UNIVERSITÀ DEGLI STUDI DI PADOVA

MASTER DEGREE IN PHYSICS


# *A Fermionic Swampland Conjecture*

## SUPERGRAVITY AND THE SWAMPLAND


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*"Eclipse first, the rest nowhere"*
ウマ娘

# *Abstract*

In this thesis we study the Fermionic Weak Gravity conjecture in the context of gauged $\mathcal{N} = 2$ supergravity in four dimensions. The rich geometrical structure of supergravity theories provides a natural way to realize constraints coming from a consistent completion to quantum gravity, and in this work we focus on a proposed inequality relating Yukawa couplings and the scale of supersymmetry breaking. We explore the conjecture on Minkowski vacua with broken supersymmetry, both in the general case and in explicit models. We find that some physically relevant parametric limits partially realize the inequality, while other regimes remain out of predictive reach. Explicit models also suggest that it might be crucial to fix the moduli, and we give a few ways in which this work might be extended in the future.

# *Contents*

# 1 Introduction

A remarkable fact about nature is that phenomena occur over a wide range of length, energy and time scales: over the last century alone, we have managed to stretch the scope of our investigations from microscopic scales as small as one billionth of the size of an atom ($10^4$ GeV) to macroscopic scales as large as the size of our observable universe ($10^{-42}$ GeV). It is equally fascinating that phenomena at very different scales seem to be, to a great extent, *decoupled* from one another, as we need not to know the behaviour of quarks and electrons in order to describe the motion of astronomical objects, and it is precisely by leveraging this hierarchy of scales that natural sciences have been able to progress up to now.

Indeed, even in particle physics, this paradigm led to the incredibly successful theoretical framework of Effective Field Theories (EFTs), which first historical example is the Fermi theory describing weak interactions among electrons, neutrinos, protons and neutrons in $\beta$-decay. As the name suggests, an EFT describes the behaviour of some chosen local degrees of freedom (in this context, usually particles) up to a cutoff energy scale $\Lambda$, above which the theory cannot be trusted and the effective description must be replaced with a more fundamental one. In order to have a predictive framework, a certain number of free parameters must be fixed: this can be done either by theoretical means, for example by leveraging the symmetries of nature, or experimentally. As we increase the energy scale closer and closer to $\Lambda$, the value of these free parameters could point to new degrees of freedom or interactions which might reveal a deeper structure to physical phenomena. It is through this process that one can hope, one day, to achieve a fundamental theory describing all interactions.

Unfortunately, although the study of the quantum behaviour of three out of the four fundamental forces has eventually lead to the formulation of the Standard Model (SM) of particle physics, the gravitational interaction (as well as any physics beyond the SM) remains elusive to experiments. As such, the dominant way to push the search for a theory of Quantum Gravity (QG) has been through theoretical investigation. One very powerful tool, among the others, is *self-consistency*, i.e. the absence of physical contradictions: a standard example is anomaly cancellation in the context of Quantum Field Theories (QFTs), which has also been an important ingredient in the formulation of Beyond the Standard Model (BSM) theories. It is striking that at higher energy scales, such as those relevant to quantum gravity physics, it appears self-consistency may be strong enough to almost uniquely fix the theory. This is exactly what happens in the best example we have of a theory of QG, string theory, where every possible constant parameter is uniquely fixed including, remarkably, the dimension of space-time.

As we move toward the low energies, however, self-consistency becomes much less powerful, as can be understood from the many possible BSM models which have been proposed. In string theory this manifests as the existence, within our current understanding, of a huge number of resulting low-energy effective theories. Indeed, although string theory itself is fixed, its vacuum structure is not, and there are thus as many effective theories as the number of possible vacuum geometries[1], the so called *String Landscape*. It is remarkable, however, that the resulting set of theories still picks out only a subset of all the possible apparently self-consistent EFTs. Here, we used "apparently" to indicate that even though the low-energy description itself is not problematic, an inconsistency would show up should we try to couple it to QG. For starters, when the theory is coupled to gravity new anomalies will in general appear, but, interestingly, the absence of gravitational anomalies in the EFT is not enough to guarantee the quantum consistency of the theory. There are other constraints that one needs to satisfy, that simply seem to express the fact that arriving at a unitary theory of quantum gravity is not so simple. Thus, the *Swampland Program*, introduced in [1], aims to establish which constraints make up this set of consistency criteria, usually stating them in the form of conjectures. We say that an EFT belongs to the *Swampland* when it does not satisfy the Swampland constraints.

Some of the most important such proposed constraints (see for example [2] for a review) are the Weak Gravity Conjecture (WGC) [3], which roughly tells us that gravity must always be the weakest force at long distances, the No Global Symmetries Conjecture [4, 5], which tells us that in quantum gravity global symmetries are forbidden, and the Swampland Distance Conjecture (SDC) [6], which relates the cutoff of the effective theory with the behaviour of massless scalar fields. As can be understood, these conjectures are rather weak and qualitative: however, in recent times, they brought to interesting phenomenological consequences [7, 8] regarding the masses of neutrinos and dark energy, which shows their relevance in answering open questions in particle physics.

Another, not very well explored conjecture is the Fermionic Weak Gravity Conjecture (FWGC), first proposed by Palti in [9] and recently discussed also in [10, 11], which as a consequence suggests that in effective theories where supersymmetry is spontaneously broken:

$$Y M_p > m_{susy}, \tag{1.1}$$

where $Y$ is a suitable Yukawa coupling and $m_{susy}$ the supersymmetry breaking scale. This corollary of the FWGC (which for the sake of convenience we will refer to as FWGC from now on) is what will be explored in this thesis. In particular, since we deal with supersymmetric, gravitational effective theories, it will be natural to work in a supergravitational EFT. We choose $\mathcal{N} = 2$

[1] We talk about "vacuum geometries" here because vacua of string theory correspond to a certain choice of compactification geometry for the extra dimensions.

supergravity in four dimensions, since it is a balanced compromise between the fully controlled but phenomenologically distant $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 1$ supergravity, which is closer to reality but much less constrained. The goal of this work will then be to support the above conjecture by using the geometrical construction of the supergravity theory, e.g. as done in [12] in the case of the Scalar Weak Gravity Conjecture, and in doing so provide a solid starting point where it is realized rigorously. The thesis will be laid out in the following way:

- In chapter 2, we will give a quick review of the Swampland program and its phenomenological implications, starting from the black hole argument for the WGC and ending with the FWGC.

- In chapter 3, a geometrical introduction to $\mathcal{N} = 2$ supergravity in four dimensions will be given. We will start by introducing the fundamental concept of *electromagnetic duality*, which plays a major role in every $\mathcal{N} > 2$ supergravity theory. Then, we will define the Special and Quaternionic scalar manifolds, followed by a discussion on global symmetries, isometries and the gauging procedure, which is the only way to introduce a potential in theories with extended supersymmetry. Finally, we will present the relevant pieces of the $\mathcal{N} = 2$ Lagrangian, and give a basic example on how the geometrical structure of the theory can help us to test and study Swampland conjectures, following [13].

- In chapter 4 we will test the FWGC by working in full generality. First, since the conjecture deals with *broken* supersymmetry, it will be essential to cover the *super-Higgs mechanism*, the supersymmetric counterpart to the Brout-Englert-Higgs mechanism for gauge symmetries, which will be proven explicitly. This also allows us to discover the fermionic mass matrices on the vacuum. Finally, after having better contextualized the various quantities involved in the conjecture, we will check the FWGC.

- In chapter 5 we will instead test the FWGC by working with explicit examples. Since we will be constructing most of the models from scratch, it is vital to give a general strategy to compute vacua of gauged supergravities in the simplifying assumption of homogeneous scalar manifolds, which will also be our working assumption. This will grant us the ability to carry out the computation analitically. Then, we will explicitly calculate all the geometrical quantities which define the sigma models of interest:

$$\mathcal{M}_{scal} = EAdS_4 \times \frac{SU(1,1)}{U(1)} \quad \text{and} \quad \mathcal{M}_{scal} = \frac{SU(2,1)}{U(2)} \times \frac{SU(1,1)}{U(1)}, \tag{1.2}$$

  and discuss the relevant gauging of their isometries. Finally, we will explicitly compute the Yukawas, which definition has been given in the previous chapter, and check the conjecture.

- In chapter 6 we will discuss the results we obtained, future perspectives and extensions of this work.

# 2

# *Swampland and Weak Gravity*

After more than five decades of research, string theory has emerged as the most promising quantum theory of gravity. Although a plethora of string compactifications leading to four-dimensional effective field theories have been found, not every seemingly consistent EFT (e.g. from the point of view of anomaly cancellation) can be UV completed to a theory of quantum gravity. We call this collection of "bad" theories *Swampland*, while "good" theories are found in the *Landscape*. The idea of the *Swampland program* (introduced in [1], see [2, 14, 15] for reviews) is to try and give a set of constraints, in the form of conjectures, in order to discriminate between the two. Although the notion of the Swampland is in principle not restricted to string theory, Swampland conjectures are often motivated by or checked in string theory setups. Indeed, string theory provides a perfect framework to quantitatively and rigorously test the conjectures, improving our understanding of the possible string theory compactifications on the way. Interestingly, many of these conjectures are actually related, suggesting that they might simply be different facets of some more fundamental quantum gravity principles yet to be uncovered.

These constraints also serve as a great tool to investigate beyond the Standard Model physics or accessible phenomenology, cutting corners of parameter space and discarding large classes of theories. For instance, in recent years there have been developments in understanding the role of neutrinos and dark energy [7], or the "dark dimension" [16].

## 1 WEAK GRAVITY CONJECTURE

One of the first conjectured conditions that EFTs have to satisfy in order to be consistently coupled to a theory of quantum gravity, and the first one to have very strong evidence for its implications, as discussed below, is the "Weak Gravity" Conjecture, or WGC for short [3, 17]. The conjecture tries to formalize the notion that "any gauge force must be stronger than gravity", and is usually introduced heuristically in the context of black hole physics, as we will now show.

Consider a charged Reissner-Nordström black hole of (electric) $U(1)$ charge $Q$ and mass $M$. The black hole has an outer horizon $r_+$, an inner horizon $r_-$ and a temperature given by

$$T = \frac{r_+ - r_-}{4\pi r_+^2}, \tag{2.1}$$

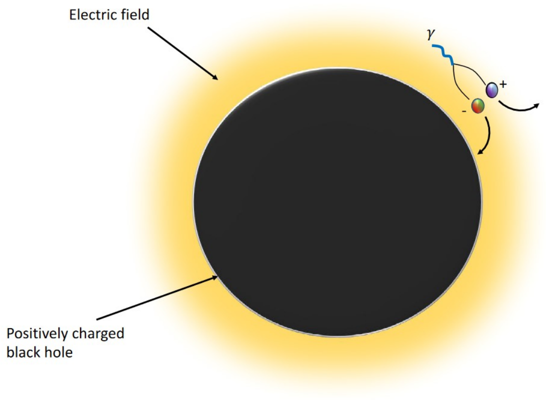


FIGURE 2.1: The process of discharge of a black hole. Heuristically, pair production due to the presence of an electric field on the black hole horizon leads to emission of a charged particle and to the discharge of the black hole.

which is known to be associated to thermal radiation [18]. If suitable charged particles are present in the theory, we expect this object to radiate them away too, leading to its discharge. Alternatively, we can also understand the discharge process by considering pair production from the vacuum on the black hole horizon. As seen in figure 2.1, the particle whose charge has the same sign as the black hole is repelled, while the other one falls towards the singularity. More precisely, depending on the masses of the emitted particles, we can identify two different regimes in which the discharge takes place and which formalize the previous two intuitions:

1. If the temperature of the black hole is much larger than the mass of the particle $T \gg m$, than the discharge process is mainly thermal, since the electric field just outside the outer horizon leads to a chemical potential term in the Boltzmann factor which discriminates between charges $\pm q$ [19, 20]:
$$\mathcal{P} \sim e^{\frac{1}{T}\left(m \pm \frac{g^2 qQ}{r_+}\right)}. \tag{2.2}$$
We see that this contribution becomes relevant when $m \sim g^2 qQ/r_+$.

2. If the temperature is much lower than the mass of the particle $T \ll m$, than we can understand the discharge process as driven by Schwinger pair production in an (almost) constant electric field, close to the horizon [20]. This is the dominant contribution when the black hole is near-extremal.

Consider now the process of black hole discharge and assume that the emitted charged particles have charges $q_i \neq 0$ and masses $m_i$. For the evaporation to be possible we must have $M > \sum_i m$

and by charge conservation $Q = \sum_i q_i$, then:

$$\frac{M}{Q} \geq \frac{1}{Q}\sum_i m_i = \frac{1}{Q}\sum_i \frac{m_i}{q_i} q_i \geq \frac{1}{Q}\left(\frac{m}{q}\right)_{min} \sum_i q_i = \left(\frac{m}{q}\right)_{min}, \tag{2.3}$$

where *min* refers to the minimum value of the ratio in parenthesis. We can pick the initial black hole to saturate the extremality bound $M = gQM_p$, where $M_p$ is the four-dimensional Planck mass and $g$ the $U(1)$ gauge coupling when the covariant derivative is normalized as $D_\mu = \partial_\mu + iqA_\mu$ and the kinetic term as $4g^{-2}F^2$. Substituting this bound back in (2.3), we obtain the Weak Gravity Conjecture.

**Weak Gravity Conjecture (WGC):** consider a theory coupled to gravity with a $U(1)$ gauge field $A$ and gauge coupling $g$:

$$S = \int d^4x \, e \left( \frac{M_p^2}{2} R - \frac{1}{4g^2} F^2 + \dots \right), \tag{2.4}$$

then there must exist a particle in the spectrum with mass $m$ and charge $q$ (usually called WGC particle) satisfying:

$$m \leq gqM_p. \tag{2.5}$$

A few remarks are in order: first of all, it is clear that this heuristic argument lies on the assumption that we expect charged black holes to discharge in order for the WGC particle to exist. However, we could easily imagine an EFT without charged particles or with a spectrum such that the charged black hole is electromagnetically stable. It is not clear as to whether the discharge process is something that should be expected to be a Swampland constraint. Proving that completely stable charged black holes carry an intrinsic inconsistency would therefore amount to the proof of the electric Weak Gravity Conjecture. Secondly, notice that the argument works and is to be trusted as long as the black hole is big enough to be captured by a (semiclassical) gravitational EFT, as its discharge and evaporation processes can then be reliably studied by means of quantum fields on a curved background.

Despite the shortcomings of this heuristic approach, the Weak Gravity Conjecture still is one of the most tested conjectures of the Swampland program, as it can be constructed in the more rigorous context of the AdS/CFT correspondence [21, 22], by studying black hole physics and BPS states [9, 23], and is also found to be implied by unitarity and positivity bounds on scattering amplitudes [3, 24–26]. Less definitive but nonetheless interesting tests have been done by looking at higher derivative corrections to Einstein gravity [27]. Moreover, (2.5) is known to be realized in all known explicit constructions in string theory [17]. All of this, together with the fact that the conjecture nicely links with other ideas in the Swampland program, gives it a solid foundation, suggesting it to point to some underlying quantum gravitational principle. As a last remark, the name "weak gravity" is due to the fact that if we use (2.5) when considering the interaction between two WGC particles

$$F_{grav} \sim \frac{m^2}{M_p^2 r^2}, \qquad F_{elec} \sim \frac{(gq)^2}{r^2}, \tag{2.6}$$

we see that it is equivalent to the statement $F_{elec} \geq F_{grav}$.

The WGC introduced up to now is usually referred to as "electric" Weak Gravity Conjecture, for obvious reasons. We can try to apply the same argument for the dual "magnetic" gauge field $dA_D = \star F$, for which charged probes in four dimensions are given by monopoles. The mass of the monopole is at least the energy stored in its magnetic field, which is linearly divergent and needs to be cut off. In this context, we can define a semiclassical radius $r_s \sim \Lambda^{-1}$ of the monopole which can be though of as the size of the core and where its effective description as a particle breaks down. In the absence of finely tuned cancellations between bare mass and field energy we then have

$$m_{mon} \gtrsim \Lambda g_{mag}^2, \tag{2.7}$$

where $g_{mag}$ is the charge of the monopole. The charge quantization condition reads $g_{mag} g \in \mathbb{Z}$, so that by neglecting integer coefficients we get to

$$m_{mon} \gtrsim \frac{\Lambda}{g^2}. \tag{2.8}$$

Then, the WGC applied to these particles reads

$$m_{mon} \leq g_{mag} M_p, \tag{2.9}$$

and by using (2.8) we are led to the magnetic version of the Weak Gravity Conjecture.

**Magnetic Weak Gravity Conjecture (MWGC):** consider a theory coupled to gravity with a $U(1)$ gauge field $A$ and gauge coupling $g$:

$$S = \int d^4x\, e \left( \frac{M_p^2}{2} R - \frac{1}{4g^2} F^2 + \ldots \right), \tag{2.10}$$

then the cutoff of the theory must be such that:

$$\Lambda \lesssim g M_p. \tag{2.11}$$

This really is a statement about the fact that in quantum gravity global symmetries are forbidden [4, 5], because sending $g \to 0$ invalidates the effective description even at low energies [28]. This conjecture also finds many different extensions, one of them giving a more precise statement about the nature of the cutoff of the theory, tying it with another well-known Swampland constraint, the Swampland Distance Conjecture [6, 8, 29]. In particular, the breakdown of the effective description in the $g \to 0$ limit is due to an infinite tower of massive states becoming light,

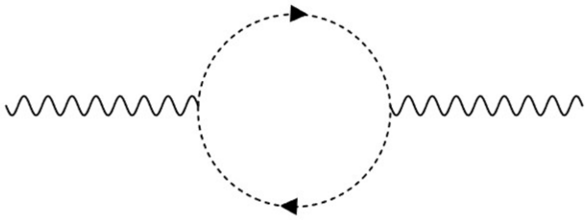

FIGURE 2.2: Self-energy diagram involving the trilinear coupling $g$ and scalar fields running in the loop.

characterized by a mass scale $m_\infty$ such that:

$$m_\infty \sim g M_p, \tag{2.12}$$

where the state at a certain "level" $n$ in the tower has mass $m \sim n^\alpha m_\infty$, $\alpha \in \mathbb{N}$. This is for example what happens in Kaluza-Klein (KK) compactifications of field theories or in more general string compactifications.

# 2 FERMIONIC WEAK GRAVITY CONJECTURE

We are now going to introduce the main focus of this thesis project, the Fermionic Weak Gravity Conjecture. Indeed, in [30] it has been proposed that (2.12) could be extended not only to scalar mediators [9], but also to spin 1/2 particles. As noted in the paper, (2.12) contains the trilinear coupling $g$, which also appears in the self-energy diagram for the vector field of the kind found in figure 2.2. As such, the conjecture can also then be interpreted in the context of the *emergence proposal* [11, 31–33], where the claim is that condition (2.12) follows from imposing that the kinetic terms for the vector fields are vanishing in the UV and that they become emergent in the IR by receiving order one contributions. Looking at (2.10), this is equivalent to say that the theory is strongly coupled in the UV and the gauge coupling $g$ becomes weakly coupled from running in the IR. This contribution is interpreted in terms of integrating out the massive tower of (this time charged) states, which charge is proportional to the level $n$ in the tower.

Following this qualitative reasoning, we could then apply it to fermions. This time the trilinear coupling of interest is the Yukawa coupling $Y$, as it appears in diagram 2.2 by replacing external photon lines with fermions lines. Then, this new Fermionic Weak Gravity Conjecture can be stated as

**Fermionic Weak Gravity Conjecture (FWGC):** consider a theory coupled to gravity containing both fermion and scalar fields:

$$S = \int d^4x\, e \left( \frac{M_p^2}{2} R - \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a + i \overline{\psi}^\alpha \not{\partial} \psi_\alpha + Y_a^{\alpha\beta} \phi^a \overline{\psi}_\alpha \psi_\beta + \dots \right) \tag{2.13}$$

then there is a massive tower of states satisfying:

$$m_\infty \sim Y M_p, \tag{2.14}$$

where $Y$ is the overall Yukawa coupling of a specific fermion with the tower.

A few considerations are in order: first of all, (2.14) can be given the same qualitative interpretation as (2.11), since sending $Y \to 0$ again invalidates the effective description. Secondly, we implicitly introduced a massive tower of states where at each level there is a fermion and a scalar: this is the first suggestion that the tower must be supersymmetric. The strength of the Yukawa coupling is also assumed to increase as we go up the tower as $Y_n \sim Y n^\alpha$, $\alpha \in \mathbb{N}$. In the case of gauge couplings ($g$) this can be related to the fact that the charge is quantized and it increases with the level, while in the case of scalar fields couplings ($\mu$) it is the mass of the state that increases up the tower. In general, we do not expect this kind of behaviour from Yukawa couplings, but if the tower is supersymmetric then the increasing Yukawa coupling is explained by the relation to either a gauge or scalar superpartner, since we will have $Y \sim g$ or $Y \sim \mu$ respectively. Another important difference is that the Yukawa coupling $Y$ is not in general normalized across all states of the theory, while in the case of $g$, for example, it is impossible to make one state parametrically weakly coupled without making all states weakly coupled (i.e. by sending $g \to 0$) due to charge quantization. In other words, a fermion having a certain weak coupling $\tilde{Y}$ with a state outside of the tower does not imply that the coupling to the tower $Y$ is small. This makes the conjecture slightly less powerful than the magnetic WGC.

Taking everything into consideration, the most controlled setting in which to try and motivate further this statement is in the context of supersymmetric theories of gravity, where towers of states of the kind described above are naturally introduced. In particular, since these towers are supersymmetric, we expect that if supersymmetry is broken at all it will be as such at lower scales than $m_\infty$, bringing us to the claim of the paper [30], and the main study object of this thesis.

**Corollary of Fermionic WGC:** consider a supersymmetric theory coupled to gravity containing both fermion and scalar fields, then if there is a vacuum which spontaneously breaks supersymmetry at scale $m_{susy}$, it will be such that

$$m_{susy} < Y M_p. \tag{2.15}$$

That is to say, supersymmetry *must* be restored at the mass scale $m_\infty \sim YM_p$ if the tower ought to be supersymmetric.

There are a few subtleties we have yet to address: in Lagrangian (2.13) we introduced many Yukawa couplings $Y_a^{\alpha\beta}$, but it is still not clear how to identify the presence of a tower of states, and in particular how to extract the overall Yukawa scale $Y$ in (2.15). As we will see later, we will try to justify our choice by means of the structure of the supergravity theory and its geometrical interpretation. Next, the scale of supersymmetry breaking should be defined as the one sensed by the tower, so the mass splitting between fermions and bosons in the tower. In general, this might be rather difficult to identify, but fortunately, in supergravity theories the scale of supersymmetry breaking on a Minkowski vacuum is such that it can be simply identified with the masses of the gravitini. Finally, here we only focused on *renormalizable* Yukawa couplings, since non-renormalizable ones (like the one possessed by gravitini or goldstini) do not require the presence of a tower of states in order to give order one one-loop contributions to the kinetic term [30].

# 3 *Supergravity as a testbed*

As mentioned in previous chapters, in this thesis we consider the Fermionic Weak Gravity Conjecture (2.15) from the point of view of supergravity theories. In particular, we will focus on $4d$ $\mathcal{N}=2$ supergravity. Indeed, supergravity theories already provide a rich playground in which to explore Swampland conjectures [10, 12, 13, 34–36], both when viewed as gravitational EFTs without reference to any UV completion or as specific models coming from string compactifications. In particular, it is clear in the literature how $\mathcal{N}=2$ supergravity models in $4d$ are linked to Type $IIA$ or $IIB$ string theories compactified on Calabi-Yau three-folds $(CY_3)$ with possible fluxes turned on (see [37] for a review of string compactifications) [38–44]. Another appealing feature of working in $\mathcal{N}=2$ supergravity, as will be discussed more in depth later, is that the vacuum structure of the theory is due to the gauging of global symmetries, making it ideal to study the WGC and the web of Swampland conjectures related to it. At the same time, $\mathcal{N}=1$ models in string theory are usually introduced as truncations of $\mathcal{N}=2$ constructions, which again justify their usage from the point of view of UV completions to the EFT. We could also consider using other $\mathcal{N}>2$ supergravities, but their structure is very rigid, as for example they only admit homogeneous spaces as scalar manifolds of the sigma model [45] while being very far from phenomenologically realistic constructions.

We will first present some general features of $\mathcal{N}=2$ supergravity theories in the first section, following the discussion and conventions of [46]. Meanwhile, the scalar geometry will be left to the second section and will follow the same reference. Then, we will describe the gauging procedure both from the point of view of isometries of the scalar geometry and through the *embedding tensor* formalism, mainly taking after [12] and [45]. The latter is particularly interesting, since it makes explicit one of the most important features of supergravity theories: duality covariance. Finally, we will describe the full $\mathcal{N}=2$ action as found in [46], focusing on the interesting sectors which we will use in later chapters. Throughout the rest of the thesis, we will work in Planck units, if not otherwise specified.

## 1 $\mathcal{N}=2$ SUPERGRAVITY IN 4D

$\mathcal{N}=2$ supergravity displays a high degree of complexity in the construction of the Lagrangian and the transformation rules of the fields, but its basic structure can be boiled down to a few intuitive geometrical principles, which then determine all of the couplings, mass matrices and vacuum energy.

1. The choice of a Special Kähler manifold $\mathcal{SM}$, which describes the self-interactions of the vector multiplets through a sigma model.

2. The choice of a Quaternionic manifold $\mathcal{QM}$, which describes the self-interactions of the hypermultiplets through a sigma model.

3. The choice of a gauge group $G$, that in the general case must be a subgroup of the isometry group of the scalar manifold $\mathcal{M}_{scalar} \equiv \mathcal{SM} \otimes \mathcal{HM}$, which can be identified with the group of global symmetries of the non-gauge sector of the Lagrangian[1], and which has an immersion in the symplectic group of electric–magnetic duality rotations (when viewed as such, it's usually called U-duality subgroup).

We can now proceed to list the full field content of a generic $\mathcal{N} = 2$ supergravity theory, while also fixing some of the conventions.

- The **gravitational multiplet**, described by the veilbein 1-form $V^a$, $(a = 0, 1, 2, 3)$ transforming in the vector representation of the structure group of the frame bundle on spacetime $SO(1, 3)$, the spin-connection 1-form $\omega^{ab}$, the $SU(2)$ doublet of gravitino 1-forms $\psi^A$, $\psi_A$ (respectively, left-chiral and right-chiral), and the graviphoton 1-form $A^0$.

- $n_V$ **vector multiplets**, each containing a vector $A^I$, $(I = 1, \ldots, n_V)$, a doublet of 0-form spinors (usually called gauginos) $\lambda^{iA}, \lambda^{i^*}_A$, transforming as vectors on $\mathcal{SM}$ and as $SU(2)$ doublets. where the upper or lower position of $A$ denotes right and left chirality respectively and a complex scalar field $z^i$, $(i = 1, \ldots, n_V)$. The scalar fields can be regarded as coordinates on the special manifold $\mathcal{SM}$ which can be freely chosen, and

$$\dim_{\mathbb{C}} \mathcal{SM} = n_V. \tag{3.1}$$

- $n_H$ **hypermultiplets**, each containing a $2n_H$ 0-form spinors (usually called hyperinos) $\zeta^\alpha$ or $\zeta_\alpha$ $(\alpha = 1, \ldots, 2n_H)$, where again the upper or lower index denotes left and right chirality respectively, and four real scalar fields $q^u$, $(u = 1, \ldots, 4n_H)$. The scalar fields can be regarded as coordinates on the Quaternionic manifold $\mathcal{QM}$ which can be freely chosen, and

$$\dim_{\mathbb{R}} \mathcal{QM} = 4n_H. \tag{3.2}$$

[1]In some instances there may be additional global symmetries that act trivially on the scalar manifolds, or the theory could be devoid of scalar fields altoghether, like pure $\mathcal{N} = 2$ supergravity. In most of these cases, the additional factors come from the non-trivial action of the $R$-symmetry group of the supersymmetry algebra on the fermion fields.

As will be explained below, any Quaternionic manifold has a holonomy group:

$$\mathcal{H}ol(\mathcal{QM}) \subset SU(2) \otimes Sp(2n_H, \mathbb{R}), \tag{3.3}$$

and the index $\alpha$ of the hyperinos transforms in the fundamental of $Sp(2n_H, \mathbb{R})$.

In the next section we are going to describe more thoroughly the scalar geometry and the gauging procedure, starting with a brief review of electromagnetic duality.

# 2 SCALAR GEOMETRY

## 2.1 *Electromagnetic duality*

Electromagnetic duality plays an important role in determining the geometry of the Special Kälher manifold, since we expect that in a supersymmetric theory the action of the duality group must extend to the entire supermultiplet (in our case, the vector multiplet) and so have some kind of action on the scalar fields.

Consider the gauge sector of a generic 2-derivative theory containing $n$ abelian vector fields $A^\Lambda$, which action is written in terms of their field strengths $F^\Lambda = dA^\Lambda$, and with arbitrary couplings to other fields $\phi^i$ (which in the case of interest are fields described by a sigma model):

$$S = \int \left( \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi) F^\Lambda \wedge \star F^\Sigma + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\phi) F^\Lambda \wedge F^\Sigma + \frac{1}{2} O_\Lambda(\phi) \wedge \star F^\Lambda \right) + S_{rest}(\phi). \tag{3.4}$$

Here, $\mathcal{I}_{\Lambda\Sigma}(\phi)$ and $\mathcal{R}_{\Lambda\Sigma}(\phi)$ are $n \times n$ matrices which might be field dependent, $\mathcal{I}$ being negative-definite to ensure unitarity. $O_\Lambda$ is a field dependent two-form which contains at most one derivative of the fields $\phi$. The vector fields by definition satisfy the Bianchi identities:

$$dF^\Lambda = 0, \tag{3.5}$$

and obey the equations of motion:

$$\nabla^\mu \frac{\partial \mathcal{L}}{\partial F^{\mu\nu\Lambda}} = 0. \tag{3.6}$$

These equations take the form of Bianchi identities if we introduce the two form

$$\tilde{G_{\Lambda\mu\nu}} = 2\frac{\partial \mathcal{L}}{\partial F^{\Lambda\mu\nu}}, \tag{3.7}$$

where $2\tilde{G_{\Lambda\mu\nu}} = \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}_\Lambda$. Now, equation (3.6) can be rewritten as:

$$dG_\Lambda = 0. \tag{3.8}$$

With these definitions, the system of Bianchi identities and equations of motion seems invariant under arbitrary constant $GL(2n, \mathbb{R})$ transformations,

$$\mathbb{F}' = S\mathbb{F}, \quad \mathbb{F} \equiv \begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix}, \tag{3.9}$$

however, the rotations must also preserve the definitions of $\tilde{G_{\Lambda\mu\nu}}$ in equation (3.7). This can be checked to restrict the allowed rotations to the symplectic group $S \in Sp(2nv, \mathbb{R})$ [46, 47].

