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Final Dissertation

The de Sitter conjecture and the string theory swampland

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## Contents

1 Introduction ..... 3
2 The string theory swampland ..... 5
2.1 The swampland and the landscape ..... 5
2.2 The conjectures ..... 6
2.2.1 Weak Gravity Conjecture ..... 8
2.2.2 Swampland Distance Conjecture ..... 9
2.2.3 The AdS stability conjecture ..... 9
2.2.4 The dS conjecture ..... 11
3 KK reduction and quantum potential ..... 13
3.1 Kaluza-Klein reduction ..... 13
3.1.1 Circle $S^{1}$ case ..... 14
3.1.2 Torus $T^{2}$ case ..... 16
3.2 1-loop quantum potential ..... 23
3.2.1 Massive scalar case ..... 24
3.2.2 Massive vector case ..... 27
4 Application to the SM ..... 31
$4.1 \quad S^{1}$ case ..... 31
$4.2 \quad T^{2}$ case ..... 33
4.2.1 Vacuum conditions ..... 35
5 SM constraints from swampland conjectures ..... 43
5.1 3D case ..... 44
5.1.1 Only neutrinos ..... 44
5.1.2 Neutrinos + Axion ..... 46
5.2 2D case ..... 47
5.2.1 Only neutrinos ..... 47
5.2.2 Neutrinos + Axion ..... 50
5.2.3 Majorana neutrinos - NH and IH ..... 51
6 Conclusions ..... 55

## Chapter 1

## Introduction

At a fixed energy scale the number of equivalent effective theories at our disposal to describe physical processes can be remarkable. A way that has been used as a guide to choose among features equivalent at a certain energy scale is the behaviour at high energies, in the ultraviolet (UV), of the same effective theories. For example renormalizability has been widely used, namely only theories whose UV divergences can be absorbed by appropriate counterterms are considered. More recently the compatibility with quantum gravity is being used, i.e. those theories that, after the coupling to gravity, present features incompatible with the known theories of quantum gravity are discarded.

The quantum gravity theory usually used for the compatibility method is string theory because it is the only quantum gravity theory that has passed all the required tests until now. It is a theory that, in the supersymmetric case, is defined in 11 dimensions that need to be decreased to the classic 4. This is performed compactifying the extra dimensions, but the great variety of ways to do it produces a landscape of possible different effective theories.

To such landscape of theories descending from string theory we can counterpose the set of all the remaining low energy theories that do not originate from it: the swampland. This gives the name Swampland program to the research aimed to select between different effective theories using the compatibility with quantum gravity. The main idea behind it is to use some criteria, called swampland conjectures, to distinguish between effective theories that are completable into string theory and those that are not. This method is explained in more details in the second chapter.

In this thesis we are going to analyze and use to swampland conjectures: the Anti de Sitter (AdS) conjecture and the de Sitter (dS) conjecture, these forbids respectively non-susy stable AdS vacua and stable (or mildly unstable) dS vacua, putting in the swampland all the effective theories that contain them.

String theory is not the only theory that, when compactfied to lower dimensions, originate a landscape of vacua. In fact even ordinary quantum field theories do it when coupled to gravity, for example in [1] it was shown that compactifying the Standard Model (SM) coupled to gravity in 4 dimensions on a circle $S^{1}$ and on a torus $T^{2}$ creates a landscape of lower dimensional vacua. They are obtained through the 1-loop quantum corrections of the terms appearing after the compactification of the SM terms. Such corrections will create a potential for the scalar fields parameterizing the circle and the torus and its minima become the vacuum points. The potential and thus its vacua are mainly influenced by the lightest SM degrees of freedom, in particular their existence depends greatly on the values of the neutrino masses.

Later in $[2,3,4]$ the AdS conjecture was applied to the SM lower dimensional vacua. If a consistent dimensional reduction of a theory results to be incompatible with quantum gravity then also the higher dimensional one should be. This allowed to put constraints on the neutrino masses because only the that do not generate stable AdS vacua are permitted.

This thesis focuses on refining some of the previous results, following especially [3] which analyzed the
landscape of SM vacua and applied the AdS conjecture, with the application of the dS conjecture as well. In chapter 2 the swampland program as well as the conjectures we are going to use are presented and quickly reviewed. We show how they are derived and the other conjectures they originate from, the Weak Gravity Conjecture and the Distance Conjecture.

In the third chapter the dimensional reduction is explicitly performed on the Einstein-Hilbert action to obtain the kinetic terms for the fields parameterizing the compactified dimensions. Moreover, the quantum corrections generating the potential are presented and they are computed for two particular cases.

In chapter 4 the consistency of the dimensional truncation is checked and the vacuum conditions are derived. We look at the equations of motion originated from the compactified actions, we evaluate them at the background and eventually we perturb them. This gives the conditions that the potential needs to fulfill to have a stable vacuum.

In the fifth chapter we evaluate numerically the potential to check what are the neutrino masses that make rise to AdS or dS vacua. The analysis is performed for different cases, being the neutrino mass terms Dirac or Majorana or with Normal or Inverted Hierarchy.

Eventually in chapter 6 the work done is summed up and the results, in particular of chapter 5, are showed in a more schematic way.

## Chapter 2

## The string theory swampland


#### Abstract

The construction of a consistent quantum theory of gravity is one of the major goals in modern theoretical physics. This is not at all a simple task, even though many progresses are being made. One of the aspects that is currently being widely explored [5, 6, 7] consists in the discrimination of the theories that can be completed at high energies in a quantum gravity theory (in particular string theory) from those that can not. To achieve this goal the effort rotates around the development of some conjectures that can provide effective methods and constraints to use and impose on the analysed theories. Their name comes from the fact that they are not formally proven. In particular in the thesis the focus will be on the AdS stability conjecture [8] and the de Sitter conjecture $[9,10]$.


### 2.1 The swampland and the landscape

The modern approach to study quantum field theories is to treat them as Effective Field Theories, which describe the physics at a given energy scale, but may break down when higher energies are involved. These theories derive from complete theories eliminating (integrating out) the heaviest degrees of freedom: as we consider lower and lower energies some degrees of freedom become less important than others, therefore they can be integrated out to create a theory more suitable to describe the physics at those energy scales. There can be identical EFTs coming from different complete UV theories, because as the energy decreases, i.e. moving towards the infrared (IR), the irrelevant operators, which were dominating in the UV, will lose their importance.
What is depicted in the above paragraph is the general procedure to get an EFT from a UV complete theory. However we do not know what is the physics in the UV, thus we can only suppose how a UV should look like. Actually in physics the majority of the calculations is performed using EFTs and even the quantum field theory describing the Standard Model (SM) is thought as an EFT. Indeed a further hint of its EFT nature is that, when the SM is coupled to gravity, we know it is going to be inconsistent at high energies, and therefore, as any other EFT, we expect it to be completed by adding new degrees of freedom. However, without principles that could guide us, this becomes a very challenging task.
The use of EFTs is actually more unavoidable than it could transpire from the above paragraphs: we can not disregard them, because we do not know what is the physics at energy scales beyond the ones we are able to test in the experiments and therefore the possible UV completions of any EFT are numerous if not infinite. The EFTs become then an essential tool to carry out calculations and to make predictions. In fact a real plethora of EFTs for very disparate fields of physics has been developed throughout the years.
Given the great number of EFTs in the IR, we would need a mechanism to discriminate between them. That could be derived from the coupling of a theory in the UV to gravity since this imposes very binding constraints. Indeed among the various quantum theories of gravity only one has passed, at least until now, all the tests required: string theory. So we could expect that any consistent EFT should originate from it.
String theory is one of the most known and used candidate quantum theories of gravity. Nevertheless
it presents some problematic features when it comes to the application to the infrared physics. In the supersymmetric case it is a theory defined in 10 dimensions, thus we need a method for the dimensional reduction to get back to 4 dimensions. The number of possible vacua, and therefore different theories derivable from it, is huge because all the viable compactifications to get to a 4 D theory are numerous. In fact, for example, the number of different vacua in a particular scenario has been estimated to be of the order of $10^{27200}$ [11]. Due to these huge numbers the past approach based on the rigorous construction of an EFT from string theory has proven itself futile in charting all the possible theories originated from it. So it would appear that, if too many self consistent EFTs can be derived from string theory, it has little utility when the IR physics is involved: we could simply take any effective theory in agreement with the observations to perform calculations and it would be almost impossible to make unique predictions about the results of future experiments. Moreover, even if all the string theory vacua were known there is no clear selection mechanism to discriminate between them. Recently, to overcome such difficulties, there was a shift in the attitude of the community and a new method is being developed: the swampland program. It is based on the idea that not everything can descend from string theory, even though its landscape of vacua is huge, it is not infinite. Therefore not all the EFTs that get coupled to gravity can be UV completed in a consistent way.
The swampland idea consists in testing the self consistency of a certain theory when coupled to gravity. Self consistency can be a very good tool to fix theories, in particular at high energies, such as the ones occurring when gravity effects become not negligible. Here self consistency can be used to have almost a unique theory. Unfortunately, at low energies it is far less efficient as it can be seen by looking at the number of self consistent EFTs originating from string theory.
If a UV theory is known, then how to obtain an EFT from that is widely understood. Vice versa it is not at all trivial to reach a more complete theory, or more specifically a quantum gravity theory, from an effective one. An EFT is valid only below a certain cut-off $\Lambda$, when the energy is increased and reaches $\Lambda$ then the theory must be modified, for instance adding new degrees of freedom. But there can be several choices and so, from the same EFT, it is possible to get to very different UV completions. Besides, when gravity gains relevance, the theory is likely to break down unless the changes made to keep it consistent are very deep. This is not possible for all the EFTs. It is then from here that the idea of the swampland is born.

Definition 1. The Swampland is the set of effective field theories that do not fulfill the requirements to be consistently completed in a quantum gravity theory.

Definition 2. The Landscape consists of all the low energy theories that can descend from a quantum theory of gravity.

This is illustrated in Figure 2.1. These definitions remain pointless unless we can identify some criteria that allow to distinguish the theories that belong to the swampland from the ones that belong to the landscape. That is the reason why the main purpose of the swampland program is to build those criteria and to learn how to apply and take advantage of them.

### 2.2 The conjectures

In order to fulfill the task of discriminating the belonging of an EFT to the swampland or to the landscape a certain number of criteria has been proposed. They are called Swampland Conjectures because they have not been formally proven even if more and more evidence for their validity is being gathered. The conjectures can be roughly split in two categories: the string derived ones and the string inspired ones. The former consist in imposing some of the same characteristics of the known and rigorously constructed string theory vacua on the EFTs. So these type of criteria constrain the EFTs that present characteristics not consistent with the ones obtainable from string theory. The latter do not use directly the string theory vacua, but bind the EFTs on the basis of the structures and features that should appear at higher energies. From this it can be seen that the level of rigour of the swampland conjectures is somewhat subjective. For the string derived conjectures it could be questioned that maybe they would not satisfy other vacua that we are not technically able to


Figure 2.1: The swampland and the landscape in the theory space. The yellow area below the cone represents the set of theories that can derive from string theory.
construct to a certain level of rigour, while the string inspired conjectures are loosely proven for their own construction.

Since the conjectures are designed for a theory coupled to gravity, it means that when $M_{p} \rightarrow \infty$ (namely when gravity disappears) also the bounds of the conjectures must vanish. This should be true also in reverse: as gravity gets stronger, the bounds get stricter. This is the reason of the cone above the Landscape in Figure 2.1: as the energy increases the bounds lead towards a more limited number of UV completions, arriving in the end at a very restricted multiplicity of theories.

The conjectures have usually a common structure. As mentioned above the construction of an EFT valid at higher energies proceeds integrating in the new degrees of freedom needed to keep the theory consistent. This will give a new cut-off higher than the preceding one. The procedure continues until a cut-off that separates the quantum gravity description from the non-gravitational one is reached. To proceed further in a consistent way the EFT needs to be deeply modified. If this does not happen and then gravity is added, a new quantum gravity cut-off is generated and, if it lies below the normal cut-off of the theory, the EFT or, at least, its last modification used to increase the cut off will belong to the swampland. Clearly the most interesting situation appears to be the one where the new cut-off results to be very low, so that the entire theory must be discarded.
The conjectures are born exploiting very different aspects of string theory, therefore one would not expect them to be connected, but such connections are increasingly being found as it can be seen in Figure 2.2. Hence they could be different aspects of the same underlying and still unknown characteristic of string theory.
A big part of the effort of the swampland program is put not only in the development of the conjectures, but also in testing them. This is often done with string theory constructions, but the result remains limited to the particular setup chosen, like the type of compactification. The remedy is to use physics that is independent of the model like Black Hole physics, although this usually leads to weaker constraints.

In this thesis two conjectures are applied and presented: the Anti de Sitter (AdS) stability conjecture and de Sitter (dS) conjecture. However, in order to derive them, we need to cite and explain also the Weak Gravity Conjecture (WGC) and the Swampland Distance Conjecture (SDC).


Figure 2.2: Some of the swampland conjectures and the connections among them. Image from [6].

### 2.2.1 Weak Gravity Conjecture

This conjecture [12] starts with the idea that gravity is the weakest force and then turns it into a principle affirming that in any EFT consistent with quantum gravity there must exist an elementary particle whose repulsive gauge force is greater then their attractive gravity force. Formally it states that in a $U(1)$ gauge theory there is a particle that satisfies

$$
\begin{equation*}
\frac{q g}{m} \geq 1 \tag{2.1}
\end{equation*}
$$

in Planck units, where $q$ is the quantized charge, $g$ is the gauge coupling and $m$ is the mass.
A justification for this conjecture comes from black hole ( BH ) physics. A charged BH has to satisfy the condition

$$
\begin{equation*}
M \geq Q \tag{2.2}
\end{equation*}
$$

to avoid the appearance of a naked singularity. An extremal black hole saturates the inequality, namely it has $M=Q$. The bound follows from the analysis of their decay. Imagine that one of the products is another BH , this will have to fulfill the condition (2.2). Due to charge and mass conservation the other decay product must have $M \leq Q$ and therefore it can not be a black hole, but a particle, see figure 2.3. We get eventually at the formulation (2.1) of the Weak Gravity Conjecture.


Figure 2.3: The decay of an extremal BH , which creates another black hole respecting the extremality bound, will necessarily produce one particle with charge greater than its mass. Image from [6].

The conjecture can be generalized with the same line of reasoning to black branes and therefore to a theory in $d$ dimensions with a $p$-form [13]. In this case the WGC postulates the existence of a charged ( $p-1$ )-brane with tension $T$ and charge $Q$ that satisfies

$$
\begin{equation*}
\frac{p(d-p-2)}{d-2} T^{2} \leq Q^{2}\left(M_{p}^{d}\right)^{d-2} \tag{2.3}
\end{equation*}
$$

To get to the AdS instability conjecture we need to use a sharpened version of the WGC [8] that substitutes the inequality in equation (2.1) with a strict inequality in theories without supersymmetry.

This is justified saying that if the equality is satisfied at a certain point, only supersymmetry can protect it at all loop orders.

### 2.2.2 Swampland Distance Conjecture

This conjecture was first proposed in [14]. In string theory the magnitude of any parameter is controlled by the vacuum expectation value (vev) of a type of scalar fields called Moduli. They live in a metric space named moduli space. Modifying their values, namely moving in the moduli space, allows to change the values of the parameters of the theory.
In string theory all the symmetries are associated to gauge degrees of freedom and then to gauge couplings because global symmetries are forbidden [15]. The main heuristic argument for such prohibition is based on black hole physics. Let us consider a theory with a $U(1)$ global symmetry and a Schwarzschild black hole created putting together an arbitrary number of particles charged under that symmetry. The subsequent global charge will not affect the metric and so neither the black hole horizon. Semi-classically the black hole will lose mass through the Hawking radiation, but it will not lose its global charge. Thus deducing the charge of a black hole of known mass becomes impossible. We may associate an infinite entropy to such uncertainty because an infinite number of states can give rise to the same black hole geometry. However this would be in contradiction with the expectation that the entropy of a black hole is finite. Such inconsistency has led to the idea that no global symmetries should be present in a quantum gravity theory.
Given that moving in the moduli space modifies the values of the couplings, in principle we may reach regions of that space where a gauge coupling goes to zero, recovering a global symmetry. This happens because a symmetry that is originally global is made consistent in quantum gravity by coupling the respective current to some gauge field, but if the coupling vanishes then the global symmetry is restored. This is therefore problematic. The issue can be solved establishing the responsible point in moduli space at an infinite field distance so that it becomes not reachable. However, as long as we are approaching that point, its problematic nature should slowly be revealed. The SDC states that this is always the case when getting further in the moduli space in any direction as it is thought to be non compact, therefore any path would lead towards infinity. Hence moving in any direction would slowly and continuously break down the EFT description. Formally the conjecture is defined comparing the theory at a point $p$ and at a point $p_{0}$ on the moduli space with $d\left(p, p_{0}\right)>T$ for any positive $T$. In $p$ we have an infinite tower of light state with mass

$$
\begin{equation*}
m \sim e^{-\alpha T} \tag{2.4}
\end{equation*}
$$

with $\alpha>0$. Then in the limit $T \rightarrow \infty$ the number of state with mass smaller than a fixed value becomes infinite. This means that, when going to infinity in the moduli space, an infinite tower of light states appears at low energies invalidating the EFT since we should consider and integrate in all of those states, which is impossible.

### 2.2.3 The AdS stability conjecture

This conjecture was first introduced in [8] and states that no non-supersymmetric stable AdS vacua can exist in a consistent theory of quantum gravity. Therefore if such a vacuum appears in an EFT, then such a theory or, if possible, the values of the parameters that allowed its presence must be discarded. It is usually simple to create instabilities without supersymmetry, but it is only with the presence of gravity that this seems to happen all the time.
The way the AdS conjecture originates from the WGC involves the charges of the branes, and so their fluxes, in a particular theory. Let us consider a particle in a quantum field theory. Moving in time this particle will draw a path called world-line that in space-time is a one dimensional object. If the particle is charged it will couple to a 1-form $A_{\mu}$, namely a one dimensional potential. With $A_{\mu}$ we can build a field strength $F_{\mu \nu}$ and therefore an electromagnetic field which will generate a flux. If, instead of a particle, we had a string, the argument would be the same, but with a 2 -form $B_{\mu \nu}$ and a field strength $F_{\mu \nu \rho}$. If we had a three dimensional object instead of a string we would have a 3 -form and a
field strength with 4 indices and so on for a $p$ dimensional $p$-brane. Clearly, as the flux of the electric and magnetic fields in the particle case depends on the number of charges, i.e. of particles, the flux in higher dimension will depend on the number of branes.

The presence of the fluxes will create a backreaction on the geometry of space-time [16]. Therefore if we had an $A d S$ space, its curvature and so its cosmological constant would depend on the fluxes and consequently on the number of branes. This becomes more clear considering the example of an extremal Reissner-Nordström black hole in which the charge is equal to the mass. Here the Schwarzchild radius is proportional to the charge and in the near-horizon limit, due to the presence of the backreaction, the geometry becomes $A d S_{2} \times S^{2}$. Both the radius of $A d S$ and the one of the sphere depend on the total charge of the BH , thus also the cosmological constant of $A d S$ will depend on the charge. The same process happens for some configurations of $p$-branes and $A d S_{p+2}$.

The AdS conjecture works with a mechanism called fragmentation of AdS space [17], which works in an $A d S_{p+2}$ space supported by fluxes. As mentioned above the cosmological constant depends on the charges of the branes. Therefore if we created a couple brane-antibrane in the same way as we can create a couple particle-antiparticle and then the antibrane were annihilated on the other branes, we would be left in a situation like the one in Figure 2.4 with a portion of space surrounded by the brane.


Figure 2.4: In AdS the nucleation of a bubble with a $p$-brane as border with charge greater than tension forces the decay of the false vacuum of AdS to the true vacuum.

Inside the bubble whose border is the brane we have less flux than outside due to the brane itself that is charged and therefore even the cosmological constant inside would be greater than the one outside. This difference leads to an energy gap and, if the tension of the brane is not big enough to balance the discrepancy, the bubble will inflate until it will fill the entire space because the energy gained during the process is higher than the one needed to win the brane tension.

