UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”
Corso di Laurea Magistrale in Fisica

Tesi di Laurea

Membrane-mediated vacuum decays, three-forms and the Weak Gravity Conjecture

Relatore
Prof. Luca Martucci

Laureando
Andrea Sangiovanni

Anno accademico 2017/2018

Contents

Introduction

When taking a look around our universe we know from precision measurements that space-time is expanding, dragging around planets, stars and galaxies with it, thus furnishing decisive proof for the theory of the Big Bang. No more than two decades ago it was discovered that such an expansion proceeds at a slightly accelerating rate: the cause for this phenomenon, still unknown, may be the so called *dark energy*, an entity that would permeate all space, foreshadowed by Einstein in his famous general relativity equations via the introduction of the cosmological constant[1]. Despite this constant evolution of the universe, at a quick glance it may seem reasonable to think that its "state", that is its particle content and the values of the coupling constants, cannot undergo an abrupt change. The first blow to this picture was given in the sixties by the Higgs mechanism hypothesis, that predicted a transition phase due to symmetry breaking.

In the late seventies, in addition, it was first laid down in a mathematically precise way the idea that, even if our universe resides in a minimum of its potential, it may not be so stable after all, thanks to the pioneering work of Coleman and De Luccia ([30][31]). In fact, if another minimum with lower energy exists, the universe may be prone to reach it via a quantum tunneling process, thus changing its state: if many successive minima with lower energy are present, this transition process could go on for a long time, coming to an end when the universe reaches the lowest vacuum state, or when the transition to another minimum becomes impossible for energetic reasons.

The intriguing (and, in some way, daunting) concept that the universe may be in an unstable state, or that it was so in the distant past, has sparked many fruitful ideas to solve conundrums that regard our present observations. In this respect, great attention has been devoted to the cosmological constant problem, devising relaxation mechanisms that could in principle explain the astonishingly small rate of acceleration that our universe is subject to. In the work of Brown and Teitelboim [41], dating back to the late eighties, a four-form kinetic term was added to the usual Einstein-Hilbert action in four dimensions, yielding

[1] We recall that Einstein's equations with a cosmological constant read: $R_{mn} - \frac{1}{2}Rg_{mn} + \Lambda g_{mn} = 8\pi G T_{mn}$

something like (neglecting the appropriate coefficients)[2]:

$$S = S_{GRAV} + \int F^{mnrs} F_{mnrs} \tag{1}$$

The core idea of their mechanism relies on the fact that in four-dimensions this field strength is non-dynamical and can be expressed in terms of a single constant. This means that using its equation of motion the four-form field strength F^{mnrs} can be written as:

$$F^{mnrs} = c\epsilon^{mnrs}, \tag{2}$$

where c is a constant. It follows that the term $\int F^{mnrs} F_{mnrs}$ contributes with a factor $\propto c^2$ to the vacuum energy of the theory, effectively acting as a cosmological constant. Brown and Teitelboim further showed that the value of c can change if a membrane coupled to the three-form potential[3] of F^{mnrs} appears. This mechanism is completely analogous to the discharge of an electric field between two plates due to the appearance of pairs of electrons inside the field, a phenomenon known as Schwinger process. We see then its relevance for the vacuum instability problem: if the universe sits in a vacuum with some value of the cosmological constant c, the nucleation of a spherical membrane can cause it to tunnel to another minimum, with cosmological constant c', and this event could be repeated until a stable vacuum is reached. On the inside of the membrane there is the new ("true") vacuum state, whereas the outside remains in the old ("false") one.
Although affected by a few problems, Brown and Teitelboim's approach has received conspicuous attention, producing more sophisticated hypotheses regarding the cosmological constant relaxation in contexts such as string theory, for example through the work of Bousso and Polchinski [42].

As a matter of fact, string theory is a natural realm in which to study vacuum transitions processes, thanks to the variety of potentials in four dimensions that it can give rise to. More specifically, the starting point when studying string theory is a ten-dimensional action, that apparently has no relation to our four-dimensional world. In the low energy-limit, however, it is possible to get rid of the six extra dimensions (via a compactification technique) and to obtain a supergravity effective theory in four spacetime dimensions, that will possess some scalar potential with its corresponding extrema. The crucial point is that the properties of the extra dimensions and of the fields that live on them greatly influence the shape of the scalar potential in four dimensions, ultimately determining the precise location of its extrema. In the course of the thesis we will consider models in which a contribution to the potential is given by the fluxes of some gauge field-strengths, the generalization to higher

[2]Here and in the rest of the thesis we employ the notation of Wess and Bagger [7]:

- m $(\mu, \dot{\mu})$ for bosonic (fermionic) curved indices
- a $(\alpha, \dot{\alpha})$ for bosonic (fermionic) flat indices

[3]The three-form potential enters the field strength as: $F^{mnrs} = \partial^{[m} A^{nrs]}$.

dimension of the familiar electromagnetic field-strength. Just like the latter can have a non-trivial flux through a surface, the generalized gauge field-strengths can thread the extra dimensions with some flux, directly influencing the structure of the scalar potential in four dimensions. Concretely speaking, a different value of the flux of some field strength through the extra dimensions causes a different scalar potential in four dimensions.

The main objective of the thesis is to take into account a specific model of string compactification (described in [25]) and to study, in the context of the resulting four-dimensional supersymmetric effective theory, the transitions among vacuum states characterized by different values of the fluxes, by means of the nucleation of a membrane that separates the true from the false vacuum. This means that on one side of the membrane there will be a potential defined by some values of the fluxes e, whereas on the other side they will assume different values e'.

In order to take on this study in a mathematically precise way, it is appropriate to employ a formalism that allows a dynamical treatment of the values of the fluxes, as a consequence of the fact that during the transition they change from e to e'. In a few recent articles (we refer for example to [20] and [21]) this has been achieved by introducing new gauge three-forms that substitute the values of the fluxes e, that completely disappear from the potential. In any case, the original theory can be recovered by imposing the equations of motion for the new three-forms, re-obtaining a potential dependent on the fluxes.

If we stick to using the new gauge three-forms, instead, the updated transition mechanism works in this way: just like in the Brown-Teitelboim approach, the gauge three-forms can be coupled to membranes, that will therefore be characterized by some quantized charge q. This is exactly the higher dimensional alias of how the electromagnetic potential A_m can be coupled to a charged particle like the electron. Then, by using the equations of motion for the three-forms, we will show that the quantized charge q causes a jump $e' - e = q$ in the values of the fluxes on the two sides of the membrane, in this way completely describing the vacuum transition.

The advantage of the new formalism, that in a nutshell substitutes the fluxes with gauge three-forms, is that it provides a natural way to add membranes into the theory, introducing a coupling between them and the three-forms. At the same time, the gauge three-forms have field strengths that are four-forms and consequently, as in (2), they are non-dynamical: in our case the constant c that appears in the equation of motion (2) is the value of the fluxes e.

We will hence concentrate on re-writing the model of [25] in the new formulation, showing that all the properties of the original theory are maintained (for example, showing that the scalar potential written with the three-forms is compatible with the one written with the explicit values of the fluxes). Then, we will focus on the transitions among different AdS vacua mediated by membranes, trying to understand what energetic constraints come into play. In particular, other than a charge the membranes possess a tension (intuitively speaking, the mass per unit area) that cannot be arbitrarily large if we want the transition to be allowed: it will be then of crucial importance to understand how to correctly assess this tension in order to test the feasibility of the vacuum decay in a concrete case. To this end, we will have to introduce various approximations (chiefly the so-called thin-wall approximation) and take into account the contribution that the fields of the model give to the membrane tension. This study is particularly relevant in light of the claim by Ooguri and Vafa (presented in [34]) that all non-supersymmetric AdS vacua are unstable.

Finally, having computed the charge and the tension of a membrane that mediates some decay among different vacuum states, we will be able to employ these results to the weak gravity conjecture, that roughly speaking states that the tension and the charge should be equal when the underlying state is supersymmetric. It is in this application that the three-forms formalism shows its power: it allows us to write kinetic terms for the three forms, similar to the canonical Yang-Mills term (namely, the kinetic term for the one-form electromagnetic potential A_m):

$$\mathcal{L}_{\text{Three-forms}} \propto -\frac{1}{g^2} F^{mnrs} F_{mnrs} \quad \longleftrightarrow \quad \mathcal{L}_{\text{electro-magnetic}} = -\frac{1}{4(q_e)^2} F^{mn} F_{mn}, \tag{3}$$

where q_e is the charge of the electron and g is the elementary charge possessed by the membrane that we will try to compute. The physical charge of the membrane, that shall be compared to the tension in order to examine the WGC, will hence be a multiple of the elementary charge g.

To shed some further light on this topic we will also consider some simple models different from the one described in [25], trying to draw some conclusions about the applicability of the weak gravity conjecture to our case, that involves charged membranes.

The thesis will be subdivided in the following way:

In **chapter 1** we will introduce some basic supersymmetry and supergravity notions, fundamental for the formalism that we will develop later.

In **chapter 2** we will show how a generic ten-dimensional action with fluxes deriving from string theory can be compactified to yield a four dimensional supergravity effective theory, as well as how to compute its scalar potential.

In **chapter 3** the gauge three-forms' formalism devised in [20] and [21], that resides in a supergravity context, will be exhibited.

Chapter 4 will treat the concrete application of the gauge three-forms' formalism to the model of [25].

In **chapter 5** we will study the extrema of our model and the transitions among AdS vacua with different values of the fluxes, taking into account the due approximations.

In **chapter 6**, finally, we will apply the results of chapter 4 and 5 to the weak gravity conjecture.

In **Appendix A** a few notions of complex geometry, especially useful for the content of chapter 2, will be recalled.

CHAPTER 1

Supersymmetry and Supergravity

Supersymmetry was first introduced in the 70's as an additional symmetry that could be shared by the particles that compose our universe: roughly speaking, it states that every boson (fermion) possesses a fermionic (bosonic) "supersymmetric partner". Of course as of now we do not know any of these partners (e.g. there is no clue of a bosonic partner for the electron), and so if supersymmetry is a real property of our world such additional particles should be extremely heavy (at least above the electroweak scale) or interact very feebly with known matter, even though research in this direction at the LHC and other experiments has been so far inconclusive [38][39].
Supersymmetry is a rigid theory, namely it acts in the context of Minkowskian spacetime: if it is made local it can be shown to encompass diffeomorphisms, so that it naturally includes gravity in its description. The theories that exhibit local supersymmetry are called "supergravity theories". In the next sections we will review both rigid and local supersymmetry, in order to pave the way for the formalism that we will develop in chapter 3, that resides in a supergravity context.

1.1 Rigid $\mathcal{N} = 1$ Supersymmetry in 4 dimensions

The Coleman-Mandula theorem states, under a few reasonable assumptions, that the Lie algebra of a quantum field theory containing the Poincaré group $\mathcal{P}$ can at most be of the form $\mathcal{P} \oplus \mathcal{A}$, where the generators of $\mathcal{A}$ are Lorentz scalars, as a result of the direct sum. This is valid provided that the theory is not composed only of massless particles (in that case further symmetries, such as conformal invariance, can arise).
It was subsequently shown by Haag, Lopuszanski and Sohnius that, including anticommutators in the definition of the algebra, the symmetry of the theory can be uniquely extended to include the so called supersymmetry. The generators of supersymmetry are Q^I_α, where α is a spinorial index, while $I = 1, ..., \mathcal{N}$ labels the number of generators. The Q^I_α form a graded algebra together with the Lorentz group generators (M_{mn},P_m). As a consequence, the Q^I_α belong to the representation $\left(\frac{1}{2}, 0\right)$ of the Lorentz group (it can be shown that higher

representations are forbidden, see e.g. [1]), as exemplified by the relation:

$$[M_{mn}, Q^I_\alpha] = i(\sigma_{mn})_\alpha^\beta Q^I_\beta \qquad [P_m, Q^I_\alpha] = 0 \tag{1.1}$$

The supersymmetry generators, instead, combine in the following way:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^{IJ} \tag{1.2}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^* \tag{1.3}$$

Dotted indices belong to the conjugate representation (dotted and undotted indices are raised and lowered by means of $\epsilon_{\dot\alpha\dot\beta}$ and $\epsilon_{\alpha\beta}$, respectively). Z^{IJ} are Lorentz scalars called central charges, i.e. they commute with the whole supersymmetry algebra. More importantly, they can be non-vanishing only if $\mathcal{N} > 1$. We have omitted, instead, possible tensorial central charges.
Physically speaking, it can be shown that the supersymmetry generators raise or lower the spin of half a unit, providing a link between bosons and fermions. It is therefore natural to expect that supersymmetrically invariant actions must contain equal numbers of bosonic and fermionic degrees of freedom.

The simplest example of such an action is the Wess-Zumino model, that contains a complex scalar ϕ, a Weyl spinor ψ and an auxiliary complex scalar F (essential if supersymmetry is required to close also off-shell). Both fermions and bosons have a total of 2 complex degrees of freedom, as prescribed. The action is[1] (see for instance [2]):

$$S_{WZ} = \int \mathrm{d}^4x \, \left(\partial_m\phi^*\partial^m\phi + i\bar\psi\bar\sigma^m\partial_m\psi + F^*F\right) \tag{1.4}$$

The corresponding supersymmetry transformations are as follows (ϵ is an infinitesimal Weyl spinor parametrizing the transformation):

$$\begin{aligned} \delta\phi &= \epsilon\psi \\ \delta\psi_\alpha &= -i(\bar\sigma^m\epsilon^\dagger)_\alpha\partial_m\phi \\ \delta F &= -i\epsilon^\dagger\bar\sigma^m\partial_m\psi \end{aligned} \tag{1.5}$$

The most convenient way to generalize this approach and obtain automatically supersymmetry-invariant actions is to take advantage of the superspace formalism [3]. It is customary to introduce the Grassmann numbers θ_α,$\bar\theta_{\dot\alpha}$, which enjoy the usual anticommutation, derivation and integration rules. In addition they must have $-\frac{1}{2}$ mass dimension so as to give rise to sensible actions. A generic function (named superfield) of the spacetime coordinates and the

[1] The convention $\bar\sigma^m = (1, -\sigma^i)$ where σ^i are the usual Pauli matrices has been used.

Grassmann numbers can therefore be expanded as:

$$Y(x,\theta,\bar\theta) = f(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^m\bar\theta A_m(x) + \\ + \theta\theta\bar\theta\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta d(x) \tag{1.6}$$

In this formalism the supersymmetry charges can be shown to be:

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m \qquad \bar Q_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m, \tag{1.7}$$

where $\sigma^m = (1, \sigma^i)$, with 1 the identity and σ^i the Pauli matrices.
In order to reduce the number of components of $Y(x,\theta,\bar\theta)$, so as to fit the remaining ones in a representation of the supersymmetry algebra, constraints have to be introduced. The chiral constraint makes use of covariant derivatives with respect to the supersymmetry transformations, defined as follows:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial x^m} \qquad \bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\beta\sigma^m_{\dot\alpha\beta}\frac{\partial}{\partial x^m} \tag{1.8}$$

Their anticommutation rule is (while all other anticommutators among the D's and the Q's vanish):

$$\{D_\alpha, \bar D_{\dot\beta}\} = -2\sigma^m_{\alpha\dot\beta}P_m \tag{1.9}$$

Hence a chiral field will satisfy the condition:

$$\bar D_{\dot\alpha}\Phi = 0 \tag{1.10}$$

In order to find the most general component expansion of a chiral field it is useful to introduce new coordinates y^m, defined as:

$$y^m = x^m + i\theta\sigma^m\bar\theta \qquad \bar y^m = x^m - i\theta\sigma^m\bar\theta \tag{1.11}$$

The advantage of utilizing these coordinates resides in their properties:

$$\bar D_{\dot\alpha}\theta_\beta = 0 = \bar D_{\dot\alpha}y^m \qquad D_\alpha\bar\theta_\beta = 0 = D_\alpha\bar y^m \tag{1.12}$$

Using the chirality condition (1.10) it can be seen that a chiral superfield can depend only on θ and y^m, whereas an explicit dependence on $\bar\theta$ is forbidden. Therefore its component expansion will be:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2F(y) \tag{1.13}$$

It can be hence shown that the corresponding component expansion for the variables x^m is:

$$\Phi(x,\theta,\bar\theta) = \phi + \sqrt{2}\theta\psi + \theta^2F + i\theta\sigma^m\bar\theta\partial_m\phi - \frac{i}{\sqrt{2}}\theta^2\partial_m\psi\sigma^m\bar\theta + \frac{1}{4}\theta^2\bar\theta^2\Box\phi \tag{1.14}$$

Furthermore it is useful to see how to extract the various components, starting from the superfield Φ:

$$\begin{aligned} \Phi|_{\theta=\bar{\theta}=0} &= \phi \\ \frac{1}{\sqrt{2}} D_\alpha \Phi|_{\theta=\bar{\theta}=0} &= \psi_\alpha \\ -\frac{1}{4} D^2 \Phi|_{\theta=\bar{\theta}=0} &= F \end{aligned} \tag{1.15}$$

As an aside it is relevant to observe that the projector $-\frac{1}{4}D^2$ corresponds, up to a total derivative, to the integration measure $\int \mathrm{d}^2\theta$.
Another kind of supersimmetrically-invariant restriction that can be imposed is a reality condition on a so called vector superfield V:

$$V = \bar{V} \tag{1.16}$$

The component expansion of V is rather long but, as is valid for the chiral field, the number of fermionic degrees of freedom equals the number of bosonic ones. It is immediately shown that the chiral and vector superfields correspond respectively to the chiral and vector multiplets deriving from the supersymmetry representations.
Supersymmetric actions will be built out of these and other superfields, in a suitable way:

$$\mathcal{S} = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} X, \tag{1.17}$$

with X a generic real superfield. Using the invariance of the Grassmann integration measure upon translation and the explicit expression of the supersymmetry transformations it is easily shown that (1.17) is indeed invariant.
The most general lagrangian density furnishing the chiral fields with kinetic terms can be assembled starting from a Kähler potential, i.e. a function of some chiral fields and their conjugates $K(\Phi^i, \bar{\Phi}^i)$:

$$\mathcal{L}_K = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} K(\Phi, \bar{\Phi}) \tag{1.18}$$

To make things more clear, we show the simplest possible example of a Kähler potential involving just one scalar field:

$$K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi \tag{1.19}$$

The reason for the name of the Kähler potential is that its second derivative $g_{i\bar{\jmath}}$ (also called Kähler metric and often denoted $K_{i\bar{\jmath}}$) is present, after component expansion, in the scalar and spinorial kinetic terms: in particular this means that the scalar fields constitute coordinates on the Kähler manifold with metric $K_{i\bar{\jmath}}$. For a discussion of such manifolds we refer to appendix A. In any case, the Kähler metric is defined as:

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} \tag{1.20}$$

The lagrangian density $\mathcal{L}_K$, in addition, enjoys an invariance under Kähler transformations (where Λ and $\bar{\Lambda}$ are holomorphic functions):

$$K(\phi^i, \bar{\phi}^i) \longrightarrow K(\phi^i, \bar{\phi}^i) + \Lambda(\phi^i) + \bar{\Lambda}(\bar{\phi}^i) \tag{1.21}$$

A further term, giving non-derivative "interactions" among the components of the chiral superfield, can be added to the lagrangian:

$$\mathcal{L}_W = \int \mathrm{d}^2\theta \, W(\Phi^i) + \int \mathrm{d}^2\bar{\theta} \, \bar{W}(\bar{\Phi}^i) \tag{1.22}$$

W is called superpotential and, in order to make sure that the lagrangian is invariant under supersymmetry, it must necessarily be a holomorphic function of Φ^i (that is, it cannot contain any $\bar{\Phi}^i$). In fact this implies that W is a chiral superfield, i.e.

$$\bar{D}_{\dot{\alpha}} W = \frac{\partial W}{\partial \Phi^i} \bar{D}_{\dot{\alpha}} \Phi^i + \frac{\partial W}{\partial \bar{\Phi}^i} \bar{D}_{\dot{\alpha}} \bar{\Phi}^i = 0, \tag{1.23}$$

where the first term is zero because the Φ^i are chiral and the second vanishes because of the holomorphicity of W.
Consequently the most general form of the superpotential is:

$$W(\Phi^i) = \sum_i \sum_{n=1}^{\infty} a_{n,i} \Phi_i^n \tag{1.24}$$

The total lagrangian for a set of chiral fields $\{\Phi^i\}$ will hence be $\mathcal{L}_{\text{chiral}} = \mathcal{L}_K + \mathcal{L}_W$. If we wish to include a gauge interaction for the chiral fields a coupling with the superfield V must be introduced. The corresponding modified "super Yang-Mills" lagrangian is[2]:

$$\mathcal{L}_{SYM} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \bar{\Phi} e^V \Phi + \int \mathrm{d}^2\theta \, W(\Phi^i) + \int \mathrm{d}^2\bar{\theta} \, \bar{W}(\bar{\Phi}^i) + \frac{1}{4g^2} \mathrm{Tr} \int \mathrm{d}^2\theta \, W^\alpha W_\alpha \tag{1.25}$$

where g is the gauge coupling and W_α is the "generalized" field strength defined as:

$$W_\alpha = -\frac{1}{4} \bar{D}^2 (e^{-V} D_\alpha e^V) \tag{1.26}$$

The lagrangian $\mathcal{L}_{SYM}$ is then invariant under the action of a non-abelian symmetry group

[2]Neglecting Fayet-Iliopulos terms which arise if the symmetry group is not semi-simple.

generated by $\{T^a\}$, whose explicit transformations are:

$$\begin{cases} e^V \longrightarrow e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \\ \Phi \longrightarrow e^{i\Lambda}\Phi, \end{cases} \tag{1.27}$$

with V and Λ taking values in the Lie algebra of the symmetry group.
In this context it is common to adopt the Wess-Zumino gauge, resulting in the following component expansion of V (A_m is a vector boson, λ is a fermion, the gaugino, and D is an auxiliary field):

$$V = \theta\sigma^m\bar{\theta}A_m + +i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \qquad \Rightarrow \qquad e^V = 1 + V + \frac{1}{2}V^2 \tag{1.28}$$

After expanding $\mathcal{L}_{SYM}$ in components we obtain, for the bosonic sector of the theory:

$$\begin{aligned} \mathcal{L}_{SYM} = \mathrm{Tr}\left[-\frac{1}{4g^2}F_{mn}F^{mn} + \frac{1}{2}D^2\right] + \overline{D_m\phi}D^m\phi + \bar{F}_iF^i + \\ - \frac{\partial W}{\partial\phi^i}F^i - \frac{\partial\bar{W}}{\partial\bar{\phi}_i}\bar{F}_i + \bar{\phi}_iT^a\phi^iD^a, \end{aligned} \tag{1.29}$$

where F_{mn} is the field strength of A_m, F^i are the auxiliary fields of the chiral superfields Φ^i and D^a are the auxiliary fields of V. The equations of motion for the auxiliary fields read:

$$\bar{F}^i = \frac{\partial W}{\partial\phi^i} \qquad D^a = -\bar{\phi}_iT^a\phi^i \tag{1.30}$$

When the equations (1.30) are inserted into the bosonic lagrangian (1.29) the scalar potential V is obtained:

$$V(\phi,\bar{\phi}) = \frac{\partial W}{\partial\phi^i}\frac{\partial\bar{W}}{\partial\bar{\phi}_i} + \frac{1}{2}\sum_a |\bar{\phi}_i(T^a)^i_j\phi^j|^2 \tag{1.31}$$

As a consequence the supersymmetric vacua of the theory will have to be found by setting to zero the scalar potential and looking for possible solutions. This is so because it can be shown that a vacuum is supersymmetric if and only if its energy is zero. In fact, recalling the supersymmetry algebra (1.3) (for example with $\mathcal{N} = 1$) and considering a vacuum state $|\Omega\rangle$ we see that:

$$\langle\Omega|P^0|\Omega\rangle \simeq \sum_{\alpha,\dot{\alpha}} \left(||Q_\alpha|\Omega\rangle||^2 + ||\bar{Q}_{\dot{\alpha}}|\Omega\rangle||^2\right) \geq 0 \tag{1.32}$$

Hence the vacuum is supersymmetric (namely, Q_α and $\bar{Q}_{\dot{\alpha}}$ act trivially on it) only if its energy P^0 vanishes.

1.2 $\mathcal{N} = 1$ Supergravity in 4 dimensions

The starting point in deriving the minimal supergravity action consists in making the supersymmetry transformation parameter ϵ local, i.e. dependent on the spacetime coordinates,

$\epsilon(x)$. Doing so and considering the Wess-Zumino action (1.4) it can be seen [4] that it is necessary to introduce a vectorial spinor ψ_m^α with an appropriate variation depending on the derivative of ϵ, if we wish that the action is still supersymmetry-invariant. ψ_m^α transforms under the $(1, \frac{1}{2})$ representation of the Lorentz group, and it is called gravitino. Moreover, to maintain the equality between fermionic and bosonic degrees of freedom, a graviton g_{mn} has to be introduced. Hence gravity is naturally present in locally supersymmetric theories.

As a matter of fact it is more convenient to restate the theory through Cartan formalism. In this perspective the metric tensor satisfies:

$$g_{mn}(x) = e_m^a(x) e_n^b(x) \eta_{ab}, \tag{1.33}$$

where η_{ab} is the flat Lorentz metric and the a, b indices are subject to the action of local Lorentz transformations $\Lambda^a{}_b$:

$$e_m^{a'}(x) = \Lambda^a{}_b(x) e_m^b(x) \tag{1.34}$$

Flat and curved indices are related via the vielbeins; for example in the case of gamma matrices:

$$\gamma_m = e_m^a \gamma_a \tag{1.35}$$

A covariant derivative (with respect to local Lorentz transformations) can be defined acting on spinors such as the gravitinos as:

$$D_m \psi_n = \partial_m \psi_n + \frac{1}{4} \omega_m{}^{ab} \gamma_{ab} \psi_n \tag{1.36}$$

where $\frac{1}{4}\gamma_{ab}$ are the generators of the Lorentz group and $\omega^a{}_b$ is the spin connection (i.e. a connection on the spinor bundle), and, as in the case of the Christoffel connection, it is possible to define the Ricci curvature tensor (we are using the conventions of appendix A), related to the usual one by means of the vielbeins:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{1.37}$$

The full Riemann tensor, furthermore, satisfies the usual antisymmetry properties:

$$R_{[cd}{}^a{}_{b]} = 0 \tag{1.38}$$

Covariant derivatives satisfy, analogously to the Yang Mills case, the familiar relation with the curvature tensor (that is, the "field strength" of the gauge field [6]):

$$[D_m, D_n] = \frac{1}{4} R_{mn}{}^{ab} \gamma_{ab} \tag{1.39}$$

Moving to flat indices a further term, the torsion, appears:

$$[D_a, D_b] = T^c{}_{ab} D_c + R_{ab}{}^{cd} \frac{1}{4} \gamma_{cd} \qquad \text{with} \qquad T^c{}_{ab} \equiv 2 e_a^m e_b^m D_{[m} e_{n]}^c \tag{1.40}$$

Mathematically speaking, a vanishing torsion implies that the spin connection can be fully computed starting from the vielbeins, in the following way:

$$\omega_m^{ab}[e] = \frac{1}{2} e_{cm}(\Omega^{abc} - \Omega^{bca} - \Omega^{bac}) \qquad \text{with} \qquad \Omega_{abc} = e_a^m e_b^n (\partial_m e_{nc} - \partial_n e_{mc}) \tag{1.41}$$

Instead, if the torsion is non-vanishing, an additional term (the so called "contorsion tensor"), dependent on the torsion, must be added:

$$\hat{\omega}_m^{ab}[e] = \omega_m^{ab}[e] + K^a{}_m{}^b \tag{1.42}$$

This will be seen to happen in the supergravity case. The supergravity lagrangian will therefore contain a kinetic term for the graviton (the Einstein-Hilbert lagrangian $\mathcal{L}_{EH}$), a kinetic term for the gravitino (the Rarita-Schwinger lagrangian $\mathcal{L}_{RS}$) and a term quartic in the gravitino to enforce supersymmetry:

$$\begin{aligned}\mathcal{L}[e,\psi] &= \mathcal{L}_{EH}[e] + \mathcal{L}_{RS}[e,\psi] + \mathcal{L}_{\psi^4}[e,\psi] = \\ &= -\frac{1}{4}|e| e_a^m e_b^n R_{mn}{}^{ab} + \frac{1}{2}\epsilon^{mnrs}\bar{\psi}_m \gamma_n \gamma_5 D_r \psi_s + \mathcal{L}_{\psi^4}[e,\psi],\end{aligned} \tag{1.43}$$

where $|e|$ is the determinant of the vielbein. The quartic term is proportional to the contorsion tensor:

$$\mathcal{L}_{\psi^4}[e,\psi] = -\frac{1}{4}|e|(K_a{}^{ac}K_b{}^b{}_c + K^{abc}K_{cab}), \tag{1.44}$$

where:

$$K^a{}_m{}^b = -i(\bar{\psi}^{[a}\gamma^{b]}\psi_m + \frac{1}{2}\bar{\psi}^a\gamma_m\psi^b) \tag{1.45}$$

The corresponding supersymmetry variations are:

$$\delta e_m^a = -i\bar{\epsilon}_\alpha \gamma^a \psi_m^\alpha \qquad \delta\psi_m^\alpha = D_m \epsilon^\alpha \tag{1.46}$$

It can be noted that the variation of the "gauge field", that is the gravitino, is proportional to the transformation parameter, like in the Yang Mills case. It is possible to exhibit a more compact form of the action considering the spin connection in the non-vanishing torsion case, namely $\hat{\omega}_m^{ab}[e]$, and thus absorbing the quartic terms. With this expedient the lagrangian and the supersymmetry transformations become:

$$\begin{aligned}&\hat{\mathcal{L}}[e,\psi] = -\frac{1}{4}|e| e_a^m e_b^n R_{mn}{}^{ab} + \frac{1}{2}\epsilon^{mnrs}\bar{\psi}_m\gamma_n\gamma_5\hat{D}_r\psi_s \\ &\text{with:} \qquad \hat{D}_m\psi_n = \partial_m\psi_n + \frac{1}{4}\hat{\omega}_m{}^{ab}\gamma_{ab}\psi_n \\ &\delta e_m^a = -i\bar{\epsilon}_\alpha\gamma^a\psi_m^\alpha \qquad \delta\psi_m^\alpha = \hat{D}_m\epsilon^\alpha\end{aligned} \tag{1.47}$$

It can be shown that the action of two consecutive supersymmetry transformations on the vielbein results in a combination of a diffeomorphism (parametrized by ξ^m), a local Lorentz transformation (parametrized by $\Lambda^a{}_b$), and a supersymmetry transformation (dependent on

ϵ), thus ensuring the closure of the algebra, that is:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e^a_m = \xi^n \partial_n e^a_m + e^a_n \partial_m \xi^n + \Lambda^a{}_b e^b_m + \epsilon \gamma^a \psi_m \tag{1.48}$$

As far as the gravitino is concerned the supersymmetry algebra closure is guaranteed by its equation of motion. If, instead, we do not impose the equations of motions auxiliary fields must be introduced to match the degrees of freedom (d.o.f.). In fact, the vielbein has 6 off-shell d.o.f.: starting from 16 components, 4 can be fixed via diffeomorphisms, whereas 6 are set by local Lorentz transformations. The gravitino, instead, has a 12 off-shell d.o.f.: the number of components is $2^2 \times 4$, of which 4 get canceled by local supersymmetry transformations depending on ϵ. The six missing bosonic degrees of freedom are provided by an auxiliary vector field b_m and a complex scalar M. The supersymmetry transformations (1.47) must then be modified to accommodate the new fields. The complete supergravity multiplet is therefore composed of the vielbein, the gravitino and the auxiliary vector and complex scalar:

$$e^a_m \qquad \psi^\alpha_m \qquad b_m \qquad M \tag{1.49}$$

Like in the case of rigid supersymmetry, convenience in the formulation of locally supersymmetric actions in more involved cases implies the use of the superspace formalism: the indices $A = (a, \alpha, \dot{\alpha})$ will denote flat space and $M = (m, \mu, \dot{\mu})$ the curved space ones. The basic objects used to build the supergravity action in the superspace formalism are the super-vielbein $E^A_M(x, \theta)$ and the super-spin connection Ω^{AB}_M. Super-Einstein transformations parametrized by ξ^M act on these objects: the components $\xi^m(x)|_{\theta=0}$ are the diffeomorphisms, whereas the $\xi^\mu|_{\theta=0} = \epsilon^\mu$ are local supersymmetry transformations. In addition there are the local Lorentz transformations Λ^{AB}: stemming from the fact that the spin is a Casimir of the Lorentz group representations, Λ^{AB} cannot contain terms mixing bosons and fermions. Further requiring that no new generators appear, the super-Lorentz transformations matrix takes the form (with $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$):

$$\Lambda^{AB} = \begin{pmatrix} \Lambda^{ab} & 0 & 0 \\ 0 & -\frac{1}{4}(\sigma_{ab})_{\alpha\beta}\Lambda^{ab} & 0 \\ 0 & 0 & -\frac{1}{4}(\sigma_{ab})_{\dot{\alpha}\dot{\beta}}\Lambda^{ab} \end{pmatrix} \tag{1.50}$$

Reasoning in an analogue way for the spin connection (which, as said, can be thought as the connection for the Lorentz transformations) the following result is obtained:

$$\Omega^{AB}_M = \begin{pmatrix} \Omega^{ab}_M & 0 & 0 \\ 0 & -\frac{1}{4}(\sigma_{ab})_{\alpha\beta}\Omega^{ab}_M & 0 \\ 0 & 0 & -\frac{1}{4}(\sigma_{ab})_{\dot{\alpha}\dot{\beta}}\Omega^{ab}_M \end{pmatrix} \tag{1.51}$$

It is useful then to define super-covariant derivatives (remembering that the index M splits into m and $\mu, \dot{\mu}$):

$$\mathcal{D}_M = \partial_M + \frac{1}{4}\Omega_M^{ab}\gamma_{ab} \tag{1.52}$$

Just as in the regular case, the super-torsion T_{MN}^P and the super-curvature tensor R_{MN}^{mn} can be appropriately defined through the commutator (for bosonic components) and anticommutator (for fermionic components) of the flat covariant derivatives (linked to the curved ones by means of the super-vielbein):

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b] &= T_{ab}^C \mathcal{D}_C + \frac{1}{4}\mathcal{R}_{ab}^{cd}\gamma_{cd} \\ \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= T_{\alpha\beta}^C \mathcal{D}_C + \frac{1}{4}\mathcal{R}_{\alpha\beta}^{cd}\gamma_{cd} \end{aligned} \tag{1.53}$$

$\mathcal{R}$ is the supergravity multiplet, whose bosonic components are:

$$\mathcal{R} = -\frac{1}{6}M + \theta^2\left[-\frac{1}{2}R + \frac{2}{3}MM^* + \frac{1}{3}b^a b_a - ie_a^m \mathcal{D}_m b^a\right], \tag{1.54}$$

where M is the complex auxiliary field and b_a is the vector auxiliary field with flat index. R is the usual Ricci curvature, that depends on the vielbein and the spin connection, and is defined as:

$$R = e_a^m e_b^n (\partial_m {\omega_n}^{ab} - \partial_n {\omega_m}^{ab} + {\omega_n}^{ac}{\omega_{mc}}^b - {\omega_m}^{ac}{\omega_{nc}}^b) \tag{1.55}$$

In order to match the superspace description with the regular one a convenient gauge choice of the super-Einstein transformations can be made:

$$E_m^a(x)|_{\theta=0} = e_m^a \qquad E_m^\alpha(x)|_{\theta=0} = \psi_m^\alpha \qquad \Omega_m^{ab}(x)|_{\theta=0} = \omega_m^{ab} \tag{1.56}$$

Moreover a few components of the torsion must be adjusted to accommodate various requirements and fix some components [7]: the most relevant are a consistency condition, $T_{\alpha\beta}^{\dot\gamma} = T_{\alpha\beta}^a = 0$, which guarantees that the chiral constraint $\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$ is well defined, and the super-conformal choice $T_{\alpha a}^a = 0$, ensuring that no non-sense equation of motion (such as $E = 0$) appears.

The next task ahead is to find an invariant measure suitable for the supergravity action. It can be shown that the appropriate integration measure is:

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta \ \ \mathrm{sdet}E_M^A \equiv \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta \ \ E, \tag{1.57}$$

where $\mathrm{sdet}E_M^A$ is the super-determinant (also called Berezinian) of E_M^A, a generalization of the usual determinant for matrices in superspace. Finally, it can be shown that in the case of rigid superspace $\mathrm{sdet}E_M^A = 1$, so that the correct measure is recovered.

The wish is to construct a theory resembling the action (1.25) with proper modifications to accommodate curved superspace: in order to introduce superpotential terms a chiral invariant measure has to be found. It can be shown that the following quantity suits this requirement:

$$\mathcal{E} = \frac{\bar{\mathcal{D}}^2 E}{\mathcal{R}} \tag{1.58}$$

The chiral invariant measure will therefore be defined as:

$$\int \mathrm{d}^4x \mathrm{d}^2\theta \, 2\mathcal{E} \tag{1.59}$$

The last thing left to generalize is the projector $-\frac{1}{4}D^2$ appearing in equation (1.15); using the commutation rules (1.53) it can be shown that the appropriate projector in supergravity is $-\frac{1}{4}(\mathcal{D}^2 - 8\mathcal{R})$. As a result the two can be interchanged when passing from rigid to curved superspace. For example the new field strength W_α for the vector multiplet V will be:

$$W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8\mathcal{R}) e^V \mathcal{D}_\alpha e^{-V} \tag{1.60}$$

Combining everything together the most general supergravity action (with no more than second derivatives) describing chiral superfields Φ^i coupled to a vector superfield V is [8]:

$$\begin{aligned} \mathcal{S}_{SG} = -3 \int \mathrm{d}^4x \mathrm{d}^4\theta \, E \, e^{-\frac{1}{3}K\left(\Phi_i, \bar{\Phi}_i e^{2V}\right)} + \\ + \int \mathrm{d}^4x \mathrm{d}^2\theta \, 2\mathcal{E} \, \left(W(\Phi_i) + \bar{W}(\bar{\Phi}_i) + \frac{1}{16g^2}\mathrm{Tr}(W^\alpha W_\alpha) \right), \end{aligned} \tag{1.61}$$

where g is the usual gauge coupling (that, in full generality, could also depend on the Φ^i). The term which differs the most from (1.25) is the one involving the Kähler potential: the reason why it appears lies in requiring the correct normalization of the Einstein action and the kinetic terms (even though, in order to recover the standard Einstein-Hilbert action, a so-called Weyl rescaling will be needed, as we will show later on). In order to partially justify its presence it is useful to re-introduce the Newton coupling constant $k^2 = 8\pi G$ and to expand the exponential, hence obtaining for the Kähler potential term :

$$-\frac{3}{k^2} \int \mathrm{d}^4x \mathrm{d}^4\theta \, E + \int \mathrm{d}^4x \mathrm{d}^4\theta \, E \, K\left(\Phi_i, \bar{\Phi}_i e^{2V}\right) + \mathcal{O}(k^2) \tag{1.62}$$

It can be seen that, sending $k \to 0$, i.e. in the low-energy limit, the $\mathcal{O}(k^2)$ terms vanish, and what remains is the Einstein and Rarita-Schwinger action (contained in the first term) and the regular Kähler potential for the gauge-coupled chiral superfields.
The component expansion for the bosonic sector of the lagrangian (1.61) is (with ϕ^i being the lowest component of the chiral superfield Φ^i):

$$|e|^{-1}\mathcal{L}_{SG} = -\frac{1}{2}R - K_{i\bar{j}}\tilde{\mathcal{D}}_m\phi^i(\tilde{\mathcal{D}}^m\phi^j)^* - \frac{1}{4}F_{mn}F^{mn} - V, \tag{1.63}$$

where $|e|$ is the determinant of the vielbein and $\tilde{\mathcal{D}}$ is the covariant derivative with respect to the gauge vector A_m. The explicit expression for the scalar potential V is, if we put to zero the terms proportional to the auxiliary fields D (that we will not consider in the following):

$$V = e^K \left(K^{i\bar{j}} D_i W D_{\bar{j}} W - 3|W|^2 \right), \tag{1.64}$$

where:

$$K^{i\bar{j}} \equiv \left(\frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} \right)^{-1} \qquad \text{and} \qquad D_i W \equiv \frac{\partial W}{\partial \phi^i} + W \frac{\partial K}{\partial \phi^i} \tag{1.65}$$

D_i is in fact a covariant derivative with respect to the Kähler transformations denoted by:

$$K\left(\Phi_i, \bar{\Phi}_i e^{2V}\right) \to K\left(\Phi_i, \bar{\Phi}_i e^{2V}\right) + \Lambda(\Phi_i) + \bar{\Lambda}(\bar{\Phi}_i) \tag{1.66}$$

Among the minima of the potential V, the supersymmetric ones are found imposing that the supersymmetric variations of the fermions appearing in the chiral superfield Φ and the vector superfield V evaluated at the minimum vanish. The reason for this is that, in the vacuum states that correspond to the minima, all the fermionic fields of the theory must vanish. This means that the variations for the scalars and the vielbein (that depend only on the fermions) vanish identically, whereas the fermions' variation could possibly give a contribution. As a result, if we impose that these variations vanish, the following supersymmetricity condition (in the absence of D terms) is obtained:

$$D_i W = 0 \tag{1.67}$$

It is worth observing from (1.64) that supersymmetric vacua imply Minkowski (i.e. null curvature) or Anti-de Sitter (AdS, with negative curvature) spacetime.

CHAPTER 2

Flux Compactifications

The idea of the existence of extra dimensions, beyond the four familiar ones we can experience in everyday life, dates back to the beginning of the past century and is due to the German physicist Theodor Kaluza, who tried to ease the conflict between general relativity and electromagnetism and to obtain a unified theory. Since then this seemingly simple and yet revolutionary concept (not restricting ourselves to the apparently arbitrary number of spacetime dimensions) has been applied in a variety of ways, as well as on a multiplicity of different problems. One of the most prominent fields where the guess that there exist extra dimensions that we cannot directly perceive has been put to an extensive use is string theory, and as a matter of fact this has been one of the most recognizable features of string theory as a whole. Remarkably this characteristic is not artificially added into the theory, but it is implemented in a mathematically precise way by quantum consistency requirements.

At a first glance, nevertheless, our world shows only four dimensions, and consistency with this basic premise is an inescapable urgency for all theories that aspire to have some connection with reality. A possible solution is to postulate the existence of “large” extra dimensions, but with the crucial difference that the access to them is severely restricted for most of the particles that populate the standard model. This is the case of the large fifth dimension of the Randall-Sundrum model [19], that can be explored only by the gravitons (this idea is used to explain why gravity appears to be the “weakest” force in our universe), whilst all the other particles are confined to the usual four dimensions.

Another approach, that is often adopted by string theory, is to assume that the supernumerary dimensions are so small that we have not been able to sense their presence yet: they must in a sense be extremely “small” and therefore unaccessible to the present particle accelerator energy (and, as far as we know, even much above it). Even though they cannot be directly probed, the geometrical properties of the extra dimensions are fundamental in determining the features of the particles and the forces that characterize the universe (if the string theory description is true, ça va sans dire), and therefore understanding how this influence is exerted is essential in order to achieve a satisfying physical theory.

