UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Factorisation of scattering on the string worldsheet

Thesis supervisor

Prof. Alessandro Sfondrini

Thesis co-supervisor

Dr. Davide Polvara

Candidate

Nicola Baglioni

Academic Year 2022/2023

Contents

Introduction

The study of the AdS/CFT [1] correspondence has held a central role in theoretical physics for now more than twenty years. In this thesis we focus on an instance of this correspondence that is the duality between gravity (superstring) theories on backgrounds involving three-dimensional anti-de Sitter space (AdS_3) and supersymmetric two-dimensional conformal field theories (CFT_2). The problem of determining the string spectrum has been recently tackled using exact methods collectively going under the name of integrability techniques. This approach has been shown to be very useful in the investigation especially of the AdS_5/CFT_4 case [2], that is in the case of a $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions and type IIB superstring theory in $AdS_5 \times S^5$ spacetime. In recent years a lot of work has been also done in the AdS_3 case that we treat here. In ref. [3] it was shown that the theory describing free strings on maximally supersymmetric AdS_3 backgrounds supported by a mixture of Ramond-Ramond (RR) and Neveu-Schwarz-Neveu-Schwarz (NSNS) fluxes is classically integrable. The consequence is that it has been possible to bootstrap a factorised S matrix of the string non-linear sigma model on the two-dimensional string worldsheet. This symmetry-based bootstrap approach is able to provide exact predictions that must be checked perturbatively on either side of the gauge/string duality. The aim of this thesis is to perturbatively verify the properties of the worldsheet S-matrix for the bosonic sector of the string in $AdS_3 \times S^3 \times T^4$ supported by mixed RR and NSNS fluxes in light-cone gauge.

Let us briefly expand on the role of integrability in this framework. Integrability is usually a feature of $d = 2$ quantum field theories. The S-matrix of such theories enjoys a series of properties. An important one is that a generic n-to-n particles process must factorise into a sequence of two-to-two processes. In general these conditions largely simplify the job of exactly solving the theory. This becomes relevant in the AdS/CFT case if we observe that the dynamics of the string is described by a non-linear σ model (NLSM) living on the string worldsheet, which is in fact a two-dimensional manifold; the latter can be adequately decompactified in order to properly define the asymptotic states on which the S-matrix acts.

The machinery of integrability has produced important results on specific regions of the parameter space and is a very promising tool for the study of the correspondence. Let us mention very quickly how integrability fits in the picture. For a more detailed discussion, an excellent introduction to its applicability in the context of AdS/CFT is presented in ref. [4].

On the gauge side we can suppose to take a SYM theory with gauge group $SU(N_c)$. The number of colours N_c can be taken to infinity. We can define an effective coupling $\lambda = g_{YM}^2 N_c$, where g_{YM} is the original coupling constant, and take the $N_c \to \infty$ limit while keeping λ fixed. This is the so-called *planar limit*. In this limit the only Feynman diagrams whose contribution is not suppressed are those that do not present any intersection of the internal lines; these diagrams are those that can be drawn on a plane, hence the name of the limit. Integrability has proven to be of use especially in this planar limit.

The AdS/CFT correspondence relates the string couplings, namely the effective string tension T and the string coupling constant g_{str}, to λ and N_c as:

$$\lambda = 4\pi^2 T^2, \qquad\qquad \frac{1}{N_c} = \frac{g_{str}}{4\pi^2 T^2}. \tag{1}$$

From the relations in (1) it is clear that the planar limit taken in the gauge side would correspond to a free string theory in the string side of the duality. Even though the string is free, we have a theory living in the worldsheet of the string that is not free. This theory can be actually identified with an interacting field theory whose couplings are proportional to the inverse of T. Therefore this shows how the gauge/string duality also manifests itself as a weak/strong duality. In fact in the weak regime ($\lambda << 1$) the gauge theory is perturbative and the worldsheet theory is not while in the strong regime ($\lambda >> 1$) the opposite happens.

An exact approach using the integrability of the theory on the worldsheet produces results at every value of the coupling λ (or T) and it is therefore paramount for the understanding of a region of the parameter space that could not otherwise be probed. Therefore integrability is able to connect the regime of perturbative gauge theory with the regime of perturbative string theory. Integrability has produced results in agreement with the perturbative expectations both for the gauge theory and for the string theory, meaning at both the large and small λ regimes.

The plan of this thesis is the following. In the first two chapters we introduce some of the properties of the scattering for integrable quantum field theories (IQFTs) in $(1 + 1)$-dimensions.

In particular, in the *first chapter* we start with a brief overview of the properties of scattering processes for IQFTs. We introduce the factorisation of the S-matrix in the sense of Zamolodchikov [5].

In the *second chapter* we present some examples of integrable quantum field theories such as the sinh-Gordon and Bullough-Dodd models. We focus on their scattering properties and we perform some tree-level calculations to show in practice how factorised scattering looks like.

In the second part of this work we delve into the non-linear σ model we mentioned before. The NLSM is defined on the worldsheet of the bosonic string propagating in the $AdS_3 \times S^3 \times T^4$ background supported by a mixture of RR and NSNS fluxes. The ratio of these two fluxes is parametrised by the parameter $q \in [0, 1]$ where 0 corresponds to pure RR and 1 to pure NSNS. This parameter will appear in the starting string action in front of a differential two-form, the so called Kalb-Ramond field B. The parameter q together with the overall string tension T are two free parameters in the theory.

In the *third chapter* we derive the NLSM on the worldsheet. We start from the bosonic string action and impose the light-cone gauge-fixing procedure, aimed at removing the spurious degrees of freedom on the worldsheet. Then we go into detail on the calculations needed to obtain the worldsheet Hamiltonian. The perturbative regime for this theory corresponds to the large string tension regime ($T >> 1$). As we then show this corresponds to a 'small field' expansion and therefore we truncate the Hamiltonian up to the desired perturbative order.

The *fourth chapter* is entirely dedicated to the analysis of two-particle interactions in the string non-linear sigma model at the tree level. We study the properties of the interaction vertices and compare them to the expectations for an integrable theory. In addition we show how to compute scattering amplitudes and we obtain the two-to-two particles S-matrix at the tree level.

The *fifth chapter* focuses on three-to-three particle processes. We expect their amplitudes to factorise into products of two-to-two ones. Even though the discussion is limited at the tree level we show that this property is satisfied at the desired order in $\frac{1}{T}$. We present some of the calculations of these properties in full details.

The *sixth chapter* contains the conclusions.

Chapter 1

S-matrix in 1+1 dimensions

1.1 Integrable theories in d=2

The intention of the present work is to study a model that belongs to a specific class of theories, the so called *integrable* quantum field theories in $d = 2$. Here we will omit a serious discussion of integrability *per se* and just illustrate some of the properties of integrable theories that are at the center of the discussions in the following chapters. Historically integrability was first studied in the context of Hamiltonian systems. These systems can be classified in terms of their symmetries. The existence of such symmetries constrains the dynamics of the model and as a result if the system has enough independent symmetries the equations of motion can be integrated explicitly. We will not delve any deeper into Hamiltonian integrability, for some references see *e.g.* [6]. In the field-theoretical framework the situation is slightly different. Since the number of degrees of freedom become infinite, so does the number of symmetries that are needed to exactly solve the system. The consequence is that a general mathematical definition of integrable system does not exist. In most cases a theory is said to be integrable simply when it possesses a large enough number of symmetries.

The relevant objects associated with each symmetry are the conserved charges of the model. Integrable quantum field theories therefore possess an infinite tower of conserved charges in involution, meaning commuting with each other. In this work we are not interested in specifying the nature of these charges and in most cases we will settle with just assuming their existence. A large set of techniques has been employed to study such models and in some cases they can be solved exactly both at the classical and quantum level. More details on classical and quantum integrability can be found in ref. [7] and [8, 9].

As previously mentioned, we will focus on integrable quantum field theories in $d = 2$, meaning one space and one time dimension. The choice of $1 + 1$ dimensions is not casual. In fact for this type of models the scattering processes of particles moving in one spacial dimension have some very special features that we will illustrate in the following. In this regard the most important object is the S-matrix, that is the operator containing all of the information on the scattering of the theory. In the rest of the chapter we will introduce some properties of such operator for $d = 2$ integrable theories.

1.2 S-matrix and its properties

In this section we want to discuss some properties specific to the integrable S-matrix in two dimensions. The idea is to keep the discussion simple, avoiding the more technical points. Let us start by introducing the S-matrix. It is the operator connecting the initial particles asymptotic states with the final ones. These states are usually defined respectively in the far past and the far future, where we are allowed to require the constituents of the states to be so far apart from each other that they are virtually free. The S-matrix encodes all of the information regarding whatever happens in between the initial and final states. The asymptotic m-particle states will be written as:

$$|p_1, p_2, \ldots, p_m\rangle^{in/out}_{a_1,a_2,\ldots,a_m} \tag{1.1}$$

The indices a_i keep track of possible flavours. Call the set of charges of the model $\{\hat{Q}_s\}_{s=1,\ldots,\infty}$. For now the index s just labels the infinite tower of charges in involution that we assume our model possesses. Since they commute with each other they diagonalise the asymptotic states and we can write:

$$\hat{Q}_s|p_1, p_2, \ldots, p_m\rangle^{in/out}_{a_1,a_2,\ldots,a_m} = \sum_i (q_s)^{a_i}|p_1, p_2, \ldots, p_m\rangle^{in/out}_{a_1,a_2,\ldots,a_m}. \tag{1.2}$$

Here $(q_s)^{a_i}$ are the eigenvalues of the single particle states. We should observe here that the additivity of the eigenvalues in equation (1.2) is not obvious. The equality is actually verified only when the charges are defined as integrals of local densities. In the following discussion we will mostly ignore such issues. For more details about these aspects see *e.g.* ref. [10],[11].

Let us see what properties descend directly from the existence of an infinite number of such conserved charges. First of all it is not hard to imagine that the scattering of particles in an integrable theory must obey the following two constraints:

- Equality of the set of initial and final momenta
- Absence of particle production

Let us explain what we mean by these two. Looking at equation (1.2) and knowing that the charges commute with the S-matrix (since they commute with the Hamiltonian[1]) it is clear that in general an eigenstate will evolve into a superposition of states with the same eigenvalue. This corresponds to a conservation equation for each charge. Hence when the model is integrable it must obey an infinite number of equations, corresponding to the infinite number of charges. Schematically, taking an n particles in-state and m particles out-state we have:

$$(q_s)^{a_1} + (q_s)^{a_2} + \cdots + (q_s)^{a_n} = (q_s)^{a_{n+1}} + (q_s)^{a_{n+2}} + \cdots + (q_s)^{a_{n+m}}, \qquad s = 1, \ldots, \infty. \tag{1.3}$$

Notice also that at least some of these charges will depend on the momenta (e.g. energy, momentum itself). The result is that these equations completely constrain the possible kinematic configurations that can result from an interaction process. The only remaining solution is the one that identically satisfies the equations (1.3), namely a scattering where the set of final momenta is the same as the set of initial ones, hence the *equality of the set of initial and final momenta* condition. At the most the set of momenta can be reshuffled amongst the outgoing particles. This argument also constrains the possible interactions to be those where the masses of the initial particles are the same as those of the final ones. Only the internal quantum numbers are allowed to change.

These first two conditions on the S-matrix, although very constraining, do not tell us much on the shape of the non-vanishing remnant of interaction. The last property we want to introduce, peculiar only to $d = 2$ theories, is the so called *factorisation of the S-matrix.*

1.2.1 Factorisation of scattering

In the last section we argued that only $n \to n$ interactions can happen in an integrable theory. For integrable theories in $d = 2$ these n particles scatterings factorise into a sequence of 2 particles ones.

[1] We did not write an expression for the S-matrix yet it is easy to imagine that it will depend someway on the Hamiltonian of the system.

As a consequence the full scattering matrix can be constructed just by understanding 2 particles interactions. We do not provide a rigorous proof of this fact and instead we present an argument that might give a more intuitive sense of how factorisation happens. The discussion will follow a logic similar to that in ref. [12] and first proposed in ref. [5].

The example we will be focusing on is that of a $3 \to 3$ particles scattering but the logic can be easily generalized to a $n \to n$ one.

Suppose to take a general model. We could observe for example the conservation of momentum, produced by the invariance of the theory under space translations. Suppose to identify a particle with its wavepacket. The action of the translation operator $\hat{T}$ on the wavepackets is:

$$\Psi_i(x) = \int dp e^{-a(p-p_i)^2} e^{ip(x-x_i)} \xrightarrow{\hat{T}_b = e^{ib\hat{p}}} \Psi_i(x) = \int dp e^{-a(p-p_i)^2} e^{ip(x-(x_i-b))} \tag{1.4}$$

Clearly all the particles are shifted spacially by the same amount. On the other hand, since in an integrable model we have an infinite tower of conserved charges suppose to have one that transforms in an higher-spin representation of the Lorentz group [2], call it $\hat{P}_s$. It generates a symmetry that will still shift the wavepackets, although in general not in a constant way. The shift might depend on the momentum of the particles. This can be seen by defining the shift as a momentum dependent phase, giving:

$$\Psi_i(x) = \int dp e^{-a(p-p_i)^2} e^{ip(x-x_i)+i\phi(p)}. \tag{1.5}$$

For a generic $\phi(p)$ the translating contribution is found expanding the phase in series around p_1. This corresponds to the approximation of the Gaussian integral close to the peak p_1 and therefore the particle's wavepacket will be shifted by

$$b(p) = -\frac{\partial \phi(p)}{\partial p}_{|p=p_i}$$

which in general depends on the momentum. The action of these charges will then modify each particle's trajectory in a different way. By applying these transformations, which we have assumed to be symmetries of the model, we can take completely different processes and relate their amplitudes; this is the observation on which the factorisation of scattering is built on. In fact, using the commutation of the $\hat{P}_s$ with the S-matrix for each value of the number c it holds:

$$\langle p_1, p_2, p_3| \, e^{ic\hat{P}_s} \, S e^{\,-ic\hat{P}_s} \, |p_4, p_5, p_6\rangle = \langle p_1, p_2, p_3| \, S \, |p_4, p_5, p_6\rangle. \tag{1.6}$$

This basically means that the same amplitude describes processes with trajectories that intersect in completely different points. Observe now the processes in figure 1.1; these are not Feynman diagrams but just depictions of macroscopic trajectories with two-particle interactions (grey dots) and three-particle interactions (red dot) with time flowing upwards. From the previous argument the processes in figure 1.1 should all yield the same result, meaning that three-particle interactions happen as a sequence of two-particle ones. This is exactly what happens in the case of integrable quantum field theories, provided they have a sufficient number of higher-spin conserved charges. More than one proof of this property can be found in the literature (*e.g.* ref. [13]), differing mainly on the number of higher-spin charges required.

Suppose now that the two-particle S-matrix of the processes in figure 1.1 is diagonal. For example we can require the S-matrix of the made-up process $a(p) + b(q) \to c(p) + d(q)$ to be diagonal if [14]:

$$S_{ab}^{cd}(p,q) = \delta_a^c \delta_b^d S_{ab}(p,q). \tag{1.7}$$

[2]We call s the 'spin' of an operator $\hat{P}_s$ when it satisfies the relation $[M, \hat{P}_s] = s\hat{P}_s$, where M is the generator of Lorentz boosts in $d = 2$.

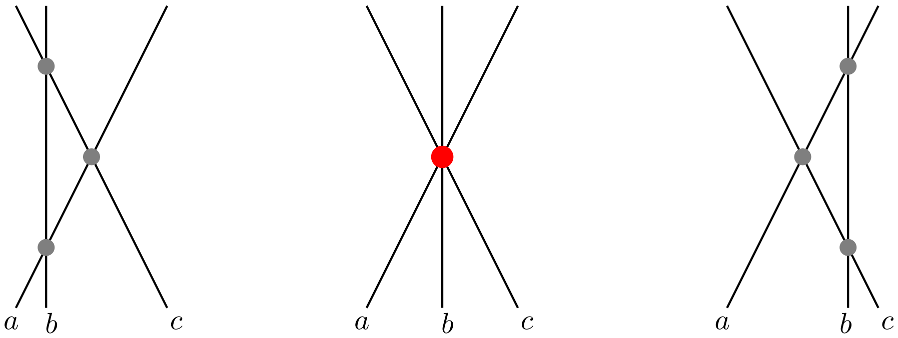


Figure 1.1: Depiction of three different possible trajectories of three particles interactions, initial state at the bottom to final state at the top. These processes must all give the same amplitude in an integrable theory.

Where a, b, c, d label the particles while p, q are the momenta. Then the equivalence of the processes in figure 1.1 can be written schematically as:

$$S_{ab}S_{ac}S_{cb} = S_{cb}S_{ac}S_{ab} = S_{abc}. \tag{1.8}$$

Where S_{abc} corresponds to a three-particle S-matrix. This equation is important for two reasons. Firstly, the second equality tells us exactly how the $3 \to 3$ interaction factorises. The three-particle process factorises into 3 two-to-two processes. In general we can use the same argument to shift n trajectories and it is easy to deduce that an N-particle process will be made of $\frac{N(N-1)}{2}$ two-particle processes. Secondly the first equality, meaning the relation between the diagram on the left and on the right, is essentially the content of the Yang-Baxter equation [3] which is a general constraint on the S-matrix for an integrable theory. A detailed discussion of Y-B equation is beyond the scope of this work, for more details see *e.g.* ref. [15].

It is worthwhile observing that the mechanism of shifting the trajectories produces these results only for $d = 2$. When there are more spacial dimensions it is still possible to shift the trajectories but now in such a way that the lines representing the particles never intersect. This is actually covered by Coleman-Mandula theorem [16], which guarantees that, for theories in $d > 2$, the existence of higher-spin charges is sufficient to trivialize the S-matrix to $S = \mathbb{1}$. The specialty of the $d = 2$ integrable case is therefore that the geometry of the model does not allow trajectories to not intersect yet they will intersect only two at a time.

1.3 On the exact two-particle S-matrix

The properties that we argued in the previous section greatly simplify the problem of determining the S-matrices of integrable $d = 2$ theories. In particular we just need to understand how the two-particle interactions work in order to obtain the full S-matrix of the theory. As a consequence the bootstrap approach to finding the S-matrix can be very powerful. This approach consists in trying to determine the shape of the S-matrix by imposing general consistency relations and symmetries. In this section we briefly mention some of the properties that are required by the two-particle S-matrix; some of these properties will be used in the next chapters.

Let us restrict to a relativistic S-matrix. In this case it is useful to parametrise the momenta, and therefore the S-matrix, in terms of rapidities (defined in section 2.1). Since the non-linear sigma model we investigate next is non-relativistic we will have to make some adjustments.

[3] Y-B is an operatorial equation, the form shown in the first equality in equation (1.8) is obviously trivially satisfied in the diagonal case.

