - k_3)_\delta + g_{\beta\delta}(k_4 - k_2)_\gamma + g_{\gamma\delta}(k_3 - k_4)_\beta\right] \tag{8.13c}$$



$$= i\sqrt{2} G^{CD}(k_3 - k_4)_\beta\,, \tag{8.13d}$$



$$= \mp\sqrt{2} g_{\beta\delta}\, \mu\, Q^C \quad (\tilde{k}_4 - \tilde{k}_2 = \pm\tilde{\ell})\,. \tag{8.13e}$$

In the Feynman rules (1.11), (8.13) all the momenta are outgoing.

8.2 Amplitude decomposition

The one-loop amplitude for this process is decomposed as follows,

$$\begin{aligned}
A_{4,H} = \frac{1}{(4\pi)^{2-\epsilon}}\Bigg[& \big(c_{1|2|3|H;\,0}\, I_{1|2|3|H} + c_{1|2|H|3;\,0}\, I_{1|2|H|3} + c_{1|H|2|3;\,0}\, I_{1|H|2|3}\big) + \\
& + \big(c_{12|3|H;\,0}\, I_{12|3|H} + c_{12|H|3;\,0}\, I_{12|H|3} + c_{1|23|H;\,0}\, I_{1|23|H} + c_{1|H|23;\,0}\, I_{1|H|23} + \\
& + c_{2|H|31;\,0}\, I_{2|H|31} + c_{H|2|31;\,0}\, I_{H|2|31} + c_{1|2|3H;\,0}\, I_{1|2|3H} + c_{1|2H|3;\,0}\, I_{1|2H|3} + \\
& + c_{1H|2|3;\,0}\, I_{1H|2|3}\big) + \big(c_{12|3H;\,0}\, I_{12|3H} + c_{23|H1;\,0}\, I_{23|H1} + c_{H2|31;\,0}\, I_{H2|31}\big) + \\
& + c_{123|H;\,0}\, I_{123|H}, \Bigg] + \mathcal{R}_H
\end{aligned} \tag{8.14}$$

and

$$\begin{aligned}
\mathcal{R}_H = \frac{1}{(4\pi)^{2-\epsilon}}\Bigg[& \big(c_{1|2|3|H;\,4}\, I_{1|2|3|H}\left[\mu^4\right] + c_{1|2|H|3;\,4}\, I_{1|2|H|3}\left[\mu^4\right] + c_{1|H|2|3;\,4}\, I_{1|H|2|3}\left[\mu^4\right]\big) \\
& + \big(c_{12|3|H;\,2}\, I_{12|3|H}\left[\mu^2\right] + c_{12|H|3;\,2}\, I_{12|H|3}\left[\mu^2\right] + c_{1|23|H;\,2}\, I_{1|23|H}[\mu^2] + c_{1|H|23;\,2}\, I_{1|H|23}\left[\mu^2\right] \\
& + c_{2|H|31;\,2}\, I_{2|H|31}\left[\mu^2\right] + c_{H|2|31;\,2}\, I_{H|2|31}\left[\mu^2\right] + c_{1|2|3H;\,2}\, I_{1|2|3H}\left[\mu^2\right] + c_{1|2H|3;\,2}\, I_{1|2H|3}\left[\mu^2\right] \\
& + c_{1H|2|3;\,2}\, I_{1H|2|3}\left[\mu^2\right]\big) + \big(c_{12|3H;\,2}\, I_{12|3H}\left[\mu^2\right] + c_{23|H1;\,2}\, I_{23|H1}\left[\mu^2\right] \\
& + c_{H2|31;\,2}\, I_{H2|31}\left[\mu^2\right] + c_{123|H;\,2}\, I_{123|H}\left[\mu^2\right]\big)\Bigg],
\end{aligned} \tag{8.15}$$

The expressions for the MIs appearing in Eq.(8.14) are [64]

$$
\begin{aligned}
I_{i|j|k|H} &= I_{j|i|H|k} = I_{i|H|k|j} = \frac{2r_\Gamma}{s_{ij}s_{jk}}\frac{1}{\epsilon^2}\left[(-s_{ij})^{-\epsilon} + (-s_{jk})^{-\epsilon} - \left(-m_H^2\right)^{-\epsilon}\right] - \\
&- \frac{2r_\Gamma}{s_{ij}s_{jk}}\left[\mathrm{Li}_2\left(1-\frac{m_H^2}{s_{ij}}\right) + \mathrm{Li}_2\left(1-\frac{m_H^2}{s_{jk}}\right) + \frac{1}{2}\log^2\frac{s_{ij}}{s_{jk}} + \frac{\pi^2}{6}\right],
\end{aligned} \tag{8.16a}
$$

$$
I_{ij|k|H} = I_{ij|H|k} = I_{k|H|ij} = I_{H|k|ij} = I_{k|ij|H} = \frac{r_\Gamma}{\epsilon^2}\frac{(-s_{ij})^{-\epsilon} - (-m_H^2)^{-\epsilon}}{(-s_{ij}) - (-m_H^2)}, \tag{8.16b}
$$

$$
I_{i|j|kH} = I_{kH|ij} = I_{i|kH|j} = \frac{r_\Gamma}{\epsilon^2}(-s_{ij})^{-1-\epsilon}, \tag{8.16c}
$$

$$
I_{ij|Hk} = I_{Hk|ij} = I_{ij|kH} = \frac{r_\Gamma}{\epsilon(1-2\epsilon)}(-s_{ij})^{-\epsilon}, \tag{8.16d}
$$

$$
I_{123|H} = \frac{r_\Gamma}{\epsilon(1-2\epsilon)}\left(-m_H^2\right)^{-\epsilon}, \tag{8.16e}
$$

$$
\begin{aligned}
I_{i|j|k|m}\left[\mu^4\right] &= \frac{4\epsilon(\epsilon-1)}{a_0^2(2\epsilon-3)(2\epsilon-1)} I_{i|j|k|m} \\
&+ \frac{a_1(\epsilon-1)}{a_0(2\epsilon-3)}\left[I_{j|k|mi}\left[\mu^2\right] - \frac{2\epsilon I_{j|k|mi}}{a_0(2\epsilon-1)}\right] \\
&+ \frac{a_2(\epsilon-1)}{a_0(2\epsilon-3)}\left[I_{ij|k|m}\left[\mu^2\right] - \frac{2\epsilon I_{ij|k|m}}{a_0(2\epsilon-1)}\right] \\
&+ \frac{a_3(\epsilon-1)}{a_0(2\epsilon-3)}\left[I_{i|jk|m}\left[\mu^2\right] - \frac{2\epsilon I_{i|jk|m}}{a_0(2\epsilon-1)}\right] \\
&+ \frac{a_4(\epsilon-1)}{a_0(2\epsilon-3)}\left[I_{i|j|km}\left[\mu^2\right] - \frac{2\epsilon I_{i|j|km}}{a_0(2\epsilon-1)}\right] = -\frac{1}{6} + \mathcal{O}(\epsilon),
\end{aligned} \tag{8.16f}
$$

$$
\begin{aligned}
I_{ij|k|H}\left[\mu^2\right] &= I_{ij|H|k}\left[\mu^2\right] = I_{k|H|ij}\left[\mu^2\right] \\
&= I_{H|k|ij}\left[\mu^2\right] = I_{k|ij|H}\left[\mu^2\right] \\
&= \frac{-r_\Gamma}{2(1-\epsilon)(1-2\epsilon)}\frac{(-s_{ij})^{1-\epsilon} - (-m_H^2)^{1-\epsilon}}{(-s_{ij}) - (-m_H^2)} = -\frac{1}{2} + \mathcal{O}(\epsilon),
\end{aligned} \tag{8.16g}
$$

$$
\begin{aligned}
I_{i|j|kH}\left[\mu^2\right] &= I_{kH|ij}\left[\mu^2\right] = I_{i|kH|j}\left[\mu^2\right] \\
&= \frac{-r_\Gamma\,(-s_{ij})^{-\epsilon}}{2(1-\epsilon)(1-2\epsilon)} = -\frac{1}{2} + \mathcal{O}(\epsilon),
\end{aligned} \tag{8.16h}
$$

$$
\begin{aligned}
I_{ij|Hk}\left[\mu^2\right] &= I_{Hk|ij}\left[\mu^2\right] = I_{ij|kH}\left[\mu^2\right] \\
&= \frac{r_\Gamma\,(-s_{ij})^{1-\epsilon}}{2(3-2\epsilon)(1-2\epsilon)} = -\frac{1}{6}s_{ij} + \mathcal{O}(\epsilon),
\end{aligned} \tag{8.16i}
$$

$$
I_{123|H}\left[\mu^2\right] = \frac{r_\Gamma\,(-m_H^2)^{1-\epsilon}}{2(3-2\epsilon)(1-2\epsilon)} = -\frac{1}{6}m_H^2 + \mathcal{O}(\epsilon), \tag{8.16j}
$$

The factor r_Γ is defined as

$$r_\Gamma \equiv \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\,, \tag{8.17}$$

while coefficients a read as follows,

$$a_0 \equiv \sum_{s=1}^{4} a_s\,, \qquad a_s \equiv \sum_{t=1}^{4}\left(S^{-1}_{i|j|k|m}\right)_{st}\,, \tag{8.18}$$

in terms of the cut-dependent matrix

$$\left(S_{i|j|k|m}\right)_{st} \equiv -\frac{1}{2}\left(v^{(s)}_{i|j|k|m} - v^{(t)}_{i|j|k|m}\right)^2\,, \tag{8.19}$$

where

$$\begin{aligned} v^{(1)}_{i|j|k|m} &= 0\,, & v^{(2)}_{i|j|k|m} &= p_i\,, \\ v^{(3)}_{i|j|k|m} &= p_i + p_j\,, & v^{(4)}_{i|j|k|m} &= -p_m\,. \end{aligned} \tag{8.20}$$

In (8.14), the contribution generating the rational terms have been collected in $\mathcal{R}$ and $\mathcal{R}_H$, respectively, hence distinguished by the so-called cut-constructible terms. We remark that within the FDF this distinction is pointless and has been performed only to improve the readability of the formulas. Indeed within the FDF the two contributions are computed simultaneously from the same cuts.

The coefficients c's entering in the decompositions(8.14) can be obtained by using the generalized unitarity techniques for quadruple, triple, and double cuts. We observe that single-cut techniques are not needed because of the absence of (d-dimensional) massive particles in the loop. In general, the cut $C_{i_1\cdots i_k}$, defined by the conditions $D_{i_1} = \cdots = D_{i_k} = 0$, allows for the determination of the coefficients $c_{i_1\cdots i_k;\,n}$.

8.3 $A(1^+, 2^+, 3^+, H)$

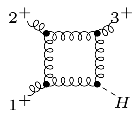
In this section we present the calculation of coefficients about one loop amplitude of Higgs + 3 partons precess, where each parton has the same helicity. We take in exams only the positive helicity because the case with all minus helicity is trivially determinate putting square bracket instead angle brackets in our results. In the next section we will tackle cases where external gluons have different helicity. However, before to do calculations about coefficient is useful write down leading-order amplitude about process

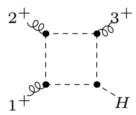
$$A^{tree}_{4,H}(1^+, 2^+, 3^+, H) = \frac{-i m_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{8.21}$$

We will see that the coefficients can be written in term of (8.21).

8.3.1 Box Coefficients

We now show the calculations about the box coefficients. Using FDF regularization scheme, the gluon loop in d dimension split in a gluon loop with a mass μ and a scalar loop with the properties seen in section 6.1.

$$C_{1|2|3|H} = N_c \Bigg($$


$$+$$


$$\Bigg) \tag{8.22}$$

where N_c is the number of colors. To extract the box coefficient we have to consider a quadruple-cut. Building blocks of Integrand of quadruple cut are tree-level three point functions. Building blocks of first term are

- 3 gluons $\quad A_{gg^\bullet g^\bullet}$
- Higgs+2 gluons $\quad A_{Hg^\bullet g^\bullet}$
- gluon+2 scalars $\quad A_{gs^\bullet s^\bullet}$
- Higgs+2 scalar $\quad A_{Hs^\bullet s^\bullet}$

where particles with dot represents the internal legs. They are

$$A^{\nu_1\nu_2}_{gg^\bullet g^\bullet}(k_1^+, L_1, L_2) = \frac{i}{\sqrt{2}}\Big(g^{\alpha\nu_1}(k_1^{\nu_2} - L_1^{\nu_2}) + g^{\alpha\nu_3}(L_2^{\nu_1} - k_1^{\nu_1}) + g^{\nu_1\nu_2}(L_1^{\alpha} - L_2^{\alpha})\Big)\varepsilon_\alpha(k_1) \tag{8.23a}$$

$$A^{\nu_1\nu_2}_{Hg^\bullet g^\bullet}(L_1, L_2) = -2i(g^{\nu_1\nu_2}(-\mu^2 - L_1 \cdot L_2) + L_1^{\nu_2}L_2^{\nu_1}) \tag{8.23b}$$

$$P^{\nu_1\nu_2}_{cut}(k) = g^{\nu_1\nu_2} - \frac{k^{\nu_1}k^{\nu_2}}{\mu^2} \tag{8.23c}$$

$$A^{AB}_{gs^\bullet s^\bullet}(k_1^+, L_1, L_2) = \frac{i}{\sqrt{2}}\Big(L_1^{\nu} - L_2^{\nu}\Big)G^{AB}\varepsilon_\nu(k_1) \tag{8.23d}$$

$$A^{AB}_{Hs^\bullet s^\bullet}(L_1, L_2) = 2i(\mu^2 Q^A Q^B + L_1 \cdot L_2 G^{AB}) \tag{8.23e}$$

$$P^{AB}_{cut} = \hat{G}^{AB} \tag{8.23f}$$

where $P^{\nu_1\nu_2}_{cut}$ and P^{AB}_{cut} are respectively the gluon and scalar cut propagators. Then the gluon integrand yields

$$\begin{aligned}
\mathcal{I}^{gluon}_{1|2|3|H} = {} & A^{\delta\nu}_{gg^\bullet g^\bullet}\Big(k_1^+, -\ell, \ell - k_1\Big)P^{\delta\epsilon}_{cut}(\ell)\times \\
& \times A^{\mu\epsilon}_{gg^\bullet g^\bullet}\Big(k_2^+, -\ell - k_2, \ell\Big)P^{\mu\rho}_{cut}(\ell + k_2)\times \\
& \times A^{\eta\rho}_{gg^\bullet g^\bullet}\Big(k_3^+, -\ell - k_2 - k_3, \ell + k_2\Big)P^{\eta\kappa}_{cut}(\ell + k_2 + k_3)\times \\
& \times A^{\kappa\tau}_{Hg^\bullet g^\bullet}\Big(\ell + k_2 + k_3, -\ell + k_1\Big)P^{\tau\nu}_{cut}(\ell - k_1)
\end{aligned} \tag{8.24}$$

