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Cosmic Microwave Background anomalies: models and interpretation

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Abstract

The current standard cosmological model –the so called LambdaCDM model – provides an excellent fit to a variety of cosmological data, in primis the temperature anisotropies and polarization of the Cosmic Microwave Background (CMB) radiation. However, as confirmed by the latest Planck satellite data, on the largest angular scales some "anomalies" in the behaviour of the CMB fluctuations have been reported. Despite the fact that their statistical significance remains at the 3-sigma level, they have been independently measured previously also by the WAMP satellite, and at the moment a compelling explanation in terms of systematics and or foregrounds does not exist. This might open a window into new physics, probably related to the early Universe, since the largest scales where these anomalies have been reported are the most sensitive to the initial conditions of our universe (e.g. some of these anomalies would hint to a small deviation from statistical isotropy, which is indeed one of the pillars the standard cosmological model is based on). The Thesis aims, first of all, at providing a review of some of the most important large-scale anomalies that have been reported in the CMB data. Then it aims at a detailed study of the various cosmological mechanisms that have been proposed so far to explain such features, especially those related to inflationary models (i.e. to an early epoch of accelerated expansion – inflation–the universe went through, which gave rise to the first density perturbations, the seeds for the subsequent formation of all the structures we see in the Universe). An original goal of the Thesis work would be in particular to explore possibly new solutions to these puzzles, trying to invoke models of inflation characterised by some amount of primordial non-Gaussianity. Particular attention would be dedicated in this context to the so called stochastic approach to inflation, where the primordial quantum fluctuations of the fields present during the inflationary epoch are studied via stochastic equations of motion.

Chapter 1

Introduction

The anomalies in the CMB are an interesting puzzle, even today an inflationary model that reproduces all the anomalies is not present. In this work we will focus on the hemispherical power asymmetry. We will study both the models proposed so far in the literature to explain the anomalies and a possible explicit two-fields model which addresses the problem with the stochastic approach. A brief summary of the topics covered:

In chapter (2) we present the most important components of the $\Lambda - CDM$ model, and the properties that characterize it. Further references in Modern Cosmology (Scott Dodelson) and The Early Universe (Kolb & Turner).

In chapter (3) we analyze the Cosmic Microwave Background (CMB) anomalies. These anomalies are in conflict with the standard model of Cosmology. Departures from the model were studied first by Ferreira et al. (1998), Pando et al. (1998) and in turn, refuted by Banday et al. (2000), Komatsu et al. (2002). Other studies on such departures were made in WMAP CMB measurements by Bennett et al. (2003) and recently in the Planck data (Planck Collaboration XXIII 2014, Planck Collaboration XVI 2016).

In chapter (4) we present the non Gaussian landscape picture and show some mechanisms that can reproduce the large scale hemispherical asymmetry, based on this model. References on this are Schmidt et al. (2013), Byrnes & Tarrant (2015), Byrnes et al. (2016), Adhikari et al. (2016). These papers are based on early proposals by Gordon et al (2005), Erickcek et al. (2008), Dvorkin et al. (2008).

In chapter (5) we outline the basic ideas and properties of the stochastic approach to inflation, references on this approach are due to Vilenkin (1983) and Starobinskii (1986). To obtain confidence with the method and the evolution equations that govern the motion of the fields in the stochastic approach, we discuss a useful two fields model.

In chapter (6) we introduce the model chosen in this thesis. We then discuss and justify an approximation of which our model is endowed. Thanks to this, approximated solutions of the system of Langevin equations are found; also the classical behavior and the noise term are presented.

In chapter (7) the definition of the ζ variable is presented and the two point function is computed at the leading order in the approximation introduced in (6).

In chapter (8) conclusions are shown and future perspectives discussed.

Chapter 2

Λ CDM model

The ΛCDM (Lambda cold dark matter) or Lambda-CDM model is a parametrization of the Big Bang cosmological model in which the universe contains three major components: first, a cosmological constant denoted by Lambda and associated with dark energy; second, the postulated cold dark matter (abbreviated CDM); and third, ordinary matter. It is frequently referred to as the standard model of Big Bang cosmology because it is the simplest model that provides a reasonably good account of the following properties of the cosmos:

1. The existence and structure of the cosmic microwave background (CMB);
2. The large-scale structure in the distribution of galaxies;
3. The abundances of hydrogen (including deuterium), helium, and lithium;
4. The accelerating expansion of the universe observed in the light from distant galaxies and supernovae.

The model assumes that general relativity is the correct theory of gravity on cosmological scales. It emerged in the late 1990s as a concordance cosmology, after a period of time when disparate observed properties of the universe appeared mutually inconsistent and there was no consensus on the makeup of the energy density of the universe. The ΛCDM model can be extended by adding cosmological inflation, quintessence and other elements that are current areas of speculation and research in cosmology. As stated previously the letter Λ (lambda) represents the cosmological constant, which is currently associated with a vacuum energy or dark energy in empty space that is used to explain the contemporary accelerating expansion of space against the attractive effects of gravity. A cosmological constant has negative pressure, $p = -\rho c^2$ [1], which contributes to the stress-energy tensor that, according to the general theory of relativity, causes accelerating expansion.
The fraction of the total energy density of our ("compatible flat") universe that is dark energy, Ω_Λ, is estimated to be 0.669 ± 0.038 based on the 2018 Dark Energy Survey results using Type Ia Supernovae or 0.6847 ± 0.0073 based

[1] We choose as an example an equation of state with $w = -1$, we underline that to have an expansion the request is $w < -\frac{1}{3}$.

on the 2018 release of Planck satellite data. Another important constituent is dark matter, that is postulated in order to account for gravitational effects observed in very large-scale structures (the "flat" rotation curves of galaxies, the gravitational lensing of light by galaxy clusters and enhanced clustering of galaxies) that cannot be accounted for by the quantity of observed matter. Cold dark matter as currently hypothesized is:

1. Non-baryonic:
   It consists of matter other than protons and neutrons (and electrons, by convention, although electrons are not baryons).

2. Cold:
   It's kinetic energy is far less than the mass energy at the epoch of freeze-out (thus neutrinos are excluded, being non-baryonic but not cold).

3. Dark:
   It is not charge under $U(1)_{EM}$, so it can not interact with photons.

4. Collision-less:
   The dark matter particles interact with each other and other particles only through gravity and possibly the weak force.

5. Stable or long-lived:
   It has a constant decay time longer than the age of the Universe, so it can form a relic abundance that can reach us.

Dark matter constitutes about 26.8% of the mass-energy density of the universe. The remaining 4.8% comprises all ordinary matter observed as atoms, chemical elements, gas and plasma, the stuff of which visible planets, stars and galaxies are made. Also, the energy density budget includes a very small fraction of relic neutrinos and radiation, the so called cosmic microwave background (CMB), discovered in 1965.
The CMB offers us a look at the universe when it was only 300,000 years old. The photons in the cosmic microwave background last scattered off electrons at redshift 1100; since then they have traveled freely through space. They are therefore the most powerful probes of the early universe. In fact up to now there has been spent a lot of effort to study in detailed the structure of the CMB. In particular from our first 25 years of surveying the CMB we learned that the early universe was very smooth. No anisotropies were detected in the CMB. We are now moving on. We have discovered anisotropies in the CMB, indicating that the early universe was not completely smooth. Recently some statistical properties (anomalies) in the CMB were found, in tension with the standard model of cosmology that up to now gave an excellent fit to the CMB anisotropies. Is in fact the main purpose of this work to search for some new solutions of these interesting puzzles, trying to invoke some models of inflation. Before discussing the models which explain some of the anomalies present in the CMB, is instructive to explain what the anomalies are.

Chapter 3

CMB anomalies

The CMB is well described by a black-body function with $T = 2.725K$. Another observable quantity inherent in the CMB is the variation in temperature from one part of the sky to another. Since the first detection of these temperature anisotropies by the COBE satellite, there has been intense activity to map the sky at increasing level of sensitivity and angular resolution.
WMAP and Planck satellites have led to a stunning confirmation of the "Standard model of cosmology". Nevertheless some departures from such model were found, first in WMAP and recently in Planck data. Such departures lead to several claims of unexpected statistical properties (anomalies) in the CMB fluctuations. While many of these are significant only at the $2-3\sigma$ level and could easily be the result of statistical flukes, it is still interesting to speculate whether they may share a common physical cosmological origin. First we list the six most debated anomalies and then we will investigate whether non-Gaussianiy alone may be the origin of one of these anomalies. The six relevant anomalies are [4]:

1. Local estimates of the angular power spectrum on large scales where the WMAP first-year data indicated an asymmetry of power between two hemispheres on the sky (Eriksen et al. 2004, Hansen et al. 2004). This hemispherical asymmetry has subsequently been modelled by a dipolar modulation of an isotropic sky (Eriksen et al. 2007, Hoftuft et al. 2009), and detected at the $2-3\sigma$ level detection for scales $l < 60$ in Planck Collaboration XVI (2016).

2. While the dipolar modulation is detected only on large scales, the spatial distribution of power on the sky has been shown to be correlated over a much wider range of multipoles (Hansen et al. 2009, Axelsson et al. 2013, Planck Collaboration XVI 2016). By estimating the power spectrum in local patches on the sky for a given multipole range, one can create a map of the corresponding power distribution. Even for an isotropic and Gaussian sky, such a map always exhibits a random dipole component. However, it has been shown that the directions of these dipole components from multipoles between $l = 2$ to $l = 1500$ are significantly more aligned in the Planck data than in random Gaussian simulations. The directions of these dipoles are close to the direction of the best fit large

scale dipolar modulation of the anomaly 1. But note that anomalies 1 and 2 are very different: 1 is present at large scales as an anomalously large dipolar modulation amplitude, whereas anomaly 2 is present at smaller scales where the amplitude of the observed dipolar modulation is consistent with that expected in the random Gaussian simulations, yet the preferred directions of the dipolar power distribution are aligned between multipoles.

3. In Vielva et al. (2004) it was shown that the wavelet coefficients for angular scales of about $\simeq 10^\circ$ on the sky have an excess kurtosis, while the skewness is consistent with zero. The excess kurtosis was shown to originate from a cold spot in the southern Galactic hemisphere. However, when masking the spot with a disk of 5° radius, the kurtosis of the map was found to be consistent with Gaussian simulations. The position of the cold spot on the sky is in the hemisphere where the dipolar modulation in 1 is positive. Must also be noticed that the cold spot is surrounded by a symmetric hot ring.

4. The Planck and WMAP power spectrum of CMB temperature anisotropy at large scales ($l < 30$) appears to trend significantly below the values consistent with the best fit cosmological model. In particular, the quadrupole is very low and a dip in the spectrum is observed around $l \simeq 21$. The low large-scale spectrum could well be a statistical fluctuation at these scales where the cosmic variance is large, but the significance is still at the $2-3\sigma$ level.

5. The quadrupole and octopole appear to be aligned and similarly dominated by their respective high-m components (Tegmark et al. 2003).

6. The C_l for the lowest even multipoles has been found to be consistently lower than for odd multipoles. The significance of this parity anomaly has been reported to be at the $2-3\sigma$ level (Planck Collaboration XVI 2016).

The correlation between some of these anomalies was studied by Muir et al. (2018) and hence these anomalies were shown to a large degree to be statistically independent. Most of the effort made in the study of the anomalies concentrate only on the hemispherical power asymmetry or, at most, in explaining two anomalies simultaneously. Such models are based on earlier proposals which stated that the properties of the observed CMB sky could be modelled by the presence of a long-wavelength fluctuation field that modulates otherwise isotropic and Gaussian fluctuations.
In particular, Adhikari et al.[1] have undertaken a systematic and general study of the power asymmetry expected in the CMB if the primordial perturbations are non-Gaussian and exist on scales larger than those we can observe. The analysis focused both on local and non-local models of primordial non-Gaussianity and the method developed is quite general for describing deviations from statistical isotropy in a finite subvolume of an otherwise isotropic

(but non-Gaussian) large volume. When local non-Gaussianity is invoked, the observed scale dependence of the power asymmetry anomaly can be recovered by the introduction of two bispectral indices. In Byrnes et al.[8], is computed the response of the two-point function to a long-wavelength perturbation in models characterized by a local-like bispectrum. However, in all of these works only the effects of the second order terms f_{NL} in the primordial non-Gaussianity have been studied in detail and the main focus was on the large-scale power asymmetry. Only recently, in Adhikari et al. [2], it was shown that large scale power asymmetry may arises in models with local trispectra with strong scale dependent τ_{NL} amplitude. Alternative inflationary models have also been proposed to explain other CMB anomalies, such as the lack of power at large angular scales and the CMB multipoles alignment [18]. Typically in this case, the models rely on deviations from the usual slow-roll phase in a period immediately before the observable 60 e-folds. In fact, the anomalies on the largest scales could provide hints about the conditions that led to the inflationary dynamics (in the observable window) given that they appear on the largest scales that will ever be observable.
However, the majority of those inflationary models proposed to date to explain the CMB anomalies have encountered some difficulties. In fact, it seems with our journey knowledge very difficult to construct an inflationary model that can reproduce all anomalies and satisfying all today's bounds. Nevertheless recently work on this was made by Bartolo et al. [4], even if an inflationary model does not appear. Therefore, in this work, we prefer to consider one anomaly, the hemispherical power asymmetry, and look for toy-models that can naturally reproduce it. In particular, inspired by the papers of Adhikari [1], [2] and Byrnes [8], [9], [11] we search for non-Gaussian models, where the non-Gaussianity is the origin of the above mentioned anomaly in the data. In this perspective it seems natural to review in more detail the fundamental papers of Adhikari and Byrnes to better understand the property that unites the two articles, the non-Gaussian Landscape.

Chapter 4

Non-Gaussian Landscape

Inflation is the leading scenario explaining the generation of primordial perturbations [10]. The observable primordial perturbations probe physics during the last $\alpha_{obs} \sim 60$ e-foldings of inflation after the horizon exit of the largest observable modes. Inflation may however have lasted much longer, so that the observable part of the universe would constitute only a fraction of the entire inflating patch. Long-wavelength perturbations generated before the horizon exit of the largest observable scales average to constants over our observable patch. Since these contributions are different in different parts of the inflating region, variances in the physical properties of patches smaller than the entire inflating region are generated. This leads to a landscape picture where the statistics of perturbations within a horizon patch depend on the location of the patch itself. In a general model we can have $\alpha_{tot} > \alpha_{obs}$; this implies that it is not possible to make firm predictions for the observable signatures in our horizon patch but one is led to consider probabilities of different signatures. It turns out that, especially in non-Gaussian models, the differences between the entire inflating patch and a random patch with the size of our observable universe can be significant. Indeed one can calculate the distribution of expected deviations from isotropy in our observed sky from any model with isotropic but non-Gaussian primordial fluctuations. This means that the observation of a violation of the isotropy in our observable universe is "spontaneous", due to the fact that we concentrate in a subvolume of a larger universe with non-Gaussian fluctuations. But a violation of isotropy, even if apparent, means a privileged direction. This is a property that characterizes the hemispherical power asymmetry as the latter is parameterized as a dipole modulation. In fact the observed C.M.B power asymmetry can be reproduced in models endowed of non-Gaussian primordial fluctuations, in particular is no more needed the enhancing of the amplitude of superhorizon fluctuations. In fact the most common method for introducing a dipolar power modulation is to postulate the existence of a large-amplitude superhorizon fluctuation in a spectator field during inflation that then alters the power spectrum on smaller scales via local-type non-Gaussianity: the Erickcek-Kamionkowski-Carroll (EKC) mechanism [17]. Now we review the paper of Adhikari where he studied the effects of non-Gaussian primordial perturbations on the CMB, starting with the local model.

4.1 Local Model

We assume that, at some early time (after reheating but prior to the release of the cosmic microwave background radiation), a large volume of the Universe (V_l) contains adiabatic fluctuations described by isotropic but non-Gaussian statistics [1]. Since observations are made in a smaller volume, $V_s << V_l$ that corresponds in size to our presently observable Hubble volume, we have to understand how the statistics of fluctuations in this volume are influenced by the fluctuations in the large volume. To start we will consider the usual local model with constant f_{NL} for simplicity.
We can assume that the Bardeen potential Φ is a non-Gaussian field described by the local model:

$$\Phi(\mathbf{x}) = \phi(\mathbf{x}) + f_{NL}\big(\phi(\mathbf{x})^2 - < \phi(\mathbf{x})^2 > \big) \quad , \tag{4.1}$$

where $\phi(\mathbf{x})$ is a Gaussian random field. When the large volume is only weakly non-Gaussian, the power spectrum observed in our sky, $P_{\Phi,s}(k, \mathbf{x})$, will be related to the mean power spectrum in the large volume, $P_\phi(k)$, by:

$$P_{\Phi,s}(k, \mathbf{x}) = P_\phi(k)\left[1 + 4f_{NL}\int \frac{d^3\mathbf{k}_l}{(2\pi)^3}\phi(\mathbf{k}_l)e^{i\mathbf{k}_l\cdot\mathbf{x}}\right] \quad , \tag{4.2}$$

where the radial integration for k_l is confined to $|\mathbf{k}_l| < \frac{\pi}{r_{cmb}}$, being the CMB spectrum the quantity of interest. The power spectrum P_ϕ and the amplitude of non-Gaussianity f_{NL} appearing on the right-hand side of Eq.(4.2) are those defined in the large volume.
We refer to the paper [1] for the definition of the inhomogeneous power spectrum; here we only outline the main steps for the derivation of formula (4.2). To obtain this equation, Adhikari starts from the two point function for the variable Φ. In particular, as stated above, thanks to the weak non Gaussianity only the linear term in f_{NL} is significant in the power spectrum of the Φ variable. When averaging on the entire universe the result is the isotropic power spectrum P_ϕ but, if we consider the mean value on a small volume, one of the variables present in the expectation value is no more stochastic and takes a particular value. This explains the presence of the field $\phi(\mathbf{k}_l)$ appearing in the integral. The power spectrum in Eq.(4.2) can depend on the position $\mathbf{x}$ within V_s because an individual realization (local value) of the fluctuations $\phi(\mathbf{k}_l)$ can be nonzero, since the variable is no more stochastic. If we consider the average statistics in the large volume (equivalent to averaging over all regions of size V_s), then the term proportional to f_{NL} in Eq.(4.2) averages to zero since $< \phi(\mathbf{k}_l) >_{V_l} = 0$. In that case we recover the isotropic power spectrum of V_l.
To see the effects of the in-homogeneous power spectrum on the CMB we can start expanding the power spectrum through a multipole expansion:

$$P_{\Phi,s}(k, \hat{n}) = P_\phi(k)\left[1 + f_{NL}\sum_{LM} g_{LM}Y_{LM}(\hat{n})\right] \quad , \tag{4.3}$$

where Y_{LM} is a spherical harmonic and $\hat{n}$ is the direction of observation on the last scattering surface. To find the expansion coefficients g_{LM} we make use of the plane wave expansion:

$$e^{i\mathbf{k}_l\cdot\mathbf{x}} = 4\pi \sum_{LM} i^L j_L(k_l x) Y^*_{LM}(\hat{k}_l) Y_{LM}(\hat{n}) \quad , \tag{4.4}$$

where j_L is a spherical Bessel function of the first kind and $\mathbf{x} = x\hat{n}$ specifies the position of the observed fluctuation: for the CMB $x = r_{cmb}$ is the comoving distance to the last scattering surface. Eq.(4.2) then implies that:

$$g_{LM} = 16\pi i^L \int_{|\mathbf{k}_l|<\frac{\pi}{x}} \frac{d^3\mathbf{k}_l}{(2\pi)^3} \phi(\mathbf{k}_l) j_L(k_l x) Y^*_{LM}(\hat{k}_l) \quad . \tag{4.5}$$

The quantity g_{LM} has a fixed value in any single volume V_s, but when averaged over all small volumes in V_l, $< g_{LM} >_{V_l} = 0$ due to the property of the $\phi(\mathbf{k}_l)$ seen before. The expected covariance, on the other hand, is non-zero:

$$\begin{aligned} < g_{LM} g^*_{L_1M_1} >_{V_l} = 256\pi^2 (-1)^{L_1} i^{L+L_1} \int_{|\mathbf{k}_l|<\frac{\pi}{x}} \frac{d^3\mathbf{k}_l}{(2\pi)^3} j_L(k_l x) Y^*_{LM}(\hat{k}_l) \times \\ \times \int_{|\mathbf{k}_{1l}|<\frac{\pi}{x}} \frac{d^3\mathbf{k}_{1l}}{(2\pi)^3} j_{L_1}(k_{1l} x) Y_{L_1M_1}(\hat{k}_{1l}) < \phi(\mathbf{k}_l)\phi^*(\mathbf{k}_{1l}) >_{V_l} . \end{aligned} \tag{4.6}$$

We exploit the following relation:

$$< \phi(\mathbf{k}_l)\phi^*(\mathbf{k}_{1l}) >_{V_l} = (2\pi)^3 \delta(\mathbf{k}_l - \mathbf{k}_{1l}) P_\phi(k_l) \quad ,$$

and we integrate over $\mathbf{k}_l$.Using the orthogonality condition of the spherical harmonics we get:

$$\begin{aligned} < g_{LM} g^*_{L_1M_1} >_{V_l} &= \frac{32}{\pi} \delta_{LL_1} \delta_{MM_1} \int_0^{\frac{\pi}{x}} dk_l k_l^2 j_L^2(k_l x) P_\phi(k_l) \\ &= 64\pi \delta_{LL_1} \delta_{MM_1} \int_0^{\frac{\pi}{x}} \frac{dk_l}{k_l} j_L^2(k_l x) \mathcal{P}_\phi(k_l) \quad , \end{aligned} \tag{4.7}$$

where in the last line we have defined the dimensionless power spectrum as: $\mathcal{P}_\phi(k) = \frac{k^3 P_\phi(k)}{2\pi^2}$.
We have again used the subscript V_l to indicate the ensemble average over the values of g_{LM} in the full volume V_l. Due to the dependence of the upper limit of integration from the size of the small volume, both the individual values of g_{LM} and their variances depend on it. We can now study the monopole and dipole contributions from non-Gaussian cosmic variance to the modulated component of the power spectrum in a small volume.