In the end, if we apply the WGC, we obtain, following [13], at least one $p$-brane that, using the refined version of the WGC, will have charge greater than its tension. Thus the fragmentation of the AdS space can happen and it will result in an instability because, when the bubble is nucleated, we can imagine that the system inside is decayed into the true vacuum. Then the reaching of the Anti de Sitter border by bubble will mark the decay from the false vaccum into the true vacuum of the entire space

### 2.2.4 The dS conjecture

The history of our universe can be split in three different eras: radiation domination, matter domination and dark energy domination which is the one we live in now because the $70 \%$ of the total energy is dark energy. The domination of the dark energy, i.e. of the cosmological constant, is characterized by an accelerated expansion of the universe, thus the beginning of this era has caused the shift of the metric from a Friedmann-Robertson-Walker one to a de Sitter one with a positive cosmological constant.
Hence our universe would seem to be approaching a de Sitter phase. Nevertheless the construction of dS vacua in string theory has proven to be harder than expected, indeed no attempt has been fully successful so far. This difficulty could be due to technical problems intrinsic in such a construction or it could be a hint of a deeper feature of quantum gravity that forbids somehow dS vacua.
Starting from the idea mentioned above, in [9] a new conjecture has been proposed. It takes the name of dS conjecture. The intention is to forbid dS vacua imposing an inferior bound to the magnitude of the gradient of the scalar potential in an EFT, such as

$$
\begin{equation*}
|\nabla V|>A \tag{2.5}
\end{equation*}
$$

for some $A>0$. But with this definition some problems arise since it forbids any kind of vacua, even $A d S$ and Minkowski ones. To address the issue the conjecture has been formally defined to be

$$
\begin{equation*}
M_{p} \frac{|\nabla V|}{V} \geq c \tag{2.6}
\end{equation*}
$$

where $c$ is a constant of order 1 . In this way the constant $A$ stops to be a constant and start depending on the scalar field $\phi$ whose potential is $V$. In this way $A d S$ vacua, for which $V$ is negative, and Minkwski vacua, for which $V=0$, are allowed. Moreover, as it should be for any conjecture, when gravity is lifted, namely $M_{p} \rightarrow \infty$ then the condition is automatically fulfilled.
However, a natural objection to the dS conjecture arises: its application and therefore the absence of dS vacua would seem to prevent the accelerating expansion of the universe. This problem can be solved employing the same tools used for the inflation in the early universe, that is the expansion is driven by a scalar field rolling down the not too steep slope of a potential. This kind of models are known under the name of Quintessence [18]. Hence, if this were the case of present day expansion, putting dS vacua in the swampland would not be problematic. Current observations require $\nabla V$ and $V$ to be of the same order and small in Planck units, and, in order to have the accelerated expansion, we should demand the parameter $\omega$ in the equation

$$
\begin{equation*}
P=\omega \rho \tag{2.7}
\end{equation*}
$$

to be $\omega<-1 / 3$. This is compatible with equation (2.6) as long as $c<\sqrt{2}[9]$.
However the dS conjecture so formulated is questionable. Indeed some problematic examples have been found. For instance, in [19], it was pointed out that the Higgs potential in the Standard Model gives

$$
\begin{equation*}
\frac{|\nabla V|}{V} \sim 10^{-55} \tag{2.8}
\end{equation*}
$$

In this case the computed value differs by many orders of magnitude from the expected one that was of order one. Such result can be explained with the existence of more scalar fields in the SM potential, but we would be forced to discard a theory that has proven to work perfectly so far.
In [10] a sharpened version of the conjecture based on entropy arguments has been proposed. It was shown that such a refinement evades all the adduced counter examples. It is formulated saying that in a quantum gravity theory a scalar potential has to satisfy either equation (2.6) or

$$
\begin{equation*}
\min \left(\nabla_{i} \nabla_{j} V\right) \leq-c^{\prime} \frac{V}{M_{p}^{2}} \tag{2.9}
\end{equation*}
$$

where $c^{\prime}$ is a positive constant of order one and $\min \left(\nabla_{i} \nabla_{j} V\right)$ is the smallest eigenvalue of the Hessian of the potential. Such condition, if satisfied, makes the fluctuations of the scalar field making up the
potential not-negligible. This invalidates the following semi-classical argument which leads to derive the first condition of the conjecture (2.6), so that if (2.9) is satisfied, then we expect (2.6) not to be respected. All the examples found so far that do not respect the first condition do respect the second one.
The justification of the sharpened version is based on the use of the Gibbons-Hawking entropy [20] of $d S$ space which relates the entropy to the radius $R$ of the event horizon

$$
\begin{equation*}
S_{G H} \sim R^{2} \sim \frac{1}{\Lambda} \tag{2.10}
\end{equation*}
$$

where $\Lambda$ is the cosmological constant. $\Lambda$ is then linked to the dimension of the Hilbert space. With an accelerated expansion, that is with $c<\sqrt{2}$ in eq.(2.6), we get an apparent horizon with

$$
\begin{equation*}
R=\frac{1}{\sqrt{V}} \tag{2.11}
\end{equation*}
$$

The existence of the apparent horizon enables us to use the Bousso bound [21] that states that the entropy on a hypersurface with area $A$ is bounded by

$$
\begin{equation*}
S \leq \frac{A}{4} \tag{2.12}
\end{equation*}
$$

The second condition of the dS conjecture, namely eq.(2.9), is needed because, as explained in [10], this semi-classical description of the entropy breaks down if there is a too strong instability and the apparent horizon inside the event horizon essential to compute the entropy can not exist.
The first condition, instead, is obtained computing the entropy explicitly. Let us parameterize the number of particle species appearing gradually for the SDC as we move towards infinity in the moduli space as

$$
\begin{equation*}
N(\phi) \sim n(\phi) e^{b \phi} \tag{2.13}
\end{equation*}
$$

with $\frac{d n}{d \phi} \geq 0$ since the increase of states in the SDC is monotonic and with $b \sim \mathcal{O}(1)$. We expect the entropy of the tower of states to increase with the number of particle species $N$. We can then parameterize the entropy with $N$ and with the radius $R$ of the apparent horizon as

$$
\begin{equation*}
S_{\text {tower }} \sim N^{\gamma} R^{\delta} \tag{2.14}
\end{equation*}
$$

where also $\gamma, \delta \sim \mathcal{O}(1)[10]$. Now we can apply the Bousso bound inside the horizon. Note that in the infinite distance limit $N$ grows exponentially and so $R$ must change in an opposite way in order not to violate the limit. Furthermore as $N$ increases the bound is expected to be saturated so that we have, using eq.(2.11),

$$
\begin{equation*}
V \sim \frac{1}{R^{2}} \sim N^{\frac{-2 \gamma}{2-\delta}} \tag{2.15}
\end{equation*}
$$

From this we can recover eq.(2.6) noting that

$$
\begin{equation*}
\left|\frac{\partial V}{\partial \phi}\right| \sim \frac{2 \gamma}{2-\delta} N^{\frac{-2 \gamma}{2-\delta}-1} \frac{\partial n}{\partial \phi} e^{b \phi}+\frac{2 \gamma b}{2-\delta} N^{\frac{-2 \gamma}{2-\delta}} \tag{2.16}
\end{equation*}
$$

Thus, since $\frac{d n}{d \phi} \geq 0$ and $2-\delta \geq 0$ because $R$ was exponentially suppressed, we have that

$$
\begin{equation*}
\left|\frac{\partial V}{\partial \phi}\right| \geq c V=\frac{2 \gamma b}{2-\delta} N^{\frac{-2 \gamma}{2-\delta}} \tag{2.17}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
c=\frac{2 \gamma b}{2-\delta} \sim \mathcal{O}(1) \tag{2.18}
\end{equation*}
$$

We have then found the first of the conditions (2.6) of the dS conjecture.

## Chapter 3

## KK reduction and quantum potential


#### Abstract

An interesting consequence of the AdS and the dS conjectures is the possibility to constrain some parameters in an EFT so that stable non-supersymmetric AdS vacua or stable (and mildly unstable) dS vacua do not appear. Let us take the Standard Model: it is not defined on an AdS or dS space so the conjecture seems at first sight to be inapplicable here, however if the SM were consistent with string theory, then also its compactifications would. Such compartifications, which can be performed over a great variety of manifolds, allow the rise of a potential and, possibly, of a forbidden AdS or dS vacuum for some new scalar fields $[1,2,3,4]$. This permits to constrain the masses and the type (Dirac or Majorana) of the neutrinos since it will turn out that they are the massive degrees of freedom most responsible for the aforementioned potential.

We are going to use the compactification in the case of a circle and a torus [22]. The compactification of the Einsten-Hilbert (EH) action gives rise to kinetic terms for the scalar fields parameterizing the manifold. Instead, the potential we need for the possible appearance of the vacua is obtained from the 1-loop quantum corrections of the compactified terms referring to the different particle species in the standard model [23, 24]. We are going to see how to obtain it explicitly in the compactification from 5 to 4 dimensions for a massive scalar field and a massive vector field.


### 3.1 Kaluza-Klein reduction

The technique of compactification consists in considering one or more of the spatial dimensions to be cyclic so that every point and element in a certain direction presents itself again and again. Such a situation becomes then indistinguishable from a circle (or another geometrical figure, depending on what compact manifold we are compactifying on).

Historically it was firstly used by Theodor Kaluza in a famous article (of which [25] is a modern translation) with further contributions by Oskar Klein [26]. He proved the possibility of deriving the Einstein-Maxwell lagrangian, combining electromagnetism and general relativity, from a 5 dimensional EH action, that is from general relativity in five dimensions. This fact was then known as KaluzaKlein miracle. Later on the compactification technique has been widely used until it became one of the fundamental features of theories born to live in a number of dimensions higher than four.

Practically, to perform the calculation, the components of the metric representing the cyclic dimensions need to be parameterized with suitable fields. Then the higher dimensional fields are expanded in Fourier modes, among which there is a massless one that, in our case, is the only we will keep in the lower dimensional EFT. Next, computing explicitly the EH action makes new kinetic terms for this fields comes out while from quantum corrections a potential comes out (see section 3.2).

In the following the compactification on a circle and on a torus of the EH action plus a cosmological constant using Cartan's formalism is performed explicitly. It would clearly be possible to compute the Ricci scalar directly without the use of the vielbeins, but that would take much longer and would be more in danger of making mistakes.

### 3.1.1 Circle $S^{1}$ case

In this case the $z$ direction gets compactified on a circle, so that

$$
\begin{equation*}
z \sim z+2 \pi R \tag{3.1}
\end{equation*}
$$

where R is the radius of the circle. This means that all the points with z-coordinate multiple of $2 \pi R$ are identified with each other, see Figure 3.1.
Regarding the parameterization of the metric, a four dimensional metric can always be parameterized as

$$
\hat{g}_{M N}=\left(\begin{array}{cc}
g_{\mu \nu}+A_{\mu} A_{\nu} & A_{\mu}  \tag{3.2}\\
A_{\nu} & \phi
\end{array}\right)
$$

where a hat indicates the four dimensional quantities. Here and in what follows there is a distinction between spacetime indices and vielbein indices we have to bear in mind: the former ones are the capital latin letters $M$ and $N$ that take the values $0,1,2, z$ and the greek ones that run from 0 to 2 ; the latter ones are $I, J, K, L$ that can be $0,1,2,3$ and $i, j, k, l$ that are $0,1,2$. Note that with spacetime indices the compactified directions is indicated with a $z$ while in vielbein indices with a 3. Besides, all the five dimensional quantities that could be confused with the four dimensional ones are indicated with a hat. Let us go back to the metric, in order to simplify the calculations, the vectors can be omitted since they will not affect the scalar part of the resultant theory. Another smart choice to make everything easier is to change the definition of $\phi$, that is usually called dilaton, and to adjust the parameterization of $\hat{g}_{M N}$ as

$$
\hat{g}_{M N}=\left(\begin{array}{cc}
e^{2 \beta \phi} g_{\mu \nu} & 0  \tag{3.3}\\
0 & e^{2 \alpha \phi}
\end{array}\right)
$$

In this case computing the determinant is particularly easy, it reads

$$
\begin{equation*}
\operatorname{det}(\hat{g})=e^{2(\alpha+d \beta)} \operatorname{det}(g) \tag{3.4}
\end{equation*}
$$

where $d$ is the number of non compactified dimensions (in our case $d=3$ ).
Before starting to compute the EH action, there is another remark that has to be made: the dilaton will be considered independent of the compactified dimension. This corresponds to a reduction of the particle content of the theory. In fact a scalar field in our case can be expanded in fourier modes as

$$
\begin{equation*}
\phi\left(x^{\mu}, z\right)=\sum_{n=-\infty}^{+\infty} \phi^{(n)}\left(x^{\mu}\right) e^{i n z / R} \tag{3.5}
\end{equation*}
$$

Given that for a massless scalar field the equation of motion is $\square \phi=0$, then the fields $\phi^{(n)}$ have to satisfy

$$
\begin{equation*}
\left(\square-\frac{n^{2}}{R^{2}}\right) \phi^{(n)}=0 \tag{3.6}
\end{equation*}
$$

therefore we have a massless scalar field for $n=0$ and an infinite tower of massive fields for $n \neq 0$. Usually the radius of the circle is set to be very small, otherwise we would see it, hence from eq.(3.6) we see that the masses of the resultant scalar fields become big [22], thus they are irrelevant for the low energy physics and can be safely neglected. We are reducing the particle content of the theory because an infinity of massive fields has been thrown away, then we are effectively truncating the theory to the $\phi$ massless sector. Such truncation is said to be consistent if the equations of motion computed from the original action and then truncated are equal to the ones derived from the truncated action [27]. We will check such consistency for the truncations we are going to perform in the next chapter. The truncation process, which gives rise to a new theory with a different particle content, needs not to be confused with the derivation of an EFT as depicted in the previous chapter. In the latter some heavy degrees of freedom, irrelevant under a certain energy scale, are integrated out, which means that they will modify the theory by adding new interaction terms between the remaining fields, so that they replace the interactions that were earlier mediated by the same heavy fields. In this way a solution of the EFT can only be an approximated solution of the original theory. On the other hand
the truncation does not change the interactions between the remaining fields and, if consistent, the solutions to the new equations of motion are solutions even to the original ones.

Let us go forward to the actual calculation. The EH action plus a cosmological constant is [28]

$$
\begin{equation*}
S_{E H}=\int d^{4} x \sqrt{-\hat{g}}\left[\frac{1}{16 \pi \hat{G}}(\hat{R}-2 \Lambda)\right] . \tag{3.7}
\end{equation*}
$$

It is quite convenient to use the vielbeins, i.e. to choose basis vectors such that

$$
\begin{equation*}
d s^{2}=\hat{g}_{M N} d x^{M} d x^{N}=\eta_{I J} E^{I} E^{J}, \tag{3.8}
\end{equation*}
$$

where here and in the following the signature of the flat space metric $\eta$ is $(-,+,+,+)$. Hence we choose

$$
\begin{equation*}
E^{i}=e^{\beta \phi} e^{i}, \quad E^{3}=e^{\alpha \phi} d z \tag{3.9}
\end{equation*}
$$

where the lowercase latin letters stands for the vielbein indices and $e^{i}$ indicates the three dimensional vielbeins. In order to get the Ricci scalar we firstly need the spin connection and the curvature 2 -form (which is indicated with a tilde to distinguish it from the Ricci tensor). The former can be extrapolated from

$$
d E^{I}=-\hat{\omega}^{I}{ }_{J} \wedge E^{J} \Rightarrow\left\{\begin{array}{l}
d E^{i}=-\hat{\omega}_{j}^{i} \wedge E^{j}-\hat{\omega}_{3}^{i} \wedge E^{3}  \tag{3.10}\\
d E^{3}=-\hat{\omega}_{j}{ }_{j} \wedge E^{j}
\end{array}\right.
$$

So in our case we get

$$
\begin{align*}
& d E^{i}=\beta e^{-\beta \phi} \partial_{j} \phi E^{j} \wedge E^{i}-\omega_{j}^{i} \wedge E^{j}, \\
& d E^{3}=\alpha e^{-\beta \phi} \partial_{j} \phi E^{j} \wedge E^{3} . \tag{3.11}
\end{align*}
$$

From this we can obtain the spin connection, that is

$$
\begin{align*}
& \hat{\omega}^{i}{ }_{j}=\omega^{i}{ }_{j}+\beta e^{-\beta \phi}\left(\partial_{j} \phi E^{i}-\partial^{i} \phi E_{j}\right),  \tag{3.12}\\
& \hat{\omega}^{3}{ }_{i}=\alpha e^{-\beta \phi} \partial_{i} \phi E^{3} .
\end{align*}
$$

Now we need the curvature 2 -form that is defined as

$$
\tilde{R}_{J}^{I}=d \hat{\omega}^{I}{ }_{J}+\hat{\omega}^{I}{ }_{K} \wedge \hat{\omega}^{K}{ }_{J} \Rightarrow\left\{\begin{array}{l}
\tilde{R}_{j}^{i}=d \hat{\omega}_{j}{ }_{j}+\hat{\omega}_{k}^{i} \wedge \hat{\omega}^{k}{ }_{j}+\hat{\omega}_{3}^{i} \wedge \hat{\omega}^{3}{ }_{j},  \tag{3.13}\\
\tilde{R}_{i}^{3}=d \hat{\omega}_{i}{ }_{i}+\hat{\omega}_{k}^{3} \wedge \hat{\omega}_{i}^{k} .
\end{array}\right.
$$

After a lot of algebra we can write the result in our case as

$$
\begin{align*}
\tilde{R}_{j}^{i}= & r_{j}^{i}+E^{k} \wedge E^{l} e^{-2 \beta \phi}\left(\beta \delta_{[l}^{i} \partial_{k]} \partial_{j} \phi+\beta^{2} \eta_{j[l} \partial_{k]} \phi \partial^{i} \phi-\beta^{2} \partial^{m} \phi \partial_{m} \phi \delta_{[k}^{i} \eta_{l] j}-\beta^{2} \delta_{[l}^{i} \partial_{k]} \phi \partial_{j} \phi\right. \\
& \left.-\beta \eta_{j[l} \partial_{k]} \partial^{i} \phi\right)-\beta e^{-\beta \phi}\left(\partial_{k} \phi \omega_{j}^{k} \wedge E^{i}+\partial_{k} \phi \omega^{i k} \wedge E_{j}\right),  \tag{3.14}\\
\tilde{R}_{i}^{3}= & E^{j} \wedge E^{3} e^{-2 \beta \phi}\left(\alpha(\alpha-2 \beta) \partial_{i} \phi \partial_{j} \phi+\alpha \partial_{i} \partial_{j} \phi+\alpha \beta \eta_{i j} \partial_{k} \phi \partial^{k} \phi\right)+\alpha e^{-\beta \phi} \partial_{j} \phi E^{3} \wedge \omega_{i}^{j},
\end{align*}
$$

where $r^{i}{ }_{j}$ is the 3 dimensional curvature 2 -form. In order to get the Ricci scalar with the formula

$$
\begin{equation*}
\hat{R}=\eta^{I J} R^{K}{ }_{I K J}=2 R_{i 3}^{3}{ }^{i}+R_{i j}^{j}{ }_{i j}{ }^{i}, \tag{3.15}
\end{equation*}
$$

we need the Riemann tensor that is given by

$$
\begin{equation*}
\tilde{R}^{I}{ }_{J}=\frac{1}{2} d x^{M} \wedge d x^{N} R_{M N}{ }^{I}{ }_{J}, \tag{3.16}
\end{equation*}
$$

so that, turning the spacetime indices $M$ and $N$ to vielbein indices, we have

$$
\left\{\begin{array}{l}
\tilde{R}^{3}{ }_{i}=\frac{1}{2} E^{k} \wedge E^{j} R_{k j}{ }^{3}{ }_{i}+E^{j} \wedge E^{3} R_{j 3}{ }^{3}{ }_{i},  \tag{3.17}\\
\tilde{R}^{i}{ }_{j}=\frac{1}{2} E^{k} \wedge E^{l} R_{k l}{ }^{i}{ }_{j}+E^{k} \wedge E^{3} R_{k 3}{ }^{i}{ }_{j} .
\end{array}\right.
$$

From this we need only the two terms appearing in eq.(3.15) that are

$$
\begin{align*}
& R_{j i}^{i}{ }_{j}^{j}=\delta_{i}^{k} \eta^{l j} R_{k l}{ }^{i}{ }_{j}=e^{-2 \beta \phi} R+2 e^{-2 \beta \phi}\left(\beta(1-d) \partial_{i} \partial^{i} \phi-\beta^{2}(1-d)(2-d)(\partial \phi)^{2}\right) \\
& R_{i 3}^{3}{ }_{i 3}=-\eta^{i j} R_{i j}^{3}{ }^{3}=e^{-2 \beta \phi}\left(\alpha(\beta(2-d)-\alpha)(\partial \phi)^{2}-\alpha \partial_{i} \partial^{i} \phi\right)+\alpha e^{-2 \beta \phi} \partial_{k} \phi \omega_{j}^{k j}, \tag{3.18}
\end{align*}
$$

where $R$ is the 3 dimensional Ricci scalar. So for the four dimensional one we get

$$
\begin{equation*}
\hat{R}=e^{-2 \beta \phi} R+e^{-2 \beta \phi}\left(2 \alpha(\beta(2-d)-\alpha)-\beta^{2}(d-1)(d-2)\right)(\partial \phi)^{2}-2 e^{-2 \beta \phi}(\alpha+\beta(d-1)) \square \phi . \tag{3.19}
\end{equation*}
$$

Inserting this expression in eq.(3.7), we obtain

$$
\begin{align*}
S & =\frac{1}{16 \pi \hat{G}} \int d^{3} x \int_{0}^{2 \pi R} d z \sqrt{-g} e^{(\alpha+d \beta) \phi}\left(e^{-2 \beta \phi} R-2 \Lambda+e^{-2 \beta \phi}(2 \alpha(\beta(d-2)-\alpha)\right.  \tag{3.20}\\
& \left.\left.-\beta^{2}(d-1)(d-2)\right)(\partial \phi)^{2}-2 e^{-2 \beta \phi}(\alpha+\beta(d-1)) \square \phi\right) .
\end{align*}
$$

Here the coefficient in front of the new Ricci scalar does not resemble the EH action, thus we can choose appropriately $\alpha$ and $\beta$ to move in what is called the Einstein frame, namely when the coefficient of the Ricci scalar is 1 ,

$$
\begin{equation*}
\alpha=-\beta(d-2), \quad \beta^{2}=\frac{1}{2(d-1)(d-2)} . \tag{3.21}
\end{equation*}
$$

Then, choosing $d=3$ we get $\alpha=-\frac{1}{2}$ and $\beta=\frac{1}{2}$. With this choice and the redefinition of the gravitational constant in three dimensions [29] as

$$
\begin{equation*}
G=\frac{\hat{G}}{2 \pi R}, \tag{3.22}
\end{equation*}
$$

we arrive at the final form of the action

$$
\begin{equation*}
S=\frac{1}{16 \pi G} \int d^{3} x \sqrt{-g}\left(R-2 e^{\phi} \Lambda-\frac{1}{2}(\partial \phi)^{2}\right), \tag{3.23}
\end{equation*}
$$

where an irrelevant total derivative $\square \phi$ has been neglected.
In the end, from the EH action plus a cosmological constant we obtained the kinetic term for the extra scalar field parameterizing the metric. Moreover, note that we can add to this action any other expression coming from the original action, as it will be the case of the quantum potential presented in section 3.2.

### 3.1.2 Torus $T^{2}$ case

In the previous section the compactification on the circle did not need a particular geometrical construction since the identification of points in a single dimension, i.e. on a circle, does not present very complex aspects, see Figure 3.1.
Otherwise, when two dimensions are involved, the construction gets slightly more involved as different and more complex manifold can be implicated. We are going to use a torus whose construction can be found in [29].


Figure 3.1: In the compactified direction every point gets identified with the one with coordinates $x+2 \pi R$. The resultant geometrical figure is a circle. Image from [29].

## Torus parameterization

The most straightforward possibility is the rectangular torus, see Figure 3.2. Consider a complex plane $z$ and a rectangular region with sides $L_{1}$ and $i L_{2}$. Identifying the points on the horizontal segments with each other and doing the same with the vertical ones as follows

$$
\begin{equation*}
z \sim z+L_{1}, \quad z \sim z+i L_{2}, \tag{3.24}
\end{equation*}
$$

leads to the torus. The area shaped by the rectangle is called fundamental domain of the rectangular torus because it includes all the inequivalent rectangular tori, namely all the other tori can be obtained from this one.


