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Moduli spaces of gauge theories in 3 dimensions

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A mia madre e mio padre.

Abstract

The objective of this thesis is to study the moduli spaces of pairs of mirror theories in theree dimensions with $\mathcal{N} = 4$. The original conjecture of $3d$ mirror symmetry was motivated by the fact that in these pairs of theories the Higgs and Coulomb branches are swapped. After a brief introduction to supersymmetry we will first focus on the Higgs branch. This will be investigated through the Hilbert series and the plethystic program. The methods used for the Higgs branch are very well known in literature, more difficult is the case of the Coulomb branch since it receives quantum corrections. We will explain how it is parametrized in term of monopole operators and having both Higgs and Coulomb branches for theories with different gauge groups we will be able to show how mirror symmetry works in the case of ADE theories. We will show in which cases these Yang-Mills vacua are equivalent to one instanton moduli spaces.

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Chapter 1

Introduction

The history of our understanding of the laws of Nature is an history of unification. The first example is probably Newton's law of universal gravitation, which states that there is just one equation describing the attraction among planets and between one planet and an ordinary object like an apple. Maxwell equations unify electromagnetism with special relativity. Quantumelectrodynamics unifies electrodynamics with quantum mechanics. Electroweak interaction is the unified description of electromagnetism and the weak interaction. And so on and so forth, till the formulation of the Standard Model (SM) which describes in an unified way all known non-gravitational interactions. Supersymmetry (and its local version, supergravity), is a candidate to complete this long journey. It is a way not just to describe in a unified way all known interactions, but in fact to describe matter and radiation all together.

The Standard Model was developed during the second half of the 20th century and it took the contribution of many physicists to be completed. The current formulation was finalized in the 70's upon experimental confirmation of the existence of quarks. Since then, the top quark was discovered in 1995, the tau neutrino in 2000, and more recently the Higgs boson in 2013. These discoveries have given further credence to the Standard Model.

The SM is an example of a quantum field theory (QFT) which treats particles as excited states of an underlying physical field. Writing a theory in the QFT framework permits to get theories which are automatically consistent with both Quantum Mechanics and Special Relativity. In this context, quantum mechanical interactions between particles are described by interaction terms between the corresponding underlying fields.

The construction of the SM was essentially guided by the symmetries that we wanted the theory to have. In fact, each field in SM is subject to a particular kind of symmetry, called gauge symmetry. The presence of a gauge symmetry implies that there are degrees of freedom in the mathematical formalism which do not correspond to changes in the physical state.

So, our model of particle phenomenology is a gauge field theory based on the gauge group $SU_C(3) \times SU(2)_L \times U(1)_Y$. The theory is commonly viewed as containing a fundamental set of particles comprising leptons, quarks, gauge bosons and the Higgs particle. The electroweak symmetry $SU_L \times U(1)_Y$ is spontaneously broken by a Higgs mechanism to the electromagnetic symmetry $U(1)_{EM}$, leaving one Higgs boson.

The Standard Model is renormalizable and mathematically self-consistent, however, despite having huge and continuous successes in providing experimental predictions, it does leave some unexplained phenomena. In particular, although the physics of special relativity is incorporated, general relativity is not, and the Standard Model fails at energies or distances where the graviton is expected to emerge. Therefore in a modern field theory context, it is seen as an effective field theory.

The typical scale of the SM, the electroweak scale, is

$$M_{ew} \sim 250\, GeV \iff L_{ew} \sim 10^{-16}\, mm.$$

The SM is very well tested up to such energies. This cannot be the end of the story though. In fact, at high enough energies, as high as the Planck scale M_{pl}, gravity becomes comparable with other forces and cannot be neglected in elementary particle interactions. At some point, we need a quantum theory of gravity.

The Higgs potential reads

$$V(H) \sim \mu^2|H|^2 + \lambda|H|^4 \text{ where } \mu^2 < 0.$$

Experimentally, the minimum of such potential is at around 174 GeV. This implies that the bare mass of the Higgs particle is roughly around 100 GeV, but we still have to take into account radiative corrections. Scalar masses are subject to quadratic divergences in perturbation theory. The UV cut-off should then be naturally around the TeV scale in order to protect the Higgs mass, and the SM should then be seen as an effective theory valid at energies inferior to the TeV scale. The question is why the Higgs boson is so much lighter than the Planck mass: one would expect that the large quantum contributions to the square of the Higgs boson mass would inevitably make the mass huge, comparable to the scale at which new physics appears, unless there is an incredible fine-tuning cancellation between the quadratic radiative corrections and the bare mass.

New physics, if any, may include many new fermionic and bosonic fields, possibly coupling to the SM Higgs. Each of these fields will give radiative contribution to the Higgs mass of the kind above, hence, no matter what new physics will show-up at high energy, the natural mass for the the Higgs field would always be of order the UV cut-off of the theory, generically around $\sim M_{pl}$. We would need a huge fine-tuning to get it stabilized at $\sim 100\, GeV$ (we now know that the physical Higgs mass is at 125 GeV). This is known as the hierarchy problem: the experimental value of the Higgs mass is unnaturally smaller than its natural theoretical value.

In principle, there is a very simple way out of this. This resides in the fact that scalar couplings provide one-loop radiative contributions which are opposite in sign with respect to fermions. Therefore, if the new physics is such that each quark and lepton of the SM were accompanied by two complex scalars having the same Higgs couplings of the quark and lepton, then all radiative contributions would automatically cancel, and the Higgs mass would be stabilized at its tree level value. Such conspiracy, however, would be quite ad hoc, and not really solving the fine-tuning problem mentioned above; rather, just rephrasing it.

A solution comes from the existence of a symmetry imposing to the theory the correct matter content (and couplings) for such cancellations to occur. This is exactly what supersymmetry is: in a supersymmetric theory there are fermions and bosons (and couplings) just in the right way to provide exact cancellation for the radiative corrections. In summary, supersymmetry is a very natural and economic way (though not the only possible one) to solve the hierarchy problem.

Known fermions and bosons cannot be partners of each other. For one thing, we do not observe any degeneracy in mass in elementary particles that we know. Hence, in a supersymmetric world, each SM particle should have a (yet not observed) supersymmetric partner.

The problem is that the world we already had direct experimental access to, is not supersymmetric. If at all realized, supersymmetry should be a spontaneously broken symmetry in the vacuum state chosen by Nature. However, in order to solve the hierarchy problem without fine-tuning this scale should be lower then (or about) 1 TeV. Including lower bounds from present day experiments, it turns out that the SUSY breaking scale should be in the following energy range

$$100\,GeV \leq \text{ SUSY breaking scale } \leq 1000\,GeV.$$

There is another reason to believe in supersymmetry and it is possibly stronger, from a phenomenological point of view, then that provided by the hierarchy problem. The three SM gauge couplings run according to the renormalization group (RG) equations.

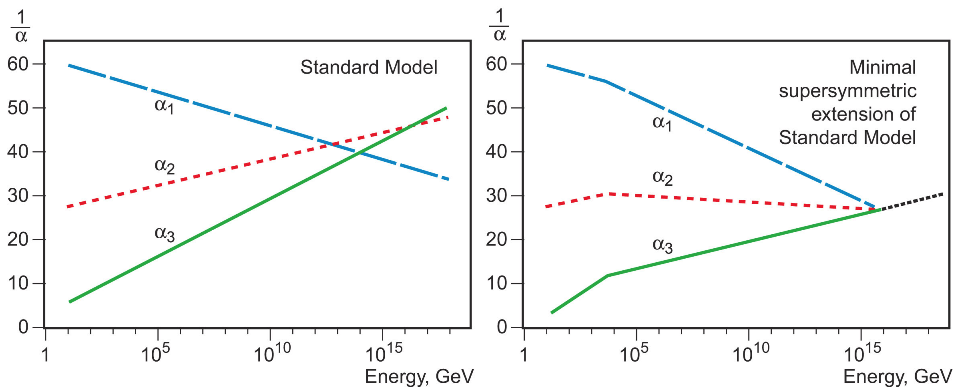


Figure 1.1: Running of the coupling constants for $SU(3)_C$ in green, $SU(2)_L$ in red and $U(1)_Y$ in blu. On the left we have the Standard Model. On the right the case of the Minimally Supersymmetric Standard Model is presented.

At the EW scale there is a hierarchy between them, but RG equations make this hierarchy change with the energy scale. In fact, supposing there are no particles other than the SM ones, at a much higher scale, $M_{GUT} \sim 10^{15}\,GeV$, the three couplings tend to meet. This naturally calls for a Grand Unified Theory (GUT), where the three interactions are unified in a single one, two possible GUT gauge groups being SU(5) and SO(10). The symmetry breaking pattern one should have in mind would then be as follows

$$SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_{em}.$$

The first problem arising with this idea is the proton decay problem: this model makes the proton not fully stable and it turns out that its expected lifetime in such GUT framework is violated by present experimental bounds. On a more theoretical side, if we do not allow for new particles besides the SM ones to be there at some intermediate scale, the three gauge couplings only approximately meet. Remarkably, making the GUT supersymmetric (SGUT) solves all of these problems. If one just allows for the minimal supersymmetric extension of the SM spectrum, known as MSSM, the three gauge couplings exactly meet, and the GUT scale is raised enough to let proton decay rate being compatible with present experimental bounds.

From all these reasons, it is evident the theoretical interest in building supersymmetric theories. Even if we adopt supersymmetry, still there are many questions that the physical community would like to be able to answer. An important issue regards Quantum Chromodynamics (QCD), which is the theory of strong interactions, the fundamental force describing the interactions between quarks and gluons which make up hadrons. QCD is a type of quantum field theory based on $SU(3)$ part of the SM symmetry group. The beta function describing the running of QCD coupling constant is negative and this produces two different behaviors in the IR and in the UV. What happens in the IR is that we have the confinement. This implies that the force between quarks increases as they are separated. Although analytically unproved, confinement is widely believed to be true because it explains the failure in finding free quarks. On the other side, in the UV we have asymptotic freedom which means that at very high-energy scales quarks and gluons interact very weakly creating a quark-gluon plasma.

The weak coupling regime is when the energy range is such that the coupling constant is smaller than one and the theory can be treated perturbatively, instead the strong coupling regime takes place when the coupling constant is bigger than one. The perturbative approach to quantum field theory is able to tell us basically everything we want to know from the theory since it tells us all the probabilities for the interactions. These probabilities are approximated as a perturbative expansion in the coupling constant. So, if we are in the weakly coupled regime, corrections became smaller and smaller in terms of powers of the parameter α (the coupling constant). The perturbative approximation is computed through Feynman diagrams.

For the strongly coupled regime the story is completely different. In this case we cannot rely on perturbative expansion and we have to develop new methods to study what happens. An interesting answer to the problem comes from considering certain pairs of theories. For a theory in the pair the way of doing computations in the strong coupling is given by translating these computations into different computations in the weakly coupled theory where we can rely on perturbative expansions.

This approach to the problem goes back to the works by Claus Montonen and David Olive in the late 1970s, the interested reader can look at [38] for the original paper. Their idea dates back to an observation that can be done yet in classical electromagnetism. An important property of sourceless Maxwell equations is their invariance under the transformation that simultaneously replaces the electric field by the magnetic field. In other words, given a pair of electric and magnetic fields that solve Maxwell's equations, it is possible to describe a new physical setup in which these electric and magnetic fields are essentially interchanged, and the new fields will again give a solution of Maxwell's equations. It is natural to ask whether there is an analog in gauge theory of the symmetry interchanging the electric and magnetic fields in Maxwell's equations. This is Montonen-Olive duality which applies to a very special type of gauge theory called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Roughly, if one of the theories has a gauge group G, then the dual theory has gauge group ${}^{L}G$ where ${}^{L}G$ denotes the Langlands dual group which is in general different from G. An important quantity in quantum field theory is complexified coupling constant. This is a complex number defined by the formula

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2},$$

where θ is the theta angle, a quantity appearing in the Lagrangian that defines the theory, and g is the coupling constant. For example, in the Yang-Mills theory that describes the electromagnetic field, this number g is simply the elementary charge e carried by a single proton. In addition to exchanging the gauge groups of the two theories, Montonen-Olive duality transforms a theory with complexified coupling coupling constant τ to a theory with complexified constant $-1/\tau$.

After this, in 1995 Nathan Seiberg introduced another duality [42] which goes after his name. Unlike Montonen-Olive duality, which relates two versions of the maximally supersymmetric gauge theory in four-dimensional spacetime, Seiberg duality relates less symmetric theories called $\mathcal{N} = 1$ supersymmetric gauge theories. The two $\mathcal{N} = 1$ theories appearing in Seiberg duality are not identical, but they give rise to the same physics at large distances in the IR. Like Montonen-Olive duality, Seiberg duality generalizes the symmetry of Maxwell's equations that interchanges electric and magnetic fields.

The objective of this thesis is to study a particular kind of duality which takes place in three dimensions and which is actually quite different from the dualities mentioned before, its name is *mirror symmetry*. In fact, mirror symmetry is not a duality between a strongly coupled and a weakly coupled theory but between two strongly coupled theories, where the same conformal fixed point in the IR can be given distinct gauge theory descriptions in the UV. Anyway, these theories play an important role in our general understanding of dualities and for this reason it is worthy studying them.

The theories involved in mirror symmetry are quite special since we have to make some simplifying assumptions. First of all we restrict to three dimensions, two spatial plus a temporal one, but this is not enough. So, in addition to the small number of dimensions we want our theories to have an high amount of supersymmetry. The more the theory is symmetric, the more its dynamic is constrained. From this point of view putting an high amount of supersymmetry is again a simplifying assumption. In this thesis we'll focus on theories with $\mathcal{N} = 4$ supersymmetry. Once again, what we would like to find in the future are dualities between $\mathcal{N} = 1$ theories since the MSSM is an $\mathcal{N} = 1$ theory, but for the moment this is out our reach. Our intention is to proceed gradually from simpler dualities to more difficult and realistic ones learning step by step how to get them. Hopefully, learning to build dual theories will let us compute the dual theory to QCD and in such a way we will be able to compute everything also in the strong-coupling regime by making calculations in the weakly coupled dual theory.

Anyway, three dimensions and $\mathcal{N} = 4$ supersymmetry is still not enough to identify mirror pairs, we'll also have to put ourselves in the fixed point of the renormalization group, in the IR. This means that the couplings of the dual theories are taken to be infinite since for such theories in three dimensions the strongly coupled regime is in the IR. Putting ourselves in the fixed point of renormalization group gives us theories with a further symmetry which is called conformal symmetry. Basically these theories are scale invariant plus they have other generators which give us the whole conformal symmetry algebra.

This is the last symmetry we require because at this point people were able to build couples of theories that they called *mirror theories*. For our purposes, we'll focus on quiver gauge theories, which are theories whose gauge groups are described by quiver diagrams.

In this case some mirror couples are known and the matching of their moduli spaces provided the original motivation for the conjecture. Duality, in this case, exchanges the Higgs and Coulomb branches. This property of mirror symmetry concerning the space of vacua is the property we are interested in. In the following we will compare the moduli spaces for theories with different gauge groups in order find mirror pairs. These mirror pairs are known in literature and basically our work will consist in explicitly check some results outlined in [16]. Furthermore, in the final part of this thesis we propose some new calculations which let us identify the geometry of the Coulomb branch of certain other theories.

Chapter 2

SUSY algebra

In this chapter we want to recall some important mathematical notions about Lie algebras. Furthermore, we want to generalize Lie algebras to super Lie algebras, which are maybe less familiar to the reader. At the end, the generic extended supersymmetric algebra is given in its general form.

A Lie algebra is a vector space endowed with also an operation of product, which we indicate with $[\cdot,\cdot]$, between two vectors in the vector space. The product is antisymmetric i.e. $[X,Y] = -[Y,X]$ and the Jacobi identity holds:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{2.1}$$

A super Lie algebra, sometimes called a graded algebra, has elements which belong to the direct sum

$$V = V_B + V_F, \tag{2.2}$$

where V_B is the bosonic vector space and V_F is the fermionic one. A generic element in this direct sum vector space is

$$X = a^k B_k + \theta^r F_r. \tag{2.3}$$

a^k is a real number while θ^r is a Grassman number. B_k and F_r are elements of the basis of the two vector spaces.

A super Lie algebra must satisfy some additional properties. There are two kinds of product, indicated in two different ways depending on the two kinds of generators involved. We have $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$. Given F_1 and F_2 generators of V_F, and B_1, B_2 generators of V_F, the following relations hold:

$$[B_1, B_2] = -[B_2, B_1], \tag{2.4}$$
$$\{F_1, F_2\} = \{F_2, F_1\}, \tag{2.5}$$
$$[B_1, F_2] = -[F_2, B_1]. \tag{2.6}$$

Furthermore, we have the following axioms:

$$\begin{aligned} \{V_F, V_F\} &\subset V_B, \\ [V_B, V_B] &\subset V_B, \\ [V_F, V_B] &\subset V_F. \end{aligned} \tag{2.7}$$

Finally, there is the super Jacobi identity. There are compact ways to write it, and essentially it is really similar to the ordinary Jacoby identity but with some changes in the signs. In fact, in this case some generators can be bosonic while others are fermionic and the signs are related to this fact. It is sufficient to write four equations to explicitly define what happens:

$$\begin{aligned}
[B_1,[B_2,B_3]]+[B_2,[B_3,B_1]]+[B_3,[B_1,B_2]]&=0,\\
[B_1,[B_2,F]]+[B_2,[F,B_1]]+[F,[B_1,B_2]]&=0,\\
[B,\{F_1,F_2\}]+\{F_1,[F_2,B]\}-\{F_2,[B,F_1]\}&=0,\\
[F_1,\{F_2,F_3\}]+[F_2,\{F_3,F_1\}]+[F_3,\{F_1,F_2\}]&=0.
\end{aligned} \tag{2.8}$$

In our case the bosonic generators are taken to be the usual Poincaré generators, P_μ for the translations and $M_{\mu\nu}$ for boosts and rotations. Now we introduce some new fermionic generators which are called supercharges and are usually written as Q^I_α. In general there can be N supercharges but we also define $\mathcal{N}=\frac{N}{d_s}$ where d_s is the dimension of the smallest irreducible spinorial representation of $SO(1,d-1)$. In general we refer to theories with $\mathcal{N}>1$ as extended supersymmetric theories.

We are now ready to present the super-Poincaré algebra which is the extension of the Poincaré algebra to incorporate the supersymmetry algebra and one could easily verify that it satisfys all the axioms of supersymmetric Lie algebras.

$$\begin{aligned}
\left[P_\mu,P_\nu\right]&=0,\\
\left[M_{\mu\nu},P_\rho\right]&=i\eta_{\rho\mu}P_\nu-i\eta_{\rho\nu}P_\mu,\\
\left[M_{\mu\nu},M_{\rho\sigma}\right]&=i\left(\eta_{\mu\rho}M_{\nu\sigma}-\eta_{\mu\sigma}M_{\nu\rho}-\eta_{\nu\rho}M_{\mu\sigma}+\eta_{\mu\sigma}M_{\nu\rho}\right),\\
[B_l,B_m]&=if^n_{lm}B_n,\\
\left[P_\mu,B_l\right]&=0,\\
\left[M_{\mu\nu},B_l\right]&=0,\\
\left[P_\mu,Q^I_\alpha\right]&=0,\\
\left[P_\mu,\overline{Q}^I_{\dot\alpha}\right]&=0,\\
\left[M_{\mu\nu},Q^I_\alpha\right]&=i(\sigma_{\mu\nu})_\alpha^{\ \beta}Q^I_\beta,\\
\left[M_{\mu\nu},\overline{Q}^{I\dot\alpha}\right]&=i(\overline{\sigma}_{\mu\nu})^{\dot\alpha}_{\dot\beta}\overline{Q}^{I\dot\beta},\\
\{Q^I_\alpha,\overline{Q}^J_{\dot\beta}\}&=2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ},\\
\{Q^I_\alpha,Q^J_\beta\}&=\{\overline{Q}^I_{\dot\alpha},\overline{Q}^J_{\dot\beta}\}=0.
\end{aligned} \tag{2.9}$$

In the previous equation we introduced also the generators for the internal symmetry (in general can be global plus gauge) B_l which are Lorentz scalars. They commute with the usual bosonic generators as expected since the full algebra is the direct product of the Poincaré algebra and the algebra G spanned by the B_l:

$$SO(1,3)\times G.$$

Furthermore, we do not consider here *central charge extensions* of SUSY algebra since in this thesis we are interested in massless representations and in that case the representations of central charges would be trivially realized.

Chapter 3

$\mathcal{N} = 1$ Supersymmetry

In this chapter we want to introduce two kind of fields which are invariant under $\mathcal{N} = 1$ supersymmetry transformations, the vector and chiral superfields. These are the building blocks for $\mathcal{N} = 1$ supersymmetric theories, if one is not interested in describing gravitational interactions.

Chiral superfields describe matter and we will construct the most general supersymmetric action for them. We will also introduce super Yang-Mills theories which are nothing but the supersymmetric version of Yang-Mills. We will use vector superfields to describe pure Yang-Mills theories and we model radiation in this way. Then, we will couple the two sectors with the final goal of deriving the most general $\mathcal{N} = 1$ supersymmetric action describing the interaction of radiation with matter.

Finally, we focus on their moduli space which is the set of vacuum states of the theory and we will also make some examples.

3.1 Massless supermultiplets

We are interested here in massless supermultiplets, since, in all the theories we deal with, we study massless particles which gain masses via the Higgs mechanism. For $\mathcal{N} = 1$ the R-symmetry group is just U(1) so we suppress the indices I, J for the fermionic generators.

Given a massless particle, we boost to a frame in which $P^\mu = (E, 0, 0, E)$. In this case SUSY algebra gives:

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2E\sigma^0_{\alpha\dot\beta} + 2E\sigma^3_{\alpha\dot\beta} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

and we have also

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0. \tag{3.2}$$

Usually, one defines the operators $a_\alpha \equiv \frac{Q_\alpha}{2\sqrt{E}}$ and $a^\dagger_\alpha \equiv \frac{\overline{Q}_{\dot\alpha}}{2\sqrt{E}}$ which satisfy the fermionic harmonic oscillator algebra

$$\begin{aligned} &\{a_1, a_1^\dagger\} = 1,\ \{a_1, a_2^\dagger\} = 0,\ \{a_2, a_1^\dagger\} = 0,\ \{a_2, a_2^\dagger\} = 0, \\ &\{a_\alpha, a_\beta\} = 0,\ \{a^\dagger_\alpha, a^\dagger_\beta\} = 0. \end{aligned} \tag{3.3}$$

These relations correspond to the 1-d clifford algebra $\{a_1, a_1^\dagger\} = 1$. Now, we can build the supermultiplets by using the operators defined above. Let us start with a state $|E, \lambda\rangle$ called Clifford Vacuum (this is different from the vacuum of the theory, which is the state of minimal energy). This vacuum $|E, \lambda\rangle$ is the lowest weight state of an irreducible SUSY representation, defined to be the state of the supermultiplet which is annihilated by a_α for $\alpha = 1, 2$. From $|E, \lambda\rangle$ we can build another state, $a_\alpha^\dagger |E, \lambda\rangle$. While $a_1^\dagger |E, \lambda\rangle \equiv |1\rangle$ with $\langle 1|1\rangle$ is a good state, $a_2^\dagger |E, \lambda\rangle \equiv |2\rangle$ is not. In fact $\langle 2|2\rangle = 0$ and we discard it. From 3.3 we have the anticommutativity of $a_\alpha^\dagger$ and this tells us that we cannot act more than once on $|1\rangle$ to get other states. Anyway, if we act with a_1 we get again the clifford vacuum as a consequence of 3.3. Now, let us write $|1\rangle \equiv |E, \lambda'\rangle$.

$$[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \Rightarrow [J_3, \overline{Q}_{\dot{1}}] = \frac{1}{2}\overline{Q}_{\dot{1}}, \tag{3.4}$$

and this tells us that acting with $a_1^\dagger$ on $|E, \lambda\rangle$ we get $|E, \lambda'\rangle$ with $\lambda' = \lambda + \frac{1}{2}$, the helicity of the state has been increased by $\frac{1}{2}$.

In the end, all supermultiplets of $\mathcal{N} = 1$ SUSY in $d = 4$ are given by a couple of particles $\{|E, \lambda\rangle, \left|E, \lambda + \frac{1}{2}\right\rangle\}$. In particular we are interested in two of these representations, the chiral multiplet and the vector multiplet.

- The degrees of freedom the scalar multiplet consist in a scalar and a spinor and are usually written like:

  $$\left(-\frac{1}{2}, 0\right),$$

  where each entry is the helicity of the two states and as expected they differ by $\frac{1}{2}$. For our field theories in $4d$ we also require CPT invariance, so we have to add to the previous irrep its conjugate in a direct sum. This gives

  $$\left(-\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).$$

  The Chiral supermultiplet is where the matter sits in a $\mathcal{N} = 1$ theory. Calling the whole supermultiplet Φ, we have found that it contains a scalar called ϕ and a spinor denoted ψ. We express it by writing

  $$\Phi = (\phi, \psi).$$

- A vector supermultiplet is composed of a vector and a spinor. If we also add its CPT conjugate, like we did before, we get

  $$\left(-1, -\frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right).$$

  We have a vector field field V_μ and a spinor field λ. Calling the whole supermultiplet V we write

  $$V = (V_\mu, \lambda).$$

3.2 Super Yang-Mills

In this part we want to introduce the reader to the construction of $\mathcal{N} = 1$ supersymmetric gauge theories. In the introduction to the subject, we'll be rather sketchy but we want to introduce all the fundamental ingredients which will be necessary throughout this thesis. Some good references for this part are [9], [46], [2].

3.2.1 $\mathcal{N} = 1$ Superspace formalism

First of all, we introduce a formalism that permits to write SUSY invariant lagrangians.

Let us introduce four grassman variables θ_α, $\overline{\theta}_{\dot{\alpha}}$ and define new variables $z^m = (x^\mu, \theta, \overline{\theta})$ which are the coordinates in Superspace. A superfield will be a generic function defined on superspace. Now, we notice that SUSY transformations can be written in terms of "translations in superspace". To understand what this means, let us consider an ordinary translation of a scalar field $\phi(x)$. We know that there exist an operator $\mathcal{P}_\mu$ such that $\phi(x+a) = e^{-ia^\mu \mathcal{P}_\mu}\phi(x)e^{ia^\mu \mathcal{P}_\mu}$. When a is small we can expand both sides. On one hand:

$$\phi(x+a) = (1 - ia^\mu \mathcal{P}_\mu + \cdots)\phi(x)(1 + ia^\mu \mathcal{P}_\mu + \cdots) = \phi(x) - ia^\mu [\mathcal{P}_\mu, \phi(x)] + \cdots .$$

On the other

$$\phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x) + \cdots .$$

Thus

$$[\mathcal{P}_\mu, \phi(x)] = i\partial_\mu \phi(x) \equiv P_\mu \phi(x).$$

Hence, we have two objects playing the role of momentum: $\mathcal{P}_\mu$ which is the generator of translations and $P_\mu \equiv i\partial_\mu$ which gives the representation of $\mathcal{P}_\mu$ in field space. These two objects are often written using the same symbol.

Supersymmetry is the generalization of this to superspace. In analogy with usual translations, we now define translations in superspace by a quantity $(a, \xi, \overline{\xi})$ as

$$\begin{aligned} &\theta \to \theta + \xi, \\ &\overline{\theta} \to \overline{\theta} + \overline{\xi}, \\ &x_\mu \to x_\mu + i\theta\sigma_\mu\overline{\xi} - i\xi\sigma_\mu\overline{\theta}, \end{aligned} \tag{3.5}$$

where one should notice the crucial addition of the fermionic bilinears to the shift of x_μ. They are of course needed so that the composition of two fermionic translations yields a translation in x_μ. The operators generating such transformation on the superfields are $\mathcal{P}$, $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, so, given a scalar super field Y, we have:

$$Y(x_\mu + a_\mu + i\theta\sigma_\mu\overline{\xi} - i\xi\sigma_\mu\overline{\theta}, \theta + \xi, \overline{\theta} + \overline{\xi}) = e^{-ia\mathcal{P} + \xi\mathcal{Q} - \overline{\xi}\overline{\mathcal{Q}}} Y(x_\mu, \theta, \overline{\theta}) e^{ia\mathcal{P} - \xi\mathcal{Q} + \overline{\xi}\overline{\mathcal{Q}}}. \tag{3.6}$$

Not surprisingly, the operators $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are also represented by differential operators acting on superfields, as one can check:

$$[\mathcal{Q}_\alpha, Y] = Q_\alpha Y, \;\; [\overline{\mathcal{Q}}_{\dot{\alpha}}, Y] = \overline{Q}_{\dot{\alpha}} Y,$$

where on the left hand side we have a commutator or anticommutator depending on the spin of Y. The distinction we made, between $\mathcal{Q}$ acting as operators on superfield and Q

acting in the field representation, will be particularly useful when will come to talk about the chiral ring.

Having seen that SUSY is a translation in superspace, any integral over superspace of a superfield Y like

$$\int d^4x d^2\theta d^2\overline{\theta} Y, \tag{3.7}$$

will be manifestly supersymmetric because the integration measure is translational invariant by construction and Y by definition.