In order for these duality transformations to be consistent, the matrices $\mathcal{I}, \mathcal{R}$ and $O$ should also transform non-trivially. In particular, if we define a symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$, sometimes called the *period* matrix, and the self-dual[2] combination $O^{+}$ as

$$\begin{aligned} \mathcal{N}_{\Lambda\Sigma} &= \mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma}, \\ O^{+}_{\Lambda} &= \frac{1}{2}(O_{\Lambda} - i\tilde{O}_{\Lambda}), \end{aligned} \tag{3.10}$$

they must transform non-trivially as:

$$\begin{gathered} \begin{cases} \mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}, \\ O^{+'} = O^{+}(A + B\mathcal{N})^{-1}, \end{cases} \\ \text{where } \; S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}) \end{gathered} \tag{3.11}$$

Thus, while the duality group acts on the vector field strengths, in order for it to be realized consistently we expect there to be also an action of the diffeomorphism group of the scalar manifold, such that it realizes the following homomorphism:

$$i_{\delta} : \; Diff(\mathcal{M}_{scal}) \longrightarrow Sp(2n, \mathbb{R}), \tag{3.12}$$

and transforms the symplectic matrices as in equation (3.11). In particular, there should be a suitable embedding of the isomorphism group into the symplectic group such that Bianchi identities and *all* equations of motions are invariant. This subset of the full symplectic group is usually called $U$-duality group, or duality symmetry group. Notice that the invariance here it's not in the Lagrangian, but in the equations of motion, as already stressed.

As a last remark, before introducing vector and hyper scalars geometries, notice that in a generic $\mathcal{N} = 2$ supergravity theory the total number of abelian gauge fields is $n = n_V + 1$, where the extra one is the graviphoton.

[2]Notice in fact that $\tilde{O}^{+} = iO^{+}$

## 2.2 *Special Kähler manifolds*

In the following, we will simply call "Special Kähler" a Special Kähler manifold of local type, which is the one relevant in supergravity applications and which characterization was first given in [48]. First, notice that since the vector multiplet sigma model can be covered by patches of complex coordinates and holomorphic transition functions between them, it defines a complex manifold $\mathcal{SM}$. Assume now the manifold is Kähler, so that we can globally define a closed two form $K$, called fundamental form: this form can be expressed in terms of the hermitian metric $g_{ij^*}$, which is in turn expressed as exterior powers of a real function $\mathcal{K}$ called Kähler potential, as

$$g = \bar{\partial}\partial\mathcal{K}. \tag{3.13}$$

Here, $\partial$ and $\bar{\partial}$ denote the holomorphic and anti-holomorphic exterior derivatives (with respect to which we define the Dolbeault cohomology of the complex manifold).

Let's then define a flat (i.e. with vanishing bundle curvature) holomorphic (in terms of transition functions) vector bundle $\mathcal{SV} \to \mathcal{SM}$ with structure group $Sp(2n_V + 2, \mathbb{R})$ and a complex $U(1)$ line bundle $\mathcal{L} \to \mathcal{SM}$. We say that the manifold is of Kähler-Hodge type if the first Chern class of the line bundle is the cohomology class of the fundamental form. That is, given a $U(1)$ connection $Q$, we have:

$$dQ = K \tag{3.14}$$

where the equality is modulo exact forms.

We call **Special Kähler Manifold (of local type)** a Kähler manifold $\mathcal{SM}$ equipped with a tensor bundle $\mathcal{H} = \mathcal{SV} \otimes \mathcal{L} \to \mathcal{SM}$, such that $\mathcal{SM}$ is of Kähler-Hodge type with respect to $\mathcal{L}$ and the Kähler potential can be written in terms of holomorphic sections of this tensor bundle $\Omega \in \Gamma(\mathcal{H}, \mathcal{SM})$ as:

$$\Omega^M = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}, \quad \mathcal{K} = -\log i \left( X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda \right) \equiv -\log i \, \langle \bar{\Omega} | \Omega \rangle \,, \tag{3.15}$$

where we defined $\langle \, | \, \rangle$ as a symplectic invariant inner product. The transition functions between patches $i$ and $j$ look like:

$$\Omega_j = e^{f_{ij}} M_{ij} \Omega_i \tag{3.16}$$

where $f_{ij}$ are holomorphic functions associated to Kähler transformations of $\mathcal{K}$ between different patches

$$\mathcal{K}_j = \mathcal{K}_i + f_{ij} + \bar{f}_{ij} \tag{3.17}$$

and $M_{ij}$ is a constant $Sp(2n_V+2,\mathbb{R})$ matrix (which acts on the index $M = 1,\dots,2n_V+2$ of $\Omega$), both subject to the usual cocycle condition on triple overlaps.

We can also describe the tensor bundle $\mathcal{H}$ by introducing non-holomorphic symplectic sections

$$V^M = e^{\mathcal{K}/2}\Omega^M = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \tag{3.18}$$

so that equation (3.15) can be written as the normalization condition

$$1 = i\left\langle V \middle| \overline{V} \right\rangle . \tag{3.19}$$

Moreover, we can see that these section, even though not holomorphic, are covariantly holomorphic with respect to a Kähler covariant derivative with weight $p = 1/2$, which naturally appears when trying to construct Lagrangians for supergravity theories [47]:

$$\nabla_{i^*} V \equiv \left(\partial_{i^*} - \frac{1}{2}\partial_{i^*}\mathcal{K}\right) V = 0 = \nabla_i \overline{V}, \tag{3.20}$$

as can be explicitly checked, while taking the other covariant derivative we can define:

$$U_i \equiv \nabla_i V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{i\Sigma} \end{pmatrix} = (U_{i^*})^*. \tag{3.21}$$

An alternative is to define the symplectic sections $V^M$ by solving the set of differential equations:

$$\nabla_i V = U_i, \tag{3.22}$$
$$\nabla_i U_j = iC_{ijk} g^{kl^*} U_{l^*}, \tag{3.23}$$
$$\nabla_{i^*} U_j = g_{i^*j} V, \tag{3.24}$$
$$\nabla_{i^*} V = 0, \tag{3.25}$$

which among other things can be checked to yield (3.19). Moreover, the second equation defines a set of totally symmetric covariantly holomorphic sections of $\mathcal{TM}^3 \otimes \mathcal{L}^2$ ($\mathcal{T}$ denotes here the tangent bundle), $C_{ijk}$ through which another definition of special geometry can be given [49]. This set of equations is actually what one gets from the $\mathcal{N} = 2$ solution of Bianchi identities in superspace [46]. The $C_{ijk}$ tensor also obeys the following identities:

$$\nabla_{[k} C_{i]jl} = \nabla_{[k^*} C_{i^*]j^*l^*} = \nabla_{k^*} C_{ijl} = \nabla_k C_{i^*j^*l^*} 0, \tag{3.26}$$
$$\mathcal{R}_{i^*jl^*k} = g_{l^*j} g_{ki^*} + g_{l^*l} g_{ji^*} - C_{i^*l^*s^*} C_{tkj} g^{ts^*}, \tag{3.27}$$

where $\mathcal{R}$ is the Riemann curvature tensor on $\mathcal{SM}$.

Finally, we can define the *period* matrix by the following relations:

$$\overline{M}_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^\Lambda, \qquad h_{i\Sigma} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma \tag{3.28}$$

which can be solved by introducing the following $(n_V + 1) \times (n_V + 1)$ matrices

$$f_I^\Lambda = \begin{pmatrix} f_i^\Lambda \\ \overline{L}^\Lambda \end{pmatrix}, \quad h_{\Lambda I} = \begin{pmatrix} h_{i\Lambda} \\ \overline{M}_\Lambda, \end{pmatrix} \tag{3.29}$$

and defining:

$$\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda I} \cdot (f^{-1})^I_\Sigma, \tag{3.30}$$

Following this definition, one finds that the period matrix transforms as it should under symplectic duality transformations in equation (3.11). Indeed, this is why the structure of special geometry was introduced: the existence of the symplectic bundle $\mathcal{H} \longrightarrow \mathcal{SM}$ is required in order to be able to pull-back the action of the diffeomorphism to a certain symplectic transformation of the fibres and implement the homomorphism $i_\delta$. From the previous relations, it is straightforward to derive a set of constraints that these symplectic sections must satisfy, for example in order to ensure the symmetry of the period matrix. Among these relations we quote the following [50]:

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} L^\Lambda \overline{L}^\Sigma = -\frac{1}{2}, \tag{3.31}$$

$$\langle V|U_i\rangle = \langle V|U_{i^*}\rangle = \langle U_i|U_j\rangle = 0, \tag{3.32}$$

$$U^{\Lambda\Sigma} \equiv f_i^\Lambda f_{j^*}^\Sigma g^{ij^*} = -\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1|\Lambda\Sigma} - \overline{L^\Lambda} L^\Sigma, \tag{3.33}$$

$$g_{ij^*} = -i\,\langle U_i|U_{j^*}\rangle = -2 f_i^\Lambda \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f_{j^*}^\Sigma, \tag{3.34}$$

$$C_{ijk} = \langle \nabla_i U_j|U_k\rangle = f_i^\Lambda \partial_j \overline{\mathcal{N}}_{\Lambda\Sigma} f_k^\Sigma = (N - \overline{N})_{\Lambda\Sigma} f_i^\Lambda \partial_j f_k^\Sigma. \tag{3.35}$$

Notice that the positivity of the Kähler metric also implies $\mathrm{Im}\mathcal{N} < 0$ which is required by unitarity, and its symmetry is also implied by the second equation [47]. This equation also ensures the presence of a prepotential $F(X)$ (if $n_V > 1$) in at least one symplectic frame, that is an homogenous, degree two function of the sections $X^\Lambda$ such that:

$$F_\Lambda = \frac{\partial F(X)}{\partial X^\Lambda}. \tag{3.36}$$

Since, differently from the case of rigid supersymmetry, this prepotential is not always defined, we will give a prepotential-free formulation of the action in the following section. As a last remark, notice that in the prepotential formulation it is tempting to try and give the $X^\Lambda$ the role of coordinates. In supergravity, however, they are one more than the actual coordinates $z^i$ of the special manifold we are working with, and due to how these are patched together in (3.16), they can not be interpreted as regular coordinates, but as homogeneous *projective* coordinates, such that we can define a frame where

$$z^i = X^i / X^0, \quad X^0 = const, \tag{3.37}$$

which is usually called "special" frame. This is why the alternative name "projective" Special Kähler geometry is also used.

## 2.3 *Quaternionic manifolds*

The geometry of the hypermultiplet sector is heavily influenced by their transformation properties under the $SU(2) \subset U(2)$ R-symmetry subgroup. To see this, we first look at their holonomies. In particular, one simple way to derive the holonomy group of a scalar manifold in supersymmetric theories is to consider which endomorphisms of the tangent space preserve the structure of the supersymmetry transformation rules [47]. In this case, the holonomy group of the manifold $\mathcal{QM}$ gets restricted to:

$$\mathcal{H}ol(\mathcal{QM}) = SU(2) \times H, \quad H \subset Sp(2n_H, \mathbb{R}), \tag{3.38}$$

where the $SU(2)$ factor comes, as anticipated, from R-symmetry.

We call a **Quaternionic** manifold $\mathcal{QM}$ a $4n_H$-dimensional smooth manifold admitting a restricted holonomy group:

$$\mathcal{H}ol(\mathcal{QM}) = SU(2) \times H, \quad H \subset Sp(2n_H, \mathbb{R}), \tag{3.39}$$

and where these structures define a principal $SU(2) \times USp(2n_H)$ frame bundle, with non-vanishing $SU(2)$ curvature. As a consequence of the restricted holonomy, the manifold admits a triplet of fundamental two forms $K^x$, called *hyper-Kähler* forms, such that they are covariantly closed with respect to the $SU(2)$ bundle $\mathcal{SU} \longrightarrow \mathcal{QM}$ connection 1-form $\omega^x$:

$$\nabla K^x \equiv dK^x + \epsilon^{xyz}\omega^y \wedge K^z = 0, \tag{3.40}$$

and are proportional to its bundle curvature

$$\Omega^x = \lambda K^x. \tag{3.41}$$

We see that this kind of geometrical construction generalizes the complex $U(1)$ line bundle of Kähler-Hodge manifolds to the non-abelian case of $SU(2)$. Indeed, the triplet of fundamental two forms is this time defined in terms of a triplet of complex structures $J^x$ which obey the Quaternionic algebra:

$$(J^x) : \mathcal{T}(\mathcal{QM}) \to \mathcal{T}(\mathcal{QM}), \qquad J^x J^y = -\delta^{xy}\mathbb{1} + \epsilon^{xyz} J^z, \tag{3.42}$$

with respect to which the metric is hermitian, and the metric $h_{uv}$ of the Quaternionic manifold, as:

$$K^x_{uv} = h_{uw}(J^x)^w_v, \tag{3.43}$$

similarly to the previous section. Notice, however, that equation (3.40) implies that the manifold is *not* Kähler, since $dK^x \neq 0$[3]. This means that an $\mathcal{N} = 2$ supergravity theory cannot be rewritten simply in terms of $\mathcal{N} = 1$ fields and couplings, but it must be non-trivially truncated [51, 52].

Introducing $SU(2)$ indices $A = 1, 2$ and $Sp(2n_H)$ indices $\alpha = 1, \ldots, 2n_H$ that run in the fundamental of both groups, we can introduce vielbein 1-forms:

$$U^{A\alpha}_u, \quad h_{uv} = U^{A\alpha}_u U^{B\beta}_v \mathbb{C}_{\alpha\beta}\epsilon_{AB}, \tag{3.44}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are respectively the $Sp(2n_H)$ and $SU(2)$ flat metrics. Conventions on raising and lowering indices using these metrics can be found in A. The veilbeins respect the compatibility condition with the Levi-Civita connection on the manifold, and so are covariantly closed with respect to the $su(2)$ valued connection $\omega^x$ and the $sp(2n_H)$ valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\nabla U^{A\alpha} \equiv dU^{A\alpha} + \frac{i}{2}\omega^x \sigma_x^{AB} \wedge U^\alpha_B + \Delta^{\alpha\beta} \wedge C_{\beta\gamma} U^{A\gamma} = 0, \tag{3.45}$$

where $\sigma_x^{AB} = \epsilon^{AC}(\sigma_x)_C{}^B$ and $(\sigma_x)_C{}^B$ are the standard Pauli matrices. Furthermore $U^{A\alpha}$ satisfies the reality condition:

$$U_{A\alpha} \equiv (U^{A\alpha})^* = \epsilon AB \mathbb{C}_{\alpha\beta} U^{B\beta}, \tag{3.46}$$

while the inverse veilbein is defined as $U^{A\alpha}_u U^v_{A\alpha} = \delta^v_u$. We can use these quantites to flatten a pair of indices of the Riemann tensor:

$$\mathcal{R}^{uv}_{st} U^{A\alpha}_u U^{B\beta}_v = -\frac{i}{2}\Omega^x_{st}(\sigma_x)^{AB}\mathbb{C}^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{st}\epsilon^{AB}, \tag{3.47}$$

where $\mathbb{R}^{\alpha\beta}_{st}$ is the field strength of the $Sp(2n_H)$ connection and:

$$\mathbb{R}^{\alpha\beta}_{st} = \lambda\epsilon_{AB} U^{\alpha A}_{[s} U^{\beta B}_{t]} + \epsilon_{AB} U^{\gamma A}_s U^{\delta B}_t C^{\alpha\rho} C^{\beta\sigma} \Omega_{\gamma\delta\rho\sigma}. \tag{3.48}$$

The previous equations imply that Quaternionic manifolds are Einstein spaces such that the Ricci tensor reads:

$$\mathcal{R}_{uv} = \lambda(2 + n_H)h_{uv}. \tag{3.49}$$

Finally, we note the following interesting identities:

$$\Omega^x_{uv} U^u_{A\alpha} U^v_{B\beta} = i\lambda \mathbb{C}_{\alpha\beta}(\sigma_x)_{AB}, \tag{3.50}$$

[3] Confusingly, in some references, these manifolds are also referred to as *Quaternionic Kähler*, even though they are not Kähler.

$$\frac{i}{2}\Omega^x(\sigma_x)_{AB} = \lambda U^{A\alpha} \wedge U^B_\alpha, \tag{3.51}$$

$$h^{st}\Omega^x_{us}\Omega^y_{tw} = -\lambda^2\delta^{xy}h_{uw} + \lambda\epsilon^{xyz}\Omega^z_{uw}. \tag{3.52}$$

# 3 GAUGING

As mentioned in the introduction to the chapter, the gauging procedure requires choosing a certain number of generators among the isometries of the product manifold:

$$\mathcal{M}_{scal} = \mathcal{SM} \times \mathcal{QM}. \tag{3.53}$$

Since the Special Kähler manifold is involved, we expect in general to have a non-trivial embedding into the symplectic group, since our gauge group of choice resides in the U-duality group of the theory, $G \subset G_U \subset Sp(2n_V + 2)$ and moreover the number of its generators must be less than the number of vectors, $\text{dim}G \leq n_V + 1$. We can break down the gauging procedure in three steps:

1. Choice of the gauge generators among the generators of $G_U$, which determines the action of the gauge group on the fields, their representation, and the structure constant, as well as its embedding in the electromagnetic duality group.

2. Introduction of gauge curvatures and covariant derivatives: for example, the action of an isometry on scalar fields defines a covariant derivative as

$$\phi^i \to \phi^i + \epsilon^I k^i_I, \quad D_\mu\phi^i = \partial_\mu\phi^i + A^I_\mu k^i_I, \tag{3.54}$$

   while we must require that the usual Maxwell-type gauge transformations are replaced by:

$$\delta A^I_\mu = \partial_\mu\epsilon^I + f^I_{JK}A^J_\mu\epsilon^K, \tag{3.55}$$

   where $f$ are the structure constant of the (possibly non-abelian) gauge group of choice.

3. Restoration of supersymmetry, which gets broken by the above modifications. This entails introducing new $\mathcal{O}(\epsilon)$ terms in the supersymmetry transformation rules of the fermions, usually called "fermionic shifts". This, in turn, also enforces the introduction of Yukawa-like terms (which will be particularly important in this thesis)

$$\mathcal{L}_{yuk} \sim \mathcal{O}(\epsilon), \tag{3.56}$$

   and potential terms

$$V(\phi) \sim \mathcal{O}(\epsilon^2) \tag{3.57}$$

   in the supergravity Lagrangian.

Before proceeding further, let us notice this important fact. When choosing the generators inside $G_U$, which is the symmetry group of the equations of motion and Bianchi identities, their symplectic embedding might map them to transformations which are not part of the global symmetry group of a given Lagrangian $G_{\mathcal{L}} \subset G_U$ (which is the way we usually perform gaugings). In this case we talk about "magnetic" gaugings, because some of the gauge vector fields have $G_\Lambda$ as a field strength instead of $F^\Lambda$, as we will see later. We instead talk about "electric" gauging if $G \subset G_{\mathcal{L}} \subset G_U$. One non-trivial statement of great importance is the fact that given a certain gauge group, we can always perform $Sp(2n_V+2)$ transformations (which in general change the Lagrangian if $S \notin G_U$) to find (at least one) electric realization. In the next sections, then, we will focus on this restricted case while we will briefly discuss the fully general case when introducing the embedding tensor formalism in 3.3.

## 3.1 *Holomorphic momentum maps on Special Kähler manifolds*

Consider the set of isometries of $\mathcal{SK}$ that leave the hermitian metric $g_{ij^*}$ invariant. Since the manifold is complex, we expect these isometries to be holomorphic, i.e $k^i(z,z^*) = k^i(z)$ (since they must act on the coordinates without mixing $z$ and $\bar{z}$):

$$z^i \to z^i + \epsilon^\Lambda k^i_\Lambda(z), \tag{3.58}$$

and satisfy the Killing equations with respect to holomorphic indices:

$$\nabla_i k_j + \nabla_j k_i = 0, \quad \nabla_{i^*} k_j + \nabla_j k_{i^*} = 0. \tag{3.59}$$

Moreover, they are covariantly holomorphic with respect to the Levi-Civita connection:

$$\nabla_i k_j = 0. \tag{3.60}$$

As already mentioned, in the case of Special Kähler manifolds the isometry group has a natural embedding in the symplectic group, so that:

$$\mathcal{L}_\Lambda V \equiv k^i_\Lambda \partial_i V + k^{i^*}_\Lambda \partial_{i^*} V = T_\Lambda V \tag{3.61}$$

where $\mathcal{L}_\Lambda$ denotes the Lie derivative with respect to $k_\Lambda$, $T_\Lambda \in Sp(2n_V+2)$ and where we neglected possible contributions of the kind $f_\Lambda(z)V$, since they usually do not appear [46]. If the gauging is to be electric, the action of $T_\Lambda$ must be block diagonal, and in particular must be such that the sections transform in the adjoint representation of $G$, thus leading to:

$$(T_\Lambda)_\Sigma{}^\Delta = \begin{pmatrix} f^\Delta_{\Lambda\Sigma} & 0 \\ 0 & -f^\Delta_{\Lambda\Sigma} \end{pmatrix}. \tag{3.62}$$

Remembering that the Kähler potential is expressed as a symplectic invariant, we also have $\mathcal{L}_\Lambda \mathcal{K} = 0$, which means that both the metric and the complex structure (because of holomorphicity) is invariant. As such, also the Kähler two-form is invariant

$$\mathcal{L}_\Lambda K = 0 = i_\Lambda dK + d i_\Lambda K = d i_\Lambda K, \tag{3.63}$$

where we used the relation between $\mathcal{L}$, $d$ and the interior product $i$ and the fact that $dK = 0$. Then, locally, we can find a set of real functions $P^0_\Lambda$ such that:

$$i_\Lambda K = dP^0_\Lambda, \tag{3.64}$$

called *Killing* prepotentials or momentum maps. In holomorphic coordinates, the previous equation can be rewritten as:

$$k^i_\Lambda = i g^{ij^*} \partial_{j^*} P^0_\Lambda. \tag{3.65}$$

If the Lie-algebra of the gauge group is semi-simple [53], we can write the following identity:

$$i g_{ij^*} k^i_{[\Lambda} k^j_{\Sigma]} = -\frac{1}{2} f_{\Lambda\Sigma}^{\Delta} P^0_\Delta, \tag{3.66}$$

which follows from the realization of the gauge algebra in terms of Poisson brackets of $P^0$ [46]. We can find other constraints that these Killing vectors and prepotentials must satisfy in order for the gauging to be consistent. Among the others [53] we find:

$$k^i_\Lambda f^\Sigma_i = i P^0_\Lambda L^\Sigma - f^\Sigma_{\Lambda\Delta} L^\Delta, \tag{3.67}$$

$$k^i_\Lambda L^\Lambda = k^{i^*}_\Lambda \overline{L}^\Lambda = P^0_\Lambda L^\Lambda = P^0_\Lambda \overline{L}^\Lambda = 0, \tag{3.68}$$

$$f^\Sigma_i k^i_\Lambda \overline{L}^\Lambda = -(f^\Sigma_i k^i_\Lambda \overline{L}^\Lambda)^*. \tag{3.69}$$

## 3.2 *Triholomorphic momentum maps on Quaternionic manifolds*

We now turn to the (this time real) isometries of the Quaternionic manifold $\mathcal{QM}$. Since in $\mathcal{N} = 2$ theories the scalar manifold factorizes, in order for the gauging to be consistent we must have an action by triholomorphic isometries $k^u$ of the same gauge group $G$ that acts on the Special Kähler manifold $\mathcal{SM}$. In other words, we must have that the vector

$$k_\Lambda = k^i_\Lambda \partial_i + k^{i^*}_\Lambda \partial_{i^*} + k^u_\Lambda \partial_u \tag{3.70}$$

is a Killing vector of the product metric on $\mathcal{QM} \times \mathcal{SM}$:

$$\hat{g} = \begin{pmatrix} g_{ij^*} & 0 \\ 0 & h_{uv} \end{pmatrix}. \tag{3.71}$$

Notice however that this time $\mathcal{L}_{k^u_\Lambda} V = 0$, which reflects the fact that these isometries have trivial duality action. This fact is particularly important when discussing abelian gaugings of Quaternionic isometries, since their action on the Special Kähler manifold as well as their symplectic embedding is trivial. Triholomorphicity here means that the Killing vector fields leave the HyperKähler structure invariant up to $SU(2)$ rotations:

$$\mathcal{L}_\Lambda K^x = \epsilon^{xyz} K^y W^z_\Lambda, \tag{3.72}$$

where $W$ is an $SU(2)$ compensator, which cannot be removed in virtue of the non-triviality of the $\mathcal{SU}$-bundle. However, analogously to the previous case, we can define a triplet of 0-form prepotential $P^x_\Lambda$ such that the derivative in equation (3.64) is replaced by a covariant derivative with respect to the $SU(2)$ connection $\omega^x$, and using (3.41):

$$i_\Lambda \Omega^x = -\nabla P^x_\Lambda, \tag{3.73}$$

which in components reads:

$$2k^u_\Lambda \Omega^x_{uv} = -\nabla_u P^x_\Lambda. \tag{3.74}$$

The Poissonian realization of the Lie Algebra this time leads to the equivariance condition:

$$\lambda^{-1} \Omega^x_{uv} k^u_\Lambda k^v_\Sigma + \frac{1}{2} \epsilon^{xyz} P^y_\Lambda P^z_\Sigma = \frac{1}{2} f^\Delta_{\Lambda\Sigma} P^x_\Delta \tag{3.75}$$

We also report here useful identities involving quaternionic prepotentials[4]:

$$P^x_\Lambda = -\frac{1}{2n_H \lambda} \nabla^u k^v_\Lambda \Omega^x_{uv}, \tag{3.76}$$

$$U^A_{\alpha u} U^{B\alpha}_v = -\frac{i}{2\lambda} \Omega^{AB}_{uv} - \frac{1}{2} \epsilon^{AB} h_{uv}, \tag{3.77}$$

$$\nabla_u k_{v\Lambda} = -\frac{1}{2\lambda} \Omega^x_{uv} P^x_\Lambda - \frac{1}{2} U^{\alpha A}_{[u} U^{\beta B}_{v]} \epsilon_{AB} M_{\alpha\beta}, \tag{3.78}$$

where $M_{\alpha\beta}$ is the mass matrix of the hyperinos and which will be introduced later.

## 3.3 *Embedding tensor formalism*

From all previous considerations, it is clear that the choice of a gauge algebra breaks the duality covariance of the original (ungauged) theory. This can be seen from the fact that only some selected vectors enter the gauging procedure, changing the equations of motion and breaking invariance under the U-duality group. Indeed, the gauging procedure assigns specific charges to the field content of our theory, and a symplectic rotation implies reassigning these charges. The most we can do is to reinstate symplectic covariance by means of the *embedding tensor* $\Theta^\alpha_M$, a $(2n_V + 2) \times \dim \mathfrak{g}_U$

[4]Notice that the last equation has typos in [54], indeed the minus sign in front of the first term is required for consistency with the other two equations, as can be checked by direct computation.

matrix which expresses the gauge algebra generators as a combination of the U-duality ones:

$$X_M \equiv \Theta^{\alpha}_{M} t_{\alpha}, \tag{3.79}$$

and which gets treated as a spurion field. We immediately see that this implies $\dim(G) = \mathrm{rank}(\Theta)$, which is clearly a highly redundant description, since we must have $\dim(G) < n_V + 1$. Nonetheless, the advantage of this formalism is that it allows to recast all the consistency conditions on the choice of the gauge group into $G_U$-covariant (and thus independent of the symplectic frame) constraints on $\Theta$. These constraints are:

1. **Gauge invariance.** The embedding tensor should define an $X_M$ realizing the gauge algebra[5], that is:

$$[X_M, X_N] = -X^P_{MN} X_P, \tag{3.80}$$

   where $X^P_{[MN]} \equiv (X_{[M})_{N]}{}^P$ play the role of structure constants. The key difference is that the implied constraint

$$X^P_{(MN)} X_P = 0 \tag{3.81}$$

   is usually non-trivial when magnetic gaugings are present. Notice that we expect that $X^P_{MN}$, in the electric frame and for common gaugings, to reduce to the form (3.62).

2. **Locality.** Since at most $n_V + 1$ vectors have a mutually local Lagrangian description, this must always be true in any symplectic frame, independently of the gauging. This is expressed by:

$$\Theta^{\alpha}_{M} \Theta^{\beta}_{N} \mathbb{C}^{MN} = 0. \tag{3.82}$$

   This equation, since symplectic invariant, also tells us that there also should be a frame where the gauging is realized purely electrically, since we can always choose $\Theta^{\alpha}_{M} = (\Theta^{\Lambda\alpha}, 0)$, $\Lambda = 1, \ldots, n_V + 1$. This second constraint is dependent from the first one in $\mathcal{N} > 2$ theories.

3. **Supersymmetry.** The following constraint is a linear one, coming from supersymmetry. It restricts the allowed representations of $\Theta$ with a projection:

$$X_{(MNP)} = X^Q_{(MN} \mathbb{C}_{P)Q} = 0, \tag{3.83}$$

   which in general removes the heighest weight one.

