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High Performance Multivariable Spectral Estimator with Bounded McMillan Degree

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Abstract

Spectral estimation is a prominent issue: besides the classical applications in information engineering, such as signal processing, communications and model identification, techniques that are able to deal with this matter can be useful in a widespread variety of fields, from biology to seismology.

To the present, multivariable spectral estimation with complexity constraints on the acceptable solutions is a challenging issue. There are at least two reasons that make this subject significant. Firstly, nowadays the ability of dealing with non scalar processes is necessary, since they play a key role in practice. Secondly, the requirement that the complexity of the solution is bounded is indispensable from the viewpoint of concrete implementation.

In this work, spectral estimation problem is recast in the form of a matricial generalized moment problem with a complexity constraint expressed in the form of a bound on the McMillan degree of the obtained interpolants. Interpolation data are achieved via a proper filterbank and an information-theoretic index is introduced in order to measure the distance between spectral density functions. This framework naturally leads to formulate the issue of interest in the form of a constrained optimization problem. This can be efficiently tackled by means of duality theory, because the corresponding dual problem turns out to be particularly suitable for a solving strategy based on Newton-type algorithmic approach.

An existence theorem for the dual problem is deduced, and the global convergence of the proposed algorithm is proved.

Index Terms

Multivariable spectral approximation, spectral approximation, relative entropy, generalized moment problem, convex optimization, matricial Newton algorithm.

Contents

Nomenclature

$\operatorname{dom} f(\cdot)$	The domain of the function $f(\cdot)$
$\mathbb{C}^{n\times n}$	The set of complex-valued $n \times n$ matrices
$\mathcal{C}(\mathbb{T}; \mathcal{H}(n))$	The space of the continuous functions defined on the unit circle $\mathbb{T}$ with values in $\mathcal{H}(n)$
$\mathcal{H}(n)$	The set of the Hermitian matrices in $\mathbb{C}^{n\times n}$
$\mathcal{N}(\mu_\rho, \Sigma_\rho)$	Gaussian random vector with mean μ_ρ and covariance Σ_ρ
$\mathcal{S}_+^{m\times m}(\mathbb{T})$	The set of $\mathbb{C}^{m\times m}$-valued bounded and coercive spectral densities, page 17
$\nabla f(\cdot)$	The gradient of f
$\operatorname{epi}(f)$	The epigraph of the function f
∂S	Given the set S, ∂S denotes its boundary
$\operatorname{Tr} A$	The trace operator
$A^\dagger$	The Moore-Penrose pseudoinverse of the matrix A
$H_f(\cdot)$	The Hessian of f
$S_{\mathbf{x}}(e^{j\cdot})$	Power spectral density of the stochastic process $\mathbf{x}$

Chapter 1

Multivariate Spectral Estimation: Introduction

One of the most important features of modern information engineering is that it makes use of statistics as a general framework, providing mathematic tools for modeling reality. The cornerstone of this approach is the concept of statistical process, that allows us to deal with *random signals*. Indeed, statistical processes represent an effective mathematical model for random signals, whose introduction outperforms the traditional deterministic viewpoint, since it allows, for instance, an easier description of mechanisms such as information transmission or the presence of disturbance in control systems. On the one hand, the process of conveying information is strictly linked to the idea of randomness: a signal, whose values can be completely predicted on the basis of its present state, carries no *new* information. On the other hand, control theory is motivated by the necessity of achieving and preserving the desired behavior for a system, counteracting the effects of disturbances. An effective manner to perform this task is to regard disturbances, whose exact trend is, by definition, unknown, as random signals.

Of course, these simple examples do not deplete the potentialities of statistical processes, that can be productively employed in a widespread variety of application fields. In particular, they can also be employed to provide a manageable model when the deterministic description of a phenomenon is excessively complex. As a result, they find applications also in economics, meteorology and medicine, for instance.

This work deals with a problem associated with the description of random processes: the estimation of its spectrum (also known as the power spectral density) according to available finite data records obtained from some realizations of the process itself. Deferring mathematical details to the subsequent section, an idea of what this problem involves is briefly introduced next. A meaningful way to interpret a

random process is to consider it as a family of ordinary signals, parameterized by a random variable. Once that the value of the random variable is chosen (exactly as it happens in extractions from a ballot-box), a realization of the stochastic process is attained. Without loss of generality, realizations are assumed to be signals defined on a time-domain, so the obtained samples can be labeled by a time index. From the observed values is possible to estimate parameters of interest such as the mean value of the process and its correlation function. This function provides a measure of how any two of the samples are interlaced. Recall that, under *stationarity* assumptions, the correlation depends only on the difference of the time indexes of the samples. If these assumptions hold, it is possible to introduce a description of the process in the frequency domain, considering the Fourier-transform of the correlation function, that is the previously introduced power spectral density.

However, in order to obtain the spectrum of a process, one should have at its disposal *all* the correlation lags, whose number is infinite. This is a consequence of the definition of the Fourier-transform. Moreover, the correlation lags should be known *exactly*. As a consequence, a real setting originates a duplex problem: on the one hand, the number of correlation lags available is ineluctably finite; on the other hand, only an estimate of them is available.

These considerations describe the essence of the spectrum estimation problem: it consists in estimating how the total power of a process is distributed over frequency, given a finite record of a stationary data sequence. This is one of the fundamental matters in information engineering, with widespread applications in communications, control systems and signal processing. Think about speech analysis: spectral techniques can be adopted in speech compression and recognition. However, it is restrictive to concentrate only on this context. Spectral analysis finds applications also in many other fields. In economics, astronomy and meteorology it can reveal hidden periodicities in the phenomena of interest; in mechanics it is a useful tool in order to deal with vibration monitoring and reduction; in seismology it can be employed with the purpose of investigating the nature of the seismic events, so it may help to predict them. Finally, in medicine a meaningful example of the benefits that stem from the application of spectral analysis is given by the diagnostic value of electrocardiogram and electroencephalogram.

This work focuses on a particular setting, the challenging case in which the aim is to estimate the spectrum of a multivariate random process, providing a rational solution of bounded complexity. These requirements are necessary, from a system-theoretic point of view, in order to assure that the solution, i.e. the estimated spectrum, is concretely serviceable in practice. Actually, a rational transfer function can be realized by a state-space model, that is particularly convenient for an

algorithmic implementation. However, an excessive complexity could make it unemployable in realistic situations, so the necessity of a complexity bound comes forth. Since the MIMO (Multiple Input Multiple Output) case is concerned, this bound will be expressed in terms of the McMillan degree of the solution. The McMillan degree is a measure of the complexity of a system, as discussed in Appendix A.

A solution will be proposed, in the spirit of a convex optimization approach that has been developed in the last decade, in the context of generalized moment problems. It provides a fertile perspective, since it tackles the classical problem of analytic interpolation by means of an innovative point of view, that allows to deal with the complexity constraints.

This approach will be presented in the state of-the-art section, after a brief review on some mathematical background and notation.

The outline of the thesis is the following. Chapter 2 is devoted to introduce some basic mathematical background and the state of-the-art approaches to spectral estimation, with particular regard to the multivariable framework. Chapter 3 describes the model that will be employed in order to tackle the issue of interest. Moreover, the feasibility of the problem is investigated. Chapter 4 is the crucial part of the thesis. A new approach to multivariate spectral estimation with a constraint on the complexity of the estimate is exposed. The problem, that is recast in the form of a convex optimization, is reduced to the minimization of a proper function. To begin with, it is proven that the solution, if it exists, is unique. Then the existence of such a solution is established. Finally, a Newton-type algorithm is introduced, and its convergence to the optimal solution is proven. Chapter 5 shows the results of some numerical simulations of the proposed approach. Chapter 6 proposes a brief summary of the achieved results and outlines some hints for future improvements and investigations.

Chapter 2

Preliminaries and State of-the-art

2.1 Mathematical Background

This section provides a brief review on some useful mathematical facts about stochastic processes. The aim is to recall basic definitions and to fix the notation that will be used throughout the thesis. A more detailed exposition about this subject can be found, for instance, in [4] and [13].

Definition 2.1.1 (Stochastic process)**.** Given the probability space $\mathcal{S} = (\Omega, \mathcal{A}, \mathbb{P})$, where Ω is the sample space, $\mathcal{A}$ is the set of the events defined over Ω and $\mathbb{P}$ is a probability measure, a (discrete time) scalar stochastic process is a function $\mathbf{x} : \mathbb{Z} \times \Omega \to \mathbb{R}$, such that $\mathbf{x}(t, \cdot)$ is a random variable defined on $\mathcal{S}$, for each $t \in \mathbb{Z}$.

The complete statistic description of a process is provided by the family of the *probability distributions* $\{F_n \, s.t. \, n \in \mathbb{N}\}$, that are defined as

$$F_n(x_1, \ldots, n_n; t_1, \ldots, t_n) := \mathbb{P}\left[\mathbf{x}(t_1) \leq x_1, \ldots, \mathbf{x}(t_n) \leq x_n\right], \quad x_i \in \mathbb{R},\, t_i \in \mathbb{Z}. \tag{2.1}$$

If F_n is absolutely continuous, or if generalized functions are considered, it is possible to define the corresponding *probability density function* f_n, defined as

$$f_n(x_1, \ldots, n_n; t_1, \ldots, t_n) := \frac{\partial^n F_n(x_1, \ldots, n_n; t_1, \ldots, t_n)}{\partial x_1 \ldots \partial x_n} \quad x_i \in \mathbb{R},\, t_i \in \mathbb{Z}. \tag{2.2}$$

Usually, a complete knowledge of the statistical description of the process is not available. Therefore, it is useful to introduce the second order statistical description, that corresponds to the knowledge of *mean* and *correlation* or *covariance*. It the case of Gaussian processes the second order description provides the complete statistical description. Introducing the expectation operator $\mathbb{E}[\cdot]$, the mean is defined as the

function $m : \mathbb{Z} \to \mathbb{C}$ given by

$$m(t) := \mathbb{E}[\mathbf{x}(t)] = \int_{\mathbb{R}} x f_1(x,t)dt \quad t \in \mathbb{Z}. \tag{2.3}$$

Correlation $r(t,s)$ and covariance $\sigma(t,s)$ are functions that map $\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{C}$, in the general case of complex-valued processes. They are defined as

$$r(t,s) := \mathbb{E}[\mathbf{x}(t)\mathbf{x}^*(s)], \quad t,s \in \mathbb{Z}. \tag{2.4}$$

where $\mathbf{x}^*(t)$ denotes the complex conjugate of $\mathbf{x}(t)$, and

$$\sigma(t,s) := \mathbb{E}[(\mathbf{x}(t) - m(t))(\mathbf{x}^*(s) - \bar{m}(t))], \quad t,s \in \mathbb{Z}. \tag{2.5}$$

They both have the property of being positive semidefinite functions. A function $g(\cdot,\cdot)$ is positive semidefinite if and only if

$$\sum_{h,k=1}^{n} a_h g(t_h,t_k) {a_k}^* \geq 0, \quad a_1,\ldots,a_n \in \mathbb{C},\, t_1,\ldots,t_n \in \mathbb{Z}.$$

A stochastic process is said to be *strictly stationary* if, $\forall\, \tau,\, n \in \mathbb{Z}$, the following equation holds:

$$F_n(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau) = F_n(x_1,\ldots,x_n;t_1,\ldots,t_n), \quad x_i \in \mathbb{R}, t_i \in \mathbb{Z}. \tag{2.6}$$

A less tight condition is known as second order (or weak, or wide sense) stationarity, since it involves only the second order probability description, requiring that the mean is a constant function and the correlation (or equivalently, the covariance) depends on the difference between the time indexes of its arguments rather than their values. In formulae, weak stationarity is equivalent to the two following identities:

1. $m(t) = m$, $\forall\, t \in \mathbb{Z}$;
2. $r(t,s) = r(t-s,0)$, $\forall\, t,s \in \mathbb{Z}$ (similarly for the covariance function).

With abuse of notation, since it actually depends on a unique argument, $r(t-s,0)$ is then expressed as $r(k)$, or r_k, with $k = t-s$.

At last, it is now possible to introduce the spectrum (or power spectral density) $S(\cdot)$ of a (weakly) stationary stochastic process, defined as the Fourier-transform of the correlation function $r(\cdot)$:

$$S(e^{j\theta}) = \sum_{\tau=-\infty}^{\infty} r(\tau)e^{-j\theta\tau}, \quad \theta \in [-\pi,\pi]. \tag{2.7}$$

Bochner's theorem allows to conclude that, since $r(\cdot)$ is a positive semidefinite function, its Fourier-transform $S_{\mathbf{x}}(e^{j\cdot})$ is real and non-negative.

A multivariate stochastic process is a vector whose elements are scalar stochastic processes. If it takes complex values, the real and the imaginary part of each of its components are required to be random variables defined on the same probability space. In the multivariate case, the mean becomes a vector function, whereas correlation and covariance are given by positive semidefinite matrices $R(\tau) = \mathbb{E}[\mathbf{x}(t+\tau)\mathbf{x}^*(t)]$ and $\Sigma(\tau) = \mathbb{E}[(\mathbf{x}(t+\tau) - m)(\mathbf{x}(t) - m)^*]$, where * denotes the transpose and conjugate operator. If the weak stationarity assumption holds, it is possible to define the power spectral density $S(e^{j\omega})$. Since $R(\tau) = R^*(-\tau)$, it holds that $S(e^{j\theta}) = S^*(e^{j\theta})$ i.e. the spectrum displays the Hermitian symmetry. In addition, since the correlation matrix keeps the property of being a positive semidefinite function, it holds that $S(e^{j\theta}) \geq 0 \, \forall \theta \in [-\pi, \pi]$. For each value of $\theta \in [-\pi, \pi]$, the power spectral density is a positive semidefinite matrix.

A result that will significantly be employed in the following sections is that it is possible to attain an additive decomposition of the spectrum. Let the function $f(z)$ be defined by:

$$f(z) = \frac{1}{4\pi} \int_{-\pi}^{\pi} S(e^{j\theta}) \frac{z + e^{j\theta}}{z - e^{j\theta}} d\theta. \tag{2.8}$$

It is a *positive real* function, i.e. it is analytic with positive real part in $|z| > 1$. The power spectral density can be expressed as

$$S(e^{j\theta}) = 2\Re\left[f(e^{j\theta})\right] \tag{2.9}$$

and f admits the following series representation for $|z| > 1$:

$$f(z) = \frac{1}{2} r_0 + \sum_{k=1}^{+\infty} r_k z^{-k} \tag{2.10}$$

In addition, some results on spectral factorization are recalled (see [14], for an exhaustive description). Consider a spectral density $S \in \mathcal{S}_+^{m\times m}(\mathbb{T})$, where $\mathcal{S}_+^{m\times m}(\mathbb{T})$ is the set of $C^{m\times m}$-valued bounded and coercive spectral densities. A measurable function W with values in $\mathbb{C}^{m\times p}$ is said to be a spectral factor of S if

$$W(e^{j\theta}) W^*(e^{j\theta}) = S(e^{j\theta}), \quad \text{a.e. on } \mathbb{T}.$$

It is possible to prove that W is full row rank and that necessarily $p \geq m$. Since W is bounded on $\mathbb{T}$, it belongs to the set of $L_2^{m\times p}$ functions, i.e. $\int \operatorname{Tr} WW^* < \infty$.

Moreover, since any $S \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ satisfies the Szegö condition

$$\int_{-\pi}^{\pi} \log\det S(e^{j\theta})\frac{d\theta}{2\pi} > -\infty,$$

it admits a spectral factor W in $H_2^{m\times m}$, i.e. the Hardy space of functions in $L_2^{m\times m}$ that have an analytic extension in $|z| > 1$. These can be interpreted as the Z-transform of causal signals. A square factor that exhibits these properties is called *outer* or *minimum phase* spectral factor. Let $W_1 \in L_2^{m\times m}$ and $W_2 \in L_2^{m\times p}$ be spectral factors of $S \in \mathcal{S}_+^{m\times m}(\mathbb{T})$. Then $U := W_1^{-1}W_2$ is an *all-pass* function, i.e. $U(e^{j\theta})U^*(e^{j\theta}) = I\,\forall\,\theta \in [-\pi,\pi]$.

It is also useful to recall some notions about linear transformations of stochastic processes, that will be widely employed in the following sections. To begin with, it is worthwhile to remind that a linear time invariant (LTI) BIBO system transforms the input process $\mathbf{x}$, that is required to be weakly stationary, in the jointly (weakly) stationary output process $\mathbf{y}$, defined by

$$\mathbf{y}(t) = \sum_{s=-\infty}^{\infty} H(t-s)\mathbf{x}(s), \quad t \in \mathbb{Z}, \tag{2.11}$$

where $H(\cdot)$ is the impulse response of the system, that is BIBO stable if and only if $\sum_{s=-\infty}^{\infty}|H(s)| < \infty$. The transfer function of the system is denoted by $\hat{H}(e^{j\cdot})$. Actually, in order to obtain the wide sense stationarity of the output process, a weaker condition suffices:

$$M_{\mathbf{y}} = \int_{-\pi}^{\pi} \hat{H}(e^{j\theta})S_{\mathbf{x}}(e^{j\theta})\hat{H}^*(e^{j\theta})\frac{d\theta}{2\pi} < \infty. \tag{2.12}$$

If this condition holds, from eq. (2.11) it is possible to obtain a statistical (second order) description of the output process $\mathbf{y}$. Here it is sufficient to recall the expression of the spectrum of the output process, given by

$$S_{\mathbf{y}}(e^{j\theta}) = \hat{H}(e^{j\theta})S_{\mathbf{x}}(e^{j\theta})\hat{H}^*(e^{j\theta}), \quad \theta \in [-\pi,\pi]. \tag{2.13}$$

In the introduction a system-theoretic index that describes the complexity of MIMO systems, the so-called McMillan degree, has been mentioned. The reader is deferred to appendix A for the theory that leads to its definition. To the purpose of this work, it is sufficient to recall that the McMillan degree is the dimension of each state space minimal realization of a given rational transfer matrix.

2.2 State of-the-art

Spectral estimation has a long history, that dates back to early applications of Fourier analysis. Many techniques have been developed: some of them require only the process of interest to be stationary and do not need further knowledge about the structure of the process to be analyzed. This is the case of non-parametric approaches. They aim at estimating the spectrum by means of Fourier transform of a finite sequence of sample covariance lags, which are obtained from a finite data record. The ancestor of such techniques is the periodogram method. Other techniques have been derived from it, allowing to improve the performances in terms of the variance of the estimate. On the other hand, when additional information about the process of interest is available, it can be convenient to recast the problem of spectral estimation, so that it amounts to estimate the parameters of a proper model, that takes into consideration the knowledge about the process. This is a parametric approach, that has brought to the development of a variety of techniques, whose efficacy is related to the specific context. For example, the issue of spectral lines analysis has been a fertile framework, in which many high resolution methods have been worked out. The reader is deferred to [21] for an exhaustive analysis of the aforementioned techniques.

The method that is proposed in this work draws inspiration from a new approach that has succeeded in the last decade. It follows in the footsteps of recent results about generalized moment problems with complexity bound (see [3] and references therein, for instance). A typical problem of this kind is the following:

Problem 2.2.1. Given a set of distinct points in the outside of the unit circle, say

$$Z := \{z_0, \ldots, z_n\}$$

and a set of values in the right half of the complex plain, denoted by

$$W := \{w_0, \ldots, w_n\},$$

find a parameterization of all the functions $f(z)$ that they

1. fulfill the interpolation conditions, i.e. $f(z_k) = w_k$ for $k = 0, \ldots, n$;

2. are *positive-real*, i.e. they are analytic and have nonnegative real part outside the unit circle;

3. are rational of McMillan degree at most n.

Whereas condition 1) and 2) originate the Nevanlinna-Pick interpolation problem, that is a classical issue in complex analysis with important applications in systems and control theory (e.g. in robust stabilization and control), the third condition represents a genuine novelty. Classical theory dealing with Nevanlinna-Pick interpolation provides effective techniques that allow to compute rational solutions, but do not suggest a policy that is capable of obtaining the complete parameterization of all the rational solution with bounded degree. The fact that the interpolant is rational with bounded degree, however, is of capital importance in practical implementation. These considerations are the starting point for the definition of a new class of techniques. A leading role in this framework is played by convex optimization and duality theory. Indeed, a cost function, that in the case of interest is an index that measures the distance (or the pseudo-distance) between two spectral densities, is introduced. A convex functional, the so-called *primal Lagrangian function*, is then obtained by considering the cost function and the interpolation conditions. Unconstrained minimization of the primal function allows to discriminate the solutions which are compatible with the interpolation constraints. It is interesting to anticipate that information theory suggests many interesting metrics. Initial methods usually included the solution given by the maximum entropy spectrum as a special case, for instance. Whilst the primal problem is infinite dimensional, since the solution is given by a function, the dual one is finite dimensional. Moreover, it is possible to prove that a solution exists and it is unique. In principle, it can be computed trough gradient descent methods.

An implementation of this approach with regard to spectral estimation, focusing on the scalar case, has been exhaustively presented for the first time in [2]. The result is the so-called *Tunable High-Resolution Estimator*, (THREE) which is made up of three elements, as the name suggests:

- a bank of filters;
- the theory for parameterizing the set of the solutions that are consistent with the interpolation data provided by the filters;
- an algorithm that is capable of extracting a spectrum belonging to the previously introduced set.

In the following, it will be clarified how filters can be used to collect interpolation data about the spectral density function. Traditional methods need to collect as many covariance lags as possible, exploiting the fact that they provide the coefficients of the series expansion of $f(z)$ introduced in equation (2.9). THREE method, on the contrary, requires only to compute zeroth-order lags, assuring robustness and better

performances in the presence of short data records. In [2], the estimation problem is recast so that it turns out to be equivalent to fix the optimal values of the parameters of an ARMA model. It is proved that the MA parameters (also known as spectral zeros) can be chosen arbitrarily, as much as the poles of the filterbank. The poles of the ARMA model, however, are univocally defined once the MA part of the model and the poles of the filter are fixed. The particular case in which the poles and the spectral zeros coincide can be solved in closed form by means of linear equations. It correspond to the central solution of the Nevanlinna-Pick interpolation. Any other choice of the spectral zeros implies that a convex optimization problem has to be solved, whose features are the ones that have been introduced before. In particular, the cost function that is employed by THREE is a generalization of the entropy gain. Let Ψ be defined as

$$\Psi(z) = \frac{\rho(z)\rho(z^{-1})}{\tau(z)\tau(z^{-1})},$$

where $\rho(z)$ and $\tau(z)$ are monic polynomial with roots given by the spectral zeros and the filterbank poles, respectively. The cost function is defined as

$$\mathbb{I}_\Psi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[f(e^{j\theta}) + f(e^{-j\theta})]\Psi(e^{j\theta})d\theta.$$

The special case in which $\rho(z) = \tau(z)$ (i.e. $\Psi = 1$), that has been introduced before, leads to the solution that maximized the entropy gain of the spectrum. The dual problem is suitable to be solved by a Newton-type algorithm. A very interesting property of THREE is its effectiveness in discovering spectral lines and steep variation in the spectrum of interest. The reason is that the selection of the poles of the filter in the proximity of any arc of the unit circle allows to improve the resolution of the power spectrum in the corresponding frequency range.

The problem of the choice of the spectral zeros in the THREE method has naturally led the authors to consider the possibility to include the presence of some *a priori* information about the spectrum in the problem formulation. This topic has been developed in [10]. Available information, that could be achieved by means of a coarse estimation of the spectrum of interest, is expressed in the form of a spectral density Ψ to be approximated by the solution $\hat{\Phi}$. In order to measure the distance between spectral densities, an information-theoretic criterion is employed, namely the *Kullback-Leibler* (pseudo-)distance (also known as *relative entropy* or *information divergence*, see [11] and [6]):

$$D_{KL}(\Psi||\Psi) = \int \Psi \log \frac{\Psi}{\Phi}. \tag{2.14}$$

A state-space realization of the filterbank used to collect the interpolation data is given by

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{y}(k), \tag{2.15}$$

where (A, B) is a reachable pair, A is asymptotically stable (i.e. it has all its eigenvalues in the unit circle), B is assumed to be full column rank, for simplicity, and the filter is fed by the scalar, zero-mean stochastic process $\mathbf{y}$, whose power spectral density is to be estimated. The transfer function that describes the filter is given by

$$G(z) = (zI - A)^{-1}B. \tag{2.16}$$

The covariance of the output

$$\Sigma = \mathbb{E}\left[x_k x_k^*\right], \tag{2.17}$$

where * denotes the transpose and complex conjugate, can be estimated with traditional techniques from the observation of the steady state output of the filter. The problem of interest is then formalized as follows.

Problem 2.2.2. Given Σ and $\Psi \in \mathcal{S}_+(\mathbb{T})$, find $\hat{\Phi}$ that solves: minimize

$$D_{KL}(\Psi||\Phi)$$
$$\text{over } \{\Phi \in \mathcal{S}_+(\mathbb{T}) \mid \int G\Phi G^* = \Sigma\}$$

A key role is played by the operator $\Gamma : \mathcal{C}(\mathbb{T}, \mathcal{H}(m)) \to \mathcal{H}(n)$, such that:

$$\Gamma(\Phi) := \int G\Phi G^*. \tag{2.18}$$

Recall that $\mathcal{H}(n)$ denotes the set of the $\mathbb{C}^{n\times n}$-valued Hermitian matrices and $\mathcal{C}(\mathbb{T}, \mathcal{H}(m))$ the space of the continuous functions with values in $\mathcal{H}(m)$, defined on the unit circle. Clearly, the covariance Σ has to belong to Range(Γ) in order to make the problem feasible. It is interesting to point out the the minimization is performed in respect of $\Phi(e^{j\theta})$, that is the second argument of the function defined by eq. (2.14). This choice is imposed by the desire of obtaining maximum entropy solution as a special case (when $\Psi = 1$, as it has been previously described). Moreover, given that Ψ is rational, the solution of the approximation problem turns to be rational with a bound on its McMillan degree, that cannot exceed $\deg \Psi + 2n$, where n is equal to the degree of the transfer function realized by the filterbank. In this framework, under the hypothesis that the conditions of feasibility hold (the proof can be found in [9], for instance), it is possible to prove existence and uniqueness of the solution of the dual problem. A suitable Newton-type algorithm is described. However, it is affected by numerical instability due to the fact that the gradient tends to infinity in the vicinity of the boundary. An alternative approach (see for instance [19] and

[15]) is based on the iteration of a non-linear map on a bounded subset. In [19] a local convergence result is obtained and the successful thorough simulations motivate a conjecture about its global validity. In [15] a new proof of the existence and uniqueness of the solution of the Kullback-Leibler scalar approximation problem is presented. Moreover, the non-linear map algorithm is proved to be equivalent to a modified gradient descent method with fixed step size.