2.1 The Kaluza-Klein dimensional reduction

The chief strategy to tackle this task, that goes by the name of Kaluza-Klein (KK) dimensional reduction method, is to start from a ten-dimensional action (in the following we will be exclusively concerned with string theories with a total of 10 spacetime dimensions, 3+1 usual ones and 6 extra)[1], that describes the theory in its full extension, and to assume that the fields that compose the theory depend on the extra dimensional coordinates in the simplest possible way. In general we can consider small fluctuations of the fields around some background value[2], calling x the 4d coordinates and y the extra ones:

$$\Phi(x, y) = \Phi_{bg}(x, y) + \delta\Phi(x, y) \tag{2.1}$$

The fluctuations $\delta\Phi$ must therefore satisfy, just like $\Phi_{bg}(x, y)$, the ten-dimensional equations of motion, that will be something like:

$$\mathcal{O}_{TOT}\, \delta\Phi(x, y) = (\mathcal{O}_x + \mathcal{O}_y)\, \delta\Phi(x, y) = 0, \tag{2.2}$$

where $\mathcal{O}_{TOT}$ is a differential operator (for example, a Laplacean) that can be split into a part acting only on the standard 4 dimensions ($\mathcal{O}_x$) and another pertaining only to the extra dimensions ($\mathcal{O}_y$). Furthermore the fluctuations in the ten dimensional theory can be expanded in a Fourier series:

$$\delta\Phi(x, y) = \sum_{n=0}^{\infty} \phi^{(n)}(x) Y^{(n)}(y), \tag{2.3}$$

where the functions $Y^{(n)}(y)$ satisfy:

$$\mathcal{O}_y Y^{(n)}(y) \propto m_{(n)}^2 Y^{(n)} \tag{2.4}$$

The coefficients $m_{(n)}$ are nothing but the masses of the fields $\phi^{(n)}(x)$ that appear in the decomposition (2.3); in fact inserting (2.3) into (2.2) the following relations holds:

$$\mathcal{O}_{TOT}\, \delta\Phi(x, y) = (\mathcal{O}_x + \mathcal{O}_y) \sum_n \phi^{(n)}(x) Y^{(n)}(y) = \sum_n \left(\mathcal{O}_x - m_{(n)}^2\right) \phi^{(n)}(x) Y^{(n)}(y) \tag{2.5}$$

Most importantly it can be shown that the masses $m_{(n)}$ are inversely proportional to the size of the extra dimensions: as a result, the smaller they are, the larger the masses will be. From now on the indices belonging to 4d spacetime will be the middle latin ones m,n, etc. whereas the extra indices are denoted by i,j, etc. Using this convention the four dimensional part of the metric must be expanded in the following way, so as to preserve all the properties

[1] Actually the starting point could also be M-theory with 11 dimensions.

[2] The field Φ can be a scalar, a vector, a fermion etc., we have suppressed the possible indices for convenience.

of the Ricci scalar:

$$g_{mn}(x,y) = g_{mn}(x) \left(\frac{\det g_{ij}(x,y)}{\det g^{(0)}_{ij}(y)} \right)^{-\frac{1}{d-2}} \tag{2.6}$$

Once this Fourier expansion has been performed, the idea of the KK reduction is to insert the expressions like (2.3) in the total action and to integrate over the extra dimensions, *de facto* eliminating them. This is a key point in our discussion: in the low energy limit it can be shown that the massive modes in the expansion (2.3), that is all of the $\phi^{(n)}$ expect for $\phi^{(0)}$, can be integrated out and play no role in the resulting theory. The "memory" of the additional dimensions, as a consequence, will remain in the new theory in the form of the zero modes (that is, the massless ones) that appear in the expansion (2.3). The final action that is obtained after this process is properly defined in four dimensions and it is in all respects an *effective theory*, because it has been obtained, and it is valid, only in the low energy limit.

With the tools of the appendix A at hand it is now possible to take on the study of type IIa superstring theory compactified on Calabi-Yau threefolds in the presence of fluxes. Before delving into the details, it is necessary to explain what we have just said: type IIa superstring theory describes the behaviour of strings in a supersymmetric context, namely with two supersymmetric charges ($\mathcal{N} = 2$), whence the "II". In order for this description to be consistent it is required that the theory is stated in 10 (9 spatial plus one temporal) dimensions, and as a result all of the fields in the action will be expressed in 10d. These fields arise from the quantization of closed superstrings, and in general are organized in a tower of states with increasing mass. Such masses, however, are proportional to the inverse of the string length scale, that being extremely small (of order of the Planck length) results in enormous masses for the massive states. It is then natural, if we want to study low energy applications, to concentrate our attention exclusively on the massless states that develop from string quantization, and to neglect all of the massive states. Moreover we will add to the picture the fluxes of some field-strengths (similar to the fluxes of the electro-magnetic field strength F_{mn} that we are more familiar with), that will contribute with further terms to the action.

The next step will consist in reducing the number of supercharges from 2 to 1, in order to make contact with more realistic models: this will be done by projecting out some of the components of the fields, by means of a so-called "orientifold projection", that will be explained more extensively in the following.

Once we have stated this 10-dimensional theory, the wish is to dimensionally reduce it and obtain a four-dimensional effective theory. As we have briefly outlined before, this is achieved by applying the Kaluza-Klein reduction method. In the present case we will compactify the theory on a Calabi-Yau three-fold, that is a complex manifold with three complex dimensions (as a result, it will be described by three complex coordinates). The ten total dimensions will be then split into two parts: the ordinary four dimensions M_4 and the Calabi-Yau manifold X_6 (that possesses six real dimensions). 10d spacetime will hence be the direct product of these two spaces:

$$M_{10} = M_4 \times X_6 \tag{2.7}$$

The appropriate formalism to treat such a compactification and the complex geometry of the Calabi-Yau manifold makes use of p-forms, the generalization of the differentials that are employed in one-dimensional integrals. The p-forms have a natural domain of integration, the p-cycles, that are submanifolds of dimension p of a more ample space, such as the Calabi-Yau manifold. All of these notions are more accurately defined in appendix A, but throughout this chapter we will explain the most relevant parts when necessary.
If we want to compactify the 10d-theory we will have to express all of the fields in the action as a combination of a part that lives in four dimensions and another that pertains to the six extra ones, so as to be able to perform the integration on X_6 and reduce to only four dimensions. After this procedure we will finally obtain a four dimensional effective theory with $\mathcal{N} = 1$ supersymmetry. Summing up the whole procedure we can draw this schematic diagram:

$$M_4 \times X_6 \quad (\mathcal{N} = 2) \quad \xrightarrow{\text{Orientifold projection}} \quad M_4 \times X_6 \quad (\mathcal{N} = 1) \quad \xrightarrow{\text{KK reduction}} \quad M_4 \quad (\mathcal{N} = 1)$$

2.2 The 10d action

As we have said, the starting point is a theory living in 10 spacetime dimensions. Following Grimm's convention [13] we will denote fields in 10d with a hat, whereas in 4d they will be devoid of it.
Quantization of the closed superstring with appropriate boundary conditions - Ramond (R) and Neveu-Schwarz (NS) - gives rise to a varied massless bosonic spectrum, encoded in the so-called R-R and NS-NS sectors (while the R-NS and NS-R contain the fermions, which we will not care about, as they can be recovered by supersymmetry) [14]. The NS-NS sector contains a scalar field, the dilaton $\hat{\phi}$, the graviton g_{MN} (capital indices span from 0 to 9) and an antisymmetric two form $\hat{B}_2$. In the R-R sector a one-form $\hat{C}_1$ and a three-form $\hat{C}_3$ are present. All of these p-forms can intuitively be seen as the generalization of the electromagnetic potential that we are more familiar with. Furthermore, we employ the formalism of Romans [15] that introduces a mass m^0 for the NS-NS two-form $\hat{B}_2$. The parameter m^0 can be also seen as a constant "zero-form"[3] F_0.
The most general ten-dimensional action that includes the mentioned fields is (setting the

[3]Using F_0 proves useful if the democratic approach, that adds other forms $\hat{C}_5$, $\hat{C}_7$ and $\hat{C}_9$, related to the conventional ones via an expression involving the Hodge duality, is used. For further details we refer to [40].

gravitational coupling constant to 1):

$$\begin{aligned}S_{10\mathrm{d}} = \int -\frac{1}{2}\hat{R} * 1 - \frac{1}{4}d\hat{\phi}\wedge *d\hat{\phi} - \frac{1}{4}e^{-\hat{\phi}}d\hat{H}_3\wedge *d\hat{H}_3 - \frac{1}{2}e^{\frac{3}{2}\hat{\phi}}d\hat{F}_2\wedge *d\hat{F}_2 \\ - \frac{1}{2}e^{\frac{1}{2}\hat{\phi}}d\hat{F}_4\wedge *d\hat{F}_4 - \frac{1}{2}e^{\frac{5}{2}\hat{\phi}}(m^0)^2 * 1 + \mathcal{L}_{\text{Chern-Simons}}\end{aligned} \tag{2.8}$$

$\mathcal{L}_{\text{Chern-Simons}}$ is a topological term, given by:

$$\begin{aligned}\mathcal{L}_{\text{Chern-Simons}} = -\frac{1}{2}\Big[\hat{B}_2\wedge d\hat{C}_3\wedge d\hat{C}_3 - (\hat{B}_2)^2\wedge d\hat{C}_3\wedge d\hat{C}_1 + \frac{1}{3}(\hat{B}_2)^3\wedge(d\hat{C}_1)^2 \\ -\frac{m^0}{3}(\hat{B}_2)^3\wedge d\hat{C}_3 + \frac{m^0}{4}(\hat{B}_2)^4 d\hat{C}_1 + \frac{(m^0)^2}{20}(\hat{B}_2)^5\Big]\end{aligned} \tag{2.9}$$

The field strengths $\hat{H}_3$, $\hat{F}_2$ and $\hat{F}_4$ respectively read:

$$\begin{aligned}\hat{H}_3 &= \hat{H}_3^{bg} + d\hat{B}_2 \\ \hat{F}_2 &= \hat{F}_2^{bg} + d\hat{C}_1 + m^0\hat{B}_2 \\ \hat{F}_4 &= \hat{F}_4^{bg} + d\hat{C}_3 - \hat{C}_1\wedge\hat{H}_3 - \frac{m^0}{2}(\hat{B}_2)^2,\end{aligned} \tag{2.10}$$

where $\hat{H}_3^{bg}$, $\hat{F}_2^{bg}$ and $\hat{F}_4^{bg}$ are the background fluxes of the respective forms. These fluxes of the p-forms through some p-cycle in the extra dimensions can be thought as nothing but the analogue of an electric or magnetic flux through a closed surface that we usually consider in the four standard dimensions. p-cycles, in fact, are submanifolds of dimension p in the extra dimensions and have no boundary, just like a sphere in the ordinary four dimensions. We see then that the field strengths in (2.10) are the sum of the derivative of their potential (just like in electromagnetism $F = dA$) plus a fixed contribution of a non vanishing background flux in the extra dimensions.

2.3 Compatification of type IIa theories

We can now proceed in compactifying the theory, by means of Kaluza-Klein dimensional reduction, on a Calabi-Yau manifold with 3 complex dimensions, using the following block-diagonal metric ansatz:

$$g_{MN} = \begin{pmatrix} g_{mn}(x) & 0 \\ 0 & g_{i\bar{j}}(z,\bar{z}), \end{pmatrix} \tag{2.11}$$

where m and n span the four dimensional spacetime, while $i, j = 1, 2, 3$ run over the Calabi-Yau manifold described by the coordinates z^i and $\bar{z}^i$. Writing down (2.11) we are assuming that, at least at zero order, the two spaces M_4 and X_6 do not interfere with each other, namely that there are no non-diagonal terms.

The next step is to express the fields in (2.8) splitting their dependence on the four standard dimensions (labelled by x) and on the extra ones (labelled by z and $\bar{z}$). In particular, we should expand the fields of the theory in a basis of the space $M_4 \times X_6$, something like:

$$\Phi(x, z, \bar{z}) = \eta(x) + \psi(x) \wedge \rho(z, \bar{z}) + \chi(z, \bar{z}), \tag{2.12}$$

where η, ψ, ρ and χ are some functions of the respective coordinates. Moreover, if Φ is a p-form, all of the products on the right hand side should also give rise to a p-form. More specifically ρ and χ should belong to some basis of the compactification manifold X_6. How can such a basis be found? In general the elements of a basis defined on X_6 are (p, q)-forms, where p refers to the coordinates z and q to the $\bar{z}$, and whose natural domain of integration is a (p, q)-cycle. As an example we could write a (1,1)-form ω:

$$\omega = \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{2.13}$$

Of course we could have built a (2,1)-form, or a (3,0)-form, etc. The question, therefore, is how many independent forms of each type (that is, for every possible p, q, that can go from 1 to 3) exist on the complex manifold X_6 we are considering. It is the geometrical properties of this manifold that precisely determine the number of basis elements for each kind of (p, q)-forms. More specifically, every independent (p, q)-form[4] is a so-called *cohomology class*, with well defined properties that are explored with further detail in appendix A. The set of *all* of the (p, q)-cohomology classes is the cohomology group $H^{p,q}$. As a result, the number of basis elements for each cohomology group of a manifold X_6 is defined to be the *Hodge number* $h^{p,q}$. For example, if $h^{1,1} = 3$, it means that the manifold X_6 possesses three independent $(1, 1)$-forms, and that we should expand the fields of the theory accordingly. At this point it is crucial to note that every (p, q)-form is also a $(p + q)$-form. If, for example, we wanted to build a 3-form field using the expansion (2.12), we could use a sum of (3,0)- and (2,1)-forms on the right hand side, because both of them are 3-forms. The total number of independent $(p + q)$ forms for some given p and q with fixed sum is the so-called Betti number b^{p+q} (for a more precise definition we always refer to Appendix A).

In this regard, it can be shown that the Calabi-Yau manifold X_6 under consideration is completely characterized by the Hodge numbers $h^{3,0} = 1$, $h^{2,1}$ and $h^{1,1}$. We further note, using (6.82), that:

$$b^3 = 2(h^{3,0} + h^{2,1}) = 2 + 2h^{2,1} \tag{2.14}$$

We can therefore introduce bases of harmonic forms (recalling from appendix A that they are in correspondence with the cohomology classes) for each of the involved cohomology groups:

- ω_A for $H^{1,1}(X_6)$ $(A = 1, ..., h^{1,1})$
- $\tilde{\omega}_A$ for $H^{2,2}(X_6)$, dual to ω_A (so that the index A runs over the same range as for ω_A).
- $(\alpha_{\hat{K}}, \beta^{\hat{K}})$ for $H^3(X_6)$ $(\hat{K} = 0, ..., h^{2,1})$

[4]More precisely, every independent equivalence class of (p, q)-forms, where two (p, q)-forms are equivalent if they differ by an exact form.

$\alpha_{\hat{K}}$ and $\beta^{\hat{K}}$ also satisfy the symplectic basis conditions:

$$\int \alpha_{\hat{K}} \wedge \beta^{\hat{L}} = \delta_{\hat{K}}^{\hat{L}} \qquad \int \alpha_{\hat{K}} \wedge \alpha^{\hat{L}} = \int \beta_{\hat{K}} \wedge \beta^{\hat{L}} = 0, \tag{2.15}$$

whereas the duality condition on the ω_A and the $\tilde{\omega}_A$ is:

$$\int \omega_A \wedge \tilde{\omega}_B = \delta_{AB} \tag{2.16}$$

Expanding the RR and NSNS fields in their appropriate bases and restricting to the zero modes (as we have mentioned that the massive modes do not contribute to the effective theory) gives:

$$\begin{aligned} &\hat{C}_1(x) = c_1(x) \qquad \hat{B}_2 = B_2(x) + b^A(x)\omega_A \\ &\hat{C}_3(x) = c_3(x) + A^A(x) \wedge \omega_A + \xi^{\hat{K}}(x)\alpha_{\hat{K}} - \tilde{\xi}_{\hat{K}}(x)\beta^{\hat{K}}, \end{aligned} \tag{2.17}$$

where $c_1(x)$, $A^A(x)$, $B_2(x)$ and $c_3(x)$ are respectively 1-, 1-, 2- and 3-forms defined on four dimensional spacetime, whereas $b^A(x)$, $\xi^{\hat{K}}(x)$ and $\tilde{\xi}_{\hat{K}}(x)$ are scalars always living in 4d.
The background fluxes for the field strengths (2.10) can be accordingly expanded as:

$$\hat{H}_3^{bg} = q^{\hat{K}}\alpha_{\hat{K}} - p_{\hat{L}}\beta^{\hat{L}} \qquad \hat{F}_2^{bg} = -m^A\omega_A \qquad \hat{F}_4^{bg} = e^A\tilde{\omega}_A \tag{2.18}$$

We see consistently that $\hat{H}_3^{bg}$, being a 3-form, has been expanded in a basis of the cohomology group H^3, that indeed includes the 3-forms $(\alpha_{\hat{K}}, \beta^{\hat{K}})$; the same goes for $\hat{F}_2^{bg}$ (expanded in a basis of 2-forms ω_A) and $\hat{F}_4^{bg}$ (expanded in a basis of 4-forms $\tilde{\omega}_A$).
Additional zero modes that must be taken care of in the compactification arise from the deformations of the metric $g_{i\bar{j}}$. These fluctuations can be divided into two categories: the ones deforming the components g_{ij} and $g_{\bar{i}\bar{j}}$, and the ones transforming the mixed components $g_{i\bar{j}}$ [17]. It can be shown that the deformations of $g_{i\bar{j}}$ are nothing but fluctuations of the Kähler form J, defined in appendix A; as a result the corresponding moduli space M^K is:

$$M^K = H^{1,1}(X_6) \tag{2.19}$$

The term *moduli*, therefore, indicates the fields that parametrize some given deformations, in this case the ones of the Kähler form. The moduli space is the set of all possible deformations of a given kind, and from a mathematical point of view it is a Kähler space.
As a consequence of (2.19) there will be $h^{1,1}$ parameters v^A describing M^K. Their Kähler potential K^K, that will enter the dimensionally reduced action, is:

$$K^K = -\ln\left(\frac{4}{3}\int_{X_6} J \wedge J \wedge J\right) \equiv -\ln(8\kappa), \tag{2.20}$$

where κ is one of the intersection numbers, that are defined as follows:

$$\begin{aligned} \kappa_{ABC} &= \int_{X_6} \omega_A \wedge \omega_B \wedge \omega_C & \kappa_{AB} &= \int_{X_6} \omega_A \wedge \omega_B \wedge J = \kappa_{ABC} v^C \\ \kappa_A &= \int_{X_6} \omega_A \wedge J \wedge J = \kappa_{ABC} v^B v^C & \kappa &= \frac{1}{6} \int_X J \wedge J \wedge J = \frac{1}{6} \kappa_{ABC} v^A v^B v^C \end{aligned} \tag{2.21}$$

Intuitively speaking the intersection numbers express how many times some p-cycles defined on the manifold X_6 (in our case, the ω_A and J) intersect each other. The deformations δg_{ij}, instead, are linked to a change of the complex structure Y, that can in turn be encoded in deformations of the nowhere-vanishing holomorphic and harmonic 3-form (6.85). These deformations turn out to be described by a (2,1)-cohomology class. That is to say that the moduli space M^{CS} of complex structure deformations has dimension $h^{2,1}$ and is:

$$M^{CS} = H^{2,1}(X_6) \tag{2.22}$$

The $h^{2,1}$ parameters associated with M^{CS} are the complex scalars z^K, with K running from 1 (not 0 as for $\hat{K}$) to $h^{2,1}$. In order to define them more precisely we introduce the coordinates $Z^{\hat{K}}$ and $\mathcal{F}_{\hat{K}}$ the so-called *periods* of $\Omega(z)$:

$$Z^{\hat{K}}(z) \equiv \int_{X_6} \Omega \wedge \beta^{\hat{K}} \qquad \mathcal{F}_{\hat{K}}(z) \equiv \int_{X_6} \Omega \wedge \alpha_{\hat{K}} \tag{2.23}$$

As a result $\Omega(z)$ can be expanded using (2.15):

$$\Omega(z) = Z^{\hat{K}}(z) \alpha_{\hat{K}} - \mathcal{F}_{\hat{K}}(z) \beta^{\hat{K}} \tag{2.24}$$

The Kähler potential associated to the complex structure moduli is:

$$K^{CS} = -\ln \left(i \int_{X_6} \Omega \wedge \bar{\Omega} \right) = -\ln i \left(\bar{Z}^{\hat{K}} \mathcal{F}_{\hat{K}} - Z^{\hat{K}} \bar{\mathcal{F}}_{\hat{K}} \right) \tag{2.25}$$

The $2(h^{2,1}+1)$ coordinates $Z^{\hat{K}}$ and $\mathcal{F}_{\hat{K}}$ are not entirely independent if Ω defines the complex structure. First of all the $\mathcal{F}_{\hat{K}}$ can be expressed as functions of the $Z^{\hat{K}}$: using a corollary of the Poincaré duality (defined in equation (6.64)) it can be shown that:

$$\mathcal{F}_{\hat{K}} = \frac{1}{2} \partial_{\hat{K}} \left(Z^{\hat{L}} \mathcal{F}_{\hat{L}} \right) \tag{2.26}$$

Integrating on both sides gives:

$$\mathcal{F} = \frac{1}{2} Z^{\hat{K}} \mathcal{F}_{\hat{K}}, \tag{2.27}$$

where $\mathcal{F}$ is the *prepotential* of the Calabi-Yau manifold under scrutiny. Another dependent coordinate among the $Z^{\hat{K}}$ is eliminated making use of the Kähler invariance enjoyed by Ω

and K^{CS} ($h(z)$ is a holomorphic function):

$$\Omega(z) \to \Omega(z)e^{-h(z)} \qquad K^{CS}(z) \to K^{CS}(z) + h(z) + \bar{h}(\bar{z}) \tag{2.28}$$

In this way it is possible to fix, for example, $Z^0 = 1$, being left with just $h^{2,1}$ coordinates $z^K \equiv \frac{Z^K}{Z^0}$, as required by (2.22).
The bosonic fields built so far can be assembled into supersymmetric $\mathcal{N} = 2$ multiplets of the following form (omitting the fermions):

- 1 gravitational multiplet containing the bosonic components (g_{mn}, c_1)
- $h^{1,1}$ vector multiplets containing the bosonic components (A^A, v^A, b^A)
- $h^{2,1}$ hypermultiplets containing the bosonic components $(z^K, \xi^K, \tilde{\xi}_K)$
- 1 tensor multiplet containing the bosonic components $(B_2, \phi, \xi^0, \tilde{\xi}_0)$

A useful simplification can be made re-assembling and rescaling some of the fields:

$$t^A \equiv b^A + iv^A \qquad e^D \equiv \frac{e^{\hat{\phi}}}{\sqrt{\kappa}}, \tag{2.29}$$

where the t^A are the coefficients of the basis expansion of the so-called complexified Kähler form J_c:

$$J_c \equiv \hat{B}_2 + iJ = t^A\omega_A \tag{2.30}$$

2.4 The orientifold projection

Before writing down the complete action we perform an orientifold projection, in order to reduce the number of supersymmetries from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. The general idea behind this process is to introduce a projection operator $\mathcal{O}$ and to apply it to the fields of the theory we have built in the previous section, retaining only the ones that remain unaffected. We will then see that the remaining degrees of freedom naturally accommodate into multiplets of $\mathcal{N} = 1$ supersymmetry.
The discrete symmetry operator $\mathcal{O}$ is composed of an involution σ (i.e. such that $\sigma^2 = 1$; it acts non-trivially only on the Calabi-Yau manifold X_6), the world-sheet parity operator Ω_p, that flips the orientation of the strings, and $(-1)^{F_L}$, with F_L the number of fermions in the left-moving sector (arising from the quantization of the closed superstring):

$$\mathcal{O} = \Omega_p(-1)^{F_L}\sigma \tag{2.31}$$

Further properties of σ include antiholomorphicity, isometry and the action on the Kähler form as (denoting with σ^* the pullback of σ on the cotangent bundle where J lives):

$$\sigma^* J = -J \tag{2.32}$$

The locus of the points left fixed by the action of σ is called orientifold plane, a non-dynamical object that is commonly indicated with On, where n is its spatial dimension. As a consequence of the fact that σ acts trivially on the standard four dimensions the orientifold plane will at least be an $O3$ (so that it has 3 spatial and 1 temporal dimension). The consistency requirement (imposed by the Hodge numbers) $J \wedge J \wedge J \propto \Omega \wedge \bar{\Omega}$ implies that:

$$\sigma^* \Omega = e^{2i\theta} \bar{\Omega}, \tag{2.33}$$

with $e^{2i\theta}$ a generic constant phase.
As we have said the core strategy of the orientifold projection is to eliminate the fields that are not invariant under the action of the operator $\mathcal{O}$. First of all we note that $\hat{C}_1$ and $\hat{C}_3$ are the only odd fields under $(-1)^{F_L}$, whereas $\hat{B}_2$ and $\hat{C}_3$ are the only odd ones with respect to the world-sheet parity operator Ω_p. It then turns out that the $\mathcal{O}$-invariant 10-dimensional fields must satisfy:

$$\begin{aligned} &\sigma^* \hat{\phi} = \hat{\phi} \qquad \sigma^* g_{MN} = g_{MN} \qquad \sigma^* \hat{B}_2 = -\hat{B}_2 \\ &\sigma^* \hat{C}_1 = -\hat{C}_1 \qquad \sigma^* \hat{C}_3 = \hat{C}_3 \end{aligned} \tag{2.34}$$

Correspondingly the homology groups $H^{1,1}(X_6)$, $H^{2,2}(X_6)$ and $H^3(X_6)$ split under the action of σ in even and odd subspaces. As a result also their basis elements will get reduced. Denoting even subspaces with the subscript $+$ and odd ones with $-$ we get [13]:

Cohomology Group	$H^{1,1}_+$	$H^{1,1}_-$	$H^{2,2}_+$	$H^{2,2}_-$	H^3_+	H^3_-
Dimension	$h^{1,1}_+$	$h^{1,1}_-$	$h^{1,1}_-$	$h^{1,1}_+$	$h^{2,1}+1$	$h^{2,1}+1$
Basis	ω^+_α	ω^-_i	$\tilde{\omega}^+_i$	$\tilde{\omega}^-_\alpha$	$\alpha_{\hat{K}}$	$\beta^{\hat{K}}$

We note that the dimensions of $H^{2,2}_+$ and $H^{2,2}_-$ are swapped with respect to the ones of $H^{1,1}_+$ and $H^{1,1}_-$: this is a consequence of the fact that the volume of the manifold is proportional to $\int_{X_6} J \wedge J \wedge J$ which is odd for (2.32); moreover, the base $\tilde{\omega}_A$ is the Hodge dual of ω_A, so that the only way to obtain an odd function over the whole manifold when computing $\int_{X_6} \omega_A \wedge \tilde{\omega}_B$ is:

$$\int \omega^+_\alpha \wedge \tilde{\omega}^-_\beta = \delta^\alpha_\beta \qquad \int \omega^+_\alpha \wedge \tilde{\omega}^+_i = 0 \tag{2.35}$$

We see then that we must have $h^{1,1}_+ = h^{2,2}_-$ and $h^{1,1}_- = h^{2,2}_+$.

In order to simplify the following computations a specific choice of the symplectic basis (2.15) can be made, considering the $\alpha_{\hat{K}}$ to be even under $\mathcal{O}$, and the $\beta^{\hat{K}}$ to be odd. In any case the reasoning could be made with a generic basis choice and the end result would not be affected.

Bearing in mind (2.34) we see that J, $\hat{B}_2$ and $\hat{C}_1$ must be expanded in an odd basis if we want to meet the requirement that only fields even under $\mathcal{O}$ survive the projection, and all the rest in an even basis. It can be noted, besides, that $\hat{C}_1$ gets entirely projected out, as a consequence of the fact that no odd (neither even for that matter) harmonic one-forms exist, recalling (6.86), and that σ acts as the identity on four-dimensional spacetime. The corresponding expansions are hence:

$$\begin{aligned} &J = v^i \omega_i^- \qquad \hat{B}_2 = b^i \omega_i^- \qquad i = 1, ..., h_-^{1,1} \\ &\hat{C}_3 = c_3(x) + A^\alpha \wedge \omega_\alpha^+ + \xi^{\hat{K}} \alpha_{\hat{K}} \qquad \alpha = 1, ..., h_+^{1,1} \end{aligned} \tag{2.36}$$

As it was done previously, it is convenient to assemble the real scalar fields components of J and $\hat{B}_2$ into new complex fields:

$$t^i \equiv b^i + iv^i \tag{2.37}$$

Correspondingly we define the complexified Kähler form as:

$$J_c \equiv \hat{B}_2 + iJ \tag{2.38}$$

The intersection numbers (2.21), being proportional to the odd volume form, are non vanishing only if they contain one or three odd basis elements of the form ω_i^-. As a result the vanishing components are:

$$\kappa_{\alpha\beta\gamma} = \kappa_{\alpha ij} = \kappa_{\alpha i} = \kappa_\alpha = 0 \tag{2.39}$$

The surviving ones maintain the relations among themselves described in (2.21).
The background field strengths are also involved in the projection (with symmetry properties determined by their gauge fields), resulting in:

$$\hat{H}_3^{bg} = -p_{\hat{K}} \beta^{\hat{K}} \qquad \hat{F}_2^{bg} = -m^i \omega_i^- \qquad \hat{F}_4^{bg} = e^i \tilde{\omega}_i^+ \tag{2.40}$$

We immediately see that when these field strengths are integrated on appropriate p-cycles, dual to the basis p-forms that appear in the field strengths' expansion, the result is nothing but the values of the fluxes (i.e. the coefficients in the basis expansion). For example, as regards $\hat{H}_3^{bg}$:

$$\int_{\alpha_{\hat{L}}} \hat{H}_3^{bg} = \int_{\alpha_{\hat{L}}} -p_{\hat{K}} \beta^{\hat{K}} = -p_{\hat{L}} \int_{\alpha_{\hat{L}}} \beta^{\hat{L}} = -p_{\hat{L}}, \tag{2.41}$$

where we have used the symplectic relations (2.15). This definition is in agreement with the intuition that these values of the fluxes are the analogue of a more conventional electric or magnetic flux through a surface.
Putting (2.36) and (2.40) together the complete field strengths turn out to be:

$$\begin{aligned} &\hat{H}_3 = -p_{\hat{K}} \beta^{\hat{K}} + db^i \wedge \omega_i^- \\ &\hat{F}_2 = -m^i \omega_i^- + m^0 b^i \omega_i^- \\ &\hat{F}_4 = e^i \tilde{\omega}_i^+ + dc_3(x) + dA^\alpha \wedge \omega_\alpha^+ + d\xi^{\hat{K}} \wedge \alpha_{\hat{K}} + \left(b^i m^j - \frac{1}{2} m^0 b^i b^j \right) \kappa_{ijk} \tilde{\omega}_+^k \end{aligned} \tag{2.42}$$

As far as the 3-form Ω is concerned, the condition (2.33) implies:

$$\mathrm{Im}(e^{-i\theta}Z^{\hat{K}})=0 \qquad \mathrm{Re}(e^{-i\theta}\mathcal{F}_{\hat{K}})=0 \tag{2.43}$$

The Kähler transformation (2.28) is analogously reduced to a simple real rescaling, that is:

$$\Omega(z)\to\Omega(z)e^{-\mathrm{Re}(h(z))} \qquad K^{CS}(z)\to K^{CS}(z)+2\mathrm{Re}(h(z)) \tag{2.44}$$

In this context, instead of employing this invariance tout court to fix one of the periods of Ω, it is useful to "hide" it in a complex compensator (i.e. a non-physical field) subject to the transformation law [18]:

$$Z\equiv e^{D-i\theta-\frac{K^{CS}}{2}} \qquad Z\to Ze^{\mathrm{Re}(h(z))} \tag{2.45}$$

We hence define a rescaled Ω as:

$$Z\Omega=\mathrm{Re}(ZZ^{\hat{K}})\alpha_{\hat{K}}-i\mathrm{Im}(Z\mathcal{F}_{\hat{L}})\beta^{\hat{L}} \tag{2.46}$$

The purpose of this rescaling is to define a new combination of fields Ω_c that is nothing but a chiral multiplet formed by the remnants of the orientifold projection of Ω and a part of $\hat{C}_3$:

$$\Omega_c=\xi^{\hat{K}}\alpha_{\hat{K}}+2i\mathrm{Re}(Z\Omega) \tag{2.47}$$

A basis expansion shows that:

$$\Omega_c=2n^{\hat{K}}\alpha_{\hat{K}} \quad \text{with} \quad n^{\hat{K}}=\frac{1}{2}\int\Omega_c\wedge\beta^{\hat{K}}=\frac{1}{2}\left(\xi^{\hat{K}}+2i\mathrm{Re}(ZZ^{\hat{K}})\right) \tag{2.48}$$

The new Kähler potential that arises employing $Z\Omega$ is:

$$K^{Q}=-2\mathrm{ln}\left[2\int_{X_6}\mathrm{Re}(Z\Omega)\wedge*\mathrm{Re}(Z\Omega)\right] \tag{2.49}$$

In the end, we have shown that the total moduli space of the theory possesses the following structure:

$$M^{K}\times M^{Q}, \tag{2.50}$$

where M^K is the moduli space spanned by the deformations of the complexified Kähler form J_c (that is, by the fields b^i and v^i), whereas M^Q is described by the dilaton D, the axion ξ and by a submanifold defined by the complex structure moduli z^K.
Summarizing all the (bosonic components of the) multiplets of $\mathcal{N}=1$ supersymmetry that remain after the orientifold projection we obtain:

- 1 gravitational multiplet (with bosonic component g_{mn})
- $h^{1,1}_{+}$ vector multiplets (with bosonic component A^{α})

- $h^{1,1}_{-}$ chiral multiplets (with bosonic component t^i)

- $h^{2,1}+1$ chiral multiplets (of which 1 unphysical, with bosonic component $n^{\hat{K}}$)

We consistently note that the bosonic component c_1 has disappeared from the gravitational multiplet, leaving the familiar $\mathcal{N}=1$ supersymmetry spectrum.
Taking into account the reasoning made so far it is finally possible to perform the Kaluza-Klein reduction and expand the resulting components; inserting (2.42) and (2.36) in (2.8) and integrating over the Calabi-Yau manifold X_6 we get [9],[13]:

$$S_{\text{4d}} = S_{\text{kinetic}} + V \tag{2.51}$$

With:

$$\begin{aligned} S_{\text{kinetic}} &= \int -\frac{1}{2}R*1 - G_{ij}dt^i\wedge *dt^j + \frac{1}{2}\text{Im}\mathcal{N}_{\alpha\beta}F^\alpha\wedge *F^\beta + \frac{1}{2}\text{Re}\mathcal{N}_{\alpha\beta}F^\alpha\wedge F^\beta \\ &\quad - dD\wedge *dD - G_{KL}dq^K\wedge *dq^L + \frac{e^{2D}}{2}\text{Im}\mathcal{M}_{\hat{K}\hat{L}}d\xi^{\hat{K}}\wedge *d\xi^{\hat{L}} \\ V &= \frac{e^{2D}}{4\kappa}\int_{X_6} H_3\wedge *H_3 - \frac{e^{4D}}{2}\left(e_A - \bar{\mathcal{N}}_{AC}m^C\right)\left(\text{Im}\mathcal{N}^{-1}\right)^{AB}\left(e_B - \mathcal{N}_{BD}m^D\right), \end{aligned} \tag{2.52}$$

where $e_A = (e_0 - \xi^K p_K, e_i)$ and $m^A = (m^0, m^i)$ and $F^\alpha = dA^\alpha$. The constant e_0 that has appeared can be viewed as the dual of the three-form c_3, that in four dimensions carries no propagating degrees of freedom. Adding to the initial action a Lagrange multiplier of the form $e_0 dc_3$, in fact, the three-form c_3 can be eliminated from the expression, giving the formula (2.52).
The metric G_{ij} is nothing but the Kähler metric of the Kähler potential (2.20), and it will therefore be:

$$G_{ij} = \frac{\partial^2 K^K}{\partial t^i \partial \bar{t}^j} = -\frac{3}{2}\left(\frac{\kappa_{ij}}{\kappa} - \frac{3}{2}\frac{\kappa_i\kappa_j}{\kappa}\right) \tag{2.53}$$

Analogously G_{KL} is the Kähler metric for the complex structure Kähler potential (that is $G^{TOT}_{K\bar{L}} = \frac{\partial^2 K^Q}{\partial z^K \partial \bar{z}^L}$) appropriately restricted taking into account the orientifold projection (for more details, we refer to [13]).
$\text{Im}\mathcal{M}_{\hat{K}\hat{L}}$ instead depends on the complex structure deformations moduli in the following way:

$$\text{Im}\mathcal{M}_{\hat{K}\hat{L}} = -\text{Im}\mathcal{F}_{\hat{K}\hat{L}} + 2\frac{(\text{Im}\mathcal{F}_{\hat{K}\hat{L}})\text{Re}(ZZ^{\hat{M}})(\text{Im}\mathcal{F}_{\hat{M}\hat{N}})\text{Re}(ZZ^{\hat{N}})}{\text{Re}(ZZ^{\hat{N}})(\text{Im}\mathcal{F}_{\hat{N}\hat{M}})\text{Re}(ZZ^{\hat{M}})} \tag{2.54}$$

In addition the integral appearing in V corresponds to:

$$\int_{X_6} H_3\wedge *H_3 = -p_{\hat{K}}\left(\text{Im}\mathcal{M}\right)^{-1\hat{K}\hat{L}} p_{\hat{L}} \tag{2.55}$$

The matrix $\mathcal{N}$ appearing in the potential V, instead, has the following components:

$$\mathcal{N} = \begin{pmatrix} -\frac{1}{3}\kappa_{ijk}b^i b^j b^k - i\kappa + i\left(\kappa_{ij} - \frac{\kappa_i\kappa_j}{4\kappa}\right) b^i b^j & \frac{1}{2}\kappa_{ijk}b^j b^k - i\left(\kappa_{ij} - \frac{\kappa_i\kappa_j}{4\kappa}\right) b^j \\ \frac{1}{2}\kappa_{ijk}b^j b^k - i\left(\kappa_{ij} - \frac{\kappa_i\kappa_j}{4\kappa}\right) b^j & -\kappa_{ijk}b^k + i\left(\kappa_{ij} - \frac{\kappa_i\kappa_j}{4\kappa}\right) \end{pmatrix} \tag{2.56}$$

$$\mathrm{Im}\mathcal{N}^{-1} = \begin{pmatrix} -\frac{1}{\kappa} & -\frac{b^i}{\kappa} \\ -\frac{b^i}{\kappa} & \kappa^{ij} - \frac{b^i b^j}{\kappa} - \frac{v^i v^j}{2\kappa} \end{pmatrix} \tag{2.57}$$

The potential V can be rewritten in the canonical supergravity formalism (1.64) by means of the Kähler potentials (2.20) and (2.49) and the associated superpotentials:

$$\begin{aligned} W^K &= e_0 + \int_{X_6} J_c \wedge F_4 - \frac{1}{2}\int_{X_6} J_c \wedge J_c \wedge F_2 - \frac{m^0}{6}\int_{X_6} J_c \wedge J_c \wedge J_c = \\ &= e_0 + e_i t^i + \frac{1}{2}\kappa_{ijk} m^i t^j t^k - \frac{m^0}{6}\kappa_{ijk} t^i t^j t^k \\ W^Q &= -2n^{\hat{K}} p_{\hat{K}} \end{aligned} \tag{2.58}$$

With this step we have concluded the task that was set at the beginning of the chapter: starting from a $\mathcal{N} = 2$ ten-dimensional theory, we have ended up with a $\mathcal{N} = 1$ four-dimensional action, described by the kinetic and potential terms (2.52). In the next chapter we will develop an alternative formalism to deal with the values of the fluxes that appear explicitly in the superpotential W^K (2.58). In chapter 4, instead, we will apply the dimensional reduction outlined in this chapter to obtain an effective theory in a concrete case, properly stated with the new formalism of chapter 3, with the superpotentials (2.58) being the main protagonists.

CHAPTER 3

An alternative formulation for the p-forms fluxes

In the previous chapter we have shown how a completely general superstring theory of type IIa can be dimensionally reduced to obtain a four dimensional supergravity effective theory. In this treatment the contribution of non-vanishing fluxes of the p-forms present in the theory has been taken into account. The values of these fluxes enter in the potential (2.52) of the theory and will determine the location of its extrema. Until now we have treated the fluxes as mere constants: after all, they are nothing but the coefficients of the basis expansion (2.40). In the following, however, we will be concerned in studying the transitions among different minima of the potential (2.52), labeled by varying values of the fluxes. In other words, the transitions that will be studied will occur between a minimum of the potential with some values of the fluxes (e_A, m^A) and another minimum with values $(e^{'}_A, m^{A'})$. The "jump" in the fluxes, that is $(\Delta e_A, \Delta m^A) = (e^{'}_A - e_A, m^{A'} - m^A)$, is caused, as we will see explicitly in chapter 5, by the presence of a $(p+2)$-brane that extends along two of the usual spatial directions, thus separating 3d space in two different regions, and that wraps some p-cycle in the internal manifold. If, for example, a 4-brane wraps a 2-cycle in the internal space, one of the fluxes e_i of the field strength F_4 will change. In general, a $(p+2)$-brane wrapping a p-cycle results in a jump in the value of the flux of the field strength that is a $(p+2)$-form. To see how this comes about it is useful to employ a more familiar example, namely a charged particle accelerating in an electric field in two spacetime dimensions. In chapter 5, nevertheless, the full case with the membranes will be thoroughly explained.

Let us consider then the action in two-dimensional flat space for a particle with charge e and mass m (say an electron) immersed in an electromagnetic field F_{mn}. This field strength derives from the potential A_m by means of the usual relation $F_{mn} = \partial_m A_n - \partial_n A_m$; there will be a kinetic term for the particle and for the field, as well as the coupling between the two[1]:

$$S = -m \int \mathrm{d}s \sqrt{-\dot{y}^m \dot{y}_m} - \frac{1}{4} \int \mathrm{d}^2 x \sqrt{-g} F_{mn} F^{mn} + e \int \mathrm{d}s \, \dot{y}^m A_m \tag{3.1}$$

The parameter s parametrizes the world line of the particle and, observing that the whole action is reparametrization-invariant, it can be chosen to be the particle proper time. y^m

[1] We use the conventional mostly positive metric employed in general relativity.

describes the world line of the particle (that in this case is composed of 1 spatial dimension and 1 temporal one) and $\dot{y}^m = \frac{\mathrm{d}y^m}{\mathrm{d}s}$ is the proper velocity of the particle. g, instead, is the determinant of the metric. With these choices the equations of motion of the action become:

$$\begin{aligned} a^m &= \frac{e}{m} F^{mn} \dot{y}_n \\ \partial_m(\sqrt{-g} F^{mn}) &= -e \int \mathrm{d}s \, \delta^{(2)}(x^m - y^m(s)) \dot{y}^n(s), \end{aligned} \tag{3.2}$$

where $a^m = \frac{\mathrm{d}\dot{y}^m}{\mathrm{d}s}$ is the acceleration of the particle. The interesting thing to note is that in two dimensions the antisymmetric tensor F_{mn} has only one independent component, denoted with E. As a result the following decomposition is always valid, with ϵ_{mn} being the Levi-Civita symbol in two dimensions:

$$F_{mn} = \frac{E \epsilon_{mn}}{\sqrt{-g}} \tag{3.3}$$

Having in mind the higher dimensional case of the membranes that will be treated next, an analogue procedure can be carried out in an arbitrary number of dimensions, let it be d, provided that the field strength under scrutiny is a d-form. If that is the case, in fact, only one independent component fully characterizes the field strength.
Using (3.3) the equations of motion become:

$$\begin{aligned} a^m &= \frac{e}{m} \frac{E \epsilon^{mn}}{\sqrt{-g}} \dot{y}_n \\ (\partial_m E) \epsilon^{mn} &= -e \int \mathrm{d}s \, \delta^{(2)}(x^m - y^m(s)) \dot{y}^n(s) \end{aligned} \tag{3.4}$$

Adopting the convention $\epsilon^{01} = 1$ it is ensured that a particle with positive charge immersed in a positive electromagnetic field accelerates towards increasing x^1.

The direct consequence of (3.4) is that the value of the electric field E jumps when crossing the world-line of the particle. By integration of the second equation, indeed, it can be immediately seen that the term containing the Dirac delta causes a jump in the value of E: as a result, once a boundary value for the electromagnetic field has been chosen, its evolution is completely determined in all spacetime, except for the world-line where the Dirac delta is defined.