In particular, this last constraint comes from requiring the cancellation of the $\mathcal{O}(\epsilon)$ terms we introduced in the modification of the supersymmetry rules due to minimal couplings. These

[5]We expect this realization to be faithful only when there are no inert matter fields under symplectic transformations. This is the case for $\mathcal{N} > 2$ supergravity, since they always sit in the same multiplet as the vector fields. As already mentioned, however, in $\mathcal{N} = 2$ supergravity we have Quaternionic isometries with trivial (and as such, unfaithfully realized) duality action.

minimal couplings can be introduced again as:

$$D_\mu \equiv \partial_\mu - A_\mu^M \Theta_M^\alpha t_\alpha = \partial_\mu - A_\mu^\Lambda \Theta_\Lambda^\alpha t_\alpha - A_{\Lambda\mu}\Theta^{\Lambda\alpha} t_\alpha, \tag{3.84}$$

from which it is clear that the gauging is electric iff $\Theta^{\Lambda\alpha} = 0$ and the field strengths are the local Lagrangian ones $F^\Lambda$. As a last remark, let's briefly motivate the introduction of tensor fields in supergravity theories, which is of great importance when considering the derivation of these models from string compactifications [37, 45]. When building the field strength of the vectors:

$$\mathbb{F}^M = dA^M + X_{NP}^M A^N \wedge A^P. \tag{3.85}$$

It is suggestive to compute the covariant exterior derivative of this quantity to notice that $D\mathbb{F}^M \propto X_{(PQ)}^M \neq 0$. It is not shocking, though, as the true field strengths are defined by $F = F^M X_M$, and from (3.81) we get $DF^M X_M = DF = 0$. In order to fix this problem, the introduction of tensor fields is required, and the gauge algebra is not realized as a Lie algebra but as a free differential algebra [55]. These tensor fields are seen as dual to some of the original fields in the ungauged formulation, and this is to be expected since the gauging procedure should not add new degrees of freedom.

# 4 LAGRANGIAN AND SUPERSYMMETRY TRANSFORMATION RULES

Our next task is to write down the $\mathcal{N} = 2$ supergravity Lagrangian, which full derivation can be found in [46, 54]. Although it won't be reported here, it is worth mentioning that the second step of the gauging procedure, discussed in the previous chapter, can be performed more geometrically by identifying all connections and curvatures on the sigma model manifolds and promoting them to "gauged connections", i.e. adding a piece proportional to the prepotentials or killing vectors. This allows to sistematically introduce all correct gauge covariant derivatives. Moreover, without loss of generality, we will assume the gauging is done in the electric frame. We will then later relax this assumption when searching for explicit models in Chapter 5.

Before proceeding further, we will specify how fermions are introduced on the sigma model as sections of suitable bundles, which helps identifying the various covariant derivatives. Then, we will be ready to write down both the Lagrangian sectors and the supersymmetric transformation rules of interest.

1. The **gravitino** field $\psi_\mu^A$ transforms as a spinor valued 1-form on spacetime and as a section of the $\mathcal{L} \otimes \mathcal{SU}$ bundle.

2. The **gaugino** field $\lambda^{iA}$ transforms as a spinor valued 0-form on spacetime and as a section of $\mathcal{L} \otimes \mathcal{T}(\mathcal{SM}) \otimes \mathcal{SU}$, where $\mathcal{T}$ denotes again the tangent bundle.

3. The **hyperino** field $\zeta^{\alpha}$ transforms as a spinor valued 0-form on spacetime and as a section of $\mathcal{T}(\mathcal{QM}) \otimes \mathcal{SU}^{-1}$ where this notation is used to define the bundle that one obtains from $\mathcal{T}(\mathcal{QM})$ by deleting the $SU(2)$ part of the structure group.

The $\mathcal{N} = 2$ supergravity action is given schematically by:

$$S = \int d^4x\, e \left( \mathcal{L}_{kin} + \mathcal{L}_{4f} + \mathcal{L}_g \right), \tag{3.86}$$

where $\mathcal{L}_{kin}$ contains the kinetic terms of the fields as well as Pauli-like couplings involving derivatives of scalar fields, $\mathcal{L}_{4f}$ contains four-fermions terms which we will not be interested about and $\mathcal{L}_g$ contain $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$ terms introduced by the gauging. To keep track of these terms, we will explicitly introduce $g$ factors in the Lagrangian as book-keeping device. It will be useful later to split this last contribution in fermionic mass matrices, gravitino mixing matrices and potential, as $\mathcal{L}_g = \mathcal{L}_{mass} + \mathcal{L}_{mix} + V(q, z, \bar{z})$. Explicitly, we have:

$$\begin{aligned} \mathcal{L}_{kin} \supset & -\frac{1}{2}R + g_{ij^*}\nabla_\mu z^i \nabla^\mu \bar{z}^{j^*} + h_{uv}\nabla_\mu q^u \nabla^\mu q^v + \frac{\epsilon^{\mu\nu\lambda\sigma}}{e}\left(\overline{\psi}^A_\mu \gamma_\sigma \nabla_\nu \psi_{\lambda A} - \overline{\psi}_{\mu A}\gamma_\sigma \nabla_\nu \psi^A_\lambda\right) \\ & - \frac{i}{2} g_{ij^*}\left(\overline{\lambda}^{iA}\gamma^\mu \nabla_\mu \lambda^{j^*}_A + \overline{\lambda}^{j^*}_A \gamma^\mu \nabla_\mu \lambda^{iA}\right) - i\left(\overline{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha + \overline{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha\right) \\ & + i\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right), \end{aligned} \tag{3.87}$$

$$g^{-1}\mathcal{L}_{mass} \supset 2S_{AB}\overline{\psi}^A_\mu \gamma^{\mu\nu}\psi^B_\nu + M^{\alpha\beta}\overline{\zeta}_\alpha \zeta_\beta + M^\alpha_{iB}\overline{\zeta}_\alpha \lambda^{iB} + M_{iAlB}\overline{\lambda}^{iA}\lambda^{lB} + h.c. \tag{3.88}$$

$$g^{-1}\mathcal{L}_{mix} \supset 2iN^A_\alpha \overline{\zeta}^\alpha \gamma_\mu \psi^\mu_A + i g_{ij^*}W^{iAB}\overline{\lambda}^{j^*}_A \gamma_\mu \psi^{\mu B} + M^\alpha_{iB}\overline{\zeta}_\alpha \lambda^{iB} + h.c. \tag{3.89}$$

$$\begin{aligned} g^{-2}\delta^A_B V(z, \bar{z}, q) &= g_{ij^*}W^{iAC}W^{j^*}_{BC} + 2N^A_\alpha N^\alpha_B - 12S^{AC}S_{BC} \\ &= \left(g_{ij^*}k^i_\Lambda k^{j^*}_\Sigma + 4h_{uv}k^u_\Lambda k^v_\Sigma\right)\overline{L}^\Lambda L^\Sigma + \left(g^{ij^*}f^\Lambda_i f^\Sigma_{j^*} - 3\overline{L}^\Lambda L^\Sigma\right)P^x_\Lambda P^x_\Sigma, \end{aligned} \tag{3.90}$$

where $\nabla$ always denotes a suitable (gauge) covariant derivative with respect to the spin connection and bundle connections [46]. Just to give an example, and because it will be used later, we report the covariant derivative for the gravitino field:

$$\nabla\psi_A = d\psi_A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi_A + \frac{i}{2}\hat{\mathcal{Q}} \wedge \psi_A + \hat{\omega}_A{}^B \wedge \psi_B, \tag{3.91}$$

where $\hat{\mathcal{Q}}$ and $\hat{\omega}$ denote gauged 1-form $\mathcal{L}$ and $\mathcal{SU}$ connections:

$$\hat{\mathcal{Q}} = Q + gA^\Lambda P^0_\Lambda, \tag{3.92}$$

$$\hat{\omega} \equiv \hat{\omega}^x \sigma_x = (\omega^x + gA^\Lambda P^x_\Lambda)\sigma_x. \tag{3.93}$$

In the above Lagrangians we also introduced the fermionic mass matrices:

$$M^{\alpha\beta} = -\mathcal{U}_u^{\alpha A}\mathcal{U}_v^{\beta B}\epsilon_{AB}\nabla^{[u}k_\Lambda^{v]}L^\Lambda; \quad M_{\alpha\beta} \equiv (M^{\alpha\beta})^*, \tag{3.94}$$

$$M_{iB}^{\alpha} = -4\mathcal{U}_{Bu}^{\alpha}k_\Lambda^u f_i^\Lambda; \quad M_{i^*\alpha}^{B} \equiv -(M_{iB}^{\alpha})^*, \tag{3.95}$$

$$M_{iAlB} \equiv M_{iAlB}^{(1)} + M_{iAlB}^{(2)} = \epsilon_{AB}k_{[i|\Lambda}f_{l]}^\Lambda + \frac{i}{2}(\sigma_x)_A{}^C\epsilon_{BC}\nabla_i f_l^\Lambda P_\Lambda^x, \tag{3.96}$$

where $M_{iAlB}$ contains typos in [46], while the correct expression can be found in [54], and fermionic "shift" matrices:

$$S_{AB} = S_{BA} = \frac{i}{2}(\sigma_x)^C{}_A\epsilon_{BC}P_\Lambda^x L^\Lambda; \quad S^{AB} \equiv (S_{AB})^*, \tag{3.97}$$

$$W^{iAB} \equiv \epsilon^{AB}k_\Lambda^i\overline{L}^\Lambda + i(\sigma_x)_C{}^B\epsilon^{CA}P_\Lambda^x g^{ij^*}f_{j^*}^\Lambda; \quad W_{AB}^{j^*} \equiv (W^{iAB})^*, \tag{3.98}$$

$$N_\alpha^A = 2\mathcal{U}_{u\alpha}^A k_\Lambda^u\overline{L}^\Lambda; \quad N_A^\alpha \equiv -(N_\alpha^A)^*. \tag{3.99}$$

which are so called because they enter the $\mathcal{O}(g)$ susy transformation rules of the fermions:

$$\delta\psi_{A\mu} = \nabla_\mu\epsilon_A + igS_{AB}\gamma_\mu\epsilon^B + \dots \tag{3.100}$$

$$\delta\lambda^{iA} = \dots + gW^{iAB}\epsilon_B + \dots \tag{3.101}$$

$$\delta\zeta_\alpha = \dots + gN_\alpha^A\epsilon_A + \dots. \tag{3.102}$$

It is also worth mentioning the supersymmetric transformation of the veilbeins:

$$\delta V_\mu^a = -i\overline{\psi}_{A\mu}\gamma^a\epsilon^A - i\overline{\psi}_\mu^A\gamma^a\epsilon_A. \tag{3.103}$$

In equation (3.87) the normalization of the kinetic term of the quaternions follows from the choice $\lambda = -1$ in (3.41), which will be used from now on. In the same equation the kinetic terms of the vectors have been written in terms of the self dual and anti self dual combinations:

$$\mathcal{F}_{\mu\nu}^{\pm} = \frac{1}{2}\left(F_{\mu\nu}^\Lambda \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma}\right). \tag{3.104}$$

Since they will be important when discussing Yukawa couplings, we also report here the *gradient flow* equations, which express covariant derivatives of fermionic shifts in terms of mass matrices, and which in [54] contain typos in the vector multiplet sector:

$$\nabla_i S_{AB} = \frac{1}{2}g_{ij^*}W_{(AB)}^{j^*}, \tag{3.105}$$

$$\nabla_{i^*}S_{AB} = 0, \tag{3.106}$$

$$\nabla_i N_A^\alpha = \frac{1}{2}M_{iA}^\alpha, \tag{3.107}$$

$$\nabla_{i^*}N_A^\alpha = 0, \tag{3.108}$$

$$\nabla_i W^{jAB} = 2S^{AB}\delta_i^j + \epsilon^{AB}g^{jl^*}g_{im^*}k_\Lambda^{m^*}f_{l^*}^\Lambda, \tag{3.109}$$

$$\nabla_{j^*} W^{iAB} = g^{im^*}\left(2M^{AB}_{j^*m^*} + 2\epsilon^{AB} g_{lm^*} f^{\Lambda}_{j^*} k^l_{\Lambda} - \epsilon^{AB} g_{lj^*} f^{\Lambda}_{m^*} k^l_{\Lambda}\right), \tag{3.110}$$

$$\nabla_u W^{j^*}_{AB} = -\frac{1}{2} g^{ij^*} M^{\alpha}_{i(B} U_{A)\alpha u}, \tag{3.111}$$

$$\nabla_u S_{AB} = -\frac{1}{2} U_{\alpha u (A} N^{\alpha}_{B)}, \tag{3.112}$$

$$\nabla_u N^{\beta}_{A} = -4 U^{\beta B}_{u} S_{AB} - U_{uA\alpha} M^{\alpha\beta}. \tag{3.113}$$

As a last remark, the expression for the potential (3.90) in terms of the fermionic shift matrices is called *supersymmetric Ward identity* and its general form is valid in any supergravity theory.

# 5 AN EXAMPLE: DE SITTER AND GRAVITINO MASS

As a working example on how the geometric constraints on $\mathcal{N} = 2$ supergravity can help us attack and test Swampland conjectures, we review here recent work on the subject [13]. The paper shows how charged gravitini cannot have a small or vanishing Lagrangian mass on a de Sitter (dS) background while respecting the (magnetic) Weak Gravity Conjecture (2.11). In particular, assuming without restrictions that the gauging is electric, from the construction of the previous sections we will need three ingredients: the kinetic terms of the vectors, in order to identify the gauge coupling, the gravitini-gauge vectors minimal couplings in order to identify the charge, and the value of the vacuum energy when the gravitino mass is vanishing.

Rewriting the vector kinetic term in (3.87) in terms of the usual field strengths, instead of self dual or anti self dual ones, we get:

$$\mathcal{L}_{kin} \supset \frac{1}{4}\mathcal{I}_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma\mu\nu}, \qquad \mathcal{I}_{\Lambda\Sigma} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}, \tag{3.114}$$

where we recall $\mathcal{I}$ to be negative definite, scalar dependent and where $F^{\Lambda}_{\mu\nu} = 2\partial_{[\mu} A^{\Lambda}_{\nu]} + f^{\Lambda}_{\Sigma\Delta} A^{\Sigma}_{\mu} A^{\Delta}_{\nu}$. Since $-\mathcal{I}$ is positive definite (and symmetric) we can define veilbeins of the kind:

$$-\mathcal{I}_{\Lambda\Sigma} = \delta_{ij} E^i_{\Lambda} E^j_{\Sigma}, \qquad E^i_{\Lambda} E^{\Lambda}_{j} = \delta^i_j, \tag{3.115}$$

and insert them in the kinetic term, getting to:

$$\mathcal{L}_{kin} \supset -\frac{1}{4}\delta_{ij} F^i_{\mu\nu} F^{j\mu\nu}, \tag{3.116}$$

which are the kinetic terms for the "canonical" vectors $v^i = E^i_{\Lambda} A^{\Lambda}$. These vectors will include the $U(1)$ factor we are interested in exploring through the WGC.

Next, we are interested in identifying the $U(1)$ charge of the gravitini. To do so, we explicitly rewrite the relevant terms in their covariant derivative (3.91) as:

$$-i\mathcal{L}^{3/2}_{kin} = -\overline{\psi}^A_\mu \gamma^{\mu\rho\sigma}\mathcal{D}_\rho \psi_{A\sigma} - \frac{i}{2}\overline{\psi}^A_\mu \gamma^{\mu\rho\sigma} v^i_\rho \left(P^0_i \delta^B_A + (\sigma_x)_A{}^B P^x_i\right)\psi_{B\sigma}, \tag{3.117}$$

where we defined $P^\bullet_i = P^\bullet_\Lambda E^\Lambda_i$ and we used the $\epsilon \cdot \gamma$ duality property (A.13) to introduce $\gamma^{\mu\nu\rho}$. Notice that the kinetic term is canonically normalized. When choosing the veilbein, we can always perform a rotation such that our $U(1)$ gauge vector of interest $u_\mu$ is along a particular direction, say:

$$u_\mu \equiv v^{i=1}_\mu, \tag{3.118}$$

so that its minimal coupling to the gravitino field can be rewritten simply as:

$$-i\overline{\psi}^A_\mu \gamma^{\mu\rho\sigma} u_\rho Q_A{}^B \psi_{B\sigma}, \quad \text{with} \quad 2Q_A{}^B \equiv P^0_1 \delta^B_A + (\sigma_x)_A{}^B P^x_1 \psi_{B\sigma}. \tag{3.119}$$

Notice that the $Q$ matrix is expressed as a linear combination of $(\mathbb{1}, \sigma_x)$ and as such is hermitian. We can diagonalize it by a unitary transformation which we perform on the gravitini, such that in this basis the minimal couplings read:

$$-i\overline{\psi}^1_\mu \gamma^{\mu\rho\sigma} u_\rho q_1 \psi_{1\sigma} - i\overline{\psi}^2_\mu \gamma^{\mu\rho\sigma} u_\rho q_2 \psi_{2\sigma}, \tag{3.120}$$

where $q_1$ and $q_2$ are the physical charges (gauge coupling times integer charge) between gauge bosons and gravitini. As such, (2.11) can be recasted simply as:

$$\Lambda < q_1 \quad \text{and} \quad \Lambda < q_2 \qquad (\text{Planck units}). \tag{3.121}$$

Let's now turn to the potential, which if the gravitini mass is vanishing can be written using (3.33) as:

$$V = -\frac{1}{2}\mathcal{I}^{-1|\Lambda\Sigma}\left[P^0_\Lambda P^0_\Sigma + P^x_\Lambda P^x_\Sigma\right] + 4h_{uv}k^u_\Lambda k^v_\Sigma \overline{L}^\Lambda L^\Sigma, \tag{3.122}$$

and in particular:

$$V \geq \frac{1}{2}\delta^{ij}\left[P^0_i P^0_j + P^x_i P^x_j\right]. \tag{3.123}$$

Now, realizing that:

$$\begin{aligned}
\delta^{ij}\left[P^0_i P^0_j + P^x_i P^x_j\right] &= \frac{1}{2}\delta^{ij}\left(P^0_i \delta^B_A + (\sigma_x)_A{}^B P^x_i\right)\left(P^0_j \delta^A_B + (\sigma_y)_B{}^A P^y_j\right) \\
&\geq \frac{1}{2}\delta^{ij}\left(P^0_1 \delta^B_A + (\sigma_x)_A{}^B P^x_1\right)^2 = \mathrm{Tr}[QQ] = \mathrm{Tr}\Big[U^\dagger Q U U^\dagger Q U\Big],
\end{aligned} \tag{3.124}$$

we can write the following inequality:

$$V \geq q_1^2 + q_2^2 \quad \Rightarrow \quad V \geq q_1^2 \;\; \text{and} \;\; V \geq q_2^2 \quad \Rightarrow \quad V \geq \Lambda^2. \tag{3.125}$$

This in turn can be cast by considering the Hubble scale:

$$H \geq \Lambda/\sqrt{3}, \tag{3.126}$$

which means that the tree-level dS critical points will receive large quantum corrections and cannot be trusted. This challenges the consistency of such dS vacua, and is a manifestation of the Dine-Seiberg problem [56]. Moreover, this kind of reasoning can be extended to the case of a parametrically small gravitini mass matrix (3.97). In this case the potential gets another contribution of the form

$$V_{gmass} = -4\overline{L}^{\Lambda} L^{\Sigma} P^x_{\Lambda} P^x_{\Sigma}. \tag{3.127}$$

These masses are parametrically small when compared to the Hubble scale, which means that

$$\overline{L}^{\Lambda} L^{\Sigma} P^x_{\Lambda} P^x_{\Sigma} \ll H^2 \tag{3.128}$$

and the potential is still above the UV cutoff $\Lambda$, as before. Notice that if the gravitini are uncharged the same conclusion cannot be reached, but at the same time the WGC cannot be applied and it is not clear if the theory should fall in the Swampland or not. In the original paper, the authors also provide a wide range of examples of how the above reasoning is realized explicitly, which however won't be discussed here, since we want to focus on the basic reasoning. We saw nonetheless how the geometrical and constrained structure of $\mathcal{N} = 2$ supergravity could be used to test Swampland conjectures. In the next chapters we will try to apply a similar philosophy to the calculation of Yukawa couplings.

# 4 *Testing the Fermionic WGC: general relations*

On the line of [12, 13, 34, 35], we will first try and motivate (2.15) in the full general theory, while the construction of explicit models will be addressed in Chapter 5. In particular, the main objective of this chapter is to give a precise, geometrical meaning to the Yukawa $Y$ and the scale of supersymmetry breaking $m_{susy}$ appearing in the Fermionic Weak Gravity Conjecture, and to express them in terms of relevant quantities of the bundle we have constructed on the scalar manifold $\mathcal{P} \longrightarrow M_{scal}$. The hope is to gain insight on how the conjecture could be realized in terms of hierarchies of such quantities. Since this kind of construction is not present in the literature, in the following we provide an original approach to the conjecture. En passant, we will also prove the super-Higgs mechanism for $\mathcal{N} = 2$ theories, which can be again regarded as original work.

Since we are talking about *broken* (local) supersymmetry, we will first have to introduce the super-Higgs mechanism and correctly identify the physical, fermionic mass matrices. Then, from this analysis, the computation of the Yukawa couplings, albeit involved, will be straightforward. To keep the logic simple, we will focus on calculating these quantities on Minkowski backgrounds, since in this case the Lagrangian masses are the physical (tree-level) masses and the scale of supersymmetry breaking is easily identified. The next section will be devoted to this last, broad topic of supersymmetry breaking, and will mainly follow [45].

## 1 VACUA AND SUPERSYMMETRY BREAKING

A (Lorentz-preserving) vacuum of a supergravity theory is a solution of the equations of motion that is maximally symmetric in the gravitational sector, and as such can be described by a cosmological constant $\Lambda$. Due to maximal symmetry and Lorentz invariance, only scalar fields[1] can take on take on a certain (constant and possibly non vanishing) vacuum expectation value (vev):

$$\langle \phi_s(x) \rangle = \phi_0^s. \tag{4.1}$$

[1] In general, we can also have fermionic bilinears $\langle \overline{\psi}\psi \rangle$ with non-vanishing vevs, but since they won't be relevant to the discussion, we will neglect them in the following.

This value describes a critical point of the scalar potential,

$$\left.\frac{\partial V(\phi)}{\partial \phi^s}\right|_{\phi^s=\phi_0^s} = 0, \tag{4.2}$$

and the cosmological constant is given by

$$\Lambda = V(\phi_0). \tag{4.3}$$

A vacuum $\phi_0$ can be supersymmetric, namely can preserve an amount of supersymmetry. In this case there should exist a local supersymmetry parameter $\epsilon_A(x)$ along which the supersymmetry variation, evaluated on the solution, of the fermionic fields vanish. This follows from the fact that if $\bar{\epsilon}Q\,|0\rangle \equiv \bar{\epsilon}_A Q^A\,|0\rangle = 0$ then

$$\delta_\epsilon f = \langle 0|\,[\bar{\epsilon}Q, f]\,|0\rangle = 0, \tag{4.4}$$

where $Q$ are the supercharges generating the odd part of the supersymmetry algebra. When appearing in the Lie bracket, $f$ is meant as a field operator. Notice that we can restrict $f$ to be a fermionic field, since the supersymmetric variation of all scalar fields is trivially vanishing on the vacuum. Indeed, by construction, in any supergravity theory we schematically have $\delta_\epsilon(\text{Bosons}) = \epsilon(\text{Fermions})$, which vev vanishes.

Then, we can translate the above equation to explicit conditions for $\psi_\mu^A, \lambda^{iA}$ and $\zeta^\alpha$. That is, evaluating (3.100),(3.101) and (3.102) on the vacuum:

$$\begin{aligned}
\delta\psi_{A\mu} &= \mathcal{D}_\mu \epsilon_A + igS_{AB}\gamma_\mu \epsilon^B = 0, \\
\delta\lambda^{iA} &= gW^{iAB}\epsilon_B = 0, \\
\delta\zeta_\alpha &= gN_\alpha^A \epsilon_A = 0,
\end{aligned} \tag{4.5}$$

where we neglected all contributions from vector or fermion fields and where the full covariant derivative reduces to the Lorentz-covariant one, since the pullback on spacetime of the scalar manifold bundle connections always involves expressions like $\partial_\mu \phi^i \nabla_i(\dots)$, and on the vacuum we have $\partial_\mu \langle \phi(x) \rangle = 0$. Equation (4.5) is known as the Killing spinor equation and can have at most $\mathcal{N}$ solutions $\tilde{\epsilon}_a$, $a = 1, \dots, n \leq \mathcal{N}$. In this case we say that the vacuum preserves $n \leq \mathcal{N}$ supersymmetries. In particular, in the case of $\mathcal{N} = 2$ supergravity, the vacuum can preserve all supersymmetries, no supersymmetries, or one supersymmetry, and we talk about $\mathcal{N} = 2$, $\mathcal{N} = 0$ or $\mathcal{N} = 1$ vacua respectively. It will be useful to work out the integrability condition of the first of (4.5), which reads

$$\begin{aligned}
0 = \mathcal{D}_{[\mu}\delta\psi_{\nu]A} &= \mathcal{D}_{[\mu}\mathcal{D}_{\nu]}\epsilon_A + igS_{AB}\gamma_{[\nu}\mathcal{D}_{\mu]}\epsilon^B \\
&= \frac{1}{8}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma}\epsilon_A - g^2\gamma_{\mu\nu}S_{AB}S^{BC}\epsilon_C,
\end{aligned} \tag{4.6}$$

and where we used the following commutator of covariant derivatives, also reported in (A.1):

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = \frac{1}{4}\gamma_{ab}R^{ab}{}_{\mu\nu}. \tag{4.7}$$

**$\mathcal{N} = 2$ vacuum.** Let's assume first the vacuum preserves two supersymmetries: this implies

$$W^{iAB} = 0, \quad N^A_\alpha = 0, \tag{4.8}$$

and from the integrability condition, using gamma contraction identities:

$$-12g^2 S_{AB}S^{BC} = \frac{1}{8}R_{\mu\nu\rho\sigma}\gamma^{\mu\nu}\gamma^{\rho\sigma}\delta^C_A, \tag{4.9}$$

where again all expressions are understood as evaluated on the vacuum. Then, using again gamma identities, permutation properties of the Riemann tensor indices and the fact that $R_{\mu\nu}|_{vac} = -\Lambda g_{\mu\nu}$, the previous equation can be recast as:

$$\Lambda\delta^C_B = -12g^2 S_{AB}S^{BC} = -12g^2(S^\dagger S)_A{}^C \equiv -12g^2(S^2)_A{}^C, \tag{4.10}$$

which is indeed what we get by looking at the potential in (3.90) and using (4.8) in (4.3). For ease of notation, in the previous expression we defined the matrix $S^2 = S^\dagger S$. From this it is also clear that since $S^2$ is positive semidefinite, supersymmetric vacua can be only Minkowski ($\Lambda = 0$) or anti de Sitter (AdS, $\Lambda < 0$). In the Minkowski case, we immediately see that $S^2 = 0$, and hence gravitini remain massless.

**$\mathcal{N} = 0$ vacuum.** Let's now consider a vacuum with no supersymmetry: in this case

$$W^{iAB} \neq 0, \quad N^A_\alpha \neq 0, \tag{4.11}$$

which means that in general the cosmological constant will be a certain combination of fermionic shifts:

$$\Lambda\delta^A_B = g_{ij^*}W^{iAC}W^{j^*}_{BC} + 2N^A_\alpha N^\alpha_B - 12S^{AC}S_{BC}. \tag{4.12}$$

In the Minkowski case, we get the special condition:

$$g_{ij^*}W^{iAC}W^{j^*}_{BC} + 2N^A_\alpha N^\alpha_B - 12S^{AC}S_{BC} = 0, \tag{4.13}$$

and following (4.11) the gravitini mass matrix will be non-vanishing. In particular, since we must have no solution to the Killing spinor equations, the matrices $W^2$ and $N^2$ appearing above need not be degenerate. Thus, the $S$ matrix will have two non-zero eigenvalues, which are the two non-vanishing masses of the gravitini.

$\mathcal{N} = \mathbf{1}$ **vacuum.** Finally, we turn to the case of partial supersymmetry breaking. In this case, the Killing spinor equations admit one solution, which without loss of generality we can rotate to be along the first component $\epsilon_1$. Then, we have:

$$W^{iA1} = 0, \quad N_{\alpha}^{1} = 0. \tag{4.14}$$

Looking instead at the integrability condition, now we have:

$$\Lambda\delta_A^1 = -g^2 12 S_{AB} S^{B1}, \tag{4.15}$$

which is satisfied by:

$$g^2 S^2 = \begin{pmatrix} -\frac{1}{12}\Lambda & 0 \\ 0 & g^2 S_{2A} S^{A2} \end{pmatrix}. \tag{4.16}$$

Again, in the case of a Minkowski vacuum we can see that the form of the $S$ matrix is simple and we have one massive and one massless gravitino. Notice that since the two $SU(2)$ doublet gravitini are treated differently, this vacuum will inevitably need to (partially) break R-symmetry. This can be achieved minimally by one hypermultiplet and one vector, and implies that a theory with only vector multiplets can not display $\mathcal{N} = 1$ vacua [43, 57, 58].