The scalar integrand is

$$\begin{aligned}\mathcal{I}^{scalar}_{1|2|3|H} = & \frac{i}{\sqrt{2}}\Big(-\ell^\nu - (\ell-k_1)^\nu\Big) G^{AB}\varepsilon^+_\nu(k_1)\hat{G}^{BC} \\ & \times \frac{i}{\sqrt{2}}\Big((-\ell-k_2)^\rho - (\ell)^\rho\Big) G^{CD}\varepsilon^+_\rho(k_2)\hat{G}^{DE} \\ & \times \frac{i}{\sqrt{2}}\Big((-\ell-k_2-k_3)^\sigma - (\ell+k_2)^\sigma\Big) G^{EF}\varepsilon^+_\sigma(k_3)\hat{G}^{FG} \\ & \times 2i\Big(\mu^2 Q^G Q^H + (\ell-k_1)\cdot(-\ell-k_2-k_3)G^{GH}\Big)\hat{G}^{HA}\end{aligned} \tag{8.25}$$

using the rules of scalar algebra we can show these relations [25]

$$Q^A\hat{G}^{AB}Q^B = 0\,, \tag{8.26}$$

$$Q^A\hat{G}^{AB}G^{BC}\hat{G}^{CD}Q^D = 0\,, \tag{8.27}$$

$$Q^A\hat{G}^{AB}G^{BC}\hat{G}^{CD}G^{DE}\hat{G}^{EF}Q^F = 0 \tag{8.28}$$

$$\mathrm{tr}\left(G\,\hat{G}\,G\,\hat{G}\,G\,\hat{G}\,G\,\hat{G}\right) = -1. \tag{8.29}$$

and eq. (8.25) becomes

$$\begin{aligned}\mathcal{I}^{scalar-loop}_4 = & -2\sqrt{2}\ell\cdot\varepsilon(k_1)\ell\cdot\varepsilon(k_2)(-2k_2\cdot\varepsilon(k_3) - 2\ell\cdot\varepsilon(k_3)) \\ & \Big(k_1\cdot k_2 + k_1\cdot k_3 + k_1\cdot\ell - k_2\cdot\ell - k_3\cdot\ell - \ell^2 + \mu^2\Big)\end{aligned} \tag{8.30}$$

We know from section (5.4.2) we have to evaluate the integrand by the two solutions of quadruple cut.

$$\ell^2 = \mu^2, \qquad \big(\ell+k_2\big)^2 = \mu^2, \qquad \big(\ell+k_2+k_3\big)^2 = \mu^2, \qquad \big(\ell-k_1\big)^2 = \mu^2, \tag{8.31}$$

A useful choise of basis to build ℓ^μ is,

$$e_1^\nu = k_1^\nu, \quad e_2^\nu = k_2^\nu, \quad e_3^\nu = \frac{\langle k_1|\gamma^\nu|k_2]}{2}, \quad e_4^\nu = \frac{\langle k_2|\gamma^\nu|k_1]}{2}. \tag{8.32}$$

that simplify solutions showed in (5.4.2). The two solutions are

$$\ell_1^\nu = \mu^2 a_+ e_3^\nu - \frac{1}{a_+}e_4^\nu \tag{8.33}$$

$$\ell_2^\nu = \mu^2 a_- e_3^\nu - \frac{1}{a_-}e_4^\nu \tag{8.34}$$

with

$$a_\pm = \frac{2[k_3|k_1]}{s_{k_1k_2}[k_3|k_2]\Big(1 \pm \sqrt{\frac{4\mu^2 s_{k_1k_3}}{s_{k_1k_2}s_{k_2k_3}} + 1}\Big)} \tag{8.35}$$

The value of coefficient of the box diagram is

$$C_{1|2|3|H}(1^+,2^+,3^+,H) = \frac{1}{2}\frac{im_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}s_{12}s_{23} = -N_c\frac{1}{2}A^{tree}_{4,H}(1^+,2^+,3^+,H)s_{12}s_{23} \tag{8.36}$$

Taking into account eq. (8.36), we find the other possible Higgs configurations by shifting the label of each particle in the following way

$$1 \to 3 \tag{8.37}$$
$$2 \to 1 \tag{8.38}$$
$$3 \to 2 \tag{8.39}$$

So the others box coeefcients are

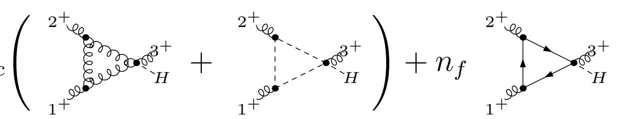
$$C_{1|2|H|3} = -N_c \frac{1}{2} A^{\text{tree}}_{4,H} s_{13} s_{12} \tag{8.40}$$
$$C_{1|H|2|3} = -N_c \frac{1}{2} A^{\text{tree}}_{4,H} s_{23} s_{13} \tag{8.41}$$

8.3.2 Triangle coefficient

For these triangle contributions we need to distinguish two possible configurations. Configuration with one massive channel, in which the Higgs and a gluon are together in a same corner and configuration with two massive channel, where two gluons are in a same corner.

One Massive Channel

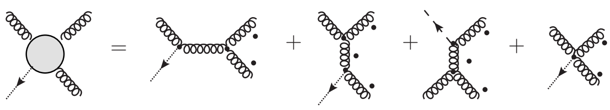
The former configuration is represented by

$C_{1|2|3H} = N_c$ 

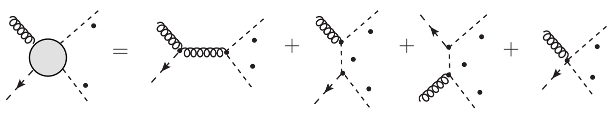
(8.42)

with n_f number of fermions. Building blocks are the ones used for box plus $A_{Hgg^\bullet g^\bullet}$, $A_{Hgs^\bullet s^\bullet}$ and $A_{Hgq^\bullet \bar{q}^\bullet}$

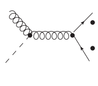
$$A_{gq^\bullet \bar{q}^\bullet}(k^+) = -i\gamma^\alpha \varepsilon^+_\alpha(k) \tag{8.43}$$

$A^{\nu_1\nu_2}_{Hgg^\bullet g^\bullet}(k, L_1, L_2) =$ 

(8.44)

$A^{AB}_{Hgs^\bullet s^\bullet}(k, L_1, L_2) =$ 

(8.45)

$A_{Hgq^\bullet \bar{q}^\bullet}(k^+, L_1, L_2) =$ 
(8.46)

and add also the cut fermion propagator

$$P_{cut}(k) = \not{k} + i\mu\gamma^5 \tag{8.47}$$

These tree-level amplitude is built easily using Feynman rule, paying attention to prescription to use the massive propagator where loop momentum flows. The integrand of these diagrams are

$$\begin{aligned}
\mathcal{I}^{\text{gluon}}_{1|2|3H} &= A^{\delta\nu}_{gg^\bullet g^\bullet}\Big(k_1^+, -\ell, \ell - k_1\Big) P^{\delta\epsilon}_{cut}(\ell)\times \\
&\times A^{\mu\epsilon}_{gg^\bullet g^\bullet}\Big(k_2^+, -\ell - k_2, \ell\Big) P^{\mu\rho}_{cut}(\ell + k_2)\times \\
&\times A^{\tau\rho}_{Hgg^\bullet g^\bullet}\Big(k_3^+, -\ell + k_1, \ell + k_2\Big) P^{\tau\nu}_{cut}(\ell - k_1)
\end{aligned} \tag{8.48}$$

$$\begin{aligned}
\mathcal{I}^{\text{scalar}}_{1|2|3H} &= \frac{i}{\sqrt{2}}\Big(-\ell^\nu - (\ell - k_1)^\nu\Big) G^{AB}\varepsilon_\nu(k_1)\hat{G}^{BC} \\
&\times \frac{i}{\sqrt{2}}\Big((-\ell - k_2)^\rho - (\ell)^\rho\Big) G^{CD}\varepsilon_\rho(k_2)\hat{G}^{DE} \\
&\times A^{EF}_{Hgs^\bullet s\bullet}\Big(k_3^+, \ell + k_2, -\ell + k_1\Big)\hat{G}^{FA}
\end{aligned} \tag{8.49}$$

$$\begin{aligned}
\mathcal{I}^{\text{fermion}}_{1|2|3H} &= \text{Tr}\Big\{A_{gq^\bullet\bar{q}\bullet}(k_1^+) P_{cut}(\ell) A_{gq^\bullet\bar{q}\bullet}(k_2^+) P_{cut}(\ell + k_2)\times \\
&\times A_{Hgq^\bullet\bar{q}\bullet}(k_3^+, \ell + k_2, -\ell + k_1) P_{cut}(\ell - k_1)\Big\}
\end{aligned} \tag{8.50}$$

The triple-cut conditions are

$$\ell^2 = \mu^2, \qquad \big(\ell + k_2\big)^2 = \mu^2, \qquad \big(\ell - k_1\big)^2 = \mu^2, \tag{8.51}$$

In order to evaluate the triple-cut, (8.32) is a good basis to decompose ℓ. The solutions of triple-cut in this basis are

$$\ell_1^\nu = te_3^\nu - \frac{\mu^2}{ts_{12}}e_4^\nu \tag{8.52}$$

$$\ell_2^\nu = te_4^\nu - \frac{\mu^2}{ts_{12}}e_3^\nu \tag{8.53}$$

where t is a free parameter. As showed in section 5.4 the coefficient is obtained evaluating the integrand with the two solution and taking the limit $t \to \infty$. The coefficient will be the average between the coefficients to zero order of this t expansion.

$$C_{1|2|3H} = \mu^2\frac{2i}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}s_{13}s_{23} = -2(N_c - n_f)\mu^2 A^{\text{tree}}_{4,H}\frac{s_{13}s_{23}}{m_H^4} \tag{8.54}$$

Using permutation as before we can calculate the coefficients of integral with same topology of this.

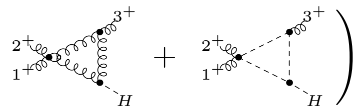
$$C_{1H|2|3} = -2(N_c - n_f)\mu^2 A^{\text{tree}}_{4,H}\frac{s_{13}s_{13}}{m_H^4} \tag{8.55}$$

$$C_{1|2H|3} = -2(N_c - n_f)\mu^2 A^{\text{tree}}_{4,H}\frac{s_{12}s_{23}}{m_H^4} \tag{8.56}$$

Two Masses Channel

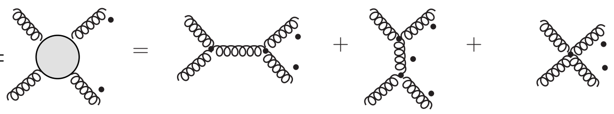
The two mass channel configuration is represented by

$$C_{12|3|H} = N_c \Bigg(\quad + \quad \Bigg), \tag{8.57}$$

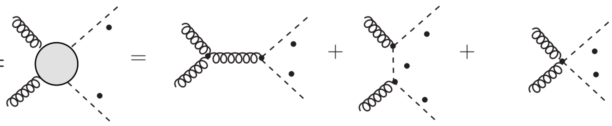


To build this integrand we are need to the tree level amplitude of four gluons, $A_{ggg^\bullet g^\bullet}$ and $A_{ggs^\bullet s^\bullet}$

$$A^{\lambda_1\lambda_2}_{ggg^\bullet g^\bullet}(k_1^+, k_2^+, L_1, L_2) = \tag{8.58}$$



$$A^{AB}_{ggs^\bullet s^\bullet}(k_1^+, k_2^+, L_1, L_2) = \tag{8.59}$$



and the integrands are

$$\begin{aligned}
\mathcal{I}^{loop}_{12|3|H} = & A^{\mu\nu}_{ggg^\bullet g^\bullet}(k_1^+, k_2^+, -\ell + k_3, \ell + k_H) P^{\mu\rho}_{cut}(\ell - k_3) \times \\
& \times A^{\kappa\rho}_{gg^\bullet g^\bullet}\Big(k_3^+, -\ell, \ell - k_3\Big) P^{\kappa\lambda}_{cut}(\ell) \times \\
& \times A^{\sigma\lambda}_{Hgg^\bullet g^\bullet}\Big(-\ell - k_H, \ell\Big) P^{\sigma\nu}_{cut}(\ell + k_H)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{scalar}_{12|3|H} = & A^{AB}_{ggs^\bullet s^\bullet}(k_1^+, k_2^+, -\ell + k_3, \ell + k_H) \hat{G}^{BC} \times \\
& \times A^{CD}_{gs^\bullet s^\bullet}(k_3, -\ell, \ell + k_3) \hat{G}^{DE} A^{EF}_{Hs^\bullet s^\bullet}(-\ell - k_4, \ell) \hat{G}^{FA}
\end{aligned}$$

The triple-cut condition are

$$\ell^2 = \mu^2, \qquad \big(\ell + k_3\big)^2 = \mu^2, \qquad \big(\ell - k_2\big)^2 = \mu^2, \tag{8.60}$$

In this case, (8.32) is not a good basis for ℓ in order to obtain the triple-cut solution. The better choice is

$$e_1^\nu = k_H^{\flat\nu}, \quad e_2^\nu = k_3^\nu, \quad e_3^\nu = \frac{\langle k_H^\flat|\gamma^\nu|k_3]}{2}, \quad e_4^\nu = \frac{\langle k_3|\gamma^\nu|k_H^\flat]}{2}. \tag{8.61}$$

with

$$k_H^\nu = k_H^{\flat\nu} + \bar{k}_H^\nu, \qquad \bar{k}_H^\nu = \frac{m_H^2}{2k_H \cdot k_3} k_3^\nu \tag{8.62}$$

in this basis the solutions of the triple cut are

$$l_1^\nu = -\frac{m_H^2}{2k_H^\flat \cdot k_3} k_3^\nu + t e_3^\nu - \frac{\mu^2}{2t k_H^\flat \cdot k_3} e_4^\nu \tag{8.63}$$

$$l_2^\nu = -\frac{m_H^2}{2k_H^\flat \cdot k_3} k_3^\nu + t e_4^\nu - \frac{\mu^2}{2t k_H^\flat \cdot k_3} e_3^\nu \tag{8.64}$$

Evaluating the triangle integrand by these solution and performing the coefficient as before, we have

$$C_{12|3|H} = C_{12|H|3} = N_c \frac{1}{2} A_{4,H}^{\text{tree}} \big(s_{13} + s_{23}\big) \tag{8.65}$$

The coefficients of the others configuration are obtained using (8.39),

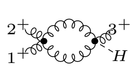
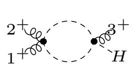
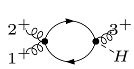
$$C_{2|H|31} = C_{H|2|31} = N_c \frac{1}{2} A_{4,H}^{\text{tree}} \big(s_{12} + s_{23}\big) \tag{8.66}$$

$$C_{1|H|23} = C_{H|1|23} = N_c \frac{1}{2} A_{4,H}^{\text{tree}} \big(s_{12} + s_{13}\big) \tag{8.67}$$

Clearly in these formulas we using the property that Higgs is colorless, so amplitudes are invariant to exchange gluon corner by Higgs corner.