In the case of translations in ordinary space we were done at this point. In superspace we still have to perform an extra step because the generic superfield Y above forms a reducible representation of SUSY algebra, that is, it is possible to restrict the form of Y to contain only a subset of the original fields that are still mapped into each other by SUSY.

3.2.2 Chiral superfields

A chiral superfield can be constructed by noticing that there exist two differential operators:

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu, \\ \overline{D}_{\dot{\alpha}} &= -\overline{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu, \end{aligned} \tag{3.8}$$

which anticommute with all the supercharges. This means that if Y is a superfield, so are $D_\alpha Y$ and $\overline{D}_{\dot{\alpha}} Y$ and it is consistent with SUSY to set one of them to zero. Setting both would result in requiring $Y = const$, so we call chiral superfield a superfield Φ obeying $\overline{D}_{\dot{\alpha}} Y = 0$ (analogously, $D_\alpha Y = 0$ defines an antichiral superfield). The solution to $\overline{D}_{\dot{\alpha}}\Phi = 0$ can be obtained by observing that the (complex) combination $y_\mu = x_\mu + i\theta\sigma_\mu\overline{\theta}$ obeys $\overline{D}_{\dot{\alpha}} y_\mu = 0$ and so does trivially θ_α. Thus

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),$$

is the general expression for a chiral superfield.

A SUSY invariant action for the chiral superfield Φ is:

$$\int d^4x d^2\theta d^2\overline{\theta}\overline{\Phi}\Phi. \tag{3.10}$$

By expanding Φ and $\overline{\Phi}$ and integrating over θ and $\overline{\theta}$ we obtain precisely:

$$\int d^4x d^2\theta d^2\overline{\theta}\overline{\Phi}\Phi = \cdots = \int d^4x \left(-\partial_\mu\overline{\phi}\partial^\mu\phi - i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + \overline{F}F\right). \tag{3.11}$$

From this we notice that a chiral superfield has almost the same content of a chiral supermultiplet. It contains a scalar field ϕ and a spinor ψ, but it also has another scalar field F. Imposing the equation of motion for the theory given by the above lagrangian we would see that actually F is not a propagating field and the content of our theory on-shell becomes just the content of the chiral supermultiplet.

3.2.3 Vector superfields

To introduce gauge interactions we need a second type of superfield, the vector superfield V. In our search for irreducible representations of SUSY we impose

$$V = V^{\dagger}. \tag{3.12}$$

We can expand in components V, displaying the Lie algebra index, in the following way:

$$\begin{aligned} V^a(x,\theta,\overline{\theta}) = C^a(x) + i(\theta\chi^a(x) - \overline{\theta}\overline{\chi^a}) - \theta\sigma^\mu\overline{\theta}V^a_\mu + \frac{i}{2}\theta\theta\,(M^a(x) + iN^a(x)) + \\ - \frac{i}{2}\overline{\theta\theta}\,(M^a(x) - iN^a(x)) + i\theta\theta\overline{\theta}\left(\overline{\lambda^a}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi^a(x)\right) + \\ - i\overline{\theta\theta}\theta\left(\lambda^a(x) + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi^a}(x)\right) + \frac{1}{2}\left(\mathcal{D}^a(x) - \frac{1}{2}\Box C^a(x)\right), \end{aligned} \tag{3.13}$$

where C^a, V^a_μ and $\mathcal{D}^a$ are real and the shifts in λ^a and $\mathcal{D}^a$ by derivatives of χ^a and C^a are for later convenience. If T^a are the generators of the Lie algebra ($[T^a, T^b] = if^{abc}T^c$), one can write a matrix valued object $V = V^a T^a$ obeying $V = V^\dagger$ because V^a are real and T^a Hermitian. We choose the normalization $Tr(T^a T^b) = \delta^{ab}$ for our generators. One can perform a supergauge transformation which brings this superfield in the so called Wess-Zumino gauge, in which it takes the easier form

$$V_{WZ}(x,\theta,\overline{\theta}) = -\theta\sigma^\mu\overline{\theta}V^a_\mu(x) + i\theta\theta\overline{\theta}\overline{\lambda^a}(x) - i\overline{\theta\theta}\theta\lambda^a(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}\mathcal{D}^a(x). \tag{3.14}$$

In the rest of this thesis we will always assume that vector superfields are in the Wess-Zumino gauge.

We give now the expression for the lagrangian kinetic term of a vector superfield. First, it's necessary to introduce the chiral superfield strength. It turns out that the right expression for it is

$$W_\alpha = \overline{D}^2\left(e^{-V}D_\alpha e^V\right). \tag{3.15}$$

The antichiral superfield $\overline{W}_{\dot{\alpha}}$ has a similar definition, obtained from $\overline{W}_{\dot{\alpha}} = (W_\alpha)^*$. The lagrangian for the free fields is:

$$\begin{aligned} S_{SYM} &= \frac{1}{8g^2}\int d^4x d^2\theta\,(W^{\alpha\,a}W^a_\alpha) = \\ &= \int d^4x\left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - i\overline{\lambda}^a\overline{\sigma}^\mu\nabla_\mu\lambda^a + \frac{1}{2}\mathcal{D}^a\mathcal{D}^a + \frac{-i}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu\,a}\right), \end{aligned} \tag{3.16}$$

where, in view of coupling the pure SYM Lagrangian with matter, it is convenient to introduce the coupling constant g explicitly, like we have done. In this case also the fields must be rescaled but we prefer to omit these technicalities here and just give the final expressions, a good reference for this is [9].

From this expression for the vector superfield it is clear that it encloses almost the same degrees of freedom of a vector supermultiplet: here we have a vector field V_μ, a gaugino field λ, its complex conjugate field $\overline{\lambda}$ and an auxiliary field $\mathcal{D}$. One can prove that on-shell, $\mathcal{D}$ does not propagate, and is fixed by the other fields of the theory. Furthermore, on shell, the dynamics of $\overline{\lambda}$ is fixed by the dynamics of λ i.e. they are no longer independent fields.

This leaves us with only a vector field and a weyl spinor on-shell, we conclude that a vector superfield carries the same d.o.f. of the vector supermultiplet.
The last piece in the summation 3.16 gives the Bianchi identity upon variation, in fact $\tilde{F}$ is the dual field strength of F. Usually we are not interested in this piece which turns out to be imaginary, so we sum S_{SYM} with its complex conjugate in order to cancel the imaginary part and we are left with an ordinary real action.

3.2.4 Supersymmetric gauge theories for $\mathcal{N} = 1$

Consider now the matter described by a chiral superfield Φ transforming in some representation (reducible or irreducible) of the Lie algebra. The usual gauge transformation is

$$\Phi \to \Phi' = e^{-i\Lambda}\Phi,$$

where $\Lambda = \Lambda^a T^a$ must be a chiral superfield because otherwise Φ' would no longer be chiral. If we assume that a vector superfield transforms like

$$e^V \to e^{V'} = e^{-i\overline{\Lambda}} e^V e^{i\Lambda},$$

we can build an interaction term, between the chiral and the vector superfield, which is also invariant under the gauge transformations just described and it reads:

$$\int d^4x d^2\theta d^2\overline{\theta}\ \overline{\Phi} e^V \Phi. \tag{3.17}$$

We can still add another ingredient: the superpotential $\mathcal{W}$ for the matter fields. The superpotential $\mathcal{W}(\Phi)$ is just an holomorphic function of its argument, the chiral superfield Φ, which is invariant under the gauge group G. So, the proposed term for describing interactions in a theory of a chiral superfield is

$$\mathcal{L}_{int} = \int d^2\theta \mathcal{W}(\Phi) + \int d^2\overline{\theta}\overline{\mathcal{W}}(\overline{\Phi}), \tag{3.18}$$

where the hermitian conjugate has been added to make the whole thing real.
The holomorphicity of the superpotential ensures it to be a chiral superfield and the integral 3.18 to be supersymmetric invariant. $\mathcal{W}$ should not contain covariant derivatives since $D_\alpha \Phi$ is not a chiral superfield, given that D_α and $\overline{D}_{\dot{\alpha}}$ do not commute. Therefore, the superpotential should have an expression like

$$\mathcal{W}(\Phi) = \sum_{n=1}^{\infty} a_n \Phi^n\,. \tag{3.19}$$

Actually, if renormalizability is an issue, the superpotential should be at most cubic. Indeed, the F-term of $\mathcal{W}$ has dimension $[\mathcal{W}] + 1$. Since our chiral superfield Φ has dimension one, it follows that to avoid non-renormalizable operators the highest power in the expansion 3.19 should be $n = 3$.

We introduce again the coupling constant g that just gives a few rescalings for the fields, but finally the Super Yang-Mills action coupled to matter is written:

$$\begin{aligned} S =& \frac{1}{16g^2}\int d^4x d^2\theta Tr\left(W^\alpha W_\alpha\right) + \frac{1}{16g^2}\int d^4x d^2\overline{\theta} Tr\left(\overline{W}^{\dot{\alpha}}\overline{W}_{\dot{\alpha}}\right) + \\ &+ \int d^4x d^2\theta d^2\overline{\theta}\ \overline{\Phi} e^{2gV}\Phi + \int d^4x d^2\theta \mathcal{W}(\Phi) + \int d^4x d^2\overline{\theta}\overline{\mathcal{W}}(\overline{\Phi}). \end{aligned} \tag{3.20}$$

3.3 Classical moduli space of vacua for $\mathcal{N} = 1$ theories

The moduli space of vacua of a generic field theory is defined to be the set $\mathcal{M}$ of all inequivalent states of minimal energy. Such states are called vacua.

Since the kinetic terms in the Hamiltonian are quadratic in the derivatives of the fields, in a vacuum configuration, all fields have to be constant over space-time. Furthermore, it can be shown that Lorentz invariance forces all fields, except for scalar ones, to be not only constant, but to vanish everywhere.

Now, let us focus on the classical supersymmetric theory that we built in the previous sections. We try to find the scalar field configurations which minimize the energy of the system. The third piece in (3.20) can be expanded in terms of its component fields obtaining

$$-\overline{\nabla_\mu \phi}\nabla^\mu \phi - i\overline{\psi}\overline{\sigma}^\mu \nabla_\mu \psi + \overline{F}F + i\sqrt{2}g(\overline{\phi}\lambda\psi - \overline{\psi}\overline{\lambda}\phi) + g\overline{\phi}\mathcal{D}\phi, \tag{3.21}$$

where the indices are contracted in the only possible way and $\nabla_\mu = \partial_\mu - igV^a_\mu T^a_R$. The contribution from the super potential reads

$$-\frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial\phi_i \partial\phi_j}\psi_i\psi_j + \frac{\partial\mathcal{W}}{\partial\phi_i}F_i + c.c.\,. \tag{3.22}$$

We can collect all terms contributing to the scalar potential

$$\mathcal{V} = -\frac{1}{2}\mathcal{D}^{a2} - g\overline{\phi}T^a\phi\mathcal{D}^a - \overline{F}^i F_i - \frac{\partial\mathcal{W}}{\partial\phi_i}F_i - \frac{\partial\overline{\mathcal{W}}}{\partial\overline{\phi}^i}\overline{F}^i. \tag{3.23}$$

The auxiliary fields $\mathcal{D}$ and F appear only in these terms and we can solve for them as

$$\mathcal{D}^a = -g\overline{\phi}T^a\phi,\ F_i = -\frac{\partial\overline{\mathcal{W}}}{\partial\overline{\phi}^i},\ \overline{F}^i = -\frac{\partial\mathcal{W}}{\partial\phi_i}\,. \tag{3.24}$$

Substituting the solutions in the 3.23 gives

$$\begin{aligned}\mathcal{V} &= \frac{g^2}{2}\left(\overline{\phi}T^a\phi\right)^2 + \frac{\partial\overline{\mathcal{W}}}{\partial\overline{\phi}^i}\frac{\partial\mathcal{W}}{\partial\phi_i} \\ &= \frac{1}{2}\mathcal{D}^{a2} + \overline{F}^i F_i\,,\end{aligned} \tag{3.25}$$

where the last line is evaluated at the solution.

The potential $\mathcal{V}$ is obviously nonnegative, and the classical SUSY vacua are described by $\mathcal{V} = 0$, i.e. by the simultaneous vanishing of the $\mathcal{D}$ and F terms, referred to as D-flatness and F -flatness conditions:

$$\begin{aligned}\mathcal{D}^a &= -g\overline{\phi}T^a\phi = 0, \\ F_i &= \frac{\partial\mathcal{W}}{\partial\phi_i} = 0.\end{aligned} \tag{3.26}$$

The space of solutions to the D-flatness and F-flatness conditions is known as the classical moduli space of the theory.

To make explicit what we said about the classical moduli space, in what follows we'll consider an example. Before we do that, however, we would like to rephrase our definition of moduli space, presenting an alternative (but equivalent) way to describe it. Suppose we are considering a theory without superpotential. For such a theory the space of D-flat directions coincides with the moduli space. The space of D-flat directions is defined as the set of scalar field VEVs satisfying the D-flat conditions

$$\mathcal{M}_{cl} = \{\langle\phi_i\rangle \mid \mathcal{D}^a = 0\}/gauge\ transformations. \tag{3.27}$$

Generically it is not at all easy to solve the above constraints and find a simple parametrization of $\mathcal{M}_{cl}$. An equivalent, though less transparent, definition of the space of D-flat directions can help in this respect. It turns out that the same space can be defined as the space spanned by all (single trace) gauge invariant operator VEVs made out of scalar fields, modulo classical relations between them

$$M_{cl} = \{\langle Gauge\ invariant\ operators\rangle \equiv X_r(\phi_i)\}/classical\ relations. \tag{3.28}$$

The latter parametrization is very convenient since, up to classical relations, the construction of the moduli space is unconstrained. In other words, the gauge invariant operators provide a direct parametrization of the space of scalar fields VEVs satisfying the D-flatness conditions in 3.26. So, the moduli space can be parametrized by gauge invariants, possibly subject to algebraic conditions among them. More precisely, the gauge invariants have the structure of a ring, called the chiral ring, with a number of invariants being the generators of the ring. This number (modulo the possible relations among the generators) defines the dimension of the moduli space. Notice that if a superpotential is present, this is not the end of the story: F-equations will put extra constraints on the $X_r(\phi_i)$'s and may lift part of (or even all) the moduli space of supersymmetric vacua. In section 3.6 we will consider a concrete model with no superpotential term.

The second parametrization of the moduli spaces, obtained building the GIOs (Gauge Invariant Operators), is the one we'll adopt in section 3.6. This procedure can also be generalized taking into account non trivial F-constraints. In appendix A the reader can find a brief summary of how the moduli space is interpreted in algebraic geometry; the general procedure to extract the vacuum geometry in presence of F-terms is also pointed out.

3.4 Primary decomposition and Hilbert series

In appendix A the reader can find a more detailed description of the sense in which we say that the moduli spaces of our supersymmetric theories are algebraic varieties. Here we just say that owing to the polynomial nature of D-constraints and F-constraints the moduli space can be thought of as the set of points on which certain polynomials simultaneously vanish. This is exactly the description of an algebraic or affine variety.

Once we accept that the vacua can be described in the context of algebraic geometry, we have many geometric tools at our disposal for analysing their structure. Two of the most fundamental concepts are the following.

3.4.1 Extracting Irreducible Pieces

The moduli space may not be a single piece, but rather, may comprise various components, which are said to be the irreducible pieces. An irreducible piece is just a piece that cannot be expressed as union of other algebraic varieties, we talk more diffusely about it in

Appendix A. This is a well recognised feature in supersymmetric gauge theories. The different components are typically called branches of the moduli space, such as Coulomb or Higgs branches. It is an important task to identify the different components since the massless spectrum on each component has its own unique features.

We are thus naturally led to look for a process to extract the various irreducible components of the vacuum space. Such an algorithm exists and, in the mathematics literature, is called primary decomposition of the ideal corresponding to the moduli space. We explain in appendix A what a prime ideal is. The important fact to be understood here is just that there is a bijection between a certain class of ideals, the prime ideals, and the irreducible pieces of the algebraic variety. Identifying these ideals let us understand which are the pieces composing our variety.

Algorithms to perform primary decomposition have been extensively studied in computational algebraic geometry (the interested reader can look at [30], [35], [24]) and from them we can computationally extract all the geometrical pieces composing our vacuum.

3.4.2 The Hilbert Series

The Hilbert series is a key to the problem of counting GIOs in a gauge theory. Mathematically, it is also an important quantity that characterises an algebraic variety. Although it is not a topological invariant as it depends on the embedding under consideration, it nevertheless encodes many important properties of the variety once the embedding is known.

We recall that for a variety $\mathcal{M}$ in $\mathbb{C}[x_1, \dots, x_k]$ (see Appendix A), the Hilbert series is the generating function for the dimension of the graded pieces:

$$HS(t; \mathcal{M}) = \sum_{i=-\infty}^{+\infty} (dim_{\mathbb{C}} \mathcal{M}_i)\, t^i \,, \tag{3.29}$$

where $\mathcal{M}_i$, the i-th graded piece of $\mathcal{M}$ can be thought of as the number of independent degree i (Laurent) polynomials on the variety $\mathcal{M}$. A useful property of $HS(t)$ is that it is a rational function in t and can be written in two ways:

$$HS(t; \mathcal{M}) = \begin{cases} \frac{Q(t)}{(1-t)^k} \ Hilbert\ series\ of\ the\ first\ kind; \\ \frac{P(t)}{(1-t)^{dim \mathcal{M}}} \ Hilbert\ series\ of\ the\ second\ kind. \end{cases} \tag{3.30}$$

Importantly, both $P(t)$ and $Q(t)$ are polynomials with integer coefficients. The powers of the denominators are such that the leading pole captures the dimension of the embedding space and the manifold, respectively.

3.5 Quivers for $\mathcal{N} = 1$ theories

Quiver gauge theories are a special kind of super Yang-Mills theories which can be described by oriented graphs called quivers. What is incorporated in the graph is the gauge group and matter content of the theory.

Depending on the amount of supersymmetry, the quivers are built in different ways. The importance of quiver gauge theories was outlined in the paper [19], where it was pointed out that quiver gauge theories arise on the worldvolume of Type II Dp-branes in a background of D(p+4)-branes probing a certain singularity of the space.

Let us start saying what a quiver is. Quivers are a particular kind of directed graphs in which arrows starting end ending on the same node are also allowed. Depending on the amount of SUSY in the theory, we can have different quivers. So we have quivers for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SUSY for example. Actually, $\mathcal{N} = 4$ theories in $d = 3$ will have the same kind of quivers we have for $\mathcal{N} = 2$ and $d = 4$. Basically, quiver diagrams consist of nodes, to which we assign vector multiplets transforming in the adjoint of the gauge group, and links, to which we assign hypermultiplets.

First of all, let us consider here quivers for $\mathcal{N} = 1$. We also introduce a different kind of node, which pictorially we denote with a square. Arrows between any kind of nodes are allowed. The purpose of the formalism lies in the possibility of reading off the gauge group and the matter content of a super Yang-Mills theory from the graph itself. The rules are simple and can be summarized as follows:

- each of the n round nodes on the quiver diagram corresponds to a factor G_i of the gauge group $G = G_1 \times \cdots \times G_n$;
- square nodes are associated to the global symmetries of the theory, which are the flavours, and can be factorized as in the previous case;
- each of the arrows corresponds to a $\mathcal{N} = 1$ chiral superfield transforming in the fundamental representation of the target group and in the antifundamental representation of the source;
- a number contained in the node is a shorthand notation for the unitary group, which means that m in a round node is an $U(m)$ gauge group factor. The same holds for a global symmetry factor.

These rules do not fix the SYM lagrangian completely. What still needs to be specified is the superpotential, since this information is not present in the quiver for $\mathcal{N} = 1$ theories. Quivers can be arbitrary complicated but they give an immediate, pictorial representation of the theory of our interest. An example could be the following.

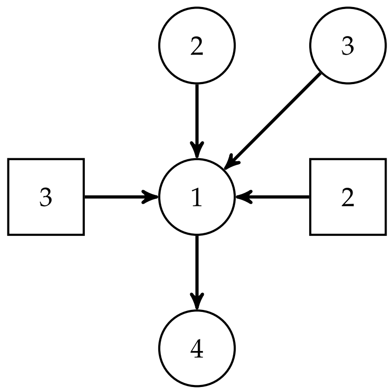
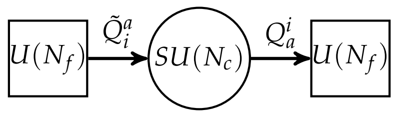


Figure 3.1: A generic quiver diagram. The gauge group is $U(4) \times U(3) \times U(2) \times U(1)$. There is also a global symmetry $U(3) \times U(2)$ which rotates between the flavours of the matter content of the theory.

The next example describes a very much studied theory: the supersymmetric QCD (SQCD).



$$\mathcal{W} = 0$$

Figure 3.2: Quiver diagram for SQCD. Q and $\tilde{Q}$ correspond to quarks and antiquarks. Gauge group and global symmetries are generalized to the case of N_c colours and N_f flavours.

This theory has quarks Q_a^i and antiquarks $\tilde{Q}_i^a$, with flavour indices $i = 1, \ldots, N_f$ and colour indices $a = 1, \ldots, N_c$. Thus, there is a total of $2N_cN_f$ chiral degrees of freedom from the quarks and antiquarks. Denoting the fundamental representation with $\square$ and with $\mathbf{1}$ the trivial representation of the group, the transformations rules are summed up in the following table.

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	U(1)$_Q$	U(1)$_{\tilde{Q}}$
Q_a^i	$\overline{\square}$	$\mathbf{1}$	$\square$	1	0
$\tilde{Q}_i^a$	$\square$	$\overline{\square}$	$\mathbf{1}$	0	-1

Table 3.1: Transformation rules for Q and $\tilde{Q}$. The global symmetries for Q and $\tilde{Q}$ are $U(N_f)_R$ and $U(N_f)_L$ but they can be factorized in $SU(N_f)_R$ and $SU(N_f)_L$ times the two $U(1)$ baryonic global symmetries $U(1)_Q$ and U(1)$_{\tilde{Q}}$.

The convention for gauge and flavour indices that we have adopted is that lower indices are for an object transforming in the fundamental and upper indices for an object transforming in the anti-fundamental.

The lagrangian corresponding to this quiver is written in superspace formalism like

$$\begin{aligned} S =& \frac{1}{8g^2}\int d^4xd^2\theta Tr\left(W^\alpha W_\alpha\right) + \frac{1}{8g^2}\int d^4xd^2\overline{\theta} Tr\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right) + \\ &+ \int d^4xd^2\theta d^2\overline{\theta}\,\overline{Q}e^{2gV}Q + \int d^4xd^2\theta d^2\overline{\theta}\,\overline{\tilde{Q}}e^{2gV}\tilde{Q}. \end{aligned} \tag{3.31}$$

3.6 Moduli space of SQCD

In the previous section we showed the kind of lagrangian that we have for SQCD. Now we want to study the classical moduli space for such a theory.

In the following, we will adopt an highest weight notation to label representations. We will use this notation throughout the thesis, so, we recall here some of the basic irreps we are going to use for SQCD, in order to introduce the notation. Dynkin labels for $SU(n)$ have $n-1$ entries $[n_1, n_2, \dots, n_{n-1}]$, the fundamental representation for such group reads $[1, 0, \dots, 0]$ while the antifundamental is $[0, \dots, 0, 1]$ in this convention. The adjoint representation is $[1, 0, \dots, 0, 1]$. For a product group $SU(n) \times SU(n)$ we use the notation $[\dots ; \dots]$ where the n-tuple to the left of ";" is the representation of the left $SU(n)$ and likewise for the $SU(n)$ on the right.

To discuss this theory is useful to divide our dissertation into cases.

3.6.1 The case of $N_f < N_c$

Using the two $SU(N_f)$ flavours symmetries and the (global part of the) gauge symmetry $SU(N_c)$, one can show that on the moduli space given by the the matrices Q and $\tilde{Q}$ can be put, at most, in the following form

$$Q = \begin{pmatrix} v_1 & 0 & \cdots & 0 & \cdots \\ 0 & v_2 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & v_{N_f} & \cdots \end{pmatrix}, \tag{3.32}$$

and similarly for $\tilde{Q}^T$. This means that at a generic point of the moduli space the gauge group is partially broken to $SU(N_c - N_f)$. Thus there are

$$(N_c^2 - 1) - ((N_c - N_f)^2 - 1) = 2N_cN_f - N_f^2, \tag{3.33}$$

broken generators. The total number of degrees of freedom of the system is, of course, unaffected by this spontaneous symmetry breaking and the massive gauge bosons each eat one degree of freedom from the chiral matter via the Higgs effect (see Section 3.7). Therefore, of the original $2N_cN_f$ chiral supermultiplets, only N_f^2 singlets are left massless. Hence, the dimension of the moduli space of vacua is

$$dim\left(\mathcal{M}_{N_f<N_c}\right) = N_f^2. \tag{3.34}$$

Following the way of parametrizing the moduli space suggested in section 3.3, we can describe the remaining N_f^2 light degrees of freedom in a gauge invariant way by an

$N_f \times N_f$ matrix field given by a single trace in the gauge group indices obtaining a meson:

$$M^i_j = Q^i_a \tilde{Q}^a_j \qquad (mesons). \tag{3.35}$$

The color indices are summed since we took the trace over the gauge group, so the M^i_j are gauge invariant. 3.35 constitute the only GIOs in this case, we have no baryons. Since Q and $\tilde{Q}$ transform respectively in $[1,0,\dots;0,\dots,0]$ and $[0,\dots,0;0,\dots,1]$ of the $SU(N_f)_L \times SU(N_f)_R$ global symmetry, it follows that M transforms in the bifundamental representation $[1,0,\dots;0,\dots,1]$ of the $SU(N_f)_L \times SU(N_f)_R$ global symmetry. For $N_f < N_c$ theory, there are no relations (constraints) between mesons. Phrasing this geometrically, and noting the dimension from 3.34 we have that the moduli space is freely generated: there are no relations among the generators. The space $\mathcal{M}_{N_f<N_c}$ is, in fact, nothing but $\mathbb{C}^{N_f^2}$.

GIOs composed of k quarks and k antiquarks must be of the form: $M^{i_1}_{j_1} \cdots M^{i_k}_{j_k}$. Because of the symmetry under the interchange of any two M's, this product transforms in the representation $Sym^k[1,0,\dots;0,\dots,1]$ of the $SU(N_f)_L \times SU(N_f)_R$ global symmetry. The total dimension of these representations is

$$\binom{N_f^2+k-1}{k}. \tag{3.36}$$

From what we said we are now able to give the generating function or unrefined Hilbert series:

$$HS^{N_f<N_c}(t) = \sum_{k=0}^{\infty} \binom{N_f^2+k-1}{k} t^{2k} = \frac{1}{(1-t^2)^{N_f^2}}. \tag{3.37}$$

We note that this expression does not depend on the colors N_c and it is the Hilbert series for $\mathbb{C}^{N_f^2}$ with weight 2 for each meson.

3.6.2 The case of $N_f \geq N_c$

In this case, at a generic point in the moduli space, the $SU(N_c)$ gauge symmetry is broken completely,

$$Q = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & v_{N_c} \\ 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{3.38}$$

and

$$\tilde{Q}^T = \begin{pmatrix} \tilde{v}_1 & 0 & \cdots & 0 \\ 0 & \tilde{v}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \tilde{v}_{N_c} \\ 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{3.39}$$

The number of remaining massless chiral supermultiplets (i.e. the dimension of the moduli space) is given by

$$dim\left(\mathcal{M}_{N_f \geq N_c} = 2N_cN_f - (N_c^2 - 1)\right). \tag{3.40}$$

We can describe the light degrees of freedom in a gauge invariant way by the following basic generators:

$$\begin{aligned} M^i_j &= Q^i_a \tilde{Q}^a_j && (mesons); \\ B^{i_1 \ldots i_{N_c}} &= Q^{i_1}_{a_1} \ldots Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \ldots a_{N_c}} && (baryons); \\ \tilde{B}^{i_1 \ldots i_{N_c}} &= \tilde{Q}^{a_1}_{i_1} \ldots \tilde{Q}^{a_{N_c}}_{i_{N_c}} \epsilon_{a_1 \ldots a_{N_c}} && (antibaryons). \end{aligned} \tag{3.41}$$

We observe that for $N_f \geq N_c$, under the global $SU(N_f)_L \times SU(N_f)_R$ the mesons M transform in the bifundamental $[1, 0, \ldots ; 0, \ldots, 1]$ representation, the baryons B and antibaryons $\tilde{B}$ transform respectively in

$$[0, \ldots, 1_{N_c;L}, 0 \ldots, 0; 0, \ldots, 0],$$

and

$$[0, \ldots, 0; 0, \ldots, 1_{N_c;R}, 0 \ldots, 0].$$

In the above, $1_{j;L}$ denotes a 1 in the j-th position from the left, and $1_{j;R}$ denotes a 1 in the j-th position from the right. The total number of basic generators for the GIOs coming from the three contributions in 3.41 is therefore

$$N_f^2 + \binom{N_f}{N_c} + \binom{N_f}{N_f - N_c} = N_f^2 + 2\binom{N_f}{N_c}. \tag{3.42}$$

We emphasise that the basic generators in 3.41 are not independent, but they are subject to the following constraints. Since the product of two epsilon tensors can be written as an antisymmetrised sum of Kronecker deltas, it follows that

$$B^{i_1 \ldots i_{N_c}} \tilde{B}_{j_1 \ldots j_{N_c}} = M^{[i_1}_{j_1} \cdots M^{i_{N_c}]}_{j_{N_c}}. \tag{3.43}$$

We can rewrite this constraint more compactly as

$$(*B)\tilde{B} = *(M^{N_c}), \tag{3.44}$$

where $(*B)_{i_{N_c+1} \ldots i_{N_f}} = \frac{1}{N_c!} \epsilon_{i_1 \ldots i_{N_f}} B^{i_1 \ldots i_{N_c}}$. Another constraint follows from the fact that any product of M's, B's and $\tilde{B}$'s antisymmetrised on $N_c + 1$ (or more) upper or lower flavour indices must vanish:

$$M \cdot *B = M \cdot *\tilde{B} = 0, \tag{3.45}$$

where a " $\cdot$ " denotes a contraction of an upper with a lower flavour index. It can be shown (see for example [3]) that all other constraints follow from the basic ones 3.44 and 3.45.