The mechanism with the full set of ten dimensions is completely analogous: a generalized Dirac delta with domain on the world-volume of a charged membrane implicates a jump for the flux of the corresponding field strength. As we have mentioned before, this can be seen as the effect of a brane (with 2 extended dimensions and some others extra) wrapping an appropriate p-cycle in the extra dimensions, and causing the flux to jump: the crucial problem with this approach is that only a qualitative description of the process can be displayed, given the complexity of the complete theory in 10 dimensions. An alternative strategy, that we will show to give precise quantitative results, stems from the effective theory in four dimensions that results from the Kaluza-Klein reduction. Our objective will be to introduce membranes (that is, objects that extend along 2 space dimensions separating the whole space into two sub-regions) into the effective theory, causing the fluxes to jump when going through the membranes themselves.

In this view, however the fluxes that appear in (2.52) are nothing but constants characterizing the potential: the field strengths from which they have been generated do not appear in the kinetic term of (2.52). In other words, there is no term analogous to $-\frac{1}{4}\int \mathrm{d}^2x\sqrt{-g}F_{mn}F^{mn}$ in the case of the electron, and no coupling to a source (that is, the membrane) like $e\int \mathrm{d}s\, \dot{y}^m A_m$. It is therefore necessary, if one desires to work only in the four dimensional framework, to adopt another approach and to associate, in some way we will define, a field strength to the fluxes (e_A, m^A).

A new formalism that precisely allows this kind of strategy in 4d supergravity has been recently developed in [20]: in this approach the values of the fluxes (e_A, m^A) that appear in the superpotentials (2.58) are completely eliminated from the theory, at the cost of the appearance of appropriate gauge field strengths, equipped with their respective kinetic term. In this way a coupling term with a membrane can be naturally introduced into the theory [21], employing the potentials of the field strength we just mentioned. The original theory, moreover, can be recovered using the equations of motion for the field strengths: it will be then shown that the values of the fluxes will correspond to the vacuum expectation value of the field strengths. There is of course one caveat: the number of degrees of freedom of the theory must remain unchanged. As a consequence, the field strengths that substitute the values of the fluxes must be 4-forms (and their potentials 3-forms), so that they contain only one degree of freedom, just like the electromagnetic field in two spacetime dimensions that we took under scrutiny before. The 4-form field strengths, therefore, will display a decomposition similar to equation (3.3), but in four dimensions, as we will show in chapter 4 when dealing with a concrete case.

The importance of gauge three-forms, nevertheless, is not limited to their applications to the membrane coupling: in the Kaloper-Sorbo model [43][44] of inflation, for example, an axion ϕ is coupled to the field strength of a gauge three-form A_{mnr} in the following way:

$$\begin{aligned} S_{KS} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2\cdot 4!}F_{mnrs}F^{mnrs} - \frac{1}{2}(\partial\phi)^2 + \frac{\mu}{4!}\phi\epsilon^{mnrs}F_{mnrs}\right] + \\ + S_{\text{boundary}}, \end{aligned} \tag{3.5}$$

where μ is some coupling constant, F_{mnrs} is the four-form field strength and S_{boundary} is an additional term needed to make the variational principle well defined for the three-form A_{mnr} (upon which we will talk about more extensively in a moment). If the equation of motion for the three-form is imposed a non-trivial scalar potential for the axion is produced:

$$V_{KS} = \frac{1}{2}(c + \mu\phi)^2, \tag{3.6}$$

where c is the constant value assumed by F_{mnrs}, the analogous of E in (3.3): the study of the scalar potential and of its different branches (that depend on the value of c that we choose) sets the stage for the comprehension of the axion's slow-roll dynamics towards its minima. In the following, however, we will not take into account such models and we will concentrate exclusively on the procedure that introduces gauge three-forms in lieu of the values of the fluxes. A final remark, that will be extremely useful for our subsequent discussion, regards the boundary term. In order to simplify the reasoning (that can anyway be performed in full generality [43]), we put the scalar ϕ to zero and consider only the the remaining action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2\cdot 4!}F_{mnrs}F^{mnrs}\right) + S_{\text{boundary}} \tag{3.7}$$

The corresponding equation of motion for the three-form A_{mnr} is:

$$F^{mnrs} = c\epsilon^{mnrs}, \tag{3.8}$$

Thus implying that F^{mnrs} is in fact fixed.
Generally speaking, this equation of motion is deduced variating with respect to A_{mnr}, and then considering vanishing variations at the boundary to discard the surface terms that appear, that is:

$$\delta A_{mnr}|_{\text{boundary}} = 0 \tag{3.9}$$

However, it is clear that this condition is not gauge invariant, and it may not be satisfied for non-trivial boundary conditions. In order to overcome this difficulty, we would like to impose a gauge-invariant condition on the field strength:

$$\delta F_{mnrs}|_{\text{boundary}} = 0 \tag{3.10}$$

If we wish to fulfill the requirement (3.10) the boundary term in (3.7) must take the following form:

$$S_{\text{boundary}} = \frac{1}{6}\int \mathrm{d}^4x\sqrt{-g}\,\partial_m\left(F^{mnrs}A_{nrs}\right) \tag{3.11}$$

In this way it can be shown (see for example [47]) that the equations of motion of the original theory remain unaffected by the presence of the boundary term, with the advantage that only surface integrals that depend on δF_{mnrs} appear, consistently with (3.10). Most importantly, it can be shown that when the equations of motion (3.8) are imposed the boundary term gives to the action a contribution that is twice as large as that of $-\frac{1}{2\cdot 4!}F_{mnrs}F^{mnrs}$, but with the opposite sign, that is:

$$\left[-\frac{1}{2\cdot 4!}F_{mnrs}F^{mnrs} + S_{\text{boundary}}\right]_{\text{on-shell}} = \frac{c^2}{2} - c^2 = -\frac{c^2}{2} \tag{3.12}$$

In this way we obtain a net contribution to the cosmological constant that is compatible with the Einstein equations of motion (see [47] for further details). This last feature of the boundary terms will prove crucial for our later discussions (even though in more involved cases) in order to make sure that the scalar potential of our theories possesses the correct sign.

In the previous chapter we have shown that the action of the IIa superstring theory, suitably compactified, can be written in four spacetime dimensions in the standard supergravity formalism, making use of the superpotentials (2.58). These superpotentials give rise to the proper scalar potential through the relation (1.64), and as stated previously they depend on the values of the fluxes, that up to here are mere constants. The final objective of the next sections, therefore, will be to rewrite these superpotentials (along with the other terms in the supergravity action) in such a way that all traces of the values of the fluxes have vanished, leaving room for new gauge field strengths. Before dealing with the complete case, however, we start from simpler settings, first in rigid supersymmetry and then extending the analysis in supergravity. The superpotential we will consider throughout the analysis, depending on a set of superfields Φ^A and T, as well as on the values of the fluxes, will be of the form:

$$W(\Phi) = e_A\Phi^A + m^A\mathcal{G}_{AB}\Phi^B + \hat{W}(\Phi, T) \tag{3.13}$$

$\mathcal{G}_{AB}$ and $\hat{W}(\Phi, T)$ are some holomorphic functions, with the parts containing e_A and m^A being subject to the dualization procedure. $\hat{W}(\Phi, T)$, instead, describes also some additional superfields T not involved in the dualization procedure: this term, therefore, will be crucial in keeping track of their contribution to the action. More precisely, taking into account $\hat{W}(\Phi, T)$ it will be possible to include in the theory also fluxes that we wish to mantain constant (that is, not dualized)[2]. In addition it will be shown, when the time comes, that the superpotentials (2.58) can be written in the form (3.13), and so that the dualization approach can be effectively applied.
We can now proceed in exposing in deeper detail the outlined strategy, always referring to [20] and [21].

3.1 Flux dualization in rigid supersymmetry

Single three-form multiplets

As a first example, let's take into account a supersymmetric $\mathcal{N} = 1$ theory with the matrix $\mathcal{G}_{AB}$ appearing in (3.13) taken to be a constant, and $\text{Im}\mathcal{G}_{AB} = 0$. The superpotential can hence be rewritten as:

$$e_A\Phi^A + m^A\text{Re}\mathcal{G}_{AB}\Phi^B \equiv r_A\Phi^A \tag{3.14}$$

The chiral superfields Φ^A and T enjoy a component expansions of the likes of (1.15).
The objective of this section is to exhibit a dualization procedure that allows to substitute the values of the fluxes r^A with appropriate field strengths of some gauge three-form potentials $A^A_{(3)}$, appearing in the chiral multiplets Y^A, called *single three-forms multiplets*, that we will shortly define.

[2]The ratio for this special treatment will come from the tadpole condition, that will be explored in full detail later on.

Recalling the notions of the first chapter, the lagrangian in the full superspace can be written as:

$$\mathcal{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \left(\int \mathrm{d}^2\theta \left[r_A \Phi^A + \hat{W}(\Phi, T) \right] + \text{h.c} \right), \tag{3.15}$$

where $K(\Phi, \bar{\Phi})$ is the Kähler potential of the theory (which will give the kinetic terms once expanded in components) and h.c. denotes the hermitian conjugate, necessary to obtain an hermitian lagrangian. As usual, the integration of the Kähler potential is upon the whole superspace, whilst for the superpotential it is over only half of it.

It is now possible to lay out more explicitly the dualization procedure: the general idea is to substitute the real constants r_A with auxiliary chiral superfields, let them be X_A, and to add a Lagrange multiplier, containing a real superfield U^A, so as to compel X_A to satisfy particular conditions. These conditions are nothing but the fact that, if the real superfields U^A are integrated out, the original theory with the constants r_A must be recovered. On the other hand, the path that we will follow is to integrate out the chiral superfields X_A, obtaining a relation between the Φ^A and the U^A: in this way, substituting the original superfields Φ^A, nothing is left of the values of the fluxes r_A, and the new theory thus obtained depends exclusively on U^A and its components. The gauge field strengths (or, more precisely, their potentials) we mentioned when we first introduced this procedure are included in the θ^2 component of the real superfield U^A.

When dealing with more involved cases the main points of the strategy will be exactly the same, even if the Lagrange multiplier terms will have a different structure, as well as the components of the new fields.

Practically speaking the lagrangian (3.15) is superseded by:

$$\begin{aligned} \mathcal{L}_{\text{new}} = &\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \left(\int \mathrm{d}^2\theta \left[X_A \Phi^A + \hat{W}(\Phi, T) \right] + \text{h.c} \right) + \\ &+ i \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, (X_A - \bar{X}_A) U^A \end{aligned} \tag{3.16}$$

The new term in the second line is the Lagrangian multiplier, and integrating out U^A the following condition is obtained:

$$X_A - \bar{X}_A = 0 \tag{3.17}$$

Recalling that the X_A are chiral superfields, that is $\bar{D}_{\dot{\alpha}} X_A = D_\alpha \bar{X}_A = 0$, it can be seen that (3.17) entails that $X_A = r_A$, with r_A real constants, just as the ones we started from, showing that the initial lagrangian (3.15) and the new one (3.16) are indeed equivalent.

The new formulation with no trace of the r_A, instead, is obtained integrating out X_A with vanishing variations at the boundary, that is:

$$\delta X_A|_{\text{bd}} = 0 \tag{3.18}$$

The result is:

$$\Phi^A = \frac{i}{4} \bar{D}^2 U^A \equiv Y^A \tag{3.19}$$

The components of the real superfield U^A are:

$$\begin{aligned} U^A|_{\theta=\bar\theta=0} &= u^A & -\frac{1}{8}\bar\sigma_m^{\alpha\dot\alpha}[D_\alpha,\bar D_{\dot\alpha}]U^A|_{\theta=\bar\theta=0} &= A^A_m \\ \frac{i}{4}\bar D^2 U^A|_{\theta=\bar\theta=0} &= \bar\phi^A & \frac{1}{16}D^2\bar D^2 U^A|_{\theta=\bar\theta=0} &= -D^A + i\partial^m A^A_m, \end{aligned} \tag{3.20}$$

where ϕ^A are complex scalars, u^A and D^A are real scalars and A^A_m are real vectors. Since A^A_m are one-forms their Hodge duals $A^A_{(3)}$ are three-forms, and it is exactly these three-forms that appear through their field strength in the chiral superfields Y^A. It can be shown, in fact, that the component expansion of Y^A (employing the coordinates defined in (1.13)) is:

$$Y^A = y^A + \theta\sqrt{2}\chi^A + \theta^2({}^*F^A_{(4)} + iD^A), \tag{3.21}$$

where D^A are the real auxiliary fields we have just introduced and ${}^*F^A_{(4)}$ are the (Hodge duals of the) field strengths of the three-forms $A^A_{(3)}$ that we have just mentioned. The chiral superfields Y^A are also called *single three-form multiplets*: the reason behind the name is that their θ^2 component contains the field strength of a single real three-form.
The field strengths $F^A_{(4)}$ are linked to their respective gauge potentials:

$$F^A_{(4)} = dA^A_{(3)} \tag{3.22}$$

These field strengths are, as usual, gauge invariant under a transformation of the potential:

$$A^A_{(3)} \longrightarrow A^A_{(3)} + d\Lambda^A_{(2)} \tag{3.23}$$

This kind of gauge transformation is deduced from a more general transformation of the real superfields U^A. The superfield Y^A, in fact, is manifestly invariant under the transformation:

$$U^A \longrightarrow U^A + L^A, \tag{3.24}$$

where L^A are *real* linear superfields, namely they satisfy:

$$D^2 L^A = \bar D^2 L^A = 0 \tag{3.25}$$

It can be shown that, when (3.24) is expanded in components, it gives precisely the ordinary gauge transformation (3.23).
One last step is required to bring the dualization procedure to completion, that is to eliminate the auxiliary fields X^A from the action. This is achieved by variating the action (3.15) with respect to the fields Φ^A, employing the boundary condition:

$$\delta\Phi^A|_{\mathrm{bd}} = 0 \tag{3.26}$$

In this way we obtain the following expression for the X_A:

$$X_A = \frac{1}{4}\bar{D}^2 K_A - \hat{W}_A(Y,T), \tag{3.27}$$

where the subscripts in K_A and $\hat{W}_A(Y,T)$ indicate derivation with respect to Φ^A, which is then substituted with Y^A according to (3.19). The derivative $\frac{1}{4}\bar{D}^2$ appears because the integration measure $\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$ can be split, as mentioned in the section devoted to supersymmetry, into:

$$\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} = -\frac{1}{4}\int \mathrm{d}^2\theta \bar{D}^2 \tag{3.28}$$

This decomposition is valid up to a total derivative, and in fact we will soon see that this subtlety will play an important role for boundary terms.
Finally substituting the expressions (3.19) and (3.27) in the modified lagrangian (3.16) we obtain a new lagrangian with no sign of the constants r_A, nor of the auxiliary fields X^A:

$$\hat{\mathcal{L}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(Y,\bar{Y}) + \left(\hat{W}(Y,T) + \text{h.c}\right) + \mathcal{L}_{\text{bd}}, \tag{3.29}$$

with $\mathcal{L}_{\text{bd}}$ being a boundary term of the form:

$$\mathcal{L}_{\text{bd}} = i\int \mathrm{d}^2\theta \left(\int \mathrm{d}^2\bar{\theta} + \frac{1}{4}\bar{D}^2\right)\left[\left(\frac{1}{4}\bar{D}^2 K_A - \hat{W}_A(Y,T)\right)U^A\right] + \text{h.c} \tag{3.30}$$

It is evident that (3.30) is a total derivative, and the reason why the sum $\left(\int \mathrm{d}^2\bar{\theta} + \frac{1}{4}\bar{D}^2\right)$ does not vanish, as should happen according to (3.28), is that if the boundary conditions are non-trivial total derivatives may not vanish anymore, invalidating the equality (3.28). Keeping track of the boundary terms is not a futile exercise, but, as we will see later, it is crucial in order to obtain a scalar potential of the theory with the correct sign. Furthermore, the requirement that the condition (3.17) can be obtained without imposing particular conditions on U^A completely fixes the form of the Lagrange multiplier contained in the lagrangian (3.16). The vanishing variation condition (3.26) imposed on the chiral superfields Φ^A instead implies, through the relation (3.19), the following condition on Y^A:

$$\delta Y^A|_{\text{bd}} = \frac{i}{4}\left(\bar{D}^2 \delta U^A\right)|_{\text{bd}} = 0 \tag{3.31}$$

The final step of the procedure, that reveals its relevance, is to expand the components of the new lagrangian (3.29) and to extract the bosonic and fermionic sector, as well as the explicit expression of the boundary term. Restricting to the bosonic sector, it can be shown that the component lagrangian is:

$$\begin{aligned}\hat{\mathcal{L}}_{\text{bos}} = K_{A\bar{B}}\left[D^A - i\partial_m(*A^{Am}_{(3)})\right]\left[D^B + i\partial_n(*A^{Bn}_{(3)})\right] + \\ + \left[i\hat{W}\left(D^A - i\partial_m(*A^{Am}_{(3)})\right) + \text{h.c.}\right] + \mathcal{L}^{\text{bos}}_{\text{bd}},\end{aligned} \tag{3.32}$$

with:

$$
\begin{aligned}
\mathcal{L}_{\rm bd}^{\rm bos} = &- \partial_m \left[i(^*A_{(3)}^{Am})(K_{B\bar{A}} - K_{A\bar{B}})D^B + (^*A_{(3)}^{Am})(K_{B\bar{A}} + K_{A\bar{B}})\partial_n(^*A_{(3)}^{Bn}) \right] + \\
&- \partial_m \left[(^*A_{(3)}^{Am})\hat{W}_A + (^*A_{(3)}^{Am})\bar{\hat{W}}_{\bar{A}} \right]
\end{aligned}
\tag{3.33}
$$

This boundary term makes sure that the variation of the action with respect to $A_{(3)}^A$ is well defined, just as the simpler example we displayed in (3.11).
We can see that in the bosonic lagrangian there are kinetic terms, proportional to the Kähler metric $K_{A\bar{B}}$, for the auxiliary fields D^A and the gauge three-forms $A_{(3)}^A$, and consistently the fluxes r^A do not appear anymore.
We proceed now in evaluating a slightly more involved case of dualization in rigid supersymmetry, that will be explored in the next section.

Double three-form multiplets

With the help of the strategy outlined in the previous section we can now proceed to face a more general case of dualization in rigid supersymmetry. In the following we will consider a completely general matrix $\mathcal{G}_{AB}$, that we recall being holomorphic. This will have repercussions on the dualization procedure: in order to substitute the fluxes in the superpotential two sets of real gauge three-forms (or, equivalently, one set of complex gauge three-forms), belonging to some chiral multiplets S^A, will have to be introduced. This is why the S^A will be called *double three-form multiplets*.

In addition to the fields that will be subject to dualization, that is the Φ^A, the $\hat{W}$ component of the superpotential will exhibit a dependence on a set of spectator superfields T, that will remain untouched by the whole procedure. As a result the notation will be:

$$
\mathcal{G}_{AB} = \mathcal{G}_{AB}(\Phi) \qquad \hat{W} = \hat{W}(\Phi, T) \tag{3.34}
$$

In order to simplify the problem it is assumed that $\mathcal{G}_{AB}$ is symmetric: this feature will be justified later on in a more ample context, grounded on physical and geometrical motives. The claimed generality of the reasoning, therefore, will not be nullified. The real and imaginary parts of the matrix $\mathcal{G}_{AB}$ will be called:

$$
\mathrm{Re}\mathcal{G}_{AB} \equiv \mathcal{N}_{AB} \qquad \mathrm{Im}\mathcal{G}_{AB} \equiv \mathcal{M}_{AB} \tag{3.35}
$$

Henceforth we will employ the condition:

$$
\det(\mathcal{M}_{AB}) \neq 0 \tag{3.36}
$$

This means that $\mathcal{M}_{AB}$ is invertible: if that is not the case a slightly different procedure, that will be explained before closing this section, must be utilized.
The initial lagrangian, with explicit appearance of the values of the fluxes, is the generaliza-

tion of (3.15) to the full non-constant $\mathcal{G}_{AB}$ case:

$$\mathcal{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \left(\int \mathrm{d}^2\theta \left[e_A \Phi^A + m^A \mathcal{G}_{AB}(\Phi)\Phi^B + \hat{W}(\Phi, T) \right] + \mathrm{h.c} \right) \tag{3.37}$$

Analogously to the previous case we introduce new "fluxes" $r_A(\Phi)$:

$$r_A(\Phi) = e_A + m^B \mathcal{G}_{AB}(\Phi) \tag{3.38}$$

The crucial difference from the case of single three-forms is that the r_A *depend* in a manifest way on the chiral superfields. We can now replace them with new auxiliary chiral superfields X_A, introducing a Lagrange multiplier to make sure that the original theory can be recovered straightforwardly:

$$\begin{aligned} \mathcal{L}_{new} = & \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \left(\int \mathrm{d}^2\theta \left[X_A \Phi^A + \hat{W}(\Phi, T) \right] + \mathrm{h.c} \right) \\ & - \frac{1}{4} \left(\int \mathrm{d}^2\theta \left[\Sigma_A \mathcal{M}^{AB} (X_A - \bar{X}_A) \right] + \mathrm{h.c.} \right) \end{aligned} \tag{3.39}$$

$\mathcal{M}^{AB}$ is nothing but the inverse of the imaginary part of the matrix G_{AB}, whereas Σ_A is a set of complex linear superfields; in other words, they satisfy:

$$\bar{D}^2 \Sigma_A = 0 \tag{3.40}$$

We can now take advantage of the fact that:

$$\bar{D}^2 \bar{D}_{\dot{\alpha}} = 0 = D^2 D_\alpha \tag{3.41}$$

This is a consequence of the anticommutation properties (1.9) of the superspace coordinates θ, implying that when three or more of them are multiplied the result is exactly zero. With this in mind the complex linear superfields Σ_A can be rewritten as covariant derivatives of a completely general Weyl spinor superfield Ψ_A^α:

$$\Sigma_A = \bar{D}_{\dot{\alpha}} \bar{\Psi}_A^{\dot{\alpha}} \tag{3.42}$$

Integrating out the Weyl spinor $\bar{\Psi}_A^{\dot{\alpha}}$ gives rise to the condition:

$$D_\alpha \left(\mathcal{M}^{AB} (X_B - \bar{X}_B) \right) = 0 \tag{3.43}$$

Just as in (3.17) the previous equation, along with its complex conjugate, implies that the term inside brackets is a real constant[3]

$$\mathcal{M}^{AB} (X_B - \bar{X}_B) = m^A \tag{3.44}$$

[3]More precisely, we have A such constants, one for each term.

The auxiliary fields X_A can then be written as:

$$\begin{aligned} X_A &= \mathrm{Re} X_A + i\mathcal{M}_{AB} m^B = \mathrm{Re} X_A + i\mathrm{Im}\mathcal{G}_{AB} m^B = \\ &= \mathrm{Re}\left(X_A - \mathcal{G}_{AB} m^B\right) + \mathcal{G}_{AB} m^B \end{aligned} \tag{3.45}$$

The fields X^A, besides, are chiral, and as a result it can be proven that:

$$\mathrm{Re}\left(X_A - \mathcal{G}_{AB} m^B\right) = e_A, \tag{3.46}$$

with e_A real constants. Summing up, the chiral superfields X_A must be equal to:

$$X_A = e_A + \mathcal{G}_{AB} m^B \tag{3.47}$$

Inserting this equation into (3.39) we immediately see that the original lagrangian (3.37) is recovered, as required.
On the other hand, exactly like in the previous section, the path to get rid of the values of the fluxes e_A and m^A is to integrate out the auxiliary fields X_A, obtaining:

$$\Phi^A = \frac{1}{4}\bar{D}^2\left[\mathcal{M}^{AB}\left(\Sigma_B - \bar{\Sigma}_B\right)\right] \equiv S^A \tag{3.48}$$

The S^A are chiral superfields, and are called (generalized[4]) *double three-form multiplets*, because in its bosonic component expansion there will be two real three-forms' field strengths. The three-form potentials, therefore, appear in the component expansion of the Σ_A, that reads:

$$\begin{aligned} &\Sigma_A|_{\theta=\bar{\theta}=0} = \sigma_A && \frac{1}{2}\bar{\sigma}^{m\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}]\Sigma_A|_{\theta=\bar{\theta}=0} = (^*C_{(3)A})^m \\ &-\frac{1}{4}D^2\Sigma_A|_{\theta=\bar{\theta}=0} = \bar{s}_A && \frac{1}{16}D^2\bar{D}^2\Sigma_A|_{\theta=\bar{\theta}=0} = 0 \\ &\frac{1}{16}\bar{D}^2 D^2\Sigma_A|_{\theta=\bar{\theta}=0} = \frac{i}{2}\partial_m(^*C_{(3)A})^m, \end{aligned} \tag{3.49}$$

where σ_A and s_A are scalars, and $C_{(3)A}$ is a complex three-form containing the two real three-forms we have previously talked about.
It is important to note that, even if we used a compact notation, the matrix $\mathcal{M}^{AB}$ in (3.48) depends on the chiral fields, that is:

$$\mathcal{M}^{AB} = \mathcal{M}^{AB}(\Phi) \tag{3.50}$$

As a consequence equation (3.48) is non-linear and, generically, the S^A cannot be explicitly extracted as a function of the Σ_A. Analogously to the case of single three-forms the lagrangian written in terms of the new chiral superfields S^A enjoys a generalized gauge invariance,

[4]Properly speaking the double three-form multiplets appear in the dualization of a superpotential with constant, yet non-vanishing, $\mathrm{Im}\mathcal{G}_{AB}$. This case, that has not been treated in the main text, can be found in [20].

parametrized by two linear real superfields $\tilde{L}_A$ and L^B:

$$\Sigma_A \longrightarrow \Sigma_A + \tilde{L}_A + \mathcal{G}_{AB} L^B \tag{3.51}$$

This transformation can be specified to the usual gauge transformation acting on the three-forms potential and leaving the field strengths, that are four-forms, invariant. We will show explicitly this property, along with the component expansions of S^A, in the next section, that will be focused on the more physically relevant case of supergravity dualization.
Going on with the dualization procedure, the equations of motion for Φ^A read:

$$X_A = \frac{1}{4}\bar{D}^2 K_A - \hat{W}_A(S,T), \tag{3.52}$$

where we have used the same conventions of the previous section: the only difference is that the single-three form multiplets Y^A have been superseded by the double three-form multiplets S^A. Finally, plugging (3.52) and (3.48) into (3.39) a new lagrangian, with no sign of the values of the fluxes e_A and m^A, is obtained:

$$\hat{\mathcal{L}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(S,\bar{S}) + \Big(\hat{W}(S,T) + \mathrm{h.c}\Big) + \mathcal{L}_{\mathrm{bd}} \tag{3.53}$$

As usual $\mathcal{L}_{\mathrm{bd}}$ is a boundary term, that is a total derivative, of the form:

$$\mathcal{L}_{\mathrm{bd}} = \int \mathrm{d}^2\theta \left(\mathrm{d}^2\bar{\theta} + \frac{1}{4}\bar{D}^2\right)\left[\left(\frac{1}{4}\bar{D}^2 K_A - \hat{W}_A(S,T)\right)\mathcal{M}^{AB}(\Sigma_B - \bar{\Sigma}_B)\right] \tag{3.54}$$

The work exposed until now has been conducted assuming the invertibility of the matrix $\mathcal{M}_{AB}$, the imaginary part of $\mathcal{G}_{AB}$. If that is not the case, however, a distinction between eigenvectors with null and non-vanishing eigenvalues must be made: appropriately mixing single and double three-form dualization the usual strategy can be successfully carried out [20].
In the next section we will review how to perform the dualization procedure in the case of local supersymmetry, i.e. supergravity.

3.2 Flux dualization in Supergravity

With the sequence of steps defined until now it is possible to adapt the dualization procedure to a $\mathcal{N}=1$ supergravity context, which will be more relevant for the continuation of this thesis. In the next chapters, in fact, we will take into account supergravity effective theories in four dimensions of the kind outlined in chapter 2, that is deduced from the compactification and orientifold projection of a $\mathcal{N}=2$ ten-dimensional theory containing the fields that appear from string quantization. One of our main goals will hence be to rewrite the 4d effective theory with no explicit appearance of the values of the fluxes e_A and m^A: the physical reason behind this procedure is to couple in a natural way the gauge three-form potentials

that substitute the fluxes to membranes, as explained at the beginning of this chapter. As a result in this section we will devote more attention to the component expansions of the superfields and of the lagrangians, in order to be able to work with them more easily.
In the following the case of a supergravity multiplet coupled to a set of $i = 1, ..., N$ chiral fields Φ^i will be taken under scrutiny: its action is contained in equation (1.61) when setting the vector superfields to zero:

$$\mathcal{S}_{SG} = -3 \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, e^{-\frac{1}{3}K\left(\Phi,\bar{\Phi}\right)} + \left(\int \mathrm{d}^4x \mathrm{d}^2\theta\, 2\mathcal{E}\, W(\Phi) + \text{h.c.} \right) \tag{3.55}$$

The standard supergravity multiplet has the components (1.49), which are all present when working off-shell:

$$e_m^a \qquad \psi_m^\alpha \qquad A_m \qquad M \tag{3.56}$$

When we first exposed the supersymmetry formalism we emphasized the importance of the invariance of the action with respect to a Kähler transformation:

$$K(\Phi,\bar{\Phi}) \longrightarrow K(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \tag{3.57}$$

When making the supersymmetry transformations local, however, we did not mention this invariance again: the reason is that further transformations are required to ensure that the supergravity action is invariant under (3.57).
First of all the superpotential $W(\Phi)$ must transform as:

$$W(\Phi) \longrightarrow e^{-\Lambda(\Phi)} W(\Phi) \tag{3.58}$$

In this way the potential (1.64) remains invariant. Nevertheless this is not sufficient: an additional symmetry, known as Weyl transformation, must be included in order to fully implement the Kähler invariance in the action. More precisely, as we are working in the superspace formalism, we will talk about super-Weyl transformations. The action of the super-Weyl transformations on the bosonic and fermionic components of the super-vielbein E_Λ^M is respectively defined as [22] (recalling that a is a flat spatial index, and α a flat spinorial one):

$$E_M^a \longrightarrow e^{\mathcal{Y}+\bar{\mathcal{Y}}} E_M^a \qquad E_M^\alpha \longrightarrow e^{2\bar{\mathcal{Y}}-\mathcal{Y}} \left(E_M^\alpha - \frac{i}{4} E_M^a \sigma_a^{\alpha\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{Y}} \right), \tag{3.59}$$

where $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ are arbitrary chiral superfields parametrizing the transformation.
In a similar manner also the chiral fields Φ^i undergo a Weyl transformation of the form:

$$\Phi^i \longrightarrow e^{w\mathcal{Y}} \Phi^i, \tag{3.60}$$

where w is a number, the so-called Weyl weight of the chiral superfield. The supergravity projector instead transforms as:

$$\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) \longrightarrow e^{-4\mathcal{Y}} \left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) e^{2\bar{\mathcal{Y}}}, \tag{3.61}$$

whereas the transformations for the full and chiral superspace measures E and $\mathrm{d}^2\theta\,\mathcal{E}$ are:

$$E \longrightarrow e^{2(\mathcal{Y}+\bar{\mathcal{Y}})}E \qquad \mathrm{d}^2\theta\,\mathcal{E} \longrightarrow e^{6\mathcal{Y}}\mathrm{d}^2\theta\,\mathcal{E} \tag{3.62}$$

Combining the just mentioned super-Weyl transformations it can be shown that the total variation of the supergravity action is [7]:

$$\delta S_{SG} = \int \mathrm{d}^4x\mathrm{d}^2\theta\, 2\mathcal{E}\left[\frac{3}{4}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)(\mathcal{Y}+\bar{\mathcal{Y}})e^{-\frac{K(\Phi,\bar{\Phi})}{3}} + 6\mathcal{Y}W(\Phi)\right] + \text{h.c.} \tag{3.63}$$

At the same time a Kähler transformation parametrized by Λ induces a variation:

$$\delta S_{SG} = \int \mathrm{d}^4x\mathrm{d}^2\theta\, 2\mathcal{E}\left[-\frac{1}{8}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)(\Lambda+\bar{\Lambda})e^{-\frac{K(\Phi,\bar{\Phi})}{3}} - \Lambda W(\Phi)\right] + \text{h.c.} \tag{3.64}$$

We immediately see that if $\Lambda = 6\mathcal{Y}$ the two variations cancel out, making the action invariant. In order to obtain this cancellation, however, we have chosen a particular relation between the transformation parameters Λ and $\mathcal{Y}$.

Another approach that naturally allows to encompass super-Weyl transformations is to construct an action that is super-Weyl invariant from the beginning: in order to achieve this, however, it is indispensable to introduce an additional chiral superfield. Before dealing with our case, let us examine how this mechanism works in a simpler setting.
First of all, we take into account the usual Einstein-Hilbert action:

$$S_{EH} = \int \mathrm{d}^4x\sqrt{-g}\,\frac{1}{2}R \tag{3.65}$$

We would like to make it invariant under a Weyl transformation, that acts on the metric g_{mn} as:

$$\delta g_{mn} = -2\lambda(x)g_{mn}, \tag{3.66}$$

where $\lambda(x)$ is a local dilation factor. It is evident that (3.65) is not invariant under such a transformation: consequently we modify it by adding a coupling and kinetic term for a real scalar field ϕ:

$$S'_{EH} = \int \mathrm{d}^4x\sqrt{-g}\,\left(\frac{1}{12}R\phi^2 + \frac{1}{2}\partial_m\phi\partial^m\phi\right), \tag{3.67}$$

with ϕ enjoying the following Weyl transformation law:

$$\delta\phi = \lambda(x)\phi \tag{3.68}$$

It can be shown that combining the transformations (3.66) and (3.68) the modified action (3.67) remains invariant. On the other hand, if we wish to recover the standard Einstein-Hilbert action we must gauge fix the Weyl transformation and eliminate the auxiliary scalar field setting it to:

$$\phi = \sqrt{6} \tag{3.69}$$

Recapping the overall strategy we identify the following steps: write the original action, which is not Weyl-invariant; add terms that comprise an auxiliary field with an appropriate transformation law, thus making the action Weyl-invariant; when necessary, gauge-fix the symmetry recovering the original action. The advantage of this procedure is that it gives a natural way to implement Weyl-symmetry and that it lies in the framework of the study of conformal theories [48].

As regards our concrete case we start with the supergravity action (3.55), that involves N chiral fields Φ^i. We wish to make this action invariant under the super-Weyl transformations (3.59),(3.60),(3.61) and (3.62).

Analogously to the procedure we have just described for the Einstein-Hilbert action, we consider an enlarged set of $N+1$ chiral multiplets $\mathcal{Z}^A$ (with $A = 0, ..., N$), that are a function of the original chiral fields Φ^i and of an additional field Z, called Weyl compensator, that plays the exact same role of ϕ in (3.67). The chiral fields $\mathcal{Z}^A$ have the following components:

$$\mathcal{Z}^A = z^A + \sqrt{2}\theta\psi^A + \theta^2 F_Z^A, \tag{3.70}$$

where, as usual, z^A are complex scalar fields, ψ^A are Weyl fermions and F_Z^A are non-propagating complex auxiliary fields, and they all depend on the old chiral superfields Φ^i, as well as on the Weyl compensator Z. The $\mathcal{Z}^A$, moreover, are subject to super-Weyl transformations:

$$\mathcal{Z}^A \longrightarrow e^{-6\mathcal{Y}}\mathcal{Z}^A \tag{3.71}$$

As a result, the new super-Weyl invariant action is:

$$\mathcal{S}'_{SG} = -3\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, E\, |Z|^{\frac{2}{3}}\, e^{-\frac{1}{3}K(\mathcal{Z},\bar{\mathcal{Z}})} + \left(\int \mathrm{d}^4x\mathrm{d}^2\theta\, 2\mathcal{E}\, W(\mathcal{Z}) + \text{h.c.}\right) \tag{3.72}$$

The task we want to carry out in the next sections is to substitute the values of the fluxes from the action (3.72)[5] with appropriate gauge field strengths, in the same way as in the rigid supersymmetry case. We note that (3.72) contains the $N+1$ superfields $\mathcal{Z}^A$, and we will treat all of them on an equal footing. After having completed the substitution, however, we will want to recover the original theory with a supergravity multiplet coupled to only N physical chiral multiplets. As a consequence of its definition, in fact, the Weyl compensator Z is unphysical and is a mere mathematical device to make the action Weyl invariant. In order to get rid of Z we will carry out an appropriate gauge-fixing of the super-Weyl invariance, analogous to (3.69). In order to do this the dependence of the $\mathcal{Z}^A$ on the chiral compensator Z will be made explicit in the following way:

$$\mathcal{Z}^A = Zf^A(\Phi), \tag{3.73}$$

where the f^A are some functions of the physical superfields Φ^i that do not change under a super-Weyl transformation. It is easy to see that this decomposition is invariant through a

[5]Employing also slightly different kinetic terms.

redefinition of the splitting between the chiral compensator and the physical fields:

$$Z \longrightarrow e^{-g(\Phi)} Z \qquad f^A(\Phi) \longrightarrow e^{g(\Phi)} f^A(\Phi), \tag{3.74}$$

where such an invariance corresponds to the Kähler transformation (3.57) of the original action (3.55).
The advantage of this strategy is that, dualizing the $\mathcal{Z}^A$ before gauge-fixing Z, we will be able to employ exactly the same steps as we did in rigid supersymmetry.

Single three-form multiplets

After this discussion we can face a simple warm-up case of the supergravity multiplet coupled to a single superfield Z, the Weyl compensator. In other words, we are describing a theory that contains only the components of the supergravity multiplet as physical degrees of freedom, with the addition of the Weyl compensator, so that the action is Weyl invariant. The lagrangian reads:

$$\mathcal{L} = -3 \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E \left(Z\bar{Z}\right)^{\frac{1}{3}} + \left(\int \mathrm{d}^2\theta\, 2\mathcal{E}\, rZ + \text{h.c.} \right), \tag{3.75}$$

where r is a real constant, whose role is completely analogous to the values of the fluxes e_A and m^A we considered in the previous sections.
Similarly to the rigid supersymmetry case the lagrangian (3.92) is modified with the substitution of the constant r with a chiral field X, and the addition of an appropriate Lagrange multiplier:

$$\begin{aligned} \mathcal{L}_{new} = -3 \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E \left(Z\bar{Z}\right)^{\frac{1}{3}} + \Bigg(\int \mathrm{d}^2\theta 2\mathcal{E}\, XZ + \\ + \int \mathrm{d}^2\theta 2\mathcal{E}\, \frac{1}{8} \left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) \left(X + \bar{X}\right) U + \text{h.c.} \Bigg) \end{aligned} \tag{3.76}$$

The term on the second line is the Lagrange multiplier, whose resemblance to its analogous in the case of rigid supersymmetry, i.e. $i \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, (X - \bar{X})U$, is manifest. The only difference is that this time the Lagrange multiplier has been written with an integral over chiral superspace, hence the presence of the projector $\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)$. In order to guarantee that the lagrangian remains super-Weyl invariant the new real superfield U must transform as:

$$U \longrightarrow e^{-2(\mathcal{Y}+\bar{\mathcal{Y}})} U \tag{3.77}$$

The original theory is recovered if U is integrated out, implying that the auxiliary chiral field X is a real constant r, exactly like in the rigid supersymmetry case.
On the other hand, if the variation with respect to X is performed, a dependence of the

chiral compensator Z on the real superfield U is obtained:

$$Z = -\frac{1}{4}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) U \equiv Y, \tag{3.78}$$

where Y are called *single three-form multiplets*, just like in the rigid supersymmetry case. The components of the real superfield U are analogous to the ones reported in (3.20), provided that the supersymmetry covariant derivative D is substituted by the supergravity covariant derivative $\mathcal{D}$. In particular the Hodge dual of a three-form $A_{(3)}$ appears in one of the components of U:

$$-\frac{1}{8}\bar{\sigma}_m^{\alpha\dot{\alpha}}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]U^A|_{\theta=\bar{\theta}=0} = ({}^*A_{(3)})_m \tag{3.79}$$

The lagrangian is invariant by a change in the real superfield U parametrized by an arbitrary linear superfield L:

$$U \longrightarrow U + L \tag{3.80}$$

This transformation reads for the gauge three-forms:

$$A_{(3)} \longrightarrow A_{(3)} + d\Lambda_{(2)}, \tag{3.81}$$

where $\Lambda_{(2)}$ is a generic two-form. This gauge transformation makes sure that the gauge three-form $A_{(3)}$ enters in the new chiral field Y only via its field strength.
In order to obtain a lagrangian devoid of the constant r, as well as of the auxiliary field X, the equations of motion for Z must be used:

$$X = -\frac{1}{4}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)\left(Z^{-\frac{2}{3}}\bar{Z}^{\frac{1}{3}}\right) \tag{3.82}$$

Finally, substituting this equation along with (3.78) into $\mathcal{L}_{\text{new}}$ the dualized lagrangian is obtained:

$$\hat{\mathcal{L}} = -3\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\left(Y\bar{Y}\right)^{\frac{1}{3}} + \mathcal{L}_{\text{bd}} \tag{3.83}$$

We emphasize again that $\hat{\mathcal{L}}$ depends exclusively on the single three-form Y, and in particular on the Hodge dual of the field strength of the three-form $A_{(3)}$. The lagrangian is still super-Weyl invariant, as a consequence of the fact that Y retains the transformation properties of Z:

$$Y \longrightarrow e^{-\mathcal{Y}}Y \tag{3.84}$$

The boundary term, where X is evaluated according to equation (3.100) is:

$$\mathcal{L}_{\text{bd}} = \int \mathrm{d}^2\theta\, 2\mathcal{E}\,\frac{1}{8}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)\left[(X - \bar{X})U\right] + \text{h.c.} \tag{3.85}$$

At this point the dualized lagrangian can be gauge fixed, so as to eliminate the chiral compensator and to obtain the minimal supergravity formulation: in this respect it is convenient to use the Weyl invariance to set the single three-form multiplet to 1:

$$Y = 1 \tag{3.86}$$

This choice implies that the lowest component of Y must be:

$$Y|_{\theta=\bar{\theta}=0} = 1 \tag{3.87}$$

Furthermore the highest component must satisfy:

$$-\frac{1}{4}\mathcal{D}^2 Y|_{\theta=\bar{\theta}=0} = 0 \tag{3.88}$$

This puts a constraint on the complex scalar supergravity auxiliary field M:

$$\mathrm{Im}M = -{}^*dA_{(3)} = -{}^*F_{(4)} \tag{3.89}$$

As a consequence M can be written as:

$$M = \mathrm{Re}M - i{}^*F_{(4)} \tag{3.90}$$

The net result is that the supergravity multiplet, once composed by the vielbein, gravitino, real auxiliary vector b_m and complex auxiliary scalar M (1.49), gets modified by the introduction, in the imaginary part of M, of the gauge three-form field strength:

$$e_m^a \qquad \psi_m^\alpha \qquad b_m \qquad \mathrm{Re}M \qquad {}^*F_{(4)} \tag{3.91}$$

We note that of course the number of degrees of freedom has remained unchanged, since the Hodge dual of a four-form (that is, the field strength) is nothing but a real scalar in four dimensions.
After this analysis we can proceed to study the dualization procedure taking into account the coupling of the supergravity multiplet to physical chiral fields in the non-linear case[6].

Double three-form multiplets

In this section we consider the usual supergravity multiplet coupled to $N+1$ chiral superfields $\mathcal{Z}^{\mathcal{A}}$ (so that A runs from 0 to N): N among these are physical multiplets, whereas the remaining one is the chiral compensator Z. In addition we introduce a set of spectators chiral superfields T, that are not subject to the dualization procedure, and are inert under super-Weyl transformations.