The analytic structure of the S-matrix can be studied through its dependence on the Mandelstam variable $s = (p_1 + p_2)^2$ where $p_1,\ p_2$ are the two-momenta of the two initial particles. We call the two-particle S-matrix by $S(s)$. It is clear that physical configurations must satisfy $s > (m_1 + m_2)^2$, nevertheless $S(s)$ can be analytically extended to the complex plane. Once we continue $S(s)$ into the complex plane we can work on the problem of determining the analytic structure of the S-matrix using complex analysis techniques. For relativistic theories we can study the properties of the S-matrix as a function of the difference of the rapidities of the two particles, $\theta = \theta_1 - \theta_2$. It turns out that $S(\theta)$ is a meromorphic function on the θ-plane. An introductory discussion of its analytic properties, also considering the contribution to the scattering of bound states, is presented in ref. [12].

The shape of the S-matrix is also clearly dependent on the (infinite) symmetries that the integrable model enjoys. On top of this we require the S-matrix to be unitary and to satisfy the factorisation property illustrated in section 1.2.1. At the level of the two-particle S-matrix this means that it must satisfy Yang-Baxter equation, which schematically corresponds to the first equality in equation (1.8). All of these conditions produce a number of equations that the exact S-matrix must satisfy. These are reviewed for generic models in ref. [12] and [14]. The S-matrix for a model similar to what we study in the rest of this work is constructed in ref. [17] using the bootstrap approach.

Finally we want to mention crossing symmetry, which we use to simplify some calculations in the next chapters; this symmetry also imposes strong constraints on the structure of the S-matrix. A more detailed discussion of these constraints can be found *e.g.* in ref. [12], [17]. For a relativistic theory crossing symmetry stems from the known fact that the scattering amplitude for a process with a *particle* in the initial (final) state is the same as the one for the same process but with an *anti-particle* carrying opposite momentum and energy in the final (initial) state [18].

In the case of the non-relativistic string sigma model discussed in the next sections these relations still hold as a consequence of the creation and annihilation operators expansion of the fields that we will show are solutions of the free equations of motion. These symmetries will allow us to connect the amplitudes of different processes between one another, as we show in section 4.2.

Chapter 2

Perturbative calculations for $d = 2$ models

In this chapter we want to discuss two integrable models in $d = 2$. These are the sinh-Gordon model and the Bullough-Dodd model. The integrability of these models is discussed respectively in ref. [19] and [20]. The intention is to use these simpler models to present some of the mechanisms that we will also observe for the non-linear sigma model which is the focus of the next chapters. We discuss scattering amplitudes at the first order of perturbation theory, therefore extensively using the machinery of Feynman diagrams. Here we will just present the calculations and presuppose the connection between Feynman diagrams and the perturbative expansion or how to obtain the scattering amplitude from the Feynman rules. Some of this will be discussed briefly in chapter 4 when we introduce the two-particle S-matrix ($S_{2\to 2}$) of the string sigma model. Part of this chapter is inspired by the first chapter in Prof. P. Dorey 's review [12].

2.1 Sinh-Gordon model

A notable example of integrable field theory in $d = 2$ is the *sinh-Gordon model*, with Lagrangian:

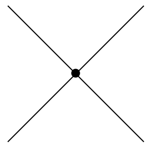
$$\mathcal{L}_{sG} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \lambda^{-2}\left[\cosh(\lambda\phi) - 1\right]. \tag{2.1}$$

Evidently, the interacting potential admits an infinite polynomial expansion in ϕ. For our purposes we will just focus on the first few terms of this expansion; these are all even powers of the fields as one can tell from the parity of the potential. We can solve the scattering processes of such theory perturbatively if we take λ to be a perturbative coupling constant ($\lambda << 1$). It is easy to show that truncating the expansion of the hyperbolic cosine up to the sixth order in the fields we are left with the Lagrangian of an interacting real boson with mass $m = 1$:

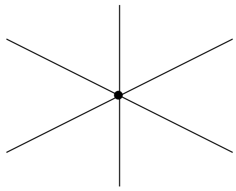
$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\lambda^2}{6!}\phi^6. \tag{2.2}$$

Since the goal is to perform perturbative calculation we need to know how to construct Feynman diagrams for this model. From the Lagrangian, one can directly obtain the Feynman rules:

$$= \frac{i}{p^2 - 1 + i\epsilon}.$$



$= -i\lambda.$



$= -i\lambda^2.$

Figure 2.1

To simplify the calculation it is convenient to use light-cone coordinates for the momenta, defining:

$$(p, \bar{p}) = (p^0 + p^1, p^0 - p^1). \tag{2.3}$$

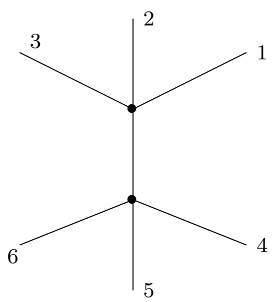
Where p^0 and p^1 are respectively energy and momentum. This way the on-shell condition is $p^2 = p\bar{p} = 1$. At this point, for a given particle i involved in the scattering process, we can parametrize $(p_i, \bar{p}_i) = (a_i, {a_i}^{-1})$, $a_i > 0$ for physical particles. For a relativistic theory such as this one it is often useful to introduce rapidities:

$$p_i^0 = \cosh\theta_i, \quad p_i^1 = \sinh\theta_i. \tag{2.4}$$

θ_i is the rapidity of particle i and, as can be checked from equation (2.3), we have $a_i = e^{\theta_i}$. The parametrisation by means of a_i will be useful when employing crossing relations on Feynman diagrams. Observe in fact that once we have written the amplitude with this parametrization it is possible to pass to the crossed amplitude by simply sending $a_i \to -a_i$.

We want to compute the $3 \to 3$ scattering amplitude at tree-level; this can be immediately obtained from crossing the $6 \to 0$, meaning the process where all the particles are incoming. We need to sum over all connected diagrams with 6 external legs. Besides the (constant) contribution from the 6-leg vertex we only have the topology in Figure 2.2.

$D(a_1, a_2, a_3) =$



$$= \frac{-i\lambda^2}{(p_1 + p_2 + p_3)^2 - 1 + i\epsilon} = \frac{-i\lambda^2 a_1 a_2 a_3}{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3) + i\epsilon(a_1 a_2 a_3)}.$$

Figure 2.2: Feynman diagram with all momenta incoming.

It is easy to see that there are in total 10 of these diagrams contributing to the process. This number depends on the number of different ways of labelling the external legs. Since all the particles involved in the process are identical we can choose one of the two vertices in figure 2.2 and pick 3 momenta out of 6 in $\binom{6}{3} = 20$ ways and, since the two vertices are identical, the number must be halved in order to not count the same diagrams twice.

Once we have found all the possible diagrams we need to impose 2-momentum conservation in order to obtain the physical amplitude. This is the same as requiring the conservation of left and right light-cone momenta, namely the equations:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0, \quad \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} = 0. \tag{2.5}$$

Here, as also in the diagram above, all momenta are assumed incoming. It is also evident from these expressions that $a_i \to -a_i$ substitutes an incoming particle with an outgoing one and this corresponds to crossing a particle since ϕ is a real field.

Now define the amplitude for a generic process (since we have not yet fixed the signs of the parameters) as:

$$A_6\left(\{a_i\}\right) = \frac{1}{2} \sum_{\substack{i,\,j,\,k\,=\,1 \\ i\,\neq\,j\,\neq\,k}}^{6} D(a_i, a_j, a_k) - i\lambda^2. \tag{2.6}$$

This is actually only contribution from the Feynman rules and we are neglecting the external legs factors in front. Notice that in equation (2.6) we added the constant 6-vertex contribution to the sum of *tree* diagrams. Proceed constraining the amplitude A_6 with equation (2.5) and naively put $\epsilon = 0$ from the start. The result of the calculation can be checked to be [12]:

$$A_6 = i\lambda^2 - i\lambda^2 = 0. \tag{2.7}$$

For a generic theory this would be a very surprising result. In fact not only the sum of diagrams with a propagator adds up to a constant: it exactly cancels the single 6-leg vertex and all of this is independent on the choice of the external momenta. This means that the scattering amplitude for every process involving six external particles is zero, since this result does not depend on the sign of each a_i.

From integrability we expect this result actually to hold true for every inelastic process and this can in principle be checked by expanding the original Lagrangian and summing all the contributions. However we did not take into account the case when $i\epsilon$ becomes relevant, namely when the propagators become on-shell. For configurations of momenta satisfying this condition the amplitude is generally non-zero. These configurations, as we see shortly, correspond to $3 \to 3$ scattering processes where the initial and final set of momenta of the 3 particles are exactly the same; these are the configurations that we discussed in section 1.2.

This kind of calculations also provide a way to perturbatively verify the factorisation of the scattering matrix of the model, meaning the factorisation of $n \to n$ processes into a product of amplitudes corresponding to $2 \to 2$ processes. The mechanism explaining how this happens perturbatively is sketched more in detail in the following section.

2.1.1 Sketch of Factorisation for sinh-Gordon model

Since we just claimed that the scattering amplitude corresponds to the sum of their contributions, it is interesting to understand the mechanism of factorisation of the amplitude in terms of Feynman diagrams. In the previous section it was argued that tree-level amplitudes of the *sinh-Gordon model* are non-zero only when the set of momenta is conserved in the scattering. At the tree level the reason for this property can be traced back to the shape of the propagator.

Let us consider diagrams shown in figure 2.2. They can all be put in the form:

$$D_i = \frac{iA_i}{B_i + i\epsilon}.$$

The numerator is the product of two four-vertices. In this case simply:

$$A_i = -\lambda^2. \tag{2.8}$$

The denominator is due to the propagator. For example recall from figure 2.2 that $B_1 \propto (a_1 + a_2)(a_1 + a_3)(a_2 + a_3)$. As already mentioned each B_i will be zero only if at least one of the incoming and outgoing particles have same momenta. As we show shortly the factorisation at tree-level is mainly due to the propagator and hence sinh-Gordon theory constitutes the simplest example where A_i are just constant. This obviously can also work when A_i are momentum-dependent as will be the case for the sigma model on the string worldsheet in chapter 5.

With some algebra we can write D_i as [1]:

$$\frac{iA_i}{B_i + i\epsilon} = A_i \left[i\frac{B_i}{B_i^2 + \epsilon^2} + \frac{\epsilon}{B_i^2 + \epsilon^2} \right]. \tag{2.9}$$

In these expressions the $i\epsilon$ is eventually sent to zero. We can observe that those inside the square brackets are the limiting expressions of distributions, namely the principal value of $\frac{1}{B_i}$, that is $P\left(\frac{1}{B_i}\right)$, (on the left) and the $\delta(B_i)$ (actually with a π proportionality, on the right).

Now, when taking the amplitude we must add the six-vertex contribution (here V_6) and we have something like:

$$\delta(p_{tot})\delta(E_{tot})(-i\lambda^2)\left[\left(P\left(\frac{1}{B_1}\right) + \cdots + P\left(\frac{1}{B_{10}}\right) + \frac{iV_6}{\lambda^2}\right) - i\pi\left(\delta(B_1) + \cdots + \delta(B_{10})\right)\right]. \tag{2.10}$$

The calculation presented in the last section, where $i\epsilon = 0$ was assumed, shows that the bracket containing the principal values and the six-vertex is null. Finally from equation (2.5) one can check that if one of the incoming momenta is also one of the outgoing ones then all incoming and outgoing momenta are the same. Therefore $\delta(B_i)\delta(p_{tot})\delta(E_{tot})$ factors exactly impose that the initial set of momenta stays the same in the final state. Consequently the scattering does not depend on the V_6 term but only on the $2 \to 2$ amplitudes.

It is also possible to explicitly show the factorisation of the three particle interaction by isolating couples of Feynman diagrams whose sum produce the delta-function responsible for the conservation of the initial set of momenta. This is presented for similar models in ref. [21]. In what follows we highlight the main steps of such derivation since this mechanism can be generalised to different integrable models and will also be applicable to the string sigma model studied in the rest of the thesis.

Let us concentrate on the diagrams shown in figure 2.3.

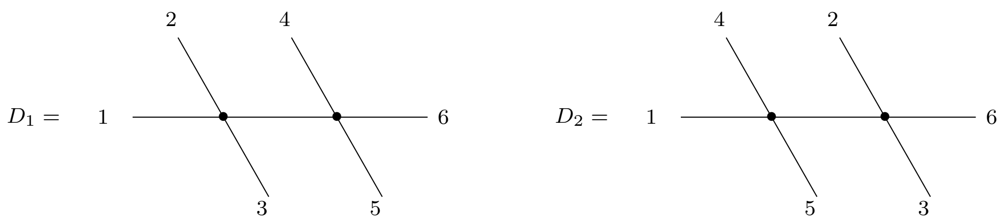


Figure 2.3: Couple of diagrams diverging for $a_2 \to -a_3$.

[1] This is essentially Sokhotski–Plemelj theorem. This decomposition was first used to argue the factorisation of scattering in the seminal Zamolodchikov-Zamolodchikov paper [5].

We intend the external momenta to be all incoming. We want to see what happens close to the configurations allowed by integrability, that is those where the set of final momenta is the same as the set of initial ones. These are represented by points in the phase space; concentrate for example in the region of the phase space close to the configurations:

$$a_1 = -a_6, \quad a_2 = -a_3, \quad a_4 = -a_5. \tag{2.11}$$

First note that this configuration is the one corresponding to the scattering $\phi(p_1) + \phi(p_2) + \phi(p_4) \to \phi(p_1) + \phi(p_2) + \phi(p_4)$. Observe that in total 6 of the 10 diagrams become singular at this configuration, and the two in figure 2.3 are among the 6. Therefore the sum of the two diagrams in the limit $a_2 \to -a_3$ will be:

$$-i\lambda^2 \left[\frac{-a_1 a_2^2}{(a_2 + a_1)(a_1 - a_2)(a_2 + a_3) + a_1 a_2^2 i\epsilon} + \frac{+a_1 a_2^2}{(a_2 - a_1)(-a_1 - a_2)(a_2 + a_3) + a_1 a_2^2 i\epsilon} \right]. \tag{2.12}$$

This is just:

$$i\lambda^2 a_1 a_2^2 \left[\frac{1}{(a_2 + a_3)(a_1^2 - a_2^2) + i a_1 a_2^2 \epsilon} - \frac{1}{(a_2 + a_3)(a_1^2 - a_2^2) - i a_1 a_2^2 \epsilon} \right]. \tag{2.13}$$

From this expression we can recover a delta-function using the limit:

$$\lim_{\epsilon \to 0^+} \left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = -2\pi i \delta(x). \tag{2.14}$$

Eventually the sum of the two diagrams yields:

$$D_1 + D_2 = -2\pi\lambda^2 \frac{a_1 a_2^2}{|a_1^2 - a_2^2|} \delta(a_2 + a_3) = -\pi\lambda^2 \frac{1}{|\sinh\theta_{12}|} \delta(\theta_2 + \theta_3). \tag{2.15}$$

In the rightmost member the expression is written in terms of the rapidities, where θ_{12} is the difference of the rapidities of particles 1 and 2.

To finally compute the amplitude of the $3 \to 3$ process we should also add all the other diagrams. Since other 4 of them diverge it is easy to show that we can obtain similar expressions as in equation (2.15) for two other couples of diagrams. Finally we should plug back the total energy and momentum conservation alongside the external factors. The expressions of the overall factors perfectly agree with the factorisation of the amplitude [21]. Considering only the contribution in (2.15) the result for the tree-level amplitude is:

$$S_{124}^{D_1+D_2}(\theta_1, \theta_2, \theta_4) = S_{12}(\theta_1, \theta_2) S_{14}(\theta_1, \theta_4) \times \delta(\theta_1 + \theta_6)\delta(\theta_2 + \theta_3)\delta(\theta_4 + \theta_5). \tag{2.16}$$

Considering also the remaining two couples of singular diagrams we can apply the same logic to show that we have in total three contributions (the deltas multiplying everything are removed here):

$$S_{12}(\theta_1, \theta_2) S_{14}(\theta_1, \theta_4) + S_{12}(\theta_1, \theta_2) S_{24}(\theta_2, \theta_4) + S_{24}(\theta_2, \theta_4) S_{14}(\theta_1, \theta_4). \tag{2.17}$$

This is the tree-level contribution due to only one of the possible kinematical configurations that preserve the initial set of momenta. In general since the particles in the scattering are identical we have 6 points in the phase space where the S-matrix exhibits this behaviour.

In conclusion observe that the full factorisation equation, equation (1.8), would yield schematically:

$$S_{124} = \left(\mathbb{1} + S_{12}^{tree} + \cdots\right)\left(\mathbb{1} + S_{24}^{tree} + \cdots\right)\left(\mathbb{1} + S_{41}^{tree} + \cdots\right) \tag{2.18}$$

At the relevant order of perturbation theory, namely λ^2, this equation produces exactly three terms each corresponding to the product of 2 two-particle amplitudes. Therefore the perturbative result we found reproduces perfectly the expected factorisation of the S-matrix.

2.2 Bullough–Dodd model

Bullough-Dodd model is another notable example of an integrable theory in $d = 2$. The process we are interested in computing for this model consists in total of 5 externa particles, meaning we need to consider Feynman diagrams with 5 external legs. Since the total number of particles in the process is odd the initial and final states must have a different number of particles and hence the integrability of the theory dictates that the amplitude must be zero.

We want to carry out an explicit calculation of a tree-level amplitude and verify the absence of particle production. This kind of calculation becomes very cumbersome and hence not easy to perform analytically; in the following we will show how to deal with this problem.