Figure 3.2: Left: fundamental domain of the torus. Center: first identification of the vertical lines in the fundamental domain. Right: second identification of the horizontal lines. Image from [29].

Actually the important parameter describing the rectangular torus does not consist in $L_{1}$ and $L_{2}$, but in their ratio. In fact performing the conformal transformation $z \rightarrow z / L_{1}$ leads to new equations for the identifications

$$
\begin{equation*}
z \sim z+1, \quad z \sim z+i T \tag{3.25}
\end{equation*}
$$

where $T=L_{2} / L_{1}$. Thus, to construct a rectangular torus we need just one parameter. But this does not saturate all the ways to build a torus. Indeed we could have a twisted domain. To build it let us take two not parallel segments $\omega_{1}$ and $\omega_{2}$ in the upper half of the complex plane and construct a parallelogram as in Figure 3.3. Identifying its opposite sides with each other we can build a torus

$$
\begin{equation*}
z \sim z+\omega_{1}, \quad z \sim z+\omega_{2} . \tag{3.26}
\end{equation*}
$$

As we just did to get to eq.(3.25), we can introduce a new parameter

$$
\begin{equation*}
\tau=\frac{\omega_{2}}{\omega_{1}}=\tau_{1}+i \tau_{2}, \tag{3.27}
\end{equation*}
$$

with the additional condition that $\tau_{2}>0$ so that it belongs to the upper half plane. Now, performing a conformal transformation similar to the one we used previously $z \rightarrow z / \omega_{1}$, we obtain the new identification rules

$$
\begin{equation*}
z \sim z+1, \quad z \sim z+\tau, \quad \text { with } \tau_{2}>0 \tag{3.28}
\end{equation*}
$$

The torus with twisted domain allows the identification between points that will not be exactly one above the other as can be seen in Figure 3.4. This does not change in any way the characteristics


IZ

Figure 3.3: Twisted domain for a torus with the parameters $\omega_{1}$ and $\omega_{2}$. Image from [29].


Figure 3.4: Identification of points on the opposite sides of the parallelogram for a torus with twisted domain. Image from [29].
of the torus, the different identification does not concern the geometry, but only the formal way to connect the various points.

In this case we need two parameters to build the most general torus. Anyway, in the next calculations we will need to introduce also the area of the torus to correctly parameterize the metric.

One last important point to analyse is the possible set of values for the parameter $\tau$ defininf the torus. It is not the entire upper half complex plane since it presents some symmetries that allow to map some tori into others. The first of them is the identification

$$
\begin{equation*}
\tau \sim \tau+1 \tag{3.29}
\end{equation*}
$$

As it can be seen in Figure 3.5 two tori with such an identification are actually the same.


Figure 3.5: Identifying $\tau \sim \tau+1$ leads to the same torus. Image from [29].
The second symmetry consists in two consecutive conformal transformations

$$
\begin{equation*}
\tilde{z}=\frac{z}{\tau}, \quad \quad z^{\prime}=\tilde{z}-\frac{1}{\tau} \tag{3.30}
\end{equation*}
$$

This leads to an equivalent torus as depicted in Figure 3.6.
The overall result of the two conformal transformation is to identify

$$
\begin{equation*}
\tau \sim-\frac{1}{\tau} . \tag{3.31}
\end{equation*}
$$



Figure 3.6: With two consecutive conformal transformation it is possible to show that identifying $\tau \sim-1 / \tau$ leads to an equivalent torus. Image from [29].

We got therefore the second symmetry of the torus which relates points with $|\tau|>1$ with points with $|\tau|<1$. Now the fundamental domain for the parameter $\tau$, namely $F_{0}$, gets restricted to a more narrow region, see Figure 3.7. There would be more than one region to be a valid candidate, but only the following contains exactly once all the possible tori [29]

$$
\begin{equation*}
F_{0}=\left\{-\frac{1}{2}<\tau_{1} \leq \frac{1}{2}, \quad \tau_{2}>0,|\tau| \geq 1 \text { and if }|\tau|=1 \text { then } \tau_{1} \geq 0\right\} \tag{3.32}
\end{equation*}
$$



Figure 3.7: The fundamental domain of the torus in the $\tau$ plane. Only with $\tau_{1}>0$ the border is included, because it is identified with the border in the other half. Image from [29].

Note that the two symmetries of the torus have generators

$$
\begin{equation*}
T \tau=\tau+1, \quad S \tau=-\frac{1}{\tau} \tag{3.33}
\end{equation*}
$$

and, together, they form a group called modular group or projective special linear group $\operatorname{PSL}(2, \mathbb{Z})$, that consists in all the 2 by 2 matrices with integer elements, unit determinant and that satisfies $A=-A$ where $A$ is one of the matrices. These properties come in since the combination $g$ of the transformations of eq.(3.33) becomes

$$
\begin{equation*}
g \tau=\frac{a \tau+b}{c \tau+d}, \quad \text { with } a, b, c, d \in \mathbb{Z} \text { and } a d-b c=1 . \tag{3.34}
\end{equation*}
$$

## Torus compactification

The compactification of two spatial dimensions $y$ and $z$ on a torus makes use of the above construction, indeed the metric can be parameterized as

$$
\hat{g}_{M N}=\left(\begin{array}{ccc}
g_{\mu \nu} & & \mathbb{O}_{2}  \tag{3.35}\\
& \frac{A}{\tau_{2}} & \frac{A \tau_{1}}{\tau_{2}} \\
\mathbb{O}_{2} & \frac{A \tau_{1}}{\tau_{2}} & \frac{A|\tau|^{2}}{\tau_{2}}
\end{array}\right)
$$

where $A$ is the area of the torus. Note that now the greek indices run from 0 to 1 and we refer to the compactified dimensions in vielbein indices as 2 and 3 while in spacetime indices as $y$ and $z$. All the comments made in the case of the circle remain valid, in fact even here it is better to use Cartan's formalism. In order to find the vielbeins let us write down the line element

$$
\begin{equation*}
d s^{2}=g_{\mu \nu} d x^{\mu} d x^{\nu}+\frac{A}{\tau_{2}} d y^{2}+2 \frac{A \tau_{1}}{\tau_{2}} d y d z+\frac{A}{\tau_{2}}|\tau|^{2} d z^{2}=\eta_{I J} E^{I} E^{J} \tag{3.36}
\end{equation*}
$$

where $I, J=0,1,2,3$. Hence we can choose the third and fourth vielbeins to be

$$
\begin{equation*}
E^{2}=\sqrt{\frac{A}{\tau_{2}}}\left(d y+\tau_{1} d z\right), \quad \quad E^{3}=\sqrt{A \tau_{2}} d z \tag{3.37}
\end{equation*}
$$

Now, proceeding as before we look for the spin connections using the adapted version eq.(3.10)

$$
\begin{align*}
& d E^{i}=-\omega_{j}^{i} \wedge E^{j} \\
& d E^{2}=\left(\frac{\partial_{i} A}{2 A}-\frac{\partial_{i} \tau_{2}}{2 \tau_{2}}\right) E^{i} \wedge E^{2}+\frac{\partial_{i} \tau_{1}}{\tau_{2}} E^{i} \wedge E^{3}  \tag{3.38}\\
& d E^{3}=\left(\frac{\partial_{i} A}{2 A}+\frac{\partial_{i} \tau}{2 \tau_{2}}\right) E^{i} \wedge E^{3}
\end{align*}
$$

where $i, j=0,1$. So that the spin connections read

$$
\begin{align*}
& \hat{\omega}_{3}^{2}=-\frac{\partial_{i} \tau_{1}}{2 \tau_{2}} E^{i} \\
& \hat{\omega}_{i}^{2}=\left(\frac{\partial_{i} A}{2 A}-\frac{\partial_{i} \tau_{2}}{2 \tau_{2}}\right) E^{2}+\frac{\partial_{i} \tau_{1}}{2 \tau_{2}} E^{3}  \tag{3.39}\\
& \hat{\omega}_{i}^{3}=\left(\frac{\partial_{i} A}{2 A}+\frac{\partial_{i} \tau_{2}}{2 \tau_{2}}\right) E^{3}+\frac{\partial_{i} \tau_{1}}{2 \tau_{2}} E^{2}
\end{align*}
$$

Now using the form for the $T^{2}$ of eq.(3.13) we can obtain the curvature 2 -form

$$
\begin{align*}
\tilde{R}_{j}^{i} & =r_{j}^{i}-\left(\frac{\partial^{i} \tau_{2} \partial_{j} \tau_{1}}{2 \tau_{2}^{2}}-\frac{\partial^{i} \tau_{1} \partial_{j} \tau_{2}}{2 \tau_{2}^{2}}\right) E^{3} \wedge E^{2}  \tag{3.40}\\
\tilde{R}_{i}^{2} & =\left(\frac{\nabla_{j} \partial_{i} A}{2 A}-\frac{\nabla_{j} \partial_{i} \tau_{2}}{2 \tau_{2}}-\frac{\partial_{i} A \partial_{j} A}{4 A^{2}}+\frac{3}{4} \frac{\partial_{i} \tau_{2} \partial_{j} \tau_{2}}{\tau_{2}^{2}}-\frac{\partial_{i} A \partial_{j} \tau_{2}}{4 \tau_{2} A}+\frac{\partial_{i} \tau_{2} \partial_{j} A}{4 \tau_{2} A}-\frac{\partial_{i} \tau_{1} \partial_{j} \tau_{1}}{4 \tau_{2}^{2}}\right) E^{j} \wedge E^{2} \\
& +\left(\frac{\partial_{i} A \partial_{j} \tau_{1}}{4 A \tau_{2}}-\frac{3}{4} \frac{\partial_{i} \tau_{2} \partial_{j} \tau_{1}}{\tau_{2}^{2}}+\frac{\nabla_{j} \partial_{i} \tau_{1}}{2 \tau_{2}}-\frac{\partial_{i} \tau_{1} \partial_{j} \tau_{2}}{4 \tau_{2}^{2}}+\frac{\partial_{i} \tau_{1} \partial_{j} A}{4 \tau_{2} A}\right) E^{j} \wedge E^{3}  \tag{3.41}\\
\tilde{R}_{i}^{3} & =\left(\frac{\nabla_{j} \partial_{i} A}{2 A}+\frac{\nabla_{j} \partial_{i} \tau_{2}}{2 \tau_{2}}-\frac{\partial_{i} A \partial_{j} A}{4 A^{2}}-\frac{\partial_{i} \tau_{2} \partial_{j} \tau_{2}}{4 \tau_{2}^{2}}+\frac{\partial_{i} A \partial_{j} \tau_{2}+\partial_{i} \tau_{2} \partial_{j} A}{4 A \tau_{2}}+\frac{3}{4} \frac{\partial_{i} \tau_{1} \partial_{j} \tau_{1}}{\tau_{2}^{2}}\right) E^{j} \wedge E^{3}  \tag{3.42}\\
& +\left(\frac{\nabla_{j} \partial_{i} \tau_{1}}{2 \tau_{2}}-\frac{\partial_{i} \tau_{1} \partial_{j} \tau_{2}}{2 \tau_{2}^{2}}+\frac{\partial_{i} A \partial_{j} \tau_{1}+\partial_{i} \tau_{1} \partial_{j} A}{4 A \tau_{2}}-\frac{\partial_{i} \tau_{1} \partial_{j} \tau_{2}+\partial_{j} \tau_{1} \partial_{i} \tau_{2}}{4 \tau_{2}^{2}}\right) E^{j} \wedge E^{2}
\end{align*}
$$

$\tilde{R}^{2}{ }_{3}=\left(\frac{\nabla_{j} \partial_{i} \tau_{1}}{2 \tau_{2}}-\frac{\partial_{j} \tau_{2} \partial_{i} \tau_{1}}{2 \tau_{2}^{2}}\right) E^{i} \wedge E^{j}+\left(\frac{\partial_{i} A \partial^{i} A}{4 A^{2}}-\frac{\partial^{i} \tau_{2} \partial_{i} \tau_{2}}{4 \tau_{2}^{2}}-\frac{\partial^{i} \tau_{1} \partial_{i} \tau_{1}}{4 \tau_{2}^{2}}\right) E^{3} \wedge E^{2}$,
where $r^{i}{ }_{j}$ is the curvature two-form relative to the non compactified dimensions. Now we can extract the Riemann tensor from a formula similar to eq.(3.16)
$\tilde{R}^{i}{ }_{j}=\frac{1}{2} E^{K} \wedge E^{L} R_{K L}{ }^{i}{ }_{j}=\frac{1}{2} E^{k} \wedge E^{l} R_{k l}{ }^{i}{ }_{j}+E^{2} \wedge E^{l} R_{2 l}{ }^{i}{ }_{j}+E^{3} \wedge E^{l} R_{3 l}{ }^{i}{ }_{j}+E^{3} \wedge E^{2} R_{32}{ }^{i}{ }_{j}$,
$\tilde{R}^{2}{ }_{i}=\frac{1}{2} E^{K} \wedge E^{L} R_{K L}{ }^{2}{ }_{i}=E^{j} \wedge E^{2} R_{j 2}{ }^{2}{ }_{i}+E^{j} \wedge E^{3} R_{j 3}{ }^{2}{ }_{i}$,
$\tilde{R}^{3}{ }_{i}=\frac{1}{2} E^{K} \wedge E^{L} R_{K L}{ }^{3}{ }_{i}=E^{j} \wedge E^{2} R_{j 2}{ }^{3}{ }_{i}+E^{j} \wedge E^{3} R_{j 3}{ }^{3}{ }_{i}$,
$\tilde{R}^{2}{ }_{3}=\frac{1}{2} E^{K} \wedge E^{L} R_{K L}{ }^{2}{ }_{3}=E^{i} \wedge E^{j} R_{i j}{ }^{2}{ }_{3}+E^{3} \wedge E^{2} R_{32}{ }^{2}{ }_{3}$.
Comparing these ones with the previous ones we can read the the components of the Riemann tensor. The ones we are going to need are
$R^{k}{ }_{i k j}=R_{i j}$,
$R_{i 2}{ }^{i}{ }_{2}=\eta^{i j} R_{2 j}{ }^{2}{ }_{i}=-\frac{\square A}{2 A}+\frac{\square \tau_{2}}{2 \tau_{2}}+\frac{(\partial A)^{2}}{4 A^{2}}-\frac{3}{4} \frac{\left(\partial \tau_{2}\right)^{2}}{\tau_{2}^{2}}+\frac{\partial_{i} A \partial^{i} \tau_{2}}{2 A \tau_{2}}+\frac{\left(\partial \tau_{1}\right)^{2}}{4 \tau_{2}^{2}}$,
$R^{3}{ }_{232}=-\frac{(\partial A)^{2}}{4 A^{2}}+\frac{\left(\partial \tau_{2}\right)^{2}}{4 \tau_{2}^{2}}+\frac{\left(\partial \tau_{1}\right)^{2}}{4 \tau_{2}^{2}}$,
$R^{i}{ }_{3 i 3}=-\eta^{i j} R_{j 3}{ }^{3}{ }_{i}=-\frac{\square A}{2 A}-\frac{\square \tau_{2}}{2 \tau_{2}}+\frac{(\partial A)^{2}}{4 A^{2}}+\frac{\left(\partial \tau_{2}\right)^{2}}{4 \tau_{2}^{2}}-\frac{\partial_{i} A \partial^{i} \tau_{2}}{2 A \tau_{2}}-\frac{3}{4} \frac{\left(\partial \tau_{1}\right)^{2}}{\tau_{2}^{2}}$,
where $R_{i j}$ is the two dimensional Ricci tensor in vielbein indices. Let us then proceed to the full Ricci tensor. We will need only three of its four components (still in vielbein indices) $R_{i j}, R_{22}$ and $R_{33}$. They are
$\hat{R}_{i j}=R^{K}{ }_{i K j}=R^{k}{ }_{i k j}+R^{2}{ }_{i 2 j}+R_{i 3 j}^{3}$,
$R_{22}=R^{K}{ }_{2 K 2}=R^{i}{ }_{2 i 2}+R^{3}{ }_{232}$,
$R_{33}=R^{K}{ }_{3 K 3}=R^{i}{ }_{3 i 3}+R^{2}{ }_{323}$.
Hence, inserting the components of the Riemann tensor we find
$\hat{R}_{i j}=R_{i j}-\frac{\nabla_{j} \partial_{i} A}{A}+\frac{\partial_{i} A \partial_{j} A}{2 A^{2}}-\frac{\partial_{i} \tau_{1} \partial_{j} \tau_{1}}{2 \tau_{2}^{2}}-\frac{\partial_{i} \tau_{2} \partial_{j} \tau_{2}}{2 \tau_{2}^{2}}$,
$R_{22}=-\frac{\square A}{2 A}+\frac{\square \tau_{2}}{2 \tau_{2}}-\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}+\frac{\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}+\frac{\partial^{i} A \partial_{i} \tau_{2}}{2 A \tau_{2}}$,
$R_{33}=-\frac{\square A}{2 A}-\frac{\square \tau_{2}}{2 \tau_{2}}+\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\partial^{i} A \partial_{i} \tau_{2}}{2 A \tau_{2}}$.
Now we only need one final step to get to the Ricci scalar

$$
\begin{equation*}
\hat{R}=\eta^{I J} \tilde{R}_{I J}=\eta^{i j} \tilde{R}_{i j}+\tilde{R}_{22}+\tilde{R}_{33}=R-\frac{2 \square A}{A}+\frac{(\partial A)^{2}}{2 A^{2}}-\frac{\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}} . \tag{3.58}
\end{equation*}
$$

We can finally insert everything in the EH action (3.7). Noting that $\sqrt{-\hat{g}}=A \sqrt{-g}$ and redefining the gravitational constant similarly to the $S^{1}$ case we arrive at

$$
\begin{equation*}
S_{2}=\frac{1}{16 \pi G_{2}} \int d^{2} x \sqrt{-g}\left(A R+\frac{(\partial A)^{2}}{2 A}-\frac{A}{2 \tau_{2}^{2}}\left(\partial \tau_{1}\right)^{2}-\frac{A}{2 \tau_{2}^{2}}\left(\partial \tau_{2}\right)^{2}-2 \Lambda A\right) \tag{3.59}
\end{equation*}
$$

where a total derivative has been neglected. The form of the compactified action is not unique, but it is defined up to a conformal transformation of the type $g_{\mu \nu}=A^{2 \alpha} \tilde{g}_{\mu \nu}$. To compute it let us proceed like in [30]. The new Christoffel symbol is

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho}=\tilde{\Gamma}_{\mu \nu}^{\rho}+\frac{\alpha}{A}\left(\delta_{\nu}^{\rho} \tilde{\nabla}_{\mu} A+\delta_{\mu}^{\rho} \tilde{\nabla}_{\nu} A-g_{\mu \nu} \tilde{\nabla}^{\rho} A\right)=\tilde{\Gamma}_{\mu \nu}^{\rho}+\delta \Gamma_{\mu \nu}^{\rho} \tag{3.60}
\end{equation*}
$$

where all the quantities with a tilde are relative to the metric $\tilde{g}_{\mu \nu}$. Note that, while $\Gamma_{\mu \nu}^{\rho}$ is not a tensor, $\delta \Gamma_{\mu \nu}^{\rho}$ is, therefore when we apply to it the covariant derivative we get more Christoffel symbols. Hence, after using the usual formula for the Riemann tensor, many terms can be inserted in the covariant derivatives applied to $\delta \Gamma$ and we obtain the simpler formula

$$
\begin{align*}
R_{\beta \mu \nu}^{\alpha} & =\tilde{R}_{\beta \mu \nu}^{\alpha}+\tilde{\nabla}_{\mu} \delta \Gamma_{\beta \nu}^{\alpha}-\tilde{\nabla}_{\nu} \delta \Gamma_{\beta \mu}^{\alpha}+\delta \Gamma_{\beta \nu}^{\tau} \delta \Gamma_{\tau \mu}^{\alpha}-\delta \Gamma_{\beta \mu}^{\tau} \delta \Gamma_{\tau \nu}^{\alpha} \\
& =\tilde{R}_{\beta \mu \nu}^{\alpha}+\frac{\alpha}{A}\left(\delta_{\nu}^{\alpha}\left(\tilde{\nabla}_{\mu} \tilde{\nabla}_{\beta} A-\frac{\tilde{\nabla}_{\mu} A \tilde{\nabla}_{\beta} A}{A}\right)+\delta_{\beta}^{\alpha}\left(\tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} A-\frac{\tilde{\nabla}_{\mu} A \tilde{\nabla}_{\nu} A}{A}\right)\right. \\
& -\tilde{g}_{\beta \nu}\left(\tilde{\nabla}_{\mu} \tilde{\nabla}^{\alpha} A-\frac{\tilde{\nabla}_{\mu} A \tilde{\nabla}^{\alpha} A}{A}\right)-\delta_{\mu}^{\alpha}\left(\tilde{\nabla}_{\nu} \tilde{\nabla}_{\beta} A-\frac{\tilde{\nabla}_{\nu} A \tilde{\nabla}_{\beta} A}{A}\right)  \tag{3.61}\\
& \left.-\delta_{\beta}^{\alpha}\left(\tilde{\nabla}_{\nu} \tilde{\nabla}_{\mu} A-\frac{\tilde{\nabla}_{\nu} A \tilde{\nabla}_{\mu} A}{A}\right)+\tilde{g}_{\beta \mu}\left(\tilde{\nabla}_{\nu} \tilde{\nabla}^{\alpha} A-\frac{\tilde{\nabla}_{\nu} A \tilde{\nabla}^{\alpha} A}{A}\right)\right)
\end{align*}
$$

Contracting two indices we get the Ricci tensor

$$
\begin{equation*}
R_{\mu \nu}=R_{\mu \alpha \nu}^{\alpha}=\tilde{R}_{\mu \nu}-\alpha \tilde{g}_{\mu \nu} \frac{\tilde{\square} A}{A}+\alpha \tilde{g}_{\mu \nu} \frac{(\partial A)^{2}}{A^{2}} \tag{3.62}
\end{equation*}
$$

and then the Ricci scalar

$$
\begin{equation*}
R=g^{\mu \nu} R_{\mu \nu}=A^{-2 \alpha}\left(\tilde{R}-\frac{2 \alpha}{A} \tilde{\square} A+\frac{2 \alpha}{A^{2}}(\partial A)^{2}\right) \tag{3.63}
\end{equation*}
$$

Inserting in (3.59) we have

$$
\begin{align*}
S_{2}=\frac{1}{16 \pi G_{2}} \int d^{2} x \sqrt{-\tilde{g}} A^{2 \alpha+1} & \left(A^{-2 \alpha} \tilde{R}-2 \alpha A^{-2 \alpha-1} \square A+A^{-2 \alpha-2}\left(2 \alpha+\frac{1}{2}\right)(\partial A)^{2}\right.  \tag{3.64}\\
& \left.-\frac{A^{-2 \alpha}}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)-2 \Lambda\right)
\end{align*}
$$

The two dimensional case is a special one, indeed we can not move to the Einstein frame, i.e. the one with 1 as coefficient of the Ricci scalar, no matter what value of $\alpha$ we choose. We can, however, simplify the action choosing $\alpha=-1 / 4$ and neglecting the total derivative. We reach then the form

$$
\begin{equation*}
S_{2}=\frac{1}{16 \pi G_{2}} \int d^{2} x \sqrt{-\tilde{g}}\left(A \tilde{R}-\frac{A}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)-2 \Lambda A^{1 / 2}\right) \tag{3.65}
\end{equation*}
$$

which agrees with [4]. As in the $S^{1}$ case the action can be enriched by the terms appearing in the original theory and by the ones obtainable with quantum corrections.