## 2 SUPER-HIGGS MECHANISM

It turns out there is a mechanism akin to the Brout-Englert-Higgs mechanism [59] for spontaneous (local) supersymmetry breaking: as we have seen, when the background is of Minkowski type, the masses of the gravitini can be taken to be associated to the scale of supersymmetry breaking

$$m_{susy}^4 = 12 g^2 S_{AB} S^{AB}, \tag{4.17}$$

just as the masses of the vector fields are related to the gauge symmetry breaking scale in the Higgs mechanism. This is not a coincidence: the gravitino can be taken to be the "gauge" fields of local supersymmetry, as can be seen from the comparison of its transformation rule in pure supergravity with the gauge transformation of an abelian vector field:

$$\begin{aligned} \delta\psi_{\mu A} &= \partial_\mu \epsilon_A(x) \\ \delta A_\mu &= \partial_\mu \lambda(x). \end{aligned} \tag{4.18}$$

As with massless vectors, counting the on-shell (real) degrees of freedom (dof) of the massless gravitino leads to $\#dof = 2$, while in the massive case we expect the on-shell dofs to be $\#dof = 4$. As such, the gravitino should acquire two new spin-1/2 polarizations from a "goldstino" through a mechanism, suitably called *super-Higgs mechanism*. Proving this at the Lagrangian level on arbitrary curved backgrounds is non-trivial and requires dealing with the various constraints coming from supersymmetry. There has been recent work in the context of $\mathcal{N} = 1$ theories in [60], and in this thesis we will provide an extension in $\mathcal{N} = 2$ supergravity, assuming an $\mathcal{N} = 0$ vacuum. As in the paper, the main strategy is to properly get rid of gravitino-fermions mixings by redefining the former with a supersymmetric transformation. A new Lagriangian will then follow which will make explicit the decoupling of the goldstino (we will call this gauge the "unitary gauge", similarly to the Higgs case). In the following, we will implicitly evaluate all expressions on the vacuum, and all covariant derivatives can again be considered to be only the Lorentz-covariant ones. Moreover, we will set $g = 1$ in the Lagrangian for ease of notation.

## 2.1 *Gravitino redefinition*

We start by identifying the terms responsible for gravitini-fermions mixing in equation (3.89). We can define the combination of the spin-1/2 fermions

$$v^A = ig_{ij^*}W^{iBA}\lambda_B^{j^*} + 2iN_\alpha^A\zeta^\alpha, \tag{4.19}$$

$$v_A = -ig_{ij^*}W_{BA}^{j^*}\lambda^{Bi} - 2iN_A^\alpha\zeta_\alpha. \tag{4.20}$$

Notice that the fermion $v^A$, which in the following will be called goldstino, has the same chirality as the gravitino, namely

$$\gamma_5 v^A = -v^A, \qquad \gamma_5 v_A = v_A. \tag{4.21}$$

The first step is to decouple the goldstino $v^A$ from the gravitino, namely we want to have $\mathcal{L}_{mix} = 0$. To this purpose, we redefine the gravitino as

$$\psi_{\mu A} = \Psi_{\mu A} + iS_{AB}\gamma_\mu X^B{}_D v^D + X_A{}^D\mathcal{D}_\mu v_D, \tag{4.22}$$

$$\psi_\mu^A = \Psi_\mu^A + iS^{AB}\gamma_\mu X_B{}^D v_D + X^A{}_D\mathcal{D}_\mu v^D, \tag{4.23}$$

$$\bar\psi_{\mu A} = \bar\Psi_{\mu A} - iS_{AB}X^B{}_D\bar V^D\gamma_\mu + X_A{}^D\mathcal{D}_\mu\bar v_D, \tag{4.24}$$

$$\bar\psi_\mu^A = \bar\Psi_\mu^A - iS^{AB}X_B{}^D\bar v_D\gamma_\mu + X^A{}_D\mathcal{D}_\mu\bar v^D, \tag{4.25}$$

where $\Psi_\mu^A$ is the new gravitino and $X_A{}^B$ is to be determined by asking that $\mathcal{L}_{mix} = 0$. Notice that we used the charge conjugation rule $(\bar\psi^A\gamma^\mu\chi_B)^\dagger = -\bar\psi_A\gamma^\mu\chi^B$, which is responsible for the additional minus in the second term of the gravitino redefinition. We perform this substitution everywhere in the Lagrangian, up to two fermions.

1. **Gravitino mass term.** Inserting the gravitino redefinition into the gravitino mass term we get

$$\begin{aligned} 2S_{AB}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu + h.c. = {} & 2S_{AB}\bar{\Psi}^A_\mu\gamma^{\mu\nu}\Psi^B_\nu \\ & + 12iS_{AB}S^{BC}X_C{}^D(\bar{\Psi}^A_\mu\gamma^\mu v_D) \\ & + 4S_{AB}X^B{}_E(\bar{\Psi}^A_\mu\gamma^{\mu\nu}\mathcal{D}_\nu v^E) \\ & + 24S_{AB}S^{AF}S^{BC}X_F{}^D X_C{}^E(\bar{v}_D v_E) \\ & - 12iS_{AB}S^{AF}X_F{}^D X^B{}_E(\bar{v}_D\gamma^\mu\mathcal{D}_\mu v^E) \\ & - 2S_{AB}X^A{}_D X^B{}_E(\bar{v}^D\gamma^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu v^E) + h.c. \end{aligned} \tag{4.26}$$

2. **Gravitino goldstino mixing term.** Inserting the gravitino redefinition into the gravitino goldstino mixing term we get

$$\begin{aligned} \mathcal{L}_{mix} &= \bar{v}^A\gamma^\mu\psi_{\mu A} + h.c. \\ &= \bar{v}^A\gamma^\mu\bar{\Psi}_{\mu A} + 4iS_{AB}X^B{}_D(\bar{v}^A v^D) + X_A{}^D(\bar{v}^A\gamma^\mu\mathcal{D}_\mu v_D) + h.c. \\ &= \bar{\Psi}^{\mu A}\gamma^\mu\bar{v}_A - 4iS^{AB}X_B{}^D(\bar{v}_A v_D) - X^A{}_D(\bar{v}_A\gamma^\mu\mathcal{D}_\mu v^D) + h.c. \end{aligned} \tag{4.27}$$

3. **Gravitino kinetic term.** Inserting the gravitino redefinition into the gravitino kinetic term we get

$$\begin{aligned} \epsilon^{\mu\nu\lambda\sigma}\bar{\psi}^A_\mu\gamma_\sigma\mathcal{D}_\nu\psi_{\lambda A} + h.c. = {} & \epsilon^{\mu\nu\lambda\sigma}\bar{\Psi}^A_\mu\gamma_\sigma\mathcal{D}_\nu\Psi_{\lambda A} \\ & - 4S_{AB}X^B{}_E(\bar{\Psi}^A_\mu\gamma^{\mu\nu}\mathcal{D}_\nu v^E) \\ & + 2iX_A{}^B(\bar{\Psi}^A_\mu\gamma^{\mu\nu\lambda}\mathcal{D}_\nu\mathcal{D}_\lambda v_B) \\ & + 6iS_{AB}S^{AF}X_F{}^D X^B{}_E(\bar{v}_D\gamma^\mu\mathcal{D}_\mu v^E) \\ & + 4S_{AB}X^A{}_D X^B{}_E(\bar{v}^D\gamma^{\mu\nu}\mathcal{D}_\mu D_\nu v^E) \\ & - \epsilon^{\mu\nu\lambda\sigma}X^A{}_D X_A{}^E(\bar{v}^D\gamma_\sigma\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\lambda v_E) + h.c. \end{aligned} \tag{4.28}$$

   In obtaining these expressions we have systematically integrated by parts, neglected higher-order contributions due to torsion $\mathcal{D}e$ and dropped terms in which the spacetime derivative was acting on functions of the scalar fields, such as $S_{AB}$ and $X^B_A$, since these are constant on the vacuum.

When summing these three contributions, the third line of the gravitino mass term cancels against the second line of the gravitino kinetic term, the fifth line of the gravitino mass term adds up to the fourth line of the kinetic term, the last line of the gravitino mass term cancels again the fifth line of the gravitino kinetic term, but that contribution survives with a factor 2. We find:

$$\begin{aligned}
\epsilon^{\mu\nu\lambda\sigma}\bar{\psi}^A_\mu\gamma_\sigma\mathcal{D}_\nu\psi_{\lambda A}+2S_{AB}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu+\bar{v}^A\gamma^\mu\psi_{\mu A} = {} & \epsilon^{\mu\nu\lambda\sigma}\bar{\Psi}^A_\mu\gamma_\sigma\mathcal{D}_\nu\Psi_{\lambda A}+2S_{AB}\bar{\Psi}^A_\mu\gamma^{\mu\nu}\Psi^B_\nu \\
& +12iS_{AB}S^{BC}X_C{}^D(\bar{\Psi}^A_\mu\gamma^\mu v_D)+\bar{\Psi}^A_\mu\gamma^\mu v_A \\
& -6iS_{AB}S^{AF}X_F{}^D X^B{}_E(\bar{v}_D\gamma^\mu\mathcal{D}_\mu v^E)-X^D{}_E(\bar{v}_D\gamma^\mu\mathcal{D}_\mu v^E) \\
& +24S_{AB}S^{AF}S^{BC}X_F{}^D X_C{}^E(\bar{v}_D v_E)-4iS^{AB}X^B{}_D(\bar{v}_A v_D) \\
& +2iX_A{}^B(\bar{\Psi}^A_\mu\gamma^{\mu\nu\lambda}\mathcal{D}_\nu\mathcal{D}_\lambda v_B) \\
& +2S_{AB}X^A{}_D X^B{}_E(\bar{v}^D\gamma^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu v^E) \\
& -\epsilon^{\mu\nu\lambda\sigma}X^A{}_D X_A{}^E(\bar{v}^D\gamma_\sigma\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\lambda v_E)+h.c..
\end{aligned} \tag{4.29}$$

Notice that we have two kinds of gravitino-goldstino mixing term, namely $\psi\gamma^\mu v$ and $\psi\gamma^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu v$. The latter is non-zero only on a curved background, since on flat space $D_\mu=\partial_\mu$ and flat derivatives commute. To cancel this term, we need to perform a redefinition of the graviton. This will generate another term of the kind $\psi\gamma^\mu v$, multiplied by the scalar potential $V$, which will be cancelled by an appropriate choice of $X_A^B$.

## 2.2 *Graviton redefinition and cancellation of mixing terms*

First, let us recall the commutator of two spacetime covariant derivatives:

$$[\mathcal{D}_\mu,\mathcal{D}_\nu]v_A=\frac{1}{4}\gamma_{ab}R^{ab}{}_{\mu\nu}v_A. \tag{4.30}$$

Then, contracting with gamma matrices, we find

$$\gamma^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu v_A=-\frac{1}{4}Rv_A, \tag{4.31}$$

$$\gamma^{\mu\nu\rho}\mathcal{D}_\nu\mathcal{D}_\rho v_A=\frac{1}{2}\left(R^{\mu\nu}\gamma_\nu-\frac{1}{2}R\gamma^\mu\right)v_A. \tag{4.32}$$

Then, we redefine the graviton as

$$e^a_\mu=\tilde{e}^a_\mu-\left(iX_A{}^B\bar{\Psi}^A_\mu\gamma^a v_B+h.c.\right) \tag{4.33}$$

and thus, using $\delta e=e\,e^\mu_a\delta e^a_\mu$ and $\delta(eR)=-2e\left(R^\mu_a-\frac{1}{2}Re^\mu_a\right)\delta e^a_\mu$ (we call $\delta e^a_\mu\equiv\tilde{e}^a_\mu-e^a_\mu$), we have that the Einstein-Hilbert term and the potential term (which are the only ones giving a two-fermion contribution) change as:

$$\begin{aligned}
-\frac{1}{2}eR(e) - eV &= -\frac{1}{2}\tilde{e}R(\tilde{e}) - \tilde{e}V + \tilde{e}\left(R^{\mu}_{a} - \frac{1}{2}R\tilde{e}^{\mu}_{a}\right)\delta e^{a}_{\mu} - \tilde{e}\,V\,\tilde{e}^{\mu}_{a}\delta e^{a}_{\mu} \\
&= -\frac{1}{2}\tilde{e}R(\tilde{e}) - \tilde{e}V + \tilde{e}\left(R^{\mu}_{a} - \frac{1}{2}(R+2V)\tilde{e}^{\mu}_{a}\right)\delta e^{a}_{\mu} \\
&= -\frac{1}{2}\tilde{e}R(\tilde{e}) - \tilde{e}V - \tilde{e}\left(iX_{A}{}^{B}\bar{\Psi}^{A}_{\mu}\left(R^{\mu\nu}\gamma_{\nu} - \frac{1}{2}R\gamma^{\mu}\right)v_{B} + h.c.\right) \\
&+ \tilde{e}\left(iVX_{A}{}^{B}\bar{\Psi}^{A}_{\mu}\gamma^{\mu\nu}v_{B} + h.c.\right)
\end{aligned} \tag{4.34}$$

One can check that the term before the last one above cancels against the fifth line of (4.29), which reads

$$+2iX_{A}{}^{B}(\bar{\Psi}^{A}_{\mu}\gamma^{\mu\nu\lambda}\mathcal{D}_{\nu}\mathcal{D}_{\lambda}v_{B}) + h.c. = iVX_{A}{}^{B}\bar{\Psi}^{A}_{\mu}\left(R^{\mu\nu}\gamma_{\nu} - \frac{1}{2}R\gamma^{\mu}\right)v_{B} + h.c. \tag{4.35}$$

Then, the only gravitino-goldstino mixing terms remaining are of the type $\bar{\Psi}\gamma v$ and read

$$12iS_{AM}S^{MC}X_{C}{}^{B}\bar{\Psi}^{A}_{\mu}\gamma^{\mu}v_{B} + \bar{\Psi}^{A}_{\mu}\gamma^{\mu}v_{A} + iX_{A}{}^{B}\bar{\Psi}^{A}_{\mu}\gamma^{\mu}v_{B} + h.c. \equiv 0. \tag{4.36}$$

By asking that this vanishes, we find the matrix $X_{A}{}^{B}$:

$$X_{A}{}^{B} = \left(-iV\delta^{A}_{B} - 12iS_{BC}S^{CA}\right)^{-1}, \tag{4.37}$$

$$X^{A}{}_{B} = \left(iV\delta^{B}_{A} + 12iS^{BC}S_{CA}\right)^{-1}. \tag{4.38}$$

Using equation (4.12), the expression for $X_{A}{}^{B}$ can be written as

$$X_{A}{}^{B} = \left(-iV\delta^{A}_{B} - 12iS_{BC}S^{CA}\right)^{-1} = \left(-ig_{ij^{*}}W^{iAC}W^{j^{*}}_{BC} - 2iN^{A}_{\alpha}N^{\alpha}_{B}\right)^{-1}. \tag{4.39}$$

This goes as the inverse of the supersymmetry breaking scale and as such is ill defined on a supersymmetric vacuum, as expected. Turning the logic around, one could derive the scalar potential and the Ward identity of any gauge supergravity in any dimension by asking that gravitino-goldstino mixing terms cancel out as above. After all gravitino goldstino mixing terms have cancelled, we are left with

$$\begin{aligned}
\epsilon^{\mu\nu\lambda\sigma}\bar{\psi}^{A}_{\mu}\gamma_{\sigma}\mathcal{D}_{\nu}\psi_{\lambda A} + 2S_{AB}\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu} + \bar{v}^{A}\gamma^{\mu}\psi_{\mu A} &= \epsilon^{\mu\nu\lambda\sigma}\bar{\Psi}^{A}_{\mu}\gamma_{\sigma}\mathcal{D}_{\nu}\Psi_{\lambda A} + 2S_{AB}\bar{\Psi}^{A}_{\mu}\gamma^{\mu\nu}\Psi^{B}_{\nu} \\
&- 6iS_{AB}S^{AF}X_{F}{}^{D}X^{B}{}_{E}(\bar{v}_{D}\gamma^{\mu}\mathcal{D}_{\mu}v^{E}) - X^{D}{}_{E}(\bar{v}_{D}\gamma^{\mu}\mathcal{D}_{\mu}v^{E}) \\
&+ 24S_{AB}S^{AF}S^{BC}X_{F}{}^{D}X_{C}{}^{E}(\bar{v}_{D}v_{E}) - 4iS^{AB}X_{B}{}^{D}(\bar{v}_{A}v_{D}) \\
&+ 2S_{AB}X^{A}{}_{D}X^{B}{}_{E}(\bar{v}^{D}\gamma^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}v^{E}) \\
&- \epsilon^{\mu\nu\lambda\sigma}X^{A}{}_{D}X_{A}{}^{E}(\bar{v}^{D}\gamma_{\sigma}\mathcal{D}_{\mu}\mathcal{D}_{\nu}\mathcal{D}_{\lambda}v_{E}) + h.c.
\end{aligned} \tag{4.40}$$

This is almost the final result, but first we have to get rid of the higher derivatives terms in the last two lines. To this purpose, we need to redefine the graviton once more.

## 2.3 *Second graviton redefinition and cancellation of higher derivative terms*

First, let us rewrite the higher derivative terms in a more convenient form

$$\begin{aligned}\mathcal{L}_{hd} &= 2S_{AB}X^A{}_DX^B{}_E(\bar{v}^D\gamma^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu v^E) - \epsilon^{\mu\nu\lambda\sigma}X^A{}_DX_A{}^E(\bar{v}^D\gamma_\sigma\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\lambda v_E) + h.c. \\ &= -\frac{R}{2}S_{AB}X^B{}_DX^A{}_E(\bar{v}^Dv^E) - \frac{i}{2}X^A{}_EX_A{}^E\bar{v}^D\left(R^{\mu\nu}\gamma_\nu - \frac{1}{2}R\gamma^\mu\right)\mathcal{D}_\mu v_E\end{aligned} \tag{4.41}$$

To cancel them, we redefine the graviton as

$$\tilde{e}^a_\mu = E^a_\mu + \left(\frac{1}{2}S_{AB}X^B{}_DX^A{}_E(\bar{v}^Dv^E)\tilde{e}^a_\mu - \frac{i}{2}X^A{}_DX_A{}^E\bar{v}^D\gamma^a\mathcal{D}_\mu v_E + h.c.\right) \tag{4.42}$$

such that ($\delta\tilde{e}^a_\mu = E^a_\mu - \tilde{e}^a_\mu$):

$$\begin{aligned}-\frac{1}{2}\tilde{e}R(\tilde{e}) - \tilde{e}V &= -\frac{1}{2}ER(E) - EV + \left(R^\mu_a - \frac{1}{2}(R+2V)E^\mu_a\right)\delta\tilde{e}^a_\mu \\ &= -\frac{1}{2}ER(E) - EV \\ &+ E\left[\frac{R}{2}S_{AB}X^B{}_DX^A{}_E(\bar{v}^Dv^E) + \frac{i}{2}X^A{}_DX_A{}^E\bar{v}^D\left(R^{\mu\nu}\gamma_\nu - \frac{1}{2}R\gamma^\mu\right)\mathcal{D}_\mu v_E\right. \\ &\left.+ VS_{AB}X^B{}_DX^A{}_E(\bar{v}^Dv^E) - \frac{i}{2}VX^A{}_DX_A{}^E\bar{v}^D\gamma^\mu\mathcal{D}_\mu v_E + h.c.\right].\end{aligned} \tag{4.43}$$

The terms in the third line cancel the precisely against $\mathcal{L}_{hd}$. The terms in the fourth line give new contributions to the mass and to the kinetic term of the spin $1/2$ fermions.

## 2.4 *Final lagrangian after redefinitions*

After all of these redefinitions and cancellations, the old Lagrangian $\mathcal{L}(e,\psi,v,\text{mix})$ in terms of the old variables becomes the same lagrangian in terms of the new variables but without gravitino goldstino mixing term, $\mathcal{L}(E,\Psi,v,\text{no mix})$ plus additional, new contributions to the mass and the kinetic term of $v$. Schematically

$$\mathcal{L}(e,\psi,v,\text{mix}) = \mathcal{L}(E,\Psi,v,\text{no mix}) + \mathcal{L}_{new}(v), \tag{4.44}$$

where

$$\begin{aligned}\mathcal{L}_{new} &= \left(-6iS_{MC}S^{MF}X_F{}^BX^C{}_A + X_A{}^B - \frac{i}{2}VX^C{}_AX_C{}^B\right)\bar{v}_B\gamma^\mu\mathcal{D}_\mu v^A + \\ &+ \left(24S^{AB}S_{AF}S_{BC}X^F{}_DX^C{}_E + 4iS_{DC}X^C_E + VS_{AB}X^B{}_DX^A{}_E\right)\bar{v}^Dv^E + h.c.\end{aligned} \tag{4.45}$$

After some manipulations, using (4.37), this becomes

$$\mathcal{L}_{new} = \frac{1}{2}X_A{}^B\bar{v}_B\gamma^\mu\mathcal{D}_\mu v^A - 2iX_B{}^DS^{BE}\bar{v}_Dv_E + h.c. \tag{4.46}$$

What is left to check is that this new Lagrangian makes the decoupling of the goldstino explicit, as promised in the introduction.

## 2.5 *Decoupling of the goldstino*

**Decoupling from the kinetic matrix.** First, we want to check that the kinetic matrix for the spin 1/2 fermions has a zero eigenvector, namely the goldstino. After the redefinitions above, this matrix reads

$$\begin{aligned}\mathcal{L}_{kin}(v) &= \mathcal{L}_{old} + \mathcal{L}_{new} \\ &= -\frac{i}{2} g_{ij^*}\bar{\lambda}^{iA}\gamma^\mu D_\mu \lambda^{j^*}_A - i\bar{\zeta}_\alpha \gamma^\mu D_\mu \zeta^\alpha + \frac{1}{2} X_A{}^B \bar{v}_B \gamma^\mu D_\mu v^A + h.c. \\ &\equiv \mathcal{M}_{I\bar{J}} \bar{\theta}^I \gamma^\mu D_\mu \theta^{\bar{J}},\end{aligned} \tag{4.47}$$

where we introduced

$$\mathcal{M}_{I\bar{J}} = \begin{pmatrix} -\frac{i}{2} g_{ij^*}\delta^A_B + g_{ik^*} g_{lj^*} W^{k^*}_{BM} W^{lAN} X_N{}^M & g_{ik^*} X_N{}^M W^{k^*}_{MB} N^N_\alpha \\ g_{lj^*} X_N{}^M W^{lAN} N^\beta_M & -i\delta^\beta_\alpha + 2X_N{}^M N^N_\alpha N^\beta_M \end{pmatrix} \tag{4.48}$$

and also

$$\theta^I = \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \end{pmatrix}, \qquad \theta^{\bar{J}} = \begin{pmatrix} \lambda^{j^*}_A \\ \zeta^\alpha \end{pmatrix}. \tag{4.49}$$

We can check that the full kinetic matrix (4.48) admits the zero eigenvector

$$\mathcal{G}^{\bar{J}} = \begin{pmatrix} W^{j^*}_{AQ}\eta^Q \\ N^\alpha_Q \eta^Q \end{pmatrix}, \tag{4.50}$$

where $\eta^Q$ is a constant spinor. Then, $\mathcal{G}^{\bar{J}}$ is the goldstino. We have

$$\mathcal{M}_{I\bar{J}}\mathcal{G}^{\bar{J}} = \begin{pmatrix} -\frac{i}{2} g_{ij^*} W^{j^*}_{BQ} + \frac{1}{2} g_{ik^*} g_{lj^*} W^{k^*}_{BM} W^{lMN} W^{j^*}_{PQ} X_N{}^M + g_{ik^*} W^{k^*}_{BM} N^N_\alpha N^\alpha_Q X_N{}^M \\ \left(g_{lj^*} X_N{}^M W^{lAN} N^\beta_M\right) W^{j^*}_{AQ} - i\left(\delta^\beta_\alpha + 2iX_N{}^M N^N_\alpha N^\beta_M\right) N^\alpha_Q \end{pmatrix} \tag{4.51}$$

The first line vanishes since

$$\begin{aligned} -\frac{i}{2} g_{ij^*} W^{j^*}_{BQ} &+ \frac{1}{2} g_{ik^*} g_{lj^*} W^{k^*}_{BM} W^{lMN} W^{j^*}_{PQ} X_N{}^M + g_{ik^*} W^{k^*}_{BM} N^N_\alpha N^\alpha_Q X_N{}^M \\ &= -\frac{i}{2} g_{ij^*} W^{j^*}_{BQ} + \frac{1}{2} g_{ik^*} W^{k^*}_{BM} \underbrace{\left(g_{lj^*} W^{lAN} W^{j^*}_{AQ} + 2N^N_\alpha N^\alpha_Q\right)}_{i(X^{-1})^Q{}_N} X_N{}^M \\ &= -\frac{i}{2} g_{ij^*} W^{j^*}_{BQ} + \frac{i}{2} g_{ik^*} W^{k^*}_{BQ} = 0. \end{aligned} \tag{4.52}$$

The second line vanishes since

$$\begin{aligned}
&\left(g_{Ij^*} X_N{}^M W^{IAN} N^\beta_M\right) W^{j^*}_{AQ} - i\left(\delta^\beta_\alpha + 2iX_N{}^M N^N_\alpha N^\beta_M\right) N^\alpha_Q \\
&\quad = g_{Ij^*} X_N{}^M W^{IAN} N^\beta_M W^{j^*}_{AQ} - N^\alpha_Q + 2X_N{}^M N^N_\alpha N^\alpha_Q N^\beta_M \\
&\quad = X_N{}^M N^\beta_M \left(g_{Ij^*} W^{IAN} N^\beta_M - i(X_N{}^M)^{-1} + 2N^N_\alpha N^\alpha_Q\right) = 0.
\end{aligned} \tag{4.53}$$

**Decoupling from the mass matrix.** Next, we want to check that also the mass matrix of the spin 1/2 fermions has the same zero eigenvector. After the redefinitions above, this matrix reads

$$\begin{aligned}
\mathcal{L}_{mass}(v) &= \mathcal{L}_{mass,old} + \mathcal{L}_{mass,new} \\
&= M_{iAlB}\bar\lambda^{iA}\lambda^{lB} + M^{\alpha\beta}\bar\zeta_\alpha\zeta_\beta + M^\alpha_{iB}\bar\zeta_\alpha\lambda^{iB} - 2iX_B{}^D S^{BE}\bar v_D v_E + h.c. \\
&= \mathcal{M}_{IJ}\bar\theta^I\bar\theta^J + h.c.
\end{aligned} \tag{4.54}$$

where $\theta^I$ have been introduced before, while

$$\mathcal{M}_{IJ} = \begin{pmatrix} M_{iMlN} + 2iX_B{}^D S^{BE} g_{ij^*} W^{j^*}_{MD} g_{lk^*} W^{k^*}_{NE} & \frac{1}{2}M^\beta_{iM} + 4iX_B{}^D S^{BE} g_{ij^*} W^{j^*}_{MD} N^\beta_E \\ \frac{1}{2}M^\alpha_{iM} + 4iX_B{}^D S^{BE} g_{lk^*} W^{k^*}_{ME} N^\alpha_D & M^{\alpha\beta} + 8iX_B{}^D S^{BE} N^\alpha_D N^\beta_E \end{pmatrix}. \tag{4.55}$$

Given the goldstino $\mathcal{G}^I$ defined in (4.50), we want to check that

$$\mathcal{M}_{IJ}\mathcal{G}^J \equiv 0, \tag{4.56}$$

which in components explicitly reads:

$$\begin{aligned}
\mathcal{M}_{1J}\mathcal{G}^J &= M_{iMlN} W^{lNQ} + 2iX_B{}^D S^{BE} g_{ij^*} W^{j^*}_{MD} g_{lk^*} W^{k^*}_{NE} W^{lNQ} \\
&\quad + \frac{1}{2} M^\beta_{iN} N^Q_\beta + 4iX_B{}^D S^{BE} g_{ij^*} W^{j^*}_{MD} N^\beta_E N^Q_\beta \equiv 0,
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
\mathcal{M}_{2J}\mathcal{G}^J &= \frac{1}{2} M^\alpha_{iM} W^{lMQ} + 4iX_B{}^D S^{BE} g_{lk^*} W^{k^*}_{ME} N^\alpha_D W^{lMQ} \\
&\quad + M^{\alpha\beta} N^Q_\beta + 8iX_B{}^D S^{BE} N^\alpha_D N^\beta_E N^Q_\beta \equiv 0.
\end{aligned} \tag{4.58}$$

Using (4.37) we can simplify this expression in such a way that all dependence from $X_A{}^B$ drops and renaming some of the indices we are left with

$$\mathcal{M}_{IJ}\mathcal{G}^J = \begin{pmatrix} M_{jCmD} W^{mDE} + \frac{1}{2} M^\beta_{jC} N^E_\beta - 2g_{ij^*} W^{j^*}_{CA} S^{EA} \\ \frac{1}{2} M^\alpha_{mD} W^{mDE} + M^{\alpha\beta} N^E_\beta - 4S^{AE} N^\alpha_A \end{pmatrix}. \tag{4.59}$$

The vector above has to vanish identically. In what follows, we check that it does on the vacuum, and in particular as a consequence of

$$\begin{pmatrix} \partial_k V(z,\bar z,q) \\ \partial_u V(z,\bar z,q) \end{pmatrix} = 0. \tag{4.60}$$