8.3.3 Bubble coefficient

For the double cut, we need to consider two topologies for this calculation, a diagram with Higgs + gluon corner and diagram with 3 gluons corner. The coefficient of latter integral is zero [14]. To compute the integrand of bubble diagram we have to merge together the amplitude of four gluon with the amplitude of Higgs + 3 gluons using two cutted propagators, as did above.
The coefficient to calculate is

$C_{12|3H} = N_c \Bigg($  $+$  $\Bigg) + n_f$  (8.68)

Using the building blocks seen in previous section, the gluon-loop integrand is

$$\mathcal{I}_{12|3H}^{gluon} = A_{ggg^\bullet g^\bullet}^{\mu\nu}(k_1^+, k_2^+, -\ell, \ell - k_1 - k_2) P_{cut}^{\mu\rho} \times (\ell) A_{Hgg^\bullet g^\bullet}^{\sigma\rho}(k_H, k_3^+, -\ell + k_2 + k_1, \ell) P_{cut}^{\sigma\nu}(\ell - k_1 - k_2) \tag{8.69}$$

the scalar-loop integrand is

$$\mathcal{I}_{12|3H}^{scalar} = A_{ggs^\bullet s^\bullet}^{AB}(k_1^+, k_2^+, -\ell, \ell - k_1 - k_2) \hat{G}^{BC} A_{Hgs^\bullet s^\bullet}^{CD}(k_H, k_3^+, -\ell + k_2 + k_1, \ell) \hat{G}^{DE} \tag{8.70}$$

and the fermion loop integrand is

$$\mathcal{I}^{fermion}_{12|3H} = \text{Tr}\Big\{ A^{AB}_{ggq^\bullet \bar{q}^\bullet}(k_1^+, k_2^+, -\ell, \ell - k_1 - k_2) P_{cut}(\ell) \times$$
$$\times A_{Hgq^\bullet \bar{q}^\bullet}(k_H, k_3^+, -\ell + k_2 + k_1, \ell) P_{cut}(\ell - k_1 - k_2) \Big\} \tag{8.71}$$

The double-cut is given by

$$\ell^2 = \mu^2, \qquad \big(\ell - k_1 - k_2\big)^2 = \mu^2, \tag{8.72}$$

In this case for ℓ^ν can be chosen this basis

$$e_1^\nu = k_1^{\flat\nu}, \quad e_2^\nu = \chi^\nu, \quad e_3^\nu = \frac{\langle k_1^\flat|\gamma^\nu|\chi]}{2}, \quad e_4^\nu = \frac{\langle \chi|\gamma^\nu|k_1^\flat]}{2}. \tag{8.73}$$

where χ is a complex massless momentum and $k_1^\flat$ is a massless momentum related to external momenta by

$$k_{12}^\nu = k_1^{\flat\nu} + \bar{k}_1^\nu \qquad \bar{k}_1^\nu = \frac{S_{12}}{\gamma_1}\chi^\nu \tag{8.74}$$

and

$$k_{12} = k_1 + k_2, \qquad S_{12} = k_1 \cdot k_2 \qquad \gamma_1 = 2k_{12} \cdot \chi \tag{8.75}$$

The solution of double cut is then

$$\ell_1^\nu = y e_1^\nu + a e_2^\nu + t e_3^\nu + b e_4^\nu \tag{8.76}$$

with

$$a = \frac{S_{12}}{\gamma_1}(1 - y) \qquad b = \frac{1}{\gamma_1 t}\Big(y(1-y)S_{12} - \mu^2\Big) \tag{8.77}$$

and y and t are free parameters. As seen in section 5.4, the bubble coefficient has a pure bubble contributions, obtained using (5.71) and performing expansions as explained, and the triangles contributions of coefficients of all triangle diagrams that can be obtained from the bubble diagram. Note that an useful way to obtain the triangle integrands of sub-triangle diagrams is multiply the bubble integrand to denominator of propagator that identify the sub-triangle. In fact evaluating this integrand in the corespondent triple-cuts solutions survives only the term where the propagator is simplified. The three sub-triangle integrand are identified from these denominators

$$(\ell + k_3)^2, \qquad ,(\ell + k_H)^2, \qquad (\ell - k_2)^2 \tag{8.78}$$

at which correspond the cut conditions imposing $D = \mu^2$ The triple-cut solutions are obtained from (5.71) adding one of the cut conditions in (8.78). With the third conditions y has two value

$$y_+(L) = \frac{B_1(L) - \sqrt{B_1(L)\text{^}2 + 4B_0(L)B_2(L)}}{2B_2(L)} \tag{8.79}$$

$$y_-(L) = \frac{B_1(L) - \sqrt{B_1(L)\text{^}2 + 4B_0(L)B_2(L)}}{2B_2(L)} \tag{8.80}$$

with

$$B2(L) = S_{12}\langle\chi|L|k_1^\flat] \tag{8.81}$$

$$B1(L) = \gamma_1 t\langle k_1^\flat|L|k_1^\flat] - S_{12}t\langle\chi|L|\chi] + S_{12}\langle\chi|L|k_1^\flat] \tag{8.82}$$

$$B0(L) = \gamma_1 t^2\langle k_1^\flat|L|\chi] - \mu^2\langle\chi|L|k_1^\flat] + \gamma_1 t L^2 + S_{12}t\langle\chi|L|\chi] \tag{8.83}$$

and $L = k_3, k_H, k_2$ respectively to triangle diagrams determined by (8.78). Now we have all ingredients to evaluate the bubble coefficient using algorithm (8.1),

$$C_{12|3H} = 4(N_c - n_f)\mu^2 A_{4,H}^{\text{tree}}\frac{s_{13}s_{23}}{s_{12}m_H^4} \tag{8.84}$$

The permutation of indices as (8.39) give us the coefficients of the others configurations.

$$C_{1H|23} = 4(N_c - n_f)\mu^2 A_{4,H}^{\text{tree}}\frac{s_{12}s_{13}}{s_{23}m_H^4} \tag{8.85}$$

$$C_{31|2H} = 4(N_c - n_f)\mu^2 A_{4,H}^{\text{tree}}\frac{s_{12}s_{23}}{s_{13}m_H^4} \tag{8.86}$$

8.3.4 Full-Amplitude

The value of the one loop-amplitude $A(1^+, 2^+, 3^+, H)$ is obtained merging together all ingredients in (8.14). The amplitude is

$$\begin{aligned}
A_4^{\text{1-loop}}(1^+, 2^+, 3^+, H) = \frac{1}{(4\pi)^{2-\epsilon}} A_{4,H}^{\text{tree}}\Big\{ N_c\Big[-\frac{1}{2}s_{12}s_{23}I_{1|2|3|H} \\
-\frac{1}{2}s_{13}s_{12}I_{1|2|H|3} - \frac{1}{2}s_{23}s_{13}I_{1|H|2|3} + \frac{1}{2}(s_{13}+s_{23})\big(I_{12|3|H} + I_{12|H|3}\big) + \\
+\frac{1}{2}(s_{12}+s_{23})\big(I_{2|H|31} + I_{H|2|31}\big) + \frac{1}{2}(s_{12}+s_{13})\big(I_{1|H|23} + I_{H|1|23}\big)\Big] + \\
\big(N_c - n_f\big)\Big[-2\frac{s_{12}s_{13}}{m_H^4}I_{1H|2|3}[\mu^2] - 2\frac{s_{12}s_{23}}{m_H^4}I_{1|2|3H}[\mu^2] - 2\frac{s_{13}s_{23}}{m_H^4}I_{1|2H|3}[\mu^2] + \\
4\frac{s_{12}s_{13}}{s_{23}m_H^4}I_{1H|23}[\mu^2] + 4\frac{s_{13}s_{23}}{s_{12}m_H^4}I_{12|3H}[\mu^2] + 4\frac{s_{12}s_{23}}{s_{13}m_H^4}I_{31|2H}[\mu^2]\Big]\Big\}
\end{aligned} \tag{8.87}$$

where $N_c = 3$ is the number of colors, n_f is the number of light fermions. Expliciting the value of the coefficients and master integral we have

$$
\begin{aligned}
&A_4^{\text{1-loop}}(1^+,2^+,3^+,H) \\
&= \frac{r_\Gamma}{(4\pi)^{2-\epsilon}} A_{4,H}^{\text{tree}} \Bigg\{ N_c \Bigg(-\frac{1}{\epsilon^2}\left[(-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} + (-s_{13})^{-\epsilon} + \frac{\pi^2}{2}\right] \\
&+ \left[\mathrm{Li}_2\left(1-\frac{m_H^2}{s_{12}}\right) + \mathrm{Li}_2\left(1-\frac{m_H^2}{s_{23}}\right) + \mathrm{Li}_2\left(1-\frac{m_H^2}{s_{13}}\right)\right] + \\
&+ \left[\frac{1}{2}\log^2\frac{s_{12}}{s_{23}} + \frac{1}{2}\log^2\frac{s_{12}}{s_{13}} + \frac{1}{2}\log^2\frac{s_{23}}{s_{13}}\right]\Bigg) + \\
&+ \frac{1}{3}(N_c - n_f)\frac{s_{12}s_{13} + s_{12}s_{23} + s_{13}s_{23}}{m_H^4}\Bigg\}
\end{aligned}
\tag{8.88}
$$

The result agrees with the one presented in Ref. [64].

8.4 $A(1^-,2^+,3^+,H)$

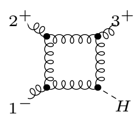
The principle to calculation of the coefficients about the process with the minus helicity is the same seen in the previous section. The only difference is that now we are need to distinguish the configuration according to position of minus helicity. We are need more than a permutation of legs indices. In this case the amplitude at leading-order holds

$$A_{4,H}^{\text{tree}}(1^-,2^+,3^+,H) = \frac{i[23]^4}{[12][23][31]} \tag{8.89}$$

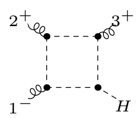
8.4.1 Box Coefficients

Considering a minus polarization vector for gluon 1 we obtain easily (as above) the integrand of box.

$C_{1|2|3|H} = N_c \Bigg($



$+$



$\Bigg)$ (8.90)

Using the quadruple cut solution we obtain

$$C_{1|2|3|H} = -N_c \frac{1}{2} A_{4,H}^{\text{tree}} s_{12}s_{23} \tag{8.91}$$

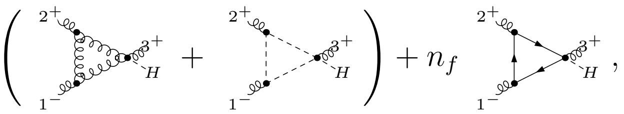
in this case, thanks symmetry of integrand, the others coefficients are determinated performing the permutation of indices (8.39)

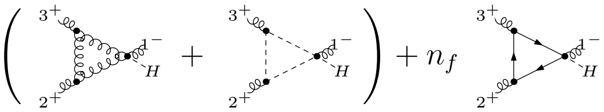
$$C_{1|2|H|3} = -N_c \frac{1}{2} A_{4,H}^{\text{tree}} s_{13}s_{12} \tag{8.92}$$

$$C_{1|H|2|3} = -N_c \frac{1}{2} A_{4,H}^{\text{tree}} s_{23}s_{13} \tag{8.93}$$

8.4.2 Triangle Coefficients

For diagrams with one massive channel we have to treat two kind of configurations

$C_{1|2|3H} = N_c$  (8.94)

$C_{1H|2|3} = N_c$  (8.95)

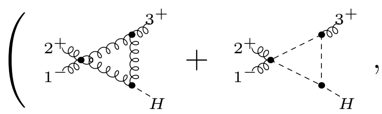
in order to distinguish the configuration in which Higgs in a corner together with a plus or minus gluon. The coefficients of the former are 0

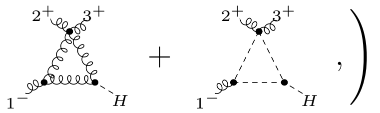
$$C_{1|2|3H} = C_{1|2H|3} = 0 \tag{8.96}$$

the latter is

$$C_{1H|2|3} = -2(N_c - n_f)\mu^2 A^{\text{tree}}_{4,H} \frac{s_{12}s_{13}}{s^2_{23}} \tag{8.97}$$

The triple cuts with two massive channels has in account two kind of configuration, as before.

$C_{12|3|H} = N_c$  (8.98a)

$C_{1|23|H} =$  (8.98b)

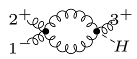
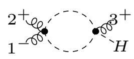
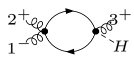
The others are obtained by permutation starting from these. Performing the integrands and using the well know triple-cut solutions, the result is

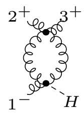
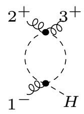
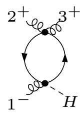
$$C_{12|3|H} = N_c \frac{1}{2} A^{\text{tree}}_{4,H} \left(s_{13} + s_{23}\right) \tag{8.99}$$

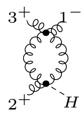
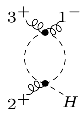
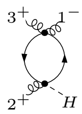
$$C_{1|23|H} = N_c \frac{1}{2} A^{\text{tree}}_{4,H} \left(s_{13} + s_{13}\right) \tag{8.100}$$

8.4.3 Bubble coefficients

For the bubble diagrams, the configurations are

$C_{12|3H} = N_c \Bigg($  $+$  $\Bigg) + n_f$  (8.101a)

$C_{23|H1} = N_c \Bigg($  $+$  $\Bigg) + n_f$  (8.101b)

$C_{H2|31} = N_c \Bigg($  $+$  $\Bigg) + n_f$  (8.101c)

and the results are

$$C_{12|3H} = 0 \tag{8.102}$$

$$C_{23|H1} = 4(N_c - n_f)\mu^2 A^{\text{tree}}_{4,H} \frac{s_{12}s_{13}}{s_{23}^3} \tag{8.103}$$

$$C_{H2|31} = 0 \tag{8.104}$$

The cut $C_{123|H}$ does not give any contribution.

8.4.4 Full-Amplitude

Taking into account the coefficient and using (8.14), amplitude $A(1^-, 2^+, 3^+, H)$ is

$$\begin{aligned} & A_4{}^{\text{1-loop}}(1^-, 2^+, 3^+, H) \\ & = \frac{r_\Gamma}{(4\pi)^{2-\epsilon}} A^{\text{tree}}_{4,H} \Bigg\{ N_c \Bigg(-\frac{1}{\epsilon^2}\Big[(-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} + (-s_{13})^{-\epsilon} + \frac{\pi^2}{2}\Big] \\ & + \Big[\text{Li}_2\left(1 - \frac{m_H^2}{s_{12}}\right) + \text{Li}_2\left(1 - \frac{m_H^2}{s_{23}}\right) + \text{Li}_2\left(1 - \frac{m_H^2}{s_{13}}\right)\Big] + \\ & + \Big[\frac{1}{2}\log^2\frac{s_{12}}{s_{23}} + \frac{1}{2}\log^2\frac{s_{12}}{s_{13}} + \frac{1}{2}\log^2\frac{s_{23}}{s_{13}}\Big]\Bigg) + \\ & + \frac{1}{3}(N_c - n_f)\frac{s_{12}s_{13}}{s_{23}^2}\Bigg\} \end{aligned} \tag{8.105}$$

The result agrees with the one presented in Ref. [33].

Chapter 9

Master Integrals

A perturbative approach to the quantitative description of the scattering of particles in quantum field theory involves the computation of Feynman diagrams. For a given number of external particles - the legs of diagram - fixed by the process under study, and a given order in perturbation theory, the skeletons of diagrams are built up by joining the edges of legs and propagators into vertexes, forming tree patterns and closed loops. Beyond the tree level, each Feynman diagram represents an integral which has, in general, a tensorial structure, induced by the tensorial nature of the interacting fields. Therefore, the result of its evaluation must be a linear combination of the tensors provided by the theory and by the kinematics of the process under study. The coefficients of this linear combination, usually called form factors, can be always extracted from each Feynman diagram, before performing any evaluation, by means of suitably chosen projectors. These form factors are scalar integrals closely connected to the original Feynman diagram: the numerator of their integrand may contain all the possible scalar products formed by external momenta and loop variables; whereas its denominator is formed by the denominators of propagators present in the diagram itself. Due to the bad convergence of loop integrals in four dimensions, regularization prescriptions are mandatory. Hereafter the integrals are regularized within the framework of t'Hooft-Veltman continuous-dimensional regularization scheme Accordingly, the dimension D of an extended integration space is used as a regulator for both infrared (IR) and ultraviolet (UV) divergences, which finally do appear as poles in (D - 4) when D goes to 4. The aim of a precise calculation is to compute Feynman diagrams for any value of the available kinematic invariants. Except in case of simple configurations (e.g. very few legs and/or few scales), quite generally, approximations have to be taken by limiting the result to a specific kinematics domain, and, thus, looking for a hierarchy among the scales, to get rid of the ones which anyhow would give a negligible contribution in that domain. The puzzling complexity of the Feynman diagrams calculation arises because of two different sources: either multi-leg or multi-loop processes. In recent years the progress in the evaluation of higher loop radiative corrections in quantum field theory has received a strong boost, due to the optimization and automatizing of various techniques [65]. In this work we review one of the most effective computational tools which have been developed in the framework of the dimensional regularization: the method of differential equations for Feynman integrals [28, 29].

The computational strategy is threefold.

- In a preliminary stage, by exploiting some remarkable properties of the dimensionally regularized integrals, namely integration-by-parts identities (IBP), Lorentz invariance identities (LI), and further sets of identities due to kinematic symmetry specific of each diagram, one establishes several relations among the whole set of scalar integrals associated to the original Feynman diagram. By doing so, one reduces the result, initially demanding for a large number of scalar integrals (from hundreds to billions, according to the case), to a combination of a minimal set (usually of the order of tens) of independent functions, the so called master integrals (MI's).