Counting the number of quarks and antiquarks in the basic constraints and considering the representations in which they transform one can find that for $N_f \geq N_c$, under the global $SU(N_f)_L \times SU(N_f)_R$ constraint 3.44 transforms as

$$[0, \ldots, 0, 1_{N_c;L}, 0, \ldots, 0; 0, \ldots, 0, 1_{N_c;R}, 0, \ldots, 0].$$

Similarly, in 3.45 the first constraint transforms as

$$[0, \dots, 0, 1_{(N_c+1);L}, 0, \dots, 0; 0, \dots, 0, 1],$$

and the second as

$$[1, 0, \dots, 0; 0, \dots, 0, 1_{(N_c+1);R}, 0, \dots, 0].$$

Indeed, the dimension of the representation corresponding to the constraint 3.44 is $\binom{N_f}{N_c}^2$, and the dimension of each of the representations corresponding to the constraints 3.45 is $N_f\binom{N_f}{N_c+1}$. Thus, there are $\binom{N_f}{N_c}^2 + 2N_f\binom{N_f}{N_c+1}$ basic constraints.

Because of these constraints, the space $\mathcal{M}_{N_f \geq N_c}$ are not freely generated and give interesting geometries which we'll study in the next section for a special case.

3.6.3 The case $N_f = N_c$

The special case of $N_f = N_c$ deserves some special attention.

From 3.34, the total number of basic generators for the GIOs, coming from the three contribution in 3.41, is $N_f^2 + 2$. From 3.40 the dimension of the moduli space is

$$dim\left(\mathcal{M}_{N_f=N_c}\right) = N_f^2 + 1. \tag{3.46}$$

There is one constraint 3.44 that can be reduced to a single hypersurface:

$$det(M) = (*B)(*\tilde{B}), \tag{3.47}$$

where we have used the identity $det(M) = (1/N_c!) * *(M^{N_c})$. This agrees with the fact that this constraint transforms in the trivial representation of $SU(N_f)_L \times SU(N_f)_R$. Since, in this case, the dimension of the moduli space equals the number of the basic generators minus the number of constraints, we arrive to another important conclusion: the moduli space $\mathcal{M}_{N_f=N_c}$ is a complete intersection in the sense that the dimension of the affine variety is given by the number of generators, minus the number of relations. It is in fact a single hypersurface in $\mathbb{C}^{N_f^2+2}$.

An interesting question to consider, is to determine the number of independent GIOs that can be constructed from the basic generators 3.41 subject to the constraints 3.44 and 3.45. In the case $N_f = N_c$, where the only constraint is 3.47, the generating function can be easily computed from the knowledge that the moduli space is a complete intersection. There are N_c^2 mesonic generators of weight t^2 and two baryonic generators of weight t^{N_c}, subject to a relation of weight t^{2N_c}. As a result the unrefined Hilbert series takes the form

$$HS^{N_f=N_c}(t) = \frac{1 - t^{2N_c}}{(1-t^2)^{N_c^2}(1-t^{N_c})^2}. \tag{3.48}$$

This is the Hilbert series of the hypersurface 3.47.

3.7 The SuperHiggs mechanism

In the following, by considering the SQCD example again, we would like to show how the superfield degrees of freedom rearrangement explicitly works upon higgsing. It is expected that upon supersymmetric higgsing a full vector superfield becomes massive, eating up a chiral superfield.

Let us consider again SQCD with gauge group $SU(N_c)$ and N_f flavours and focus on a point of the moduli space where all scalar field's VEVs v_i are 0 but v_1. At this point of the moduli space the gauge group is broken to $SU(N_c - 1)$ while flavour symmetries to $SU(N_f - 1)_L \times SU(N_f - 1)_R$. The number of broken generators is

$$N_c^2 - 1 - [(N_c - 1)^2 - 1] = 2(N_c - 1) + 1 ,$$

which just corresponds to the statement that if we decompose the adjoint representation of $SU(N_c)$ into $SU(N_c - 1)$ representations we get

$$Adj_{N_c} = 1 + \square + \overline{\square} + Adj_{N_c - 1} \to G^A = X^0, X_1^\alpha, X_2^\alpha, T^a ,$$

where G^A are the generators of $SU(N_c)$, T^a those of $SU(N_c - 1)$ and the X are the generators of the coset $SU(N_c)/SU(N_c - 1)$ ($A = 1, 2, \ldots, N_c^2 - 1$, $a = 1, 2, \ldots, (N_c - 1)^2 - 1$ and $\alpha = 1, 2, \ldots, N_c - 1$).

Upon this decomposition, the matter fields matrices can be rewritten schematically as

$$Q = \begin{pmatrix} \omega^0 & \psi \\ \omega & Q' \end{pmatrix}, \tag{3.49}$$

and

$$\tilde{Q} = \begin{pmatrix} \tilde{\omega}^0 & \tilde{\psi} \\ \tilde{\omega} & \tilde{Q}' \end{pmatrix}, \tag{3.50}$$

where, with respect to the surviving gauge and flavour symmetries, ω^0 and $\tilde{\omega}^0$ are singlets, ω and $\tilde{\omega}$ are flavour singlets but carry the fundamental (resp. anti-fundamental) representation of $SU(N_c - 1)$, ψ (resp. $\tilde{\psi}$) are gauge singlets and transform in the fundamental representation of $SU(N_f - 1)_L$ (resp. $SU(N_f - 1)_R$), and finally Q' (resp. $\tilde{Q}'$) are in the fundamental (resp. anti-fundamental) representation of $SU(N_c - 1)$ and in the fundamental representation of $SU(N_f - 1)_L)$ (resp. $SU(N_f - 1)_R$).

By expanding the scalar fields around their VEVs (which are all vanishing but v_1), one could plug all back into the SQCD Lagrangian and find the fermion and scalar masses so generated, together with the massive gauge bosons and corresponding massive gauginos. On top of this, there will remain a set of massless fields, belonging to the massless vector superfields spanning $SU(N_c - 1)$ and the massless chiral superfields Q' and $\tilde{Q}'$. We refrain to perform this calculation explicitly and just want to show that the number of bosonic and fermionic degrees of freedom, though arranged differently in the supersymmetry algebra representations, are the same before and after higgsing.

Let us just focus on the bosonic degrees of freedom, since due to supersymmetry, the same result holds for the fermionic ones. For $v_1 = 0$ we have a fully massless spectrum. As far as bosonic degrees of freedom are concerned we have $2(N_c^2 - 1)$ of them coming from the gauge bosons and $4N_cN_f$ coming from the complex scalars. All in all there are $2(N_c^2 - 1) + 4N_cN_f$ bosonic degrees of freedom. For $v_1 \neq 0$ things are more complicated. As for the vector superfield degrees of freedom, we have $(N_c - 1)^2 - 1$ massless ones, which correspond to $2[(N_c-1)^2-1]$ bosonic degrees of freedom, and $1+2(N_c-1)$ massive ones which correspond to $4+8(N_c-1)$ bosonic degrees of freedom. As for the matter fields, some of them are not there since they have been eaten by the (by now massive) vectors. These are ω and $\tilde{\omega}$, which are eaten by the vector multiplets associated to the generators X_1^α and X_2^α, and the combination $\omega^0-\tilde{\omega}^0$ which is eaten by the vector multiplet associated to the generator X^0. We have already taken them into account, then. The massless chiral superfields Q' and $\tilde{Q}'$ provide $2(N_c-1)(N_f-1)$ bosonic degrees of freedom each, the symmetric combination $S=\omega^0+\tilde{\omega}^0$ another 2 bosonic degrees of freedom, and finally the massless chiral superfields ψ and $\tilde{\psi}$ $2(N_f-1)$ each. All in all we get

$$2[(N_c-1)^2-1]+4+8(N_c-1)+4(N_c-1)(N_f-1)+2+4(N_f-1)=2(Nc^2-1)+4NF,$$

which are exactly the same as those of the un-higgsed phase.

3.8 Quantum moduli space of vacua and chiral ring

In a quantized theory, a classical field becomes an operator $\mathcal{O}$ depending on spacetime. In this case we are interested in a particular functional of the the field-operator, the energy functional which can be expressed like

$$\hat{E}[\mathcal{O}]=\int d^3xT^{00}[\mathcal{O}].$$

This functional, takes a field-operator, and gives back another operator $\hat{E}[\mathcal{O}]$ acting on the Fock space of states of the quantum field theory. Therefore, we say that a state is a vacuum state if the expectation value of $\hat{E}[\mathcal{O}]$ on such state is minimal. For what we said, only scalar fields can assume non-vanishing vacuum expectation values but like in the classical case this is not the best choice for a parametrization of the moduli space.

In a quantized supersymmetric theory, in order to describe the moduli space, it is convenient to introduce a new class of operators: the chiral operators. We saw that a chiral superfield is defined to be a special type of superfield which is defined by

$$\overline{D}_{\dot{\alpha}}X(x,\theta,\bar{\theta})=0,$$

where $\overline{D}_{\dot{\alpha}}$ is the covariant superspace derivative. This implies that the lowest component of this superfield is annihilated by $\overline{\mathcal{Q}}_{\dot{\alpha}}$, namely

$$[\overline{\mathcal{Q}}_{\dot{\alpha}},\varphi(x)]=0.$$

The same facts are true for the gluino λ_α which is the lowest component of the chiral superfield W_α. This fermionic field satisfies

$$\{\overline{\mathcal{Q}}_{\dot{\alpha}},\lambda_\alpha(x)\}=0.$$

In general a chiral operator $\mathcal{O}(x)$ is a gauge invariant operator such that it is annihilated by all the supercharges of one chirality.

$$[\overline{\mathcal{Q}}_{\dot{\alpha}},\mathcal{O}(x)\}=0.$$

Before going on studying the properties of such operators, we should discuss how to construct them in a gauge theory. Here we have at our disposal the gauge-variant objects

$\phi(x)$ and $\lambda_\alpha(x)$ that are annihilated by $\overline{\mathcal{Q}}_{\dot{\alpha}}$. One could thus construct gauge-invariant composite operators, such as $tr(\lambda^2(x))$, that are naively annihilated by $\overline{\mathcal{Q}}_{\dot{\alpha}}$ by using the Leibniz rule.

Now that we have defined those chiral operators, we can immediately derive the important property that the v.e.v of any time ordered product of chiral operators is independent on their spacetime position. Consider for instance the product of two bosonic chiral operators at different spacetime points $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$ and take a derivative with respect of the spacetime coordinates x^μ.

$$\begin{aligned}&\frac{\partial}{\partial x^\mu}\left\langle 0|T\left(\mathcal{O}_1(x)\mathcal{O}_2(y)\right)|0\right\rangle = \\ &= \left\langle 0\left|T\left(\frac{\partial}{\partial x^\mu}\mathcal{O}_1(x)\mathcal{O}_2(y)\right)\right|0\right\rangle + \delta^0_\mu\left\langle 0|[\mathcal{O}_1(x),\mathcal{O}_2(y)]|0\right\rangle\delta(x^0-y^0).\end{aligned} \tag{3.51}$$

Now, both terms vanish separately. The first one because

$$\begin{aligned}\left\langle 0\left|\frac{\partial}{\partial x^\mu}\mathcal{O}_1(x)\mathcal{O}_2(y)\right|0\right\rangle &= -i\left\langle 0|[\mathcal{P}_\mu,\mathcal{O}_1(x)]\mathcal{O}_2(y)|0\right\rangle = \\ &= \frac{i}{2}\overline{\sigma}^{\dot{\alpha}\alpha}_\mu\left\langle 0|[\{\mathcal{Q}_\alpha\overline{\mathcal{Q}}_{\dot{\alpha}}\}\mathcal{O}_1(x)]\mathcal{O}_2(y)|0\right\rangle = \\ &= \frac{i}{2}\overline{\sigma}^{\dot{\alpha}\alpha}_\mu\left\langle 0|\{\overline{\mathcal{Q}}_{\dot{\alpha}}[\mathcal{Q}_\alpha,\mathcal{O}_1(x)]\}\mathcal{O}_2(y)|0\right\rangle = \\ &= \frac{i}{2}\overline{\sigma}^{\dot{\alpha}\alpha}_\mu\left\langle 0|\{\overline{\mathcal{Q}}_{\dot{\alpha}}[\mathcal{Q}_\alpha,\mathcal{O}_1(x)]\mathcal{O}_2(y)\}|0\right\rangle = \\ &= 0.\end{aligned} \tag{3.52}$$

where we used the SUSY algebra, the Jacobi identity and the chirality of the operators in order to bring $\overline{\mathcal{Q}}_{\dot{\alpha}}$ to act on the vacuum which is assumed to be supersymmetric, and therefore annihilated by the supercharges. The second term in 3.51 is also zero because the equal time commutator vanishes since the same arguments used above to show that the first term vanishes can be applied. Given that the v.e.v. of chiral operators is independent of their position, one can go to the limit of large separation and apply cluster decomposition to obtain:

$$\langle 0|T(\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n))|0\rangle = \langle\mathcal{O}_1\rangle\cdots\langle\mathcal{O}_n\rangle,$$

where there is no longer any need to specify the positions. This is the important property of factorization of correlation functions involving chiral operators.Therefore we have found that the v.e.v of a generic chiral operator $\mathcal{O}(x)$ does not depend on x.

Suppose the vacuum is supersymmetric, then objects of the type $\{\overline{\mathcal{Q}}_{\dot{\alpha}},...\}$ do not contribute to the expectation values. Therefore one can define an equivalence relation between chiral operators. Two chiral operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ are equivalent if there exist a gauge invariant operator $X_{\dot{\alpha}}(x)$ such that

$$\mathcal{O}_1(x) = \mathcal{O}_2(x) + [\overline{Q}^{\dot{\alpha}}, X_{\dot{\alpha}}\}. \tag{3.53}$$

The set of equivalence classes of chiral operators under 3.53 forms a ring, known as the chiral ring.

A very important fact, is the existence of a map from the chiral ring to the ring of holomorphic functions over the moduli space. This map takes a chiral operator $\mathcal{O}$ and sends it to the an holomorphic function over the moduli space $f : \mathcal{M} \to \mathbb{C}$. This map is injective, which means that each chiral operator defines a different holomorphic function over the Moduli Space. However, it has not been proved yet that this map is surjective, although it is generally believed so. As a working assumption, one conjectures that this map is indeed bijective and therefore invertible, and it is also a isomorphism of rings. In this way we have a correspondence between the chiral ring and the moduli space seen as an algebraic variety upon which a set of functions vanishes.

Chapter 4

$\mathcal{N} = 2$ Supersymmetry

The more a theory is supersymmetric, the more its field content and interactions are constrained. Typically the number of copies of supersymmetry is a power of 2, i.e. 1, 2, 4, 8. In four dimensions, a spinor has four degrees of freedom and thus the minimal number of supersymmetry generators is four and this is known as minimal supersymmetry ($\mathcal{N} = 1$ SUSY).

The maximal possible number of supersymmetry generators is 32. Theories with more than 32 supersymmetry generators automatically have massless fields with spin greater than 2. It is not known how to make massless fields with spin greater than two interact, so the maximal number of supersymmetry generators considered is 32. This corresponds to an $\mathcal{N} = 8$ supersymmetry theory in four dimensions.

4.1 Massless supermultiplets for $\mathcal{N} = 2$

In this case, equation 2.9 gives

$$\{Q_\alpha^A, \overline{Q}_{\dot{\beta}B}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \ , \tag{4.1}$$

where, respect to $\mathcal{N} = 1$, we reintroduce the indices A, $B = 1, 2$. Once again, we boost to a frame in which $P^\mu = (E, 0, 0, E)$. We already saw that the anticommuting relations for the second components are equal to zero. As a result, acting with $\overline{Q}_{\dot{2}B}$ on the clifford vacuum gives rise again to zero-normed states. The non trivial anticommuting relations involve the first components:

$$\{Q_1^A, \overline{Q}_{\dot{1}B}\} = 4E\delta^A_B. \tag{4.2}$$

Now, we define two operators $a_A \equiv \frac{1}{\sqrt{4E}} Q_1^A$ and $a_B^\dagger \equiv \frac{1}{\sqrt{4E}} \overline{Q}_{\dot{1}B}$ which satisfy the Clifford algebra

$$\{a_A, a_B^\dagger\} = \delta_{AB}. \tag{4.3}$$

To construct a representation, we can start by choosing a state annihilated by all a_A, known as the Clifford vacuum; such state carries some irrep of the Poincaré algebra. It has energy E and some helicity λ, and we call it $|E, \lambda\rangle$ or for shortness $|\lambda\rangle$. This state can be either bosonic or fermionic, and should not be confused with the actual vacuum of the theory, which is the state of minimal energy.

In a similar way to what we discussed before, we notice that $a_B^\dagger$ increases helicity of $\frac{1}{2}$ while a_A decreases it of the same quantity. The full representation is obtained acting on $|E, \lambda\rangle$ with the creation operators $a_A^\dagger$ as follows:

$$|\lambda\rangle\,,\; a_1^\dagger\,|\lambda\rangle\,,\; a_2^\dagger\,|\lambda\rangle\,,\; a_1^\dagger a_2^\dagger\,|\lambda\rangle\,. \tag{4.4}$$

- If we start with the Clifford vacuum $\left|-\frac{1}{2}\right\rangle$ we can build the hypermultiplet representation of SUSY with $\mathcal{N}=2$. Our first guess is

  $$\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right), \tag{4.5}$$

  which is a shorthand notation for the representation with helicity-states in parenthesis. As always half of the states have integer helicity (bosons), half of them half-integer helicity (fermions).

  Actually, this is not the end because we must add the CPT conjugate

  $$\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right). \tag{4.6}$$

  The degrees of freedom are those of two Weyl fermions and two complex scalars. This is where matter sits in a $\mathcal{N}=2$ supersymmetric theory. In $\mathcal{N}=1$ language this representation corresponds to two chiral multiplets with opposite chirality (CPT flips the chirality). Notice that in principle this representation enjoys the CPT self-conjugate condition. However, a closer look shows that an hypermultiplet cannot be self-conjugate (that's why we added the CPT conjugate representation). The way the various states are constructed out of the Clifford vacuum, shows that under $SU(2)$ R-symmetry the helicity 0 states behave as a doublet while the fermionic states are singlets. If the representation were CPT self-conjugate the two scalar degrees of freedom would have been both real.

- If we start instead with the Clifford vacuum $|0\rangle$ we build the vector multiplet in $\mathcal{N}=2$.
  So, we get

  $$\left(0, \frac{1}{2}, \frac{1}{2}, 1\right), \tag{4.7}$$

  where we recognize a scalar, two spinors and one vector. Again we add its CPT-conjugate:

  $$\left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}, \frac{1}{2}, 1\right). \tag{4.8}$$

  This representation corresponds to two one $\mathcal{N}=1$ vector multiplet and one $\mathcal{N}=1$ chiral multiplet.

4.2 $\mathcal{N}=2$ super Yang-Mills

In this section we would like to construct the most general $\mathcal{N}=2$ supersymmetric action in four dimensions. We will follow the same logic of the previous section, but we will not develop the corresponding $\mathcal{N}=2$ superspace approach, in fact $\mathcal{N}=2$ superspace formulation isn't very well known yet. Instead, we will use the $\mathcal{N}=1$ superspace formalism and see which specific properties does more supersymmetry impose on an otherwise generic $\mathcal{N}=1$ Lagrangian.

We have two kinds of $\mathcal{N}=2$ multiplets, the vector multiplets and the hypermutiplets. What we noticed at the level of representations of the supersymmetry algebra on states holds also at the field level. In particular, using a $\mathcal{N}=1$ language, a $\mathcal{N}=2$ vector superfield can be seen as the direct sum of a vector superfield V and a chiral superfield Φ. Similarly, in terms of degrees of freedom a hypermultiplet can be constructed out of two $\mathcal{N}=1$ chiral superfields, H_1 and H_2. Schematically, we have

$$\begin{aligned} [\mathcal{N}=2\ \textit{vector multiplet}] &\quad : \quad V=(\lambda_\alpha, V_\mu, \mathcal{D}) \oplus \Phi = (\phi, \psi_\alpha, F), \\ [\mathcal{N}=2\ \textit{hypermultiplet}] &\quad : \quad H_1=(H_1, \psi_{1\alpha}, F_1) \oplus \overline{H}_2 = (\overline{H}_2, \overline{\psi}_{2\dot\alpha}, \overline{F}_2), \end{aligned} \tag{4.9}$$

where we notice that H_1 and $\overline{H}_2$ transform in the same representations of internal symmetries, while H_2 transforms in the complex conjugate representation. The same is true for V and Φ which both transform in the adjoint representation of the gauge group.

Let us start considering a pure SYM, with gauge group G. There are two minimal requirements we should impose. As already stressed, the vector multiplet should transform in the adjoint representation of the gauge group. Moreover, the gauge and matter couplings could not be independent since under the compact part of the R-symmetry, $SU(2)_R$, all bosonic fields V_μ, $\mathcal{D}$, F and ϕ are singlets, but λ_α and ψ_α transform as a doublet. This is because Q^1_α, Q^2_α transform under the fundamental representation of $SU(2)_R$, and the same should hold for λ_α and ψ_α (recall that they are obtained acting with the two supersymmetry generators on the Clifford vacuum).
The action is

$$\begin{aligned} S =& \frac{1}{16g^2}\int d^4x d^2\theta Tr\left(W^\alpha W_\alpha\right) + \frac{1}{16g^2}\int d^4x d^2\overline{\theta} Tr\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right) + \\ &+ \int d^4x d^2\theta d^2\overline{\theta} Tr\, \overline{\Phi} e^{2gV}\Phi. \end{aligned} \tag{4.10}$$

Compared to $\mathcal{N}=1$ lagrangians describing matter coupled SYM theory, the above Lagrangian is special in many respects. A necessary and sufficient condition for $\mathcal{N}=2$ supersymmetry is the existence of a $SU(2)_R$ rotating the two generators Q^1_α, Q^2_α into each other. Owing to this, the Lagrangian has no superpotential, $\mathcal{W}=0$. Indeed, a superpotential would give ψ interactions and/or mass terms, that are absent for λ. This is clearly forbidden by the $SU(2)_R$ symmetry.

Let us now consider adding hypermultiplets. In this case the scalar fields H_1 and $\overline{H}_2$ form a $SU(2)_R$ doublet. Hypermultiplets cannot interact between themselves since no cubic $SU(2)$ invariant is possible.

The action for a $\mathcal{N}=2$ gauge theory which couples a vector multiplet to an hypermultiplet finally reads

$$\begin{aligned} S_{\mathcal{N}=2} =& \frac{1}{16g^2}\int d^4x d^2\theta Tr\left(W^\alpha W_\alpha\right) + \frac{1}{16g^2}\int d^4x d^2\overline{\theta} Tr\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right) + \\ &+ \int d^4x d^2\theta d^2\overline{\theta} Tr\, \overline{\Phi} e^{2gV}\Phi + \int d^4x d^2\theta d^2\overline{\theta}\left(\overline{H}_1 e^{2gV} H_1 + H_2 e^{2gV}\overline{H}_2\right) + \\ &+ \int d^4x d^2\theta \sqrt{2} g H_1 \Phi H_2 + h.c.\,. \end{aligned} \tag{4.11}$$

The F-term coupling the chiral multiplets H_1, H_2 with the chiral multiplet Φ belonging to the $\mathcal{N} = 2$ vector multiplet is there because of $\mathcal{N} = 2$ supersymmetry (it is somehow the supersymmetric partner of the kinetic terms which couple the hypermultiplet with V). This coupling is what we refer to when we speak about superpotential $\mathcal{W}$ in $\mathcal{N} = 2$ theories and this also holds for $\mathcal{N} = 4$ in three dimensions.

4.3 Moduli space for $\mathcal{N} = 2$ theories

The moduli spaces for $\mathcal{N} = 2$ theories come in two contributions depending on the fields we have in the theory. One contribution is given by the $\mathcal{N} = 2$ vector multiplet which contains a scalar field in the $\mathcal{N} = 1$ chiral multiplet (in the adjoint representation of the gauge group). The second contribution comes from the hypermultiplet, which brings two scalars (in some representation of the gauge group).

As regards the vector multiplet, while we do not have a superpotential, we do have a potential, which comes from D-terms. Indeed, the auxiliary fields equations of motion are in this case

$$F^a = 0\,, \qquad D^a = -g[\phi, \overline{\phi}]^a\,,$$

since the auxiliary fields F^a appear only in the non-dynamical kinetic term $\overline{F}_a F^a$ and therefore are trivial. The potential hence reads

$$\mathcal{V}_{vec}(\phi) = \frac{1}{2} D^a D^a = \frac{1}{2} g^2 Tr[\phi, \overline{\phi}]^2\,. \tag{4.12}$$

Instead, for the hypermultiplet we can eliminate the auxiliary fields F_1 and F_2 and after some algebra the scalar potential for the hypermultiplets can be recasted as a D-term contribution only and reads

$$\mathcal{V}_{matt}(H_1, H_2) = \frac{1}{2} D^2 = \frac{1}{2} g^2 |\overline{H}_1 T^a_R H_1 - \overline{H}_2 T^a_R H2|^2. \tag{4.13}$$

The two contributions are summed in a theory with lagrangian like in 4.11 giving:

$$\mathcal{V}_{\mathcal{N}=2}(\phi, H_1, H_2) = \frac{1}{2} g^2 Tr[\phi, \overline{\phi}]^2 + \frac{1}{2} g^2 |\overline{H}_1 T^a_R H_1 - \overline{H}_2 T^a_R H_2|^2. \tag{4.14}$$

The subspace of the supersymmetric vacuum moduli in which $H_1 = H_2 = 0$ is called the Coulomb branch, since a number of Abelian gauge fields usually remain in the infrared there. The subspace where $\phi = 0$ is called the Higgs branch. Some people in the field reserve the word Higgs branch also for the branch where the gauge group is completely broken but we'll follow the convention stated before. Finally, the branches in which both the hypermultiplet's scalars H_1, H_2 and the vector multiplet's scalar ϕ are nonzero are called mixed branches.

4.4 Quivers for $\mathcal{N} = 2$ theories

We can consider a $\mathcal{N} = 2$ lagrangian just as a special case of a $\mathcal{N} = 1$ lagrangian. Therefore, a $\mathcal{N} = 2$ quiver is just a special type of a $\mathcal{N} = 1$ quiver. We define a $\mathcal{N} = 2$ quiver as a quiver such that

- For a generic arrow H_1, having as a source the node A and as a target the node B, there is also an arrow H_2 having as a source B and as a target A.

- For a generic node $A(B)$ there is an arrow $\Phi_A(\Phi_B)$ having as a source and as a target $A(B)$.

We give an example for such a quiver in the next figure.

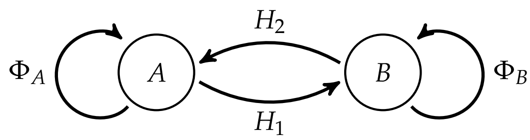


Figure 4.1: Generic $\mathcal{N} = 2$ quiver diagram opened in $\mathcal{N} = 1$ notation. The gauge group is $A \times B$ and it is a theory of one hypermultiplet coupled to two vector fields.

The rule for associating a lagrangian to its quiver is the same as the one given before for $\mathcal{N} = 1$ quivers. For each node in the $\mathcal{N} = 2$ quiver, we now have a $\mathcal{N} = 1$ vector multiplet (V_A or V_B in figure) and a $\mathcal{N} = 1$ chiral multiplet (Φ_A and Φ_B). In the same way, the arrows between different nodes correspond to the two chiral multiplets (H_1 and H_2) which compose a $\mathcal{N} = 2$ hypermultiplet and they transform according to the rules for $\mathcal{N} = 1$ quivers.

In order to avoid writing numerous lines in a single quiver, people use a shorthand notation to write down $\mathcal{N} = 2$ quivers. The prescription is the following:

- instead of two opposite directed arrows between two nodes, we just draw an unoriented line between the two nodes;

- for each node, we avoid writing down the line that starts and ends on that node.