### 3.2 1-loop quantum potential

The application on the Standard Model of the conjectures showed in the previous chapter requires the existence of a scalar potential presenting Anti de Sitter or de Sitter vacua. The compactification of the EH action generated some kinetic terms for the fields parameterizing the circle and the torus. To obtain a potential for them we need to carry out the compactification on the other fields in the Standard Model and to compute quantum corrections up to, at least, first order in $\hbar$.

Fluctuations are an intrinsic feature of all quantum theories. Most calculations are performed exploiting such feature in perturbation theory, namely we look for solutions around a constant minimum. This means that as we increase the order of the perturbation the results are more and more precise. Computing higher order corrections is often a quite difficult task. One way to obtain the quantum potential we need (i.e. the part of the correction not involving derivatives on the fields) is to use the quantum effective action $\Gamma[\phi]$ and the background field method $[23] . \Gamma[\phi]$ is such that replacing the normal action $S[\phi]$ with it and using the tree-level rules gives the full quantum theory. Clearly now the problem has shifted towards the computation of $\Gamma[\phi]$. Let us consider a QFT with a scalar field $\phi$ and an external source $J$, we have the partition function

$$
\begin{equation*}
Z[J]=e^{\frac{i}{\hbar} W[J]}=\int D \phi \exp \left[\frac{i}{\hbar} \int d^{4} x(\mathcal{L}[\phi]+J(x) \phi(x))\right] \tag{3.66}
\end{equation*}
$$

where $W[J]$ is the connected generating functional, also called Wilsonian effective action. The effective action is obtained from $W[J]$ by means of a Legendre transform

$$
\begin{equation*}
\Gamma[\Phi]=W[J]-\int d^{4} x \frac{\delta W[J]}{\delta J(x)} J(x), \quad \frac{\delta W[J]}{\delta J(x)}=\Phi(x) \tag{3.67}
\end{equation*}
$$

where $\Phi(x)$ is the quantum average of $\phi(x)$. The seeked effective potential $V(\hat{\phi})$ is defined starting from $\Gamma[\Phi]$ and setting $\Phi$ at a constant value $\hat{\phi}$

$$
\begin{equation*}
\Gamma[\hat{\phi}]=-V(\hat{\phi}) \int d^{4} x \tag{3.68}
\end{equation*}
$$

Practically we expand eq.(3.66), as in [31], replacing $\phi(x)=\Phi(x)+\eta(x)$ where $\eta(x)$ represents the fluctuations over the quantum average. So we get

$$
\begin{align*}
\int d^{4} x(\mathcal{L}[\phi]+J(x) \phi(x)) & =\int d^{4} x(\mathcal{L}[\Phi]+J(x) \Phi(x))+\int d^{4} x \eta(x)\left(\left.\frac{\delta \mathcal{L}[\phi]}{\delta \phi(x)}\right|_{\phi=\Phi}+J(x)\right)  \tag{3.69}\\
& +\left.\frac{1}{2} \int d^{4} x d^{4} y \eta(x) \frac{\delta^{2} \mathcal{L}[\phi]}{\delta \phi(x) \delta \phi(y)}\right|_{\phi=\Phi} \eta(y)+\ldots
\end{align*}
$$

Clearly the path integral, which was performed over the full $\phi(x)$, will be now computed only for $\eta(x)$ since $\Phi(x)$ does not fluctuate. Therefore combining eq.(3.66) with eq.(3.67) and inserting the expansion we get an expression for the quantum effective action. Keeping only terms up to second order in $\eta$ in eq.(3.69) provides the first order quantum corrections to $\Gamma$

$$
\begin{equation*}
\Gamma[\Phi]=\int d^{4} x \mathcal{L}[\Phi]-i \hbar \log \int D \eta \exp \left[\left.\frac{i}{2 \hbar} \int d^{4} x d^{4} y \eta(x) \frac{\delta^{2} \mathcal{L}[\phi]}{\delta \phi(x) \delta \phi(y)}\right|_{\phi=\Phi} \eta(y)\right] . \tag{3.70}
\end{equation*}
$$

The term with only one $\eta$ is neglected because when performing the path integral it will generate only terms containing the source that is eventually set to zero, making all the contribution vanish.

This procedure gives the full $\Gamma$, but we are interested only in the quantum corrections to the scalar potential, therefore it is sufficient to set the quantum average to a constant value as in eq.(3.68).
What we did until now is the general procedure, however for the compactification case it is analogous: the compactification is explicitly performed for the different fields of the Standard model and then
the quantum effective action is computed using eq.(3.70). In the following we set $\hbar=1$. For the $S^{1}$ case the formula we are going to use in Chapter 5 comes from [1]. Here the metric is parameterized in a slightly different way and, besides, periodic boundary conditions have been specified

$$
\begin{equation*}
V(R)=\frac{2 \pi r^{3} \Lambda}{R^{3}}+\sum_{i}(2 \pi R) \frac{r^{3}}{R^{3}}(-1)^{2 s_{i}} n_{i} \rho_{i}(R) \tag{3.71}
\end{equation*}
$$

where $R$ is the radius of the circle, $r$ is a scale to be set later, the sum over $i$ indicates all the species in the theory, $s$ their spin, $n$ their degrees of freedom and $\rho$ is

$$
\begin{equation*}
\rho(R)=\mp \sum_{n=1}^{\infty} \frac{2 M^{4}}{(2 \pi)^{4}} \frac{K_{2}(2 \pi R M n)}{(2 \pi R M n)^{2}} \tag{3.72}
\end{equation*}
$$

here the minues stands for bosons and the plus for fermions. With respect to the $T^{2}$ case we are going to use the formula from [2]

$$
\begin{equation*}
V\left(A, \tau_{1}, \tau_{2}, M_{i}\right)=4 \pi^{2} A\left[\sum_{i} \rho_{i}\left(A, \tau_{1}, \tau_{2}, M_{i}\right)+\Lambda\right] \tag{3.73}
\end{equation*}
$$

where $A, \tau_{1}$ and $\tau_{2}$ are the parameters of the torus, the sum runs, as before, on the different particle species and $\rho$ is

$$
\begin{align*}
& \rho\left(A, \tau_{1}, \tau_{2}, M\right)=-\frac{1}{(2 \pi)^{4} A^{2}}\left[\frac{2(\sqrt{A} M)^{3 / 2}}{\tilde{\tau}_{2}^{1 / 4}} \sum_{p=1}^{\infty} \frac{1}{p^{3 / 2}} K_{3 / 2}\left(2 \pi p \sqrt{A} M \sqrt{\tilde{\tau}_{2}}\right)\right. \\
& \quad+2 \tilde{\tau}_{2} A M^{2} \sum_{p=1}^{\infty} \frac{1}{p^{2}} K_{2}\left(\frac{2 \pi p \sqrt{A} M}{\sqrt{\tilde{\tau}_{2}}}\right)  \tag{3.74}\\
& \left.\quad+4 \sqrt{\tilde{\tau}_{2}} \sum_{n, p=1}^{\infty} \frac{1}{p^{3 / 2}}\left(n^{2}+\frac{A M^{2}}{\tilde{\tau}_{2}}\right)^{3 / 4} \cos \left(2 \pi \tilde{\tau}_{1} p n\right) K_{3 / 2}\left(2 \pi p \tilde{\tau}_{2} \sqrt{n^{2}+\frac{A M^{2}}{\tilde{\tau}_{2}}}\right)\right]
\end{align*}
$$

where $\tilde{\tau}_{i}=\tau_{i} /|\tau|^{2}$. Now let us see explicitly how to get to eq.(3.72) with the compactification from 5 to 4 dimensions for two specific cases.

### 3.2.1 Massive scalar case

Let us begin with the calculation of the effective potential for a massive scalar field in the compactification from 5 to 4 dimensions. We have that

$$
\begin{equation*}
0 \leq x_{5} \leq 2 \pi R, \quad \phi\left(x^{\mu}, x_{5}\right)=\sum_{n=-\infty}^{+\infty} \phi^{(n)}\left(x^{\mu}\right) e^{i n x_{5} / R} \tag{3.75}
\end{equation*}
$$

For simplicity the metric is parameterized as

$$
\begin{equation*}
d s^{2}=g_{\mu \nu} d x^{\mu} d x^{\nu}+\phi d x_{5} d x_{5} \tag{3.76}
\end{equation*}
$$

The usual action for the scalar field is

$$
\begin{equation*}
S=\int d^{5} x \phi\left(x^{\mu}, x_{5}\right)\left(\square-M^{2}\right) \phi\left(x^{\mu}, x_{5}\right) \tag{3.77}
\end{equation*}
$$

To compute the functional trace we will obtain during the evaluation of eq.(3.70) we need, like in [4], to set the normalization of the wavefunctions

$$
\begin{equation*}
\psi_{n}=\frac{e^{i n x_{5} / R}}{\sqrt{2 \pi R \sqrt{\phi}}} \quad \Longrightarrow \quad \int_{0}^{2 \pi R} d x_{5} \sqrt{g_{S_{1}}} \psi_{m}^{*} \psi_{n}=\int_{0}^{2 \pi R} d x_{5} \frac{\sqrt{\phi}}{2 \pi R \sqrt{\phi}} e^{i(n-m) x_{5} / R}=\delta_{m n} \tag{3.78}
\end{equation*}
$$

So that the functional trace will be

$$
\begin{align*}
\operatorname{tr}_{5 D}\left(-\partial_{5 D}^{2}+m^{2}\right) & =\sum_{n} \int_{0}^{2 \pi R}\left\langle n \mid x_{5}\right\rangle\left\langle x_{5}\right| \operatorname{tr}\left(-\partial_{4 D}^{2}-\phi^{-1} \partial_{x_{5}}^{2}\right)|n\rangle \\
& =\sum_{n} \int_{0}^{2 \pi R} \psi_{n}^{*} \psi_{n} \operatorname{tr}\left(-\partial_{4 D}^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right) d x_{5}  \tag{3.79}\\
& =\frac{2 \pi R}{2 \pi R \sqrt{\phi}} \sum_{n} \operatorname{tr}\left(-\partial_{4 D}^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right) .
\end{align*}
$$

Let us apply the backgorund field method explained in the prior section, i.e. let us split $\phi$ in constant background and quantum fluctuations parts $\phi=\hat{\phi}+\eta$. Then the quantum effective action is obtained through the formula

$$
\begin{align*}
\Gamma[\hat{\phi}] & =-2 \pi R \operatorname{Vol}_{4 d} V(\hat{\phi})-i \log \int D \eta \exp \left(\left.\frac{i}{2} \int d^{5} x d^{5} y \eta(x) \frac{\delta^{2} S}{\delta \phi(x) \delta \phi(y)}\right|_{\phi=\hat{\phi}} \eta(y)\right)  \tag{3.80}\\
& =-2 \pi R \operatorname{Vol}_{4 d} V(\hat{\phi})-i \log \int D \eta \exp \left(\frac{i}{2} \int d^{5} x \eta(x)\left(\square-M^{2}\right) \eta(x)\right) .
\end{align*}
$$

We need to evaluate the second term: we use a Fourier transform and expand it in the five dimensional modes

$$
\begin{align*}
& \frac{1}{2} \int \frac{d^{4} k d^{4} \tilde{k} d^{4} x d x_{5}}{(2 \pi)^{4}} e^{i(k+\tilde{k}) x} \sum_{n, m} e^{i(n+m) x_{5} / R} \tilde{\eta}^{(n)}(k)\left(-\tilde{k}^{2}-\frac{m^{2}}{\phi R^{2}}-M^{2}\right) \tilde{\eta}^{(m)}(\tilde{k})  \tag{3.81}\\
& =-\frac{2 \pi R}{2} \int d^{4} k \sum_{n} \tilde{\eta}^{(n)}(k)\left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right) \tilde{\eta}^{(-n)}(-k)
\end{align*}
$$

where we used eq.(3.78) and

$$
\begin{equation*}
\int d^{4} x e^{i(k+\tilde{k}) x}=(2 \pi)^{4} \delta^{(4)}(k+\tilde{k}) . \tag{3.82}
\end{equation*}
$$

Solving the path integral gives

$$
\begin{align*}
\Gamma[\hat{\phi}] & =-2 \pi R \operatorname{Vol}_{4 d} V(\hat{\phi})-i \log \operatorname{det}\left(\square-M^{2}\right)^{-1 / 2} \\
& =-2 \pi R \operatorname{Vol}_{4 d} V(\hat{\phi})+\frac{i}{2} \operatorname{tr} \log \left(-\square+M^{2}\right) \\
& =-2 \pi R \operatorname{Vol}_{4 d} V(\hat{\phi})+\frac{i}{2} \frac{2 \pi R}{2 \pi R \sqrt{\phi}} \sum_{n} \int d^{4} k d^{4} x\langle x| \log \left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right)|k\rangle\langle k \mid x\rangle  \tag{3.83}\\
& =-2 \pi R \operatorname{Vol}_{4 d} V(\hat{\phi})-\frac{1}{2} \frac{2 \pi R V o l_{4 d}}{2 \pi R \sqrt{\phi}} \sum_{n} \int \frac{d^{4} k}{(2 \pi)^{4}} \log \left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right),
\end{align*}
$$

where in the last equality a Wick rotation has been performed. Comparing with eq.(3.68) we can read the quantum potential

$$
\begin{equation*}
V_{e f f}(\phi)=\frac{1}{2} \sum_{n} \frac{1}{2 \pi R \sqrt{\phi}} \int \frac{d^{4} k}{(2 \pi)^{4}} \log \left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right) . \tag{3.84}
\end{equation*}
$$

Starting with any particle species leads to this same formula multiplied by the number of degrees of freedom of that particular species [4]. Such a formula is affected by a divergence, therefore a
regularization procedure is needed. Let us use the zeta function regularization

$$
\begin{align*}
V_{e f f}(\phi, M) & =-\left.\frac{1}{2} \frac{d}{d s} \sum_{n} \frac{1}{2 \pi R \sqrt{\phi}} \int \frac{d^{4} k}{(2 \pi)^{4}}\left(k^{2}+\frac{n^{2}}{\phi R}+M^{2}\right)^{-s}\right|_{s=0} \\
& =-\left.\frac{d}{d s} \frac{1}{64 \pi^{3} R \sqrt{\phi}} \sum_{n} \int_{0}^{\infty} d k k^{3}\left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right)^{-s}\right|_{s=0} \\
& =-\frac{d}{d s} \frac{1}{64 \pi^{3} R \sqrt{\phi}} \sum_{n} \frac{\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right)^{2-s}}{(2-s)(1-s)}=-\frac{d}{d s} \frac{\left(\phi R^{2}\right)^{s-5 / 2}}{64 \pi^{3}(2-s)(1-s)} F\left(s-2 ; 0, \phi R^{2} M^{2}\right) \tag{3.85}
\end{align*}
$$

where $F(s ; a, c)$ is, as in [24],

$$
\begin{equation*}
F(s ; a, c)=\sum_{n=-\infty}^{+\infty} \frac{1}{\left[(n+a)^{2}+c^{2}\right]^{s}} \tag{3.86}
\end{equation*}
$$

This function, since it is a sum on all the values of $n$, is periodic for $a$ of period 1 , thus it is possible to expand it in Fourier series of coefficients $c_{p}$ and period $T=1$

$$
\begin{align*}
F(s ; a, c) & =\sum_{p} e^{i 2 \pi a} c_{p}=\sum_{p} e^{i 2 \pi a} \frac{1}{T} \int_{T} d y e^{-i 2 \pi y} F(s ; y, c) \\
& =\sum_{p} e^{i 2 \pi p a} \int_{0}^{1} d y e^{-i 2 \pi p y} \sum_{n} \frac{1}{\left[(n+y)^{2}+c^{2}\right]^{s}} \tag{3.87}
\end{align*}
$$

Exchanging the sum and the integral and making the substitution $z=n+y$ leads to

$$
\begin{equation*}
F(s ; a, c)=\sum_{p} e^{2 \pi i p a} \sum_{n} \int_{n}^{n+1} d z e^{-2 \pi i p z} e^{2 \pi i p n} \frac{1}{\left[z^{2}+c^{2}\right]^{s}} \tag{3.88}
\end{equation*}
$$

Sewing together the integrals so that the domain of integration is $\mathbb{R}$ gives

$$
\begin{equation*}
F(s ; a, c)=\sum_{p} e^{2 \pi i p a} \int_{-\infty}^{+\infty} d z e^{-2 \pi i p z} \frac{1}{\left[z^{2}+c^{2}\right]^{s}} \tag{3.89}
\end{equation*}
$$

where $e^{i 2 \pi p n}=1$ with $p, n \in \mathbb{Z}$ has been used. To proceed further we make use of

$$
\begin{equation*}
\frac{1}{z^{s}}=\frac{1}{\Gamma(s)} \int_{0}^{\infty} d t t^{s-1} e^{-z t} \tag{3.90}
\end{equation*}
$$

so that

$$
\begin{align*}
F(s ; a, c) & =\sum_{p} e^{2 \pi i p a} \int_{-\infty}^{+\infty} d z e^{-2 \pi i p z} \frac{1}{\Gamma(s)} \int_{0}^{\infty} d t t^{s-1} e^{-\left(z^{2}+c^{2}\right) t} \\
& =\frac{\sqrt{\pi}}{\Gamma(s)} \sum_{p} e^{2 \pi i p a} \int_{0}^{\infty} d t t^{s-3 / 2} e^{-c^{2} t-p^{2} \pi^{2} / t}  \tag{3.91}\\
& =\frac{\sqrt{\pi}}{\Gamma(s)}|c|^{1-2 s}\left(\int_{0}^{\infty} d u u^{s-3 / 2} e^{-u}+2 \sum_{p=1}^{\infty} \cos (2 \pi p a) \int_{0}^{\infty} d u u^{s-3 / 2} e^{-u+c^{2} p^{2} \pi^{2} / u}\right)
\end{align*}
$$

to get to the third line we used the substitution $u=c^{2} t$. In the second term of the last line we can recognise a modified Bessel function of the second kind

$$
\begin{equation*}
K_{-\nu}(z)=K_{\nu}(z)=2^{\nu-1} z^{-\nu} \int_{0}^{\infty} d t \frac{e^{-t-z^{2} / 4 t}}{t^{1-s}} \tag{3.92}
\end{equation*}
$$

and a Euler Gamma function in the first one. Hence we have

$$
\begin{equation*}
F(s ; a, c)=\frac{\sqrt{\pi}}{\Gamma(s)}|c|^{1-2 s}\left[\Gamma\left(s-\frac{1}{2}\right)+4 \sum_{p=1}^{\infty}(\pi p|c|)^{s-1 / 2} \cos (2 \pi p a) K_{s-\frac{1}{2}}(2 \pi p|c|)\right] . \tag{3.93}
\end{equation*}
$$

The first term makes rise to contributions that are then reabsorbed in a counter term [24], therefore we can neglect it. Reminding eq.(3.85) we need to derive this expression with respect to $s$. Since $\lim _{s \rightarrow-2} \frac{1}{\Gamma(s)}=0$, we have to derive only the factor $\Gamma(s)$ and then set $s$ to -2

$$
\begin{equation*}
\lim _{s \rightarrow-2} \frac{d}{d s} \frac{1}{\Gamma(s)}=\lim _{s \rightarrow-2}\left(-\frac{\psi^{(0)}(s)}{\Gamma(s)}\right)=2, \tag{3.94}
\end{equation*}
$$

where $\psi^{(0)}(s)$ is said digamma function. Thus we obtain

$$
\begin{equation*}
\lim _{s \rightarrow-2} \frac{d}{d s} F(s ;, a, c)=8 \sqrt{\pi}|c|^{5} \sum_{p=1}^{\infty}(\pi p|c|)^{-5 / 2} \cos (2 \pi p a) K_{-5 / 2}(2 \pi p|c|) . \tag{3.95}
\end{equation*}
$$

In our case the parameters are $a=0$ and $c=\sqrt{\phi} L M$. Inserting everything in eq.(3.85) we obtain the final formula of the quantum effective potential for the massive scalar field which agrees with [1]

$$
\begin{equation*}
V_{e f f}(\phi, M)=-\sum_{p=1}^{\infty} \frac{2 M^{5}}{(2 \pi)^{5}} \frac{K_{5 / 2}(2 \pi p \sqrt{\phi} L M)}{(\sqrt{\phi} L M)^{5 / 2}} . \tag{3.96}
\end{equation*}
$$