- The second phase consists of the actual evaluation of the MI's. By using the set of identities previously obtained, it is also possible to write Differential Equations in the kinematic invariants which are satisfied by the MI's themselves. When possible, these equations can be solved exactly in D dimensions. Alternatively, they can be Laurent-expanded around suitable values of the dimensional parameter up to the required order, obtaining a system of chained differential equations for the coefficients of the expansions, which, in the most general case, are finally integrated by Euler's variation of constants method.

- The third phase consist in a better way to solve differential equations of master integrals. By using IBP and LI identity is possible to choose a basis of MI that are uniform in trascendentality. This choice allow to write systems of differential equations with a good property. After a rotation (Magnus) is possible to factor the D dependence from kinematics parameters. Now the solutions have a form of a Magnus series [66].

9.1 Integration-by-parts Identities

Integration-by-parts identities (IBP-Id's) are among the most remarkable properties of dimensionally regularized integrals and they were first proposed in the eighties by Chetyrkin and Tkachov [27]. The basic idea underlying IBP-Id's is an extension to D-dimensional spaces of Gauss' theorem. For each of the integrals of a diagram one can write the vanishing of the integral of a divergence given by,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0. \tag{9.1}$$

In the above identities the index i runs over the number of loops ($i = 1, 2, ..., \ell$), and the vector v_μ can be any of the $(\ell+g-1)$ independent vectors of the problem:$k_1, \cdots, k_\ell, p_1, \cdots, p_{g-1}$; in such way, for each integrand, $\ell(\ell+g-1)$ IBP-Id's can be established. When evaluating explicitly the derivatives, one obtains a combination of integrands with a total power of the irreducible scalar products equal to $(s-1)$, s and $(s+1)$ and total powers of the propagators in the denominator equal to $(t+r)$ and $(t+r+1)$, therefore involving, besides the integrals of the class $I_{r,s,t}$, also the classes $I_{t,r,s-1}$, $I_{t,r+1,s}$ and $I_{t,r+1,s+1}$. Simplifications between reducible scalar products and propagators in the denominator may occur, lowering the powers of the propagators. During that simplification, some propagator might disappear, generating an integral belonging to a *subtopology*, with $t-1$ propagators.

9.1.1 Example

In this section we show as IBP works doing some examples about integrals of one-loop diagrams. Considering a massive tadpole integral

$$\int d^D q \frac{1}{D} \tag{9.2}$$

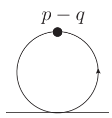
with $D = q^2 - m^2$. In order to apply Eq. (9.1) we have only one choice for v^μ, namely $v^\mu = q^\mu$. Eq. (9.1) reads as

$$\int d^d q \frac{\partial}{\partial q^\mu}\left(q_\mu \frac{1}{D}\right) = 0 \tag{9.3}$$

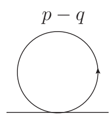
and we obtain

$$\int d^D q \frac{1}{D^2} = \frac{D-2}{2m^2} \int \frac{1}{D} \tag{9.4}$$

Representing a square propagator with a dot, we can do the IBP identity in terms of Feynman diagrams



$$= \frac{D-2}{2m^2}$$



The function (9.2) is a irreducible integral, or *master integral.* In fact because Eq. (9.1) acts as a derivative, starting from a tadpole with a generic exponent n in the denominator we obtain a relation with a tadpole characterized by an exponent $n+1$. So Eq. (9.4) is an IBP identity, cause it is a relation between a function ad a master integral.
Now we consider the one-loop self energy of a fermion

$$\int d^D q \frac{1}{D_1 D_2} \tag{9.5}$$

with

$$D_1 = q^2 \qquad D_2 = (p-q)^2 - m^2 = q^2 - 2p\cdot q \tag{9.6}$$

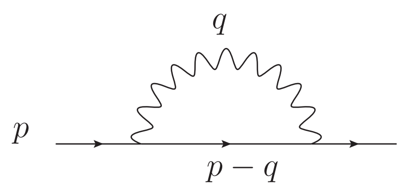


Figure 9.1: Fermion self-energy

Equation (9.1) read as

$$\int d^D q \frac{\partial}{\partial q^\mu}\Big(v_\mu \frac{1}{D_1 D_2}\Big) = 0 \qquad\qquad v_\mu = p_\mu, q_\mu \tag{9.7}$$

We discuss the two possible choices of v^μ

$v_\mu = p_\mu$**.** Considering integrand of eq. (9.7),

$$\frac{\partial}{\partial q^\mu}\Big(p_\mu \frac{1}{D_1 D_2}\Big) = 0 \tag{9.8}$$

Developing derivative and using (9.6),

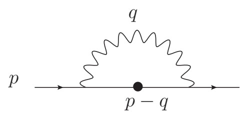
$$\begin{aligned} \frac{\partial}{\partial q^\mu}\Big(p_\mu \frac{1}{D_1 D_2}\Big) &= -\frac{2p\cdot q}{D_1^2 D_2} - \frac{2p\cdot q}{D_1 D_2^2} + 2\frac{p^2}{D_1 D_2^2} \\ &= -\frac{D_1 - D_2}{D_1^2 D_2} - \frac{D_1 - D_2}{D_1 D_2^2} + 2\frac{m^2}{D_1 D_2^2} = \\ &= +\frac{1}{D_1^2} - \frac{1}{D_2^2} + 2\frac{m^2}{D_1 D_2^2} = 0 \end{aligned} \tag{9.9}$$

Recovering equation with integral we obtain

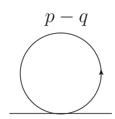
$$2m^2 \int d^D q \frac{1}{D_1 D_2^2} = \int d^D q \frac{1}{D_2^2} \tag{9.10}$$

where we used property $\int d^D q \frac{1}{D_1^2} = 0$ From Eq. (9.4) we now that the right side of (9.10) is reducible, so we have

$$\int d^D q \frac{1}{D_1 D_2^2} = \frac{D-2}{2m^4} \int d^D q \frac{1}{D_2} \tag{9.11}$$



$$= \frac{D-2}{2m^4}$$



$v_\mu = q_\mu$**.** Now we works on integrand

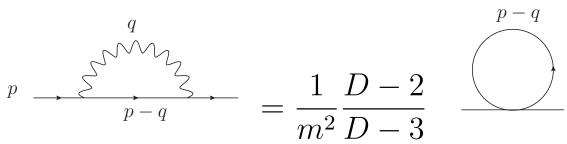
$$\frac{\partial}{\partial q^\mu}\Big(q_\mu \frac{1}{D_1 D_2}\Big) = 0 \tag{9.12}$$

Using relation $\frac{\partial}{\partial q^\mu} q_\mu = d$ and performing algebraic operations as before we have

$$(D-3)\int d^D q \frac{1}{D_1 D_2} = \int d^D q \frac{1}{D_2^2} \tag{9.13}$$

and the relation (9.4) allows us to write the diagram of a self-energy of a fermion in terms of the tadpole master integral

$$\int d^D q \frac{1}{D_1 D_2} = \frac{1}{m^2}\frac{D-2}{D-3}\int d^D q \frac{1}{D_2} \tag{9.14}$$



9.2 Lorentz Invariance Identities

Another class of identities can be derived by exploiting a general properties of the integrals, namely their nature as Lorentz scalars. If we consider an infinitesimal Lorentz transformation on the external momenta, $p_i \to p_i + \delta p_i$, where $\delta p_i = \omega_{\mu\nu} p_{i,\nu}$ with $\omega_{\mu\nu}$ a totally antisymmetric tensor, we have

$$I(p_i + \delta p_i) = I(p_i). \tag{9.15}$$

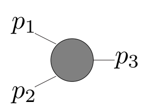
Because of the antisymmetry of $\omega_{\mu\nu}$ and because

$$I(p_i + \delta p_i) = I(p_i) + \sum_n \frac{\partial I(p_i)}{\partial p_{n,\mu}} = I(p_i) + \omega_{\mu\nu}\sum_n p_{n,\nu}\frac{\partial I(p_i)}{\partial p_{n,\mu}}, \tag{9.16}$$

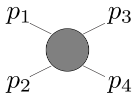
we can write the following relation

$$\sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right) I(p_i) = 0. \tag{9.17}$$

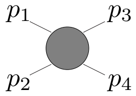
Eq. (9.17) can be contracted with all possible antisymmetric combinations of the external momenta $p_{i,\mu} p_{j,\nu}$, to obtain other identities for the considered integrals.
In case of integral associated to any *vertex* topologies with two independent external momenta, p_1 and p_2, we can build up the identity

$$\left[(p_1\cdot p_2)\left(p_{1,\mu}\frac{\partial}{\partial p_{1,\mu}} - p_{2,\mu}\frac{\partial}{\partial p_{2,\mu}} + p_2^2 p_{1,\mu}\frac{\partial}{\partial p_{2,\mu}} - p_1^2 p_{2,\mu}\frac{\partial}{\partial p_{1,\mu}}\right)\right]$$  $$= 0 \tag{9.18}$$

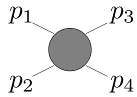
In the case of a richer kinematics, like in the case of integrals associated to *box* topologies with three independent external momenta, p_1, p_2 and p_3, we ca write down three LI-id's

$$(p_{1,\mu}p_{2,\mu} - p_{1,\nu}p_{2,\mu}) \sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right)$$ 

$$= 0, \qquad (9.19)$$

$$(p_{1,\mu}p_{3,\mu} - p_{1,\nu}p_{3,\mu}) \sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right)$$ 

$$= 0, \qquad (9.20)$$

$$(p_{2,\mu}p_{3,\mu} - p_{2,\nu}p_{3,\mu}) \sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right)$$ 

$$= 0\ . \qquad (9.21)$$

9.3 Differential Equations for Master Integrals

The outcome of the *reduction* procedure, previously discussed, is a collection of identities thanks to which any expression, demanding originally for the evaluation of a very large number of integrals, is simplified and written as linear combination of few MI's with rational coefficients. The completion of the analytic achievement of the result proceeds with the evaluation of the yet unknown MI's. As we will see in a moment, the same collection of identities is as well necessary to write Differential Equations satisfied by the MI's. Once all the MI's of a given topology are identified, the problem of their calculation arises. Exactly at this stage of the computation, *differential equations* enter the game. The use of differential equations in one of the internal masses was first proposed out by Kotikov [27],then extended to more general differential equations in any of Mandelstam variables by Remiddi [28].
Let us point out the basic idea of the method. To begin with, consider any scalar integral defined as

$$M(s_1, s_2, \cdots, s_{\mathcal{N}}) = \int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{S_1^{n_1} \ldots S_q^{n_q}}{\mathcal{D}_1 \ldots \mathcal{D}_t}, \qquad (9.22)$$

where $\{s_1, s_2, \cdots, s_{\mathcal{N}}\}$ is any set of kinematic invariants of the topology and $\mathcal{N}$ is the number of such invariants. Let us denote the set $\{s_1, s_2, \cdots, s_{\mathcal{N}}\} = \mathbf{s}$ and consider the following quantities

$$O_{jk}(\mathbf{s}) = p_{j,\mu} \frac{\partial M(\mathbf{s})}{\partial p_{k,\mu}} \quad (j, k = 1, 2, \cdots, g-1), \qquad (9.23)$$

where $g-1$ is the number of independent external momenta. By the chain differentiation rule we have

$$O_{jk}(\mathbf{s}) = p_{j,\mu} \cdot \sum_{\alpha=1}^{\mathcal{N}} \frac{\partial s_\alpha}{\partial p_{k,\mu}} \frac{\partial M(\mathbf{s})}{\partial s_\alpha} = \sum_{\alpha=1}^{\mathcal{N}} \left(p_{j,\mu} \cdot \frac{\partial s_\alpha}{\partial p_{k,\mu}} \right) \frac{\partial M(\mathbf{s})}{\partial s_\alpha}. \qquad (9.24)$$

According to the available number of the kinematic invariants, the *r.h.s.* of Eq. (9.23) and the *r.h.s.* of Eq. (9.24) may be equated to form the following system

$$\sum_{\alpha=1}^{\mathcal{N}} \left(p_{j,\mu} \cdot \frac{\partial s_\alpha}{\partial p_{k,\mu}} \right) \frac{\partial M(\mathbf{s})}{\partial s_\alpha} = p_{j,\mu} \frac{\partial M(\mathbf{s})}{\partial p_{k,\mu}}, \tag{9.25}$$

which can be solved in order to express $\frac{\partial M(\mathbf{s})}{\partial s_\alpha}$ in terms of $p_{j,\mu}\frac{\partial M(\mathbf{s})}{\partial p_{k,\mu}}$, so the corresponding identity, can be finally read as a *differential equation* for M.

The system is formed by a set of *first-order differential equations* (ODE), one for each MI, say M_j, whose general structure reads like the following,

$$\frac{\partial}{\partial s_\alpha} M_j(D, \mathbf{s}) \;\; = \;\; \sum_k A_k(D, \mathbf{s}) \; M_k(D, \mathbf{s}) + \sum_h B_h(D, \mathbf{s}) \; N_h(D, \mathbf{s}) \tag{9.26}$$

where $\alpha = 1, \cdots, \mathcal{N}$, is the label of the invariants, and N_k are MI's of the subtopologies. Note that the above equations are exact in D, and the coefficients A_k, B_k are rational factors whose singularities represent the thresholds and the pseudothresholds of the solution.

The coefficients of the differential equations (9.26) are in general singular at some kinematic points (thresholds and pseudothresholds), and correspondingly, the solutions of the equations can show singular behaviors in those points, while the unknown integral might have not. The boundary conditions for the differential equations are found by exploiting the known analytical properties of the MI's under consideration, imposing the regularity or the finiteness of the solution at the *pseudothresholds* of the MI. This qualitative information is sufficient for the quantitative determination of the otherwise arbitrary integration constants, which naturally arise when solving a system of differential equations.

Chapter 10

Magnus Exponential and Differential Equations

The MI's are functions of the kinematic invariants constructed from the external momenta, of the masses of the external particles and of the particles running in the loops, as well as of the number of space-time dimensions. Remarkably, the existence of the aforementioned relations forces the MI's to obey linear systems of first-order differential equations in the kinematic invariants, which can be used for the determination of their expression. In the most general case, MI are finally integrated by using the method of Euler's variation of constants. The nested structure of the Laurent expansion of the linear system leads to an iterative structure for the solution that, order-by-order in $\epsilon = (4-D)/2$, is written in terms of repeated integrals, starting from the kernels dictated by the homogeneous solution. The transcendentality of the solution is associated to the number of repeated integrations and increases by one unit as the order of the ϵ-expansion increases. The solution of the system, namely the MI's, is finally determined by imposing the boundary conditions at special values of the kinematic variables, properly chosen either in correspondence of configurations that reduce the MI's to simpler integrals or in correspondence of pseudo-thresholds. In this latter case, the boundary conditions are obtained by imposing the regularity of the MI's around unphysical singularities, ruling out divergent behavior of the general solution of the systems. For any given scattering process the set of MI's is not unique, and, in practice, their choice is rather arbitrary. Usually MI's are identified after applying the Laporta reduction algorithm [26]. Afterward, convenient manipulations of the basis of MI's may be performed. Proper choices of MI's can simplify the form of the systems of differential equations and, hence, of their solution, although general criteria for determining such optimal sets are not available. An important step in this direction has been recently taken in Ref. [30], where Henn proposes to solve the systems of DE's for MI's with algebraic methods. The key observation is that a *good* choice of MI's allows one to cast the system of DE's in a *canonical form*, where the dependence on ϵ, is factorized from the kinematic. The integration of a system in canonical form trivializes and the analytic properties of its general solution are manifestly inherited from the matrix associated to the system, which is the kernel of the representation of the solutions in terms of repeated integrations. As pointed out in [30], finding an algorithmic procedure which, starting from a generic set of MI's, leads to a set MI's fulfilling a canonical system of DE's is a formidable task. In practice, the quest for the suitable basis of MI's is determined by qualitative properties required for the solution, such as finiteness in the $\epsilon \to 0$ limit, and homogeneous transcendentality, which turn into quantitative tools like the unit leading singularity criterion and the dlog representation in terms of Feynman parameters [67].