Mono-pole modulation (L=0)

The power-spectrum amplitude shift A_0 in the parametrization of Eq.(4.3) is:

$$P_\Phi(k) = P_\phi(k)[1 + A_0] = P_\phi(k)\left[1 + f_{NL}\frac{g_{00}}{2\sqrt{\pi}}\right] \quad , \tag{4.8}$$

where A_0 can be either positive or negative but has a lower bound $A_0 \geq -1$. From the previous discussion, it is clear that g_{00} is Gaussian distributed with zero mean and variance given by:

$$< g_{00}^2 >= 64\pi \int \frac{dk_l}{k_l} \left[\frac{sin(k_l x)}{k_l x}\right]^2 \mathcal{P}_\phi(k_l) \quad . \tag{4.9}$$

Therefore, the distribution of the monopole power modulation amplitude A_0 also follows a normal distribution for small values of A_0[1], with zero mean and standard deviation given by:

$$\sigma_{f_{NL}}^{mono} = \frac{1}{2\sqrt{\pi}} |f_{NL}|^2 < g_{00}^2 >^{\frac{1}{2}} \quad . \tag{4.10}$$

The expression for $< g_{00}^2 >$ is sensitive to the infrared limit of the integral. That is, all super-Hubble modes can contribute and hence an infrared cut off must be used. In particular, the observed value of f_{NL} is, in general, shifted from the mean value in the large volume.

For a constant f_{NL}, the effect of the monopole modulation is to change the power-spectrum amplitude on all scales and therefore is not observationally distinguishable from the "bare" value of the power-spectrum amplitude.

The probability distribution for the monopole modulation, as stated above, follows a normal distribution, namely:

$$p_N(A_0, \sigma^{mono}) = \frac{1}{\sqrt{2\pi}\sigma^{mono}} \exp\left(-\frac{A_0^2}{2\sigma_{mono}^2}\right) \quad , \tag{4.11}$$

where the variance has contributions from the Gaussian realization and the non-Gaussian coupling to the realization of long wavelength modes: $\sigma_{mono}^2 = \sigma_{f_{NL}}^2 + \sigma_G^2$. The term σ_G is the contribution coming from the Gaussian realization. In computing $\sigma_{f_{NL}}^{mono}$ the necessary infrared cutoff has already been set by the value of N_{extra} that can be related to the number of superhorizon e-folds of inflation. This is, in our simple example, the prediction for the distribution of monopole shift due to non Gaussian modulation. What we see is that the variance of the distribution of monopole modulations receives a contribution from the non-Gaussian coupling to the long wavelenght modes. This contribution is of fundamental importance since the f_{NL} factor is not observable and therefore it can be increased as to widen the probability distribution and thus making the presence of high monopole values shift less anomalous with respect to the Gaussian case. This degree of freedom will be very useful in the case of the dipole power asymmetry, as we will see.

Dipole modulation L=1

The dipole modulation of the power spectrum in the parametrization of Eq. (4.3) is given by:

$$P_\Phi(k, \hat{n}) = P_\phi(k) \left[1 + f_{NL} \sum_{M=-1}^{M=1} g_{1M} Y_{1M}(\hat{n})\right] \quad . \tag{4.12}$$

[1]Follows from the weakly non-Gaussianity assumption in the large volume.

Since we are interested in the dipole modulation of the observed power spectrum in the CMB sky, the above equation should be obtained from Eq.(4.2) by absorbing the (unobservable) monopole shift to the observed power spectrum. On the right-hand side of the previous equation $P_\phi(k)$ is the observed isotropic power spectrum and f_{NL} is the observed amplitude of local non-Gausianity within our Hubble volume. This holds also for higher correlation functions, for example for the bispectrum we have [1]:

$$< \Phi(k_1)\Phi(k_2)\Phi(k_3) >= (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)B^{obs}(k_1, k_2, k_3) \quad , \tag{4.13}$$

where:

$$B^{obs}(k_1, k_2, k_3) = 2f_{NL}(1 + f_{NL}g_{00})P_\phi(k_1)P_\phi(k_2) + \text{sym} \quad . \tag{4.14}$$

Since the observed isotropic power spectrum is $P_\Phi^{obs} = P_\phi(1 + f_{NL}g_{00})$, inserting this expression in the previous equation one obtains the shift of the f_{NL} from the bare to the observed value, namely $f_{NL}^{obs} = f_{NL}/(1 + f_{NL}g_{00})$ [2].
Returning to the monopole modulation, the g_{1M} coefficients are Gaussian distributed with zero mean and variance:

$$< g_{1M}g_{1M}^* >= 64\pi \int \frac{dk_l}{k_l} \left[\frac{sin(k_l x)}{(k_l x)^2} - \frac{cos(k_l x)}{k_l x} \right]^2 \mathcal{P}_\phi(k_l) \quad . \tag{4.15}$$

If we pick a direction $\vec{d_i}$ in which to measure the dipole modulation A_i such that:

$$P_\Phi(k) = P_\phi(k)[1 + 2A_i cos(\theta)] \quad , \tag{4.16}$$

where $cos(\theta) = \hat{d} \cdot \hat{n}$, then the contribution to the dipole from the non-Gaussianity is $A_i^{NG} = \frac{1}{4}\sqrt{\frac{3}{\pi}} f_{NL} g_{10}$, which is normally distributed with zero mean and standard deviation:

$$\sigma_{f_{NL}} = \frac{1}{4}\sqrt{\frac{3}{\pi}} |f_{NL}| < g_{10}^2 >^{\frac{1}{2}} \quad . \tag{4.17}$$

The above discussion about the distribution of the dipole asymmetry A_i assumes that we measure A_i in a fixed direction $\vec{d}$. However, we have no a priori choice of direction $\vec{d}$ in most situations. This is especially true when considering a power asymmetry that is generated by the random realization of superhorizon perturbations as opposed to a single exotic perturbation mode.
This means that possible observations of dipole power modulations are necessarily reported using the amplitude of dipole modulation in the direction of the maximum modulation. We can obtain this amplitude considering any three orthonormal directions (d_1, d_2, d_3) on the CMB sky and measuring the corresponding three dipole modulation amplitudes (A_1, A_2, A_3) along the three selected directions for each sky. The amplitude of modulation for the CMB

[2] Abuse of notation f_{NL}^{obs} is the f_{NL} in (4.12).

sky is then given by $A = \sqrt{A_1^2 + A_2^2 + A_3^2}$. Clearly then, A follows the χ distribution with three degrees of freedom:

$$p_\chi(A, \sigma) = \sqrt{\frac{2}{\pi}} \frac{A^2}{\sigma^3} \exp\left(-\frac{A^2}{2\sigma^2}\right) \quad , \tag{4.18}$$

where $\sigma^2 = \sigma^2_{f_{NL}} + \sigma^2_G$. What we can do now is to test the analytic calculations for monopole and dipole power modulations using numerical realizations of CMB maps. In particular with (4.11) and (4.18) we predict the possible values of dipole and monopole modulations that could be present in the CMB. In particular in this formulas, the presence of $\sigma_{f_{NL}}$ plays the role of widen the distribution and makes the high value of the dipole modulation less anomalous respect to the Gaussian case. Clearly the values of $\sigma_{f_{NL}}$ depends on the particular f_{NL} and so equations (4.11) and (4.18). But we can turn over this view and use the observed asymmetry for parameter estimation of the non-Gaussian amplitude f_{NL}. Combining f_{NL} constraints from bispectrum and power asymmetry measurements we can evaluate the Bayesian evidence for $f_{NL} \neq 0$. In our simple examples, the observed CMB power asymmetry only provides weak evidence for non-Gaussianity on large scales [1].

4.2 Beyond Local Model

Up to now we have discussed in detail the effect of local-type non-Gaussianity with constant f_{NL} that couples superhorizon modes across the CMB sky to the observable modes. Of course, the anomaly is found to be scale dependent and so this explanation can be considered only an approximation. In particular, to get closer to reality, both the amplitude of the asymmetry and the amplitude of non-Gaussianity must sharply decrease on smaller scales.
A more detailed study is necessary for scale dependent modulations, as the asymmetry we want to reproduce is strongly scale dependent. To address the question of how to generate a scale dependent modulations, we will first demonstrate that such modulations are a generic feature of non-Gaussian models other than the local model. In fact one can easily extend the inhomogeneous power spectrum calculation in the presence of local non-Gaussianity to other bispectrum shapes. We can start considering that a Fourier mode of the Bardeen potential is given by [1]:

$$\Phi(\vec{k}) = \phi(\vec{k}) + \frac{f_{NL}}{2} \int \frac{d^3\vec{q}_1}{(2\pi)^3} \int d^3\vec{q}_2 \phi(\vec{q}_1)\phi(\vec{q}_2) N_2(\vec{q}_1, \vec{q}_2, \vec{k}) \delta(\vec{k} - \vec{q}_1 - \vec{q}_2) + \dots \quad ,$$

where, as before, ϕ is a Gaussian field. The kernel N_2 can be chosen to generate any desired bispectrum and the dots represent higher order terms in powers of ϕ. Considering only the generic quadratic term, the power spectrum in sub-volumes can be computed as in the case of the local bispectrum, and we get:

$$P_{\Phi,S}(k, \vec{x}) = P_\phi(k) \left[1 + 2f_{NL} \int \frac{d^3\vec{k}_l}{(2\pi)^3} \phi(\vec{k}_l) N_2(\vec{k}_l, -\vec{k}, \vec{k}) e^{\imath \vec{k}_l \cdot \vec{x}}\right] \quad .$$

From the form of the above equation, one can see that the k dependence of the kernel N_2 is carried by the modulated component of the power spectrum in the small volume. As an example, the kernels for local, equilateral and orthogonal bispectrum templates are [1], [18]:

$$\begin{aligned} N_2^{local} &= 2 \quad ; \\ N_2^{ortho} &= \frac{4k_l^2 - 2kk_l}{k^2} \quad ; \\ N_2^{equil} &= 2\frac{k_l^2}{k^2} \quad . \end{aligned} \tag{4.19}$$

To match the observed scale dependence of the power asymmetry anomaly, the strength of coupling of subhorizon modes to the long wavelength background must be scale dependent. The relevant scale dependence in this context can be fully parametrized by introducing two bispectral indices that capture the scale dependence in our observable volume and a more general coupling strength to the long wavelength modes:

$$P_{\Phi,S}(k,\vec{x}) = P_\phi(k)\left[1 + 4f_{NL}(k_0)\left(\frac{k}{k_0}\right)^{n_f}\int\frac{d^3\vec{k}_l}{(2\pi)^3}\left(\frac{k}{k_0}\right)^{\alpha}\phi(\vec{k}_l)e^{\imath\vec{k}_l\cdot\vec{x}}\right] \quad .$$

Here $n_f < 0$ turns off any power asymmetries on shorter scales. The parameter $\alpha < 0$ enhances the sensitivity of the model to infrared modes. In principle, additional data could eventually constrain all of the parameters introduced above. However, here we will restrict our attention to the case with just one additional parameter. We take a local-shape bispectrum with an amplitude that depends on the scale of the short wavelength mode as:

$$f_{NL}(k) = f_{NL}^0\left(\frac{k}{k_0}\right)^{n_{f_{NL}}} \quad .$$

In terms of the parameters in the previous equation $n_f = n_{f_{NL}}$ and $\alpha = 0$.
When local-type non-Gaussianity has a scale-independent amplitude, the power spectrum amplitude is modulated similarly at all scales, thereby making the effect unobservable. However, the scale-dependent case is more interesting as it makes the power modulation scale dependent and therefore an observable effect. This can be easily seen by generalizing equation (4.8), for a scale dependent modulation:

$$P_\Phi(k) = P_\phi(k)\left[1 + f_{NL}(k)\frac{g_{00}}{2\sqrt{\pi}}\right] \quad .$$

As previously discussed, g_{00} is normally distributed with zero mean and the variance $< g_{00}^2 >$ requires a cutoff to limit contributions from arbitrarily large modes.
For scale-dependent non-Gaussianity, there is a scale-dependent power modulation, which can generically be interpreted as shifting the spectral index in the small volume away from the mean value in the large volume.

In this paper we have seen how the presence of non Gaussian primordial perturbations that exist, with no special features, on scales larger than those we can observe, modulate the power spectrum observed in sub-volume. We have seen different modulations that can be either scale independent or dependent. In the case of a scale dependent non-Gaussianity of the localtype this is sufficient to generate a dipole power asymmetry at large scales without invoking exotic superhorizon fluctuations. In particular the model considers only the second order term f_{NL} as primordial non Gaussianity, neglecting possible higher order. The value for f_{NL} chosen to make the observed power asymmetry no longer anomalous is consistent with the rather weak large-scale constraint, but is well above the scale-independent bound [1].
At this point one can wonder what are the effects of possible higher order modulations. To address this question we follow Adhikari, who showed that large scale power asymmetry may arise in models with local trispectra with strong scale dependent τ_{NL} amplitude [2]. So in the next section we review in detail his paper.

4.3 Hemispherical power asymmetry from scale dependent tri-spectrum

We have just seen the general relationship between statistical anisotropies observed in a finite volume when the curvature fluctuations on larger scale are coupled to those on smaller scales [2]. Moreover there has been a lot of efforts to model the observed statistical anisotropies in the CMB. The models can roughly can be categorized into two groups in which:

1. There is an explicit breaking of statistical isotropy , which means a preferred direction in the Universe.

2. The statistical isotropy breaking is spontaneous due to some stochastic modulating field or primordial non-Gaussianity.

In his work [2], Adhikari, uses a framework where the observed power asymmetry arises spontaneously as the result of looking at a sub-volume of a larger space whose fluctuations are described by isotropic but non-Gaussian statistics. In a non-Gaussian model, the dipolar modulation of the Fourier space two-point function is described by the collapsed limit of the Fourier space four-point function (the trispectrum) of primordial fluctuations.
Moreover he studied the effect of a scale-dependent trispectrum in the CMB fluctuations by calculating the induced nonGaussian covariance of modulation estimators. This approach is similar to the previous one, in fact previously we have seen the link between the presence in the variance of a non Gaussian modulation and the resulting less anomaly of a presence of a power modulation in the CMB. Still this approach is more general, since such formalism allows to simultaneously consider the effect on the modulations expected in the CMB polarization and forecast the improvement in trispectrum constraints when adding polarization data. To see that this formalism, namely a modulation from a scale-dependent primordial trispectrum, can explain the hemispherical power asymmetry we start from the multipole moments of temperature or polarization fluctuations, which depend on the primordial potential $\Phi(\vec{k})$ as follows:

$$a^x_{lm} = 4\pi(-\imath)^l \int \frac{d^3\vec{k}}{(2\pi)^3}\Phi(\vec{k})g^x_l(k)Y^*_{lm}(\hat{k}) \quad , \tag{4.20}$$

where $g^x_l(k)$ is the CMB transfer function with $x = T, E$ describing temperature and E-mode polarization fluctuations respectively.
Here we will write down the general expressions for the covariances of modulation estimators in the presence of a trispectrum. To see this we start defining the dipole modulation estimators using l, $l+1$ correlations as follows:

$$\Delta\hat{X}^{wx}_0(l) = \frac{1}{(2l+1)\sqrt{C^{ww}_l C^{xx}_{l+1}}}\sum_{m=-l}^{l} a^{w*}_{lm}a^x_{l+1,m} \quad ; \tag{4.21}$$

$$\Delta\hat{X}^{wx}_1(l) = \frac{1}{(2l+1)\sqrt{C^{ww}_l C^{xx}_{l+1}}}\sum_{m=-l}^{l} a^{w*}_{lm}a^x_{l+1,m+1} \quad ; \tag{4.22}$$

where w, x can be either T, E and C_ls are the CMB angular power spectrum of the best-fit cosmology. Similar estimators can be defined for higher-order modulations, by considering l, $l+2$ correlations: for example, for quadrupolar modulation. If the primordial fluctuations are Gaussian, the covariance of the dipole modulation estimators is given by:

$$< \Delta\hat{X}_M^{wx*}(l)\Delta\hat{X}_{M'}^{yz}(l') >_G = \frac{\delta_{M,M'}\delta_{l,l'}}{2l+1}\frac{C_l^{wy}C_{l+1}^{xz}}{\sqrt{C_l^{ww}C_{l+1}^{xx}C_{l'}^{yy}C_{l'+1}^{zz}}} \quad , \tag{4.23}$$

where $M, M' = 0, 1$. The two estimators $\Delta\hat{X}_0(l)$ and $\Delta\hat{X}_1(l)$ contain three degree of freedom that determine the amplitude and direction of the dipole modulation. The estimator $\Delta\hat{X}_M$ measures a possible dipole modulation present in the primordial power spectrum in models that generate a CMB power asymmetry by explicitly changing the power spectrum. In this case the means of $\Delta\hat{X}_M$ are non-zero: $< \Delta\hat{X}_M(l) > \neq 0$.
The situation is different in models where the primordial fluctuations have significant non-Gaussianity. In this case it is possible that global isotropy is respected, i.e. $< \Delta\hat{X}_M(l) > = 0$, but the expected cosmic variance of CMB dipolar modulation increases. In fact the increment is due to the presence of the non Gaussian contribution.
We can calculate this contribution to the covariance that depends on a particular configuration of the CMB trispectrum:

$$< \Delta\hat{X}_M^{wx*}(l)\Delta\hat{X}_{M'}^{yz}(l') >_{nG} = \delta_{M,M'}\frac{\sum_{m,m'} < a_{lm}^w a_{l+1,m+M}^{x*} a_{l'm'}^{y*} a_{l'+1,m'+M'}^z >_c}{(2l+1)(2l'+1)\sqrt{C_l^{ww}C_{l+1}^{xx}C_{l'}^{yy}C_{l'+1}^{zz}}} \quad , \tag{4.24}$$

where the subscript c indicates connected part of the trispectrum. To compute the CMB four-point function we follow the method in W. Hu (2001) [5], [6], which constructs the CMB trispectrum from a "reduced trispectrum" that automatically enforces the trispectrum to have rotation, parity and permutation symmetries. The CMB four-point function can be written using Wigner-3j symbols, as:

$$\begin{aligned} < a_{l1m1}^w a_{l2m2}^x a_{l3m3}^y a_{l4m4}^z > = \sum_{LM} P_{yl_3zl_4}^{wl_1xl_2}(L) \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \times \\ \times \begin{pmatrix} l_3 & l_4 & L \\ m_3 & m_4 & M \end{pmatrix} (-1)^M + (l_2 \leftrightarrow l_3) + (l_2 \leftrightarrow l_4) \end{aligned} \tag{4.25}$$

where:

$$\begin{aligned} P_{yl_3zl_4}^{wl_1xl_2}(L) =& \mathcal{T}_{yl_3zl_4}^{wl_1xl_2}(L) + (-1)^{L+l_1+l_2}\mathcal{T}_{yl_3zl_4}^{xl_2wl_1}(L) + (-1)^{L+l_3+l_4}\mathcal{T}_{zl_4yl_3}^{wl_1xl_2}(L) + \\ &+ (-1)^{l_1+l_2+l_3+l_4}\mathcal{T}_{zl_4yl_3}^{xl_2wl_1}(L) \quad . \end{aligned} \tag{4.26}$$

The reduced CMB trispectrum $\mathcal{T}$ depends on the model of primordial trispectrum. In this work, we will consider a scale-dependent local τ_{NL} trispectrum:

[7]

$$T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \tau_{NL} \left(\frac{k_2 k_4}{k_p^2}\right)^n P(k_1)P(k_3)P(|\vec{k}_1 - \vec{k}_2|) + \text{permutations} \quad ,$$

where the index n describes the scale dependence of the trispectrum amplitude of the otherwise local-type trispectrum, and k_p is the pivot at which the amplitude is τ_{NL}; we take $k_p = 0.05 Mpc^{-1}$. Similar to the calculation for the constant τ_{NL} trispectrum, we obtain [6]:

$$\mathcal{T}^{wl_1xl_2}_{yl_3zl_4}(L) = \tau_{NL} h_{l_1 l_2 L} h_{l_3 l_4 L} \int dr_1 r_1^2 \alpha^w_{l_1}(r_1, n) \beta^x_{l_2}(r_1) \int dr_2 r_2^2 \alpha^y_{l_3}(r_2, n) \times \times \beta^z_{l_4}(r_2) F_L(r_1, r_2) \quad (4.27)$$

where:

$$h_{l_1 l_2 L} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \quad . \quad (4.28)$$

We define:

$$\alpha^w_l(r, n) = \frac{2}{\pi} \int dk k^2 \left(\frac{k}{k_p}\right)^n g^w_l(k) j_l(kr) \quad ; \quad (4.29)$$

$$\beta^x_l(r) = 4\pi \int \frac{dk}{k} P(k) g^w_l(k) j_l(kr) \quad ; \quad (4.30)$$

$$F_L(r_1, r_2) = 4\pi \int \frac{dK}{K} P(K) j_L(K r_1) j_L(K r_2) \quad , \quad (4.31)$$

and the j_l are spherical Bessel functions. Note that the angular power spectrum can be written as an integral over the comoving distance r, using the quantities $\alpha_l(r)$ and $\beta_l(r)$ defined above:

$$C^{wx}_l = \int dr r^2 \alpha^w_l(r, 0) \beta^x_l(r) \quad . \quad (4.32)$$

Adhikari numerically evaluated the reduced trispectrum of equation (4.27) allowing him to compute the non-Gaussian covariances, for the dipole modulation estimators [2]. The full covariance matrix for dipole modulation estimators (including the f_{sky} scaling for partial sky coverage and the noise power spectra) is given by:

$$\begin{aligned} \mathbf{C} =& < \Delta \hat{X}^{wx*}_M(l) \Delta \hat{X}^{yz}_{M'}(l') > \\ =& \frac{1}{(2l+1) f_{sky}} \frac{\delta_{M,M'}}{\sqrt{C^{ww}_l C^{xx}_{l+1} C^{yy}_{l'} C^{zz}_{l'+1}}} \Bigg[\delta_{l,l'} \tilde{C}^{wy}_l \tilde{C}^{xz}_{l+1} + \\ &+ \frac{1}{2l'+1} \sum_{m,m'} < a^w_{lm} a^{x*}_{l+1,m+M} a^{y*}_{l'm'} a^z_{l'+1,m'+M'} >_c \Bigg] \quad , \end{aligned} \quad (4.33)$$

where w, x, y, z can be T, E, while $M, M' = 0, 1$ and $\tilde{C}^{wy}_l = C^{wy}_{l,cmb} + C^{wy}_{l,noise}$. The noise power spectrum for Planck is approximated using the specifications

for two channels as in S.Galli [21], with $f_{sky} = 0.65$. For numerical evaluations, Adhikari used camb A. Lewis [22] to obtain the transfer functions $g_l(k)$ using Planck 2015 best-fit cosmological parameters P.A.R. Ade [23].
In the rest of the work [2], Adhikari has numerically evaluated these covariances obtaining a fiducial set of scale dependent trispectrum parameters[3] that can explain the observed hemispherical power asymmetry at large scales and studied how including polarization and higher-order modulations could improve model constraints. Moreover he discussed the non-Gaussian covariance of angular power spectra generated by a scale-dependent primordial trispectrum and how it could bias the reconstruction of the spectral index of the power spectrum. So, by using a scale-dependent local-type trispectrum which has a large collapsed-limit signal i.e. in which long-wavelength modes are significantly coupled to small scale modes, the following primary results are obtained:

1. Two of the large-scale CMB temperature anomalies — the hemispherical power asymmetry and the power deficit at large scales — can be well-modeled by such model;

2. Such a trispectrum has other modulating effects on the temperature and polarization fluctuations that can be used to improve constraints on the scale-dependent trispectrum parameters;

3. If we require the trispectrum amplitude and parameters be large enough to explain both the hemispherical power asymmetry and the power deficit at large scales, then we find that the non-Gaussian covariance between the measured angular power spectra of the CMB can be large enough to significantly bias the inference of cosmological parameters.