2.2.1 Multivariable Spectral Estimation

The multivariate case is much more challenging. In [10] the authors propose to employ the matricial Kullback-Leibler-von Neumann generalization of the scalar relative entropy as an index for optimization. This is defined by

$$D_{KL}(\Psi||\Phi) := \mathrm{Tr} \int \Psi \left(\log \Psi - \log \Phi\right),$$

where the function $\log(\cdot)$ denotes now the matricial logarithm, as defined in [7]. Even though it is possible to derive a closed expression for the solution in the simple case in which $\Psi = I$, in [16], it is shown that the general case is much harder. Indeed, the variational analysis cannot be carried out easily in the case of minimization in respect of the second argument. As a consequence, it seems that Kullback-Leibler distance cannot be generalized nicely to the multivariable case. On the contrary, another index, the Hellinger distance, that stems from statistics, turns out to be very effective. In the scalar setting, it is defined by

$$D_H(\Psi||\Phi) := \left[\int \left(\sqrt{\Phi} - \sqrt{\Psi}\right)^2\right]^{\frac{1}{2}}$$

The proposed multivariate extension is given by

$$\tilde{D}_H(\Psi||\Phi) := \left[\inf\left\{||W_\Psi - W_\Psi||_2^2 \,:\, W_\Psi, W_\Phi \in L_2^{m\times m},\, W_\Psi W_\Psi^* = \Psi,\, W_\Phi W_\Phi^* = \Phi\right\}\right]^{\frac{1}{2}} \tag{2.19}$$

It is proved that the infimum is a actually a minimum, that it provides a *bona fide* distance (relative entropy is not, conversely) and that it coincides with the L^2 distance between the set of all the spectral factors of Ψ and Φ, respectively. The correspondent multivariable problem has the same structure of the scalar problem 2.2.2:

Problem 2.2.3. Given Σ and $\Psi \in \mathcal{S}_+(\mathbb{T})^{m\times m}$, find $\hat{\Phi}$ that solves:

$$\begin{gathered}\text{minimize } D_H(\Psi||\Phi) \\ \text{over } \{\Phi \in \mathcal{S}_+(\mathbb{T})^{m\times m} \mid \textstyle\int G\Phi G^* = \Sigma\}\end{gathered}$$

Papers [16] and [17] describe a Newton-type matricial algorithm for the dual problem, that exploits some key systems theoretic ideas such as spectral factorization and the connected systematic usage of Lyapunov and Riccati equations in order to solve complex integrals. In particular, [17] tackles the issue of existence providing an exhaustive proof. Moreover, further results that prove the global convergence of the matricial Newton algorithm are achieved. In the scalar case $m = 1$ the usage of the Hellinger distance allows the computation of a solution with McMillan degree higher than the one that is attained via Kullback-Leibler approach. Indeed, in the Hellinger case the resulting upper bound on the degree is given by $\deg \Psi + 4n$, instead of $\deg \Psi + 2n$. This method, however, was the only one allowing to deal with the multivariate case efficiently, when it was introduced. Nevertheless, the fact that the degree is $2n$ higher can be very significant in practice. This work aims at proposing an alternative method that is able to manage the multivariate case, providing a lower upper bound on the complexity of the achieved solution. It will be shown that this bounds amounts to $\deg \Psi + 2n$, as in the Kullback-Leibler case approach.

Another significant contribution is given in [18], where the focus is on the *well-posedness* of the previously introduced constrained minimization problems. In particular, the scalar case with the Kullback-Leibler distance and the multivariable one with the Hellinger distance are tackled. A problem is said to be *well posed*, in the sense of Hadamard, if the following requirements are fulfilled:

1. a solution exists;
2. the solution is unique;
3. the solution depends continuously on the data.

In the case of interest, the purpose is to prove that the solution of the problem depends continuously on the covariance Σ. The main result of that paper is that both the problems 2.2.2 and 2.2.3 are well posed, for $\Sigma > 0$ and variations $\delta\Sigma$ belonging to Range(Γ), where Γ is defined as in equation (2.18). The importance of this result is clear in the concrete framework in which the real value of Σ can only be *estimated*, for instance by averaging on some records that are observed after the filter has reached the steady-state. A notable consequence of the results on well-posedness is that, if the estimate $\hat{\Sigma}$ of Σ is strongly consistent, i.e.

$$\lim_{N\to\infty} \hat{\Sigma}(x_1, \ldots, x_N) = \Sigma \quad \text{almost surely,}$$

then the solutions obtained starting from the estimate $\hat{\Sigma}$ converge to the one corresponding to Σ almost surely. This result holds not only for problems 2.2.2 and

2.2.3, but whenever the functional of the primal problem depends continuously on the Lagrange parameter Λ, whose optimal choice is the aim of the dual problem solution.

The results provided in [18] are significant in the real setting: the covariance of the filter output Σ has to be estimated and, since its estimate $\hat{\Sigma}$ is not the true variance, the feasibility condition could be violated. Therefore, a method is required in order to obtain the best approximation of the estimated covariance that makes the problem feasible. A simple-minded approach could be to project the estimate onto Range(Γ). This method, however, could make the estimate lose positive definiteness. Therefore, another approach has been proposed in [8]. The issue of the choice of the best approximation of the estimated state covariance is recast in the form of a stand-alone optimization problem, in the same spirit of the previously introduced ones. As a measure of distance between positive definite covariances in Range(Γ), an information criterion is employed, that is the same multivariate extension of the Kullback-Leibler distance that will be analyzed in Chapter 3. Duality theory is applied, leading to a convex optimization problem that can be solved via a matricial Newton algorithm. Simulations show that this method significantly outperforms the results that are obtained via simple projection of the estimated covariance on Range(Γ) and, in case, by the addition of a positive term in order to make the approximant positive definite. This is true both in the scalar and the multivariable framework (described by problem 2.2.2 and 2.2.3 on page 13, respectively). It is significant to point out that this method turns out to be more effective, even in the case that the starting covariances it generates are quite similar to the ones obtained via the projection method.

Chapter 3

Multivariate Spectral Estimation: Problem Statement and Preliminary Considerations

3.1 Problem Statement and Mathematical Model

Consider a zero-mean, wide-sense stationary $m-$dimensional process $\mathbf{y}(t)$, with $t \in \mathbb{Z}$, whose spectral density $\Phi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ is unknown. The aim is to estimate Φ. Denoting the unit circle by $\mathbb{T}$, recall that $\mathcal{S}_+^{m\times m}(\mathbb{T})$ is the set of the bounded and coercive[1] spectral density functions with values in $\mathbb{C}^{m\times m}$.

The method that is proposed is based on three elements:

1. information given in the form of the steady-state covariance Σ of the output of a proper bank of filters, fed by the process $\mathbf{y}$ (see figure 3.1);

2. a *prior* spectral density $\Psi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$;

3. a suitable index that measures the distance between spectral densities.

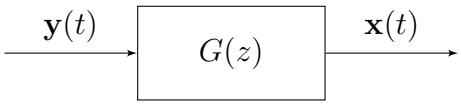


Figure 3.1: Filterbank fed by the input process $\mathbf{y}(t)$. The matrix Σ represents the covariance of the output process $\mathbf{x}(t)$.

Soon it will be clear that the role of the filterbank is to provide interpolation data for the spectral estimation. Regarding the spectral density Ψ, it allows to take into account the presence of *a priori* information, a contingency that is frequent in

[1]Recall that $\Phi(e^{j\theta})$ is said to be *coercive* if there exists a positive constant k such that $\Phi(e^{j\theta}) - kI > 0$ a.e. on $\mathbb{T}$

practice, for example when a coarse estimate of Φ is available. Since, in general, Ψ is not consistent with the interpolation conditions imposed by the filterbank, an approximation problem arises. It follows that it is necessary to introduce an index that measures how far the estimate $\hat{\Phi}$ is from the *prior* Ψ.

The purpose of this section is to analyze the ingredients that have been previously introduced. It is worthwhile to recall that this kind of model draws inspiration from the new convex optimization approach that has been developed, for instance, in [2], [10], [16] and [17], as already summarized in the state of-the-art section 2.2. Let us face these issues one at a time, starting from the role of the bank of filters.

3.1.1 The Bank of Filters

To begin with, a state-space realization of the bank of filters is introduced. It can be expressed by means of the difference equation

$$x(t+1) = Ax(t) + By(t), \quad t \in \mathbb{Z}, \tag{3.1}$$

whose corresponding transfer function is

$$G(z) = (zI - A)^{-1}B, \tag{3.2}$$

where $A \in \mathbb{C}^{n\times n}$ is a stability matrix (i.e. it has all its eigenvalues inside the unit circle $\mathbb{T}$), $B \in \mathbb{C}^{n\times m}$ is full rank, $n \geq m$ and (A, B) is a reachable pair. Moreover, although most of the theory could be developed in complete generality, only the case in which Ψ is rational will be considered. In fact, this is the case of interest in all practical situations. Actually, the results that will be achieved can be extended to the general case, at the price of a moderate extra burden.

As it was previously hinted, the filterbank provides the interpolation data. Indeed, a straightforward consequence of (2.7) and (2.13) is that the estimate of the spectrum of interest Φ has to satisfy the constraint

$$\int G\Phi G^* = \Sigma, \tag{3.3}$$

where Σ is the covariance matrix of the output $\mathbf{x}$ in (3.1). Two examples can make this result more significant, showing how the problem of interest has reference to classical issues such as *Nevanlinna-Pick interpolation* and the *covariance extension problem*. Before dealing with them, it is convenient to write the transfer function

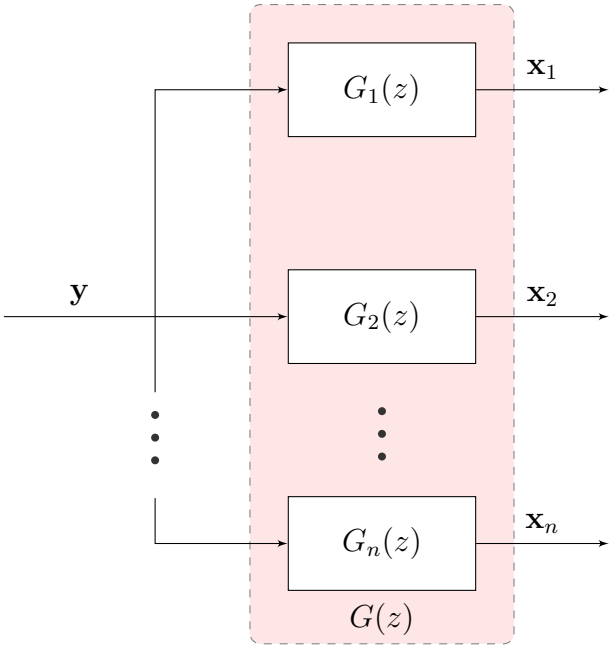


Figure 3.2: Block diagram of the bank of filters

$G(z)$ in the form

$$G(z) = \begin{bmatrix} G_1(z) \\ G_2(z) \\ \vdots \\ G_n(z) \end{bmatrix}.$$

whose corresponding block diagram representation is shown in figure 3.2.

Example 3.1.1. For the sake of simplicity, assume that the wide-sense stationary stochastic process $y(t)$ is scalar. Suppose $G_k(z)$ to be a first order stable filter, i.e.

$$G_k(z) = \frac{z}{z - p_k}, \qquad |p_k| < 1. \tag{3.4}$$

Feeding the filter with $y(t)$ the output stationary process $x(t)$ is attained. The difference equation that allows to compute $x(t)$ on the basis of the past and present values of the input process is the following:

$$x(t) = y(t) + p_k x(t).$$

As a result, it is possible to conclude that $x(t) = \sum_{h=0}^{\infty} {p_k}^h y(t-h)$. If p_k is real, the

output process is real too and its covariance can be expressed as $\mathbb{E}\left[x^2(t)\right]$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[x^2(t)\right] &= \mathbb{E}\left[\left(\sum_{h=0}^{\infty} {p_k}^h y(t-h)\right)^2\right] \\
&= \mathbb{E}\left[\left(y(t) + p_k y(t-1) + {p_k}^2 y(t-2) + \dots\right)^2\right] \\
&= \mathbb{E}[y^2(t)]\left(1 + p_k^2 + p_k^4 + \dots\right) + 2p_k\mathbb{E}\left[y(t)y(t-1)\right]\left(1 + p_k^2 + p_k^4 + \dots\right) \\
&\quad + 2{p_k}^2\mathbb{E}\left[y(t)y(t-2)\right]\left(1 + p_k^2 + p_k^4 + \dots\right) + \dots \\
&= r_0\left(1 + p_k^2 + p_k^4 + \dots\right) + 2p_k r_1\left(1 + p_k^2 + p_k^4 + \dots\right) \\
&\quad + 2{p_k}^2 r_2\left(1 + p_k^2 + p_k^4 + \dots\right) + \dots \\
&= \frac{2}{1 - {p_k}^2}\left(\frac{1}{2}r_0 + r_1 + r_2 + \dots\right) \\
&= \frac{2}{1 - {p_k}^2} f({p_k}^{-1}),
\end{aligned}$$

where the last equality can be written thanks to the additive decomposition described by eq. (2.9). The knowledge of the covariance provides the interpolation condition

$$f({p_k}^{-1}) = \frac{1}{2}\left(1 - {p_k}^2\right)\mathbb{E}\left[x^2(t)\right].$$

In [2] it was shown that an adequate choice of the filterbank poles can improve the resolution of the estimate in the corresponding frequency range. This approach differs from the traditional one, in which the spectrum of interest is estimated on the basis of the series expansion of $f(z)$ near infinity (since its coefficients are given by the covariance lags). If p_k is a complex pole, the covariance of the output process is given by

$$\mathbb{E}\left[x(t)x^*(t)\right] = \frac{1}{1 - |p_k|^2}\left(f({p_k}^{-1}) + f(\bar{p_k}^{-1})\right),$$

where $\bar{p_k}$ is the complex conjugate of p_k.

Assume $G_k(z)$ has the same structure of (3.4), for each $k = 1, \dots, n$. A state space realization could be given by the matrices:

$$A = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The covariance matrix of the n-dimensional output process attained by means of

such a filterbank is

$$\begin{aligned}\Sigma &= \mathbb{E}\left[x(t)\bar{x}(t)\right] \\ &= \begin{bmatrix} \frac{w_1+\bar{w}_1}{1-p_1\bar{p}_1} & \frac{w_1+\bar{w}_2}{1-p_1\bar{p}_2} & \dots & \frac{w_1+\bar{w}_n}{1-p_1\bar{p}_n} \\ \frac{w_2+\bar{w}_1}{1-p_2\bar{p}_1} & \frac{w_2+\bar{w}_2}{1-p_2\bar{p}_2} & \dots & \frac{w_2+\bar{w}_n}{1-p_2\bar{p}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n+\bar{w}_1}{1-p_n\bar{p}_1} & \frac{w_n+\bar{w}_2}{1-p_n\bar{p}_2} & \dots & \frac{w_n+\bar{w}_n}{1-p_n\bar{p}_n} \end{bmatrix},\end{aligned} \tag{3.5}$$

where, by means of (2.10), $w_k = f({p_k}^{-1}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{-j\theta}+p_k}{e^{-j\theta}-p_k}\Phi(e^{j\theta})d\theta$, $k = 1, \dots, n$. The covariance has the form of a Pick matrix. The problem of parameterizing the set of spectral density functions that satisfy (3.3) can be recast in the form a classical *Nevanlinna-Pick interpolation problem* (see section 2.2 on page 9 for a brief description of this issue). Recall that, since $\Sigma \geq 0$, such a problem admits a solution and if the covariance is a positive definite matrix, there are infinitely many solutions.

Example 3.1.2. Assume that the filter realizes the transfer function

$$G(z) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{n-2} \\ z^{n-1} \end{bmatrix},$$

whose state-space realization can be obtained by means of the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The k-th component of output of the filter, fed by $y(t)$, is given by its time-delayed version $y(t-k+1)$. Therefore, the covariance of the output process is equal to

$$\Sigma = \begin{bmatrix} r_0 & r_1 & \dots & r_{n-1} \\ \bar{r}_1 & r_0 & \dots & r_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{r}_{n-1} & \bar{r}_{n-2} & \dots & r_0 \end{bmatrix}.$$

The class of functions that satisfy (3.3), as in the previous case, is nonempty since

$\Sigma \geq 0$ and has infinitely many elements if $\Sigma > 0$. Such functions are given by all the possible extensions of the covariance sequence given by $r_0, \ldots, r_{n-1}$. In this case, the issue of finding a proper interpolant is known as the *covariance extension problem*. Such a problem is of paramount importance in many control applications.

3.1.2 On the Prior Density and the Necessity for a Distance Measure

As it was previously hinted, the proposed mathematical model allows to take into account a prior spectral density Ψ. In practice, a rough estimation of the spectral density of interest is often available (by means of traditional techniques such as periodogram, for instance). In [2], where an ARMA model is sought in order to work out the problem of spectral estimation, it is suggested that the *a priori* information could be used to determine an adequate choice for the spectral zeros. These can be fixed arbitrarily, but it is proved that choosing them near the effective spectral zeros improves the resolution (in terms of spectral lines and steep variations detection). In subsequent works, drawing inspiration from the same convex optimization approach, such as [10] and [16], it is proposed to represent *a priori* information in the form of a spectral density $\Psi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$. Since, in general, Ψ is not consistent with the constraint (3.3), that can be imposed once the state covariance is known, an approximation problem arises. The aim is to estimate Φ, so that the estimation procedure generates a spectral density that satisfies (3.3) and is as close as possible, in a sense that has to be defined, to the prior Ψ.

Therefore, the choice of the index employed to measure how far a spectral density is from another one tuns out to be a key point. This issue has been of crucial importance in the most recent developments of the approach that was firstly proposed in [2]. In particular, it widely depends on the choice of the distance measure whether or not it is possible to apply the method to the multivariate context. For instance, the Kullback-Leibler (pseudo-)distance (2.14) has been effectively employed in the scalar framework, as shown in [10]. However, its natural multivariate extension has turned to be very challenging to deal with (the reader is deferred to [16] for the analysis leading to this result). In [16] and [17] this issue has been resolved by means of an *ad hoc* multivariate extension of the Hellinger distance, see (2.19). It is worthwhile to recall that the proposed extension is such that the distance of two multivariate spectral densities amounts to the L_2 distance between the sets of their spectral factors. Such a method, however, provides an upper bound on the complexity of the estimate that is equal to $\deg \Psi + 4n$, whereas in the Kullback-Leibler case it amounts to $\deg \Psi + 2n$. The remainder can be very significant in practical

applications.

This work aims at proposing an approach that is effective in the multivariable context and, at the same time, provides an upper bound on complexity that is equal to $\deg \Psi + 2n$. Such an approach is based on the choice of a different distance measure, that is inspired to the scalar Kullback-Leibler distance, too. In order to explain how this index was attained, a brief review is exposed. Assume ρ and σ are probability density functions (of a continuous random variable). Then their Kullback-Leibler divergence is defined as

$$D_{KL}(\rho||\sigma) = \int \rho \log \frac{\rho}{\sigma}. \tag{3.6}$$

This distance index arises in the context of hypothesis testing as explained in [11] and in [6, chapter 11], for instance. $D_{KL}(p(x|H_0)||p(x|H_1))$ can be interpreted as the expected discrimination information for H_0 over H_1, i.e. the mean information per sample for discriminating in favour of a hypothesis H_0 against a hypothesis H_1, when hypothesis H_0 is true.
Kullback-Leibler distance has widespread applications in a variety of fields, from information theory to quantum mechanics. It is also known as *relative entropy*, *gain of information* or *Kullback-Leibler divergence*, and many generalizations and *ad hoc* formulations are available. Actually, relative entropy is not a proper distance measure, because it is not symmetric and it does not obey to the triangular inequality. Nevertheless, it is possible to prove that it is nonnegative and that it vanishes if and only if $\rho = \sigma$ (a.s.). Indeed, since $\log(\cdot)$ is a convex function, $-\log(\cdot)$ is convex and the Jensen's inequality holds:

$$\begin{aligned} D_{KL}(\rho||\sigma) &= \int \rho \log \frac{\rho}{\sigma} \\ &= -\int \rho \log \frac{\sigma}{\rho} \\ &= \mathbb{E}_\rho[\log \frac{\sigma}{\rho}] \\ &\geq \log \mathbb{E}_\rho[\frac{\sigma}{\rho}] \\ &\geq \log \int \sigma \\ &\geq 0. \end{aligned} \tag{3.7}$$

From Jensen's inequality it follows that $D_{KL}(\rho||\sigma) = 0$ if and only if $\rho = \sigma$ a.e. This is sufficient, in many applications, in order to ascribe a distance character to this index.

In [10] relative entropy is generalized to the case of scalar spectral density func-

tions. Given Ψ and Φ in $\mathcal{S}_+^{m\times m}(\mathbb{T})$, it has been defined as

$$D_{KL}(\Psi||\Phi) = \int \Psi \log \frac{\Psi}{\Phi}. \tag{3.8}$$

Since the normalization condition $\int \rho = \int \sigma = 1$ does not necessarily hold for spectral densities, the positivity of (3.8) has to be analyzed. From the inequality $\log(x) \leq x - 1$ and the monotonicity of the integral, it follows that:

$$\begin{aligned} D_{KL}(\Psi||\Phi) &= \int \Psi \log \frac{\Psi}{\Phi} \\ &= -\int \Psi \log \frac{\Phi}{\Psi} \\ &\geq \int \Psi \left(1 - \frac{\Phi}{\Psi}\right) \\ &\geq \int \Psi - \int \Phi \end{aligned} \tag{3.9}$$

If $\int \Psi = \int \Phi$, (3.9) allows to conclude that $D_{KL}(\Psi||\Phi) \geq 0$. Moreover, equality holds if and only if $\Psi = \Phi$. If $\lambda := \int \Phi$ is known, it is possible to rescale Ψ to $\tilde{\Psi}$ so that the minimum possible value of $D_{KL}\left(\tilde{\Psi}||\Phi\right)$ over the set $\mathcal{S}_+(\mathbb{T})$ is zero, that is achieved if and only if there exists $\hat{\Phi} \in \mathcal{S}_+(\mathbb{T})$ such that $\hat{\Phi} = \tilde{\Psi}$. Therefore, the minimization is performed with regards to $\tilde{\Psi}$, instead of Ψ. It means that the solution would turn out to be the best approximation of the *shape* of Ψ rather of Ψ itself. However, this is usually what is sought in practical situations (think about speech processing, for instance). In [10], it is proved that if the matrix A is singular, λ can be easily computed as

$$\lambda = \frac{v^*\Sigma v}{\|v^*B\|^2}.$$

Moreover, even though A is non singular, the problem of minimizing $D_{KL}(\Psi||\Phi)$ is still significant. Indeed, given $\hat{\Phi}$ that minimizes $D_{KL}\left(\Psi||\Phi\right)$, it is possible to normalize Ψ to $\tilde{\Psi}$ *a posteriori*, so that $\tilde{\Psi} = \frac{\int \hat{\Phi}}{\int \Psi}\Psi$ is such that $\int \tilde{\Psi} = \int \Phi$. As a

consequence,

$$\begin{aligned} D_{KL}(\tilde{\Psi}||\Phi) &= \int \tilde{\Psi} \log \frac{\tilde{\Psi}}{\Phi} \\ &= \Psi \frac{\int \hat{\Phi}}{\int \Psi} \left[\log \Psi + \log \int \hat{\Phi} - \log \int \Psi - \log \Psi \right] \\ &= \frac{\int \hat{\Phi}}{\int \Psi} \int \left[\Psi \log \frac{\Psi}{\Phi} \right] - \frac{\int \hat{\Phi}}{\int \Psi} \int \left[\Psi \log \frac{\Psi}{\hat{\Phi}} \right] \\ &= \frac{\int \hat{\Phi}}{\int \Psi} \int \left[\Psi \log \frac{\Psi}{\Phi} \right] + \int \hat{\Phi} \log \frac{\int \hat{\Phi}}{\int \Psi} \\ &= \alpha D_{KL}(\Psi||\Phi) + \beta. \end{aligned} \tag{3.10}$$

where $\alpha = \frac{\int \hat{\Phi}}{\int \Psi} > 0$ and $\beta = \int \hat{\Phi} \log \frac{\int \hat{\Phi}}{\int \Psi}$ are constants. Therefore, the minimizing Φ would be the same, justifying the usage of $D_{KL}(\Psi||\Phi)$ as an index to measure how far Φ is from the prior.