In this convention, the previous quiver is simply written as follows.

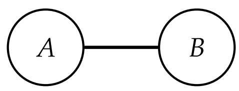


Figure 4.2: $\mathcal{N} = 2$ notation for the quiver in figure 4.1.

We could also put a flavour node, which we indicate with a square node, and for a theory with gauge group $U(1)$ and n flavours we have the following quiver.

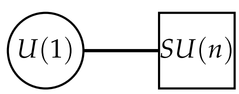


Figure 4.3: Quiver diagram for $\mathcal{N} = 2$ SQED.

Sometimes, for $\mathcal{N} = 2$ quivers we will adopt the convention of just writing the number n inside the gauge nodes to indicate the unitary group $U(n)$. An example of this notation will be the neckless quiver diagram which we'll see again in chapter 7.

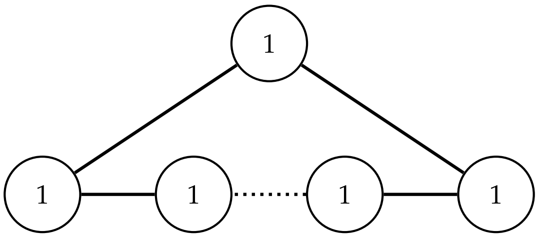


Figure 4.4: Quiver diagram for n copies of $U(1)$ in the gauge group.

Chapter 5

Quiver gauge theories in 3d with $\mathcal{N} = 4$

We finally turn to super Yang-Mills theories with $\mathcal{N} = 4$ in three dimensions, two of them spacelike and one timelike.

First of all, we have to understand how we can recover a theory in $3d$ with $\mathcal{N} = 4$ SUSY starting from a $\mathcal{N} = 2$ theory in $4d$. This is achieved through dimensional reduction.

Then, we'll see that owing to the small number of dimensions, there is a new kind of symmetries, which are called topological symmetries owing to their topological nature. In fact, they depend on the dimension of spacetime and they are fundamental for the theories we are going to study. Indeed, in the following sections we will look for operators which are non trivially charged under these symmetries. These play a crucial role in the study of the Coulomb branch in Chapter 7.

Another important aspect of these theories is the fact that photons can be dualyzed to scalars. This is a special feature of $3d$ theories and tells us that we could equivalently describe the gauge fields in terms of scalars. This is essential for us since we are interested in the moduli space of vacua of these theories.

5.1 Dimensional reduction

The main focus of this thesis are $\mathcal{N} = 4$ theories defined on a $d = 2 + 1$ dimensional Minkowski spacetime. These theories are particular interesting because of their topological properties and because of the high amount of SUSY which constrains them.

As regards SUSY, these theories can be firstly characterized by their number of supercharges, which is equal to 8. In general, we are able to move from a theory with a given number of supercharges to another in lower dimensions but with the same number of supercharges by a particular procedure which is called dimensional reduction.

Let us recall that $\mathcal{N} = N/d_s$. In four dimensions the smallest irreducible spinor is a Majorana spinor with real dimension equal to four. This means that for $\mathcal{N} = 2$ and $d = 4$ we have a theory of 8 real supercharges. As we already said, also for $\mathcal{N} = 4$ and $d = 3$ we have 8 supercharges. In this case the real dimension of the minimal spinorial representation is 2 and this accounts for the total of 8 supercharges.

In both cases we have the same number of supercharges and our aim is to show that a theory in $d = 3$ $\mathcal{N} = 4$ can be obtained from one in $d = 4$ with $\mathcal{N} = 2$ through dimensional reduction. Actually, this is not the only way to obtain it. It is generally proved that all supersymmetric gauge theories with 8 supercharges can be obtained from dimensional reductions of a theory with 8 supercharges in $d = 6$. So, both $d = 3$ $\mathcal{N} = 4$ and $d = 4$ $\mathcal{N} = 2$ can be obtained in such a way. Depending on our purposes it's sometimes more convenint to get the field content of $d = 3$ $\mathcal{N} = 4$ reducing one time from $d = 4$ $\mathcal{N} = 2$ instead than reducing three times from $d = 6$ $\mathcal{N} = 1$. Other times is better starting from $d = 6$ $\mathcal{N} = 1$ if we are for example interested in symmetries.

Dimensional reduction consists in neglecting the dependence of one or more coordintes in the fields' arguments. This compactification of one, or more, of the spacetime dimensions, also reflects on the representations of the Poincaré group carried by the fields themselves. To start with the dimensional reduction we must choose a branching rule. We start noticing that $SO(3) \subset SO(4)$, so the issue is to decide how we want to restrict a representation, let us say the fundamental, of the group $SO(4)$ to its subgroup $SO(3)$. The standard way to do it is to embed the fundamental of $SO(3)$ in the fundamental of $SO(4)$, what is left from a simple counting of the dimensions is a trivial representation:

$$[1,1]_{SO(4)} \simeq [2]_{SO(3)} \oplus [0]_{SO(3)} \,. \tag{5.1}$$

The previous equation gives all the informations we need to compute how the other representations of $SO(4)$ behave when we do the dimensional reduction. We are now ready to list the dimensional reductions for the fields we are interested in:

$$\begin{aligned} &[0,0]_{SO(4)} \mapsto [0]_{SO(3)}; \\ &[1,0]_{SO(4)} \mapsto [1]_{SO(3)}; \\ &[0,1]_{SO(4)} \mapsto [1]_{SO(3)}; \\ &[1,1]_{SO(4)} \mapsto [2]_{SO(3)} + [0]_{SO(3)}. \end{aligned} \tag{5.2}$$

Therefore the field components of the supermultiplets decompose as follows:

- $\phi(x) \mapsto \phi(x)$: the scalar field in $d = 4$ dimension is mapped to a scalar field in $d = 3$;
- $\psi_{d=4}(x) \mapsto \psi_{d=3}(x)$: the spinor field in $d = 4$ dimension is mapped to the spinor field in $d = 3$;
- $\overline{\psi}_{d=4}(x) \mapsto \psi_{d=3}(x)$: the spinor of the other chirality is mapped to the spinor field in $d = 3$;
- $V_\mu(x) = V_i(x) + V_0(x)$: the vector field is mapped a scalar field and a three dimensional vector.

From these rules, one reconstructs the field content of $d = 3$ $\mathcal{N} = 4$ gauge theories knowing how the vector multiplets and hypermultiplets in $d = 4$ $\mathcal{N} = 2$ are done. In table 5.1 we summed up this field content.

Table 5.1: Field content and charges of the supersymmetry multiplets involved in three dimensional mirror symmetry.

$\mathcal{N}=4$	$\mathcal{N}=2$	Components	Spin	$SU(2)_L \times SU(2)_R$	G
vector	vector	V_μ	1		
multiplet	multiplet (V)	λ_α	1/2		
		η	0	$\{\eta, Re\phi, Im\phi\}$ in $(0,1)$	
		$\mathcal{D}$	0	$\{\lambda_\alpha, \xi_\alpha\}$ in $(\frac{1}{2},\frac{1}{2})$	adjoint
	chiral	ϕ	0	$\{\mathcal{D}, ReF_\Phi, ImF_\Phi\}$ in $(1,0)$	
	multiplet(Φ)	ξ_α	1/2		
		F_Φ	0		
hyper	chiral	h_1	0		
multiplet	mutiplet(H_1)	$\psi_{1\alpha}$	1/2	$\{h_1, h_2^\dagger\}$ in $(\frac{1}{2},0)$	R
		F_1	0	$\{h_1^\dagger, h_2\}$ in $(0,\frac{1}{2})$	
	chiral	h_2	0	$\{\psi_{1\alpha}, \psi_{2\alpha}\}$ in $(\frac{1}{2},\frac{1}{2})$	
	multiplet(H_2)	$\psi_{2\alpha}$	1/2	F_1, F_2 integrated out	R*
		F_2	0		

The results above are also obtained from a threefold dimensional reduction from $d=6$ $\mathcal{N}=(1,0)$. In six dimensions, the fields are the gauge field V and Weyl fermions ψ in the adjoint representation of the gauge group G. There is an $SU(2)_R$ symmetry that acts only on the fermions; the fermions and supercharges transform as doublets of $SU(2)_R$. Upon dimensional reduction to three dimensions (that is, taking the fields to be independent of three coordinates $x_{4,5,6}$) one obtains a theory with the following additional structures. The last three components of V in three dimensions become the scalar fields ϕ_i, $i=1,2,3$, in the adjoint representation. These scalars transform in the vector representation under the group of rotations of the $x_{4,5,6}$; we will call the double cover of this group $SU(2)_L$. Note that in reduction to four dimensions, only two such scalars appear, and instead of $SU(2)_L$, one gets only a $U(1)$ symmetry of rotations of the $x_{5,6}$ plane. This symmetry is often called $U(1)_R$. Finally, three dimensional Euclidean space $\mathbb{R}^3$ has a group of rotations whose double cover we will call $SU(2)_E$. Under $SU(2)_L \times SU(2)_R \times SU(2)_E$, the fermions transform as $[1;1;1]$, as do the supercharges (so that $SU(2)_L$ is a group of R-symmetries just like $SU(2)_R$), while the scalars transform as $[2;0;0]$.

The two types of dimensional reduction, from $d=4$ or $d=6$ are completely equivalent. In our discussion the first reduction let us focus more on the field content while the second on the R-symmetries.

5.2 Hidden symmetries

The small number of dimensions of these theories gives rise to some interesting effects. A new type of symmetries arise which are called Hidden Symmmetries. These symmetries are topological in nature: they depend explicitly on the dimension of spacetime, and are at the foundations of techniques we use to study their Coulomb Branch.

Suppose that we are studying the quantum electrodynamics in three dimensions: the gauge group is $U(1)$. The field strength 2-form F satisfies both the equations of motion

$$d * F = 0, \tag{5.3}$$

and the Bianchi identity

$$dF = 0. \tag{5.4}$$

Where the $*$ operation is the Hodge dual of k-forms. In $d = 3$, taking a coordinate chart, the Hodge dual of $F^{\mu\nu}$ is

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}, \tag{5.5}$$

but it's more convenient to always work in form notation so the previous is rewritten

$$J = *F. \tag{5.6}$$

Quite surprisingly, the conservation of this current is just given by Bianchi identity:

$$d^\dagger J = *d*J = 0 \iff d*J = 0 \iff d**F = 0 \iff dF = 0. \tag{5.7}$$

This implies that the physical theory under study enjoys a symmetry not explicitly readable from the lagrangian. This topologically conserved current presupposes the existence of a global $U(1)_J$ symmetry which is not explicit in the Lagrangian. For a general gauge group G of rank r, the theories thus hide a global $U(1)^r_J$.

Because of the fact that this current does not arise from an explicit invariance of the lagrangian, we call such a symmetry an Hidden Symmetry. It is also called a Topological Symmetry for the fact that it depends explicitly on the fact we are in $d = 3$ or even sometimes ANO symmetry, because the states of the Fock space which carry a non-zero topological charge, in the Higgs Phase, are a certain class of solitons called Abrikosov–Nielsen–Olsen vortices.

In the following, we will define a certain class of operators which carry a non-zero ANO charge, and that in the Higgs Phase create such ANO vortex states. These operators are called Monopole Operators.

5.3 Dualization of the photons

In three dimensions the photon has $3 - 2 = 1$ polarizations. Therefore, as firstly pointed out by Polyakov in [40], the action of a free photon can be explicitly written in terms of a free scalar field, which is called dual photon, and usually denoted with γ. Consider the equation of motion of F, namely $d*F = 0$, and regard it as a Bianchi identity for the 1-form $J = *F$, the hodge dual of F, defined above. Notice that J is a closed form since $dJ = d*F = 0$ on shell. We use this equation to infer that, due to Poincaré Lemma, in a contractible topological space such as $\mathbb{R}$, J is also an exact form, and therefore it exists a 0-form γ such that $d\gamma = J$. Therefore, we see that the Bianchi identity and the equation of motion are swapped, when one interprets them for the photon $F = dA$ and for the dual photon γ. The bianchi identity for F becomes the equation of motion $d*J = 0$, which is $d*d\gamma = 0$ if written in terms of the dual photon γ.

Notice also that under a gauge transformation F does not change. The dual photon does not change too since it is defined starting from F, therefore the $U(1)$ gauge group acts trivially on the dual photon γ.

5.4 Moduli spaces for 3d $\mathcal{N} = 4$ theories

5.4.1 The classical Coulomb branch

The $\mathcal{N} = 4$ vector multiplet in three dimensions contains a complex scalar ϕ coming from the $\mathcal{N} = 1$ chiral multiplet and a real scalar η from the $\mathcal{N} = 1$ vector multiplet in the reduction from four to three dimensions. It's easier to reorganize these degrees of freedom in three real scalars like we saw in table 5.1. We can rename $\phi_3 \equiv \eta$, $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ in order to obtain $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ which transform as $[1;0]$ under the R-symmetry group. Their potential energy is proportional to

$$\sum_{i<j} Tr[\phi_i, \phi_j]^2, \tag{5.8}$$

where $i, j = 1, 2, 3$. Here, ϕ_i is actually expanded upon the set of generators T^a of the gauge algebra $\phi_i = \phi_i^a T^a$.

This potential indicates that a supersymmetric vacuum exists, since flat directions $V = 0$ can be achieved by a set of commuting ϕ^i. Therefore the scalars must take values in the Cartan subalgebra of the gauge group. For a gauge group G of rank r, $U(1)^r \subseteq G$ is its Cartan subalgebra and hence fields which take value in $U(1)^r$ can be written

$$\phi^i = diag(x_1^{(i)}, \ldots, x_r^{(i)}).$$

This means that along flat directions the scalars acquire a nonzero vacuum expectation value: the gauge group G is broken by the adjoint Higgs mechanism to its maximal torus $U(1)^r$. We refer to this as complete Higgsing. The acquisition of a VEV for the scalars in the vector multiplet is precisely the statement that there is a moduli space: the Coulomb branch.

If G has rank r, the space of zeroes of V, up to gauge transformation, has real dimension $3r$. In addition to the ϕ^i, there are then r massless photons. Since a photon is dual to a scalar in three space-time dimensions, there are in all $4r$ massless scalars, $3r$ components of ϕ^i and r duals of the photons. The Coulomb branch $\mathcal{M}_C$ has thus real dimension $4r$ or r if we express it in quaternionic units. The classical Coulomb branch is the moduli space

$$\mathcal{M}_C = \left(R^3 \times S^1\right)^r,$$

where the ϕ^i parametrise the non-compact space $\mathbb{R}^{3r}$ while dual photons γ^i parametrise the compact space S^r.

Despite what we just said the treatment of the Coulomb branch is not so simple since the Coulomb branch is renormalized. What we found is just the classical Coulomb branch but we are always interested in quantum theories. The classical Coulomb branch has to be quantum mechanically corrected by loop corrections and instanton effects. In chapter 7 we'll see how the problem is solved thanks to monopole operators.

5.4.2 The classical Higgs branch

The VEVs of the scalars in the hypermultiplets also parametrize an hyperKähler space (see for example [39] for a mathematical introduction to hyperKähler geometry): the Higgs branch.

Notice that non-zero VEVs for the hypermultiplets means the gauge group is broken completely. Consequently the dimension of this space is given by the number of $\mathcal{N} = 4$ hypermultiplets ($dim(R)$) minus the number of gauge fields that become massive due to complete Higgsing($|G|$).

This is only valid when no superpotential is present, that is, for $\mathcal{N} = 4$ in $3d$. If $U(k)$ is the gauge group, all k^2 generators are broken, i.e. k^2 d.o.f. become massive and need to be subtracted from the hyper degrees of freedom. For a general group G we have in quternionc units:

$$dim_{\mathbb{H}}(\mathcal{M}_H) = dim(R) - |G|. \tag{5.9}$$

Since supersymmetry invariance dictates that no other cross terms between scalars in the vector multiplet and scalars in hypermultiplet may show in the Lagrangian, the total moduli space will be a product $\mathcal{M}_C \times \mathcal{M}_H$. This is very important because the difference with the $\mathcal{N} = 2$ $4d$ moduli space is now manifest: in the $\mathcal{N} = 2$ case we had some mixed branches that we don't have here.

Chapter 6

The Higgs branch of 3d $\mathcal{N} = 4$ quiver gauge theories

In section 3.6 we saw how the moduli space of SQCD can be parametrized in term of the gauge invariant chiral operators modulo classical relations between them. This procedure is much simpler than taking the VEV of the scalar fields modulo gauge group transformations. Anyway building the ring of the chiral and gauge invariant operators can also become more and more cumbersome when we deal with high ranked gauge groups or with a lot of matter fields.

To solve this problem it's necessary to introduce a more general procedure to enumerate gauge invariants and encoding global symmetries. This procedure exists and involves the use of the plethystic exponential and Molien-Weyl projection formula. The general procedure consists of some steps that we summarize below.

- First of all we read out of the quiver the assignment of the representations in which the hyermultiplets transform. We consider here all the symmetries involved, both global and gauge ones.
- We build the Hilbert Series counting the operators made out of symmetric products of the hypermultiples, and grade them with respect to the symmetries found in the previous step. The mathematical tool we'll use to take these symmetric product is the plethystic exponential.
- Then we project the Hilbert series into the sector of gauge invariant chiral operators, finding in this way a new Hilbert series which only counts elements of the chiral ring of gauge invariant operators. In fact, the Molien-Weyl projection acts "averaging" over all the operators which are not in general gauge invariant leaving us with just the invariant ones.
- Finally we can extract the number, the gradings of the generators and the relations of the chiral ring of the Higgs branch out of this last series.

The interested reader will find lots of details and examples of how this procedure works in practical cases in [8]. After discussing the techniques we introduced in these section we'll focus on the Higgs branches of two theories, namely $U(1)$ with n flavours and $SU(2)$ with n flavours. In the following we'll follow very closely [8].

6.1 The plethystic exponential

As we said, we need a way to compute the symmetrizations of the hypermultiplets in order to project in a second moment on the sector of gauge invariant operators.

Our goal would be to construct the chiral ring of gauge invariant operators, actually it's often too hard to compute the whole chiral ring and we limit ourselves to just count the different chiral operators in a graded way. From just these informations we could be able to identify the corresponding affine variety of the moduli spaces using the techniques developed in recent years in the field of computational algebraic geometry. We pointed out some of these methods in Appendix A where some reference are given to software packages used to carry out the calculations.

What we try to do instead is to find the number of generators and degree/number of the relations between them. In order to do that, we introduce a function that counts the symmetrized products of a given set of objects: the plethystic exponential.

Suppose we are given a set of monomials, $\{a, b, c\}$ to start with, and we ask ourselves how many monomials at degree two we can find which are symmetric in the exchange of the factors they have. The answear is simple:

$$\{a^2, b^2, c^2, ab, ac, bc\},$$

there are six monomials in this set.

We could repeat our question for the symmetric monomials of degree three. This time we have:

$$\{a^3, b^3, c^3, a^2b, ab^2, a^2c, ac^2, b^2c, bc^2, abc\}.$$

This set has cardinality ten.

We could go on and a nice guess would be trying to find the number of monomials at each degree k. In this case one would get from combinatorics that we have $\binom{3+k-1}{k}$. In general always from combinatorics we can prove that the k-times symmetrized product of n basic factors is $\binom{n+k-1}{k}$.

The plethystic exponential is a way to incorporate this procedure in a unique function. First of all we notice that the generator of symmetrization for n variables can be written:

$$\prod_i \frac{1}{1-x_i},$$

where the fraction is intended to be expanded in a geometric series. In the case of $n = 3$ with which we are familiar this gives

$$\frac{1}{1-a}\frac{1}{1-b}\frac{1}{1-c} = (1 + a + a^2 + a^3 + \cdots)(1 + b + b^2 + b^3 + \cdots)(1 + c + c^2 + c^3 + \ldots),$$

and if we limit ourselves to the first terms in the infinite series, regrouping them together opportunely we have the following expansion:

$$\begin{aligned} 1 + (a + b + c) + (a^2 + b^2 + c^2 + bc + ac + bc) + \\ +(a^3 + b^3 + c^3 + a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + abc) + \cdots . \end{aligned} \tag{6.1}$$

We recognize that at each degree in the monomials we have the k-th symmetric product of the three basic factors a, b and c that we call generators. We could also refine this method taking as generators at, bt and ct where we just use t to count the degree of the symmetrization. In this case we say t is the fugacity which counts the order of symmetrization. In fact, the formal expansion in this case would be

$$1 + (a + b + c)t + (a^2 + b^2 + c^2 + bc + ac + bc)t^2 + \cdots .$$

We could prove that once we have the character of a representation, the characters for the k-th symmetrized representation is the representation that has, for characters, the k-th symmetrization of the starting one. Let us make an example and evaluate the 2-nd rank symmetric product of $[1]_{SU(2)}$. We have

$$Sym^2([1]_{SU(2)}) = Sym^2(z + \frac{1}{z}) = z^2 + z \cdot \frac{1}{z} + \frac{1}{z^2} = z^2 + 1 + \frac{1}{z^2} = [2]_{SU(2)}.$$

Now we want to rewrite the generator of symmetrizations in a more compact way introducing a new function which we call plethystic exponential:

$$\begin{aligned} \prod_{i=1}^{n} \frac{1}{1 - x_i} &= e^{log \prod_{i=1}^{n} \frac{1}{1-x_i}} = \\ &= e^{-\sum_{i=1}^{n} log(1-x_i)} = \\ &= e^{\sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{x_i^k}{k}} = \\ &= e^{\sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{x_i^k}{k}} = \\ &= e^{\sum_{k=1}^{\infty} f(x_i^k) k}. \end{aligned} \tag{6.2}$$

This leads us straightly to the definition of the plethystic exponential. Given a function $f(t_1, ..., t_n) : \mathbb{R}^n \to \mathbb{R}$ such that $f(0, 0, \dots, 0) = 0$, we define the plethystic exponential of f to be the function

$$PE[f(t_1, ..., t_n)] = \exp\left(\frac{\sum_{k=1}^{\infty} f(t_1^k, ..., t_n^k)}{k} \right). \tag{6.3}$$

This is the function that generates the symmetrizations that we were searching for.

We also notice that the plethystic exponential enjoys the usual sum to product property of the ordinary exponential, which is

$$PE[f(t) + g(t)] = PE[f(t)]PE[g(t)],$$

for any couple of functions f and g which vanish at the origin.

As an example let us compute $PE[3t] = \dfrac{1}{(1-t)^3}$ and expand it in Maclaurin series.

$$PE[3t] = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \cdots . \tag{6.4}$$

We can see that the coefficient in front of t^k gives us the number of symmetrized objects at that degree.

Finally, here we graded the set of all monomials, via a grading given by the ordinary degree. However, the same procedure works for any grading of any ring. In particular we wish to grade the ring of chiral operators, with a grading given by physical properties of the operators themselves. This grading is given by the charges these operators carry with respect of physical symmetries, i.e. the operators quantum numbers such as the conformal dimension, the topological ANO charge, etc. .

6.2 Molien-Weyl projection

Irreducible representations of a compact Lie group G can be constructed starting from a highest weight space and applying negative roots to a highest weight vector. One crucial thing that this construction does not easily tell us is what the character of this irreducible representation will be. The character would tell us not just which weights occur in the representation, but with what multiplicities they occur (this multiplicity is one for the highest weight, but in general can be larger). The importance of knowing the characters of the irreducibles is that, given an arbitrary representation, we can then compute its decomposition into irreducibles.

As a vector space, the character ring $R(G)$ has a distinguished basis given by the characters $\chi_i(g)$ of the irreducible representations. Recall that the orthogonality relations for characters are essentially the same as in the finite group case, with the sum over group elements replace by an integral

$$\int \overline{\chi_i(g)}\chi_j(g)dg = \delta^{ij}, \tag{6.5}$$

where i and j are labels for irreducible representations and dg is the standard Haar measure, normalized so that the volume of G is 1. For an arbitrary representation V, once we know its character χ we can compute the multiplicities n_i of the irreducibles in the decomposition

$$V = \bigoplus n_i V_i,$$

as

$$n_i = \int \overline{\chi_i(g)}\chi(g)dg.$$

Another way of thinking of this is that as a vector space $R(G)$ has an inner product $< \cdot, \cdot >_G$ and the characters of the representations form an orthonormal basis.

When dealing with a set of polynomials of formal series expansions, on which a compact lie group acts, the Haar measure dg can be written in terms of an integration over fugacties. We avoid a complete derivation of this technical result, but the interested reader can refer to [22]. We compute now the Haar measure for $U(1)$ and for $SU(2)$ because we will use them in the following to explicitly compute the Hilbert Series for the Higgs branch of theories with gauge group $U(1)$ and $SU(2)$:

$$\begin{aligned} \int d\mu_{U(1)} &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}, \\ \int d\mu_{SU(2)} &= \frac{1}{2\pi i}\oint_{|z|=1} dz\left(\frac{1-z^2}{z}\right). \end{aligned} \tag{6.6}$$

The variable z appearing on this expression of the Haar measure is the fugacity associated to the gauge symmetry.

The idea is to integrate the Hilbert series, which can be interpreted as a character ring, over the whole gauge group. In fact the character for the trivial representation in this ring is just $[0] = 1$ and consequently $\overline{\chi_i(g)} = 1$ in equation 6.5. Suppose G is a group, for every element $g \in G$ there exists the inverse element $g^{-1} \in G$. If we suppose that G acts nontrivially on an addendum $a(t,z)$ of the Hilbert series $HS(t,z)$, when $g \in G$ acts on $a(t,z)$ we find $a(t,gz)$. In the same way, when $g^{-1} \in G$ acts on $a(t,z)$, we find $a(t,g^{-1}z)$.

So, summing over all possible $g \in G$, or integrating over the continuous set of $g \in G$, gives zero since the contribution of the action of g will cancel out with the contribution of g^{-1}:

$$\int d\mu_G \; a(t, gz) = 0. \tag{6.7}$$

Suppose now, on the other hand, that G acts trivially on another addendum $b(t,z)$ of the Hilbert Series. Now, all $g \in G$ are represented by the identity 1 in the group. Therefore when $g \in G$ acts on $b(t,z)$ we find $b(t, gz) = b(t,z) = b(t, g^{-1}z)$. Therefore, integrating over the gauge group now gives

$$\int d\mu_G \; b(t, gz) = Vol(G) \cdot b(t,z), \tag{6.8}$$

where $Vol(G)$ is the volume of the gauge group and we can define the Haar measure in order that it is equal to one. In this case, integrating the Hilbert Series over the whole gauge group sets to zero all the terms which contain factors transforming nontrivially under G and leaves only the terms which count singlets of the gauge group. In our physical case these singlets are precisely the gauge invariant operators we want to count.

6.3 Two examples of Higgs branches

In this section we want to focus on two special types of theories whose mirror theories are already well known in literature. They are $U(1)$ with n flavours and $SU(2)$ with n flavours. The dual theories are based on the ADE quivers and we will talk about them in Chapter 7.

Now we just want to focus on the Higgs branches of the theories afore mentioned, and we'll notice in passing that they correspond to the reduced moduli space of one G instanton (G is the group corresponding through the McKay correspondence to the ADE quiver). In Chapter 7 we will compare the result of this section with those of the Coulomb branches of the dual theories in ordered to find the original examples of mirror symmetry outlined already in [32].

6.3.1 $U(1)$ with n flavours

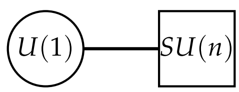


Figure 6.1: Quiver for $U(1)$ with n flavours in its $\mathcal{N} = 2$ form. The $U(1)$ component of the $U(n)$ flavour symmetry can be absorbed in the $U(1)$ gauge group leaving just a $SU(n)$ flavour symmetry.

It's useful to open this quiver, writing it in its $\mathcal{N} = 1$ form, since we want to see explicitly which are the fields involved and fix the notation.

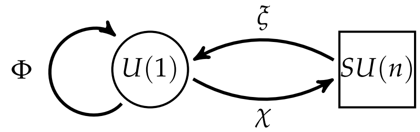


Figure 6.2: Quiver for $U(1)$ with n flavours in its $\mathcal{N} = 1$ form.

The gauge and global symmetries acting on the Higgs branch are summed up in the following table.

		$U(n)_f$	
	$U(1)$	$SU(n)_f$	$U(1)_f$
	z	$x_1, \ldots, x_{n-1}$	q
Φ	0	$[0, \ldots, 0]$	0
ξ^i	1	$[0, \ldots, 1]$	-1
χ_i	-1	$[1, \ldots, 0]$	1

Table 6.1: Representations assigment for the different matter fields in the $U(1)$ theory with n flavours. The label $[n_1, ..., n_r]$ always represents the Dynkin label highest weight.