### 3.2.2 Massive vector case

Let us now consider a more complicated situation: a massive vector in the compactification from 5 to 4 dimensions. Its number of degrees of freedom is equal to the dimension of the space-time minus 1 that is fixed by the gauge choice. After the compactification and an appropriate definition of fields a 5 d massive vector gives rise to a 4 d massive vector ( 3 degrees of freedom) and to a massive scalar ( 1 degree of freedom). Thus we should obtain eq.(3.84) multiplied by 4 . Initially the equation of motion of the 5 d field is

$$
\partial_{M} F^{M N}=M^{2} A^{N} \Longrightarrow\left\{\begin{array}{l}
\partial_{\mu} F^{\mu 5}=M^{2} A^{5}  \tag{3.97}\\
\partial_{\mu} F^{\mu \nu}+\partial_{5} F^{5 \nu}=M^{2} A^{\nu}
\end{array}\right.
$$

where $F^{M N}$ is the field strength and $A^{N}$ the vector field. The space is compactified as in the massive scalar case, therefore we can expand the field in the five dimensional modes

$$
\left\{\begin{array}{l}
A^{\mu}(x, y)=\sum_{n=-\infty}^{+\infty} A^{\mu(n)}(x) e^{i n y / R}  \tag{3.98}\\
A^{5}(x, y)=\sum_{n=-\infty}^{+\infty} \sqrt{\frac{\phi}{M^{2}}\left(\frac{n^{2}}{\phi R^{2}}+M^{2}\right)} \varphi^{(n)}(x) e^{i n y / R}
\end{array}\right.
$$

The choice of the fields for the expansion is not random, in fact it will make the final result neater. In terms of the newly defined fields the equations of motion read

$$
\left\{\begin{array}{l}
\frac{\sqrt{\frac{n^{2}}{\phi R^{2}}+M^{2}}}{\sqrt{\phi M^{2}}} \square \varphi^{(n)}-\frac{i n}{\phi R} \partial_{\mu} A^{\mu(n)}=\sqrt{\frac{M^{2}}{\phi}\left(\frac{n^{2}}{\phi R^{2}}+M^{2}\right)} \varphi^{(n)}  \tag{3.9.9}\\
\partial_{\mu} F^{\mu \nu(n)}-\frac{i n}{\sqrt{\phi R^{2} M^{2}}} \sqrt{M^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}} \partial^{\nu} \varphi^{(n)}=\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right) A^{\nu(n)}
\end{array}\right.
$$

Proceeding as in the usual derivation of the Proca equation for a massive vector field we take the divergence of the second equation in (3.99)

$$
\begin{equation*}
\partial_{\mu} A^{\mu(n)}=-\frac{\frac{i n}{\sqrt{\phi R^{2} M^{2}}} \square \varphi^{(n)}}{\sqrt{\frac{n^{2}}{\phi R^{2}}+M^{2}}}, \tag{3.100}
\end{equation*}
$$

and we insert it in the first equation of (3.99) getting

$$
\begin{equation*}
\square \varphi^{(n)}=\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right) \varphi^{(n)} \tag{3.101}
\end{equation*}
$$

that is, as we wanted, the equation of motion for a 4 d massive scalar field. To get also the equation for the 4 d massive vector we have to redefine $A^{\mu}$

$$
\begin{equation*}
A^{\mu(n)}=V^{\mu(n)}-\frac{\frac{i n}{\sqrt{\phi R^{2} M^{2}}} \partial^{\mu} \varphi^{(n)}}{\sqrt{\frac{n^{2}}{\phi R^{2}}+M^{2}}} \tag{3.102}
\end{equation*}
$$

so that $\partial_{\mu} V^{\mu(n)}=0$ as it should be for a massive vector. In fact using this redefinition in the second equation of (3.99) we have

$$
\begin{equation*}
\partial_{\mu} F^{\mu \nu}=\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right) V^{\nu(n)} \tag{3.103}
\end{equation*}
$$

We have therefore found the correct definition of the new 4 dimensional fields. Now we can use them to compute the quantum potential we were looking for. The action of a massive vector field is

$$
\begin{align*}
S & =\int d^{4} x d y\left[-\frac{1}{4} F_{M N} F^{M N}-\frac{1}{2} M^{2} V_{M} V^{M}\right]  \tag{3.104}\\
& =\int d^{4} x d y\left(-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{2} F_{\mu 5} F^{\mu 5}-\frac{M^{2}}{2} A_{\mu} A^{\mu}-\frac{M^{2}}{2} A_{5} A^{5}\right)
\end{align*}
$$

With the above field definition we have that

$$
\begin{equation*}
F_{\mu 5}=\sum_{n=-\infty}^{+\infty} e^{i n y / R}\left(\partial_{\mu} \varphi^{(n)}-\frac{i n}{R} V_{\mu}^{(n)}-\frac{\frac{n^{2}}{\phi R^{2}} \partial_{\mu} \varphi^{(n)}}{\frac{n^{2}}{\phi R^{2}}+M^{2}}\right) \tag{3.105}
\end{equation*}
$$

Therefore

$$
\begin{align*}
S & =\int d^{4} x d y \sum_{n, m} e^{i(n+m) y / R}\left[-\frac{1}{4} F_{\mu \nu}^{(n)} F^{\mu \nu(n)}\right. \\
& -\frac{1}{2 \phi}\left(\sqrt{\left.\frac{M^{2} \phi}{M^{2}+\frac{n^{2}}{\phi R^{2}}} \partial_{\mu} \varphi^{(n)}-\frac{i n}{R} V_{\mu}^{(n)}\right)\left(\sqrt{\frac{M^{2} \phi}{M^{2}+\frac{m^{2}}{\phi R^{2}}}} \partial^{\mu} \varphi^{(m)}-\frac{i m}{R} V^{\mu(m)}\right)}\right. \\
& -\frac{M^{2}}{2 \phi} \sqrt{\frac{\phi}{M^{2}}\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right)} \varphi^{(n)} \sqrt{\frac{\phi}{M^{2}}\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right)} \varphi^{(m)}  \tag{3.106}\\
& \left.-\frac{M^{2}}{2}\left(V_{\mu}^{(n)}-\frac{\frac{i n}{\sqrt{\phi R^{2} M^{2}}} \partial_{\mu} \varphi^{(n)}}{\sqrt{\frac{n^{2}}{\phi R^{2}}+M^{2}}}\right)\left(V^{\mu(m)}-\frac{\frac{i m}{\sqrt{\phi R^{2} M^{2}}} \partial^{\mu} \varphi^{(m)}}{\sqrt{\frac{m^{2}}{\phi R^{2}}+M^{2}}}\right)\right] \\
& =\sum_{n} \int d^{4} x\left[-\frac{1}{4} F_{\mu \nu}^{(n)} F^{\mu \nu(-n)}-\frac{1}{2}\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right) V_{\mu}^{(n)} V^{\mu(-n)}\right. \\
& \left.-\frac{1}{2} \partial_{\mu} \varphi^{(n)} \partial^{\mu} \varphi^{(-n)}-\frac{1}{2}\left(M^{2}+\frac{n^{2}}{\phi R^{2}}\right) \varphi^{(n)} \varphi^{(-n)}\right] .
\end{align*}
$$

Now we can clearly recognise the action of a 4 d massive vector field with mass $M^{2}+\frac{n^{2}}{\phi R^{2}}$ and of a scalar field with the same mass. Going to the momentum space gives

$$
\begin{align*}
S & =-\int d^{4} k \sum_{n}\left\{\tilde{V}_{\mu}^{(n)}(k)\left[\frac{1}{2} \eta^{\mu \nu}\left(k^{2}+\frac{n^{2}}{\phi L^{2}}+M^{2}\right)-\frac{1}{2} k^{\mu} k^{\nu}\right] \tilde{V}_{\nu}^{(-n)}(-k)\right.  \tag{3.107}\\
& \left.+\tilde{\varphi}^{(n)}(k)\left(k^{2}+\frac{n^{2}}{\phi L^{2}}+M^{2}\right) \tilde{\varphi}^{(-n)}(-k)\right\} .
\end{align*}
$$

In momentum space the zero divergence condition of the massive vector becomes $k^{\mu} V_{\mu}=0$. Such condition will make the first term contribute to the final effective potential with a factor $3 \log \left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right)$ since the functional trace of eq.(3.83) is evaluated using the eigenvalues of the matrix constructed with this first term. The second one, instead, will contribute with $\log \left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right)$.

The result for the other particles is similar to the one of the massive scalar and massive vector. In fact in general the formula for the potential is [4]

$$
\begin{equation*}
V_{S^{1}}=(-1)^{2 s_{p}+1} \frac{n_{p}}{2} \sum_{n} \frac{1}{2 \pi R \sqrt{\phi}} \int \frac{d^{4} k}{(2 \pi)^{4}} \log \left(k^{2}+\frac{n^{2}}{\phi R^{2}}+M^{2}\right), \tag{3.108}
\end{equation*}
$$

which, when compactifying from 4 to 3 dimensions, turns into eq.(3.72) after calculations similar to the ones presented in the paragraph 3.2.1. Here $s_{p}$ is the spin of the considered particle and $n_{p}$ its degrees of freedom. In the following we will consider the graviton (with spin 2 and 2 degrees of freedom), the photon (spin 1 and 2 degrees of freedom), the neutrinos (fermions with spin $1 / 2$ and 2 or 4 degrees of freedom depending on their nature being Majorana or Dirac) and eventually an axion (a scalar boson with spin 0 and 1 degree of freedom).

## Chapter 4

## Application to the SM

In order to apply the conjectures on the results obtained in the previous chapter we need to study the actions (3.23) and (3.59) with the addition of the potential from the 1-loop quantum corrections. This is the same approach followed by $[1,2,3,4]$. The first step is to check the consistency of the dimensional truncation performed in the third chapter, i.e. we have to check if the equations of motion (EoM) computed from the 4 d action and the compactified are equal to the ones obtained directly from the lower dimensional action [27]. The next point is to analyse the background solutions so that we can understand what are the conditions to have a vacuum and which kind of vacuum it is. Eventually we study the stability of such vacua perturbing the background.

## 4.1 $S^{1}$ case

As a warm-up let us begin with the equations of motion for the action compactified on the circle. Let us compute them before the compactification. They can be split in two: the ones that include only the non-cyclic dimensions and the one along the compactified dimension

$$
\begin{align*}
& \hat{R}_{\mu \nu}-\frac{1}{2} \hat{g}_{\mu \nu} \hat{R}+\Lambda \hat{g}_{\mu \nu}=0  \tag{4.1}\\
& \hat{R}_{z z}-\frac{1}{2} \hat{g}_{z z} \hat{R}+\Lambda \hat{g}_{z z}=0 .
\end{align*}
$$

We do not consider the equation with indices $\mu z$ because $g_{\mu z}=0$ and the vielbein choice we made is

$$
E_{M}^{I}=\left(\begin{array}{cc}
e^{\beta \phi} e_{\mu}{ }^{i} & 0  \tag{4.2}\\
0 & e^{\alpha \phi}
\end{array}\right)
$$

therefore we get that

$$
\begin{equation*}
\hat{R}_{\mu z}=E_{\mu}{ }^{I} E_{z}{ }^{J} \hat{R}_{I J}=E_{\mu}{ }^{i} E_{z}{ }^{3} \hat{R}_{i 3}, \tag{4.3}
\end{equation*}
$$

but this component of the Ricci tensor is zero. This can be understood becasue

$$
\begin{equation*}
\hat{R}_{j 3}=R_{j i 3}^{i}=-\eta^{i k} R_{k i}{ }^{3}{ }_{j}, \tag{4.4}
\end{equation*}
$$

and, using eq.(3.17) we can read the value from eq.(3.14). Going back to the equations of motion note that these are exactly the Einstein field equations reflecting the chosen parameterization of the metric. The compactification needs now to be made explicit. We need then to compute the Ricci tensor

$$
\begin{equation*}
\hat{R}_{i j}=R_{i K j}^{K}=R_{i k j}^{k}+R_{i 3 j}^{3} . \tag{4.5}
\end{equation*}
$$

These two components of the Riemann tensor can be read again from eq.(3.14) using eq.(3.17)

$$
\begin{align*}
& R^{k}{ }_{i k j}= R_{i k j}^{l} \delta_{l}^{k}=e^{-2 \beta \phi} R_{i j}+2 e^{-2 \beta \phi}\left(\beta \delta_{[i}^{k} \partial_{k]} \phi \partial_{j} \phi-\beta \eta_{j[i} \partial_{k]} \partial^{k} \phi+\beta^{2} \eta_{j i} \partial_{k]} \phi \partial^{k} \phi-\beta^{2} \delta_{[i}^{k} \partial_{k]} \phi \partial_{j} \phi\right. \\
&\left.-\beta^{2} \delta_{[k}^{k} \eta_{i] j} \partial^{m} \phi \partial_{m} \phi\right)-2 \beta e^{-2 \beta \phi}\left(\delta_{[i}^{k} \omega_{k] j}^{l} \partial_{l} \phi+\eta_{j[i} \omega_{k]}{ }^{k l} \partial_{l} \phi\right), \\
& R_{i 3 j}^{3}=-e^{-2 \beta \phi}\left(\alpha(\alpha-2 \beta) \partial_{i} \phi \partial_{j} \phi+\alpha \partial_{i} \partial_{j} \phi+\alpha \beta \eta_{i j} \partial_{k} \phi \partial^{k} \phi-\alpha \omega_{j}{ }_{i}^{k} \partial_{k} \phi\right) . \tag{4.6}
\end{align*}
$$

So that, after specifying that in 3 dimensions $\alpha=-\frac{1}{2}$ and $\beta=\frac{1}{2}$, the Ricci tensor is

$$
\begin{equation*}
\hat{R}_{i j}=e^{-\phi}\left(R_{i j}-\frac{1}{2} \partial_{i} \phi \partial_{j} \phi-\frac{1}{2} \eta_{i j} \square \phi\right) . \tag{4.7}
\end{equation*}
$$

However in eq.(4.1) the Ricci tensor with spacetime indices appears. Since we chose as vielbeins (4.2) then the Ricci tensor we were looking for is

$$
\begin{equation*}
\hat{R}_{\mu \nu}=E_{\mu}{ }^{i} E_{\nu}{ }^{i} \hat{R}_{i j}=R_{\mu \nu}-\frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi-\frac{1}{2} g_{\mu \nu} \square \phi . \tag{4.8}
\end{equation*}
$$

From this we can easily get the Ricci scalar

$$
\begin{equation*}
\hat{R}=e^{-\phi}\left(R-\frac{1}{2}(\partial \phi)^{2}-2 \square \phi\right) \tag{4.9}
\end{equation*}
$$

Then, knowing that $\hat{g}_{\mu \nu}=e^{\phi} g_{\mu \nu}$, the first equation of (4.1) becomes

$$
\begin{equation*}
R_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}+e^{\phi} \Lambda g_{\mu \nu}+\frac{1}{4} g_{\mu \nu}(\partial \phi)^{2}-\frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi=0 \tag{4.10}
\end{equation*}
$$

The same reasoning can be done for $\hat{R}_{z z}$, we start from the Ricci tensor with flat indices

$$
\begin{equation*}
\hat{R}_{33}=R_{3 I 3}^{I}=R_{3 i 3}^{i}=R_{i 3}^{3}{ }^{i}, \tag{4.11}
\end{equation*}
$$

where the last expression can be read directly from the second line of eq.(3.18) fixing $\alpha$ and $\beta$

$$
\begin{equation*}
\hat{R}_{33}=\frac{1}{2} e^{-\phi} \square \phi \tag{4.12}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\hat{R}_{z z}=E_{z}{ }^{3} E_{z}{ }^{3} \hat{R}_{33}=\frac{1}{2} e^{-2 \phi} \square \phi \tag{4.13}
\end{equation*}
$$

Therefore, knowing that $\hat{g}_{z z}=e^{-\phi}$, the second equation of motion becomes

$$
\begin{equation*}
R-2 \square \phi-\frac{1}{2}(\partial \phi)^{2}-2 \Lambda e^{\phi}=0 \tag{4.14}
\end{equation*}
$$

This last equation can be further developed taking the trace of eq.(4.10)

$$
\begin{equation*}
R-\frac{1}{2}(\partial \phi)^{2}=6 \Lambda e^{\phi} \tag{4.15}
\end{equation*}
$$

and inserting it into eq.(4.14)

$$
\begin{equation*}
\square \phi=2 \Lambda e^{\phi} . \tag{4.16}
\end{equation*}
$$

To check the consistency of the truncation we have make sure that both eq.(4.10) and eq.(4.16) are exactly the same even after the compactification. Let us start from the action (3.23), if we vary the metric we obtain

$$
\begin{gather*}
R_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}-\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu \nu}}(\sqrt{-g} \mathcal{L})=0 \\
\Downarrow  \tag{4.17}\\
R_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}+e^{\phi} \Lambda g_{\mu \nu}+\frac{1}{4} g_{\mu \nu}(\partial \phi)^{2}-\frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi=0
\end{gather*}
$$

that is precisely what we wanted. The same happens for the dilaton $\phi$

$$
\begin{equation*}
\frac{\delta \mathcal{L}}{\delta \phi}-\partial_{\mu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} \phi}=0 \quad \Longrightarrow \quad \square \phi=2 \Lambda e^{\phi} \tag{4.18}
\end{equation*}
$$

Thus in the circle case the truncation is consistent.

## $4.2 T^{2}$ case

Let us do the procedure again for the torus case. The equations of motion before the compactification are

$$
\begin{equation*}
\hat{R}_{M N}-\frac{1}{2} \hat{g}_{M N} \hat{R}+\Lambda \hat{g}_{M N}=0 \tag{4.19}
\end{equation*}
$$

To make the compactification explicit we are going to need eqs.(3.55, 3.56, 3.57, 3.58) and the last component of the Ricci tensor we previously neglected, namely

$$
\begin{equation*}
R_{32}=R_{3 i 2}^{i}=-\eta^{i j} R_{j 3}{ }^{2}{ }_{i} . \tag{4.20}
\end{equation*}
$$

This component of the Ricci tensor can be read from eq.(3.41) using eq.(3.45), therefore

$$
\begin{equation*}
R_{32}=-\frac{\square \tau_{1}}{2 \tau_{2}}+\frac{\partial_{i} \tau_{1} \partial^{i} \tau_{2}}{\tau_{2}^{2}}-\frac{\partial_{i} \tau_{1} \partial^{i} A}{2 A \tau_{2}} . \tag{4.21}
\end{equation*}
$$

To switch the vielbein indices with the spacetime ones let us remind the choice we made

$$
E_{M}^{I}=\left(\begin{array}{ccc}
e_{\mu}{ }^{i} & 0^{0} & 0_{2}  \tag{4.22}\\
\mathbb{O}_{2} & \sqrt{\frac{A}{\tau_{2}}} & 0 \\
& \sqrt{\frac{A}{\tau_{2}}} \tau_{1} & \frac{1}{\sqrt{A \tau_{2}}}
\end{array}\right) .
$$

Then we have

$$
\begin{equation*}
\hat{R}_{\mu \nu}=e_{\mu}{ }^{i} e_{\nu}{ }^{j} \hat{R}_{i j}=R_{\mu \nu}-\frac{\nabla_{\nu} \partial_{\mu} A}{A}+\frac{\partial_{\mu} A \partial_{\nu} A}{2 A^{2}}-\frac{\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}}{2 \tau_{2}^{2}}-\frac{\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}}{2 \tau_{2}^{2}}, \tag{4.23}
\end{equation*}
$$

$R_{y y}=E_{y}{ }^{2} E_{y}{ }^{2} R_{22}+E_{y}{ }^{3} E_{y}{ }^{3} R_{33}+2 E_{y}{ }^{3} E_{y}{ }^{2}=-\frac{\square A}{2 \tau_{2}}+\frac{A \square \tau_{2}}{2 \tau_{2}^{2}}+\frac{A}{2 \tau_{2}^{3}}\left(\left(\partial \tau_{1}\right)^{2}-\left(\partial \tau_{2}\right)^{2}\right)+\frac{\nabla^{\mu} A \nabla_{\mu} \tau_{2}}{2 \tau_{2}^{2}}$,
$R_{y z}=E_{y}{ }^{2} E_{z}{ }^{2} R_{22}+E_{y}{ }^{2} E_{z}{ }^{3} R_{23}=\frac{A}{\tau_{2}} \tau_{1}\left(-\frac{\square A}{2 A}+\frac{\square \tau_{2}}{2 \tau_{2}}-\frac{1}{2} \frac{\left(\partial \tau_{2}\right)^{2}}{\tau_{2}^{2}}+\frac{1}{2} \frac{\left(\partial \tau_{1}\right)^{2}}{\tau_{2}^{2}}+\frac{\nabla^{\mu} A \nabla_{\mu} \tau_{2}}{2 A \tau_{2}}\right)$

$$
\begin{equation*}
+A\left(\frac{\nabla^{\mu} \tau_{1} \nabla_{\mu} \tau_{2}}{\tau_{2}^{2}}-\frac{\square \tau_{1}}{2 \tau_{2}}-\frac{\nabla^{\mu} \tau_{1} \nabla^{\mu} A}{2 A \tau_{2}}\right) \tag{4.25}
\end{equation*}
$$

$$
\begin{equation*}
R_{z z}=E_{z}{ }^{2} E_{z}{ }^{2} R_{22}+E_{z}{ }^{3} E_{z}{ }^{3} R_{33}+2 E_{z}{ }^{2} E_{z}^{3} R_{32}=2 A \tau_{1}\left(\frac{\nabla_{\mu} \tau_{1} \nabla^{\mu} \tau_{2}}{\tau_{2}^{2}}-\frac{\square \tau_{1}}{2 \tau_{2}}-\frac{\nabla_{\mu} \tau_{1} \nabla^{\mu} A}{2 A \tau_{2}}\right) \tag{4.26}
\end{equation*}
$$