**Top component of 4.59 )** Let's first decompose the top component of the vector of interest in different contributions, by using the definitions of mass and shift matrices:

$$M^{(1)}_{jCmD}W^{m[DE]} = -\frac{1}{2}\delta^E_C k^m_\Sigma \overline{L}^\Sigma k_{j\Lambda} f^\Lambda_m + \frac{1}{2}\delta^E_C k^m_\Sigma \overline{L}^\Sigma k_{m\Lambda} f^\Lambda_j, \tag{4.61}$$

$$\begin{aligned} M^{(2)}_{jCmD}W^{m(DE)} &= \frac{1}{2}(\sigma_x\sigma_y)_C{}^E g^{ml^*} f^\Sigma_{l^*} P^y_\Sigma \nabla_j f^\Lambda_m P^x_\Lambda \\ &= \frac{1}{2}\delta^E_C g^{ml^*} f^\Sigma_{l^*} P^x_\Sigma \nabla_j f^\Lambda_m P^x_\Lambda + \frac{i}{2}(\sigma_z)_C{}^E \epsilon^{xyz} g^{ml^*} f^\Sigma_{l^*} P^y_\Sigma \nabla_j f^\Lambda_m P^x_\Lambda, \end{aligned} \tag{4.62}$$

$$M^{(1)}_{jCmD}W^{m(DE)} = i(\sigma_x)_C{}^E k_{[j|\Sigma} f^\Sigma_{m]} g^{ml^*} f^\Lambda_{l^*} P^x_\Lambda, \tag{4.63}$$

$$M^{(2)}_{jCmD}W^{m[DE]} = \frac{i}{2}(\sigma_x)_C{}^E \nabla_j f^\Lambda_m P^x_\Lambda k^m_\Sigma \overline{L}^\Sigma, \tag{4.64}$$

$$\begin{aligned} 2S^{AE} g_{i^*j} W^{i^*}_{(CA)} &= g_{i^*j}(\sigma_y\sigma_x)_C{}^E P^x_\Sigma \overline{L}^\Sigma g^{i^*l} f^\Lambda_l P^y_\Lambda \\ &= g_{i^*j}\delta^E_C P^x_\Sigma \overline{L}^\Sigma g^{i^*l} f^\Lambda_l P^x_\Lambda + i(\sigma_z)_C{}^E g_{i^*j}\epsilon^{yxz} P^x_\Sigma \overline{L}^\Sigma g^{i^*l} f^\Lambda_l P^y_\Lambda, \end{aligned} \tag{4.65}$$

$$2S^{AE} g_{i^*j} W^{i^*}_{[CA]} = i g_{i^*j}(\sigma_x)_C{}^E P^x_\Sigma \overline{L}^\Sigma k^{i^*}_\Lambda L^\Lambda, \tag{4.66}$$

$$\begin{aligned} \frac{1}{2}M^\beta_{jC} N^E_\beta &= -4U^E_{v\beta} U^\beta_{uC} k^v_\Lambda \overline{L}^\Lambda k^u_\Sigma f^\Sigma_j \\ &= \delta^E_C 2h_{uv} k^v_\Lambda \overline{L}^\Lambda k^u_\Sigma f^\Sigma_j - 2i(\sigma_x)_C{}^E \Omega^x_{vu} k^v_\Lambda \overline{L}^\Lambda k^u_\Sigma f^\Sigma_j, \end{aligned} \tag{4.67}$$

where we used $\sigma_x\sigma_y = \delta_{xy}\mathbb{I} + i\epsilon^{xyz}\sigma_z$ and (3.77). We see that we have terms proportional to $\delta$ and terms proportional to $\sigma$. Let's focus on the former:

$$\begin{aligned} \frac{\delta^E_C}{2}\Big(&-k^m_\Sigma \overline{L}^\Sigma k_{j\Lambda} f^\Lambda_m + g_{ml^*} k^m_\Sigma \overline{L}^\Sigma k^{l^*}_\Lambda f^\Lambda_j + g^{ml^*} f^\Sigma_{l^*} P^x_\Sigma \nabla_j f^\Lambda_m P^x_\Lambda \\ &-2P^x_\Sigma \overline{L}^\Sigma f^\Lambda_j P^x_\Lambda + 4h_{uv} k^v_\Lambda \overline{L}^\Lambda k^u_\Sigma f^\Sigma_j\Big). \end{aligned} \tag{4.68}$$

If now we compute the fully covariant derivative of the potential with respect to holomorphic coordinates on $\mathcal{SM}$, we get:

$$\begin{aligned} \nabla_k V = {}& g_{ij^*} k^i_\Lambda k^{j^*}_\Sigma \overline{L}^\Lambda f^\Sigma_k + 4h_{uv} k^u_\Lambda k^{j^*}_\Sigma \overline{L}^\Lambda f^\Sigma_k + k_{k\Lambda} f^\Lambda_{j^*} k^{j^*}_\Sigma L^\Sigma \\ &+ g^{ij^*} \nabla_k f^\Sigma_j P^x_\Lambda P^x_\Sigma - 2f^\Lambda_k \overline{L}^\Sigma P^x_\Lambda P^x_\Sigma = \partial_k V, \end{aligned} \tag{4.69}$$

where we used (3.68) so that through the killing equation we can write

$$\begin{aligned} \nabla_k k^i_\Lambda \overline{L}^\Lambda &= g^{il^*} \nabla_k k_{l^*\Lambda} \overline{L}^\Lambda = -g^{il^*} \nabla_{l^*} k_{k\Lambda} \overline{L}^\Lambda \\ &= -g^{il^*} \nabla_{l^*}(k_{k\Lambda} \overline{L}^\Lambda) + g^{il^*} k_{k\Lambda} f^\Lambda_{l^*} = g^{il^*} k_{k\Lambda} f^\Lambda_{l^*}. \end{aligned} \tag{4.70}$$

Then, if we use the Special property (3.69), we see terms proportional to $\delta$ are also proportional to the covariant derivative of the potential, and hence vanish on the vacuum. Switching to the terms proportional to $\sigma$, which are:

$$\begin{aligned} i(\sigma_z)_C{}^E\Big(&\frac{1}{2}\epsilon^{xyz} g^{ml^*} f^\Sigma_{l^*} P^y_\Sigma \nabla_j f^\Lambda_m P^x_\Lambda + k_{[j|\Sigma} f^\Sigma_{m]} g^{ml^*} f^\Lambda_{l^*} P^z_\Lambda + \frac{1}{2}\nabla_j f^\Lambda_m P^z_\Lambda k^m_\Sigma \overline{L}^\Sigma \\ &-\epsilon^{yxz} P^x_\Sigma \overline{L}^\Sigma f^\Lambda_j P^y_\Lambda - g_{i^*j} P^z_\Sigma \overline{L}^\Sigma k^{i^*}_\Lambda L^\Lambda - 2\Omega^z_{vu} k^v_\Lambda \overline{L}^\Lambda k^u_\Sigma f^\Sigma_j\Big), \end{aligned} \tag{4.71}$$

we can use the equivariance condition (3.75) and some of the geometric properties of these quantities to rewrite them as

1. Using the definition of $U^{\Lambda\Sigma}$ found in (3.33) and (3.24):

$$\begin{aligned}&\frac{1}{2}\epsilon^{xyz}g^{ml^*}f^{\Sigma}_{l^*}P^y_{\Sigma}\nabla_j f^{\Lambda}_m P^x_{\Lambda}=\frac{1}{2}\epsilon^{xyz}P^x_{\Lambda}P^y_{\Sigma}\left(\nabla_j U^{[\Lambda\Sigma]}-g^{ml^*}\nabla_j f^{[\Sigma}_{l^*}f^{\Lambda]}_m\right)\\&=\frac{1}{2}\epsilon^{xyz}P^x_{\Lambda}P^y_{\Sigma}\left(-\overline{L}^{[\Lambda}f^{\Sigma]}_j-\overline{L}^{[\Sigma}f^{\Lambda]}_j\right)=0.\end{aligned} \tag{4.72}$$

2. Using the equivariance condition (3.75):

$$-\epsilon^{yxz}P^x_{\Sigma}\overline{L}^{\Sigma}f^{\Lambda}_j P^y_{\Lambda}-2\Omega^z_{vu}k^v_{\Lambda}\overline{L}^{\Lambda}k^u_{\Sigma}f^{\Sigma}_j=f^{\Delta}_{\Lambda\Sigma}P^z_{\Delta}\overline{L}^{\Lambda}f^{\Sigma}_j. \tag{4.73}$$

3. Finally, using (3.67) and (3.68):

$$\begin{aligned}\frac{1}{2}\nabla_j f^{\Lambda}_m P^z_{\Lambda}k^m_{\Sigma}\overline{L}^{\Sigma}&=\frac{1}{2}\nabla_j(f^{\Lambda}_m k^m_{\Sigma}\overline{L}^{\Sigma})P^z_{\Lambda}-\frac{1}{2}f^{\Lambda}_m P^z_{\Lambda}\nabla_j k^m_{\Sigma}\overline{L}^{\Sigma}\\&=-\frac{1}{2}f^{\Lambda}_{\Sigma\Delta}f^{\Delta}_j\overline{L}^{\Sigma}P^z_{\Lambda}-\frac{1}{2}f^{\Lambda}_m P^z_{\Lambda}g^{ml^*}f^{\Sigma}_{l^*}k_{j\Sigma}.\end{aligned} \tag{4.74}$$

We are then left with:

$$i(\sigma_z)_C{}^E\left(-\frac{1}{2}k_{m\Sigma}f^{\Sigma}_j g^{ml^*}f^{\Lambda}_{l^*}P^z_{\Lambda}+\frac{1}{2}f^{\Delta}_{\Lambda\Sigma}P^z_{\Delta}\overline{L}^{\Lambda}f^{\Sigma}_j-P^z_{\Lambda}g^{ml^*}f^{[\Sigma}_{l^*}f^{\Lambda]}_m k_{j\Sigma}-g_{i^*j}P^z_{\Sigma}\overline{L}^{\Sigma}k^{i^*}_{\Lambda}L^{\Lambda}\right). \tag{4.75}$$

If we use (3.67) and, since (3.68) holds:

$$P_{\Sigma}f^{\Sigma}_j=\nabla_j(P_{\Sigma}L^{\Sigma})-L^{\Sigma}\nabla_j P_{\Sigma}=-iL^{\Sigma}k_{j\Sigma}, \tag{4.76}$$

we are finally left with:

$$\begin{aligned}&i(\sigma_z)_C{}^E\left(-P^z_{\Lambda}g^{ml^*}f^{[\Sigma}_{l^*}f^{\Lambda]}_m k_{j\Sigma}-\frac{1}{2}k_{j\Lambda}P^z_{\Sigma}\overline{L}^{\Sigma}L^{\Lambda}\right)\\&=i(\sigma_z)_C{}^E P^z_{\Lambda}k_{j\Sigma}\left(-U^{[\Lambda\Sigma]}-\frac{1}{2}\overline{L}^{\Lambda}L^{\Sigma}\right)\\&=-\frac{i}{2}(\sigma_z)_C{}^E P^z_{\Lambda}k_{j\Sigma}\overline{L}^{\Sigma}L^{\Lambda}=0,\end{aligned} \tag{4.77}$$

due to (3.68). Thus, the first component completely vanishes on the vacuum.

**Bottom component of 4.59)** Let's switch now to the second component and compute the derivative of the potential with respect to the coordinates on $\mathcal{HM}$:

$$\begin{aligned}\nabla_s V&=4\nabla_s k_{\Lambda u}k^u_{\Sigma}\overline{L}^{\Lambda}L^{\Sigma}+4k_{\Lambda u}\nabla_s k^u_{\Sigma}\overline{L}^{\Lambda}L^{\Sigma}+\left(U^{\Lambda\Sigma}-3\overline{L}^{\Lambda}L^{\Sigma}\right)(\nabla_s P^x_{\Lambda}P^x_{\Sigma}+P^x_{\Lambda}\nabla_s P^x_{\Sigma})\\&=4\nabla_s k_{\Lambda u}k^u_{\Sigma}\overline{L}^{\Lambda}L^{\Sigma}+\left(U^{\Lambda\Sigma}-3\overline{L}^{\Lambda}L^{\Sigma}\right)\nabla_s P^x_{\Lambda}P^x_{\Sigma}+h.c.=\partial_s V\end{aligned} \tag{4.78}$$

If we now use (3.74) and (3.78) we get:

$$\begin{aligned}\nabla_s V &= 2k^u_\Sigma \Omega^x_{su} P^x_\Lambda \overline{L}^\Lambda L^\Sigma - 2U_{\alpha A[s} U_{u]\beta B} \epsilon^{AB} M^{\alpha\beta} k^u_\Sigma L^\Sigma + \left(U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma\right) \nabla_s P^x_\Lambda P^x_\Sigma + h.c. \\ &= -M^{\alpha\beta} N^A_\beta U_{\alpha A s} + \left(U^{\Lambda\Sigma} - 2\overline{L}^\Lambda L^\Sigma\right) \nabla_s P^x_\Lambda P^x_\Sigma + h.c.\end{aligned} \tag{4.79}$$

Let's then compute the following quantities:

$$\begin{aligned}4U_{s\alpha E} S^{AE} N^\alpha_A &= -\frac{i}{2} (\sigma_x)^{AE} U_{s\alpha E} U^{\alpha A u} k^u_\Sigma L^\Sigma \\ &= -4\Omega^x_{su} k^u_\Sigma P^x_\Lambda \overline{L}^\Lambda L^\Sigma = -2\nabla_s P^x_\Sigma P^x_\Lambda \overline{L}^\Lambda L^\Sigma,\end{aligned} \tag{4.80}$$

$$\begin{aligned}\frac{1}{2} U_{s\alpha E} M^\alpha_{mD} W^{mDE} &= -2U_{s\alpha E} U^\alpha_{uD} k_\Lambda f^\Lambda_i \left(\epsilon^{DE} k^i_\Sigma \overline{L}^\Sigma + i(\sigma_x)^{DE} P^x_\Sigma g^{ij^*} f^\Sigma_{j^*}\right) \\ &= -2h_{us} k^u_\Lambda f^\Lambda_i k^i_\Sigma \overline{L}^\Sigma - 2\Omega^x_{su} k^u_\Lambda P^x_\Sigma U^{\Lambda\Sigma} \\ &= -2h_{us} k^u_\Lambda f^\Lambda_i k^i_\Sigma \overline{L}^\Sigma - U^{\Lambda\Sigma} \nabla_s P^x_\Lambda P^x_\Sigma,\end{aligned} \tag{4.81}$$

where we used (3.74), (3.77), (3.78) and the fact that $\sigma$ is traceless. Finally, if we also use the fact that $f^\Lambda_i k^i_\Sigma \overline{L}^\Sigma$ is imaginary due to (3.69), we see that

$$\nabla_s V = -M^{\alpha\beta} N^E_\beta U_{\alpha E s} + 4U_{s\alpha E} S^{AE} N^\alpha_A - \frac{1}{2} U_{s\alpha E} M^\alpha_{mD} W^{mDE} + h.c. \tag{4.82}$$

Hence, multiplying the expression by the inverse veilbein, we see that the bottom component of the initial vector is proportional to $U^{sE\alpha}\nabla_s V$ and similarly to the top one vanishes on the vacuum. This concludes the proof of the super-Higgs mechanism. Before moving on, it is worth noticing that the above calculation easily extends to the case of $\mathcal{N}=1$ vacua displaying partial supersymmetry breaking, as some of the components of the fermionic shifts simply vanish.

## 3 YUKAWA COUPLINGS

From now on, *we will assume to be working on a Minkowski vacuum*, and still consider the case of full supersymmetry breaking. This means that the fermionic mass matrix (4.48) in the unitary gauge and for $V=0$ reads:

$$\mathcal{M}_{I\bar{J}} = \begin{pmatrix} M_{jCmD} - \frac{1}{6}(S^{-1})^{AB} g_{i^*j} W^{i^*}_{CA} g_{l^*m} W^{l^*}_{DB} & \frac{1}{2} M^\beta_{jC} - \frac{1}{3}(S^{-1})^{AB} g_{i^*j} W^{i^*}_{CA} N^\beta_B \\ \frac{1}{2} M^\alpha_{mD} - \frac{1}{3}(S^{-1})^{AB} g_{l^*m} W^{l^*}_{DB} N^\alpha_A & M^{\alpha\beta} - \frac{2}{3}(S^{-1})^{AB} N^\alpha_A N^\beta_B \end{pmatrix}. \tag{4.83}$$

Renormalizable Yukawa couplings are defined by expanding this matrix around the vacuum and taking the linear term, that is taking its partial derivative and evaluating it on the vacuum. Before doing so, we must make sure that the manifold $\mathcal{M}_{scal}$ is parametrized with coordinates that yield canonical kinetic terms for the scalars and fermions on the vacuum $(z^i_0, q^u_0)$. That is, we must choose:

$$\hat{h}_{uv}(q_0) = \delta_{uv}, \quad \hat{g}_{ij^*}(z_0, \bar{z}_0) = \delta_{ij^*}. \tag{4.84}$$

On the other hand, from a geometrical standpoint, we would like $Y$ appearing in (2.15) to be a scalar on $\mathcal{M}_{scalar}$. To achieve this, the most sensible thing to do is to define the following quantity:

$$|Y|^2_{sf_1f_2} \equiv |\nabla_s \mathcal{M}_{f_1f_2}|^2|_{(q,z)=(q_0,z_0)} \tag{4.85}$$

where $\mathcal{M}_{f_1f_2}$ is the mass matrix term related to fermions $f_1$ and $f_2$ and $\nabla_s$ is a suitable covariant derivative with respect to the coordinate $s$ on $\mathcal{M}_{scal}$. It follows that the Yukawas we will explore are the ones defined in Riemann normal coordinates on the vacuum. Indeed, in these coordinates the previous scalar quantity reads:

$$|\hat{Y}|^2_{sf_1f_2} = |\partial_s \mathcal{M}_{f_1f_2}|^2|_{(q,z)=(q_0,z_0)}, \tag{4.86}$$

which is respecting both (4.84) and the general definition of Yukawa coupling as partial derivatives of fermionic mass matrices.

It is worth noticing that all bundle connections on $\mathcal{QM}$ vanish in Riemann normal coordinates, as it happens with the spin connection on spacetime, since also in this case the vielbeins respect the compatibility condition (3.45). In special geometry, however, the covariant derivative also includes Kähler weights:

$$\nabla_i = \left(\partial_i + \frac{p}{2}\partial_i \mathcal{K}\right), \tag{4.87}$$

so the Yukawa couplings in the vector multiplet sector are not only the ones computed in Riemann normal coordinates, but in a patch where the Kähler potential is redefined through a Kähler transformation as:

$$\hat{\mathcal{K}} = \mathcal{K} + h(z) + \bar{h}(\bar{z}), \quad \text{where} \quad \partial_i K|_{z=z_0} = -\partial_i h(z)|_{z=z_0}, \tag{4.88}$$

so that $\hat{\nabla}_i = \hat{\partial}_i$ on the vacuum.

Having made clear which are the Yukawas captured by the quantity (4.85), we are now ready to compute them. The first step is to take covariant derivatives of the fermionic mass matrix and using the gradient flow relations (3.105-3.113) to simplify the expressions. We first give an example of an explicit computation, taking $\nabla_s \mathcal{M}^{\alpha\beta}$, while we then simply report the results for all the other Yukawas. Start by considering the relevant component of $M_{I\bar{J}}$:

$$M^{\alpha\beta} - \frac{2}{3}(S^{-1})^{AB} N^\alpha_A N^\beta_B. \tag{4.89}$$

Taking the covariant derivative with respect to $q^s$ we get:

$$Y^{\alpha\beta}_s \equiv \nabla_s M^{\alpha\beta} + \frac{2}{3}(S^{-1})^{AX}(S^{-1})^{BY} \nabla_s S_{XY} N^\alpha_A N^\beta_B - \frac{4}{3}(S^{-1})^{AB} \nabla_s N^\alpha_A N^\beta_B, \tag{4.90}$$

where, since this term in the Lagrangian is contracted with $\bar{\zeta}_\alpha \zeta_\beta$, we also used its symmetry properties to group some of the terms. Using now the gradient flow equations we have:

$$\nabla_s M^{\alpha\beta} - \frac{1}{3}(S^{-1})^{AX}(S^{-1})^{BY} U_{s\gamma(X} N^\gamma_{Y)} N^\alpha_A N^\beta_B + \frac{16}{3} U^{\alpha B}_s N^\beta_B + \frac{4}{3}(S^{-1})^{AB} U_{s\gamma A} M^{\alpha\gamma} N^\beta_B. \tag{4.91}$$

Then, let's explicitly compute the first term:

$$\nabla_s M^{\alpha\beta} = -U^{\alpha A}_u U^{\beta B}_v \epsilon_{AB} \nabla_s \nabla^u k^v_\Lambda L^\Lambda = -\epsilon_{AB} U^{\alpha A}_u U^{\beta B}_v \mathcal{R}^{vu}{}_{st} k^t_\Lambda L^\Lambda, \tag{4.92}$$

where we used the identity relating second derivatives of killing vectors to the curvature of $\mathcal{QM}$ [61]. We can now decompose the curvature in terms of the $SU(2) \times Sp(2n_H)$ ones using (3.47) and getting to:

$$\nabla_s M^{\alpha\beta} = \epsilon_{AB} \left( \frac{i}{2} \Omega^x_{st} (\sigma_x)^{BA} C^{\beta\alpha} - \mathbb{R}^{\beta\alpha}_{st} \epsilon^{BA} \right) k^t_\Lambda L^\Lambda = 2\mathbb{R}^{\beta\alpha}_{st} k^t_\Lambda L^\Lambda. \tag{4.93}$$

Finally:

$$\begin{aligned} Y^{\alpha\beta}_s = {} & 2\mathbb{R}^{\beta\alpha}_{st} k^t_\Lambda L^\Lambda - \frac{1}{3}(S^{-1})^{AX}(S^{-1})^{BY} U_{s\gamma(X} N^\gamma_{Y)} N^\alpha_A N^\beta_B \\ & + \frac{16}{3} U^{\alpha B}_s N^\beta_B + \frac{4}{3}(S^{-1})^{AB} U_{s\gamma A} M^{\alpha\gamma} N^\beta_B. \end{aligned} \tag{4.94}$$

## 3.1 *List of Yukawas*

In a similar fashion, we can calculate all other Yukawa couplings, which we report together with the one we already computed in the following equations:

$$\begin{aligned} Y^{\alpha\beta}_s \equiv \nabla_s \mathcal{M}^{\alpha\beta} = {} & 2\mathbb{R}^{\beta\alpha}_{st} k^t_\Lambda L^\Lambda - \frac{1}{3}(S^{-1})^{AX}(S^{-1})^{BY} U_{s\gamma(X} N^\gamma_{Y)} N^\alpha_A N^\beta_B \\ & + \frac{16}{3} U^{\alpha B}_s N^\beta_B + \frac{4}{3}(S^{-1})^{AB} U_{s\gamma A} M^{\alpha\gamma} N^\beta_B, \end{aligned} \tag{4.95}$$

$$\begin{aligned} Y^\alpha_{s|mD} \equiv \nabla_s \mathcal{M}^\alpha_{mD} = {} & \frac{1}{2} \nabla_s M^\alpha_{mD} - \frac{1}{6}(S^{-1})^{AX}(S^{-1})^{BY} U_{s\gamma(X} N^\gamma_{Y)} g_{ml^*} W^{l^*}_{DB} N^\alpha_A \\ & + \frac{1}{6}(S^{-1})^{AB} M^\gamma_{m(D} U_{B)\gamma s} N^\alpha_A + \frac{4}{3} g_{ml^*} W^{l^*}_{DF} U^{\alpha F}_s + \frac{1}{3}(S^{-1})^{AB} g_{ml^*} W^{l^*}_{DB} U_{sA\gamma} M^{\gamma\alpha}, \end{aligned} \tag{4.96}$$

$$\begin{aligned} Y_{k|jCmD} \equiv \nabla_k \mathcal{M}_{jCmD} = {} & \nabla_k M_{jCmD} + \frac{1}{12}(S^{-1})^{AX}(S^{-1})^{BY} g_{kj^*} W^{j^*}_{(XY)} g_{jl^*} W^{l^*}_{CA} g_{mi^*} W^{i^*}_{DB} \\ & - \frac{2}{3}(S^{-1})^{AB} M_{kCjA} g_{mn^*} W^{n^*}_{DB} - \frac{1}{3}(S^{-1})^{AB} \left( 2\epsilon_{CA} f^\Lambda_k k_{j\Lambda} - \epsilon_{CA} f^\Lambda_j k_{k\Lambda} \right) g_{mn^*} W^{n^*}_{DB}, \end{aligned} \tag{4.97}$$

$$\begin{aligned} Y^\beta_{k|jC} \equiv \nabla_k \mathcal{M}^\beta_{jC} = {} & \frac{1}{2} \nabla_k M^\beta_{jC} - \frac{1}{6}(S^{-1})^{AX}(S^{-1})^{BY} g_{kj^*} W^{j^*}_{(XY)} g_{jl^*} W^{l^*}_{CA} N^\beta_B \\ & - \frac{2}{3}(S^{-1})^{AB} M_{kCjA} N^\beta_B - \frac{1}{3}(S^{-1})^{AB} \left( 2\epsilon_{CA} f^\Lambda_k k_{j\Lambda} - \epsilon_{CA} f^\Lambda_j k_{k\Lambda} \right) N^\beta_B, \end{aligned} \tag{4.98}$$

$$Y_{k^*|jCmD} \equiv \nabla_{k^*} \mathcal{M}_{jCmD} = \nabla_{k^*} M_{jCmD} - \frac{1}{3} g_{jk^*} g_{mi^*} W^{i^*}_{DC} - \frac{1}{3} \epsilon_{CA} (S^{-1})^{AB} k_{k^*\Lambda} f^\Lambda_j g_{mi^*} W^{i^*}_{DB}, \tag{4.99}$$

$$Y^{\alpha}_{k^*|mD} \equiv \nabla_{k^*}\mathcal{M}^{\alpha}_{mD} = \frac{2}{3}N^{\alpha}_{D}g_{mk^*} - \frac{1}{3}(S^{-1})^{AB}\epsilon_{DB}k_{k^*\Lambda}f^{\Lambda}_{m}N^{\alpha}_{A}. \tag{4.100}$$

where we either used or left implicit the following expressions:

$$\nabla_s M^{\alpha\beta} = 2\mathbb{R}^{\beta\alpha}_{st}k^t_{\Lambda}L^{\Lambda}, \tag{4.101}$$

$$\nabla_s M^{\alpha}_{mD} = -2U^{u\alpha}_{D}\Omega^{x}_{su}P^{x}_{\Lambda}f^{\Lambda}_{m} - U^{u\alpha}_{D}\left(\Omega^{x}_{ts}\Omega^{x}_{vu} + h_{uv}h_{ts}\right)\nabla^t k^{v}_{\Lambda}f^{\Lambda}_{m}, \tag{4.102}$$

$$\nabla_k M_{jCmD} = i\epsilon_{CD}k_{[j|\Lambda}C_{m]kl}g^{l^*l}f^{\Lambda}_{l^*} - \frac{1}{2}(\sigma_x)_{CD}P^{x}_{\Lambda}\left(\nabla_k C_{jml}g^{l^*l}f^{\Lambda}_{l^*} + C_{jmk}\bar{L}^{\Lambda}\right), \tag{4.103}$$

$$\nabla_k M^{\beta}_{jC} = -4iC_{kjl}g^{ll^*}f^{\Lambda}_{l^*}U^{\beta}_{uC}k^{u}_{\Lambda}, \tag{4.104}$$

$$\nabla_{k^*}M^{\alpha}_{mD} = 2N^{\alpha}_{D}g_{k^*m}, \tag{4.105}$$

$$\nabla_{k^*}M_{jCmD} = \epsilon_{CD}k_{[j|\Lambda}g_{m]l^*}L^{\Lambda} + \epsilon_{CD}\nabla_{k^*}k_{[j|\Lambda}f^{\Lambda}_{m]} - \frac{i}{2}(\sigma_x)_{CD}C_{k^*z^*l^*}C_{jml}g^{ll^*}g^{zz^*}f^{\Lambda}_{z}P^{x}_{\Lambda}, \tag{4.106}$$

and where we used most of the geometrical constraints in Chapter 3. In the previous equations, we recall that $C_{ijk}$ is the covariantly holomorphic section defined on $\mathcal{SM}$, $\Omega$ the $su(2)$-valued curvature on $\mathcal{QM}$ and $\mathbb{R}$ the $sp(2n_H)$-valued curvature on $\mathcal{QM}$.

## 3.2 *Computation of squares*

The next step is to calculate the squares of these quantities, and again use all available constraints in order to try and construct (2.15), where we also recall that we defined:

$$m^4_{susy} = 12S_{AB}S^{AB} \equiv 12\,\mathrm{Tr}\,S^2 \overset{(4.13)}{=} g_{ij^*}W^{iAB}W^{j^*}_{AB} + 2N^{\alpha}_{A}N^{A}_{\alpha} = 3P^{x}_{\Lambda}P^{x}_{\Sigma}L^{\Lambda}\bar{L}^{\Sigma}. \tag{4.107}$$

From the above expressions it is clear that we will not get inequality (2.15) directly, because, for example, we expect that the squares will involve mixed products of shift and mass matrices with no clear sign. Even in this case, we can still try to make sense of these quantities in specific parametric limits, as will be shown below. Before moving on, notice that we have not calculated $\nabla_s\mathcal{M}_{jCmD}$ and $\nabla_k\mathcal{M}^{\alpha\beta}$. This is because if we hope to capture conjecture (2.15), then we need to look for fermions which couple to *supersymmetric* partners through $Y$, which is not the case in the Yukawas we have left out.

**4.100.** Let's start by computing the square of the simplest equation, which involves the Yukawa coupling between a hyperino and supersymmetric vector multiplets $Y^{\alpha}_{k^*|mD}$. Computing the square, schematically, we have

$$|Y^{\alpha}_{k^*|mD}|^2 = (1\mathrm{st})^2 + (2\mathrm{nd})^2 + 1\mathrm{st}\,\times\,2\mathrm{nd} \geq (1\mathrm{st})^2 + 1\mathrm{st}\,\times\,2\mathrm{nd} \tag{4.108}$$

where we can neglect the second term squared since it involves terms of the kind $U^{\Lambda\Sigma}k_\Lambda K_\Sigma$ which do not interest us. Then, realizing that:

$$g^{mk^*}k_{k^*\Lambda}f_m^\Lambda \overset{(3.67)}{=} 0 \tag{4.109}$$

we are left with:

$$|Y^\alpha_{k^*|mD}|^2 \geq \frac{4}{9}N^\alpha_D N^D_\alpha n_V, \tag{4.110}$$

where we used $g_{mk^*}g^{mk^*} = n_V$. We see that this indeed leads to the conjecture if we take the parametric limit:

$$g_{ij^*}W^{iAB}W^{j^*}_{AB} \ll 2N^\alpha_D N^D_\alpha, \tag{4.111}$$

which will be informally written as $W^2 \ll N^2$ in the following, leading to:

$$|Y^\alpha_{k^*|mD}|^2 \gtrsim \frac{4}{9}m^4_{susy}. \tag{4.112}$$

As to why we might recover the conjecture in this case, notice that the limit we employed corresponds to giving the shift of the FWGC fermion (this time a hyperino) the dominant contribution in breaking supersymmetry. We will use this line of thought to parse the other squares in the following.