The extension to the multivariate setting proposed in [10] is based on the Kullback-Leibler-Von Neumann distance. This index is usually related to quantum mechanics. Let ρ and σ be Hermitian matrices with unit trace. These are employed in statistical quantum systems theory to describe the state of an $n-$level system. The quantum relative entropy of ρ with respect of σ is given by

$$S(\rho||\sigma) = \mathrm{Tr}\left[\rho \left(\log \rho - \log \sigma\right)\right]. \tag{3.11}$$

it follows from Klein's inequality that $S(\rho||\sigma) \geq 0$, and that $S(\rho||\sigma) = 0$ if and only if $\rho = \sigma$. This is the so-called Von Neumann-Umegaki extension of the Kullback-Leibler relative entropy. It suggests the following distance-type index between two multivariate spectral densities Ψ and Φ:

$$D_{KLvNU}(\Psi||\Phi) = \mathrm{Tr} \int \left[\Psi \left(\log \Psi - \log \Phi\right)\right]. \tag{3.12}$$

Its features are analyzed in [16]. In particular, the issue of nonnegativity is tackled. It is proved that the index is nonnegative if $\int \mathrm{Tr}\, \Psi = \int \mathrm{Tr}\, \Phi$. Under this assumption it vanishes if and only if $\Psi = \Phi$. Therefore, in the same spirit of what has been done in the case of the scalar version, in general it is necessary to rescale Ψ *a posteriori*, considering the minimization of $D_{KLvN}(\tilde{\Psi}||\Phi)$ with $\tilde{\Psi} = \frac{\int \mathrm{Tr}\, \hat{\Phi}}{\int \mathrm{Tr}\, \Psi} \Psi$. As previously hinted, the variational analysis of the primal functional arising from (3.12) shows that an explicit form for the solution $\hat{\Phi}$ in terms of Ψ it cannot be obtained. This prevents us from using this distance for our estimation purposes.

Therefore, the necessity of resorting to other indexes, in order to successfully

deal with the multivariate spectral estimation problem, arises. This work proposes a different usage of the relative entropy index, that gives encouraging results for spectral estimation. As usual, assume Ψ, $\Phi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$. Then, the distance-measure between spectral densities that will be considered is given by

$$\begin{aligned} D(\Phi\|\Psi) &:= \int \frac{1}{2}\left[\log\frac{\det\Psi}{\det\Phi} + \mathrm{Tr}(\Psi^{-1}\Phi) - m\right] \\ &= \int_{-\pi}^{\pi} \frac{1}{2}\left[\log\frac{\det\Psi(e^{j\theta})}{\det\Phi(e^{j\theta})} + \mathrm{Tr}(\Psi(e^{j\theta})^{-1}\Phi(e^{j\theta})) - m\right]\frac{d\theta}{2\pi}. \end{aligned} \tag{3.13}$$

The choice of this index draws inspiration from the expression assumed by the relative entropy when two Gaussian probability density functions, relating to a (real) random vector, are compared. Recall that the probability density function of a Gaussian random vector is defined once its mean and covariance are known. In particular, $\mathbf{x} \sim \mathcal{N}(\mu_\rho, \Sigma_\rho)$, with $\Sigma_\rho \in \mathbb{R}^{n\times n}$, exhibits the density function

$$\rho(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma_\rho)}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu_\rho)^\top {\Sigma_\rho}^{-1}(\mathbf{x}-\mu_\rho)\right], \tag{3.14}$$

Assume σ to describe a random vector $\mathbf{x} \sim \mathcal{N}(\mu_\sigma, \Sigma_\sigma)$. Since the second order description is equivalent to the complete statistical description of a Gaussian random vector, is predictable that the relative entropy will depend only on means and covariance matrices. Firstly, the relative entropy is written in the form of a difference of expectations.

$$\begin{aligned} D_{KL}(\rho||\sigma) &= \int \rho\log\frac{\rho}{\sigma} \\ &= \int \rho\log\rho - \int\rho\log\sigma \\ &= \mathbb{E}_\rho\left[\log\rho\right] - \mathbb{E}_\rho\left[\log\sigma\right]. \end{aligned}$$

Secondly, the explicit expression of the probability density functions is inserted, dealing separately with the two addends.

$$\begin{aligned} \mathbb{E}_\rho\left[\log\rho\right] &= \mathbb{E}_\rho\left[-\frac{1}{2}(\mathbf{x}-\mu_\rho)^\top{\Sigma_\rho}^{-1}(\mathbf{x}-\mu_\rho) - \frac{1}{2}\log(2\pi)^n\det\Sigma_\rho\right] \\ &= -\frac{1}{2}\mathbb{E}_\rho\left[\mathrm{Tr}\,(\mathbf{x}-\mu_\rho)(\mathbf{x}-\mu_\rho)^\top{\Sigma_\rho}^{-1}\right] - \frac{1}{2}\log(2\pi)^n\det\Sigma_\rho \\ &= -\frac{n}{2} - \frac{1}{2}\log(2\pi)^n\det\Sigma_\rho. \end{aligned}$$

Notice that the linearity of the trace operator has been exploited in order to replace

$\mathbb{E}[\mathrm{Tr}]$ with $\mathrm{Tr}[\mathbb{E}]$. Regarding the second expectation term,

$$\begin{aligned}
\mathbb{E}_\rho[\log\sigma] &= \mathbb{E}_\rho\left[-\frac{1}{2}(\mathbf{x}-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}(\mathbf{x}-\mu_\sigma) - \frac{1}{2}\log(2\pi)^n \det\Sigma_\sigma\right] \\
&= -\frac{1}{2}\mathbb{E}_\rho\left[\mathrm{Tr}\,(\mathbf{x}-\mu_\sigma)(\mathbf{x}-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}\right] - \frac{1}{2}\log(2\pi)^n\det\Sigma_\sigma \\
&= -\frac{1}{2}\mathbb{E}_\rho\left[\mathrm{Tr}\,(\mathbf{x}-\mu_\rho+\mu_\rho-\mu_\sigma)(\mathbf{x}-\mu_\rho+\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}\right] - \frac{1}{2}\log(2\pi)^n\det\Sigma_\sigma \\
&= -\frac{1}{2}\mathbb{E}_\rho\left[\mathrm{Tr}\,(\mathbf{x}-\mu_\rho)(\mathbf{x}-\mu_\rho)^\top {\Sigma_\sigma}^{-1} + 2\,\mathrm{Tr}\,(\mathbf{x}-\mu_\rho)(\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}\right. \\
&\quad \left.+(\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}(\mu_\rho-\mu_\sigma)\right] - \frac{1}{2}\log(2\pi)^n\det\Sigma_\sigma \\
&= -\frac{1}{2}\left[\mathrm{Tr}\,\Sigma_\rho{\Sigma_\sigma}^{-1} + (\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}(\mu_\rho-\mu_\sigma)\right] - \frac{1}{2}\log(2\pi)^n\det\Sigma_\sigma.
\end{aligned}$$

where it has been used the fact that

$$\mathbb{E}_\rho\left[\mathrm{Tr}\,(\mathbf{x}-\mu_\rho)(\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}\right]$$

is null, because $\mathbb{E}_\rho[(\mathbf{x}-\mu_\rho)] = 0$. Finally, the relative entropy can be written as

$$\begin{aligned}
D_{KL}(\rho||\sigma) &= \mathbb{E}_\rho[\log\rho] - \mathbb{E}_\rho[\log\sigma] \\
&= -\frac{n}{2} - \frac{1}{2}\log(2\pi)^n\det\Sigma_\rho + \frac{1}{2}\left[\mathrm{Tr}\,\Sigma_\rho{\Sigma_\sigma}^{-1} + (\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}(\mu_\rho-\mu_\sigma)\right] \\
&\quad + \frac{1}{2}\log(2\pi)^n\det\Sigma_\sigma \\
&= \frac{1}{2}\left[\mathrm{Tr}\,\Sigma_\rho{\Sigma_\sigma}^{-1} + (\mu_\rho-\mu_\sigma)^\top {\Sigma_\sigma}^{-1}(\mu_\rho-\mu_\sigma) + \log\frac{\det\Sigma_\sigma}{\det\Sigma_\rho} - n\right].
\end{aligned} \tag{3.15}$$

Starting from this result, the new distance measure between spectral density functions (3.13) has been defined. The argument can be interpreted as the relative entropy between two Gaussian densities, such that, for all $e^{j\theta} \in \mathbb{T}$, $\Psi(e^{j\theta})$ and $\Phi(e^{j\theta})$ play the role of covariance matrices. Therefore integration on the unit circle is introduced. In addition, the term depending on the mean values disappears, because the stochastic processes of interest are assumed to have zero mean. It is worthwhile to recall that the case that will be dealt with corresponds to the choice of considering $D(\Phi||\Psi)$. As it will be explained in section 4.2, see remark 4.2.1 on page 38, variational analysis shows that the alternative choice $D(\Psi||\Phi)$ is not convenient.

Remark 3.1.1. The nonnegativity of the distance measure (3.13) is patent, because it is a natural consequence of the fact that relative entropy is nonnegative and that, pointwise, the spectral densities $\Phi(e^{j\theta})$ and $\Psi(e^{j\theta})$ behave exactly like the covariance

matrices Σ_σ and Σ_ρ in (3.15): they are hermitian and positive definite. Since the integrand in (3.13) is nonnegative, such is the distance measure. Moreover, no rescaling of the prior spectral density is required.

3.2 Problem Formalization and Feasibility

It is now possible to present a precise formulation of the issue of interest:

Problem 3.2.1. (Approximation problem) Let $\Psi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ and let $\Sigma \in \mathbb{C}^{m\times m}$ such that $\Sigma = \Sigma^* > 0$. Find $\hat{\Phi}$ that solves

$$\text{minimize } \mathrm{D}(\Phi||\Psi)$$

$$\text{over } \left\{\Phi \in \mathcal{S}_+^{\mathrm{m}\times\mathrm{m}}(\mathbb{T}) | \int \mathrm{G}\Phi\mathrm{G}^* = \Sigma\right\}$$

In order to tackle the problem, a convex optimization approach, in the spirit of [10] and [16], will be developed. To begin with, the first issue to face is the one of feasibility. The question to answer for is whether, given a bank of filters of the form (2.15) and the steady-state covariance of its output, Σ, a coercive spectral density function $\Phi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ exists such that (3.3) holds, i.e. $\int G\Phi G^* = \Sigma$. A convenient way to deal with this problem is to introduce a suitable operator. Let $\mathcal{H}(n) = \{M \in \mathbb{C}^{n\times n} \,|\, M = M^*\}$ and $\mathcal{C}(\mathbb{T}; \mathcal{H}(m))$ the space of the continuous functions defined on the unit circle with values in $\mathcal{H}(m)$. The operator $\Gamma : \mathcal{C}(\mathbb{T}; \mathcal{H}(m)) \to \mathcal{H}(n)$ is defined by

$$\Gamma(\Phi) := \int G\Phi G^*. \tag{3.16}$$

As regards the range of the above defined operator, i.e. the set $\text{Range}\,(\Gamma) := \{M \in \mathcal{H}(n) \,|\, \exists \Phi \in \mathcal{C}(\mathbb{T}; \mathcal{H}(m))$ such that $\int G\Phi G^* = M\}$, the following theorem holds (the proof can be found in [17] and [9]):

Theorem 3.2.1. *Consider $\Sigma \in \mathcal{H}(n)$ and a system described by (3.1), where A is (asymptotically) stable, B is full column rank and $(A; B)$ is a reachable pair. Then:*

1. *Σ is in* Range(Γ) *if and only if there exists $H \in \mathbb{C}^{m\times n}$ such that*

$$A - A\Sigma A^* = BH + H^*B^*. \tag{3.17}$$

2. *Σ is in* Range(Γ) *if and only if it satisfies the rank condition*

$$\text{rank}\begin{bmatrix} \Sigma - A\Sigma A^* & B \\ B^* & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}. \tag{3.18}$$

3. Let Σ be positive definite. The following facts are equivalent:

- *There exists $H \in \mathbb{C}^{m\times n}$ that solves* (3.17).
- *There exists $\Phi \in \mathcal{C}(\mathbb{T}; \mathcal{H}(m))$, $\Phi > 0$, such that $\Gamma(\Phi) = \Sigma$.*
- *There exists $\Phi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ such that $\int G\Phi G^* = \Sigma$.*

As a consequence, for $\Sigma \in \text{Range}\,(\Gamma)$ s.t. $\Sigma > 0$, the Problem 3.2.1 is feasible, because there exists a bounded and coercive spectral density Φ such that the constraint (3.3) is satisfied.

Remark 3.2.1. Assume that the hypotheses of Theorem 3.2.1 hold and consider an Hermitian matrix $\Sigma > 0$. It is interesting to derive an expression for a matrix $H \in \mathbb{C}^{m\times n}$ which solves equation (3.17). Let Φ be a spectral density such that $\int G\Phi G^* = \Sigma$. Recall that the transfer function realized by the system (3.1) has the form (2.16). Therefore, $G(z) = AG(z)z^{-1} + Bz^{-1}$. Then, it is possible to write

$$\begin{aligned}
\Sigma &= \int G\Phi G^* \\
&= \int \left(AG(e^{j\theta})e^{-j\theta} + B\right)\Phi(e^{j\theta})\left(AG(e^{j\theta})e^{-j\theta} + B\right)^* \\
&= \int AG(e^{j\theta})\Phi(e^{j\theta})G^*(e^{j\theta})A^* + \int B\Phi(e^{j\theta})G^*(e^{j\theta})A^* \\
&\quad + \int AG(e^{j\theta})\Phi(e^{j\theta})B^* + \int B\Phi(e^{j\theta})B^* \qquad (3.19)\\
&= A\Sigma A^* + B\int \left[\frac{1}{2}\Phi(e^{j\theta})B^* + \Phi(e^{j\theta})G^*(e^{j\theta})A^*\right]\frac{d\theta}{2\pi} \\
&\quad + \int \left[\frac{1}{2}B\Phi(e^{j\theta}) + AG(e^{j\theta})\Phi(e^{j\theta})\right]B^* \\
&= A\Sigma A^* + BH + H^*B^*.
\end{aligned}$$

In view of (3.19), an expression of $H \in \mathbb{C}^{m\times n}$ such that (3.17) holds is obtained. It is worthwhile to analyze it, in order to make clearer its relation with the statistical description of the input process. Recalling that $G(z) = (zI - A)^{-1}B =$

$\sum_{k=1}^{\infty} A^{k-1}Bz^{-k}$, it is possible to write:

$$\begin{aligned} H &= \int \left[\frac{1}{2}\Phi(e^{j\theta})B^* + \Phi(e^{j\theta})G^*(e^{j\theta})A^*\right]\frac{d\theta}{2\pi} \\ &= \int \left[\frac{1}{2}\Phi(e^{j\theta})B^* + \Phi(e^{j\theta})(Be^{-j\theta} + ABe^{-j2\theta} + A^2Be^{-j3\theta} + \dots)^*A^*\right]\frac{d\theta}{2\pi} \\ &= \int \left[\frac{1}{2}\Phi(e^{j\theta})B^* + \Phi(e^{j\theta})(B^*e^{j\theta} + B^*A^*e^{j2\theta} + B^*(A^*)^2e^{j3\theta} + \dots)A^*\right]\frac{d\theta}{2\pi} \\ &= \int \left[\frac{1}{2}\Phi(e^{j\theta})B^* + \Phi(e^{j\theta})e^{j\theta}B^*A^* + \Phi(e^{j\theta})e^{j2\theta}B^*(A^*)^2 + \dots\right]\frac{d\theta}{2\pi} \\ &= \begin{bmatrix}\frac{1}{2}r_y(0) & r_y(1) & r_y(2) & \dots\end{bmatrix}\begin{bmatrix} B^* \\ B^*A^* \\ B^*(A^*)^2 \\ \vdots \end{bmatrix}, \end{aligned} \tag{3.20}$$

where the result follows from the fact that covariance lags $r_y(k)$ (recall that $\mathbf{y}$ is assumed to be zero-mean) can be evaluated on the basis of the spectral density $\Phi(e^{j\theta})$ via inverse Fourier transform:

$$r_y(k) = \int_{-\pi}^{\pi} \Phi(e^{j\theta})e^{jk\theta}\frac{d\theta}{2\pi}.$$

Another condition equivalent to the facts of Theorem 3.2.1, in the case in which $\Sigma > 0$, involves geometric considerations. It states that:

Proposition 3.2.1. *A positive definite matrix* $\Sigma >\in \mathcal{H}(n)$ *belongs to* Range (Γ), *i.e. there exists* $\Phi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ *such that* $\int G\Phi G^* = \Sigma$, *if and only if*

$$(I - \Pi_B)(\Sigma - A\Sigma A^*)(I - \Pi_B) = 0$$

where $\Pi_B = B(B^*B)^{-1}B^*$ *is the orthogonal projection onto* Range B.

The proof can be found in [8]. It is reported here for the sake of completeness.

Proof. As regards necessity, it easily follows from algebraic manipulations applied

to equation (3.17). Indeed,

$$\begin{aligned}(I-\Pi_B)(\Sigma - A\Sigma A^*)(I-\Pi_B) &= (I-\Pi_B)(BH+H^*B^*)(I-\Pi_B)\\ &= BH - BH\Pi_B + H^*B^* - H^*B^*\Pi_B - \Pi_B BH\\ &\quad - \Pi_B H^*B^* + \Pi_B BH\Pi_B + \Pi_B H^*B^*\Pi_B\\ &= BH - BHB(B^*B)^{-1}B^* + H^*B^*\\ &\quad - H^*B^* - BH - B(B^*B)^{-1}B^*H^*B^*\\ &\quad + BHB(B^*B)^{-1}B^* + B(B^*B)^{-1}B^*H^*B^*\\ &= 0.\end{aligned} \tag{3.21}$$

As for sufficiency, it is possible to achieve the result by means of (3.18). To begin with, a new matrix is defined

$$T := \begin{bmatrix} C & B \end{bmatrix} \in \mathbb{C}^{n\times n},$$

where $C \in \mathbb{C}^{n\times(n-m)}$ has full column rank and it is such that $\operatorname{Range} C \perp \operatorname{Range} B$ (i.e. $\forall x \in \mathbb{C}^{(n-m\times 1)}$, $y \in \mathbb{C}^{m\times 1}$, it holds that the scalar product $\langle Cx, By\rangle = x^*C^*By$ is null). The matrix C is chosen in order to guarantee that T is invertible. Moreover, since C is full column rank and its range is orthogonal to $\operatorname{Range} B$, it is possible to express to express it as $C = (I-\Pi_B)V$, where $V \in \mathbb{C}^{n\times(n-m)}$ has full column rank. From the necessity part and the new expression of C it follows that

$$C^*(\Sigma - A\Sigma A^*)C = 0.$$

Consider now the matrix

$$\begin{bmatrix} \Sigma - A\Sigma A^* & B \\ B^* & 0 \end{bmatrix},$$

a congruent matrix Δ is obtained via

$$\Delta := \begin{bmatrix} T^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma - A\Sigma A^* & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} T^*\left(\Sigma - A\Sigma A^*\right)T & T^*B \\ B^*T & 0 \end{bmatrix}.$$

As a consequence,

$$\Delta = \begin{bmatrix} C^*\left(\Sigma - A\Sigma A^*\right)C & C^*\left(\Sigma - A\Sigma A^*\right)B & C^*B \\ B^*\left(\Sigma - A\Sigma A^*\right)C & B^*\left(\Sigma - A\Sigma A^*\right)B & B^*B \\ B^*C & B^*B & 0 \end{bmatrix} = \begin{bmatrix} 0 & \star & 0 \\ \star & \star & B^*B \\ 0 & B^*B & 0 \end{bmatrix}. \tag{3.22}$$

Since B^*B is invertible (recall that B is assumed to have full column rank) and matrix rank is not changed by multiplication by an invertible matrix, it follows that

$$\operatorname{rank}\begin{bmatrix}\Sigma - A\Sigma A^* & B\\ B^* & 0\end{bmatrix} = \operatorname{rank}\Delta = 2m.$$

Finally, result (3.18) allows to conclude that Σ belongs so $\operatorname{Range}\Gamma$. □

Next, a result on the dimension of $\operatorname{Range}(\Gamma)$ is proved.

Proposition 3.2.2. *The* real *dimension of the linear space* $\operatorname{Range}(\Gamma)$ *is equal to* $m(2n-m)$

This result is proved in [8], along the same lines that follow:

Proof. To begin with, it is worthwhile to notice that the real dimension of $\operatorname{Range}(\Gamma)$ amounts to the real dimension of the linear space of the matrices that can be written in the form $BH + H^*B^*$, with $H \in \mathbb{C}^{m\times n}$. This is a consequence of Theorem 3.2.1. A change of basis in the state space of the system (3.1) does not affect $\operatorname{Range}(\Gamma)$. Under the hypothesis that B has full column rank, the change of basis can be chosen so that it is possible to write B such as $B := \begin{bmatrix}I_m\\ 0\end{bmatrix}$. Therefore, partitioning H so that $H = \begin{bmatrix}H_1 & H_2\end{bmatrix}$, with $H_1 \in \mathbb{C}^{m\times m}$ and $H_2 \in \mathbb{C}^{m\times(n-m)}$, each matrix that can be expressed as $BH + H^*B^*$, has the form

$$M := BH + H^*B^* = \begin{bmatrix}I_m\\ 0\end{bmatrix}\begin{bmatrix}H_1 & H_2\end{bmatrix} + \begin{bmatrix}H_1^*\\ H_2^*\end{bmatrix}\begin{bmatrix}I_m & 0\end{bmatrix}$$

As a consequence, M can be written as

$$M = \begin{bmatrix}H_1 + H_1^* & H_2\\ H_2^* & 0\end{bmatrix}$$

Since the element in position $(1,1)$ is real, whereas H_2 can assume complex values, in general, the resulting *real* dimension is given by $m^2 + 2\left[m(n-m)\right] = m(2n-m)$. □

A simplifying assumption that will be employed from now on is that $\Sigma = I$. However, this is not an oversimplification. The hypothesis that the covariance matrix is positive definite already holds. Therefore, it is sufficient to consider the matrix transformation induced by $\Sigma^{\frac{1}{2}}$. Apply the coordinate change only to the matrices A

and B, so that:

$$\tilde{A} \leftarrow \Sigma^{-\frac{1}{2}} A \Sigma^{\frac{1}{2}};$$
$$\tilde{B} \leftarrow \Sigma^{-\frac{1}{2}} B;$$

Now the system has the transfer function $\tilde{G}(e^{j\theta}) = \Sigma^{-\frac{1}{2}} G(e^{j\theta})$. As a consequence

$$\int \tilde{G} \Phi \tilde{G}^* = I.$$

Henceforth, $\tilde{A}, \tilde{B}, \tilde{G}$ will be referred as A, B, G and the covariance matrix will be assumed to be the identity matrix.

Chapter 4

A Convex Optimization Approach to Multivariate Spectral Estimation

4.1 Convex Optimization Approach: Motivations

This work burgeons along the same lines developed in [10], focusing on the multivariate framework such as [16] and [17]. In the following, we assume that the feasibility condition 3.17 holds. The aim is to recast the issue of multivariate spectral estimation so that it can be efficiently worked out by means of suitable algorithmic procedures. To this purpose, the formalization that has been introduced so far has two key elements: a prior spectral density and a distance measure. This has been done because it naturally gives rise to a convex optimization problem, that can be dealt with by means of traditional tools such as duality theory and variational analysis. It will be proven that the solution to Problem 3.2.1 exists and it is unique. Moreover, it can be computed via consolidated techniques, such as a Newton type algorithm. Of course, this should be implemented in a smart manner, so as to achieve the best performances.

Preliminary, it is worthwhile to recall that the choice of the duality theory approach is motivated by the fact that, in the case of interest, it allows to solve a challenging problem (the starting one, that henceforth will be called primal problem) through the solution of a simpler one (the corresponding dual problem). Indeed, the primal problem aims at finding the optimal approximant directly. Therefore it is infinite-dimensional, because the minimization of a proper functional, that will be introduced soon, takes place over the infinite dimensional space $\mathcal{S}_+^{m\times m}(\mathbb{T})$. On the contrary, the corresponding dual problem is finite dimensional, because the optimization is carried out over a subset of $\mathcal{H}(n)$. Since the previously introduced formalization allows to solve the dual problem manageably and the solution of the

primal one can be straightforwardly obtained once the solution of the dual problem is known, this method turns out to be really effective and suitable for a numeric implementation.

In order to make the following part clearer, some fundamental notions about duality theory are recalled in Appendix B.

4.2 Variational Analysis

In this section the focus is on the Lagrangian and the dual Lagrangian function arising from Problem 3.2.1 on page 28. As it has been already mentioned, the choice of the distance measure allows to elaborate the variational analysis, so that a closed form for the optimal solution of the primal problem is obtained. In addition, the dual problem is deeply analyzed, and it is proved that the solution is unique, if it exists. The issue of existence is much more challenging and it will be dealt with separately. Moreover, the policy that will be proposed in order to prove that Problem 3.2.1 admits a solution, leads to meaningful achievements in the light of an algorithmic implementation of the estimation procedure. Indeed, the key result stated in Theorem 4.3.1 allows to conclude both on the existence of a solution for the dual problem and the global convergence of a Newton-type algorithm that can be employed to compute it.