The most general lagrangian (with no more than second derivatives) that can be con-

[6]As already mentioned, a careful analysis of all cases is present in [20].

structed with these fields, maintaining the super-Weyl invariance, is:

$$\mathcal{L} = -3\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \Omega(\mathcal{Z}, \bar{\mathcal{Z}}, T, \bar{T}) + \left(\int \mathrm{d}^2\theta\, 2\mathcal{E}\, W(\mathcal{Z}, T) + \mathrm{h.c.}\right) \tag{3.92}$$

We can choose a specific form for the kinetic term Ω, that contains the usual Kähler potential K:

$$\Omega(\mathcal{Z}, \bar{\mathcal{Z}}, T, \bar{T}) = |Z|^{\frac{2}{3}} e^{-\frac{1}{3}K(\mathcal{Z}, \bar{\mathcal{Z}}, T, \bar{T})} \tag{3.93}$$

The structure of Ω implies that, once the chiral compensator Z has been gauge-fixed in a suitable way, the usual supergravity kinetic term (1.61) is recovered. The super-Weyl invariance, besides, requires that the kinetic and superpotential terms satisfy a few specific homogeneity properties with respect to a rescaling of the superfields $\mathcal{Z}^A$:

$$\Omega(\lambda\mathcal{Z}, \bar{\lambda}\bar{\mathcal{Z}}, T, \bar{T}) = |\lambda|^{\frac{2}{3}}\Omega(\mathcal{Z}, \bar{\mathcal{Z}}, T, \bar{T}) \qquad W(\lambda\mathcal{Z}, \lambda T) = \lambda W(\mathcal{Z}, T) \tag{3.94}$$

As we did for the rigid supersymmetry case we choose a specific structure for the superpotential:

$$W(\mathcal{Z}, T) = e_A \mathcal{Z}^A + m^A \mathcal{G}_{AB}(\mathcal{Z})\mathcal{Z}^B + \hat{W}(\mathcal{Z}, T), \tag{3.95}$$

where it can be noted that the whole dependence on the spectator chiral fields is contained in $\hat{W}(\mathcal{Z}, T)$. In order to make contact with the supergravity effective theories presented in the previous chapter, that possess superpotentials such as (2.58), it is favorable to restrict to the case where $\mathcal{G}_{AB}$ is the second derivative of a so-called *prepotential* $\mathcal{G}$:

$$\mathcal{G}_{AB} \equiv \frac{\partial^2 \mathcal{G}}{\partial \mathcal{Z}^A \partial \mathcal{Z}^B}, \tag{3.96}$$

effectively making $\mathcal{G}_{AB}$ symmetric. The justification for this restriction comes from the fact that matrices $\mathcal{G}_{AB}$ of the kind (3.96) derive from certain models of string compactification, as we will show in chapter 4.

It is an immediate consequence of (3.94) that $\mathcal{G}_{AB}$ and the prepotential must satisfy:

$$\mathcal{G}_{AB}(\lambda\mathcal{Z}) = \lambda\mathcal{G}_{AB}(\mathcal{Z}) \qquad \mathcal{G}(\lambda\mathcal{Z}) = \lambda^2\mathcal{G}(\mathcal{Z}) \tag{3.97}$$

Implying that:

$$\mathcal{G}_{AB}(\mathcal{Z})\mathcal{Z}^B = \mathcal{G}_A(\mathcal{Z}) \tag{3.98}$$

As in the case of rigid supersymmetry (3.35) we define:

$$\mathrm{Re}\mathcal{G}_{AB} \equiv \mathcal{N}_{AB} \qquad \mathrm{Im}\mathcal{G}_{AB} \equiv \mathcal{M}_{AB} \tag{3.99}$$

Since the matrix $\mathcal{G}_{AB}$ depends on the chiral multiplets, the auxiliary fields X_A depend upon them too:

$$X_A = e_A + m^B \mathcal{G}_{AB}(\mathcal{Z}) \tag{3.100}$$

Adding the Lagrange multiplier to (3.92) and using the substitution (3.100) yields:

$$\begin{aligned}\mathcal{L}_{\text{new}} = -3 \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \Omega(\mathcal{Z}, \bar{\mathcal{Z}}, T, \bar{T}) + \\ \left(\int \mathrm{d}^2\theta\, 2\mathcal{E}\, \left[X_A \mathcal{Z}^A - \frac{1}{4} \left(\bar{\mathcal{D}}^2 - 8\mathcal{R} \right) \left[\mathcal{M}^{AB} (X_A - \bar{X}_A) \Sigma_B \right] + \hat{W}(\mathcal{Z}, T) \right] + \text{h.c.} \right)\end{aligned} \tag{3.101}$$

$\mathcal{M}^{AB}$ is the inverse of the imaginary part of $\mathcal{M}_{AB}$, whereas Σ_A is a linear superfield that satisfies:

$$\left(\bar{\mathcal{D}}^2 - 8\mathcal{R} \right) \Sigma_A = 0 \tag{3.102}$$

As in the rigid supersymmetry context it can be written as the supergravity covariant derivative of a generic Weyl spinor superfield:

$$\Sigma_A = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Psi}_A^{\dot{\alpha}} \tag{3.103}$$

Performing the variation of the lagrangian with respect to Ψ_A^{α} gives an expression for X_A that, suitably adjusted by means of the procedure described in the previous sections, gives the original theory, with X_A depending on a set of real constants e_A and m^A:

$$X_A = e_A + \mathcal{G}_{AB} m^B \tag{3.104}$$

On the other hand the dualization strategy goes on by integrating out exactly the auxiliary fields X_A, obtaining:

$$\mathcal{Z}^A = \frac{1}{4} \left(\bar{\mathcal{D}}^2 - 8\mathcal{R} \right) \left[\mathcal{M}^{AB} (\Sigma_B - \bar{\Sigma}_B) \right] \equiv S^A \tag{3.105}$$

As we see this expression, that is non-linear in the fields S^A (since the matrix $\mathcal{M}^{AB}$ on the right side depends on them), perfectly coincides with (3.48), as long as the supersymmetry projector is replaced by the supergravity one. The chiral fields S^A, as usual, are called *double three-form multiplets*, and as we will see explicitly in a few lines they encode the field strength of a complex three-form or, analogously, of two real independent three-forms. In the same manner the three-form potentials lie in the components of the fields Σ_A.
The new lagrangian is invariant under a gauge transformation of the Σ_A, parametrized by linear superfields $\tilde{L}_A$ and L^A, inducing an analogous transformation for the three-forms, that will be displayed later:

$$\Sigma_A \longrightarrow \Sigma_A + \tilde{L}_A + \mathcal{G}_{AB} L^B \tag{3.106}$$

Varying the lagrangian with respect to the $\mathcal{Z}^A$, instead, gives an expression for the X_A:

$$X_A = \frac{1}{4} \left(\bar{\mathcal{D}}^2 - 8\mathcal{R} \right) \left[\Omega_A + \frac{\partial \mathcal{M}^{BC}}{\partial S^A} (X_B - \bar{X}_B)(\Sigma_C - \bar{\Sigma}_C) \right] - \hat{W}_A, \tag{3.107}$$

where the subscript on Ω_A and $\hat{W}_A$ indicates derivation with respect to $\mathcal{Z}^A$, that is then replaced by the double three-form multiplets S^A.
Inserting (3.105) and (3.107) into (3.101) a new lagrangian, with no explicit sign of the constants e_A and m^A, is produced:

$$\hat{\mathcal{L}} = -3\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \Omega(S,\bar{S},T,\bar{T}) + \left(\int \mathrm{d}^2\theta\, 2\mathcal{E}\, \hat{W}(S,T) + \text{h.c.}\right) + \mathcal{L}_{\text{bd}} \tag{3.108}$$

The boundary term reads:

$$\mathcal{L}_{\text{bd}} = \int \mathrm{d}^2\theta \left(\mathrm{d}^2\bar{\theta} + \frac{1}{4}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)\right)\left[X_A \mathcal{M}^{AB}(\Sigma_B - \bar{\Sigma}_B)\right] \tag{3.109}$$

Until now we have done nothing but rephrasing the procedure carried out in the rigid supersymmetry case with only a few minor changes, such as replacing the supersymmetry projector with the supergravity one. In the following, however, in order to study extensively a concrete model of flux compactification, we will be interested in obtaining an explicit expression for the bosonic sector of the lagrangian (3.108): it is necessary, therefore, to delve deeper into the details and expand $\hat{\mathcal{L}}$ into its components.

First of all, taking advantage of the redundancy (3.106) we use the Wess-Zumino gauge, that puts some constraints on the components of the linear superfields Σ_A:

$$\begin{aligned}
\Sigma_A|_{\theta=\bar{\theta}=0} &= 0\\
\mathcal{D}^2\Sigma_A|_{\theta=\bar{\theta}=0} &= -4\mathsf{s}_A\\
\bar{\sigma}_m^{\alpha\dot{\alpha}}\left[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\right]\Sigma_A|_{\theta=\bar{\theta}=0} &= -2\left((^*\tilde{A}_{(3)A})_m + \mathcal{G}_{AB}(^*A^A_{(3)})_m\right)\\
\mathcal{D}^2\bar{\mathcal{D}}^2\Sigma_A|_{\theta=\bar{\theta}=0} &= 8i\mathcal{D}_m\left((^*\tilde{A}_{(3)A})^m + \mathcal{G}_{AB}(^*A^A_{(3)})^m\right) + 16\bar{M}\mathsf{s}_A,
\end{aligned} \tag{3.110}$$

where M is the lowest component of the supergravity multiplet $\mathcal{R}$, and $\tilde{A}_{(3)A}$ and $A^A_{(3)}$ are two sets of $N+1$ *real* three-forms, that appear via their respective field strengths $\tilde{F}_{(4)A}$ and $F^A_{(4)}$ in the double three-form multiplets S^A. This is a consequence of the fact that the transformations (3.106) take a specific (and more familiar) form when applied to them (with $\tilde{\Lambda}_{(2)A}$ and $\Lambda^A_{(2)}$ arbitrary two-forms):

$$\tilde{A}_{(3)A} \longrightarrow \tilde{A}_{(3)A} + d\tilde{\Lambda}_{(2)A} \qquad A^A_{(3)} \longrightarrow A^A_{(3)} + d\Lambda^A_{(2)} \tag{3.111}$$

Using the relation (3.105) it can be shown that the lowest components of S^A, that we call s^A, are related to the s_A that appear in (3.110) via the matrix $\mathcal{M}^{AB}$ (recalling that it depends on the multiplets S^A, and therefore on their lowest components s^A):

$$s^A = \mathcal{M}^{AB}(s,\bar{s})\mathsf{s}_B \tag{3.112}$$

As a consequence of the fact that this equation is non-linear in general it is not possible to make the dependence of s^A on the scalars s_A explicit: as a result it is more convenient to use the s^A as independent fields in the following steps.
The θ^2 component of the chiral superfields S^A, that is the auxiliary field component, reads:

$$F_S^A \equiv -\frac{1}{4}\mathcal{D}^2 S^A|_{\theta=\bar\theta=0} = \bar M s^A + \frac{i}{2}\mathcal{M}^{AB}\left[{}^*\tilde F_{(4)B} + \bar{\mathcal{G}}_{BC}{}^*F_{(4)}^C + 2\mathrm{Re}\left(\bar{\mathcal{G}}_{BCD}\bar F_S^D \bar s^C\right)\right], \quad (3.113)$$

where $\mathcal{G}_{ABC}$ corresponds to $\frac{\partial \mathcal{G}_{AB}}{\partial \mathcal{Z}^C}$, computed in $\mathcal{Z}^A = S^A$.
It can be seen, as expected, that the complex auxiliary field F_S^A includes the (Hodge duals of the) field strengths of the gauge three-forms. The homogeneity properties (3.94), however, impose that:

$$\bar{\mathcal{G}}_{ABC}\bar s^C = 0 \quad (3.114)$$

Consequently the reduced expression for the auxiliary fields F_S^A is:

$$F_S^A = \bar M s^A + \frac{i}{2}\mathcal{M}^{AB}\left({}^*\tilde F_{(4)B} + \bar{\mathcal{G}}_{BC}{}^*F_{(4)}^C\right) \quad (3.115)$$

At this point, having completed the dualization procedure, it is possible to fix the Weyl invariance and reduce the set of chiral fields S^A only to the physical ones. In this regard we employ the decomposition (3.73):

$$S^A = S f^A(\Phi), \quad (3.116)$$

where Φ^i are a set of N physical fields, and S is the chiral compensator. The most convenient gauge fixing imposes:

$$S = 1 \qquad f^0(\Phi) = 1 \quad (3.117)$$

With this choice it is evident that the components of S^0 satisfy (as usual we neglect the fermionic components):

$$S^0|_{\theta=\bar\theta=0} = 1 \qquad -\frac{1}{4}\mathcal{D}^2 S^0|_{\theta=\bar\theta=0} = F_S^0 = 0 \quad (3.118)$$

These constraints impose a restriction on the single terms that compose F_S^0, whereas the auxiliary fields F_S^i of the physical multiplets remain the same:

$$\begin{aligned} \bar M &= -\frac{i}{2}\mathcal{M}^{0B}\left({}^*\tilde F_{(4)B} + \bar{\mathcal{G}}_{BC}{}^*F_{(4)}^C\right) \\ F_S^i &= \bar M s^i + \frac{i}{2}\mathcal{M}^{iB}\left({}^*\tilde F_{(4)B} + \bar{\mathcal{G}}_{BC}{}^*F_{(4)}^C\right) \end{aligned} \quad (3.119)$$

The first line of the previous equation is precisely of the same form of (3.89) in the case of single three-forms. In that example only half of the supergravity multiplet auxiliary field M (more specifically its imaginary part) was fixed in terms of the field strength of a *real* gauge three-form. In the present case, however, since the dualization procedure has produced two sets of *real* three-forms, the degrees of freedom counting entails that both the real and imaginary part of M are fixed in terms of $\tilde{A}_{(3)A}$ and $A^A_{(3)}$.

3.3 Recap of the dualization procedure

At this point it is useful to sum up the gist of the dualization procedure in the most general supergravity case: we start from a theory that involves a superpotential that depends on some chiral fields $\mathcal{Z}^A$ (that comprise a chiral compensator Z) and on the fluxes e_A and m^A, substituting them with auxiliary fields X_A and adding a Lagrange multiplier to the action. Then the superfields $\mathcal{Z}^A$ are substituted by the double-three form multiplets S^A, that contain the field strengths of two sets of real gauge three-form potentials $\tilde{A}_{(3)A}$ and $A^A_{(3)}$, that are in correspondence with the fluxes e_A and m^A. Finally, a gauge-fixing allows to eliminate the Weyl compensator Z, yielding the constraints (3.119) on the components of the S^A. The last step, to which we shall proceed now, is to write the new action in terms of the double three-form multiplets S^A, that will be the starting point in order to extract its bosonic components.
In order to be more specific and display explicit expressions for the just mentioned relations in the next chapter we will restrict to the case of the superpotentials in the first line of (2.58), that appear in the four dimensional supergravities that derive from the compactification of a 10d effective theory.
The superpotential in (2.58), describing a generic set of $i = 1, ..., N$ chiral superfields, is a particular case of a superpotential of the form:

$$W(\Phi) = e_0 + e_i\Phi^i + \frac{1}{2}\kappa_{ijk}m^i\Phi^j\Phi^k - \frac{m^0}{6}\kappa_{ijk}\Phi^i\Phi^j\Phi^k \tag{3.120}$$

We do not display an analogous conversion for the complex structure superpotential, that appears in the second line of (2.58), because we will treat the values of the fluxes present therein to be fixed by the tadpole condition, upon which we will shed some light in the next chapter. Anyway, dualization of all the fluxes is feasible and displayed in full detail in [23]. From now on, therefore, we will indicate the part of the superpotential that does not undergo dualization with $\hat{W}(T)$, as we have done previously, where T are the spectator chiral superfields. In addition we do not consider a coupling between the $\mathcal{Z}^A$ and the T, so that $\hat{W}$ depends only on T.
If we want to rephrase (2.58) in the formalism developed in [20] and [21] that we have used until now a chiral compensator Z must be added, thus forming a set of $N + 1$ superfields $\mathcal{Z}^A = Zf^A(\Phi)$. Recalling that the index A splits into 0 and i we choose the functions $f^i(\Phi)$ to be simply equal to the chiral fields themselves, and $f^0(\Phi)$ to be the identity; as a result we have:

$$\mathcal{Z}^0 = Z \qquad \mathcal{Z}^i = \Phi^i \tag{3.121}$$

With these fields the potential (3.120) can be restated as:

$$\begin{aligned} W(\mathcal{Z},T) &= e_0\mathcal{Z}^0 + e_i\mathcal{Z}^i + \frac{1}{2\mathcal{Z}^0}\kappa_{ijk}m^i\mathcal{Z}^j\mathcal{Z}^k - \frac{m^0}{6(\mathcal{Z}^0)^2}\kappa_{ijk}\mathcal{Z}^i\mathcal{Z}^j\mathcal{Z}^k + \hat{W}(T) = \\ &= e_A\mathcal{Z}^A + \frac{1}{2Z}\kappa_{ijk}m^i\mathcal{Z}^j\mathcal{Z}^k - \frac{m^0}{6Z^2}\kappa_{ijk}\mathcal{Z}^i\mathcal{Z}^j\mathcal{Z}^k + \hat{W}(T) \end{aligned} \tag{3.122}$$

The homogeneity condition $W(\lambda\mathcal{Z},\lambda T) = \lambda W(\mathcal{Z},T)$ is manifestly satisfied, provided that $\hat{W}(\lambda T) = \lambda\hat{W}(T)$. Furthermore this superpotential can be rewritten in the more compact form (3.95) employing a suitable prepotential $\mathcal{G}(\mathcal{Z})$, as it can be immediately verified taking its second derivative with respect to $\mathcal{Z}^A$:

$$\mathcal{G}(\mathcal{Z}) = \frac{1}{6\mathcal{Z}^0}\kappa_{ijk}\mathcal{Z}^i\mathcal{Z}^j\mathcal{Z}^k \tag{3.123}$$

The gauge fixing condition we choose is equal to (3.117), therefore setting:

$$\mathcal{Z}^0 = Z = 1 \tag{3.124}$$

We make one last assumption, that is that the kinetic term $\Omega(\mathcal{Z},\bar{\mathcal{Z}},T,\bar{T})$ has the form (that will be concretely realized in the model we will take into consideration in the next chapter):

$$\Omega(\mathcal{Z},\bar{\mathcal{Z}},T,\bar{T}) = e^{-\frac{1}{3}K(\mathcal{Z},\bar{\mathcal{Z}})-\frac{1}{3}K(T,\bar{T})} \tag{3.125}$$

The resulting dualized lagrangian, obtained with the general method outlined before (recalling that S^A are the double three-form multiplets), is:

$$\hat{\mathcal{L}} = -3\int \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, E\, e^{-\frac{1}{3}K(S,\bar{S})-\frac{1}{3}K(T,\bar{T})} + \left(\int \mathrm{d}^2\theta\, 2\mathcal{E}\,\hat{W}(T) + \text{h.c.}\right) + \mathcal{L}_{\text{bd}}, \tag{3.126}$$

where $\mathcal{L}_{\text{bd}}$ is exactly the same as (3.109).

So far we have maintained almost full generality in dualizing the initial lagrangian: the only concessions consisted in assuming a specific form for the superpotential (3.120), that is typical of supergravity effective theories arising from flux compactification as showed in Chapter 2, and for the kinetic part (3.125), again justified for the same reasons.

In the next chapter we will instead delve deeper into the analysis and, after having chosen a specific model, expand (3.126) into its components.

CHAPTER 4

A specific model of flux compactification

In the first part of this chapter we will apply the procedure that allows to make the values of the fluxes "dynamical", that has been discussed at length in the preceding pages, to a concrete model of flux compactification, first described in [24]. The starting point of this work is to consider a paper by Narayan and Trivedi [25], which, taking into account the model of [24], found the extrema of the scalar potential of the theory and studied the transitions among them mediated by membranes. We will try, therefore, to explicitly see how their results can be rederived using the new formalism developed in [20] and [21]. Far from being a mere restatement of known results, this analysis will allow to ponder the consequences of the properties of the membranes involved in the transitions on statements such as the weak gravity conjecture, upon which we will talk about more extensively in due time.

4.1 The compactification space

On a more concrete stance, the model considered by [24] and [25] is the compactification of a 10-dimensional action (2.8) on a topological space of the form:

$$T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3) \tag{4.1}$$

We recall (see for example [14]) that the field content of the 10-dimensional theory consists of the NSNS sector, composed of the dilaton $\hat{\phi}$, the two-form $\hat{B}_2$ and the graviton g_{MN}, as well as of the RR sector, containing the one-form $\hat{C}_1$ and the three-form $\hat{C}_3$. As usual the ten-dimensional fields have been denoted with a hat.

The compactification space (4.1) consists of a six-dimensional torus, product of three-lower dimensional torii $T^2 \times T^2 \times T^2$, modded out by the action of two rotation groups $\mathbb{Z}_3$. This structure implies that $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is not, mathematically speaking, a manifold, but another topological space known as *orbifold*.

In general an orbifold is defined as the quotient space X/G of a smooth manifold X and one (or more) of its discrete isometry groups[1] G. From a geometrical perspective an orbifold

[1]Where G can therefore be the direct product of more than one group.

is a manifold in which all the points that are connected by an isometry (that is, that lie on the same orbit of the isometry group) are identified: we can think, therefore, that these points are "shrunk" into a single one, giving the resulting orbifold. A very simple example is given by the circumference S^1 with the points that lie on the opposite sides of a given axis identified (so that the isometry group is simply the reflection $\mathbb{Z}_2$): this orbifold hence is $S^1/\mathbb{Z}_2$, and appears as a segment of length 1. It can be noted that the points that lie exactly on the symmetry axis defined by $\mathbb{Z}_2$ are not affected by the identification procedure, and as a result they are called *singular points*. The relevance of orbifolds resides in the fact that they can be seen as simple "degenerate" examples of smooth Calabi-Yau manifolds, because of the presence of the singular points. As we have said smooth Calabi-Yau three-folds are complex *manifolds*, that is, they admit an atlas of charts with holomorphic transition functions mapping them to some open subset of $\mathbb{C}^3$, whereas orbifolds do not possess such a family of maps (more precisely, the transition functions do not satisfy the holomorphicity condition), and as such are not manifolds. A link between Calabi-Yau manifolds and specific orbifolds, however, can be established by "smoothing out" the singularities of the orbifold, thus trying to recover a smooth metric. This is achieved by excising a ball of radius r around the considered singularity: using the definition (4.1) of the orbifold we are studying it can be seen that the boundary of such ball is $S^5/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ (S^5 because it must be a six-dimensional ball, modded out by the same groups as the whole orbifold). The following step is to replace the excised ball with a smooth Ricci-flat Kähler manifold with the same boundary as the ball. This guarantees that the complete manifold can be approximated to a Calabi-Yau one (recalling from Appendix A that Calabi-Yau spaces are nothing but Ricci-flat Kähler manifolds). The replacement manifold, however, possesses a non-trivial topology which contributes to the moduli space of the theory, according to its Hodge numbers, in exactly the same way as what we have seen in chapter 2. This technique is usually known as the "blow up" of the singularities. The original orbifold can then be recovered by reducing the radius of the excised ball to zero, re-obtaining the initial singularity.
In the continuation, however, we will not display explicitly the blow-up moduli, as we will not take them into account in the study of the extrema of the scalar potential of the theory, following the work of [25].

We define three complex coordinates z_A, with $A = 1, 2, 3$, to parametrize the six dimensional torus T^6: each of them corresponds to a single torus T^2. As a result they must satisfy the periodicity conditions:

$$z_A \simeq z_A + 1 \qquad z_A \simeq z_A + \alpha, \tag{4.2}$$

where $\alpha = e^{i\frac{\pi}{3}}$.
This choice allows us to note that the torus T^6 enjoys a symmetry given by a rotation group $\mathbb{Z}_3$, called T, acting on the coordinates in this way:

$$(z_1, z_2, z_3) \longrightarrow (\alpha^2 z_1, \alpha^2 z_2, \alpha^2 z_3) \tag{4.3}$$

Analyzing this constraint it can be shown that the symmetry T admits 27 fixed points. Another $\mathbb{Z}_3$ symmetry Q, however, is present:

$$(z_1, z_2, z_3) \longrightarrow (\alpha^2 z_1 + \frac{1+\alpha}{3}, \alpha^4 z_2 + \frac{1+\alpha}{3}, z_3 + \frac{1+\alpha}{3}) \tag{4.4}$$

Modding out the action of Q it can be seen that only 9 fixed points remain. Furthermore, by means of the blow-up procedure discussed above, these 9 singularities can be shown to give rise to 9 complex moduli, given by the combination of the metric and the $\hat{B}_2$ moduli. The orbifold (4.1), then, is obtained modding out the action of the symmetries T and Q on T^6, indeed yielding $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$.
This Calabi-Yau manifold has Hodge numbers $h^{1,2} = h^{2,1} = 0$ and $h^{1,1} = h^{2,2} = 12$: as a result its Euler characteristic (6.61) is $\chi = 24$. We have seen in chapter 2 that the number of complex structure moduli is given by $h^{1,2}$; in the compactification we are considering, therefore, the complex structure moduli are absent. The moduli that parametrize the deformations of the Kähler metric, instead, are in correspondence with $h^{1,1}$, so that there will be a total of 12. Among these, 9 are the blow up moduli we have just discussed: as a consequence taking the orbifold limit fixes the value of 9 of the 12 Kähler moduli. The remaining three real moduli parametrize the size of the torii T^2 and, combined with 3 further real moduli coming from the zero mode expansion of the NSNS 2-form B_2, form a total of 3 complex moduli. An additional complex modulus, finally, comes from the axion that originates from the compactification of the RR form $\hat{C}_3$, paired up with the dilaton, as in section 2.4.

The preceding discussion does not consider the orientifold projection and the presence of background fluxes. These ingredients will be included in the next section, following the path sketched in chapter 2 in order to obtain a four dimensional effective theory.

4.2 The field content of the effective theory

Following [25] we define a basis of 2-forms, where each 2-form corresponds to one of the T^2 torii (there are $h^{1,1} = 3$ generators in total):

$$\omega_A = (\kappa\sqrt{3})^{\frac{1}{3}} i \mathrm{d}z^A \wedge \mathrm{d}\bar{z}^A, \tag{4.5}$$

where $A = 1, 2, 3$ (the indices are not summed) and κ is a normalization constant. The dual basis, composed of four-forms, is:

$$\tilde{\omega}^A = \left(\frac{3}{\kappa}\right)^{\frac{1}{3}} \mathrm{d}z^B \wedge \mathrm{d}\bar{z}^B \wedge \mathrm{d}z^C \wedge \mathrm{d}\bar{z}^C, \tag{4.6}$$

where the indices B and C are different from A. It is therefore evident that ω_A and $\tilde{\omega}^A$ satisfy the duality condition, analogous to equation (2.16):

$$\int_{T^6/(\mathbb{Z}_3\times\mathbb{Z}_3)} \omega_A \wedge \tilde{\omega}^B = \delta_A^B \tag{4.7}$$

These specific form of the bases have been chosen because of their invariance properties: in fact the wedge product $\mathrm{d}z^A \wedge \mathrm{d}\bar{z}^A$ is the only one left unaffected by the action of the two symmetry groups T and Q.
The orientifold projector, that reduces the number of supersymmetries from $\mathcal{N}=2$ to $\mathcal{N}=1$, is the same as the one used in section 2.4:

$$\mathcal{O} = \Omega_p(-1)^{F_L}\sigma \tag{4.8}$$

In this specific case σ is a reflection mapping the complex coordinates of the compactification manifold to minus their complex conjugate; it is evident therefore that $\sigma^2 = 1$, as required:

$$\sigma : z^A \longrightarrow -\bar{z}^A \tag{4.9}$$

It can be shown that the involution σ leaves invariant a three-cycle in the internal manifold (that therefore is even under σ), that is the compact part of an orientifold plane[2] $O6$ that also extends along the three canonical spacetime directions.
In order to write down the metric of T^6 we note that symmetry under the Q transformation (4.4) implies that $g_{AB} = g_{\bar{A}\bar{B}} = g_{A\bar{B}} = 0$, for $A \neq B$. Furthermore, invariance under T (4.3) entails that the metric is diagonal. Consequently it can be written as:

$$\mathrm{d}s^2 = \sum_{A=1}^{3} \gamma_A \mathrm{d}z^A \wedge \mathrm{d}\bar{z}^A \tag{4.10}$$

The γ_A are moduli parametrizing the sizes of the three T^2, and they can be rewritten more conveniently as:

$$v_A = \frac{1}{2(\kappa\sqrt{3})^{\frac{1}{3}}}\gamma_A \tag{4.11}$$

The harmonic and holomorphic three-form (6.85) reads:

$$\Omega(z) = 3^{\frac{1}{4}} i\, \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \tag{4.12}$$

Its normalization is chosen so as to satisfy:

$$i \int_{T^6/(\mathbb{Z}_3\times\mathbb{Z}_3)} \Omega \wedge \bar{\Omega} = 1 \tag{4.13}$$

[2]We will talk about $O6$ planes more extensively when dealing with the tadpole condition.

Ω is a three-form defined on the compactification space and as such can be expanded in a basis of the cohomology group H^3, that according to equation (6.82) has a number of basis elements equal to:

$$b^3 = h^{1,2} + h^{2,1} + h^{3,0} + h^{0,3} = 2h^{1,2} + 2 \tag{4.14}$$

As a consequence of the fact that the Hodge number $h^{2,1}$ vanishes for the orbifold we are considering, so that there are no complex structure moduli, the expansion (2.24) for the three-form Ω contains only two basis generators, the ones corresponding to $h^{3,0}$ and its dual:

$$\Omega = \frac{1}{\sqrt{2}} (\alpha_0 + i\beta_0) \tag{4.15}$$

When subject to the orientifold projector $\mathcal{O}$ it can be shown that α_0 is even (so that it survives the projection), whereas β_0 is odd (getting projected out). This results from the parity properties of Ω under the action of σ:

$$\sigma : \quad \Omega \longrightarrow \bar{\Omega} \tag{4.16}$$

Furthermore α_0 and β_0 satisfy the symplectic basis relation (2.15):

$$\int_{T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)} \alpha_0 \wedge \beta_0 = 1 \tag{4.17}$$

Using the same notation of chapter 2 we denote the even and odd generators of $H^{1,1}$ and $H^{2,2}$ as:

$$\begin{aligned} &\text{Even generators:} \quad \omega_\alpha^+ , \ \tilde{\omega}_i^+ \\ &\text{Odd generators:} \quad \omega_i^- , \ \tilde{\omega}_\alpha^- \end{aligned} \tag{4.18}$$

It turns out, however, that the number of odd generators is exactly 3, the same as the elements of the basis (4.5): we can therefore use the same basis expansions for the fields, changing the notation and setting $A = i$. It can be noted explicitly, in fact, that the basis (4.5) is odd under the involution σ, using the definition (4.9) and the antisymmetry of the wedge product of one-forms:

$$\sigma : \quad \mathrm{d}z^A \wedge \mathrm{d}\bar{z}^A \longrightarrow -\mathrm{d}z^A \wedge \mathrm{d}\bar{z}^A \tag{4.19}$$

Employing these tools it can be seen that the two-form B_2 must be expanded in a basis of σ-odd two-forms, because of its parity properties under the operator Ω:

$$\hat{B}_2 = b^i \omega_i^- \equiv b^i \omega_i \tag{4.20}$$

As a result its corresponding field strength $\hat{H}_3$ (that in general enjoys the expansion (2.10)) with the addition of a background flux, reads:

$$\hat{H}_3 = -p\beta_0 + db^i \wedge \omega_i \tag{4.21}$$

The RR sector one-form $\hat{C}_1$ is odd under the reflection σ, and because of the fact that the Hodge numbers $h^{1,0} = h^{0,1}$ vanish, it does not survive the orientifold projection. In fact, one could think of considering $\hat{C}_1$ as an ordinary one-form, with no part in the internal manifold (precisely because there are no basis elements to do this). In this way, though, $\hat{C}_1$ would not be odd under σ anymore, recalling that the involution acts as the identity on the standard four dimensions. As a result, the only way out is to admit that $\hat{C}_1$ is not there at all.
The RR three-form $\hat{C}_3$, instead, is even under σ, and therefore can be expanded into the even three-form α_0 as:

$$\hat{C}_3 = \xi\alpha_0 \tag{4.22}$$

In principle, as in (2.17), $\hat{C}_3$ could feature in its expansion the $h^{1,1}_+$ even generators ω^+_α, times some one-form living in the usual four dimensions. As we have seen, however, there are no even basis elements for $H^{1,1}$, and as a result the expansion for $\hat{C}_3$ can be nothing but (4.22). The corresponding field strength $\hat{F}_4$, then, can be written as (considering also the contribution of the Romans mass m^0, as done in (2.42)):

$$\hat{F}_4 = e_i\tilde{\omega}^i + d\xi \wedge \alpha_0 - \frac{m^0}{2}\hat{B}_2 \wedge \hat{B}_2, \tag{4.23}$$

where $\tilde{\omega}_i$ are the even basis elements of $H^{2,2}$, that we recall being in correspondence, via a duality relation, with the odd basis elements of $H^{1,1}$.

For the time being we do not consider the contribution of background fluxes for $\hat{F}_2$ and $\hat{F}_6$, as we will later show that their presence can be accounted for by shifting other fields of the theory; the resulting expansion is:

$$\hat{F}_2 = m^0\hat{B}_2 \tag{4.24}$$

To complete the discussion of the field content of the model we must include also the dilaton field $\hat{\phi}$, and combine the moduli b^i arising from $\hat{B}_2$ and the v^i deduced from the form of the metric into three complex moduli t^i. These new fields and the dilaton (that can be interchangeably written with or without hat, as it is a scalar in the internal manifold) can be written accordingly to (2.29):

$$t^i \equiv b^i + iv^i \qquad e^D \equiv \frac{e^{\hat{\phi}}}{\sqrt{vol}}, \tag{4.25}$$

where vol is the intersection number defined in (4.29). The v^i, defined in (4.11), can be seen as the coefficients in the basis expansion of the Kähler form J, that, being odd under

σ depends on the generators ω_i, reads:

$$J = v^i \omega_i \tag{4.26}$$

Furthermore it is convenient to pair up the dilaton D with the axion ξ coming from the basis expansion of $\hat{C}_3$, defining:

$$n = \frac{1}{2}\xi + \frac{i}{\sqrt{2}} e^{-D} \tag{4.27}$$

In the case of the specific compactification we are considering, moreover, it is possible to display a more explicit form for the intersection numbers: for example, considering the basis expansion (4.5), it is straightforward to note that, for the antisymmetry of the wedge product, the only non-vanishing triple intersection number is:

$$\kappa_{123} = \int_{T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)} \omega_1 \wedge \omega_2 \wedge \omega_3 \equiv \kappa \tag{4.28}$$

Here we are employing a bit of an abuse of notation, denoting the triple intersection number with κ: as a matter of fact this proves useful because all the other intersection numbers can be written in terms of $\kappa_{123} \equiv \kappa$ by means of the relations (2.21):

$$\begin{aligned}
&\int_{T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)} J \wedge J \wedge J = \frac{1}{6}\kappa_{ijk} v^i v^j v^k = \kappa_{123} v^1 v^2 v^3 \equiv \kappa v^1 v^2 v^3 \equiv vol \\
&\kappa_1 = 2\kappa v^2 v^3 \qquad \kappa_2 = 2\kappa v^1 v^3 \qquad \kappa_3 = 2\kappa v^1 v^2 \\
&\kappa_{12} = \kappa v^3 \qquad \kappa_{13} = \kappa v^2 \qquad \kappa_{23} = \kappa v^1,
\end{aligned} \tag{4.29}$$

where in the first line we have defined the volume of the compactification.

Recapping what we have discussed so far, the fields that are present in the four-dimensional effective theory, obtained by inserting the basis expansions we have displayed into (2.8) and performing a dimensional reduction, are:

- Three complex moduli t^i: their real parts are the axions b^i; their imaginary parts are the v^i.

- One complex modulus n: its real part is the axion ξ, whereas its imaginary part depends on the dilaton D.

The Kähler potential for the t^i moduli can be computed using the expression (2.20), obtaining:

$$\begin{aligned}
K^K &= -\ln\left(\frac{4}{3}\int_{T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)} J \wedge J \wedge J\right) = -\ln\left[\frac{4}{3}\kappa_{ijk}\left(\frac{t^i - \bar{t}^i}{2i}\right)\left(\frac{t^j - \bar{t}^j}{2i}\right)\left(\frac{t^k - \bar{t}^k}{2i}\right)\right] = \\
&= -\ln\left(\frac{4}{3}\frac{\kappa}{6} v^1 v^2 v^3\right) = -\ln\left(8\kappa v^1 v^2 v^3\right)
\end{aligned} \tag{4.30}$$

It is important to note that this Kähler potential depends exclusively on the imaginary parts of the t^i moduli.
As far as the complex structure potential is concerned, instead, we have already discussed the fact there are no complex structure moduli, due to the vanishing $h^{2,1}$. Nevertheless, using the rescaled Ω defined in (2.46) and its Kähler potential (2.49) we obtain:

$$K^Q = -\ln e^{-4D} = 4D \tag{4.31}$$

The total Kähler potential is then:

$$K \equiv K^K + K^Q = -\ln\left(8\kappa v^1 v^2 v^3\right) + 4D \tag{4.32}$$

The corresponding superpotentials can be computed from equation (2.58):

$$\begin{aligned} W^K &= e_i t^i - \frac{m^0}{6}\kappa_{ijk}t^i t^j t^k \\ W^Q &= -p\xi - \sqrt{2}ipe^{-D} = -2pn, \end{aligned} \tag{4.33}$$

We have mentioned before that we have not considered the contribution of the fluxes of $\hat{F}_6$ and $\hat{F}_2$: let's assume that they are present and that the superpotential W^K takes the form:

$$W^K = e_0 + e_i t^i + \frac{\kappa_{ijk}}{2}m^i t^j t^k - \frac{m^0}{6}\kappa_{ijk}t^i t^j t^k \tag{4.34}$$

If we shift the fields in the following way:

$$\begin{aligned} t^i &\longrightarrow t^i - \frac{m^i}{m^0} \\ \xi &\longrightarrow \xi - \frac{e_0}{p} - \frac{e_i m^i}{p} - \frac{\kappa_{ijk}}{3}\frac{m^i m^j m^k}{p(m^0)^2} \end{aligned} \tag{4.35}$$

The superpotential W^K reduces to the one in (4.33), with e_i substituted by $\hat{e}_i$:

$$\hat{e}_i \equiv e_i + \frac{\kappa_{ijk}m^j m^k}{2m^0} \tag{4.36}$$

Further noting that the shifts (4.35) do not change the imaginary parts v^i and D that appear in the Kähler potential (4.32), it can be concluded that working with the reduced superpotential (4.33) is a sensible course of action.
Before embarking on the substitution of the fluxes and the computation of the scalar potential a few remarks about the fluxes' normalization and the tadpole condition are in order.

Flux quantization and the tadpole condition

The values e_i, m^0 and p of the background fluxes for the field strengths that we have introduced before are not arbitrary: they must in fact satisfy a precise constraint, imposed by the generalized Dirac quantization condition [14][27]:

$$\frac{1}{(2\pi)^{p-1}\alpha'^{(p-1)/2}}\int_{\Sigma_p}\hat{F}_p\in\mathbb{Z}, \tag{4.37}$$

where $\hat{F}_p$ is a $p-$form belonging either to the NSNS or the RR sector, Σ_p is a $p-$cycle in the internal space and α' is the string coupling constant, related to the string length scale l_S by:

$$\alpha'=\frac{1}{2}{l_S}^2 \tag{4.38}$$

This in turn implies that:

$$\int_{\Sigma_p}\hat{F}_p=2(\kappa_{10})^2\mu_{8-p}f_p=(2\pi)^{p-1}\alpha'^{(p-1)/2}f_p, \tag{4.39}$$

where $f_p\in\mathbb{Z}$ and μ_{8-p}, instead, is the charge of an $8-p$ brane, related to the parameter α' by [26]:

$$\mu_{8-p}=(2\pi)^{p-8}\alpha'^{-(9-p)/2} \tag{4.40}$$

The integers related to the field strengths $\hat{F}_4$, $\hat{F}_0$ and $\hat{H}_3$ are respectively f_4, f_0 and h_3. The values of the fluxes of the considered compactification can then be written as:

$$e_i=\frac{1}{\sqrt{2}}\left(2\pi\sqrt{\alpha'}\right)^3 f_{i,4}\qquad m^0=\frac{f_0}{2\sqrt{2}\pi\sqrt{\alpha'}}\qquad p=(2\pi)^2\alpha' h_3 \tag{4.41}$$

The extra factors of $\sqrt{2}$ in the denominators of e_i and m^0 arise from the conventions of [24], that consider an additional $\sqrt{2}$ coefficient for the RR potentials and F_0.

Having settled this aspect, it is now necessary to address another issue, the tadpole cancellation condition, that imposes that some of the values of the fluxes satisfy precise constraints.

Physically speaking it is required that, given a $p-$form potential and its field strength, the total flux of the external derivative of the field strength along a compact $(p+1)-$cycle vanishes. If, in fact, the "field lines" of a given field strength can escape freely in a non-compact space, such as the three extended spatial dimensions, on the contrary they must necessarily be closed when threading a compact surface. From a mathematical perspective, this condition comes from the examination of the Bianchi identities for the RR field strengths derived from (2.10), that read, using the fact that $d^2=0$:

$$\begin{aligned}
d\hat{F}_2&=m^0 d\hat{B}_2+\text{source terms}=m^0\hat{H}_3+\text{source terms}\\
d\hat{F}_4&=-d\hat{C}_1\wedge\hat{H}_3-\frac{m^0}{2}d(\hat{B}_2)^2
\end{aligned} \tag{4.42}$$

where the source terms will be analyzed in a few lines.