The F-terms are obtained taking the derivative of the superpotential with respect to the supermultiples and putting it equal to zero. The superpotential for our theory is $\mathcal{W} = \chi_i \Phi \xi^i$ where fundamental and antifundamental indices are properly contracted. We have

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial \chi_i} &= \Phi \xi^i = 0, \\ \frac{\partial \mathcal{W}}{\partial \xi^i} &= \chi_i \Phi = 0, \\ \frac{\partial \mathcal{W}}{\partial \Phi} &= \chi_i \xi^i = 0. \end{aligned} \tag{6.9}$$

We notice that there are two possibilities:

- $\chi_i = 0$, $\xi^i = 0$ and $\langle \Phi \rangle$ can assume any value. This corresponds to the Coulomb branch, where the expectation value of the scalars in the vector multiplet take non-zero value. In this case the gauge group is unbroken, so the photon is massless and the electric potential of a point charge is $1/r$ at large r. For this reason, this branch of moduli space is called the "Coulomb branch".
- $\Phi = 0$ and $\chi_i \xi^i = 0$ so that ξ and χ don't have to separately vanish but they can assume non-zero value as long as their contraction is zero. This corresponds to the Higgs branch where scalars in the hypermultiplets take non-zero expectation values and the gauge group is completely broken.

As explained before, one cannot calculate the Coulomb branch in this manner since it receives numerous quantum corrections, which means that the superpotential would need loop renormalisation and instanton corrections. On the other hand the Higgs Branch is not renormalised, hence we get no contribution from quantum mechanics, the classical result is valid. Here we focus only on the Higgs branch, leaving the computation for the Coulomb branch for later.

The space of solutions $\Phi = 0$, with $\chi_i \xi^i = 0$ is called the F-flat space, or sometimes the Master Space and it is usually denoted by $\mathcal{F}^\flat$.

$$\mathcal{F}^\flat = \{\Phi = 0, \langle \chi_i \rangle = any, \langle \xi^i \rangle = any | \chi_i \xi^i = 0\}. \tag{6.10}$$

Now, we want to compute the generating function encoding generators and relations for the $\mathcal{F}^\flat$ space, namely its Hilbert series. We will call this $g_{1,n}^{\mathcal{F}^\flat}$. To incorporate all the information in the Hilbert series we proceed noticing that

1. we can fix the degree of the generators ξ and χ to be t,
2. the degree of the relation $\chi_i \xi^i = 0$ appears at the order t^2.

This, plus the transformation rules for the matter fields, gives

$$g_{1,n}^{\mathcal{F}^\flat} = \left(1 - t^2\right) PE \left([1, 0, \ldots, 0] wt + [0, 0, \ldots, 1] \frac{t}{w}\right), \tag{6.11}$$

where

- $w = q/z$ is a redefined fugacity, to absorb the $U(1)$ factor of the $U(n)$ global symmetry, with fugacity q, into the gauge local $U(1)$ with fugacity z.
- $PE\left([1, \ldots, 0] wt\right)$ counts the symmetric products of χ and $PE\left([0, \ldots, 1] t/w\right)$ the symmetric products of ξ.
- The $(1 - t^2)$ prefactor accounts for the relation occurring at degree two in the generators.

Recall that the characters of $SU(n)$ are given by

$$[1, \ldots, 0] = x_1 + \frac{1}{x_{n-1}} + \sum_{k=2}^{n-1} \frac{x_k}{x_{k-1}}, \tag{6.12a}$$

$$[0, \ldots, 1] = \frac{1}{x_1} + x_{n-1} + \sum_{k=2}^{n-1} \frac{x_{k-1}}{x_k}, \tag{6.12b}$$

and therefore, using the properties of the plethystic exponential, we can write

$$\begin{aligned} & g_{1,n}^{\mathcal{F}^\flat}(t, x_1, \cdots, x_{n-1}, w) = \\ & = \frac{(1 - t^2)}{(1 - x_1 wt)\left(1 - \frac{1}{x_{n-1}} wt\right)} \prod_{k=2}^{n-1} \frac{1}{1 - \frac{x_k}{x_{k-1}} wt} \times \\ & \times \frac{1}{\left(1 - \frac{1}{x_1} \frac{t}{w}\right)\left(1 - x_{n-1} \frac{t}{w}\right)} \prod_{k=2}^{n-1} \frac{1}{1 - \frac{x_{k-1}}{x_k} \frac{t}{w}}. \end{aligned} \tag{6.13}$$

Now we would like to perform the Molien-Weyl projection, in order to find the Hilbert Series which just counts the gauge invariant operators and not all of them. To do that we perform an integration over the $U(1)$ gauge group. The Haar measure for $U(1)$ is

$$HS_{SU(n)}(t, x_1, \cdots, x_{n-1}) = \frac{1}{2\pi i} \int_{|w|=1} \frac{dw}{w} g_{1,n}^{\mathcal{F}^\flat}(t, x_1, \cdots, x_n, w), \tag{6.14}$$

where we restrict to the unit circle, since the radius of convergence for t is 1 and therefore only the poles within such circle must be considered. They are given by

$$w_1 = \frac{t}{x_1}, \quad w_2 = \frac{x_1}{x_2}t, \quad \cdots \quad w_{n-1} = \frac{x_{n-2}}{x_{n-1}}t, \quad w_n = tx_{n-1}. \tag{6.15}$$

After performing the contour integration, one finds

$$HS_{SU(n)}(t, x_1, \cdots, x_{n-1}) = \sum_{p=0}^{\infty} [p, 0, \cdots, 0, p]_{SU(n)} t^{2p}. \tag{6.16}$$

The dimensions of the $[p, 0, \ldots, 0, p]$ representations are theoretically found by setting the characters $x_i = 1$. From a practical point of view we can use the Weyl dimension formula to get the result. In any case, the unrefined Hilbert series which counts the gauge invariant operators at a given degree, is given by

$$HS_{SU(n)}(t) = \frac{\sum_{p=0}^{n-1} \binom{n-1}{p}^2 t^{2p}}{(1-t^2)^{2(n-1)}}. \tag{6.17}$$

From this we can infer the dimensionality of the Higgs Branch, since the pole at $t = 1$ is of order $2(n-1)$ according to equation 5.9. Therefore

$$\dim_{\mathbb{C}} \mathcal{M}_H = 2\,(n-1)\,. \tag{6.18}$$

This perfectly agrees with the general formula 5.9 for the quaternionic dimension of the Higgs branch in a 3d $\mathcal{N} = 4$ theory.

6.3.2 $SU(2)$ with n flavours

Consider now the quiver gauge theory summarized in the following graph.

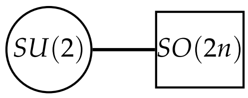


Figure 6.3: Quiver for $SU(2)$ with n flavours in its $\mathcal{N} = 2$ form.

Like we did also in the previous case we want to rewrite this $\mathcal{N} = 2$ quiver diagram in the $\mathcal{N} = 1$ notation in order to establish a notation for the fields and transformation laws.

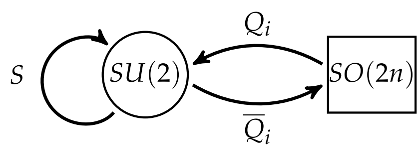


Figure 6.4: Quiver for $SU(2)$ with n flavours in its $\mathcal{N} = 1$ form.

In this case the superpotential is given by

$$\mathcal{W} = Q \cdot S \cdot Q = Q_a^i \epsilon^{ab} S_{bc} \epsilon^{cd} Q_d^i, \tag{6.19}$$

The F-terms are obtained by taking derivatives with respect to the scalars in the multiplets. The Higgs branch occurs when the scalars coming from the $\mathcal{N} = 4$ vector multiplet vanish while the ones from the hypermultiplets take non-zero VEVs.

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial Q_f^i} &= 2\epsilon^{fb}\epsilon^{cd} S_{bc} Q_d^i, \\ \frac{\partial \mathcal{W}}{\partial S_{bc}} &= \epsilon^{ab}\epsilon^{cd} Q_a^i Q_d^i = Q_b^i Q_c^i + Q_c^i Q_b^i, \end{aligned} \tag{6.20}$$

Then $\mathcal{F}^\flat_{1,SO(2n)} = \{S = 0, \langle Q_b^i \rangle \neq 0 | Q_b^i Q_c^i + Q_c^i Q_b^i = 0\}$, and the condition on the Q_b^i implies that the second symmmetric product of two of them has to vanish. This relation is of order squared in the fields, and trasforms as the $[2]_{SU(2)}$. This character will appear as the prefactor of the Plethystic exponential.

$$\begin{aligned} g^{\mathcal{F}^\flat}_{1,SO(2n)}(t, x_1, \ldots x_n, z) = \left(1 - z^2 t^2\right)\left(1 - t^2\right)\left(1 - z^{-2} t^2\right) \times \\ \times PE\left[[1, 0, \ldots, 0]_{SO(2n)}\left(z + z^{-1}\right)\right]. \end{aligned} \tag{6.21}$$

The charatcters for the fundamental representation of $SO(2n)$ is:

$$[1, \ldots, 0]_{SO(2n)}(x_a) = \sum_{a=1}^{n}\left(x_a + x_a^{-1}\right). \tag{6.22}$$

Using this, we can rewrite the plethystic exponential as follows

$$\begin{aligned} g^{\mathcal{F}^\flat}_{SO(2n)}(t, x_1, \ldots x_n, z) = \left(1 - z^2 t^2\right)\left(1 - t^2\right)\left(1 - z^{-2} t^2\right) \times \\ \times \prod_{a=1}^{n}\left(\frac{1}{1 - z x_a t}\frac{1}{1 - z^{-1} x_a t}\frac{1}{1 - z x_a^{-1} t}\frac{1}{1 - z^{-1} x_a^{-1} t}\right). \end{aligned} \tag{6.23}$$

This Hilbert series counts all the operators, disregarding the fact that they are gauge invariant or not.

In order to retain only the gauge invariant ones, we perform the Molien-Weyl projection and obtain the Higgs branch Hilbert series.

$$\begin{aligned} HS(t, x_1, \cdots, x_n) &= \int d\mu_{SU(2)} g^{\mathcal{F}^\flat}_{SO(2n)}(t, x_1, \cdots x_n, z) = \\ &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1 - z)\left(1 - \frac{1}{z}\right) g^{\mathcal{F}^\flat}_{SO(2n)}(t, x_1, \cdots x_n, z) = \\ &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1 - z)\left(1 - \frac{1}{z}\right)\left(1 - z^2 t^2\right)\left(1 - t^2\right)\left(1 - z^{-2} t^2\right) \times \\ &\times \frac{1}{(1 - t)^\delta}\prod_{a=1}^{n}\left(\frac{1}{1 - z x_a t}\frac{1}{1 - z^{-1} x_a t}\frac{1}{1 - z x_a^{-1} t}\frac{1}{1 - z^{-1} x_a^{-1} t}\right) = \\ &= \sum_{p=0}^{\infty}[0, p, 0, \cdots, 0]_{SO(2n)} t^{2p}. \end{aligned} \tag{6.24}$$

Chapter 7

The Coulomb branch of a 3d $\mathcal{N} = 4$ quiver gauge theories

We are interested here in the Coulomb branch of the moduli space of three dimensional $\mathcal{N} = 4$ superconformal field theories, for this branch, unlike the Higgs one we have quantum corrections to take into account.

The traditional way to describe it, is giving the vacuum expectation value for the three scalars in the $\mathcal{N} = 4$ vector multiplets. For a generic VEV, the gauge group G is completely broken down to $U(1)^r$ and in this way all matter fields and W-bosons are massive. The low energy dynamics on such a point of the Coulomb branch is described by an effective field theory of r abelian vector multiplets, which can be dualized into twisted hypermultiplets by the dualization of the photons we saw before.

The hyperkähler metric on the Coulomb branch can be computed semiclassically by integrating out the massive hypermultiplets and W-boson vector multiplets at one loop. This 1-loop description is only reliable in a weakly coupled regions where the fields that have been integrated out are very massive, well above the renormalization cutoff scale Λ which dictates the range of validity of the effective field theory. In particular, nowadays it is not known how to dualize a non-abelian vector multiplet.

Our strategy consists in avoiding the dualization of free abelian vector multiplets by considering 't Hooft monopole operators as outlined in [21]. In particular, we will count the number of bare and dressed monopole operators, grading them with their conformal dimension Δ, and the topological symmetry $Z(\hat{G})$ in case this in non-trivial.

7.1 Monopole Operators

It can be proved that the relevant chiral operators necessary to describe the Coulomb branch of the moduli space are monopole operators. Therefore, in this chapter we define a certain class of local operators in three dimensional conformal field theories that are not polynomial in the fundamental fields and create topological disorder. The importance of those operators relies on the fact that they are "soliton creating operators" [14], [45], for which they carry nonvanishing charge under the topological symmetry.

In this part of the exposition we follow closely [5], [10]. In the abelian case we can consider the easy case of QED in three dimensions. This theory is not supersymmetric and the gauge group is abelian. In three dimensions we have a Weyl fermion in the irrep $[1]_{SO(3)}$ which means we have two-components complex spinors. Fermions appear in N_f flavours and we take the euclidean space action given by:

$$S_{QED} = \int d^3x \left(\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \psi_l^\dagger (i\sigma_j D_j) \psi_l \right),$$

where A is the $U(1)$ gauge field, $F = dA$ is its field strength and ψ_j are the two component complex Weyl fermions with j that runs from 1 to the number of flavours N_f. Since the gauge coupling e has mass dimension $\frac{1}{2}$, the theory is super-renormalizable and free in the UV while is strongly coupled in the infrared. The theory posses an interesting conserved current, the dual of the field strength

$$J^\mu = \frac{1}{4\pi^2} \epsilon^{\mu\nu\rho} F_{\nu\rho}.$$

We look for operators possessing a non vanishing charge under such a topological symmetry $U(1)_J$. In the Higgs phase they are called vortex operators (ANO operators) and they create Abrikosov-Nielsen-Olesen vortices when acting on a state as it is shown in [1].

As it was shown in the series of papers [11] and [17] an operator that carries a nontrivial charge under the topological symmetry group can be defined by requiring that the gauge fields have a singularity at the insertion point in the path integral, and such singularity should be that of a Dirac monopole field

$$A^{N,S}(r) = \frac{m}{2}(1 - cos(\theta)) d\phi.$$

The two opposite signs correspond to two different charts covering the two hemispheres of $\mathbb{S}^2$ that surround the insertion point. The magnetic charge m is subject to the usual Dirac quantization condition. In order for the gauge fields to have such singularities, one must also define some operators V_m which must be inserted exactly at the gauge field singularity, acting therefore as some "monopole creating operators".

Let us now restrict ourselves to the case in which the theory has followed the renormalization group flow all the way down to the infrared fixed point. Here, all the gauge couplings have run to infinity, and the theory is superconformal. In a generic CFT there exists a concept called Operator-State Correspondence. This is a map that associates to each local operator, a state in the Fock space of the theory. This works because a conformal theory on $\mathbb{R}^3$ can be radially quantized, i.e. written as a theory on $\mathbb{R} \times \mathbb{S}^2$. Thus monopole operators of the original theory carrying GNO charge m are in one-to-one mapping to states on the radially quantized theory with flux m through the sphere. We call these states t'Hooft monopoles.

In the following we will prove that the flux of a 't Hooft monopole is quantized by a generalization of the usual Dirac quantization condition. Then, we will exploit again the radial quantization and the operator-state correspondence, in order to compute the conformal dimension of these operators, which is equal to the energy of the corresponding state in the radially quantized theory. We now prove that a 't Hooft-Polyakov monopole's magnetic charge satisfies a quantization condition which forces it to belong to the weight lattice of the dual group of the gauge group.

Consider the principal bundle $\mathcal{P}(\mathbb{R}^3 - \{0\}, G)$. As a topological space, the base space of this bundle has the same homotopy type of the base space of the bundle $\mathcal{P}(\mathbb{S}^2, G)$.

Thanks to the theorem of equivalence of homotopic bundles the two bundles define physically equivalent gauge theories, and we should then consider $\mathcal{P}(\mathbb{S}^2, G)$. An open cover of charts for the manifold $\mathbb{S}^2$ can be taken as the couple of open sets:

$$\begin{aligned} U_N &= \{\phi \in [0, 2\pi], \theta \in [0, \frac{\pi}{2} + \epsilon]\}, \\ U_S &= \{\phi \in [0, 2\pi], \theta \in [\frac{\pi}{2} - \epsilon, \pi]\}. \end{aligned} \tag{7.1}$$

The gauge connection is given, in the two patches, by

$$\begin{aligned} A_N &\simeq \frac{m}{2}\,(1 - \cos\theta)\,d\phi, \\ A_S &\simeq \frac{m}{2}\,(-1 - \cos\theta)\,d\phi. \end{aligned} \tag{7.2}$$

Where A_N is defined in patch covering the upper hemisphere surrounding the origin, and A_S is defined in the lower one. Here m belongs to the the Lie Algebra $\mathfrak{g}$ of G. The transition function between the two patches of the bundle is given by

$$\begin{aligned} &t_{NS} : U_N \cap U_S \to G, \\ &\phi \mapsto \exp(i\Phi(\phi)). \end{aligned} \tag{7.3}$$

The two fields A_N and A_S are related by a Yang-Mills gauge transformation:

$$A_N = t_{NS}^{-1} A_S t_{NS} - i t_{NS}^{-1} d t_{NS}. \tag{7.4}$$

Computing the exterior derivative of the transition function gives

$$dt_{NS} = i t_{NS} d\Phi. \tag{7.5}$$

Putting the last two equations together implies

$$d\Phi = A_N - t_{NS}^{-1} A_S t_{NS}. \tag{7.6}$$

Integrating this one finds

$$\Phi = \int_0^{2\pi} d\varphi A_N - t_{NS}^{-1} A_S t_{NS} = 2\pi m. \tag{7.7}$$

Therefore, by requiring the transition function between the patches to be smooth and single-valued, one finds the Dirac quantization condition:

$$\exp(2\pi i m) = 1_G. \tag{7.8}$$

This condition requires m to belong to the weight lattice of $\hat{G}$, the Langland dual of the group G [20].

7.2 Hilbert series for a 3d $\mathcal{N} = 4$ theory

The monopole operators are the chiral operators relevant for the description of the Coulomb branch of the moduli space of vacua. An important assumption on which relies our good counting is that for a given GNO charge m there will be a unique monopole operator. Till now there is no proof of this fact. However, there are strong indications that this is indeed correct since all the computations for the Coulomb branches we have made agree perfectly with the perturbative computations known in literature.

First of all, let us resume some fundamental facts. The GNO magnetic charges of both bare and dressed monopole operators belong to the weight lattice of $\hat{G}$, the GNO dual of the gauge group G. In addition to this, we are looking at invariant monopole operators so we must consider the fact that the Weyl group $\mathcal{W}_{\hat{G}}$ acts on m. It takes m in a Weyl chamber to another Weyl chamber. In order to avoid an annoying overcounting we restrict m to take values in a single Weyl chamber of the wight lattice $\Gamma^*_{\hat{G}}$ [20].

Earlier, we mentioned the GNO dual of a group, actually it's enough to introduce the concept of dual root system and say that m belongs to the Weyl chamber of the dual weight lattice of G. The dual weight lattice is certainly well defined if we have a notion for the dual root system of a generic Lie group. In fact, if we know a root system it's easy to build its associated weight lattice. This takes us to give the definition of a dual root system. If Φ is a root system in V, the coroot $\alpha^\vee$ of a root α is defined by

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)}\, \alpha.$$

The set of coroots also forms a root system $\Phi^\vee$ in V, called the dual root system (or sometimes inverse root system). By definition, $(\alpha^\vee)^\vee = \alpha$, so that Φ is the dual root system of $\Phi^\vee$. The lattice in V spanned by $\Phi^\vee$ is also called the coroot lattice. Both Φ and $\Phi^\vee$ have the same Weyl group W and, for s in W,

$$(s\alpha)^\vee = s(\alpha^\vee).$$

Finally if Δ is a set of simple roots for Φ, then $\Delta^\vee$ is a set of simple roots for $\Phi^\vee$.

Now we are ready to introduce the formula which counts the conformal dimension for monopole operators of GNO charge m, in the infrared CFT. This formula was firstly conjectured by Gaiotto and Witten in [23] and then it was proved in [10] and [7]. It reads

$$\Delta(m) = -\sum_{\alpha \in \Delta^+} |\alpha(m)| + \frac{1}{2} \sum_i \sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)|. \tag{7.9}$$

The first sum is over the set of all positive roots $\alpha \in \Delta^+$ and represents the contribution coming from the $\mathcal{N} = 4$ vector multiplets. The second term is instead the contribution coming from the matter hypermultiplets. The sum is over the weights of the matter representations.

Gaiotto and Witten [23] also proposed a classification of 3d $\mathcal{N} = 4$ theories. A theory is termed: good if all monopole operators have $\Delta > \frac{1}{2}$; ugly if there all monopole operators have $\Delta \geq \frac{1}{2}$, but some of them saturate the unitarity bound $\Delta = \frac{1}{2}$; bad if there exist monopole operators with $\Delta < \frac{1}{2}$, violating the unitarity bound. In the bad case Δ is not the conformal dimension of the infrared CFT. In the ugly case the monopole operators saturating the unitarity bound are free decoupled fields. In the following we focus on good or ugly theories and leave the reader, for a treatment of bad theories, to [4], [47].

In the paper [16], the authors propose their general formula for the Hilbert series of the Coulomb branch of a $3d$ $\mathcal{N}=4$ good or ugly theory, which enumerates gauge invariant operators modulo F-terms:

$$HS_G(t) = \sum_{m \in \Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} t^{\Delta(m)} P_G(t,m). \tag{7.10}$$

The physical interpretation of this general formula is simple. The Coulomb branch of the moduli space is parametrized by bare and dressed gauge invariant monopole operators which are $\mathcal{N}=2$ chiral multiplets. We therefore enumerate these monopole operators, grading them by their quantum numbers under the global symmetry group, which consists of topological symmetries and dilatation (or superconformal R-symmetry). The sum is over all GNO monopole sectors, belonging to a Weyl chamber of the weight lattice $\Gamma^*_{\hat{G}}$ of the GNO dual group to the gauge group G. $t^{\Delta(m)}$ counts bare monopole operators according to their conformal dimension 7.9, which depends on the gauge group and matter content of the gauge theory. Finally $P_G(t,m)$ is a classical factor which counts the gauge invariants of the residual gauge group H_m, which is unbroken by the GNO magnetic flux m, according to their dimension. This classical factor accounts for the dressing of the bare monopole operator by the complex scalar $\phi \in \mathfrak{h}_m$, the lowest component of the chiral multiplet inside the $\mathcal{N}=4$ vector multiplet.. The classical factor is expressed as

$$P_G(t,m) = \prod_{i=1}^{r} \frac{1}{1-t^{d_i(m)}}, \tag{7.11}$$

where $d_i(m)$, $i=1,\ldots,r$ are the degrees of the Casimir invariants of the residual gauge group H_m left unbroken by the GNO magnetic flux m.

Let us make an example to clarify what the residual gauge group is, and choose $G=U(n)$. Explicitly, we associate to the magnetic flux $\vec{m}$ a partition of N $\lambda(\vec{m}) = (\lambda_j(\vec{m}))_{j=1}^N$, with $\sum_j \lambda_j(\vec{m}) = N$ and $\lambda_i(\vec{m}) \geq \lambda_{i+1}(\vec{m})$, which encodes how many of the fluxes m_i are equal. The residual gauge group which commutes with the monopole flux is $\prod_{i=1}^N U(\lambda_i(\vec{m}))$. The classical factor is given by 7.11 and reads:

$$P_{U(N)}(t;\vec{m}) = \prod_{j=1}^{N} Z^U_{\lambda_j(\vec{m})},$$

where

$$Z^U_k = \prod_{i=1}^{k} \frac{1}{1-t^i}, \quad k \geq 1, \tag{7.12}$$

$$Z^U_0 = 1. \tag{7.13}$$

Monopole operators also may or may not be charged under the topological symmetry group, the center of the GNO dual group $Z(\hat{G}) = \Gamma^*_{\hat{G}}/\Lambda_r(\hat{\mathfrak{g}})$, which is a quotient of the weight lattice $\Gamma^*_{\hat{G}}$ of $\hat{G}$ by the root lattice $\Lambda_r(\hat{\mathfrak{g}})$ of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ (or the coroot lattice of $\mathfrak{g}$). We refer again to [20] for an excellent and more detailed explanation. So, if the gauge group G is not simply connected there is a nontrivial topological symmetry $Z(\hat{G})$ under which monopole operators may be charged. Let z be a fugacity valued in the topological symmetry group and $J(m)$ the topological charge of a monopole operator of GNO charge m. The Hilbert series of the Coulomb branch 7.10 can be refined to

$$HS_G(t) = \sum_{m \in \Gamma^*_{\hat{G}} / \mathcal{W}_{\hat{G}}} z^{J(m)} t^{\Delta(m)} P_G(t, m). \tag{7.14}$$

7.3 Simple orbifolds and Hilbert series

Let us start by constructing the simplest two dimensional complex orbifold, we will see later how this is related to the moduli space of vacua.

The way to do it, is to take the complex plane $\mathbb{C}^2$ and to quotient it by the central symmetry of the origin. What we obtain is by definition an orbifold and we write it like $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. To identify the algebraic description of this quotient orbifold one takes z_1 and z_2 to be coordinates of $\mathbb{C}^2$ and the action of the parity group is taken to be

$$(z_1, z_2) \to (-z_1, -z_2).$$

One can identify all the monomial functions of z_1 and z_2 that are invariant under the action of parity. In this case these are very easy to find:

$$f(z_1, z_2) = z_1^i z_2^j, \tag{7.15}$$

$$i - j = 0 \ mod\ 2. \tag{7.16}$$

The three $\mathbb{Z}_2$ invariant monomials of degree two are:

$$\begin{aligned} X &= z_1^2, \\ Y &= z_2^2, \\ Z &= z_1 z_2. \end{aligned} \tag{7.17}$$

X, Y, Z are three complex variables which we use to describe the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ algebraically. They are called generators of the (infinitely many) $\mathbb{Z}_2$ invariant polynomials and they are related to each other by the constraint:

$$XY = Z^2. \tag{7.18}$$

This is just an algebraic curve in $\mathbb{C}^3$ and it describes exactly the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. If one thinks of the variables X, Y, Z as real coordinates, 7.18 is just the equation of a cone. The orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ is often called a "complex cone" and the singularity of this space is referred to as a conical singularity.

Consider therefore the simplest case of a moduli space $\mathbb{C}^2$ with variables z_1 and z_2. These two complex variables can be thought of as just the VEV of two complex scalars and in this sense they parametrize the vacua of the theory. In this case there is no group of symmetry to quotient over, the Hilbert series just counts monomials of degree k in the two variables. Choosing fugacities t_1, t_2, the Hilbert series is written:

$$
\begin{aligned}
HS(t_1, t_2; \mathbb{C}^2) &= \sum_{i,j=0}^{\infty} t_1^i t_2^j = \\
&= 1 + t_1^2 + t_2^2 + t_1 t_2 + ... \\
&= \frac{1}{1-t_1} \frac{1}{1-t_2} \\
&= PE[t_1 + t_2].
\end{aligned}
\tag{7.19}
$$

A natural $U(2)$ symmetry acts on $\mathbb{C}^2$. This gives us the idea to for a so called fugacity map:

$$
t_1 = tx, \; t_2 = \frac{t}{x}, \tag{7.20}
$$

then

$$
\begin{aligned}
HS(t_1, t_2; \mathbb{C}^2) &= PE\left[t(x + \frac{1}{x})\right] \\
&= PE\left[\chi([1]_{SU(2)})t\right] \\
&= \sum_{k=0}^{\infty} \chi([k]_{SU(2)}) t^k.
\end{aligned}
\tag{7.21}
$$

We see that the Hilbert series for $\mathbb{C}^2$ corresponds to the function that generates k^{th} rank symmetric products of the fundamental representation of $SU(2)$. An usual trick consists in unrefining the Hilbert series, this simply means we put $t_i = t$ in 7.19:

$$
\begin{aligned}
HS(t; \mathbb{C}^2) &= \frac{1}{(1-t)^2} \\
&= \sum_{k=0}^{\infty} \binom{2+k-1}{k} t^k \\
&= \sum_{k=0}^{\infty} dim\left([k]_{SU(2)}\right) t^k.
\end{aligned}
\tag{7.22}
$$

7.22 gives us information on the moduli space. In fact, the order of the pole is the dimension of the underlying moduli space and the dimension of $[k]_{SU(2)}$ tells us how many different monomials are at that degree. So, in the trivial example of the variety $\mathbb{C}^2$ the pole is of order two in accordance with the dimension of the space itself.