We have now all the terms we need. Let us expand eq.(4.19) in the different components

$$
\begin{align*}
\hat{R}_{\mu \nu}-\frac{1}{2} \hat{g}_{\mu \nu} \hat{R}+\Lambda \hat{g}_{\mu \nu}= & R_{\mu \nu}-\frac{\nabla_{\mu} \partial_{\nu} A}{A}+\frac{\partial_{\mu} A \partial_{\nu} A}{2 A^{2}}-\frac{1}{2 \tau_{2}^{2}}\left(\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}+\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}\right) \\
& -\frac{1}{2} g_{\mu \nu}\left(R-\frac{2 \square A}{A}+\frac{(\partial A)^{2}}{2 A^{2}}-\frac{\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}\right)+\Lambda g_{\mu \nu}=0,  \tag{4.27}\\
R_{y y}-\frac{1}{2} g_{y y} \hat{R}+\Lambda g_{y y}= & R-\frac{\square A}{A}-\frac{\square \tau_{2}}{\tau_{2}}+\frac{(\partial A)^{2}}{2 A^{2}}-\frac{3\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}+\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{2}}{A \tau_{2}}-2 \Lambda=0,  \tag{4.28}\\
R_{y z}-\frac{1}{2} g_{y z} \hat{R}+\Lambda g_{y z}= & R-\frac{\square A}{A}-\frac{\square \tau_{2}}{\tau_{2}}+\frac{\square \tau_{1}}{\tau_{1}}+\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}-\frac{3\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{2}}{A \tau_{2}}  \tag{4.29}\\
& -\frac{2}{\tau_{1} \tau_{2}} \nabla_{\mu} \tau_{1} \nabla^{\mu} \tau_{2}+\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{1}}{A \tau_{1}}+\frac{(\partial A)^{2}}{2 A^{2}}-2 \Lambda=0,
\end{align*}
$$

$$
\begin{align*}
R_{z z}-\frac{1}{2} g_{z z} \hat{R}+\Lambda g_{z z}= & \frac{A}{2 \tau_{2}}|\tau|^{2}\left(R-\frac{\square A}{A}-2 \Lambda+\frac{(\partial A)^{2}}{2 A^{2}}\right)-\left(\frac{A}{2}+\frac{A \tau_{1}^{2}}{2 \tau_{2}^{2}}\right) \square \tau_{2}+\frac{A \tau_{1}}{\tau_{2}} \square \tau_{1} \\
& +\left(\frac{A \tau_{1}^{2}}{4 \tau_{2}^{3}}-\frac{3 A}{4 \tau_{2}}\right)\left(\partial \tau_{2}\right)^{2}+\left(\frac{A}{4 \tau_{2}}-\frac{3 \tau_{1}^{2}}{4 \tau_{2}^{3}}\right)\left(\partial \tau_{1}\right)^{2}+\frac{\tau_{1}}{\tau_{2}} g^{\mu \nu} \partial_{\mu} A \partial_{\nu} \tau_{1}  \tag{4.30}\\
& +\left(\frac{1}{2}-\frac{\tau_{1}^{2}}{2 \tau_{2}^{2}}\right) g^{\mu \nu} \partial_{\mu} A \partial_{\nu} \tau_{2}-\frac{2 A \tau_{1}}{\tau_{2}^{2}} g^{\mu \nu} \partial_{\mu} \tau_{1} \partial_{\nu} \tau_{2}=0 .
\end{align*}
$$

We have then found the equations of motion before the compactification. Now they have to be compared with the ones derived from the action (3.59). Let us vary the metric, we get [28]

$$
\begin{align*}
\delta_{g_{\mu \nu}} S & =\frac{1}{16 \pi G} \int d^{2} x \sqrt{-g}\left[A\left(R_{\mu \nu}+\nabla_{\mu} \nabla_{\nu}+g_{\mu \nu} \nabla^{2}\right) \delta g^{\mu \nu}-\frac{1}{2} g_{\mu \nu}\left(A R+\frac{(\partial A)^{2}}{2 A}\right.\right. \\
& \left.\left.-\frac{A}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)-2 \Lambda A\right) \delta g^{\mu \nu}+\left(\frac{\partial_{\mu} A \partial_{\nu} A}{2 A}-\frac{A}{2 \tau_{2}^{2}}\left(\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}+\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}\right)\right) \delta g^{\mu \nu}\right]=0 \tag{4.31}
\end{align*}
$$

Integrating by parts the first term and setting the boundary terms to zero we extract the equation of motion

$$
\begin{align*}
& \left(R_{\mu \nu}+\nabla_{\mu} \nabla_{\nu}+g_{\mu \nu} \square\right) A-\frac{1}{2} g_{\mu \nu}\left(A R+\frac{(\partial A)^{2}}{2 A}-\frac{A}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)-2 \Lambda A\right)  \tag{4.32}\\
& +\frac{\partial_{\mu} A \partial_{\nu} A}{2 A}-\frac{A}{2 \tau_{2}^{2}}\left(\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}+\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}\right)=0
\end{align*}
$$

that is equal to eq.(4.27). Let us derive also the equations of motion for the other fields:

- varying $\tau_{1}$ we get

$$
\begin{equation*}
A \square \tau_{1}+\nabla_{\mu} A \nabla^{\mu} \tau_{1}-\frac{2 A}{\tau_{2}} \nabla_{\mu} \tau_{2} \nabla^{\mu} \tau_{1}=0 \tag{4.33}
\end{equation*}
$$

- varying $\tau_{2}$

$$
\begin{equation*}
A \square \tau_{2}+\frac{A}{\tau_{2}}\left(\left(\partial \tau_{1}\right)^{2}-\left(\partial \tau_{2}\right)^{2}\right)+\nabla_{\mu} A \nabla_{\nu} \tau_{2}=0 \tag{4.34}
\end{equation*}
$$

- varying $A$

$$
\begin{equation*}
R+\frac{(\partial A)^{2}}{2 A^{2}}-\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\left(\partial \tau_{1}\right)^{2}}{2 \tau_{1}^{2}}-\frac{\square A}{A}-2 \Lambda=0 . \tag{4.35}
\end{equation*}
$$

These last three look very different from the previous ones but, actually, they are the same if combined properly

$$
\begin{align*}
& (4.28)=(4.35)-\frac{1}{A \tau_{2}}(4.34) \\
& (4.29)=(4.35)-\frac{1}{A \tau_{2}}(4.34)+\frac{1}{A \tau_{2}}(4.33)  \tag{4.36}\\
& (4.30)=\frac{A}{2 \tau_{2}}|\tau|^{2}(4.35)+\left(\frac{1}{2}-\frac{\tau_{1}^{2}}{2 \tau_{2}^{2}}\right)(4.34)+\tau_{1} \tau_{2}(4.33)
\end{align*}
$$

The equations of motion before and after the compactification match and therefore the truncation from 4 to 2 dimension on a torus is consistent.

### 4.2.1 Vacuum conditions

We have found the equations of motion for the compactifications on a circle and on a torus. In order to apply the conjectures we need to derive a set of conditions that allow us to identify the vacua and to recognize their stability or instability. First of all the potential coming from the quantum corrections has to be included in the action, therefore we make the substitution

$$
\begin{equation*}
M_{p}^{2} \Lambda \longrightarrow V\left(A, \tau_{1}, \tau_{2}\right) \tag{4.37}
\end{equation*}
$$

in eq.(3.59) for the $T^{2}$ case, where $M_{p}=\frac{1}{\sqrt{8 \pi G}}$ is the reduced Planck mass. The $S^{1}$ is much simpler since the potential depends only on the radius of the circle $R$, therefore it is theoretically easy to to find the critical points: they are just the ones where the first derivative of the potential vanishes. Even stability and instability are simply set by the second derivative. Then the nature of the vacua, i.e. Minkowski, de Sitter or Anti de Sitter, is fixed by the value of the potential computed at the critical points, respectively zero, positive or negative.

As far as the torus is concerned the situation gets more complicated, we have now a function depending on three variables. Let us start looking at the equations of motion we derived. They can be further simplified. Firstly note that the Einstein tensor in two dimensions vanishes identically [32]. Indeed the Riemann tensor can be written as

$$
\begin{equation*}
R_{\alpha \beta \gamma \delta}=\frac{R}{2}\left(g_{\alpha \delta} g_{\beta \gamma}-g_{\alpha \gamma} g_{\beta \delta}\right) \tag{4.38}
\end{equation*}
$$

and therefore the Ricci tensor is

$$
\begin{equation*}
R_{\mu \nu}=R_{\mu \gamma \nu}^{\gamma}=\frac{1}{2} g_{\mu \nu} R . \tag{4.39}
\end{equation*}
$$

Such property sets the Einstein tensor to zero identically

$$
\begin{equation*}
G_{\mu \nu} \equiv R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R=0 \tag{4.40}
\end{equation*}
$$

This characteristic of two dimensional gravity and the substitution (4.37) we made lead to slightly different equations of motion. For $A$ we have

$$
\begin{equation*}
R+\frac{(\partial A)^{2}}{2 A^{2}}-\frac{\square A}{A}-\frac{\left(\partial \tau_{1}\right)^{2}}{2 \tau_{2}^{2}}-\frac{\left(\partial \tau_{2}\right)^{2}}{2 \tau_{2}^{2}}-\frac{2}{M_{p}^{2}}\left(V+A \frac{\partial V}{\partial A}\right)=0 \tag{4.41}
\end{equation*}
$$

for $\tau_{1}$

$$
\begin{equation*}
\square \tau_{1}+\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{1}}{A}-\frac{2 \nabla_{\mu} \tau_{2} \nabla^{\mu} \tau_{1}}{\tau_{2}}-\frac{2 \tau_{2}^{2}}{M_{p}^{2}} \frac{\partial V}{\partial \tau_{1}}=0 \tag{4.42}
\end{equation*}
$$

for $\tau_{2}$

$$
\begin{equation*}
\square \tau_{2}+\frac{1}{\tau_{2}}\left(\left(\partial \tau_{1}\right)^{2}-\left(\partial \tau_{2}\right)^{2}\right)+\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{2}}{A}-\frac{2 \tau_{2}^{2}}{M_{p}^{2}} \frac{\partial V}{\partial \tau_{2}}=0 \tag{4.43}
\end{equation*}
$$

for the metric

$$
\begin{align*}
& g_{\mu \nu} \square A-\nabla_{\mu} \partial_{\nu} A-\frac{A}{2} g_{\mu \nu}\left(\frac{(\partial A)^{2}}{2 A^{2}}-\frac{1}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)-\frac{2}{M_{p}^{2}} V\right)+\frac{\partial_{\mu} A \partial_{\nu} A}{2 A}  \tag{4.44}\\
& -\frac{A}{2 \tau_{2}^{2}}\left(\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}+\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}\right)=0 .
\end{align*}
$$

This last one can be simplified even further, let us take its trace

$$
\begin{equation*}
\frac{\square A}{A}=-\frac{2}{M_{p}^{2}} V\left(A, \tau_{1}, \tau_{2}\right) \tag{4.45}
\end{equation*}
$$

and inserting it back into eq.(4.44) gives

$$
\begin{equation*}
\frac{A}{2} g_{\mu \nu}\left(-\frac{2}{M_{p}^{2}} V-\frac{(\partial A)^{2}}{2 A^{2}}+\frac{1}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)\right)-\nabla_{\mu} \partial_{\nu} A+\frac{\partial_{\mu} A \partial_{\nu} A}{2 A}-\frac{A}{2 \tau_{2}^{2}}\left(\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}+\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}\right)=0 \tag{4.46}
\end{equation*}
$$

These equations describe the evolution of four fields: $A, \tau_{1}, \tau_{3}$ and the metric. The latter is more easily handled if we trade $g_{\mu \nu}$ with a conformal factor $\sigma$. In fact in two dimensions any Riemannian manifold is conformally flat [33], namely it is always possible to write

$$
\begin{equation*}
g_{\mu \nu}=e^{2 \sigma} \eta_{\mu \nu} \tag{4.47}
\end{equation*}
$$

where $\eta$ is the Minkowski flat metric (with $(-,+)$ signature in our case). We need to write explicitly the equations of motions in terms of $\sigma$, indeed both the Ricci scalar and the d'Alambert operator, containing the metric, need to be written again. The Christoffel symbols become

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho}=\frac{1}{2} g^{\mu \lambda}\left(\partial_{\mu} g_{\nu \lambda}+\partial_{\nu} g_{\mu \lambda}-\partial_{\lambda} g_{\mu \nu}\right)=\delta_{\nu}^{\rho} \partial_{\mu} \sigma+\delta_{\mu}^{\rho} \partial_{\nu} \sigma-\eta_{\mu \nu} \eta^{\rho \lambda} \partial_{\lambda} \sigma \tag{4.48}
\end{equation*}
$$

thus the Ricci tensor

$$
\begin{equation*}
R_{\mu \nu}=\partial_{\rho} \Gamma_{\mu \nu}^{\rho}-\partial_{\mu} \Gamma_{\rho \nu}^{\rho}-\Gamma_{\mu \lambda}^{\rho} \Gamma_{\rho \nu}^{\lambda}+\Gamma_{\lambda \rho}^{\lambda} \Gamma_{\mu \nu}^{\rho}=-\eta_{\mu \nu} \square_{\eta} \sigma \tag{4.49}
\end{equation*}
$$

where $\square_{\eta}$ is the d'Alambert operator intended with the flat metric. Eventually we get to the Ricci scalar

$$
\begin{equation*}
R=g^{\mu \nu} R_{\mu \nu}=-2 e^{-2 \sigma} \square_{\eta} \sigma \tag{4.50}
\end{equation*}
$$

The expression $\square_{g} f$ with the metric $g$ and where $f$ is a generic scalar function becomes

$$
\begin{equation*}
\square_{g} f=\nabla^{\mu} \partial_{\mu} f=e^{-2 \sigma} \eta^{\mu \nu}\left(\partial_{\mu} \partial_{\nu} f-\Gamma_{\mu \nu}^{\rho} \partial_{\rho} f\right)=e^{-2 \sigma} \square_{\eta} f \tag{4.51}
\end{equation*}
$$

From now on we will write for simplicity $\square=\square_{\eta}$. Inserting these results into the equations of motion gives
$\square \sigma-\frac{(\partial A)^{2}}{4 A^{2}}+\frac{\left(\partial \tau_{1}\right)^{2}}{4 \tau_{2}^{2}}+\frac{\left(\partial \tau_{2}\right)^{2}}{4 \tau_{2}^{2}}+\frac{A}{M_{p}^{2}} e^{2 \sigma} \frac{\partial V}{\partial A}=0$,
$\square \tau_{1}+\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{1}}{A}-\frac{2 \nabla_{\mu} \tau_{2} \nabla^{\mu} \tau_{1}}{\tau_{2}}-\frac{2 \tau_{2}^{2}}{M_{p}^{2}} e^{2 \sigma} \frac{\partial V}{\partial \tau_{1}}=0$,
$\square \tau_{2}+\frac{1}{\tau_{2}}\left(\left(\partial \tau_{1}\right)^{2}-\left(\partial \tau_{2}\right)^{2}\right)+\frac{\nabla_{\mu} A \nabla^{\mu} \tau_{2}}{A}-\frac{2 \tau_{2}^{2}}{M_{p}^{2}} e^{2 \sigma} \frac{\partial V}{\partial \tau_{2}}=0$,
$\frac{1}{2} \eta_{\mu \nu}\left(-\frac{2}{M_{p}^{2}} e^{2 \sigma} V-\frac{(\partial A)^{2}}{2 A^{2}}+\frac{1}{2 \tau_{2}^{2}}\left(\left(\partial \tau_{1}\right)^{2}+\left(\partial \tau_{2}\right)^{2}\right)\right)$
$-\frac{1}{A}\left(\partial_{\mu} \partial_{\nu} A-\partial_{\mu} \sigma \partial_{\nu} A-\partial_{\nu} \sigma \partial_{\mu} A+\eta_{\mu \nu} \nabla_{\rho} \sigma \nabla^{\rho} A\right)$
$+\frac{\partial_{\mu} A \partial_{\nu} A}{2 A^{2}}-\frac{1}{2 \tau_{2}^{2}}\left(\partial_{\mu} \tau_{1} \partial_{\nu} \tau_{1}+\partial_{\mu} \tau_{2} \partial_{\nu} \tau_{2}\right)=0$,
and in the trace (4.45) leads to

$$
\begin{equation*}
\frac{\square A}{A}=-\frac{2}{M_{p}^{2}} e^{2 \sigma} V \tag{4.56}
\end{equation*}
$$

Since $V$ depends on three variables $A, \tau_{1}$ and $\tau_{2}$ we need to understand what are the conditions on $V$ when those fields are constant, that is when they are at a critical point. Let us then define the background solutions as $A, \tau_{1}, \tau_{2}=$ constant, then the equations of motion become

$$
\begin{gather*}
R=\frac{2 A}{M_{p}^{2}} \frac{\partial V}{\partial A} \Longrightarrow \square \sigma=-\frac{A}{M_{p}^{2}} e^{2 \sigma} \frac{\partial V}{\partial A}  \tag{4.57}\\
\frac{\partial V}{\partial \tau_{1}}=0  \tag{4.58}\\
\frac{\partial V}{\partial \tau_{2}}=0  \tag{4.59}\\
V=0 \tag{4.60}
\end{gather*}
$$

Therefore, to have a vacuum solution the potential has to fulfill such conditions. From eq.(4.57) it is possible to understand what is the type of space we are analyzing: $\frac{2 A}{M_{p}^{2}} \frac{\partial V}{\partial A}$ is the curvature of the background space, therefore

- if $\frac{\partial V}{\partial A}=0$ the space is flat, namely Minkowski,
- if $\frac{\partial V}{\partial A}<0$ the curvature is negative, thus the space is Anti de Sitter,
- if $\frac{\partial V}{\partial A}>0$ the curvature is positive, thus the space is de Sitter.

The next step is to understand how the fields in the background evolve after a small perturbation because the conjectures can be applied only to stable vacua. The three cases for the three different spaces need to be studied separately.

## Minkowski

In Minkowski space we have that $g_{\mu \nu}=\eta_{\mu \nu}$, so, through eq.(4.47), we have

$$
\begin{equation*}
\sigma=0 \tag{4.61}
\end{equation*}
$$

Let us perturb the constant background value of the fields with small fluctuations, i.e. $\sigma=0+\delta \sigma$, $\tau_{1}=$ constant $+\delta \tau_{1}, \tau_{2}=$ constant $+\delta \tau_{2}$ and $A=$ constant $+\delta A$. From the equations of motions we get

$$
\begin{align*}
& \square \delta \sigma=-\frac{A}{M_{p}^{2}}\left(\delta A \partial_{A} \partial_{A} V+\delta \tau_{1} \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{A} \partial_{\tau_{2}} V\right),  \tag{4.62}\\
& \square \delta \tau_{1}=\frac{2}{M_{p}^{2}} \tau_{2}^{2}\left(\delta A \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{\tau_{1}} \partial_{\tau_{2}} V\right),  \tag{4.63}\\
& \square \delta \tau_{2}=\frac{2}{M_{p}^{2}} \tau_{2}^{2}\left(\delta A \partial_{A} \partial_{\tau_{2}} V+\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{2}} V+\delta \tau_{2} \partial_{\tau_{2}} \partial_{\tau_{2}} V\right), \tag{4.64}
\end{align*}
$$

$\partial_{\mu} \partial_{\nu} \delta A=0$.
The last one can be explicitly solved as

$$
\begin{equation*}
\delta A(x, t)=c_{1} x+c_{2} t+c_{3}, \tag{4.66}
\end{equation*}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are three arbitrary constants. This type of equation is telling us that we can set $\delta A=0[4]$. The solution that includes $x$ is clearly non-normalizable and if, for example, we choose the reference frame where $x=t$, then eq.(4.66) describes only non-normalizable solutions. To take care of this we need to fix the values of $c_{1}$ and $c_{2}$ to zero and we get $\delta A=c_{3}$. In the end we can just redefine the background value of $A$ to reabsorb $c_{3}$ so that eventually $\delta A=0$. With such result the remaining three equations of motion become

$$
\begin{align*}
\square \delta \sigma & =-\frac{A}{M_{p}^{2}}\left(\delta \tau_{1} \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{A} \partial_{\tau_{2}} V\right),  \tag{4.67}\\
\square \delta \tau_{1} & =\frac{2}{M_{p}^{2}} \tau_{2}^{2}\left(\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{\tau_{1}} \partial_{\tau_{2}} V\right),  \tag{4.68}\\
\square \delta \tau_{2} & =\frac{2}{M_{p}^{2}} \tau_{2}^{2}\left(\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{2}} V+\delta \tau_{2} \partial_{\tau_{2}} \partial_{\tau_{2}} V\right) . \tag{4.69}
\end{align*}
$$

The two dimensional one is a special case, in fact there are not degrees of freedom associated to the metric as we have seen showing that the Einstein tensor vanishes identically. Then even $\delta \sigma$ should not be a real degree of freedom, but it should, instead, be possible to write it in terms of the other fields $\delta \tau_{1}$ and $\delta \tau_{2}$. Then, let us solve firstly the eqs. $(4.68,4.69)$, they can be recast as

$$
\begin{equation*}
\square \overrightarrow{\delta \tau}=\frac{2 \tau_{2}^{2}}{M_{p}^{2}} m^{2} \overrightarrow{\delta \tau} \tag{4.70}
\end{equation*}
$$

where

$$
\overrightarrow{\delta \tau}=\binom{\delta \tau_{1}}{\delta \tau_{2}} \quad \text { and } \quad m^{2}=\left(\begin{array}{ll}
\partial_{\tau_{1}} \partial_{\tau_{1}} V & \partial_{\tau_{1}} \partial_{\tau_{2}} V  \tag{4.71}\\
\partial_{\tau_{1}} \partial_{\tau_{2}} V & \partial_{\tau_{2}} \partial_{\tau_{2}} V
\end{array}\right)=\left(\begin{array}{ll}
m_{11}^{2} & m_{12}^{2} \\
m_{12}^{2} & m_{22}^{2}
\end{array}\right) .
$$