What we have seen so far, is that in both papers of Adhikari [1], [2], the main idea to explain the hemispherical power asymmetry is to use the non-Gaussian landscape picture, namely the influence of long-wavelength modes to the observed statistic in sub-volumes. In fact in both articles these modes are used to modulate the variance of the expected dipole modulation making less anomalous the presence of a power asymmetry in the CMB data. Nevertheless both papers don't address the issue of an inflationary model that can explain the power asymmetry, in fact in both articles they assume the form of the Bardeen potential or directly a particular trispectrum. But if we would find an inflationary model that gives us a particular form of the Bardeen potential or a particular trispectrum we will be able to predict, thanks to the acquired skills, the most probable value of the power modulation expected in a random patch of our universe, with typical size our Hubble volume. This give us a complete and robust recipe to tackle the problem of the hemispherical power asymmetry. So, now what we want to do is try to see if there could be some inflationary model that can give us the trispectrum used by Adhikari [2]. The choice of the model is not simple, due to both the presence of a plethora of models in literature, and the fact that is difficult to find a one to one connection between

[3]Fiducial parameters $\tau_{NL} = 2 \cdot 10^4$ and $n = -0.68$.

the previous trispectrum and a class of models that can generate it.
We will be obliged to choose a particular model, as a general calculation is a hard obstacle; hence is useful to review some papers where an inflationary model is used to explain the power asymmetry. In this way we will start learning which models can work and which fail in reproducing the power asymmetry. Fundamental works on this arguments were made by Byrnes [8], [9], [11].

4.4 Response function and OPE expansion

In the letters [8] and [9] Byrnes reports results for a special set of inflationary scenarios which can accommodate the hemispherical asymmetry. Byrnes starts from the work of Aiola et al.[3], who demonstrated that the asymmetry could be approximately fit by a position-dependent power-spectrum at the last-scattering surface of the form:

$$\mathcal{P}^{obs}(k) = \frac{k^3 P(k)}{2\pi^2} \left(1 + 2A(k)\hat{p}\cdot\hat{n}\right) \quad , \tag{4.34}$$

where $\hat{p}$ represents the direction of maximal asymmetry, $\hat{n}$ is the line-of-sight from Earth, and $A(k)$ is an amplitude which Aiola et al. [3] found to scale roughly like $k^{-0.5}$. Byrnes tries to see how an inflationary model can produce an asymmetry which replicates this scale dependence. Furthermore in his works [8] and [9], Byrnes, choosing a primordial bispectrum which is compatible with the modulation $A(k)$, provides an analysis of the CMB temperature bispectrum generated by this scale- and shape-dependent primordial bispectrum. To do this, he introduced an explicit model which can be contrived to match all current observations and also serves as an useful example showing the complications which are encountered. To learn how we can generate the asymmetry, we start denoting the field with scale-dependent fluctuations by σ, and take it to substantially dominate the bispectrum for the observable curvature perturbation ζ. The ζ two-point function can depend on σ, or alternatively on any combination of σ and other Gaussian fields. The question to be resolved is how $P(k)$, the power spectrum of curvature perturbations, responds to a long-wavelength background of σ modes denoted $\delta\sigma(\vec{x})$.
We can answer to the previous question thanks to the fact that this response can be computed using the operator product expansion ("OPE") [8], and is expressed in terms of the ensemble averaged two- and three-point functions of the inflationary model. We focus on models in which the primary effect is due to the amplitude of the long-wavelength background rather than its gradients. Since the perturbation is small it is possible to write:

$$P(k,\vec{x}) = P(k)\left(1 + \delta\sigma(\vec{x})\rho_\sigma(k) + \dots\right) \quad . \tag{4.35}$$

We call $\rho_\sigma(k)$ the "response function". The OPE gives [8] :

$$\rho_\sigma(k) \simeq \frac{1}{P(k)} \left[\Sigma^{-1}(k_L)\right]_{\sigma\lambda} B^\lambda(k,k,k_L) \quad \text{if} \quad k >> k_L \quad , \tag{4.36}$$

where a sum over λ is implied, and $\Sigma^{\alpha\beta}$ and B^β are spectral functions for certain mixed two- and three point correlators of ζ with the light fields of the inflationary model, which we collectively denote $\delta\phi^\alpha$[4]:

$$< \delta\phi^\alpha(\vec{k}_1)\delta\phi^\beta(\vec{k}_2) > = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)\Sigma^{\alpha\beta} \quad ;$$
$$< \delta\phi^\alpha(\vec{k}_1\zeta(\vec{k}_2)\zeta(\vec{k}_3) > = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)B^\alpha \quad .$$

[4]In our framework the sum in (4.36) includes only the σ fields.

In the special case of a slow-roll model in which a single field generates all perturbations, it can be shown that the right-hand side of (4.36) is related to the reduced bispectrum:

$$\rho_\sigma(k) = \frac{12}{5} f_{NL}(k, k, k_L) \qquad k >> k_L \quad ,$$

where f_{NL} is defined by:

$$\frac{6}{5} f_{NL}(k_1, k_2, k_3) = \frac{B(k_1, k_2, k_3)}{P(k_1)P(k_2) + 2\text{cyclic perms}} \quad .$$

To proceed further, in (4.36) appear long wavelength background modes that we model in this way [9]:

$$\delta\sigma(\vec{x}) \approx E\mathcal{P}_\sigma^{\frac{1}{2}}(k_L) \cos\left(\vec{k}_L \cdot \vec{x} + \theta\right) \quad ,$$

where E labels the 'exceptionality' of the amplitude, with $E = 1$ being typical and $E >> 1$ being substantially larger than typical. We assume the wavenumber k_L to be fixed. The phase θ will vary between realizations, and the Earth is located at $\vec{x} = 0$.

The last-scattering surface is at comoving radius x_{ls}; evaluating (4.36) on this surface at physical location $\vec{x} = x_{ls}\hat{n}$ and assuming $\alpha = \frac{x_{ls} k_L}{2\pi} < 1$ so that the wavelength associated with k_L is somewhat larger than x_{ls}, we get:

$$P(k, \vec{x}) = P(k) \left(1 - C(k) + 2A(k) \frac{\vec{x} \cdot \hat{k}_L}{x_{ls}} + \ldots \right) \quad . \tag{4.37}$$

To obtain the last formula, we expand the argument of the back-ground modes, since by assumptions the argument is less than 1.

The quantities $A(k)$ and $C(k)$ are determined in terms of the response ρ_σ and long-wavelength back-ground by:

$$A(k) = \pi\alpha E \mathcal{P}_\sigma^{\frac{1}{2}}(k_L) \rho_\sigma(k) \sin\theta \quad ;$$
$$C(k) = -A(k) \frac{\cos\theta}{\pi\alpha \sin\theta} \quad .$$

Both A and C share the same scale-dependence, so it is possible to use $C(k)$ to explain the lack of power on large scales. If so, the model could simultaneously explain two anomalies—although this would entail a stringent constraint on α in order that $C(k)$ does not depress the power spectrum too strongly at small l. After having presented the general formulas for the determination of the response function and, thanks to this, the prediction of the particular scale dependence of the modulation $A(k)$ it's time now to construct explicitly some inflationary models that, due to the fact that have a large and negative η-parameter, can generate a scale dependent modulation that match the scale dependence found by Aiola et al. Starting with the single field model (we have that when one field dominates the two- and three-point functions of ζ) the bispectrum is equal in squeezed and equilateral configurations [24]. Therefore:

$$\rho_\sigma = \frac{12}{5} f_{NL}(k, k, k_L) = \frac{12}{5} f_{NL}(k, k, k) \quad ,$$

and since the scale dependence of the asymmetry is determined by the scale dependence of the response function we have in this case that the asymmetry scales in the same way as the equilateral configuration $f_{NL}(k,k,k)$. If the scaling is not too large it can be computed using [7]:

$$\frac{d\ln|f_{NL}|}{d\ln k} = \frac{5}{6f_{NL}}\sqrt{\frac{r}{8}}\frac{M_P^3 V'''}{3H^2} \quad ,$$

where $r < 0.1$ is the tensor-to-scalar ratio. To achieve strong scaling we require $\frac{m_P^3 V'''}{3H^2} >> 1$. But within a few e-foldings this will typically generate an unacceptably large second slow-roll parameter η_σ, defined by:

$$\eta_\sigma = \frac{M_P^2 V''}{3H^2} \quad .$$

Therefore it will spoil the observed near scale-invariance of the power spectrum. As a specific example, a self-interacting curvaton model was studied in [11]. This gave rise to many difficulties, including logarithmic running of $f_{NL}(k,k,k)$ with k, which is not an acceptable fit to the scale dependence of $A(k)$. Even worse, because the scale dependence of f_{NL} is large only when f_{NL} is suppressed below its natural value, both the trispectrum amplitude g_{NL} and the quadrupolar modulation of the power spectrum were unacceptable. So in conclusion, the usual models of inflation with a single fields are not good to reproduce the hemispherical asymmetry, at least with this approach, due to the many difficulties cited above. We must therefore think of other possible scenarios in which the asymmetry can be reproduced. The immediate extension is to study multi-field models.

In multiple-source scenarios there is more flexibility. If different fields contribute to the power spectrum and bispectrum it need not happen that a large η_σ necessarily spoils scale-invariance. In these scenarios ρ_σ no longer scales like the reduced bispectrum, but rather like its square-root. Therefore:

$$\frac{d\ln A}{d\ln k} \approx \frac{1}{2}\frac{d\ln|f_{NL}(k,k,k)|}{d\ln k} \approx \frac{d\ln\left(\frac{\mathcal{P}_\sigma}{\mathcal{P}}\right)}{d\ln k} \approx 2\eta_\sigma - (n_s - 1) \quad , \tag{4.38}$$

where $\mathcal{P}$ is the dimensionless power spectrum, $n_s - 1 \simeq -0.03$ is the observed scalar spectral index and η_σ was defined previously. If we can achieve a constant $\eta_\sigma \approx -0.25$ while observable scales are leaving the horizon, then it is possible to produce an acceptable power-law for $A(k)$, further references [8], [29]. A simple potential with large constant η_σ is:

$$W(\phi,\sigma) = V(\phi)\left(1 - \frac{1}{2}\frac{m_\sigma^2\sigma^2}{M_P^4}\right) \quad .$$

The inflaton ϕ is taken to dominate the energy density and therefore drives the inflationary phase. Initially σ lies near the hilltop at $\sigma = 0$, so its kinetic energy is subdominant and $\epsilon \approx M_P^2 V_\phi^2/V^2$; here ϵ is the conventional slow-roll parameter. As inflation proceeds σ will roll down the hill satisfying the following equation:

$$\frac{d\sigma(N)}{dN} = -\frac{W_\sigma}{3H^2} = \frac{1}{3H^2}V(\phi)\frac{\sigma m_\sigma^2}{M_P^4} \quad .$$

On the right hand side, a part for the σ field, we recognize the η_σ parameter. The integration of the above equation yields:

$$\sigma(N) = \sigma_* e^{-\eta_\sigma N} \quad ,$$

where '*' denotes evaluation at the initial time and N measures the number of subsequent e-folds. To keep the σ energy density subdominant we must prevent it rolling to large field values, which implies that σ_* must be chosen to be very close to the hilltop. But the initial condition must also lie outside the diffusion dominated regime, meaning the classical rolling should be substantially larger than quantum fluctuations in σ. This requires $|\frac{d\sigma}{dN}| >> \frac{H_*}{2\pi}$. In combination with the requirement that σ remains subdominant in the observed power spectrum, we find that σ_* should be chosen so that $|\sigma_*| > \sqrt{\epsilon_* \mathcal{P}} M_P / |\eta_\sigma|$.
For typical values of $\epsilon = 10^{-2}$ and $\eta_\sigma = -0.25$ this requires $|\sigma(60)| > 100 M_P$ which is much too large. The problem can be ameliorated by reducing ϵ_*, but then σ contributes significantly to ϵ during the inflationary period. This reduces the bispectrum amplitude to a tiny value, or causes σ to contaminate the power spectrum and spoil its scale invariance [8]. To avoid these problems, we consider a potential in which the effective mass of the σ field makes a rapid transition. An example is:

$$W = W_0 \left(1 + \frac{\eta_\phi}{2} \frac{\phi^2}{M_P^2}\right) \left(1 + \frac{\eta_\sigma(N)}{2} \frac{\sigma^2}{M_P^2}\right) \quad ,$$

where $\eta_\sigma(N)$ is chosen to be -0.25 while observable scales exit the horizon, later running rapidly to settle near -0.08. We take the transition to occur roughly 16 e-folds after the largest observable scales exited the horizon [8]. The field ϕ will dominate the Gaussian part of ζ and its mass should be chosen to match the observed spectral index.
The most urgent question is whether the bispectrum amplitude is compatible with present constraints for f_{NL}^{local},f_{NL}^{equi}, which are weighted averages over the bispectrum amplitude on groups of related configurations. At present, the strongest constraints apply to f_{NL}^{local} which averages over modestly squeezed configurations. To determine the response of these estimators Byrnes constructs a Fisher estimate [8]. He numerically computes $\sim 5 \times 10^6$ bispectrum configurations for the above potential covering the range from $l \sim 1$ to $l \sim 7000$ and uses these to predict the observed angular temperature bispectrum. For a choice of parameter values which generate the correct amplitude and scaling of $A(k)$, he finds that a Planck-like experiment would measure order-unity values,

$$\hat{f}_{NL}^{local} = 0.25 \quad ; \qquad \hat{f}_{NL}^{equi} = 0.6 \quad ; \qquad \hat{f}_{NL}^{ortho} = -1.0 \quad .$$

These estimates are one to two orders of magnitude smaller than possible estimates based on the EKC mechanism [17] and are easily compatible with present-day constraints. We presented this works because is a viable scenario for which seek an inflationary explanation of the asymmetry. In fact in this paper much less tension with observation than would be expected on the basis of EKC mechanism is involved, also the difficulty that one can have in

constructing a model in agreement with the stringent bound from Planck are caught. In fact, to build a successful model we have been forced to make a number of arbitrary choices, including the initial and final values of the σ mass, and the location and rapidity of the transition. Anyway the line of the thesis is to use the stochastic approach to inflation and an implementation of the η procedure is not simple in this framework, so we do not pursue in the following this scenario. From this paper we learn that a single field model of inflation is not capable to reproduce the correct power asymmetry, so we are forced to use multi-field scenario. In particular using multi-field scenario we evade another very important problem. In fact in the adiabatic single field models it is well known that the long-wavelength contributions amount to simply shifting the local time coordinate. The situation is different if more than one light dynamical scalar is present during inflation and in this case the long-wavelength modes are no more gauge modes and the previous problem is not present.
So the next step is to study the properties of a multi-fields system, the simple one consist in two fields, in stochastic inflation. In the next chapter we explain the main properties of the stochastic approach to inflation and present an article with whom we try our-self with a two fields model in this scenario [13].

Chapter 5

Stochastic Inflation

The simpler models of inflation are implemented through the use of a scalar field whose dynamics is governed by a classical Klein-Gordon equation [20]. This field is moving on a expanding back-ground field metric that we choose in this case to be a FRLW metric with a De Sitter expansion.
Not only the classical evolution is important but also the quantum nature of the scalar field leads to very interesting physics and the stochastic approach is the natural framework to study it. The approach consists in splitting the scalar field in momentum space into long and short wavelength modes.
Starting from the Heisenberg operator equation of motion for the scalar field, the evolution of the long wavelength part satisfies a classical, but stochastic, equation of motion. The quantum effects, in the form of short wavelength modes, build up a noise term, as we will explain below. The field satisfies:

$$\nabla_\mu \nabla^\mu \phi + \frac{\partial V}{\partial \phi} = 0 \quad .$$

The scalar field may be written $\phi = \phi_{long} + \phi_{short}$, or more explicitly:

$$\phi = \phi_{long}(\vec{x},t) + \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \theta(k - \epsilon a H) \left(a_k \phi_k(t) e^{-\imath \vec{k}\cdot\vec{x}} + a_k^\dagger \phi_k^*(t) e^{\imath \vec{k}\cdot\vec{x}} \right) \quad , \tag{5.1}$$

where $\phi_{long}(\vec{x},t)$ contains only modes such that $k << aH$; $a_k^\dagger$, a_k are the usual creation and annihilation operators, ϵ is a constant much smaller than 1, and θ stands for the step function.
Usually inflation takes place in regions where the scalar field potential is not very steep and thus the short wavelength part satisfies the mass less equation:

$$\nabla_\mu \nabla^\mu \phi_{short} = 0 \quad .$$

Thus the short wavelength modes can be taken as:

$$\phi_k = \frac{H}{\sqrt{2k}} \left(\frac{1}{aH} + \frac{\imath}{k} \right) \exp\left(\frac{\imath k}{aH} \right) \quad .$$

From this, Starobinskii derives the equation of motion for the long wavelength "coarse-grained" part which, in the slow-rolling approximation, reads:

$$\dot{\phi}_{long}(\vec{x},t) = -\frac{1}{3H} \frac{\partial V}{\partial \phi_{long}} + f(\vec{x},t) \quad , \tag{5.2}$$

where the spatial gradient of ϕ_{long} has been neglected since it is sub-dominat for modes $k << aH$ and hence the evolution of the coarse-grained field can be followed in each domain independently [14], [15] where:

$$f(\vec{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \delta(k-k_s)\dot{k}_s \left(a_k \phi_k(t) e^{-\imath \vec{k}\cdot\vec{x}} + a_k^\dagger \phi_k^*(t) e^{\imath \vec{k}\cdot\vec{x}} \right) \quad , \tag{5.3}$$

where $k_s = \epsilon a H$ stands for the inverse of the coarse-grained domain radius.
According to this picture the universe is described by the value of the field in different coarse-grained domains of comoving size $k_s^{-1} \simeq (\epsilon a_1 H_1)^{-1}$.
With the inflationary expansion, an initial domain gets divided into sub domains, after one Hubble time-step. The magnitude of the coarse-grained field ϕ in these smaller domains is determined by both the classical and the noise parts appearing in the right-hand side of equation (5.2). The classical force "pushes" the field down the potential in all the subdomains, whereas the stochastic force acts with different strengths and arbitrary direction in different subdomains. This process repeats itself in all subsequent Hubble time-steps.
The results of this process is that in the new subdomains the field takes on different values: in those where the classical "drag" overwhelms the stochastic "push" the field goes down the potential and inflation will eventually end, yielding a domain like our present universe.
However, in those domains where the opposite happens there will be regions where the stochastic force is greater than the classical one. In this case we can obtain the so called Linde's eternal inflation picture, for more information [27].

A very crucial step in order to be able to obtain the correlation function in the stochastic approach, consists in calculate the auto-correlation function for the noise term [12]. The auto-correlation is:

$$\begin{aligned} < f(\vec{x},t) f(\vec{y},t') >= \frac{1}{(2\pi)^3} \int d^3k \int d^3p \delta(p-k_s(t'))\dot{k}_s(t')\delta(k-k_s(t)) \times \\ \times \dot{k}_s(t) < \left(a_k \phi_k(t) e^{-\imath \vec{k}\cdot\vec{x}} + a_k^\dagger \phi_k^*(t) e^{\imath \vec{k}\cdot\vec{x}} \right) \left(a_p \phi_p(t') e^{-\imath \vec{p}\cdot\vec{y}} + a_p^\dagger \phi_p^*(t') e^{\imath \vec{p}\cdot\vec{y}} \right) > \quad , \end{aligned} \tag{5.4}$$

the only term that survive is $a_k, a_p^\dagger$, using the commutation relation for the annihilation operators this term gives a delta Dirac term $\delta(\vec{k}-\vec{p})$. Integrating in $\vec{p}$ we get:

$$\begin{aligned} < f(\vec{x},t) f(\vec{y},t') >= \frac{1}{(2\pi)^3} \int d^3k \delta(k-k_s(t'))\dot{k}_s(t')\delta(k-k_s(t))\dot{k}_s(t) \times \\ \times \phi_k(t)\phi_k^*(t') e^{\imath \vec{k}\cdot(\vec{x}-\vec{y})} \quad , \end{aligned} \tag{5.5}$$

applying the Dirac delta on itself we get $\delta(k_s(t)-k_s(t'))$. This term can be treated using the property of the Dirac delta and we get $\frac{\delta(t-t')}{\epsilon H^2 a}$. This implies, thanks to the delta, the fact that $t = t'$. Finally we obtain:

$$< f(\vec{x},t) f(\vec{y},t') >= \frac{1}{(2\pi)^3} \frac{\delta(t-t')}{\epsilon H^2 a} (\epsilon H \dot{a})^2 \int d^3k \delta(k-k_s(t)) |\phi_k(t)|^2 e^{\imath \vec{k}\cdot(\vec{x}-\vec{y})} \quad . \tag{5.6}$$

We used the form of the short-wave length to compute the modulus square. Finally we obtain [1]:

$$< f(\vec{x},t) f(\vec{y},t') >= \frac{H^3}{4\pi^2} j_0(k_s(t)|\vec{x}-\vec{y}|)\delta(t-t') \quad , \tag{5.7}$$

this is the form of the correlation in the noise for spatially separated points. To be noticed the white noise property in time. The previous relations are quite general, they hold for a general scalar field, minimally coupled to gravity, undergoing a slow rollover transition towards a state of lower energy. The choice for the scale factor is not crucial for the results. More general accelerated expansion laws can be considered; the previous equations will continue to hold provided that we include some numerical coefficients characteristic of the considered expansion [16].
If we are interested on one particular domain, say with comoving coordinate $\vec{x}$, then only the delta term remains and at this point one could attempt to study the probability distribution $P(\phi, t)$ for the coarse-grained field ϕ by means of the Fokker-Planck equation. To be complete, due to the time dependence of the Hubble parameter, it has been proposed that a more fundamental time variable, in the Langevin equation, is given by $\alpha \propto \ln(a) = \int dt H(t)$ [25].