Let us start from the following Lagrangian:

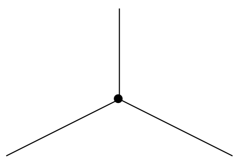
$$\mathcal{L}_{BD} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \lambda^{-2}\left[e^{2\lambda\phi} + 2e^{-\lambda\phi} - 3\right]. \tag{2.19}$$

This is the Lagrangian for the Bullough–Dodd model. We expect it to show the same scattering properties as the sinh -Gordon theory. In this case the cancellation of the sum of diagrams contributing at tree-level to the $5 \to 0$ interaction is even more remarkable since we have in total 3 different topologies of Feynman diagrams. Let us see this in detail. We must once again expand the Lagrangian up to fifth order in the fields, finding:

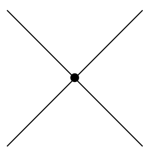
$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\phi)^2 - \frac{\lambda}{3!}\phi^3 - \frac{3\lambda^2}{4!}\phi^4 - \frac{5\ \lambda^3}{5!}\phi^5. \tag{2.20}$$

This time the Feynman rules are:

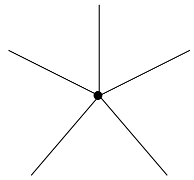
$$= \frac{i}{p^2 - 1 + i\epsilon}.$$



$$= -i\lambda.$$



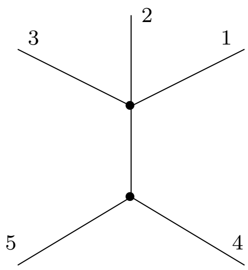
$$= -3i\lambda^2.$$



$$= -5i\lambda^3.$$

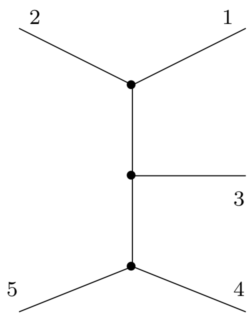
Figure 2.4: Feynman rules

In this case the relevant topologies for processes involving 5 particles are shown in figures 2.5 and 2.6. We parametrise the momenta as for the sinh-Gordon model, as in equation (2.4). For determining the multiplicity of diagrams of the type shown in figure 2.5 we can again just argue that we can choose 3 out of 5 external momenta in $\binom{5}{3} = 10$ without double counting this time since the two vertices are different. For the second topology the argument would be the following: choose 1 out of 5 labels for the vertex with one external leg and then choose 2 out of 4 particles for the other two identical diagrams (avoiding double counting). The end result is $5 \times \binom{4}{2} \times \frac{1}{2} = 15$ total diagrams.



$$= \frac{-3i\lambda^3}{(p_1 + p_2 + p_3)^2 - 1 + i\epsilon} = \frac{-3i\lambda^3 a_1 a_2 a_3}{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3) + i(a_1 a_2 a_3)\epsilon}.$$

Figure 2.5: First topology.



$$= \frac{-i\lambda^3}{[(p_1 + p_2)^2 - 1 + i\epsilon]\,[(p_1 + p_2 + p_3)^2 - 1 + i\epsilon]}$$

$$= \frac{-i\lambda^3\,(a_1 a_2 a_3)\,(a_1 a_2)}{[(a_1 + a_2)(a_2 + a_3)(a_1 + a_3) + i\,(a_1 a_2 a_3)\,\epsilon]\,\left[a_1^2 + a_2^2 + a_1 a_2 + i\,(a_1 a_2)\,\epsilon\right]}.$$

Figure 2.6: Second topology.

To obtain the scattering amplitude we need to constrain the 5 external momenta by the two equations of conservation:

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0, \quad \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} = 0. \tag{2.21}$$

In this case we are always allowed to ignore the $+i\epsilon$ prescription in the propagators since the equations in (2.21) never allow for an internal momentum to be on-shell. Consequently for this amplitude the only check of integrability is the absence of particles production.

Due to the quantity of diagrams the explicit calculation is too large to do by hand. It was performed using Mathematica. Here we show a convenient way to present it in a compact form. Suppose that the constraints in equation (2.21) leave the amplitude only depending on a_1, a_2, a_3. In order to reduce the length of the expressions it is useful to define the following symmetric polynomials (for more details we refer the reader to ref. [22]) of the external momenta:

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = a_1 a_2 + a_2 a_3 + a_1 a_3, \quad s_3 = a_1 a_2 a_3. \tag{2.22}$$

First we express the sum of diagrams of the first topology in figure 2.5 in terms of these polynomials.

Suppose we call $\{(\mathcal{D}_1)\}_{i=1,\ldots,10}$ the set of diagrams of such kind. A quick check shows that the sum, divided by the coupling factor for convenience, can be written in terms of these polynomials as:

$$\frac{\sum_{i=1}^{10} (\mathcal{D}_1)_i}{-i\lambda^3} = \frac{6s_1^4 s_2 s_3 - 6s_1^3 s_2^3 - 6s_1^3 s_3^2 + 6s_1^2 s_2^2 s_3 + 6s_1 s_2^4 + 9s_1 s_2 s_3^2 - 6s_2^3 s_3}{-s_1^4 s_2 s_3 + s_1^3 s_2^3 + s_1^3 s_3^2 - s_1^2 s_2^2 s_3 - s_1 s_2^4 + s_2^3 s_3} = \frac{N_1}{D_1}. \tag{2.23}$$

It is useful for the final summation to isolate the numerator N_1 and denominator D_1 of the sum. Similarly for the contributions of the diagrams in figure 2.6 we have:

$$\frac{N_2}{D_2} = \frac{\sum_{i=1}^{15} (\mathcal{D}_2)_i}{-i\lambda^3}. \tag{2.24}$$

In this case the expressions of numerator and denominator are still quite large:

$$\begin{aligned} N_2 = &-4096 s_1 s_3^4 s_2^{13} + 4096 s_3^5 s_2^{12} + 12288 s_1^3 s_3^4 s_2^{12} - 12288 s_1^2 s_3^5 s_2^{11} - 12288 s_1^5 s_3^4 s_2^{11} - 36864 s_1 s_3^6 s_2^{10} \\ &+ 4096 s_1^7 s_3^4 s_2^{10} + 86016 s_1^3 s_3^6 s_2^9 + 20480 s_1^6 s_3^5 s_2^9 - 61440 s_1^5 s_3^6 s_2^8 - 12288 s_1^8 s_3^5 s_2^8 - 73728 s_1^4 s_3^7 s_2^7 \\ &+ 86016 s_1^6 s_3^7 s_2^6 + 12288 s_1^9 s_3^6 s_2^6 - 12288 s_1^8 s_3^7 s_2^5 - 36864 s_1^7 s_3^8 s_2^4 - 4096 s_1^{10} s_3^7 s_2^4 + 4096 s_1^9 s_3^8 s_2^3. \end{aligned}$$

$$\begin{aligned} D_2 = &-4096 s_1 s_3^4 s_2^{13} + 4096 s_3^5 s_2^{12} + 12288 s_1^3 s_3^4 s_2^{12} - 12288 s_1^2 s_3^5 s_2^{11} - 12288 s_1^5 s_3^4 s_2^{11} + 4096 s_1^7 s_3^4 s_2^{10} \\ &+ 12288 s_1^3 s_3^6 s_2^9 + 20480 s_1^6 s_3^5 s_2^9 - 24576 s_1^5 s_3^6 s_2^8 - 12288 s_1^8 s_3^5 s_2^8 + 12288 s_1^6 s_3^7 s_2^6 + 12288 s_1^9 s_3^6 s_2^6 \\ &- 12288 s_1^8 s_3^7 s_2^5 - 4096 s_1^{10} s_3^7 s_2^4 + 4096 s_1^9 s_3^8 s_2^3. \end{aligned}$$

Once we have found these individual contributions we can sum everything together to get:

$$\frac{N_1 D_2 + N_2 D_1}{D_1 D_2} = -5. \tag{2.25}$$

Once we plug back the coupling factor the result is $5i\lambda^3$. This contribution is opposite to the 5-leg vertex and this shows that the amplitude for whatever process involving 5 particles is always null at the tree-level. Similar yet increasingly more complicated calculations can be carried out for the higher-order vertices obtained from the expansion of the Lagrangian in (2.19) still yielding a vanishing sum of diagrams at tree-level.

Chapter 3

The non-linear sigma model

In the rest of this work we will study a non-linear sigma model. This usually refers to a class of models whose Lagrangian can be written as:

$$\mathcal{L}_{NLSM} = \frac{1}{2}\,(\partial_\mu \phi^a)\,(\partial^\mu \phi^b) g_{ab}(\phi) - V(\phi). \tag{3.1}$$

Here ϕ^a are scalar fields with $a = 1, \ldots, m$ and constitute a map from the base space (from now on referred to as Σ) to the target space $\mathcal{M}$, such that $\dim(\mathcal{M}) = m$. Hence we can take the symmetric tensor g as the metric on the target space. The model is referred to as *non-linear* when the target space is a curved manifold. In such case $g_{ab}(\phi)$ depends explicitly on the coordinates on $\mathcal{M}$, namely the fields, producing additional interaction terms in the above Lagrangian.

Sigma models offer a tremendously useful framework in the context of integrable field theories. Many of them have been shown to be integrable classically (for example see ref. [10]) and in some cases we are able to deform an integrable sigma model in such ways as to produce a whole continuum of integrable theories [23]. One of the possible target spaces that was shown to be classically integrable is $AdS^3 \times S^3 \times T^4$ and this is precisely the target space we will focus on in the rest of this thesis.

Coincidentally a Lagrangian satisfying the form presented in equation (3.1) is also well suited to describe theories of bosonic strings. In fact, the theory that will be investigated in the following describes a bosonic closed string of circumference $2r$ propagating in a 10-dimensional spacetime $\mathcal{M}$. The string is a one-dimensional object, that is it has zero 'width', meaning that the surface describing its points in spacetime is parametrised by 2 coordinates and is called the string worldsheet Σ. As the letter suggest the string worldsheet will be the base space and the spacetime into which it is embedded will be the target manifold. Since the string is closed the worldsheet is parametrised by coordinates (τ, σ) with $\sigma \in (-r, r)$, respectively time and space coordinates on the worldsheet. Therefore the base space is topologically a cylinder [1].

In this chapter we are interested in the light-cone gauge-fixed string sigma model defined on the worldsheet. We abstain from any in depth discussion on the actual string theory but most of what we touch on can be found in standard references such as [24],[25].

[1] This clearly poses the problem of defining asymptotic particle states which we will circumvent by working in the *decompactification limit*.

3.1 Bosonic strings with mixed flux backgrounds

First we start by introducing the generic action:

$$S = -\frac{T}{2}\int_{-\infty}^{+\infty} d\tau \int_{-r}^{+r} d\sigma \left[\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\right]. \tag{3.2}$$

There are two different sets of indices, $\mu, \nu = 0, \ldots, 9$ and $a, b = \tau, \sigma$. X^μ are functions of (τ, σ) and can be thought of as coordinates of the string in spacetime with metric defined through the symmetric tensor $G_{\mu\nu}$. The signature is chosen to be mostly positive. h^{ab} is the metric on the worldsheet. In the following we will just write $h^{ab}\sqrt{-h} = h^{ab}\sqrt{-\det(h)} = \gamma^{ab}$. T is the string tension.

This expression for the action enjoys a number of symmetries that here we just mention (see *e.g.* ref. [24] for a serious discussion) in order to justify the need for the gauge-fixing that will be examined in detail in a moment. In fact, this description of the string is clearly redundant. In the same way as for the relativistic particle action, the action in equation (3.2) is independent on the parametrisation. In addition since the worldsheet is 2-dimensional the action is also invariant under a Weyl transformation on the worldsheet metric h_{ab}. These redundancies are to be treated as a gauge freedom and as a consequence a convenient gauge can be chosen.

On the other hand this action can be interpreted as that of a two-dimensional field theory if one thinks of the X^μ as fields instead of coordinates. This way we have 10 bosons living in the worldsheet coupled to the G-field. We usually talk about a non-linear σ model when the latter is the metric on a curved target space, producing additional field-dependent terms in the Lagrangian. Not all the bosonic fields are necessarily physical. In fact the gauge freedom means that some of these fields might not be. This will be investigated in the next sections when fixing the light-cone gauge.

What is presented in equation (3.2) is not actually the action of the model we will study. We introduce another term containing a differential 2-form, called Kalb-Ramond form, that is added to the previous action. With this addition the full Lagrangian looks like:

$$\mathcal{L} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{3.3}$$

$B_{\mu\nu}$ is an anti-symmetryc field and ϵ^{ab} is the Levi-Civita symbol with $\epsilon^{01} = 1$. On the other hand $G_{\mu\nu}$ is the metric on $AdS^3 \times S^3 \times T^4$, which we introduce shortly. The model is said to be a mixed flux model because the B-field depends explicitly on a parameter q which interpolates between RR and NSNS fluxes in the range $q \in [0, 1]$. This and similar models have been already studied in recent years and the mixed flux model was itself shown to be integrable at the classical level (some references are [3],[26]). However such theories are still not completely understood. The intention of the rest of this work is to study the mixed flux model using a perturbative approach and verify the properties of the scattering that descend from the integrability of the model.

In the next section we introduce the expression of the metric on $AdS_3 \times S^3 \times T^4$ and of the B-field and once we have introduced all the ingredients we move to light-cone gauge-fixing and the derivation of the NLSM action.

3.1.1 $AdS_3 \times S^3 \times T^4$ metric and B-field

We will work on target background $AdS^3 \times S^3 \times T^4$, whose metric is:

$$\begin{aligned} ds^2 = &-\Big(\frac{4+z_1^2+z_2^2}{4-z_1^2-z_2^2}\Big)^2 dt^2 + \Big(\frac{4}{4-z_1^2-z_2^2}\Big)^2 (dz_1^2+dz_2^2) \\ &+ \Big(\frac{4-y_1^2-y_2^2}{4+y_1^2+y_2^2}\Big)^2 d\phi^2 + \Big(\frac{4}{4+y_1^2+y_2^2}\Big)^2 (dy_1^2+dy_2^2) \\ &+ \sum_{j=5}^{8} dx_j dx_j. \end{aligned} \tag{3.4}$$

The coordinates on $AdS_3 \times S^3 \times T^4$ are $X^\mu = \{t, \phi, z_1, z_2, y_1, y_2, x_5, x_6, x_7, x_8\}$. The sector describing AdS_3 has coordinates (t, z_1, z_2) while (ϕ, y_1, y_2) are coordinates in S^3 and finally (x_5, x_6, x_7, x_8) are those on T^4. The torus sector is flat while the rest has non-zero curvature. One notices immediately that the metric is independent on the two coordinates t, ϕ. This will be relevant later.

The Kalb-Ramond 2-form is chosen to have the expression in coordinates:

$$B = \frac{32q}{(4-z_1^2-z_2^2)^2}\left[z_1 dz_2 \wedge dt + z_2 dt \wedge dz_1\right] + \frac{32q}{(4+y_3^2+y_4^2)^2}\left[y_3 dy_4 \wedge d\phi + y_4 d\phi \wedge dy_3\right]. \tag{3.5}$$

It does not have degrees of freedom on the directions on the torus, but only on $AdS_3 \times S^3$.

3.2 Light-cone gauge fixing

As anticipated the description of the model via the Lagrangian in equation (3.3) is characterised by some non-physical degrees of freedom. A way of fixing this redundancy is to impose the light-cone gauge; this procedure allows us to understand what are the actual physical degrees of freedom. The choice of the light-cone gauge is not unique but it is convenient since it simply relates the string Hamiltonian with the Hamiltonian of the model on the worldsheet. In the following this procedure is explained in detail.

From now on we define

$$\dot{X}^\mu \equiv \partial_\tau X^\mu \quad \text{and} \quad \acute{X}^\mu \equiv \partial_\sigma X^\mu. \tag{3.6}$$

Introducing the conjugate momenta

$$p_\mu = \frac{\delta \mathcal{L}}{\delta(\dot{X}^\mu)} = -T\Big(\gamma^{\tau\beta} G_{\mu\nu} + \epsilon^{\tau\beta} B_{\mu\nu}\Big)\partial_\beta X^\nu \tag{3.7}$$

The action associated with the Lagrangian (3.3) can be rewritten in the first-order form:

$$S = \int d\tau d\sigma \Big(p_\mu \dot{X}^\mu + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2\Big), \tag{3.8}$$

where

$$C_1 = p_\mu \acute{X}^\mu, \tag{3.9a}$$
$$C_2 = G^{\mu\nu} p_\mu p_\nu + T^2 G_{\mu\nu} \acute{X}^\mu \acute{X}^\nu + 2T G^{\mu\nu} B_{\nu\kappa} p_\mu \acute{X}^\kappa + T^2 G^{\mu\nu} B_{\mu\kappa} B_{\nu\lambda} \acute{X}^\kappa \acute{X}^\lambda. \tag{3.9b}$$

To eventually obtain the action for the model we will have to impose $C_1 = 0$ and $C_2 = 0$, which are called Virasoro constraints. These two constraints will be used to remove the longitudinal modes of the string x_-, x_+, p_-, p_+. Therefore in the first-order formalism the worldsheet metric does not appear explicitly in the calculations. Evidently this is not the only viable strategy. It is convenient for us because we are not interested in the expression of the worldsheet metric or of the longitudinal fields. We could adopt a different approach and instead solve for those objects, as outlined for example in ref. [27]. On the other hand in our derivation of the gauge-fixed worldsheet theory, which closely resembles the one presented in ref. [28] (where instead the background spacetime is $AdS_5 \times S^5$), we will only need the $G_{\mu\nu}$ and $B_{\mu\nu}$ fields.

Before solving the two equations we need to fix the light-cone gauge. In order to fix the gauge we will use the following light-cone coordinates:

$$x_- = \phi - t \quad , \quad x_+ = a\phi + (1-a)t \tag{3.10}$$

together with

$$\begin{aligned} z &= \frac{1}{\sqrt{2}}(z_1 + i z_2) \quad , \quad y = \frac{1}{\sqrt{2}}(y_1 + i y_2)\,, \\ u &= \frac{1}{\sqrt{2}}(x_5 + i x_6) \quad , \quad v = \frac{1}{\sqrt{2}}(x_7 + i x_8)\,. \end{aligned} \tag{3.11}$$

The expressions for the metric and 2-form components in these coordinates are reported in appendix A. The conjugate momenta relative to the light-cone coordinates in (3.10) are:

$$p_+ = p_\phi + p_t, \qquad p_- = (1-a)p_\phi - a p_t. \tag{3.12}$$

In these definitions $a \in [0,1]$. The light-cone gauge, which we impose in one moment, will be attained through a condition on x_+. Therefore the parameter a is simply interpolating between different gauges. For explicit calculations it is usually convenient to fix a value of a. However most of the results in this work are true for a generic value of it. Evidently, observables (such as the quantised energy spectrum of the string) will not be dependent on a.

The action in equation (3.8) now takes the form:

$$S = \int d\tau d\sigma \Big(p_+\dot{x}_+ + p_-\dot{x}_- + p_z\dot{z} + p_{\bar{z}}\dot{\bar{z}} + p_y\dot{y} + p_{\bar{y}}\dot{\bar{y}} + p_u\dot{u} + p_{\bar{u}}\dot{\bar{u}} + p_v\dot{v} + p_{\bar{v}}\dot{\bar{v}} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}C_1 + \frac{1}{2T\gamma^{\tau\tau}}C_2 \Big)\,. \tag{3.13}$$

The invariance of the string model under shifts of t and ϕ coordinates produces two conserved charges:

$$E = -\int_{-r}^{r} d\sigma p_t, \qquad J = \int_{-r}^{r} d\sigma p_\phi. \tag{3.14}$$

Clearly these are the total energy of the string and the angular momentum in the ϕ direction.