To solve the equation we can use the mass eigenstates diagonalizing the matrix $m^{2}$, therefore we need its eigenvalues

$$
\begin{equation*}
\lambda_{1,2}=\frac{m_{11}^{2}+m_{22}^{2} \pm \sqrt{\left(m_{11}^{2}-m_{22}^{2}\right)^{2}+4 m_{12}^{4}}}{2} \tag{4.72}
\end{equation*}
$$

that we are going to call $M_{11}^{2}$ and $M_{22}^{2}$. From the relative eigenvectors

$$
\begin{align*}
& \vec{v}_{1}=\left(\begin{array}{c}
2 m_{12}^{2} \\
\left.m_{22}^{2}-m_{11}^{2}+\sqrt{\left(m_{11}^{2}-m_{22}^{2}\right)^{2}+4 m_{12}^{4}}\right), ~
\end{array}\right.  \tag{4.73}\\
& \vec{v}_{2}=\left(\begin{array}{c}
2 m_{12}^{2} \\
\left.m_{22}^{2}-m_{11}^{2}-\sqrt{\left(m_{11}^{2}-m_{22}^{2}\right)^{2}+4 m_{12}^{4}}\right)
\end{array}\right. \tag{4.74}
\end{align*}
$$

we can define the diagonalizing matrix $S=\left(\vec{v}_{1}, \vec{v}_{2}\right)$. Applying everything to eq.(4.70) we get

$$
\begin{equation*}
\square \overrightarrow{\delta \tau}=\frac{2 \tau_{2}^{2}}{M_{p}^{2}} S M^{2} S^{-1} \overrightarrow{\delta \tau} \tag{4.75}
\end{equation*}
$$

that becomes

$$
\begin{equation*}
\square \delta \overrightarrow{\tau^{ \pm}}=\frac{2 \tau_{2}^{2}}{M_{p}^{2}} M^{2} \delta \vec{\tau}^{ \pm} \tag{4.76}
\end{equation*}
$$

where we defined

$$
M^{2}=\left(\begin{array}{cc}
M_{11}^{2} & 0  \tag{4.77}\\
0 & M_{22}^{2}
\end{array}\right)
$$

and

$$
\delta \overrightarrow{\tau^{ \pm}}=\binom{\delta \tau^{+}}{\delta \tau^{-}}=S^{-1} \overrightarrow{\delta \tau}=\frac{1}{\operatorname{det} S}\left(\begin{array}{lc}
m_{22}^{2}-m_{11}^{2}-\sqrt{\left(m_{11}^{2}-m_{22}^{2}\right)^{2}+4 m_{12}^{4}} & -2 m_{12}^{2}  \tag{4.78}\\
m_{11}^{2}-m_{22}^{2}-\sqrt{\left(m_{11}^{2}-m_{22}^{2}\right)^{2}+4 m_{12}^{4}} & 2 m_{12}^{2}
\end{array}\right)\binom{\delta \tau_{1}}{\delta \tau_{2}}
$$

In order to insert the solutions for $\delta \tau_{1}$ and $\delta \tau_{2}$ we can invert the above relations, getting

$$
\left\{\begin{array}{l}
\delta \tau_{1}=2 m_{12}^{2}\left(\delta \tau^{+}+\delta \tau^{-}\right)  \tag{4.79}\\
\delta \tau_{2}=2 \delta \tau^{+}\left(M_{11}^{2}-m_{11}^{2}\right)+2 \delta \tau^{-}\left(M_{22}^{2}-m_{22}^{2}\right)
\end{array}\right.
$$

We can now write the solutions for eq.(4.76). Let us begin with the case $M^{2}=0$. The equation is more easily solved in the light cone coordinates

$$
\begin{equation*}
x^{+}=\frac{t+x}{\sqrt{2}}, \quad x^{-}=\frac{t-x}{\sqrt{2}}, \quad d s^{2}=-2 d x^{+} d x^{-} \tag{4.80}
\end{equation*}
$$

With these coordinates eq.(4.68) and eq.(4.69) become

$$
\left\{\begin{array} { l } 
{ \square \delta \tau _ { 1 } = 0 }  \tag{4.81}\\
{ \square \delta \tau _ { 2 } = 0 }
\end{array} \Longrightarrow \left\{\begin{array} { l } 
{ \partial _ { + } \partial _ { - } \delta \tau _ { 1 } = 0 } \\
{ \partial _ { + } \partial _ { - } \delta \tau _ { 2 } = 0 }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\delta \tau_{1}=f_{1}\left(x^{+}\right)+g_{1}\left(x^{-}\right) \\
\delta \tau_{2}=f_{2}\left(x^{+}\right)+g_{2}\left(x^{-}\right)
\end{array}\right.\right.\right.
$$

where $f_{1,2}$ and $g_{1,2}$ are arbitrary functions of one variable. Now it is possible to put these results into eq.(4.67) and integrate twice to get $\delta \sigma$

$$
\begin{align*}
\delta \sigma=\int d x^{+} d x^{-} \frac{\partial}{\partial x^{+}} \frac{\partial}{\partial x^{-}} \delta \sigma= & \frac{\partial_{A} \partial_{\tau_{1}} V}{4}\left(x^{-} \int d x^{+} f_{1}\left(x^{+}\right)+x^{+} \int d x^{-} g_{1}\left(x^{-}\right)\right)  \tag{4.82}\\
& \frac{\partial_{A} \partial_{\tau_{2}} V}{4}\left(x^{-} \int d x^{+} f_{2}\left(x^{+}\right)+x^{+} \int d x^{-} g_{2}\left(x^{-}\right)\right)
\end{align*}
$$

We clearly see that in two dimensions $\sigma$, i.e. gravity, is not a real degree of freedom, but it can be written in terms of other degrees of freedom. Let us move on to the case $M \neq 0$, the solutions to eq.(4.76) are of the type

$$
\begin{equation*}
\delta \tau^{+}=e^{i k x-i \omega t} \tag{4.83}
\end{equation*}
$$

with the additional condition

$$
\begin{equation*}
\omega^{2}=k^{2}+\frac{2 \tau_{2}^{2}}{M_{p}^{2}} M_{11}^{2} \tag{4.84}
\end{equation*}
$$

the same will be true for $\delta \tau^{-}$with $M_{22}^{2}$. We proceed now as in the above, let us use the light cone coordinates

$$
\begin{equation*}
\delta \tau^{+}=e^{\frac{i}{\sqrt{2}}\left[x^{+}(k-\omega)-x^{-}(k+\omega)\right]}, \tag{4.85}
\end{equation*}
$$

and inserting everything in eq.(4.67) gives
$\delta \sigma=\int d x^{+} d x^{-}\left[\frac{m_{12}^{2}}{2} \partial_{A} \partial_{\tau_{1}} V\left(\delta \tau^{+}+\delta \tau^{-}\right)+\frac{\delta \tau^{+}}{2}\left(M_{11}^{2}-m_{11}^{2}\right) \partial_{A} \partial_{\tau_{2}} V+\frac{\delta \tau^{-}}{2}\left(M_{22}^{2}-m_{22}^{2}\right) \partial_{A} \partial_{\tau_{2}} V\right]$.
Even in this case it is clear that $\delta \sigma$ is not a real degree of freedom. The perturbed equations of motion are therefore reduced to two. The last point we need to evaluate concerns the stability of the vacua. We can see it from the solutions (4.83), in fact we obtain a stable solution only if $M_{11}^{2}>0$ (or $M_{22}^{2}>0$ in the case of $\delta \tau^{-}$).

## de Sitter

A two dimensional de Sitter space can be built embedding a 2 d hyperboloid with equation

$$
\begin{equation*}
-X_{0}^{2}+X_{1}^{2}+X_{2}^{2}=R^{2} \tag{4.87}
\end{equation*}
$$

where $R$ is said de Sitter radius, in a 3d Minkowski space [34]. There can be many different set of valid coordinates to describe the hyperboloid, in the following we are going to use

$$
\begin{equation*}
X_{0}=\frac{1}{2 t}\left(t^{2}-x^{2}-R^{2}\right), \quad X_{1}=\frac{1}{2 t}\left(t^{2}-x^{2}+R^{2}\right), \quad X_{2}=\frac{R x}{t}, \tag{4.88}
\end{equation*}
$$

where $t \in(0, \infty)$. With this choice the squared line element is

$$
\begin{equation*}
d s^{2}=-d X_{0}^{2}+d X_{1}^{2}+d X_{2}^{2}=\frac{R^{2}}{t^{2}}\left(-d t^{2}+d x^{2}\right) \tag{4.89}
\end{equation*}
$$

Comparing this with eq.(4.47) we see that

$$
\begin{equation*}
e^{2 \sigma}=\frac{R^{2}}{t^{2}} \tag{4.90}
\end{equation*}
$$

Therefore eq.(4.57) gives

$$
\begin{equation*}
\frac{\partial V}{\partial A}=\frac{M_{p}^{2}}{A R^{2}}>0 \tag{4.91}
\end{equation*}
$$

as it should be because the curvature of de Sitter space is positive. Then we can perturb the equations of motion just like in the Minkowski case

$$
\begin{align*}
& \square \delta \sigma=\frac{1}{t^{2}}-\frac{A R^{2}}{M_{p}^{2} t^{2}}\left(\delta A \partial_{A} \partial_{A} V+\delta \tau_{1} \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{A} \partial_{\tau_{2}} V\right),  \tag{4.92}\\
& \square \delta \tau_{1}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} t^{2}}\left(\delta A \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{\tau_{2}} \partial_{\tau_{1}} V\right),  \tag{4.93}\\
& \square \delta \tau_{2}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} t^{2}}\left(\delta A \partial_{A} \partial_{\tau_{2}} V+\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{2}} V+\delta \tau_{2} \partial_{\tau_{2}} \partial_{\tau_{2}} V\right),  \tag{4.94}\\
& \eta_{\mu \nu} \frac{\delta A}{t^{2}}+\partial_{\mu} \partial_{\nu} \delta A+\frac{1}{t} \delta_{\mu}^{0} \partial_{\nu} \delta A+\frac{1}{t} \delta_{\nu}^{0} \partial_{\mu} \delta A+\eta_{\mu \nu} \frac{\partial_{0} \delta A}{t}=0 . \tag{4.95}
\end{align*}
$$

The last one can be expanded in its components

$$
\left\{\begin{array}{l}
\partial_{0} \partial_{1} \delta A+\frac{\partial_{1} \delta A}{t}=0  \tag{4.96}\\
\partial_{0}^{2} \delta A+\frac{\partial_{0} \delta \delta}{t}-\frac{\delta A}{t^{2}}=0 \\
\partial_{1}^{2} \delta A+\frac{\partial_{0} \delta A}{t}+\frac{\delta A}{t^{2}}=0 .
\end{array}\right.
$$

The solution to this set of equations is

$$
\begin{equation*}
\delta A(x, t)=C_{1}\left(t-\frac{x^{2}}{t}\right)+\frac{C_{2} x}{t}+\frac{C_{3}}{t} . \tag{4.97}
\end{equation*}
$$

Similarly to the Minkowski case we can set $\delta A=0$ [4]. In fact we can rewrite the solution in terms of the embedding coordinates

$$
\begin{align*}
\delta A(x, t) & =\left(C_{1}-\frac{C_{3}}{R^{2}}\right)\left(\frac{t}{2}-\frac{x^{2}}{2 t}-\frac{R^{2}}{2 t}\right)+\left(C_{1}+\frac{C_{3}}{R^{2}}\right)\left(\frac{t}{2}-\frac{x^{2}}{2 t}+\frac{R^{2}}{2 t}\right)+\frac{C_{2}}{R} \frac{R x}{t}  \tag{4.98}\\
& =A X_{0}+B X_{1}+C X_{2} .
\end{align*}
$$

Using the symmetries of the embedding space, namely $S O(1,2)$, we can reabsorb the constant $C$ together with $A$ or $B$, in any case what remains contains a factor $x^{2}$ that is not normalizable, therefore the remaining constant has to be put to zero ending in $\delta A=0$. With such condition the equations of motion become

$$
\begin{gather*}
\square \delta \sigma=\frac{1}{t^{2}}\left(1-\frac{A R^{2}}{M_{p}^{2}}\left(\delta \tau_{1} \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{A} \partial_{\tau_{2}} V\right)\right),  \tag{4.99}\\
\square \delta \tau_{1}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} t^{2}}\left(m_{11}^{2} \delta \tau_{1}+m_{12}^{2} \delta \tau_{2}\right),  \tag{4.100}\\
\square \delta \tau_{2}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} t^{2}}\left(m_{12}^{2} \delta \tau_{1}+m_{22}^{2} \delta \tau_{2}\right), \tag{4.101}
\end{gather*}
$$

where $m_{i j}$ is defined likewise to the Minkowski case. Proceeding as before and with analogous notation we can write the mass eigenstates

$$
\begin{equation*}
\square \overrightarrow{\delta \tau}^{ \pm}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} t^{2}} M^{2} \overrightarrow{\delta \tau}^{ \pm} \tag{4.102}
\end{equation*}
$$

Even for the de Sitter case we can show that $\sigma$ is not a real degree of freedom, but it can be written in terms of $\delta \tau_{1}$ and $\delta \tau_{2}$ up to an integration. Let us focus on $\delta \tau^{+}$(the $\delta \tau^{-}$case is analogous). An infinity of solutions for eq.(4.102) is found assuming the separation of variables, namely that there exists a solution such that

$$
\begin{equation*}
\delta \tau^{+}=\phi(x) \varphi(t) \tag{4.103}
\end{equation*}
$$

Given that two expressions are equal between them only if they are equal to a common parameter $z^{2}$, we can write eq.(4.102) as

$$
\begin{equation*}
\frac{\partial_{1}^{2} \phi(x)}{\phi(x)}=\frac{\partial_{0}^{2} \varphi(t)}{\varphi(t)}+\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} t^{2}} M_{11}^{2}=z^{2} . \tag{4.104}
\end{equation*}
$$

We obtain then

$$
\left\{\begin{array} { l } 
{ \partial _ { 1 } ^ { 2 } \phi ( x ) = \phi ( x ) z ^ { 2 } }  \tag{4.105}\\
{ \partial _ { t } ^ { 2 } \varphi ( t ) = ( z ^ { 2 } - \frac { 2 \tau _ { 2 } ^ { 2 } R ^ { 2 } } { M _ { p } ^ { 2 } t ^ { 2 } } M _ { 1 1 } ^ { 2 } ) \varphi ( t ) }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\phi(x)=e^{z x} \\
\varphi(t)=c_{1} \sqrt{t} I_{\nu}(-i z t)+c_{2} \sqrt{t} K_{\nu}(-i z t)
\end{array}\right.\right.
$$

where $I_{\nu}$ and $K_{\nu}$ are Bessel functions of the first and second order with $\nu=\frac{1}{2} \sqrt{1-\frac{8 \tau_{2}^{2} R^{2}}{M_{p}^{2}} M_{11}^{2}}$. Every linear combination of these particular solutions is still a solution, therefore a more general solution is

$$
\begin{equation*}
\delta \tau^{+}(x, t)=\sqrt{t} \int\left(f(z) e^{z x} I_{\nu}(-i z t)+g(z) e^{z x} K_{\nu}(-i z t)\right) d z \tag{4.106}
\end{equation*}
$$

Now we can put this solution in the expression for $\delta \sigma$ and integrate exactly in the same way we have done before in the Minkowski case. Even for the de Sitter case, like in the Minkowski one, the stability of the vacua is given by the mass squared in eq.(4.102): we have stability if $M^{2}>0$ [4].

## Anti de Sitter

The two dimensional AdS space is built in built in a similar way to the dS space [35]. We can embed a hyperboloid with equation

$$
\begin{equation*}
-X_{0}^{2}-X_{1}^{2}+X_{2}^{2}=R^{2} \tag{4.107}
\end{equation*}
$$

where $R$ is the Anti de Sitter radius, in a flat space with signature $(-,-,+)$. A viable set of coordinates respecting eq.(4.107) is

$$
\begin{equation*}
X_{0}=\frac{1}{2 x}\left(x^{2}-t^{2}+R^{2}\right), \quad X_{1}=\frac{1}{2 x}\left(x^{2}-t^{2}-R^{2}\right), \quad X_{2}=\frac{R t}{x}, \tag{4.108}
\end{equation*}
$$

where $x \in(0, \infty)$ [36]. The two dimensional case is a special one, in fact, comparing this choice of coordinates with the dS one, we can see that they are the same up to the exchange of time and space. Actually in 2d AdS and dS share the same symmetry group with time and space swapped, the former being $S O(2,1)$ and the latter $S O(1,2)$. Therefore we expect all the solutions we are going to find to be similar to the dS ones up to a symmetry transformation.

With the chosen coordinates the metric becomes

$$
\begin{equation*}
d s^{2}=-d X_{0}^{2}-d X_{1}^{2}+d X_{2}^{2}=\frac{R^{2}}{x^{2}}\left(-d t^{2}+d x^{2}\right), \tag{4.109}
\end{equation*}
$$

and comparing with eq.(4.47) we see that

$$
\begin{equation*}
e^{2 \sigma}=\frac{R^{2}}{x^{2}} . \tag{4.110}
\end{equation*}
$$

Using this into eq.(4.57) gives

$$
\begin{equation*}
\frac{\partial V}{\partial A}=-\frac{M_{p}^{2}}{A R^{2}}, \tag{4.111}
\end{equation*}
$$

which is negative as it should be since it is AdS. The perturbed equations of motion become

$$
\begin{gather*}
\square \delta \sigma=-\frac{1}{x^{2}}-\frac{A R^{2}}{M_{p}^{2} x^{2}}\left(\delta A \partial_{A} \partial_{A} V+\delta \tau_{1} \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{A} \partial_{\tau_{2}} V\right),  \tag{4.112}\\
\square \delta \tau_{1}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} x^{2}}\left(\delta A \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{\tau_{2}} \partial_{\tau_{1}} V\right),  \tag{4.113}\\
\square \delta \tau_{2}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} x^{2}}\left(\delta A \partial_{A} \partial_{\tau_{2}} V+\delta \tau_{1} \partial_{\tau_{1}} \partial_{\tau_{2}} V+\delta \tau_{2} \partial_{\tau_{2}} \partial_{\tau_{2}} V\right),  \tag{4.114}\\
\eta_{\mu \nu} \frac{\delta A}{x^{2}}-\partial_{\mu} \partial_{\nu} \delta A-\frac{1}{x} \delta_{\mu}^{1} \partial_{\nu} \delta A-\frac{1}{x} \delta_{\nu}^{1} \partial_{\mu} \delta A+\eta_{\mu \nu} \frac{\partial_{1} \delta A}{x}=0 . \tag{4.115}
\end{gather*}
$$

The last one can be expanded in its components

$$
\left\{\begin{array}{l}
\partial_{0} \partial_{1} \delta A+\frac{\partial_{0} \delta A}{x}=0  \tag{4.116}\\
\partial_{0}^{2} \delta A+\frac{\partial_{1} \delta A}{x}+\frac{\delta A}{x^{2}}=0 \\
\partial_{1}^{2} \delta A+\frac{\partial_{1} \delta A}{x}-\frac{\delta A}{x^{2}}=0,
\end{array}\right.
$$

and a solution to it is

$$
\begin{equation*}
\delta A(x, t)=C_{1}\left(x-\frac{t^{2}}{x}\right)+\frac{C_{2} t}{x}+\frac{C_{3}}{x}, \tag{4.117}
\end{equation*}
$$

where $C_{1}, C_{2}$ and $C_{3}$ are the integration constants. Also in the AdS case we have to set $\delta A=0$ [4], because all the terms in it are non-normalizable. With such condition the remaining equations of motion get modified as

$$
\begin{equation*}
\square \delta \sigma=-\frac{1}{x^{2}}\left(1+\frac{A R^{2}}{M_{p}^{2}}\left(\delta \tau_{1} \partial_{A} \partial_{\tau_{1}} V+\delta \tau_{2} \partial_{A} \partial_{\tau_{2}} V\right)\right), \tag{4.118}
\end{equation*}
$$

$$
\begin{align*}
& \square \delta \tau_{1}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} x^{2}}\left(m_{11}^{2} \delta \tau_{1}+m_{12}^{2} \delta \tau_{2}\right)  \tag{4.119}\\
& \square \delta \tau_{2}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} x^{2}}\left(m_{12}^{2} \delta \tau_{1}+m_{22}^{2} \delta \tau_{2}\right) \tag{4.120}
\end{align*}
$$

The procedure is exactly the same as in the dS case: we shift to the mass eigenstates for eq.(4.119) and eq.(4.120) and we find a possible solution. Then it is inserted into eq.(4.118) to prove that gravity is not a real degree of freedom. So we have

$$
\begin{equation*}
\square \overrightarrow{\delta \tau}^{ \pm}=\frac{2 \tau_{2}^{2} R^{2}}{M_{p}^{2} x^{2}} M^{2} \overrightarrow{\delta \tau}^{ \pm} \tag{4.121}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta \tau^{+}(x, t)=\sqrt{x} \int\left(f(z) e^{z t} I_{\nu}(-i z x)+g(z) e^{z t} K_{\nu}(-i z x)\right) d z \tag{4.122}
\end{equation*}
$$

with the same notation of the dS case.
The stability of the solutions in the AdS case is different from the others. For the AdS it is valid the Breitenlöhner-Freedman bound [37] that states that a solution can be stable even if the mass of the fields is negative, as long as it does not go below a certain fixed value. In our two dimensional case [4] we have

$$
\begin{equation*}
m^{2} \geq-\frac{1}{4 R^{2}} \tag{4.123}
\end{equation*}
$$

where $m^{2}$ is the mass of the field. Turning to our equation (4.121) we have

$$
\begin{equation*}
\square_{A d S} \overrightarrow{\delta \tau}^{ \pm}=\frac{2 \tau_{2}^{2}}{M_{p}^{2}} M^{2} \overrightarrow{\delta \tau}^{ \pm}, \tag{4.124}
\end{equation*}
$$

so the mass is $\frac{2 \tau_{2}^{2}}{M_{p}^{2}} M^{2}$. We can also get $R$ from the eq.(4.111) so that the final condition on the eigenvalues of the mass matrix built from the second derivatives on the potential is

$$
\begin{equation*}
M^{2} \geq \frac{A}{8 \tau_{2}^{2}} \frac{\partial V}{\partial A} \tag{4.125}
\end{equation*}
$$

The derivation of the bound focuses on positivity of energy of the system. The energy can be obtained from the stress-energy tensor and then imposing its positivity gives the aforementioned bound due to a reality constraint on the parameters of the theory. Being the energy positive forces the fluctuations to vanish fast enough to spatial infinity so that other positive terms can dominate the energy [37].

## Chapter 5

## SM constraints from swampland conjectures

In this section we are going to apply the AdS and dS conjectures to the standard model in order to find some constraints on the neutrino masses, i.e. we are going to discard all the masses that generate an AdS or a dS vacuum. In the previous chapters we collected all the elements we needed, in particular we have the potentials obtained through the first order quantum corrections for the 3D and 2D cases of chapter 3 and the conditions to have the vacua for Minkowski, de Sitter and Anti de Sitter. We can then apply the conjectures as it was done in $[2,3,4]$ for both analysed cases $S^{1}$ and $T^{2}$.