10.1 Magnus series expansion

Consider a generic linear matrix differential equation

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ . \tag{10.1}$$

If $A(x)$ commutes with its integral $\int_{x_0}^{x} d\tau A(\tau)$, *e.g.* in the scalar case, the solution can be written as

$$Y(x) = e^{\int_{x_0}^{x} d\tau A(\tau)} \ Y_0 \ . \tag{10.2}$$

In the general non-commutative case, one can use the *Magnus theorem* [31] to write the solution as,

$$Y(x) = e^{\Omega(x,x_0)} \ Y(x_0) \equiv e^{\Omega(x)} \ Y_0 \, , \tag{10.3}$$

where $\Omega(x)$ is written as a series expansion, called *Magnus expansion*,

$$\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ . \tag{10.4}$$

The first three terms of the expansion (10.4) read as follows:

$$\begin{aligned}
\Omega_1(x) &= \int_{x_0}^{x} d\tau_1 A(\tau_1) \ , \\
\Omega_2(x) &= \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \ [A(\tau_1), A(\tau_2)] \ , \\
\Omega_3(x) &= \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \ [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \ .
\end{aligned} \tag{10.5}$$

We remark that if A and its integral commute, the series (10.4) is truncated at the first order, $\Omega = \Omega_1$, and we recover the solution (10.2). As a notational aside, in the following we will use the symbol $\Omega[A](x)$ to denote the Magnus expansion obtained using A as kernel.

Magnus series is related to the Dyson series [68], and their connection can be obtained starting from the Dyson expansion of the solution of the system (10.1),

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x) \ , \quad Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \ A(\tau_1)A(\tau_2)\cdots A(\tau_n) \ , \tag{10.6}$$

in terms of the *time-ordered* integrals Y_n. Comparing Eq. (10.3) and (10.6) we have

$$\sum_{j=1}^{\infty} \Omega_j(x) = \ \log \left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right) \ , \tag{10.7}$$

and the following relations

$$\begin{aligned}
Y_1 &= \Omega_1 \ , \\
Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2 \ , \\
Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3 \ , \\
&\ \vdots \\
Y_n &= \Omega_n + \sum_{j=2}^{n} \frac{1}{j} Q_n^{(j)} \ .
\end{aligned} \tag{10.8}$$

The matrices $Q_n^{(j)}$ are defined as

$$Q_n^{(j)} = \sum_{m=1}^{n-j+1} Q_m^{(1)} Q_{n-m}^{(j-1)} \ , \quad Q_n^{(1)} \equiv \Omega_n \ , \quad Q_n^{(n)} \equiv \Omega_1^n \ . \tag{10.9}$$

In the following, we will use both Magnus and Dyson series. The former allows us to easily demonstrate how a system of DE's, whose matrix is linear in ϵ, can be cast in the canonical form. The latter can be more conveniently used for the explicit representation of the solution.

10.2 Differential equations for Master Integrals

We consider a linear system of first order differential equations

$$\partial_x f(\epsilon, x) = A(\epsilon, x) \ f(\epsilon, x) \ , \tag{10.10}$$

where f is a vector of MI's, while x is a variable depending on kinematic invariants and masses. We suppose that A depends linearly on ϵ,

$$A(\epsilon, x) = A_0(x) + \epsilon A_1(x) \ , \tag{10.11}$$

and we change the basis of MI's via the Magnus series obtained by using A_0 as kernel,

$$f(\epsilon, x) = B_0(x)\; g(\epsilon, x)\;, \qquad B_0(x) \equiv e^{\Omega[A_0](x,x_0)}\,. \tag{10.12}$$

B_0 obeys the equation,

$$\partial_x B_0(x) = A_0(x) B_0(x)\;, \tag{10.13}$$

which, implies that the new basis g of MI's fulfills a system of differential equations in the *canonical* factorized form [66],

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x)\,. \tag{10.14}$$

The matrix $\hat{A}_1$ is related to A_1 by a similarity map,

$$\hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)\;, \tag{10.15}$$

and does not depend on ϵ. The solution of Eq. (10.14) can be found by using the Magnus theorem with $\epsilon \hat{A}_1$ as kernel

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\;, \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x,x_0)}\,, \tag{10.16}$$

where the vector g_0 corresponds to the boundary values of the MI's. Therefore, the solution of the original system Eq. (10.10) finally reads,

$$f(\epsilon, x) = B_0(x) B_1(\epsilon, x) g_0(\epsilon)\,. \tag{10.17}$$

It is worth to notice that $\Omega[\epsilon \hat{A}_1]$ in Eq.(10.16) depends on ϵ, while $\Omega[A_0]$ in Eq.(10.12) does not.

Let us remark that the previously described two-step procedure is equivalent to solving, first, the *homogeneous* system

$$\partial_x f_{\rm H}(\epsilon, x) = A_0(x) f_{\rm H}(\epsilon, x)\,, \tag{10.18}$$

whose solution reads,

$$f_{\rm H}(\epsilon, x) = B_0(x) g(\epsilon)\;, \tag{10.19}$$

and, then, to find the solution of the full system by Euler constants' variation. In fact, by promoting g to be function of x,

$$f_{\rm H}(\epsilon, x) \to f(\epsilon, x) = B_0(x) g(\epsilon, x)\;, \tag{10.20}$$

and by requiring f to be solution of Eq.(10.10)), one finds that $g(\epsilon, x)$ obeys the differential equation(10.14). The matrix B_0, implementing the transformation from the linear to the canonical form, is simply given as the product of two matrix exponentials. Indeed one can split A_0 into a diagonal term, D_0, and a matrix with vanishing diagonal entries N_0,

$$A_0(x) = D_0(x) + N_0(x)\;. \tag{10.21}$$

The transformation B is then obtained by the composition of two transformations

$$B(x) = e^{\Omega[D_0](x,x_0)} e^{\Omega[\hat{N}_0](x,x_0)} = e^{\int_{x_0}^{x} d\tau \; D_0(\tau)} e^{\Omega[\hat{N}_0](x,x_0)} \; , \tag{10.22}$$

where $\hat{N}_0$ is given by

$$\hat{N}_0(x) = e^{-\int_{x_0}^{x} d\tau \; D_0(\tau)} \; N_0(x) \; e^{\int_{x_0}^{x} d\tau \; D_0(\tau)} \tag{10.23}$$

In the last step of Eq.(10.22) we have used the commutativity of the diagonal matrix D_0 with its own integral. The leftmost expansion performs a transformation that "rotates" away D_0, while the second expansion gets rid of the $\mathcal{O}(\epsilon^0)$ contribution coming from $\hat{N}_0$, i.e. coming from the image of N_0 under the first transformation. In the examples hereby discussed it is possible, by trials and errors, to find a set of MI's obeying a system of DE's linear in ϵ. Moreover in these cases one finds that $\Omega[\hat{N}_0]$ contains just the first term of the series, except for the non-planar box, where also the second order is non vanishing.

Chapter 11

Harmonic Polylogarithms

It has been known for a long time that the analytic evaluation of integrals in perturbative quantum field theory gives rise to the Euler dilogarithm $\mathrm{Li}_2(x)$ and its generalizations, Nielsen's polylogarithms. Going to higher orders in perturbation theory, it was recently realized that Nielsen's polylogarithms are insufficient to evaluate all integrals appearing in Feynman graphs at two loops and beyond. This limitation can only be overcome by the introduction of further generalizations of Nielsen's polylogarithms. This generalization is made by the harmonic polylogarithms (HPLs) [32], appearing in loop integrals involving two mass scales, and two-dimensional harmonic polylogarithms (2dHPLs) [69], appearing in loop integrals involving three mass scales. These functions are already now playing a central role in the analytic evaluation of Feynman graph integrals [70]. 2dHPLs and multiple polylogarithms also appear for example if generalized hypergeometric functions are expanded in their indices around integer values [71].

The harmonic polylogarithms of weight w and argument x are identified by a set of w indices, grouped into a w-dimensional vector $\vec{m}_w$ and are indicated by $\mathrm{H}(\vec{m}_w; x)$. More explicitly, for $w = 1$ one defines

$$
\begin{aligned}
\mathrm{H}(0;x) &= \ln x \ , \\
\mathrm{H}(1;x) &= \int_0^x \frac{dx'}{1-x'} = -\ln(1-x) \ , \\
\mathrm{H}(-1;x) &= \int_0^x \frac{dx'}{1+x'} = \ln(1+x) \ .
\end{aligned}
\tag{11.1}
$$

For their derivatives, one has

$$
\frac{d}{dx}\mathrm{H}(a;x) = f(a;x) \ , \tag{11.2}
$$

where the index a can take the 3 values $0, +1, -1$ and the 3 rational fractions $f(a;x)$

are given by

$$\begin{aligned} f(0;x) &= \frac{1}{x} , \\ f(1;x) &= \frac{1}{1-x} , \\ f(-1;x) &= \frac{1}{1+x} . \end{aligned} \tag{11.3}$$

Note the (minor) asymmetry of (11.1), in contrast with the higher symmetry of (11.2).
For $w > 1$, let us elaborate slightly the notation for the w-dimensional vectors $\vec{m}_w$. Quite in general, let us write

$$\vec{m}_w = (a, \vec{m}_{w-1}) , \tag{11.4}$$

where $a = m_w$ is the leftmost index (taking of course one of the three values $0, 1, -1$), and $\vec{m}_{w-1}$ stands for the vector of the remaining $(w-1)$ components. Further, $\vec{0}_w$ will be the vector whose w components are all equal to the index 0. The harmonic polylogarithms of weight w are then defined as follows:

$$\mathrm{H}(\vec{0}_w;x) = \frac{1}{w!}\ln^w x , \tag{11.5}$$

while, if $\vec{m}_w \neq \vec{0}_w$

$$\mathrm{H}(\vec{m}_w;x) = \int_0^x dx' \; f(a;x') \; \mathrm{H}(\vec{m}_{w-1};x') . \tag{11.6}$$

Quite in general the derivatives can be written in the compact form

$$\frac{d}{dx}\mathrm{H}(\vec{m}_w;x) = f(a;x)\mathrm{H}(\vec{m}_{w-1};x) , \tag{11.7}$$

where, again, $a = m_w$ is the leftmost component of $\vec{m}_w$.

In analogy with (11.15), if $\vec{1}_w, \vec{(-1)}_w$ are the vectors whose components are all equal to 1 or -1, we have by applying recursively the definitions

$$\begin{aligned} \mathrm{H}(\vec{1}_w;x) &= \frac{1}{w!}(-\ln(1-x))^w , \\ \mathrm{H}(\vec{(-1)}_w;x) &= \frac{1}{w!}\ln^w(1+x) . \end{aligned} \tag{11.8}$$

Let us now have a look at the first few values of the indices. For $w = 2$ one has the 9 functions

$$
\begin{aligned}
\mathrm{H}(0,0;x) &= \frac{1}{2!}\ln^2 x \ , \\
\mathrm{H}(0,1;x) &= \int_0^x \frac{dx'}{x'}\mathrm{H}(1;x') = -\int_0^x \frac{dx'}{x'}\ln(1-x') \ , \\
\mathrm{H}(0,-1;x) &= \int_0^x \frac{dx'}{x'}\mathrm{H}(-1;x') = \int_0^x \frac{dx'}{x'}\ln(1+x') \ , \\
\mathrm{H}(1,0;x) &= \int_0^x \frac{dx'}{1-x'}\mathrm{H}(0;x') = \int_0^x \frac{dx'}{1-x'}\ln x' \ , \\
\mathrm{H}(1,1;x) &= \int_0^x \frac{dx'}{1-x'}\mathrm{H}(1;x') = -\int_0^x \frac{dx'}{1-x'}\ln(1-x') \ , \\
\mathrm{H}(1,-1;x) &= \int_0^x \frac{dx'}{1-x'}\mathrm{H}(-1;x') = \int_0^x \frac{dx'}{1-x'}\ln(1+x') \ , \\
\mathrm{H}(-1,0;x) &= \int_0^x \frac{dx'}{1+x'}\mathrm{H}(0;x') = \int_0^x \frac{dx'}{1+x'}\ln x' \ , \\
\mathrm{H}(-1,1;x) &= \int_0^x \frac{dx'}{1+x'}\mathrm{H}(1;x') = -\int_0^x \frac{dx'}{1+x'}\ln(1-x') \ , \\
\mathrm{H}(-1,-1;x) &= \int_0^x \frac{dx'}{1+x'}\mathrm{H}(-1;x') = \int_0^x \frac{dx}{1+x'}\ln(1+x') \ .
\end{aligned}
\tag{11.9}
$$

Those 9 functions can all be expressed in terms of logarithmic and dilogarithmic functions; indeed, if

$$
\mathrm{Li}_2(x) = -\int_0^x \frac{dx'}{x'}\ln(1-x')
\tag{11.10}
$$

is the usual Euler's dilogarithm, one finds

$$
\begin{aligned}
\mathrm{H}(0,1;x) &= \mathrm{Li}_2(x) \ , \\
\mathrm{H}(0,-1;x) &= -\mathrm{Li}_2(-x) \ , \\
\mathrm{H}(1,0;x) &= -\ln x \ln(1-x) + \mathrm{Li}_2(x) \ , \\
\mathrm{H}(1,1;x) &= \frac{1}{2!}\ln^2(1-x) \ , \\
\mathrm{H}(1,-1;x) &= \mathrm{Li}_2\left(\frac{1-x}{2}\right) - \ln 2 \ln(1-x) - \mathrm{Li}_2\left(\frac{1}{2}\right) \ , \\
\mathrm{H}(-1,0;x) &= \ln x \ln(1+x) + \mathrm{Li}_2(-x) \ , \\
\mathrm{H}(-1,1;x) &= \mathrm{Li}_2\left(\frac{1+x}{2}\right) - \ln 2 \ln(1+x) - \mathrm{Li}_2\left(\frac{1}{2}\right) \ , \\
\mathrm{H}(-1,-1;x) &= \frac{1}{2!}\ln^2(1+x) \ .
\end{aligned}
\tag{11.11}
$$

The two-dimensional harmonic polylogarithms, a generalization of the harmonic polylogarithms $\mathrm{H}(\vec{a},x)$, have been introduced for the analytic evaluation of a class of two-loop, off-mass-shell scattering Feynman graphs in massless QCD.

The 2dHPLs family which we consider here is obtained by the repeated integration, in the variable y, of rational factors chosen, in any order, from the set $1/y$, $1/(y-1)$, $1/(y+z-1)$, $1/(y+z)$, where z is another independent variable (hence the 'two-dimensional' in the name). It is clear that the set of rational factors might be further extended or modified; for the harmonic polylogarithms discussed above the set of rational factors was for instance $1/y$, $1/(y-1)$, $1/(y+1)$, involving only constants and no other variable besides y.

More precisely and in full generality, let us define the rational factor as [69]

$$\mathrm{g}(a;y) = \frac{1}{y-a} \ , \tag{11.12}$$

where a is the *index*, which can depend on z, $a = a(z)$; the rational factors which we consider for the 2dHPLs are then

$$\begin{aligned}
\mathrm{g}(0;y) &= \frac{1}{y} \ , \\
\mathrm{g}(1;y) &= \frac{1}{y-1} \ , \\
\mathrm{g}(1-z;y) &= \frac{1}{y+z-1} \ , \\
\mathrm{g}(-z;y) &= \frac{1}{y+z} \ .
\end{aligned} \tag{11.13}$$

With the above definitions the index takes one of the values $0, 1, (1-z)$ and $(-z)$.