We can find relations in terms of the light-cone momenta:

$$P_+ = \int_{-r}^{r} d\sigma p_+ = J - E, \qquad P_- = \int_{-r}^{r} d\sigma p_- = (1-a)J + aE. \tag{3.15}$$

Additionally there is a condition that stems from the nature of the string itself. It is called the level-matching condition and it is a consequence of the fact that the strings in question are closed. Therefore it needs to hold that:

$$x^\mu(r) - x^\mu(-r) = 0.\text{[2]} \tag{3.16}$$

This means that we are choosing a string with zero winding number, the reason being that only in this case the large string expansion, discussed in the next section, will be justified. All of the relations

[2] Compact directions, such as the ϕ angle on S^3 can have non-zero winding number: $\phi(r) - \phi(-r) = 2\pi m$.

and conditions above will still hold after fixing the gauge and will therefore constrain the physical sector of the Hilbert space.

Let us impose the uniform light-cone gauge:

$$x_+ = \tau \quad , \quad p_- = 1. \tag{3.17}$$

First observe a few things. From equations (3.15), integrating the constant 1 we obtain:

$$2r = (1-a)J + aE. \tag{3.18}$$

The radius of the string is expressed in terms of physical quantities. This suggests that the light-cone gauge-fixing breaks the symmetry under rescalings on the worldsheet [17].

Also, taking the constraint $C_1 = 0$ we find $\acute{x}_- = -p_a \acute{X}^a$,where the latin indices now and in the following run from 0 to 7, which will be the labels of the 8 physical fields. From this fact it follows that:

$$0 = \Delta x_- = \int_{-r}^{r} x'_- = -\int_{-r}^{r} d\sigma p_\mu x'^\mu = P_{w.s.}. \tag{3.19}$$

One the right hand side is the definition of the worldsheet total momentum whose conservation is a consequence of the theory's invariance under σ traslations. Then from equation (3.16) we see that the physical states are required to have 0 total momentum.

At this point the action becomes:

$$S = \int d\tau d\sigma \Big(p_+ + p_z \dot{z} + p_{\bar{z}} \dot{\bar{z}} + p_y \dot{y} + p_{\bar{y}} \dot{\bar{y}} + p_u \dot{u} + p_{\bar{u}} \dot{\bar{u}} + p_v \dot{v} + p_{\bar{v}} \dot{\bar{v}} \Big) \,. \tag{3.20}$$

Notice that $\dot{x}_-$ in the action was neglected being a total derivative. It is clear from the action that

$$p_+ = -\mathcal{H} \tag{3.21}$$

where $\mathcal{H}$ is the worldsheet hamiltonian density. This is expected since the worldsheet Hamiltonian generates the τ translations. Also, from the equations in (3.15) it is easy to relate $\mathcal{H}$ to the target space energy from which we can find the spectrum of the quantised string theory.

The goal now is to find the worldsheet Hamiltonian as a function of the z, y, u, v fields. To do so we need to solve for p_+ the two Virasoro constraints, as shown in a moment.

However before going through the calculation let us summarise what was already found. Through the gauge fixing procedure we have reduced the number of d.o.f. to 8, having removed x_+ and x_-. These remaining physical degrees of freedom will be represented by 4 complex fields z, y, u, v.

3.2.1 Decompactification limit and large string tension expansion

After we have fixed the gauge we can start the analysis of the model. The goal is to study the excited states of the string for generic (finite) length. This can be attained through integrability techniques once the scattering matrix on the worldsheet is known. However the string sigma model is defined on a 2-dimensional cylinder and the the notion of asymptotic particles states is ill-defined. To have a theory that is instead defined on the 2-dimensional plane we need to consider the *decompactification limit*, that is a limit where the string length goes to infinity.

From equations (3.15) and (3.17) it is clear that after gauge fixing:

$$P_- = 2r = (1-a)J + aE.$$

The decompactification limit is taken when $P_- \to \infty$, keeping the string tension constant. This limit effectively unfolds the worldsheet from a cylinder to a plane. Observe that this does not affect the gauge fixed Lagrangian but only changes the integration boundaries of the action. Note also that we need to have P_+ finite since it is the Hamiltonian of the gauge fixed theory. Then from the equations in (3.15) it is clear that both E and J will go to infinity in the decompactification limit. What is most relevant for our purposes, the resulting theory in this limit is a $d = 2$ quantum field theory defined on the (τ, σ) plane.

There is one last step to take before actually solving the Virasoro constraints. The perturbative approach works when we can split the Lagrangian in a free, solvable, component and weakly coupled interaction component. Although not immediate to see, this treatment can be applied to the string sigma model if we assume the string tension large enough. Suppose the action takes the form:

$$S = \int d\sigma d\tau \mathcal{L}_{gf}. \tag{3.22}$$

Then we can perform the rescaling $\sigma \to T\sigma$. It is clear from the Virasoro constraints obtained from equations (3.9) that under the rescaling p_+ is independent on T, and hence so is the Lagrangian. As a consequence the rescaled action becomes:

$$S = T \int d\sigma d\tau \mathcal{L}^{res}_{gf}.$$

Then through a simple field redefinition

$$X \to \frac{X}{\sqrt{T}}$$

we get a canonically normalized kinetic term. After this redefinition the lagrangian can be expanded in powers of the fields. Each of the interaction terms then carries a power of the string tension as:

$$S = \int d\sigma d\tau \big[\mathcal{L}^{(2)} + T^{-\frac{1}{2}}\mathcal{L}^{(3)} + T^{-1}\mathcal{L}^{(4)} + \cdots\big]. \tag{3.23}$$

This manipulation makes it clear that when the string tension is large enough the interaction terms have small couplings. This shows that by taking the large string regime we can work perturbatively and also that the coupling of each operator becomes smaller as the number of fields in the operator grows.

3.2.2 Solving for the Hamiltonian

Now we can pass to the calculation to obtain the gauge-fixed Hamiltonian. As anticipated, after gauge-fixing, p_+ can be obtained by solving the Virasoro constraints. The two terms that we need to equate to zero are:

$$C_1 = p_\mu \acute{X}^\mu = 0, \tag{3.24a}$$

$$C_2 = G^{\mu\nu}p_\mu p_\nu + T^2 G_{\mu\nu}\acute{X}^\mu\acute{X}^\nu + 2TG^{\mu\nu}B_{\nu\kappa}p_\mu\acute{X}^\kappa + T^2G^{\mu\nu}B_{\mu\kappa}B_{\nu\lambda}\acute{X}^\kappa\acute{X}^\lambda = 0. \tag{3.24b}$$

In the following, we will take $T = 1$. We can forget about the string tension since, as observed in the last section, the expansion in large T will correspond to an expansion in small fields. The first equation after imposing the gauge-fixing conditions in equations (3.17) becomes:

$$\acute{x}_- = -p_a\acute{X}^a. \tag{3.25}$$

We now substitute this expression in (3.24b) and find $\mathcal{H}$ by solving the constraint $C_2 = 0$:

$$a_1\mathcal{H}^2 - a_2\mathcal{H} + a_3 = 0 \tag{3.26}$$

where

$$\begin{aligned} a_1 &= G^{++}, \\ a_2 &= 2G^{+-} + 2G^{++}B_{+a}\acute{X}^a + 2G^{+-}B_{-a}\acute{X}^a, \\ a_3 &= G^{--} + G^{ab}p_a p_b + G_{ab}\acute{X}^a\acute{X}^b + G_{--}p_a p_b\acute{X}^a\acute{X}^b + 2G^{--}B_{-a}\acute{X}^a \\ &+ 2G^{-+}B_{+a}\acute{X}^a + 2G^{ac}B_{-c}p_a p_b\acute{X}^b + G^{cd}B_{-c}B_{-d}p_a p_b\acute{X}^a\acute{X}^b \\ &+ G^{++}B_{+a}B_{+b}\acute{X}^a\acute{X}^b + G^{--}B_{-a}B_{-b}\acute{X}^a\acute{X}^b + 2G^{+-}B_{+a}B_{-b}\acute{X}^a\acute{X}^b. \end{aligned} \tag{3.27}$$

In principle we could solve this second order equation exactly obtaining, as expected, the Hamiltonian in terms of only the physical fields and their derivatives. In particular out of the two solutions of the quadratic equation (3.26), namely

$$\mathcal{H}_\pm = \frac{a_2}{2a_1} \pm \sqrt{\left(\frac{a_2}{2a_1}\right)^2 - \frac{a_3}{a_1}}$$

the one with the + should be chosen; this in order to have a positive definite energy. However in this form the Hamiltonian would not be fit for quantization. Instead we will solve equation (3.26) iteratively to the desired order in the fields. Here are the main steps of the procedure.

We start observing, as it can be checked from the expressions in appendix A, that by setting all the physical fields to 0 we obtain:

$$a_1 = -1 + 2a \quad , \quad a_2 = 2 \quad , \quad a_3 = 0. \tag{3.28}$$

Therefore at the second order in the fields (we write Φ for a generic field to simplify the notation), the solution is entirely captured by

$$\mathcal{H}^{(2)} = \frac{a_3}{a_2} + O(\Phi^4). \tag{3.29}$$

Clearly the assumption is that the lowest order for the Hamiltonian is the second so that the term with $\mathcal{H}^2$ only contributes at the fourth order. In fact, iterating the equation by substituting $\mathcal{H}^{(2)}$ in the squared Hamiltonian term we find an equation for the fourth order Hamiltonian:

$$\mathcal{H}^{(4)} = \frac{a_3}{a_2} + \frac{a_1}{a_2}\frac{a_3^2}{a_2^2} + O(\Phi^6). \tag{3.30}$$

Finally at the sixth order, by applying the same steps, the solution is:

$$\mathcal{H}^{(6)} = \frac{a_3}{a_2} + \frac{a_1}{a_2}\left(\frac{a_3}{a_2} + \frac{a_1}{a_2}\frac{a_3^2}{a_2^2}\right)^2 + O(\Phi^8). \tag{3.31}$$

We will stop at this order of the expansion as we are eventually interested in studying tree-level scattering processes with at most 6 particles. As it can be checked by expanding the terms defined in equations (3.27) only even powers of the fields and their derivatives will contribute to the Hamiltonian. Clearly to obtain the Hamiltonian at the sixth order the leading term in equation (3.31) must be expanded in powers and only the powers up to the sixth order must be kept. The calculation is quite cumbersome and was performed using Mathematica. The Hamiltonian up to the fourth and the one containing the sixth powers of the fields are shown in appendix B.

3.3 The free theory

The expansion in the fields that was just introduced allows us to quantise the theory identifying the quadratic term as the free fields. The interaction terms will be treated as a perturbation of the free theory. We should first look for the Euler-Lagrange equations for the free theory. In general the Lagrangian can be found starting from the Hamiltonian and inverting the Legendre transform. Although it is not a trivial procedure, if we are only looking for the quadratic term it is easy to show that the Lagrangian is:

$$\mathcal{L}^{(2)} = \dot{z}\dot{\bar{z}} + \dot{y}\dot{\bar{y}} + \dot{u}\dot{\bar{u}} + \dot{v}\dot{\bar{v}} - \acute{z}\acute{\bar{z}} - \acute{y}\acute{\bar{y}} - \acute{u}\acute{\bar{u}} - \acute{v}\acute{\bar{v}} - y\bar{y} - z\bar{z} - iq\left(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}}\right). \tag{3.32}$$

From equation (3.32) we see that parity on the worldsheet is not a symmetry of the free theory, and this can be checked if we transform the Lagrangian under $\sigma \to -\sigma$. We can however check that a new discrete symmetry arises combining worldsheet parity together with the $q \to -q$ transformation.

The equations of motion for the z and $\bar{z}$ fields are then:

$$\begin{aligned} \left(\partial_\tau^2 - \partial_\sigma^2 + 2iq\partial_\sigma + 1\right) z &= 0, \\ \left(\partial_\tau^2 - \partial_\sigma^2 - 2iq\partial_\sigma + 1\right) \bar{z} &= 0. \end{aligned} \tag{3.33}$$

The same equations hold for the y and $\bar{y}$ fields. One notices immediately that the Lagrangian and hence the EOM are not invariant under 2-dimensional (worldsheet) Lorentz transformations. This is expected since the condition imposed to fix the gauge (equation (3.17)) breaks Lorentz symmetry. Therefore the gauge-fixed theory is not relativistic. Finally we can identify the two remaining fields as massless bosons, since they obey the equations:

$$\begin{aligned} \left(\partial_\tau^2 - \partial_\sigma^2\right) u &= 0, \qquad \left(\partial_\tau^2 - \partial_\sigma^2\right) v = 0 \\ \left(\partial_\tau^2 - \partial_\sigma^2\right) \bar{u} &= 0, \qquad \left(\partial_\tau^2 - \partial_\sigma^2\right) \bar{v} = 0. \end{aligned} \tag{3.34}$$

The dispersion relations in terms of the spatial momentum p_1 can be argued from the EOM. The actual expansion into creation and annihilation operators is shown in the following section. For the massive modes the dispersion relations are given by:

$$\begin{aligned} \omega(p_1) \equiv \omega_z(p_1) = \omega_y(p_1) &= \sqrt{p_1^2 - 2qp_1 + 1}, \\ \bar{\omega}(p_1) \equiv \omega_{\bar{z}}(p_1) = \omega_{\bar{y}}(p_1) &= \sqrt{p_1^2 + 2qp_1 + 1}. \end{aligned} \tag{3.35}$$

Note that in order to have real positive energies it needs to hold that $q \in \mathbb{R}$ and $|q| < 1$. We focus on this range for the parameter q.

On the other hand the more standard dispersion relation holds for the massless modes:

$$\omega_u(p_1) = \omega_{\bar{u}}(p_1) = \omega_v(p_1) = \omega_{\bar{v}}(p_1) = |p_1|.$$

This is the full description of the free bosonic spectrum of the theory: we started from a relativistic theory with 10 degrees of freedom and we are left with a non-relativistc theory where only 8 modes are physical. As shown above these physical d.o.f. are packaged into 4 complex bosons of which 2 massless and 2 with a non trivial dispersion relation (depending also on q).

3.3.1 Quantisation

The quantisation of the theory follows the usual procedure. Since the dispersion relation for the fields z and y is not the standard relativistic one it is worth showing the main steps. The solutions to the EOM can be expressed as momentum space integrals. For the fields z and $\bar{z}$:

$$\begin{aligned}\bar{z} &= \int \frac{dp_1}{\sqrt{(2\pi)}}\left[\frac{e^{-i(\bar{\omega}(p_1)\tau - p_1\sigma)}}{\sqrt{\bar{\omega}\,(p_1)}}a_{\bar{z}}(p_1) + \frac{e^{+i(\omega(p_1)\tau - p_1\sigma)}}{\sqrt{\omega\,(p_1)}}a^\dagger_z(p_1)\right],\\ z &= \int \frac{dp_1}{\sqrt{(2\pi)}}\left[\frac{e^{-i(\omega(p_1)\tau - p_1\sigma)}}{\sqrt{\omega\,(p_1)}}a_z(p_1) + \frac{e^{+i(\bar{\omega}(p_1)\tau - p_1\sigma)}}{\sqrt{\bar{\omega}\,(p_1)}}a^\dagger_{\bar{z}}(p_1)\right].\end{aligned} \tag{3.36}$$

Clearly the $a_z(p), a_{\bar{z}}(p)$ coefficients and their hermitian conjugates will become the particle creation and annihilation operators of the quantised theory. For the fields in equations (3.36) to be solutions of the EOM the dispersion relations must be the ones already presented in equations (3.35). The same relations hold for the fields y and $\bar{y}$.

Now let us show briefly show the quantisation procedure for the z field. The quantisation is attained by imposing:

$$\big[z(\tau,\sigma), P_z(\tau,\sigma')\big] = i\delta(\sigma-\sigma'). \tag{3.37}$$

In the free theory the momenta relative to the z particles are:

$$P_z = \dot{\bar{z}}, \qquad P_{\bar{z}} = \dot{z}. \tag{3.38}$$

and similarly for the other modes. Using the solutions in equations (3.36) one can check the relation for the creation and annihilation operators:

$$\left[a_z(p), a^\dagger_z(q)\right] = \delta(p-q). \tag{3.39}$$

As usual this procedure converts the fields to operators acting on a Hilbert space; multi-particle states are constructed by applying creation operators to the vacuum state of the theory, that is defined by

$$a_z(p)|0\rangle = 0, \quad \forall p.$$

This must hold for each particle type. Single and multi-particle states are defined as:

$$a^\dagger_z(p)|0\rangle = |z(p)\rangle\ , \qquad a^\dagger_z(p)a^\dagger_z(q)|0\rangle = |z(p)z(q)\rangle \quad \text{and so on.} \tag{3.40}$$

Having quantised the theory we can also check the expression for quantities like energy and momentum in terms of ladder operators. Again restricting to the fields z and $\bar{z}$ we find:

$$\begin{aligned}H &= \int dp\left[\omega(p)a^\dagger_z(p)a_z(p) + \bar{\omega}(p)a^\dagger_{\bar{z}}(p)a_{\bar{z}}(p)\right],\\ P &= \int dp\left[pa^\dagger_z(p)a_z(p) + pa^\dagger_{\bar{z}}(p)a_{\bar{z}}(p)\right].\end{aligned} \tag{3.41}$$

For example, in the free theory the zero worldsheet momentum condition derived in equation (3.19) means that a multi-particle state $|p_1,\ldots,p_n\rangle$ is physical if $p_1+\cdots+p_n=0$.

For later convenience we can also derive the Feynman propagator of the theory, which will be necessary to perform perturbative calculations in chapter 5.

The massless modes propagate as a relativistic particle with $m = 0$ hence the propagator can be written in momentum space as:

$G_u(p) =$ $= i\left(p_0^2 - p_1^2 + i\epsilon\right)^{-1}.$

Figure 3.1: Massless propagator.

The other particles have q-dependent expression for the propagator. Knowing the dispersion relation is non-standard so it will be the Feynman propagator itself. We find it by requiring the propagator to be the Green's function of the equations of motion:

$$\left(\partial_{\tau_1}^2 - \partial_{\sigma_1}^2 + 2iq\partial_{\sigma_1} + 1\right) G(x_1, x_2) = i\delta^{(2)}(x_1 - x_2), \tag{3.42}$$

where $x_i = (\tau_i, \sigma_i)$ and $G(x_1, x_2)$ is the propagator. As per usual it is convenient to solve the equation in momentum space and after a few simple steps we find the expression for the z propagator in momentum space as:

$\tilde{G}_z(p) =$ $= i\left(p_0^2 + 2p_1 q - p_1^2 - 1 + i\epsilon\right)^{-1}.$

Figure 3.2: Propagator for z particles.

We note that the propagator in figure 3.2 is oriented and the inversion of the momentum p_1 for the z propagator gives as a result the $\bar{z}$ propagator as it happens for a charged particle/anti-particle pair in a relativistic theory. From now on when using Feynman diagrams in the context of the string sigma model we will use the colour green for the lines representing massless particles and blue for the lines representing z, $\bar{z}$, y, $\bar{y}$.