To deal with analysis, it is necessary to introduce the scalar product between two n-dimensional Hermitian matrices A, B. It will defined by

$$\langle A, B \rangle := \mathrm{Tr}(AB^*). \tag{4.1}$$

This choice represents the natural extension of the standard scalar product between vectors $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^k$:

$$\langle \mathbf{a}, \mathbf{b} \rangle := \mathbf{a}^* \mathbf{b}.$$

Indeed, by defining $\mathbf{a}$ and $\mathbf{b}$ as the $(n \times n)$-dimensional vectors that are obtained by putting in a column the columns of A and B, respectively, it turns out that

$$\langle A, B \rangle = \mathrm{Tr}\left(AB^*\right) = \mathbf{a}^* \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle .$$

Now it is possible to write the Lagrangian function corresponding to Problem

3.2.1. Remind that the simplifying assumption $\Sigma = I$ holds (see Section 3.2).

$$\begin{aligned} L_\Psi(\Phi, \Lambda) &= \int \left[\log \frac{\det(\Psi)}{\det(\Phi)} + \mathrm{Tr}(\Psi^{-1}\Phi) \right] + \left\langle \Lambda, \int G\Phi G^* - I \right\rangle \\ &= \int \left[\log \frac{\det(\Psi)}{\det(\Phi)} + \mathrm{Tr}(\Psi^{-1}\Phi) + \mathrm{Tr}(\Lambda G\Phi G^*) \right] - \mathrm{Tr}\,\Lambda. \end{aligned} \tag{4.2}$$

The first addend comes from the distance measure to minimize, see eq. (3.13). The second takes into account the constraint (3.3). $\Lambda \in \mathcal{H}(n)$ is the Lagrange multiplier and plays the role of the coefficients λ_i encountered in Appendix B. Each $\Lambda \in \mathcal{H}(n)$ can be decomposed in only one way as the sum of a term in Range (Γ) and a term that is orthogonal to Range (Γ). The orthogonal direct sum takes the form:

$$\Lambda = \Lambda_\Gamma \oplus \Lambda_\perp, \quad \Lambda_\Gamma \in \mathrm{Range}\,(\Gamma),\ \Lambda_\perp \in (\mathrm{Range}\,(\Gamma))^\perp.$$

Λ belongs to $(\mathrm{Range}\,(\Gamma))^\perp$ if and only if, $\forall\, \Phi \in \mathcal{C}(\mathbb{T}; \mathcal{H}(m)$, it holds that

$$\begin{aligned} 0 &= \left\langle \Lambda, \int G\Phi G^* \right\rangle \\ &= \mathrm{Tr}\left[\int G\Phi G^*\Lambda \right] \\ &= \mathrm{Tr}\left[\int G^*\Lambda G\Phi \right], \end{aligned}$$

where the cyclic property of the trace has been employed. As a consequence, $\forall\, \Lambda \in (\mathrm{Range}\,(\Gamma))^\perp$, the following identity is satisfied:

$$G^*(e^{j\theta})\Lambda G(e^{j\theta}) \equiv 0, \quad \forall \theta \in [-\pi, \pi]\,. \tag{4.3}$$

Moreover, for each $\Lambda \in (\mathrm{Range}\,(\Gamma))^\perp$, $\mathrm{Tr}\,[\Lambda] = \langle \Lambda, I\rangle = 0$, because $I \in \mathrm{Range}\,(\Gamma)$ in views of the feasibility assumption. Hence, a term $\Lambda_\perp \in (\mathrm{Range}\,(\Gamma))^\perp$ gives no contribution to the Lagrangian (4.2). Therefore, the Lagrange parameter Λ can be assumed to belong to Range(Γ).

The purpose is now to pursue the *unconstrained* minimization of the functional (4.2). In this setting the role of partial derivatives is played by the directional derivatives (also knowns as *Gateaux derivatives*). We remark that $L_\Psi(\cdot, \cdot)$, defined as in (4.2), is strictly convex in Φ. This is also a result of the choice of distance measure. Moreover, the Lagrangian function in (4.2) is differentiable in Φ, in the sense of directional derivatives, i.e. it has continuous directional derivatives in whatever direction $\delta\Phi \in \mathcal{C}(\mathbb{T})$. As a consequence, the unconstrained minimization is realized by imposing that the first variation is zero in each direction. For $\delta\Phi \in \mathcal{C}(\mathbb{T})$, the

first variation is given by

$$\delta L(\Phi, \Lambda; \delta\Phi) = \int \left[-\operatorname{Tr}(\Phi^{-1}\delta\Phi) + \operatorname{Tr}(\Psi^{-1}\delta\Phi) + \operatorname{Tr}(G^*\Lambda G\delta\Phi) \right]. \tag{4.4}$$

This results comes from the properties of the trace operator and the fact that

$$\delta \log\det(X, \delta X) = \operatorname{Tr}(X^{-1}\delta X). \tag{4.5}$$

Therefore, (4.4) is zero $\forall\, \delta\Phi \in C(\mathbb{T})$ if and only if

$$\Phi = \left[\Psi^{-1} + G^*\Lambda G \right]^{-1}. \tag{4.6}$$

This value is defined as Φ°. Afterwards, it will be interesting to consider also some alternative forms for the expression of Φ°:

$$\Phi^\circ = \left[\Psi^{-1} + G^*\Lambda G \right]^{-1}; \tag{4.7}$$

$$\Phi^\circ = W_\Psi (I + G_1^*\Lambda G_1)^{-1} W_\Psi^*; \tag{4.8}$$

$$\Phi^\circ = \Psi - \Psi G^*(\Lambda^{-1} + G\Psi G^*)G\Psi; \tag{4.9}$$

where $G_1(e^{j\theta})$ is defined by

$$G_1(e^{j\theta}) := G(e^{j\theta}) W_\Psi(e^{j\theta}), \tag{4.10}$$

with W_Ψ being a stable and minimum phase spectral factor of Ψ. Note that W_Ψ exists because $\Psi \in \mathcal{S}_+^{m\times m}(\mathbb{T})$ (see Section 2.1).

Since Φ° is required to be a bounded spectral density, i.e. it has to be integrable on the unit circle $\mathbb{T}$, *a posteriori* we require that the minimization takes place over a proper subset of Range (Γ). To this aim, the second expression provided by (4.7) highlights that one has to ask that $\Lambda \in \mathcal{L}_+$, where

$$\mathcal{L}_+ := \left\{ \Lambda \in \mathbb{C}^{n\times n} \,|\, \Lambda = \Lambda^*,\, I + G_1^*\Lambda G_1 > 0 \,\forall\, e^{j\theta} \in \mathbb{T} \right\}. \tag{4.11}$$

Therefore, in this setting the natural set for the Lagrangian multipliers Λ turns out to be

$$\mathcal{L}_+^\Gamma := \mathcal{L}_+ \cap \text{Range}\,(\Gamma). \tag{4.12}$$

To sum up, the main result is that for each $\Lambda \in \mathcal{L}_+^\Gamma$ there exists a unique Φ° that minimizes the Lagrangian functional. Its expression in terms of the Lagrangian multiplier Λ is given by (4.7).

Remark 4.2.1. As previously hinted in Subsection 3.1.2, a possible alternative in the

definition of the distance measure could have been

$$\tilde{D}(\Psi\|\Phi) = \int \frac{1}{2}\left[\log\frac{\det\Phi}{\det\Psi} + \mathrm{Tr}(\Phi^{-1}\Psi) - m\right]. \tag{4.13}$$

This choice, however, turns out to be difficult to deal with in the light of a convex optimization approach. Indeed, the corresponding Lagrangian function is defined by

$$\tilde{L}_\Psi(\Phi,\Lambda) = \int \left[\log\frac{\det(\Phi)}{\det(\Psi)} + \mathrm{Tr}(\Phi^{-1}\Psi) + \mathrm{Tr}(\Lambda G\Phi G^*)\right] - \mathrm{Tr}\,\Lambda. \tag{4.14}$$

and the first directional derivative can be written as

$$\delta\tilde{L}_\Psi(\Phi,\Lambda;\delta\Phi) = \int \mathrm{Tr}\left\{\left[\Phi^{-1} - \Phi^{-1}\Psi\Phi^{-1} + G^*\Lambda G\right]\delta\Phi\right\}. \tag{4.15}$$

The condition that has to be satisfied by stationary points, i.e. $\delta\tilde{L}_\Psi(\Phi,\Lambda;\delta\Phi) = 0$, for all $\delta\Phi \in C(\mathbb{T})$, is equivalent to ask that

$$\Phi^{-1} - \Phi^{-1}\Psi\Phi^{-1} + G^*\Lambda G = 0. \tag{4.16}$$

This equality allows to obtain an expression of stationary points in terms of Λ, that is not as simple as the one provided when the distance (3.13) is considered. Nevertheless, such an expression is computed for the sake of completeness. Consider equation (4.16). By pre-multiplication and post-multiplication by Φ, it is possible to rewrite it as

$$\Phi - \Psi + \Phi G^*\Lambda G\Phi = 0 \tag{4.17}$$

Assume W_Λ such that $G^*\Lambda G = {W_\Lambda}^* W_\Lambda$. Then

$$\begin{aligned} 0 &= \Phi - \Psi + \Phi G^*\Lambda G\Phi \\ &= \underbrace{W_\Lambda\Phi{W_\Lambda}^*}_{\Phi_\Lambda} - \underbrace{W_\Lambda\Psi{W_\Lambda}^*}_{\Psi_\Lambda} + W_\Lambda\Phi{W_\Lambda}^*W_\Lambda\Phi{W_\Lambda}^* \\ &= \Phi_\Lambda - \Psi_\Lambda + {\Phi_\Lambda}^2. \end{aligned} \tag{4.18}$$

Equation (4.18) is solved by choosing

$$\Phi_\Lambda = -\frac{1}{2}I + \left(\frac{1}{4}I + \Psi_\Lambda\right)^{\frac{1}{2}},$$

as can be proven by substitution:

$$-\frac{1}{2}I + \left(\frac{1}{4}I + \Psi_\Lambda\right)^{\frac{1}{2}} + \Psi_\Lambda + \frac{1}{4}I - \left(\frac{1}{4}I + \Psi_\Lambda\right)^{\frac{1}{2}} + \frac{1}{4}I + \Psi_\Lambda = 0$$

Therefore, we can obtain the following expression for the optimal solution Φ°:

$$\Phi^\circ = -\frac{1}{2}(G^*\Lambda G)^{-1} + {W_\Lambda}^{-1}\left(\frac{1}{4}I + W_\Lambda \Psi {W_\Lambda}^*\right){W_\Lambda}^{-*}.$$

The next step is to consider the dual Lagrangian function, that can be written as $\inf_\Phi L(\Phi,\Lambda) = L(\Phi^\circ,\Lambda)$, because the previous considerations allow to conclude that the infimum is actually a minimum. The reasoning proceeds along the same line exposed in Appendix B. To begin with, let us consider the expression of $L(\Phi^\circ,\Lambda)$:

$$\begin{aligned}
L(\Phi^\circ(\Lambda),\Lambda) &= \int \log \frac{\det(\Psi)}{\det(W_\Psi\left[I + G_1^*\Lambda G_1\right]^{-1} W_\Psi^*)} \\
&\quad + \operatorname{Tr}\left[\Psi^{-1} W_\Psi \left(I + G_1^*\Lambda G_1\right)^{-1} W_\Psi^*\right] \\
&\quad + \operatorname{Tr}\left[\Lambda G W_\Psi \left(I + G_1^*\Lambda G_1\right)^{-1} W_\Psi^* G^*\right] - \operatorname{Tr}\Lambda \\
&= \int \log\det(I + G_1^*\Lambda G_1) + \operatorname{Tr}\left[\left(I + G_1^*\Lambda G_1\right)^{-1}\right] \\
&\quad + \operatorname{Tr}\left[\Lambda G_1 \left(I + G_1^*\Lambda G_1\right)^{-1} G_1^*\right] - \operatorname{Tr}\Lambda \\
&= \int \log\det(I + G_1^*\Lambda G_1) + \operatorname{Tr}\left[\left(I + G_1^*\Lambda G_1\right)^{-1}\left(I + G_1^*\Lambda G_1\right)\right] - \operatorname{Tr}\Lambda \\
&= \int \log\det(I + G_1^*\Lambda G_1) + n - \operatorname{Tr}\Lambda.
\end{aligned} \tag{4.19}$$

Instead of maximizing $L(\Phi^\circ(\Lambda),\Lambda)$, we will focus on the minimization of a simpler functional arising from it, that is defined as:

$$J(\Lambda) := \int \left[\operatorname{Tr}\Lambda - \log\det(I + G_1^*\Lambda G_1)\right] = \int \operatorname{Tr}\left[\Lambda - \log(I + G_1^*\Lambda G_1)\right]. \tag{4.20}$$

Hereafter, we will refer to $J(\cdot)$ as the *dual function*.

The main result that will be achieved here and in the following sections is that the solution that minimizes the dual function exists and it is unique. Moreover, a Newton-type algorithm with backtracking, that solves the problem efficiently, is proposed. To this purpose, the starting point is to prove that the dual function is *strictly convex*. This allows to conclude that the optimal solution, if it exists, is unique. As regards existence, it is a more challenging issue, that is deferred to the following section. In order to analyze the problem of convexity, a straightforward procedure is to evaluate the first and second variation of $J(\Lambda)$. They are given by

$$\delta J_\Psi(\Lambda;\delta\Lambda) = \int \left\{\operatorname{Tr}\left[\delta\Lambda\right] - \operatorname{Tr}\left[\left(I + G_1^*\Lambda G_1\right)^{-1} G_1^*\delta\Lambda G_1\right]\right\} \tag{4.21}$$

and

$$\delta^2 J_\Psi(\Lambda; \delta\Lambda_1, \delta\Lambda_2) = \int \mathrm{Tr}\left[(I + G_1^* \Lambda G_1)^{-1} G_1^* \delta\Lambda_2 G_1 (I + G_1^* \Lambda G_1)^{-1} G_1^* \delta\Lambda_1 G_1\right]. \tag{4.22}$$

These expressions have been obtained by exploiting (4.5) and the fact that, denoting the matrix inversion operator by $R : M \mapsto M^{-1}$, its first derivative in direction δM is given by

$$\delta R\,(M, \delta M) = -M^{-1} \delta M M^{-1}. \tag{4.23}$$

It is now possible to define the linear functional $\nabla J_{\Psi,\Lambda}(\cdot) := \delta J_\Psi(\Lambda; \cdot)$ as the *gradient* of J_Ψ at Λ. Instead, the bilinear form $H_\Lambda(\cdot, \cdot) := \delta^2 J_\Psi(\Lambda; \cdot, \cdot)$ is the *Hessian* of J_Ψ at Λ. In order to prove that the dual function admits a unique minimum on $\mathcal{L}_+^\Gamma$, we will first establish that the dual function is strictly convex. Recall that a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *strictly convex* if A is a convex set and, for each $x, y \in A$ s.t. $x \neq y$, given $\lambda \in (0, 1)$, the following inequality holds:

$$f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y).$$

Two preliminary results are given by the following lemmata. The first is a result on the continuity of the map from an Hermitian matrix to its minimum eigenvalues. The second one extends Lemma 5.2 in [17]. They allow to prove the continuity of the derivatives of the dual function.

Lemma 4.2.1. *Let $H \in \mathcal{H}(n)$ and m be its minimum eigenvalue. The map $H \mapsto m$ is continuous.*

The following proof is set out in [17].

Proof. Let $a(s)$ be the characteristic polynomial of H, i.e. $a(s) := \det(sI - H)$. Since its coefficients are obtained by means of sums and products of the elements of H, the map from H to the vector of the coefficients of its characteristic polynomial is continuous. Moreover, the roots of a monic polynomial depend continuously on its coefficients. As a consequence the map from the Hermitian matrix H to its minimum real eigenvalue is continuous, because it is a compositions of continuous maps. □

Lemma 4.2.2. *Let $Q_\Lambda(z) := I + G_1^*(z) \Lambda G_1(z)$. Consider a sequence $\Lambda_n \in \mathcal{L}_+^\Gamma$ converging to $\Lambda \in \mathcal{L}_+^\Gamma$. Then $Q_{\Lambda_n}^{-1}$ are well defined and converge uniformly to Q_Λ^{-1} on $\mathbb{T}$.*

Taking into account the fact that Ψ is bounded, the proof is extremely similar to the one offered in [17]. It provides the existence of a uniform upper bound on Q_Λ^{-1}, too.

Proof. Since $\Lambda \in \mathcal{L}_+^\Gamma$, it also belongs to $\mathcal{L}_+$, so that Q_Λ is a positive definite continuous matrix on $\mathbb{T}$. From positive definiteness and Lemma 4.2.1 it follows that there exists a continuous function $m(e^{j\theta})$ such that for all $e^{j\theta} \in \mathbb{T}$, $Q_\Lambda(e^{j\theta}) \geq m(e^{j\theta})I$. Let $m_\Lambda := \min_\theta m(e^{j\theta})$. Therefore, the inequality $Q_\Lambda(e^{j\theta}) \geq m_\Lambda I$ holds for each value $e^{j\theta} \in \mathbb{T}$. Recall that $G_1(e^{j\theta}) = G(e^{j\theta})W_\Psi(e^{j\theta})$ and let $\delta\Lambda \in B(0,\varepsilon)$, the closed ball of radius ε centered in 0. It is possible to write that

$$||G_1^*\delta\Lambda G_1|| = ||W_\Psi^* G^*\delta\Lambda G W_\Psi|| \leq \varepsilon M_{G_1},$$

where M_{G_1} is defined by

$$M_{G_1} = \max_\theta \|W_\Psi^*(e^{j\theta})G^*(e^{j\theta})\|\|G(e^{j\theta})W_\Psi(e^{j\theta})\|.$$

Since the prior density Ψ is bounded, so is its minimum phase spectral factor W_Ψ and, therefore, M_{G_1} is bounded, too. Choose $\varepsilon < \frac{m_\Lambda}{M_{G_1}}$, so that $||G_1^*\delta\Lambda G_1|| < m_\Lambda$. As a consequence, the set of the matricial functions $I + G_1^*(\Lambda + \delta\Lambda)G_1$, with $(\delta\Lambda, \theta) \in B(0,\varepsilon)\times[-\pi,\pi]$ is compact and does not contain any singular matrix. Since the matrix inversion operator is continuous at any nonsingular matrix, $Q_{\Lambda+\delta\Lambda}^{-1}(e^{j\theta})$ admits a uniform bound $M(\Lambda,\varepsilon)$ on $B(0,\varepsilon) \times [-\pi,\pi]$. For n sufficiently large, $(\Lambda - \Lambda_n) \in B(0,\varepsilon)$ because $\Lambda_n \to \Lambda$. The last result is that $Q_{\Lambda_n}^{-1} \to Q_\Lambda^{-1}$ uniformly on $\mathbb{T}$. Indeed,

$$\begin{aligned}\sup_\theta ||Q_{\Lambda_n}^{-1} - Q_\Lambda^{-1}|| &= \sup_\theta ||Q_{\Lambda_n}^{-1}\left[G_1^*(\Lambda - \Lambda_n)G_1\right]Q_\Lambda^{-1}|| \\ &\leq M^2 \sup_\theta ||G_1^*(\Lambda - \Lambda_n)G_1|| \\ &\leq M^2\varepsilon M_{G_1}.\end{aligned}$$

□

On the basis of these result, it is possible to establish the following theorem:

Theorem 4.2.1. *The dual functional $J_\Psi(\Lambda)$ is $\mathcal{C}^2(\mathcal{L}_+^\Gamma)$ and* strictly *convex on $(\mathcal{L}_+^\Gamma)$.*

Proof. To begin with, we prove the continuity of the first variation of $J_\Psi(\Lambda)$, by exploiting the previously introduced result on the convergence of the sequence $Q_{\Lambda_n}^{-1}$. We recall that a real function $f(x)$ is said to be continuous in $x_0 \in X$ if, given that

the sequence $x_n \to x_0$, the following result holds:

$$\lim_{n\to+\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x_0).$$

Considering a sequence $M_n \in \text{Range}\,(\Gamma)$, such that $M_n \to 0$, by Lemma 4.2 it is possible to conclude that ${Q_{\Lambda+M_n}}^{-1}$ converges uniformly to ${Q_\Lambda}^{-1}$, that is upper bounded. Applying element-wise the bounded convergence theorem, it turns out that

$$\lim_{n\to\infty} \int \text{Tr}\left[{Q_{\Lambda+M_n}}^{-1} G_1^* \delta\Lambda G_1\right] = \int \text{Tr}\left[Q_\Lambda^{-1} G_1^* \delta\Lambda G_1\right].$$

As a consequence, $J_\Psi(\Lambda)$ is $\mathcal{C}^1(\mathcal{L}_+^\Gamma)$. It is easy to see that the same result holds for $\delta^2 J_\Psi(\Lambda; \delta\Lambda_1, \delta\Lambda_2)$, too. Therefore, $J_\Psi(\Lambda)$ is $\mathcal{C}^2(\mathcal{L}_+^\Gamma)$. Recall that J_Ψ is said to be $\mathcal{C}^k(\mathcal{L}_+^\Gamma)$ if and only if it is continuous in each point of $\mathcal{L}_+^\Gamma$ and has continuous directional derivatives of any order up to k, in whatever directions $\{\delta\Lambda_1, \ldots, \delta\Lambda_k\} \in \text{Range}\,(\Gamma)$.

The next point to investigate is the *strict* convexity of the dual functional $J_\Psi(\Lambda)$. Recall that a function $f : S \subset \mathbb{R}^N \to \mathbb{R}$ that is $\mathcal{C}^2(S)$, where S is open, is strictly convex if and only if its Hessian is positive definite at each $x \in S$.

$$\begin{aligned} H_\Lambda(\delta\Lambda, \delta\Lambda) &= \delta^2 J_\Psi(\Lambda; \delta\Lambda\delta\Lambda) \\ &= \int \text{Tr}\left[(I + G_1^*\Lambda G_1)^{-1} G_1^* \delta\Lambda G_1 (I + G_1^*\Lambda G_1)^{-1} G_1^* \delta\Lambda G_1\right] \\ &= \int \text{Tr}\left[Q_\Lambda^{-\frac{1}{2}} G_1^* \delta\Lambda G_1 Q_\Lambda^{-1} G_1^* \delta\Lambda G_1 Q_\Lambda^{-\frac{1}{2}}\right]. \end{aligned} \tag{4.24}$$

Looking over the integrand of equation (4.24), it is possible to recognize a positive-definite quadratic form:

$$H_\Lambda(\delta\Lambda, \delta\Lambda) = \int \text{Tr}\left[X^* Q_\Lambda^{-1} X\right], \quad X = G_1^* \delta\Lambda G_1 Q_\Lambda^{-\frac{1}{2}}$$

Indeed, for $\Lambda \in \mathcal{L}_+^\Gamma$, $Q_\Lambda > 0$. The integral vanishes if and only if the integrand is identically zero. Moreover, since $G_1^*\Lambda G_1 = W_\Psi^* G^* \Lambda G W_\Psi = 0 \,\forall\, \theta \in [-\pi, \pi]$, if and only if $\Lambda \in \text{Range}(\Gamma)^\perp$, it follows that the integrand is identically zero, if and only if $\delta\Lambda = 0$. Hence the Hessian is positive-definite and the dual functional is strictly convex. □

This result allows to conclude that the solution that minimizes the dual function, if it exists, is unique. The next step is to prove existence. Then it will be possible to introduce an algorithmic procedure to solve the problem arising from the minimization of the dual function (4.20).

4.3 Existence of the Solution

In this section the matter of the existence of a solution for the dual problem is tackled. It is worthwhile to anticipate that our approach allows to successfully settle both this issue and the one of the convergence of the algorithmic procedure that will be exposed in the next section. As regards existence, the key result that is achieved is that, even though the set $\mathcal{L}_+^\Gamma$ is open and unbounded, the search for the optimal solution can be restricted to a compact subset.

The purpose is to prove that the function $J_\Psi(\Lambda)$ is inf-compact, i.e. $\forall\, \alpha \in \mathbb{R}$, the set $\left\{\Lambda \in \mathcal{L}_+^\Gamma \,|\, J_\Psi(\Lambda) \leq \alpha\right\}$ is compact. The proof is inspired by the one offered in [15]. At first, define $\bar{\mathcal{L}}_+^\Gamma$ as the closure of $\mathcal{L}_+^\Gamma$, i.e. the set

$$\bar{\mathcal{L}}_+^\Gamma = \left\{\Lambda = \Lambda^* \in \mathbb{C}^{n\times n} \,|\, \Lambda \in \text{Range}(\Gamma),\, I + G_1^*\Lambda G_1 \geq 0,\, \forall e^{j\theta} \in \mathbb{T}\right\}$$

Given that, for Λ belonging to the boundary $\partial\mathcal{L}_+^\Gamma$, the Hermitian matrix $I + G_1^*\Lambda G_1$ is singular, it will be useful to define on $\bar{\mathcal{L}}_+^\Gamma$ the sequence of functions:

$$J_\Psi^n(\Lambda) = \int \text{Tr}[\Lambda + \log(I + G_1^*\Lambda G_1 + \frac{1}{n}I)]. \tag{4.25}$$

Three lemmata will lead to the final result, stated in Theorem 4.3.1. Preliminary, recall that a real-valued function f is said to be lower semicontinuous at x_0 if, $\forall\, \varepsilon > 0$, there exists a neighborhood U of x_0 such that, $\forall\, x \in U$, $f(x) \geq f(x_0) - \varepsilon$. Recall also that, given $f : \mathbb{C}^{n\times n} \to \mathbb{R}$, its *epigraph* $\text{epi}(f)$ is defined by

$$\text{epi}(f) := \left\{(x, a) \in \mathbb{C}^{n\times n} \times \mathbb{R} |\, a \geq f(x)\right\}.$$

Moreover, f is a lower semicontinuous function if and only if its epigraph is closed.