Let us go back to the case of $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. As in the previous case, we can still arrange the coordinates in a doublet [1] of $SU(2)$, but now we must consider the quotientation for $\mathbb{Z}_2$. z_1 and z_2 are not invariant under this symmetry so [1] cannot be a generator for all $\mathbb{Z}_2$ invariant polynomials. The first invariant polynomials are at order two and they are 2-nd rank symmetric product of z_1 and z_2. In group theoretic language this is $Sym^2[1] = [2]$ and it has dimension three. This irrep represents the three $\mathbb{Z}_2$ invariant monomials in 7.17. Notice that there are no monomials with odd numbers of weights of [1]. All higher order invariants will be symmetric products of these, e.g.

$$
\begin{aligned}
Sym^2[2] &= [4] + [0], \\
Sym^3[2] &= [6] + [2], \\
&\cdots .
\end{aligned}
\tag{7.23}
$$

These also contain information about the relations between generators. We know that the first of this relations is at order two in the product of generators and is expressed by 7.18. This is precisely the role of the singlet in the 2-nd rand symmetric product. If we take higher symmetric products we will find more invariants (the highest weight [2k] in the decomposition) and more relations (all the other representations in the decomposition). The Hilbert series is then easy to write as a summation:

$$\begin{aligned} HS(t_1,t_2;\mathbb{C}^2) &= \sum_{i,j=0;\, i-j=0\, mod\, 2}^{\infty} t_1^i t_2^j = \\ &= 1 + t_1^2 + t_2^2 + t_1 t_2 + \dots . \end{aligned} \tag{7.24}$$

It's now very easy to show that if we substitute $t_1 = tx$ and $t_2 = \frac{t}{x}$ then the previous Hilbert series can be rewritten in terms of characters of $SU(2)$ as:

$$HS(t^2, x;\mathbb{C}^2) = \sum_{k=0}^{\infty} \chi([2k]) t^{2k}, \tag{7.25}$$

where the fugacity t^2 rather than t reflects the degree of the generators. If we unrefine the series by setting the fugacity $x = 1$ it follows that:

$$\begin{aligned} HS(x, t^2; \frac{\mathbb{C}^2}{\mathbb{Z}_2}) &= \sum_{k=0}^{\infty} [2k] t^{2k} \\ &= \frac{1-t^4}{(1-t^2)^3} \\ &= (1-t^4) PE\left[3t^2\right] \\ &= (1-t^4) PE\left[dim(\chi([2]))t^2\right]. \end{aligned} \tag{7.26}$$

Since the dimension matches we refine the series and write:

$$\begin{aligned} HS(t^2; \frac{\mathbb{C}^2}{\mathbb{Z}_2}) &= \sum_{k=0}^{\infty} (2k+1) t^{2k} \\ &= \frac{1-t^4}{(1-t^2)^3} \\ &= (1-t^4) PE\left[[2]t^2\right] \\ &= \frac{1-(t^2)^2}{(1-x^2t^2)(1-t^2)(1-\frac{t^2}{x^2})}, \end{aligned} \tag{7.27}$$

and this is the closed formula for the Hilbert series that represents $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$.

The three factors in the denominator signify that the ring of monomials, that are invariant under $\mathbb{Z}_2$, is generated by three monomials. The numerator encodes the relation between these generators: the power of t signifies that this relation is at order 4, or at order 2 in the generators. As we already said the pole of the Hilbert series gives the dimension of the space. 7.27 can be still simplified and if we do it we notice that there is a pole of order 2 which gives a moduli space of complex dimension 2, as expected from the parent space $\mathbb{C}^2$.

Notice that in this case the dimension of the space is $d = g - r$, where g is the number of generators and r is the number of relations. A moduli space whose dimension obeys such a rule is called complete intersection. Space that are complete intersections enjoy some nice properties: they have a finite number of relations between the three generators. The Hilbert series can be written simply and information about generators and their relations can be easily extracted. For a classification of complete intersection spaces and their associated Hilbert series one can refer to [31].

7.4 U(1) gauge group and n flavours

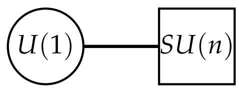


Figure 7.1: Quiver for $U(1)$ with n flavours.

The preceding diagram is the quiver for SQED (supersymmetric quantum electrodynamics) with n flavours. Now we want to find the Coulomb branch of the moduli space of vacua for this theory.

As we already said the dimension formula for the monopole operators is:

$$\Delta(m) = -\sum_{\alpha\in\Delta^+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)|. \tag{7.28}$$

In this case the gauge group is abelian which means that the set of positive roots is empty and the first sum is ruled out. This leaves us with

$$\Delta(m) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)|. \tag{7.29}$$

The sum is over n hypermultiplets in the fundamental representation of the gauge group and this means that:

$$\Delta(m) = \frac{1}{2}n|\rho(m)|. \tag{7.30}$$

The fundamental representation of $U(1)$ is labelled by q^1 and its weight is clearly 1. This gives $|\rho(m)| = |1\cdot m| = |m|$.

We obtain that $\Delta(m) = \frac{n}{2}|m|$ and the final result for our Hilbert series is:

$$HS_{U(1),n} = \frac{1}{1-t}\sum_{m\in\mathbb{Z}} z^m t^{\frac{n}{2}|m|}. \tag{7.31}$$

Evaluating the sum just requires some work but at the end we find that:

$$HS_{U(1),n} = \frac{1-t^n}{(1-t)(1-zt^{\frac{n}{2}})(1-z^{-1}t^{\frac{n}{2}})}. \tag{7.32}$$

We see from this Hilbert series that the Coulomb branch of $3d$ $\mathcal{N} = 4$ SQED with n electrons is a complete intersection generated by the complex scalar Φ of fugacity t, the monopole V_{+1} of magnetic flux $+1$ and fugacity $zt^{\frac{n}{2}}$ and the monopole V_{-1} of magnetic flux -1 and fugacity $z^{-1}t^{\frac{n}{2}}$. They are subject to a single relation $V_{-1}V_{+1} = \Phi^n$ at dimension n and topological charge 0.

Let us now look at what we found from a purely mathematical point of view. If we identify $a = \varphi$, $b = V_1$, $c = V_{-1}$ we can think of each product of these operators as a monomials in a, b, c. If we take sum and products of these we get a ring and a, b, c are called generators of the ring. Anyway, the ring is not freely generated, but there is a relation between the generators which must be taken into account.

The relation reads $a^n = bc$ and now algebraic geometry comes into play. In the appendix A I will give some useful notions which will be quite formal but can help to clarify the meaning of the formalism that is used throughout this thesis. In the following I will try to give the intuitive meaning to the notation introduced.

So, going back to our ring generated by three coordinates a, b, c with one relation, we have a powerful way to write the ring incorporating the identification provided by the relation. This way is taking the quotient

$$\frac{\mathbb{C}[a,b,c]}{\langle bc = a^n \rangle}. \tag{7.33}$$

At this stage, there is a general procedure to work out which affine variety corresponds to the coordinate ring given by 7.33, we refer the reader to [26] for further details but lots of details are also given in Appendix A.

The case we are discussing is quite simple and there is no need to use computational algebraic geometry to evaluate the affine variety. Let us start with the even simpler case $n = 2$.

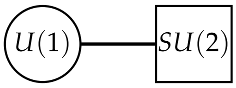


Figure 7.2: Quiver for $U(1)$ with 2 flavours.

In this case we can refer to the discussion we had for the orbibold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. If we specialize the Hilbert series 7.32 to the case $n = 2$ this assumes the same form of equation 7.27. If we replace in 7.32 t with t^2 and z with x^2 we get the same expression. The two replacements are just redefinitions of the fugacities so the two Hilbert series are indeed the same and they correspond to $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$.

A straightforward generalization of this simple case for n generic consists in proving that to the coordinate ring $\mathbb{C}[a,b,c]/\langle bc = a^n \rangle$ corresponds the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_n}$. This is actually easy to show, just take the action of $\mathbb{Z}_n$ on $\mathbb{C}^2$ to be

$$\begin{cases} z_1 \mapsto e^{i\frac{2\pi}{n}} z_1 \\ z_2 \mapsto e^{i\frac{2\pi}{n}} z_2 \end{cases} ,$$

where we just multiplied z_1 and z_2 by the n-th root of 1. We can form three monomials invariant under the action of the group which are the generators for the coordinate ring, they are $a = z_1 z_2$, $b = z_1^n$ and $c = z_2^n$. These are also subject to the relation $a^n = bc$ and we obtained just the coordinate ring $\mathbb{C}[a,b,c]/\langle bc = a^n \rangle$ as we wanted.

The easy case of SQED has outlined a nice correspondence between two ways of characterizing the moduli space of vacua either as a coordinate ring or an affine variety.

7.5 $U(1)$ gauge group with copies of charge $q_1, \ldots, q_n$

We want now to generalize the previous case in which we had n flavours, to the possibility of having copies with different charges.

Like in the previous case, in the formula for the conformal dimension, the first sum is ruled out and we are left with

$$\begin{aligned}
\Delta(m) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i \in \{q_i\}_{i=1,\ldots,n}} |\rho_i(m)| \\
&= \frac{1}{2}(n_{q_1}|q_1 \cdot m| + n_{q_2}|q_2 \cdot m| + \ldots + n_{q_n}|q_n \cdot m|) \\
&= \frac{1}{2}(n_{q_1}|q_1| + n_{q_2}|q_2| + \ldots + n_{q_n}|q_n|)|m| \equiv \frac{1}{2}\mathcal{N}|m|,
\end{aligned} \tag{7.34}$$

where n_{q_i} stands for the number of copies of hypermultiplets in the q_i rep of $U(1)$.
We already know which Hilbert series we have in this case:

$$HS_{U(1),q_1,\ldots,q_n} = \frac{1-t^{\mathcal{N}}}{(1-t)(1-zt^{\frac{\mathcal{N}}{2}})(1-z^{-1}t^{\frac{\mathcal{N}}{2}})}. \tag{7.35}$$

The orbifold corresponding to the previous Hibert series can be written like

$$\mathcal{M}_C = \frac{\mathbb{C}^2}{\mathbb{Z}_{\mathcal{N}}} = \frac{\mathbb{C}^2}{\mathbb{Z}_{\sum_j n_{q_j}|q_j|}}. \tag{7.36}$$

The theory that we have just considered cannot be represented by a quiver of the ones previously described. In the quivers we saw before, each line corresponds to two chiral multiplets transforming in some fundamental and antifundamental representation of the groups given by the nodes. In this case we are instead considering matter fields which transform in representations which are different from the fundamental or the antifundamental. We could for sure develop a new graphical formalism for such quivers in order to account for these new possibilities but this is not useful to our discussion and we won't do it.

7.6 $U(1) \times U(1)$ gauge group: the $\hat{A}_2$ quiver

This theory is described by the following quiver.

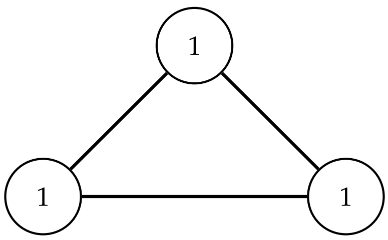


Figure 7.3: Quiver diagram for $U(1) \times U(1) \times U(1)/U(1)$. The resulting diagram is the same of the extended Dynkin diagram of the algebra A_2 which is indicated $\hat{A}_2$.

In the quiver drawn above there seem to be three $U(1)$ factors in the gauge group but this is not the case since there is an overall $U(1)$ factor which can be decoupled. Let us start from the three $U(1)$ factors and then we will see how the decoupling is obtained.

The magnetic fluxes are labeled by three integers (m_0, m_1, m_2), the monopole operators have conformal dimension dependent on these magnetic charges. The matter sector contributes to the conformal dimension in the following way:

$$\Delta(m_0, m_1, m_2) = \frac{1}{2}(|m_0 - m_1| + |m_1 - m_2| + |m_2 - m_0|). \tag{7.37}$$

The formula is invariant for $m_i \to m_i + a$ and this indicates that one of the $U(1)$ factors is decoupled. We can remove the decoupled $U(1)$ with a gauge fixing by setting the flux m_0 to zero, $m_0 = 0$. Therefore, the theory has two $U(1)$ topological symmetries corresponding to the non trivial $U(1)$ factors.

As always we introduce fugacities z_1 corresponding to the topological charge m_1 and z_2 corresponding to the charge m_2 and we write the Hilbert series that grades monopole operators according to their conformal dimensions and topological charges:

$$\begin{aligned} HS_{\hat{A}_2}(t, z_1, z_2) &= \frac{1}{(1-t)^2} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} z_1^{m_1} z_2^{m_2} t^{\frac{1}{2}(|m_1|+|m_1-m_2|+|m_2|)} \\ &= 1 + t\left(1 + 1 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) + \dots, \end{aligned} \tag{7.38}$$

and now we notice that there is an interesting redefintion of fugacities that could be done. Such redefiniton is technically called fugacity map and in our case is $z_1 = \frac{y_1^2}{y_2}$ and $z_2 = y_2^2 y_1$. If we substitute in the preceding we get:

$$\begin{aligned} HS_{\hat{A}_2}(t, y_1, y_2) &= 1 + \left(y_1 y_2 + \frac{y_2^2}{y_1} + \frac{y_1^2}{y_2} + \frac{y_1}{y_2^2} + \frac{y_2}{y_1^2} + \frac{1}{y_1 y_2} + 2\right) t + \dots \\ &= 1 + [1,1]_{SU(3)} t + \dots . \end{aligned} \tag{7.39}$$

From this we conjecture that the following holds:

$$HS_{\hat{A}_2}(t, y_1, y_2) = \sum_{k=0}^{\infty} [k,k]_{SU(3)} t^k. \tag{7.40}$$

This conjecture can be proven with perturbative methods but actually here we'll just show that it holds for the unrefined Hilbert series since dimensions match and we'll assume it is also an identity in term of characters, like it is. The unrefined Hilbert series in this case reads:

$$\begin{aligned} HS_{\hat{A}_2(t)} &= \frac{1}{(1-t)^2} \sum_{m_1=-\infty}^{+\infty} \sum_{m_1=-\infty}^{+\infty} t^{\frac{1}{2}(|m_1|+|m_1-m_2|+|m_2|)} \\ &= \frac{1+4t+t^2}{(1-t)^4} \\ &= \sum_{k=0}^{\infty} (k+1)^3 t^k. \end{aligned} \tag{7.41}$$

This proves our claim, in fact, $dim\left([k,k]_{SU(3)}\right) = (k+1)^3$.

The current theory fully displays an $SU(3)$ symmetry enhancement due to the quantum effects that we have in the Coulomb branch. In fact, it's a well known fact that the Coulomb branch of this theory is the reduced moduli space of one instanton of $SU(3)$.

We can already notice here that the Coulomb branch of this theory is exactly the same moduli space that we found for a theory with $U(1)$ gauge group and 3 flavours when we studied its Higgs branch. We just found a first example of mirror theories!

7.7 $U(1) \times U(1) \times U(1)$ gauge group: the $\hat{A}_3$ quiver

We add now another $U(1)$ factor and we study the theory obtained in such a way. Its quiver diagram is the following one.

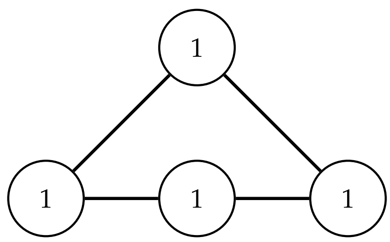


Figure 7.4: Quiver diagram for $U(1) \times U(1) \times U(1) \times U(1)/U(1)$. The resulting diagram is the same of the extended Dynkin diagram of the algebra A_3 which is indicated $\hat{A}_3$.

This time we immediately remove the decoupled $U(1)$ factor and we get the conformal dimension for the monopole operators

$$\Delta(m_1, m_2, m_3) = \frac{1}{2}(|m_1| + |m_1 - m_2| + |m_2 - m_3| + |m_3|). \tag{7.42}$$

The Hilbert series in this case reads:

$$HS_{\hat{A}_3}(t, z_1, z_2, z_3) = \frac{1}{(1-t)^3} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} z_1^{m_1} z_2^{m_2} z_3^{m_3} t^{\Delta(m_1,m_2,m_3)}. \tag{7.43}$$

To carry out the calculation we decided to work with the unrefined Hilbert series given by

$$HS_{\hat{A}_3}(t) = \frac{1}{(1-t)^3} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} t^{\frac{1}{2}(|m_1|+|m_1-m_2|+|m_2-m_3|+|m_3|)}. \tag{7.44}$$

The unrefined Hilbert series is easier to handle and at the end we get an expression which involves the dimension of some well known representations. From that we guess the representations involved.

A more exact calculation would require to compute exactly the characters of the representations involved, anyway such calculations are done with perturbative methods while here a direct calculation takes us to the final result.

The calculations even for the unrefined Hilbert series require a lot of work so we proceed giving just the final result.

$$\begin{aligned}
HS_{\hat{A}_3}(t) &= \frac{1}{(1-t)^3} \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} t^{\frac{1}{2}(|m_1|+|m_1-m_2|+|m_2-m_3|+|m_3|)} \\
&= \dots \\
&= \frac{t^3+9t^2+9t+1}{(1-t)^6} \\
&= (t^3+9t^2+9t+1) \sum_{k\geq 0} \binom{k+5}{5} t^k \\
&= \dots \\
&= 1+15t+84t^2+\sum_{k\geq 3} \frac{(k+1)^2(k+2)^2(2k+3)}{12} t^k.
\end{aligned} \tag{7.45}$$

In this final expression we can recognize the dimensions of some well known irreps. Take the Dynkin diagrams to be the ones for $SU(4)$ and notice that $dim([1,0,1]) = 15$, $dim([2,0,2]) = 84$ and in general $dim([k,0,k]) = \frac{(k+1)^2(k+2)^2(2k+3)}{12}$. The last dimension can be computed directly from the Weyl dimension formula which gives the dimension for a general irrep of $SU(4)$. Once again we have that the current theory displays an SU(4) symmetry enhancement due to the quantum corrections that we have in the Coulomb branch.

7.8 The general case of a $\hat{A}_{n-1}$ quiver gauge theory

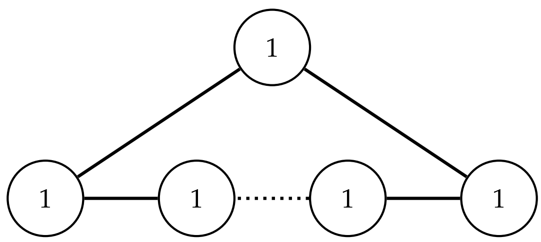


Figure 7.5: Quiver diagram for $U(1) \times U(1) \times \cdots \times U(1)/U(1)$. The resulting diagram is the same of the extended Dynkin diagram of the algebra A_{n-1} which is indicated $\hat{A}_{n-1}$.

In this case the magnetic fluxes are labeled by n integers $(m_0, \dots, m_{n-1})$. The $\mathcal{N} = 2$ hypermultiplets are associated to the links as we saw in section 4.4 and they are charged in the fundamental and antifundamental of two adjacent $U(1)$ factors. Under $U(1)^n$ they have charge $(1,-1,0,\dots,0)$, $(0,1,-1,0,\dots,0)$,..., $(-1,0,\dots,0,1)$.

The dimension formula is

$$\Delta(m_i) = \frac{1}{2} \sum_{i=0}^{n-1} |m_i - m_{i+i}|, \tag{7.46}$$

where $m_n = m_0$. Once again one of the $U(1)$ is decoupled and the gauge group actually is $U(1)^n/U(1)$. So, in general we have a theory with $n-1$ $U(1)$ topological symmetries corresponding to the non trivial $U(1)$ factors.

We can introduce fugacities z_i for the $n-1$ topological symmetries. The refined Hilbert series reads

$$HS_{U(1)^n/U(1)}(t, z_i) = \frac{1}{(1-t)^{n-1}} \sum_{m_1,\ldots,m_{n-1}\in\mathbb{Z}^{n-1}} z_1^{m_1} \cdots z_{n-1}^{m_{n-1}} t^{\Delta(0,m_1,\ldots,m_{n-1})}. \tag{7.47}$$

The $n-1$ topological symmetries are enhanced to $SU(n)$ by quantum effects as previously discussed. From the point of view of the Hilbert series we can see the enhancement by promoting the z_i to fugacities for the Cartan subgroup of $SU(n)$.

The fugacity map

$$z_1 = y_1^2 y_2,\ z_2 = \frac{y_2^2}{y_1 y_3},\ \cdots\ z_{n-1} = y_{n-1}^2 y_{n-2}, \tag{7.48}$$

let us resum 7.47 and we obtain an expansion in terms of characters of $SU(n)$

$$HS_{U(1)^n/U(1)}(t, z_i) = \sum_{k=0}^{\infty} [k, 0, \ldots, 0, k] t^k, \tag{7.49}$$

where $[k_1, \ldots, k_{n-1}]$ is the generic Dynkin label for $SU(n)$ and here represents the character corresponding to the irreducible representation itself. This expression manifestly demonstrates the presence of an enhanced global symmetry $SU(n)$.

7.9 ADE classification

The ADE classification takes its name from the root systems corresponding to the semisimple Lie algebras A_n, D_n, E_6, E_7, E_8. These ADE root systems have the property of being simply laced which means that roots can only form angles of 90° or 120° between them.

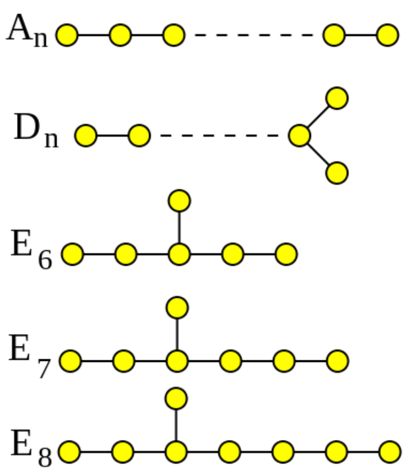


Figure 7.6: The simply laced Dynkin diagrams

ADE root systems have other interesting features and applications in the context of this thesis. For example they are also used to to classify binary polyhedral groups (we'll call them Γ_G), which are the finite subgroups of $SU(2)$. This correspondence between the finite subgroups of $SU(2)$ and root systems is known as McKay correspondence.

After discussing this, we will talk about du Val singularities and ALE spaces (Asymptotically Locally Euclidean).

Finally we'll introduce some supersymmetric gauge theories in $3d$ and $\mathcal{N} = 4$ whose quivers are called ADE quivers.

In the following we will try to avoid any inessential rigorousness. We will just try to state the main mathematical results which find their application in the study of 3d mirror symmetry. We'll have to do it in order to make this thesis more concise but the interested reader can find lots of more details in [22], [43].

7.9.1 Invariant ring of polynomials

To know which are these finite subgroups of $SU(2)$ one can exploit the usual homomorfism between $SU(2)$ and $SO(3)$. Since the subgroups of $SO(3)$ are classified we also have all the subgroups of $SU(2)$ and the list is the following one:

- $\mathbb{Z}_n$ is the cyclic group of order n;
- Dic_n is the binary dihedral group of of order $4n$;
- $\mathbb{BT}$ is the binary tetrahedral group of order 24;
- $\mathbb{BO}$ is the binary octahedral group of order 48;
- $\mathbb{BI}$ is the binary icosahedral group of order 120.

In discussing orbifolds as spaces constructed by quotients of manifolds over finite groups, a prominent role was played by invariant polynomial functions under the action of the group. In section 7.3 we saw how to build the polynomial ring for the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ and we want now to extend that result to the subgroups of $SU(2)$ which we denote Γ_G (in the following this notation will become clearer).

We recall that a ring is simply a set with addition and multiplication. The set of polynomials is clearly a ring since adding and multiplying polynomials together returns a polynomial. We denote the ring of polynomials in n variables with coefficients in $\mathbb{C}$ as $\mathbb{C}[x_1, \ldots, x_n]$ with the same notation used in Appendix A.

We can also think the coordinates $x_1, \ldots, x_n$ we introduced just like coordinates of an n-dimensional vector space V. In this case we denote with $\mathbb{C}[V]$ the ring of polynomials generated out of the coordinates $\{x_i\}_{i=1,\ldots,n}$ with respect to the chosen basis in V. Therefore, $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$.

A polynomial function $g \in \mathbb{C}[V]$ is said to be homogeneous of degree d if $g(\alpha v) = \alpha^d f(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$. Then, we can write $\mathbb{C}[V]$ as a graded $\mathbb{C}$-ring. Let $\mathbb{C}[V]_d$ be the subspace of homogeneous polynomial of degree d. Then

$$\mathbb{C}[V] = \bigoplus_d \mathbb{C}[V]_d. \tag{7.50}$$

Now consider a subgroup of G of $GL(V)$, the general linear group of linear transformations on V, embedded by a suitable homomorphism. G acts on V in the usual way:

$G \times V \to V$, $(g, v) \mapsto gv$. We define now polynomials invariant under the action of a group G. A polynomial function $f \in \mathbb{C}[V]$ is G-invariant if $f(gv) = f(v)\ \forall g \in G$ and $v \in V$. The G-invariant polynomial functions form a subalgebra, called the invariant ring. This is usually written $\mathbb{C}[V]^G$. This is by inheritance a graded algebra:

$$\mathbb{C}[V]^G = \bigoplus_d \mathbb{C}[V]^G_d. \tag{7.51}$$

It is a theorem due to Hilbert that the invariant ring is finitely generated, i.e. one can find elements in $\mathbb{C}[V]^G$ and relations among them from which the hole ring can be constructed. The branch of mathematics that studies what this generators and relations are, is the classical invariant theory.

So, let us consider the subgroups of $SU(2)$, Γ_G. The invariant ring is generated in this case by three elements a, b, c which obey a relation $\rho(a, b, c) = 0$, in formula (see Appendix A) we have:

$$\mathbb{C}[V]^{\Gamma_G} = \frac{\mathbb{C}[a, b, c]}{\langle\rho\rangle}. \tag{7.52}$$

Moreover we always have an isomorphism

$$\begin{aligned} &\chi : \mathbb{C}^2 \to \mathbb{C}^3 \\ &(z_1, z_2) \mapsto (a(z_1, z_2), b(z_1, z_2), c(z_1, z_2)), \end{aligned} \tag{7.53}$$

which preserves the quotient structure and we can write the invariant ring in terms of the quotient space built by letting $\mathbb{C}^2$ being acted upon by Γ_G. This gives another characterization for the affine variety as an orbifold. Adding the orbifold characterization 7.52 can be rewritten as

$$\mathbb{C}[V]^{\Gamma_G} = \frac{\mathbb{C}[a, b, c]}{\langle\rho\rangle} \simeq \frac{\mathbb{C}^2}{\Gamma_G}. \tag{7.54}$$

In the following, we list all the subgroups in Γ_G end the corresponding relations for the characterization of the moduli space as an affine variety:

- $\mathbb{Z}_n$ corresponds to $a^2 + b^2 - c^n = 0$ affine variety;
- Dic_{n-2} corresponds to $a^2 + b^2 c + c^{n-1} = 0$;
- $\mathbb{BT}$ corresponds to $a^2 + b^3 + c^4 = 0$;
- $\mathbb{BO}$ corresponds to $a^2 + b^3 + bc^3 = 0$;
- $\mathbb{BI}$ corresponds to $a^2 + b^3 + c^5 = 0$.

We notice that the characterization we just found must agree with the result of section 7.4. In section 7.4 we showed that the affine variety corresponding to the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_n}$ was $XY = Z^n$ if we choose the variables $X, Y, Z \in \mathbb{C}$ to describe it. So, if we define $a = X + iY$, $b = X - iY$ and $c = Z$ we get the relation $a^2 + b^2 = c^n$ listed above.

7.9.2 The McKay correspondence

McKay graphs are oriented graphs that encode the irreducible representations of a finite group G (weighted quivers). Here we consider McKay graphs for the finite subgroups Γ_G of $SU(2)$. These graphs are unoriented, unlike more general groups.

Let R be the faithful representation obtained by the embedding $\Gamma \hookrightarrow SU(2)$ and let χ_i, $i = 1, 2, \ldots, d$ be characters of the irreducible representations $\{R_i\}$ of Γ. The McKay graph for Γ is defined to be a quiver diagram with d vertices, one for each R_i, and n_{ij} lines from R_i to R_j where n_{ij} is the number of times R_j appears in the decomposition

$$R \otimes R_i = \bigoplus_j n_{ij} R_j. \tag{7.55}$$

For the subgroups of SU(2), it's easy to prove that $n_{ij} = n_{ji}$ and the graph is unoriented. The McKay correspondence states that there is a one-to-one correspondence between McKay graphs for the subgroups of $SU(2)$ and the affine Dynkin diagrams of simply laced semi simple Lie groups. The correspondence is explicitly seen in the form of the McKay graph.

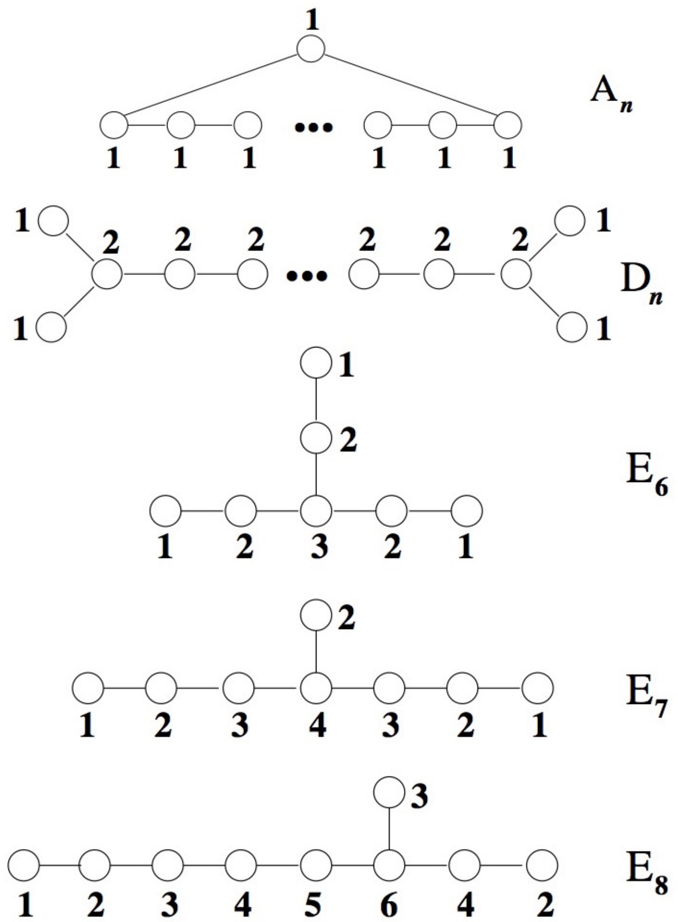


Figure 7.7: The Mckay graphs for the subgroups of $SU(2)$. Chosen a finite subgroup of $SU(2)$, inside each node is the dimension of the representation corresponding to that node. From the top we have the McKay graph for $\mathbb{Z}_n$ with n representations, then the one for Dic_{n-2}, $\mathbb{BT}$, $\mathbb{BO}$, $\mathbb{BI}$.