3.3.2 Obtaining the Lagrangian

As shown in the previous sections we chose to solve the constraints in the most convenient way as to find the Hamiltonian. However in order to compute scattering amplitudes it is much easier to read off the Feynman rules from the Lagrangian. In this section we illustrate the procedure to pass from the Hamiltonian to the Lagrangian description, in particular in the case in which we wish to stop the expansion at the sixth order in the fields.

The connection between Lagrangian and Hamiltonian is notoriously a Legendre transformation. Schematically:

$$\mathcal{L}(\dot{\phi}) = p \cdot \dot{\phi} - \mathcal{H}(p) \quad \text{where} \quad p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \tag{3.43}$$

The dependency on the momentum in the string sigma model Hamiltonian is not only in the kinetic (quadratic) term so we do not have an easy formula to pass from the interacting Lagrangian to the interacting Hamiltonian.

In this section we show how to derive a useful formula that allows us to go from the Hamiltonian to the Lagrangian framework up to a specific order in the fields. Let us work with a simple model. Take a Lagrangian dependent on a single field ϕ and denote it:

$$\mathcal{L}(\phi, \phi', \dot{\phi}) = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \cdots \tag{3.44}$$

Here and in the following we use the notation such that the terms $\mathcal{L}^{(n)}$ only display the nth powers of the field and its derivatives. We are also assuming that the Lagrangian only contains even powers.

The same notation is used for the Hamiltonian which instead depends on the momentum (that we call p) in place of the time derivatives of the field $\dot{\phi}$:

$$\mathcal{H}(\phi, \phi', p) = \mathcal{H}^{(2)} + \mathcal{H}^{(4)} + \mathcal{H}^{(6)} + \cdots \tag{3.45}$$

Hence the $\mathcal{H}^{(n)}$ only contains nth powers of ϕ, ϕ' and p combined.

What we start with is the Hamiltonian as a function of the momenta. To get the Lagrangian in terms of velocities we could just invert Hamilton's $\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p}$ and plug $p(\dot{\phi})$ into equation (3.43). This was done perturbatively using Mathematica to get an expression of $\mathcal{L}^{(4)}$ and $\mathcal{L}^{(6)}$. On the other hand, if one is not looking for an exact expression but instead wants to truncate the expressions at a certain order (the sixth in our case) the inversion of the Legendre transformation can be avoided altogether. Here we show a simple calculation to obtain the Lagrangian from the Hamiltonian at the fourth and sixth orders. We just start from the Legendre transformation and try to isolate the terms with different powers of the fields.

Suppose to rewrite the Legendre transform in equation (3.43) expressing everything in terms of the velocities:

$$\mathcal{L} = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{H}_{|_{p=\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}}. \tag{3.46}$$

Assuming a canonical kinetic term we have:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} + \frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}} + \frac{\partial \mathcal{L}^{(6)}}{\partial \dot{\phi}} + \ldots \tag{3.47}$$

We can expand the Hamiltonian around $\dot{\phi}$:

$$\mathcal{H}_{|_{p=\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}} = \mathcal{H}_{|_{p=\dot{\phi}}} + \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{|_{p=\dot{\phi}}} \cdot \left(\frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}} + \frac{\partial \mathcal{L}^{(6)}}{\partial \dot{\phi}} + \ldots\right) + \frac{1}{2}\left(\frac{\partial^2 \mathcal{H}}{\partial p^2}\right)_{|_{p=\dot{\phi}}} \cdot \left(\frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}} + \frac{\partial \mathcal{L}^{(6)}}{\partial \dot{\phi}} + \ldots\right)^2 + \cdots \tag{3.48}$$

Isolating explicitly each order of the powers field and leaving only the contribution from the Hamiltonian on the RHS of equation (3.43) we find:

$$\begin{aligned} \mathcal{L}^{(2)} - \dot{\phi}^2 &= -\mathcal{H}^{(2)}_{|_{p=\dot{\phi}}}, \\ \mathcal{L}^{(4)} - \frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}}\dot{\phi} &= -\mathcal{H}^{(4)}_{|_{p=\dot{\phi}}} - \left(\frac{\partial \mathcal{H}^{(2)}}{\partial p}\right) \cdot \left(\frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}}\right)_{|_{p=\dot{\phi}}}. \end{aligned} \tag{3.49}$$

In equations (3.49) we kept the terms of the Hamiltonian in equation (3.48) so that we have only second order in the field in the first line, then fourth order in the second. The first line is what we implicitly already used to obtain the free Lagrangian that was already explored in detail in section 3.3.

Evaluate now the second line. Only fourth order terms must be kept; after inspecting the terms of the expansion in equation (3.48) and since $\mathcal{H}^{(2)} = \frac{1}{2}p^2 + \ldots$, at 4th order we have:

$$\left(\frac{\partial \mathcal{H}^{(2)}}{\partial p}\right) \cdot \left(\frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}}\right)_{|_{p=\dot{\phi}}} = \frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}}\dot{\phi}. \tag{3.50}$$

Such term cancels the term on the LHS of the second line of equation (3.49). Therefore the first useful result is:

$$\mathcal{L}^{(4)} = -\mathcal{H}^{(4)}|_{p=\dot{\phi}}. \tag{3.51}$$

If we go to the next order in the fields, the sixth, from the expansion of the Hamiltonian we obtain:

$$\mathcal{L}^{(6)} - \frac{\partial \mathcal{L}^{(6)}}{\partial \dot{\phi}}\dot{\phi} = -\mathcal{H}^{(6)}_{|_{p=\dot{\phi}}} - \left(\frac{\partial \mathcal{H}^{(4)}}{\partial p}\right)\cdot\left(\frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}}\right)_{|_{p=\dot{\phi}}} - \left(\frac{\partial \mathcal{H}^{(2)}}{\partial p}\right)\cdot\left(\frac{\partial \mathcal{L}^{(6)}}{\partial \dot{\phi}}\right)_{|_{p=\dot{\phi}}} - \frac{1}{2}\left(\frac{\partial \mathcal{L}^{(4)}}{\partial \dot{\phi}}\right)^2. \tag{3.52}$$

The terms containing the derivative of the sextic Lagrangian cancel as before. Applying now equation (3.51) to the sixth order expansion in the fields of the Legendre transformation in equation (3.52) we get:

$$\mathcal{L}^{(6)} = -\mathcal{H}^{(6)}|_{p=\dot{\phi}} + \left(\frac{\partial \mathcal{H}^{(4)}}{\partial p}\right)\cdot\left(\frac{\partial \mathcal{H}^{(4)}}{\partial p}\right)_{|_{p=\dot{\phi}}} - \frac{1}{2}\left(\frac{\partial \mathcal{H}^{(4)}}{\partial p}\right)^2_{|_{p=\dot{\phi}}}. \tag{3.53}$$

And we finally obtain:

$$\mathcal{L}^{(6)} = -\mathcal{H}^{(6)}|_{p=\dot{\phi}} + \frac{1}{2}\left(\frac{\partial \mathcal{H}^{(4)}}{\partial p}\right)^2_{|_{p=\dot{\phi}}}. \tag{3.54}$$

Let us summarize what was found. Somewhat unexpectedly the fourth order Lagrangian is found as just the opposite of the fourth order Hamiltonian where the momenta are substituted with the velocities. This is similar to the velocity-independent result where $\mathcal{L}^{int} = -\mathcal{H}^{int}$ to all orders [18]. The expression for the sixth order Lagrangian is less trivial and receives contribution from the sixth order Hamiltonian and from the fourth order one (through the last term in equation (3.54)).

This formula can be easily adapted to find the Lagrangian of the string sigma model. In this case we consider 8 different momenta $p_z, p_{\bar{z}}, \dots, p_v, p_{\bar{v}}$ relative to complex fields, instead of p. In the expression for $\mathcal{L}^{(6)}$ it is sufficient to rewrite the squared term as:

$$\left(\frac{\partial \mathcal{H}^{(4)}}{\partial p_z}\frac{\partial \mathcal{H}^{(4)}}{\partial p_{\bar{z}}} + \frac{\partial \mathcal{H}^{(4)}}{\partial p_y}\frac{\partial \mathcal{H}^{(4)}}{\partial p_{\bar{y}}} + \frac{\partial \mathcal{H}^{(4)}}{\partial p_u}\frac{\partial \mathcal{H}^{(4)}}{\partial p_{\bar{u}}} + \frac{\partial \mathcal{H}^{(4)}}{\partial p_v}\frac{\partial \mathcal{H}^{(4)}}{\partial p_{\bar{v}}}\right) \tag{3.55}$$

and eventually substitute the momenta with the relative velocities. In conclusion the quartic Lagrangian, can be obtained straightforwardly from the quartic Hamiltonian (shown in appendix B) and has a relatively simple form. However the sextic Lagrangian becomes more complicated and cannot be expressed fully in a practical form. In order to perform perturbative calculations we can look for symmetries of such interacting Lagrangians in order to isolate all the different interaction terms.

3.4 Interacting Lagrangian

From the previous considerations we are able to find an expression for both the fourth order and the sixth order Lagrangians. In the first case the expression is manageable and using the formula in equation (3.51) we can find the fourth order Lagrangian.

It can be expressed in a rather compact form as:

$$
\begin{aligned}
\mathcal{L}^{(4)} &= -(|z|^2 - |y|^2)(|\dot{z}|^2 + |\dot{y}|^2 + |\dot{u}|^2 + |\dot{v}|^2 + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2) \\
&- |z|^2|\acute{z}|^2 + |y|^2|\acute{y}|^2 + |z|^2|\dot{z}|^2 - |y|^2|\dot{y}|^2 - \frac{iq}{2}(|z|^2 - |y|^2)(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}}) \\
&+ \frac{iq}{2}(\acute{z}\dot{\bar{z}} + \acute{\bar{z}}\dot{z} + \acute{y}\dot{\bar{y}} + \acute{\bar{y}}\dot{y} + \acute{u}\dot{\bar{u}} + \acute{\bar{u}}\dot{u} + \acute{v}\dot{\bar{v}} + \acute{\bar{v}}\dot{v})(\bar{z}\dot{z} - z\dot{\bar{z}} - \bar{y}\dot{y} + y\dot{\bar{y}}) \\
&- \frac{iq}{2}(|\dot{z}|^2 + |\dot{y}|^2 + |\dot{u}|^2 + |\dot{v}|^2 + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2)(\bar{z}\acute{z} - z\acute{\bar{z}} - \bar{y}\acute{y} + y\acute{\bar{y}}) \\
&- \frac{2a-1}{2}(|\dot{z}|^2 + |\dot{y}|^2 + |\dot{u}|^2 + |\dot{v}|^2 + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2)^2 \\
&+ \frac{2a-1}{2}\Big((|z|^2 + |y|^2)^2 + (\acute{z}\dot{\bar{z}} + \acute{\bar{z}}\dot{z} + \acute{y}\dot{\bar{y}} + \acute{\bar{y}}\dot{y} + \acute{u}\dot{\bar{u}} + \acute{\bar{u}}\dot{u} + \acute{v}\dot{\bar{v}} + \acute{\bar{v}}\dot{v})^2\Big) \\
&- \frac{iq}{2}(2a-1)(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}})(|\dot{z}|^2 + |\dot{y}|^2 + |\dot{u}|^2 + |\dot{v}|^2 + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2 - |z|^2 - |y|^2) \\
&+ \frac{iq}{2}(2a-1)(\acute{z}\dot{\bar{z}} + \acute{\bar{z}}\dot{z} + \acute{y}\dot{\bar{y}} + \acute{\bar{y}}\dot{y} + \acute{u}\dot{\bar{u}} + \acute{\bar{u}}\dot{u} + \acute{v}\dot{\bar{v}} + \acute{\bar{v}}\dot{v})(\bar{z}\dot{z} - z\dot{\bar{z}} + \bar{y}\dot{y} - y\dot{\bar{y}})\,.
\end{aligned}
\tag{3.56}
$$

Fist notice that in the expression above the terms containing the gauge parameter a are isolated from the rest. The choice $a = \frac{1}{2}$ greatly simplifies the Lagrangian in this case. Additionally observe that the expression of the Lagrangian has some symmetries that reduce the number of different interaction terms. In particular it is completely symmetric when exchanging u and v. The gauge independent part is antisymmetric under z and y exchange while the gauge dependent part is symmetric. We can also check from equation (3.56) that the same discrete symmetries observed in section 3.3 still hold for the quartic interactions.

At the sixth order the expression becomes much more complicated and it is more convenient to isolate only the relevant terms without reporting the Lagrangian in full. Let us first do so for the quartic Lagrangian in (3.56). We can rewrite it as:

$$
\mathcal{L}^{(4)} = \sum_{X=z,y} \mathcal{L}^{(4)}_X + \sum_{\mu=u,v} \mathcal{L}^{(4)}_\mu + \sum_{\substack{X=z,y \\ \mu=u,v}} \mathcal{L}^{(4)}_{X\mu} + \mathcal{L}^{(4)}_{zy} + \mathcal{L}^{(4)}_{uv}. \tag{3.57}
$$

We can additionally split each term based on the gauge parameter a. When using it for perturbative calculations it is easier these terms separated in the way we are reporting them. We have:

$$
\begin{aligned}
\mathcal{L}^{(4)}_X &= \pm A^{(4)}_X + (a - \frac{1}{2})B^{(4)}_X \qquad \text{where } + \text{ is for } z \text{ and } - \text{ for } y, \\
\mathcal{L}^{(4)}_{zy} &= C^{(4)}_{zy} + (a - \frac{1}{2})D^{(4)}_{zy}, \\
\mathcal{L}^{(4)}_{X\mu} &= \pm E^{(4)}_{X\mu} + (a - \frac{1}{2})F^{(4)}_{X\mu} \qquad \text{where } + \text{ is for } z \text{ and } - \text{ for } y.
\end{aligned}
\tag{3.58}
$$

And the expressions are:

$$
\begin{aligned}
A_X^{(4)} =& -2|X|^2|\acute{X}|^2 + \frac{iq}{2}\left((|X|^2+|\acute{X}|^2)(X\acute{\bar{X}} - \bar{X}\acute{X}) - \dot{\bar{X}}^2 X\acute{X} + \dot{X}^2\bar{X}\acute{\bar{X}}\right),\\
B_X^{(4)} =& |X|^4 - (\dot{X}^2 - \acute{X}^2)(\dot{\bar{X}}^2 - \acute{\bar{X}}^2) + iq\left((|\acute{X}|^2 - |X|^2)(X\acute{\bar{X}} - \bar{X}\acute{X}) - \dot{\bar{X}}^2 X\acute{X} + \dot{X}^2\bar{X}\acute{\bar{X}}\right),\\
C_{zy}^{(4)} =& -|z|^2(|\acute{y}|^2+|\dot{y}|^2) + |y|^2(|\acute{z}|^2+|\dot{z}|^2) - i\frac{q}{2}\left[(|z|^2-|\acute{z}|^2-|\dot{z}|^2)(\bar{y}\acute{y} - y\acute{\bar{y}})\right]\\
&+ i\frac{q}{2}\left[(|y|^2-|\acute{y}|^2-|\dot{y}|^2)(\bar{z}\acute{z} - z\acute{\bar{z}}) + (\acute{z}\dot{\bar{z}}+\acute{\bar{z}}\dot{z})(-\bar{y}\dot{y} + y\dot{\bar{y}}) + (\acute{y}\dot{\bar{y}}+\acute{\bar{y}}\dot{y})(\bar{z}\dot{z} - z\dot{\bar{z}})\right]\\
D_{zy}^{(4)} =& -\Big(2(|\dot{z}|^2+|\acute{z}|^2)(|\dot{y}|^2+|\acute{y}|^2) - 2|z|^2|y|^2 - 2(\acute{z}\dot{\bar{z}}+\acute{\bar{z}}\dot{z})(\acute{y}\dot{\bar{y}}+\acute{\bar{y}}\dot{y}) + iq\Big((\bar{z}\acute{z} - z\acute{\bar{z}})(|\dot{y}|^2+|\acute{y}|^2-|y|^2)\\
&+ (\bar{y}\acute{y} - y\acute{\bar{y}})(|\dot{z}|^2+|\acute{z}|^2-|z|^2) - (\acute{z}\dot{\bar{z}}+\acute{\bar{z}}\dot{z})(\bar{y}\dot{y} - y\dot{\bar{y}}) - (\acute{y}\dot{\bar{y}}+\acute{\bar{y}}\dot{y})(\bar{z}\dot{z} - z\dot{\bar{z}})\Big)\Big),\\
E_{X\mu}^{(4)} =& -|X|^2(|\dot{\mu}|^2+|\acute{\mu}|^2) + i\frac{q}{2}(\acute{\mu}\dot{\bar{\mu}}+\acute{\bar{\mu}}\dot{\mu})(\bar{X}\dot{X} - X\dot{\bar{X}}) - i\frac{q}{2}(|\dot{\mu}|^2+|\acute{\mu}|^2)(\bar{X}\acute{X} - X\acute{\bar{X}}),\\
F_{X\mu}^{(4)} =& -\Big(2(|\dot{X}|^2+|\acute{X}|^2)(|\dot{\mu}|^2+|\acute{\mu}|^2) - 2(\acute{X}\dot{\bar{X}}+\acute{\bar{X}}\dot{X})(\acute{\mu}\dot{\bar{\mu}}+\acute{\bar{\mu}}\dot{\mu}) + iq(\bar{X}\acute{X} - X\acute{\bar{X}})(|\dot{\mu}|^2+|\acute{\mu}|^2)\\
&- iq(\acute{\mu}\dot{\bar{\mu}}+\acute{\bar{\mu}}\dot{\mu})(\bar{X}\dot{X} - X\dot{\bar{X}})\Big).
\end{aligned}
\tag{3.59}
$$

On the other hand for the exclusively massless terms we have:

$$
\begin{aligned}
\mathcal{L}_{\mu}^{(4)} =& -\frac{2a-1}{2}\Big((|\dot{\mu}|^2+|\acute{\mu}|^2)^2 - (\acute{\mu}\dot{\bar{\mu}}+\acute{\bar{\mu}}\dot{\mu})^2\Big),\\
\mathcal{L}_{uv}^{(4)} =& -\frac{2a-1}{2}\Big(2(|\dot{u}|^2+|\acute{u}|^2)(|\dot{v}|^2+|\acute{v}|^2) - 2(\acute{u}\dot{\bar{u}}+\acute{\bar{u}}\dot{u})(\acute{v}\dot{\bar{v}}+\acute{\bar{v}}\dot{v})\Big).
\end{aligned}
\tag{3.60}
$$

This concludes the description of the fourth order interaction. As one can see from equation (3.57) the Lagrangian only contains 5 different types of interaction terms. Therefore to this order the theory displays only 5 different interaction vertices, shown in appendix D, somewhat simplifying the classification of $2 \to 2$ particle processes. This task is the main focus of the next chapter.