**4.95.** We now turn to the fully hypermultiplet sector of the Yukawas. This time, we can straight out neglect positive terms involving the squares of the mass matrix $M^{\alpha\beta}$ and curvature squares. We have:

$$|Y^{\alpha\beta}_s|^2 \geq (\text{2nd})^2 + (\text{3rd})^2 + \text{mixed terms}, \tag{4.113}$$

where among the other terms, if we work in the same parametric limit as before $W^2 \ll N^2$ we note the following non-trivial results:

$$(\text{2nd})^2 \approx 36\,\text{Tr}\,S^2, \tag{4.114}$$

$$(\text{3rd})^2 \approx \frac{1024}{3}n_H\,\text{Tr}\,S^2, \tag{4.115}$$

$$(\text{2nd} \times \text{3rd}) \approx \frac{32}{3}\,\text{Tr}\,S^2, \tag{4.116}$$

$$(\text{1st} \times \text{3rd}) \approx 32(1+2n_H)\,\text{Tr}\,S^2, \tag{4.117}$$

$$(\text{3rd} \times \text{4th}) = 0, \tag{4.118}$$

$$\begin{aligned}(\text{1st} \times \text{2nd}) \approx &+\frac{4}{3}C^{\gamma'\gamma}C^{\alpha\rho}C^{\beta\sigma}\epsilon^{CX}N^\delta_C N^Y_{\gamma'}N^A_\alpha N^B_\beta(S^{-1})_{AX}(S^{-1})_{BY}\Omega_{\delta\gamma\rho\sigma}\\ &-4\epsilon^{AX}\epsilon^{BY}S_{YA}S_{XB}\end{aligned} \tag{4.119}$$

where the $\approx$ has been used each time we neglected some parametrically small contribution. Carrying out all calculations, we have:

$$|Y_s^{\alpha\beta}|^2 \gtrsim \frac{1}{9}\,(304n_H + 32)\,m_{susy}^4 - \Big[4\epsilon^{AX}\epsilon^{BY}S_{YA}S_{XB} + M\text{-mixed terms} \\ -\frac{4}{3}C^{\gamma'\gamma}C^{\alpha\rho}C^{\beta\sigma}\epsilon^{CX}N_C^{\delta}N_{\gamma'}^{Y}N_{\alpha}^{A}N_{\beta}^{B}(S^{-1})_{AX}(S^{-1})_{BY}\Omega_{\delta\gamma\rho\sigma} + h.c.\Big]\,, \tag{4.120}$$

where $M$-mixed terms contain all terms involving $M^{\alpha\beta}$ and which do not further reduce upon using all geometric identities. We see that this case is less clear: even though the scale of supersymmetry breaking appears with a positive coefficient, there are still contributions with no clear sign. The second term can be simplified by evaluating it in a basis where the gravitino mass matrix is made diagonal trough an $SU(2)$ rotation, which can always be done [45], and reduces to:

$$-\,\mathrm{Re}[4\epsilon^{AX}\epsilon^{BY}S_{YA}S_{XB}] = 8\,\mathrm{Re}[s_1 s_2], \tag{4.121}$$

where $s_1$ and $s_2$ are the two eigenvalues of the $S$ matrix. We see that this term vanishes only in the case of partial supersymmetry breaking. The same is true for the third term, since $S$ is invertible only in the broken (1-d) subspace $S_{22}$, assuming a diagonal matrix, $\epsilon$ is antisymmetric and (4.14) holds. Thus, the conjecture is realized in this sector only in the very restricting case of partial susy breaking, $W \ll N$ and when the masses of the hyperinos are parametrically smaller than the scale of supersymmetry breaking, so that we can neglect mixed terms altogether. We will informally write this condition as $M \ll m_{susy}$.

**4.99.** The next Yukawa we turn to is the one of the form $\bar{z}\bar{\lambda}\lambda$. Following the same logic as before, we try to compute all quantities in the parametric limit $W^2 \ll N^2$, which is the suitable one in this sector. We can immediately neglect the first term squared and the third squared, and by explicit calculation:

$$(2\mathrm{nd})^2 \approx \frac{12}{9}n_V\,\mathrm{Tr}\,S^2, \tag{4.122}$$

$$(1\mathrm{st}\ \times\ 2\mathrm{nd}) = \frac{1}{3}(n_V - 1)k_{m|\Lambda}k_{\Sigma}^{m}L^{\Lambda}\bar{L}^{\Sigma}, \tag{4.123}$$

$$(2\mathrm{nd}\ \times\ 3\mathrm{rd}) = 0, \tag{4.124}$$

$$(1\mathrm{st}\ \times\ 3\mathrm{rd}) = -\frac{1}{6}(S^{-1})_{DB}W^{mDB}k_{m|\Sigma}k_{\Lambda}^{j^*}f_j^{\Sigma}L^{\Lambda} \\ -\frac{1}{6}(S^{-1})_{DB}W^{mDB}\left(\nabla_{k^*}k_{\Lambda}^{j^*}f_m^{\Lambda} - \nabla_{k^*}k_{m|\Lambda}f_j^{\Lambda}g^{jj^*}\right)f_{j^*}^{\Sigma}k_{\Sigma}^{k^*} \\ +\frac{i}{6}(\sigma_x)^{A}{}_{D}(S^{-1})_{AB}W^{mDB}C_{jml}C_{k^*z^*l^*}g^{ll^*}g^{zz^*}g^{jj^*}f_z^{\Lambda}f_{j^*}^{\Sigma}k_{\Sigma}^{k^*}P_{\Lambda}^{x}, \tag{4.125}$$

which, upon neglecting the non-negative real contribution (4.123) leads us to

$$\begin{aligned} |Y_{k^*|jCmD}|^2 \gtrsim \frac{n_V}{9} m_{susy}^4 - 2\,\mathrm{Re}\Big[ & \frac{1}{6}(S^{-1})_{DB} W^{mDB} k_{m|\Sigma} k_\Lambda^{j^*} f_j^\Sigma L^\Lambda \\ & - \frac{1}{6}(S^{-1})_{DB} W^{mDB} \left( \nabla_{k^*} k_\Lambda^{j^*} f_m^\Lambda - \nabla_{k^*} k_{m|\Lambda} f_j^\Lambda g^{jj^*} \right) f_{j^*}^\Sigma k_\Sigma^{k^*} \\ & + \frac{i}{6} (\sigma_x)^A{}_D (S^{-1})_{AB} W^{mDB} C_{jml} C_{k^* z^* l^*} g^{ll^*} g^{zz^*} g^{jj^*} f_z^\Lambda f_{j^*}^\Sigma k_\Sigma^{k^*} P_\Lambda^x \Big] . \end{aligned} \tag{4.126}$$

Again, we see no clear inequality only involving terms similar to the first one, since there are also terms with no fixed sign.

**4.96.** Now focusing on $Y^\alpha_{s|mD}$, we can employ the parametric limit $N \ll W$, since this time the FWGC fermion is a gaugino. Neglecting the first and last term squared and subleading contribution proportional to $N$, we can compute

$$(4\text{th})^2 \approx 128 n_H \,\mathrm{Tr}\, S^2, \tag{4.127}$$

$$\begin{aligned} (1\text{st} \times 4\text{th}) + h.c. &= -4 n_H g_{mm^*} W^{m(DF)} W^{m^*}_{(DF)} \\ &\geq -4 n_H g_{mm^*} W^{mDF} W^{m^*}_{DF} \approx -48 n_H \,\mathrm{Tr}\, S^2, \end{aligned} \tag{4.128}$$

$$(1\text{st} \times 5\text{th}) = -\frac{1}{6}(S^{-1})_{AB} W^{mDB} U_D^{u\alpha} U^{sA\gamma} \left( \Omega^x_{ts} \Omega^x_{vu} + h_{uv} h_{ts} \right) \nabla^t k_\Lambda^v f_m^\Lambda M_{\gamma\alpha}, \tag{4.129}$$

$$(4\text{th} \times 5\text{th}) = 0, \tag{4.130}$$

and get to

$$|Y^\alpha_{s|mD}|^2 \gtrsim \frac{20}{3} n_H m_{susy}^4 - \frac{1}{3}\,\mathrm{Re}\left[ (S^{-1})_{AB} W^{mDB} U_D^{u\alpha} U^{sA\gamma} \left( \Omega^x_{ts} \Omega^x_{vu} + h_{uv} h_{ts} \right) \nabla^t k_\Lambda^v f_m^\Lambda M_{\gamma\alpha} \right] . \tag{4.131}$$

The last term can be further massaged, but no useful identities can be used to reduce it to a simpler form. Nonetheless, the conjecture can be recovered if we assume that this term is negligible on the vacuum. This happens if again $M \ll m_{susy}$ since in this case we have, schematically, $S^{-1} W M \ll S^{-1} W m_{susy} \overset{N^2 \ll W^2 \sim S^2}{\ll} m_{susy}$ and can be neglected with respect to the first term.

**4.98, 4.99.** Finally, we group together the Yukawas $Y_{k|jCmD}$ and $Y^\alpha_{k|jC}$ since their squares involve further difficulties, model-dependent quantities and mixed terms that cannot be further reduced. We do not report the full expressions, but we comment on the nature of the problematic steps:

1. One of the most problematic terms is the one which we get when considering $\nabla_k \nabla_j f_m^\Lambda \sim \nabla_k C_{jml} g^{ll^*} f_{l^*}^\Lambda$. There are no geometric identities involving $\nabla_k C_{jml}$ (other than its symmetry properties) and we ought to keep it until the end as a model-dependent quantity.

2. All mixed terms involving mass matrix $M_{jCmD}$ are not vanishing, and at most can be neglected by following a similar reasoning as in the previous cases.

3. Working in the usual parametric regime, we are still not able to recover the scale of supersymmetry breaking, as can be understood from (4.98), where neglecting contributions proportional to $W$ means killing off the second term. Then, as can be verified explicitly, squares of other terms or mixed ones do not yield $m_{susy}^4$ as in previous cases. On the other hand, if we try to work away from this regime, we ought to keep all shift matrices as is, without being able to reconstruct $\operatorname{Tr} S^2$. Similarly, we encounter the same difficulties also in (4.97): as an example, we can try to compute its second term squared, which just yields

$$(2nd)^2 = U^{\Lambda\Sigma} P_\Lambda P_\Sigma, \tag{4.132}$$

and no other term can be combined with it in order to give something proportional to $W^2$.

## 3.3 *Summary of results and comments*

We have seen that given our geometrical interpretation of $m_{susy}$ and $Y$, we can't seem to be able to satisfactorily recover the fermionic WGC in full generality. However, this statement has a certain degree of confidence depending on which Yukawa we focus on. In particular:

1. For the Yukawas involving an hyperino and (the matter sector of) vector multiplets, $Y^{\alpha}_{k^*|mD}$, the conjecture follows only by assuming that the process of supersymmetry breaking is dominated by the hypermultiplets, that is:

$$|Y^{\alpha}_{k^*|mD}|^2 \gtrsim \frac{4}{9} m_{susy}^4 \quad \text{if} \quad W \ll N. \tag{4.133}$$

This might indicate that, qualitatively, (2.15) is realized when the tower belongs to a sector of the theory which doesn't break supersymmetry too much with respect to other sectors, or when the FWGC fermions (in this case the hyperini) bring the dominant contribution to supersymmetry breaking. We will see that this last suggestion is more suitable to cover the purely vector or purely hypermultiplet Yukawas.

2. The same reasoning can be followed for the Yukawa coupling a gaugino to an hypermultiplet, $Y^{\alpha}_{s|mD}$. In this case not only we have to implement the suitable limit $N \ll W$ , but also assume that the hyperini original mass matrix $M^{\alpha\beta}$ is such that $M^2 \ll m_{susy}^2$. We then get to:

$$|Y^{\alpha}_{s|mD}|^2 \gtrsim \frac{20}{3} n_H m_{susy}^4. \tag{4.134}$$

This time, the new condition can be interpreted as the fact that in the unbroken phase the hyperini need to be massless. Their mass is then generated predominantly by the super-Higgs mechanism. This is reasonable if we notice that the original paper [30] only focuses on susy breaking and is agnostic with respect to the initial (non-physical) masses of the fermions. We can thus give an emphasis to the role of supersymmetry breaking by neglecting the original mass matrices.

3. For what concerns the Yukawa involving hypermultiplets only $Y_s^{\alpha\beta}$, we can employ the same working assumptions as the previous case, $W \ll N$, getting to:

$$|Y_s^{\alpha\beta}|^2 \gtrsim \frac{1}{9}(304n_H + 32)m_{susy}^4 + \frac{4}{3}(NNNNS^{-1}\Omega - 3\epsilon\epsilon SS + h.c.) \tag{4.135}$$

where the non-vanishing terms have been reported only schematically. In this case, being the sign of the extra terms not clear, it is not definitive whether the conjecture holds or not.

4. Focusing then on the coupling involving $\bar{z}\bar{\lambda}\lambda$, in the usual parametric limits, we get:

$$\begin{aligned}|Y_{k^*|jCmD}|^2 \gtrsim \frac{n_V}{9}m_{susy}^4 - 2\,\mathrm{Re}\Bigg[&\frac{1}{6}(S^{-1})_{DB}W^{mDB}k_{m|\Sigma}k_\Lambda^{j^*}f_j^\Sigma L^\Lambda \\ &-\frac{1}{6}(S^{-1})_{DB}W^{mDB}\left(\nabla_{k^*}k_\Lambda^{j^*}f_m^\Lambda - \nabla_{k^*}k_{m|\Lambda}f_j^\Lambda g^{jj^*}\right)f_{j^*}^\Sigma k_\Sigma^{k^*} \\ &+\frac{i}{6}(\sigma_x)^A{}_D(S^{-1})_{AB}W^{mDB}C_{jml}C_{k^*z^*l^*}g^{ll^*}g^{zz^*}g^{jj^*}f_z^\Lambda f_{j^*}^\Sigma k_\Sigma^{k^*}P_\Lambda^x\Bigg].\end{aligned} \tag{4.136}$$

Even in this case, we can say nothing about the conjecture due to the extra terms.

5. For the couplings involving $z$, that is $Y_{k|jC}^\beta$ and $Y_{k|jCmD}$, the situation is even less clear, since there are difficulties even in setting up an inequality involving the scale of supersymmetry breaking.

Even if the general analysis remains unclear, there seems to be hints that the conjecture could be realized, at least in certain parametric regimes. The next step will be to further investigate explicit models in order to better understand this possibility. Another possibility, which however will not be explored in this thesis, is that identifying which states make up the tower is key to checking (2.15), and we encounter difficulties because we are simply summing over all Yukawas. This cannot be checked by working entirely in the EFT, from the bottom-up, but requires making reference to UV completions in string-theoretic models, as briefly done in [11, 30].

# 5 *Testing the Fermionic WGC: explicit examples*

In the previous chapter, we established that, in full generality, it remains unclear whether the conjecture is realized or not. At most, we provided parametric limits in which some of the Yukawas seem to respect inequality (2.15) and which have a clear, qualitative physical interpretation. In order to explore the more general scenario, however, it is instructive to consider explicit supergravity models: on the one hand, these models must be simple enough to allow analytical computations, but on the other, they ought to be sufficiently general. In order to satisfy both requirements, we stick to abelian gaugings of quaternionic isometries in two different scalar manifold geometries:

$$\mathcal{M}^{(1)}_{scal} = \frac{SU(1,1)}{U(1)} \times \frac{SO(4,1)}{SO(4)}, \quad \mathcal{M}^{(2)}_{scal} = \frac{SU(1,1)}{U(1)} \times \frac{SU(2,1)}{SU(2)}, \tag{5.1}$$

which all yield models with one hypermultiplet and one vector multiplet. The homogeneous manifold $SO(4,1)/SO(4)$ is nothing but 4d hyperbolic space, and in the following we will simply call it $EAdS_4$, while $SU(2,1)/SU(2)$ is also called *universal* hypermultiplet since manifolds of this kind appear in all Type IIA/B string theory compactifications on $CY_3$ [62, 63]. We will also look at one particular $U(1) \times U(1)$ gauging with scalar geometry $SU(1,1)/U(1) \times EAdS_4$, realized by Ferrara, Girardello and Porrati [57], which has the nice property of displaying both fully broken and partially broken supersymmetry in a frame without prepotential, avoiding a notable no-go theorem [64, 65].

The abelian gaugings we will look at will be mostly $U(1)$ gaugings, which we will construct using some of the properties of homogeneous manifolds, as will be shown in the next section, following [66–69]. Each $U(1)$ model will be studied by means of two different symplectic embeddings, given by two different prepotentials:

$$F(X)^{(1)} = -iX^0X^1 \quad \text{and} \quad F(X)^{(2)} = \frac{(X^1)^3}{X^0}. \tag{5.2}$$

The $U(1) \times U(1)$ model will be studied in a frame without prepotential and in a frame where a prepotential of the kind

$$F(X) = -\frac{i}{2}[(X^0)^2 - (X^1)^2] \tag{5.3}$$

exists, as done in [52]. Finally, after dealing with the construction of the models and the analysis of the vacua, we will select the Minkowski, supersymmetry breaking ones in order to check the conjecture by explicit computation.

## 1 VACUA AND COSETS

The scalar potential of supergravity theories is an arbitrarily complicated function of the (real) scalar fields $\phi^i$, which number depends on the number of supermultiplets and supersymmetries (in the $\mathcal{N} = 2$ case, we have $i = 1, \ldots, 4n_H + 2n_V$). This means that finding its critical points involves solving a system of coupled algebraic equations collectively expressed as

$$\frac{\partial V(\phi)}{\partial \phi^i} = 0. \tag{5.4}$$

Finding solutions to such a complicated system of equations is challenging and one usually restricts one's attention to subsectors of the scalars, invariant under specific symmetry groups, in order to simplify the task. Also in this case solutions can often be provided only numerically. If the manifold is homogeneous, however, as it happens in every $\mathcal{N} > 2$ supergravity theory or in specific $\mathcal{N} = 2$ models, there is an alternative [66–69], which relies on the fact that homogeneous manifolds are *coset spaces*, meaning they are realized as

$$\mathcal{M}_{scal} = \frac{G}{H}. \tag{5.5}$$

Here, $G$ is the isometry group of the space and $H < G$ is the subgroup of $G$ leaving an arbitrary point $x$ fixed, and is called the *isotropy subgroup* of the manifold. By definition, there is a transitive action of $G$ on the space, and every point can be mapped to any other by a $G$ transformation. The idea is then to take condition (5.4) and map it to the origin ($\phi = 0$), where all field dependence drops and we are left with a system of second order equations in the gauging parameters, as will be now shown in the $\mathcal{N} = 2$ case.

Recall that the gauging parameters can be encoded in a duality covariant object by using of the embedding tensor ${\Theta_M}^\alpha$ and transforming it like a spurion under duality and isometry actions. We can moreover introduce it in the potential by re-expressing the killing vectors and momentum maps as:

$$\begin{aligned} k^\bullet_M &= k^\bullet_\alpha {\Theta_M}^\alpha, \\ P^\bullet_\Lambda &= P^\bullet_\alpha {\Theta_M}^\alpha, \end{aligned} \tag{5.6}$$

where we used the index $M$ instead of $\Lambda$ since we now allow our gaugings to be magnetic. We can thus consider the potential as a function of the coordinates and the embedding tensor:

$$V = V(\phi, \Theta). \tag{5.7}$$

In general, we expect a $G$ transformation to have an embedding (even if trivial) in the duality subgroup acting on the symplectic sections: this means that when acting on the coordinates with a $G$ transformation, we should also act on $\Theta$ with the corresponding duality action as

$$\Theta' = U\Theta, \tag{5.8}$$

since if that is the case the combinations

$$\Theta_M{}^\alpha V^M, \quad \Theta_M{}^\alpha U_i^M \tag{5.9}$$

are duality invariant and the scalar potential remains unchanged in form. This means that, if we go from $\phi'$ to $\phi$ with a $G$ transformation, we are allowed to write the equality:

$$V(\phi', \Theta) = V(\phi, \Theta'). \tag{5.10}$$

Furthermore we can see that, up to similarity transformations of the isometry generators (which of course doesn't affect the gauge group) $t_\alpha \to U t_\alpha U^{-1}$, the only action we really need to take into account is the left action $\Theta_M^\alpha \to (\Theta')_M^\alpha = U_M^N \Theta_N^\alpha$. The same reasoning applies also to derivatives of the scalar potential, and in particular to equation (5.4), which on the origin is simply a quadratic function of the embedding tensor:

$$\frac{\partial V}{\partial \phi^i}(0, \Theta') = 0, \tag{5.11}$$

and can be solved together with the constraints (3.81),(3.82) defining a consistent gauging. In short, rather than fixing the gauging and then performing a scan of all possible critical points of the scalar potential and then scan among the possible gaugings, one can simply solve a set of quadratic conditions on the embedding tensor and then read the resulting values of $\Theta$ which define at the same time the original gauge group, the value of the cosmological constant and the masses at the critical point. The procedure is shown graphically in the case of $\mathcal{N} = 8$ supergravity in 5.1.

Since we are mainly interested in $U(1)$ gaugings, the strategy of using the embedding tensor is highly redundant, as our gauge group rank is one. In order to simplify the picture, we can also trade the system of first and second order equations involving $\Theta$ with a system of (at most) fourth order equations in fewer variables by using the *rank factorization*[45] of $\Theta$. Given the rank $r = \dim(G_g)$, we can indeed write:

$$\Theta_M^\alpha = \sum_{I=1}^{r} \xi_M^I \xi_I^\alpha, \tag{5.12}$$

and in the particularly simple case of $U(1)$ gaugings $\Theta$ factorizes as:

$$\Theta_M^\alpha = \xi_M \xi^\alpha, \tag{5.13}$$

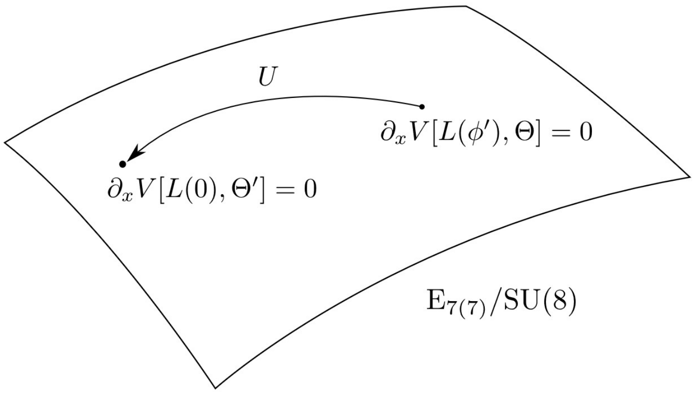


FIGURE 5.1: Stationary points of the $\mathcal{N} = 8$ scalar potential found for $\phi = \phi'$ can be translated to the origin of the scalar manifold by an isometry transformation $U \in E_{7(7)}$ and by a redefinition of the embedding tensor $\Theta' = U\Theta$.

and the closure and locality constraints, which can be cast as:

$$\xi_M \mathbb{C}^{MN} \xi_N = 0, \quad \xi^\alpha \xi^\beta [k_\alpha, k_\beta] = 0 \tag{5.14}$$

are naively satisfied. This is the strategy we will employ to find vacua of models involving $U(1)$ gaugings, while the $U(1) \times U(1)$ model will be taken from the literature. Before moving on, notice that we can interpret the $\xi^I_M$ as our freedom in the choice of symplectic frame, and the $\xi^\alpha_I$ as the freedom of choosing the gauge algebra among the isometry generators.

## 2 EADS $\times$ SU(1,1)/U(1) AND SU(2,1)/U(2) $\times$ SU(1,1)/U(1) MODELS

In this section, we will better characterize the Quaternionic and Special Kähler geometries we will later use to search for compatible vacua among the various models and check the fermionic WGC.

### 2.1 *EAdS$_4$ geometry*

Four dimensional hyperbolic space can be parametrized by a set of coordinates $(z^0, z^1, z^2, z^3)$ in the *Poincarè half-space* metric given by:

$$h_{uv} = \frac{1}{2(z^0)^2} \delta_{uv}, \quad z_0 > 0. \tag{5.15}$$

It is straightforward to check that the $SU(2) \times Sp(2)$ vielbeins are given by:

$$U^{\alpha A} = \frac{1}{2z^0}\epsilon^{\alpha\beta}(dz^0 - i\sigma^x dz^x)^A_\beta, \tag{5.16}$$

and satisfy property (3.44). We also report the $SU(2)$ bundle connection and curvature which were calculated in [57]:

$$\omega^x_u = \frac{1}{z^0}\delta^x_u, \quad \Omega^x_{0u} = -\frac{1}{2(z^0)^2}\delta^x_u, \quad \Omega^x_{yz} = \frac{1}{2(z^0)^2}\epsilon^{xyz}. \tag{5.17}$$

It is time to turn to the isometries and momentum maps of hyperbolic space, among which we ought to choose which to gauge. We give here a comprehensive treatment. The (continuous, connected) isometry group of the quaternionic manifold is the conformal group of $EAdS_4$, $SO(1,4)$, which generators are given by:

$$\begin{aligned} &P_i\,, \quad \text{translations}\\ &J_i\,, \quad \text{rotations}\\ &D\,, \quad \text{dilatations}\\ &S_i\,, \quad \text{special conformal transformations} \end{aligned} \tag{5.18}$$

where $i = 1,2,3$, giving in total 10 isometries. Since we chose the upper $\mathbb{R}^4$ parametrization, their realizations as killing vectors is particularly simple:

**Translations)** By direct inspection of the metric, we notice these isometries to be realized as proper translations on the three-dimensional constant $z^0$ slices:

$$k^u_{t_i} = \delta^u_i\,, \quad i = 1,2,3. \tag{5.19}$$

**Rotations)** As above, they are simply realized as:

$$\begin{aligned} k_{r_1} &= z^2\partial_{z^3} - z^3\partial_{z^2},\\ k_{r_2} &= z^3\partial_{z^1} - z^1\partial_{z^3},\\ k_{r_3} &= z^1\partial_{z^2} - z^2\partial_{z^1}. \end{aligned} \tag{5.20}$$

**Dilatations)** Dilatations are realized as:

$$k^u_\Lambda = z^u \tag{5.21}$$

**Special conformal transformations)** Finally, these are less intuitive and are given by:

$$k^u_{s_i} = \delta^{ui}\delta_{lm}z^l z^m - 2z^u z^i \ , \quad i = 1,2,3. \tag{5.22}$$

We can indeed verify that the killing vectors satisfy the following Lie algebra:

$$\begin{aligned}
&[k_{r_i}, k_{r_j}] = -\epsilon_{ijk}k_{r_k}, \\
&[k_{r_i}, k_{t_j}] = -\epsilon_{ijk}k_{t_k}, \\
&[k_{r_i}, k_{s_j}] = -\epsilon_{ijk}k_{s_k}, \\
&[k_{r_i}, k_\lambda] = 0, \\
&[D, k_{t_i}] = -k_{t_i}, \\
&[D, k_{s_i}] = k_{s_i}, \\
&[k_{t_i}, k_{s_j}] = 2\delta_{ij}k_\lambda - 2\epsilon_{ijk}J_k
\end{aligned} \tag{5.23}$$

which is the one of the conformal group with generators multiplied by $-i$. We are now ready to calculate moment maps using (3.76):

$$P^x = \frac{1}{2}\nabla_{[u}k_{v]}(\Omega^x)^{uv} = \frac{1}{2}\partial_{[u}k_{v]}(\Omega^x)^{uv}, \quad x = 1,2,3. \tag{5.24}$$

Explicit computation yields:

$$\begin{aligned}
&P^x_{t_i} = \frac{1}{z^0}k^x_{t_i}, \\
&P^x_{r_i} = \frac{1}{z^0}k^x_{r_i} + \frac{1}{2}\partial_l k_{r_i|m}\epsilon^{xlm} \\
&P^x_{s_i} = \frac{1}{z^0}k^x_{s_i} + 2\epsilon^{ixk}z^k - 2z^0\delta^x_i \\
&P^x_\lambda = \frac{1}{z^0}k^x_\lambda.
\end{aligned} \tag{5.25}$$

It is also useful to identify the isotropy subalgebra of the isometry algebra, which is $so(4)$. We can notice that if we redefine the following generators:

$$k^\pm_{a_i} = \frac{1}{2}(k_{s_i} \pm k_{t_3}) \tag{5.26}$$

then we discover the following subalgebra:

$$\begin{aligned}
&[k_{r_i}, k_{r_j}] = -\epsilon_{ijk}k_{r_k}, \\
&[k_{r_i}, k^\pm_{a_j}] = -\epsilon_{ijk}k_{a_k}, \\
&[k^\pm_{a_i}, k^\pm_{a_j}] = \pm\epsilon_{ijk}k_{r_k}.
\end{aligned} \tag{5.27}$$

which is exactly the one of $so(4)$ in the case of $-$, while of $so(1,3)$ in the case of $+$. Moreover, since the Lie bracket of $k^+_{a_i}$ and $k^-_{a_i}$ closes on dilatations, we can identify $k_\lambda$ as the 4th boost of $so(1,4)$.