Lemma 4.3.1. *The pointwise limit $J_\Psi^\infty(\Lambda)$, defined as $J_\Psi^\infty(\Lambda) := \lim_{n\to\infty} J_\Psi^n(\Lambda)$, exists and is a lower semicontinuous and convex function defined over $\bar{\mathcal{L}}_+^\Gamma$, with values in the extended reals.*

Proof. The additive term $\frac{1}{n}I$ ensures that, for each n, $J_\Psi^n(\Lambda)$ is a continuous and convex function of Λ on the closed set $\bar{\mathcal{L}}_+^\Gamma$. From the properties of $J_\Psi^n(\Lambda)$, it follows that $\text{epi}(J_\Psi^n(\Lambda))$ is a closed and convex subset of $\mathbb{C}^{n\times n} \times \mathbb{R}$. In addition, the pointwise sequence is monotonically increasing, since $J_\Psi^n(\Lambda) < J_\Psi^{n+1}(\Lambda)$. Therefore, it converges to $J_\Psi^\infty(\Lambda)$, that is equal to $\sup_n J_\Psi^n(\Lambda)$. Since the intersection of closed sets is closed and the intersection of convex sets is convex, $\text{epi}\, J_\Psi^\infty(\Lambda) = \cap_n \text{epi}\, J_\Psi^n(\Lambda)$ is closed and convex. As a consequence, $J_\Psi^\infty(\Lambda)$ is lower semicontinuous and convex. The proof of the latter implication can be found in [20]. □

Lemma 4.3.2. *Assume that the feasibility condition* (3.17) *holds.*
Let $\mathcal{B} := \left\{\Lambda \in \partial\mathcal{L}_+^\Gamma \,|\, \det\left[G_1^*\Lambda G_1 + I\right] = 0,\, \forall e^{j\theta} \in \mathbb{T}\right\}$ *and consider its complement set* $\mathcal{B}^c := \left\{\Lambda \in \partial\mathcal{L}_+^\Gamma \,|\, \Lambda \notin \mathcal{B}\right\}$. *Then:*

1. $J_\Psi^\infty(\Lambda)$ *is bounded below on* $\bar{\mathcal{L}}_+^\Gamma$*;*

2. $J_\Psi^\infty(\Lambda) = J_\Psi(\Lambda)$ *on* $\mathcal{L}_+^\Gamma$*;*

3. $J_\Psi^\infty(\Lambda)$ *is finite over* $\mathcal{B}^c$*.*

Proof.

1. In order to prove the existence of a lower bound for $J_\Psi^\infty(\Gamma)$ on $\bar{\mathcal{L}}_+^\Gamma$, let us consider $\mathrm{Tr}(\Lambda)$. If the feasibility condition (3.17) holds, there exists Φ_I in $\mathcal{C}_+(\mathbb{T})$ such that $\int G\Phi_I G^* = I$. Therefore,

$$\begin{aligned}\mathrm{Tr}\left[\Lambda\right] &= \mathrm{Tr}\left[\int G\Phi_I G^*\Lambda\right] \\ &= \mathrm{Tr}\left[\int G^*\Lambda G\Phi_I\right] \\ &= \mathrm{Tr}\left[\int W_\Psi^* G^*\Lambda G W_\Psi W_\Psi^{-1}\Phi_I W_\Psi^{-*}\right] \\ &= \mathrm{Tr}\left[\int G_1^*\Lambda G_1\Xi\right],\end{aligned} \tag{4.26}$$

   where the cyclic property of the trace has been employed and the auxiliary spectral density Ξ has been defined as

$$\Xi := W_\Psi^{-1}\Phi_I W_\Psi^{-*}. \tag{4.27}$$

   It follows that

$$\begin{aligned}\mathrm{Tr}\left[\Lambda\right] &= \mathrm{Tr}\left[\int G_1^*\Lambda G_1\Xi + \Xi - \Xi\right] \\ &= \mathrm{Tr}\left[\int (G_1^*\Lambda G_1 + I)\Xi\right] - \mathrm{Tr}\left[\int \Xi\right].\end{aligned} \tag{4.28}$$

   Let Δ be such that $(G_1^*\Lambda G_1 + I) = \Delta^*\Delta$ (equation (4.52) provides an explicit expression to compute the factor Δ). Therefore it is possible to write

$$\mathrm{Tr}\left[(G_1^*\Lambda G_1 + I)\Xi\right] = \mathrm{Tr}\left[\Delta\Xi\Delta^*\right].$$

   Given that $\Xi = W_\Psi^{-1}\Phi_I W_\Psi^{-*}$ is a coercive spectrum, because both Φ_I and Ψ belong to $\mathcal{S}_+^{m\times m}(\mathbb{T})$, $\exists\,\mu \,|\, \Xi(e^{j\theta}) \geq \mu I \,\forall\, e^{j\theta} \in \mathbb{T}$. Finally, recall that the trace

and the integral are monotonic operators. Therefore

$$\begin{aligned} \operatorname{Tr}[\Lambda] &= \operatorname{Tr}\left[\int (G_1^* \Lambda G_1 + I)\Xi\right] - \operatorname{Tr}\left[\int \Xi\right] \\ &\geq \mu \operatorname{Tr}\left[\int (G_1^* \Lambda G_1 + I)\right] - \operatorname{Tr}\left[\int \Xi\right]. \end{aligned} \tag{4.29}$$

As a consequence,

$$\begin{aligned} J_\Psi^n(\Lambda) &= \int \operatorname{Tr}\left[\Lambda - \log(I + G_1^* \Lambda G_1 + \frac{1}{n} I)\right] \\ &\geq -\int \operatorname{Tr}[\Xi] + \int \operatorname{Tr}\left[\mu(I + G_1^* \Lambda G_1) - \log(I + G_1^* \Lambda G_1 + \frac{1}{n} I)\right]. \end{aligned} \tag{4.30}$$

In order to find a lower bound for $J_\Psi^n(\Lambda)$, it is useful to consider the problem of minimizing the auxiliary function

$$J_{\Psi,r}^n(\Lambda) = \int \operatorname{Tr}\left[\mu(I + G_1^* \Lambda G_1) - \log(I + G_1^* \Lambda G_1 + \frac{1}{n} I)\right]. \tag{4.31}$$

Let $\{x_i\}$ be the eigenvalues of $(I + G_1^* \Lambda G_1)$. Then,

$$\begin{aligned} J_{\Psi,r}^n(\Lambda) &= \int \operatorname{Tr}\left[\mu(I + G_1^* \Lambda G_1) - \log(I + G_1^* \Lambda G_1 + \frac{1}{n} I)\right] \\ &= \int \mu \sum_{i=1}^m x_i - \sum_{i=1}^m \log\left(x_i + \frac{1}{n}\right) \\ &= \int i\,(x_1, \ldots, x_m)\,, \end{aligned} \tag{4.32}$$

where $i(x_1, \ldots, x_m) := \mu \sum_{i=1}^m x_i - \sum_{i=1}^m \log\left(x_i + \frac{1}{n}\right)$. Moreover,

$$\frac{\partial}{\partial x_i}\left[i(x_1, \ldots, x_m)\right] = \mu - \frac{1}{x_i + \frac{1}{n}} \quad \forall\, i.$$

The minimum is attained by choosing $x_i = \frac{1}{\mu} - \frac{1}{n}$, $\forall\, i$. Therefore,

$$i(x_1, \ldots, x_m) \geq m - \frac{\mu m}{n} + m \log \mu$$

Going back to the function $J_\Psi^n(\Lambda)$, the final result that $J_\Psi^n(\Lambda)$ is bounded below over $\bar{\mathcal{L}}_+^\Gamma$ follows:

$$\begin{aligned} J_\Psi^n(\Lambda) &\geq \underbrace{-\int \operatorname{Tr}[\Xi]}_{\alpha} + \underbrace{m - \frac{\mu m}{n} + m \log \mu}_{\beta} \\ &\geq \alpha + \beta. \end{aligned} \tag{4.33}$$

2. Beppo Levi's Theorem allows to conclude that $J_\Psi^\infty(\Lambda) = J_\Psi(\Lambda)$ in $\mathcal{L}_+^\Gamma$:

$$\begin{aligned} J_\Psi^\infty(\Lambda) &= \int \mathrm{Tr}[\Lambda] + \int \mathrm{Tr}\left[\lim_{n\to\infty} \log(I + G_1^*\Lambda G_1 + \frac{1}{n}I)\right] \\ &= J_\Psi(\Lambda). \end{aligned} \tag{4.34}$$

3. Finally, since for $\Lambda \in \mathcal{B}^c$ we have that the rational function $\det(I + G_1^*\Lambda G_1)$ is not identically zero, its logarithm is integrable over $\mathbb{T}$, so $J_\Psi^\infty(\Lambda)$ is finite. Instead, $J_\Psi^\infty(\Lambda) = +\infty$ for $\Lambda \in \mathcal{B}$.

□

Lemma 4.3.3. *If the feasibility hypothesis holds, then*

$$\lim_{\|\Lambda\|\to+\infty} J_\Psi(\Lambda) = +\infty. \tag{4.35}$$

Proof. From equation (4.28) it is clear that $\mathrm{Tr}[\Lambda] > -\mathrm{Tr}[\int \Xi]$, so $\mathrm{Tr}[\Lambda]$ it is bounded below (because Ξ is bounded). Consider a sequence $\{\Lambda_k\} \in \mathcal{L}_+^\Gamma$, such that

$$\lim_{k\to\infty} \|\Lambda_k\| = +\infty.$$

Let Λ_k^0 be $\frac{\Lambda_k}{\|\Lambda_k\|}$. Since $\mathcal{L}_+^\Gamma$ is convex and $\Lambda = 0$ belongs to $\mathcal{L}_+^\Gamma$, $\forall\, \alpha \in [0,1]$, $\alpha\Lambda \in \mathcal{L}_+^\Gamma$. Therefore $\Lambda_k^0 \in \mathcal{L}_+^\Gamma$ for sufficiently large k. Let $\eta := \liminf \mathrm{Tr}[\Lambda_k^0]$ [1]. Since

$$\mathrm{Tr}\,\Lambda_k^0 = \frac{1}{\|\Lambda_k\|}\mathrm{Tr}\,\Lambda_k > -\frac{1}{\|\Lambda_k\|}\mathrm{Tr}\left[\int \Xi\right] \to 0,$$

for $\|\Lambda_k\| \to \infty$, it holds that $\eta \geq 0$. There exists a subsequence such that the limit of its trace is η. Given that Λ_k^0 belongs to the surface of the unit ball, which is compact, the subsequence contains a subsubsequence $\left\{\Lambda_{k_m}^0\right\}$ that is convergent. Define

$$\Lambda_\infty := \lim_{m\to\infty} \Lambda_{k_m}^0.$$

The next step is to prove that $\Lambda_\infty \in \mathcal{L}_+^\Gamma$. To this aim, notice that Λ_∞ is the limit of a convergent sequence in the finite-dimensional linear space $\mathrm{Range}(\Gamma)$. Therefore it belongs to $\mathrm{Range}(\Gamma)$. Moreover, recall that the primary sequence $\{\Lambda_k\}$ has elements

[1] Recall that, given a sequence of real numbers $\{x_k\}$, $\liminf x_k$ is defined as follows:

$$\liminf x_k := \lim_{k\to\infty}\left\{\inf_{m\geq n} x_m\right\}.$$

Moreover, given an arbitrary sequence of real numbers, $\{x_k\}$, there exists a subsequence converging to $\liminf x_k$.

belonging to $\mathcal{L}_+^\Gamma$. It means that, for each Λ_k, $(I + G_1^* \Lambda_k G_1) > 0$. As a consequence, it holds that, for each l,

$$\left(\frac{1}{\|\Lambda_{k_l}\|} I + G_1^* \Lambda_{k_l}^0 G_1 \right) > 0 \quad \text{on } \mathbb{T} \, .$$

Taking the pointwise limit for $l \to \infty$, it results that $G_1^* \Lambda_\infty G_1$ is positive semidefinite, and so $(I + G_1^* \Lambda_\infty G_1)$ is strictly positive definite on $\mathbb{T}$. Therefore, $\Lambda_\infty \in \mathcal{L}_+^\Gamma$.

The next step is to prove that $\operatorname{Tr} \Lambda_\infty > 0$. If the feasibility condition (3.17) holds, there exists Φ_I such that $I = \int G \Phi_I G^*$. Therefore, it is possible to write:

$$\begin{aligned} \operatorname{Tr} \Lambda_\infty &= \operatorname{Tr} \int G \Phi_I G^* \Lambda_\infty \\ &= \int \operatorname{Tr} \left[G^* \Lambda_\infty G \Phi_I \right] \\ &= \int \operatorname{Tr} \left[W_\Psi^{-*} W_\Psi^* G^* \Lambda_\infty G W_\Psi W_\Psi^{-1} \Phi_I \right] \\ &= \int \operatorname{Tr} \left[G_1^* \Lambda_\infty G_1 \underbrace{W_\Psi^{-1} \Phi_I W_\Psi^{-*}}_{\Xi} \right] , \end{aligned} \tag{4.36}$$

where Ξ is defined as in equation (4.27). Therefore,

$$\begin{aligned} \operatorname{Tr} \Lambda_\infty &= \int \operatorname{Tr} \left[G_1^* \Lambda_\infty G_1 \Xi \right] \\ &= \int \operatorname{Tr} \left[\Xi^{\frac{1}{2}} G_1^* \Lambda_\infty G_1 \Xi^{\frac{1}{2}} \right] . \end{aligned} \tag{4.37}$$

Since $G_1^* \Lambda_\infty G_1 \geq 0$, in order to prove that the trace of Λ_∞ is positive, in view of (4.37) it is sufficient to show that $G_1^* \Lambda_\infty G_1$ is not identically zero. Assume by contrast that this condition is not satisfied, so $G_1^* \Lambda_\infty G_1 \equiv 0$. As a consequence,

$$\begin{aligned} 0 &\equiv G_1^* \Lambda_\infty G_1 \\ &= W_\Psi^* G^* \Lambda_\infty G W_\Psi \end{aligned} \tag{4.38}$$

Therefore, $G^* \Lambda_\infty G \equiv 0$. However, this means that $\Lambda_\infty \in \operatorname{Range}(\Gamma)^\perp$. But it has already been proved that $\Lambda_\infty \in \operatorname{Range}(\Gamma)$. Moreover, $\Lambda_\infty \neq 0$, since it belongs to the surface of the unit ball. This is a contradiction, because $\operatorname{Range}(\Gamma) \cap \operatorname{Range}(\Gamma)^\perp = \{0\}$. Therefore, $G_1^* \Lambda_\infty G_1$ is not identically zero, and from (4.37) it follows that $\eta = \operatorname{Tr} \Lambda_\infty > 0$.

Hence, there exists K such that $\operatorname{Tr} \Lambda_k^0 > \frac{\eta}{2}$ for all $k > K$. Notice that $G_1^* G_1$ is positive definite on $\mathbb{T}$ (and indeed coercive). Moreover, $G_1^* \Lambda_k^0 G_1 \leq G_1^* G_1$, because

Λ_k^0 belongs to the unit ball. Therefore,

$$\begin{aligned}
\liminf_{k\to\infty} J_\Psi(\Lambda_k) &= \liminf_{k\to\infty} \int \mathrm{Tr}\left[\Lambda_k - \log(I + G_1^*\Lambda_k G_1)\right] \\
&= \liminf_{k\to\infty} \mathrm{Tr}\left[\|\Lambda_k\|\Lambda_k^0\right] - \int \mathrm{Tr}\left[\log\left(I + G_1^*\|\Lambda_k\|\Lambda_k^0 G_1\right)\right] \\
&= \liminf_{k\to\infty} \mathrm{Tr}\left[\|\Lambda_k\|\Lambda_k^0\right] - \int \mathrm{Tr}\left[\log\left[\|\Lambda_k\|\left(\frac{1}{\|\Lambda_k\|}I + G_1^*\Lambda_k^0 G_1\right)\right]\right] \\
&\geq \liminf_{k\to\infty} \|\Lambda_k\|\frac{\eta}{2} - \int \mathrm{Tr}\left[\log\|\Lambda_k\|\right] - \int \mathrm{Tr}\left[\log\left(\frac{1}{\|\Lambda_k\|}I + G_1^* G_1\right)\right] \\
&= \liminf_{k\to\infty} \frac{\eta}{2}\left(\|\Lambda_k\| - \frac{2}{\eta}\int \mathrm{Tr}\left[\log\|\Lambda_k\|\right]\right) - \int \mathrm{Tr}\left[\log\left(\frac{1}{\|\Lambda_k\|}I + G_1^* G_1\right)\right] \\
&= +\infty.
\end{aligned} \tag{4.39}$$

□

Lemmata (4.3.1), (4.3.2) and (4.3.3) imply that $J_\Psi(\Lambda)$ is inf-compact over $\bar{\mathcal{L}}_+^\Gamma$. As a consequence, employing Weierstrass' Theorem we can conclude that there exists a minimum point $\Lambda^\circ \in \bar{\mathcal{L}}_+^\Gamma$. Besides, more can be proved: such a minimum point belongs to $\mathcal{L}_+^\Gamma$. This is the content of the following theorem.

Theorem 4.3.1. *If the feasibility condition (3.17) holds, the problem of minimizing $J_\Psi(\Lambda)$ over $\mathcal{L}_+^\Gamma$ admits a unique solution $\Lambda^\circ \in \mathcal{L}_+^\Gamma$.*

Proof. Since $J_\Psi(\Lambda)$ is inf-compact over $\bar{\mathcal{L}}_+^\Gamma$, it admits a minimum point Λ° there. Obviously, $\Lambda^\circ \notin \mathcal{B}$, because for $\Lambda \in \mathcal{B}$, $J_\Psi(\Lambda) = +\infty$. Suppose $\Lambda^\circ \in \mathcal{B}^c$. By Lemma 4.3.2 it follows that $J_\Psi(\Lambda^\circ)$ is finite. By convexity of $\bar{\mathcal{L}}_+^\Gamma$, $\forall\, \varepsilon \in [0,1]$, $\Lambda^\circ + \varepsilon(I - \Lambda^\circ) \in \bar{\mathcal{L}}_+^\Gamma$, since the feasibility condition (3.17) ensures that $I \in \mathcal{L}_+^\Gamma$. The one-sided directional derivative is:

$$\begin{aligned}
\delta J_{\Psi_+}(\Lambda^\circ; I - \Lambda^\circ) &= \lim_{\varepsilon \searrow 0}\left[\frac{J_\Psi\left(\Lambda^\circ + \varepsilon\left(I - \Lambda^\circ\right)\right) - J_\Psi(\Lambda^\circ)}{\varepsilon}\right] \\
&= \mathrm{Tr}\left[I - \Lambda^\circ\right] - \int \mathrm{Tr}\left[\left(I + G_1^*\Lambda^\circ G_1\right)^{-1} G_1^*\left(I - \Lambda^\circ\right) G_1\right] \\
&= \mathrm{Tr}\left[I - \Lambda^\circ\right] - \int \mathrm{Tr}\left[\left(I + G_1^*\Lambda^\circ G_1\right)^{-1}\left(G_1^* G_1 - G_1^*\Lambda^\circ G_1 + I - I\right)\right] \\
&= \mathrm{Tr}\left[I - \Lambda^\circ\right] - \int \mathrm{Tr}\left[\left(I + G_1^*\Lambda^\circ G_1\right)^{-1}\left(I + G_1^* G_1\right) - I\right] \\
&= -\infty,
\end{aligned} \tag{4.40}$$

because, for each $\Lambda \in \mathcal{B}^c$, $(I + G_1^*\Lambda G_1)$ is singular. As a consequence, the minimum point cannot belong to $\partial\mathcal{L}_+^\Gamma$. Therefore the optimal solution $\Lambda^\circ \in \mathcal{L}_+^\Gamma$. □

Therefore, the existence of a solution for Problem 3.2.1 is proven. Indeed, as a consequence of the previous theorem, the Lagrangian multiplier Λ°, defined as

$$\Lambda^\circ := \min_{\Lambda \in \mathcal{L}_+^\Gamma} J_\psi(\Lambda),$$

exists. Uniqueness has been already proven in Section 4.2. Given Λ°, the corresponding spectral density Φ° is unique and can be computed as in equation (4.6), providing the solution of the spectral estimation problem.

4.4 Proposed Algorithm

Our purpose is to minimize the dual function (4.20). Once the solution is found, it is straightforward to solve the multivariate spectral approximation Problem 3.2.1, by means of equation (4.6). The proposed approach is based on Newton method, that is described in Appendix C. Consequently the most important steps to analyze are the following:

1. Find the *search direction* $\Delta\Lambda_i$;

2. Compute the Newton *step length* t.

They are dealt with separately in the following subsections. As regards the choice of a starting point for the minimizing sequence $\{\Lambda_i\}$, it is possible to choose $\Lambda_0 = 0$.

Search direction

Even though the problem is finite dimensional, the computation of the search direction is more demanding than a matrix inversion, because in the case of interest a matricial expression of the Hessian and the gradient, allowing to compute the search direction Δx as

$$\Delta x = -{H_x}^{-1} \nabla f_x,$$

is not available. In order to find the Newton step $\Delta\Lambda_i$ s.t.

$$H_{\Lambda_i}(\Delta\Lambda_i, \cdot) = -\nabla J_{\Psi,\Lambda_i}(\cdot),$$

one has to solve the equation:

$$\int G_1(I + G_1^* \Lambda_i G_1)^{-1} G_1^* \Delta\Lambda_i G_1 (I + G_1^* \Lambda_i G_1)^{-1} G_1^* = \int G_1(I + G_1^* \Lambda_i G_1)^{-1} G_1^* - I$$

The idea is to consider a set of generators of Range(Γ). This set can be obtained easily, by recalling that $\Sigma_k \in \text{Range}(\Gamma) \Leftrightarrow \exists\, H_k \in \mathbb{C}^{m\times n}\, s.t.\, \Sigma_k - A\Sigma_k A^* = BH_k + {H_k}^* B^*$. Therefore, considering a base $\{H_1, \ldots, H_N\}$ for $\mathbb{C}^{m\times n}$, a corresponding set of generators $\{\Sigma_1, \ldots, \Sigma_N\}$ can be found. Note that these are not necessarily independent. The procedure goes along as follows:

1. Compute
$$Y = \int G_1(I + G_1^*\Lambda_i G_1)^{-1} G_1^* - I \tag{4.41}$$
2. For each generator Σ_k, compute
$$Y_k = \int G_1(I + G_1^*\Lambda_i G_1)^{-1} G_1^*\Sigma_k G_1(I + G_1^*\Lambda_i G_1)^{-1} G_1^* \tag{4.42}$$
3. Find $\{\alpha_k\}$ s.t. $Y = \sum_k \alpha_k Y_k$;
4. Update $\Delta\Lambda_i = \sum_k \alpha_k \Sigma_k$.

The most challenging step is to compute Y and Y_k. A solution is to employ the spectral factorization techniques: they allow to compute integrals (4.41) and (4.42) by means of the evaluation of the steady state covariance of the output of adequate linear systems.

In the light of the problem of interest, the following results in linear stochastic systems theory are particularly significant. They are exposed in the form of lemmata.

Lemma 4.4.1. *Let A be a stability matrix and assume that $W(z)$ is a minimal realization of the spectral factor $\Phi(z)$, i.e. $\Phi(z) = W(z)W^*(z)$, such that $W(z) = C\,(zI - A)^{-1}\,B + D$. Let Π be the unique solution of the Lyapunov equation*

$$P = APA^* + BB^*.$$

Then, the following facts hold:

1. $\int \Phi(e^{j\theta})\frac{d\theta}{2\pi} = C\Pi C^* + DD^*$.
2. $Z(z) = C(zI - A)^{-1}\,(A\Pi C^* + BD^*) + \frac{1}{2}\,(C\Pi C^* + DD^*)$ *is a realization of the causal part of $\Phi(z)$, i.e. $\Phi(z) = Z(z) + Z^*(z)$, with $Z(z)$ analytic outside the unit circle $\mathbb{T}$.*

Lemma 4.4.2. *Let $F(z) = C\,(zI - A)^{-1}\,B + D$ be a square transfer function, where D is invertible. A realization of the inverse transfer function is given by*

$$F^{-1}(z) = -D^{-1}C\left(zI - \left(A - BD^{-1}C\right)\right)^{-1} BD^{-1} + D^{-1}.$$

Lemma 4.4.3. *Let $P = P^*$ be a matrix belonging to $\mathbb{C}^{n\times n}$. Then*

$$\begin{bmatrix} B^* \left(z^{-1}I - A^*\right)^{-1} & I \end{bmatrix} \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} (zI - A)^{-1} B \\ I \end{bmatrix} = 0$$

Lemma 4.4.4. *Let A be a stability matrix and $H(z) = C\left(zI - A\right)^{-1} B + D$ be a minimal realization. Let P be a solution of the Lyapunov equation*

$$P = A^*PA + C^*C. \tag{4.43}$$

Let $\begin{bmatrix} K \\ J \end{bmatrix}$ be an orthonormal basis of the kernel of the matrix $\begin{bmatrix} A^*P^{\frac{1}{2}} & C^* \end{bmatrix}$, i.e.

$$\begin{bmatrix} A^*P^{\frac{1}{2}} & C^* \end{bmatrix} \begin{bmatrix} K \\ J \end{bmatrix} = 0, \quad \begin{bmatrix} K^* & J^* \end{bmatrix} \begin{bmatrix} K \\ J \end{bmatrix} = I. \tag{4.44}$$

Let $G := P^{-\frac{1}{2}}K$ and define $H_1(z)$ as

$$H_1(z) := \left(D^*C + B^*PA\right)\left(zI - A\right)^{-1} G + B^*PG + D^*J. \tag{4.45}$$

Then, $H^*(z)H(z) = H_1(z)H_1^*(z)$.

Lemmata 4.4.1 and 4.4.2 are classic results in stochastic linear system theory. As regards Lemmata 4.4.3 and 4.4.4, the proof can be found in Appendix D on page 85.

By means of the previous results, it is possible to efficiently tackle the problem of evaluating the integrals (4.41) and (4.42). At first, one should try to factorize $Q_\Lambda(z) = (I + G_1^*(z)\Lambda G_1(z))$. To this purpose, it is worthwhile to introduce some considerations about the transfer function $G_1(z) = G(z)W_\Psi(z)$. Assume that a realization of the (canonical) minimum phase spectral factor $W_\Psi(z)$ is given by

$$W_\Psi(z) = C_\Psi(zI - A_\Psi)^{-1}B_\Psi + D_\Psi, \tag{4.46}$$

so that $\Psi = W_\Psi W_\Psi{}^*$ Then, recalling that $G(z)$ is defined by equation (2.16), G_1 can be interpreted as a series of two systems. This is shown in figure 4.1.