Let us now find a similar expression as the one in equation (3.57) for the interaction Lagrangian with six fields. Starting from the interacting Hamiltonian the Lagrangian is found by using the formula in equation (3.54). The calculation is quite large and it was performed using Mathematica. The resulting expression can be split into different pieces as:

$$
\begin{aligned}
\mathcal{L}^{(6)} =& \sum_{X=z,y}\mathcal{L}_X^{(6)} + \sum_{\mu=u,v}\mathcal{L}_{\mu}^{(6)} + \sum_{\substack{X=z,y\\ \mu=u,v}}\mathcal{L}_{XX\mu}^{(6)} + \sum_{\substack{X=z,y\\ \mu=u,v}}\mathcal{L}_{X\mu\mu}^{(6)} + \mathcal{L}_{zzy}^{(6)} + \mathcal{L}_{zyy}^{(6)} + \mathcal{L}_{uuv}^{(6)} + \mathcal{L}_{uvv}^{(6)} +\\
&+ \sum_{\mu=u,v}\mathcal{L}_{zy\mu}^{(6)} + \sum_{X=z,y}\mathcal{L}_{uvX}^{(6)}.
\end{aligned}
\tag{3.61}
$$

The labels of each piece in equation (3.61) indicate all the types of fields involved in the term keeping in mind each field must appear with its conjugate since the Lagrangian is hermitean. For example $\mathcal{L}_{X\mu\mu}^{(6)}$ contains one z and one $\bar{z}$ and two μ and two $\bar{\mu}$ fields.

Here are reported 10 pieces however only 8 of them are truly different interaction terms since we can define the term $\mathcal{L}^{(6)}_{uvv}$ from $\mathcal{L}^{(6)}_{uuv}$ and also $\mathcal{L}^{(6)}_{zyy}$ from $\mathcal{L}^{(6)}_{zzy}$ using the symmetries that we mentioned before.

These terms can be written as:

$$\begin{aligned}
\mathcal{L}^{(6)}_{X} &= A^{(6)}_{X} \pm (2a-1)B^{(6)}_{X} + (2a-1)^2 C^{(6)}_{X} \quad \text{where + is for } z \text{ and } - \text{ for } y,\\
\mathcal{L}^{(6)}_{\mu} &= (2a-1)^2 D^{(6)}_{\mu},\\
\mathcal{L}^{(6)}_{\mu\mu X} &= E^{(6)}_{\mu\mu X} \pm (2a-1)F^{(6)}_{\mu\mu X} + (2a-1)^2 G^{(6)}_{\mu\mu X} \quad \text{where + is for } z \text{ and } - \text{ for } y,\\
\mathcal{L}^{(6)}_{\mu XX} &= H^{(6)}_{\mu XX} \pm (2a-1)I^{(6)}_{\mu XX} + (2a-1)^2 J^{(6)}_{\mu XX} \quad \text{where + is for } z \text{ and } - \text{ for } y,\\
\mathcal{L}^{(6)}_{uvv} &=(2a-1)^2 K^{(6)}_{uvv},\\
\mathcal{L}^{(6)}_{yzz} &= L^{(6)}_{yzz} + (2a-1)M^{(6)}_{yzz} + (2a-1)^2 N^{(6)}_{yzz},\\
\mathcal{L}^{(6)}_{yyz} &= L^{(6)}_{yyz} - (2a-1)M^{(6)}_{yyz} + (2a-1)^2 N^{(6)}_{yyz},\\
\mathcal{L}^{(6)}_{zy\mu} &= O^{(6)}_{zy\mu} \pm (2a-1)P^{(6)}_{zy\mu} + (2a-1)^2 Q^{(6)}_{zy\mu} \quad \text{where + is for } z \text{ and } - \text{ for } y,\\
\mathcal{L}^{(6)}_{uvX} &= R^{(6)}_{uvX} \pm (2a-1)S^{(6)}_{uvX} + (2a-1)^2 T^{(6)}_{uvX} \quad \text{where + is for } z \text{ and } - \text{ for } y.
\end{aligned} \tag{3.62}$$

We again split the interactions in contributions of the powers 0, 1, 2 in the factor $(2a-1)$. It is evident from the expressions in equation (3.62) that the z and y interaction terms are the same except in the pieces with the first power of $(2a-1)$, where they have opposite sign. Conversely under the exchange of $u \longleftrightarrow v$ (and same for the conjugates) the Lagrangian does not change. This is expected since the directions on the torus are interchangeable and are not affected by the gauge-fixing procedure in section 3.2. Hence the u and v interactions are the same.

In terms of spacetime symmetries we note that the interacting Lagrangians in equations (3.57) and (3.61) preserve the ones of the free theory observed in section 3.3. Therefore each term has an even number of time derivatives making the theory invariant under time inversion $(\tau \to -\tau)$. This is not the case for spacial derivatives, meaning the theory is not parity invariant. However the interaction terms retain a sort of CP symmetry, meaning the invariance under $(\sigma \to -\sigma)$ and $(q \to -q)$ combined.

The expressions of the terms in equation (3.62) can all be found in appendix C.

Chapter 4

$S_{2\to 2}$ of the string sigma model

In the previous chapter we introduced the string action in section 3.1 and we performed the gauge-fixing (section 3.2) which allowed us to identify the physical fields on the worldsheet. We also found a perturbative expansion of the Lagrangian in section 3.4. Therefore we have all the ingredients to study the scattering properties of the interacting sigma model living on the worldsheet.

In this chapter we concentrate on the information contained in the $\mathcal{L}^{(4)}$ term. As we will see in a moment this term by itself completely describes all $2 \to 2$ processes at tree-level of perturbation theory and starting from the Lagrangian we are able to study in detail the interactions involving 4 external particles. In the following we report some notable findings of our analysis of the four-point vertices. The analysis is aimed at checking the properties of the interactions and comparing them with the general constraints imposed by integrability discussed in chapter 1. This has been done leaving the a gauge parameter free. Computing all the relevant scattering amplitudes we can construct $S_{2\to 2}$ at tree-level. This is of great importance since, assuming the theory to be quantum integrable, the two-particle scattering amplitudes become the only building blocks of the full S-matrix. Furthermore two-particle amplitudes obtained perturbatively offer an important check for the exact S-matrix computed through bootstrap techniques [17] and can give us insights on the integrability side.

4.1 On the calculation of $S_{2\to 2}$

The S-matrix of an interacting theory is an operator containing all the information on the scattering of particles in such theory. It is defined as the operator connecting the asymptotic initial and final particle states. In order to do perturbative calculations we need to outline the general ideas involved in the perturbative expansion of the S-matrix. However we will only mention some facts that are used in the following sections and not go into details about perturbation theory applied to quantum field theories. A general discussion on the perturbative approach to QFTs (pQFTs) can be found *e.g.* in ref. [18].

In the context of pQFTs we define the S-matrix as the following operator:

$$\boldsymbol{S} = \mathbb{1} + i\boldsymbol{T} = \mathcal{T}\left[e^{-i\int d\sigma d\tau :\mathcal{H}_{int}:}\right]. \tag{4.1}$$

The perturbative treatment consists in expanding the time-ordered exponential in equation (4.1) since, recalling the large string tension regime we are working in (section 3.2), the perturbative order increases with the number of fields considered. In this thesis we are interested in computing all tree-level 2-to-2 particles processes and (in the next chapter) some 3-to-3 processes, again at first order in perturbation theory. Each of these processes are associated to different terms in the expansion of equation (4.1). The time-ordering operation is notably handled with the aid of Wick's Theorem [18].

The whole perturbative apparatus is operatively simplified by the use of Feynman diagrams. For the $2 \to 2$ processes the relevant term in the expansion is:

$$\boldsymbol{T}_{2\to2} = -\int d\sigma d\tau : \mathcal{H}^{(4)} : . \tag{4.2}$$

This is diagrammatically represented by vertices with four external legs. On the other hand when 6 particles are involved in the scattering the contributing terms are:

$$\boldsymbol{T}_{3\to3} = -\int d\sigma d\tau : \mathcal{H}^{(6)} : + \frac{i}{2!}\mathcal{T}\left[\left(\int d\sigma d\tau : \mathcal{H}^{(4)} :\right)^2\right]. \tag{4.3}$$

Firstly, this shows that for the purposes of this work the Hamiltonian can be truncated up to the sixth order in the fields. Diagrammatically the two terms in equation (4.3) generate two different topologies of diagrams. The term containing the sextic Hamiltonian is a six-point vertex. Conversely in the other term the time-ordered product gives us a number of diagrams with 2 vertices connected by one propagator.

The above discussion justifies why the Feynman diagrams can be used to compute scattering amplitudes. Following the definition in equation (4.1) it is the Hamiltonian that should be used to compute the Feynman rules associated to each Feynman diagrams whilst in the previous chapter we spent some time deriving the interaction Lagrangian exactly for the purpose of deriving the Feynman rules. We want to spend a few words on this Lagrangian v. Hamiltonian choice.

This is a peculiar situation because due to the presence of time derivatives in the interacting terms $\mathcal{H}^{(int)} \neq -\mathcal{L}^{(int)}$. The evaluation of the integrals with the Hamiltonian (for example in equation (4.3)) requires the use of the canonical formalism, writing the fields in the free theory's creation and annihilation operators expansion, then computing contractions using Wick's theorem. However the Feynman rules can be read off more easily from the Lagrangian in an algorithmic way (see for example [18] for the case with derivative couplings). Evidently the Lagrangian formalism stems from the path integral formulation of the model, notoriously providing an alternative way of constructing the quantum theory ([18],[29]).

When the interaction Lagrangian and Hamiltonian are not the same, namely when there are couplings with time derivatives, the derivation of the Feynman rules using the Hamiltonian becomes much more involved. In particular the use of Feynman diagrams is intimately related to the Lagrangian approach and is instead less suited to the Hamiltonian one in this case. The relation between the two procedures is explained more in detail for example in sec. 6.2 and 7 of ref. [29].

4.1.1 External legs factors

After this premise it is clear that the first step of every calculation is deriving the Feynman rules for the interaction vertices. The 4-point vertices are presented in appendix D. If we are looking to derive scattering amplitudes we should also add the *external legs* factors, which are independent on the interaction. In general the amplitudes can be obtained by adding these factors to the Feynman rules:

1. each external particle (labeled by i) carries a factor $\frac{1}{\sqrt{2\omega_i}}$;
2. conservation of energy and momentum enforced by $\delta\left(\sum_i p_i\right)\delta\left(\sum_i \omega_i\right)$.

Since we will present the amplitudes for $2 \to 2$ processes let us see this case. Observe that the $\delta(\omega_{tot})$ will be rewritten, using the known properties of the δ-functions, as:

$$\delta\left(\sum_{i=1}^{n} \omega_i\right) = \sum_{j=1}^{2} \frac{\delta\left(p - p_j^*\right)}{\left|\frac{\partial \omega_1}{\partial p} + \frac{\partial \omega_2}{\partial p} + \cdots + \frac{\partial \omega_n}{\partial p}\right|} \tag{4.4}$$

Where in equation (4.4) we solve the constraints with respect to the momentum p and we find two solutions since they are second degree equations. When the dispersion relations are the standard relativistic ones the factor is notoriously:

$$\frac{\omega_1 \omega_2}{|\omega_1 \cdot p_2 - \omega_2 \cdot_1|}.$$

For the particles with dispersion relations as in equation (3.35) this factor will become instead:

$$\frac{\omega_1 \omega_2}{|\omega_1 \cdot (p_2 \mp q) - \omega_2 \cdot (p_1 \mp q)|}. \tag{4.5}$$

The sign is $-$ or $+$ respectively for the z, y particles and for $\bar{z}$, $\bar{y}$ particles as can be checked from the dispersion relations. For example if particle 1 is z and particle 2 is $\bar{y}$ we have a factor:

$$\frac{\omega_1 \bar{\omega}_2}{|\omega_1 \cdot (p_2 + q) - \bar{\omega}_2 \cdot (p_1 - q)|}. \tag{4.6}$$

4.2 Properties of vertices

The Feynman rules for the interaction vertices with four external legs are presented in figures D.1 and D.2. Notice that these rules are written in the convention where all the particles are incoming, meaning for a $4 \to 0$ process. From the integrability of the theory we expect only $2 \to 2$ processes with the same sets of momenta at the initial and final states to be non-vanishing. These properties have been checked successfully for all the 5 different interaction vertices. In this section we present some examples of the calculations that we carried out; notice that here for simplicity we just use the contributions to the amplitudes due to the Feynman diagrams without reporting the external legs factors.

Take for example the vertex with both z and y particles. Here we show the procedure for the case $a = \frac{1}{2}$. Use notation where index 0 is for the energy and 1 is for the momentum. The Feynman rules reported in appendix D can be associated to amplitudes of processes where 4 particles annihilate each others as:

$$\begin{aligned}\langle 0|\boldsymbol{T}|z(p')y(p)\bar{z}(k')\bar{y}(k)\rangle \propto\ & i(p_0k_0 + p_1k_1 - p_0'k_0' - p_1'k_1') + \frac{iq}{2}\left[(p_1'k_0' + k_1'p_0')(k_0 - p_0)\right] \\ & + \frac{iq}{2}\left[+(p_1k_0 + k_1p_0)(p_0' - k_0') + (1 + p_0'k_0' + p_1'k_1')(p_1 - k_1)\right] \\ & - \frac{iq}{2}\left[(1 + p_0k_0 + p_1k_1)(p_1' - k_1')\right].\end{aligned} \tag{4.7}$$

From the free solutions for the z and $\bar{z}$ fields presented in section 3.3 we see that we can assume the amplitudes to satisfy crossing symmetry; the only difference with a usual particle/anti-particle crossing is that when we cross, for example, $z(p')$ into $\bar{z}(p')$, besides just inverting the sign of energy and momentum, we must also use in place of p_0' the dispersion relation for $\bar{z}$ (equation (3.35)), which differs from the one of z by the sign of q.

First note that all the processes that can be obtained from the vertex in (4.7) of the form $4 \to 0$ or $3 \to 1$ vanish simply because the energy and momentum conservation cannot be satisfied. This is actually true for all the possible processes that do not contain massless particles.

Now we can use crossing symmetry to analyse a process such as:

$$y + \bar{y} \to z + \bar{z}\,.$$

We can show that this process has vanishing amplitude. Using crossing symmetry we can write:

$$\langle 0|\boldsymbol{T}|y(p)\bar{y}(k)z(p')\bar{z}(k')\rangle = \langle \bar{z}(-p')z(-k')|\boldsymbol{T}|y(p)\bar{y}(k)\rangle. \tag{4.8}$$

Hence we can just invert the signs and obtain :

$$\begin{aligned}\langle \bar{z}(p')z(k')|T|y(p)\bar{y}(k)\rangle \propto\; & i(p_0k_0 + p_1k_1 - p'_0k'_0 - p'_1k'_1)\\ & + \frac{iq}{2}\big[(p'_1k'_0 + k'_1p'_0)(k_0 - p_0) - (p_1k_0 + k_1p_0)(p'_0 - k'_0)\\ & + (1 + p'_0k'_0 + p'_1k'_1)(p_1 - k_1) + (1 + p_0k_0 + p_1k_1)(p'_1 - k'_1)\big].\end{aligned} \tag{4.9}$$

Recall that the dispersion relations for the process above are:

$$p'_0 = \sqrt{p'^2_1 + 2qp'_1 + 1} \quad k'_0 = \sqrt{k'^2_1 - 2qk'_1 + 1} \quad p_0 = \sqrt{p_1^2 - 2qp_1 + 1} \quad k_0 = \sqrt{k_1^2 + 2qk_1 + 1}. \tag{4.10}$$

Energy and momentum conservation give the following constraints:

$$\begin{cases} p'_1 + k'_1 = p_1 + k_1 \\ (p'^2_1 + 2qp'_1 + 1)^{\frac{1}{2}} + (k'^2_1 - 2qk'_1 + 1)^{\frac{1}{2}} = (p_1^2 - 2qp_1 + 1)^{\frac{1}{2}} + (k_1^2 + 2qk_1 + 1)^{\frac{1}{2}} \end{cases} \tag{4.11}$$

The system in equation (4.11) has only two solutions:

1. $\{k_1 = p'_1, p_1 = k'_1\}$ with energies $\{k_0 = p'_0, p_0 = k'_0\}$,
2. $\{k'_1 = k_1 + 2q, p'_1 = p_1 - 2q\}$ with energies $\{k_0 = k'_0, p_0 = p'_0\}$.

As it can be explicitly checked in equation (4.9) in both cases the amplitude is null. This result can be easily shown to be true for generic values of a. In general we used the same logic to show that *all inelastic* processes vanish at tree-level.

Another relevant result can be extracted from the same interaction vertex. We can take into account the process:

$$y(p) + z(k) \to y(p') + z(k')\,,$$

which is not supposed to vanish. Integrability only constrains the set of momenta to be conserved in the process. In this $2 \to 2$ case the conservation of energy and momentum is actually sufficient to fix the momenta. In fact, repeating the same steps as before we find that kinematics has solutions:

1. $\{k_1 = k'_1, p_1 = p'_1\}$ with energies $\{k_0 = k'_0, p_0 = p'_0\}$,
2. $\{k_1 = p'_1, p_1 = k'_1\}$ with energies $\{k_0 = p'_0, p_0 = k'_0\}$.

The first solution gives a non-zero amplitude. Ths expression, together with the rest of the non-trivial amplitudes, is presented in the next section 4.3. Notice that this process just leaves z and y with the same momenta they started with. On the other hand the second configuration would invert the momenta but the amplitude in this case cancels. It can be shown that this is true for any other 2 into 2 amplitude at tree-level, hence making the S-matrix *reflectionless*.

Take now an inelastic process involving massless modes:

$$\bar{u} + u \to z + \bar{z}\,.$$

The vertex contribution is:

$$\langle 0|\boldsymbol{T}|u(p)z(p')\bar{u}(k)\bar{z}(k')\rangle \propto +i\left[p_0k_0 + p_1k_1 - \frac{q}{2}(p_0k_0 + p_1k_1)(p_1' - k_1') + \frac{q}{2}(p_1k_0 + k_1p_0)(p_0' - k_0')\right]. \tag{4.12}$$

First, in this process the energy and momentum conservation equations do not allow the massless particles to be heading in the same verse, meaning that the only solutions are those for p_1 and k_1 with different signs [1]. In fact we have:

$$\begin{cases} k_1 + p_1 = k_1' + p_1' \\ |k_1| + |p_1| = (p_1'^2 + 2qp_1' + 1)^{\frac{1}{2}} + (k_1'^2 - 2qk_1' + 1)^{\frac{1}{2}} \end{cases} \tag{4.13}$$

The consequence is that the structure of this vertex already ensures the vanishing of all the inelastic scattering. In fact, since massless modes have energy $p_0 = |p_1|$ it is immediate to show that, when p_1 and k_1 have opposite sign

$$p_0k_0 + p_1k_1 = 0, \quad p_1k_0 + k_1p_0 = 0. \tag{4.14}$$

When we compare this result with the vertex shown in equation (4.12) we see that it vanishes automatically when the particles scattering have momenta of opposite signs. This automatically cancels also the $1 \to 3$ vertices that are kinematically possible in this case.