Correspondingly, the 2dHPLs at weight $w = 1$ (*i.e.* depending, besides the variable y, on a single further argument, or *index*) are defined to be

$$\begin{aligned}
\mathrm{G}(0;y) &= \ln y \ , \\
\mathrm{G}(1;y) &= \ln(1-y) \ , \\
\mathrm{G}(1-z;y) &= \ln\left(1 - \frac{y}{1-z}\right) \ , \\
\mathrm{G}(-z;y) &= \ln\left(1 + \frac{y}{z}\right) \ .
\end{aligned} \tag{11.14}$$

The 2dHPLs of weight w larger than 1 depend on a set of w indices, which can be grouped into a w-dimensional *vector* of indices $\vec{a}$. By writing the vector as $\vec{a} = (a, \vec{b})$, where a is the leftmost component of $\vec{a}$ and $\vec{b}$ stands for the vector of the remaining $(w-1)$ components, the 2dHPLs are then defined as follows: if all the w components of $\vec{a}$ take the value 0, $\vec{a}$ is written as $\vec{0}_w$ and

$$\mathrm{G}(\vec{0}_w;y) = \frac{1}{w!}\ln^w y \ , \tag{11.15}$$

while, if $\vec{a} \neq \vec{0}_w$,

$$\mathrm{G}(\vec{a};y) = \int_0^y \mathrm{d}y' \ \mathrm{g}(a;y') \ \mathrm{G}(\vec{b};y') \ . \tag{11.16}$$

In any case the derivatives can be written in the compact form

$$\frac{\partial}{\partial y}\mathrm{G}(\vec{a};y) = \mathrm{g}(a;y)\mathrm{G}(\vec{b};y)\ , \tag{11.17}$$

where, again, a is the leftmost component of $\vec{a}$ and $\vec{b}$ stands for the remaining $(w-1)$ components.

From (11.15) and (11.16), one arrives immediately at a multiple (or repeated) integral representation of the 2dHPL:

$$\mathrm{G}(\vec{m}_w;y) = \int_0^y \mathrm{d}t_1 \mathrm{g}(m_1;t_1) \int_0^{t_1} \mathrm{d}t_2 \mathrm{g}(m_2;t_2) \ldots \int_0^{t_{w-1}} \mathrm{d}t_w \mathrm{g}(m_w;t_w)\ , \tag{11.18}$$

valid for $m_w \neq 0$, and

$$\mathrm{G}(\vec{m}_w;y) = \int_0^y \mathrm{d}t_1 \mathrm{g}(m_1;t_1) \int_0^{t_1} \mathrm{d}t_2 \mathrm{g}(m_2;t_2) \ldots \int_0^{t_{v-1}} \mathrm{d}t_v \mathrm{g}(m_v;t_v) \mathrm{G}(\vec{0}_{w-v};t_v)\ , \tag{11.19}$$

valid for $\vec{m}_w = (\vec{m}_v, \vec{0}_{w-v})$ with $\vec{m}_v \neq \vec{0}_v$.

The definition is essentially the same as for the harmonic polylogarithms, if allowance is made for the greater generality of the 'indices', which can now depend on the second variable z. Let us, however, stress an important difference between the present definitions and the notation already used above, where the rational factors were indicated by $\mathrm{f}(a,x)$ and the harmonic polylogarithms by $\mathrm{H}(\vec{a},x)$; we have indeed

$$\begin{aligned} \mathrm{f}(1;x) &= -\mathrm{g}(1;x)\ , \\ \mathrm{f}(1-z;x) &= -\mathrm{g}(1-z;x)\ , \\ \mathrm{f}(z;x) &= \mathrm{g}(-z;x)\ , \end{aligned} \tag{11.20}$$

while there is no change when $a = 0$:

$$\mathrm{f}(0;x) = \mathrm{g}(0;x)\ . \tag{11.21}$$

Also for $a = -1$ one would have $\mathrm{f}(-1;x) = \mathrm{g}(-1;x)$, but we will not consider this case here as it never appears together with the other values of the indices $(1-z), (-z)$ The same applies between the harmonic polylogarithms H previously introduced and the 2dHPLs, as any H-function goes into the corresponding G-function, with the following correspondence rules: the indices $(1), (1-z)$ of H remain unchanged as indices of G, but each occurrence of $(1), (1-z)$ gives a change of sign between H and G; any index (z) of H goes into an index $(-z)$ of G (which keeps the same sign as H). One has for instance

$$\begin{aligned} \mathrm{H}(z,1-z;y) &= -\mathrm{G}(-z,1-z;y)\ , \\ \mathrm{H}(0,z,1-z,1;y) &= \mathrm{G}(0,-z,1-z,1;y)\ , \end{aligned} \tag{11.22}$$

and so on. The main advantage of the new notation is the (obvious) continuity in z of the g's and the G's; one has for instance

$$\lim_{z\to 1} \mathrm{g}(1-z;y) = \mathrm{g}(0;y) \ , \tag{11.23}$$

to be compared with

$$\lim_{z\to 1} \mathrm{f}(1-z;y) = -\mathrm{f}(0;y) \ , \tag{11.24}$$

and the same applies to any index of a G-function (when the limit exists). Note, however, that the positivity for positive value of the argument is lost – so that, for instance, one has $\mathrm{G}(0,1;1) = -\pi^2/6$, to be compared with the more elegant relation $\mathrm{H}(0,1;1) = \pi^2/6$.

11.1 Shuffle Algebra

Algebra and reduction equations of the 2dHPLs are the same as for the ordinary HPLs [32]. Let us start by the integration by parts (ibp) identities. From the very definition,

$$\begin{aligned}
\mathrm{H}(m_1\cdots m_q;x) &= \int_0^x dx'\ f(m_1;x')\mathrm{H}(m_2\cdots m_q;x') \\
&= \mathrm{H}(m_1;x)\mathrm{H}(m_2\cdots m_q;x) - \int_0^x dx'\ \mathrm{H}(m_1;x')f(m_2;x')\mathrm{H}(m_3\cdots m_q;x') \\
&= \mathrm{H}(m_1;x)\mathrm{H}(m_2\cdots m_q;x) - \mathrm{H}(m_2m_1;x)\mathrm{H}(m_3\cdots m_q;x) \\
&+ \mathrm{H}(m_3m_2m_1;x)\mathrm{H}(m_4\cdots m_q;x) - \cdots - (-1)^p\mathrm{H}(m_q\cdots m_1;x) \ . \qquad (11.25)
\end{aligned}$$

The above identity can be immediately verified, independently of its derivation, by the 'standard methods': it holds at $x = 0$; when differentiating with respect to x, one obtains a number of terms which are immediately seen to cancel out pairwise; therefore, the relation is true. This relation shows that in the case that $\vec{m}_q$ is symmetric and q is even the H-function reduces to products of lower weight functions.

Another important set of identities expresses the product of any two H-functions of weight w_1 and w_2 as a linear combination of H-functions of weight $w = w_1 + w_2$. Let us start from the case $w_1 = 1$; the identity reads

$$\begin{aligned}
\mathrm{H}(a;x)\mathrm{H}(m_p,\cdots,m_1;x) &= \mathrm{H}(a,m_p\cdots,m_1;x) \\
&+ \mathrm{H}(m_p,a,m_{p-1}\cdots,m_1;x) \\
&+ \mathrm{H}(m_p,m_{p-1},a,m_{p-2}\cdots m_1;x) \\
&+ \cdots \\
&+ \mathrm{H}(m_p,\cdots,m_1,a;x) \ . \qquad (11.26)
\end{aligned}$$

It can be established by induction in p. For $p = 1$ it is almost trivial, corresponding to (11.25) for $q = 2$. Assume then that it holds for $p-1$; take the identity for $p-1$, multiply by $f(m_p;x)$ and integrate over x. In the *r.h.s.* we can do the integral and obtain all necessary terms except for the one starting with a. The *l.h.s.* can be integrated by parts to give the proper *l.h.s.* term plus another term that can be integrated and gives indeed the missing term. This completes the proof.

(11.26) can be generalized to the product of two H-functions $\mathrm{H}(\vec{p};x)\mathrm{H}(\vec{q};x)$; if p,q are the dimensions of $\vec{p},\vec{q}$ (or, which is the same, the weights of the two H-functions), the product is equal to the sum of $(p+q)!/p!q!$ terms, each term being an H-function of weight $(p+q)$ with coefficient $+1$, obtained by choosing p indices in all possible ways (hence the binomial coefficients) and filling them from left to right with the components of $\vec{p}$ without changing their order, while the remaining q places contain the components of $\vec{q}$, again without altering their order. This can be expressed with the formula

$$\mathrm{H}(\vec{p};x)\mathrm{H}(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} \mathrm{H}(\vec{r};x) \tag{11.27}$$

in which $\vec{p}\uplus\vec{q}$ represents all mergers of $\vec{p}$ and $\vec{q}$ in which the relative orders of the elements of $\vec{p}$ and $\vec{q}$ are preserved.

As an example, for $p=2, \vec{p}=(a,b)$ and $q=3, \vec{q}=(r,s,t)$ one has

$$\begin{aligned} \mathrm{H}(a,b;x)\mathrm{H}(r,s,t;x) &= \mathrm{H}(a,b,r,s,t;x)+\mathrm{H}(a,r,b,s,t;x) \\ &+ \mathrm{H}(a,r,s,b,t;x)+\mathrm{H}(a,r,s,t,b;x) \\ &+ \mathrm{H}(r,a,b,s,t;x)+\mathrm{H}(r,a,s,b,t;x) \\ &+ \mathrm{H}(r,s,a,b,t;x)+\mathrm{H}(r,a,s,t,b;x) \\ &+ \mathrm{H}(r,s,a,t,b;x)+\mathrm{H}(r,s,t,a,b;x)\ , \end{aligned} \tag{11.28}$$

as can be easily checked, again, by the ‘standard method’.

11.2 HPL’s of Different Arguments

In this section we will look at the identities which can be established for suitable changes of the argument [32]. The common feature is that any H-function of weight w and argument x can be expressed as an homogeneous expression of the same weight w, involving either H-functions depending on a same argument, say t, related to x by the considered change, or constants corresponding to H-functions of special constant values of the arguments (typically 1).

Important relations between HPLs and 2dHPLs can be obtained from fundamental theorem of integral calculus

$$F(x) = F(x_0) + \int_{x_0}^{x} dt \frac{dF(t)}{dt} \tag{11.29}$$

that applied to HPLfunctions begins

$$\begin{aligned} \mathrm{H}(a,b,\ldots;x) &= \mathrm{H}(a,b,\ldots;0) + \int_0^x \mathrm{d}t \frac{d\mathrm{H}(a,b,\ldots;t)}{dt} \\ &= \mathrm{H}(a,b,\ldots;0) + \int_0^x \mathrm{d}t f(a,t)\mathrm{H}(b,\ldots;t) \end{aligned} \tag{11.30}$$

or in more general way

$$\mathrm{H}(a,b,\ldots;y(x)) = \mathrm{H}(a,b,\ldots;y(0)) + \int_0^x \mathrm{d}t f(a,y(t))\mathrm{H}(b,\ldots;y(t))y'(t) \tag{11.31}$$

An example of weight $w = 2$ is

$$\begin{aligned} \mathrm{H}(0,1;x^2) &= 2\int_0^x \frac{dt'}{t'}\mathrm{H}(1;t'^2) \\ &= 2\int_0^x \frac{dt'}{t'}\left(\mathrm{H}(1;t') - \mathrm{H}(-1;t')\right) \\ &= 2\mathrm{H}(0,1;x) - 2\mathrm{H}(0,-1;x) \end{aligned} \tag{11.32}$$

where we used the identity at lower weight $w = 1$

$$H(1,x^2) = H(1,x) - H(-1,x), \tag{11.33}$$

which can be easily derived from the definition (11.1) and property of logarithm $\ln(1-x^2) = \ln(1-x) + \ln(1+x)$.

Therefore, in general, using Eq.(11.29), relations between HPL's of different arguments can be obtained in a bottom-up approach, recursively, starting from weight $w = 1$, up to the deiserd weight.

Chapter 12

Master Integrals for Higgs plus jet at One-loop

In this section, we present the calculation the MI's for Higgs + 1 jet production at one-loop. These integrals have been calculated by Gehrmann and Remiddi using the differential equation method [72]. We compute a slightly different set of MI's by employing Magnus series expansion in order to solve the system of differential equations that they fulfill. In the first step, we identify find a Laporta basis of MI, by generating the IBP-id's using the code Reduze2 [36, 73]. In the next step, we construct a *good* basis of MI's obeying a system of differential equations whose matrix is linear in the dimensional parameter, $\epsilon = (4 - D)/2$. Finally, we apply Magnus exponential to build the canonical set of MI's

12.1 Master Integrals

In the case of the one-loop $gg \to gH$, the Laporta reduction algorithm, implemented in the code Reduze2, allow us to identify four master integrals, belonging these topologies:

- s-channel Bubble: $I_1(D, n_1, n_3) = \int d^D k \frac{1}{D_1^{n_1} D_3^{n_3}}$
- t-channel Bubble: $I_2(D, n_2, n_4) = \int d^D k \frac{1}{D_2^{n_2} D_4^{n_4}}$
- m_H^2-channel Bubble: $I_3(D, n_1, n_4) = \int d^D k \frac{1}{D_1^{n_1} D_4^{n_4}}$
- one-mass Box: $I_4(D, n_1, n_2, n_3, n_4) = \int d^D k \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}}$

12.1.1 Bubble Integrals

Let us observe that I_1, I_2, and I_3 are different permutation of a same graph, namely they all correspond to a massless two-denominator integral with external momentum p, with t $p^2 = s, t, m_H^2$ in the three cases, respectively.

Feynman Parameters

The one-loop bubble functions can be computed analytically using Feynman parameters [40, 44], and its expression for arbitrary powers of denominators reads,

$$I(D,n_1,n_2,p^2) = \pi^{D/2}(-p^2)^{D/2-n_1-n_2}\frac{\Gamma(-D/2+n_1+n_2)\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)}\,. \tag{12.1}$$

where p is momentum flowing through the bubble. The Laporta integral correspond to the case $n_1 = n_2 = 1$. Its series expansion for $\epsilon \to 0$ reads,

$$\begin{aligned}I(4-2\epsilon,1,1,p^2) = {}& \frac{\pi^2}{\epsilon} + (2\pi^2 - \gamma\pi^2 - \pi^2\ln(\pi) - \pi^2\ln(-p^2))+\\ &+\frac{1}{12}(48\pi^2 - 24\gamma\pi^2 + 6\gamma^2\pi^2 - \pi^4 - 24\pi^2\ln(\pi)+\\ &+12\gamma\pi^2\ln(\pi) + 6\pi^2\ln(\pi)^2 - 24\pi^2\ln(-p^2) + 12\gamma\pi^2\ln(-p^2)+\\ &+12\pi^2\ln(\pi)\ln(-p^2) + 6\pi^2\ln(-p^2)^2)\epsilon + \mathcal{O}(\epsilon^2)\end{aligned} \tag{12.2}$$

We can observe that this function is not homogeneous in trascendentality. In fact, if we consider γ, π and logarithms to have weight $w = 1$, in eq.(12.2), each order of expansion in ϵ contains terms with different weight: for example π^2 has $w = 2$, and $\gamma\pi^2$ has $w = 3$. Therefore, we say that $I(D, 1, 1)$ is not tanscendentally homogeneous, or equivalently it is not transcendentally uniform. If we consider another types of bubble integrals, either with a squared denominator, $I(D, 2, 1, p^2)$, or with a shifted value of dimensions, $I(D - 2, 1, 1, p^2)$, and look at their series expansion in ϵ,

$$\begin{aligned}I(4-2\epsilon,2,1,p^2) = {}& \frac{\pi^2}{p^2\epsilon} + \frac{1}{p^2}\Big(-\gamma\pi^2 - \pi^2\ln(\pi) - \pi^2\ln(-p^2)\Big)+\\ &+\frac{\epsilon}{12p^2}\Big(6\gamma^2\pi^2 - \pi^4 + 12\gamma\pi^2\ln(\pi) + 6\pi^2\ln^2(\pi)+\\ &+12\gamma\pi^2\ln(-p^2) + 12\pi^2\ln(\pi)\ln(-p^2) + 6\pi^2\ln^2(-p^2)\Big) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{12.3}$$

$$\begin{aligned}I(2-2\epsilon,1,1,p^2) = {}& \frac{2\pi}{p^2\epsilon} - \frac{1}{p^2}\Big(2(\gamma\pi + \pi\ln(\pi) + \pi\ln(-p^2))\Big)+\\ &+\frac{\epsilon}{6p^2}\Big(6\gamma^2\pi - \pi^3 + 12\gamma\pi\ln(\pi) + 6\pi\ln^2(\pi)+\\ &+12\gamma\pi\ln(-p^2) + 12\pi\ln(\pi)\ln(-p^2) + 6\pi\ln(-p^2)^2\Big) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{12.4}$$

we realize that they indeed are transcendetally uniform! Moreover we observe that $I(D - 2, 1, 1, p^2) = 2I(D, 2, 1, p^2)$, as one could also realize from the closed form expression in eq.(12.1).