## 2.2 *Universal hypermultiplet geometry*

We now switch to the case of $SU(2,1)/U(2)$. We can find the metric, killing vectors and prepotentials by using the coset construction given in [70] and which we report here. Let us write the generators of the $su(2,1)$ algebra as follows:

$$su(2,1) = \mathrm{Span}(J^x, J_0, H_0, T_a, T_\bullet), \quad a = 1,2, \tag{5.28}$$

where $J^x$ are the quaternionic $SU(2)$ structure generators and $J_0$ is the $U(1)$ generator commuting with them. The four remaining generators $H_0, T_a, T_\bullet$ generate the Borel subalgebra. The matrix representation of the generators in the fundamental of $SU(2,1)$ is:

$$\begin{gathered}
J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}, \\
J^0 = \begin{pmatrix} -\frac{4i}{3} & 0 & 0 \\ 0 & \frac{2i}{3} & 0 \\ 0 & 0 & \frac{2i}{3} \end{pmatrix}, \quad H_0 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad T_\bullet = -\frac{i}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \\
T_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -1-i & 0 \\ -1+i & 0 & 1-i \\ 0 & -1-i & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1-i & 0 \\ 1+i & 0 & -1-i \\ 0 & 1-i & 0 \end{pmatrix},
\end{gathered} \tag{5.29}$$

the invariant matrix defining the fundamental representation is $\mathrm{diag}(+1, -1, -1)$. The commutation relations among the generators of the Borel subalgebra are:

$$[H_0, T_\bullet] = T_\bullet, \quad [H_0, T_M] = \frac{1}{2} T_M, \quad [T_1, T_2] = T_\bullet. \tag{5.30}$$

The parametrization of the manifold in terms of coordinates $(u, \chi, z^1, z^2)$ is defined by a coset representative of the form:

$$\mathbb{L}(q) = e^{-\chi T_\bullet}\, e^{\sqrt{2} z^M T_M}\, e^{2uH_0}. \tag{5.31}$$

The metric is thus given by:

$$ds^2 = du^2 + \frac{e^{-4u}}{4}(d\chi + Z^T \mathbb{C} dZ)^2 + \frac{e^{-2u}}{2} dZ^T dZ, \tag{5.32}$$

where $\mathbb{C}$ is the completely antisymmetric $2 \times 2$ matrix. In the following we will give the explicit form of the killing vectors describing the 8 (continuous, connected) isometries of the manifold:

$$
\begin{aligned}
k_{J^1} &= \frac{(z^1+z^2)}{2}\partial_u + \frac{1}{2}[\mathrm{Im}(\mathcal{E})(z^1+z^2)(\mathrm{Re}(\mathcal{E})+1)(z^1-z^2)]\partial_\chi \\
&+ \frac{1}{2}[1-\mathrm{Im}(\mathcal{E})-\mathrm{Re}(\mathcal{E})+z^1(z^1+z^2)+z^2(z^1-z^2)]\partial_{z^1} \\
&+ \frac{1}{2}[1+\mathrm{Im}(\mathcal{E})-\mathrm{Re}(\mathcal{E})+z^1(z^2-z^1)+z^2(z^1+z^2)]\partial_{z^2}, \\
k_{J^2} &= \frac{(z^2-z^1)}{2}\partial_u + \frac{1}{2}[\mathrm{Im}(\mathcal{E})(z^2-z^1)(\mathrm{Re}(\mathcal{E})+1)(z^1+z^2)]\partial_\chi \\
&+ \frac{1}{2}[-1-\mathrm{Im}(\mathcal{E})+\mathrm{Re}(\mathcal{E})+z^1(z^2-z^1)+z^2(z^1+z^2)]\partial_{z^1} \\
&+ \frac{1}{2}[1-\mathrm{Im}(\mathcal{E})-\mathrm{Re}(\mathcal{E})-z^1(z^1+z^2)-z^2(z^1-z^2)]\partial_{z^2}, \\
k_{J^3} &= -\frac{\chi}{2}\partial_u + \frac{1}{2}[-\mathrm{Im}(\mathcal{E})^2+\mathrm{Re}(\mathcal{E})^2-1]\partial_\chi \\
&+ \frac{1}{2}[-\mathrm{Im}(\mathcal{E})z^1+\mathrm{Re}(\mathcal{E})z^2-3z^2]\partial_{z^1} \\
&+ \frac{1}{2}[-\mathrm{Im}(\mathcal{E})z^2-\mathrm{Re}(\mathcal{E})z^1+3z^1]\partial_{z^2}, \\
k_{J_0} &= -\chi\partial_u + [-\mathrm{Im}(\mathcal{E})^2+\mathrm{Re}(\mathcal{E})^2-1]\partial_\chi \\
&+ [-\mathrm{Im}(\mathcal{E})z^1+\mathrm{Re}(\mathcal{E})z^2+z2]\partial_{z^1} + [-\mathrm{Im}(\mathcal{E})z^2-\mathrm{Re}(\mathcal{E})z^1-z^1]\partial_{z^2}, \\
K_{T_\bullet} &= -\partial_\chi, \quad k_{T_i} = -\frac{1}{\sqrt{2}}\epsilon_{ij}z^j\partial_\chi + \frac{1}{\sqrt{2}}\partial_{z^i}, \quad k_{H_0} = \frac{1}{2}\partial_u + \chi\partial_\chi + \frac{1}{2}z^i\partial_{z^i},
\end{aligned}
\tag{5.33}
$$

where we used the following complex quantities:

$$
\mathcal{E} = e^{2u} + |\mathcal{Z}|^2 + i\chi, \quad \mathcal{Z} = \frac{z^1+iz^2}{\sqrt{2}}. \tag{5.34}
$$

Finally, we can find all momentum maps simply by using the fact that in quaternionic homogeneous manifolds the following definition holds:

$$
P^x_\alpha = \frac{1}{2}\,\mathrm{Tr}\left[J^x\mathbb{L}^{-1}t_\alpha\mathbb{L}\right], \tag{5.35}
$$

where $t_\alpha$ are the isometry generators in matrix form we defined above. Since their explicit form is missing in [70], we compute them here:

$$
\begin{aligned}
P_{J^1} &= \frac{1}{2}\Big\{e^{-u}\left(-1-e^{2u}+\frac{1}{2}(z^1-z^2)^2-2z^1z^2\right), -e^{-u}\left(\chi-(z^1)^2+(z^2)^2\right), \\
&\frac{1}{4}e^{-2u}\left(2z^1+2\chi(z^1+z2)-(z^1)^3+e^{2u}(z^1-z^2)-2z^2+z^2(z^1)^2-z^1(z^2)^2+(z^2)^3\right)\Big\}, \\
P_{J^2} &= \frac{1}{2}\Big\{e^{-u}\left(\chi-(z^1)^2+(z^2)^2\right), e^{-u}\left(-1-e^{2u}+\frac{1}{2}(z^1+z^2)^2+2z^1z^2\right), \\
&\frac{3}{2}(z^1+z^2)-\frac{1}{4}e^{-2u}\left((z^1)^3-2(1+\chi)z^2+z^2(z^1)^2+(z^2)^3+z^1(-2+2\chi+(z^2)^2\right)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
P_{J^3} = \Big\{ & \frac{1}{8}e^{-u}\left((z^1)^3-(z^1)^2z^2+z^2(6+2\chi+2e^{2u}-(z^2)^2)+z^1(-6+2\chi-2e^{2u}+(z^2)^2)\right), \\
& \frac{1}{8}e^{-u}\left((z^1)^3+(z^1)^2z^2+z^1(-6-2\chi-2e^{2u}+(z^2)^2)+z^2(-6+2\chi-2e^{2u}+(z^2)^2)\right), \\
& \frac{1}{32}e^{-2u}\Big(-4-4\chi^2-4e^{4u}+12(z^1)^2-(z^1)^4+12(z^2)^2-2(z^1)^2(z^2)^2-(z^2)^4 \\
& +12e^{2u}(-2+(z^1)^2+(z^2)^2)\Big)\Big\}, \\
P_{J_0} = \Big\{ & \frac{1}{4}e^{-u}\left(z^1\left(2\chi-2e^{2u}+(z^2)^2+2\right)-z^2\left(-2\chi-2e^{2u}+(z^2)^2+2\right)+(z^1)^3-(z^1)^2z^2\right), \\
& \frac{1}{4}e^{-u}\left(z^1\left(-2\chi-2e^{2u}+(z^2)^2+2\right)+z^2\left(2\chi-2e^{2u}+(z^2)^2+2\right)+(z^1)^3+(z^1)^2z^2\right), \\
& \frac{1}{16}e^{-2u}\left(-4\chi^2+4e^{2u}\left(3(z^1)^2+3(z^2)^2+2\right)-4e^{4u}-\left((z^1)^2+(z^2)^2+2\right)^2\right)\Big\}, \\
P_{T_1} = & \left\{-\frac{e^{-u}}{2\sqrt{2}},\frac{e^{-u}}{2\sqrt{2}},-\frac{e^{-2u}z^2}{2\sqrt{2}}\right\}, \quad P_{T_2} = \left\{-\frac{e^{-u}}{2\sqrt{2}},-\frac{e^{-u}}{2\sqrt{2}},\frac{e^{-2u}z^1}{2\sqrt{2}}\right\}, \\
P_{T_\bullet} = & \left\{0,0,-\frac{1}{4}e^{-2u}\right\}, \quad P_{H_0} = \left\{-\frac{1}{4}e^{-u}(z^1+z^2),\frac{1}{4}e^{-u}(z^1-z^2),\frac{1}{4}\chi e^{-2u}\right\}.
\end{aligned} \tag{5.36}
$$

## 2.3 *Special SU(1,1)/U(1) geometry*

Let's finally turn to the Special Kähler manifold $SU(1,1)/U(1)$, parametrized by the complex coordinate $w$. During our analysis, we will use a total of four different parametrizations and choices of symplectic sections:

1. A symplectic section given $X^0=1, X^1=w$ and prepotential $F(X)=-iX^0X^1$,
2. A symplectic section given $X^0=1, X^1=w$ and prepotential $F(X)=(X^1)^3/X^0$,
3. A symplectic section given $X^0=1, X^1=w$ and prepotential $F(X)=-\frac{i}{2}[(X^0)^2-(X^1)^2]$,
4. A symplectic section given by $e^{-\mathcal{K}/2}V^M=\left(-\frac{1}{2},\frac{i}{2},iw,w\right)$ without prepotential, which can be obtained by a symplectic transformation of the first section.

As we will see, the choices 1,3 and 4 lead to the same homogeneous manifold, while 2 gives rise to a different one, although equipped with the same coset structure [71]. We will use the first two choices in the $U(1)$ gaugings, while the last two will be reserved to the Ferrara model. When a prepotential exists, the symplectic sections read $e^{-\mathcal{K}/2}V^M=\{1,w,\partial_{X^0}F(X),\partial_{X^1}F(X)\}$. From these, we can calculate the Kähler potential, the metric, the connection and the symmetric $C_{ijk}$ sections following the identities given in Chapter 3. We get, respectively:

1. The full symplectic section and Kähler potential reads:

$$
\mathcal{K}(w,w^*)=-\log\left[2(w+w^*)\right], \quad V^M=e^{\mathcal{K}/2}\left(1,w,-iw,-i\right), \quad \mathrm{Re}(w)>0. \tag{5.37}
$$

The metric is thus given in the *Poincarè half-space* form and the connection follows as:

$$g_{zz^*} = \frac{1}{(w+w^*)^2}, \quad \Gamma^z_{zz} = \frac{-2}{(w+w^*)}, \tag{5.38}$$

while:

$$U_z^M = e^{\mathcal{K}/2}\left\{\frac{-1}{(w+w^*)}, 1-\frac{w}{(w+w^*)}, -i\left(1-\frac{w}{(w+w^*)}\right), \frac{i}{(w+w^*)}\right\}, \tag{5.39}$$

and finally:

$$C_{zzz} = 0, \quad \nabla_z U_z^M = 0. \tag{5.40}$$

2. The full symplectic section and Kähler potential reads:

$$\mathcal{K}(w,w^*) = -\log\left[i(w-w^*)^3\right], \quad V^M = e^{\mathcal{K}/2}\left(1, w, -w^3, 3w^2\right), \quad \mathrm{Im}(w) > 0. \tag{5.41}$$

The metric is thus given in the *Poincarè half-space* form and the connection follows as:

$$g_{zz^*} = -\frac{3}{(w-w^*)^2}, \quad \Gamma^z_{zz} = \frac{-2}{(w-w^*)}, \tag{5.42}$$

while:

$$U_z^M = e^{\mathcal{K}/2}\left\{\frac{-3}{(w-w^*)}, 1-\frac{3w}{(w-w^*)}, -3w^2-\frac{3w^3}{(w-w^*)}, 6w-\frac{9w^2}{(w-w^*)}\right\}, \tag{5.43}$$

and finally:

$$\begin{aligned} C_{zzz} &= \frac{6i}{(w-w^*)^3}, \\ \nabla_z C_{zzz} &= \frac{18i}{(w-w^*)^4}, \\ \nabla_z U_z^M &= e^{\mathcal{K}/2}\left\{-\frac{6}{(w-w^*)^2}, -\frac{2\,(w^*+2w)}{(w-w^*)^2}, \frac{6w^2w^*}{(w-w^*)^2}, -\frac{6w\,(2w^*+w)}{(w-w^*)^2}\right\}. \end{aligned} \tag{5.44}$$

3. The full symplectic section and Kähler potential reads:

$$\mathcal{K}(w,w^*) = -\log\left[2(1-|w|^2)\right], \quad V^M = e^{\mathcal{K}/2}\left(1, w, -i, iw\right) \quad |w| < 1. \tag{5.45}$$

The metric is thus given in the *Poincarè disk* form and the connection follows as:

$$g_{zz^*} = \frac{1}{(1-|w|^2)^2}, \quad \Gamma^z_{zz} = \frac{-2w^*}{(1-|w|^2)}, \tag{5.46}$$

while:

$$U_z^M = e^{\mathcal{K}/2}\left\{\frac{w^*}{1-|w|^2}, 1+\frac{|w|^2}{1-|w|^2}, -\frac{iw^*}{1-|w|^2}, i\left(1+\frac{w^*}{1-|w|^2}\right)\right\}, \tag{5.47}$$

and finally:

$$C_{zzz} = 0, \quad \nabla_z U_z^M = 0. \tag{5.48}$$

4. The last symplectic section without prepotential defines a Kähler potential:

$$V^M = e^{\mathcal{K}/2}\left(-\frac{1}{2}, \frac{i}{2}, iw, w\right), \quad \mathcal{K} = -\log(w+w^*), \tag{5.49}$$

with metric and connection defined as the first one. Moreover:

$$U_z^M = e^{\mathcal{K}/2}\left\{\frac{1}{2\,(w+w^*)}, -\frac{i}{2\,(w+w^*)}, i\left(1-\frac{w}{w+w^*}\right), 1-\frac{w}{w+w^*}\right\}, \tag{5.50}$$

while all other relevant quantities vanish as in (5.40).

As remarked above, parametrizing the coset structure through a cubic prepotential leads to a different manifold, as can be noticed from the fact that the $C_{zzz}$ sections are non-vanishing. As a last remark, notice that the isometries of this space will not be needed, as the gauging will interest the quaternionic isometries only.

## 2.4 *Gaugings*

As mentioned, we will need to construct all $U(1)$ gaugings from scratch, while the Ferrara model (in both symplectic embeddings) will be taken from the literature. In order to efficiently search for vacua, we will first use the transitive property of homogeneous manifolds in order to translate them to the origin, as illustrated in section 1. Let us then write the chosen $U(1)$ killing vector and the corresponding momentum map using the rank-decomposed embedding tensor:

$$k_M^u = \xi_M \zeta^\alpha k_\alpha^u, \quad P_M^x = \xi_M \zeta^\alpha P_\alpha^x, \tag{5.51}$$

where the index $\alpha$ runs over the isometries of the manifolds. The potential then reads:

$$V = 4h_{uv}k_\alpha^u k_\beta^v \zeta^\alpha \zeta^\beta \xi_M \xi_N V^M \bar{V}^N + (g^{zz^*} U_z^M U_{z^*}^N - V^M V^N)\zeta^\alpha \zeta^\beta \xi_M \xi_N P_\alpha^x P_\beta^x. \tag{5.52}$$

We can now solve for the gauging parameters the following two conditions

$$V|_{q=0} = 0, \quad \partial_u V|_{q=0} = 0, \tag{5.53}$$

defining a Minkowski critical point on the origin of our manifolds, which for the $EAdS_4$ and universal model are respectively:

$$(z^0, z^1, z^2, z^3) = (1, 0, 0, 0) \quad \text{and} \quad (u, \chi, z^1, z^2) = (0, 0, 0, 0). \tag{5.54}$$

For the Special $SU(1,1)/U(1)$ with quadratic and cubic prepotentials we choose:

$$w = 1, \quad w = i, \quad \text{and} \quad w = 0 \tag{5.55}$$

respectively for $F(X) = -iX^0X^1$, $F(X) = (X^1)^3/X^0$, and $2F(X) = i[(X^0)^2 - (X^1)^2]$, while the section without prepotential used in the Ferrara model has again origin $w = 1$. Finally, we will need to solve the fourth-order system of equations (5.53) in the gauging parameters, which will be done using the Mathematica package *Singular*[1]. Singular is a computer algebra system for polynomial computations, and the routine we will use is based on the technique called *relinearization* [72], which has been used in a similar way in [73] to find vacua of maximal supergravity in five dimensions.

# 3 GAUGINGS AND VACUA OF EADS × SU(1,1)/U(1)

## 3.1 *Ferrara model*

The Ferrara model [52, 57] is a no-scale supergravity model displaying partial or fully broken supersymmetry. It can be constructed by gauging two of the three translation isometries of the quaternionic manifold, which we choose to be:

$$k_M = \{gk_{t_1}, g'k_{t_2}, 0, 0\}, \quad P^x_M = \frac{1}{z^0}k^x_M\Lambda. \tag{5.56}$$

We will now compute this model in the two mentioned symplectic sections:

**Symplectic section without prepotential)** The explicit form of the shift matrices follows as:

$$\begin{aligned} W^{z^*}_{AB} &= -i(w+w^*)^{1/2}\frac{1}{z^0}X_{AB}, \\ N^{\alpha}_{A} &= -i(w+w^*)^{-1/2}\frac{1}{z^0}\epsilon^{\alpha\beta}X_{\beta A}, \\ S_{AB} &= -\frac{i}{2}(w+w^*)^{-1/2}\frac{1}{z^0}X_{AB}, \end{aligned} \tag{5.57}$$

where

$$X_{AB} = \frac{g}{2}(\sigma^1)_{AB} - i\frac{g'}{2}(\sigma^2)_{AB} = \begin{pmatrix} \frac{g'-g}{2} & 0 \\ 0 & \frac{g'+g}{2} \end{pmatrix}. \tag{5.58}$$

[1] https://www.singular.uni-kl.de/

We immediately see that if $g = \pm g'$, then one gravitino is massless and supersymmetry is partially broken. Otherwise it is fully broken. The potential can be checked to be identically vanishing, so that we have sliding vevs for all scalar fields defining moduli spaces of $N = 0$ or $N = 1$ Minkowski vacua. We can also explicitly compute mass matrices:

$$\begin{aligned} M_{zCzD} &= -\frac{i}{b^0}(w+w^*)^{-5/2}X_{CD}, \\ M^{\beta}_{zC} &= \frac{2i}{b^0}(w+w^*)^{-3/2}\epsilon^{\alpha\beta}X_{\beta A}, \\ M^{\alpha\beta} &= i(w+w^*)^{-1/2}\frac{1}{b^0}X^{\alpha\beta}. \end{aligned} \tag{5.59}$$

where we used the fact that $k^z = 0$ and we made use of the gradient flow equations to ease the calculations.

**Symplectic section with prepotential)** For convenience, we take $g = g'$ as in [52], and we set it to $g = 1$. Let's again compute all shift matrices by applying their definitions:

$$\begin{aligned} S_{AB} =& \frac{i}{2\sqrt{2}}\frac{1}{z^0}(1-|w|^2)^{-1/2}\left[(\sigma_2)_{AB} + w(\sigma_3)_{AB}\right], \\ W^{z^*}_{AB} =& \frac{i}{\sqrt{2}z^0}(1-|w|^2)^{1/2}\left[w^*(\sigma_2)_{AB} + (\sigma_3)_{AB}\right], \\ N^{\alpha}_{A} =& \frac{i}{\sqrt{2}z^0}(1-|w|^2)^{-1/2}\left[(\sigma_2)_{AB} + w(\sigma_3)_{AB}\right]. \end{aligned} \tag{5.60}$$

We can then compute mass matrices through the gradient flow equations or their definition:

$$\begin{aligned} M_{zCzD} =& -\frac{i}{\sqrt{2}z^0}w^*(1-|w|^2)^{-\frac{5}{2}}\left[w^*(\sigma_2)_{AB} + (\sigma_3)_{AB}\right], \\ M^{\alpha}_{zA} =& \frac{i\sqrt{2}}{z^0}(1-|w|^2)^{-\frac{3}{2}}\epsilon^{\alpha\beta}\left[w^*(\sigma_2)_{\beta A} + (\sigma_3)_{\beta A}\right], \\ M^{\alpha\beta} =& -\frac{2i}{z^0}(1-|w|^2)^{-1/2}\left[(\sigma_2)^{\alpha\beta} + w(\sigma_3)^{\alpha\beta}\right]. \end{aligned} \tag{5.61}$$

As in the previous symplectic embeddings, all shift matrices squared cancel against each other and the potential is identically vanishing, defining a no-scale model. Let's then discuss the form of the gravitino mass matrix:

$$S_{AB} = \frac{i}{2\sqrt{2}}\frac{1}{z^0}(1-|w|^2)^{-1/2}\begin{pmatrix} i & w \\ w & i \end{pmatrix}, \tag{5.62}$$

with eigenvalues $\lambda = i \pm w$, which means that all vacua are non-supersymmetric since $|w| < 1$. Notice that $g$ would have appeared as an overall normalization. Considering two different $g, g'$ would have simply meant introducing partial supersymmetric vacua at $z = \pm g/g'$. We neglect this case since it has been already covered by the previous symplectic embedding.

| Vacuum | Gaugings parameters | Type | Scalar masses $\{q^u, \mathrm{Re}(w), \mathrm{Im}(w)\}$ |
|---|---|---|---|
| $s_1$ | $\xi^i = 0$, $\xi^5 = \pm\xi^4$, $\xi_M = (\xi_3\xi_4/\xi_2, \xi_2, \xi_3, \xi_4)$ | $\mathcal{N} = 0$ | $\{0,0,0,0,0,0\}$ |
| $s_2$ | $\xi^i = (\xi^4, 0, 0)$, $\xi^5 = 0$ | $\mathcal{N} = 2$ | $\{m^2_{s_2}, m^2_{s_2}, m^2_{s_2}, m^2_{s_2}, 0, 0\}$ |

TABLE 5.1: Gaugings and vacua of the $EAdS_4 \times SU(1,1)/U(1)$ model, with symplectic embedding given by the prepotential $F(X) = -iX^0X^1$.

### 3.2 *U(1) gaugings*

Before solving (5.53) in the 14 total variables $\xi_M$, $\xi^\alpha$, we can further reduce the number of gauging parameters by again using the homogeneity of the quaternionic manifold. Indeed, since we are interested in gauging a single $U(1)$ vector, we can always act with the isotropy subgroup, in this case $SO(4)$, at the origin and on $k \equiv k_\alpha\xi^\alpha$ in order to rotate it in a chosen non-compact direction, which we choose to be $k_\lambda$. Moreover, by looking at the isometries (5.19)-(5.22), (5.26) and the commutation relations (5.23), (5.27) we see that we also have an $SO(3) \subset SO(4)$ generated by $k_{r_i}$ commuting with $k_\lambda$, so that by leaving it fixed we can rotate the remaining 6 generators $k^-_{a_i}, k_{r_i}$ (which clearly transform as $SO(3)$ vectors) to eliminate two more parameters. We then choose:

$$k = \xi^i k_{r_i} + \xi^4 k^-_{a_1} + \xi^5 k_\lambda, \tag{5.63}$$

and solve the system of equation (5.53) in 9 total parameters. We report in the tables all Minkowski vacua we find by using the above parametrization.

#### *Symplectic section with quadratic prepotential*

Let's start with the symplectic embedding defined by the prepotential $F(X) = -iX^0X^1$, which vacua can be found in table 5.1 and where we defined:

$$m^2_{s_2} = 2(\xi^4)^2\left[(\xi_1+\xi_2)^2 + (\xi_3+\xi_4)^2\right]. \tag{5.64}$$

Let's now describe the gaugings we obtained one by one.

**$s_1$:** solution $s_1$ of system (5.53) describes a fully-fledged no scale model with an identically vanishing potential. The vacua it describes are, in general, non supersymmetric, since the momentum map on the origin reads:

$$P^x_\alpha\xi^\alpha = (\pm\xi^5, 0, 0), \tag{5.65}$$

| Vacuum | Gauging parameters | Type | Scalar masses $\{q^u, \mathrm{Re}(w), \mathrm{Im}(w)\}$ |
|---|---|---|---|
| $s_3^a$ | $\xi^5 = 0$, $\xi_M = (\xi_2^2/(9\xi_4), \xi_2, -\xi_4^2/\xi_2, \xi_4)$ | $\mathcal{N} = 0$ | $\{(m^-_{s_3^a})^2, (m^-_{s_3^a})^2, (m^+_{s_3^a})^2, (m^+_{s_3^a})^2, 0, 0\}$ |
| $s_3^b$ | $\xi^5 = 0$, $\xi_M = (\xi_1, 0, 0, 0)$ | $\mathcal{N} = 0$ | $\{(m^-_{s_3^b})^2, (m^-_{s_3^b})^2, (m^+_{s_3^b})^2, (m^+_{s_3^b})^2, 0, 0\}$ |
| $s_4$ | $\xi^i = (\xi^4, 0, 0)$, $\xi^5 = 0$ | $\mathcal{N} = 2$ | $\{m^2_{s_4}, m^2_{s_4}, m^2_{s_4}, m^2_{s_4}, 0, 0\}$ |
| $s_5$ | $\xi^i = (\xi^4, 0, 0)$, $\xi_M = (3\xi_4, -\xi_3, \xi_3, \xi_4)$ | $\mathcal{N} = 2$ | $\{m^2_{s_5}, m^2_{s_5}, m^2_{s_5}, m^2_{s_5}, m^2_{s_5}, 0\}$ |

TABLE 5.2: Gaugings and vacua of the $EAdS_4 \times SU(1,1)/U(1)$ model, with symplectic embedding given by the prepotential $F(X) = (X^1)^3/X^0$.

and the gravitino mass matrix has two non-vanishing eigenvalues. One of the massless modes is a would-be goldstone since the $U(1)$ factor gets broken:

$$k = (\pm\xi^5, 0, 0, 0). \tag{5.66}$$

Being a no-scale model, these quantities depend on the scalar fields $(z^0, z^1, z^2, z^3)$, but, at least in this case, all points of moduli space belong to the same class of vacua described above.

**s$_2$:** this solution parametrizes a class of models yielding supersymmetric vacua with fixed quaternionic scalars and massive modes. There are two flat directions along the scalar fields $\mathrm{Re}(w), \mathrm{Im}(w)$. This means that the spectrum organizes as one massive short $\mathcal{N} = 2$ hypermultiplet and one massless $\mathcal{N} = 2$ vector supermultiplet. As such, the $U(1)$ factor in left unbroken on the vacuum.

***Symplectic section with cubic prepotential***

Let's now switch to the the symplectic section with prepotential $F(X) = (X^1)^3/X^0$, which vacua can be found in table 5.2 and where we defined:

$$\begin{aligned}
\left(m^{\pm}_{s_3^a}\right)^2 &= \frac{(\xi_2^2 + 9\xi_4)^3}{81\xi_2^2\xi_4^2}\left(\sum_i(\xi^i)^2 + (\xi^4)^2 \pm \sqrt{4(\xi^1\xi^4)^2 + \left(\sum_i(\xi^i)^2 + (\xi^4)^2\right)^2}\right), \\
\left(m^{\pm}_{s_3^b}\right)^2 &= \xi_1^2\left(\sum_i(\xi^i)^2 + (\xi^4)^2 \pm \sqrt{4(\xi^1\xi^4)^2 + \left(\sum_i(\xi^i)^2 + (\xi^4)^2\right)^2}\right), \\
m^2_{s_4} &= 2\xi_4^2\left(\xi_1^2 + (\xi_2 + \xi_3)^2 - 6\xi_1\xi_4 + 9\xi_4^2\right), \\
m^2_{s_5} &= \frac{16}{3}\xi^5\left(\xi_3^2 + 9\xi_4^2\right).
\end{aligned} \tag{5.67}$$

Again, let's comment on the gaugings we found one by one.

$\mathbf{s_3^a}$ **and** $\mathbf{s_3^b}$**:** these gaugings describe a class of non-supersymmetric models with all quaternionic scalars fixed and flat directions along the Special scalars. For a choice of gauging parameters such that $m_{s_3}^- = 0$, these vacua develop new flat directions in the quaternionic sector (as can be checked by noticing that some of the scalars disappear from the potential). The gravitino mass matrix is, in general, non-vanishing since

$$P_\alpha^x \xi^\alpha = (\xi^1 - \xi^4, \xi^2, \xi^3), \tag{5.68}$$

and the $U(1)$ factor is left unbroken as $k = 0$. For a particular choice of $\xi^\alpha$, we can make the vacua supersymmetric, as described by the next gauging.

$\mathbf{s_4}$**:** this gauging describes a class of supersymmetric vacua with fixed quaternionic scalars and flat Special directions. Again, the spectrum reorganizes as one massive short hypermultiplet and one massless vector supermultiplet. The quaternionic scalars can be made massless with a particular choice of gauging, but they do not disappear from the potential, and as such remain fixed. Indeed, it can be checked that positive quartic self-interactions between these scalars are present.

$\mathbf{s_5}$**:** this last gauging describes a class of supersymmetric vacua with fixed scalars and a would-be goldstone coming from the higgsing of the $U(1)$ factor:

$$k = (\xi^5, 0, 0, 0). \tag{5.69}$$

As such, this model displays a spectrum composed of one massive long vector $\mathcal{N} = 2$ supermultiplet of mass $m_{s_5}$. As in the previous case, we can make the scalars massless and un-higgs the $U(1)$ factor by choosing $\xi^5 = 0$, but the model does not develop new flat directions in the quaternionic sector since positive quartic interactions remain present.