After having introduced the main properties of the stochastic approach to inflation, namely the division of the field in short and long components, and the evolution of the latter through a stochastic equation of motion of the Langevin type where the noise term is made through the short part of the field. Clearly up to now, all arguments were made strictly apply to a single field model. When more than one fields are present, we have a system of Langevin equations that are coupled together, rather that two separated equations.
As we saw previously, a model of at least two fields is necessary if we want to reproduce the power asymmetry. So is natural, to move the first steps also with the stochastic approach, to review the paper of Matarrese [13], where he studied a two fields model of inflation in the stochastic approach. Even if in this paper, the Fokker-Planck equation is solved for a particular model that we will see, and Matarrese did not concentrate on the Langevin equation, we reviewed this paper because from this we can capture the difficulty in finding a general solution, and the potential used in this work is very similar to our, and we will understand why this happens.
As stated above in Matarrese's article they concentrate on the Fokker-Planck equation; in our work we will rather concentrate on spatial correlations which could be derived from the Langevin-type equation. For completeness one can study correlations between different coarse-grained domains using the functional form of the Fokker-Planck equation, even if this road is very complex and not very useful.

[1] We can neglect the ϵ term present in the final formula, since ϵ is small by assumption.

5.1 Stochastic inflation in a simple two-fields model

Inflationary models with more than one scalar fields have been studied with different aims in recent years. In the work [13] the model chosen consists in a two fields model where one is a free massless field and the inflaton has an exponential potential, therefore leading to power-law inflation. In spite of its simplicity, the model displays a number of interesting features, such as the multiplicative effects produced by the inflaton fluctuations on the motion of the massless field, which is expected to occur also in more complicated multiple-field models. Moreover the main statistical properties of the distribution at the different scales of interest are studied. The joint probability for the two fields is always found to be non-Gaussian; however, in order that the non-Gaussian features are quantitatively relevant it is necessary that the system starts its stochastic evolution from a state with an energy density comparable to the Planck one.
To see this interesting property of the model we can proceed as follow: for the multiplicative effects we have to derive the equations which govern the dynamics of a two scalar-fields system during inflation within the stochastic approach, specializing our analysis to the axion (massless field) dynamics in a power-law inflation. Then, for the non-Gaussian property of the model, we have to solve the Fokker-Planck equation which governs the evolution of the joint probability distribution for our two-field system. Starting from the multiplicative features, as we have seen previously for the evolution of a scalar field, either the inflaton or any other scalar field in the theory, is described by a Langevin-type equation for a coarse-grained variable. This is usually written in terms of the proper time t, but as previously mentioned a better time variable to use is given by $\alpha = \ln(a/a_0)$ [25]. In terms of the number of e-folding and considering the stochastic evolution inside a single coarse grained domain, we have the system:

$$\begin{aligned} \frac{d\phi}{d\alpha} &= -\frac{\partial_\phi V}{3H^2} + \frac{H}{2\pi}\eta_\phi(\alpha) \;\; ; \\ \frac{d\chi}{d\alpha} &= -\frac{\partial_\chi V}{3H^2} + \frac{H}{2\pi}\eta_\chi(\alpha) \;\; ; \end{aligned} \tag{5.8}$$

where $H^2 = (8\pi/3m_P^2)V(\phi,\chi)$ with m_P the Planck mass; η_ϕ and η_χ are Gaussian noises with zero mean and correlation functions $<\eta_\phi(\alpha)\eta_\phi(\alpha')>= \delta(\alpha-\alpha')$ equal for χ, as previously seen for a single coarse domain.
Moreover the two noise terms are assumed to be an-correlated. The two Langevin equations are said to be of multiplicative type since the coefficients of the noise terms depend on the random variable itself. The previous system can be rewitten in the following way:

$$\begin{aligned} \frac{\partial\phi}{\partial\tau} &= -f_\phi(\phi,\chi) + g_\phi(\phi,\chi)\eta_\phi(\tau) \;\; ; \\ \frac{\partial\chi}{\partial\tau} &= -f_\chi(\phi,\chi) + g_\chi(\phi,\chi)\eta_\chi(\tau) \;\; ; \end{aligned} \tag{5.9}$$

where $\tau = t, \alpha$ and the f terms denote the classical force and the g terms the amplitude of the stochastic noise. The 4 terms are given in the previous equation for the case $\tau = \alpha$ and change in the case $t = \tau$. From them we can derive the associated Fokker-Planck equation for the joint probability $\mathcal{P}_{\phi\chi}$. In the so called Stratonovich approach this is given by:

$$\frac{\partial \mathcal{P}_{\phi\chi}}{\partial \tau} = \frac{\partial}{\partial \phi}\left(f_\phi \mathcal{P}_{\phi\chi} + \frac{g_\phi}{2}\frac{\partial}{\partial \phi}(g_\phi \mathcal{P}_{\phi\chi})\right) + \frac{\partial}{\partial \chi}\left(f_\chi \mathcal{P}_{\phi\chi} + \frac{g_\chi}{2}\frac{\partial}{\partial \chi}(g_\chi \mathcal{P}_{\phi\chi})\right) \quad . \tag{5.10}$$

Note that there are no cross-derivative terms because the noise terms for ϕ and χ are statistically independent. The individual probabilities for ϕ and χ can be obtained from $\mathcal{P}_{\phi\chi}$ integrating it with respect to χ or ϕ, respectively.
In general, the Fokker-Planck equation for the joint probability is very difficult to solve for an arbitrary potential $V(\phi, \chi)$; the system of Langevin equations is also very hard to solve and in general the equations are coupled. However, both considerably simplify in particular cases. In our case we take the χ field to be a massless field whose contribution to the total energy density is negligible; in such a case H, the Hubble rate, is function of ϕ only. Moreover the classical force term for χ, f_χ, in the Langevin equations, vanishes and the remaining factors become only functions of ϕ. Thus, the Langevin equation for ϕ becomes χ independent. On the contrary, the diffusion coefficient for χ is a function of ϕ. This means that the equation for χ depends on the ϕ field. In other words the χ field is affected by the ϕ field but the contrary does not happen. In the Matarrese's paper the inflaton field ϕ has the following potential:

$$V(\phi) = M^4 \exp\left(-\frac{\lambda\phi}{\sigma}\right) \quad ,$$

where $\sigma = m_P/\sqrt{8\pi}$ and we have defined ϕ so that $\phi_f = 0$ at the end of inflation. As originally derived by Lucchin and Matarrese [28], this kind of potential leads to power-law inflation. The second field χ is taken to be a massless field during inflation. This model exactly describes the dynamics of the axion in a power-law inflation. Moreover, Matarrese et al. discuss the constraints that the isocurvature axion perturbation model imposes on the parameters of the theory. They find that for a set of parameters, this model is cosmologically interesting [13]. Without wasting time, we immediately focus on the equations to be solved for our system:

$$\begin{aligned} \frac{d\phi}{d\alpha} &= \lambda\sigma + \left(\frac{M^4}{4(3-\lambda^2/2)\pi^2\sigma^2}\right)^{1/2} e^{-(\lambda\phi/2\sigma)}\eta_\phi(\alpha) \quad ; \\ \frac{d\chi}{d\alpha} &= \left(\frac{M^4}{4(3-\lambda^2/2)\pi^2\sigma^2}\right)^{1/2} e^{-(\lambda\phi/2\sigma)}\eta_\chi(\alpha) \quad . \end{aligned} \tag{5.11}$$

It can also be useful to define the classical configuration, that obtained when the noise terms are set to zero:

$$\phi_{cl}(\alpha) = \lambda\sigma(\alpha - \alpha_f) \quad ; \qquad \chi_{cl}(\alpha) = \chi_0 \quad .$$

The system of equations can be solved exactly thanks to the following change of variable:

$$\phi \to \Phi = \frac{4\pi\sigma^2\sqrt{6-\lambda^2}}{M^2\lambda}\exp\left(\frac{\lambda\phi}{2\sigma} - \frac{\lambda^2\alpha}{2}\right) \quad ,$$

in fact, thanks to this the first equation of the previous system is reduced to a non multiplicative equation with vanishing force term and hence easily solvable. Moreover in this system of coordinate, namely (Φ, χ), performing the following change of time variable $\theta = \frac{1-e^{-(\lambda^2\alpha)}}{\lambda^2}$ and introducing the dimensionless variable $\zeta = \lambda\chi/2\sigma$ the Fokker-Planck equation can be solved exactly. In fact the equation reduces to:

$$\frac{\partial \mathcal{P}_{\Phi\zeta}}{\partial\theta} = \frac{\partial^2 \mathcal{P}_{\Phi\zeta}}{\partial\Phi^2} + \frac{1}{\Phi^2}\frac{\partial^2 \mathcal{P}_{\Phi\zeta}}{\partial\zeta^2} \quad . \tag{5.12}$$

Must be noticed that the ζ diffusion coefficient becomes singular as $\Phi \to 0$, this will cause the divergence of the statistical moments of the ζ or χ field, as we will see. The general solution of this equation is given by a linear superposition of modes of the type:

$$\mathcal{P}_{\Phi\zeta} = K(\Phi_0)\int_0^\infty \int_{-\infty}^{+\infty} dk \frac{d\mu}{2\pi} C(\mu) e^{(-\mu^2\theta)} e^{-\imath k(\zeta-\zeta_0)} \mathcal{F}_{k\mu}(\Phi) \quad ,$$

where $C(\mu)$ is an arbitrary function of μ that can be set using the initial condition and in the same way also the overall K constant. In particular $\mathcal{F}_{k\mu}(\Phi)$ satisfies the equation:

$$\frac{\partial^2 \mathcal{F}_{k\mu}(\Phi)}{\partial^2\Phi^2} + \left(\mu^2 - \frac{k^2}{\Phi^2}\right)\mathcal{F}_{k\mu} = 0 \quad ,$$

whose solution is $\mathcal{F}_{k\mu}(\Phi) \approx \sqrt{\Phi}\mathcal{C}_\nu(\mu\Phi)$, having denoted by $\mathcal{C}_\nu$ any set of solutions of the Bessel equation and $\nu = \pm\sqrt{k^2 + \frac{1}{4}}$. Among these solutions, we must choose those which satisfy the correct boundary conditions. The boundary condition have to be imposed at $\Phi = 0$ and our lack of knowledge of the physics at the Planck scale makes it hard to determine it. There are two possibility: first a reflecting boundary condition that preserve the over all normalization of the probability, the other possibility is an absorbing boundary condition which has a non vanishing probability flux toward the Planck energy. Between this two possibilities we choose the first one, the reflecting boundary condition. This corresponds to $\partial_\Phi \mathcal{P}_{\Phi\zeta}|_{\Phi=0} = 0$, which is satisfied only by the Bessel functions $J_\nu(\mu\Phi)$; with $\nu > 1/2$ or $\nu = -1/2$. The initial condition is $\mathcal{P}_{\Phi\zeta}(\theta = 0) = \delta(\Phi - \Phi_0)\delta(\zeta - \zeta_0)$, with this we set the constant of the linear super position. We obtain for the joint probability distribution:

$$\mathcal{P}_{\Phi\zeta} = \int_0^\infty \frac{d\mu}{2\pi} e^{-(\mu^2\theta)} \mu\sqrt{\Phi\Phi_0}\Bigg(\int_{-\infty}^\infty dk e^{-\imath k(\zeta-\zeta_0)} J_{\nu_k}(\mu\Phi_0) J_{\nu_k}(\mu\Phi) + \\ - \frac{1}{\delta(0)}\left[J_{1/2}(\mu\Phi_0)J_{1/2}(\mu\Phi) - J_{-1/2}(\mu\Phi_0)J_{-1/2}(\mu\Phi)\right]\Bigg) \quad , \tag{5.13}$$

where $\delta(0)$ formally is introduced because of the normalization and $\nu_k = \sqrt{k^2+1/4}$. The integration over μ can be performed [26]:

$$\mathcal{P}_{\Phi\zeta} = \frac{\sqrt{\Phi\Phi_0}}{4\pi\theta} \exp\left(-\frac{\Phi^2+\Phi_0^2}{4\theta}\right)\left\{\int_{-\infty}^{\infty} dk e^{-\imath k(\zeta-\zeta_0)} I_{\nu_k}\left(\frac{\Phi\Phi_0}{2\theta}\right) + \right. \\ \left. -\frac{1}{\delta(0)}\left[I_{1/2}\left(\frac{\Phi\Phi_0}{2\theta}\right) - I_{-1/2}\left(\frac{\Phi\Phi_0}{2\theta}\right)\right]\right\} \quad , \tag{5.14}$$

where I_{ν_k} denote the modified Bessel functions. It can be seen, by integrating the previous equation with respect to ζ , that the individual probability for Φ is [26]:

$$\mathcal{P}_{\Phi} = \frac{1}{\sqrt{4\pi\theta}}\left[\exp\left(-\frac{(\Phi-\Phi_0)^2}{4\theta}\right) + \exp\left(-\frac{(\Phi+\Phi_0)^2}{4\theta}\right)\right] \quad , \tag{5.15}$$

which corresponds to a Gaussian process with reflecting boundary condition at $\Phi = 0$, as expected. Of course, due to the nonlinear transformation from Φ back to the original field variable, the ϕ distribution is non-Gaussian. For the individual probability for ζ we have [26]:

$$\mathcal{P}_{\zeta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-\imath k(\zeta-\zeta_0)} \exp\left(-\frac{\Phi_0^2}{4\theta}\right)\left(\frac{\Phi_0^2}{4\theta}\right)^{1/4+\nu_k/2} \frac{\Gamma(3/4+\nu_k/2)}{\Gamma(1+\nu_k)} \times \\ \times M\left(\frac{3}{4}+\frac{\nu_k}{2}; 1+\nu_k; \frac{\Phi_0^2}{4\theta}\right) + \frac{1}{2\pi\delta(0)}\mathrm{erfc}\left(\frac{\Phi_0}{2\sqrt{\theta}}\right) \quad , \tag{5.16}$$

where M denotes the Kummer function and erfc the complementary error function. The last term in the previous equation corresponds to a uniform probability contribution of infinitesimal amplitude which arises as a result of the imposition of the reflecting boundary condition at $\Phi = 0$ where the diffusion coefficient of ζ diverges. As a consequence, all the even moments of χ are infinite and the odd ones vanish. Nevertheless, we can build up physically meaningful finite quantities by considering the dimensionless ratios:

$$\frac{<(\chi-\chi_0)^{2n}>}{<(\chi-\chi_0)^2>^n} = \frac{3}{2n+1}\left[\mathrm{erfc}\left(\frac{\Phi_0}{2\sqrt{\theta}}\right)\right]^{1-n} \quad ,$$

for any $\alpha > 0$. This result implies that the axion distribution is non Gaussian at any time $\alpha > 0$ and for any set of initial conditions. Even though distributions with infinite moments are perfectly well defined one might wonder whether these infinities can be somehow regularized. This is the case in which the probability with absorbing boundary condition is considered. Matarrese studied the moment of the χ field using the probability distribution coming from absorbing boundary condition. This is obtained from the previous distribution neglecting the Dirac delta term. His primary goal was to study the χ field distribution inside our observable Universe, which is the relevant one for the problem of structure formation. In this case Matarrese expanded the argument of the Kummer function present in the probability density function for ζ, since for scale inside our observable Universe holds $\frac{\Phi_0^2}{4\theta} >> 1$. He found

that, for scales inside our observable Universe, the distribution looks pretty Gaussian. On the other hand, on much larger scales the value of Φ_0 can be chosen in such a way that the initial energy density associated to the inflaton in the coarse-grained domain is as large as the Planck energy density, which makes the factor $\frac{\Phi_0^2}{4\theta}$ of order unity, giving rise again to a truly non-Gaussian behavior.
Moreover in the rest of the work Matarrese et al. integrate numerically the system of the Langevin equations and they evaluate the joint probability distribution for the fields as an average over the realizations: this allows them to study the statistical properties of the model for different initial conditions.
We have seen how this article is fundamental for those who study the dynamics of a two-field system in the stochastic approach. What we've learned from Matarrese's article is that the big-scale statistic for the two fields ϕ and χ is non-Gaussian. In the previous chapters we have seen how this property is fundamental for a model that tries to explain the hemispherical power asymmetry, moreover this model does not have the defect that the long wavelenght modes are gauge modes being a multi-field model. Furthermore, the choice of the exponential potential leads to analytical results which should not be underestimated for a non-numerical master's thesis. However this model has a flaw, that of being a single clock model. In fact our main aim is to explain the power asymmetry that affects the larger scales, which can be studied by delta N formula and the correlation functions on these scales can be obtained with this method. But this poses a problem because the χ subdominant field will not enter in H and therefore will not contribute to the correlation functions. This leads us to consider an extension of this model, which can be done by adding a mass term to the χ field and abandoning the assumption of single clock in a way that we will see in the next chapter.

Chapter 6

Proposed Model

We start from a model with two fields, in particular we choose the following potential:

$$V(\phi, \tilde{\chi}) = M^4 \exp\left(\frac{-\lambda\phi}{\sigma}\right) + \frac{m^2\tilde{\chi}^2}{2} \quad , \qquad \sigma = \frac{m_p}{\sqrt{8\pi}} \quad , \tag{6.1}$$

where the $\tilde{\chi}$ field is supposed massive but light[1]. In this way, in usual slow roll approximation, the $\tilde{\chi}$ field gives a small but non negligible contribution to the Hubble parameter and through the Langevin equation of motion couples directly to the ϕ field (the inflaton). The contribution of the $\tilde{\chi}$ field to H is supposed to be smaller respect than the contribution from ϕ. To compare the two contribution, we have to redefine the $\tilde{\chi}$ field in this way $\chi = \frac{\lambda\tilde{\chi}}{\sigma}$. From this redefinition we get a new "mass" term $m_\chi^2 = m^2\frac{\sigma^2}{\lambda^2}$. To obtain that the contribution from ϕ is the dominant one we have to impose $M^4 >> m_\chi^2$, since the fields take similar value. We will see where this condition appears in the rest of the work. The Langevin equations for the two coarse grained fields are the following :

$$\begin{aligned} \frac{d\phi}{d\alpha} &= -\frac{\partial_\phi V}{3H^2} + \frac{H}{2\pi}\eta_\phi \quad ; \\ \frac{d\chi}{d\alpha} &= -\frac{\partial_\chi V}{3H^2} + \frac{H}{2\pi}\eta_\chi . \end{aligned} \tag{6.2}$$

where H is the Hubble parameter. This system of equations is valid in the slow regime. In particular, in this approximation, the Hubble rate becomes $H^2 = \frac{V}{3\sigma^2}$. Moreover α is the number of e-foldings and the $\vec{x}$ dependence is suppressed for convenience. Inserting the potential in the above system we get:

$$\begin{aligned} \frac{d\phi}{d\alpha} = \lambda\sigma M^4 \exp\left(-\frac{\lambda\phi}{\sigma}\right)\frac{1}{M^4\exp\left(\frac{-\lambda\phi}{\sigma}\right) + \frac{m_\chi^2\chi^2}{2}} + \frac{1}{2\sigma\sqrt{3}\pi}\times \\ \times\sqrt{M^4\exp\left(\frac{-\lambda\phi}{\sigma}\right) + \frac{m_\chi^2\chi^2}{2}}\,\eta_\phi \quad ; \end{aligned} \tag{6.3}$$

[1]Moreover λ is constant and dimensionless, yet the scale M is constant but dimension-full.

$$\frac{d\chi}{d\alpha} = -\frac{m_\chi^2\chi}{M^4\exp\left(\frac{-\lambda\phi}{\sigma}\right)+\frac{m_\chi^2\chi^2}{2}} + \frac{1}{2\sigma\sqrt{3}\pi}\sqrt{M^4\exp\left(\frac{-\lambda\phi}{\sigma}\right)+\frac{m_\chi^2\chi^2}{2}}\eta_\chi \quad . \tag{6.4}$$

Performing the following non linear change of variable $\Phi = \exp(\frac{\lambda\phi}{\sigma 2})$, the system becomes:

$$\begin{aligned}\frac{d\Phi}{d\alpha} &= \frac{\lambda^2}{2}\Phi\frac{1}{1+\frac{m_\chi^2\chi^2\Phi^2}{2M^4}} + \frac{\Phi\lambda}{2\sigma}\frac{M^2}{2\sigma\sqrt{3}\pi\Phi}\sqrt{1+\frac{m_\chi^2\chi^2\Phi^2}{2M^4}}\eta_\phi \quad ;\\ \frac{d\chi}{d\alpha} &= -\frac{\chi\Phi^2}{1+\frac{m_\chi^2\Phi^2\chi^2}{2M^4}}\frac{m_\chi^2}{M^4} + \frac{M^2}{2\sigma\sqrt{3}\pi\Phi}\sqrt{1+\frac{m_\chi^2\chi^2\Phi^2}{2M^4}}\eta_\chi \quad .\end{aligned} \tag{6.5}$$

At the denominator there is the term $\frac{m_\chi^2}{M^4}$ that is small by assumption; this suggests an expansion in the equation of motion. As a result we get the following approximate expression:

$$\begin{aligned}\frac{d\Phi}{d\alpha} &= \frac{\lambda^2\Phi}{2}\left(1-\frac{m_\chi^2\chi^2\Phi^2}{2M^4}\right) + \frac{\lambda M^2}{4\sigma^2\sqrt{3}\pi}\left(1+\frac{m_\chi^2\chi^2\Phi^2}{4M^4}\right)\eta_\phi \quad ;\\ \frac{d\chi}{d\alpha} &= -\frac{m_\chi^2}{M^4}\chi\Phi^2\left(1-\frac{m_\chi^2\Phi^2\chi^2}{2M^4}\right) + \frac{M^2}{2\sigma\sqrt{3}\pi\Phi}\left(1+\frac{m_\chi^2\Phi^2\chi^2}{M^4 4}\right)\eta_\chi \quad .\end{aligned} \tag{6.6}$$