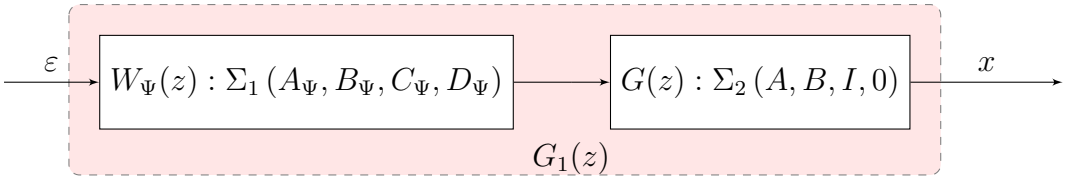


Figure 4.1: Representation of the transfer function $G_1(z)$ as a series of systems

Therefore, a state space realization for $G_1(z)$ is given by the following system:

$$\begin{cases} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_\Psi & 0 \\ BC_\Psi & A \end{bmatrix}}_{A_1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_\Psi \\ BD_\Psi \end{bmatrix}}_{B_1} \varepsilon(t) \\ y(t) = \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{C_1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases} \tag{4.47}$$

As a consequence, the quantity $Q_\Lambda(z) = I + G_1^*(z)\Lambda G_1(z)$ can be expressed in the form of the matrix product

$$Q_\Lambda(z) = \begin{bmatrix} B_1^*(z^{-1}I - A_1^*)^{-1} & I \end{bmatrix} \begin{bmatrix} C_1^*\Lambda C_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (zI - A_1)^{-1}B_1 \\ I \end{bmatrix}. \tag{4.48}$$

By Lemma 4.4.3, it is possible to conclude that, $\forall P = P^* \in \mathbb{C}^{n\times n}$ the following expression holds:

$$Q_\Lambda(z) = \begin{bmatrix} B_1^*(z^{-1}I - A_1^*)^{-1} & I \end{bmatrix} \underbrace{\begin{bmatrix} A_1^*PA_1 - P + C_1^*\Lambda C_1 & A_1^*PB_1 \\ B_1^*PA_1 & B_1^*PB_1 + I \end{bmatrix}}_{X} \begin{bmatrix} (zI - A_1)^{-1}B_1 \\ I \end{bmatrix} \tag{4.49}$$

The linear matrix inequality

$$X \geq 0 \tag{4.50}$$

is solvable for $P = P^* > 0$ if and only if the following ARE is solvable for $P = P^* > 0$:

$$P = A_1^*PA_1 - A_1^*PB_1(B_1^*PB_1 + I)^{-1}B_1^*PA_1 + C_1^*\Lambda C_1 \tag{4.51}$$

Taking P as the stabilizing solution of (4.51), X can be written as

$$X = \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix},$$

where $N = (B_1^* P B_1 + I)^{\frac{1}{2}}$ and $M = (B_1^* P B_1 + I)^{-\frac{1}{2}} B_1^* P A_1$. As a consequence, the minimum phase spectral factor

$$\Delta_\Lambda(z) = \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (zI - A_1)^{-1} B_1 \\ I \end{bmatrix} \tag{4.52}$$

is such that:

$$Q_\Lambda(z) = \Delta_\Lambda^*(z) \Delta_\Lambda(z). \tag{4.53}$$

It follows that $Q_\Lambda^{-1}(z) = {\Delta_\Lambda}^{-1}(z) \Delta_\Lambda^{-*}(z)$. Applying this result to the computation of (4.41) and (4.42), it turns out that:

$$\begin{aligned} Y + I &= \int G_1 (I + G_1^* \Lambda_i G_1)^{-1} G_1^* \\ &= \int \underbrace{G_1 \Delta_\Lambda^{-1}}_{W_Y} \Delta_\Lambda^{-*} G_1^* \\ &= \int W_Y W_Y^* \end{aligned} \tag{4.54}$$

As concerns the evaluation of Y_k, if the generator Σ_k is positive definite, it can be achieved as follows:

$$\begin{aligned} Y_k &= \int G_1 (I + G_1^* \Lambda_i G_1)^{-1} G_1^* \Sigma_k G_1 (I + G_1^* \Lambda_i G_1)^{-1} G_1^* \\ &= \int G_1 \Delta_\Lambda^{-1} \left(\Delta_\Lambda^{-*} G_1^* \Sigma_k^{\frac{1}{2}} \Sigma_k^{\frac{1}{2}} G_1 \Delta_\Lambda^{-1} \right) \Delta_\Lambda^{-*} G_1^* \\ &= \int G_1 \Delta_\Lambda^{-1} \left(H_k^* H_k \right) \Delta_\Lambda^{-*} G_1^* \\ &= \int G_1 \Delta_\Lambda^{-1} \underbrace{W_{H_k} W_{H_k}^*}_{H_k^* H_k} \Delta_\Lambda^{-*} G_1^* \\ &= \int \underbrace{W_{Y_k}}_{G_1 \Delta_\Lambda^{-1} W_{H_k}} {W_{Y_k}}^*. \end{aligned} \tag{4.55}$$

In the last but one inequality, Lemma 4.4.4 has been employed in order to compute a left factor, W_{H_k}, starting from the right one, H_k.

Remark 4.4.1. Notice that the generators Σ_k are not necessarily positive definite. If Σ_k is not positive defined, in order to evaluate the integral (4.42) by means of the previously introduced spectral factorization techniques, the following procedure can be applied. Let $-\lambda_k \leq 0$ be the minimal eigenvalue of Σ_k. Then, $\Sigma_k + (1 + \lambda_k) I > 0$.

Therefore, equation (4.55) can be written as follows:

$$\begin{aligned}
Y_k &= \int G_1(I + G_1^*\Lambda_i G_1)^{-1} G_1^* \Sigma_k G_1 (I + G_1^*\Lambda_i G_1)^{-1} G_1^* \\
&= \int G_1 \Delta_\Lambda^{-1} \Delta_\Lambda^{-*} G_1^* \left[\underbrace{\Sigma_k + (1+\lambda_k) I}_{\geq 0} - (1+\lambda_k) I \right] G_1 \Delta_\Lambda^{-1} \Delta_\Lambda^{-*} G_1^* \\
&= \int G_1 \Delta_\Lambda^{-1} \Delta_\Lambda^{-*} G_1^* \left[\Sigma_k + (1+\lambda_k) I \right] G_1 \Delta_\Lambda^{-1} \Delta_\Lambda^{-*} G_1^* \\
&\quad - (1+\lambda_k) \int G_1 \Delta_\Lambda^{-1} \Delta_\Lambda^{-*} G_1^* G_1 \Delta_\Lambda^{-1} \Delta_\Lambda^{-*} G_1^* \\
&= \int G_1 \Delta_\Lambda^{-1} \left(H_{k,1}^* H_{k,1} \right) \Delta_\Lambda^{-*} G_1^* \\
&\quad - (1+\lambda_k) \int G_1 \Delta_\Lambda^{-1} \left(H_{k,2}^* H_{k,2} \right) \Delta_\Lambda^{-*} G_1^* \\
&= \int G_1 \Delta_\Lambda^{-1} \left(W_{H_{k,1}} W_{H_{k,1}}^* \right) \Delta_\Lambda^{-*} G_1^* \\
&\quad - (1+\lambda_k) \int G_1 \Delta_\Lambda^{-1} \left(W_{H_{k,2}} W_{H_{k,2}}^* \right) \Delta_\Lambda^{-*} G_1^* \\
&= \int \underbrace{W_{Y_{k,1}}}_{G_1 \Delta_\Lambda^{-1} W_{H_{k,1}}} W_{Y_{k,1}}{}^* - (1+\lambda_k) \int \underbrace{W_{Y_{k,2}}}_{G_1 \Delta_\Lambda^{-1} W_{H_{k,2}}} W_{Y_{k,2}}{}^*.
\end{aligned} \tag{4.56}$$

Again, Lemma 4.4.4 has been employed in order to compute a left factor, $W_{H_{k,i}}$, starting from the right one, $H_{k,i}$, for $i = 1, 2$.

In conclusion, once Y and Y_k are written as in equations (4.54) and (4.55) (or in (4.56) if the generator Σ_k is not positive definite), Lemma 4.4.1 allows to evaluate the above integrals by means of appropriate state space realizations of $W_Y(z)$ and $W_{Y_k}(z)$ (or $W_{Y_{k,i}}(z)$ for $i = 1, 2$).

Step length

With regards to the backtracking line search, it proceeds until the two following conditions hold:

$$\Lambda_i + t_i^k \Delta\Lambda_i \in \mathcal{L}_+^\Gamma; \tag{4.57}$$

$$J_\Psi(\Lambda_i + t_i^k \Delta\Lambda_i) < J_\Psi(\Lambda_i) + \alpha t_i^k \nabla J_{\Psi,\Lambda_i} \Delta\Lambda_i. \tag{4.58}$$

The first one can be evaluated by testing if $Q_{\Lambda_i + t_i^k \Delta\Lambda_i}$ admits a factorization of the kind introduced in the previous subsection. If equation (4.51) is not solvable for $P = P^* > 0$, then t_i is halved ($t_i^{k+1} = \frac{1}{2} t_i^k$) until the it gets solvable.

The second condition, however, requires to compute

$$J_\Psi(\Lambda) = \mathrm{Tr} \int \left[\Lambda - \log(I + G_1^* \Lambda G_1) \right]. \tag{4.59}$$

The evaluation of

$$\mathrm{Tr}\left[\int \log(I + G_1^* \Lambda G_1)\right] = \int \log\det(I + G_1^* \Lambda G_1) \tag{4.60}$$

can be attained straightforwardly in the light of a fundamental result in statistical filtering and estimation theory, known as Wiener-Masani theorem. Its main result is to provide a tool for determining whether a stochastic process is *purely non-deterministic* (p.n.d. in the following) on the basis of the analysis of its spectral density function. As a spin off, it establishes an equation that can be employed to evaluate (4.60). Before stating the theorem, let us introduce some preliminaries. After the theorem statement, some remarks about it and the techniques leading to prove it will be exposed. Notice that in this part the symbol Λ will have a different meaning, given in Lemma 4.4.6. The reader is deferred to [12] for an exhaustive exposition and the rigorous proofs of all the results.

To begin with, assume y to be a zero mean real stationary process or rank m. define y_N by

$$y_N(t) := \begin{bmatrix} y(t)' & y(t-1)' & \dots & y(t-N)' \end{bmatrix}'. \tag{4.61}$$

The covariance matrix of y_N is given by

$$T_N := \mathbb{E}\left[y_N(t) y_N(t)'\right] = \left[\Sigma(t-j)\right]_{i,j}, \tag{4.62}$$

where $\Sigma(\tau) = \mathbb{E}\left[y_N(t+\tau) y_N(t)'\right]$, as usual. T_N will be called the *covariance matrix of order N* of the process y. Notice that it is symmetric positive semidefinite $m(N+1) \times m(N+1)$ matrix and has a block Toeplitz structure. The first important result about it is given in the form of the following lemma.

Lemma 4.4.5. *Assume y to be a full rank process. If T_N is singular for some N, then the process y is purely deterministic.*

As regards T_N, it can be easily obtained through a bordering scheme, defined by

$$T_N = \begin{bmatrix} \Sigma(0) & B \\ B' & T_{N-1} \end{bmatrix},$$

where $B = \begin{bmatrix} \Sigma(1) & \Sigma(2) & \dots \Sigma(N) \end{bmatrix}$. Another very interesting consideration arises from the introduction of the finite memory predictor of $y(t)$ of memory N:

$$\hat{y}_N(t) = \mathbb{E}\left[y(t) | y(t-1), y(t-2), \dots, y(t-N)\right].$$

The one-step prediction error vector $e_N := y(t) - \hat{y}_N(t)$ is called also the *innovation*

of memory N of the process y and, by definition, it is orthogonal to the subspace generated by the y_N, i.e. $\mathbf{H}(y(t-1),\dots,y(t-N))$. As a consequence,

$$\mathrm{Var}\left\{\begin{bmatrix} e_N(t) \\ \mathbf{y}^{N-1} \end{bmatrix}\right\} = \begin{bmatrix} \Lambda_N & 0 \\ 0 & T_{N-1} \end{bmatrix},$$

where $\Lambda_N := \mathrm{Var}\, e_N(t)$. Since $\hat{y}(t) := BT_{N_1}{}^{\dagger}\mathbf{y}^{N-1}$, it follows that

$$\begin{aligned} \begin{bmatrix} \Lambda_N & 0 \\ 0 & T_{N-1} \end{bmatrix} &= \begin{bmatrix} I_m & -BT_{N_1}{}^{\dagger} \\ 0 & I_{mN} \end{bmatrix} \begin{bmatrix} \Sigma(0) & B \\ B & T_{N-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -BT_{N_1}{}^{\dagger} & I_{mN} \end{bmatrix} \\ &:= U_N T_N U_N{}', \end{aligned} \tag{4.63}$$

where $T_{N_1}{}^{\dagger}$ is the Moore-Penrose pseudoinverse of T_{N_1}. Therefore,

$$T_N = U_N{}^{-1} \begin{bmatrix} \Lambda_N & 0 \\ 0 & T_{N-1} \end{bmatrix} U_N'{}^{-1}.$$

Since U_N is an upper triangular matrix with determinant equal to 1, it follows that

$$\det T_N = \det \Lambda_N \det T_{N-1} \quad N = 1, 2, \dots$$

and a recursion formula is established:

$$\det T_N = \prod_{k=1}^{N} \det \Lambda_k \det \Lambda_0, \quad N = 1, 2, \dots \tag{4.64}$$

Notice that Λ_0 can be assumed to be Σ_0. An important result arising from eq. (4.64) is the following lemma.

Lemma 4.4.6. *Assume the process y to have full rank m. The finite memory prediction error $e_N(t)$ converges, when $N \to \infty$, to a stationary process e, whose variance $\Lambda := \mathbb{E}\left[e(t)e(t)'\right]$ is equal to*

$$\Lambda = \lim_{N\to\infty} \Lambda_N. \tag{4.65}$$

Moreover,

$$\lim_{N\to\infty} \frac{1}{N} \log\det T_N = \log\det \Lambda. \tag{4.66}$$

Finally, the process is p.n.d. if and only if Λ is nonsingular.

It is interesting to analyze the relation between the the spectrum of the covariance matrix of order N of the process y, that will be referred to as $\sigma(T_N)$, and the spectral

density Φ. Recall that, given a matrix A and its eigenvalues $\{\lambda_{A_k}\}$, the following equations hold:

$$\operatorname{Tr} A = \sum_k \lambda_{A_k}, \quad \det A = \prod_k \lambda_{A_k}.$$

Since the diagonal blocks of the $m(N+1) \times m(N+1)$ matrix T_N are equal to the $m \times m$ matrix $\Sigma(0)$, it follows that

$$\frac{1}{N+1} \operatorname{Tr} T_N = \operatorname{Tr} \Sigma(0) = \int_{-\pi}^{\pi} \operatorname{Tr} \Phi(e^{j\theta}) \frac{d\theta}{2\pi}, \tag{4.67}$$

where the second equation is obtained by bringing the trace operator, which is linear, under the integral sign and recalling the relation established by eq. (2.7). The spectral distribution is assumed to be absolutely continuous. Dividing both the members of eq. (4.67) by m, an interesting result is attained:

$$\frac{1}{m(N+1)} \sum_{k=1}^{m(N+1)} \lambda_k(T_N) = \int_{-\pi}^{\pi} \frac{1}{m} \sum_{k=1}^{m} \lambda_k(\theta) \frac{d\theta}{2\pi}. \tag{4.68}$$

Therefore, the arithmetic mean of the eigenvalues of T_N is equal to the average over the unit circle of the arithmetic mean of the eigenvalues of the spectral density. A much more general result, in the same spirit of the former one, is now exposed, in order to introduce the main result of this section. It is known as the matrix version of the Szegö theorem. It will presented in the form of a lemma:

Lemma 4.4.7. *Assume that the process y is full rank and with continuous spectral distribution. Let $\Phi \in \mathcal{C}(\mathbb{T}; \mathcal{H}(m))$ be its spectral density and $\{\lambda_1(\theta), \lambda_2(\theta), \ldots, \lambda_m(\theta)\}$ be the set of the eigenvalues of the matrix $\Phi(e^{j\theta})$. Then, for any continuous real valued function F with compact support on the positive half line $[0, +\infty)$:*

$$\lim_{N \to \infty} \frac{1}{N} \Sigma_{\lambda \in \sigma(T_N)} F(\lambda) = \int_{-\pi}^{\pi} \sum_{k=1}^{m} F(\lambda_k(\theta)) \frac{d\theta}{2\pi} \tag{4.69}$$

Actually, another step is needed in order to dispose of all the elements that are needed to prove the Wiener-Masani theorem. Indeed, the previous lemma is valid for functions F that are continuous with compact support on the positive half line. However, the function that is involved in the theorem is the logarithm, that does not have compact support on the right half plane, nor it is continuous at zero. Nevertheless, it is possible to prove that eq. (4.69) holds also in this case. The basic idea of the proof is to represent the logarithm as a difference between two adequate functions, that are chosen so that it is possible to prove that they obey to eq. (4.69). As a consequence, this equation hold also for $F = \log$. At last, the main result is

stated:

Theorem 4.4.1 (Wiener-Masani)**.** *Assume the stationary stochastic process y is full rank and has an absolutely continuous spectral distribution with spectral density Φ. Moreover, recall that T_N is the covariance matrix of order N of the process y, as defined by eq.* (4.62) *and that Λ is the covariance of the one-step ahead optimal prediction error, defined by eq.* (4.65)*. Then*

$$\lim_{N\to\infty} \frac{1}{N} \log\det T_N = \int_{-\pi}^{\pi} \log\det \Phi(e^{j\theta}) \frac{d\theta}{2\pi}, \tag{4.70}$$

so that

$$\log\det \Lambda = \int_{-\pi}^{\pi} \log\det \Phi(e^{j\theta}) \frac{d\theta}{2\pi}. \tag{4.71}$$

Therefore, y is p.n.d. if and only if

$$\int_{-\pi}^{\pi} \log\det \Phi(e^{j\theta}) \frac{d\theta}{2\pi} > -\infty \tag{4.72}$$

Proof. Once that it is established that choosing $F(\cdot) = \log(\cdot)$ Lemma 4.4.7 still holds, an heuristic proof is the following. Firstly, recalling the properties of the logarithm, it holds that

$$\frac{1}{N} \sum_{\lambda\in\sigma(T_N)} \log\lambda = \frac{1}{N} \log \sum_{\lambda\in\sigma(T_N)} \lambda = \frac{1}{N} \log\det T_N$$

Secondly,

$$\sum_{\lambda(\theta)\in\sigma(\Phi(e^{j\theta}))} \log\lambda(\theta) = \log \prod_{\lambda(\theta)\in\sigma(\Phi(e^{j\theta}))} \lambda(\theta) = \log\det \Phi(e^{j\theta})$$

Finally, Lemma 4.4.6 allows to conclude that y is p.n.d. if and only if $\log\det\Lambda = \int_{-\pi}^{\pi} \log\det\Phi(e^{j\theta}) \frac{d\theta}{2\pi} > -\infty$. □

With regard to the computation of the integral (4.60), the most significant result offered by the previous theorem is that it establishes that $\int \log\det\Phi$ is equal to $\log\det\Lambda$, where Λ is the variance of the prediction error of the one-step ahead predictor of the process y, whose spectral density is Φ. Assume that the spectral factorization $\Phi = \Delta^*\Delta$ holds. As a consequence of Binet's theorem,

$$\begin{aligned} \log\det\Phi &= \log\det\left[\Delta^*\Delta\right] \\ &= \log\det\left[\Delta\Delta^*\right]. \end{aligned} \tag{4.73}$$

The variance of one-step ahead prediction error, Λ, corresponds to $\Delta(0)$. In order to

compute it, it is possible to consider a realization of the spectral factor Δ. Suppose that such a realization is given by the state space system $\Sigma(A, B_-, C, D_-)$. Then, Λ is given by $D_-D_-{}^\top$. Indeed, recall that Δ is the canonical minimum phase spectral factor of Q_Λ, and it has been obtained by solving (4.51). Therefore,

$$\int \log\det(I + G_1^*\Lambda G_1) = \log\det\left(D_-D_-{}^\top\right)$$

In conclusion, spectral factorization techniques play a key role in the backtracking line search, because they allow to check whether the candidate step length t_k is acceptable. Indeed, they are widely employed in order to evaluate both the condition (4.57) and (4.58).

4.5 Convergence Analysis

In this section, the key problem of the convergence of the proposed algorithm is tackled. As it has been already anticipated, this issue is tightly linked with the matter of the existence of a solution for the dual problem.

Recall that the underlying hypotheses for the convergence result attained in Section C.2, with respect to the dual functional $J_\Psi(\cdot)$, are the following:

1. $J_\Psi(\cdot)$ is twice continuously differentiable;
2. The starting point $\Lambda_0 \in \mathcal{L}_+^\Gamma$ and the sublevel set $S = \left\{\Lambda \in \mathcal{L}_+^\Gamma | J_\Psi(\Lambda) \leq J_\Psi(\Lambda_0)\right\}$ is closed;
3. $J_\Psi(\cdot)$ is *strongly* convex, i.e. $\exists\ m$ s.t. $\nabla^2 J_\Psi(\Lambda) > mI$, $\forall\ \Lambda \in S$. Moreover, $\exists$ M s.t. $\nabla^2 J_\Psi(\Lambda) < MI$, $\forall\ \Lambda \in S$;
4. The Hessian is Lipschitz continuous in S, i.e. $\exists L$ such that:

$$\left\|\nabla^2 J_\Psi(\Lambda_1) - \nabla^2 J_\Psi(\Lambda_2)\right\|_2 < L\left\|\Lambda_2 - \Lambda_1\right\|_2 \quad \forall \Lambda_1, \Lambda_2 \in S$$

Under these hypotheses, the proposed algorithm converges to the solution $\Lambda^\circ \in \mathcal{L}_+^\Gamma$ that minimizes the dual function J_Ψ. Moreover, after a certain number of iterations the backtracking line search always select the full step (i.e. $t = 1$). During the last stage the rate of convergence is quadratic, since

$$\|\Lambda_{i+1} - \Lambda^\circ\| \leq C\|\Lambda_i - \Lambda^\circ\|^2, \quad \exists \text{ costant } C$$

The continuous differentiability of the dual function has already been proved in 4.2 on page 36. Theorem 4.3.1 states that the sublevel sets of the the dual function

J_Ψ are compact, and hence closed (recall that, in a finite dimensional vector space, a set is compact if and only if it is closed and bounded). Moreover, it is possible to conclude straightforwardly on *strong* convexity and Lipschitz continuity of the Hessian. Indeed, let us consider the sublevel set

$$S = \left\{ \Lambda \in \mathcal{L}_+^\Gamma \mid J_\Psi(\Lambda) \leq J_\Psi(\Lambda_0) \right\}.$$

Notice that, assuming that Λ_0 is the starting point, the minimizing sequence computed by the Newton algorithm with backtracking line search is such that $\forall\, k \geq 0$, $\Lambda_k \in S$. The continuity of the Hessian over $\mathcal{L}_+^\Gamma$ has already been proved in Section 4.2. Moreover, since the map from a Hermitian matrix to its minimum eigenvalue is continuous (see Lemma 4.2.1), the map from $\Lambda \in \mathcal{L}_+^\Gamma$ to the minimum eigenvalue of $H_\Lambda(\delta\Lambda, \delta\Lambda)$ is continuous, being a composition of continuous maps. Since S is compact, Weierstrass' theorem holds. Therefore, there exists m that is the minimum in the set whose elements are the eigenvalues of the Hessian $H_\Lambda(\delta\Lambda, \delta\Lambda)$, $\forall\, \Lambda \in S$. Recall that the hypothesis of *strict* convexity holds (as proved in Theorem 4.2.1). As a consequence, the Hessian H_Λ is a positive definite matrix $\forall\, \Lambda \in S$, therefore $m > 0$. In conclusion, there exists $m > 0$ such that $H_\Lambda > mI$, $\forall\, \Lambda \in S$, i.e. $J_\Psi(\Lambda)$ is *strongly* convex. As regards the Lipschitz continuity of the Hessian of $J_\Psi(\Lambda)$, it is easy to see that H_Λ is $\mathcal{C}^1(\mathcal{L}_+^\Gamma)$. Indeed, $\delta^3 J_\Psi(\Lambda; \delta\Lambda_1, \delta\Lambda_2, \delta\Lambda_3)$ can be expressed as:

$$\begin{aligned}\delta^3 J_\Psi(\Lambda; \delta\Lambda_1, \delta\Lambda_2, \delta\Lambda_3) = \int \mathrm{Tr}\Big[& (I + G_1^* \Lambda G_1)^{-1} G_1^* \delta\Lambda_3 G_1 (I + G_1^* \Lambda G_1)^{-1} \\ & \times G_1^* \delta\Lambda_2 G_1 (I + G_1^* \Lambda G_1)^{-1} G_1^* \delta\Lambda_1 G_1 \\ & + (I + G_1^* \Lambda G_1)^{-1} G_1^* \delta\Lambda_2 G_1 (I + G_1^* \Lambda G_1)^{-1} \\ & \times G_1^* \delta\Lambda_3 G_1 (I + G_1^* \Lambda G_1)^{-1} G_1^* \delta\Lambda_1 G_1 \Big].\end{aligned} \tag{4.74}$$

The continuity can be proved, along the same line developed in the proof of Theorem 4.2.1. Moreover, the result can be extended leading to the conclusion that $J_\Psi(\Lambda)$ is $\mathcal{C}^\infty(\mathcal{L}_+^\Gamma)$. Continuous differentiability implies Lipschitz continuity on a compact set. Therefore, it is possible to state that the Hessian is Lipschitz continuous on S.