The tadpole cancellation condition would then impose that the integrals of $d\hat{F}_2$ and $d\hat{F}_4$

along appropriate $p-$cycles should vanish. More precisely, the integrals should be computed respectively on a 3-cycle and a 5-cycle. As we have previously seen, however, the geometrical properties, that is the Hodge numbers, of the compactification manifold $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ imply that there are no non-trivial compact 5-cycles in the internal manifold. The only concern therefore comes from the integral of $d\hat{F}_2$, that must be performed on the only 3-cycle upon which there is a non-vanishing flux, that is β_0. On the right hand side of the second line of (4.42), in fact, the only contribution comes from $\hat{H}_3$, that has a non-trivial flux along β_0, according to (4.21). Apart from the contribution of the fluxes, there are also be additional terms sourced by $D6$-branes and $O6$-planes wrapping a three-cycle in in the internal manifold (with the remaining three spatial dimensions filling the ordinary ones). As we have said, orientifold planes are non-dynamical objects, with negative tension, that reside in the region that is left unaffected by the involution σ that defines the orientifold projection. The relation between the tension T_{Dp} and the charge μ_{Dp} of a Dp-brane and the ones of an Op-plane reads:

$$\mu_{Op} = -2^{p-5}\mu_{Dp} \qquad T_{Op} = -2^{p-5}T_{Dp}, \tag{4.43}$$

that in the case of $D6$-branes and $O6$-planes we are considering becomes:

$$\mu_{Op} = -2\mu_{Dp} \qquad T_{Op} = -2T_{Dp} \tag{4.44}$$

The reason for the appearance of the brane and orientifold plane terms is that in the action these objects can be coupled to the potential $\hat{C}_7$, with corresponding field strength $\hat{F}_8$, that is the Hodge dual of $\hat{F}_2$. The action for the $O6$ plane, for example, contains a kinetic term and the coupling to $\hat{C}_7$:

$$S_{O6} = 2\mu_6 \int_{O6} \mathrm{d}^7\xi e^{-\hat{\phi}}\sqrt{-g} - 2\sqrt{2}\mu_6 \int \hat{C}_7, \tag{4.45}$$

where μ_6 is the tension of a $D6$-brane, ξ are a set of coordinates parametrizing the $O6$-plane and g is the determinant of the metric induced on it by the world-volume metric. The factors 2 in front of the terms is due to (4.44), whereas the $\sqrt{2}$ in front of the coupling term arises from the normalization convention for the RR potentials used in [24] and that we have mentioned before.
In full generality we can then write that [27]:

$$d\hat{F}_2 = m^0\hat{H}_3 + 2\pi\sqrt{\alpha'}\left[N_{D6} - 2N_{O6}\right]\delta(\alpha_0) \tag{4.46}$$

N_{D6} is the number of $D6$-branes, N_{O6} is the number of $O6$-planes, and $\delta(\alpha_0)$ is a generalized Delta function, supported on the three-cycle α_0. This “function” acts exactly like the ordinary Delta function; in particular, when integrated on the cycle dual to α_0 it gives the unity:

$$\int_{\beta_0} \delta(\alpha_0) = 1 \tag{4.47}$$

This is exactly like, when considering the real line, $\delta(x)$ gives the unity when integrated on the "dual" of the point x, that is the whole $\mathbb{R}$ [28]. From a physical perspective this means that the orientifold plane and the $D6$-branes have support on the α_0 three-cycle, and this is in agreement with the fact, explained in the previous section, that α_0 is even under the orientifold projection.
Integrating equation (4.46) over β_0 we finally obtain:

$$0 = \int_{\beta_0} d\hat{F}_2 = m^0 \int_{\beta_0} \hat{H}_3 + 2\pi\sqrt{\alpha'}\,[N_{D6} - 2N_{O6}] \tag{4.48}$$

Examining the right hand side of the equation we note that, in the model taken in consideration, there surely is a contribution due to the $O6$-plane, because we have performed an orientifold projection. At this point it can be observed that there are two ways to ensure that the total sum on the right hand side is actually zero: one route is to set the integral of $\hat{H}_3$, that is its flux, to zero and introduce two $D6$ branes to compensate the contribution of the orientifold plane; another path is to employ the very flux of $\hat{H}_3$ in order to cancel the tadpole, without the need to add new $D6$-branes into the theory. In the following we choose this last option, and recalling that the $\hat{H}_3-$flux is $-p$ we obtain:

$$0 = \int_{\beta_0} d\hat{F}_2 = m^0 \int_{\beta_0} \hat{H}_3 - 2\pi\sqrt{\alpha'}\,2N_{O6} = -m^0 p - 2\pi\sqrt{\alpha'}\,2N_{O6} \tag{4.49}$$

Since there is only one orientifold plane we get:

$$m^0 p = -2\sqrt{2}\pi\sqrt{\alpha'} \tag{4.50}$$

Where we have reintroduced the usual $\sqrt{2}$ factor. From this equation it can be hence seen that the values of m^0 and p are not arbitrary, but depend on one another. In addition, we can rewrite the product of m^0 and p using the definitions of the integer fluxes (4.41):

$$m^0 p = \sqrt{2}\pi\sqrt{\alpha'} f_0 h_3 \tag{4.51}$$

Comparing (4.51) with (4.50) it can be seen that $f_0 h_3$ must be equal to -2. As a result, keeping in mind that they both belong the $\mathbb{Z}$, the only allowed choices are:

$$(f_0, h_3) = \pm(1, -2) \qquad (f_0, h_3) = \pm(2, -1) \tag{4.52}$$

In the following we choose to keep h_3 "fixed" (although the value it assumes is not relevant) and to let m^0 vary, in order to be able to perform the dualization procedure described in chapter 3 upon all the fluxes in the RR sector, even if, as we have just seen, the range of choices for m^0 spans only two values.

4.3 Eliminating the fluxes in the effective theory

The effective theory built so far possesses, as we have seen, a total of four complex moduli with superpotentials (4.33) and total Kähler potential (4.32).
The lagrangian of the corresponding supergravity effective theory, stated in four dimensions with the local superspace formalism, is then:

$$\mathcal{L} = -3 \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, e^{-\frac{1}{3}\left(K^K + K^Q\right)} + \left(\int \mathrm{d}^4x \mathrm{d}^2\theta\, 2\mathcal{E}\, \left(W^K + W^Q\right) + \text{h.c.} \right) \tag{4.53}$$

As we have mentioned in the previous section, we choose to eliminate the fluxes of the $p-$forms in the RR sector, e_i and m^0, in favour of new gauge three-forms, by means of the new formalism exhibited in chapter 3. The only flux in the NSNS sector, p, instead, is left untouched, acknowledging the constraints imposed by the tadpole cancellation condition. In the language of the new formalism, therefore, we treat the field associated to p, that is n (4.27) (belonging to the chiral superfield N), as a spectator field (that we usually called "T"), whereas the t^i are the lowest components of the physical fields Φ^i subject to dualization, that when supplied by the introduction of a Weyl compensator Z become a set of 3+1 fields $\mathcal{Z}^A$, defined as:

$$\mathcal{Z}^0 = Z \qquad \mathcal{Z}^i = \Phi^i \tag{4.54}$$

As a consequence our $W^Q(N)$ is the analogue of the spectator superpotential $\hat{W}(T)$ in (3.122). The superpotential $W^K(\mathcal{Z})$ can be written in the form of (3.122) setting e_0 and m^i to zero (i.e. not considering the fluxes of $\hat{F}_6$ and $\hat{F}_2$) and employing the prepotential:

$$\mathcal{G}(\mathcal{Z}) = \frac{1}{6Z}\kappa_{ijk}\Phi^i\Phi^j\Phi^k = \frac{\kappa}{Z}\Phi^1\Phi^2\Phi^3 \tag{4.55}$$

In the following, however, in order to retain full generality, we will keep also e_0 and m^i and at the end of the procedure we will put them to zero.
The kinetic term (4.71), instead, is factorized according to formula (3.125): $K^K(\Phi, \bar{\Phi})$ is the alias of $K(\mathcal{Z}, \bar{\mathcal{Z}})$, whereas $K^Q(N, \bar{N})$ is the analogue of $K(T, \bar{T})$. From now on we rename the Kähler potentials as:

$$K^K \equiv K \qquad K^Q \equiv \hat{K} \tag{4.56}$$

With these identifications in mind the procedure can be performed in exactly the same way as in chapter 3, obtaining a new lagrangian with no trace of e_i and m^0:

$$\hat{\mathcal{L}} = -3 \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, e^{-\frac{1}{3}K(S,\bar{S}) - \frac{1}{3}\hat{K}(N,\bar{N})} + \left(\int \mathrm{d}^2\theta\, 2\mathcal{E}\, W^Q(N) + \text{h.c.} \right) + \mathcal{L}_{\text{bd}}, \tag{4.57}$$

where $\mathcal{L}_{\text{bd}}$ is (3.109) with the proper substitutions to adapt it to the present model and S^A are the double three-form multiplets that have superseded the $\mathcal{Z}^A$.
The next step is to implement a gauge-fixing condition, effectively eliminating the Weyl

compensator from the lagrangian:

$$Z = 1 \tag{4.58}$$

When analyzing the components of the multiplets (3.119) we have seen that the auxiliary fields F_S^i and the lowest component M of the gravity multiplet depend on the field strengths of two sets of $A = 3 + 1$ gauge three-form potentials $\tilde{A}_{(3)A}$ and $A_{(3)}^A$. For convenience we report the expression:

$$\begin{aligned} \bar{M} &= -\frac{i}{2}\mathcal{M}^{0B}\left(*\tilde{F}_{(4)B} + \bar{\mathcal{G}}_{BC}*F_{(4)}^C\right) \\ F_S^i &= \bar{M}s^i + \frac{i}{2}\mathcal{M}^{iB}\left(*\tilde{F}_{(4)B} + \bar{\mathcal{G}}_{BC}*F_{(4)}^C\right) \end{aligned} \tag{4.59}$$

The matrices $\mathcal{G}_{AB}$ can be computed from the prepotential (4.55), restricting ourselves to the lowest components of the physical multiplets Φ^i:

$$\mathcal{G}_{AB} = \begin{pmatrix} \frac{2\kappa_{ijk}}{Z^3}t^i t^j t^k & -\frac{\kappa_{ijk}}{Z^2}t^j t^k \\ -\frac{\kappa_{ijk}}{Z^2}t^i t^k & \frac{1}{6Z}\kappa_{ijk}t^k \end{pmatrix} \tag{4.60}$$

Its imaginary part $\mathcal{M}_{AB}$ can then be recovered recalling that $t^i = b^i + iv^i$.
Plugging the explicit expressions for $\mathcal{G}_{AB}$ and $\mathcal{M}^{AB}$ into (4.59) the following conditions are obtained:

$$\begin{aligned} \mathrm{Re}M &= \frac{1}{2}\,*\mathcal{F}_{(4)}^0 \\ \mathrm{Im}M &= -2e^K*\tilde{\mathcal{F}}_{(4)0} - \frac{1}{2}K_i\,*\mathcal{F}_{(4)}^i \\ \mathrm{Re}F_S^i &= \frac{1}{4}\,*\mathcal{F}_{(4)}^0 v^i - e^K\left(K^{ij} - 2v^i v^j\right)*\tilde{\mathcal{F}}_{(4)j} \\ \mathrm{Im}F_S^i &= 2e^K\,*\tilde{\mathcal{F}}_{(4)0}v^i + \frac{1}{2}\left(*\mathcal{F}_{(4)}^i + v^i K_j\,*\mathcal{F}_{(4)}^j\right), \end{aligned} \tag{4.61}$$

where K_i and K_{ij} are defined as:

$$K_i \equiv \frac{\partial K}{\partial t^i} \qquad K_{ij} \equiv \frac{\partial^2 K}{\partial t^i \partial \bar{t}^j} \tag{4.62}$$

The two sets of 3+1 field strengths $\tilde{\mathcal{F}}_{(4)A}$ and $\mathcal{F}^A_{(4)}$ are combinations of the field strengths $\tilde{F}_{(4)A}$ and $F^A_{(4)}$, defined as:

$$\begin{aligned}
\mathcal{F}^0_{(4)} &= -F^0_{(4)} \\
\mathcal{F}^i_{(4)} &= -F^i_{(4)} + b^i F^0_{(4)} \\
\tilde{\mathcal{F}}_{(4)i} &= \tilde{F}_{(4)i} + \kappa_{ijk} b^j F^k_{(4)} - \frac{1}{2}\kappa_{ijk} b^j b^k F^0_{(4)} \\
\tilde{\mathcal{F}}_{(4)0} &= \tilde{F}_{(4)0} + b^i \tilde{F}_{(4)i} + \frac{1}{2}\kappa_{ijk} b^i b^j F^k_{(4)} - \frac{1}{6}\kappa_{ijk} b^i b^j b^k F^0_{(4)}
\end{aligned} \tag{4.63}$$

At this point the way is paved to extract the components of the new lagrangian (4.57), focusing exclusively, as usual, on the bosonic sector. In order to do this we must make use of the explicit expressions for the invariant measures E and $2\mathcal{E}$. In this regard we can rewrite (4.57) taking advantage of the relation between the superspace measures we have already mentioned:

$$\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E = -\frac{1}{8}\int \mathrm{d}^2\theta\, 2\mathcal{E}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) \tag{4.64}$$

The lagrangian that must be expanded in components hence becomes:

$$\hat{\mathcal{L}} = \int \mathrm{d}^2\theta\, 2\mathcal{E}\left[-\frac{3}{8}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) e^{-\frac{1}{3}K(S,\bar{S}) - \frac{1}{3}\hat{K}(N,\bar{N})} + \left(W^Q(N) + \text{h.c.}\right)\right] + \mathcal{L}_{\text{bd}} \tag{4.65}$$

The explicit expression for the bosonic components of $2\mathcal{E}$ is [7]:

$$2\mathcal{E} = |e|\left(1 - \theta^2 M^*\right), \tag{4.66}$$

where $|e|$ is the determinant of the vielbein and M is the supergravity multiplet complex auxiliary field, whose real and imaginary parts are constrained by the conditions (4.61).
The expansion of the bosonic components of the supergravity multiplet $\mathcal{R}$, instead, reads:

$$\mathcal{R} = -\frac{1}{6}\left[M + \theta^2\left(-\frac{1}{2}R + \frac{2}{3}|M|^2 + \frac{1}{3}b^a b_a - i e^m_a \mathcal{D}_m b^a\right)\right], \tag{4.67}$$

where R is the usual Ricci scalar, b^a is the supergravity multiplet vector auxiliary field with flat index and $\mathcal{D}_m$ is the supergravity covariant derivative with curved index.
We have previously shown that the Kähler potentials $K(S,\bar{S})$ and $\hat{K}(N,\bar{N})$ depend explicitly only on the superfields S and N, respectively. The exponential $e^{-\frac{1}{3}K(S,\bar{S}) - \frac{1}{3}\hat{K}(N,\bar{N})}$, moreover, can be expanded as:

$$e^{-\frac{1}{3}K(S,\bar{S}) - \frac{1}{3}\hat{K}(N,\bar{N})} = 1 - \frac{1}{3}K(S,\bar{S}) - \frac{1}{3}\hat{K}(N,\bar{N}) \tag{4.68}$$

The truncation to the first order term is allowed by the fact that, when reintroducing the gravitational coupling constant, higher terms in the expansion vanish in the low energy limit, as remarked in (1.62). As a consequence we only need an explicit expression for the component expansion of $\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right)\left(K(S,\bar{S}) + \hat{K}(N,\bar{N})\right)$. As far as $\hat{K}(N,\bar{N})$ is concerned this can be achieved by expanding $\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) 4D$, where D is the dilaton (a scalar), having used the expression (4.31) for the Kähler potential. As regards $K(S,\bar{S})$, instead, a bit more work is required. First of all we recall the component expansion of the gauge-fixed double three-form multiplets S^i (initially there were $A = i+1$ double three-form multiplets, but one of them has been gauge-fixed away producing the constraints (4.61)):

$$S^i = s^i + \theta^2 F_S^i \tag{4.69}$$

Then we can compute the expression for $\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) S^i$, that results from the expansion of $\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) K(S,\bar{S})$:

$$\begin{aligned}\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) S^i = -4F_S^i + \frac{4}{3}Ms^i + \theta^2 \Big[&-4e_a^m \mathcal{D}_m \left(e_n^a \partial^n s^i\right) - \frac{8}{3}M^* F_S^i + \\ &+\frac{4}{3}s^i\left(-\frac{1}{2}R - ie_a^m\mathcal{D}_m b^a + \frac{2}{3}|M|^2 + \frac{1}{3}b^a b_a\right)\Big]\end{aligned} \tag{4.70}$$

Keeping in mind the expansions of $2\mathcal{E}$, $\mathcal{R}$ and $\left(\bar{\mathcal{D}}^2 - 8\mathcal{R}\right) S^i$ we have just exhibited, the strategy now relies on inserting them into the new lagrangian (4.57), integrating over the Grassmann variables and extracting all the different terms. In particular we will get kinetic terms for the graviton (that corresponds to the vielbein e_a^m), the scalar components s^i of the double three-form multiplets S^i and the scalars D and ξ that compose the spectator superfield N, as well as terms depending on the auxiliary fields F_S^i and the spectator superpotential $W^Q(N)$. It is these last terms that deserve further inspection: we have previously shown, in fact, that the real and imaginary parts of the auxiliary fields F_S^i in (4.61) depend on the field strengths of the three-form gauge potentials $\tilde{A}_{(3)A}$ and $A_{(3)}^A$; this means that, when the equations (4.61) are substituted into the auxiliary fields lagrangian kinetic and coupling terms for the field strengths will appear. Even more interestingly, we will see in a few lines that a coupling between the spectator superpotential $W^Q(N)$ and the field strengths arises. This feature could at first seem curious, because in (4.57) no coupling term seems to be present. The crucial point is that $W^Q(N)$ is coupled to the supergravity invariant measure $\int \mathrm{d}^2\theta\, 2\mathcal{E}$, that contains in its expansion (4.66) the supergravity complex scalar auxiliary field M that in turn, according to (4.61), depends on the field strengths of $\tilde{A}_{(3)A}$ and $A_{(3)}^A$. We see then that the conditions (4.61), that originate from the gauge-fixing and that fix the components of M, imply that a coupling between $W^Q(N)$ and the field strengths arises in the expansion of the new theory. The vector auxiliary fields b^m, instead, can be integrated out in the standard manner, bearing no relationship with the procedure that substitutes the fluxes.

Having outlined the general structure of the component lagrangian, we now show how it precisely comes about.

The kinetic part $\mathcal{L}_{KIN}$ of the lagrangian reads (putting the inverse of the determinant of the vielbein on the left-hand side):

$$|e|^{-1}\mathcal{L}_{KIN} = \frac{1}{6}e^{K+\hat{K}}R - \partial n\partial n - \partial t^i \partial \bar{t}^{\bar{i}} \tag{4.71}$$

Strictly speaking we should have, instead of $\partial t^i \partial \bar{t}^{\bar{i}}$ (where t^i are the lowest components of the starting physical fields Φ^i), a kinetic term for the lowest component of the double three-form multiplets S^i: as a matter of fact the lowest components of Φ^i and S^i are exactly equal, and as a result we can write (4.71).
In addition it is important to highlight that, when we will eliminate the auxiliary fields from the theory, the kinetic terms of the scalars n and t^i will become proportional to their respective Kähler potentials, a fact that will affect the form of their equations of motion.
In order to obtain a canonically normalized Einstein and kinetic terms in four dimensions it is necessary to rescale the metric g_{mn} and pass from the so-called string frame to the Einstein frame, where the metric is g^E_{mn}:

$$g_{mn} = \frac{e^{2\phi}}{vol} g^E_{mn} \tag{4.72}$$

The appropriate Weyl rescalings of the fields that suit the canonical normalization request are:

$$\begin{aligned} e^a_m &\longrightarrow e^a_m e^{\frac{1}{6}(K+\hat{K})} \qquad & M &\longrightarrow M e^{-\frac{2}{3}(K+\hat{K})} \\ F^i_S &\longrightarrow F^i_S e^{-\frac{2}{3}(K+\hat{K})} \qquad & F_N &\longrightarrow F_N e^{-\frac{2}{3}(K+\hat{K})}, \end{aligned} \tag{4.73}$$

where F_N is the auxiliary field of the chiral superfield N that contains the scalar n.
The part of the lagrangian that contains the auxiliary fields $\mathcal{L}_{AUX}$, instead, can be computed to be [7]:

$$\begin{aligned} |e|^{-1}\mathcal{L}_{AUX} = {} & \frac{1}{3}e^{-(K+\hat{K})}\left|M + (K_{\bar{i}} + \hat{K}_{\bar{i}})F^{i*}_S + (K' + \hat{K}')F_N\right|^2 + \\ & - e^{-(K+\hat{K})}(K_{i\bar{j}} + \hat{K}_{i\bar{j}})F^i_S F^{j*}_S - e^{-(K+\hat{K})}(K'' + \hat{K}'')F_N F^*_N + \\ & - \frac{1}{3}e^{-(K+\hat{K})}b^a b_a - ie^{-(K+\hat{K})}b^m\left[(K_i + \hat{K}_i)\partial_m t^i - (K_{\bar{i}} + \hat{K}_{\bar{i}})\partial_m t^{i*}\right] + \\ & - ie^{-(K+\hat{K})}b^m\left[(K' + \hat{K}')\partial_m n - (K'^* + \hat{K}'^*)\partial_m n^*\right] + \\ & - \hat{W}M^* - \bar{\hat{W}}M + \hat{W}'F_N + \bar{\hat{W}}'F^*_N, \end{aligned} \tag{4.74}$$

where the indices i and j on the Kähler potentials denote derivation with respect to the lowest components t^i of the superfields Φ^i (while indices with a bar refer to derivation w.r.t their complex conjugates). The primes, instead, indicate derivation w.r.t n. It can be noted that the supergravity vector auxiliary field b^m appears only in the third and fourth line, and can therefore be easily integrated out. The other terms in the first two lines involve, along with the aforementioned Kähler potentials, couplings between the double-three forms auxiliary fields F^i_S and the supergravity complex scalar auxiliary field M: these terms, once substituted with the constraints (4.61) produce Yang-Mills-like kinetic terms for the three-

forms $\tilde{A}_{A(3)}$ and $A^A_{(3)}$. We have used the index $A = i + 1$ because in (4.61) also the field strengths $\tilde{F}_{0(4)}$ and $F^0_{(4)}$ are involved.
In the last line, instead, the first two terms, that represent a coupling between the spectator superpotential and the supergravity scalar, have appeared: it is precisely these terms, as hinted before, that provide a coupling between the spectator superfields N and the double three-forms field strengths, recalling that M depends on them. Comparing (4.74) with the analogous expression that appears in [7] we note that the only difference is that in [7] further terms involving derivatives of the superpotential with respect to t^i appear. In the model we are considering, however, these terms are not there because the spectator superpotential $\hat{W}(N)$ *does not* depend on the scalars t^i.

Summing up, the component lagrangian is given by two contributions: kinetic terms for gravity and the scalars appearing in $\mathcal{L}_{KIN}$ and the auxiliary fields being exhibited in $\mathcal{L}_{AUX}$. The fact that the Kähler potential in the model under scrutiny is "diagonal", i.e. the contributions of the t^i and n moduli are directly summed, implies:

$$\hat{K}_i = \hat{K}_{ij} = K' = K'' = 0 \tag{4.75}$$

Using this simplification equation (4.74) reduces to:

$$\begin{aligned} |e|^{-1}\mathcal{L}_{AUX} = & \frac{1}{3}e^{-(K+\hat{K})}\left|M + K_{\bar{i}}F^{i*}_S + \hat{K}'F_N\right|^2 + \\ & - e^{-(K+\hat{K})}K_{i\bar{j}}F^i_S F^{j*}_S - e^{-(K+\hat{K})}\hat{K}''F_N F^*_N + \\ & - \frac{1}{3}e^{-(K+\hat{K})}b^a b_a - ie^{-(K+\hat{K})}b^m\left[K_i\partial_m t^i - K_{\bar{i}}\partial_m t^{i*}\right] + \\ & - ie^{-(K+\hat{K})}b^m\left[\hat{K}'\partial_m n - \hat{K}'^*\partial_m n^*\right] + \\ & - \hat{W}M^* - \bar{\hat{W}}M + \hat{W}F_N + \bar{\hat{W}}F^*_N \end{aligned} \tag{4.76}$$

As we have emphasized before the auxiliary fields M and F^i_S get substituted by a combination of $\tilde{F}_{A(4)}$ and $F^A_{(4)}$, according to equation (4.59): as a consequence the only fields that remain to be eliminated are b^m and F_N. The equation of motion for the supergravity vector auxiliary field is:

$$b_a = -\frac{3}{2}ie^{K+\hat{K}}\left[K_i\partial_a t^i - K_{\bar{i}}\partial_a t^{i*}\right] - \frac{3}{2}ie^{K+\hat{K}}\left[\hat{K}'\partial_a n - \hat{K}'^*\partial_a n^*\right] \tag{4.77}$$

The equation of motion for the spectator multiplet auxiliary field instead receives contributions from the dualized multiplets' auxiliary fields and from the spectator superpotential:

$$F^*_N = -(\hat{K}'')^{-1}\hat{K}'\left(M + K_{\bar{i}}F^{i*}_S + \hat{W}'\right) \tag{4.78}$$

Finally, inserting (4.77) and (4.78), along with (4.61) into the Weyl rescaled $\mathcal{L}_{KIN}$ and $\mathcal{L}_{AUX}$ gives the final lagrangian:

$$\mathcal{L}_{TOT} = -\frac{1}{2}R - K_{ij}\partial t^i \partial \bar{t}^j - \hat{K}''\partial n \partial \bar{n} + \mathcal{L}_{\text{3-forms},\hat{W}}, \tag{4.79}$$

where we see that the correct normalization for the Einstein term is recovered, and a dependence of the scalar kinetic terms on the derivatives of the Kähler potentials is developed. $\mathcal{L}_{\text{3-forms},\hat{W}}$ is the contribution coming from the field strengths $\tilde{F}_{(4)A}$ and $F^A_{(4)}$ of $\tilde{A}_{A(3)}$ and $A^A_{(3)}$ and from the spectator superpotential:

$$\begin{aligned}|e|^{-1}\mathcal{L}_{\text{3-forms},\hat{W}} = -e^{K+\hat{K}}(\hat{K}'')^{-1}p^2 + \frac{e^{-(K+\hat{K})}}{16}\left(*\mathcal{F}^0_{(4)} - 8e^{K+\hat{K}}p\xi\right)^2 - 4e^{K+\hat{K}}(p\xi)^2 + \\ e^{K-\hat{K}}K^{ij}*\tilde{\mathcal{F}}_{i(4)}*\tilde{\mathcal{F}}_{j(4)} + \frac{e^{-(K+\hat{K})}}{4}K_{ij}*\mathcal{F}^i_{(4)}*\mathcal{F}^j_{(4)} + \\ + 4e^{K-\hat{K}}\left(*\tilde{\mathcal{F}}_{0(4)} + \sqrt{2}e^{\hat{K}}pe^{-D}\right)^2 + |e|^{-1}\mathcal{L}_{\text{bd}},\end{aligned} \tag{4.80}$$

where $\mathcal{L}_{\text{bd}}$ is a boundary term that arises from the reasoning made in chapter 3, corresponding in this case to:

$$\begin{aligned}|e|^{-1}\mathcal{L}_{\text{bd}} = -2\partial_m\left[(*\tilde{A}_{(3)0})^m\left(4e^{K-\hat{K}}*\tilde{\mathcal{F}}_{0(4)} + 2ie^K\hat{K}'^{-1}p\right)\right] + \\ -2\partial_m\left[(*\tilde{A})^m_{(3)i}\left(e^{K-\hat{K}}K^{ij}*\tilde{\mathcal{F}}_{j(4)} + 4b^i*\tilde{\mathcal{F}}_{0(4)} + 2ie^K b^i\hat{K}'^{-1}p\right)\right] + \\ +2\partial_m\Big\{(*A^i_{(3)})^m\Big[e^{-\hat{K}}\left(K_{ij}*\mathcal{F}^j_{(4)} - \kappa_{ijk}b^j e^K K^{kl}*\tilde{\mathcal{F}}_{l(4)} - 2\kappa_{ijk}b^j b^k*\tilde{\mathcal{F}}_{0(4)}\right) + \\ -ie^K\kappa_{ijk}b^j b^k\hat{K}'^{-1}p\Big]\Big\} + \\ +2\partial_m\Big\{(*A^0_{(3)})^m\Big[e^{-\hat{K}}\Big(\frac{e^{-K}}{16}*\mathcal{F}^0_{(4)} + \frac{e^K}{2}\kappa_{ijk}b^j b^k K^{il}*\tilde{\mathcal{F}}_{l(4)} - \frac{e^{-K}}{4}b^i K_{ij}*\mathcal{F}^j_{(4)} + \\ +\frac{2}{3}\kappa_{ijk}b^i b^j b^k e^K*\tilde{\mathcal{F}}_{0(4)}\Big) + \left(\frac{1}{2}\xi + \frac{i}{3}e^K\kappa_{ijk}b^i b^j b^k\hat{K}'^{-1}\right)\Big]p\Big\}\end{aligned} \tag{4.81}$$

We can therefore appreciate in a manifest way the fact that in (4.79) the values of the fluxes e_A and m^A have completely disappeared, leaving room for the fields strengths of the gauge three-forms $\tilde{A}_{A(3)}$ and $A^A_{(3)}$.

4.4 Extracting the scalar potential

Until now we have done nothing but apply the new formalism, substituting the values of the fluxes e_A and m^A in the superpotentials with gauge four-form field strengths. If we want to study the stability properties of vacua of the theory, however, we must come back to the canonical supergravity formulation that involves a scalar potential, as well as the kinetic terms that are already present in (4.79). In our case it is precisely the contribution of the three-forms and of the spectator superpotential, $\mathcal{L}_{\text{3-forms},\hat{W}}$ that will furnish the theory with such a potential. In order to achieve this the equations of motion for the gauge three-forms $\tilde{A}_{A(3)}$ and $A^A_{(3)}$ must be computed: inserting their solutions into $\mathcal{L}_{\text{3-forms},\hat{W}}$ the canonical scalar potential must be retrieved, if the dualization procedure is indeed consistent. This potential must be of the usual form (1.64):

$$V = e^{K+\hat{K}} \left(K^{i\bar{j}} D_i W D_{\bar{j}} W + \hat{K}''^{-1} D_n W D_{\bar{n}} W^* - 3|W|^2 \right), \tag{4.82}$$

where we have distinguished the contributions of the two Kähler potentials K and $\hat{K}$, observing that no mixed terms of the form K'_i or $\hat{K}'_i$ can appear. The only terms that involve the gauge three-forms are included in $\mathcal{L}_{\text{3-forms},\hat{W}}$ (4.80), thus constituting the starting point in order to deduce their equations of motion. Varying with respect to the gauge three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$ and integrating the resulting equations of motion gives (calling e_A and m^A the integration constants, in order to obtain the desired form):

$$\begin{aligned}
&-8e^{K-\hat{K}} * \tilde{\mathcal{F}}_{(4)0} = m^0 - 4\sqrt{2} p e^K e^{-D} \\
&-2e^{K-\hat{K}} K^{ij} * \tilde{\mathcal{F}}_{(4)j} = m^i - m^0 b^i \equiv p^i \\
&-\frac{1}{2} e^{-(K+\hat{K})} K_{ij} * \mathcal{F}^j_{(4)} = e_i + \kappa_{ijk} b^j m^k - \frac{1}{2}\kappa_{ijk} b^j b^k m^0 \equiv \rho_i \\
&-\frac{1}{8} e^{-(K+\hat{K})} * \mathcal{F}^0_{(4)} = e_0 + e_i b^i + \frac{1}{2}\kappa_{ijk} b^i b^j m^k - \frac{1}{6}\kappa_{ijk} b^i b^j b^k m^0 \equiv \rho_0 - \frac{1}{2} p\xi,
\end{aligned} \tag{4.83}$$

where we have introduced the combinations p^i, ρ_i and ρ_0.
Most importantly the boundary term (4.81), if supported by the substitution (4.83), gives a non trivial contribution to the scalar potential of the form[3]:

$$\begin{aligned}
|e|^{-1} \mathcal{L}_{\text{bd}}\big|_{\text{on-shell}} = &-32 e^{K+\hat{K}} \left(\rho_0 - \frac{1}{2} p\xi \right)^2 - 8 e^{K+\hat{K}} K^{ij} \rho_i \rho_j + \\
&- 2 e^{\hat{K}-K} K_{ij} p^i p^j - \frac{1}{4} e^{\hat{K}-K} \left(\frac{m^0}{2} + 4\sqrt{2} p e^K e^{-D} \right) m^0
\end{aligned} \tag{4.84}$$

Substituting the results (4.83) and (4.84) into $\mathcal{L}_{\text{3-forms},\hat{W}}$ the canonical form for the scalar potential is obtained:

$$\begin{aligned}
|e|^{-1} \mathcal{L}_{\text{3-forms},\hat{W}}\big|_{\text{on-shell}} = &-e^{K+\hat{K}} \hat{K}''^{-1} p^2 - 16 e^{K+\hat{K}} \left(\rho_0 + \frac{1}{2} p\xi \right)^2 - 4 e^{K+\hat{K}} K^{ij} \rho_i \rho_j + \\
&- e^{\hat{K}-K} K_{ij} p^i p^j - \frac{1}{16} e^{\hat{K}-K} (m^0)^2 - \sqrt{2} p m^0 e^{\hat{K}} e^{-D}
\end{aligned} \tag{4.85}$$

The scalar potential of the theory is then:

$$\begin{aligned}
V = -|e|^{-1} \mathcal{L}_{\text{3-forms},\hat{W}}\big|_{\text{on-shell}} = e^{K+\hat{K}} \hat{K}''^{-1} p^2 + e^{\hat{K}} \Bigg[16 e^K \left(\rho_0 - \frac{1}{2} p\xi \right)^2 + \\
+ 4 e^K K^{ij} \rho_i \rho_j + + e^{-K} K_{ij} p^i p^j + \frac{1}{16} e^{-K} (m^0)^2 + \sqrt{2} p m^0 e^{-D} \Bigg]
\end{aligned} \tag{4.86}$$

[3]It plays the same role of the boundary term in the simpler setting we have examined in equation (3.12).

What we should ask ourselves now is whether the potential (4.86) reproduces expression (4.82), which in turn is equal to (2.52). In the next section we will explicitly show that the two expressions are indeed the same, expanding them in their respective terms.

4.5 Checking the consistency of the potential

The task at hand now is an algebraic one: expand (4.85) into its constituents, collect them as much as possible and display the scalar potential in such a way that it depends only on the physical scalar fields t^i (that are composed of their real parts b^i and imaginary parts v^i), D and ξ, as well as on the value of the fluxes e_A and m^A that, thanks to the fact that we have evaluated $\mathcal{L}_{\text{3-forms},\hat{W}}$ on-shell, have reappeared into the theory. The final expression should coincide to the scalar potential obtained in chapter two from dimensional reduction (2.52).

In this regard we come back to the prescription we chose at the beginning of this chapter and set:

$$e_0 \longrightarrow 0 \qquad m^i \longrightarrow 0 \tag{4.87}$$

We do not display the whole procedure, as it would take too much space: we exhibit instead, as an example, how all the terms proportional to $(m^0)^2$ get reunited into a single one, comparing the result with what is obtained expanding (2.52). Equivalence between the other terms can be calculated in an analogue way.

First of all we recall a few general relations among the derivatives of the Kähler potential and the intersection numbers, and we display the explicit form of the Kähler metric as a function of the intersection numbers and the imaginary parts of t^i [21]:

$$K_i = -\frac{\kappa_i}{4vol} \tag{4.88}$$

$$vol = \frac{1}{3!}\kappa_{ijk}v^iv^jv^k \qquad \kappa_i = \kappa_{ijk}v^jv^k \qquad \kappa_{ij} = \kappa_{ijk}v^k \qquad \kappa_{ijk} \equiv \kappa \tag{4.89}$$

$$K_{ij} = -\frac{1}{4vol}\left(\kappa_{ij} - \frac{\kappa_i\kappa_j}{4vol}\right) \qquad K^{ij} = -4vol\left(\kappa^{ij} - \frac{v^iv^j}{2vol}\right) \tag{4.90}$$

$$K^{ij}K_{jk} = \delta^i_k \qquad K^{ij}K_k = -2v^i \qquad K_{ij}v^j = -\frac{1}{2}K_i \qquad K_iv^i = -\frac{3}{2} \tag{4.91}$$

Having (4.90) at our disposal it is easy to show that the identities (4.91) indeed hold. Another useful identity satisfied by the Kähler potential is:

$$K^{ij}K_iK_j = 3 \tag{4.92}$$

Remembering that $e^K = \frac{1}{8vol}$ and carrying out this substitution the following form for the potential derived from the three-forms (4.86) is obtained:

$$\begin{aligned} V|_{e_A=m^i=0} = \frac{e^{\hat{K}}}{2}(m^0)^2 \Bigg[& \frac{1}{18}(\kappa_{ijk}b^ib^jb^k)^2 + \left(\frac{(\kappa_{ij}b^jb^k)^2}{4vol} - \frac{1}{2}\kappa^{ij}\kappa_{ild}\kappa_{jef}b^lb^db^eb^f \right) + \\ & + \left(\frac{\kappa_i\kappa_j b^ib^j}{2vol} - 2\kappa_{ij}b^ib^j \right) + vol \Bigg] , \end{aligned} \tag{4.93}$$

where we have collected the terms with the same amount of fields b^i.
The general expression for the potential deduced directly from dimensional reduction is instead (2.52):

$$V = -\frac{e^{4D}}{2} \left(e_A - \bar{\mathcal{N}}_{AC}m^C \right) \left(\mathrm{Im}\mathcal{N}^{-1} \right)^{AB} \left(e_B - \mathcal{N}_{BD}m^D \right) , \tag{4.94}$$

where the components of the matrix $\mathcal{N}$ are the ones stated in (2.56).

We should now show that (4.93) and (4.94) coincide: the main difference to overcome lies in the fact that in (4.94) the fields v^i appear, an occurrence that is not seen in the case of (4.93). However we will see in a short notice how to ease this difficulty.

Specifying (4.94) in the case where $e_A = m^i = 0$ we obtain many terms:

$$\begin{aligned} V|_{e_A=m^i=0} = -\frac{e^{4D}}{2}(m^0)^2 \Bigg[& \frac{(\kappa_{ijk}b^ib^jb^k)^2}{18vol} - \frac{i}{3}\kappa_{ijk}b^ib^jb^k - i\frac{\kappa_{ijk}b^ib^jb^k(b^i\kappa_i)^2}{24vol^2} + i\frac{\kappa_{ijk}b^ib^jb^k\kappa_{ld}b^lb^d}{6vol} + \\ & - \frac{(b^i\kappa_i)^2}{4vol} + b^ib^j\kappa_{ij} - \frac{(\kappa_{ijk}b^ib^jb^k)^2}{12vol} - \frac{(\kappa_{ijk}b^ib^jv^k)^2}{8vol} + \frac{1}{4}\kappa_{ijk}\kappa^{il}b^jb^k\kappa_{lde}b^db^e + \\ & - i\frac{\kappa_{ijk}b^ib^jv^k\kappa_{ld}b^lv^d}{16vol^2} + i\frac{\kappa_{ijk}b^ib^j\kappa^{kd}\kappa_{ld}b^l}{8vol} + i\frac{\kappa_{ijk}b^ib^jv^k\kappa_{ld}b^lv^d}{4vol} + \\ & - \frac{i}{2}\kappa_{ijk}b^ib^j\kappa^{kl}\kappa^{ld}b^d + \frac{i}{2}\kappa_{ijk}b^ib^jb^k + i\frac{\kappa_{ijk}b^ib^jb^k(b^i\kappa_i)^2}{24vol^2} + \\ & - i\frac{\kappa_{ijk}b^ib^jb^k\kappa_{ld}b^lb^d}{6vol} + i\frac{\kappa_{ijk}b^ib^jv^k\kappa_{ld}b^lv^d}{16vol^2} + i\frac{\kappa_{ijk}b^ib^j\kappa^{kd}\kappa_{ld}b^l}{8vol} + \\ & - i\frac{\kappa_{ijk}b^ib^jv^k\kappa_{ld}b^lv^d}{4vol} + \frac{i}{2}\kappa_{ijk}b^ib^j\kappa^{kl}\kappa_{ld}b^d - \frac{(\kappa_ib^i)^2(\kappa_iv^i)^2}{32vol^3} + \\ & + \frac{(\kappa_ib^i)^2\kappa^{ij}\kappa_i\kappa_j}{16vol^2} + 2\frac{\kappa_{ij}b^iv^j\kappa_kb^k\kappa_lv^l}{8vol^2} - \frac{\kappa_{ij}b^i\kappa^{jk}\kappa_k\kappa_lb^l}{4vol} + \\ & - \frac{\kappa^{ij}\kappa_i\kappa_{jk}b^k\kappa_lb^l}{4vol} - \frac{(\kappa_{ij}b^iv^j)^2}{2vol} + b^i\kappa_{ij}\kappa^{jk}\kappa_{kl}b^l + \frac{(\kappa_ib^i)^2}{4vol} + \\ & - \kappa_{ij}b^ib^j - \frac{i}{6}\kappa_{ijk}b^ib^jb^k - vol \Bigg] \end{aligned} \tag{4.95}$$

As we can see the algebra is quite daunting for just this one term. However it turns out looking more closely that many terms cancel against each other, except for a few that can be further simplified using (4.91), that allows to throw the terms involving combinations of the intersection numbers away. In order to get rid of the terms containing the imaginary parts v^i it is convenient to exploit the identities (4.89). Furthermore, collecting the terms with the same number of b^i fields the correspondence of this potential with (4.93) becomes manifest.

In the end, summing everything together, there remain only the terms that appear in (4.93), that correctly coincide with the ones displayed by [25]. As a matter of fact, unfortunately, we have displayed until now only the $(m^0)^2$ contribution to the scalar potential: the same operation must be carried out for all the other terms, too. At the end of the calculation, realized following the same steps we have shown and employing the identities (4.89), (4.91) and (4.92), the full scalar potential is recovered:

$$\begin{aligned} V = p^2\frac{e^{2D}}{4vol} + \frac{vol(m^0)^2e^{4D}}{2} + e_i^2v_i^2\frac{e^{4D}}{2vol} + \sqrt{2}m^0pe^{3D} + e^{4D}\frac{(e_ib^i - p\xi)^2}{vol} \\ + \frac{vol(m^0)^2e^{4D}}{2}\left(\frac{b_1^2}{v_1^2} + \frac{b_2^2}{v_2^2} + \frac{b_3^2}{v_3^2}\right) - \frac{m^0e^{4D}\kappa b_1b_2b_3}{vol}\left(\frac{e_1v_1^2}{b_1} + \frac{e_2v_2^2}{b_2} + \frac{e_3v_3^2}{b_3}\right) + \\ - \frac{m^0\kappa b_1b_2b_3e^{4D}}{vol}(e_ib^i - p\xi) + \frac{m^0e^{4D}(\kappa b_1b_2b_3)^2}{2vol}\left(\frac{v_1^2}{b_1^2} + \frac{v_2^2}{b_2^2} + \frac{v_3^2}{b_3^2}\right) + \\ + \frac{m^0e^{4D}(\kappa b_1b_2b_3)^2}{2vol} \end{aligned} \tag{4.96}$$

We underline that the expression $e_i^2v_i^2$ means explicitly $e_1^2v_1^2 + e_2^2v_2^2 + e_3^2v_3^2$, without the cross-terms. The reason why such a curious term comes about lies in the structure of the term from which it originates, $4e^KK^{ij}\rho_i\rho_j$, that appears in (4.86). Recalling the definition of ρ_i and considering only its e_i contribution, in fact, the expression to evaluate is:

$$e^KK^{ij}e_ie_j, \tag{4.97}$$

that, using (4.90) and recalling that $K = -\log(8vol)$ becomes:

$$\begin{aligned} -\frac{1}{2}\left(\kappa^{ij} - \frac{v^iv^j}{2vol}\right)e_ie_j = -\frac{1}{2}\kappa^{ij}e_ie_j + \\ + \frac{1}{4vol}\left(e_1^2v_1^2 + e_2^2v_2^2 + e_3^2v_3^2 + 2e_1e_2v^1v^2 + 2e_1e_3v^1v^3 + 2e_2e_3v^2v^3\right) \end{aligned} \tag{4.98}$$

At this point let us examine in more detail the shape of $\kappa^{ij}e_ie_j$, bearing in mind the definition $\kappa^{ij} \equiv (\kappa_{ij})^{-1}$:

$$\begin{aligned} \kappa^{ij}e_ie_j = \frac{1}{\kappa_{123}v^3}e_1e_2 + \frac{1}{\kappa_{123}v^2}e_1e_3 + \frac{1}{\kappa_{123}v^1}e_2e_3 = \frac{e_1e_2v^1v^2 + e_1e_3v^1v^3 + e_2e_3v^2v^3}{\kappa_{123}v^1v^2v^3} = \\ = \frac{e_1e_2v^1v^2 + e_1e_3v^1v^3 + e_2e_3v^2v^3}{vol} \end{aligned} \tag{4.99}$$

Inserting this result in (4.98) we see then that the cross-terms cancel out and only the squares $e_1^2v_1^2 + e_2^2v_2^2 + e_3^2v_3^2$ remain in the expression of the potential.

In conclusion, this chapter has been mainly devoted to translate the model of Narayan and Trivedi [25] into the new formulation that substitutes the values of the fluxes, with the techniques of [20] and [21], and to show that the potential obtained working in the gauge three-forms' formalism exactly coincides with the one straightforwardly deduced from (2.52). In the next chapter we will reap the benefits of this work, examining the scalar potential in full detail, finding its extrema and using the information thus obtained to draw some conclusions about the weak gravity conjecture.

CHAPTER 5

Membranes and Domain Walls

In the past chapters we have seen how, starting from a ten-dimensional action with non-vanishing fluxes of the RR and NSNS sector p-forms, it is possible to perform a compactification that allows a reduction to the standard four dimensions, obtaining an effective theory with a scalar potential depending on the values of the fluxes. We have then shown how, in the specific case of the model in [25], it is possible to rewrite the effective theory in such a way that the values of the fluxes are not fixed anymore: to achieve this we have introduced some (as many as the different fluxes) gauge three-forms. The scalar potential, therefore, becomes a function of the field strengths of these three-forms, provided that they are evaluated on-shell. Consequently it is possible to find its extrema and to assess whether they are in correspondence with Minkowski or Anti-de Sitter spaces, as well as if they are supersymmetric or not. This will be the main focus of the first part of this chapter.