At the lowest order in the expansion we get the equation of motion for a theory where the χ field is "effectively" mass-less and sub dominant in H. With "effectively" mass-less we mean that the drift term for χ is suppressed with respect to the diffusion term. In other words, at the lowest order we have a single field clock model and at the leading order we perturb it through the presence of χ field. Now we can think to expand also the field in power of $\frac{m_\chi^2}{M^4}$ in this way:

$$\begin{aligned}\Phi &= \Phi_0 + \frac{m_\chi^2}{M^4}\Phi_1 \quad ;\\ \chi &= \chi_0 + \frac{m_\chi^2}{M^4}\chi_1 \quad .\end{aligned} \tag{6.7}$$

For the lowest order system we get:

$$\begin{aligned}\frac{d\Phi_0}{d\alpha} &= \frac{\lambda^2\Phi_0}{2} + \frac{\lambda M^2}{4\sigma^2\sqrt{3}\pi}\eta_\phi \quad ;\\ \frac{d\chi_0}{d\alpha} &= +\frac{M^2}{2\sigma\sqrt{3}\pi\Phi_0}\eta_\chi \quad .\end{aligned} \tag{6.8}$$

While for the leading order we obtain:

$$\begin{aligned}\frac{d\Phi_1}{d\alpha} &= \frac{\lambda^2\Phi_1}{2} - \frac{\lambda^2\chi_0^2\Phi_0^3}{4} + \frac{\lambda M^2\chi_0^2\Phi_0^2}{16\sigma^2\sqrt{3}\pi}\eta_\phi \quad ;\\ \Phi_0\frac{d\chi_1}{d\alpha} + \Phi_1\frac{d\chi_0}{d\alpha} &= -\chi_0\Phi_0^3 + \frac{M^2\Phi_0^2\chi_0^2}{8\sigma\sqrt{3}\pi}\eta_\chi \quad .\end{aligned} \tag{6.9}$$

The solution for the system (6.8) is:

$$\Phi_0(\vec{x},\alpha) = \exp\frac{\lambda^2(\alpha-\alpha_0)}{2}\Bigg(C+\int_{\alpha_0}^{\alpha} d\alpha' \exp\Big(-\frac{\lambda^2\alpha'}{2}\Big)\frac{\lambda M^2}{4\sigma^2\sqrt{3}\pi}\eta_\phi(\vec{x},\alpha')\Bigg) \quad ;$$

$$\chi_0(\vec{x},\alpha) = \int_{\alpha_0}^{\alpha} d\alpha' \frac{M^2}{2\sigma\sqrt{3}\pi\Phi_0}\eta_\chi(\vec{x},\alpha') + D \quad . \tag{6.10}$$

The solution for the system (6.9) is:

$$\Phi_1(\vec{x},\alpha) = \exp\frac{\lambda^2(\alpha-\alpha_0)}{2}\Bigg(\tilde{C}+\int_{\alpha_0}^{\alpha} d\alpha' \exp\Big(-\frac{\lambda^2\alpha'}{2}\Big)\Big(-\frac{\lambda^2\chi_0^2\Phi_0^3}{4}+ + \frac{\lambda M^2\chi_0^2\Phi_0^2}{16\sigma^2\sqrt{3}\pi}\eta_\phi(\vec{x},\alpha')\Big)\Bigg) \quad ;$$

$$\chi_1(\vec{x},\alpha) = \int_{\alpha_0}^{\alpha} d\alpha' - \chi_0\Phi_0^2 + \frac{M^2\Phi_0\chi_0^2}{8\sigma\sqrt{3}\pi}\eta_\chi - \frac{\Phi_1}{\Phi_0}\frac{M^2}{2\sigma\sqrt{3}\pi\Phi_0}\eta_\chi + \tilde{D} \quad . \tag{6.11}$$

6.1 Classical evolution

In this section we try to solve the classical evolution equation. Essentially the system of equation is composed in the following form:

$$
\begin{aligned}
H &= \frac{M^2}{\Phi\sigma\sqrt{3}}\sqrt{1+\frac{m_\chi^2\chi^2\Phi^2}{2M^4}} \quad ; \\
\frac{d\Phi}{d\alpha} &= \frac{\lambda^2}{2}\frac{\Phi}{1+\frac{m_\chi^2\chi^2\Phi^2}{2M^4}} \quad ; \\
\frac{d\chi}{d\alpha} &= -\frac{\chi\Phi^2}{1+\frac{m_\chi^2\Phi^2\chi^2}{2M^4}}\frac{m_\chi^2}{M^4} \quad .
\end{aligned}
\tag{6.12}
$$

where is used the same notation as before. Essentially this system is the same as (6.5) unless the noise term. We can proceed further in this way:

$$
\frac{d\Phi}{d\chi} = -\frac{M^4\lambda^2}{2m_\chi^2\Phi\chi} \quad . \tag{6.13}
$$

Solving the phase space evolution is simple and we get:

$$
\chi = \exp\left(-\frac{m_\chi^2\Phi^2}{\lambda^2M^4}\right) \quad . \tag{6.14}
$$

Inserting the above formula in the equation for the evolution of the Φ field we obtain:

$$
\frac{d\Phi}{d\alpha} = \frac{\lambda^2}{2}\frac{\Phi}{1+\frac{m_\chi^2\exp\left(-\frac{2m_\chi^2\Phi^2}{\lambda^2M^4}\right)\Phi^2}{2M^4}} \quad . \tag{6.15}
$$

We can solve the last equation by separating the variables:

$$
\int d\Phi\frac{1}{\Phi}+\frac{m_\chi^2}{2M^4}\Phi\exp\left(-\frac{2m_\chi^2\Phi^2}{\lambda^2M^4}\right) = \int d\alpha\frac{\lambda^2}{2} \quad . \tag{6.16}
$$

Finally we get the following solution:

$$
\ln\Phi - \frac{\lambda^2}{8}\exp\left(-\frac{m_\chi^2 2\Phi^2}{\lambda^2M^4}\right) = \frac{\lambda^2}{2}\alpha + \mathcal{C} \quad . \tag{6.17}
$$

This equation implicitly defines Φ as a function of α, and consequently through (6.14) also of χ. This equation will be useful later, when we will compute the correlation functions. The usefulness of (6.17) resides in the fact that in this way we have actually a single field model where the effect of the χ is enclosed in the exponential term in (6.17).
For future calculations is very important to obtain the classical evolution. To do this we start from equation (6.12) expanding both the left and right hand side of the two equations:

$$
\begin{aligned}
\frac{d\Phi_0}{d\alpha}+\frac{m_\chi^2}{M^4}\frac{d\Phi_1}{d\alpha} &= \frac{\lambda^2}{2}\Phi_0+\left(\frac{\lambda^2}{2}\Phi_1-\frac{\lambda^2\chi_0^2\Phi_0^3}{4}\right)\frac{m_\chi^2}{M^4} \quad ; \\
\Phi_0\frac{d\chi_0}{d\alpha}+\left(\Phi_0\frac{d\chi_1}{d\alpha}+\Phi_1\frac{d\chi_0}{d\alpha}\right)\frac{m_\chi^2}{M^4} &= -\chi_0\Phi_0^3\frac{m_\chi^2}{M^4} \quad .
\end{aligned}
$$

Then we separate the system of equations, obtaining:

$$\frac{d\Phi_0}{d\alpha} = \frac{\lambda^2}{2}\Phi_0 \quad ;$$
$$\frac{d\chi_0}{d\alpha} = 0 \quad .$$

So it is clear that χ at the classical level and lowest order is independent on time. At the leading order we get:

$$\frac{d\Phi_1}{d\alpha} = \frac{\lambda^2}{2}\Phi_1 - \frac{\lambda^2\chi_0^2\Phi_0^3}{4} \quad ;$$
$$\frac{d\chi_1}{d\alpha} = -\chi_0\Phi_0^2 \quad .$$

The solutions of the above systems are given by:

$$\Phi_0(\alpha) = C\exp\left(\frac{\lambda^2(\alpha-\alpha_0)}{2}\right) \quad ;$$
$$\chi_0(\alpha) = D \quad .$$

and

$$\begin{aligned}\Phi_1(\alpha) =& \exp\left(\frac{\lambda^2(\alpha-\alpha_0)}{2}\right)\left(\tilde{C} - \int_{\alpha_0}^{\alpha} d\alpha' \exp\left(-\frac{\lambda^2\alpha'}{2}\right)\frac{\lambda^2\chi_0^2\Phi_0^3}{4}\right)\\ =& \exp\left(\frac{\lambda^2(\alpha-\alpha_0)}{2}\right)\left(\tilde{C} + \frac{D^2C^3}{4}\exp\left(-\frac{\lambda^2\alpha_0}{2}\right)\right) + \\ &- \frac{D^2C^3}{4}\exp\left(\frac{3\lambda^2(\alpha-\alpha_0)}{2}\right)\exp\left(-\frac{\lambda^2\alpha_0}{2}\right) \quad ;\\ \text{we define} \qquad \tilde{C}_1 =& \tilde{C} + \frac{\tilde{D}^2C^3}{4} \quad \tilde{D}^2 = D^2\exp\left(-\frac{\lambda^2\alpha_0}{2}\right) \quad ;\\ \chi_1(\alpha) =& \tilde{E} - \int_{\alpha_0}^{\alpha} d\alpha'\chi_0\Phi_0^2 \quad .\end{aligned}$$

A useful check consists in verifying if the above formulas can be obtained from (6.10) and (6.11) setting the noise term to zero; a quick look shows us how this happens. To complete the system we have to find the evolution for the Hubble rate. From (6.12) we have:

$$H(\alpha) = \frac{M^2}{\Phi\sigma\sqrt{3}}\sqrt{1+\frac{m_\chi^2\chi^2\Phi^2}{2M^4}} \quad . \tag{6.18}$$

Expanding in the usual way we obtain:

$$H_0 + \frac{m_\chi^2}{M^4}H_1 = \frac{M^2}{\sqrt{3}\sigma\left(\Phi_0 + \frac{m_\chi^2}{M^4}\Phi_1\right)}\left(1+\frac{m_\chi^2\chi_0^2\Phi_0^2}{4M^4}\right) \quad . \tag{6.19}$$

Expanding again, finally we get:

$$H_0 + \frac{m_\chi^2}{M^4}H_1 = \frac{M^2}{\sqrt{3}\sigma\Phi_0}\left(1+\frac{m_\chi^2}{M^4}\left(\frac{\chi_0^2\Phi_0^2}{4} - \frac{\Phi_1}{\Phi_0}\right)\right) \quad . \tag{6.20}$$

6.2 Determination of the constants of integration

The determination of the coupling constant is a tricky task. A possible way to obtain it, is to use the slow-roll equations of motion for the Φ and χ combined with the usual slow-roll parameters. In fact the latter parameters, when their values are about 1, define the end of the inflationary phase.
At this point the value of the fields at the ending of the inflationary phase can be set with the slow-roll parameters. Thanks to this we can obtain an explicit expression for the constant of integration. The solutions of the systems of classical equations are:

$$\begin{aligned}\Phi_0(\alpha_{end}) &= C\exp\left(\frac{\lambda^2(\alpha_{end}-\alpha_0)}{2}\right) \quad ; \\ \chi_0(\alpha_{end}) &= D \quad ;\end{aligned}$$

where α_{end} is the time of the ending of the inflationary phase and α_0 is the time at which the largest scale exits the horizon during inflation.
For the leading order we get:

$$\begin{aligned}\Phi_1(\alpha_{end}) &= \exp\left(\frac{\lambda^2(\alpha_{end}-\alpha_0)}{2}\right)\left(\tilde{C}-\int_{\alpha_0}^{\alpha_{end}} d\alpha' \exp\left(-\frac{\lambda^2\alpha'}{2}\right)\frac{\lambda^2\chi_0^2\Phi_0^3}{4}\right) \\ &= \exp\left(\frac{\lambda^2(\alpha_{end}-\alpha_0)}{2}\right)\left(\tilde{C}+\frac{D^2C^3}{4}\exp\left(-\frac{\lambda^2\alpha_0}{2}\right)\right)-\frac{D^2C^3}{4}\times \\ &\times\exp\left(\frac{3\lambda^2(\alpha_{end}-\alpha_0)}{2}\right)\exp\left(-\frac{\lambda^2\alpha_0}{2}\right) \quad ;\end{aligned}$$

We define:

$$\begin{aligned}\tilde{C}_1 &= \tilde{C}+\frac{\tilde{D}^2C^3}{4} \quad \tilde{D}^2 = D^2\exp\left(-\frac{\lambda^2\alpha_0}{2}\right) \quad ; \\ \chi_1(\alpha_{end}) &= \tilde{E}-\int_{\alpha_0}^{\alpha_{end}} d\alpha'\chi_0\Phi_0^2 \quad .\end{aligned}$$

To obtain the constant of integration we start from the potential of the model because, in the slow-roll approximation, the slow-roll parameters are defined through the potential term. In fact we define the following slow-roll parameters [30]:

$$\epsilon = \epsilon^{\phi}+\epsilon^{\tilde{\chi}} \quad ; \qquad \epsilon^{\phi} = \frac{m_P^2}{2}\left(\frac{U'}{W}\right)^2 \quad ; \qquad \epsilon^{\tilde{\chi}} = \frac{m_P^2}{2}\left(\frac{V'}{W}\right)^2 \quad ; \tag{6.21}$$

where:

$$W = U(\phi)+V(\tilde{\chi}) = M^4 e^{-\frac{\lambda\phi}{\sigma}}+\frac{m^2}{2}\tilde{\chi}^2 \quad ; \qquad U' = \frac{dU}{d\phi} \quad ; \qquad V' = \frac{dV}{d\tilde{\chi}} \quad . \tag{6.22}$$

We start from this definitions of the ϕ and $\tilde{\chi}$ parameters, since are the correct ones for dimensional reasons. In fact if we would define ϵ^{ϕ} through $\frac{d}{d\Phi}$[2], the

[2] $\Phi = \exp(\lambda\phi/2\sigma)$.

slow-roll parameter becomes dimensional and no more dimensionless. Since in general we expand in power of m_χ^2/M^4, we get two contributions for each ϵ parameters, namely:

$$\epsilon = \epsilon_0 + \frac{m_\chi^2}{M^4}\epsilon_1 = \epsilon_0^\phi + \epsilon_0^{\tilde{\chi}} + \frac{m_\chi^2}{M^4}\left(\epsilon_1^\phi + \epsilon_1^{\tilde{\chi}}\right) \quad .$$

Since we have an explicit expression for Φ and χ fields, is useful to translate the above parameters in function of this fields. In this way we will get an explicit expression for the constant of integration, using the end of the inflationary phase. To achieve the form of the slow-roll parameters, we start replacing and expanding the potential term:

$$W(\Phi,\chi) = W_0(\Phi_0,\chi_0) + \frac{m_\chi^2}{M^4}W_1(\Phi,\chi) \quad ,$$

making clear the previous term:

$$W(\Phi,\chi) = \frac{M^4}{\Phi_0^2} + \frac{m_\chi^2}{M^4}\left(M^4\frac{\chi_0^2}{2} - 2M^4\frac{\Phi_1}{\Phi_0^3}\right) \quad .$$

Then we have to redefine the ϵ parameters. We start from ϵ^ϕ:

$$\epsilon^\phi = \frac{m_P^2}{2}\left(-\frac{\lambda}{2\sigma}\frac{\Phi U'(\Phi)}{W(\Phi,\chi)}\right)^2 \quad ; \qquad U'(\Phi) = \frac{dU}{d\Phi} \quad ; \qquad U(\Phi) = \frac{M^4}{\Phi^2} \quad .$$

Now we can obtain the various contributions for each ϵ term. We start from:

$$\begin{aligned}
\epsilon^\phi &= \frac{\lambda^2 m_P^2}{8\sigma^2}\Phi^2\left[\frac{-2M^4/\Phi^3}{\frac{M^4}{\Phi^2} + \frac{m_\chi^2\chi^2}{2}}\right]^2 \\
&= \frac{\lambda^2 m_P^2}{8\sigma^2}\Phi^2\left[-\frac{2}{\Phi\left(1 + \frac{m_\chi^2}{2M^4}\chi^2\Phi^2\right)}\right]^2 \\
&= \frac{\lambda^2 m_P^2}{2\sigma^2}\left(1 - \frac{m_\chi^2}{M^4}\chi_0^2\Phi_0^2\right) \quad .
\end{aligned}$$

For $\tilde{\chi}$ we get:

$$\epsilon^{\tilde{\chi}} = \frac{\lambda^2 m_P^2}{2\sigma^2}\left[\frac{m_\chi^2\chi}{\frac{M^4}{\Phi^2} + \frac{m_\chi^2\chi^2}{2}}\right]^2 = \frac{\lambda^2 m_P^2}{2\sigma^2}\left[\frac{m_\chi^2}{M^4}\chi_0\Phi_0^2\left(1 - \frac{m_\chi^2}{2M^4}\chi_0^2\Phi_0^2\right)\right]^2 = \mathcal{O}\left(\frac{m_\chi^4}{M^8}\right) \quad ;$$

$$V(\chi) = \frac{m_\chi^2\chi^2}{2} \quad ;$$

From this two contributions we obtain:

$$\begin{aligned}
\epsilon_0^\phi &= \frac{\lambda^2 m_P^2}{2\sigma^2} \quad ; \qquad & \epsilon_1^\phi &= -\frac{\lambda^2 m_P^2}{2\sigma^2}\chi_0^2\Phi_0^2 \quad ; \\
\epsilon_0^{\tilde{\chi}} &= 0 \quad ; & \epsilon_1^{\tilde{\chi}} &= 0 \quad .
\end{aligned} \tag{6.23}$$

We see that the contribution to the slow-roll parameter ϵ from the $\tilde{\chi}$ field is of higher order, in fact both $\epsilon_0^{\tilde{\chi}}$ and $\epsilon_1^{\tilde{\chi}}$ are zero. Now, to obtain the constant of integration is useful to start from the time α_{end}. This moment is the time at which the slow-roll parameter are of the order of unity. In fact the end of the inflationary phase happens when $\epsilon \approx 1$. This condition is achieved when:

$$\epsilon_0^{\phi} + \frac{m_\chi^2}{M^4}\epsilon_1^{\phi} \approx 1 \quad ;$$

Due to the fact that λ is fixed the ϵ^{ϕ} condition gives a relation between the constants of integration of χ_0 and Φ_0.
This means that one constant will be function of the other one. This degeneracy is due to the fact in a two fields model there are a proliferation of possible ending conditions and this is a consequence of the fact that, the phase-space becomes two-dimensional and there are an infinite number of possible classical trajectories in fields space. This condition is different in the single field case, in fact the end of inflation takes place at a fixed value of the inflaton field which in turn corresponds to a fixed energy density. Inserting the expression for χ_0 and Φ_0 we get from the ϵ condition, the following relation:

$$D^2C^2e^{(\lambda^2\mathcal{N})} = \frac{M^4}{m_\chi^2}\left(1 - \frac{1}{\lambda^2 4\pi}\right) \quad , \tag{6.24}$$

where $\mathcal{N} = \alpha_{end} - \alpha_0$. From this formula we see, for example, how the D constant is links to the C one, but clearly to obtain an expression for C we have to use other in formations, that may came from the spectral index for example.

6.3 Construction of the noise terms

A very crucial ingredient, for the correlation function, is the noise term. The noise term η_ϕ is a Gaussian noise term whose amplitude is fixed by the rms fluctuation of the scalar field at Hubble radius crossing, in particular it has zero mean and autocorrelation function given by:

$$< \eta_\phi(\vec{x},\alpha)\eta_\phi(\vec{x'},\alpha') >= j_0(q_s(\alpha)|\vec{x}-\vec{x'}|)\delta(\alpha-\alpha') \quad . \qquad (6.25)$$

We also define the coarse–grained domain size through the comoving wave–number $q_s(\alpha) = \epsilon H(\alpha)a(\alpha)$, being ϵ a number smaller than one and

$$H(\alpha) = H(\phi_{cl}(\Phi_{cl}(\alpha))\chi_{cl}(\alpha))$$

with $\Phi_{cl}(\alpha)$ and $\chi_{cl}(\alpha)$ the classical solutions of the Langevin equations (i.e., those obtained with the noise term "switched" off). Finally the scale–factor $a(\alpha)$ is obtained by integration of $H(\alpha)$.
Why (6.25) should be the correct formula? We can answer to this question in two steps. As previously seen the noise term is made by a combination of the fine-grained part of the inflaton field. The fine-grained term is well approximated by the convolution of a free massless field with the following window function $\theta(k - q_s(\alpha))$[3]. This choice uniquely implies the appearance of the Bessel and delta function. What remains is set by dimensional analysis; in particular since η_ϕ is dimension-less the auto correlation function must be dimension-less. From this constrain we get that the only term that can appear in (6.25) is a constant C. We set $C = 1$.
Up to now we have not mentioned the noise term for the χ field. In fact here there is a subtlety: from the equation of motion the noise term η_χ has dimension M^{-1}, since the product $H\eta_\chi$ must be dimension-less due the adimensionality of χ. So the auto-correlation function for the noise η_χ has dimension M^{-2}. The correct dimensional factor in front is obtained taking into account the initial redefinition of the $\tilde{\chi}$:

$$< \eta_\chi(\vec{x},\alpha)\eta_\chi(\vec{x'},\alpha') >= \frac{\lambda^2}{\sigma^2} j_0(q_s(\alpha)|\vec{x}-\vec{x'}|)\delta(\alpha-\alpha') \quad . \qquad (6.26)$$

In our approximation the fine-grained components of the fields are treated as free even when χ and ϕ interact; thus we assume $< \eta_\phi\eta_\chi > = 0$. Now we have all tools to calculate the correlation function of the ζ variables.

[3] θ is the step function.