Finiteness in the $\epsilon \to 0$ limit and uniform transencentality are the properies that we require to the canonical basis of MI's. As discussed in Chapter 10, we will construct them by searching for integrals that obeys differential equations in canonical form where the $\epsilon = (4-D)/2$ dependance is factorized from the kinematic.

Massless Bubbles

The s-channel bubble in the Laporta basis $I_1(D,1,1)$ obeys the following differential equation

$$\frac{d}{ds}I_1(D,1,1) = \frac{D-4}{2s}I_1(D,1,1) = -\frac{\epsilon}{s}I_1(D,1,1)\ . \tag{12.5}$$

Although it is in canonical form, we know from (12.2) that this Integral is not homogenous in transcendentality. Therefore we discard it, and consider, instead, $I_1(D,2,1)$, obeying the differential equation

$$\frac{d}{ds}I_1(D,2,1) = \frac{D-6}{2s}I_1(D,2,1) \tag{12.6}$$

This equation can be compared with the one of the Laporta integral $I_1(D',1,1)$ with shifted dimensions $D' \to D+d$

$$\frac{d}{ds}I_1(D',1,1) = \frac{D+d-4}{2s}I_1(D',1,1)\ . \tag{12.7}$$

Through this comparison we see that the Laporta Integral with a shifted Dimension $D' = D-2$ obeys the same differential equation as the one of $I_1(D,2,1)$,

$$\frac{d}{ds}I_1(D',1,1) = \frac{D-6}{2s}I_1(D',1,1) = \frac{-(1+\epsilon)}{s}I_1(D',1,1) \tag{12.8}$$

We notice that the latter equation is not canonical, but is linear in ϵ. This property is suitable for the application of the Magnus exponential. Moreover, from eq.12.4, we know that $I_1(D',1,1)$ is not finite, as its Laurent expansion in ϵ begin with a single pole. In order to have a completely finite integral, we need to multiply this integral by an ϵ prefactor. By this consideration, we arrive at the following final definition of a bubble integral

$$f_1 \equiv \epsilon I_1(D-2,1,1) = 2\epsilon I_1(D,2,1)\ , \tag{12.9}$$

which is finite when ϵ vanishes, and it has uniform transcendentality. Using IBP-id's,

$$I_1(D-2,1,1) = 2I_1(D,2,1) = -2\frac{D-3}{s}I_1(D,1,1) \tag{12.10}$$

we can relate the Laporta MI $I_1(D,1,1)$ to the new integral f_1. The massless bubble in the t- an m^2-channel follow from analogous considerations.

12.1.2 The 1-mass Box

The Differential Equation for the one-mass Box integral in the Laporta basis has the following form,

$$
\begin{aligned}
\frac{d}{ds} I_4(D,1,1,1,1) = & \frac{2(D-3)}{s(m^2-s)(m^2-s-t)} I_1(D,1,1)+ \\
& + \frac{2(D-3)}{st(m^2-s-t)} I_2(D,1,1)- \\
& - \frac{2(D-3)}{m^4 s(m^2-s)(m^2-s-t)} I_3(D,1,1)- \\
& - \frac{(D-6)(m^2-t)-2s}{2s(m^2-s-t)} I_4(D,1,1,1,1) \ .
\end{aligned}
\tag{12.11}
$$

By using eq.(12.10), we replace the Laporta bubbles $I_i (i = 1, 2, 3)$ with the corresponding $f_i (i = 1, 2, 3)$,

$$
\begin{aligned}
\frac{d}{ds} I_4(D,1,1,1,1) = & -\frac{2}{\epsilon(m^2-s)(m^2-s-t)} f_1 - \frac{2}{\epsilon s(m^2-s-t)} f_2+ \\
& + \frac{2}{\epsilon m^2 s(m^2-s)(m^2-s-t)} f_3- \\
& - \frac{(D-6)(m^2-t)-2s}{2s(m^2-s-t)} I_4(D,1,1,1,1)
\end{aligned}
\tag{12.12}
$$

We can remove the ϵ poles, by multiplying the equation byϵ^2 and, and by defining the new box function f_4 as,

$$
f_4 \equiv \epsilon^2 I_4(D,1,1,1,1) \ . \tag{12.13}
$$

Therefore, the differential equation for f_4 reads

$$
\begin{aligned}
\frac{d}{ds} f_4 = & -\frac{2\epsilon}{(m^2-s)(m^2-s-t)} f_1 - \frac{2\epsilon}{s(m^2-s-t)} f_2+ \\
& + \frac{2\epsilon}{m^2 s(m^2-s)(m^2-s-t)} f_3 - \frac{(D-6)(m^2-t)-2s}{2s(m^2-s-t)} f_4
\end{aligned}
\tag{12.14}
$$

We notice that in this differential equation the coefficients of the f_i's contain terms that are linear ϵ.

12.2 System of Differential Equations

The f_i with $i = 1, \ldots, 4$ can be defined as the *linear-*ϵ basis, because the matrix associated to their system of differential equations is linear in ϵ. The basis f_i depend on the invariants s, t, u related by $s + t + u = m^2$. We can introduce new variables x, y [74]

$$
x = \frac{s}{m^2} \qquad\qquad y = \frac{t}{m^2} \tag{12.15}
$$

The basis of MI can be cast in a vector f, defined

$$f(x,y,\epsilon) = \begin{pmatrix} \epsilon I_1(D,2,1) \\ \epsilon I_2(D,2,1) \\ \epsilon I_3(D,2,1) \\ \epsilon^2 I_4(D,1,1,1,1) \end{pmatrix} \tag{12.16}$$

which obey the following differential equations

$$\partial_x f(x,y,\epsilon) = A_x(x,y,\epsilon) f(x,y,\epsilon) \qquad \partial_y f(x,y,\epsilon) = A_y(x,y,\epsilon) f(x,y,\epsilon) \tag{12.17}$$

where

$$A_x(x,y,\epsilon) = \begin{pmatrix} -\frac{1}{x} - \frac{\epsilon}{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2\epsilon}{m^2(1-x)(1-x-y)} & -\frac{2\epsilon}{m^2 x(1-x-y)} & \frac{2\epsilon}{m^2(1-x)x(1-x-y)} & -\frac{1}{x} - \frac{(1-y)\epsilon}{x(1-x-y)} \end{pmatrix} \tag{12.18}$$

$$A_y(x,y,\epsilon) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{y} - \frac{\epsilon}{y} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2\epsilon}{m^2 y(1-x-y)} & -\frac{2\epsilon}{m^2(1-y)(1-x-y)} & \frac{2\epsilon}{m^2(1-y)y(1-x-y)} & -\frac{1}{y} - \frac{(1-x)\epsilon}{y(1-x-y)} \end{pmatrix} \tag{12.19}$$

Both systems are linear in ϵ and in both cases the $O(\epsilon^0)$ term is diagonal,

$$A_\sigma(x,y,\epsilon) = A_{\sigma,0}(x,y) + \epsilon A_{\sigma,1}(x,y) \qquad \sigma = x, y \tag{12.20}$$

The systems can be brought in the *canonical form* by performing the transformation

$$f(\epsilon,x,y) = B_0(x,y) g(\epsilon,x,y) \tag{12.21}$$

where B_0 can is obtained by a (double) Magnus exponential,

$$B_0(x,y) = e^{\int_{x_0}^{x} d\tau D_{x,0}(\tau,y)} e^{\int_{y_0}^{y} d\tau D_{y,0}(x,\tau)} \ . \tag{12.22}$$

and reads,

$$B_0(x,y) = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0 \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{xy} \end{pmatrix} \tag{12.23}$$

The new basis g fulfills the canonical systems

$$\partial_x g(\epsilon,x,y) = \epsilon \hat{A}_{x,1}(x,y)\ g(\epsilon,x,y)\ , \qquad \partial_y g(\epsilon,x,y) = \epsilon \hat{A}_{y,1}(x,y)\ g(\epsilon,x,y)\ , \tag{12.24}$$

with

$$\hat{A}_{x,1}(x,y) = \begin{pmatrix} -\frac{1}{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{m^2(1-x)} - \frac{2}{m^2(1-x-y)} & -\frac{2}{m^2(1-x-y)} & -\frac{2}{m^2(1-x)} + \frac{2}{m^2(1-x-y)} & -\frac{1}{x} - \frac{1}{1-x-y} \end{pmatrix} \tag{12.25}$$

$$\hat{A}_{y,1}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{y} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{m^2(1-x-y)} & \frac{2}{m^2(1-y)} - \frac{2}{m^2(1-x-y)} & -\frac{2}{m^2(1-y)} + \frac{2}{m^2(1-x-y)} & -\frac{1}{y} - \frac{1}{1-x-y} \end{pmatrix} \tag{12.26}$$

The two systems of DE's in Eq.(12.24) can be combined in a full differential form, along the lines of Ref. [75],

$$dg(\epsilon, x, y) = \epsilon \; d\hat{\mathcal{A}}_1(x,y) \; g(\epsilon, x, y) \; , \tag{12.27}$$

where the matrix $\hat{\mathcal{A}}_1$ fulfills the relations,

$$\partial_x \hat{\mathcal{A}}_1(x,y) = \hat{A}_{x,1}(x,y) \; , \qquad \partial_y \hat{\mathcal{A}}_1(x,y) = \hat{A}_{y,1}(x,y) \; . \tag{12.28}$$

and the integrability condition

$$\epsilon \left(\partial_x \partial_y \hat{\mathcal{A}}_1(x,y) - \partial_y \partial_x \hat{\mathcal{A}}_1(x,y) \right) + \epsilon^2 \left[\partial_x \hat{\mathcal{A}}_1(x,y), \partial_y \hat{\mathcal{A}}_1(x,y) \right] = 0 \; . \tag{12.29}$$

The matrix $\hat{\mathcal{A}}_1$ is logarithmic in the variables x and y,

$$\begin{aligned} \hat{\mathcal{A}}_1(x,y) =& M_1 \log(x) + M_2 \log(1-x) + M_3 \log(y) + M_4 \log(1-y) + \\ &+ M_5 \log(1-x-y) \; , \end{aligned} \tag{12.30}$$

with

$$M_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{m^2} & 0 & \frac{2}{m^2} & 0 \end{pmatrix},$$
$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{m^2} & \frac{2}{m^2} & 0 \end{pmatrix},$$
$$M_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{m^2} & \frac{2}{m^2} & -\frac{2}{m^2} & 0 \end{pmatrix}, \tag{12.31}$$

. The solution of the system (12.24) can be written as a Dyson series in *epsilon*. Each coefficient of the series, namely each order in *epsilon* contains 2dim-HPL's of uniform weight.

12.2.1 General solution

To find solution of eq. (12.24), we solve first the system in x, up to an unknown function of y, which is fixed by imposing the fulfillemnt of the system of differential equation in y. Pure constant terms will be fixed at the end imposing boundary conditions. Using eq. (10.16) and (12.24) we write the solution in a more explicit way

$$g(\epsilon, x) = (1 + \epsilon B_1^{(1)}(x) + \epsilon^2 B_1^{(2)}(x) + \epsilon^3 B_1^{(3)}(x) + O(\epsilon^4))g_0(\epsilon) \tag{12.32}$$

where $B_1^{(n)}$ is the n^{th}-order term of in the ϵ-expansion of the general solution, and $g_0(\epsilon)$ is a constant vector that contains all integrations constants; its explicit expansion is

$$g_0(\epsilon) = g_{00} + \epsilon g_{01} + \epsilon^2 g_{02} + \dots \tag{12.33}$$

We present the general solutions in terms of the matrices $B_1^{(n)}$, obtained integrating the matrix $\hat{A}_{x,1}(x, y)$ in x (by indefinite integration) and then imposing that the primitive is also solution of the system in y (using matrix $\hat{A}_{y,1}(x, y)$).

The first-order matrix is,

$$B_1^{(1)}(x, y) = \begin{pmatrix} -\mathrm{G}(\{0\}, x) & 0 & 0 & 0 \\ 0 & -\mathrm{G}(\{0\}, y) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2\mathrm{G}(1-y;x)}{m^2} - \frac{2\mathrm{G}(1;x)}{m^2} + \frac{2\mathrm{G}(1;y)}{m^2} & \frac{2\mathrm{G}(1-y;x)}{m^2} & \frac{2\mathrm{G}(1;x)}{m^2} - \frac{2\mathrm{G}(1-y;x)}{m^2} & \mathrm{G}(1-y;x) - \mathrm{G}(0;x) - \mathrm{G}(0;y) + \mathrm{G}(1;y) \end{pmatrix} \tag{12.34}$$

The non vanishing elements of the second-order matrix are

$$\begin{aligned}
\left[B_1^{(2)}(x, y)\right]_{11} &= G[0, 0; x], \\
\left[B_1^{(2)}(x, y)\right]_{22} &= G[0, 0; y], \\
\left[B_1^{(2)}(x, y)\right]_{41} &= \frac{1}{m^2}\Big\{-2G[0; x]G[1 - x; y] - 2G[0, 1 - x; y] + 2G[1 - x, 1 - x; y]\Big\}, \\
\left[B_1^{(2)}(x, y)\right]_{42} &= \frac{1}{m^2}\Big\{-2G[0; y]G[1; x] + 2G[1; x]G[1 - x; y] - 2G[0, 1; x]+ \\
&\quad + 2G[0, 1; y] - 2G[0, 1 - x; y] + 2G[1, 0; y] + 2G[1, 1; x]- \\
&\quad - 2G[1 - x, 0; y] - 2G[1 - x, 1; y] + 2G[1 - x, 1 - x; y]\Big\}, \\
\left[B_1^{(2)}(x, y)\right]_{43} &= \frac{1}{m^2}\Big\{-2G[0, 1; y] + 2G[0, 1 - x; y] + 2G[1 - x, 1; y] - 2G[1 - x, 1 - x; y]\Big\} \\
\left[B_1^{(2)}(x, y)\right]_{44} &= -G[0; y]G[1; x] + G[0; x](G[0; y] - G[1 - x; y])+ \\
&\quad G[1; x]G[1 - x; y] + G[0, 0; x] + G[0, 0; y] - G[0, 1; x] - G[0, 1 - x; y] \\
&\quad - G[1, 0; x] + G[1, 1; x] - G[1 - x, 0; y] + G[1 - x, 1 - x; y]
\end{aligned} \tag{12.35}$$