More interestingly, however, if we leave the part of the action that depends on the gauge three-forms as it is, it can be seen that it resembles the more familiar Yang-Mills action with several gauge fields, that in our case are the gauge three-forms. Each of these gauge fields possesses a coupling, related to the coefficient of the Yang-Mills term, just like in electromagnetism the coupling is e:

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{4e^2} F_{mn} F^{mn} \tag{5.1}$$

In our case, instead, the theory naturally displays terms of the form:

$$\mathcal{L}_{\text{generalized Yang-Mills}} \propto -\frac{1}{g^2} F_{mnrs} F^{mnrs}, \tag{5.2}$$

where g is the gauge coupling of the three-form A_{mnr}, whose field strength is F_{mnrs}.

As we have briefly outlined at the beginning of chapter 3, when motivating the use of the new formalism that substitutes the fluxes, these gauge three-forms naturally couple to objects that extend in 2+1 spacetime dimensions, just like the electromagnetic potential A_m couples to 0+1-dimensional objects (that is, the world-line of a particle). These 2+1 dimensional entities are nothing but membranes, and their coupling to the gauge three-forms will have a specific structure:

$$S_{\text{coupling}} \propto q \int \mathrm{d}^3\xi A_{(3)}, \tag{5.3}$$

where the integration is performed on three coordinates ξ^i $(i = 1, 2, 3)$ that parametrize the membrane world-volume, and $q \in \mathbb{Z}$ is the quantized charge[1], that gives how many units of the elementary charge g that we have defined in (5.2) are possessed by the membrane; this means that $Q = qg$ represents the physical charge of the membrane under the canonically normalized three-form.

On the other hand, membranes must have a kinetic term describing their dynamics, which in general is of the form:

$$S_{\text{kinetic}} \propto \int \mathrm{d}^3\xi \sqrt{-h}\, T, \tag{5.4}$$

where h is the induced metric on the membrane world-volume, and T is the tension of the membrane, that can be intuitively thought as its mass per unit area. The integration is once again performed on the membrane world-volume parametrized by ξ^i. These membranes, being charged under the gauge three-forms that have substituted the fluxes, cause a jump in the value of the fluxes when crossing the membrane itself. This fact, that will be shown explicitly, suggests in a natural way that membranes of this kind could be considered as part of domain walls separating two different extrema of the scalar potential. Rephrasing from another perspective: the scalar potential exhibits in general a diverse set of extrema, that depend on the values of the fluxes of the p-forms in the RR sector; the appearance of a membrane, if allowed, changes the values of the fluxes on one side, as a consequence of the fact that the membrane is charged with respect to the gauge three-forms that have substituted the values of the fluxes. The most direct result of this reasoning is that the properties of the charged membranes have to be carefully discussed in order to properly understand the transitions among vacua of the scalar potential with different values of the fluxes. More specifically, we will at first concentrate on the effect that a membrane interpolating between two different vacuum states has on the fields of the theory, that vary when passing from one side to the other. If, in fact, the values of the scalar fields are fixed when the model resides in one of its vacua, the back-reaction of the membrane forces the scalar fields to interpolate from their minimum value on one side of the membrane to the other minimum on the opposite side. We will show how to compute this variation and display explicit profiles of the evolution of the scalar fields. All of this work will be performed in the thin wall-approximation, that will be explained in due time.

Apart from being the mediators of the transitions among vacua of the scalar potential, the charged membranes provide an intriguing testing ground for the Weak Gravity Conjecture (WGC). This hypothesis, first proposed a little more than a decade ago [29], states[2] that there

[1] With eventually a $\sqrt{2}$ factor to match the quantization conditions (4.41).

[2] Here we have been a bit imprecise, a more accurate statement of the Weak Gravity Conjecture will be given later.

must exist particles and p-branes whose energy (or tension, in the case of the membranes) is smaller than or equal to their charge. More precisely, the equality is saturated if the state of the theory under scrutiny is supersymmetric, whereas the mass is strictly lower than the charge in non-supersymmetric cases, according to the most recent formulation of [34]. In the next chapter, therefore, we will embark on the study of the relation between the tension of the membranes interpolating the transitions, that can be deduced from expressions such as (5.4), and their charge, in principle originated by the Yang-Mills-like terms (5.2).
Before dealing with this issue, however, we come back to the scalar potential and to the quest for its extrema.

5.1 Finding the extrema of the scalar potential

The full scalar potential of the Narayan and Trivedi model, expressed in (4.96), depends on a total of eight real scalar fields: the dilaton D; the imaginary parts v^i of the Kähler moduli t^i; their real parts, the axions b^i; the axion ξ. The first approach could therefore consist in trying to find the extrema of this full-fledged scalar potential, computing its gradient with respect to (D, v^i, b^i, ξ) and setting it to zero. Unfortunately this standard procedure is not analytically feasible. An alternative strategy, elegantly explained in [25], relies on setting the axions b^i and ξ to zero, and on retaining only the other terms. In this way the scalar potential becomes:

$$V_R = \frac{e^{2D}p^2}{4vol} + \frac{vol(m^0)^2 e^{4D}}{2} + e_i^2 v_i^2 \frac{e^{4D}}{2vol} + \sqrt{2}m^0 p e^{3D}, \tag{5.5}$$

where only the dependence on the dilaton D and on the three scalars v^i remains. A possible problem, however, lingers: how can we be so sure that the extrema, in particular the minima, found using (5.5), remain stationary points when also the axions b^i and ξ are included? Looking closely at the expressions of the superpotentials of the theory (4.96) it can be noted that if we switch the signs of the axions the superpotential undergoes a transformation of the kind:

$$W \longrightarrow -\overline{W} \tag{5.6}$$

It follows that, recalling how the scalar potential is deduced from the superpotential (1.64), V remains invariant. More explicitly, it can be observed from (4.96) that switching the signs of the axions causes no appreciable effect, as they appear exclusively in quadratic combinations. Let us suppose now that we have found an extremum of (5.5): expanding the reduced potential around the point with vanishing axions ($\xi = b^i = 0 \ \ \forall i$) we can use only quadratic terms in the axions, because of the invariance of the full potential (4.96) under sign swapping. But this means precisely that the point with ($\xi = b^i = 0$) we have found remains an extremum even with the corrections included. In conclusion we can safely work with the reduced potential (5.5) and stay assured that any extremum found working in this manner retains its properties in the full theory, too.
Taking the gradient of V_R with respect to e^D (because the dilaton appears in the potential

only with this combination) and v^i gives four equations (of course the three involving derivatives w.r.t v^i have the same structure), that once solved give only one class of extrema that satisfy the requirement that e^D is larger than zero:

$$e^D = \sqrt{\frac{27}{160}\left|\frac{\kappa m^0 p^2}{e_1 e_2 e_3}\right|} \qquad v_i = \pm\sqrt{\frac{5}{3}\left|\frac{e_1 e_2 e_3}{\kappa m^0 e_i}\right|} \tag{5.7}$$

The v^i cannot, however, display arbitrary signs: exactly two of them must have a minus sign in front, otherwise we do not have an extremum of the potential; therefore the possible combinations of signs of the minimum values of v^i are:

$$(v_1, v_2, v_3) = \{(+,-,-),(-,+,-),(-,-,+)\} \tag{5.8}$$

In order to understand whether these extrema are maxima, minima or saddle points the Hessian matrix must be computed. A simple trick can be employed to simplify the calculations: the equations obtained deriving the potential with respect to v_1, v_2 and v_3 are the same, and as a result the extremal solutions retain this property, as we have shown with the combinations of signs (5.8). It is therefore possible to substitute the v^i with a single field ν, observing that since the v^i appear in (5.5) in quadratic (as in $e_i^2 v_i^2$) or cubic terms (inside $vol = \kappa v_1 v_2 v_3$) no sign ambiguities arise. Further assuming that the values of the fluxes are all equal in modulus, $|e_i| \equiv |e|$ the potential becomes:

$$V_\nu = \frac{e^{2D}p^2}{4\kappa\nu^3} + \frac{\kappa\nu^3(m^0)^2 e^{4D}}{2} + 3e^2\nu^2\frac{e^{4D}}{2\kappa\nu^3} + \sqrt{2}m^0 p e^{3D} \tag{5.9}$$

Having reduced the range of variables only to D and ν it is easy to compute the Hessian matrix, whose eigenvalues evaluated at the extrema (5.7) can be shown to be strictly positive. It follows that (5.7) are indeed minima of the reduced scalar potential, as well as extrema of the full scalar potential, recalling the discussion pertaining to the axions contribution.
Inserting the extrema (5.7) into (5.5) we obtain:

$$V_R\Big|_{\text{extrema}} = -\sqrt{\frac{4}{15}}\left(\frac{27}{160}\right)^2 \frac{p^4\kappa^{\frac{3}{2}}|m^0|^{\frac{5}{2}}}{|e_1 e_2 e_3|^{\frac{3}{2}}} \tag{5.10}$$

We see then explicitly that the extremum value of the potential is negative. This means that the vacuum state corresponding to this class of extrema is an Anti-de Sitter vacuum, that is, with negative intrinsic curvature.
It is important for our future discussion to determine whether the minima (5.7) are supersymmetric or not. As we have seen in chapter 1, the Lorentz invariance of the vacuum state imposes that all the fields of the theory must vanish, except for the scalars. The only possibly non-vanishing variations are those of the fermions:

$$\delta(\text{fermions}) \propto D_I W, \tag{5.11}$$

where the index I runs over all the fields included in the theory: in our case, the scalars t^i and n. The total superpotential is instead given by (4.33):

$$W = -p\xi - \sqrt{2}ie^{-D} + e_i t^i - \frac{m^0 \kappa_{ijk}}{6} t^i t^j t^k = -2pn + e_i t^i - \frac{m^0 \kappa_{ijk}}{6} t^i t^j t^k \tag{5.12}$$

Consequently if all of the $D_I W$ vanish the vacuum state under consideration is supersymmetric, if instead as few as one of the $D_I W$ is non-vanishing we talk about a non-supersymmetric vacuum.

Let's examine the quantities $D_I W$ one at a time; as regards the fields t^i they read:

$$D_{t^i} W = \frac{\partial W}{\partial t^i} + W \frac{\partial K}{\partial t^i} = 0, \tag{5.13}$$

where:

$$\frac{\partial W}{\partial t^i} = e_i - \frac{m^0}{2} \kappa_{ijk} t^j t^k \tag{5.14}$$

Employing the restriction $\xi = 0$ and $b^i = 0 \ \forall i$, and recalling that:

$$K = -\ln \left[\frac{4}{3} \kappa_{ijk} \left(\frac{t^i - \bar{t}^i}{2i} \right) \left(\frac{t^j - \bar{t}^j}{2i} \right) \left(\frac{t^k - \bar{t}^k}{2i} \right) \right] \tag{5.15}$$

we obtain:

$$\frac{\partial K}{\partial t^i} W = \frac{i\kappa_i}{4vol} \left(-\sqrt{2}ie^{-D} + ie_i v^i + i \frac{m^0 \kappa_{ijk}}{6} v^i v^j v^k \right), \tag{5.16}$$

where we have used the relation $\kappa_i = \kappa_{ijk} v^j v^k$. Inserting this intermediate step into (5.13) and evaluating it on the minima (5.7), as well as using the explicit expression (4.29) for κ_i in our model yields zero if and only if:

$$\mathrm{sign}(m^0 e_i) < 0 \qquad \mathrm{sign}(m^0 p) < 0 \tag{5.17}$$

As far as the scalar field n is concerned, moreover, we get:

$$D_n W = \frac{\partial W}{\partial n} + W \frac{\partial K}{\partial n} = 0, \tag{5.18}$$

with:

$$\frac{\partial W}{\partial n} = -2p \tag{5.19}$$

And (recalling that $n = \frac{1}{2}\xi + \frac{i}{\sqrt{2}} e^{-D}$ and setting ξ to zero):

$$\frac{\partial \hat{K}}{\partial n} = \frac{\partial (4D)}{\partial n} = \frac{\partial (4D)}{\partial D} \frac{\partial D}{\partial n} = 4\sqrt{2}ie^{-D} \tag{5.20}$$

We can then insert this expression into (5.18) that, evaluated on the minima, holds if:

$$\mathrm{sign}(m^0 e_i) < 0 \qquad \mathrm{sign}(m^0 p) < 0 \tag{5.21}$$

Luckily enough, taking into account the tadpole condition (4.50) we see that the condition $\mathrm{sign}(m^0 p) < 0$ is automatically satisfied. As regards $\mathrm{sign}(m^0 e_i) < 0$, on the other hand, it can be noted that once that the sign of m^0 has been chosen the e_i must all be of the same sign (positive or negative depending on the sign of m^0) if we wish to preserve supersymmetry. Actually, as we have seen in (4.52), the values of m^0 are strictly constrained, and once that we have chosen a value for p (that is considered to be fixed in our model) also m^0 is established.[3] We see then that there is an easy way to find non-supersymmetric minima: if we switch one (or more) of the values of the fluxes e_i some of the conditions (5.17) are violated, and the vacuum acquires a non-susy status. The main advantage of this approach is that the minima of the potential remain such (because the potential itself is quadratic in the fluxes e_i) and that the potential evaluated on the non-susy vacua (5.10) is exactly the same as the one of the susy ones: the only difference lies in the sign of the fluxes.

5.2 The membrane action

In the next sections we will deal with the issue of vacuum decays mediated by membranes, following the work of [21]: in order to tackle the problem in a precise way it is necessary to define the action for a membrane in the context of the four dimensional supergravity theory we have discussed in chapter 4. As we have mentioned earlier, this action in general includes a term that expresses the coupling of the membrane to the gauge three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$, as well as a kinetic term.
Taking into account the first contribution we define the following quantized charges, that correspond to the gauge three-forms in the terms analogous to (5.3) that we want to build:

$$q_A \longleftrightarrow A^A_{(3)} \qquad p^A \longleftrightarrow \tilde{A}_{(3)A} \qquad \Rightarrow \ \text{Coupling term: } q_A \int A^A_{(3)} - p^A \int \tilde{A}_{(3)A} \tag{5.22}$$

We note that the quantized charges (q_A, p^A) can hence be seen as parameters that provide a classification of the different membranes that can be introduced in the action. In addition, we observe that the membranes we are considering are originated by higher dimensional D-branes compactified on appropriate p-cycles in the internal dimensions, so that they consistently give rise to membranes in 4d.

As we have seen in chapter 1, in order to automatically ensure that the action is supersymmetry invariant the coupling term must be expressed in terms of superfields. We are compelled, therefore, to introduce two sets of *super three-forms* $\tilde{\mathcal{A}}_{(3)A}$ and $\mathcal{A}^A_{(3)}$, whose lowest

[3] Despite this fact we have dualized it anyway, in order to maintain a certain symmetry among the RR sector p-forms, as we have mentioned earlier.

components are the standard gauge three-forms:

$$\tilde{\mathcal{A}}_{(3)A}|_{\theta=\bar{\theta}=0} = \tilde{A}_{(3)A} \qquad \mathcal{A}^A_{(3)}|_{\theta=\bar{\theta}=0} = A^A_{(3)} \tag{5.23}$$

The super three-forms $\tilde{\mathcal{A}}_{(3)A}$ and $\mathcal{A}^A_{(3)}$ can be conveniently assembled into a single one:

$$\mathcal{A}_{(3)} \equiv q_A \mathcal{A}^A_{(3)} - p^A \tilde{\mathcal{A}}_{(3)A} \tag{5.24}$$

The explicit expression of $\mathcal{A}_{(3)}$, however, is rather involved and for the full details we refer to [21].
What is of chief interest in our discussion, instead, is the fact that the introduction of $\mathcal{A}_{(3)}$ allows for the construction of a supersymmetric action for the term that codes the coupling of a membrane with the gauge three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$. Correspondingly, given the coordinates ξ^i $(i = 1, 2, 3)$ that span the world-volume of the membrane, we define their extension to the whole spacetime and superspace:

$$\xi^i \longrightarrow z^M(\xi) = (\zeta^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}), \tag{5.25}$$

where θ and $\bar{\theta}$ are the usual superspace coordinates. With these conventions the coupling term (also known as Wess-Zumino action) reads:

$$S_{WZ} = \int \mathrm{d}^4x \;\; \mathcal{A}_{(3)} \wedge \delta(\mathcal{C}) \equiv \int_{\mathcal{C}} \mathcal{A}_{(3)}, \tag{5.26}$$

where $\mathcal{C}$ is the world-volume of the membrane.

The other contribution to the action of the membrane is its kinetic term: in order to obtain a supersymmetric physical spectrum on the membrane world-volume (that is, to make sure that there is the same amount of bosonic and fermionic degrees of freedom) this action must be invariant under the so called κ-symmetry, a local transformation parametrized by a spinor $\kappa^\alpha(\xi)$ (that depends on the world-volume coordinates), whose effects on the coordinates[4] are specified in full details in [21]. With this requirement it can be proven that the kinetic part of the membrane action (also called Nambu-Goto action) is:

$$S_{NG} = -2 \int \mathrm{d}^3\xi \sqrt{-h} |q_A S^A - p^A \mathcal{G}_A(S)|, \tag{5.27}$$

where h is the determinant of the metric induced on the world-volume, defined as:

$$h_{ij}(\xi) = \eta_{ab} E^a_i(\xi) E^b_j(\xi), \tag{5.28}$$

with E^a_i the pull-back of the supervielbein on the membrane world-volume:

$$E^a_i(\xi) = E^a_M(z(\xi)) \partial_i z^M(\xi), \tag{5.29}$$

[4] As well as the proof of the invariance of the action under this symmetry.

with the derivation meant with respect to the coordinates ξ^i. It is important to note that the kinetic term, as well as the coupling one, does not depend on the spectator superfield N that has not undergone the dualization procedure. In the preceding discussion, in fact, we have considered the flux p associated to N to be fixed, and as a consequence no corresponding gauge three-form (or, in other words, no charge) has appeared.
The total supersymmetry- and super Weyl-invariant action for the membrane is the sum of the Wess-Zumino and Nambu-Goto terms:

$$S_{\text{membrane}} = S_{NG} + S_{WZ} = -2\int_{\mathcal{C}} \mathrm{d}^3\xi\sqrt{-h}|q_A S^A - p^A \mathcal{G}_A(S)| + \int_{\mathcal{C}} \mathrm{d}^3\xi\, \mathcal{A}_{(3)}, \tag{5.30}$$

where all the integrations are performed on the membrane world-volume $\mathcal{C}$. As usual it is convenient, for the future discussion, to extract the bosonic components of S_{membrane}, after having fixed the Weyl invariance using (4.73). Recalling that the lowest components of S^i are the scalars t^i and that the super three-form $\mathcal{A}_{(3)}$ can be expanded according to (5.24) we obtain:

$$\begin{aligned} S_{\text{membrane}}\big|_{\text{bosonic}} \equiv S_M = -2\int_{\mathcal{C}} \mathrm{d}^3\xi\sqrt{-h}\, e^{\frac{1}{2}(K+\hat{K})}|q_A t^A - p^A\mathcal{G}_A(t)| + \\ + q_A\int_{\mathcal{C}} \mathrm{d}^3\xi\, A^A_{(3)} - p^A\int_{\mathcal{C}} \mathrm{d}^3\xi\, \tilde{A}_{(3)A} \end{aligned} \tag{5.31}$$

The first term in this expression is nothing but the generalization to a higher number of dimensions of the kinetic term of a point particle:

$$S_{KIN,\text{point particle}} = -m\int \mathrm{d}s\sqrt{-h} \tag{5.32}$$

Their structure differs only in the domain of integration, that is over a three-dimensional domain in the case of the membrane. The mass m of the point particle, instead, is substituted by the *tension* of the membrane T_M, defined as:

$$T_M \equiv 2e^{\frac{1}{2}(K+\hat{K})}|q_A t^A - p^A\mathcal{G}_A(t)| \tag{5.33}$$

It is evident, though, that the tension of the membrane is not a constant like the mass m, and instead depends explicitly on the quantized charges (q_A, p^A) as well as, more importantly, on the scalar fields t^A (recalling that $t^0 = 1$ after the gauge-fixing). The charge-dependence implies that membranes with different values of the charges have varying tensions. The dependence on the scalars t^A, instead, is more subtle: the fact that they appear inside the expression (5.33) entails that, in order to compute the tension of a given membrane, we must know the value of the scalars on the membrane itself, because the integration is performed on its world-volume $\mathcal{C}$. Though apparently an easy task, this last requirement is not trivial: if a membrane is added to the action[5] (4.79), the equations of motion for the scalars undergo

[5]We are using the term action and lagrangian in an interchangeable way, as it should not cause any confusion.

a modification: if we wish to obtain their value on the membrane world-volume they must be solved with appropriate boundary conditions. We see then that the back-reaction of the membrane on its surroundings plays an important part in determining the very tension of the membrane, that will be soon analyzed.

We can now finally display the full action that will be studied in the following: it encodes a four-dimensional supergravity theory with 4 scalar fields (3 t^i and n), two sets of 4 three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$, and a membrane coupled to the aforementioned three-forms:

$$\begin{aligned} S = &-\int \mathrm{d}^4x\, |e| \left(\frac{1}{2}R + K_{ij}\partial t^i \partial \bar{t}^j + \hat{K}'' \partial n \partial \bar{n} \right) + S_{\text{3-forms},\hat{W}} + \\ &-2\int_{\mathcal{C}} \mathrm{d}^3\xi \sqrt{-h}\, e^{\frac{1}{2}(K+\hat{K})} |q_A t^A - p^A \mathcal{G}_A(t)| + q_A \int_{\mathcal{C}} \mathrm{d}^3\xi\, A^A_{(3)} - p^A \int_{\mathcal{C}} \mathrm{d}^3\xi\, \tilde{A}_{(3)A}, \end{aligned} \tag{5.34}$$

where $S_{\text{3-forms},\hat{W}}$ is the integral over four-dimensional spacetime of (4.80). We have left full generality in the index A, that goes from 0 to 3, because, as we have seen in (4.35), the superpotential of the theory can alternatively be considered with the full set of fluxes or without e_0 and m^i, with no consequences whatsoever.

5.3 The effect of the membranes on the fluxes

With the full expression of the action we can eventually see in a mathematically precise way the crucial point we have stressed so far: the membranes, being charged under the gauge three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$, cause a change in the values of the fluxes that appear in the scalar potential of the theory that, as we have proven in the previous chapter, can be obtained from $S_{\text{3-forms},\hat{W}}$ using the equations of motion for the three-forms. These equations of motion (4.83), rewritten without the contribution of the axions b^i and ξ (recalling that they have been neglected when computing the scalar potential), read:

$$\begin{aligned} -8e^{K-\hat{K}} * \tilde{\mathcal{F}}_{(4)0} &= m^0 - 4\sqrt{2}pe^K e^{-D} \\ -2e^{K-\hat{K}} K^{ij} * \tilde{\mathcal{F}}_{(4)j} &= m^i \\ -\frac{1}{2}e^{-(K+\hat{K})} K_{ij} * \mathcal{F}^j_{(4)} &= e_i \\ -\frac{1}{8}e^{-(K+\hat{K})} * \mathcal{F}^0_{(4)} &= e_0 \end{aligned} \tag{5.35}$$

It must be underlined that these equations of motion were obtained *without* the contribution of the membrane. In order to obtain the new equations of motion that derive from (5.34), instead, it is convenient to rewrite the coupling terms between the membrane and the gauge three-forms as integrals over the whole spacetime, using a delta function:

$$q_A \int_{\mathcal{C}} A^A_{(3)} - p^A \int_{\mathcal{C}} \tilde{A}_{(3)A} = q_A \int A^A_{(3)} \wedge \delta(\mathcal{C}) - p^A \int_{\mathcal{C}} \tilde{A}_{(3)A} \wedge \delta(\mathcal{C}) \tag{5.36}$$

In this way new terms of the form $q_A\delta(\mathcal{C})$ and $p^A\delta(\mathcal{C})$ enter in the equations of motion that, once integrated, become:

$$\begin{aligned}
-8e^{K-\hat{K}}*\tilde{\mathcal{F}}_{(4)0} &= m^0 + p^0\Theta(\mathcal{C}) - 4\sqrt{2}pe^K e^{-D}\\
-2e^{K-\hat{K}}K^{ij}*\tilde{\mathcal{F}}_{(4)j} &= m^i + p^i\Theta(\mathcal{C})\\
-\frac{1}{2}e^{-(K+\hat{K})}K_{ij}*\mathcal{F}^j_{(4)} &= e_i + q_i\Theta(\mathcal{C})\\
-\frac{1}{8}e^{-(K+\hat{K})}*\mathcal{F}^0_{(4)} &= e_0 + q_0\Theta(\mathcal{C}),
\end{aligned} \tag{5.37}$$

where $\Theta(\mathcal{C})$ is a Heaviside function, resulting from the integration of $\delta(\mathcal{C})$, that is non-vanishing only on one side of the membrane, depending on the orientation of the delta function $\delta(\mathcal{C})$, that is a one-form. In fact $\delta(\mathcal{C})$ has a three-dimensional argument, and as we are working in four dimensions it must necessarily be a one-form (just like a delta function on $\mathbb{R}$ is a one form, because its argument is a point, namely a zero-form).
It is then clear that the inclusion of the membrane has drastically changed the situation: if, at first, the model resides in one of its minima with values of the fluxes (e_A, m^A), the addition of a membrane with quantized charges (q_A, p^A) changes the values on one side (in principle either in the "inside" or the "outside" of the compact membrane) to $(e_A + q_A, m^A + p^A)$, leaving the other side untouched.

Physically speaking it is evident that the mutated values must be on the inside of the membrane, as otherwise the change would require an infinite amount of energy (because, if the fluxes changed outside of the membrane, they would do so in an infinite region of space). The new values, besides, will be in correspondence with a new minimum of the potential, according to the expressions (5.7): we see therefore that the physical process we are describing consists in a transition between different vacua mediated by membranes or, in other words, in vacuum decay.

It is undeniable, at this point, that membranes play a major role in the problem of vacuum stability, and that the action (5.34) provides a foundation for the study of such transitions. It must be highlighted, however, that strictly speaking in this context we are dealing with transitions among vacua that pertain to *different* effective potentials: if the fluxes (e_A, m^A) change, in fact, also the effective scalar potential itself (5.5) is mutated (although its structure remains exactly the same, no new terms in the potential are produced by the implementation of the membrane into the action), in agreement with its expression (5.5). The study we will embark on in the next pages, therefore, does not regard transitions among different vacua of the *same* potential.

Until now we have dedicated our attention to the construction of a plausible action for some scalar fields coupled to gravity, and for a set of gauge three-forms that naturally couple to a membrane. The most reasonable question at this stage could be: how do we know that, in a semi-classical decay process, the nucleation of a membrane with charges (q_A, p^A) is actually possible? In order to provide an answer we must first lay the basis for a systematic treatment of non-perturbative vacuum transitions, following the classic work of Coleman and De Luccia [30][31], also recalled by Narayan and Trivedi in [25].

5.4 Vacuum decay

First of all, in the following we will work in the Euclidean frame of reference, as standard in the study of non-perturbative transitions, thus considering imaginary time:

$$t = i\tau \tag{5.38}$$

In this context, it is reasonable to assume that the membrane is nothing but a sphere of some radius, and that the direction transverse to its surface is labelled by the coordinate r. This choice, that in principle is arbitrary, is sensible given the fact that there is no anisotropy (or any preferred direction) in our model, being for this reason widely adopted in the literature[6] [30].
As a result, recalling that we are working in a Euclidean frame of reference, the metric of spacetime can be chosen to be O(4) invariant, reading:

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + \rho(r)^2(\mathrm{d}\Omega)^2, \tag{5.39}$$

with r the coordinate transverse to the membrane and $(\mathrm{d}\Omega)^2$ is the volume element on a unit S^3.

Let us consider now an unstable vacuum of the action (5.34), called in the following *false vacuum*, in correspondence with some values of the fluxes (e_A, m^A). Take another vacuum, termed *true vacuum*, characterized by other fluxes (e'_A, m'^A), and so associated with another scalar potential (even though of the same structure). For the moment we do not specify whether these vacua are supersymmetric or not: as we have seen in section 5.1, in fact, the only difference between the two cases resides in the signs of the fluxes e_i, whereas the value of V_R and of the moduli t^i and n remain the same. What is established, instead, is that they are *Anti-de Sitter vacua*, because in our model all the minimum values of the potential are negative (5.10).

The probability of transition Γ among these distinct vacua is [30]:

$$\Gamma = Ae^{-\frac{B}{\hbar}}, \tag{5.40}$$

where A is a normalization constant that is usually difficult to compute, and B is defined as:

$$B = S^E_{\text{true}} - S^E_{\text{false}}, \tag{5.41}$$

[6]There are however cases [21], involving BPS saturated cases, in which flat membranes can be considered (noting that they can also be thought as the limit of (5.39) for an infinite radius). In the following however we will not deal with such examples.

with S^E_{true} and S^E_{false} being the euclidean action for the true and false vacuum, respectively. We see from the definition (5.40) that, if we expand the actions that compose the coefficient B in powers of $\hbar$, every successive term will contribute more negligibly, as expected. The physical separation between the false and the true vacuum is given, in our model, by the charged membrane, that provides the shift in the fluxes with its charges $(q_A, p^A) = (e'_A - e_A, m'^A - m^A)$:

$$(e_A, m^A) \xrightarrow{\text{membrane}} (e'_A, m'^A) \tag{5.42}$$

The presence of the membrane forces the potential to change the value of the fluxes, there is no way to jump to another minimum of the same potential if we stick to charged membranes (even though this would be possible with standard Coleman-De Luccia processes not involving membranes). Oftentimes the appearance of a region of true vacuum embedded in a universe that resides in the false vacuum (with the separation between the two regions provided by the charged membrane, in our model) is called *bubble nucleation process.*

That being said, it is fundamental to compute which constraints, if any, must be imposed on the properties of the membrane if we wish that the decay process is allowed. In this regard it is convenient to work in the thin-wall approximation [30]: intuitively speaking, it assumes that the distortion caused by the membrane on the physical scalar fields is negligible, or somewhat small. It must be noted, however, that the original work of Coleman and De Luccia [30] regards scalar fields and does not involve membranes: despite this we will use their formalism considering domain walls that include also a membrane contribution.

First of all we assume that the evolution of the scalar fields respects the spherical symmetry enjoyed by the membrane. More precisely, we know that in a universe in its false vacuum the scalar fields assume a fixed value (i.e. the value that corresponds to the minimum). So they do in the region of true vacuum. When passing through the membrane of radius R, however, the two values of the scalar fields that correspond to the true and false vacuum must join for continuity reasons[7]. Recalling that the scalar fields in the model under consideration are t^i and n, we define their minimum values as:

$$\begin{aligned} t^i &= t^i_+ \quad && n = n_+ \quad \text{in the false vacuum} \\ t^i &= t^i_- \quad && n = n_- \quad \text{in the true vacuum} \end{aligned} \tag{5.43}$$

As we have hinted before, this correspondence between the values of the scalars on one side of the membrane with the ones on the other is determined by the membrane itself via the equations of motion of the scalar fields in (5.34), that we will soon display. Working in the thin wall approximation means that it is accepted that the change in the scalar fields takes place in a small region of variation of the radial coordinate r, at least compared to the radius of the membrane. In this approach such a region is called the *domain wall*: it encompasses the membrane and the part of space where the scalar fields undergo a change. Of course this is not a completely realistic case: if the scalar fields settled to their exact minimum value in

[7] If, instead, the back-reaction of the membrane is completely neglected, it is the case of the *probe approximation.*

a finite region their evolution would not be analytic. With more plausibility the minimum value is reached asymptotically at infinity: the thin wall approximation states that *most* of the variation happens in a small region. Mathematically speaking this is guaranteed if the energy difference between the two minima (say V_+ and V_-) is extremely small:

$$\epsilon = V_+ - V_- \ll 1 \tag{5.44}$$

It can be proven [30] that this assumption entails that the radius R of the membrane is very large compared to variation range of the scalar fields, and so that the approximation is indeed satisfied.
Converting (5.44) in a statement about the fluxes e_i (that determine the values V_+ and V_-), and considering m^0 and p to be fixed by the tadpole condition (4.50), it can be shown that it is equivalent to:

$$\left|\frac{e'_i - e_i}{e_i}\right| \equiv \left|\frac{\delta e_i}{e_i}\right| \ll 1, \tag{5.45}$$

where e'_i are the fluxes of V_-, whilst e_i refer to V_+.

Adopting this view the exponent B in (5.40) can be effectively computed dividing the universe in three-sub regions: the inside of the membrane, with fixed values of the scalar fields, the domain wall and the outside region (again with fixed scalars). In this way B becomes:

$$B = B_{\text{inside}} + B_{\text{domain wall}} + B_{\text{outside}} \tag{5.46}$$

Recalling how B has been defined (5.41) it can be noted that the outside contribution vanishes, because in that region the model is still in its false vacuum:

$$B_{\text{outside}} = 0 \tag{5.47}$$

In order to compute the contribution of the domain wall, which is a 3-dimensional shell, we must recall the measure of integration μ for such a geometric object:

$$\mu = 2\pi^2 \rho^3 \mathrm{d}r \tag{5.48}$$

We hence obtain:

$$B_{\text{domain wall}} = 2\pi^2 \int_{\Delta r} \rho^3 \mathrm{d}r \left(\mathcal{L}_{TOT}(t^i, n) - \mathcal{L}_{TOT}(t^i_+, n_+)\right), \tag{5.49}$$

with the integration performed over the range of variation of r, termed Δr. In $\mathcal{L}_{TOT}(t^i, n)$ we have not put any subscript on the fields, because we are taking into account the fact that the values of the scalars change over the region Δr, namely inside the domain wall.

As we have seen, in the thin-wall approximation the region of variation of the scalar fields is much smaller than the radius R of the membrane, and therefore we can assume that the factor ρ^3 is fixed at the value R^3; taking it out of the integral the following expression is obtained:

$$B_{\text{domain wall}} = 2\pi^2 R^3 \int_{\Delta r} \mathrm{d}r \left(\mathcal{L}_{TOT}(t^i, n) - \mathcal{L}_{TOT}(t^i_+, n_+)\right) = 2\pi^2 R^3 S_{DW}, \tag{5.50}$$

where the tension of the domain wall S_{DW} has been defined as:

$$S_{DW} = \int_{\Delta r} \mathrm{d}r \left(\mathcal{L}_{TOT}(t^i, n) - \mathcal{L}_{TOT}(t^i_+, n_+)\right) \tag{5.51}$$

We emphasize the fact that S_{DW} depends, via the presence of $\mathcal{L}_{TOT}(t^i, n)$, on the kinetic contribution of the scalar fields t^i and n, that is, from equation (5.34):

$$\mathcal{L}_{KIN} \propto K_{ij} \partial t^i \partial \bar{t}^{\bar{j}} + \hat{K}'' \partial n \partial \bar{n} \tag{5.52}$$

Another contribution is given by how much the scalar curvature has been modified by the presence of the membrane and by the evolution of the scalar fields. Very far from the membrane, in fact, we will find the fixed values $R(t^i_+, n_+)$ and $R(t^i_-, n_-)$. In the domain wall region, however, there is an additional contribution $\delta R(t^i, n)$ due to the fact that the scalars are not fixed. The gravitational term in S_{DW} will hence be:

$$\mathcal{L}_{GRAV} \propto \frac{1}{2}\left[R(t^i_-, n_-) - R(t^i_+, n_+) + \delta R(t^i, n)\right] \tag{5.53}$$

A further term is given by the varying part of the potential V_R, completely analogous to the term due to the scalar curvature:

$$\mathcal{L}_{POT} \propto V_- - V_+ + \delta V_R(t^i, n) \tag{5.54}$$

The last, and by far the largest according to the thin-wall approximation, contribution is given by the presence of the membrane: it adds a term proportional to a delta function centered on the membrane, that has radius R:

$$\mathcal{L}_M \propto 2e^{\frac{1}{2}(K+\hat{K})} |q_A t^A - p^A \mathcal{G}_A(t)| \delta(\mathcal{C}) \tag{5.55}$$

Formally speaking the integration in (5.51) is performed on the region of variation of the scalar fields Δr (that is somewhat arbitrary), and as a result the delta function integration would not be well-defined. We can note, however, that if $\mathcal{C} \subset \Delta r$, as in our case where the membrane is contained in the domain wall, no mathematical problem arises. The net result is that the membrane contribution to the total energy of the domain wall is equal to its tension (5.33):

$$T_M = 2e^{\frac{1}{2}(K+\hat{K})} |q_A t^A - p^A \mathcal{G}_A(t)| \tag{5.56}$$

Narayan and Trivedi further show in [25] that, if the thin-wall approximation holds, the tension of the membrane must be much greater than all of the other terms:

$$T_M \gg \int_{\Delta r} (\mathcal{L}_{KIN} + \mathcal{L}_{GRAV} + \mathcal{L}_{POT}) \tag{5.57}$$

Later on we will show explicitly that in our model this condition is indeed satisfied.
As regards the contribution of the inside of the membrane, instead, it can be shown [30], using Einstein's equations and integrating from 0 to the radius R of the membrane, that the coefficient B_{inside} is given by:

$$B_{\text{inside}} = 12\pi^2 \left[\frac{\left(1 - \frac{1}{3}R^2 V_-\right)^{\frac{3}{2}} - 1}{V_-} - \frac{\left(1 - \frac{1}{3}R^2 V_+\right)^{\frac{3}{2}} - 1}{V_+} \right] \tag{5.58}$$

The total B coefficient is then:

$$\begin{aligned} B = B_{\text{inside}} + B_{\text{domain wall}} = \\ 12\pi^2 \left[\frac{\left(1 - \frac{1}{3}R^2 V_-\right)^{\frac{3}{2}} - 1}{V_-} - \frac{\left(1 - \frac{1}{3}R^2 V_+\right)^{\frac{3}{2}} - 1}{V_+} \right] + 2\pi^2 R^3 S_{DW} \end{aligned} \tag{5.59}$$

Taking into account the fact that R is very large we can neglect the 1 addenda, obtaining:

$$B_{\text{inside}}\big|_{\text{large } R} = -\frac{4}{\sqrt{3}}\pi^2 R^3 \left(\sqrt{-V_-} - \sqrt{-V_+}\right), \tag{5.60}$$

where the minus sign in front of the minimum values of the potentials has appeared because we are dealing with AdS vacua, coherently with equations (5.10). It is important to observe that in the above expression the only incognita is the domain-wall tension S_{DW}, that will have to be carefully evaluated in the following. If the appearance of the membrane of radius R is a plausible possibility we would wish to obtain the radius, that has not been fixed yet, from (5.59) by means of the variational method, i.e. to find that a solution of the following equation exists:

$$\frac{\partial B}{\partial R} = 0 \tag{5.61}$$

In order to prove that such an extremum is there it is convenient to evaluate expression (5.58) also for small[8] R. This can be done by expanding (5.58) to second order in R (because the first order contribution vanishes), obtaining:

$$B_{\text{inside}}\big|_{\text{small } R} = -\frac{\pi^2}{2}\epsilon R^4, \tag{5.62}$$

[8]In the end we should anyway prove that the value of R that extremizes (5.61) is large enough to enforce the thin-wall approximation.

with ϵ defined in (5.44). We see then that for small R the coefficient B goes like:

$$B\Big|_{\text{small } R} = 2\pi^2 R^3 S_{DW} - \frac{\pi^2}{2}\epsilon R^4 \simeq 2\pi^2 R^3 S_{DW} \tag{5.63}$$

So that it is proportional to R^3 with a positive coefficient. On the other hand for large R (5.60) the sign of the coefficient of R^3 is not clear, as it receives both a positive and a negative contribution. Not all is lost though, as if we suppose that for large R the coefficient is negative then there must be an extremum of B somewhere in between [25]. Of course this is a sufficient condition, but we will soon show that it is also necessary. As a consequence it should be imposed that:

$$2\pi^2 S_{DW} - \frac{4}{\sqrt{3}}\pi^2\left(\sqrt{-V_-} - \sqrt{-V_+}\right) < 0 \tag{5.64}$$

Resulting in:

$$S_{DW} < \frac{2}{\sqrt{3}}\left(\sqrt{-V_-} - \sqrt{-V_+}\right) \tag{5.65}$$

This bound on the tension of the domain wall, obtained in the thin wall approximation, is extremely relevant and will play an important role in the next discussion. We can now proceed in solving equation (5.61) explicitly. Derivating B (5.59) with respect to R gives:

$$\left(1 - \frac{1}{3}R^2 V_-\right)^{\frac{1}{2}} - \left(1 - \frac{1}{3}R^2 V_+\right)^{\frac{1}{2}} = \frac{R S_{DW}}{2} \tag{5.66}$$

That once solved, using (5.44) gives the radius that extremizes B:

$$R = \frac{1}{\sqrt{\frac{V_+}{3} + \left(\frac{\epsilon}{3S_{DW}} - \frac{S_{DW}}{4}\right)^2}} \tag{5.67}$$

Re-inserting the gravitational coupling constant $k^2 = 8\pi G$ it becomes:

$$R = \frac{1}{\sqrt{\frac{k^2 V_+}{3} + \left(\frac{\epsilon}{3S_{DW}} - \frac{k^2 S_{DW}}{4}\right)^2}} \tag{5.68}$$

In this way it can be observed that in the weak coupling limit $k^2 \to 0$ the radius of the membrane is:

$$R = \frac{3S_{DW}}{\epsilon} \tag{5.69}$$

As a result if ϵ is sufficiently small R becomes extremely large, as expected.
It is immediately seen that the reality of R in (5.67) implies:

$$\sqrt{\frac{-V_+}{3}} < \left(\frac{\epsilon}{3S_{DW}} - \frac{S_{DW}}{4}\right) \tag{5.70}$$

Rearranging the terms we get:

$$\frac{S_{DW}^2}{4} + S_{DW}\sqrt{\frac{-V_+}{3}} - \frac{\epsilon}{3} < 0 \tag{5.71}$$

Inserting the inequality (5.65) in place of S_{DW} it can be proven that the above expression is indeed satisfied if and only if (5.65) holds, showing that it is indeed a necessary condition for an extremum of B to exist. In the next section we will write down and solve the equations of motion for the scalar fields of our model, so as to be able to compute explicitly the total tension of the domain wall.

5.5 The domain-wall tension

The scalar fields involved in the model introduced in section 4.2 are the three t^i and n. Previously, however, we have chosen to restrict to the special case of vanishing axions $b^i = \xi = 0$, showing that possible extrema of the potential remain so even after the inclusion of the axions. In the following, therefore, we will deal only with the imaginary parts of t^i and n, that is:

$$v^i \qquad \frac{1}{\sqrt{2}}e^{-D} \equiv \frac{1}{\sqrt{2}x}, \tag{5.72}$$

where we have made the substitution $e^D \equiv x$ for convenience in the computations.

In order to compute the equations of motion the euclidean version of the total action (5.34) and the metric (5.39) must be used. An extremely useful simplification can be made by assuming that the scalar fields depend exclusively on the radial coordinate r, for symmetry reasons. Anyway it is convenient to keep the usual cartesian coordinates, as in the thin-wall approximation the membrane can almost be considered to be flat. Of course we will have to justify this approximation at the end of the computation. Consequently we pick a coordinate z to be the one transverse to the "almost flat" membrane: as a result all of the physical fields will hence depend exclusively on z, if we choose to search for static solutions.
With these caveats the ensuing equations are obtained (first deriving with respect to the full fields t^i and n and then setting the axions to zero):

$$\begin{aligned} &\left(\frac{\partial K_{jk}}{\partial v^i}\right)\partial_z v^j \partial_z v^k - 2\partial_z\left(K_{ij}\partial_z v^j\right) = -\frac{\partial V_R}{\partial v^i} \\ &\left(\frac{\partial \hat{K}''}{\partial e^{-D}}\right)\partial_z e^{-D}\partial_z e^{-D} - 2\partial_z\left(\hat{K}''\partial_z e^{-D}\right) = -\frac{\partial V_R}{\partial e^{-D}}, \end{aligned} \tag{5.73}$$

where V_R is the axion-free potential (5.5). It is clear that in the situation we are studying there will be equations like (5.73) on both sides of the membrane, with different V_R's, depending on the value of the fluxes on each side. In order to find a solution defined on the whole spacetime it will then be necessary to join the two partial solutions on the membrane surface.