Chapter 7

Correlation functions

To compute the correlation function we have to introduce the ζ variable, that is the usual curvature perturbation on constant energy density hyper-surfaces. It's defined as follows (at the linear order):

$$\zeta(\vec{x}, \alpha) = -\psi + H(\alpha)\frac{\rho^{(1)}}{\dot{\rho}} \quad , \tag{7.1}$$

where ψ is a perturbation of the metric tensor, $\rho^{(1)}$ is the perturbation in the energy fluid and H is the Hubble rate. Since our primary goal is try to explain the hemispherical power asymmetry, we are mostly interested on largest scale probes by CMB. We can follow their evolution via the δN formula[1]. Applying the formula we get:

$$\zeta(\vec{x}, \alpha) \simeq \delta\Phi(\vec{x}, \alpha_*)\frac{\partial\alpha}{\partial\Phi_{cl,*}} + \delta\Phi^2(\vec{x}, \alpha_*)\frac{\partial^2\alpha}{\partial\Phi^2_{cl,*}} + \delta\Phi^3(\vec{x}, \alpha_*)\frac{\partial^3\alpha}{\partial\Phi^3_{cl,*}} \quad , \tag{7.2}$$

the fact that the χ field does not appear in (7.2) is due to the fact that the number of e-folding depends only on Φ thanks to (6.17)[2]. This is not the "truth" curvature perturbation, namely the one which derives from the inflaton field. In fact, as we will see in this coordinate, at the lowest order ζ, as defined in (7.2), is a random Gaussian field. Instead, due to the non linear relation between Φ and ϕ, the statistic of the inflaton field is always non-Gaussian and the same thing holds for the "truth" ζ.
So the result that we will derive from now, and for example equation (7.2), are strictly valid in the coordinate (Φ, χ).
To give an idea of the difficulty to go from Φ to ϕ we show the calculation for the linear level. We start in this way:

$$\zeta^{\phi}(\vec{x}) = \delta\phi(\vec{x}, \alpha_*)\frac{\partial\alpha}{\partial\phi_{cl,*}} \quad ,$$

[1] To be consistent we should write $\delta\alpha$ formula.
[2] $\delta\Phi = \Phi - \Phi_{cl}$, where Φ_{cl} is the classical solution of the equation of motion

then we have to express ϕ in function of Φ via $\Phi = \exp(\frac{\lambda\phi}{\sigma 2})$. We get the following identities:

$$\frac{\partial\alpha}{\partial\phi_{cl,*}} = \frac{\partial\alpha}{\partial\Phi_{cl,*}}\frac{\partial\Phi_{cl}}{\partial\phi_{cl,*}} = \frac{\partial\alpha}{\partial\Phi_{cl,*}}\frac{\lambda\Phi_{cl,*}}{2\sigma} \quad ;$$
$$\delta\phi(\vec{x}) = \frac{2\sigma}{\lambda}\ln\left(\frac{\delta\Phi(\vec{x})}{\Phi_{cl}}+1\right) \quad .$$

It is easy to see how (7.2) is highly non trivially related to ζ^{ϕ}. After this departure we return on the road for evaluating the derivative of the number of e-folding α via (6.17). We have to evaluate the right hand side of (7.2) at the initial time α_* on a hyper-surface of constant curvature. The initial time is chosen to be the time at which the scale of interest crosses the horizon during inflation.
Instead the ζ variable is defined on a final hyper-surface of constant energy at time α. For our purposes the final time is chosen to be the time at which inflation ends; in this way we can calculate the primordial correlation functions. To start, we can evaluate the derivative of the number of e-folding via equation (6.17). We get:

$$\frac{\partial\alpha}{\partial\Phi_{cl,*}} = \frac{2}{\lambda^2\Phi_{cl,*}} + \frac{m_\chi^2\Phi_{cl,*}}{M^4\lambda^2}\exp\left(-\frac{m_\chi^2 2\Phi_{cl,*}^2}{\lambda^2 M^4}\right) \quad . \tag{7.3}$$

Subsequently we expand the fields in power of $\frac{m_\chi^2}{M^4}$, obtaining in this way two contributions:

$$\frac{\partial\alpha}{\partial\Phi_{cl,*}} = \frac{2}{\lambda^2\Phi_{0,cl}} + \frac{m_\chi^2}{M^4\lambda^2}\left(\Phi_{0,cl} - \frac{2\Phi_{1,cl}}{\Phi_{0,cl}^2}\right) \quad . \tag{7.4}$$

From this we can evaluate the derivative:

$$\frac{\partial^2\alpha}{\partial\Phi_{cl,*}^2} = -\frac{2}{\lambda^2\Phi_{0,cl}^2} + \frac{m_\chi^2}{M^4\lambda^2}\left(1 + \frac{4\Phi_{1,cl}}{\Phi_{0,cl}^3}\right) \quad , \tag{7.5}$$

and finally:

$$\frac{\partial^3\alpha}{\partial\Phi_{cl,*}^3} = \frac{4}{\lambda^2\Phi_{0,cl}^3} - \frac{12m_\chi^2}{M^4\lambda^2}\frac{\Phi_{1,cl}}{\Phi_{0,cl}^4} \quad . \tag{7.6}$$

Now we move toward the calculation of the ζ variable. In the usual expansion in $\frac{m_\chi^2}{M^4}$ [3] we get two contributions: one at the lowest order and one at the leading order.

$$\begin{aligned}\zeta_0 + \frac{m_\chi^2}{M^4}\zeta_1 = &\left(\delta\Phi_{0,*} + \frac{m_\chi^2}{M^4}\delta\Phi_{1,*}\right)\left(\frac{2}{\lambda^2\Phi_{0,cl,*}} + \frac{m_\chi^2}{M^4\lambda^2}\left(\Phi_{0,cl,*} - \frac{2\Phi_{1,cl,*}}{\Phi_{0,cl,*}^2}\right)\right) + \\ &+ \left(\delta\Phi_{0,*} + \frac{m_\chi^2}{M^4}\delta\Phi_{1,*}\right)^2\left(-\frac{2}{\lambda^2\Phi_{0,cl,*}^2} + \frac{m_\chi^2}{M^4\lambda^2}\left(1 + \frac{4\Phi_{1,cl,*}}{\Phi_{0,cl,*}^3}\right)\right) + \\ &+ \left(\delta\Phi_{0,*} + \frac{m_\chi^2}{M^4}\delta\Phi_{1,*}\right)^3\left(\frac{4}{\lambda^2\Phi_{0,cl,*}^3} - \frac{12m_\chi^2}{M^4\lambda^2}\frac{\Phi_{1,cl,*}}{\Phi_{0,cl,*}^4}\right) \quad .\end{aligned} \tag{7.7}$$

[3] With $*$ we mean evaluation at α_*, with $0, 1$ we mean the expansion order, with cl the classical field.

We can then separate the two contributions:

$$\zeta_0 = \delta\Phi_{0,*}\frac{2}{\lambda^2\Phi_{0,cl,*}} - \delta\Phi^2_{0,*}\frac{2}{\lambda^2\Phi^2_{0,cl,*}} + \delta\Phi^3_{0,*}\frac{4}{\lambda^2\Phi^3_{0,cl,*}} \quad . \tag{7.8}$$

Rewriting the last formula we obtain:

$$\zeta_0 = \zeta_g + \frac{3f_{NL}}{5}\zeta_g^2 + \frac{9g_{NL}}{25}\zeta_g^3 \quad , \tag{7.9}$$

where the following parameters are used:

$$\zeta_g = \frac{2\delta\Phi_{0,*}}{\lambda^2\Phi_{0,cl,*}} \quad ;$$
$$f_{NL} = -\frac{5}{3}\frac{\lambda^2}{2} \quad ;$$
$$g_{NL} = \frac{25}{9}\frac{\lambda^4}{2} \quad .$$

This is the usual local model expansion of the ζ curvature perturbation in terms of a Gaussian field[4]. To train ourselves, we compute the power spectrum for ζ_g in order to completely determine the statistic, since correlation function of ζ_0 will be determined only via the auto-correlation function of ζ_g. The auto-correlation function for ζ_g is the following:

$$< \zeta_g(\vec{x},\alpha)\zeta_g(\vec{x}+\vec{r},\alpha') >= \frac{4}{\lambda^4\Phi_0^{cl}(\alpha_*)\Phi_0^{cl}(\alpha'_*)} < \delta\Phi_0(\vec{x},\alpha_*)\delta\Phi_0(\vec{x}+\vec{r},\alpha'_*) > \quad . \tag{7.10}$$

Inserting the expression for $\delta\Phi_0$ in the last equation we get:

$$(7.10) = \frac{4\lambda^2M^4}{16\sigma^4 3\pi^2\lambda^4C^2}\int_{\alpha_0}^{\alpha_*}\int_{\alpha_0}^{\alpha'_*} d\alpha_1 d\alpha_2 \exp\left(-\frac{\lambda^2\alpha_1}{2}\right)\exp\left(-\frac{\lambda^2\alpha_2}{2}\right)\times \\ \times < \eta_\phi(\vec{x},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2) > \tag{7.11}$$

We focus on the last term, namely the noise term:

$$< \eta_\phi(\vec{x},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2) >= \frac{1}{(2\pi)^3}\int d\vec{k}\int d\vec{k}_1 \exp(\imath\vec{k}\cdot\vec{x})\times \\ \times \exp(\imath(\vec{x}+\vec{r})\cdot\vec{k}_1) < \eta_\phi(\vec{k},\alpha_1)\eta_\phi(\vec{k}_1,\alpha_2) > \quad . \tag{7.12}$$

The Fourier transform of the auto-correlation function of the noise is given by:

$$< \eta_\phi(\vec{k},\alpha_1)\eta_\phi(\vec{k}_1,\alpha_2) >= \delta(\alpha_1-\alpha_2)\delta(\vec{k}+\vec{k}_1)\frac{2\pi^2}{kq_s(\alpha_1)}\delta(k-q_s(\alpha_1)) \quad . \tag{7.13}$$

So we get the following result for (7.10):

$$(7.10) = \frac{M^4}{6\sigma^4\lambda^2C^2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k}\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)} d\alpha_1 \exp(-\lambda^2\alpha_1)\frac{\delta(k-q_s(\alpha_1))}{q_s(\alpha_1)} \quad . \tag{7.14}$$

[4]This holds in the (Φ,χ) coordinate.

Since $q_s(\alpha)$ is a function of H and a, we have to expand it in power of $\frac{m_\chi^2}{M^4}$, obtaining:

$$q_s(\alpha) = k_0 \exp\left((\alpha-\alpha_0)\left(1-\frac{\lambda^2}{2}\right)\right) + k_0\frac{m_\chi^2}{M^4}\left(\frac{\tilde{D}^2C^2}{2}\exp\left((\alpha-\alpha_0)\times\right.\right.$$
$$\left.\left.\times\left(1+\frac{\lambda^2}{2}\right)\right) - \frac{\tilde{C}_1}{C}\exp\left((\alpha-\alpha_0)\left(1-\frac{\lambda^2}{2}\right)\right)\right) \quad , \tag{7.15}$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$. For using the well known formula for the Dirac delta function we have to calculate the zero of the function $q_s(\alpha)-k$. The function is null for $q_s(\alpha)=k$. The equation is not solvable exactly but we can find an approximate solution in the following way. First we rewrite the above equation in this way:

$$\exp\left((\alpha_+-\alpha_0)\left(1-\frac{\lambda^2}{2}\right)\right) = \frac{k}{k_0} - \frac{m_\chi^2}{M^4}\left(\frac{\tilde{D}^2C^2}{2}\exp\left((\alpha_+-\alpha_0)\left(1+\frac{\lambda^2}{2}\right)\right)+\right.$$
$$\left. - \frac{\tilde{C}_1}{C}\exp\left((\alpha_+-\alpha_0)\left(1-\frac{\lambda^2}{2}\right)\right)\right) \quad ,$$

where α_+ is the solution of $q_s(\alpha_+) = k$. At this point we have to find the solution at the lowest level. Than we insert it in formula for the zero at the leading order. At the lowest order we get:

$$\alpha_+ - \alpha_0 = \frac{1}{1-\frac{\lambda^2}{2}}\ln\left(\frac{k}{k_0}\right) \quad , \tag{7.16}$$

while at the leading order we obtain:

$$(\alpha_+-\alpha_0)\left(1-\frac{\lambda^2}{2}\right) = \ln\left(\frac{k}{k_0} - \frac{m_\chi^2}{M^4}\left(\frac{\tilde{D}^2C^2}{2}\left(\frac{k}{k_0}\right)^{\frac{1+\frac{\lambda^2}{2}}{1-\frac{\lambda^2}{2}}} - \frac{\tilde{C}_1}{C}\frac{k}{k_0}\right)\right) \quad .$$

We can expand the last formula in the usual way obtaining:

$$(\alpha_+-\alpha_0)\left(1-\frac{\lambda^2}{2}\right) = \ln\left(\frac{k}{k_0}\right) - \frac{m_\chi^2}{M^4}\left(\frac{\tilde{D}^2C^2}{2}\left(\frac{k}{k_0}\right)^{\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} - \frac{\tilde{C}_1}{C}\right) \quad . \tag{7.17}$$

This is the final formula. We need only to compute the derivative of $q_s(\alpha)$ evaluated at α_+ for the formula of the Dirac delta:

$$\left.\frac{dq_s(\alpha)}{d\alpha}\right|_{\alpha_+} = k\left(1-\frac{\lambda^2}{2}\right)\left(1+\frac{\lambda^2}{1-\frac{\lambda^2}{2}}\frac{\tilde{D}^2C^2}{2}\frac{m_\chi^2}{M^4}\left(\frac{k}{k_0}\right)^{\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right) \quad .$$

Now finally we can use the Dirac formula obtaining:

$$(7.10) = \frac{M^4 e^{-\lambda^2\alpha_0}}{6\sigma^4\lambda^2C^2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k}\frac{1}{k^2(1-\frac{\lambda^2}{2})}\left(1-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}\frac{\tilde{D}^2C^2}{2}\frac{m_\chi^2}{M^4}\left(\frac{k}{k_0}\right)^{\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right)$$
$$\times\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\left(1+\frac{\lambda^2}{1-\frac{\lambda^2}{2}}\frac{m_\chi^2}{M^4}\left(\frac{\tilde{D}^2C^2}{2}\left(\frac{k}{k_0}\right)^{\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} - \frac{\tilde{C}_1}{C}\right)\right) \quad , \tag{7.18}$$

where the radial integration are bound between $[k_0, q_s(min(\alpha_*, \alpha'_*))]$. The time α_0 is the time at witch we start to resolve the Langevin equation; for our purposes is the time at witch the largest scale crosses the horizon during inflation. From the last relation we can read the power spectrum for ζ_g:

$$P_{\zeta_g}(k) = \frac{H^2(\alpha_0)}{2\sigma^2\lambda^2} \frac{1}{k^3(1-\frac{\lambda^2}{2})} \left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} \left(1 - \frac{\tilde{C}_1}{C} \frac{m_\chi^2}{M^4} \frac{\lambda^2}{1-\frac{\lambda^2}{2}}\right) \quad , \tag{7.19}$$

where we have to include the second term at the next order in the expansion. Now we continue calculating the expression for the ζ_1 variable. We get:

$$\zeta_1(\vec{x}, \alpha_*) = \delta\Phi_{1,*} \frac{2}{\lambda^2 \Phi_{0,*}^{cl}} + \delta\Phi_{0,*} \left(\frac{\Phi_{0,*}^{cl}}{2\lambda^2} - \frac{2\Phi_{1,*}^{cl}}{\lambda^2 \Phi_{0,*,cl}^2}\right) - \frac{4}{\lambda^2 \Phi_{0,*,cl}^2} \delta\Phi_{1,*}\delta\Phi_{0,*} +$$
$$+ \delta\Phi_{0,*}^2 \left(\frac{1}{\lambda^2} + \frac{4\Phi_{1,*,cl}}{\lambda^2 \Phi_{0,*,cl}^3}\right) + 12\delta\Phi_{1,*}\delta\Phi_{0,*}^2 \frac{1}{\lambda^2 \Phi_{0,*,cl}^3} - \delta\Phi_{0,*}^3 \frac{12\Phi_{1,*,cl}}{\lambda^2 \Phi_{0,*,cl}^4} \quad . \tag{7.20}$$

At the leading order of the power spectra of the curvature perturbation we have:

$$< \zeta(\vec{x}, \alpha_*)\zeta(\vec{x}+\vec{r}, \alpha'_*) > = < \zeta_0(\vec{x}, \alpha_*)\zeta_1(\vec{x}+\vec{r}, \alpha'_*) > +$$
$$+ < \zeta_1(\vec{x}, \alpha_*)\zeta_0(\vec{x}+\vec{r}, \alpha'_*) > \quad . \tag{7.21}$$

To evaluate last formula we need the expression for $\delta\Phi_1$. This is a long expression, here we have kept only the important terms. In particular the term $1/\Phi_0$ present in χ_0 is very hard to treat. We treat it in an approximate way expanding it around it's classical value. This approximation simplify considerably the calculation since all the linear terms in η_χ are dropped because they appear only in combination with η_ϕ, and the two fields are supposed Gaussian and hence uncorrelated. To evaluate the previous power spectrum are needed only the linear terms in $\delta\Phi$ present in ζ_1 and ζ_0. The terms in $\delta\Phi^2$ and $\delta\Phi^3$ give corrections to the power spectrum. Considering the first term:

$$< \zeta_0(\vec{x}, \alpha_*)\zeta_1(\vec{x}+\vec{r}, \alpha'_*) > = \frac{4}{\lambda^4 \Phi_0(\alpha_*)\Phi_0(\alpha'_*)} < \delta\Phi_0(\vec{x}, \alpha_*)\delta\Phi_1(\vec{x}+\vec{r}, \alpha'_*) > +$$
$$+ < \delta\Phi_0(\vec{x}+\vec{r}, \alpha'_*)\delta\Phi_0(\vec{x}, \alpha_*) > \left(\frac{\Phi_0(\alpha'_*)}{2\lambda^2} - \frac{2\Phi_1(\alpha'_*)}{\lambda^2\Phi_0^2(\alpha'_*)}\right) \frac{2}{\lambda^2\Phi_0(\alpha_*)} \quad .$$

The expression for $\delta\Phi_1$ is:

$$
\begin{aligned}
&\delta\Phi_1(\vec{x},\alpha) = e^{\left(\frac{\lambda^2(\alpha-\alpha_0)}{2}\right)} \int_{\alpha_0}^{\alpha} d\alpha' e^{-\left(\frac{\lambda^2\alpha'}{2}\right)} \Bigg\{ -\frac{\lambda^2 e^{\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)}}{4} \Bigg[\int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 d\alpha_3 \times \\
&\times K^3 \exp\left(-\frac{\lambda^2}{2}(\alpha_1+\alpha_2+\alpha_3)\right) \eta_\phi(\alpha_1)\eta_\phi(\alpha_2)\eta_\phi(\alpha_3) \Bigg(D^2 + \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 Y^2 \times \\
&\times \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_0^{cl}(\beta_1)\Phi_0^{cl}(\beta_2)} - 2Y^2 \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_{0,cl}^2(\beta_1)\Phi_0^{cl}(\beta_2)} \int_{\alpha_0}^{\beta_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) K\eta_\phi(\alpha_4) \Bigg) + 3C \times \\
&\times \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 K^2 \exp\left(-\frac{\lambda^2}{2}(\alpha_1+\alpha_2)\right) \eta_\phi(\alpha_1)\eta_\phi(\alpha_2) \Bigg(D^2 + \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 Y^2 \times \\
&\times \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_0^{cl}(\beta_1)\Phi_0^{cl}(\beta_2)} - 2Y^2 \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_{0,cl}^2(\beta_1)\Phi_0^{cl}(\beta_2)} \int_{\alpha_0}^{\beta_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) K\eta_\phi(\alpha_4) \Bigg) + \\
&+ 3C^2 \int_{\alpha_0}^{\alpha'} d\alpha_1 \exp\left(-\frac{\lambda^2\alpha_1}{2}\right) K\eta_\phi(\alpha_1) \Bigg(D^2 + \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 Y^2 \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_0^{cl}(\beta_1)\Phi_0^{cl}(\beta_2)} + \\
&- 2Y^2 \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_{0,cl}^2(\beta_1)\Phi_0^{cl}(\beta_2)} \int_{\alpha_0}^{\beta_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) K\eta_\phi(\alpha_4) \Bigg) + C^3 \Bigg(\int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 Y^2 \times \\
&\times \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_0^{cl}(\beta_1)\Phi_0^{cl}(\beta_2)} - 2Y^2 \frac{\eta_\chi(\beta_1)\eta_\chi(\beta_2)}{\Phi_{0,cl}^2(\beta_1)\Phi_0^{cl}(\beta_2)} \int_{\alpha_0}^{\beta_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) K\eta_\phi(\alpha_4) \Bigg) \Bigg] + \\
&+ \frac{e^{\left(\lambda^2(\alpha'-\alpha_0)\right)}\lambda M^2}{16\sigma^2\sqrt{3}\pi} \eta_\phi(\vec{x},\alpha') \Bigg[D^2C^2 + D^2 2C \int_{\alpha_0}^{\alpha'} d\alpha_1 \exp\left(-\frac{\lambda^2}{2}\alpha_1\right) K\eta_\phi(\alpha_1) + \\
&+ D^2 \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 \ \exp\left(-\frac{\lambda^2}{2}(\alpha_1+\alpha_2)\right) K^2 \eta_\phi(\alpha_1)\eta_\phi(\alpha_2) + \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 \times \\
&\times Y^2C^2 \left(\frac{\eta_\chi(\alpha_1)\eta_\chi(\alpha_2)}{\Phi_0^{cl}(\alpha_1)\Phi_0^{cl}(\alpha_2)} - 2\frac{\eta_\chi(\alpha_1)\eta_\chi(\alpha_2)}{\Phi_{0,cl}^2(\alpha_1)\Phi_0^{cl}(\alpha_2)} \int_{\alpha_0}^{\alpha_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) K\eta_\phi(\alpha_4) \right) \\
&+ 2C \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 Y^2 \Bigg(\frac{\eta_\chi(\alpha_1)\eta_\chi(\alpha_2)}{\Phi_0^{cl}(\alpha_1)\Phi_0^{cl}(\alpha_2)} - 2\frac{\eta_\chi(\alpha_1)\eta_\chi(\alpha_2)}{\Phi_{0,cl}^2(\alpha_1)\Phi_0^{cl}(\alpha_2)} \times \\
&\times \int_{\alpha_0}^{\alpha_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) K\eta_\phi(\alpha_4) \Bigg) \int_{\alpha_0}^{\alpha'} d\beta_1 \exp\left(-\frac{\lambda^2}{2}\beta_1\right) K\eta_\phi(\beta_1) + \\
&+ \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 Y^2 \Bigg(\frac{\eta_\chi(\alpha_1)\eta_\chi(\alpha_2)}{\Phi_0^{cl}(\alpha_1)\Phi_0^{cl}(\alpha_2)} - 2\frac{\eta_\chi(\alpha_1)\eta_\chi(\alpha_2)}{\Phi_{0,cl}^2(\alpha_1)\Phi_0^{cl}(\alpha_2)} \int_{\alpha_0}^{\alpha_1} d\alpha_4 \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) \times \\
&\times \int_{\alpha_0}^{\alpha'} K\eta_\phi(\alpha_4) \Bigg) \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 \exp\left(-\frac{\lambda^2}{2}(\beta_1+\beta_2)\right) K^2 \eta_\phi(\beta_1)\eta_\phi(\beta_2) \Bigg] \Bigg\} \quad , \qquad (7.22)
\end{aligned}
$$

where the $\vec{x}$ dependence is suppressed in the noise term and we define $K = \frac{\lambda M^2}{4\sigma^2\sqrt{3}\pi}$ and $Y = \frac{M^2}{2\sigma\sqrt{3}\pi}$. We start from the first term:

$$
< \delta\Phi_1(\vec{x}+\vec{r},\alpha'_*)\delta\Phi_0(\vec{x},\alpha_*) > \frac{4}{\lambda^4\Phi_0(\alpha_*)\Phi_0(\alpha'_*)} \quad . \qquad (7.23)
$$