For example if we want to check whether the process $z \to z + u + \bar{u}$ vanishes we can consider the energy and momentum constraints:

$$\begin{cases} p_1' = p_1 + k_1' + k_1 \\ (p_1'^2 - 2qp_1' + 1)^{\frac{1}{2}} = |p_1| + |k_1| + (k_1'^2 - 2qk_1' + 1)^{\frac{1}{2}} \end{cases} \tag{4.15}$$

Suppose to take $\text{sign}(p_1) = \text{sign}(k_1)$ then we can remove the moduli from equation (4.15) and obtain:

$$\begin{cases} p_1' = k_1' + (p_1 + k_1) \\ (p_1'^2 - 2qp_1' + 1)^{\frac{1}{2}} = \pm(p_1 + k_1) + (k_1'^2 - 2qk_1' + 1)^{\frac{1}{2}} \end{cases} \tag{4.16}$$

Where in equation (4.16) we have $+$ sign when the momenta are positive and $-$ when they are negative. Either way we can sum/subtract the two equations and obtain:

$$(p_1'^2 - 2qp_1' + 1)^{\frac{1}{2}} - (k_1'^2 - 2qk_1' + 1)^{\frac{1}{2}} = \pm(p_1' - k_1'). \tag{4.17}$$

This equation has no solution for $q \in (0,1)$. The consequence is that the two outgoing massless momenta cannot have the same sign. Since the amplitude contains only terms like the one in equation (4.14) the it vanishes automatically. This same mechanism can be checked for the amplitude for generic values of a , shown in appendix D.

The vanishing of the terms in equation (4.14) can be applied to this vertex also for the amplitudes that are not forbidden by integrability. In fact if we cross a u particle the terms in equation (4.14) only change for an overall sign. Consider the elastic process:

$$z + u \to z + u\,.$$

[1] Observe that with massless particles the scattering is not even well defined if the two incoming particles have momenta directed the same way.

The energy and momentum conservation yeld:

$$\begin{cases} p_1' + p_1 = k_1' + k_1 \\ (p_1'^2 - 2qp_1' + 1)^{\frac{1}{2}} + |p_1| = |k_1| + (k_1'^2 - 2qk_1' + 1)^{\frac{1}{2}} \end{cases} \tag{4.18}$$

The only configuration where the massless momenta have the same sign is when $\{k_1 = p_1, p_1' = k_1'\}$. The other solution is more complicated. If we fix $p_1' > 0$ we can write the solution for the outgoing momenta as:

$$\begin{aligned} \{k_1 =& \frac{p_1\left[2qp_0' - 2p_1'(p_0'+2q) - 2p_1(p_0'-p_1'+q) + 2p_1'^2 + q^2 + 1\right]}{4p_1\left(p_1'-q\right) + 4p_1^2 + q^2 - 1}, \\ k_1' =& \frac{\left(q^2-1\right)p_1' + 2p_1^2\left(p_0' + 3p_1' - q\right) + 2p_1\left(p_1'p_0' - qp_0' + (p_1')^2 - 1\right) + 4p_1^3}{4p_1\left(p_1'-q\right) + 4p_1^2 + q^2 - 1}\}. \end{aligned} \tag{4.19}$$

Solution in equation (4.19) are written keeping implicit the dispersion relation $p_0' = \sqrt{p_1'^2 - 2p_1'q + 1}$. It is immediate to check that $k_1 < 0$ for every value of p_1' and q. Differently from the scattering between y and z, since u and z have different dispersion relations, here the configuration in equation (4.19) is not just the reflection of the momenta; nevertheless for this second solution it holds that $\text{sign}(p_1) = -\text{sign}(k_1)$, meaning that the amplitude cancels due to equation (4.14) as we would expect from the integrability of the theory. It is not difficult to check that the properties for the $z+u \to z+u$ and $\bar{u}+u \to z+\bar{z}$ hold for each value of a.

It is possible to check that the remaining 3 four-point interaction vertices are also purely elastic. Similar mechanisms to the ones shown for $u\bar{u} \to z\bar{z}$ work also for 4-point vertices involving only massless particles.

4.3 2-to-2 amplitudes

In this section we list all the different non-vanishing amplitudes for $2 \to 2$ processes. The computation of these amplitudes closely resembles the ones shown in section 4.2 with the addition of the external legs factors discussed in section 4.1. In the following expressions it is assumed $p > 0 > p'$. This choice ensures the correctness of the perturbative treatment for massless particles. The assumption is that states in the far past ($\tau \to -\infty$) and the far future ($\tau \to \infty$) are asymptotically free i.e. the particles are infinitely far away from each other. With this choice of ordering for the momenta the absolute value in the denominator in equation (4.5) can be removed.

Since we have 5 different interaction vertices, the 5 different amplitudes are:

$$\begin{aligned} i\boldsymbol{T}|X(p)X(p')\rangle =& \mp\frac{i\,(p+p')\,(p'\omega+p\omega')}{2\,(p-p')} - i\left(a-\frac{1}{2}\right)\left[p\omega'-p'\omega\right]|X(p')X(p)\rangle, \\ i\boldsymbol{T}|z(p)y(p')\rangle =& \frac{1}{2}i\left(p'\omega+p\omega'\right) - i\left(a-\frac{1}{2}\right)\left[p\omega'-p'\omega\right]|z(p')y(p)\rangle, \\ i\boldsymbol{T}|X(p)\mu(p')\rangle =& \pm\frac{1}{2}i\left(p'\omega+p\omega'\right) - i\left(a-\frac{1}{2}\right)\left[p\omega'-p'\omega\right]|X(p')\mu(p)\rangle, \\ i\boldsymbol{T}|\mu(p)\mu(p')\rangle =& i(2a-1)pp'|\mu(p')\mu(p)\rangle, \\ i\boldsymbol{T}|u(p)v(p')\rangle =& i(2a-1)pp'|u(p')v(p)\rangle. \end{aligned} \tag{4.20}$$

Observe that the notation follows the symmetries of the quartic Lagrangian. The X particles can be either z or y while μ can be u or v. In the first and third expressions the upper sign is for z particles and the lower one for y particles. Clearly all the other non-vanishing-amplitudes can be obtained from these 5 using crossing relations and the discrete symmetries of the quartic Lagrangian discussed in section 3.4.

We compared these results with those in ref. [17] for the case $q = 0$ and [30] for the massive scatterings, finding agreement. We mention that the amplitudes in equation (4.20) are expressed in a particular form such that they are formally the same as those for the theory in absence of the Kalb-Ramond field ($q = 0$). In fact, as first noted in [30], the expressions are the same when we substitute the correct dispersion relations in the formula. This is irrelevant for the exclusively massless amplitudes which are completely independent on the q parameter.

Chapter 5

3 particles scattering on the string worldsheet

In this chapter we study the scattering on the worldsheet involving 6 external particles. We limit the analysis to tree-level calculations. The interaction vertices for these processes can be read off from from the sextic Lagrangian which we split into the relevant terms in section 3.4; the expressions of these terms is reported in appendix C.

The study of factorisation at tree-level can be relevant in different ways. In general when the interaction factorises for a particular model it could be a hint that said model is integrable. Yet if factorisation does not happen, even simply at tree-level, we can be sure that the model is *not integrable*. For example, tree-level calculations showing amplitudes failing to factorise have been used in ref. [31] to exclude the integrability of particular models. In the following we present a general discussion of the tree-level 3-to-3 particles interactions of the string non-linear sigma model and we exhibit the factorisation of such processes.

5.1 Calculation of 3-to-3 processes

To compute the amplitudes for $3 \to 3$ processes at the tree-level we need both four-point and six-point Feynman diagrams. Accounting for the symmetries of the Lagrangian under (different) particle exchange we are able to isolate 8 different interactions. We will not write explicitly the vertices but they can easily be read off from the terms of the sextic Lagrangian shown in appendix C. From the integrability of the theory we expect the scattering to be elastic and factorise in a sequence of two-body interactions, as argued in section 1.2.1. The topologies of Feynman diagrams contributing to the amplitudes are exactly the same as those contributing to the same processes in *sinh-Gordon* theory, in section 2.1. Consequently we follow a similar logic as for *sinh-Gordon* theory.

Let us revisit the general logic to obtain $3 \to 3$ amplitudes. We start from the Feynman rules of the 6-point vertices from which we can compute a whole set of processes by the use of crossing symmetry. Besides the 6-leg vertex also some diagrams with a propagator contribute. In these diagrams the propagator connects the 4-point vertices of our sigma model, listed in appendix D. The total number of diagrams will be different depending on the process. Also in this case divergencies might only arise in a 0-measure subset of the phase space, that is when the particle in a propagator becomes on shell; exclusively in this case the $i\epsilon$ at the denominator is relevant. Therefore the strategy we adopted is (similarly to what is shown in section 2.1 and in ref. [12]) to check first that the sum of all diagrams, setting $i\epsilon$ to 0 cancels. This has been checked numerically for all the possible processes at tree-level.

This is actually enough to argue the tree-level factorisation for all the processes that do not involve massless particles, since we actually recover a case almost identical to the one in section 2.1.1. Massless processes are slightly more delicate and we keep them for last.

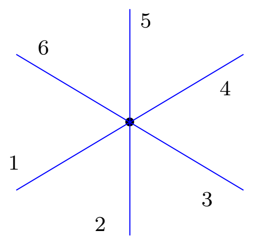
A sample calculation of such a cancellation when $i\epsilon = 0$ is shown explicitly in the next section.

5.1.1 Calculation for $z(p_1) + z(p_2) + z(p_3) \to z(p_4) + z(p_5) + z(p_6)$

Consider a process where three z particles scatter. We expect such a process to have a vanishing amplitude everywhere in the phase space except when the incoming and outgoing momenta are the same. The relevant term from the sextic Lagrangian is, from appendix C:

$$\begin{aligned} A_z^{(6)} = \mathcal{L}_z^{a=\frac{1}{2}} = &\frac{1}{4}|z|^2\left[|z|^2(|\dot{z}|^2 - 9|z'|^2) - (z'^2 - \dot{z}^2)(\bar{z}'^2 - \dot{\bar{z}}^2)\right] \\ &+ \frac{iq}{4}|z|^2\left[|z|^2(z\bar{z}' - \bar{z}z') + 2(\dot{z}^2\bar{z}\bar{z}' - \dot{\bar{z}}^2 z z') + 6|z'|^2(z\bar{z}' - \bar{z}z')\right] \\ &+ \frac{q^2}{2}|z'|^2\left[|z|^2(|\dot{z}|^2 - |z'|^2) - \frac{1}{2}z^2(\dot{\bar{z}}^2 - \bar{z}'^2) - \frac{1}{2}\bar{z}^2(\dot{z}^2 - z'^2)\right]. \end{aligned} \tag{5.1}$$

For convenience let us consider the case $a = \frac{1}{2}$. The Lagrangian in equation (5.1) gives us the Feynman rules for the 6-point vertex shown on the left hand side in figure 5.1. On the right we have the only topology stemming from the quartic Lagrangian (or Hamiltonian from the discussion in section 4.1). The Feynman rules for the 4-point vertices are reported in appendix D.



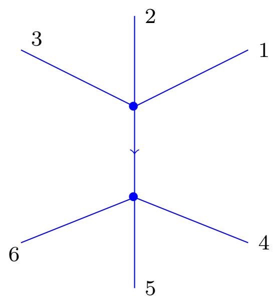


Figure 5.1: Diagrams contributing at tree-level to processes involving 6 particles in total.

Notice that these diagrams describe not only the $z\ z\ z \to z\ z\ z$ process but whatever process can be connected to it by crossing. Similarly to what we did in section 2.1 we can find a useful parametrisation of energy and momenta to be able to pass from one process to the other more easily.

Recalling the dispersion relations in equations (3.35), it holds:

$$\begin{aligned} \omega(p)^2 - (p-q)^2 &= 1 - q^2, \\ \bar{\omega}(p)^2 - (p+q)^2 &= 1 - q^2. \end{aligned} \tag{5.2}$$

The usual parametrization via the rapidities can be modified to obtain

$$\omega_i = \bar{\omega}_i = \sqrt{1-q^2}\cosh(\theta_i), \quad p_i = \pm q + \sqrt{1-q^2}\sinh(\theta_i) \tag{5.3}$$

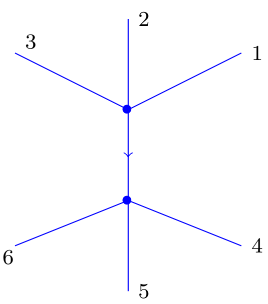
where + sign is for z and − for $\bar{z}$. Rapidities θ_i are real for physical configurations. Also in this case it is useful to redefine $e^{\theta_i} = a_i$ giving:

$$\omega_i = \bar{\omega}_i = \frac{1}{2}\sqrt{1-q^2}\left(a_i + \frac{1}{a_i}\right), \quad p_i = \pm q + \frac{1}{2}\sqrt{1-q^2}\left(a_i - \frac{1}{a_i}\right). \tag{5.4}$$

It is clear that in order to cross the amplitude with respect to the particle i we just need to invert the sign of the parameter a_i and invert the sign of q in equation (5.4). This is the same procedure that we used in section 4.2 when crossing two-particle amplitudes. In the following we assume all incoming particles when writing the Feynman rules for the diagrams. The even labels will correspond to particles z and the odd labels to antiparticles $\bar{z}$, and this justifies the orientation in the propagator in figure 5.1. As a consequence if we show the cancellation of the amplitude for whatever value of $\{a_i\}$ this means that the amplitude for every process that can be obtained by the crossing procedure also vanishes.

The expression of the diagrams with the propagator, using the parametrisation in equation (5.4) and the definition of the Feynman propagator in figure 3.2, is shown in figure 5.2. The V_4 terms are the quartic vertices attached to the propagators. Besides those and a prefactor dependent on q the expression of these diagrams is very similar to the one found in section 2.1 for the scattering in the sinh-Gordon theory.

$\mathcal{D}(a_1, a_2, a_3) =$



$$= \frac{iV_4\left[a_1, a_2, a_3, -a_1 - a_2 - a_3\right] V_4\left[a_1 + a_2 + a_3, a_4, a_5, a_6\right]}{(\omega_1 + \omega_2 + \omega_3)^2 + 2(p_1 + p_2 + p_3)q - (p_1 + p_2 + p_3)^2 - 1 + i\epsilon}$$

$$= \frac{a_1 a_2 a_3}{1 - q^2} \frac{iV_4\left[a_1, a_2, a_3, -a_1 - a_2 - a_3\right] V_4\left[a_1 + a_2 + a_3, a_4, a_5, a_6\right]}{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3) + (a_1 a_2 a_3) i \frac{\epsilon}{1-q^2}}.$$

Figure 5.2: Feynman diagram with all momenta incoming.

Before summing all the diagrams we must impose the energy and momentum conservation. Interestingly, these constraints are simply:

$$\sum_{i=1}^{6} a_i = 0, \quad \sum_{i=1}^{6} \frac{1}{a_i} = 0. \tag{5.5}$$

The q parameter does not appear in these constraints, despite energy and momentum depend on it. Therefore the energy and momentum conservation equations are the same as the ones found for the scattering in the (relativistic) sinh-Gordon theory in equation (2.5). Notice that this is not true if massless modes are involved. This is a consequence of the shape of the dispersion relation for the massive modes, together with charge conservation at the vertices.

Let us show all the steps of the calculation. Here we report the expressions where we put $q = 0$, this is just to obtain more compact expressions but the same steps lead to the same cancellation for the generic q case. First we report the Feynman rules for the 6-vertex, that we call $\mathcal{V}_6$.