As a result, all the hypotheses under which convergence of the Newton algorithm is assured, as proven in Section C.2, hold. Therefore, the proposed algorithm represents an effective computational tool in order to provide the solution of the spectral estimation Problem 3.2.1.

Chapter 5

Simulation Results

The proposed algorithm has been implemented and tested in MATLAB. The results are encouraging and suggest that our approach tackles the problem of spectral estimation efficiently, both in the scalar and the multivariate case.

To begin with, let us consider a scalar example. The procedure was run over a set of $N = 300$ samples of a process described by the following difference equation:

$$\begin{aligned} y(t) = {} & 0.5y(t-1) - 0.42y(t-2) + 0.602y(t-3) - 0.0425y(t-4) \\ & + 0.1192y(t-5) + e(t) + 1.1e(t-1) - 0.08e(t-2) - 0.15e(t-3), \end{aligned}$$

where $e(t)$ is a zero mean Gaussian white noise with unit variance. The filterbank was chosen accordingly to the covariance extension setting with six covariance lags:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Two prior spectral densities were considered. The first one is the constant spectrum $\hat{\Phi}(e^{j\theta}) = \hat{\sigma}_y^2$, where $\hat{\sigma}_y^2$ denotes the sample covariance of the process y, computed as $\hat{\sigma}_y^2 = \frac{1}{N-1}\sum_{k=1}^{N} |y_k|^2$. The second one is the estimate $\hat{\Phi}_{PEM} = W_{PEM}W_{PEM}^*$, with W_{PEM} of dimension 3, that has been obtained from the data by means of the Prediction Error Method procedure available in MATLAB System Identification toolbox. The proposed approach was applied with the tolerance of the Newton algorithm set to $\varepsilon = 10^{-16}$. The results are shown in plot 5.1. In both the cases, our method provided a good approximant Φ° of the true spectrum. As concerns the complexity of the estimates, the dimension of a minimal state-space

realization of Φ° was 10 in the first case and 16 in the second one.

Let us consider a multivariable example, now. The method was applied to the estimation of the spectrum of a process obtained by feeding a bivariate Gaussian process z, with mean 0 and variance equal to the identity matrix, to a filter $G_y(z)$ with transfer matrix

$$G_y(z) = C_y(zI - A_y)^{-1}B_y,$$

where

$$A_y = \begin{bmatrix} 0.9\cos(0.52) & 0.9\sin(0.52) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.9\sin(0.52) & 0.9\cos(0.52) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6\cos(0.58) & 0.6\sin(0.58) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6\sin(0.58) & 0.6\cos(0.58) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_y = \begin{bmatrix} 1 & 0.2633 \\ 1 & 0.8916 \\ 1 & 0.5861 \\ 1 & 0.9387 \\ 1 & 0.6181 \\ 1 & 0.2507 \\ 1 & 0.1025 \\ 1 & 0.5464 \\ 1 & 0.4426 \end{bmatrix}, C_y = \begin{bmatrix} 0.6915 & 0.3994 & 0.2056 & 0.5128 & 0.2291 & 0.6894 & 0.6569 & 0.5805 & 0.8787 \\ 0.7594 & 0.5674 & 0.9435 & 0.2762 & 0.929 & 0.7079 & 0.7790 & 0.5032 & 0.3929 \end{bmatrix}.$$

Note that each minimal state space realization of Φ has order 16. The matrices A and B of the filterbank $G(z) = (zI - A)^{-1}B$ were set to

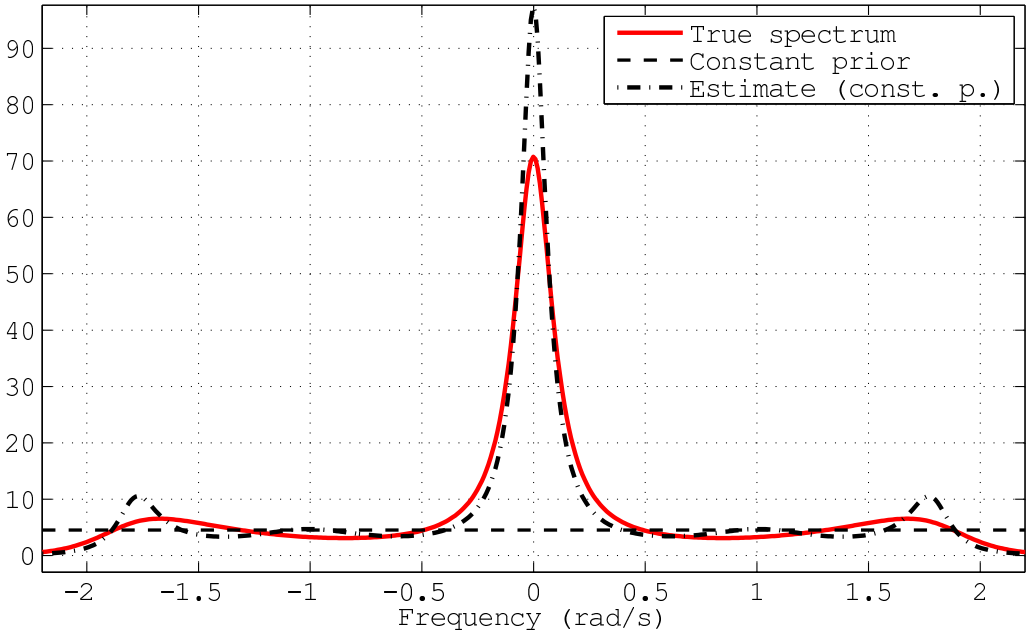
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9\cos(\frac{\pi}{4}) & 0.9\sin(\frac{\pi}{4}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9\sin(\frac{\pi}{4}) & 0.9\cos(\frac{\pi}{4}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9\cos(\frac{\pi}{2}) & 0.9\sin(\frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.9\sin(\frac{\pi}{2}) & 0.9\cos(\frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9\cos(3\frac{\pi}{4}) & 0.9\sin(3\frac{\pi}{4}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.9\sin(3\frac{\pi}{4}) & 0.9\cos(3\frac{\pi}{4}) \end{bmatrix},$$

and

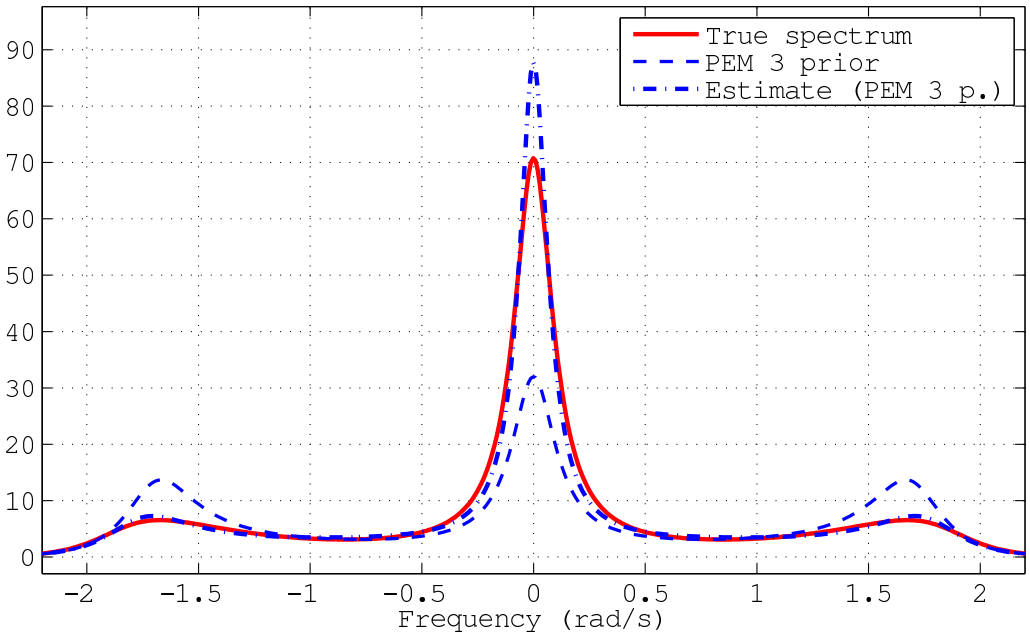
$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The considered prior was the constant spectrum equal to the sample covariance of the process y. Firstly, the procedure was run over $N = 200$ samples of the process of interest. The results provided by the proposed approach were compared to the spectrum estimate computed by the standard N4SID procedure. Such a method, available in MATLAB System and Identification toolbox, provided an estimate of the spectrum, $\hat{\Phi}_{N4SID} = W_{N4SID}W^*_{N4SID}$, by selecting W_{N4SID} of order 18. As can be observed by considering the results shown in plot 5.2, the performances of the proposed approach were similar to those achieved by the N4SID method. The dimension of a minimal realization of Φ° was 16. Finally, the method was applied to the case in which a shorter sequence of samples of the process y was available. With $N = 100$, the proposed approach outperformed the N4SID procedure, that was also affected by artifacts. This is shown in plot 5.3. As expected (see Section 3.2), the proposed approach seems to assure good performances even if the sequence of available sample of the process of interest is short.

Other aspects should be taken into consideration in future experiments. To begin with, it is worthwhile to analyze the effect of the positions of the poles of the bank of filters on the resolution of the estimate in the corresponding range of frequencies. In addition, the case in which the true covariance matrix Σ is substituted by an estimate $\hat{\Sigma}$, that is typical in practical applications, should be investigated. In particular, the method proposed in [8], that aims at finding an adequate estimate of the covariance matrix Σ, should be applied to the proposed spectral estimator. At last, the performances of the proposed estimator should be compared with the those achieved by the state of-the-art techniques described in Section 2.2.

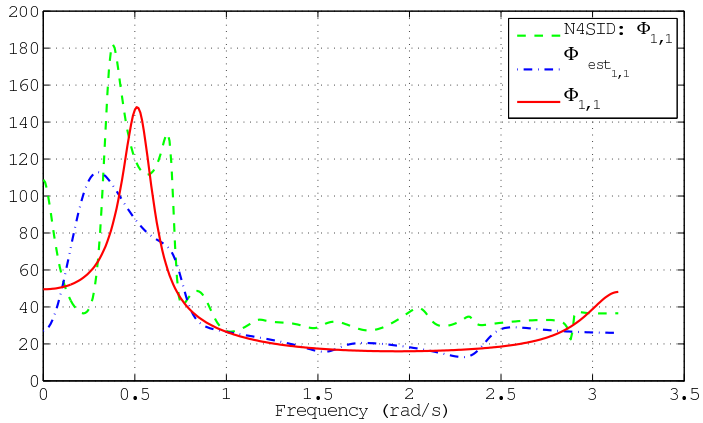


(a) *Constant prior.*

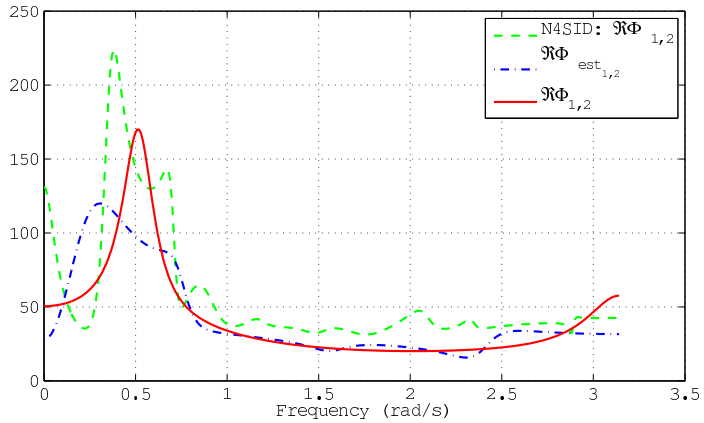


(b) *PEM 3 prior.*

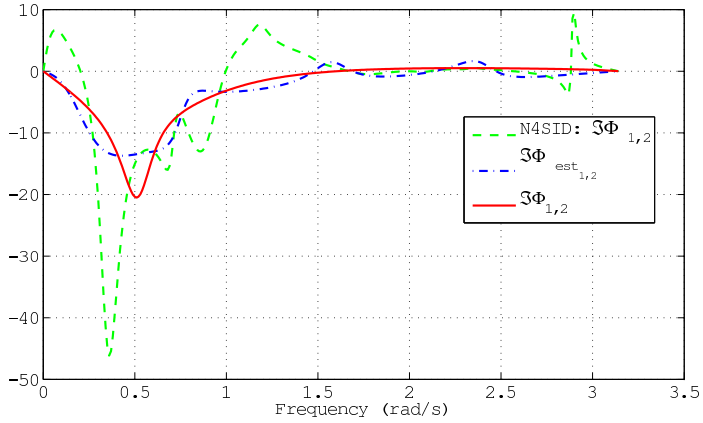
Figure 5.1: Scalar example



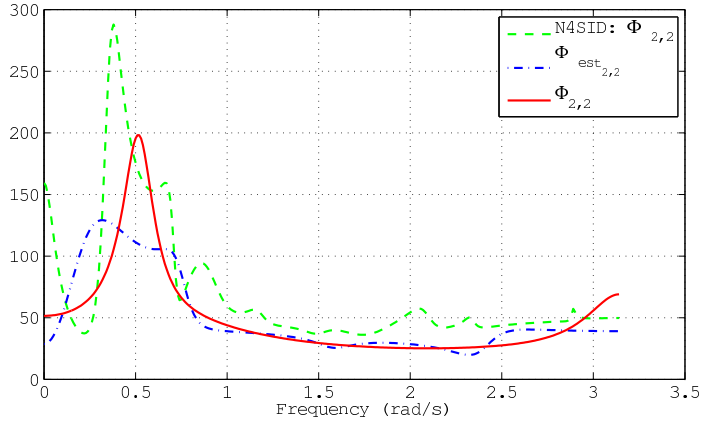
(a) $\Phi_{1,1}$.



(b) $\Re\Phi_{1,2}$.

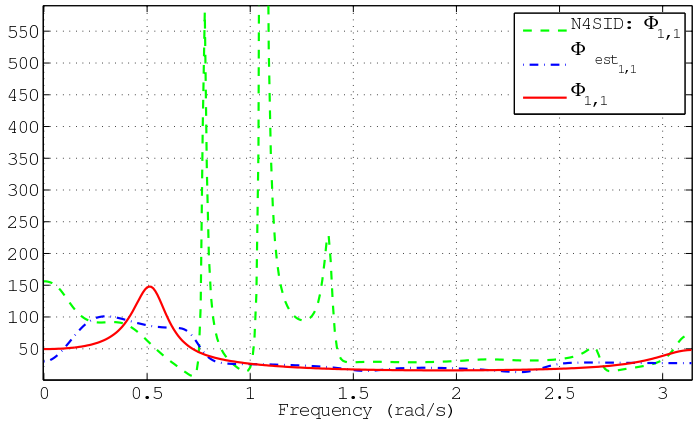


(c) $\Im\Phi_{1,2}$.

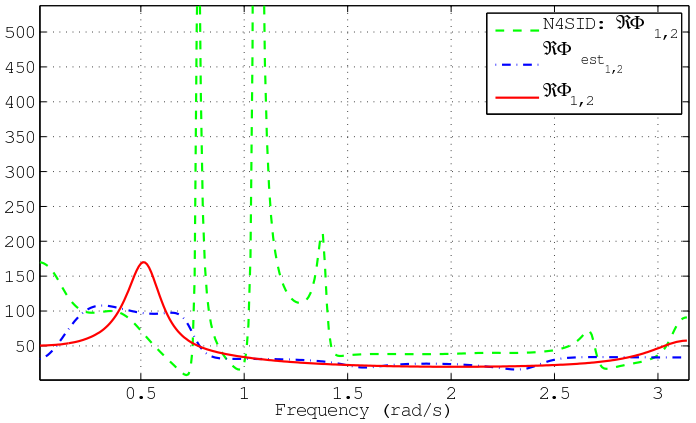


(d) $\Phi_{2,2}$.

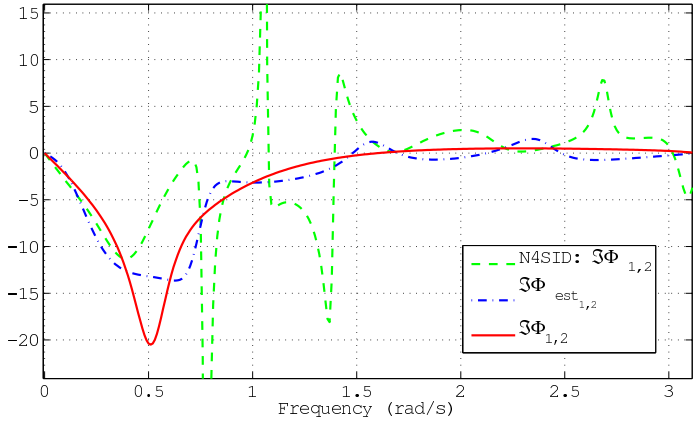
Figure 5.2: Multivariate example: N=200. The plots show the true spectrum (red solid line), the estimate provided by the proposed approach (blue dash-dot line) and the one computed by MATLAB N4SID procedure (green dashed line)



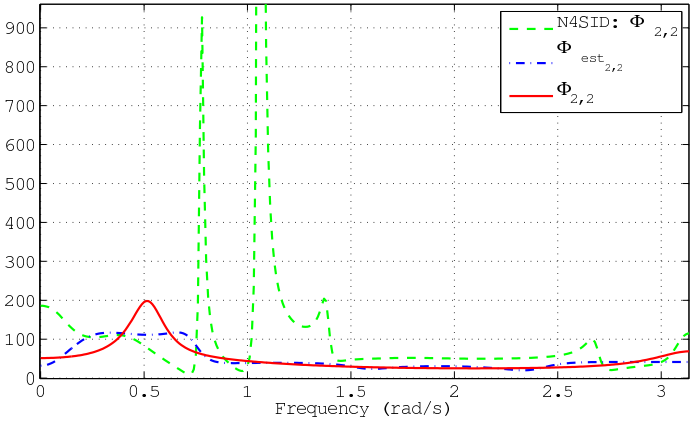
(a) $\Phi_{1,1}$.



(b) $\Re\Phi_{1,2}$.



(c) $\Im\Phi_{1,2}$.



(d) $\Phi_{2,2}$.

Figure 5.3: Multivariate example: N=100. The plots show the true spectrum (red solid line), the estimate provided by the proposed approach (blue dash-dot line) and the one computed by MATLAB N4SID procedure (green dashed line)

Chapter 6

Conclusions and Future Work

The main contribution of this work is the proposal of a new spectral estimation technique. Such an approach allows to successfully tackle the challenging issue of multivariable spectral estimation. Moreover, it provides an upper bound on the complexity of the estimate that outperforms the one offered by the state of-the-art technique for the multivariable framework described in [16] and [17].

The proposed approach draws inspiration from modern methods (described in [10] and [16]) which are based on convex optimization. As a consequence, the first step has been to recast the issue of interest in a proper form, described by Problem 3.2.1. This result has been achieved by means of the introduction of a *prior* spectrum, a bank of filters and an adequate index for measuring the (pseudo-)distance between spectral densities. Such an index represents a novelty, with regard to the state of-the-art approaches that have been analyzed in Section 2.2. Primarily, the choice of the distance measure defined by 3.13 has allowed to develop the variational analysis straightforwardly, achieving the expected results.

Indeed, once feasibility conditions have been investigated, the primal Lagrangian corresponding to Problem 3.2.1 has been analyzed. The variational analysis has allowed to obtain an easy expression of the optimal solution, Φ°, in terms of the Lagrangian multiplier Λ. Therefore, the dual problem has been tackled. Recall that such a problem is finite dimensional, while the primal one is not. In particular, the minimization of the function defined by (4.20) has been sought. Firstly, the uniqueness of the solution has been proven. Secondly, its existence has been established by means of Lemmata 4.3.1, 4.3.2, 4.3.3 and Theorem 4.3.1. Thirdly, a Newton-type algorithm with backtracking line search has been proposed in order to find the optimal solution Λ°. Many ideas offered by Linear Systems Theory have been applied: search direction is performed by recurring to spectral factorization techniques, for instance, while in backtracking line search the values assumed by the function of in-

terest are computed by means of Wiener-Masani Theorem. Finally, the convergence of the proposed algorithm has been proven. Experimental results suggest that our approach tackles both the scalar and the multivariate spectral estimation problem efficiently. Moreover, it seems to be very effective in the case of short available data records.

In the future, further numerical simulations should be performed, with the purpose of evaluating the performance of the estimator in practice, also in terms of numerical robustness. Moreover, the results obtained when the true covariance matrix Σ is substituted by an estimate $\hat{\Sigma}$ should be analyzed, with particular regard to the method proposed in [8], that could be applied to our estimator in order to achieve better performances, since in practical applications the starting covariance matrix Σ has to be estimated from the data. Another issue that requires further investigation is to determine how the position of the poles of the filterbank affects the resolution of the estimate. Indeed, the estimators described in [10] and [16], which propose an approach that is quite similar to the one that has been developed in this work, exhibit a remarkable property. If the poles are chosen in proximity of the unit circle, the resolution of the estimate in the corresponding range of frequencies is very high (as described in [2]). This result allows to detect lines and deep variations in the spectrum of interest. In addition, other distance measures between spectral densities should be taken into consideration, in order to analyze their properties and to evaluate the possibility to develop other estimation techniques along the same line that has been described in this thesis. At last, the statistical analysis of the achievable performance of the proposed estimator should be tackled.

Appendix A

The McMillan Degree

The introduction of the McMillan degree arises from the necessity of measuring the complexity of a MIMO (Multiple Input Multiple Output) system. It provides the dimension of each minimal (i.e. reachable and observable) state space realization of a given $n \times m$ transfer matrix $G(z)$.
It is interesting to introduce such an index starting from the definition of the Smith McMillan form, because it provides an overview on some of the problems that come to light in the multivariate setting. To begin with, recall that $G(z)$ is a transfer matrix, whose element $G_{i,j}$ is given by the transfer function from the $j-$th input to the $i-$th output. Assume that its rank is p, with $p \leq \min(m, n)$.

The poles and zeros of interest and their multiplicities are the ones of the matrix itself (not the poles and zeros of its individual elements). Indeed, even though the location of the poles of the matrix transfer function can be identified by inspection of its individual elements, their multiplicity is not available for direct computation. With regard to the system zeros, not only their locations, but also their existence cannot be evaluated by looking at the individual elements of the transfer matrix. Therefore, the issue of studying the poles and the zeros of the matrix transfer function is not trivial.

The Smith-McMillan form provides a powerful tool to solve this problem. In order to compute such a form, the first step is to attain the Smith form. To begin with, the matrix transfer function $G(z)$ is expressed as

$$G(z) = \frac{1}{d(z)} P(z), \tag{A.1}$$

where $d(z)$ is the least common multiple of the denominators of all the elements of $G(z)$. As a consequence, $P(z)$ turns out to be a *polynomial* matrix, i.e. a matrix

whose entries are all polynomials. By means of elementary row and column operations, $P(z)$ is transformed into an equivalent matrix $S(z)$, known as the Smith form, that is given by

$$S(z) = \text{diag}\left[\epsilon'_1(z), \epsilon'_2(z), \ldots, \epsilon'_r(z), 0, \ldots, 0\right], \tag{A.2}$$

where the polynomials ϵ'_i, known as the invariant factors of $P(s)$, are monic and exhibit the following property:

$$\epsilon'_i(z) | \epsilon'_{i+1}(z) \quad \forall z \in [0, \ldots, r-1]\,,$$

where | has to be read as "exactly divides". In order to compute the entries of the Smith form of $P(s)$ the rule to follow is the following:

$$\epsilon'_i(z) = \frac{D_i(z)}{D_{i-1}(z)}, \quad D_0(z) := 1,$$

where the polynomial $D_i(z)$ is the least common divisor of all the minors of order i that can be obtained from $P(z)$. Once that the Smith form of $P(z)$ is computed, the step to the Smith Mcmillan form of $G(z)$ is short. In the light of eq. (A.1), It is given by

$$\begin{aligned} M(z) &= \text{diag}\left[\frac{\epsilon'_1(z)}{d(z)}, \frac{\epsilon'_2(z)}{d(z)}, \ldots, \frac{\epsilon'_r(z)}{d(z)}, 0, \ldots, 0\right] \\ &= \text{diag}\left[\frac{\epsilon_1(z)}{\psi_1(z)}, \frac{\epsilon_2(z)}{\psi(z)}, \ldots, \frac{\epsilon'_r(z)}{\psi_r(z)}, 0, \ldots, 0\right], \end{aligned} \tag{A.3}$$

where

$$\frac{\epsilon_i(z)}{\psi_i(z)} = \frac{\epsilon'_i(z)}{d(z)}$$

with all the possible cancellations between $\epsilon'_i(z)$ and $d(z)$ being performed. The degree of the *pole polynomial*

$$p(z) := \prod_i^r \psi_i(z)$$

is the McMillan degree of the transfer function $G(z)$, that provides the dimension of each minimal state space realization of a systems whose transfer function is given by $G(z)$. Note that the *transmission zeros* are available by finding the roots of the polynomial

$$r(z) := \prod_i^r \epsilon_i(z).$$

They do not coincide, in general, with the zeros of the individual entries of $G(z)$.

Actually, since $M(z)$ and $G(z)$ have the same rank, r, because they are equivalent, the zeros are the values that make the rank of $G(z)$ drop below r.

Appendix B

A Brief Review on Duality Theory

Next, some fundamental notions about duality theory are recalled. The aim of this appendix is simply to provide some basic tools. They are sufficient to develop the strategy which has been proposed in section 4.2 on page 36, in order to tackle the spectral estimation problem. For an exhaustive explanation about duality theory, the reader is deferred to [1].