The non vanishing elements of the third-order matrix are

$$\left[B_1^{(3)}(x,y)\right]_{11} = -G[0,0,0;x]$$
$$\left[B_1^{(3)}(x,y)\right]_{22} = -G[0,0,0;y]$$
$$\left[B_1^{(3)}(x,y)\right]_{41} = \frac{1}{m^2}\Big\{2G[1-x;y]G[0,0;x] + G[0;x]\left(2G[0,1-x;y] - 2G[1-x,1-x;y]\right) + 2G[0,0,1-x;y] - 2G[0,1-x,1-x;y] - 2G[1-x,0,1-x;y] + 2G[1-x,1-x,1-x;y]\Big\}$$
$$\left[B_1^{(3)}(x,y)\right]_{42} = \frac{1}{m^2}\Big\{-2G[1-x;y]G[0,1;x] + 2G[1-x;y]G[1,1;x] + G[0;y]\Big(2G[0,1;x] - 2G[1,1;x]\Big) + G[1;x]\Big(2G[0,0;y] - 2G[0,1-x;y] - 2G[1-x,0;y] + 2G[1-x,1-x;y]\Big) + 2G[0,0,1;x] - 2G[0,0,1;y] + 2G[0,0,1-x;y] - 2G[0,1,0;y] - 2G[0,1,1;x] + 2G[0,1-x,0;y] + 2G[0,1-x,1;y] - 2G[0,1-x,1-x;y] - 2G[1,0,0;y] - 2G[1,0,1;x] + 2G[1,1,1;x] + 2G[1-x,0,0;y] + 2G[1-x,0,1;y] - 2G[1-x,0,1-x;y] + 2G[1-x,1,0;y] - 2G[1-x,1-x,0;y] - 2G[1-x,1-x,1;y] + 2G[1-x,1-x,1-x;y]\Big\}$$
$$\left[B_1^{(3)}(x,y)\right]_{43} = \frac{1}{m^2}\Big\{2G[0,0,1;y] - 2G[0,0,1-x;y] - 2G[0,1-x,1;y] + 2G[0,1-x,1-x;y] - 2G[1-x,0,1;y] + 2G[1-x,0,1-x;y] + 2G[1-x,1-x,1;y] - 2G[1-x,1-x,1-x;y]\Big\}$$
$$\left[B_1^{(3)}(x,y)\right]_{44} = G[1-x;y]G[0,0;x] + G[1;x]G[0,0;y] + G[0;y](-G[0,0;x] + G[0,1;x] + G[1,0;x] - G[1,1;x]) + G[1-x;y](-G[0,1;x] - G[1,0;x] + G[1,1;x]) + G[0;x](-G[0,0;y] + G[0,1-x;y] + G[1-x,0;y] - G[1-x,1-x;y]) + G[1;x](-G[0,1-x;y] - G[1-x,0;y] + G[1-x,1-x;y]) - G[0,0,0;x] - G[0,0,0;y] + G[0,0,1;x] + G[0,0,1-x;y] + G[0,1,0;x] - G[0,1,1;x] + G[0,1-x,0;y] - G[0,1-x,1-x;y] + G[1,0,0;x] - G[1,0,1;x] - G[1,1,0;x] + G[1,1,1;x] + G[1-x,0,0;y] - G[1-x,0,1-x;y] - G[1-x,1-x,0;y] + G[1-x,1-x,1-x;y] \quad (12.36)$$

Our result can be iterated to the desired order in ϵ. We note that, the solution is homogeneous in trascendetality, namely the coefficients of the ϵ expansions contain G-functions of the same weight.

12.2.2 Boundary conditions

To solve the Cauchy problem we have to consider the boundary conditions of Master Integrals. In the following we factor out an overall normalization term,

$$\frac{\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{\Gamma[1-2\epsilon]}(\pi)^{2-\epsilon} \tag{12.37}$$

for each g_i, to simplify the algebra.

The functions g_i with $i = 1,2,3$ can be obtained trivially from eq. (12.1). Therefore, we can focus on the box function g_4. The boundary condition for g_4 can be extracted directly from the differential equation of the box, imposing the regularity in the $u \to 0$ limit [72]. which can be imposed directly on the differential equation in x,

$$\begin{aligned}\partial_x g_4(x,y,\epsilon) = \epsilon\Big(&-\frac{2g_1(x,\epsilon)}{m^2(-1+x)} + \frac{2g_1(x,\epsilon)}{m^2(-1+x+y)} + \frac{2g_2(y,\epsilon)}{m^2(-1+x+y)} + \\ &+ \frac{2g_3(m^2,\epsilon)}{m^2(-1+x)} - \frac{2g_3(m^2,\epsilon)}{m^2(-1+x+y)} - \frac{g_4(x,y\epsilon)}{x} + \frac{g_4(x,y\epsilon)}{-1+x+y}\Big)\end{aligned} \tag{12.38}$$

In fact, since $u = 1-x-y$, we can multiply the left and the right hand side of (12.38) by $1-x-y$ and take the limit $x \to 1-y$. That left side vanishes, while and from the right side we have,

$$\frac{2g_1(1-y,\epsilon) + 2g_2(y,\epsilon) - 2g_3(m^2,\epsilon) + m^2 g_4(1-y,y\epsilon)}{m^2} = 0 \tag{12.39}$$

from which we derive the boundary condition for g_4 at $u = 0$ ($x = 1-y$), reading as

$$g_4(x,y,\epsilon)\Big|_{x=1-y} = \frac{-2g_1(1-y,\epsilon) - 2g_2(y,\epsilon) + 2g_3(m^2,\epsilon)}{m^2}\Big|_{x=1-y} \tag{12.40}$$

which relates g_4 to the known expressions of g_i $i = 1,2,3$. The conditions in (12.40) hold order by order in ϵ expansion, hence they can be used to fix the arbitrary integration constants of g_4. At the ϵ^0-order, we have

$$g_{00}(x,y) = \begin{pmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \\ g_4^{(0)} \end{pmatrix} \tag{12.41}$$

Terms $g_1^{(0)}$, $g_2^{(0)}$, $g_3^{(0)}$ are known from (12.1) and are

$$g_1^{(0)} = g_2^{(0)} = g_3^{(0)} = -\frac{1}{m^2}\ . \tag{12.42}$$

Using (12.40) we obtain,

$$g_4^{(0)} = \frac{2}{m^4} \tag{12.43}$$

The boundary condition at the ϵ^1-order can be read from eq. (12.32) and (12.33) as

$$\left(B_1^{(1)}(x,y)g_{00} + g_{01}\right)\epsilon \ . \tag{12.44}$$

g_{00} is now completely known, and the first three entries of g_{01} can be extracted from the bubble's expansion, as above. So considering (12.46) and imposing boundary conditions we have

$$g_4^{(1)} = -2\frac{\ln(-m^2)}{m^2} \tag{12.45}$$

The algorithm can be iterated, and at the ϵ^2-order one has,

$$\left(B_1^{(2)}(x,y)g_{00} + B_1^{(1)}(x,y)g_{01} + g_{02}\right)\epsilon \tag{12.46}$$

yielding

$$g_4^{(2)} = \frac{-\pi^2 + 3\ln\left(-m^2\right)^2}{3m^4} \tag{12.47}$$

Finally, we can write the complete solution of the $g_4(\epsilon,x,y)$ up to $O(\epsilon^3)$,

$$\begin{aligned} g_4(\epsilon,x,y) = {}& \frac{2}{m^4} - \frac{2\left(\ln\left[-m^2\right] + \mathrm{G}[0;x] + \mathrm{G}[0;y]\right)}{m^4}\epsilon + \\ & + \Big(-\frac{\pi^2}{3m^4} + \frac{2\mathrm{G}[0;x]\mathrm{G}[0;y]}{m^4} + \frac{2\mathrm{G}[0,0;x]}{m^4} + \frac{2\mathrm{G}[0,0;y]}{m^4} - \frac{2\mathrm{G}[1,0;x]}{m^4} - \\ & - \frac{2\mathrm{G}[1,0;y]}{m^4} + \frac{2\mathrm{G}[0;x]\ln\left[-m^2\right]}{m^4} + \frac{2\mathrm{G}[0;y]\ln\left[-m^2\right]}{m^4} + \\ & - \frac{\ln\left[-m^2\right]^2}{m^4}\Big)\epsilon^2 + O(\epsilon^3) \end{aligned} \tag{12.48}$$

To compare this result with literature [64], we have to invert the relation $I_4 = \epsilon^2 f_4 = \sum_{i=1}^4 (B_0)_{4i} g_i$. From (12.13) (12.22) and (12.23), we get

$$I_4(\epsilon,x,y) = \frac{1}{\epsilon^2}\frac{1}{xy}g_4(\epsilon,x,y) \ . \tag{12.49}$$

We also restore s and t variables, and recall the expressions of Gs in terms of logarithms, so that

$$\begin{aligned} I_4(\epsilon,s,t) = {}& \frac{2}{st\epsilon^2} + \frac{2\left(\ln\left(-m^2\right) - \ln(-s) - \ln(-t)\right)}{st}\frac{1}{\epsilon} + \\ & + \frac{1}{3st}\Big(-\pi^2 - 3\ln^2\left(-m^2\right) + 3\ln^2(-s) - 3\ln^2\left(\frac{s}{t}\right) + 3\ln^2(-t) - \\ & - 6\mathrm{Li}_2\left(1 - \frac{m^2}{s}\right) - 6\mathrm{Li}_2\left(1 - \frac{m^2}{t}\right)\Big) + \\ & + \frac{\left(\ln^3\left(-m^2\right) - \ln^3(-s) - \ln^3(-t)\right)}{3st}\epsilon + \\ & + \frac{\left(-\ln^4\left(-m^2\right) + \ln^4(-s) + \ln^4(-t)\right)}{12st}\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned} \tag{12.50}$$

which agrees with the expression in [64], used in (8.16a)

12.2.3 The 1-mass Box exact in D-dimension

The method used in the previous section is general and very useful specially in multiloop calculations where integrals is more complicated and series expansions is the only way to obtain a result. However in this a case we can solve analitically the differential equation for the box. Let's rewrite differential equation for g_4,

$$\partial_x g_4(x,y,\epsilon) = \epsilon\Big(-\frac{2g_1(x,\epsilon)}{m^2(-1+x)} + \frac{2g_1(x,\epsilon)}{m^2(-1+x+y)} + \frac{2g_2(y,\epsilon)}{m^2(-1+x+y)} + \\ + \frac{2g_3(m^2,\epsilon)}{m^2(-1+x)} - \frac{2g_3(m^2,\epsilon)}{m^2(-1+x+y)} - \frac{g_4(x,y,\epsilon)}{x} + \frac{g_4(x,y,\epsilon)}{-1+x+y}\Big) \tag{12.51}$$

Equation (12.51) is a linear, inhomogeneous first order differential equation of the form

$$\frac{\partial y(x)}{\partial x} + f(x)y(x) = g(x) \tag{12.52}$$

which can be solved by introducing an integrating factor

$$M(x) = e^{\int f(x)dx} \tag{12.53}$$

such that $y(x) = 1/M(x)$ solves the homogenous differential equation ($g(x) = 0$). This yields the general solution of the inhomogeneous equation as

$$y(x) = \frac{1}{M(x)}\left(\int g(x)M(x)dx + C\right) \tag{12.54}$$

where the integration constant C can be adjusted to match the boundary conditions. The bubble diagrams in the inhomogeneous term can be written in a short form

$$g_1(x,\epsilon) = A_2^{(1-loop)}(x)^{-\epsilon}, \quad g_2(y,\epsilon) = A_2^{(1-loop)}(y)^{-\epsilon}, \quad g_3(m^2,\epsilon) = A_2^{(1-loop)} \tag{12.55}$$

with

$$A_2^{(1-loop)} = \frac{(-1)^{-\epsilon}m^{-2-2\epsilon}\epsilon\Gamma\left[-2+\frac{1}{2}(4-2\epsilon)\right]\Gamma\left[-1+\frac{1}{2}(4-2\epsilon)\right]\Gamma\left[3+\frac{1}{2}(-4+2\epsilon)\right]}{\Gamma[1-\epsilon]^2\Gamma[1+\epsilon]} \tag{12.56}$$

integrating factor holds

$$M(x) = x^{\epsilon}(1-x-y)^{-\epsilon}y^{\epsilon} \tag{12.57}$$

where is imposed that the solution solve the system in y also Integrating the differential solution we obtain the solution in terms of hypergeometric functions

$$g_4(x,y,\epsilon) = -\frac{1}{m^2}2A_2^{(1-loop)}yx^{-\epsilon}(1-x-y)^{\epsilon}y^{-\epsilon}\epsilon \\ \Big(y^{-1}(-1+x+y)^{-\epsilon}B\Big[-\epsilon,1\Big]{}_2F_1\Big[1,-\epsilon,1-\epsilon,-\frac{1-x-y}{y}\Big]+ \\ + x^{1+\epsilon}(-1+y)^{-1-\epsilon}y^{-1-\epsilon}B\Big[1+\epsilon,1\Big]{}_2F_1\Big[1+\epsilon,1+\epsilon,2+\epsilon,\frac{x}{1-y}\Big]- \\ - (-\frac{x}{(1-x)(1-y)})^{1+\epsilon}B\Big[1+\epsilon,1\Big]{}_2F_1\Big[1+\epsilon,1+\epsilon,2+\epsilon,\frac{xy}{(1-x)(1-y)}\Big]\Big)\ , \tag{12.58}$$

where the parametric integral expressions of the ${}_2F_1$ is

$$
{}_2F_1[a,b,c;z] \equiv \frac{\Gamma[c]}{\Gamma[b]\Gamma[b-c]} \int_0^1 \frac{t^{b-1}t^{c-b-1}}{(1-tz)^a} dt \tag{12.59}
$$

and B is the beta function (4.7).

Chapter 13

Conclusion

We elaborated on the newest developments concerning the Feynman integral calculus for scattering amplitudes in gauge theory, and we applied these novel methods to the evaluation of the Higgs boson plus one-jet production in gluon fusion scattering amplitudes at one-loop level.

The main focus of the thesis has been the interplay of generalized-unitarity cuts in d-dimensions, for achieving the decomposition of the one-loop amplitudes in terms of a basis of Master Integrals (MI's), and of the differential equations method, for their actual determination. In particular, we exploited the generalized-cuts definition within the recently proposed four-dimensional-formulation (FDF) of the dimensional regularization scheme, in order to determine the coefficients of the MI's. Later we addressed the problem of solving the system of differential equations obeyed by the MI's, using Magnus series expansion.

We showed the potential of those methods in simplifying the heavy computational load usually required in the perturbative diagrammatic expansion.

We began from the color decomposition of the one amplitudes contributing to $gg \to Hg$, yileding the identification of the minimal set of two independent color ordered primitive amplitudes. Then we introduced the spinor-helicity formalism to simplify the algebraic manipulation of the kinematic variables. We described the generalised unitarity top-down algorithm, and gave the definition of the novel FDF diagrammatic rules, for the QCD couplings and for the effective coupling of the Higgs boson and gluon, in the infinite top-mass approximation. The determination of the coefficients of the MI's has been achieved with the powerful feature of generalized unitarity in FDF, which has the advantage of requiring purely four-dimensional ingredients, such as generalized spinors and polarization states, hence avoiding the complication of either extended Dirac algebra or higher-dimension spinor integration. The successful calculation of one-loop amplitudes for $gg \to Hg$, in an effective theory, has been a non trivial test for the FDF scheme.

The problem of Master Integrals evaluation is a fundamental question in scattering amplitude calculation. In this thesis we studied and applied the novel idea of exploiting the arbitrariness in the choice of the basis of MI's, in order to derive *simpler* systems of differential equation that can be solved by using Magnus exponential. Starting from the set of MI's identified with the Laporta algorithm, we built a system of differential equations obeying a canonical form, where the dependence on the dimensional parameter is completely factorized from the kinematic. Having cast in this form, the solution can be written as a Dyson series in $\epsilon = (4-d)/2$, in terms of repeated integrals, called generalized harmonic polylogarithm, where the coefficient of each order contains functions and constants of uniform weight.

The results obtained for the scattering amplitudes of $gg \to Hg$ agree with the result known from the literature.

The techniques and mathematical tools described in this thesis, color decomposition, spinor formalism, d-dimensional generalized-unitarity within FDF, integrand decomposition, differential equations, generalized harmonic polylogarithm, and Magnus series expansion are not limited to one-loop calculations, and have the potential to influence, if not to determine, a breakthrough in the ability of multi-loop calculation, as they did for the next-to-leading case.

With the increasing accuracy of the next experimental programme at the CERN LHC, *precision* is compulsory, both for studying the properties of the Higgs boson, and to reveal discrepancies with respect to Standard Model predictions that could indicate a door to New Physics. In this thesis we have studied, presented and applied some of the mathematical methods that can be useful to answer to the request of precision calculations that experimental particle physics forwards to the theoretical side.

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