7.9.3 Du Val singularities and ALE spaces

In section 7.9.1 we found the relation existing between Γ_G-invariant polynomials and the corresponding orbifold $\frac{\mathbb{C}^2}{\Gamma_G}$. The singularities of these special class of quotient spaces are called du Val singularities. The spaces themselves are instead called in physics ALE spaces (Asymptotically Locally Euclidean) and it's enough to know their description in terms of affine variety to be able to write the corresponding unrefined Hilbert series. We are interested here in two cases, $\mathbb{Z}_n$ and Dic_{n-2}.

- For $\mathbb{Z}_n$ the algebraic curve is $a^2+b^2-c^n=0$. We have three generators of invariants a, b and c, which are found at different degrees. Since we want to have integer powers for the generator we choose c to be of degree 2 while a and b are at order n. The relation $a^2+b^2-c^n=0$ is instead obviously at order $2n$. We are now ready to write the unrefined Hilbert series:

$$HS\left(t, \frac{\mathbb{C}^2}{\mathbb{Z}_n}\right) = \frac{1-t^{2n}}{(1-t^n)(1-t^n)(1-t^2)}. \tag{7.56}$$

- To Dic_{n-2} corresponds $a^2+b^2c+c^{n-1}=0$. We choose c to be of degree 2, b of degree $n-2$ and a of degree $n-1$. In this case we have a relation at order $2n-2$ and the Hilbert series reads:

$$HS\left(t, \frac{\mathbb{C}^2}{Dic_{n-2}}\right) = \frac{1-t^{2n-2}}{(1-t^{n-2})(1-t^{n-1})(1-t^2)}. \tag{7.57}$$

We stop here but the same procedure could be applied to the other finite subgroups of $SU(2)$, the problem is that they are related to the E quiver diagrams and we do not know mirror couples for the E series, so their case is not considered in this thesis.

7.10 Mirror symmetry, ADE quivers and ADE series

Mirror symmetry in three-dimensional gauge theories with $\mathcal{N}=4$ supersymmetry was first proposed by Kenneth Intriligator and Nathan Seiberg in their 1996 paper (see [32] for the reference) and it is a relation between pairs of three-dimensional gauge theories. For mirror theories, the Coulomb branch of the moduli space of one is the Higgs branch of the moduli space of the other. The symmetry was demonstrated using D-brane by Amihay Hanany and Edward Witten.

In section 7.4 we found that the affine variety corresponding to $U(1)$ with n flavours is $\frac{\mathbb{C}^2}{\mathbb{Z}_n}$. The mirror symmetry for this theory is the theory with the quiver diagram drawn in section 7.8, which is identical to the affine Dynkin diagram $\hat{A}_{n-1}$. From the definition of mirror symmetry we can argue that the Higgs branch for the latter theory must be equal to the Coulomb branch we found for $U(1)$ with n flavours. The Higgs branch for $\hat{A}_{n-1}$ quiver gauge theory is exact at the classical level and is just the ALE space $\mathbb{C}^2/\mathbb{Z}_n$, this fact was shown in the paper [36] by P. Kronheimer.

A supersymmetric gauge theory in $3d$ with $\mathcal{N}=4$ is said to have an ADE quiver if the quiver diagram which tells the lagrangian content of the theory is just an affine Dynkin diagram of type A, D or E (see Appendix B for the affine Dynkin diagrams). So, the $\mathcal{N}=4$

theory with gauge group based on the extended Dynkin diagram of a simply laced group G is given by the ADE series

$$K_G \equiv \left(\prod_{i=1}^{k} U(n_i) \right) /U(1), \tag{7.58}$$

where i runs over the nodes of the diagram, $k-1$ is the rank of G and n_i are the Dynkin indices of the nodes through the McKay correspondence. The matter content consists of hypermultiplets associated with the links of the extended diagram. The overall $U(1)$ factor in $\prod_{i=1}^{k} U(n_i)$ is decoupled and is factored out. The Higgs branch of the theory is exact at the classical level and is the ALE space $\mathbb{C}^2/\Gamma_G$ [37] where Γ_G is the discrete subgroup of $SU(2)$ associated with the group G by the McKay correspondence. The Coulomb branch instead receives quantum correction. By analyzing the one loop correction [32] or by studying the corresponding brane system in string theory [41], it can be identified with the reduced moduli space of one instanton of the group G.

We anticipate here that the mirror theories for the groups $G = \hat{A}_{n-1}$ and $G = \hat{D}_n$ are respectively the $\mathcal{N} = 4$ theories $U(1)$ and $SU(2)$ with n fundamental hypermultiplets. In the following we'll do a few more comments on this duality. No mirror is instead known for the E series.

7.11 A theory with $SU(2)$ gauge group and n flavours

The corresponding quiver is the one below.

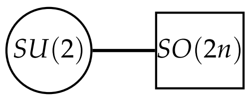


Figure 7.8: Quiver for SU(2) with n flavours.

The root system of $SU(2)$ is given by two roots, one of them is positive and the other is negative, $\Delta = \{\alpha, -\alpha\}$. There exist a standard way to build the root system for the general Lie algebra of $SU(n)$ starting from an orthonormal basis $\{e_i\}_{i=1,\dots,n}$. We will stick to it, considering the case $n = 2$ the root system is given by:

$$\begin{cases} \alpha = e_1 - e_2 \\ -\alpha = -e_1 + e_2 \end{cases} . \tag{7.59}$$

We recall here for commodity the formula for the dimension of the monopole operators 7.9:

$$\Delta(m) = -\sum_{\alpha \in \Delta^+} |\alpha(m)| + \frac{1}{2} \sum_i \sum_{\rho_i \in \mathcal{R}_i} \rho_i(m).$$

We see that the sum over the roots is restricted to only the positive ones and in our case $\Delta^+ = \{\alpha\}$. To find the GNO dual of $SU(2)$ let us compute the coroots.

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = \alpha. \tag{7.60}$$

The GNO dual of $SU(2)$ is in fact $SO(3)$. These have the same root systems and, as expected, the dual root system of $SU(2)$ is the same root system, the root system of $SO(3)$.

We are now ready to evaluate the contribution coming from the vector multiplet, we have

$$\vec{m} = (m, -m) \Rightarrow -\sum_{\alpha \in \Delta^+} |\alpha(m)| = -|\alpha(m)| = -|\vec{\alpha} \cdot \vec{m}| = -2|m|. \tag{7.61}$$

Then we have the contribution of the n matter fields in the fundamental of $SU(2)$:

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^{n} \sum_{\rho_i \in R_i} |\rho_i(m)| &= \frac{1}{2} n \sum_{\rho_i \in R_i} |\rho_i(m)| \\ &= \frac{1}{2} n \left(|(1/2, -1/2)(m, -m)| + |(-1/2, 1/2)(m, -m)| \right. \\ &= n|m|. \end{aligned} \tag{7.62}$$

Summing together the two contributions we get the final result for the conformal dimension of a coulomb operator

$$\Delta(m) = (n-2)|m|. \tag{7.63}$$

Now we must figure out the classical corrective factor of $SU(2)$, which accounts for the overcounting of the dressed monopole operators. $SU(2)$ can either be completely broken to $U(1) \times U(1)$, in the interior of a Weyl Chamber, or be unbroken, remaining just $SU(2)$ in the border. Given that the Weyl group of $SU(2)$ is $\mathbb{Z}_2$, the border of the chamber occurs for $m = 0$. For all other $m \in \mathbb{N}^*$, the group is completely broken. Therefore, the classical factor is

$$P_{SU(2)}(t, m) = \begin{cases} \frac{1}{(1-t^2)} & \text{if } m = 0 \\ \frac{1}{(1-t)} & \text{if } m \in \mathbb{N}^* \end{cases}. \tag{7.64}$$

The Hilbert Series is then

$$HS_{SU(2),n} = \sum_{m=0}^{\infty} t^{\Delta(m)} P_{SU(2)}(t, m). \tag{7.65}$$

With a direct computation one finds

$$\begin{aligned} HS_{SU(2),n} &= \frac{1}{(1-t^2)} t^{\Delta(0)} + \frac{1}{(1-t)} \sum_{m=1}^{\infty} t^{\Delta(m)} = \\ &= \frac{1}{(1-t^2)} + \frac{1}{(1-t)} \left(\sum_{m=0}^{\infty} t^{(n-2)m} - 1 \right) = \\ &= \frac{1}{(1-t^2)} + \frac{t^{n-2}}{(1-t)(1-t^{n-2})} = \\ &= \frac{1 - t^{2n-2}}{(1-t^2)(1-t^{n-2})(1-t^{n-1})}. \end{aligned} \tag{7.66}$$

From the Hilbert series written in this way, it's easy to read out the number of generators, relations and their degree. So, we have have a generator, let us say a at order $n-1$, a generator b at order $n-2$ and finally c at order 2. The relation is at order $2n-2$.

Now let us copare the Hilbert series just obtained with equation 7.57. They are the same and we can conclude that, thanks to all the correspondences previously discussed, the orbifold corresponding to the moduli space of a theory with gauge group $SU(2)$ with n flavours is just $\frac{\mathbb{C}^2}{Dic_{n-2}}$. The relation can be written in the form $a^2 + b^2 c = c^{n-1}$.

7.12 A theory with $SU(2)$ gauge group, n fundamentals and n_a adjoints

The Hilbert series of the Coulomb branch is given by

$$HS_{SU(2),n,n_a}(t) = \sum_{m=0}^{\infty} t^{\Delta(m)} P_{SU(2)}(t,m). \tag{7.67}$$

The vector multiplet contribution is still the same as for the case discussed before of $SU(2)$ with n flavours,

$$-\sum_{\alpha\in\Delta^+} |\alpha(m)| = -2|m|. \tag{7.68}$$

What changes is the following contribution:

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i}|\rho_i(m)| &= \frac{1}{2}n\left(\sum_{\rho_i\in[1]_{SU(2)}}|\rho_i(m)|\right) + \frac{1}{2}n_a\left(\sum_{\rho_i\in[2]_{SU(2)}}|\rho_i(m)|\right)\\
&= \frac{1}{2}n\{|(1/2,-1/2)\cdot(m,-m)| + |(-1/2,1/2)\cdot(m,-m)|\} + \dots\\
&\dots + \frac{1}{2}n_a\{|(1,-1)\cdot(m,-m)| + |(0,0)\cdot(m,-m)| + |(-1,1)\cdot(m,-m)|\}\\
&= \frac{1}{2}n\cdot 2|m| + \frac{1}{2}n_a\cdot 4|m|\\
&= (n+2n_a)|m|.
\end{aligned} \tag{7.69}$$

The conformal dimension of the monopole operator appears to be

$$\Delta(m) = (n+2n_a-2)|m|. \tag{7.70}$$

Comparing this result with the one of section 7.11, the only differece is that n in that case has now become $n+2n_a$. As a consequence of this, the moduli space has now become $\mathcal{M}_C = \frac{\mathbb{C}^2}{Dic_{n+2n_a-2}}$, which is the Hilbert series of the D_{n+2n_a} singularity.

7.13 A theory with $SU(2)$ gauge group and different irreps for the hypermultiplets

The next step in the generalization of the preceding case is to take n_1 hypermultiplets in the [1] irrep, n_2 in the [2] and so on.

The vector multiplet contribution is still the same

$$-\sum_{\alpha\in\Delta^+} |\alpha(m)| = -2|m|. \tag{7.71}$$

Now, let us focus on the matter content and take one irrep $[q]$ which is present in n_q copies and evaluate its contribution to the dimension formula.

$$\begin{aligned}
\frac{1}{2}n_q\sum_{\rho\in[q]}|\rho(m)| &= \frac{1}{2}n_q\{|q||m| + |q-2||m| + \dots + |2-q||m| + |-q||m|\}\\
&= \begin{cases} n_q|m|\frac{q}{2}(\frac{q}{2}+1) \; if\ q\ is\ even \\ n_q|m|(\frac{q}{2}+\frac{1}{2})^2 \; if\ q\ is\ odd \end{cases}.
\end{aligned} \tag{7.72}$$

Adding up everything together we get the dimension formula

$$\Delta(m) = \left(\sum_{q\ odd} n_q \left(\frac{q}{2} + \frac{1}{2} \right)^2 + \sum_{q\ even} n_q \frac{q}{2} \left(\frac{q}{2} + 1 \right) - 2 \right) |m|. \tag{7.73}$$

Once again we can redefine

$$N \equiv \sum_{q\ odd} n_q \left(\frac{q}{2} + \frac{1}{2} \right)^2 + \sum_{q\ even} n_q \frac{q}{2} \left(\frac{q}{2} + 1 \right).$$

and the moduli space is described by the affine variety $\mathcal{M}_C = \frac{\mathbb{C}^2}{Dic_{N-2}}$.

Quiver gauge theory is a theory in which Lagrangians are associated to quivers thorough certain rules, here we treated cases in which hypermultiplets are not simply in a fundamental and antifundamental of two groups. This generalization is easy to implement in our dimension formula 7.9 but the correspondence quiver-lagrangian is lost. Anyway we could invent new rules for the nodes in order to take into account these new possibilities and restore the correspondence between quivers and lagrangians.

7.14 A theory with $U(2)$ gauge group and n flavours

Now let us move to a new computation for $U(2)$ with n flavours.

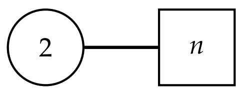


Figure 7.9: Quiver diagram for $U(2)$ with n flavours.

$U(2)$ has rank 2 and therefore the monopole operator has a magnetic charge corresponding to a two component vector $\vec{m} = (m_1, m_2)$.

In this case, $U(2)$ is not abelian and we must figure out its root system. We can think to $U(2)$ as $U(2) = U(1) \times SU(2)$ which translates to the fact that the the Lie algebra for $U(2)$ is given by direct sum of the one of $U(1)$ and the one of $SU(2)$. Therefore, the root system factorizes as a disjoint union of vector spaces $\Delta = \Delta_{U(1)} \sqcup \Delta_{SU(2)}$. The one for $U(1)$ is empty.

This leaves us with the computation of the root system of $SU(2)$ and the identification of its positive root, which is $\alpha = e_1 - e_2$ in the conventional basis. The contribution to the conformal dimension coming from the vector multiplet is

$$\Delta_{vec} = - |m_1 - m_2|. \tag{7.74}$$

We turn now to the matter contribution. The characters of the fundamental representation of $U(2)$ can be thought as the same as for $SU(2)$ but we do not impose the traceless condition this time. In this case we prefer not to use an highest weights notation, instead we label representations on the basis of their dimension. So, we indicate the fundamental representation of U(2) as **2**, and the corresponding character is $\chi(\mathbf{2}) = z_1 + z_2$, the two wights are consequently $(1, 0)$ and $(0, 1)$.

The matter contribution is

$$\Delta_{matt} = \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i \in R_i} |\rho_i(m)| = \frac{1}{2}n\left(|m_1| + |m_2|\right). \tag{7.75}$$

This leaves us with

$$\Delta(m_1, m_2) = -|m_1 - m_2| + \frac{1}{2}n\left(|m_1| + |m_2|\right). \tag{7.76}$$

The Hilbert Series is

$$HS_{U(2),n} = \sum_{m_1 \geq m_2 > -\infty} t^{\Delta(m_1,m_2)} z^{(m1+m2)} P_{U(2)}(\vec{m}, t). \tag{7.77}$$

The classical factor $P_{U(2)}(\vec{m}, t)$ is given by

$$P_{U(2)}(\vec{m}, t) = \begin{cases} \frac{1}{(1-t)(1-t^2)} & m_1 = m_2 \\ \frac{1}{(1-t)^2} & m_1 \neq m_2 \end{cases}. \tag{7.78}$$

A direct computation leads to

$$HS_{U(2),n} = \frac{(1-t^n)(1-t^{n-1})}{(1-t)(1-zt^{\frac{n}{2}})(1-z^{-1}t^{\frac{n}{2}})(1-t^2)(1-zt^{\frac{n}{2}-1})(1-z^{-1}t^{\frac{n}{2}-1})}. \tag{7.79}$$

From the Hilbert series we find that there are six generators of the chiral ring, and two relations giving for the dimensions $dim_{\mathbb{C}}\mathcal{M}_{U(2),n} = 4$.

This perfectly agrees with the general dimension of the Coulomb branch for 3d $\mathcal{N} = 4$ gauge theories which classically is $\mathcal{M} = (\mathbb{S}^1 \times \mathbb{R}^3)^r$ where r is the rank of the gauge group. So, we see a particular case of a more general fact, quantum corrections of the Coulomb branch modify the metric and the moduli space itself, but not its dimension.

Anyway the Hilbert series gives further information about the generators and relations, such as their degree. In the following table, the dimensions and topological charges of such operators are summed up.

	Δ	z
g_1	1	0
g_2	2	0
g_3	$\frac{n}{2}$	1
g_4	$\frac{n}{2}-1$	1
g_5	$\frac{n}{2}$	-1
g_6	$\frac{n}{2}-1$	-1
R_1	n	0
R_2	$n-1$	0

Table 7.1: Generators and relations classified on the basis of their conformal dimensions and topological charges.

7.15 $U(2)$ with n fundamentals and one adjoint

We move now to a more challenging calculation. In this case we are adding again a different representation of the matter multiplet, so let us see how the Hilbert series behaves in this case.

The contribution to the conformal dimension coming from the vector multiplet is again

$$\Delta_{vec} = -|m_1 - m_2|. \tag{7.80}$$

Also the contribution coming from the matter multiplet in the n fundamentals stays unchanged and it is

$$\Delta_{matt,fund} = \frac{1}{2}n\left(|m_1| + |m_2|\right). \tag{7.81}$$

We must instead figure out the contribution of the adjoint. Starting from the fundamental we have $\chi(\mathbf{2}) = z_1 + z_2$, but thinking of $U(2)$ as $SU(2) \times U(1)$ we have the expression $\chi(\mathbf{2}) = \left(x + \frac{1}{x}\right)q$. The two expression are equivalent because they are simply related to each other by a fugacity map $z_1 = xq$ and $z_2 = x^{-1}q$.

$$\begin{aligned}\chi\left(adj_{U(2)}\right) &= \chi(\mathbf{4}) = [1]q \otimes [1]q^{-1} = (z_1 + z_2)(z_1^{-1} + z_2^{-1}) = \\ &= 1 + \frac{z_1}{z_2} + \frac{z_2}{z_1} + 1.\end{aligned} \tag{7.82}$$

From the previous character we see that the weights we have to consider, for the contribution coming from the matter in the adjoint representation, are $(0,0)$, $(1,-1)$, $(-1,1)$ and $(0,0)$ again. We have

$$\begin{aligned}\Delta_{matt,adj}(m_1, m_2) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i \in R_i}|\rho_i(m)| = \frac{1}{2}\cdot 1 \cdot \left(|m_1 - m_2| + |m_2 - m_1|\right) \\ &= |m_1 - m_2|.\end{aligned} \tag{7.83}$$

The contribution Δ_{vec} and $\Delta_{matt,adj}$ cancel between themselves in the conformal dimension formula and we are left just with the n fundamentals contribution. The Hilbert series in this case reads

$$\begin{aligned}HS_{U(2),n\,fund,1\,adj} &= \sum_{m_1 \geq m_2 > -\infty} t^{\frac{1}{2}n(|m_1| - |m_2|)} P_{U(2)}(\vec{m}, t) \\ &= \cdots \\ &= \frac{1 + t^{\frac{n}{2}} + 2t^{\frac{n}{2}+1} + 2t^n + t^{n+1} + t^{\frac{3}{2}n+1}}{(1 - t^n)(1 - t)(1 - t^2)(1 - t^{\frac{n}{2}})}.\end{aligned} \tag{7.84}$$

From the expression that we have found we can immediately read out the dimension of the the moduli space from the pole. The complex dimension is four as expected from the classical considerations for the Coulomb branch.

7.16 The $\hat{D}_4$ quiver gauge theory

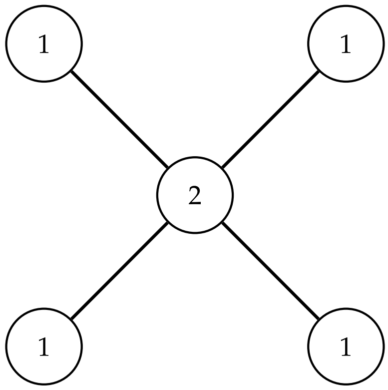


Figure 7.10: $\mathcal{N} = 2$ quiver diagram for $\hat{D}_4$ gauge theory. The nodes labelled 1 correspond to U(1) factors and the central node labelled 2 corresponds to a U(2) factor.

As explained in section 7.10 the gauge group corresponding to this ADE quiver is

$$U(1)^2 \times U(2) \times U(1)^2 / U(1).$$

The hypermultiplets are represented by the links between the nodes and are in the bi-fundamental of the nodes that they join. So we have two external hypermultiplets transforming as $(\mathbf{1}, \mathbf{2})$, one for each of the first two $U(1)$ factors on the left, and two external hypermultiplets transforming as $(\mathbf{2}, \mathbf{1})$, one for each of the last two U(1) factors on the right. The overall $U(1)$ is decoupled and is factored out. Associated with all the non trivial $U(1)$ factors there is a corresponding topological symmetry. This symmetry $U(1)^4$ is enhanced to $SO(8)$ by quantum effects as we'll see.

Before decoupling the overall $U(1)$, the magnetic fluxes would be labeled by integers m_i $(i = 0, \cdots, 3)$ for the $U(1)$ factors and a pair of integers (n_1, n_2) for the $U(2)$ factor. All fluxes are integers as requested by the Dirac quantization condition. The conformal dimension formula in this case reads

$$\Delta_{\hat{D}_4}(m_0, \ldots, m_3, n_1, n_2) = \frac{1}{2} \sum_{i=1,2} \sum_{j=0,\ldots,3} \left(|m_j - n_i|\right) - |n_1 - n_2|. \tag{7.85}$$

Like in the case of $\hat{A}_n$ the energy or conformal dimension is invariant under a shift $m_i \to m_i + s$, $n_i \to n_i + s$ for any s. To avoid this we set to zero the magnetic flux corresponding to the extended Dynkin node $m_0 = 0$, and the formula for the conformal dimension becomes

$$\Delta_{\hat{D}_4}(m_0 \equiv 0, \ldots, m_3, n_1, n_2) = \frac{1}{2} \sum_{i=1,2} \sum_{j=1,\ldots,3} \left(|m_j - n_i| + |n_1| + |n_2|\right) - |n_1 - n_2|. \tag{7.86}$$

The magnetic flux m carried by the monopole operator breaks the symmetry group G to a residual gauge group H_m. Gauge invariants under H_m are accounted for by the so called dressing function. The nonabelian factor in this case is $\tilde{H} = U(2)$. The group U(2) has two residual symmetry groups depending on whether it is broken or not:

- $\tilde{H}_1 = U(2)$ itself which has two Casimir operators with degrees 1,2. This corresponds to $n_1 = n_2$,
- $\tilde{H}_2 = U(1)^2$ which has two Casimir operators of degree 1. This corresponds to $n_1 \neq n_2$.

The classical dressing function is:

$$P_{U(2)}(t, n_1, n_2) = \begin{cases} \frac{1}{(1-t)(1-t^2)} \; if \; n_1 = n_2 \\ \frac{1}{(1-t)(1-t)} \; if \; n_1 \neq n_2 \end{cases} . \tag{7.87}$$

The Hilbert series can finally be written

$$HS_{\hat{D}_4} = \frac{1}{(1-t)^3} \sum_{n_2 \geq n_1 > -\infty} \sum_{m_1, m_2, m_3 > -\infty} P_{U(2)}(t, n_1, n_2) t^{\Delta_{\hat{D}_4}}. \tag{7.88}$$

The final result reads

$$HS_{\hat{D}_4} = \sum_{k=0}^{\infty} dim([0,k,0,0]_{SO(8)}) t^k, \tag{7.89}$$

which shows the enhancement of the global symmetry to $SO(8)$. The series 7.89 is the Hilbert series for the reduced moduli space of one instanton of SO(8) as discussed in [8].

7.17 The $\hat{D}_n$ quiver gauge theory

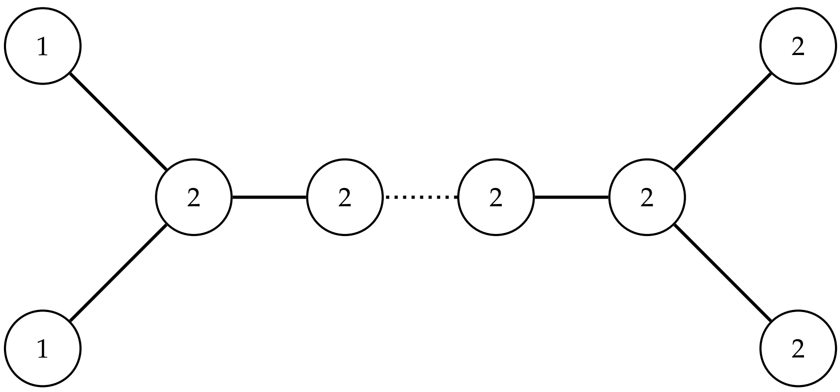


Figure 7.11: $\mathcal{N} = 2$ quiver diagram for $\hat{D}_n$ gauge theory. The nodes labelled 1 correspond to U(1) factors and the central node labelled 2 corresponds to a U(2) factor.

The general case for the $\hat{D}_n$ quiver is explained in detail in [16] and we'll follow their exposition very closely.

The gauge group is $U(1)^2 \times U(2)^{n-3} \times U(1)^2 / U(1)$ and the matter content consists of hypermultiplets associated with the link of the extended Dynkin diagram of D_n. We have hypermultiplets transforming in the representation $(\mathbf{2}_p, \mathbf{2}_{p+1})$ of neighboring $U(2)$ groups for $p = 1, \ldots, n-4$, two external hypermultiplets transforming as $(\mathbf{1}, \mathbf{2}_1)$, one for each of the first two $U(1)$ factors, and two external hypermultiplets transforming as $(\mathbf{2}_{n-3}, \mathbf{1})$, one for each of the last two $U(1)$ factors. The overall $U(1)$ is again decoupled and factored out. Associated with all the non trivial $U(1)$ factors there is a corresponding topological symmetry. In this case the symmetry $U(1)^n$ is enhanced to $SO(2n)$ by quantum effects.

Before decoupling the overall $U(1)$, the magnetic fluxes would be labeled by integers q_a ($a = 1, \dots, 4$) for the $U(1)$ factors and pairs of integers (m_1^p, m_2^p) ($p = 1, \dots, n-3$) for the U(2) factors. The dimension formula reads

$$\Delta(q_a, \vec{m}^p) = \frac{1}{2}\left(\sum_{p=1}^{n-4}\sum_{i,j=1}^{2}|m_i^p - m_j^{p+1}| + \sum_{a=1}^{2}\sum_{i=1}^{2}|m_i^1 - q_a| + \sum_{a=3}^{4}\sum_{i=1}^{2}|m_i^{n-3} - q_a|\right) \tag{7.90}$$

$$-\frac{1}{2}\sum_{p=1}^{n-3}\sum_{i,j=1}^{2}|m_i^p - m_j^p|. \tag{7.91}$$

Since the overall $U(1)$ is decoupled, the formula is invariant under a common shift of all fluxes. We can remove the decoupled U(1) by setting to zero the flux associated with the extended node $q_1 = 0$. We can also introduce fugacities z_i for the n $U(1)$ topological symmetries and associate them to the remaining nodes. The z_i are naturally associated with the simple roots of the Dynkin diagram of $SO(2n)$ and they can be promoted to fugacities for the Cartan subgroup of $SO(2n)$. In our notations, z_1 is assigned to the $U(1)$ node with flux q_2, then $z_2, \dots, z_{n-2}$ to the $U(2)$ nodes with fluxes $\vec{m}^1, \dots, \vec{m}^{n-3}$ and z_{n-1} and z_n to the last two $U(1)$ factors with fluxes q_3 and q_4. The z_i can be rewritten in a more common basis using the Cartan matrix:

$$z_1 = \frac{y_1^2}{y_2},\ z_2 = \frac{y_2^2}{y_1 y_3},\ \dots,\ z_{n-2} = \frac{y_{n-2}^2}{y_{n-3}y_{n-1}y_n},\ z_{n-1} = \frac{y_{n-1}^2}{y_{n-2}},\ z_n = \frac{y_n^2}{y_{n-2}}.$$

The refined Hilbert series reads

$$HS_{\hat{D}_n} = \frac{1}{(1-t)^3}\sum_{\substack{m_1^p \geq m_2^p > -\infty \\ q_1, q_2, q_3 > -\infty}}\prod_{i=1}^{n} z_i^a t^{\Delta(q_a, \vec{m}^p)_{q_1=0}}\prod_{p=1}^{n-3}P_{U(2)}(t, \vec{m}^p), \tag{7.92}$$

where the weights of the z_i action are

$$\vec{a} = (q_2, m_1^1 + m_2^1, \dots, m_1^{n-3} + m_2^{n-3}, q_3, q_4),$$

and $P_{U(2)}$ are the classical contribution of the Casimir invariants of the residual gauge group which commutes with the monopole flux:

$$P_{U(2)}(t, m_1, m_2) = \begin{cases} \frac{1}{(1-t)(1-t^2)}\ if\ m_1 = m_2 \\ \frac{1}{(1-t)(1-t)}\ if\ m_1 \neq m_2 \end{cases}. \tag{7.93}$$

The factor $1/(1-t)^3$ takes into account the degree of the Casimir invariants for the three remaining $U(1)$ groups.