There are various terms in the above formula, we start from the three linear terms in η_ϕ present in (7.22):

$$\frac{4}{\lambda^4 C^2}\int_{\alpha_0}^{\alpha'_*} d\bar{\alpha}\exp\left(-\frac{\lambda^2\bar{\alpha}}{2}\right)\frac{-\lambda^2}{4}\exp\left(\frac{3\lambda^2(\bar{\alpha}-\alpha_0)}{2}\right)\int_{\alpha_0}^{\bar{\alpha}} d\alpha_1\int_{\alpha_0}^{\bar{\alpha}} d\alpha_2\times$$
$$\times\int_{\alpha_0}^{\bar{\alpha}} d\alpha_3\int_{\alpha_0}^{\alpha_*} d\bar{\alpha}'\exp\left(-\frac{\lambda^2\bar{\alpha}'}{2}\right)K^4D^2\exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2+\alpha_3)}{2}\right)\times$$
$$\times<\eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x}+\vec{r},\alpha_3)\eta_\phi(\vec{x},\bar{\alpha}')> \quad .$$

To proceed further we have to evaluate the correlation term. Each fields present above is a Gaussian field, so to evaluate the previous term we use the Wick's theorem:

$$<\eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x}+\vec{r},\alpha_3)\eta_\phi(\vec{x},\bar{\alpha}')>=$$
$$=\delta(\alpha_1-\alpha_2)\delta(\alpha_3-\bar{\alpha}')j_0(q_s(\alpha_3)r)+\{\alpha_3\leftrightarrow\alpha_2\}+\{\alpha_3\leftrightarrow\alpha_1\} \quad .$$

Since the variables $\alpha_{1,2,3}$ are mute variables the three terms give the same contribution:

$$\frac{4}{\lambda^4 C^2}\int_{\alpha_0}^{\alpha'_*} d\bar{\alpha}\exp\left(-\frac{\lambda^2\bar{\alpha}}{2}\right)\frac{-\lambda^2}{4}\exp\left(\frac{3\lambda^2(\bar{\alpha}-\alpha_0)}{2}\right)\int_{\alpha_0}^{\bar{\alpha}} d\alpha_1\int_{\alpha_0}^{\bar{\alpha}} d\alpha_2\int_{\alpha_0}^{\bar{\alpha}}\times$$
$$\times\, d\alpha_3\int_{\alpha_0}^{\alpha_*} d\bar{\alpha}'\exp\left(-\frac{\lambda^2\bar{\alpha}'}{2}\right)K^4D^2\exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2+\alpha_3)}{2}\right)3\delta(\alpha_1-\alpha_2)\times$$
$$\times\,\delta(\alpha_3-\bar{\alpha}')j_0(q_s(\alpha_3)r) \quad .$$

Using the delta integration we get:

$$\frac{-3K^4D^2(2\pi)^2}{\lambda^2C^2}\int_{\alpha_0}^{\alpha'_*} d\bar{\alpha}\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)} d\bar{\alpha}'\exp\left(-\frac{\lambda^2\bar{\alpha}}{2}\right)\exp\left(\frac{3\lambda^2(\bar{\alpha}-\alpha_0)}{2}\right)\times$$
$$\times\exp\left(-\lambda^2\bar{\alpha}'\right)j_0(q_s(\bar{\alpha}')r)\int_{\alpha_0}^{\bar{\alpha}} d\alpha_1\exp\left(-\lambda^2\alpha_1\right) \quad .$$

Now, after Fourier transforming we obtain:

$$-\frac{3K^4D^2(2\pi)^2 2\pi^2}{\lambda^2C^2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k}\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)} d\bar{\alpha}'\exp(-\lambda^2\bar{\alpha}')\times$$
$$\times\frac{\delta(k-q_s(\bar{\alpha}'))}{q_s(\bar{\alpha}')}f(\alpha'_*,\alpha_0) \quad ,$$

where:

$$f(\alpha'_*,\alpha_0)=\int_{\alpha_0}^{\alpha'_*} d\bar{\alpha}\exp\left(-\frac{\lambda^2\bar{\alpha}}{2}\right)\exp\left(\frac{3\lambda^2(\bar{\alpha}-\alpha_0)}{2}\right)\int_{\alpha_0}^{\bar{\alpha}} d\alpha_1\exp\left(-\lambda^2\alpha_1\right)\Big) \quad .$$

Using the delta Dirac formula, in this case in the comoving wave number, we need only the lowest order in m_χ^2/M^4 in $q_s(\alpha)$. We get:

$$-\frac{3\pi K^4D^2e^{-\lambda^2\alpha_0}}{\lambda^2C^2}\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}f(\alpha'_*,\alpha_0) \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between $[k_0, q_s(min(\alpha_*, \alpha'_*))]$. Finally we compute the explicit form of $f(\alpha'_*, \alpha_0)$:

$$\int_{\alpha_0}^{\alpha'_*} d\bar{\alpha} \exp\left(-\frac{\lambda^2\bar{\alpha}}{2}\right) \exp\left(\frac{3\lambda^2(\bar{\alpha}-\alpha_0)}{2}\right) \frac{-1}{\lambda^2}\left(e^{-\lambda^2\bar{\alpha}} - e^{-\lambda^2\alpha_0}\right) =$$
$$= \frac{-\exp\left(-\frac{3\lambda^2\alpha_0}{2}\right)}{\lambda^2} \int_{\alpha_0}^{\alpha'_*} d\bar{\alpha}\left(1-\exp\left(\lambda^2(\bar{\alpha}-\alpha_0)\right)\right) =$$
$$= \frac{-\exp\left(-\frac{3\lambda^2\alpha_0}{2}\right)}{\lambda^2}\left(\alpha'_* - \alpha_0 - \frac{1}{\lambda^2}\left(\exp\left(\lambda^2(\alpha'_*-\alpha_0)\right)-1\right)\right) \quad .$$

α'_* in the previous equation is a function of k, since α'_* is the time at which the scale of interest k exits the horizon during inflation[5]. We now consider the next term:

$$-\frac{K^4Y^2}{C^4\lambda^2}\int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}} \exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{\alpha_*} d\gamma e^{-\frac{\lambda^2\gamma}{2}} \int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}$$
$$\times\, d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 d\beta_2 \exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2+\alpha_3)}{2}\right)\exp\left(-\frac{\lambda^2(\beta_1+\beta_2-2\alpha_0)}{2}\right)\times$$
$$\times < \eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x}+\vec{r},\alpha_3)\eta_\phi(\vec{x},\gamma)\eta_\chi(\vec{x}+\vec{r},\beta_1)\eta_\chi(\vec{x}+\vec{r},\beta_2) > \tag{7.24}$$

concentrating on the correlation term:

$$< \eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x}+\vec{r},\alpha_3)\eta_\phi(\vec{x},\gamma)\eta_\chi(\vec{x}+\vec{r},\beta_1)\eta_\chi(\vec{x}+\vec{r},\beta_2) >=$$
$$= \frac{\lambda^2}{\sigma^2}\delta(\beta_1-\beta_2)\left(\delta(\alpha_1-\alpha_2)j_0(q_s(\alpha_3)r)\delta(\alpha_3-\gamma)+\{\alpha_2\leftrightarrow\alpha_3\}+\{\alpha_2\leftrightarrow\gamma\}\right) \quad .$$

To obtain the last term we use the fact that η_ϕ and η_χ are uncorrelated fields and hence, as before, the permutation terms that appear due to Wick's theorem give the same contribution:

$$-\frac{3K^4Y^2}{C^4\sigma^2}\int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}}\exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{\alpha_*} d\gamma e^{-\frac{\lambda^2\gamma}{2}}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\times$$
$$\times\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 d\beta_2 \exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2+\alpha_3)}{2}\right)\times$$
$$\times\exp\left(-\frac{\lambda^2(\beta_1+\beta_2-2\alpha_0)}{2}\right)\delta(\beta_1-\beta_2)\delta(\alpha_1-\alpha_2)j_0(q_s(\alpha_3)r)\delta(\alpha_3-\gamma) \quad .$$

Integrating the last expression with the Dirac delta we obtain:

$$-\frac{3K^4Y^2(2\pi)^3}{C^4\sigma^2}\int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}}\exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)} d\gamma e^{-\lambda^2\gamma}\times$$
$$\times\, j_0(q_s(\gamma)r)\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'} d\alpha_1 d\beta_1 \exp\left(-\lambda^2\alpha_1\right)\exp\left(-\lambda^2(\beta_1-\alpha_0)\right) \quad .$$

[5] $\alpha'_* - \alpha_0 = \frac{1}{1-\frac{\lambda^2}{2}}\ln\left(\frac{k}{k_0}\right)$

Fourier transforming we get the following expression:

$$-\frac{6\pi^2 K^4 Y^2 e^{-\lambda^2\alpha_0}}{C^4\sigma^2}\int d\vec{k} e^{-\imath\vec{k}\cdot\vec{r}}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\tilde{f}(\alpha'_*,\alpha_0) \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between $[k_0, q_s(min(\alpha_*, \alpha'_*))]$. Moreover we have:

$$\tilde{f}(\alpha'_*,\alpha_0) = \int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}} \exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'} d\alpha_1 d\beta_1 \exp(-\lambda^2\alpha_1)\times$$
$$\times \exp(-\lambda^2(\beta_1-\alpha_0)) \quad .$$

Integrating the last formula in α_1 and β_1:

$$\tilde{f}(\alpha'_*,\alpha_0) = \frac{2e^{-\frac{3\lambda^2\alpha_0}{2}}}{\lambda^4}\int_{\alpha_0}^{\alpha'_*} d\alpha' \cosh(\lambda^2(\alpha'-\alpha_0)) - 1 =$$
$$= \frac{2e^{-\frac{3\lambda^2\alpha_0}{2}}}{\lambda^4}\left(\frac{1}{\lambda^2}\sinh(\lambda^2(\alpha'_*-\alpha_0)) - \alpha'_* + \alpha_0\right) \quad ,$$

as before α'_* is a function of k.Then we consider the next term:

$$\frac{6K^4Y^2}{C^4\lambda^2}\int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}}\exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{\alpha_*} d\gamma e^{-\frac{\lambda^2\gamma}{2}}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\times$$
$$\times\int_{\alpha_0}^{\beta_1} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d\alpha_4 \exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2)}{2}\right)\exp\left(-\frac{\lambda^2(2\beta_1+\beta_2-3\alpha_0)}{2}\right)\times$$
$$\times \exp\left(-\frac{\lambda^2\alpha_4}{2}\right) < \eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x}+\vec{r},\alpha_4)\eta_\phi(\vec{x},\gamma)\times$$
$$\times \eta_\chi(\vec{x}+\vec{r},\beta_1)\eta_\chi(\vec{x}+\vec{r},\beta_2) > \quad . \tag{7.25}$$

Following the same reasoning as before we get:

$$\frac{18K^4Y^2(2\pi)^3}{C^4\sigma^2}\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)} d\gamma e^{-\lambda^2\gamma} j_0(q_s(\gamma)r)\int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}}\times$$
$$\times \exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{\alpha'} d\alpha_1 e^{-\lambda^2\alpha_1}\int_{\alpha_0}^{\alpha'} d\beta_1 \exp\left(-\frac{3\lambda^2(\beta_1-\alpha_0)}{2}\right) \quad .$$

After Fourier transforming we obtain:

$$\frac{36\pi^2K^4Y^2e^{-\lambda^2\alpha_0}}{C^4\sigma^2}\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} g(\alpha'_*,\alpha_0) \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between $[k_0, q_s(min(\alpha_*, \alpha'_*))]$. Moreover we have:

$$g(\alpha'_*,\alpha_0) = \int_{\alpha_0}^{\alpha'_*} d\alpha' e^{-\frac{\lambda^2\alpha'}{2}}\exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2}\right)\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'} d\alpha_1 d\beta_1 \exp(-\lambda^2\alpha_1)\times$$
$$\times \exp\left(-\frac{3\lambda^2}{2}(\beta_1-\alpha_0)\right) \quad .$$

Integrating in α_1 and β_1:

$$g(\alpha'_*, \alpha_0) = e^{-\frac{3\lambda^2\alpha_0}{2}} \int_{\alpha_0}^{\alpha'_*} d\alpha' \frac{2}{3\lambda^4} \left(e^{-\frac{3\lambda^2(\alpha'-\alpha_0)}{2}} + e^{\lambda^2(\alpha'-\alpha_0)} - e^{-\frac{\lambda^2(\alpha'-\alpha_0)}{2}} - 1 \right) \quad .$$

Finally we get:

$$g(\alpha'_*, \alpha_0) = e^{-\frac{3\lambda^2\alpha_0}{2}} \frac{2}{3\lambda^4} \left(\frac{1}{\lambda^2} e^{\lambda^2(\alpha'_*-\alpha_0)} + \frac{2}{\lambda^2} e^{-\frac{\lambda^2(\alpha'_*-\alpha_0)}{2}} - \frac{2}{3\lambda^2} e^{-\frac{3\lambda^2(\alpha'_*-\alpha_0)}{2}} + \right.$$
$$\left. - \frac{7}{3\lambda^2} - \alpha'_* + \alpha_0 \right) \quad .$$

Proceeding further, the next term is:

$$< \frac{-3K^2}{\lambda^2} \int_{\alpha_0}^{\alpha'_*} \int_{\alpha_0}^{\alpha_*} d\alpha' d\alpha_1 e^{-\frac{\lambda^2\alpha_1}{2}} e^{-\frac{\lambda^2\alpha'}{2}} \exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2} \right) \int_{\alpha_0}^{\alpha'} d\bar{\alpha} e^{-\frac{\lambda^2\bar{\alpha}}{2}} \times$$
$$\times \eta_\phi(\vec{x}+\vec{r}, \bar{\alpha}) \eta_\phi(\vec{x}, \alpha_1) \left(D^2 + \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 C^{-2} Y^2 \times \right.$$
$$\left. \times \exp\left(-\frac{\lambda^2(\beta_1+\beta_2-2\alpha_0)}{2} \right) \eta_\chi(\vec{x}+\vec{r}, \beta 1) \eta_\chi(\vec{x}+\vec{r}, \beta_2) \right) > \quad .$$

Evaluating the correlation terms and omitting some steps we get:

$$-\frac{3K^2 e^{-\lambda^2\alpha_0}}{2\lambda^2} \int d\vec{k} \frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})} \left(\frac{k}{k_0} \right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} \tilde{g}(\alpha'_*, \alpha_0) \quad ,$$

where $k_0 = \epsilon H(\alpha_0) a(\alpha_0)$ and the radial integration are bound between [k_0,$q_s(min(\alpha_*, \alpha'_*))$]. Moreover we have:

$$\tilde{g}(\alpha'_*, \alpha_0) = e^{-\frac{\lambda^2\alpha_0}{2}} \left[\frac{D^2}{\lambda^2} \left(e^{\lambda^2(\alpha'_*-\alpha_0)} - 1 \right) - \frac{2\pi Y^2}{C^2\sigma^2} (\alpha'_* - \alpha_0) + \right.$$
$$\left. + \frac{2\pi Y^2}{C^2\sigma^2\lambda^2} \left(e^{\lambda^2(\alpha'_*-\alpha_0)} - 1 \right) \right] \quad .$$

We continue on the road to evaluate (7.23), the next term to calculate is:

$$\frac{2Y^2K^2}{\lambda^2C^2} \int_{\alpha_0}^{\alpha'_*} \int_{\alpha_0}^{\alpha_*} d\alpha' d\alpha_1 e^{-\frac{\lambda^2\alpha'}{2}} \exp\left(\frac{3\lambda^2(\alpha'-\alpha_0)}{2} \right) e^{-\frac{\lambda^2\alpha_1}{2}} \int_{\alpha_0}^{\alpha'} \int_{\alpha_0}^{\alpha'} d\beta_1 d\beta_2 \times$$
$$\times \exp\left(-\frac{\lambda^2(2\beta_1+\beta_2-3\alpha_0)}{2} \right) \int_{\alpha_0}^{\beta_1} d\alpha_2 e^{-\frac{\lambda^2\alpha_2}{2}} < \eta_\phi(\vec{x}+\vec{r}, \alpha_2) \eta_\phi(\vec{x}, \alpha_1) > \times$$
$$\times < \eta_\chi(\vec{x}+\vec{r}, \beta_1) \eta_\chi(\vec{x}+\vec{r}, \beta_2) > \quad , \tag{7.26}$$

for the correlation term we use the fact that the two fields are uncorrelated and the result is

$$< \eta_\phi(\vec{x}+\vec{r}, \alpha_2) \eta_\phi(\vec{x}, \alpha_1) \eta_\chi(\vec{x}+\vec{r}, \beta_1) \eta_\chi(\vec{x}+\vec{r}, \beta_2) > = \frac{\lambda^2}{\sigma^2} \delta(\beta_1 - \beta_2) \times$$
$$\times \delta(\alpha_1 - \alpha_2) j_0(q_s(\alpha_1) r) \quad .$$

Finally we obtain, omitting some steps:

$$\frac{2\pi Y^2 K^2 e^{-\lambda^2\alpha_0}}{C^2\sigma^2}\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} m(\alpha'_*,\alpha_0) \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between [k_0,$q_s(min(\alpha_*,\alpha'_*))$]. Moreover we have:

$$m(\alpha'_*,\alpha_0) = \frac{2e^{-\frac{\lambda^2\alpha_0}{2}}}{3\lambda^4}\left[2e^{-\frac{\lambda^2(\alpha'_*-\alpha_0)}{2}} + e^{\lambda^2(\alpha'_*-\alpha_0)} - 3\right] \quad .$$

Now we turn to the last 5 terms that must be calculated, the next one is:

$$\frac{K^2D^2}{\lambda^4}\int_{\alpha_0}^{\alpha'_*}\int_{\alpha_0}^{\alpha_*} d\alpha' d\bar{\alpha} e^{-\frac{\lambda^2\alpha'}{2}} e^{-\frac{\lambda^2\bar{\alpha}}{2}} e^{\lambda^2(\alpha'-\alpha_0)} < \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha}) > \quad . \tag{7.27}$$

The correlation term is then:

$$< \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha}) >= \delta(\alpha'-\bar{\alpha}) j_0(q_s(\alpha')) \quad .$$

We finally get:

$$\frac{2\pi K^2D^2}{\lambda^4}\int_{\alpha_0}^{min(\alpha'_*,\alpha_*)} d\alpha' e^{-\lambda^2\alpha_0} j_0(q_s(\alpha')r) \quad .$$

After Fourier transforming we obtain:

$$\frac{4\pi^3K^2D^2e^{-\lambda^2\alpha_0}}{\lambda^4}\int d\vec{k}\;\; \frac{e^{-\imath\vec{k}\cdot\vec{r}}}{(2\pi)^3}\frac{1}{k^3(1-\frac{\lambda^2}{2})} \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between [k_0, $q_s(min(\alpha_*,\alpha'_*))$]. We pass to next term:

$$\frac{K^4D^2}{\lambda^4C^2}\int_{\alpha_0}^{\alpha'_*}\int_{\alpha_0}^{\alpha_*} d\alpha' d\bar{\alpha} e^{-\frac{\lambda^2\bar{\alpha}}{2}} e^{-\frac{\lambda^2\alpha'}{2}} e^{\lambda^2(\alpha'-\alpha_0)} \int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'} d\alpha_1 d\alpha_2 \times$$
$$\times \exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2)}{2}\right) < \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x},\bar{\alpha}) > \quad . \tag{7.28}$$

The correlation term is:

$$< \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x}+\vec{r},\alpha_1)\eta_\phi(\vec{x}+\vec{r},\alpha_2)\eta_\phi(\vec{x},\bar{\alpha}) >=$$
$$\delta(\alpha'-\bar{\alpha}) j_0(q_s(\alpha')r)\delta(\alpha_1-\alpha_2) + \delta(\alpha'-\alpha_1) j_0(q_s(\bar{\alpha})r)\delta(\alpha_2-\bar{\alpha}) + \{1 \leftrightarrow 2\} \quad .$$

In this case only the last two terms give the same contributions; instead the first gives a different contribution:

$$\frac{(2\pi)^2K^4D^2e^{-\lambda^2\alpha_0}}{\lambda^4C^2}\Bigg[\int_{\alpha_0}^{min(\alpha'_*,\alpha_*)} d\alpha' j_0(q_s(\alpha')r)\int_{\alpha_0}^{\alpha'} d\alpha_1 e^{-\lambda^2\alpha_1} +$$
$$+ 2\int_{\alpha_0}^{\alpha'_*} d\alpha' \int_{\alpha_0}^{min(\alpha'_*,\alpha_*)} d\bar{\alpha} e^{-\lambda^2\bar{\alpha}} j_0(q_s(\bar{\alpha})r)\Bigg] \quad .$$