Suppose that the problem to solve is

Problem B.0.1. Minimize $f_0(x)$, subject to

$$\begin{aligned} f_i(x) &\leq 0; \quad i = 1, \ldots, m \\ h_i(x) &= 0; \quad i = 1, \ldots, p \end{aligned}$$

where $x \in \mathcal{D} := (\cap_{i=0}^{m} \operatorname{dom} f_i) \cap (\cap_{i=1}^{p} \operatorname{dom} h_i) \subseteq \mathbb{R}^r$.

No assumption about the convexity of $f_0(\cdot)$ are required. Indeed, it is only assumed that $\mathcal{D} \neq \emptyset$. In order to solve the problem, the *Lagrangian function* $L : \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is introduced. It is defined by

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \nu_i f_i(x) + \sum_{i=1}^{p} \lambda_i h_i(x). \tag{B.1}$$

The sets $\{\lambda_i\}$ and $\{\nu_i\}$ define the so-called *dual variables*. Now it is possible to introduce the *Lagrangian dual function* $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, such that

$$g(\nu, \lambda) := \inf_{x \in \mathcal{D}} L(x, \nu, \lambda) = \inf_{x \in \mathcal{D}} f_0(x) + \sum_{i=1}^{m} \nu_i f_i(x) + \sum_{i=1}^{p} \lambda_i h_i(x). \tag{B.2}$$

Since the dual function is the pointwise infimum of a family of affine functions in (ν, λ), it is concave, even if problem B.0.1 is not convex. Under the hypothesis that

$\nu_i \geq 0$, for all $i = 0, \ldots, m$, $g(\nu, \lambda) \leq L(\tilde{x}, \nu, \lambda) \leq f_0(\tilde{x})$, $\forall \tilde{x} \in \mathcal{D}$. In the case of interest for spectral estimation, the Lagrangian dual function obeys to the former inequality. Indeed, there are no constraints of the kind $f_i(x) \leq 0$, therefore $\{\nu_i\}$ is an empty set. Equivalently, ν_i can be assumed to be null for all the values taken by the index i. The *Lagrangian dual problem* is defined by

Problem B.0.2. Maximize $g(\nu, \lambda)$, subject to

$$\nu \geq 0$$

Denote the optimal value that solves problem B.0.1 as x^* and define $p^* = f_0(x^*)$. As regard the dual problem, let (ν^*, λ^*) be the optimal solution and $d^* = g(\nu^*, \lambda^*)$. The optimal value of the Lagrange dual problem, d^*, represents the best lower bound on p^* that can be obtained from the Lagrange dual function. Therefore, the following inequality is always satisfied:

$$d^* \leq p^*.$$

This property is called weak duality. The difference $p^* - d^*$, which is always non-negative, is called *optimality gap*. When the optimality gap is zero, i.e. $d^* = p^*$, there is *strong duality*. It means that the lower bound provided by the dual problem is as tight as possible. If strong duality holds and there exists a pair (ν^*, λ^*) such that it solves the dual problem, then each optimal solution of the primal problem minimizes $L(x, \nu^*, \lambda^*)$.

Appendix C

Basic Notions about Unconstrained Optimization

Some definitions and fundamental results about convex optimization are revised. In particular, the focus is on algorithmic procedures to tackle this issue. Indeed, the aim of this subsection is to describe and motivate some ideas that are widely employed, with reference to multivariate spectral estimation, in the present work. The reader is referred to [1] for an exhaustive exposition about convex optimization.

Preliminary, a specification is needed. For the sake of simplicity, in the following, the theory is developed with regard to the case in which the function to be minimized, f, is defined over $\mathbb{R}^r$. In the case of interest, however, the dual function whose minimization is sought, J_Ψ, is defined over a subset of the linear space Range (Γ). Starting from a base $\{H_1, \ldots, H_N\}$ of $\mathbb{R}^{m\times n}$ and solving equation (3.17) for each H_k, $k = 1, \ldots, N$, it is possible to find a set of generators $\{\Sigma_1, \ldots \Sigma_N\}$ for Range (Γ). As a consequence, Range (Γ) has a finite dimension $r \leq N$. Therefore, given a base of Range (Γ), it is possible to establish a homeomorphic correspondence between this space and $\mathbb{R}^r$. In particular, to every matrix Λ in $\mathcal{L}_+^\Gamma$ there corresponds a vector in $\mathbb{R}^r$. Similarly, to each positive definite bilinear form over Range (Γ) there corresponds a positive definite matrix in $\mathbb{R}^{r\times r}$. As a consequence, each result that can be proven in $\mathbb{R}^r$ holds in the matricial setting of interest, too.

Suppose that the following unconstrained minimization problem has to be solved:

$$\text{minimize} \quad \text{f(x)}$$

such that the function $f(x) : \mathbb{R}^r \to \mathbb{R}$ satisfies the following assumptions:

- f is *convex* and *twice continuously differentiable*;
- The problem is solvable, i.e. $\exists$ an optimal $x^\star$ such that $\inf_x f(\mathbf{x}) = \mathbf{x}^\star = p^\star$

Take into consideration algorithms computing a *minimizing sequence*, i.e. a sequence of points $x_0, x_1, \ldots \in \operatorname{dom} f$ such that $\lim_{k\to\infty} f(x_k) = p^\star$. It is necessary to attain a starting point $x_0 \in \operatorname{dom} f$ and to impose another assumption on f:

$$S = \{x \in \operatorname{dom} f | f(x) \leq f(x_0)\} \quad \text{is a closed sublevel set.}$$

An assumption that makes easier the evaluation of the convergence of the algorithms is that the function f is *strongly convex*. This means that there exists m such that:

$$H(x) = \nabla^2 f(x) > mI \quad \forall x \in S.$$

It can be proved that, under this hypothesis,

$$\left\|\nabla^2 f(x)\right\|_2 \leq (2m\varepsilon)^{\frac{1}{2}} \Rightarrow f(x) - p^\star \leq \varepsilon. \tag{C.1}$$

It should be highlighted that m is usually unknown; however, equation (C.1) suggests that when the Hessian norm is small enough the minimizing sequence is quite near to the optimal point (it provides a suboptimal condition).

Let us focus on *descent methods*. The minimizing sequence is defined such that:

$$x_{k+1} = x_k + t_k \Delta x_k,$$

where t_k is called the *step length* and it is positive (except when x_k is optimal), while Δx_k is the *search direction* and obeys to the condition

$$\nabla^\top f(x_k) \Delta x_k < 0,$$

which is necessary in order to assure that $f(x_{k+1}) \leq f(x_k)$ (i.e. the sequence is actually decreasing).

Algorithm C.1 The general descent method

1: given $x_0 \in$ dom f
2: **repeat**
3: Find descent direction Δx
4: Line search: choose $t > 0$
5: Update: $x \leftarrow x + t\Delta x$
6: **until** optimality criterion is satisfied (e.g. $\left\|f(x) - p^\star\right\| \leq \varepsilon$)

Line search can be performed in two ways:

- *Exact line search*

- *Backtracking line search*

In the former, t is chosen as $t = \arg\min_{s\geq 0} f(x + s\Delta x)$. In the latter, which is usually easier to implement, the following procedure is performed:

Algorithm C.2 Backtracking line search

1: given $\alpha \in (0, 0.5)$
2: given $\beta \in (0, 1)$
3: given Δx for f at $x \in \operatorname{dom} f$
4: $t \leftarrow 1$
5: **while** $f(x + t\Delta x) > f(x) + \alpha t \nabla^\top f(x)\Delta x$ **do**
6: $\quad t \leftarrow \beta t$
7: **end while**

In the previous algorithm, before the inequality in line 5 is evaluated, it is important to check whether $x + t\Delta x \in \operatorname{dom} f$, otherwise t is reduced until this condition holds.

C.1 Newton's Algorithm

There are many different procedures that implement the descent methods: here the focus will be on the Newton method, because the algorithm that is proposed to solve the multivariate spectral estimation employs this approach. The key idea is to consider the search direction as:

$$\Delta x_{nt} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x). \tag{C.2}$$

Under our hypothesis on f, its Hessian is surely positive defined, so $\nabla^\top f(x)\Delta x_{nt} = -\nabla^\top f(x) \left[\nabla^2 f(x)\right]^{-1} \nabla f(x) < 0$ (unless x is optimal). Therefore, the Newton step actually moves in a descent direction. This choice can be easily interpreted. For instance, given that a point $\tilde{x}$ is optimum for a convex function f if and only if $\nabla f(\tilde{x}) = 0$, by taking into account the approximation

$$\nabla f(x + v) \approx \nabla f(x) + \nabla^2 f(x)\Delta x$$

it follows that the the optimality condition is satisfied by

$$v = \Delta x_{nt} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x).$$

Alternatively, the same result can be attained by considering the second order Taylor approximation of f:

$$f(x+v) \approx \hat{f}(x+v) = f(x) + \nabla^\top f(x)v + \frac{1}{2}v^\top \nabla^2 f(x)v$$

and minimizing it. Since f is twice differentiable, the quadratic model of f will be very accurate in proximity of the value $x^\star$. It follows that when x is near $x^\star$, the point $x + t\Delta x_{nt}$ should be a very good estimate of the optimum.

Define the Newton decrement $\lambda(x)$:

$$\lambda(x) = \left(\nabla f(x)^\top \left[\nabla^2 f(x)\right]^{-1} \nabla f(x)\right)^{\frac{1}{2}}.$$

It is significant to point out that

$$-\lambda^2(x) = \nabla f(x)^\top \Delta x_{nt} = \frac{d}{dt} f(x + t\Delta x_{nt})\bigg|_{t=0}.$$

Is plays an important role in the convergence analysis.

Algorithm C.3 Newton's method

1: given a starting point $x_0 \in \text{dom}\, f$, tolerance $\varepsilon > 0$
2: **repeat**
3: Compute the Newton step Δx_{nt} and the squared Newton decrement $\lambda^2(x)$.
4: Stopping criterion: **quit** if $\frac{\lambda^2(x)}{2} \leq \varepsilon$.
5: Line search. Choose step size t by backtracking line search.
6: Update: $x \leftarrow x + t\Delta x_{nt}$.
7: **until** stopping criterion is satisfied

C.2 Convergence Analysis of Newton's Algorithm with Backtracking

Finally, the convergence analysis is faced. The underlying hypotheses are the following:

1. f is twice continuously differentiable;
2. f is *strongly* convex ($\exists\ m, M$ s.t. $mI < \nabla^2 f(x) < MI$, $\forall\ x \in S$);
3. The starting point $x_0 \in \text{dom}\, f$ and the sublevel set $S = \{x \in \text{dom}\, f | f(x) \leq f(x_0)\}$ is closed;

4. The Hessian is Lipschitz continuous in S, i.e. $\exists L$ such that:

$$\left\|\nabla^2 f(y) - \nabla^2 f(x)\right\|_2 < L \left\|y - x\right\|_2 \quad \forall x, y \in S$$

The last assumption has been added to the ones considered so far and gives a bound on the third derivative of f. It should be interpreted as a request on the quality of the approximation of f given by a quadratic model (the smaller L is, the better the quadratic model describes f).

Now, it will be proved that there exist some positive constants $\eta \, s.t. \, 0 < \eta \leq \frac{m}{L^2}$ and γ such that the Newton algorithm naturally falls into two stages: the first one, that is called *damped Newton phase*, occurs when $\left\|\nabla f(x)\right\| \geq \eta$; the second one, also known as the *pure Newton phase*, takes place when $\left\|\nabla f(x)\right\| < \eta$. The convergence is actually assured and an upper bound on the number of iterations required to converge to the solution with the required tolerance ε can be attained straightforwardly. In particular:

- If $\|\nabla f(x_k)\|_2 \geq \eta$, then

$$f(x_{k+1}) - f(x_k) \leq -\gamma. \tag{C.3}$$

  The number of iterations in this stage is upper-bounded by

$$\frac{f(x_0) - p^\star}{\gamma}.$$

- If $\|\nabla f(x_k)\|_2 < \eta$ then the backtracking line search selects $t_k = 1$ and

$$\frac{L}{2m^2}\|\nabla f(x_{k+1})\|_2 \leq \left(\frac{L}{2m^2}\|\nabla f(x_k)\|\right)^2. \tag{C.4}$$

  It follows that once $\|\nabla f(x_k)\|_2 < \eta$, $\|\nabla f(x_l)\|_2 < \eta$, $\forall l \geq k$. As a consequence,

$$\frac{L}{2m^2}\|\nabla f(x_l)\|_2 \leq \left(\frac{L}{2m^2}\|\nabla f(x_k)\|\right)^{2^{l-k}} \leq \left(\frac{1}{2}\right)^{2^{l-k}}.$$

  Since $f(x_l) - p^\star \leq \frac{1}{2m}\|\nabla f(x_l)\|_2{}^2$, the previous inequality allows to conclude that

$$f(x_l) - p^\star \leq \frac{2m^3}{L^2}\left(\frac{1}{2}\right)^{2^{l-k+1}}.$$

  So, $f(x_l) - p^\star \leq \varepsilon$ after at most $\log_2 \log_2 \frac{\varepsilon_0}{\varepsilon}$ steps, where $\varepsilon_0 = \frac{2m^3}{L^2}$.

Therefore, the total number of iterations is bounded by

$$\frac{f(x_0) - p^\star}{\gamma} + \log_2 \log_2 \frac{\varepsilon_0}{\varepsilon}.$$

Moreover, it should be highlighted that the the algorithm converges quadratically once the second condition on $\|\nabla f(x)\|_2$ holds, so that the *pure Newton phase* is also called the *quadratically convergent stage*.

Finally, these results are proved. Firstly, the *damped Newton phase* will be dealt with, showing that if $\|\nabla f(x_k)\|_2 \geq \eta$, then $f(x_{k+1}) - f(x_k) \leq -\gamma$, $\exists\, 0 < \eta < \frac{m^2}{L}$, $\gamma > 0$. Assuming that $\|\nabla f(x_k)\|_2 \geq \eta$, a lower bound on the step length chosen in the backtracking procedure is computed. Recall that f is strongly convex, so $\exists M : \|\nabla^2 f(x)\| < MI \,\forall x \in S$. It follows that:

$$\begin{aligned} f(x + t\Delta_{nt}x) &\leq f(x) + t\nabla^\top f(s)\Delta x_{nt} + \frac{1}{2}M{\|\Delta x_{nt}\|_2}^2 \\ &\leq f(x) - t\lambda^2(x) + \frac{M}{2m}t^2\lambda^2(x). \end{aligned}$$

The facts that $\lambda^2(x) = -\nabla^\top f(x)\Delta x_{nt}$ and $\Delta^\top x_{nt}\nabla^2 f(x)\Delta x_{nt} \geq m{\|\Delta x_{nt}\|_2}^2$ have been employed. As a consequence, $\tilde{t} = \frac{m}{M}$ satisfies the exit condition of backtracking line search:

$$f(x + \tilde{t}\Delta x_{nt}) \leq f(x) - \frac{1}{2}\tilde{t}\lambda^2(x) \leq f(x) - \alpha\tilde{t}\lambda^2(x)$$

Therefore, the chosen t is such that $t > \beta\frac{m}{M}$. Finally, the difference between $f(x + t\Delta x_{nt})$ and $f(x)$ is upper bounded, since, by recalling that $\lambda^2(x) \geq \frac{1}{M}{\|\nabla^2 f(x)\|_2}^2$:

$$\begin{aligned} f(x + t\Delta x_{nt}) - f(x) &\leq -\alpha t\lambda^2(x) \\ &\leq -\alpha\beta\frac{m}{M}\lambda^2(x) \\ &\leq -\alpha\beta\frac{m}{M^2}{\|\nabla f(x)\|_2}^2 \\ &\leq -\alpha\beta\eta^2\frac{m}{M^2} \\ &\leq -\gamma. \end{aligned}$$

Now, let us focus on the *quadratically convergent phase*. Assuming that $\eta \leq 3\,(1 - 2\alpha)\,\frac{m^2}{L}$, it is possible to prove that a step length that obeys to the stopping condition of backtracking line search is given by $t = 1$. The hypothesis on the Lipschitz continuity of the Hessian allows to write

$$\|\nabla^2 f(x + t\Delta x_{nt}) - \nabla^2 f(x)\|_2 \leq Lt\|\Delta x_{nt}\|_2.$$

As a consequence, given that $\tilde{f}(t) = f(x + t\Delta x_{nt})$, it can be attained that

$$|\tilde{f}''(t) - \tilde{f}''(0)| \leq Lt\|\Delta x_{nt}\|_2{}^2.$$

Since $\lambda^2(x) \geq m\|\Delta x_{nt}\|_2{}^2$, $\tilde{f}'(0) = -\lambda^2(x)$ and $\tilde{f}''(0) = \lambda^2(x)$, by means of iterated integrations it can be proved that

$$\tilde{f}(t) \leq \tilde{f}(0) - t\lambda^2(f) + \frac{t^2}{2}\lambda^2(x) + \frac{t^3 L}{6m^{\frac{3}{2}}}\lambda^3(x).$$

Assuming that $\|\nabla f(x)\|_2 \leq \eta \leq 3\,(1 - 2\alpha)\,\frac{m^2}{L}$, under the hypothesis of strong convexity it can be attained that $\lambda(x) \leq 3\,(1 - 2\alpha)\,\frac{m^{\frac{3}{2}}}{L}$. Therefore, if $t = 1$,

$$\begin{aligned} f(x + \Delta x_{nt}) &\leq f(x) - \lambda^2(x)\left[\frac{1}{2} - \frac{L}{6m^{\frac{3}{2}}}\lambda(x)\right] \\ &\leq f(x) - \alpha\lambda^2(x) \\ &\leq f(x) + \alpha\nabla^\top f(x)\Delta x_{nt}. \end{aligned}$$

So the stopping criterion is satisfied by $t = 1$. Moreover, $\|\nabla f(x_k)\|_2 \leq \eta$ imply inequality (C.4). Indeed,

$$\begin{aligned} \|\nabla f(x + t\Delta x_{nt})\|_2 &= \|\nabla f(x + t\Delta x_{nt}) - \nabla f(x) - \nabla^2 f(x)\Delta x_{nt}\|_2 \\ &= \|\int_0^1 \left(\nabla^2 f(x + t\Delta x_{nt}) - \nabla^2 f(x)\right)\Delta x_{nt} dt\|_2 \\ &\leq \frac{L}{2}\|\Delta x_{nt}\|_2{}^2 \\ &= \frac{L}{2}\|\nabla^2 f(x)^{-1}\nabla f(x)\|_2{}^2 \\ &\leq \frac{L}{2m^2}\|\nabla f(x)\|_2{}^2. \end{aligned}$$

Appendix D

Proof of Some Useful Algebraic Facts

The proof of Lemma 4.4.3 on page 52 and 4.4.4 on page 52 are now provided, based on the ones given in [5] and [16], respectively.

Proof of Lemma 4.4.3. For each $P = P^T \in \mathbb{R}^{n\times n}$,

$$\begin{aligned}
&\begin{bmatrix} B^* \left(z^{-1}I - A^*\right)^{-1} & I \end{bmatrix} \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} (zI - A)^{-1} B \\ I \end{bmatrix} \\
&= B^* \left(z^{-1}I - A^*\right)^{-1} (A^*PA - P) (zI - A)^{-1} B \\
&\quad + B^*PA (zI - A)^{-1} B + B^* \left(z^{-1}I - A^*\right)^{-1} A^*PB + B^*PB.
\end{aligned}$$

Since

$$A^*PA - P = -\left(z^{-1}I - A^*\right) P (zI - A) - A^*P (zI - A) - \left(z^{-1}I - A^*\right) PA,$$

it follows that

$$\begin{bmatrix} B^* \left(z^{-1}I - A^*\right)^{-1} & I \end{bmatrix} \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} (zI - A)^{-1} B \\ I \end{bmatrix} = 0.$$

□

Lemma 4.4.4 provides a relation between left and right spectral factors of a given rational spectral density function $\Phi(z)$. Given the right minimal spectral factor $H(z)$, it allows to compute the correspondent minimal left factor $H_1^*(z)$, so that $\Phi(z) = H^*(z)H(z) = H_1(z)H_1^*(z)$. Here this result is recalled.

Lemma 4.4.4. *Let A be a stability matrix and $H(z) = C (zI - A)^{-1} B + D$ be a minimal realization. Let P be a solution of the Lyapunov equation*

$$P = A^*PA + C^*C. \tag{D.1}$$

Let $\begin{bmatrix} K \\ J \end{bmatrix}$ *be an orthonormal basis of the kernel of the matrix* $\begin{bmatrix} A^* P^{\frac{1}{2}} & C^* \end{bmatrix}$, *i.e.*

$$\begin{bmatrix} A^* P^{\frac{1}{2}} & C^* \end{bmatrix} \begin{bmatrix} K \\ J \end{bmatrix} = 0, \quad \begin{bmatrix} K^* & J^* \end{bmatrix} \begin{bmatrix} K \\ J \end{bmatrix} = I. \tag{D.2}$$

Let $G := P^{-\frac{1}{2}} K$ *and define* $H_1(z)$ *as*

$$H_1(z) := \left(D^* C + B^* P A\right) (zI - A)^{-1} G + B^* P G + D^* J. \tag{D.3}$$

Then, $H^*(z) H(z) = H_1(z) H_1^*(z)$.

Proof of Lemma 4.4.4. Recall that, given the spectral density function $\Phi(z)$, each left spectral factor of Φ, $W(z)$, is given by the product

$$W(z) = W_1(z) Q(z),$$

where $W_1(z)$ is the minimum phase left spectral factor corresponding to $\Phi(z)$, also known as *outer* function, and $Q(z)$ is an *inner* function, i.e. such that $QQ^* = Q^*Q = I$ on $\mathbb{T}$. The aim is to prove that $H_1^* = Q^* H$, for a proper inner function Q. Let $Q := C (zI - A)^{-1} G + J$. The first step is to prove that the latter is actually an inner function. Let us evaluate

$$\begin{aligned} Q^*Q = {} & G^* \left(z^{-1} I - A^*\right)^{-1} C^* C (zI - A)^{-1} G + G^* \left(z^{-1} I - A^*\right)^{-1} C^* J \\ & + J^* C (zI - A)^{-1} G + J^* J. \end{aligned} \tag{D.4}$$

Assume that P is the solution to the algebraic Riccati equation (4.43). It is easy to check that

$$C^* C = -\left(z^{-1} I - A^*\right) P (zI - A) + \left(z^{-1} I - A^*\right) P z + z^{-1} P (zI - A). \tag{D.5}$$

Therefore, equation (D.4) can be rewritten as

$$\begin{aligned} Q^*Q = {} & -G^* P G + G^* P z (zI - A)^{-1} G + G^* \left(z^{-1} I - A^*\right)^{-1} z^{-1} P G \\ & + G^* \left(z^{-1} I - A^*\right)^{-1} C^* J + J^* C (zI - A)^{-1} G + J^* J. \end{aligned} \tag{D.6}$$

Now recall that

$$z (zI - A)^{-1} = I + A (zI - A)^{-1} \tag{D.7}$$

and

$$\left(z^{-1} I - A^*\right)^{-1} z^{-1} = I + \left(z^{-1} I - A^*\right)^{-1} A^*, \tag{D.8}$$

therefore

$$\begin{aligned} Q^*Q = {} & \left(J^*C + G^*PA\right)(zI - A)^{-1} G + \left(\left(J^*C + G^*PA\right)(zI - A)^{-1} G\right)^* \\ & + G^*PG + J^*J. \end{aligned} \tag{D.9}$$

From (D.2), it follows that Q^*Q is an inner. It means that $H^*H = H^*QQ^*H$. The last step is to prove that $Q^*H = H_1^*$. This result is obtained by means of (D.7), (D.8) and (D.5). Indeed,

$$\begin{aligned} Q^*H &= \left(G^*\left(z^{-1}I - A^*\right)^{-1} C^* + J^*\right)\left(C\left(zI - A\right)^{-1} B + D\right) \\ &= -G^*\left(z^{-1}I - A^*\right)^{-1}\left(z^{-1}I - A^*\right) P\left(zI - A\right)(zI - A)^{-1} B \\ &\quad + G^*\left(z^{-1}I - A^*\right)^{-1}\left(z^{-1}I - A^*\right) Pz\left(zI - A\right)^{-1} B \\ &\quad + G^*\left(z^{-1}I - A^*\right)^{-1} z^{-1} P\left(zI - A\right)(zI - A)^{-1} B \\ &\quad + G^*\left(z^{-1}I - A^*\right)^{-1} C^*D + J^*C\left(zI - A\right)^{-1} B + J^*D \\ &= -G^*PB + G^*Pz\left(zI - A\right)^{-1} B + G^*\left(z^{-1}I - A^*\right)^{-1} z^{-1} PB \\ &\quad + G^*\left(z^{-1}I - A^*\right)^{-1} C^*D + J^*C\left(zI - A\right)^{-1} B + J^*D \\ &= -G^*PB + G^*P\left(I + A\left(zI - A\right)^{-1}\right) B \\ &\quad + G^*\left(I + \left(z^{-1}I - A^*\right)^{-1} A^*\right) PB \\ &\quad + G^*\left(z^{-1}I - A^*\right)^{-1} C^*D + J^*C\left(zI - A\right)^{-1} B + J^+D \\ &= G^*PB + G^*PA\left(zI - A\right)^{-1} B + G^*\left(z^{-1}I - A^*\right)^{-1} A^*PB \\ &\quad + G^*\left(z^{-1}I - A^*\right)^{-1} C^*D + J^*C\left(zI - A\right)^{-1} B + J^*D \\ &= G^*\left(z^{-1}I - A^*\right)^{-1}\left(C^*D + A^*PB\right) + \left(G^*PA + J^*C\right)\left(zI - A\right)^{-1} B \\ &\quad + G^*PB + J^*D \\ &= G^*\left(z^{-1}I - A^*\right)^{-1}\left(C^*D + A^*PB\right) + G^*PB + J^*D \\ &= {H_1}^*. \end{aligned} \tag{D.10}$$

□

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