# 4 GAUGINGS AND VACUA OF SU(2,1)/U(2) × SU(1,1)/U(1)

For this next choice of $\mathcal{M}_{scal}$, we only cover $U(1)$ gaugings, which will be systematically constructed as in the previous case. Again, we can further reduce the number of gauging parameters by acting with the isotropy algebra factor $su(2) \subset u(2)$ in order to align $k$ along one of the non-compact directions, reducing the number of total variables from 12 to 9. We choose the following killing vector for the gaugings:

$$k = \xi^i k_{J^i} + \xi^4 k_{J_0} + \xi^5 k_{H_0}, \tag{5.70}$$

and solve (5.53) for 9 total parameters.

| Vacuum | Gauging parameters | Type | Scalar masses $\{q^u, \mathrm{Re}(w), \mathrm{Im}(w)\}$ |
|---|---|---|---|
| $s_1$ | $\xi^i = 0$,<br>$\xi_M = (-\xi_2, \xi_2, -\xi_4, \xi_4)$ | $\mathcal{N} = 2$ | $\{m^2_{s_1}, m^2_{s_1}, m^2_{s_1}, m^2_{s_1}, m^2_{s_1}, 0\}$ |
| $s_2$ | $\xi^i = 0$, $\xi^5 = 0$ | $\mathcal{N} = 2$ | $\{m^2_{s_2}, m^2_{s_2}, m^2_{s_2}, m^2_{s_2}, 0, 0\}$ |

TABLE 5.3: Gaugings and vacua of the $SU(2,1)/U(2) \times SU(1,1)/U(1)$ model, with symplectic embedding given by the prepotential $F(X) = -iX^0X^1$

***Symplectic section with quadratic prepotential***

As the previous section, let's start with the symplectic embedding given by the quadratic prepotential $F(X) = -iX^0X^1$, which vacua can be found in table 5.3 where we defined:

$$
\begin{aligned}
m^2_{s_1} &= 2(\xi_2^2 + \xi_4^2)\xi^5, \\
m^2_{s_2} &= 8\xi^4 \left((\xi_1 + \xi_2)^2 + (\xi_3 + \xi_4)^2\right).
\end{aligned}
\tag{5.71}
$$

The classes of vacua we find are:

**$\mathbf{s_1}$:** this class of models defines supersymmetric vacua with fixed scalars and a would-be goldstone mode that comes from the higgsing of the $U(1)$ factor, which can also be checked by noticing that:

$$
k = (\frac{\xi^5}{2}, 0, 0, 0). \tag{5.72}
$$

As such, the spectrum reorganizes in one massive long vector supermultiplet.

**$\mathbf{s_2}$:** this gauging instead defines a class of supersymmetric vacua with one massive short hypermultiplet and a massless vector multiplet. The $U(1)$ factor is indeed unbroken on the vacuum. Notice that for the choice $\xi_M = (-\xi_2, \xi_2, -\xi_4, \xi_4)$ this models reduces to the previous one with $\xi^5 = 0$ and all masses vanish. As pointed out for the previous scalar geometry, however, the model does not display truly flat directions due to the presence of positive quartic self-interactions.

***Symplectic section with cubic prepotential***

Finally, we cover the symplectic embedding of this model given by the cubic prepotential $F(X) = (X^1)^3/X^0$, which vacua can be found in table 5.4 and where we defined:

$$
\begin{aligned}
\left(m^{\pm}_{s_5^a}\right)^2 &= \frac{(\xi_2^2+9\xi_4)^3}{162\xi_2^6\xi_4^6}\left[\left(\sum_i(\xi^i)^2+4(\xi^4)^2\right)\xi_2^4\xi_4^4 \pm 4\sqrt{\xi_2^8\xi_4^8(\xi^4)^2\left(\sum_i(\xi^i)^2\right)}\right],\\
\left(m^{\pm}_{s_5^b}\right)^2 &= \frac{1}{2}\xi_3^2\left(\sum_i(\xi^i)^2+4(\xi^4)^2 \pm 4\sqrt{(\xi^4)^2\left(\sum_i(\xi^i)^2\right)}\right),\\
m^2_{s_3} &= 2\xi_4^2\left(\xi_1^2+(\xi_2+\xi_3)^2-6\xi_1\xi_4+9\xi_4^2\right),\\
m^2_{s_4} &= 2\frac{16}{3}\xi^5\left(\xi_3^2+9\xi_4^2\right).
\end{aligned}
\tag{5.73}
$$

| Vacuum | Gauging parameters | Type | Scalar masses $\{q^u, \mathrm{Re}(w), \mathrm{Im}(w)\}$ |
|---|---|---|---|
| $s_3$ | $\xi^i = 0$, $\xi_M = (\xi_1, -\xi_3, \xi_3, 0)$ | $\mathcal{N}=2$ | $\{m^2_{s_3}, m^2_{s_3}, m^2_{s_3}, m^2_{s_3}, m^2_{s_3}, 0\}$ |
| $s_4$ | $\xi^i = 0$, $\xi^5 = 0$ | $\mathcal{N}=2$ | $\{m^2_{s_4}, m^2_{s_4}, m^2_{s_4}, m^2_{s_4}, 0, 0\}$ |
| $s_5^a$ | $\xi^5 = 0$, $\xi_M = (0, 0, \xi_3, 0)$ | $\mathcal{N}=0$ | $\{(m^-_{s_5^a})^2, (m^-_{s_5^a})^2, (m^+_{s_5^a})^2, (m^+_{s_5^a})^2, 0, 0\}$ |
| $s_5^b$ | $\xi^5 = 0$, $\xi_M = (\xi_2^2/(9\xi_4), \xi_2, -3\xi_4^2/\xi^2, \xi_4)$ | $\mathcal{N}=0$ | $\{(m^-_{s_5^b})^2, (m^-_{s_5^b})^2, (m^+_{s_5^b})^2, (m^+_{s_5^b})^2, 0, 0\}$ |

TABLE 5.4: Gaugings and vacua of the $SU(2,1)/U(2) \times SU(1,1)/U(1)$ model, with symplectic embedding given by the prepotential $F(X) = (X^1)^3/X^0$

Notice that these vacua are qualitatively the same and quantitatively similar to the ones we found in the $EAdS$ model with cubic prepotential. We comment here on the main differences, if any.

**$s_3$:** these vacua are a direct counterpart to the previous section's $s_5$.

**$s_4$:** these vacua are a direct counterpart to the previous section's $s_4$.

**$s_5^a$ and $s_5^b$:** These are a counterpart to $s_3^b$ and $s_3^a$ respectively. Here as well, the $U(1)$ factor remains unbroken while the non-vanishing momentum map reads:

$$P_\alpha \xi^\alpha = (-2\xi^1, -2\xi^2, -2\xi^3). \tag{5.74}$$

# 5 FERMIONIC WGC RELATIONS

The previous sections were devoted to the construction and analysis of the various different models we chose to study. In this section, we will focus on computing the Yukawas of the $\mathcal{N} = 0$ vacua we found using master formulas (4.95)-(4.100) and check if their square can be related to the scale of supersymmetry breaking $12S^2$. To do so, we will make use of the characterization of the geometries given in section 2 as well as the explicit gauging parameters of the following vacua:

| $EAdS_4 \times SU(1,1)/U(1)$ | $SU(2,1)/U(2) \times SU(1,1)/U(1)$ |
|---|---|
| $s_1^*$ | $(s_5^a)^*$ |
| $(s_3^a)^*$ | |
| $f_1$ | |
| $f_2$ | |

TABLE 5.5: Vacua chosen to check the Fermionic WGC.

where $f_{1,2}$ refer to the Ferrara model with and without prepotential respectively, while the $*$ refers to a specific choice of gauging parameters, in order to simplify the calculations. Notice that we only picked one of the $s_3$ and $s_5$ vacua since we expect them to be qualitatively equivalent. Since the computation is involved but straightforward, we only report the final results for $|Y|^2$.

## 5.1 *Ferrara models*

Let's consider the $\mathcal{N} = 0$ $(g \neq \pm g')$ Ferrara models first. As we will later check, the conclusions will be independent of this assumption.

**No prepotential.** We begin by listing the Yukawas in the case of the symplectic section without prepotential. First, let's compute the scale of supersymmetry breaking:

$$m_{susy}^4 \equiv 12\,\mathrm{Tr}\,S^2 = \frac{3(g^2 + g'^2)}{4\,\mathrm{Re}(w)(z^0)^2}. \tag{5.75}$$

Then, we find:

$$|Y_s^{\alpha\beta}|^2 = \frac{422}{27} m_{susy}^4, \tag{5.76}$$

$$|Y_{s|zD}^{\alpha}|^2 = \frac{5258441}{108} m_{susy}^4, \tag{5.77}$$

$$|Y_{z|zC}^{\beta}|^2 = \frac{g^2[4+\mathrm{Re}(w)]^2 + g'^2[4-\mathrm{Re}(w)]^2}{108(g^2+g'^2)} m_{susy}^4, \tag{5.78}$$

$$|Y_{z|zCzD}|^2 = \frac{1}{27} m_{susy}^4, \tag{5.79}$$

$$|Y_{z^*|zD}^{\alpha}|^2 = \frac{4}{27} m_{susy}^4, \tag{5.80}$$

$$|Y_{z^*|zCzD}|^2 = \frac{1}{432} m_{susy}^4. \tag{5.81}$$

We see that there are no clear inequalities present between the Yukawas and the scale of supersymmetry breaking, since some are bigger and some smaller. The main obstacle is that the dependence on the moduli is, as in the case of (5.78), different, even among different Yukawas, meaning that the conjecture becomes moduli space dependent and has to be checked at every point. This latter problem can be fixed by looking for $\mathcal{N} = 0$ vacua with completely fixed scalars, so that we have no flat directions. Unfortunately, among the various gaugings, there are no vacua of this kind. As a last remark, notice that the calculation also holds in the case of partial supersymmetry breaking, $g = \pm g'$.

**With prepotential.** Let's now switch to the symplectic embedding with prepotential $2F(X) = -i[(X^0)^2 - (X^1)^2]$, with $g$ left as an overall gauge coupling in both translational isometries. Since we established that the conjecture is moduli space dependent, let's first evaluate it at the origin to simplify calculations, and then extend it to the rest of the scalar manifold if deemed necessary. We report the Yukawas in the following:

$$\begin{aligned}
|Y_s^{\alpha\beta}|^2 &= \frac{422}{27} m_{susy}^4, \\
|Y_{s|zD}^{\alpha}|^2 &= \frac{585301}{12} m_{susy}^4, \\
|Y_{z|zC}^{\beta}|^2 &= \frac{73}{432} m_{susy}^4, \\
|Y_{z|zCzD}|^2 &= \frac{1}{27} m_{susy}^4, \\
|Y_{z^*|zD}^{\alpha}|^2 &= \frac{4}{27} m_{susy}^4, \\
|Y_{z^*|zCzD}|^2 &= \frac{1}{27} m_{susy}^4,
\end{aligned} \tag{5.82}$$

where the scale of supersymmetry breaking is :

$$m_{susy}^4 \equiv 12\, \mathrm{Tr}\, S^2 = 3g^2. \tag{5.83}$$

We see that the strict inequality (2.15) is not obeyed, however, taking into account moduli dependence, the scale of supersymmetry breaking and these Yukawas cannot be decoupled, in the sense that $Y \to 0$ always implies $m_{susy} \to 0$. A prototypical example of this is given by:

$$\begin{aligned}
|Y_{z^*|zCzD}|^2 &= \frac{1}{9} \frac{1}{(z^0)^2} \frac{1+|w|^2}{1-|w|^2} g^2, \\
12\, \mathrm{Tr}\, S^2 &= \frac{3}{(z^0)^2} \frac{1+|w|^2}{1-|w|^2} g^2,
\end{aligned} \tag{5.84}$$

in the limit $z^0 \to \infty$. This was true also in the previous symplectic embedding, but could be simply due to the simplicity of the gaugings we are considering. If taken seriously, however, this could lead to a connection to the *Gravitino Distance Conjecture* [74, 75], as simililarly pointed out in [10] for a simple $\mathcal{N}=1$ model.

## 5.2 *U(1) gaugings*

Let's finally turn to the $U(1)$ gaugings we chose in 5.5.

$(\mathbf{s_1})^*$ In this first gauging, the conjecture will be checked on a subset of solutions of $s_1$:

$$\xi^\alpha = (0,0,0,1,1), \quad \xi_M = (0,0,g,0), \tag{5.85}$$

such that all quaternionic scalars are massive and the only modulus is thus $w$. The Yukawas read:

$$\begin{aligned}
|Y_s^{\alpha\beta}|^2 &= \frac{422}{27} m_{susy}^4, \\
|Y_{s|zD}^{\alpha}|^2 &= \frac{5258441}{108} m_{susy}^4, \\
|Y_{z|zC}^{\beta}|^2 &= \frac{1}{108} m_{susy}^4 \Big[(4-3\,\mathrm{Re}(w))^2\,\mathrm{Re}(w)^4 \\
&\quad +2(9\,\mathrm{Re}(w)(\mathrm{Re}(w)+8)+16)\,\mathrm{Re}(w)^2\,\mathrm{Im}(w)^2 + (4-3\,\mathrm{Re}(w))^2\,\mathrm{Im}(w)^4\Big], \\
|Y_{z|zCzD}|^2 &= \frac{1}{27} m_{susy}^4, \\
|Y_{z^*|zD}^{\alpha}|^2 &= \frac{4}{27} m_{susy}^4, \\
|Y_{z^*|zCzD}|^2 &= \frac{1}{27} m_{susy}^4,
\end{aligned} \tag{5.86}$$

where the scale of supersymmetry breaking is :

$$m_{susy}^4 = 12\,\mathrm{Tr}\,S^2 = 3\frac{|w|^2}{\mathrm{Re}(w)} g^2. \tag{5.87}$$

This is remarkably similar to the Yukawas of the Ferrara model with $g \pm g'$ when evaluated on the origin, the main difference being that the scalar field dependence here is way more intricate and some of the coefficients differ.

$(\mathbf{s_3^a})^*$ We repeat the same procedure for the $EAdS_4$ cubic model $s_3^a$, by choosing:

$$\xi^\alpha = (g,0,0,g/2,0), \quad \xi_M = (1,0,0,0), \tag{5.88}$$

and noticing that, again, the only modulus is $w$. The Yukawas read:

$$\begin{aligned}
|Y_s^{\alpha\beta}|^2 &= 0 \\
|Y_{s|zD}^{\alpha}|^2 &= \frac{26675893}{1024} m_{susy}^4, \\
|Y_{z|zC}^{\beta}|^2 &= 0, \\
|Y_{z|zCzD}|^2 &= \frac{53}{9} m_{susy}^4, \\
|Y_{z^*|zD}^{\alpha}|^2 &= 0, \\
|Y_{z^*|zCzD}|^2 &= \frac{5}{9} m_{susy}^4,
\end{aligned} \tag{5.89}$$

while the scale of supersymmetry breaking is :

$$m_{susy}^4 \equiv 12\,\mathrm{Tr}\,S^2 = \frac{3}{16}\frac{g^2}{\mathrm{Im}(w)^3}. \tag{5.90}$$

$(\mathbf{s_5^a})^*$ We repeat the procedure one last time, by restring the gauging to:

$$\xi^\alpha = (g, 0, 0, g, 0), \xi_M = (0, 0, 1, 0) \tag{5.91}$$

and again noticing that $w$ is the only modulus. We get the following results:

$$\begin{aligned}
|Y_s^{\alpha\beta}|^2 &= 0 \\
|Y_{s|zD}^{\alpha}|^2 &= \frac{26473361}{32} m_{susy}^4, \\
|Y_{z|zC}^{\beta}|^2 &= 0, \\
|Y_{z|zCzD}|^2 &= \frac{1}{12}\frac{53\,\mathrm{Re}(w) + 17\,\mathrm{Im}(w)}{|w|^2} m_{susy}^4, \\
|Y_{z^*|zD}^{\alpha}|^2 &= 0, \\
|Y_{z^*|zCzD}|^2 &= \frac{5}{9} m_{susy}^4,
\end{aligned} \tag{5.92}$$

where the scale of supersymmetry breaking is :

$$m_{susy}^4 \equiv 12\,\mathrm{Tr}\,S^2 = \frac{3}{4}\left(\frac{|w|^2}{\mathrm{Im}(w)}\right)^3 g^2. \tag{5.93}$$

## 5.3 *Summary and comments*

We managed to construct and analyze simple $U(1)$ gaugings using two different scalar manifold geometries: $EAdS_4 \times SU(1,1)/U(1)$ and $SU(2,1)/U(2) \times SU(1,1)/U(1)$, each with two different symplectic embeddings given by a quadratic and a cubic prepotential. Adding to this, we have taken from the literature two examples of the Ferrara model [52, 57]. We found a total of 5 compatible classes of Minkowski vacua with fully broken supersymmetry, and checked conjecture (2.15) on each one. The main obstacle we found was that, since flat directions were always present in the potential, the conjecture is moduli depedent and could be violated in some regions of moduli space. It would be interesting to construct a model where all scalars are stabilized, i.e. massive, and compute the Yukawas there. We also saw that, at least in the Ferrara model, partial supersymmetry breaking doesn't seem to play a role. As for the general analysis and from the results of this section, we can deduce that the geometrical structure of $\mathcal{N} = 2$ supergravity EFTs doesn't seem to realize the fermionic WGC, at least as defined in this thesis, and some kind of UV completion needs to be considered. For example, by identifying the true string theory moduli among the flat directions, we could try to compute the infinite geodesic distance limit of the Yukawas in explicit models and tie the conjecture to the Swampland Distance Conjecture.

# 6 Open problems and perspectives

We conclude with a brief summary and a review of possible further developments. Recall first that the goal of this thesis was to construct and motivate a proposed low-energy consequence (2.15) of the Fermionic Weak Gravity conjecture (2.14) by means of the geometrical structure of supergravity theories, and in particular by employing $\mathcal{N} = 2$ supergravity in four dimensions. The main reason to attempt this follows from recent work [12, 34, 35] in the context of the Electric and Scalar Weak Gravity conjectures, which follows a similar philosophy. Let's then go through each chapter, focusing on the results and some ways to extend them in future work.

- In chapter 2 we first introduced the notion of Swampland conjectures, discussing different formulations of the "Weak Gravity" one and supporting it by simple black hole arguments. We then introduced its magnetic version and formulated the Fermionic Weak Gravity conjecture by qualitative arguments following [30].

- In chapter 3, after having reviewed the geometrical construction of the supergravity action, we discussed the first simple example on how this can help us use and study Swampland conjectures.

- In chapter 4 we proved the super-Higgs mechanism and identified the Yukawa appearing in (2.15) schematically as:

$$|Y|^2 = |\nabla \mathcal{M}|^2, \tag{6.1}$$

where $\mathcal{M}$ are the fermionic Lagrangian mass matrices and $\nabla$ the fully covariant derivative on the scalar manifold. We computed $Y$ explicitly (assuming a Minkowski background and broken supersymmetry) to find that some parametric limits indeed realize the conjecture. For example, by assuming that the hyperini bring the dominant contribution to supersymmetry breaking:

$$|Y^{\alpha}_{k^*|mD}|^2 \gtrsim \frac{4}{9} m^4_{susy}, \quad \text{by assuming} \quad W \ll N \quad \text{on the vacuum.} \tag{6.2}$$

for the Yukawa involving hyperini, gaugini and vector multiplet scalars. This parametric limit could indicate that the conjecture might be true when the tower of states of (2.14) doesn't break supersymmetry too much, or equivalently, when the FWGC fermion (the one which couples to the supersymmetric tower, in the above case the hyperino) brings the dominant contribution to supersymmetry breaking. This setup seems to work also on other Yukawas, but in general we find that the squares also involve model-dependent mixed products with no clear sign, for example:

$$\begin{aligned} |Y_{k^*|jCmD}|^2 \gtrsim \frac{n_V}{9} m_{susy}^4 - 2\,\mathrm{Re}\Bigg[ & \frac{1}{6}(S^{-1})_{DB} W^{mDB} k_{m|\Sigma} k_\Lambda^{j^*} f_j^\Sigma L^\Lambda \\ & - \frac{1}{6}(S^{-1})_{DB} W^{mDB} \left( \nabla_{k^*} k_\Lambda^{j^*} f_m^\Lambda - \nabla_{k^*} k_{m|\Lambda} f_j^\Lambda g^{jj^*} \right) f_{j^*}^\Sigma k_\Sigma^{k^*} \\ & + \frac{i}{6}(\sigma_x)^A{}_D (S^{-1})_{AB} W^{mDB} C_{jml} C_{k^* z^* l^*} g^{ll^*} g^{zz^*} g^{jj^*} f_z^\Lambda f_{j^*}^\Sigma k_\Sigma^{k^*} P_\Lambda^x \Bigg] . \end{aligned} \tag{6.3}$$

A possible extension of this analysis would be to compute these Yukawas in extended supergravities with $\mathcal{N} > 2$ to see if mixed terms vanish. In particular, it would be interesting to start with the $\mathcal{N} = 8$ case, which is the most constrained one. In doing so, we would first have to provide a proof for the super-Higgs mechanism (at least in the case of Minkowski vacua), which can be achieved by simply repeating the construction provided here. Extended supergravities also provide a way to study the role of partially supersymmetric vacua with more complex breaking patterns. Another approach that could be taken, although different in spirit from the one presented here, is to consider explicit classes of Type IIA or IIB string theory compactifications on $CY_3$. This could lead to a rigorous justification of the parametric limits we have identified in terms of the UV theory, and in general to further constraints on the geometry of the scalar manifolds. One such example is the existence of the *c-map*, which implies that the Special Kähler geometries we need to consider are only a restricted class admitting an embedding into a Quaternionic manifold [45]. Since we have access to the UV theory, doing this could hopefully also provide a way to identify which are the states that make up the tower in (2.14).

- Finally, in chapter 5, we constructed some models from scratch and checked the ones present in the literature in order to compute the Yukawas explicitly and better quantify the model-depend terms. The hope was to gain insight on how the conjecture might be realized in very simple settings. We used an interesting technique [66–69] to scan for vacua of homogeneous manifolds and construct original models with Minkowski vacua and broken supersymmetry. For what concerns the conjecture, as we remarked in the above, mixed products are present and in general, we do not find (2.15). The main obstacle is that the models we checked always had flat directions (i.e. moduli) and, as such, the conjecture depends on which point of moduli space we consider. To avoid this, it is crucial to construct a model in which all scalars are massive and moduli space is a single point, which however requires a great degree of fine-tuning, and which we couldn't do with the present set-up. This is one of the most straightforward directions in which the work of chapter 5 could be extended. It would also be interesting to check models with more complicated moduli dependence and possibly non-abelian gaugings. In general however, as a promising first result, it seems that the scale of supersymmetry breaking and the Yukawas cannot be decoupled, so that in every model $Y \to 0$ always implies $m_{susy} \to 0$. This could be in turn explored in a future work in relation to the Gravitino Distance Conjecture [74, 75], which states that sending the gravitino mass (in our case proportional to the supersymmetry breaking scale) to zero is an infinite distance limit lying in the Swampland. Indeed, sending $Y \to 0$ should imply a breakdown of the EFT through (2.14). Finally, as in chapter 4, we could also consider explicit string compactifications, for example in order to identify which among the flat directions are true moduli and try to tie (2.15) to the Swampland Distance Conjecture.

# A

# *Notations and conventions*

## 1 SPACETIME AND SPINOR CONVENTIONS

We follow the conventions of [46]. The Lorentz metric is $\eta^{\mu\nu} = \text{diag}(+---)$, meaning that real scalar fields are normalized with $+\frac{1}{2}$ instead of $-\frac{1}{2}$. The Ricci tensor is defined as $R_{\mu\nu} = R_{\mu\rho\sigma\nu}\eta^{\rho\nu}$. Covariant derivatives on fermions are $D\psi = \left(d - \frac{1}{4}\omega^{ab}\gamma_{ab}\right)\psi$. The Riemann tensor is defined as $D\omega^{ab} = R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{cb} \equiv -\frac{1}{2}R^{ab}{}_{\mu\nu}dx^\mu dx^\nu$. The Einstein-Hilbert term here is $-\frac{1}{2}R$, while in [76] is $+\frac{1}{2}R$. We also used the identification $\sqrt{-g} = \det e^a_\mu \equiv e$. From the definition of $R^{ab}$ we can derive the following commutator of covariant derivatives:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = \frac{1}{4}\gamma_{ab}R^{ab}{}_{\mu\nu}. \tag{A.1}$$

Spinor bilinears have the following exchange symmetry properties:

| | $1$ | $\gamma^\mu$ | $\gamma^{\mu\nu}$ | $\gamma^{\mu\nu\rho}$ |
|---|---|---|---|---|
| 1 forms | $-$ | $+$ | $+$ | $-$ |
| 0 forms | $+$ | $-$ | $-$ | $+$ |

Useful gamma matrices identities are the following:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{A.2}$$

$$[\gamma_\mu, \gamma_\nu] = 2\gamma_{\mu\nu}, \tag{A.3}$$

$$\gamma^\mu\gamma_\mu = 4, \tag{A.4}$$

$$\gamma_\mu\gamma^{\mu\nu} = 3\gamma^\nu = \gamma^{\nu\mu}\gamma_\mu, \tag{A.5}$$

$$\gamma_\mu\gamma^{\mu\nu}\gamma_\nu = 12, \tag{A.6}$$

$$\gamma_\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \tag{A.7}$$

$$\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^{\nu\rho} - \gamma^\rho\eta^{\mu\nu} + \gamma^\nu\eta^{\mu\rho}, \tag{A.8}$$

$$\frac{1}{2}\left(\gamma^{\mu\nu}\gamma^{\rho\sigma} + \gamma^{\rho\sigma}\gamma^{\mu\nu}\right) = \gamma^{\mu\nu\rho\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\nu\sigma}\eta^{\mu\rho}. \tag{A.9}$$

Introducing

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \gamma_5^2 = 1, \tag{A.10}$$

(notice that this definition of $\gamma_5$ differs by a minus sign from that of [76]; this propagates also in the relations below), we have

$$\gamma^{\mu} = +\frac{i}{3!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\nu\rho\sigma}\gamma_5, \tag{A.11}$$

$$\gamma^{\mu\nu} = -\frac{i}{2!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma}\gamma_5, \tag{A.12}$$

$$\gamma^{\mu\nu\rho} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_{\sigma}\gamma_5, \tag{A.13}$$

$$\gamma^{\mu\nu\rho\sigma} = +i\epsilon^{\mu\nu\rho\sigma}\gamma_5. \tag{A.14}$$

Spinors have the following elicities:

$$\begin{aligned} \gamma_5 \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_A \end{pmatrix} &= \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_A \end{pmatrix}, \\ \gamma_5 \begin{pmatrix} \lambda_{iA} \\ \zeta^\alpha \\ \psi^A \end{pmatrix} &= - \begin{pmatrix} \lambda_{iA} \\ \zeta^\alpha \\ \psi^A \end{pmatrix}. \end{aligned} \tag{A.15}$$

Charge conjugation only raises or lowers the position of SU(2) indices, e.g. $(\bar{\chi}^A\chi^B)^\dagger = \bar{\chi}_A\chi_B$, but does not change the positions of the fermions inside fermion bilinears. However, due to the hermicity properties of the gamma matrices chosen by [46], there is a minus sign whenever the number of gamma matrices is odd, namely

$$(\chi^A\lambda^B)^\dagger = \chi_A\lambda_B, \tag{A.16}$$

$$(\chi^A\gamma^\mu\lambda_B)^\dagger = -\chi_A\gamma^\mu\lambda^B, \tag{A.17}$$

$$(\chi^A\gamma^{\mu\nu}\lambda^B)^\dagger = \chi_A\gamma^{\mu\nu}\lambda_B. \tag{A.18}$$

# 2 SCALAR GEOMETRY CONVENTIONS

We write down the formulas for the Levi-Civita connection 1-form and curvature 2-form on a Kähler manifold:

$$\Gamma^i_j = \Gamma^i_{kj}dz^k = g^{il^*}(\partial_j g_{kl^*})dz^k, \tag{A.19}$$

$$\Gamma^{i^*}_{j^*} = \Gamma^{i^*}_{k^*j^*}dz^{k^*} = g^{i^*l}(\partial_{j^*}g_{k^*l})dz^{k^*}, \tag{A.20}$$

$$\mathcal{R}^i_j = \mathcal{R}^i_{jk^*l}dz^{k^*}\wedge dz^l = \partial_{k^*}\Gamma^i_{jl}dz^{k^*}\wedge dz^l, \tag{A.21}$$

$$\mathcal{R}^{i^*}_{j^*} = \mathcal{R}^{i^*}_{j^*kl^*}dz^k\wedge dz^{l^*} = \partial_k\Gamma^{i^*}_{j^*l^*}dz^k\wedge dz^{l^*}, \tag{A.22}$$

while we report the employed conventions for the Riemann and Ricci tensors of Quaternionic manifolds:

$$R^u_v \equiv d\Gamma^u_v + \Gamma^u_w \wedge \Gamma^w_v = \mathcal{R}^u{}_{vrs} dq^r \wedge dq^s, \tag{A.23}$$

$$R_{vs} \equiv \mathcal{R}^u{}_{vus}. \tag{A.24}$$

Finally, the flat $SU(2)$ metric $\epsilon_{AB} = -\epsilon_{BA}$, $\epsilon_{12} = 1$ and the flat $Sp(2n_H, \mathbb{R})$ metric $C_{\alpha\beta} = -C_{\beta\alpha}$, where $A, B = 1, 2$ $A, B = 1, 2$ and $\alpha, \beta = 1, \ldots, 2n_H$ raise and lower indices with the following conventions:

$$\epsilon_{AB} P^B = P_A, \quad \epsilon^{AB} P_B = -P^A, \tag{A.25}$$

$$C_{\alpha\beta} P^\beta = P_\alpha, \quad C^{\alpha\beta} P_\beta = -P^\alpha, \tag{A.26}$$

while the $SU(2)$ sigma matrices $(\sigma_x)_A{}^B$ respect the following properties:

$$\epsilon_{BC} (\sigma_x)_A{}^C = (\sigma_x)_{AB} = (\sigma_x)_{BA} = -((\sigma_x)^{AB})^*. \tag{A.27}$$

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