It is easy to check at high order in t and n that the Hilbert series is

$$HS_{\hat{D}_n} = \sum_{k=0}^{\infty}[0, k, 0, \dots, 0]_{SO(2n)} t^k, \tag{7.94}$$

where $[k_1, \ldots, k_n]$ denotes the character of the $SO(2n)$ representation with Dynkin labels k_i. This expression manifestly demonstrates the presence of an enhanced global symmetry $SO(2n)$ and the series 7.94 is the Hilbert series for the reduced moduli space of one instanton of SO(8) as discussed in [8].

Chapter 8

Instanton moduli spaces

Instantons are a very prolific and fascinating subject. They have lead to new insights into a wide range of phenomena, from the structure of the Yang-Mills vacuum [44], [33], [13] to the classification of four-manifolds [18]. One of the most powerful uses of instantons in recent years is in the analysis of supersymmetric gauge dynamics.

For our introduction to the subject we'll follow two references, [45] and [39] and we'll give some details about the the tecniques that are used to study their moduli spaces through the Hilbert series [8]. Anyway, there exist lots of good reviews on the subject of instantons. The canonical reference for all the basics of the subject remains the lecture by Coleman [15].

8.1 Instantons and topology

Let us consider a pure $SU(N)$ Yang-Mills theory with action:

$$S = \frac{1}{2e^2}\int d^4x Tr F_{\mu\nu}F^{\mu\nu}, \tag{8.1}$$

where we adopt the convention of picking Hemitian generators T^m with normalization $TrT^mT^n = \frac{1}{2}\delta^{mn}$ and we write $A_\mu = A_\mu^m T^m$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. The adjoint covariant derivatives are $\mathcal{D}_\mu X = \partial_\mu X - i[A_\mu, X]$. In this section alone we work with Euclidean signature.

Motivated by the semi-classical evaluation of the path integral, we search for finite action solutions to the Euclidean equations of motion,

$$\mathcal{D}_\mu F^{\mu\nu} = 0, \tag{8.2}$$

which, in the imaginary time formulation of the theory, have the interpretation of mediating quantum mechanical tunnelling events.

The requirement of finite action means that the potential A_μ must become a pure gauge as we head towards the boundary $r \to \infty$ of spatial $\mathbb{R}^4$,

$$A_\mu \to i g^{-1}\partial_\mu g, \tag{8.3}$$

with $g(x) = e^{iT(x)} \in SU(N)$. In this way, any finite action configuration provides a map from $\partial\mathbb{R}^4 \simeq \mathbb{S}^3_\infty$ into the group SU(N). These maps are classified by homotopy theory. Two maps are said to lie in the same homotopy class if they can be continuously deformed into each other, with different classes labelled by the third homotopy group,

$$\Pi_3(SU(N)) \simeq \mathbb{Z}.$$

The integer $k \in \mathbb{Z}$ counts how many times the group wraps itself around spatial $\mathbb{S}^3_\infty$ and is known as the Pontryagin number, or second Chern class. We will sometimes speak simply of the "charge" k of the instanton. It is measured by the surface integral

$$k = \frac{1}{24\pi^2}\int_{\mathbb{S}^3_\infty} d^3S_\mu Tr(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1}(\partial_\sigma g)g^{-1}\epsilon^{\mu\nu\rho\sigma}. \tag{8.5}$$

The charge k splits the space of field configurations into different sectors. Viewing $\mathbb{R}^4$ as a foliation of concentric $\mathbb{S}^3$'s, the homotopy classification tells us that we cannot transform a configuration with non-trivial winding $k \neq 0$ at infinity into one with trivial winding on an interior $\mathbb{S}^3$ while remaining in the pure gauge ansatz. Yet, at the origin, obviously the gauge field must be single valued, independent of the direction from which we approach. To reconcile these two facts, a configuration with $k \neq 0$ cannot remain in the pure gauge form 8.3 throughout all of $\mathbb{R}^4$: it must have non-zero action.

8.1.1 An Example: SU(2)

The simplest case to discuss is the gauge group $SU(2)$ since, as a manifold, $SU(2) \simeq \mathbb{S}^3$ and $\Pi_3(\mathbb{S}^3) \simeq Z$. Examples of maps in the different sectors are

- $g^{(0)} = 1$, the identity map has winding $k = 0$;
- $g^{(1)} = (x_4 + ix_i\sigma^i)/r$ has winding number $k = 1$. Here $i = 1, 2, 3$, and the σ^i are the Pauli matrices;
- $g^{(k)} = [g^{(1)}]^k$ has winding number k.

To create a non-trivial configuration in $SU(N)$, we could try to embed the maps above into a suitable $SU(2)$ subgroup, say the upper left-hand corner of the $N \times N$ matrix. It's not obvious that if we do this they continue to be a maps with non-trivial winding since one could envisage that they now have space to slip off. However, it turns out that this doesn't happen and the above maps retain their winding number when embedded in higher rank gauge groups.

8.2 The Instanton Equations

We have learnt that the space of configurations splits into different sectors, labelled by their winding $k \in \mathbb{Z}$ at infinity. The next question we want to ask is whether solutions actually exist for different k. Obviously for $k = 0$ the usual vacuum $A_\mu = 0$ (or gauge transformations of it) is a solution. But what about higher winding with $k \neq 0$?

The first step to constructing solutions is to derive a new set of equations that the instantons will obey, equations that are first order rather than second order as in 8.2. The trick for doing this is usually referred to as the Bogomoln'yi bound. From the above considerations, we have seen that any configuration with $k \neq 0$ must have some non-zero action. The Bogomoln'yi bound quantifies this. We rewrite the action by completing the square,

$$
\begin{aligned}
S_{inst} =& \frac{1}{2e^2}\int d^4x Tr F_{\mu\nu}F^{\mu\nu} && (8.6)\\
=& \frac{1}{4e^2}\int d^4x Tr(F_{\mu\nu} \mp *F^{\mu\nu})2 \pm 2Tr F_{\mu\nu} * F^{\mu\nu} && (8.7)\\
\geq& \pm \int d^4x \partial_\mu (A_\nu F_{\rho\sigma} + \frac{2i}{3}A_\nu A_\rho A_\sigma)\epsilon^{\mu\nu\rho\sigma}, && (8.8)
\end{aligned}
$$

where the dual field strength is defined as $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and, in the final line, we've used the fact that $F_{\mu\nu} * F^{\mu\nu}$ can be expressed as a total derivative.

The final expression is a surface term which measures some property of the field configuration on the boundary $\mathbb{S}^3_\infty$. Inserting the asymptotic form $A_\nu \to ig^{-1}\partial_\nu g$ into the above expression and comparing with 8.5, we learn that the action of the instanton in a topological sector k is bounded by

$$S_{int} \geq \frac{8\pi^2}{e^2}|k|, \tag{8.9}$$

with equality if and only if

$$
\begin{aligned}
F_{\mu\nu} &= *F_{\mu\nu}(k>0),\\
F_{\mu\nu} &= - * F_{\mu\nu}(k<0).
\end{aligned} \tag{8.10}
$$

Since parity maps $k \to -k$, we can focus on the self-dual equations $F = *F$. The Bogomoln'yi argument says that a solution to the self-duality equations must necessarily solve the full equations of motion since it minimizes the action in a given topological sector. In fact, in the case of instantons, it's trivial to see that this is the case since we have

$$\mathcal{D}_\mu F^{\mu\nu} = \mathcal{D}_\mu * F^{\mu\nu} = 0, \tag{8.11}$$

by the Bianchi identity.

8.3 Collective Coordinates

The simplest solution is the $k = 1$ instanton in $SU(2)$ gauge theory. In singular gauge, the connection is given by

$$A_\mu = \frac{\rho^2(x-X)_\nu}{(x-X)^2((x-X)^2+\rho^2)}\overline{\eta}^i_{\mu\nu}(g\sigma^i g^{-1}). \tag{8.12}$$

The $\sigma^i, i = 1,2,3$ are the Pauli matrices and carry the $\mathfrak{su}(2)$ Lie algebra indices of A_μ. The $\overline{\eta}^i, \quad i = 1,2,3$ are three 4×4 anti-self-dual 't Hooft matrices which intertwine the group structure of the index i with the spacetime structure of the indices μ,ν. They are given by

$$\overline{\eta}^1 = \begin{pmatrix} o & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \overline{\eta}^2 = \begin{pmatrix} o & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \overline{\eta}^3 = \begin{pmatrix} o & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The field strength inherits its self-duality from the anti-self-duality of the $\overline{\eta}$ matrices. To build an anti-self-dual field strength, we need to simply exchange the $\overline{\eta}$ matrices in 8.12 for their self-dual counterparts,

$$\eta^1 = \begin{pmatrix} o & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \eta^2 = \begin{pmatrix} o & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \eta^3 = \begin{pmatrix} o & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We notice that the solution 8.12 is not unique: it contains a number of parameters. In the context of solitons, these are known as collective coordinates. The solution 8.12 has eight such parameters. They are of three different types:

1. 4 translations X_μ, the instanton is an object localized in $\mathbb{R}^4$, centered around the point $x_\mu = X_\mu$;
2. 1 scale size ρ, the interpretation of ρ as the size of the instanton can be seen by rescaling x and X in the above solution to demote ρ to an overall constant;
3. 3 global gauge transformations $g \in SU(2)$, this determines how the instanton is embedded in the gauge group.

8.4 Moduli space for $SU(N)$ instantons

The $SU(2)$ solution has 8 collective coordinates. We are now interested into $SU(N)$ solutions. Of course, we should keep the $4+1$ translational and scale parameters but we would expect more orientation parameters telling us how the instanton sits in the larger $SU(N)$ gauge group. Suppose we embed the above $SU(2)$ solution in the upper left-hand corner of an $N \times N$ matrix. We can then rotate this into other embeddings by acting with $SU(N)$, modulo the stabilizer which leaves the configuration untouched. We have

$$SU(N)/S[U(N-2) \times U(2)],$$

where the $U(N-2)$ hits the lower-right-hand corner and doesn't see our solution, while the $U(2)$ is included in the denominator since it acts like g in the original solution 8.12 and we don't want to overcount. Finally, the notation $S[U(p) \times U(q)]$ means that we lose the overall central $U(1) \subset U(p) \times U(q)$. The coset space above has dimension $4N-8$. So, within the ansatz embedded in $SU(N)$, we see that the $k=1$ solution has $4N$ collective coordinates. In fact, it turns out that this is all of them and the solution 8.12, suitably embedded, is the most general $k=1$ solution in an $SU(N)$ gauge group.

We are now ready to introduce the moduli space of instantons starting from the $SU(N)$ case that we just discussed. This is defined to be the space of all solutions to $F = *F$, modulo gauge transformations, in a given winding sector k and gauge group $SU(N)$. Let's denote this space as $\mathcal{I}_{k,N}$.

Coordinates on $\mathcal{I}_{k,N}$ are given by the collective coordinates of the solution. We've seen above that the $k=1$ solution has $4N$ collective coordinates or, in other words, $dim(\mathcal{I}_{1,N}) = 4N$. It can be proved that for higher k, the number of collective coordinates is:

$$dim(\mathcal{I}_{k,N}) = 4kN.$$

This has a very simple interpretation. The charge k instanton can be thought of as k charge 1 instantons, each with its own position, scale, and gauge orientation. When the instantons are well separated, the solution does indeed look like this. But when instantons start to overlap, the interpretation of the collective coordinates can become more subtle.

8.5 Hilbert series for one-instanton moduli spaces on $\mathbb{C}^2$

We have seen which is the situation for an $SU(N)$ instanton, now we want to generalize those facts to different gauge groups and in particular we want to define the Hilbert series for the moduli space of k G-instantons on $\mathbb{C}^2$, were G is the gauge group of finite rank r.

It has been proved that this moduli space has quaternionic dimension kh_G where h_G is the dual Coxeter number of the gauge group G. We are actually interested here in the moduli space of one instanton. The moduli space is reducible into a trivial $\mathbb{C}^2$ component, physically corresponding to the position of the instanton in $\mathbb{C}^2 \simeq \mathbb{R}^4$, and the remaining irreducible component of quaternionic dimension $h_G - 1$. Henceforth, we shall call this component the coherent component or the irreducible component. The Hilbert series for the coherent component takes the form

$$g^{Irr}(t; x, \dots, x_r) = \sum_{k=0}^{\infty} \chi[R_G(k)] t^{2k}, \tag{8.13}$$

where $R_G(k)$ is a series of representations of G and $\chi[R]$ is the character of the representation R. The fugacities x_i (with $i = 1, \dots, r$) are conjugate to the charges of each holomorphic function under the Cartan subalgebra of G. The moduli space of instantons is a non-compact hyperKähler space, and so there are infinitely many holomorphic functions which are graded by degrees d. Setting $x_1 = \dots = x_r = 1$, we obtain the (finite) number of holomorphic functions of degree d.

The main result of the paper [8] is that the representation $R_G(k)$ is the irreducible representation Adj^k, where Adj^k denotes the irreducible representation whose Dynkin labels are $\theta_k = k\theta$, with θ the highest root of G. So, for the A_n series $\theta_k = [k, 0, \dots, 0, k]$, for the B_n and D_n series $\theta_k = [0, k, 0, \dots, 0]$, for the C_n series $\theta_k = [2k, 0, \dots, 0]$, for E_6 $\theta_k = [0, k, 0, 0, 0, 0]$, for G_2 $\theta_k = [0, k]$, for all other exceptional groups $\theta_k = [k, 0, \dots, 0]$. By convention $R_G(0)$ is the trivial, one-dimensional, representation and $R_G(1)$ is the adjoint representation.

In the case of classical gauge groups A_n, B_n, C_n, D_n it is possible to directly verify the above statement by explicit counting of the chiral operators on the Higgs branch of a certain $\mathcal{N} = 2$ supersymmetric gauge theory with a one dimensional Coulomb branch and a $A_n, \dots, D_n$ global symmetry.

8.5.1 One $SO(8)$ instanton

An explicit counting of chiral operators in the well known $\mathcal{N} = 2$ supersymmetric gauge theory of $SU(2)$ with 4 flavours, gives the Hilbert series for the coherent component of the one $D_4 = SO(8)$ instanton moduli space (omitting the trivial component $\mathbb{C}^2$) and this is what we got in Section 7.16:

$$g^{Irr}(t; x_1, x_2, x_3, x_4) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{D_4} t^{2k}. \tag{8.14}$$

Setting these fugacities x_i to one, we get the unrefined Hilbert series:

$$\begin{aligned} g^{Irr}(t) &= \sum_{k=0}^{\infty} dim[0,k,0,0]_{D_4} t^{2k} = \\ &= \frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}}, \end{aligned} \tag{8.15}$$

where the dimension of $[0,k,0,0]$ is easily computed by the Weyl dimension formula.

Notice that summing the series we get a closed formula with a pole of order 10 at $t=1$. This means that the space is 10-complex dimensional, and is in agreement with the fact that the non-trivial component of the one-instanton moduli space for D_4 has quaternionic dimension 5 (the dual Coxeter number $h_{D_4} = 6$).

Chapter 9

Conclusion and outlook

In this thesis we discussed in depth the algebraic structure of the moduli space for theories in three dimensions with $\mathcal{N} = 4$. We focused on its algebraic characterization because this approach is for some aspect simpler than pursuing a geometric one. The algebraic structure was recovered through the Hilbert series corresponding to the Coulomb or Higgs branch. These Hilbert series basically enumerate the chiral operators grading them according to their symmetries. We exploited the conjectured bijection between holomorphic functions on the moduli space and chiral operators to characterize the chiral ring which is therefore equivalent to the ring of holomorphic functions over the variety of the vacuum.

Like we said, the Hilbert series is an important quantity that characterises an algebraic variety and we introduced it starting from simple examples like the SQCD in 4d $\mathcal{N} = 1$. So, we were able to operatively build Hilbert series starting from simpler cases to more difficult ones. From a physical point of view Hilbert series appear to be very similar to a partition functions. Even the term "fugacities" for the addends in the characters of the representations are reminiscent of this, and they can be seen as generalizations of the fugacities "z" of statistical mechanics which are related to the chemical potential.

We posed lot of attention on the Coulomb branch and the Higgs one. Our strategy for studying them was not the same but the final product was always the Hilbert series for the vacuum. Our approach was different just on the choice of the operators used to parametrize the vacuum. When dealing with the Higgs branch we took the lowest components of the chiral multiplets adding up to form the $\mathcal{N} = 2$ hypermultiplet, while we turned to monopole operators for the Coulomb branch.

This difference has also an historical motivation. In fact, in a first moment people gained information about the Coulomb branch making calculations on the Higgs branch of the dual theory since it was too difficult to cope with the quantum corrected Coulomb branch. It took a while to realize that the right chiral operators useful to directly compute the vacuum structure of the Coulomb branch were just monopole operators. Actually, the success in the understanding of monopole operators is a very recent story, we already mentioned some of the papers that brought to this breakthorough, in cronological order [1], [34], [11], [12], [6] are some of them.

Our exposition has been rather faithful to the 2013 paper [16] by Hanany, Zaffaroni and Cremonesi. We repeated some calculations and checked some other results but in addition we tried to get some new results pointing at one of the directions along which the work contained in that paper could be expanded. For this reason we computed the

Hilbert series in several cases in presence of matter multiplets which transform in representations different from the fundamental and antifundamental. This generalization is easily incorporated in the formula suggested by the authors of [16] and the summation is incorporated in the Hilbert series which is its final product.

A few words have already been spent about the theory associated with the E quiver, which could be interesting to study in order to expand the work contained in this thesis. No dual theory is known for it. The Higgs branch is the singularity $\mathbb{C}^2/\Gamma_E$ of E type. Instead, the Coulomb branch is conjectured to be the moduli space of one E instanton and this is particularly interesting because no finite dimensional version of the ADHM construction is known for instantons of type E.

Finally, we ended with a chapter entirely dedicated to the beautiful subject of instantons, in order to give an idea of what is the moduli space of one instanton that we mentioned throughout the thesis. This ending chapter was far from being a complete explanation of the fascinating subject but was useful to get another taste of the wonderful web of dualities, links and implications that characterize the exciting world of quiver gauge theories.

Appendix A

Basic notions of algebraic geometry

A.1 Affine varieties and polynomial rings

The key idea behind algebraic geometry is to describe a space (for example a manifold) in terms of the vanishing of some polynomials. For example let us start by describing the con- cept of an affine variety. An affine variety is a set of points in n-dimensional complex space $\mathbb{C}^n$ with coordinates $(x_1, \ldots, x_n)$, which satisfy a set of polynomial equations of the form $f_i(x_1, \ldots, x_n) = 0$. An affine variety is thus loosely speaking a submanifold of flat space which is defined as the locus on which a set of polynomials vanish. It can be shown that the number of these polynomial equations can always be taken to be finite to describe any particular submanifold $\mathcal{M} \subseteq \mathbb{C}^n$. The purpose of algebraic geometry is to study geometrical properties of the variety $\mathcal{M}$, in the language of commutative algebra. This is because there is another way of describing such a submanifold of flat space which turns out to be more powerful when analyzing properties of the space. Instead of defining $\mathcal{M}$ in terms of the vanishing of a finite set of polynomials, we specify it by the set of all polynomials which vanish on $\mathcal{M}$. The resulting set of polynomials, which we shall refer to as $I(\mathcal{M})$, is an example of a polynomial ring.

We recall that a ring is simply a set with addition and multiplication. The set of polynomials is clearly a ring since adding and multiplying polynomials together return a polynomial. We denote the ring of polynomials in n variables with coefficients in $\mathbb{C}$ as $\mathbb{C}[x_1, ..., x_n]$. Now, $I(\mathcal{M})$ is clearly a subset of $\mathbb{C}[x_1, ..., x_n]$. In fact, $I(\mathcal{M})$ is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. This simply means that $I(\mathcal{M})$ is a closed subset in the following sense: multiplying any element of $I(\mathcal{M})$ by an element of $\mathbb{C}[x_1, \ldots, x_n]$ remains in $I(\mathcal{M})$.

Much like normal subgroups, because $I(\mathcal{M})$ is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$, it is possible to define the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I(\mathcal{M})$. This is defined to be the ring of all polynomials in the variables $x_1, \ldots, x_n$ where two polynomials are considered equivalent if they differ by a member of the ideal $I(\mathcal{M})$. It is not difficult to verify that this resulting object is also a polynomial ring.

The final object, the quotient ring $\mathbb{C}[x_1, \ldots, xn]/I(\mathcal{M})$, encodes all of the geometrical information about $\mathcal{M}$ in an algebraically powerful package. For example, the ideals of this quotient ring are in one-to-one correspondence, in a similar manner to the correspon-

dence between $\mathcal{M}$ and $I(\mathcal{M})$ described above, with submanifolds of $\mathcal{M}$.
The most important aspect of this correspondence is the following. The smallest possible submanifolds, the points of $\mathcal{M}$, correspond to maximal ideals of the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I(\mathcal{M})$. A maximal ideal is not contained in any other ideal, except for the trivial one, namely the ring itself. A maximal ideal is also a prime ideal, which for complex algebraic geometry means that the ideal corresponds to an irreducible variety, one that cannot be decomposed as the union of other non-empty algebraic varieties.

In a polynomial ring, all ideals are finitely generated. To introduce some notation, we denote an ideal which is generated by the elements $f_1, ..., f_n$ by $F = \langle f_1, \ldots, f_n \rangle$. In other words this is the ideal which vanishes on the affine variety defined by $f_1 = \cdots = f_n = 0$. We see that this formalism first of all naturally adapts to the concept of F-terms. The vanishing of n F-terms defines the ideal F , and the variety of F-flatness corresponds to the quotient ring $\mathcal{F} = \mathbb{C}[x_1, \ldots, x_n]/F$.

Recasting the computation of the vacuum geometry into efficient, algorithmic techniques in algebraic geometry is the subject of the papers [29], [27], [28].

We recollect here the method.

1. The F-flatness conditions are an ideal of the polynomial ring $\mathbb{C}[\phi_1, \ldots, \phi_n]$:

$$\langle f_{i=1,\ldots,n} \rangle = \left\langle \left(\frac{\partial W}{\partial \phi} \right)_{i=1,\ldots,n} \right\rangle . \tag{A.1}$$

2. From the matterfields $\{\phi_1, \ldots, \phi_n\}$, we construct abasis of GIOs $\rho = \{\rho_1, \ldots, \rho_k\}$. The definitions of the GIOs in terms of the fields defines a natural ring map:

$$\mathbb{C}[\phi_1, \ldots, \phi_n] \xrightarrow{\rho} \mathbb{C}[\rho_1, \ldots, \rho_k]. \tag{A.2}$$

3. The moduli space $\mathcal{M}$ is then the image of the ring map:

$$\frac{[\phi_1, \ldots, \phi_n]}{\langle f_{i=1,\ldots,n} \rangle} \xrightarrow{\rho} \mathbb{C}[\rho_1, \ldots, \rho_k]. \tag{A.3}$$

That is to say, $\mathcal{M} \simeq Im(\rho)$ is an ideal of $\mathbb{C}[\rho_1, \ldots, \rho_k]$ which corresponds to an affine variety in $\mathbb{C}^k$. Practically, the image of the map, and thus the vacuum geometry $\mathcal{M}$, can be calculated using Grbner basis methods as implemented in the algebraic geometry software packages Macaulay 2 [30] and Singular [35].

A.2 Projective varieties and affine cones

While easy to define, affine varieties are sometimes difficult to work with in calculations. This is ultimately because non-compact embedding spaces can lead to various difficulties, such as points escaping off to infinity. One of the advantages of using projective varieties is to remove such difficulties by "compactifying" the varieties.

We recall that projective space $\mathbb{P}^n$ is the space of one-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. Points in projective space are labeled by coordinates $[x_0 : x_1 : \ldots : x_n]$, and $[x_0 : x_1 : \ldots : x_n] = [y_0 : y_1 : \ldots : y_n]$ when there is a non-zero $\lambda \in \mathbb{C}$ such that $y_i = \lambda x_i$ for all i. The equivalence classes of points define homogeneous coordinates.

Projective varieties are defined in a manner similar to their affine counterparts, but slightly more care is required in the definition. One cannot simply talk about polynomial equations on projective space. If a set of polynomial equations in the homogeneous variables is homogeneous, which is to say that all the monomials have the same total degree, then we can talk instead about the loci of zeros of such equations. This locus is invariant under the rescaling of the homogeneous coordinates. Thus we can define a projective variety as a set of points in projective space where a series of homogeneous polynomials in the projective coordinates vanishes.

All of the objects described above in connection with the affine variety $\mathcal{M}$ then have analogues in the case of a projective variety.

Since we are computing the vacuum moduli space embedded into the space of fields, which is clearly $\mathbb{C}^n$, we are interested in affine varieties. Why then have we introduced projective varieties here? Our goal is to introduce the concept of an affine cone, an object we make extensive use. The reason for adopting this setup is that projective algebraic geometry, due to compactness, is much easier to handle. We are computing the local properties of the vacuum space. It will often (but not always!) be the case that the moduli space is an affine cone over some compact projective space whose properties we can directly calculate.

The base of an affine cone is simply defined by considering an affine variety and then taking the variables $x_1, \ldots, x_n$ to be homogeneous coordinates on projective space. From the comments on projective varieties above we see that this procedure is only possible if the equations defining the original affine variety are homogeneous. The radial direction of the cone can then be thought of as the scale of the projective space (with the point at zero scale put back in). As a simple example consider the case where we have two variables x_1 and x_2 and where the defining equations of the original affine variety are trivial. The base of the resulting affine cone will then simply be $\mathbb{P}^1$, with projective coordinates $[x_1 : x_2]$. The space $\mathbb{C}^2$, with affine coordinates (x_1, x_2), is then an affine cone over $\mathbb{P}^1$. Thus flat space is an affine cone over complex projective space.

Appendix B

Affine Dynkin diagrams

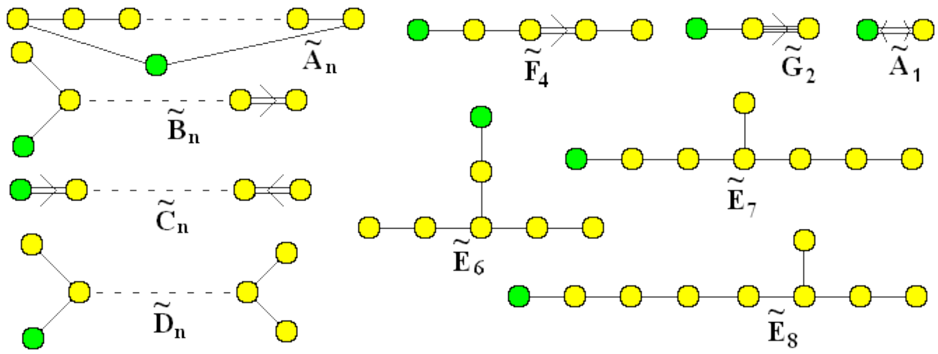


Figure B.1: The set of extended affine Dynkin diagrams, with added nodes in green

An affine Lie algebra is an infinite-dimensional Lie algebra that is constructed out of a finite-dimensional simple Lie algebra. It is a Kac-Moody algebra for which the generalized Cartan matrix is positive semi-definite and has corank 1. From purely mathematical point of view, affine Lie algebras are interesting because their representation theory, like representation theory of finite dimensional, semisimple Lie algebras is much better understood than that of general Kac-Moody algebras.

These affine Lie algebras are particularly Important in conformal field theory and string theory, but in this thesis they simply appear as quiver diagrams in the McKay corespondence. For this reason we will not give here a detailed description of affine Lie algebra and their representations since this would be beyond our scope. The interested reader could turn for example to [25].

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