The next step, as usual, is performing a Fourier transformation:

$$\frac{\pi K^4 D^2 e^{(-2\lambda^2\alpha_0)}}{\lambda^4 C^2}\Bigg[\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\frac{1}{\lambda^2}\left(-\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}+1\right)+$$
$$+2(\alpha'_*-\alpha_0)\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\Bigg] \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between [k_0,$q_s(min(\alpha_*, \alpha'_*))$]. The next term is:

$$\frac{K^2Y^2}{\lambda^4C^2}\int_{\alpha_0}^{\alpha'_*}\int_{\alpha_0}^{\alpha_*}d\alpha' d\bar{\alpha} e^{-\frac{\lambda^2\alpha'}{2}}e^{-\frac{\lambda^2\bar{\alpha}}{2}}e^{\lambda^2(\alpha'-\alpha_0)}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}d\alpha_1 d\alpha_2\times$$
$$\times\exp\left(-\frac{\lambda^2(\alpha_1+\alpha_2-2\alpha_0)}{2}\right) < \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha})\eta_\chi(\vec{x}+\vec{r},\alpha_1)\eta_\chi(\vec{x}+\vec{r},\alpha_2) > . \tag{7.29}$$

Using the fact that the fields are uncorrelated, the correlation term gives:

$$< \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha})\eta_\chi(\vec{x}+\vec{r},\alpha_1)\eta_\chi(\vec{x}+\vec{r},\alpha_2) > = \frac{\lambda^2}{\sigma^2}\delta(\alpha_1-\alpha_2)\times$$
$$\times\delta(\alpha'-\bar{\alpha})j_0(q_s(\alpha')r) \quad .$$

As usual, integrating and Fourier transforming, we get:

$$\frac{\pi e^{(-\lambda^2\alpha_0)}K^2Y^2}{\lambda^4C^2\sigma^2}\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(1-\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right) \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between [k_0, $q_s(min(\alpha_*, \alpha'_*))$]. Continuing with the calculation, the next term is:

$$-\frac{4Y^2K^4}{\lambda^4C^4}\int_{\alpha_0}^{\alpha'_*}\int_{\alpha_0}^{\alpha_*}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha_1}\int_{\alpha_0}^{\alpha'}d\alpha' d\bar{\alpha} d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 e^{-\frac{\lambda^2\alpha'}{2}}e^{-\frac{\lambda^2\bar{\alpha}}{2}}\times$$
$$\times e^{\lambda^2(\alpha'-\alpha_0)}\exp\left(-\lambda^2\frac{(2\alpha_1+\alpha_2-3\alpha_0)}{2}\right)e^{-\frac{\lambda^2\alpha_3}{2}}e^{-\frac{\lambda^2\beta_1}{2}} < \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha})\times$$
$$\times\eta_\phi(\vec{x}+\vec{r},\alpha_3)\eta_\phi(\vec{x}+\vec{r},\beta_1)\eta_\chi(\vec{x}+\vec{r},\alpha_1)\eta_\chi(\vec{x}+\vec{r},\alpha_2) > \quad . \tag{7.30}$$

We then evaluate the correlation term:

$$< \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha})\eta_\phi(\vec{x}+\vec{r},\alpha_3)\eta_\phi(\vec{x}+\vec{r},\beta_1)\eta_\chi(\vec{x}+\vec{r},\alpha_1)\eta_\chi(\vec{x}+\vec{r},\alpha_2) >=$$
$$\frac{\lambda^2}{\sigma^2}\delta(\alpha_1-\alpha_2)\Bigg(\delta(\alpha'-\bar{\alpha})j_0(q_s(\alpha')r)\delta(\alpha_3-\beta_1)+\delta(\alpha'-\alpha_3)\delta(\beta_1-\bar{\alpha})j_0(q_s(\bar{\alpha})r)+$$
$$+\{\alpha_3\leftrightarrow\beta_1\}\Bigg) \quad .$$

As before, the two last terms give the same contributions while the first term gives a different contribution:

$$-\frac{4(2\pi)^3Y^2K^4e^{(-\lambda^2\alpha_0)}}{\lambda^2C^4\sigma^2}\left[\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)}d\alpha' j_0(q_s(\alpha')r)\int_{\alpha_0}^{\alpha'}d\alpha_1\exp\left(-\frac{3\lambda^2(\alpha_1-\alpha_0)}{2}\right)\times\right.$$
$$\times\int_{\alpha_0}^{\alpha'}d\beta_1e^{-\lambda^2\beta_1}+2\int_{\alpha_0}^{\alpha'_*}d\alpha'\int_{\alpha_0}^{min(\alpha_*,\alpha'_*)}d\bar{\alpha}e^{-\lambda^2\bar{\alpha}}j_0(q_s(\bar{\alpha})r)\times$$
$$\left.\times\int_{\alpha_0}^{\alpha'}d\alpha_1\exp\left(-\frac{3\lambda^2(\alpha_1-\alpha_0)}{2}\right)\right] \quad . \tag{7.31}$$

Repeating the usual steps we obtain:

$$-\frac{8\pi^2Y^2K^4e^{(-2\lambda^2\alpha_0)}}{\lambda^2C^4\sigma^2}\left[\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\frac{2}{3\lambda^4}\left(\left(\frac{k}{k_0}\right)^{-\frac{5\lambda^2}{2-\lambda^2}}-\left(\frac{k}{k_0}\right)^{-\frac{3\lambda^2}{2-\lambda^2}}+\right.\right.$$
$$\left.\left.-\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}+1\right)+\tilde{m}(\alpha'_*,\alpha_0)\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right] \quad , \tag{7.32}$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between $[k_0, q_s(min(\alpha_*, \alpha'_*))]$. Moreover we have:

$$\tilde{m}(\alpha'_*,\alpha_0)=\frac{4}{3\lambda^2}\left(\frac{2}{3\lambda^2}\left(\exp\left(-\frac{3\lambda^2(\alpha'_*-\alpha_0)}{2}\right)-1\right)+(\alpha'_*-\alpha_0)\right) \quad .$$

Finally we consider the last term that must be calculated:

$$-\frac{Y^2K^4}{\lambda^4C^4}\int_{\alpha_0}^{\alpha'_*}\int_{\alpha_0}^{\alpha_*}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}\int_{\alpha_0}^{\alpha'}d\alpha'd\bar{\alpha}d\alpha_1d\alpha_2d\beta_2d\beta_1e^{-\frac{\lambda^2\alpha'}{2}}e^{-\frac{\lambda^2\bar{\alpha}}{2}}\times$$
$$\times e^{\lambda^2(\alpha'-\alpha_0)}\exp\left(-\lambda^2\frac{(\alpha_1+\alpha_2-2\alpha_0)}{2}\right)e^{-\frac{\lambda^2\beta_2}{2}}e^{-\frac{\lambda^2\beta_1}{2}}<\eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha})\times$$
$$\times\eta_\phi(\vec{x}+\vec{r},\beta_2)\eta_\phi(\vec{x}+\vec{r},\beta_1)\eta_\chi(\vec{x}+\vec{r},\alpha_1)\eta_\chi(\vec{x}+\vec{r},\alpha_2)> \quad . \tag{7.33}$$

Repeating the usual steps:

$$-\frac{2\pi^2Y^2K^4e^{(-2\lambda^2\alpha_0)}}{\lambda^2C^4\sigma^2}\left[\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\frac{1}{\lambda^4}\left(\left(\frac{k}{k_0}\right)^{-\frac{2\lambda^2}{1-\frac{\lambda^2}{2}}}-2\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}+1\right)+\right.$$
$$\left.+n(\alpha'_*,\alpha_0)\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right] \quad , \tag{7.34}$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between $[k_0, q_s(min(\alpha_*, \alpha'_*))]$. Moreover we have

$$n(\alpha'_*,\alpha_0)=\frac{2}{\lambda^2}\left(\frac{1}{\lambda^2}\left(\exp\left(-\lambda^2(\alpha'_*-\alpha_0)\right)-1\right)+(\alpha'_*-\alpha_0)\right) \quad .$$

To complete the calculation of the power spectrum, we need only the last term:

$$< \delta\Phi_0(\vec{x}+\vec{r},\alpha'_*)\delta\Phi_0(\vec{x},\alpha_*) > \left(\frac{\Phi_0^{cl}(\alpha'_*)}{2\lambda^2} - \frac{2\Phi_1^{cl}(\alpha'_*)}{\lambda^2\Phi_0^2(\alpha'_*)}\right)\frac{2}{\lambda^2\Phi_0^{cl}(\alpha_*)} \quad . \tag{7.35}$$

Inserting the expressions for the various terms present in the previous equation we obtain:

$$\left[\frac{1+\tilde{D}^2}{2\lambda^4}e^{\lambda^2(\alpha'_*-\alpha_0)} - \frac{2\tilde{C}_1}{\lambda^4 C^3}\right]\int_{\alpha_0}^{\alpha'_*}\int_{\alpha_0}^{\alpha_*} d\alpha' d\bar{\alpha} K^2 \exp\left(-\frac{\lambda^2(\alpha'+\bar{\alpha})}{2}\right)\times$$
$$\times < \eta_\phi(\vec{x}+\vec{r},\alpha')\eta_\phi(\vec{x},\bar{\alpha}) > \quad .$$

Integrating and performing a Fourier transformation we get:

$$\left[\frac{1+\tilde{D}^2}{4\lambda^4}e^{\lambda^2(\alpha'_*-\alpha_0)} - \frac{\tilde{C}_1}{\lambda^4 C^3}\right]K^2\int d\vec{k}\frac{e^{-\imath\vec{k}\cdot\vec{r}}}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} \quad ,$$

where $k_0 = \epsilon H(\alpha_0)a(\alpha_0)$ and the radial integration are bound between $[k_0, q_s(min(\alpha_*,\alpha'_*))]$. Completed the calculation of the power spectrum, we now proceed in the studying of the results just obtained.

Thanks to the Wiener-Khintchine theorem, the two point function can be written in function of the power spectra of the curvature perturbation at the leading order as:

$$< \zeta(\vec{x},\alpha_*)\zeta(\vec{x}+\vec{r},\alpha'_*) >= \frac{1}{(2\pi)^3}\int d^3k e^{-\imath\vec{k}\cdot\vec{r}}P_{\zeta_1}(k) \quad .$$

In (7.21) we have calculated only the first term, the second one gives an analog contribution with the replacement of α'_* with α_*. So, collecting all terms calculated so far, we obtain the full term for the power spectra $P_{\zeta_1}(k)$ at the leading order:

$$\begin{aligned}
P_{\zeta_1}(k) =& (2\pi)^3\left[\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\left(-\frac{3\pi K^4 D^2 e^{-\lambda^2\alpha_0}}{\lambda^2 C^2}f(\alpha'_*,\alpha_0) - \frac{6\pi^2 K^4 Y^2 e^{-\lambda^2\alpha_0}}{C^4\sigma^2}\times\right.\right. \\
&\times\tilde{f}(\alpha'_*,\alpha_0) + \frac{36\pi^2 K^4 Y^2 e^{-\lambda^2\alpha_0}}{C^4\sigma^2}g(\alpha'_*,\alpha_0) - \frac{3K^2 e^{-\lambda^2\alpha_0}}{2\lambda^2}\tilde{g}(\alpha'_*,\alpha_0) + \frac{2\pi Y^2 K^2 e^{-\lambda^2\alpha_0}}{C^2\sigma^2}\times \\
&\left.\times m(\alpha'_*,\alpha_0)\right) + \frac{\pi e^{-\lambda^2\alpha_0}K^2 Y^2}{\lambda^4 C^2\sigma^2}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(1-\left(\frac{k}{k}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right) - \frac{8\pi^2 Y^2 K^4 e^{-2\lambda^2\alpha_0}}{\lambda^2 C^4\sigma^2}\times \\
&\times\left[\frac{2}{3\lambda^4}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\left(\frac{k}{k_0}\right)^{-\frac{5\lambda^2}{2-\lambda^2}} - \left(\frac{k}{k_0}\right)^{-\frac{3\lambda^2}{2-\lambda^2}} - \left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} + 1\right) + \frac{\tilde{m}(\alpha'_*,\alpha_0)}{k^3(1-\frac{\lambda^2}{2})}\times\right. \\
&\left.\times\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right] - \frac{2\pi^2 Y^2 K^4 e^{-2\lambda^2\alpha_0}}{\lambda^2 C^4\sigma^2}\left[\frac{1}{\lambda^4}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\left(\frac{k}{k_0}\right)^{-\frac{2\lambda^2}{1-\frac{\lambda^2}{2}}} - 2\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right.\right. \\
&\left.\left.+1\right) + \frac{n(\alpha'_*,\alpha_0)}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right] + \left[\frac{1+\tilde{D}^2}{4\lambda^4}e^{\lambda^2(\alpha'_*-\alpha_0)} - \frac{\tilde{C}_1}{\lambda^4 C^3}\right]K^2\frac{1}{k^3(1-\frac{\lambda^2}{2})}\times \\
&\times\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}} + \frac{\pi K^4 D^2 e^{-2\lambda^2\alpha_0}}{\lambda^4 C^2}\left(\frac{1}{k^3(1-\frac{\lambda^2}{2})}\frac{1}{\lambda^2}\left(1-\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right) + 2(\alpha'_*-\alpha_0)\times\right. \\
&\left.\left.\times\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\right)\right] + \frac{4\pi^3 K^2 D^2 e^{-\lambda^2\alpha_0}}{\lambda^4}\frac{1}{k^3(1-\frac{\lambda^2}{2})} + \\
&+\text{same term but with } \alpha'_* \leftrightarrow \alpha_* \quad ,
\end{aligned} \tag{7.36}$$

where the functions in the above formula are:

$$
\begin{aligned}
f(\alpha'_*, \alpha_0) =& \frac{-\exp\left(-\frac{3\lambda^2\alpha_0}{2}\right)}{\lambda^2}\left(\alpha'_* - \alpha_0 - \frac{1}{\lambda^2}\left(\exp\left(\lambda^2(\alpha'_* - \alpha_0)\right) - 1\right)\right);\\
\tilde{f}(\alpha'_*, \alpha_0) =& \frac{2e^{-\frac{3\lambda^2\alpha_0}{2}}}{\lambda^4}\left(\frac{1}{\lambda^2}\sinh(\lambda^2(\alpha'_* - \alpha_0)) - \alpha'_* + \alpha_0\right);\\
g(\alpha'_*, \alpha_0) =& e^{-\frac{3\lambda^2\alpha_0}{2}}\frac{2}{3\lambda^4}\left(\frac{1}{\lambda^2}e^{\lambda^2(\alpha'_*-\alpha_0)} + \frac{2}{\lambda^2}e^{-\frac{\lambda^2(\alpha'_*-\alpha_0)}{2}} - \frac{2}{3\lambda^2}e^{-\frac{3\lambda^2(\alpha'_*-\alpha_0)}{2}} + \right.\\
&\left. - \frac{7}{3\lambda^2} - \alpha'_* + \alpha_0\right);\\
\tilde{g}(\alpha'_*, \alpha_0) =& e^{-\frac{\lambda^2\alpha_0}{2}}\left[\frac{D^2}{\lambda^2}\left(e^{\lambda^2(\alpha'_*-\alpha_0)} - 1\right) - \frac{2\pi Y^2}{C^2\sigma^2}(\alpha'_* - \alpha_0) + \right.\\
&\left. + \frac{2\pi Y^2}{C^2\sigma^2\lambda^2}\left(e^{\lambda^2(\alpha'_*-\alpha_0)} - 1\right)\right];\\
m(\alpha'_*, \alpha_0) =& \frac{2e^{-\frac{\lambda^2\alpha_0}{2}}}{3\lambda^4}\left[2e^{-\frac{\lambda^2(\alpha'_*-\alpha_0)}{2}} + e^{\lambda^2(\alpha'_*-\alpha_0)} - 3\right];\\
\tilde{m}(\alpha'_*, \alpha_0) =& \frac{4}{3\lambda^2}\left[\frac{2}{3\lambda^2}\left(\exp\left(-\frac{3\lambda^2(\alpha'_* - \alpha_0)}{2}\right) - 1\right) + (\alpha'_* - \alpha_0)\right];\\
n(\alpha'_*, \alpha_0) =& \frac{2}{\lambda^2}\left[\frac{1}{\lambda^2}\left(\exp\left(-\lambda^2(\alpha'_* - \alpha_0)\right) - 1\right) + (\alpha'_* - \alpha_0)\right] \quad .
\end{aligned}
\tag{7.37}
$$

$\alpha'_*(k)$ and $\alpha_*(k')$ are functions of the comoving wave-vector. In general the above two moments are different, since the time of horizon crossing is different for different scales. But for the two point function, the two scales present are the same and so the second contribution is equal to the one just calculated.
For example, to see the scale dependence of the above functions we have to replace $\alpha'_* - \alpha_0 = \frac{1}{1-\frac{\lambda^2}{2}}\ln\left(\frac{k}{k_0}\right)$.
The above functions are very different among them. For example, the linear contribution in α'_* gives a logarithmic correction to the power spectrum, the exponential term gives a power law correction and finally the sinh term present in $\tilde{f}$ gives two different power law contributions, in fact $\sinh(\ln(x)) = \frac{1}{2}(x - 1/x)$.
The most interesting contribution is the logarithmic one. In fact this correction can be understood as a modification of the spectral index of curvature perturbation.
Moreover the complete power spectra is the sum of the power spectra at the lowest and leading order in m_χ^2/M^4, namely:

$$
P_\zeta(k) = P_{\zeta_g}(k) + \frac{m_\chi^2}{M^4}P_{\zeta_1}(k) \quad , \tag{7.38}
$$

where the form of the first term is shown in equation (7.19) and the second one in (7.36). To see the effect of the logarithmic correction, we insert the form of (7.19) and we select the first term in (7.36) and only the linear term in α'_* present in f[6].

[6]This approximation is made only to see the logarithmic correction to the power spectrum

We insert the previous contributions (7.19) and (7.36), we get:

$$\frac{M^4 e^{-\lambda^2\alpha_0}}{6\sigma^4\lambda^2 C^2}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}+\frac{m_\chi^2}{M^4}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\frac{3\pi K^4 D^2 e^{-\frac{5\lambda^2\alpha_0}{2}}}{\lambda^4 C^2(1-\frac{\lambda^2}{2})k^3}\times$$
$$\times\frac{1}{1-\frac{\lambda^2}{2}}\ln\left(\frac{k}{k_0}\right) \quad .$$

We can collect the first term[7]:

$$\frac{M^4 e^{-\lambda^2\alpha_0}}{6\sigma^4\lambda^2 C^2}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}}\left(1+m_\chi^2\frac{\lambda^2}{(1-\frac{\lambda^2}{2})}\frac{D^2 e^{-\frac{3\lambda^2\alpha_0}{2}}}{128\sigma^4\pi^3}\ln\left(\frac{k}{k_0}\right)\right) \quad .$$

The previous equation can be obtained from the following power spectra:

$$\frac{M^4 e^{-\lambda^2\alpha_0}}{6\sigma^4\lambda^2 C^2}\frac{1}{k^3(1-\frac{\lambda^2}{2})}\left(\frac{k}{k_0}\right)^{-\frac{\lambda^2}{1-\frac{\lambda^2}{2}}+m_\chi^2\frac{\lambda^2}{(1-\frac{\lambda^2}{2})}\frac{D^2 e^{-\frac{3\lambda^2\alpha_0}{2}}}{128\sigma^4\pi^3}} \quad ,$$

from this we see that the spectral index is:

$$n_s = -\frac{\lambda^2}{1-\frac{\lambda^2}{2}}+m_\chi^2\frac{\lambda^2}{1-\frac{\lambda^2}{2}}\frac{D^2 e^{-\frac{3\lambda^2\alpha_0}{2}}}{2\pi m_P^4} \quad .$$

The first contribution comes from the lowest order expansion, the second one comes from the leading order and is the contribution to the curvature perturbation from the χ field. Yet the leading order correction depends on the explicit form of the D variable. The spectral index is an observational quantity, so in can be constraint by experiments. This means that ideally we could put constraints on the parameters of the theory. Why ideally? Ideally because the spectral index is a function of all the parameters present in the model namely M and m mass of the χ field enclosed in m_χ. A degeneration in the parameters evaluation is present and more observational quantities are needed for an estimation of the model parameters.

[7] $K = \frac{\lambda M^2}{4\sigma^2\sqrt{3\pi}}$

Chapter 8

Conclusions and futures perspectives

In this work we started to study the problem of anomalies present in the CMB, with particular emphasis on the hemispherical power asymmetry. We followed a step by step approach, faithfully following some very important articles for the study of the power asymmetry anomaly. These articles gave us solid modeling and statistical bases, on how long fluctuations affect statistics on small scales. In fact we saw how the properties of non-Gaussiannity and multi-source are fundamental for a model that tries to reproduce the power anomaly. Guided by these stakes, we have chosen to analyze a two-field model with separable variables, in which the inflaton has an exponential potential while the second field is characterized by having only a mass term.
At this point, once chosen the model and introduced the stochastic approach to inflation, we started studying the equation of motion for the two fields. We realised that Langevin's system of equations was too complex to be solved in a closed form. This led us to discuss an approximation of our model. In this perspective, we found an approximate solution of the system of initial equations. The accuracy of this solution is determined by the expansion parameter, i.e. $\frac{m_\chi^2}{M^4}$. In this thesis we stopped at the leading order, but future works could study the effects of the successive order or look for a general solution valid for each order.
Again in this approximation we began to study correlation functions. We have introduced the function ζ and to obtain an explicit expression in function of the fields, since we are interested in the larger scales, we have used the delta N formula. Obtained the explicit form of the function ζ and calculated the derivatives of the number of e-folding as required by the delta N formula. We have derived the two points function of ζ, both at the lowest order in $\frac{m_\chi^2}{M^4}$ and to the leading order.
We have noticed how the corrections to the leading order modify the spectral index of curvature perturbations. However the correlation function we need is the four-point one. As we have seen the presence of a non-empty trispectrum with large collapsed limit can generate the hemispherical power asymmetry. The next step would be to calculate the trispectrum and the bispectrum in

this model. Another possible interesting correction to study would be to see the effects of the terms $\delta\Phi^2$ and $\delta\Phi^3$ on the power spectrum.

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