For the complexity of the expressions for the potentials the analysis can be performed only numerically. The formulas we are going to use are (3.70) and (3.72). Both of them contains modified Bessel functions of the second kind [38] that in the large argument limit behave as

$$
\begin{equation*}
K_{\nu}(z) \sim \sqrt{\frac{\pi}{2 z}} e^{-z} \tag{5.1}
\end{equation*}
$$

In the 3 D case the argument is $2 \pi R M$, therefore if $R \gg 1 / M$, namely when the mass is large, the contribution to the potential is exponentially suppressed. Likewise in the 2 D case we have $2 \pi \sqrt{A} M$, hence when $\sqrt{A} \gg 1 / M$ the relative contribution gets exponentially suppressed. Thus as $R$ and $A$ get small we need to consider heavier and heavier particle species. However, given the big difference between the electron mass scale and the neutrino mass scale, if we obtain a vacuum in the infrared and then we move on to the point in which the electron becomes relevant, the slope of the potential has become at that point too steep to be lifted and the vacuum remains. [1]. In the following analysis we will contemplate only the particles lighter than the electron: neutrinos, photon and graviton.

The numerical analysis is performed using Mathematica. The potentials are plotted as functions of $R$ in 3D or of $A$ in 2D (after the determination of $\tau_{1}$ and $\tau_{2}$ as done in section 5.2). To identify the values of the masses that generate a vacuum we can express the $R$ or $A$ value of the critical point as function of $m_{\nu}$ and, when we are going to add an axion, also of the axion mass $m_{a}$.

In the SM neutrinos are considered massless, but by now there is enough evidence to believe that at least two of them have a tiny mass. They come in three different types: electron $\nu_{e}$, muon $\nu_{\mu}$ and tauon $\nu_{\tau}$. The experiments of solar and atmospheric neutrinos give the mass-squared difference $\Delta m_{i j}=m_{i}^{2}-m_{j}^{2}$ between the three different species, see Figure 5.1.
Such differences can be understood in two ways by using two different hierarchies between the neutrino masses: the normal one (NH) and the inverted one (IH). The former starts with $\nu_{1}$ which can be arbitrarily light, then we have [40]

- $\Delta m_{21}^{2} \approx 7,37 \times 10^{-5} \mathrm{eV}^{2}$
- $\Delta m_{31}^{2} \approx 2,53 \times 10^{-3} \mathrm{eV}^{2}$,


Figure 5.1: Mass splitting between the neutrinos. There can be two ways to apply the splitting to the three mass states: the normal hierarchy (NH) and the inverted hierarchy (IH). The colors indicate the contributions of any of the three flavors to the respective mass eigenstate. Image from [39].
the latter begins with $\nu_{3}$ and then

- $\Delta m_{32}^{2} \approx 2,46 \times 10^{-3} \mathrm{eV}^{2}$
- $\Delta m_{21}^{2} \approx 7,37 \times 10^{-5} \mathrm{eV}^{2}$.

The potentials (3.70) and (3.72) need to be evaluated for both hierarchies. The value used of the cosmological constant is $\Lambda \approx 3,25 \times 10^{-11} \mathrm{eV}^{4}$ [40].

The following results are in agreement with $[2,3,4]$, but with slightly different values for the neutrino masses, probably due to the numerical approximation.

### 5.1 3D case

Let us begin with the three dimensional case. In the formula (3.70) we set, as in [1] and[3], the scale $r$ to be $2 \pi r=10^{-9} \mathrm{eV}^{-1}$. Such choice does not change the number and the type of the critical points, but it rescales only their vacuum energy, we take it to be of the order of $1 \mathrm{GeV}^{-1}$ to have a simple normalization for our quantities. We are going to evaluate numerically the potential and to establish the mass ranges for the neutrinos to have de Sitter, Anti de Sitter o no vacua.

All the plots of the potential showed in the following have a common structure. For large $R$ the term containing the cosmological constant is going to dominate and, due to its dependence on $R$, it will tend to zero. Moreover, if they have a point of minimum, i.e. a vacuum, they present a maximum as well.

We are going to consider different cases for different particle contents. Firstly the case with photon, graviton and neutrinos (both Dirac and Majorana) and then we are going to add an axion to see how an additional beyond the standard model degree of freedom influences the appearance of vacua.

### 5.1.1 Only neutrinos

In the case with only SM degrees of freedom we will show different plots for the various types of neutrinos we can have.

## Dirac neutrinos - NH and IH

Let us begin with the Dirac neutrinos, that is they have a Dirac mass term and therefore four degrees of freedom, with a normal hierarchy for the masses. The potential is shown in Figure 5.2 for different values of the lightest neutrino mass $\nu_{1}$. The blue line presents no vacuum, while the beige one and the green one have respectively a de Sitter and an Anti de Sitter vacuum because the potential at the minimum is respectively positive and negative.


Figure 5.2: Potential as function of the circle radius. The colors indicate different values of the lightest neutrino mass for the NH.

As the mass of the lightest neutrino increases the concavity of the AdS vacuum gets deeper and deeper, therefore above the mass for which we have a Minkowski vacuum, namely when $V=0$ at the minimum, the potential presents always an AdS vacuum. On the other hand, when the mass decreases the vacuum disappears until we reach the zero mass limit shown in Figure 5.3.


Figure 5.3: Potential as function of the circle radius in the zero mass limit for the lightest neutrino.
In order to have an AdS vacuum we need to have

$$
\begin{equation*}
m_{\nu_{1}} \gtrsim 0.0083 \mathrm{eV} . \tag{5.2}
\end{equation*}
$$

Instead the mass interval to have a de Sitter vacuum is quite narrow, it is

$$
\begin{equation*}
0.0071 \mathrm{eV} \lesssim m_{\nu_{1}} \lesssim 0.0083 \mathrm{eV} \tag{5.3}
\end{equation*}
$$

Applying the AdS and the dS conjecture gives an upper limit to the Dirac lightest neutrino mass with NH

$$
\begin{equation*}
m_{\nu_{1}} \lesssim 0.0071 \mathrm{eV} \tag{5.4}
\end{equation*}
$$

If, instead of the normal hierarchy, we used the inverted hierarchy for the neutrino masses, the situation would be quite similar, but with some variations in the mass value of the lightest neutrino $\nu_{3}$, see Figure 5.4. For the blue line we have no vacuum, while for the beige one there is a de Sitter vacuum and an AdS vacuum for the green line.


Figure 5.4: Potential as function of the circle radius. The colors indicate different values of the lightest neutrino mass for the IH.

With masses respecting

$$
\begin{equation*}
m_{\nu_{3}} \gtrsim 0.00285 \mathrm{eV} \tag{5.5}
\end{equation*}
$$

we have an AdS vacuum, while the interval for the existence of the $\mathrm{d} S$ vacuum becomes

$$
\begin{equation*}
0.0023 \mathrm{eV} \lesssim m_{\nu_{3}} \lesssim 0.00285 \mathrm{eV} \tag{5.6}
\end{equation*}
$$

Then the upper limit for the lightest neutrino mass is

$$
\begin{equation*}
m_{\nu_{3}} \lesssim 0.0023 \mathrm{eV} \tag{5.7}
\end{equation*}
$$

## Majorana neutrinos - NH and IH

If the neutrinos have Majorana mass terms, then they have 2 degrees of freedom. This prevents the potential from avoiding an Anti de Sitter vacuum for both NH and IH. In fact, no matter what is the mass value, we can always find an AdS minimum and therefore, for the AdS conjecture, the Majorana neutrinos would be ruled out, see Figure 5.5.

Such result does not exclude completely the Majorana neutrinos, but only the case with the particle content we considered. If we added some other species to the infrared spectrum of the standard model the potential would be modified and the vacuum could be lifted so that we would not be able to apply the conjectures.

### 5.1.2 Neutrinos + Axion

We investigate now what would happen to the potential if we added a light scalar particle to the SM spectrum. It has just one degree of freedom and an unknown light mass $m_{a}$.

## Dirac neutrinos - NH and IH

Let us begin with the case of Dirac neutrinos. Here the differences between the NH and the IH are more evident, see Figure 5.6 and 5.7.

In the normal hierarchy case for any mass value of the axion we can always find a neutrino mass to avoid the dS or AdS vacuum. While for the inverted hierarchy, if $m_{a} \lesssim 30 \mathrm{eV}$ then we find always


Figure 5.5: Potential as function of the circle radius for Majorana neutrinos with normal hierarchy in the limit of zero mass of the lightest neutrino $\nu_{1}$.
an AdS or dS vacuum, therefore that is the lower limit for the axion mass in order to have Dirac IH masses.

## Majorana neutrinos - NH and IH

For the Majorana neutrinos the situation does not change even with an additional axion. It is not enough to lift the AdS minimum, therefore they are still ruled out.

### 5.2 2D case

Let us move to the torus case. Here the situation is slightly more complex. In order to have a minimum the conditions obtained in chapter 4 need to be fulfilled. They are

$$
\begin{equation*}
V=0, \quad \frac{\partial V}{\partial \tau_{1}}=0, \quad \frac{\partial V}{\partial \tau_{2}}=0 \tag{5.8}
\end{equation*}
$$

In fact we have 4 parameters: $A, m_{\nu}, \tau_{1}$ and $\tau_{2}$, therefore we can express everything as a function of the neutrino mass. Fortunately we can use a symmetry to simplify the problem, that is the modular invariance of the torus explained in chapter 3. Looking at the fundamental domain in Figure 3.7 we can see that there are two fixed points at $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$ and $(0,1)$. These points are extrema of the potential [41]. The idea behind such statement is that if a smooth function $f(x)$ is equal, for example, to $f(1 / x)$ then it must have an extremum at $x=1$. The last two conditions of (5.8) are automatically satisfied by such choice of $\tau_{1}$ and $\tau_{2}$.

Even the determination of the type of vacuum and its stability is harder. The former is found by the sign of $\frac{\partial V}{\partial A}$, in particular if $>0$ we have a dS vacuum, if $<0$ an AdS one, while the latter depends on the eigenvalues of the mass matrix for the $\tau_{1}, \tau_{2}$ perturbed equation of motion, particular attention needs the AdS case because the Breitenlöhner-Freedman bound is involved.
The cases analyzed are the same of the 3D situation.

### 5.2.1 Only neutrinos

## Dirac neutrinos - NH and IH

We need to check the potential for both extremal points, let us begin with $(1 / 2, \sqrt{3} / 2)$. The potential is such that when it presents a dS or an AdS vacuum it has also an AdS or dS one as it can be seen in Figure 5.8 and 5.9.

The difference between normal and inverted hierarchy consists again in different mass values for the lightest neutrino. In fact for NH there is a vacuum (i.e. the potential crosses the $A$-axis at least once)


Figure 5.6: The graphic shows the mass values of the lightest neutrino with NH and the axion that can create a dS and AdS vacuum or no vacuum.
only if

$$
\begin{equation*}
m_{\nu_{1}} \gtrsim 0.0045 \mathrm{eV} . \tag{5.9}
\end{equation*}
$$

While for IH we get that the the mass must be

$$
\begin{equation*}
m_{\nu_{3}} \gtrsim 0.00095 \mathrm{eV} \tag{5.10}
\end{equation*}
$$

Now we have to check the stability of the vacua. Given that we use both the AdS and the dS conjecture we need to analyze only one of the two vacua, because the masses excluded with one are the same of the ones excluded with the other. We look then at the first one, namely the one on the left in the graphics. It is an AdS vacuum because the first derivative of $V$ with respect to $A$ is clearly negative. To decide upon the stability we have to apply eq.(4.123), the eigenvalues of the matrix $M^{2}$ and the other expression are plotted in Figure 5.10 for NH, the situation of IH is analogous.

As we can see the eigenvalues are always above and therefore the vacuum is stable. So it is possible to set a constraint on the neutrino masses in order to avoid the appearance of the forbidden AdS vacuum

- $m_{\nu_{1}} \lesssim 0.0045 \mathrm{eV}$ for NH ,
- $m_{\nu_{3}} \lesssim 0.00095 \mathrm{eV}$ for IH .

Let us analyze the other extremal point $\left(\tau_{1}, \tau_{2}\right)=(0,1)$. The form of the potential is analogous to the other point, therefore if we take the first vacuum point and we plot something similar to Figure 5.10 to see the stability, we obtain Figure 5.11 in the NH case. Again the IH is analogous. We can observe that one of the eigenvalues is under the Breitenlöhner-Freedman bound, thus we have a saddle point and not a minimum and therefore the AdS conjecture can not be applied. Actually this point will reveal itself to be a saddle point for any configuration we are going to investigate and therefore we will not able to apply the AdS conjecture to it. On the other hand, the sharpened version of the dS conjecture contains two parts, the first forbids stable dS vacua while the second allows unstable dS vacua if the instability is strong enough. Therefore we have to look at the second intersection of the potential with the abscissa axis.

The first of the two conditions of the dS conjecture, eq.(2.6), states that the first derivative of the potential has to be bigger than the cosmological constant, clearly in our vacua we have that $\partial_{\tau_{1}} V=$ $\partial_{\tau_{2}} V=0$, thus it is not satisfied.


Figure 5.7: The graphic shows the mass values of the lightest neutrino with IH and the axion that can create a dS and AdS vacuum or no vacuum.


Figure 5.8: The potential for different lightest neutrino masses for NH. The critical points are the ones with $V=0$.

The second condition, eq.(2.9), involves the eigenvalues of the Hessian of the matrix made by the second derivatives of the potential (or squared mass) that have to be smaller than minus the cosmological constant, thus we need to have

$$
\begin{equation*}
\frac{2 \tau_{2}^{2}}{M_{p}^{2}} M^{2} \leq-\frac{2 A}{M_{p}^{2}} \frac{\partial V}{\partial A} \Longrightarrow M^{2} \geq \frac{A}{\tau_{2}^{2}} \frac{\partial V}{\partial A} \tag{5.11}
\end{equation*}
$$

If such inequality is not fulfilled then the vacua need to be discarded. As we can see in Figure 5.12 for the NH case the eigenvalues are always above the limit except for $0.0047 \mathrm{eV} \lesssim m_{\nu_{1}} \lesssim 0.006$ eV , therefore those mass values are excluded by the dS conjecture. However with the other point $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$ such values have already been excluded, hence we do not have extra conditions.

If we do the same thing for the IH case we obtain Figure 5.13 . Now all the masses above 0.001 eV need to be excluded, but again this condition does not add anything to the one obtained with the other point.


Figure 5.9: The potential for different lightest neutrino masses for IH. The critical points are the ones with $V=0$.


Figure 5.10: Comparison between the mass squared eigenvalues, i.e. the eigenvalues of the symmetric matrix built with the second derivative of the potential, and the Breitenlöhner-Freedman bound for $\left(\tau_{1}, \tau_{2}\right)=$ ( $1 / 2, \sqrt{3} / 2$ ) with NH.

## Majorana neutrinos - NH and IH

The case of Majorana neutrinos in 2D is very similar to the one in 3D. With the choice $\left(\tau_{1}, \tau_{2}\right)=$ $(1 / 2, \sqrt{3} / 2)$ an AdS vacuum is always developed for both NH and IH, no matter what is the value of the neutrino masses, see Figure 5.14. Since the conjectures would rule out all the mass values there is no need to look at the other point $(0,1)$.

### 5.2.2 Neutrinos + Axion

The addition of a massive scalar degree of freedom makes, as in 3D, the numerical analysis a little harder.

## Dirac neutrinos - NH and IH

For Dirac neutrinos, as in the case without the axion, we find always an AdS vacuum or no vacuum, in no cases we find a dS vacuum alone, in fact the form of the potential is the same as in the others cases, so with two zeros or no zeros. We can make a graphic showing the mass values of the axion and of the lightest neutrino that create or not an AdS vacuum for NH and IH with the choice $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$, see Figure 5.15 and 5.16 . Computing the eigenvalues of the mass matrix and the Breitenlöhner-Freedman bound we can check that these extrema are stable and the AdS


Figure 5.11: Comparison between the mass squared eigenvalues, i.e. the eigenvalues of the symmetric matrix built with the second derivative of the potential, and the Breitenlöhner-Freedman bound for $\left(\tau_{1}, \tau_{2}\right)=(0,1)$.
 $-\frac{A \frac{\partial V}{\partial A}}{t 2^{2}}$

- Eigenvalue 1
- Eigenvalue 2

Figure 5.12: Eigenvalues of the squared mass and limit of the second condition of the dS conjecture for the NH case at $\left(\tau_{1}, \tau_{2}\right)=(0,1)$.
conjecture is applicable. For NH any axion mass gives a bound on the neutrino mass, while for IH if $m_{a} \lesssim 0.03 \mathrm{eV}$ then no neutrino mass is allowed by the AdS conjecture.

With the other choice $\left(\tau_{1}, \tau_{2}\right)=(0,1)$ we would find again a saddle point and then the AdS conjecture would be inapplicable. Turning to the second vacuum we could apply as before the dS conjecture. However, the presence or absence of the vacua depends on the neutrino and axion masses in the same way as at the other point $(1 / 2, \sqrt{3} / 2)$, thus no new conditions can be produced.

### 5.2.3 Majorana neutrinos - NH and IH

The Majorana neutrinos case is not changed by the addition of an axion, for $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$ a stable AdS vacuum is always developed and therefore they remain excluded.


Figure 5.13: Eigenvalues of the squared mass and limit of the second condition of the dS conjecture for the IH case at $\left(\tau_{1}, \tau_{2}\right)=(0,1)$.


Figure 5.14: Potential as a function of $A$ for 2D NH Majorana neutrinos at $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$ in the zero mass limit.


Figure 5.15: The graphic shows the mass values of the lightest neutrino with NH and the axion that can create an AdS vacuum or no vacuum at $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$.


Figure 5.16: The graphic shows the mass values of the lightest neutrino with IH and the axion that can create an AdS vacuum or no vacuum at $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$.

## Chapter 6

## Conclusions

The main goal of this thesis was to test the dS and AdS swampland conjectures by applying them to an EFT. We chose it to be the Standard Model because it is a well known and experimentally tested theory, the only parameters known with very little precision are the neutrino masses $m_{\nu}$. In detail, we refined the analysis of [3], where SM compactifications to three and two dimensions had been studied, but only using the AdS conjecture.

The AdS and dS conjectures forbid respectively the AdS non-susy stable vacua and the dS stable vacua (or even unstable if the instability is not too strong) and their existence reason has roots mainly in black hole physics. The SM coupled to gravity admits a landscape of vacua when compactified to 3 dimensions on a circle or to 2 dimensions on a torus [1]. Such landscape is suitable for the application of the chosen conjectures because stable AdS or dS vacua may appear.

The reduction is performed compactifying one or two space dimensions on a manifold parameterized with suitable fields. From the Einstein-Hilbert action we got kinetic terms for those fields and from the 1-loop quantum corrections of the other SM terms we obtained a potential. This translates into the actions (3.23) and (3.59) and into the potentials (3.71) and (3.73).

In the circle case the vacuum conditions are simple because the potential depends only on one variable, thus the vacua are just minima and their type depends on the sign of the potential at the minimum: dS if positive and AdS if negative. For the 2D case the vacuum conditions are obtained by looking at the background solutions of the equations of motion for the fields parameterizing the torus, we got $\partial_{\tau_{1}} V=0, \partial_{\tau_{2}} V=0$ and $V=0$.

The first two conditions are satisfied by the two fixed points under the modular symmetry of the torus fundamental domain $\left(\tau_{1}, \tau_{2}\right)=(1 / 2, \sqrt{3} / 2)$ and $(0,1)$. The latter revealed itself to be always a saddle point and therefore the AdS conjecture can not be applicable to it.

Eventually a numerical analysis of the potential and of the possible vacua was performed. In the expressions evaluated we used only the lightest degrees of freedom because they are the ones that mostly influence the vacua of the potential. They are the photon, the graviton, the neutrinos and, successively, an axion.

Neutrinos with Majorana mass terms, for any of the considered cases, create AdS vacua, no matter the value of their mass or of the axion mass. For Dirac neutrinos we can instead get some bounds to avoid the appearance of dangerous dS and AdS vacua. Without the axion we got

- 3D Normal Hierarchy:

$$
\begin{aligned}
m_{\nu_{1}} & \lesssim 0.0071 \mathrm{eV}, \\
m_{\nu_{3}} & \lesssim 0.0023 \mathrm{eV}, \\
m_{\nu_{1}} & \lesssim 0.0045 \mathrm{eV}, \\
m_{\nu_{1}} & \lesssim 0.00095 \mathrm{eV} .
\end{aligned}
$$

These cases seem to agree with [3] up to, at most, 0.0006 eV , such discrepancy could be caused by the approximation of the numerical analysis. Adding an axion the bounds on the masses get modified

- 3D Normal Hierarchy: a bound for the mass of $\nu_{1}$ can always be found, but it will depend on the axion mass. Such bound will space between $m_{\nu_{1}} \lesssim 0.005 \mathrm{eV}$ and $m_{\nu_{1}} \lesssim 0.007 \mathrm{eV}$.
- 3D Inverted Hierarchy: in this case it is not always possible to find a bound. If $m_{a} \lesssim 0.03 \mathrm{eV}$ then no neutrino mass is allowed. Otherwise we get that $m_{\nu_{3}} \lesssim 0.002 \mathrm{eV}$.
- 2D Normal Hierarchy: the situation is similar to the 3D case with the exception that there is not a single vacuum for any mass value, but two, a dS one and a $\operatorname{AdS}$ one and therefore we can choose to use the dS conjecture or the AdS conjecture with the same effect. The bound spaces between $m_{\nu_{1}} \lesssim 0.0025 \mathrm{eV}$ and $m_{\nu_{1}} \lesssim 0.0045 \mathrm{eV}$.
- 2D Inverted Hierarchy: if $m_{a} \lesssim 0.03 \mathrm{eV}$ then we can always find a dS or AdS vacuum and therefore the Dirac neutrinos would be ruled out. Otherwise we get $m_{\nu_{3}} \lesssim 0.001 \mathrm{eV}$.

Also in these cases we find small differences with [3] for both the neutrino and the axion masses, again probably for the numerical approximation. The application of the dS conjecture to the point $\left(\tau_{1}, \tau_{2}\right)=(0,1)$ does not seem to give any additional constrain.

However, all the bounds we found are much below the experimental upper limit of $\sim 1.1 \mathrm{eV}$ [42]. The results constrain $m_{\nu}$ and, when we add the axion, $m_{a}$. If the results violated the experimental data we would have a test of the swampland conjectures, instead we got a prediction of quantum gravity on the low energy physics.

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