The most striking difference with respect to the usual scalar equations of motion is the presence of terms with derivatives of the Kähler potentials, as a consequence of the fact that they depend on the scalar fields.
In order to simplify the above expression the explicit forms of the Kähler potentials are needed. Using (4.30) and (4.31) we find:

$$\begin{aligned} K_{ij} &= \frac{1}{4}\begin{pmatrix} \frac{1}{(v_1)^2} & 0 & 0 \\ 0 & \frac{1}{(v_2)^2} & 0 \\ 0 & 0 & \frac{1}{(v_3)^2} \end{pmatrix} \\ \hat{K}'' &= \frac{1}{8}e^{2D} \end{aligned} \tag{5.74}$$

A further facilitation can be made observing that the Kähler metric for the v^i is diagonal: because of this no cross-terms between the various v^i appear, and their equations of motion are completely decoupled (except for the potential part, that we will examine in a few lines). As a result we can employ again the following substitution, forgetting the different signs of the v^i (5.8):

$$\nu \equiv v_1 = v_2 = v_3 \tag{5.75}$$

At the end of the computation, anyway, we will have to recall that at least two of the v^i must be negative, according to (5.8). As regards the equation of motion for e^{-D}, instead, we can take advantage of the chain derivation rule:

$$\frac{\partial}{\partial e^{-D}} = \frac{\partial D}{\partial e^{-D}}\frac{\partial}{\partial D} = -e^{D}\frac{\partial}{\partial D} \tag{5.76}$$

Subsequently it is convenient to express the result in terms of $x = e^D$. Proceeding in the described way the following equations appear:

$$\begin{aligned} \frac{3}{2\nu^2}\partial_z^2\nu + \frac{3}{2\nu^3}(\partial_z\nu)^2 &= \frac{\partial V_\nu}{\partial \nu} \\ \partial_z^2 x + \frac{(\partial_z x)^2}{x} &= \frac{x^2}{8}\frac{\partial V_\nu}{\partial x}, \end{aligned} \tag{5.77}$$

where V_ν has been defined in (5.9).

The next objective is to find a solution to these equations and to use the result to compute the tension of the domain wall that separates two minima of the potential V_ν with different values of the fluxes, respectively (e_A, m^A) and (e'_A, m'^A)[9]. The energy difference ϵ between the two minima must be small according to the thin-wall approximation, and as a result the fluxes cannot change arbitrarily. As a matter of fact, looking at the expression of the minimum value of the potential (5.10), p and m^0 are fixed by the tadpole condition and κ is

[9]Recalling that e_0 and m^i have not been considered in our model because of appropriate shifts of the fields (4.35).

a constant: the only fluxes that are allowed to vary are the e_i. In order to ensure that ϵ is small we must therefore impose the condition (5.45).
As we have done previously we simplify the problem by taking all the e_i to be equal in modulus:

$$e \equiv |e_i| \quad \forall i \tag{5.78}$$

Even with these facilitation, however, the equations (5.77) are not analytically solvable: the main problem resides in the complicated expression for the scalar potential V_ν. Nevertheless, the fact that the fluxes vary only slightly from one minimum to the other (5.45) implies that the minimum values of the scalar fields (5.7) do not change much when crossing the membrane. In other words, the false and the true vacuum lie approximately at the same energy level, with similar moduli of the scalar fields that label the minima. This means that it is possible to consider little perturbations of the scalar fields ν and x and to Taylor-expand around their minimum values (ν_+, x_+):

$$\nu \longrightarrow \nu_+ + \delta\nu_+ \qquad x \longrightarrow x_+ + \delta x_+ \tag{5.79}$$

with an analogous expansion for the other minima (ν_-, x_-). In the same fashion we expand the scalar potential. In the false vacuum we have (with clear analogous for the true vacuum):

$$V_\nu = V_+ + \frac{1}{2} M^2_{\nu_+}(\nu - \nu_+)^2 + \frac{1}{2} M^2_{x_+}(x - x_+)^2, \tag{5.80}$$

with the masses defined as:

$$M^2_{\nu_+} \equiv \left.\frac{\partial^2 V_\nu}{\partial \nu^2}\right|_{\nu=\nu_+} \qquad M^2_{x_+} \equiv \left.\frac{\partial^2 V_\nu}{\partial x^2}\right|_{x=x_+}, \tag{5.81}$$

where the reduced potential (5.9) has been employed. They read:

$$M^2_{\nu_+} = \frac{24057\sqrt{\frac{3}{5}}\kappa^3 (m^0)^4 p^4 \sqrt{\left|\frac{e}{k m^0}\right|}}{64000 e^6} \qquad M^2_{x_+} = \frac{33}{20}\sqrt{\frac{3}{5}\left|\frac{\kappa (m^0)^3}{e^3}\right|} \tag{5.82}$$

They are both strictly positive and therefore they automatically satisfy the Breitenlohner-Freedman bound [36], that ensures the perturbative stability of AdS vacua.
Inserting (5.79) and (5.80) into (5.77) and truncating to first order in $\delta\nu$ and δx yields:

$$\begin{aligned} &\frac{3}{2(\nu_+)^2}\partial_z^2 \delta\nu_+ - M^2_{\nu_+}\delta\nu_+ = 0 \\ &\frac{8}{(x_+)^2}\partial_z^2 \delta x_+ - M^2_{x_+}\delta x_+ = 0, \end{aligned} \tag{5.83}$$

that are generally solved by a linear superposition of functions of the form (writing also the solution for the true vacuum, labelled by the minus sign):

$$\begin{aligned} \delta\nu_+ &= a_0^+ + a_1^+ e^{b_1^+ z} + a_2^+ e^{-b_2^+ z} \qquad & \delta\nu_- &= a_0^- + a_1^- e^{b_1^- z} + a_2^- e^{-b_2^- z} \\ \delta x_+ &= c_0^+ + c_1^+ e^{d_1^+ z} + c_2^+ e^{-d_2^+ z} \qquad & \delta x_- &= c_0^- + c_1^- e^{d_1^- z} + c_2^- e^{-d_2^- z}, \end{aligned} \tag{5.84}$$

where all the b's and the d's are taken to be positive.
First of all we note that we can employ the constants $a_0^+, a_0^-, c_0^+, c_0^-$ to automatically implement the fact that the fields have a minimum value (ν_+, x_+) and (ν_-, x_-), depending on whether we consider the false or the true vacuum. Secondly, one of the exponentials in each solution must be discarded for continuity and energetic reasons. Taking a look at the expressions (5.10), in fact, we can note that the true vacuum must have a lower energetic level with respect to the false one, and as a result the modulus of its fluxes $|e'|$ must satisfy:

$$|e'| > |e| \tag{5.85}$$

It is then a consequence of (5.7) that the minimum values of the scalar fields satisfy:

$$\nu_+ < \nu_- \qquad x_+ > x_- \tag{5.86}$$

If we hope to join the two ends of the scalars' evolution on the membrane surface we must hence get rid of the descending exponential for $\delta\nu_+$ as well as of the ascending one for $\delta\nu_-$, and viceversa for δx_+ and δx_-. The solutions therefore are:

$$\begin{aligned} \delta\nu_+ &= \nu_+ + a_1^+ e^{b_1^+ z} \qquad & \delta\nu_- &= \nu_- + a_2^- e^{-b_2^- z} \\ \delta x_+ &= x_+ + c_2^+ e^{-d_2^+ z} \qquad & \delta x_- &= x_- + c_1^- e^{d_1^- z} \end{aligned} \tag{5.87}$$

Continuity of the solutions and of their derivatives on the membrane surface impose (eliminating the superscripts on the coefficients):

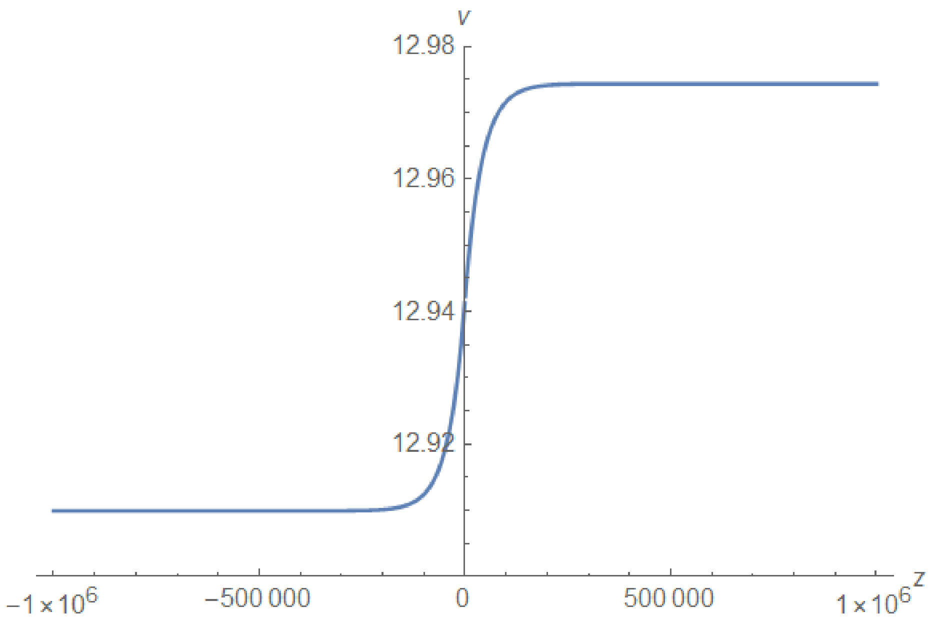
$$\begin{cases} \nu_+ + a_1 = \nu_- + a_2 \\ a_1 b_1 = -a_2 b_2 \end{cases} \qquad \begin{cases} x_+ + c_2 = x_- + c_1 \\ -c_2 d_2 = c_1 d_1 \end{cases} \tag{5.88}$$

Until here the discussion has been completely general (taking into account the due approximations): nevertheless, in order not to obtain an extremely large algebraic expression, it is convenient to proceed with a numeric example. This will allow for a compact evaluation of the tension of the domain wall, notwithstanding that the obtained results do not depend on the specific numbers we will choose. If we were to proceed with a completely implicit strategy, on the other hand, we would not be able to clearly see the significance of the result (that is, if it is close to what we expect to obtain) because of the approximations we have made throughout the analysis.

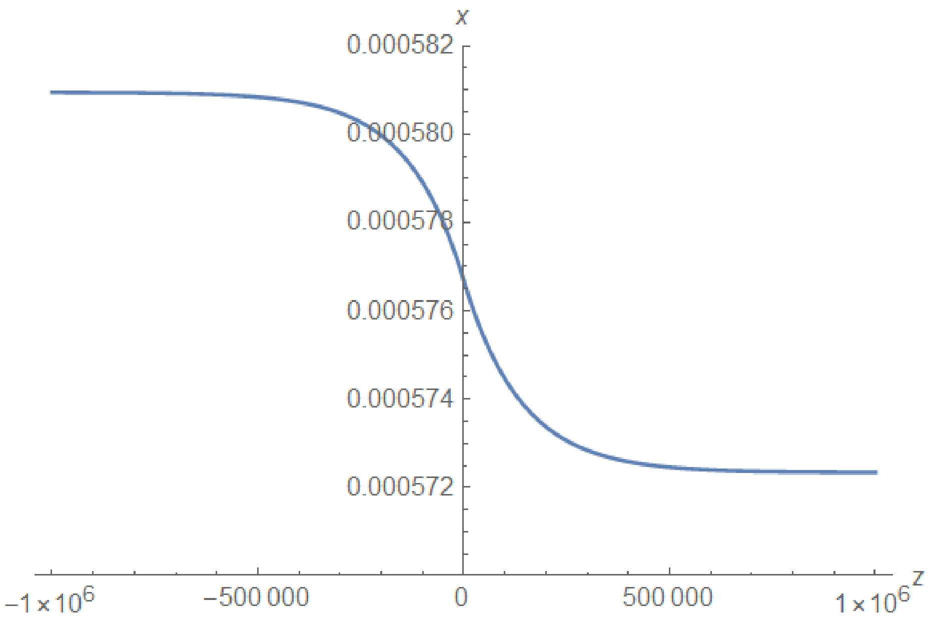
A useful choice, compatible with the quantization conditions (4.41) when using the units $2\pi\sqrt{\alpha'} \equiv 1$, is to set:

$$\begin{aligned} &|e| \longrightarrow 100/\sqrt{2} \qquad |e'| \longrightarrow 101/\sqrt{2} \qquad |\delta e| = 1/\sqrt{2} \\ &\kappa \longrightarrow 1 \qquad p \longrightarrow 1 \qquad m^0 \longrightarrow -1/\sqrt{2} \end{aligned} \tag{5.89}$$

We note that, as we are using only the moduli of the fluxes $|e|$ and $|e'|$ we are leaving open the possibility of having either supersymmetric or non-supersymmetric vacua on the two sides of the membrane. What is not examined, instead, is the transition between a non-susy and a susy vacua, that would imply $|\delta e| \simeq 2|e|$ (because one of the fluxes would have to change sign) and so lies outside of the thin-wall approximation.
These values are convenient to reproduce the behaviour of the functions and to give the possibility to display graphics in order to understand the evolution of the scalar fields. Moreover, they fulfill the constraints we have imposed so far: m^0 and p must be of opposite sign, and $\frac{|\delta e|}{|e|} \ll 1$, that corresponds to (5.44).
Solving the systems (5.88) with these values gives the following trends of the scalar fields ν and x, with the membrane located at $z = 0$, the false vacuum on the left side and the true vacuum on the right side:



Evolution of ν



Evolution of x

The values of the scalar fields on the membrane surface (i.e. $z = 0$) are:

$$\nu|_{z=0} \equiv \nu_0 \simeq 12.942 \qquad x|_{z=0} \equiv x_0 \simeq 5.767 \times 10^{-4} \tag{5.90}$$

Our main concern is to employ these values to compute the tension of the membrane, that should furnish the main contribution to the energy of the domain wall. For such purpose we should use the expression (5.33), that displays the tension as a function of the scalar fields and of the Kähler potentials, that should both be computed at $z = 0$. In the present case, with only the fluxes e_i undergoing a change, the only non-vanishing quantized charges of the membrane are the q_i:

$$q \equiv |q_i| = |\delta e| = 1/\sqrt{2} \tag{5.91}$$

Substituting the expressions of the Kähler potential in our model (4.32) we obtain:

$$T_M = 2\left[\frac{e^{2D}}{\sqrt{8\kappa v_1 v_2 v_3}}|q_i v^i|\right]_{z=0} = 3q\left[\frac{x^2}{\sqrt{2\nu}}\right]_{z=0} = 3q\frac{x_0^2}{\sqrt{2\nu_0}} \simeq 1.3867 \times 10^{-7} \tag{5.92}$$

Another contribution to the domain wall tension is given by the kinetic terms of the scalar fields (5.52):

$$T_{KIN} = \int_{-\infty}^{\infty} \mathcal{L}_{KIN} = \int_{-\infty}^{\infty}\left[\frac{3}{4\nu^2}(\partial_z\nu)^2 + \frac{4}{x^2}(\partial_z x)^2\right] \simeq 2 \times 10^{-9} \tag{5.93}$$

As we can see these quantities are negligible with respect to T_M: taking into account the approximations we have done (Taylor expanding all the terms in (5.73) and solving the differential equations in flat space) it would surely be too optimistic to take these corrections seriously and to include them. Furthermore, inserting the evolutions of the scalars (5.87) into the expressions for V_ν and the curvature R (which is computable taking the trace of the Einstein equations of motion) it can be shown that also their contribution can be neglected:

$$\int_{-\infty}^{\infty} (\mathcal{L}_{GRAV} + \mathcal{L}_{POT}) \simeq 10^{-9} \tag{5.94}$$

We have performed these integrations across all the z axis, even if actually the membrane is a sphere with a finite radius (and so a more accurate calculation should be performed in spherical coordinates). The size of the radius can be estimated using (5.67), valid in the weak coupling limit, using the tension of the domain wall that we have just computed and the definition of ϵ (5.44):

$$R \simeq 1.1 \times 10^5 \tag{5.95}$$

As we can see observing the graphs in the previous pages this is not so large a radius, compared to the region of variation of the scalars: nevertheless, taking into account the extremely small contribution of the scalars' variation to the tension of the domain wall we can accept the validity of the approximation that consists in integrating along all the z axis. In any case the reason why we performed such a calculation is to compare the tension of the domain wall with the upper bound given by the thin-wall approximation (5.65). Using the values (5.89) it becomes:

$$\frac{2}{\sqrt{3}} \left(\sqrt{-V_-} - \sqrt{-V_+} \right) \simeq 1.3864 \times 10^{-7} \tag{5.96}$$

As we can see comparing it to (5.92) the two values are extremely similar, coherently with what was found in [25]. It is difficult, however, to firmly establish if the decays we have considered are allowed or not. On the one hand, in fact, we have computed explicitly the contributions given by the variations of the scalars (5.93) and (5.94), and they are indeed positive: if we were to believe this estimate we would then conclude that this kind of decay is disallowed. On the other hand we must remember that we have made many approximations (included also the bound (5.65), derived in the thin-wall approximation framework): this fact, however, does not change the overall trend of the contributions, that remain positive. The only significant change could concern the values of scalar fields on the membrane tension, that directly influence (5.92). As a matter of fact we know for sure that the tension of the membrane is bounded below and above by the minimum values of the scalar fields, that are reached respectively at $z \to -\infty$ (for the false vacuum) and at $z \to +\infty$ (for the true vacuum). We can then write that:

$$1.3641 \times 10^{-7} < T_M < 1.4090 \times 10^{-7} \tag{5.97}$$

It is possible therefore that using a more refined approximation we could obtain a value at $z = 0$ that results in a membrane tension lower than the bound (5.65), even with the inclusion of the scalar fields' contribution: if that was the case the decay would be slightly allowed.

We must recall, in addition, that until now we have worked in a strictly classical context: the equations of motion (5.73) have been derived without keeping track of any quantum correction. As a result adopting a quantum description and computing one-loop corrections to the tension, via a Coleman-Weinberg-like mechanism, would surely provide more information about the possibility of decay.

Summing up what we have obtained, it must be observed that the decays of the non-supersymmetric AdS vacua we have studied seem to be either forbidden or only marginally allowed (depending on the corrections to the approximation we have made). On the other hand, what we expect from the claim of Ooguri and Vafa [34] is that these vacua should be unstable: in the next chapter we will try to tackle the problem from another perspective, looking at the weak gravity conjecture.

CHAPTER 6

The Weak Gravity Conjecture

In section 5.5 we have directly computed the tension of the membranes T_M that mediate the transitions among two vacua of the action (5.34). As we have emphasized, these membranes are charged under the gauge three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$ that, in the formalism we have exhibited in chapter 3, have substituted the fluxes (e_A, m^A). In the next pages we will try to investigate the relation between the tension and the charge of the membranes, in light of the possible applications to the weak gravity conjecture: the motivation for this study is that the WGC can give hints regarding the stability of the AdS vacua we have considered in the previous chapter. First proposed in [29] this hypothesis was originally motivated by the quest for a criterion to distinguish between the so called "string landscape", that is the set of consistent effective theories of quantum gravity, and the theories that do not fulfill such a criterion, that belong to the "swampland". Intuitively speaking, the WGC states that gravity is the weakest force in our universe [33]. If we consider two identical particles of mass m and charge q (related to a U(1) gauge group) we know from Newton's and Coulomb's laws that they will be subject to the following forces, if put at rest at some distance d:

$$F_{\text{gravitational}} \simeq \frac{Gm^2}{d^2} \simeq \frac{m^2}{M_{pl}^2 d^2} \qquad F_{\text{electric}} \simeq \frac{q^2}{d^2} \tag{6.1}$$

Claiming that gravity is the weakest force implies that:

$$\left(\frac{m}{M_{pl}}\right)^2 \leq q^2 \tag{6.2}$$

up to $\mathcal{O}(1)$ numerical factors.
In natural units $M_{pl} = 1$ this condition becomes:

$$\left(\frac{m}{q}\right)^2 \leq 1 \tag{6.3}$$

In this regard in our universe this inequality is undoubtedly satisfied by particles such as electrons (we do not take into account muons and tauons as they are unstable[1]): if we put two of them next to each other they will repel, as the electric force prevails. The fact that gravity is so weak of course does not prevent the formation of large-scale structure such as stars and planets, provided that they are formed by neutral objects (such as the atoms that compose the Earth) or that they are sufficiently heavy to overcome the electric repulsion (as for neutron stars).

Looking from another perspective, instead, the WGC has repercussions on, and gains plausibility from, black-hole decay dynamics. It is known that (electrically and magnetically) charged and non-rotating black holes in four spacetime dimensions satisfy the Reissner-Nordström metric:

$$\mathrm{d}s^2 = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2}{\Delta}\mathrm{d}r^2 + r^2(\mathrm{d}\Omega)^2, \tag{6.4}$$

where $(\mathrm{d}\Omega)^2$ is the usual spherical measure and Δ equals:

$$\Delta = r^2 - 2Mr + p^2 + q^2, \tag{6.5}$$

with q the electric and p the magnetic charge of the black hole. The horizons of the black hole are found imposing $\Delta = 0$, yielding the solutions (and defining $Q^2 \equiv q^2 + p^2$):

1 $M^2 > Q^2 \longrightarrow$ 2 horizons

2 $M^2 = Q^2 \longrightarrow$ 1 horizon

3 $M^2 < Q^2 \longrightarrow$ no horizons

The third solution, that would produce a naked singularity, is discarded according to the cosmic censorship conjecture. What is of interest for our discussion is the second solution, in which the mass of the black hole precisely equals its charge, which is said to be the *extremal* case. If we consider this kind of black hole it is natural to ask whether it can decay to a state of lower mass. If only particles with masses bigger than their charges existed the black hole would reduce, after a long period of decay, to an object with no mass and still some charge, an eventuality that, apart from the fact that no charged massless particles are known, is considered to be troublesome [35]. It is then plausible to assume that at least one particle state that possesses a mass lower than its charge exists.
This variety of motivations has led to the formulation of a precise statement about the existence of such particles (and p-branes) in quantum gravity theories. A precise statement of the WGC is given in [29][33][34], with the case of charged p-branes being addressed in [45]:

WGC (one kind of charged particles): In a quantum gravity theory in four dimensions there exists at least one stable particle state with mass m and charge q under a U(1) gauge

[1] More precisely, we are considering the "mild" and "strong" versions of the WGC, in which the bound is imposed on stable particles, see [29].

group that satisfies:

$$\left(\frac{m}{q}\right)^2 \leq 1 \tag{6.6}$$

WGC (one kind of charged p-branes): In a quantum gravity theory in four dimensions there exists at least one stable p-brane with tension T and charge Q related to a $(p+1)$-form gauge field that satisfies:

$$\left(\frac{T}{Q}\right)^2 \leq \mathcal{O}(1), \tag{6.7}$$

where $\mathcal{O}(1)$ is a fixed number.
In particular the bounds are saturated if the theory and the states are supersymmetric. In the case of the charged p-brane the bound is not explicitly exhibited, and we will extensively elaborate on it in the following.
In addition, we must emphasize that the statements we have given above are often referred to as the "mild WGC": a stricter version, knows as the "strong WGC", requires that the bound is satisfied by the *lightest* state of the theory.

We can now state explicitly the claim proposed by Ooguri and Vafa in [34] that motivates our discussion:

In a given theory the non-supersymmetric AdS vacuum states do not saturate the WGC bounds (6.6)(6.7) and as a consequence they are unstable.

We see then that this statement furnishes us with a direct link between the WGC and the stability properties of the vacua. We have now an alternative way to assess the possibility of decay of a given AdS vacuum: if it does not saturate the WGC bound (in our case the one involving membranes) it should be unstable. In the next sections we will try to concretely compute the WGC statement for membranes and to examine the stability of the Narayan and Trivedi model's AdS vacua we have studied in the previous chapter.

6.1 The WGC and membranes

In the model by Narayan and Trivedi that we have discussed in the previous pages we have considered 2-branes (that is, membranes) charged under two sets of 3-form gauge fields, $\tilde{A}_{(3)A}$ and $A^A_{(3)}$. We have also computed their tension explicitly, using (5.33). It is hence natural to ask: what is the relation between the tension and charge of these membranes? Do they satisfy the bound (6.7)? More specifically, we expect that membranes that interpolate between two supersymmetric vacua exactly saturate the bound, whereas if one of the vacua is non-supersymmetric we should obtain a strict inequality. This property is related to the non-perturbative stability of the vacuum state under consideration: if the the WGC bound is not saturated, we expect that the state is unstable. Furthermore, the action (5.34) we have considered previously is supersymmetric by construction, and so we expect to be able to verify the validity of the conjecture for the membranes we have introduced.

As we have recalled the tension of a given membrane can be easily computed from the expression (5.33). What of its charge instead? Of course we have at our disposal the quantized charges (q_A, p^A) that completely specify the properties of the membranes, and that define the change in flux from the false to the true vacuum. The starting point to compute the elementary charges related to $\tilde{A}_{(3)A}$ and $A^A_{(3)}$ should therefore be to take into account the coefficients of the "Yang-Mills" terms in (5.34). The part of the lagrangian that depends on the field strengths is expressed in (4.80), and reads (putting the axions and p to zero and neglecting the boundary term, that will be of no use in our discussion):

$$\begin{aligned} |e|^{-1}\mathcal{L}_{\text{3-forms},\hat{W}} = & \frac{e^{-(K+\hat{K})}}{16}\left({}^*\mathcal{F}^0_{(4)}\right)^2 + e^{K-\hat{K}}K^{ij}{}^*\tilde{\mathcal{F}}_{i(4)}{}^*\tilde{\mathcal{F}}_{j(4)}+ \\ & + \frac{e^{-(K+\hat{K})}}{4}K_{ij}{}^*\mathcal{F}^i_{(4)}{}^*\mathcal{F}^j_{(4)} + 4e^{K-\hat{K}}\left({}^*\tilde{\mathcal{F}}_{0(4)}\right)^2 \end{aligned} \tag{6.8}$$

The coefficients in the YM-like terms are then:

$$\begin{aligned} &\frac{e^{-(K+\hat{K})}}{16} \longleftrightarrow {}^*\mathcal{F}^0_{(4)} \qquad && e^{K-\hat{K}}K^{ij} \longleftrightarrow {}^*\tilde{\mathcal{F}}_{i(4)} \\ &\frac{e^{-(K+\hat{K})}}{4}K_{ij} \longleftrightarrow {}^*\mathcal{F}^i_{(4)} \qquad && 4e^{K-\hat{K}} \longleftrightarrow {}^*\tilde{\mathcal{F}}_{0(4)} \end{aligned} \tag{6.9}$$

As we can see some of the coefficients depend on K_{ij}, that generally speaking is not diagonal: as a consequence, before computing the elementary charge, we should diagonalize K_{ij} in order to obtain a single coefficient for each field strength. In this way, however, the diagonalized field strengths will be a combination of the starting ones, introducing additional subtleties. Having written the coefficients (6.9), we should compare them with the analogous of the YM coefficient for four forms:

$$\left.\frac{1}{2(d!)g^2}\right|_{d=4} = \frac{1}{2(4!)g^2}, \tag{6.10}$$

where the term on the left-hand side is the general coefficient for a d-form and g is the elementary charge (different for each field strength) that we ultimately want to compute. We recall that in our model the Kähler potentials and the Kähler metric are:

$$\begin{aligned} &K = -\ln(8\kappa v_1 v_2 v_3) \qquad && \hat{K} = 4D \\ &K_{ij} = \frac{1}{4}\begin{pmatrix} \frac{1}{(v_1)^2} & 0 & 0 \\ 0 & \frac{1}{(v_2)^2} & 0 \\ 0 & 0 & \frac{1}{(v_3)^2} \end{pmatrix} \qquad && K^{ij} = 4\begin{pmatrix} (v_1)^2 & 0 & 0 \\ 0 & (v_2)^2 & 0 \\ 0 & 0 & (v_3)^2 \end{pmatrix} \end{aligned} \tag{6.11}$$

We observe that in this particular case the metric K_{ij} is diagonal, meaning that the ${}^*\tilde{\mathcal{F}}_{i(4)}$ (as well as the ${}^*\mathcal{F}^i_{(4)}$) are decoupled if they have different indices, so that no diagonalization is required. This means that the expression (6.8) should be compared to:

$$
\begin{aligned}
|e|^{-1}\mathcal{L}_{\text{3-forms},\hat{W}} = & \frac{1}{2(4!)(g_0)^2}\left(\mathcal{F}^0_{(4)}\right)^2 + \sum_i \frac{1}{2(4!)(g^i)^2}\tilde{\mathcal{F}}_{i(4)}\tilde{\mathcal{F}}_{i(4)} + \\
& + \sum_i \frac{1}{2(4!)(g_i)^2}\mathcal{F}^i_{(4)}\mathcal{F}^i_{(4)} + \frac{1}{2(4!)(g^0)^2}\left(\tilde{\mathcal{F}}_{0(4)}\right)^2
\end{aligned}
\tag{6.12}
$$

We have purposefully removed the Hodge stars because we want to obtain the canonical terms for the field strengths. Moreover it is important to observe that, having set the axions b^i to zero, we can be sure that each of the $\tilde{\mathcal{F}}_{A(4)}$ and $\mathcal{F}^A_{(4)}$ corresponds to only one non-italics field strength, according to (4.63). Recalling the definition of the Hodge star from appendix A (6.65), we see that, for a fixed index i:

$$
*\mathcal{F}^i_{(4)} * \mathcal{F}^i_{(4)} = \frac{1}{(4!)^2}(\mathcal{F}^i_{(4)})^{mnrs}\epsilon_{mnrs}(\mathcal{F}^i_{(4)})^{tuvz}\epsilon_{tuvz} \tag{6.13}
$$

Using the properties of the Levi-Civita symbol it is easy to show that:

$$
(\mathcal{F}^i_{(4)})^{mnrs}\epsilon_{mnrs}(\mathcal{F}^i_{(4)})^{tuvz}\epsilon_{tuvz} = (4!)(\mathcal{F}^i_{(4)})^{mnrs}(\mathcal{F}^i_{(4)})_{mnrs} \tag{6.14}
$$

Using (6.14) and comparing (6.8) to (6.12) we obtain the following relations:

$$
\begin{aligned}
\frac{1}{4!}\frac{e^{-(K+\hat{K})}}{16} &\stackrel{!}{=} \frac{1}{2(4!)(g_0)^2} \\
\frac{1}{4!}e^{K-\hat{K}}K^{ii} &\stackrel{!}{=} \frac{1}{2(4!)(g^i)^2} \qquad \text{(no sum over } i\text{)} \\
\frac{1}{4!}\frac{e^{-(K+\hat{K})}}{4}K_{ii} &\stackrel{!}{=} \frac{1}{2(4!)(g_i)^2} \qquad \text{(no sum over } i\text{)} \\
\frac{1}{4!}4e^{K-\hat{K}} &\stackrel{!}{=} \frac{1}{2(4!)(g^0)^2}
\end{aligned}
\tag{6.15}
$$

Substituting the Kähler potentials and the metric (6.11) we finally obtain the elementary charges:

$$
\begin{aligned}
g_0 &= \frac{e^{2D}}{\sqrt{\kappa v_1 v_2 v_3}} \\
g^i &= e^{2D}\sqrt{\frac{\kappa v_j v_k}{v_i}} \qquad \text{where } i \neq j \neq k \\
g_i &= e^{2D}\sqrt{\frac{v_i}{\kappa v_j v_k}} \qquad \text{where } i \neq j \neq k \\
g^0 &= e^{2D}\sqrt{\kappa v_1 v_2 v_3}
\end{aligned}
\tag{6.16}
$$

We should compare these elementary charges with the membrane tension, that we recall being computable with the formula:

$$T_M = 2e^{\frac{K+\hat{K}}{2}}|q_A v^A - p^A \mathcal{G}_A(v)| = 2e^{\frac{K+\hat{K}}{2}}|\Delta W^K|, \tag{6.17}$$

where W^K is the Kähler superpotential for our model (4.34) that includes non-vanishing e_0 and m^i fluxes.
We can hence compute the tension corresponding to a membrane that possesses a unique non-vanishing quantized charge:

$$\begin{aligned}
&q_0 \neq 0 \longrightarrow T_{q_0} = q_0 \frac{e^{2D}}{\sqrt{2\kappa v_1 v_2 v_3}} \\
&p^i \neq 0 \longrightarrow T_{p^i} = p^i e^{2D}\sqrt{\frac{\kappa v_j v_k}{2v_i}} \quad \text{where } i \neq j \neq k \\
&q_i \neq 0 \longrightarrow T_{q_i} = q_i e^{2D}\sqrt{\frac{v_i}{2\kappa v_j v_k}} \quad \text{where } i \neq j \neq k \\
&p^0 \neq 0 \longrightarrow T_{p^0} = p^0 e^{2D}\sqrt{\frac{\kappa v_1 v_2 v_3}{2}}
\end{aligned} \tag{6.18}$$

Putting (6.16) and (6.18) side by side we see that they share a striking structural similarity, even though with different coefficients. In order to express them in the same units we should multiply the elementary charges (g_0, g^i, g_i, g^0) with the corresponding quantized charges (q_0, p^i, q_i, p^0). In other words, the (q_0, p^i, q_i, p^0) express "how much" elementary charge of a given kind (g_0, g^i, g_i, g^0) the membrane has[2]. In this way we should write:

$$\begin{aligned}
&Q_{q_0} = q_0 \cdot g_0 = q_0 \frac{e^{2D}}{\sqrt{\kappa v_1 v_2 v_3}} \\
&Q_{p^i} = p^i \cdot g^i = p^i e^{2D}\sqrt{\frac{\kappa v_j v_k}{v_i}} \quad \text{where } i \neq j \neq k \\
&Q_{q_i} = q_i \cdot g_i = q_i e^{2D}\sqrt{\frac{v_i}{\kappa v_j v_k}} \quad \text{where } i \neq j \neq k \\
&Q_{p^0} = p^0 \cdot g^0 = p^0 e^{2D}\sqrt{\kappa v_1 v_2 v_3}
\end{aligned} \tag{6.19}$$

The Q's hence are the true *physical charges* of the membranes.
Now, however, we face a problem: how should we compare the physical charges (6.19) and the tensions (6.18)? Naively, we could think of comparing them one by one (for example looking at Q_{q_0} and T_{q_0}), or to sum all of the charges and compare the result with the sum of the tensions. This route, unfortunately, does not yield the correct results, as the charges we have computed are basis-dependent, and therefore such a straightforward evaluation is not

[2]We make a brief recap of the names we have used: (q_A, p^A) are the *quantized charges*, they express how many units of the *elementary charges* g (i.e. the couplings) the membrane possesses; the Q are the *physical charges*, given by the multiplication of the (q_A, p^A) and the g.

allowed.
The correct path to follow when dealing with multiple charges, instead, has been outlined in papers such as [46] in the case of many charged particles, and in [45] with many charged p-branes.

6.2 The WGC with multiple charges

Before dealing with the p-branes' case that is of more interest for our discussion, we show how the generalized WGC works in the simpler case of charged particles, following [46].
Let us consider a set of N $U(1)$ gauge groups, each characterized by a charge e_a (with $a = 1, ..., N$). In general, each particle or black hole state corresponds to a vector in the space spanned by the charges e_a, in which each component of the vector expresses how much charge of a given kind is possessed by the state. Say that we have a black hole with total mass M and charge vector $\vec{Q}$; we define its charge to mass ratio as:

$$\vec{Z} = \frac{\vec{Q}}{M} \tag{6.20}$$

We wish to understand whether it can decay into a combination of particles, that in general will be composed of n_i particles for each particle species of mass m_i labeled by i. Besides, a charge vector $\vec{q_i}$ in the space spanned by the $U(1)$ groups is associated to each of the particles, that as a result have the following charge to mass ratio:

$$\vec{z_i} = \frac{\vec{q_i}}{m_i} \tag{6.21}$$

Charge and energy conservation, as well as the WGC requirement that no charge remains when the black hole has completely evaporated, entail that:

$$\vec{Q} = \sum\nolimits_i n_i \vec{q_i} \qquad M \geq \sum\nolimits_i n_i m_i \tag{6.22}$$

We further define the ratio between the mass of all the particles of species i and the mass of the black hole:

$$\sigma_i = \frac{n_i m_i}{M} \tag{6.23}$$

As a consequence of their definition the σ_i satisfy:

$$\sum\nolimits_i \sigma_i \leq 1 \tag{6.24}$$

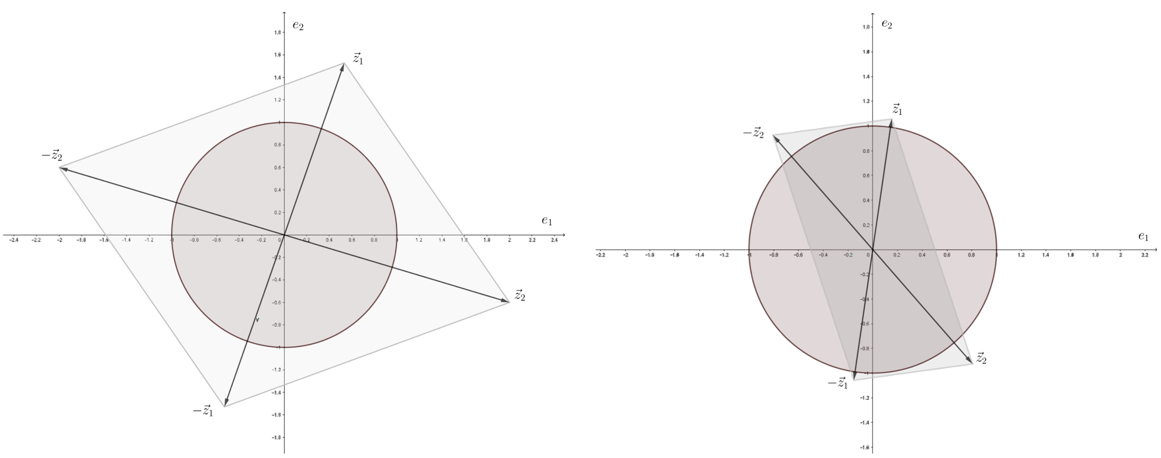
With these conventions, the total charge to mass ratio vector $\vec{Z}_{\text{particles}}$ of the particles is:

$$\vec{Z}_{\text{particles}} = \sum\nolimits_i \sigma_i \vec{z_i} \tag{6.25}$$

As a result we can see that $\vec{Z}_{\text{particles}}$ is a weighted average (with weights σ_i) of the single charge vectors $\vec{z}_i$: if we consider an extremal black hole (namely, a black hole with charge equal to its mass $|\vec{Q}| = M$), that is with $|\vec{Z}| = 1$, its decay is allowed only if $|\vec{Z}_{\text{particles}}| > 1$ for some choice of the weights σ_i. Let us suppose, in fact, that there exists a direction in charge space for which $|\vec{Z}_{\text{particles}}| < 1$ (for every possible choice of the σ_i): if we consider an extremal black hole with charge to mass vector $\vec{Z}$ along the same direction of $\vec{Z}_{\text{particles}}$ we see that it cannot decay into any combination of particles, if charge conservation holds and the WGC is true. From a geometric perspective, this means that the portion of space spanned by $\vec{Z}_{\text{particles}}$ varying the weights σ_i (called the *convex hull*) must comprise the unitary ball: the black hole states on the border of the ball are the extremal solutions, whereas the ones inside have a mass that is greater than their charge. The generalized WGC for particles is then:

Generalized WGC (particles): The convex hull spanned by the vector $\vec{z}_i$ must contain the unit ball.

It is useful to exhibit a graphic interpretation of this result in the case of two $U(1)$ groups with charges e_i $(i = 1, 2)$ and, for example, two particle species associated to vectors $\vec{z}_i$, as we can see in the following figure:



(a) *Convex hull that satisfies the WGC* (b) *Convex hull that does not satisfy the WGC*

It is then evident for geometrical reasons that, if the number of charges grows, the charge to mass ratio vectors $\vec{z}_i$ must become larger in order to satisfy the WGC, that is fulfilled when the convex hull includes the unit ball. Assuming that all of the $\vec{z}_i$ have the same modulus z and that we have as many particles species as $U(1)$ groups, it can be shown that they can contain, at most, a ball of radius $z/\sqrt{N}$, where N is the number of charges (the same as the particle species in this case). If we hope to comprise the unit ball, therefore, z must grow with N.

Inspired by this approach, we could devise a similar bound for 2-branes: taking into account the tensions T_i and the charge vectors $\vec{Q}_i$ (defined in some charge space) of a family of 2-branes labeled by i, we can define their charge to tension ratios exactly as for the particles:

$$\vec{z}_i = \frac{\vec{Q}_i}{T_i} \tag{6.26}$$

We could then suppose that these vectors must satisfy a criterion similar to their particle counterparts:

Generalized WGC (2-branes): The convex hull spanned by the charge-to-tension vectors $\vec{z}_i$ must contain a ball of radius $\mathcal{O}(1)$.

We have left, once again, unspecified the $\mathcal{O}(1)$ number that characterizes the conjecture, as we will deduce it directly in the following section. As a matter of fact there have been some attempts to generalize the weak gravity conjecture to p-branes (see e.g. [45], [49] and [50]) but there still does not seem to be a universal agreement about which exact number should bound the charge-to mass ratio. We will talk more extensively about this discussion in the conclusions.

6.3 The Narayan and Trivedi model

Furnished with a generalization of the WGC for the case of multiple charges, we can proceed in trying to compute the charge-to-tension vectors for the 2-branes that we have studied in the Narayan and Trivedi model, whose complete action is (5.34). We have computed the tensions of the membranes that display only one non-vanishing quantized charge (one among q_A, p^A), obtaining (6.18). The corresponding physical charges, deduced from the Yang-Mills-like terms in (6.8), are expressed in equation (6.19). As we have not diagonalized the Kähler metric, we can use the fact that the vectors that correspond to the single physical charges span orthogonal directions in the charge space, i.e. they do not mix with each other.
As a result, the charge-to-tension vectors for the Narayan and Trivedi model are:

$$\begin{aligned}
\vec{z}_{q_0} &= \sqrt{2} \qquad & \vec{z}_{p^0} &= \sqrt{2} \\
\vec{z}_{q_1} &= \sqrt{2} \qquad & \vec{z}_{p^1} &= \sqrt{2} \\
\vec{z}_{q_2} &= \sqrt{2} \qquad & \vec{z}_{p^2} &= \sqrt{2} \\
\vec{z}_{q_3} &= \sqrt{2} \qquad & \vec{z}_{p^3} &= \sqrt{2}
\end{aligned} \tag{6.27}$$

where in the subscript we have indicated the quantized charge that relates to the vector. We have shown on purpose all of the vectors, expanding the index $i = 1, 2, 3$, in order to emphasize the fact that there are a total of 8.

We should now compute which is the radius of the largest ball that can be contained in the convex hull spanned by the vectors (6.41), in order to assess the validity of the weak gravity conjecture. Being all equal in modulus $z = \sqrt{2}$ and orthogonal in direction, the largest ball has radius:

$$r = \frac{z}{\sqrt{N}} \overset{N=8}{=} \frac{1}{2} \tag{6.28}$$

We see then two remarkable facts: first of all, the vectors (6.41) *do not* depend on the moduli v^i and n, nor on the spacetime coordinates, they are pure numbers; secondly, the radius (6.28) is indeed an $\mathcal{O}(1)$ number, as expected from the WGC.

Unfortunately there is a problem: if we consider a membrane that mediates the decay from a non-supersymmetric vacuum to a susy one we would expect that the bound (6.7) was not saturated, and yet the expressions (6.16) and (6.18) do not depend on the susy/non-susy status of the vacuum. This is because the only difference between them is the sign of the fluxes e_i. A possible solution could hence consist in going beyond the classical level and computing quantum corrections to the charges, that we expect to be vanishing for susy vacua and negative for non-susy ones.

In any case the features we have found could at first seem rather suspect: therefore, before commenting on this result, we take into account two similar models, so as to establish whether the radius we have obtained is a mere coincidence due to the specific form of the Narayan and Trivedi model or it has some more profound significance.

6.4 Two alternative models

In this section we consider two models of a specific class, displayed in [21], that describes a set of chiral superfields $\mathcal{Z}^A$ with Kähler potential:

$$K = -\log[i\bar{f}^A\mathcal{G}_A - if^A\bar{\mathcal{G}}_A], \tag{6.29}$$

where $\mathcal{G}$ is some prepotential that must satisfy the homogeneity conditions (3.97) and f^A are the gauge fixing functions (3.116). Our objective is to compute the charges and the tensions of the 2-branes that mediate the transitions among different vacua of the potentials of the models we are going to study, in order to compute the charge-to-tension vectors that characterize the WGC.

First model

As a first specific case we choose the following function as the prepotential (writing also its derivatives):

$$\mathcal{G} = \frac{\mathcal{Z}_1^3}{\mathcal{Z}_0} \qquad \mathcal{G}_A = \left(-\frac{\mathcal{Z}_1^3}{\mathcal{Z}_0^2}, 3\frac{\mathcal{Z}_1^2}{\mathcal{Z}_0}\right) \qquad \mathcal{G}_{AB} = \begin{bmatrix} 2\frac{\mathcal{Z}_1^3}{\mathcal{Z}_0^3} & -3\frac{\mathcal{Z}_1^2}{\mathcal{Z}_0^2} \\ -3\frac{\mathcal{Z}_1^2}{\mathcal{Z}_0^2} & 6\frac{\mathcal{Z}_1}{\mathcal{Z}_0} \end{bmatrix} \tag{6.30}$$

Applying the usual gauge fixing (3.117) we obtain:

$$\begin{cases} \mathcal{Z}_0 = 1 \\ \mathcal{Z}_1 = \mathcal{Z}_0 \Phi \end{cases} \qquad \begin{cases} f^0 = 1 \\ f^1 = \Phi \end{cases} \tag{6.31}$$

The superpotential of the theory reads:

$$W = e_A \mathcal{Z}^A - m^A \mathcal{G}_A = e_0 + e_1 \Phi - 3m^1 \Phi^2 + m^0 \Phi^3, \tag{6.32}$$

whereas the Kähler potential, using (6.29) results (defining $v \equiv \text{Im}\Phi$):

$$\begin{aligned} K &= -\log[i(-\Phi^3) + i\Phi^*(3\Phi^2) - i(-\Phi^3)^* - i(\Phi)(3\Phi^*)^2] = \\ &= -\log[-i(\Phi - \Phi^*)^3] = -3\log[i\Phi - i\Phi^*] = -3\log[-2v] \end{aligned} \tag{6.33}$$

Consistency requires that $v < 0$. The action of the model describes two chiral superfields $\mathcal{Z}^A$ (the Weyl compensator Z and the physical field Φ), with $A = 0, 1$, and reads:

$$\mathcal{S}_1 = -3 \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, e^{-\frac{1}{3}K(\mathcal{Z},\bar{\mathcal{Z}}) + \hat{K}(T,\bar{T})} + \left(\int \mathrm{d}^4x \mathrm{d}^2\theta\, 2\mathcal{E}\, W(\mathcal{Z}) + \text{h.c.} \right), \tag{6.34}$$

where we have included a possible dependence on the Kähler potential $\hat{K}$ of some spectator superfields T, that *do not* contribute to the superpotential. As we have done with the Narayan and Trivedi model we wish to eliminate the fluxes (e_A, m^A) from the superpotential $W(\mathcal{Z})$, substituting them with the field strengths of two sets of gauge three-forms $\tilde{A}_{(3)A}$ and $A^A_{(3)}$.

The procedure is exactly the same as the one carried out in chapter 3 and 4: at the end of the computation, assuming that $\text{Re}\Phi = 0$ (as we did for the Narayan and Trivedi model setting the axions b^i to 0) we obtain the kinetic terms for the gauge three-forms:

$$e^{\hat{K}} \mathcal{L}_{3-\text{forms}} = \frac{v^3}{2} (^*F^0)^2 + \frac{3v}{2} (^*F^1)^2 + \frac{1}{6v} (^*\tilde{F}_1)^2 + \frac{1}{2v^3} (^*\tilde{F}_0)^2 \tag{6.35}$$

The main difference from the Narayan and Trivedi model resides in the fact that we have not considered a spectator superpotential $\hat{W}(T)$, that anyway does not influence the physical charges of the gauge three-forms.
Furthermore, the membrane contribution to the total action is the same as the one in the Narayan and Trivedi model, being dictated by supersymmetry:

$$S_{\text{membrane}} = -2 \int_{\mathcal{C}} \mathrm{d}^3\xi \sqrt{-h}\, e^{\frac{1}{2}(K+\hat{K})} |q_A t^A - p^A \mathcal{G}_A(t)| + q_A \int_{\mathcal{C}} \mathrm{d}^3\xi\, A^A_{(3)} - p^A \int_{\mathcal{C}} \mathrm{d}^3\xi\, \tilde{A}_{(3)A}, \tag{6.36}$$

where t^A are the scalar components of the double three-form multiplets, that in our case read, recalling the gauge fixing (6.31) and the fact that we have set $\text{Re}\Phi = 0$ and $\text{Im}\Phi = v$:

$$t^0 = 1 \qquad t^1 = v \tag{6.37}$$

The four quantized charges in this case are (q_0, q_1, p^0, p^1), in correspondence respectively with (e_0, e_1, m^0, m^1). Supposing that only one of the (q_0, q_1, p^0, p^1) is non-vanishing we obtain the tension of the related membranes employing (6.17):

$$\begin{aligned} q_0 \neq 0 &\longrightarrow T_{q_0} = q_0 e^{\frac{\hat{K}}{2}} \frac{2}{(-2v)^{\frac{3}{2}}} \qquad & p_0 \neq 0 &\longrightarrow T_{p^0} = p_0 e^{\frac{\hat{K}}{2}} \frac{1}{\sqrt{2}} (-v)^{\frac{3}{2}} \\ q_1 \neq 0 &\longrightarrow T_{q_1} = q_1 e^{\frac{\hat{K}}{2}} \frac{1}{(-2v)^{\frac{1}{2}}} \qquad & p^1 \neq 0 &\longrightarrow T_{p^1} = p^1 e^{\frac{\hat{K}}{2}} \frac{3}{\sqrt{2}} (-v)^{\frac{1}{2}} \end{aligned} \tag{6.38}$$

As regards the elementary charges, we should impose the same conditions that we have used in (6.15), that is comparing the coefficients that appear in (6.35) with what we expect from a Yang-Mills-like action:

$$\begin{aligned} \frac{1}{4!} e^{-\hat{K}} \frac{v^3}{2} &\overset{!}{=} \frac{1}{2(4!)(g_0)^2} \qquad & \frac{1}{4!} e^{-\hat{K}} \frac{3v}{2} &\overset{!}{=} \frac{1}{2(4!)(g_1)^2} \\ \frac{1}{4!} e^{-\hat{K}} \frac{1}{2v^3} &\overset{!}{=} \frac{1}{2(4!)(g^0)^2} \qquad & \frac{1}{4!} e^{-\hat{K}} \frac{1}{6v} &\overset{!}{=} \frac{1}{2(4!)(g^1)^2} \end{aligned} \tag{6.39}$$

Inverting the relations and multiplying by the quantized charges (q_0, q_1, p^0, p^1) we obtain the following physical charges:

$$\begin{aligned} Q_{q_0} &= q_0 e^{\frac{\hat{K}}{2}} \frac{1}{(-v)^{\frac{3}{2}}} \qquad & Q_{p^0} &= p^0 e^{\frac{\hat{K}}{2}} (-v)^{\frac{3}{2}} \\ Q_{q_1} &= q_1 e^{\frac{\hat{K}}{2}} \frac{1}{\sqrt{3}(-v)^{\frac{1}{2}}} \qquad & Q_{p^1} &= p^1 e^{\frac{\hat{K}}{2}} \sqrt{3} (-v)^{\frac{1}{2}} \end{aligned} \tag{6.40}$$

Taking the quotient of the physical charges (6.40) to their respective tensions (6.38) we obtain the charge-to-tension vectors (that, as expected, do not depend on the generic spectator Kähler potential $\hat{K}$):

$$\begin{aligned} \vec{z}_{q_0} &= \sqrt{2} \equiv \vec{z}_0 \qquad & \vec{z}_{p^0} &= \sqrt{2} \equiv \vec{z}_1 \\ \vec{z}_{q_1} &= \sqrt{\frac{2}{3}} \equiv \vec{z}_2 \qquad & \vec{z}_{p^1} &= \sqrt{\frac{2}{3}} \equiv \vec{z}_3 \end{aligned} \tag{6.41}$$

As these vectors are mutually orthogonal (because we have not mixed the charges of the membranes q_0, q_1, p^0, p^1) the border of the convex hull they span is given by $\sum_{i=0}^{3} \sigma_i z_i$, with $\sum_{i=0}^{3} \sigma_i = 1$.

In order to find the largest radius of a ball contained in the convex hull we must minimize the function $f = \sum_{i=0}^{3} (\sigma_i z_i)^2$ (that is, Pitagoras' theorem) with the constraint

$$F \equiv \sum_{i=0}^{3} \sigma_i - 1 = 0 \tag{6.42}$$

What we obtain is:

$$\begin{pmatrix} \nabla f \\ \nabla F \end{pmatrix} = \begin{pmatrix} 2\sigma_0 z_0 & 2\sigma_1 z_1 & 2\sigma_2 z_2 & 2\sigma_3 z_3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{6.43}$$

Imposing that the determinant of all the minors vanishes and employing the explicit expressions (6.41) of the $\vec{z}_i$ we get:

$$\sigma_0 = \sigma_1 \qquad \sigma_2 = \sigma_3 \qquad \sigma_0 = \frac{\sigma_2}{3} \tag{6.44}$$

As a result, substituting these relations into f we obtain the following maximum radius, which is equal to (6.28):

$$r = \frac{1}{2} \tag{6.45}$$

Second model

Another relatively simple model we can build starts with the following prepotential:

$$\mathcal{G} = \frac{\mathcal{Z}_1 \mathcal{Z}_2^2}{\mathcal{Z}_0} \qquad \mathcal{G}_A = \left(-\frac{\mathcal{Z}_1 \mathcal{Z}_2^2}{\mathcal{Z}_0^2}, \frac{\mathcal{Z}_2^2}{\mathcal{Z}_0}, \frac{2\mathcal{Z}_1 \mathcal{Z}_2}{\mathcal{Z}_0} \right) \tag{6.46}$$

We gauge fix $\mathcal{Z}_0$ to 1, obtaining:

$$\begin{cases} \mathcal{Z}_0 = 1 \\ \mathcal{Z}_1 = \mathcal{Z}_0 \Phi \\ \mathcal{Z}_2 = \mathcal{Z}_0 \Psi \end{cases} \qquad \begin{cases} f^0 = 1 \\ f^1 = \Phi \\ f^2 = \Psi \end{cases} \tag{6.47}$$

The superpotential is:

$$W = e_A \mathcal{Z}^A - m^A \mathcal{G}_A = e_0 + e_1 \Phi + e_2 \Psi + m^0 \Phi \Psi^2 - m^1 \Psi^2 - 2m^2 \Phi \Psi \tag{6.48}$$

The corresponding Kähler potential reads:

$$K = -\log[-8\mathrm{Im}\Phi (\mathrm{Im}\Psi)^2] \equiv -\log[-8vy^2], \tag{6.49}$$

where we have defined $\mathrm{Im}\Phi \equiv v$ and $\mathrm{Im}\Psi \equiv y$. The action of this model is identical to (6.34), and the derivation of the kinetic terms for the gauge three-forms follows the exact same steps as the previous case, considering spectators superfields T with a Kähler potential $\hat{K}$ and setting the real parts of Φ and Ψ to zero.
In the end, the lagrangian for the gauge three-forms that we obtain is:

$$e^{\hat{K}} \mathcal{L}_{3-\text{forms}} = \frac{vy^2}{2}(*F^0)^2 + \frac{y^2}{2v}(*F^1)^2 + v(*F^2)^2 + \frac{1}{2vy^2}(*\tilde{F}_0)^2 + \frac{v}{2y^2}(*\tilde{F}_1)^2 + \frac{1}{4v}(*\tilde{F}_2)^2 \tag{6.50}$$

Computing the tensions by putting to zero all the quantized charges except for one we obtain:

$$
\begin{aligned}
&q_0 \neq 0 \longrightarrow T_{q_0} = q_0 e^{\frac{\hat{K}}{2}} \frac{1}{y\sqrt{-2v}} && p^0 \neq 0 \longrightarrow T_{p^0} = p^0 e^{\frac{\hat{K}}{2}} \frac{y\sqrt{-v}}{\sqrt{2}} \\
&q_1 \neq 0 \longrightarrow T_{q_1} = q_1 e^{\frac{\hat{K}}{2}} \frac{\sqrt{-v}}{\sqrt{2y}} && p^1 \neq 0 \longrightarrow T_{p^1} = p^1 e^{\frac{\hat{K}}{2}} \frac{y}{\sqrt{-2v}} \\
&q_2 \neq 0 \longrightarrow T_{q_2} = q_2 e^{\frac{\hat{K}}{2}} \frac{1}{\sqrt{-2v}} && p^1 \neq 0 \longrightarrow T_{p^2} = p^2 e^{\frac{\hat{K}}{2}} \sqrt{-2v}
\end{aligned}
\tag{6.51}
$$

These should be compared with the physical charges, computed with the usual method:

$$
\begin{aligned}
&Q_{q_0} = q_0 e^{\frac{\hat{K}}{2}} \frac{1}{y\sqrt{-v}} && Q_{p^0} = p^0 e^{\frac{\hat{K}}{2}} y\sqrt{-v} \\
&Q_{q_1} = q_1 e^{\frac{\hat{K}}{2}} \frac{\sqrt{-v}}{y} && Q_{p^1} = p^1 e^{\frac{\hat{K}}{2}} \frac{y}{\sqrt{-v}} \\
&Q_{q_2} = q_2 e^{\frac{\hat{K}}{2}} \frac{1}{\sqrt{-2v}} && Q_{p^2} = p^2 e^{\frac{\hat{K}}{2}} \sqrt{-2v}
\end{aligned}
\tag{6.52}
$$

The charge-to-tension vectors are:

$$
\begin{aligned}
&z_{q_0} = \sqrt{2} && z_{p^0} = \sqrt{2} \\
&z_{q_1} = \sqrt{2} && z_{p^1} = \sqrt{2} \\
&z_{q_2} = 1 && z_{p^2} = 1
\end{aligned}
\tag{6.53}
$$

Computing the largest radius of a ball contained in the convex hull spanned by the orthogonal vectors (6.53) we get:

$$
r = \frac{1}{2} \tag{6.54}
$$

6.5 Significance and limitations of the result

In the previous section we have shown that, starting from a supersymmetric action in four dimensions of the form (5.34), containing gauge three-forms, it is possible to consider transitions among different vacua of the scalar potential mediated by membranes, and to compute the tensions and the physical charges associated to the membranes themselves. In analogy to the case of particles, we have assumed that the charge-to-tension vectors of the membranes must satisfy some form of the Weak Gravity Conjecture: namely, that their convex hull must contain a ball of some radius. We have dealt with three different models, characterized by the prepotentials of the form:

$$
\mathcal{G} = \frac{\kappa_{123}}{\mathcal{Z}_0} \mathcal{Z}_1 \mathcal{Z}_2 \mathcal{Z}_3 \qquad \mathcal{G} = \frac{\mathcal{Z}_1^3}{\mathcal{Z}_0} \qquad \mathcal{G} = \frac{\mathcal{Z}_1 \mathcal{Z}_2^2}{\mathcal{Z}_0} \tag{6.55}
$$

We note that all the three models can be deduced from the prepotential $\frac{1}{6\mathcal{Z}_0}\kappa_{ijk}\mathcal{Z}^i\mathcal{Z}^j\mathcal{Z}^k$ with an appropriate choice of the coefficients κ_{ijk}.

In all of the three cases we have found that the charge-to-tension vectors exactly include a ball of radius $\frac{1}{2}$, and, most importantly, do not depend on the moduli of the theory (respectively v^i, v, and (v, y) for the various models), nor on the spacetime coordinates. As a consequence, they are unaffected by the choice of the vacuum in which the theory lies on the two sides of the membrane, be it supersymmetric or not. This fact suggests that the WGC bound for 2-branes could be stated as:

Generalized WGC (2-branes): The convex hull spanned by the charge-to-tension vectors $\vec{z}_i$ must contain a ball of radius $\frac{1}{2}$.

This is justified by the observation that supersymmetric BPS states should saturate the bound, and that there is no discernible difference in the charge-to-tension vectors related to susy and non-susy vacua. Another reason to adopt the Generalized WGC statement is that it is in agreement with what was found by the Narayan and Trivedi model [25]. In their paper (and as we have done in another fashion in section 5.5) they show that the membrane-mediated decays between non-susy vacua are only marginally allowed (that is, they lie precisely on the threshold required for decay): this is exactly what happens with our setting, as we have seen that, no matter if the vacua are susy or non-susy, the convex hull of the charge-to-tension vectors contains the same ball of radius $\frac{1}{2}$.

There is, however, a problem: for the non-supersymmetric vacua we expect, following the WGC statement by Ooguri and Vafa [34] we have cited at the beginning of the chapter, that the bound were not saturated, and as a result that they were unstable. On the contrary, we have just said that in our setting, which admits non-susy vacua, the bound is exactly saturated. A possible way out could be the following: when deducing the 4d theory from the original 10d action, we have compactified on the orbifold (4.1), considering non-vanishing fluxes of the field strengths in the RR and NSNS sector. What we have not taken into account is the back-reaction of these fluxes upon the geometry of the compactification space, whose properties in general could be subject to a change. More specifically, the Kähler potential could be modified, thus influencing the values of the physical charges of the membranes, that explicitly depend on it via equation (6.19).

Furthermore, the generalized WGC statement we have displayed above should be compared with similar claims recently proposed in the literature. In this regard, the most precise statement regarding 2-branes has been deduced by Hebecker et alii in [45]. Employing dimensional reduction arguments they affirm that, when considering a theory in four spacetime dimensions, the convex hull spanned by the charge-to-tension vectors should contain a ball of radius $\frac{1}{\sqrt{2}}$, which is different from our result of $\frac{1}{2}$. It is not clear, however, if the two statements can be directly compared, as in the work of [45] they have had to deal with subtleties related to the dualities among the p-forms in the RR sector. As a result, further work is required to assess if the discrepancy between the two claims is real or it is merely a consequence of different conventions.

Another relevant limitation with the reasoning we have carried out in this chapter lies in the hypothesis we have made when considering the prepotentials (6.55): we have set the real parts of the physical fields to zero. This has been done with two main motivations: 1) the Kähler potentials of these models do not depend on the real part of the fields; 2) neglecting the real parts lets us write the part of the lagrangian that contains the gauge three-forms with no mixing among their field strengths. To make this point clearer, let us consider the combinations of field strengths that appear in equation (4.63) for the Narayan and Trivedi model: when setting the real parts of the moduli (that is, the axions b^i) to zero, each italics field strength "$\mathcal{F}$" corresponds to only one non-italics one "F": in this way, the coefficients of the gauge three-forms' kinetic terms that appear in equation (4.80) refer to *only one* gauge field-strength, so that each field strength is associated to only one elementary charge (namely, the charge in the YM-like term $\propto \frac{1}{g^2} F^{mnrs} F_{mnrs}$). On the other hand, were the real parts non-vanishing, if we wanted to compute the elementary charges we would have had to diagonalize the matrix associated to the quadratic forms "$\mathcal{F}^2$" and to introduce new combinations of field strengths, according to the eigenvectors of the matrix. This enormously complicates the picture, and an analytic treatment of the problem becomes apparently impossible. This fact could possibly indicate that, when considering also the real parts of the fields, the charges-to-tension ratios could become moduli-dependent, so that they must evaluated at a particular point of spacetime (presumably the location of the membrane). A related limitation of our work is that the prepotentials (6.55) yield diagonal Kähler metrics: more general prepotentials, still satisfying the homogeneity condition (3.97), produce non-diagonal metrics, that oftentimes comprise also the real parts of the physical fields. In these cases a diagonalization procedure is required so as to find appropriate field strengths that possess a unique charge, but so far our attempts, focused mainly on a prepotential of the form $\mathcal{G} = i\mathcal{Z}^0\mathcal{Z}^1$ (studied extensively in [21]) have not produced the desired results. It is then necessary to put more effort on this topic, in order to understand whether the approach we have presented is sensible also in the non-diagonal case or there is some missing ingredient.

Conclusions

The first part of the thesis has been devoted to reviewing the construction of a supergravity effective theory in four dimensions, starting from a 10d type IIa effective action. The key element in the procedure has been the presence of background fluxes for the p-forms that belong to the bosonic sector of the 10d theory, that in turn determine the characteristics of the scalar potential of the 4d effective theory. Our first task has been to rewrite the model of Narayan and Trivedi [25] in such a way that the values of the fluxes disappear from the 4d action, being replaced by appropriate gauge-three forms, by adopting the supersymmetric formulation of [20]. This has been possible because the field strength of a gauge three-form is non-dynamical in 4d, so that it can be put in correspondence with a constant (i.e. the value of a flux) via its equations of motion. Computing explicitly the scalar potential derived by the gauge three-forms it has been possible to confirm that it coincides with the standard one, showing that the new formulation that substitutes the values of the fluxes can be successfully applied to the Narayan and Trivedi model.

The second objective of the work was trying to compute the tensions of the 2-branes (that is, objects that divide ordinary space into two subregions) mediating transitions among different AdS vacua of the scalar potential, so as to understand whether, and in which cases, the decay from a vacuum state to another is allowed, and in particular if the non-supersymmetric AdS vacua of the Narayan and Trivedi model are unstable, in light of the claim of [34]. In order to do this, a supersymmetric action (5.34) involving gravity, the scalar fields of the theory, the gauge three-forms' kinetic terms and the membranes has been built, following the work of [21]. We have seen how the precise value of the membranes' tension depends on the moduli (in our case the scalar fields) evaluated at the spacetime location of the membrane. Taking into account the back-reaction of the membrane on its surroundings we have exhibited an approximate way to compute the value of the scalar fields, even though we have not been able to state with certainty if the transitions we have considered are allowed, a task for which more refined approximations are required.

The last and more relevant task of the thesis was to employ the manifestly supersymmetric formulation of [20] and [21], that yields kinetic terms for the gauge three-forms that have substituted the fluxes, to assess the validity of the weak gravity conjecture for 2-branes. As far as we know, in fact, there still isn't a general agreement in the literature about the shape that the WGC should take in the case of 2-branes in 4d. The idea employed in this work was to take inspiration from the WGC in the case of particles that possess multiple different charges, and to apply it to the case of 2-branes. More specifically, we assumed that, if the

WGC is satisfied, the vectors defined as the ratios of the charges of the membranes with respect to their tensions should contain a ball of some radius r. Coherently with the Ooguri and Vafa WGC proposal [34], supersymmetric BPS states should *precisely* contain the ball of radius r, thus saturating the WGC bound.
Practically speaking we have considered, other than the Narayan and Trivedi case, two additional models, so as to be able to perform explicit computations.
Calculating the tensions of the membranes is a relatively straightforward task, as it can be directly inferred from the membrane action contained in (5.34). A more subtle problem is to evaluate the physical charges of the membranes: in this regard the new formulation of [20] has proven its usefulness, naturally producing kinetic terms for the gauge three-forms. In analogy with the usual Yang-Mills action, we have computed the physical charges of the membranes starting from the coefficients of the kinetic terms. Finally, computing the charge-to-tension vectors, we have shown that, surprisingly, they do not depend on the values of the moduli, and that the radius r that characterizes the WGC for 2-branes should be equal to $\frac{1}{2}$. This is somewhat similar to the results of [45], even though there remain some subtleties to be more thoroughly examined.

The main advantage of the approach that was employed in the thesis is that it starts from a supersymmetric action in 4d and straightforwardly deduces the physical charges and the tensions of the membranes without further hypotheses. Nevertheless, we know for sure that there must be corrections to the computation we have performed: in our models also the membranes that mediate transitions among non-susy vacua saturate the WGC bound, which is not in agreement with the claim of Ooguri and Vafa [34]. As we have mentioned, a possible resolution of this discrepancy could be to contemplate corrections to the geometry of the compactification manifold due to the back-reaction of the fluxes on it, which are known to correct the Kähler potential of the effective field theory.

Additional limitations of our work were the relatively small variety of models considered: more contrived cases, involving non-diagonal Kähler metrics and non-vanishing real parts of the scalar fields, require further inspection.

Appendix A

In this appendix we recall a few notions of complex geometry, in particular regarding Kähler and Calabi-Yau manifolds.
We refer especially to the lectures [10], [11] and [12].

Cohomology and Homology

The first two important concepts that have to be introduced in order to characterize complex manifolds are cohomology and homology. A differential real p-form is an object belonging to the cotangent space of a manifold $\mathcal{M}$ of dimension d (written in some basis of a certain patch of the manifold), with the index p satisfying $p \leq d$ as a consequence of the antisymmetry of the wedge product:

$$A(x) \equiv \frac{1}{p!} A_{m_1 \ldots m_p}(x)\, \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p} \tag{6.56}$$

The wedge product of a p-form $A(x)$ and a q-form $B(x)$ (with $p + q \leq d$) is defined as:

$$A \wedge B = \frac{1}{p!q!} A_{[m_1 \ldots m_p} B_{m_{p+1} \ldots m_{p+q}]} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_{p+q}} \tag{6.57}$$

It can be seen that the natural domain of integration of a p-form is a submanifold of dimension p. The so-called exterior derivative d transforms a p-form into a $p+1$-form, namely:

$$dA \equiv \frac{1}{p!} \frac{\partial A_{m_1 \ldots m_p}}{\partial x^n} \mathrm{d}x^n \wedge \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p} \tag{6.58}$$

A p-form A is said to be closed if it satisfies $dA = 0$, and is called exact if $d^2A = 0$. Observing that applying the operator d twice gives the product of an antisymmetric and a symmetric tensor we see that:

$$d^2 = 0 \tag{6.59}$$

The p-th cohomology group (or De Rham cohomology group) of $\mathcal{M}$ is then defined to be the quotient space of the kernel of the operator d and its image: that is, the quotient space of closed p-forms and the forms that can be written as exterior derivatives of a $(p-1)$-form:

$$H^p(\mathcal{M}) \equiv \frac{\text{Ker } d}{\text{Im } d} \tag{6.60}$$

Stated differently, the p-th cohomology group is the set of equivalence classes of closed p-forms, where two p-forms are said to be equivalent if they differ by an exact form. It can be shown that the cohomology groups are vector spaces (that is, cohomology classes can be added and multiplied by constant) of dimensions b^p (called Betti numbers). In addition we define the Euler characteristic χ of the manifold $\mathcal{M}$ as:

$$\chi(\mathcal{M}) = \sum_{p=0}^{d} (-1)^p b^p \tag{6.61}$$

An operator that shares the same properties of d is the "boundary" operator δ, intuitively defined acting on compact submanifolds by mapping them to their boundaries. In other words, the expression $U = \delta S$ means that U is the boundary of the compact submanifold S. The objects upon which δ acts are named p-chains, and informally speaking they are submanifolds of dimension $p < d$. Addition between p-chains is interpreted as their insiemistic union, whereas multiplication by -1 as a change in the orientation of the submanifold. It can be then shown that p-chains form a vector space, just as the p-forms.
In a similar manner, therefore, we define homology groups:

$$H_p(\mathcal{M}) = \frac{\text{Ker } \delta}{\text{Im } \delta} \tag{6.62}$$

In this view H_p is the set of equivalence classes of submanifolds without borders, deemed equivalent if they differ at most by a border of another submanifold. In other terms, two p-chains S and T are in the same equivalence class if they satisfy $S = T + \delta U$, with U another submanifold.
p-chains furnish a natural domain of integration for p-forms: it is then customary to define the following kind of product between a p-forms equivalence class $[A]$ and a p-chains equivalence class $[S]$, linking cohomology to homology:

$$\langle [A], [S] \rangle = \int_S A \tag{6.63}$$

It can be shown, by means of Stokes theorem, that choosing another representative for the equivalence classes leaves the result unchanged. More importantly, this relationship between cohomology and homology can be seen to prove that the dimension of H_p is again the Betti number b^p: as a result H^p and H_p are dual vector spaces. Another important link between the two is the Poincaré duality, that, using Stokes theorem, relates a closed p-form A with a $(d-p)$-cycle S (B can be any closed $(d-p)$-form):

$$\int_{\mathcal{M}} A \wedge B = \int_S B \tag{6.64}$$

This duality implies that the Betti numbers satisfy $b^p = b^{d-p}$.
An extremely relevant operator acting on p-forms and transforming them into $(d-p)$-forms

(recalling that dim$\mathcal{M} = d$) is the Hodge star, defined as:

$$* A = \frac{\sqrt{\det(g_{mn})}}{p!(d-p)!} \epsilon_{m_1 \dots m_p n_1 \dots n_{d-p}} A^{m_1 \dots m_p} \mathrm{d}x^{n_1} \wedge \dots \wedge \mathrm{d}x^{n_{d-p}} \tag{6.65}$$

It must be noted that in order to raise the indices of the Levi-Civita tensor the notion of a Riemannian metric g_{mn} on $\mathcal{M}$ has been used. Using the Hodge star the volume of the manifold $\mathcal{M}$ can be defined as:

$$Vol = \int *1 = \int \sqrt{\det(g_{mn})}\, \mathrm{d}x^{m_1} \wedge \dots \wedge \mathrm{d}x^{m_d} \tag{6.66}$$

Providing a link between p -forms and $(d-p)$-forms the Hodge operator naturally gives rise to a product between two p-forms, say A and B, integrated over the whole manifold $\mathcal{M}$:

$$(A, B) = \int_{\mathcal{M}} A \wedge *B \tag{6.67}$$

By way of the Hodge star an adjoint operator d^* can be defined, consistently with its name:

$$(A, dB) = (d^* A, B) \tag{6.68}$$

It can be proven that its explicit form is:

$$d^* = (-1)^{dp+p+1} * d* \tag{6.69}$$

The adjoint operator d^* shares with d many of its properties, such as $(d^*)^2 = 0$, giving rise to an analogue cohomology. With the aid of d^* a "Laplacian" operator, mapping p-forms to p-forms can be defined as:

$$\Delta = d^* d + d d^* \tag{6.70}$$

A form A will be said harmonic if $\Delta A = 0$. An important theorem further states that each cohomology class in $H^p(\mathcal{M})$ contains exactly one harmonic form.
The whole bunch of definitions and results built so far can be extended in the case of a complex manifold $\mathcal{M}$ with local coordinates z^i and $\bar{z}^{\bar{i}}$, taking care to introduce complex (p, q)-forms A, two exterior derivative operators ∂ and $\bar{\partial}$ and a laplacian $\Delta_{\bar{\partial}}$:

$$\begin{aligned}
&A = A_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}(z, \bar{z})\, \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{\bar{j}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{j}_q} \\
&\partial A = \frac{\partial A_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}(z, \bar{z})}{\partial z^k}\, \mathrm{d}z^k \wedge \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{\bar{j}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{j}_q} \\
&\bar{\partial} A = \frac{\partial A_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}(z, \bar{z})}{\partial \bar{z}^k}\, \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{\bar{j}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{j}_q} \wedge \mathrm{d}\bar{z}^{\bar{k}} \\
&\Delta_{\bar{\partial}} = \overline{\partial \partial}^* + \overline{\partial}^* \overline{\partial}
\end{aligned} \tag{6.71}$$

Both of the operators ∂ and $\bar{\partial}$ give rise to a cohomology, but, as a result of the fact that the two can be proven to be isomorphic in the case of Kähler manifolds (the main topic of the next section), it is customary to use only the $\bar{\partial}$-cohomology, whose group is defined as:

$$H^{p,q}(\mathcal{M}) = \frac{\text{Ker } \overline{\partial}}{\text{Im } \overline{\partial}} \tag{6.72}$$

The vector space $H^{p,q}(\mathcal{M})$ has dimension $h^{p,q}$. Once again a theorem by Hodge and Weyl proves that each Dolbeault-cohomology class contains exactly one form harmonic with respect to the Laplacian $\Delta_{\bar{\partial}}$. The set of all the admissible $h^{p,q}$ are the Hodge numbers, and they are conventionally classified into the Hodge diamond:

$$\begin{array}{ccccccc} & & & h^{d,d} & & & \\ & & h^{d,d-1} & ... & h^{d-1,d} & & \\ h^{d,0} & ... & & ... & & ... & h^{0,d} \\ & & h^{1,0} & ... & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} \tag{6.73}$$

Kähler and Calabi-Yau manifolds

We now proceed in specifying the notions of the previous section, deriving a few other relations in the case of a particular kind of complex manifolds, that is the Kähler manifolds. A manifold is said to be complex, with d complex dimensions, if it admits an atlas of charts mapping the points of the manifold itself (labelled in some coordinate system by z^i and $\bar{z}^{\bar{i}}$, with $i = 1, ..., d$) to $\mathbb{C}^n$, with holomorphic transition functions. This equals the fact that there exists a tensor Y, called complex structure, acting on the tangent space at each point of the manifold, such that it satisfies the following conditions:

$$Y^2 = -1 \qquad Y^i_j = i\delta^i_j \qquad Y^{\bar{i}}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}} \qquad Y^i_{\bar{j}} = Y^{\bar{i}}_j = 0 \tag{6.74}$$

A Riemannian metric g is said to be compatible with the complex structure Y if this relation holds (where v and w are generic tangent vectors):

$$g(v, w) = g(Yv, Yw) \tag{6.75}$$

It can be proven that as a consequence of (6.75) the components g_{ij} and $g_{\overline{ij}}$ vanish: as a result the metric is called *Hermitian*. The so-called Kähler form J is defined as:

$$g(v, w) = J(v, Yw) \tag{6.76}$$

The considered complex manifold is also said to be a *Kähler* manifold if J is closed, i.e. if:

$$dJ = 0 \tag{6.77}$$

This statement is equivalent to saying that $g_{ij} = g_{\bar{i}\bar{j}} = 0$. The cohomology class of J is also called Kähler class. In addition, it can be shown that on some patch the non-vanishing components of the metric $g_{i\bar{j}}$ (that we will often write $K_{i\bar{j}}$ in the main text) can be locally rewritten as the derivative of a function K, called Kähler potential:

$$g_{i\bar{j}} = \frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^{\bar{j}}} \tag{6.78}$$

It is straightforward to see that the metric is invariant under the transformation (1.21).
Taking a swift aside, it can be observed that every p-form A_p belonging to the De Rham cohomology can be written as the sum of Dolbeault-cohomology forms (we use an immediate change of notation):

$$A_p = A_{p,0} + A_{p-1,1} + ... + A_{1,p-1} + A_{0,p} \tag{6.79}$$

In the case of a Kähler manifold a link between the Laplacian (6.70) and (6.71) can be established:

$$\Delta = 2\Delta_{\bar{\partial}} \tag{6.80}$$

Acting with the Laplacian on both sides of the equation and recalling the correspondence between harmonic forms and cohomology classes it can be shown that:

$$H^p(\mathcal{M}) = H^{p,0}(\mathcal{M}) \oplus H^{p-1,1}(\mathcal{M}) \oplus ... \oplus H^{0,p}(\mathcal{M}) \tag{6.81}$$

We see then that the Betti and Hodge numbers are related by:

$$b^p = h^{p,0} + h^{p-1,1} + ... + h^{0,p} \tag{6.82}$$

As a consequence of the isomorphism between ∂-cohomology and $\bar{\partial}$-cohomology, if $\mathcal{M}$ is a Kähler manifold, the Hodge numbers $h^{p,q}$ are symmetric with respect to the exchange of p and q, that is $h^{p,q} = h^{q,p}$. Furthermore, it can be proven that $H^{p,q}$ and $H^{d-p,d-q}$ are dual vector spaces, resulting in the fact that $h^{p,q} = h^{d-p,d-q}$.
A subclass of Kähler manifolds are Calabi-Yau manifolds: given the metric $g_{i\bar{j}}$, they are defined by the vanishing of the Ricci form (namely, they are Ricci-flat), that is:

$$\mathcal{R} = iR_{i\bar{j}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}} = 0 \tag{6.83}$$

This statement is equivalent to requiring that the first Chern class of $\mathcal{M}$ vanishes. For our purposes it is sufficient to know that, given a connection A on a vector bundle of $\mathcal{M}$ and its field strength $F = dA - A \wedge A$, the first Chern class c_1 is:

$$c_1 = \frac{i}{2\pi}\mathrm{Tr}F \tag{6.84}$$

Calabi and Yau proved that, given a Kähler manifold with vanishing Chern class, it is always possible to find one Ricci-flat Kähler metric in each Kähler class, and that this metric is unique. Another relevant result about Calabi-Yau manifolds with 3 complex dimensions is that there exists a unique never-vanishing holomorphic and harmonic $(3,0)$-form Ω, that can be written in some coordinate system as:

$$\Omega(z) = f(z)\, \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \tag{6.85}$$

As a result we have that in this case $h^{3,0} = 1$. If the Calabi-Yau manifold has a non-vanishing Euler characteristic χ (as for the physically relevant case) an additional condition on Hodge numbers can be proven [16]:

$$h^{1,0} = h^{2,0} = 0 \tag{6.86}$$

Bibliography

[1] P.C. West, *Introduction to rigid supersymmetric theories*, hepth/9805055

[2] S.P. Martin, *A Supersymmetry Primer*, hep-ph/9709356

[3] M. Bertolini, *Lectures on Supersymmetry*, SISSA

[4] P. van Nieuwenhuizen, *Supergravity*, Physics Reports, 68, No. 4, (1981), 189-398

[5] H. Samtleben, *Introduction to Supergravity*, Saalburg, 2007

[6] H. Nastase, *Introduction to Supergravity*, arXiv:1112.3502v3

[7] J. Wess, J.A. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 1992

[8] J.A. Bagger, *Supersymmetric Sigma Models*, Bonn-NATO Advanced Study Institute on Supersymmetry, (1984)

[9] J. Louis, A. Micu, *Type II Theories Compactified on Calabi-Yau Threefolds in the Presence of Background Fluxes*, arXiv:hep-th/0202168v3

[10] P. Candelas, *Lectures on Complex Manifolds*, Inspire 257487

[11] M. Vonk, *A mini-course on topological strings*, arXiv:hep-th/0504147v1

[12] V. Bouchard, *Lectures on complex geometry, Calabi-Yau manifolds and toric geometry*, arXiv:hep-th/0702063v1

[13] T. Grimm, J. Louis, *The effective action of Type IIA Calabi-Yau orientifolds*, arXiv:hep-th/0412277v2

[14] K. Becker, M. Becker, J.H. Schwarz, *String Theory and M Theory, A Modern Introduction*, CUP, (2006)

[15] L. J. Romans, *Massive N=2a Supergravity In Ten-Dimensions*, Phys. Lett. B 169 (1986), 374

[16] E. Malek, *Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds*, LMU München

[17] B. R. Greene, *String Theory on Calabi-Yau Manifolds*, arXiv:hep-th/9702155v1

[18] M. R. Douglas, S. Kachru, *Flux Compactifications*, arXiv:hep-th/0610102v3

[19] L. Randall, R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. 83 (17): 3370-3373 (1999)

[20] F. Farakos, S. Lanza, L. Martucci, D. Sorokin, *Three-forms in supergravity and flux compactifications*, Eur. Phys. J. C (2017) 77:602

[21] I. Bandos, F. Farakos, S. Lanza, L. Martucci, D. Sorokin, *Three-forms, dualities and membranes in four-dimensional supergravity*, J. High Energ. Phys. (2018) 2018:28

[22] P.S. Howe and R.W. Tucker, *Scale Invariance in Superspace*, Phys. Lett. B80 (1978) 138-140

[23] S. Lanza, L. Martucci, D. Sorokin, *Supersymmetric EFT for Type IIA string theory including three-form potentials*, work in progress, still unpublished

[24] O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, *Type IIA Moduli Stabilization*, JHEP 0507, 066 (2005), arXiv:hep-th/050516

[25] P. Narayan, S.P. Trivedi, *On the Stability of Non-Supersymmetric AdS Vacua*, arXiv:1002.4498v3

[26] J. Polchinski, *TASI lectures on D-branes*, arXiv:hep-th/9611050v2

[27] M. Graña, *Flux compactifications in string theory: a comprehensive review*, arXiv:hep-th/0509003v3

[28] L.I Nicolaescu, *Lectures on the Geometry of Manifolds* , University of Notre Dame

[29] N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *The String Landscape, Black Holes and Gravity as the Weakest Force*, arXiv:hep-th/0601001v2

[30] S. Coleman, F. De Luccia, *Gravitational effects on and of vacuum decay*, Phys. Rev. D 21, 3305, 1980

[31] S. Coleman, *Fate of the false vacuum: Semiclassical theory*, Phys. Rev. D 15, 2929, 1977

[32] S. Coleman, E.J. Weinberg, *Radiative corrections as the origin of spontaneous symmetry breaking*, Phys. Rev. D 8: 1888, 1973

[33] T.D. Brennan, F. Carta, C. Vafa, *The String Landscape, the Swampland, and the Missing Corner*, arXiv:1711.00864

[34] H. Ooguri, C. Vafa, *Non-supersymmetric AdS and the Swampland*, arXiv:1610.01533v3

[35] L. Susskind, *Trouble For Remnants*, arXiv:hep-th/9501106v1

[36] S. Moroz, *Below the Breitenlohner-Freedman bound in the nonrelativistic AdS/CFT correspondence*, arXiv:0911.4060v2

[37] U.H. Danielsson, G. Dibitteto, S.C. Vargas, *A swamp of non-SUSY vacua*, arXiv:1708.03293v1

[38] *Summary plots from the ATLAS Supersymmetry physics group*, https://atlas.web.cern.ch/Atlas/GROUPS/PHYSICS/CombinedSummaryPlots/SUSY/

[39] *CMS Supersymmetry Physics Results*, https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsSUS

[40] D. Cassani. *String theory compactifications with fluxes and generalized geometry*, Mathematical Physics, Université Pierre et Marie Curie - Paris VI, 2009

[41] J.D. Brown, C. Teitelboim, *Dynamical neutralization of the cosmological constant*, Physics Letters B, 195, 2, pag. 177-182, 1987

[42] R. Bousso, J. Polchinski, *Quantization of Four-form Fluxes and Dynamical Neutralization of the Cosmological Constant*, arXiv:hep-th/0004134v3

[43] N. Kaloper, L. Sorbo, *A Natural Framework for Chaotic Inflation*, arXiv:0811.1989v2

[44] N. Kaloper, A. Lawrence, L. Sorbo, *An Ignoble Approach to Large Field Inflation*, arXiv:1101.0026v1

[45] A. Hebecker, F. Rompineve, A. Westphal, *Axion Monodromy and the Weak Gravity Conjecture*, arXiv:1512.03768v3

[46] C. Cheung, *Naturalness and the Weak Gravity Conjecture*, arXiv:1402.2287v2

[47] K. Groh, J. Louis, J. Sommerfeld, *Duality and Couplings of 3-Form-Multiplets in N = 1 Supersymmetry*, arXiv:1212.4639v2

[48] D.Z. Freedman, A. Van Proeyen, *Supergravity*, Cambride University Press, 2012

[49] L.E. Ibanez, M. Montero, A. Uranga, I. Valenzuela, *Relaxion Monodromy and the Weak Gravity Conjecture*, arXiv:1512.00025v3

[50] B. Heidenreich, M. Reece, T. Rudelius, *Sharpening the Weak Gravity Conjecture with Dimensional Reduction*, arXiv:1509.06374v2