They can be written as:

$$\begin{aligned}
\mathcal{V}_6 =& i\Big[p_5p_3\omega_2\omega_4 + p_5p_3\omega_2\omega_6 + p_5p_3\omega_4\omega_6 + p_2p_4\omega_3\omega_5 + p_2p_6\omega_3\omega_5 + p_4p_6\omega_3\omega_5 \\
&+ p_1\Big(p_3\omega_2\omega_4 + p_5\omega_2\omega_4 + p_3\omega_2\omega_6 + p_5\omega_2\omega_6 + p_3\omega_4\omega_6 + p_5\omega_4\omega_6 + 9p_6 \\
&+ p_2\Big(-p_4p_5 - p_6p_5 - p_3\left(p_4 + p_6\right) + 9\Big) - p_4\left(p_3p_6 + p_5p_6 - 9\right)\Big) \\
&+ \omega_1\Big(p_4p_6\omega_3 + p_4p_6\omega_5 + p_2\left(\left(p_4 + p_6\right)\omega_3 + p_4\omega_5 + p_6\omega_5\right) - \omega_4\omega_6\omega_3 - \omega_4 - \omega_4\omega_5\omega_6 \\
&- \omega_6 - \omega_2\left(\omega_4\omega_5 + \omega_6\omega_5 + \omega_3\left(\omega_4 + \omega_6\right) + 1\right)\Big) + 9p_2p_3 + 9p_4p_3 - p_2p_4p_5p_3 - p_2p_5p_6p_3 \\
&- p_4p_5p_6p_3 + 9p_6p_3 + 9p_2p_5 + 9p_4p_5 + 9p_5p_6 - \omega_2\omega_3 - \omega_3\omega_4 - \omega_2\omega_5 - \omega_2\omega_3\omega_4\omega_5 \\
&- \omega_4\omega_5 - \omega_3\omega_6 - \omega_2\omega_3\omega_5\omega_6 - \omega_3\omega_4\omega_5\omega_6 - \omega_5\omega_6\Big].
\end{aligned} \tag{5.6}$$

Once rewritten in the parametrisation in equation (5.4) this expression becomes:

$$\begin{aligned}
\mathcal{V}_6 =& -\frac{i}{4a_1a_2a_3a_4a_5a_6}\Big[a_2a_3a_4^2a_1^2 + a_2a_3a_6^2a_1^2 + a_3a_4a_6^2a_1^2 + a_2a_5a_6^2a_1^2 - 8a_2a_3a_4a_5a_6^2a_1^2 \\
&+ a_4a_5a_6^2a_1^2 + a_2^2a_3a_4a_1^2 + a_2a_4^2a_5a_1^2 + a_2^2a_4a_5a_1^2 + 10a_2a_3a_4a_5a_1^2 + a_3a_4^2a_6a_1^2 \\
&+ a_2^2a_3a_6a_1^2 + a_2^2a_5a_6a_1^2 - 8a_2a_3a_4^2a_5a_6a_1^2 + a_4^2a_5a_6a_1^2 + 10a_2a_3a_5a_6a_1^2 - 8a_2^2a_3a_4a_5a_6a_1^2 \\
&+ 10a_3a_4a_5a_6a_1^2 + a_2a_3^2a_4^2a_1 + a_2a_4^2a_5^2a_1 + a_2^2a_4a_5^2a_1 + 10a_2a_3a_4a_5^2a_1 + a_2a_3^2a_6^2a_1 + a_2a_5^2a_6^2a_1 \\
&- 8a_2a_3a_4a_5^2a_6^2a_1 + a_4a_5^2a_6^2a_1 + a_3^2a_4a_6^2a_1 + 10a_2a_3a_4a_6^2a_1 - 8a_2a_3^2a_4a_5a_6^2a_1 + 10a_2a_4a_5a_6^2a_1 \\
&+ a_2^2a_3^2a_4a_1 - 8a_2a_3a_4a_1 + 10a_2a_3^2a_4a_5a_1 - 8a_2a_4a_5a_1 + a_2^2a_3^2a_6a_1 + a_3^2a_4^2a_6a_1 + 10a_2a_3a_4^2a_6a_1 \\
&+ a_2^2a_5^2a_6a_1 - 8a_2a_3a_4^2a_5^2a_6a_1 + a_4^2a_5^2a_6a_1 + 10a_2a_3a_5^2a_6a_1 - 8a_2^2a_3a_4a_5^2a_6a_1 + 10a_3a_4a_5^2a_6a_1 \\
&- 8a_2a_3a_6a_1 + 10a_2^2a_3a_4a_6a_1 - 8a_3a_4a_6a_1 + 10a_2a_3^2a_5a_6a_1 - 8a_2a_3^2a_4^2a_5a_6a_1 + 10a_2a_4^2a_5a_6a_1 \\
&- 8a_2a_5a_6a_1 + 10a_2^2a_4a_5a_6a_1 - 8a_2^2a_3^2a_4a_5a_6a_1 + 10a_3^2a_4a_5a_6a_1 - 8a_4a_5a_6a_1 + a_2a_3a_4^2a_5^2 \\
&+ a_2^2a_3a_4a_5^2 + a_2a_3a_5^2a_6^2 + a_3a_4a_5^2a_6^2 + a_2a_3^2a_5a_6^2 + a_3^2a_4a_5a_6^2 + 10a_2a_3a_4a_5a_6^2 + a_2a_3^2a_4^2a_5 \\
&+ a_2^2a_3^2a_4a_5 - 8a_2a_3a_4a_5 + a_3a_4^2a_5^2a_6 + a_2^2a_3a_5^2a_6 + a_2^2a_3^2a_5a_6 + a_3^2a_4^2a_5a_6 + 10a_2a_3a_4^2a_5a_6 \\
&- 8a_2a_3a_5a_6 + 10a_2^2a_3a_4a_5a_6 - 8a_3a_4a_5a_6\Big].
\end{aligned} \tag{5.7}$$

Now it is the turn of the diagrams with propagators. We put $i\epsilon = 0$. The configurations for which $i\epsilon$ must be kept will be considered in one moment. We label by $\mathcal{D}$ the only topology of these diagrams. It is convenient to isolate their numerator and denominator. For example the diagram on the right-hand side of figure 5.1 can be written as:

$$\mathcal{D}(a_1, a_2, a_3) = \frac{N(a_1, a_2, a_3)}{D(a_1, a_2, a_3)}. \tag{5.8}$$

With:

$$N(a_1, a_2, a_3) = \Big[\text{expression reported in appendix E}\Big]. \tag{5.9}$$

and

$$
\begin{aligned}
D(a_1,a_2,a_3) = & -4a_2a_3^3a_4a_5a_6a_1^4 - 4a_2^2a_3^2a_4a_5a_6a_1^4 - 4a_2a_3^4a_4a_5a_6a_1^3 - 8a_2^2a_3^3a_4a_5a_6a_1^3 \\
& -4a_2^3a_3^2a_4a_5a_6a_1^3 - 4a_2^2a_3^4a_4a_5a_6a_1^2 - 4a_2^3a_3^3a_4a_5a_6a_1^2.
\end{aligned}
\tag{5.10}
$$

This is just one of the *tree diagrams*. It turns out there are 9 of these diagrams in total. We can see this by counting all the possible different diagrams. In fact if we take one of the two vertices, it can either have 2 or 1 incoming particles in the external legs. In the former case we can choose the last leg out of 3 incoming antiparticles while for the latter we choose 2 antiparticles out of 3. Then the sum of these configurations is $\binom{3}{2}\binom{3}{1} + \binom{3}{2}\binom{3}{1} = 18$. Since the vertices are identical we halve the number not to double count, obtaining 9 diagrams.

At this point we just need to compute all the other 8 diagrams and sum them with the one with 6 legs. It is convenient to do this by summing the 9 diagrams and isolating the numerator and denominator as in figure 5.3:

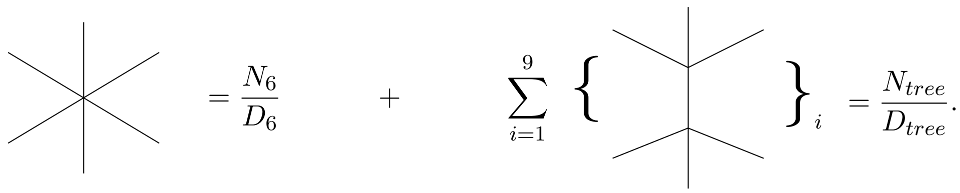


Figure 5.3: Sum of all diagrams involved in the amplitude

We therefore isolate the numerator and denominator of the six-point contribution as $\mathcal{V}_6 = N_6/D_6$ and we define the contribution from the sum of all *tree diagrams* as N_{tree}/D_{tree} .

Again in computing the amplitude it is useful to define the following symmetric polynomials:

$$
s_1 = a_2 + a_4 + a_6, \quad s_2 = a_4a_2 + a_2a_6 + a_4a_6, \quad s_3 = a_2a_4a_6. \tag{5.11}
$$

We can then solve the energy-momentum constraints for a_1 and a_3 for example. Although not immediately apparent, both the 6-leg vertex and the sum of diagrams with the propagator become independent on the last incoming particle momentum when solving equation (5.5) with respect to the other two incoming particles. Hence in our case the amplitude is independent on a_5. This simplification does not happen for $q \neq 0$.

This observation simplifies largely the calculation. In fact the expression is now much shorter and only a function of s_1, s_2, s_3:

$$
\begin{aligned}
N_{tree} = & -128is_1^2s_3^8 - 144is_3^8 + 128is_1^3s_2s_3^7 + 560is_1s_2s_3^7 - 432is_1^2s_2^2s_3^6 - 128is_2^2s_3^6 + 16is_1^3s_2^3s_3^5 \\
& + 128is_1s_2^3s_3^5, \\
D_{tree} = & 64\left(s_1s_2 - s_3\right)s_3^7.
\end{aligned}
\tag{5.12}
$$

and:

$$
\begin{aligned}
N_6 = & -i\left[s_1^2s_2^2 + 8s_2^2 - 26s_1s_3s_2 + 8s_1^2s_3^2 + 9s_3^2\right], \\
D_6 = & 4s_3^2.
\end{aligned}
\tag{5.13}
$$

As we can check, numerator and denominator in equation (5.12) can be simplified by collecting the factor $16s_3^5\,(s_1s_2 - s_3)$. This gives us:

$$\frac{N_{tree}}{D_{tree}} = \frac{i\left[s_1^2s_2^2 + 8s_2^2 - 26s_1s_3s_2 + 8s_1^2s_3^2 + 9s_3^2\right]}{4s_3^2} \tag{5.14}$$

Equation (5.14) is exactly the opposite contribution to the one from equation (5.13). This shows the cancellation of the amplitude for the 3 z scattering.

5.1.2 Factorisation

The previous calculation was performed numerically for all the 8 interactions that appear at tree-level. Therefore the amplitudes vanish on almost every kinematic configuration. At this point we can differentiate the discussion depending on whether massless particles are involved or not.

When massless particles are not involved in the scattering, as for the process we studied in the previous section, we can essentially repeat the same points that we discussed in subsection 2.1.1. The main idea is that the full cancellation does not happen only when a propagator goes on-shell. From the expression of the propagator in figure 5.2 we see that this happens if an incoming particle and an outgoing particle have the same momenta, namely if $a_i = -a_j$ (since we took all momenta incoming). When only z, $\bar{z}$ and y, $\bar{y}$ particles are involved the energy and momentum constraints are shown in equation (5.5) and are exactly the same as those of the sinh-Gordon $3 \to 3$ scattering (equation 2.5). As a consequence also for the massive scattering in the string sigma model we find that if a final particle has the same momentum as an initial one, for example $a_1 = -a_2$, then also the other momenta satisfy:

$$\{a_3 = -a_4,\ a_5 = -a_6\} \quad \text{or} \quad \{a_3 = -a_6,\ a_5 = -a_4\}. \tag{5.15}$$

This is simply a consequence of the overall energy-momentum conservation. Therefore the only configurations we neglected when putting $i\epsilon = 0$ are exactly the configurations allowed to scatter non-trivially by integrability.

This is enough to show that factorisation occurs for the massive interactions of the string sigma model. It can be shown in a similar way as we did in subsection 2.1.1 that the *tree diagrams* can be picked in couples to produce the Dirac delta-function needed to enforce the momentum conservation; therefore we can argue in the same way we did in section 2.1.1 that the $S_{3\to3}$ at the first perturbative order has the same factorised structure as in equation (2.18).

The scatterings involving massless particles are slightly different. As we already mentioned we checked numerically that also for these types of processes the amplitudes cancel when $i\epsilon = 0$. However it is worth noting that the factorisation does not happen exactly in the same way. Take for example the process:

$$u(p1) + u(p2) + z(p3) \to u(p4) + u(p5) + z(p6) \tag{5.16}$$

The six-point vertex for this process can be obtained from the $\mathcal{L}^{(6)}_{\mu XX}$ piece in appendix C, equations (C.5),(C.6),(C.7). The *tree diagrams* that contribute to this process are 8 in total, four of which have a massless propagator and the others a z particle propagator.

Notice that by crossing $z(p_3)$ we can connect the process in equation (5.16) to $u\ u \to u\ u\ z\ \bar{z}$. We expect this last process to have zero amplitude for every kinematic configuration since an integrable theory should not allow interactions to produce particles. The difference in this case with respect to interactions that do not involve massless particles is in the kinematic constraints. The energy and momentum conservation constraints for the process in equation (5.16) can be written as:

$$\begin{cases} p_1 + p_2 + \frac{1}{2}\sqrt{1-q^2}\left(a_3 - \frac{1}{a_3}\right) = p_4 + p_5 + \frac{1}{2}\sqrt{1-q^2}\left(a_6 - \frac{1}{a_6}\right) \\ |p_1| + |p_2| + \frac{1}{2}\sqrt{1-q^2}\left(a_3 + \frac{1}{a_3}\right) = |p_4| + |p_5| + \frac{1}{2}\sqrt{1-q^2}\left(a_6 + \frac{1}{a_6}\right) \end{cases} \tag{5.17}$$

The massless particles have the standard momentum and energy definitions while for the energy and momenta of the z particles are parametrised as in equation (5.4). These equations are different from the ones we have already seen in the sense that if one of the initial momenta is equal to one of the final ones the constraints do not automatically "collapse" the other momenta to the solution allowed by integrability. This fact was crucial for the factorisation of the amplitude in the sinh-Gordon case and hence also in the elastic three z scattering in the string sigma model.

Now suppose that we are in the region of the phase space close to the $p_1 = p_4$ solution. This will make some of the *tree diagrams* divergent and we expect them to diverge in couples. The divergent diagrams with the massless propagator are:

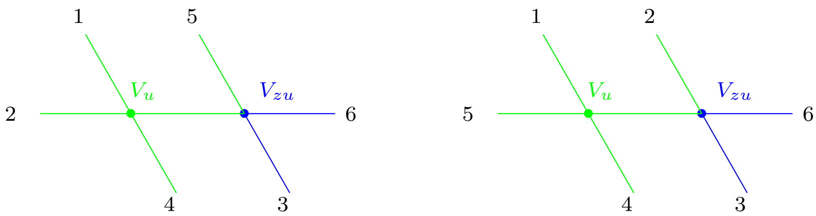


Figure 5.4

While the ones with the massive propagator are:

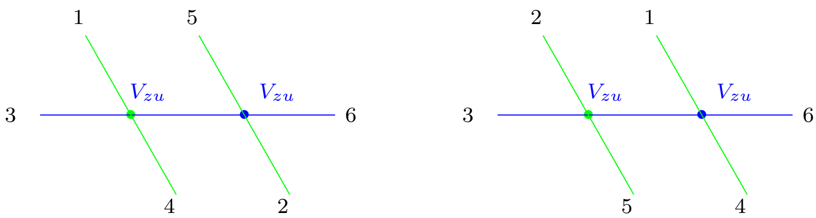


Figure 5.5

Observe that the u particles are depicted in green and the z and $\bar{z}$ particles in blue. V_u and V_{zu} refer respectively to the vertex with four u particles and the mixed vertex with u and z, both reported in appendix D.

These two couples of diagrams are able to produce the delta-functions in a way closely resembling what we already saw. Suppose to have $p_1 > 0$ and $p_2 < 0$. Notice that the only other choice can be the opposite one since the two incoming massless momenta must be opposite in sign in order to allow the scattering to happen. Taking the limit $p_1 \to p_4$ and selecting the $p_2 = p_5$ solution at this limit the sum of the propagators in figure 5.4 can be rewritten as:

$$-\frac{i}{4p_2}\left(\frac{1}{p_1 - p_4 - \frac{i\epsilon}{4p_2}} - \frac{1}{p_1 - p_4 + \frac{i\epsilon}{4p_2}}\right). \tag{5.18}$$

On the other hand from the diagrams in figure 5.5 we obtain from the sum of the propagators:

$$-\frac{ia_3}{2(1-q^2)^{1/2}}\left(\frac{1}{p_1 - p_4 - \frac{i\epsilon a_3}{2(1-q^2)^{1/2}}} - \frac{1}{p_1 - p_4 + \frac{i\epsilon a_3}{2(1-q^2)^{1/2}}}\right). \tag{5.19}$$

From the considerations in section 2.1.1 we can obtain from each couple of diagrams the delta-function enforcing $p_1 = p_4$. Therefore for this configuration of momenta the amplitude factorises in products of two 4-vertices amplitudes in the same way as for the massive case.

Until now the reasoning is similar to the one we employed for *sinh*-Gordon theory. Yet in the massless case in the string model we should immediately spot a difference. With the $p_1 = p_4$ assumption the kinematic equations in (5.17) actually yield two solutions, not only the $\{p_2 = p_5,\ a_3 = a_6\}$ one. In fact we obtain the same system that we discussed in equation (4.19). Therefore the deltas enforcing $p_1 = p_4$ would not be enough to also impose that the initial set of momenta is conserved.

This difference seems to spoil the factorisation of the interaction but actually it does not. The other solutions are taken care of once we recall the properties of the V_{zu} vertex, in particular the discussion in section 4.2. Indeed, from equations (5.17) we can check that the energy and momentum conservation, when $p_1 = p_4$ is imposed, requires either that $p_2 = p_5$, namely the only configuration for which we expect a non-zero, factorised amplitude, or that $\text{sign}(p_2) = -\text{sign}(p_5)$. Clearly all of the four diagrams in figures 5.4 and 5.5 contain the vertex V_{uz} with incoming particles 2 and 5. When the momenta have opposite signs we observed that such vertex is automatically zero. These vanishing 4-vertices are at the numerator of the apparently divergent *tree diagrams*; however the "$\frac{0}{0}$" ambiguity is prevented by the $i\epsilon$, whose limit to zero must be taken at the end. Hence all of these potentially divergent diagrams actually vanish. Hence for this process the amplitude factorises only when the initial set of momenta is equal to the final one.

We should also mention that this mechanism is also responsible for canceling processes that are not allowed by integrability. As a matter of fact, we can cross one z particle in the previous process to obtain another process where two incoming u end up into two u plus z and $\bar{z}$. Observe that the latter is described by the exact same diagrams and only the total conservation of energy and momentum differs. The kinematic constrains become instead:

$$\begin{cases} p_1 + p_2 = p_4 + p_5 + \frac{1}{2}\sqrt{1-q^2}\left(a_3 - \frac{1}{a_3}\right) + \frac{1}{2}\sqrt{1-q^2}\left(a_6 - \frac{1}{a_6}\right) \\ |p_1| + |p_2| = |p_4| + |p_5| + \frac{1}{2}\sqrt{1-q^2}\left(a_3 + \frac{1}{a_3}\right) + \frac{1}{2}\sqrt{1-q^2}\left(a_6 + \frac{1}{a_6}\right) \end{cases} \tag{5.20}$$

Here the label were kept the same so that near the configuration $p_1 = p_4$ the diagrams in figure 5.4 and 5.5 are still the divergent ones. Even in this case we do not expect any delta to appear since this process should not happen from the integrability of the theory. When $p_1 = p_4$ the equations in (5.20) reduce to the equations in (4.13) which again admit two solutions. These two solutions both require that $\text{sign}(p_2) = -\text{sign}(p_5)$. The properties of the massless vertices are in this case sufficient to cancel the numerator of the diagrams that would individually diverge otherwise. Once again we can check that each potentially divergent diagram actually vanishes due to the presence of vanishing V_{uz} or V_{uu} vertices.

Chapter 6

Conclusions

This thesis was focused on the perturbative investigation of the S-matrix of bosonic non-linear σ models defined on the worldsheet of the strings in $AdS_3 \times S^3 \times T^4$ supported by a mixture of RR and NSNS fluxes. These models have been shown to be perturbatively integrable at the tree level, as expected from the classical considerations presented in ref. [3]. In this work we have first studied the two-to-two particles processes and computed the full tree-level two-particle S-matrix. We also checked that all the scatterings involving three particles factorise.

More precisely the thesis has been structured as follows.

In the first chapter we introduced certain universal properties that integrable QFTs need to satisfy. In chapter 2 these properties have been explicitly checked at the tree-level for two notoriously integrable models. Some of these calculations have been used as training grounds in preparation for the actual string sigma model.

Then we dedicated the third chapter to the complete derivation of the NLSM starting from the string action. The model is obtained after fixing the light-cone gauge and the subsequent derivation was similar to the one in ref. [28] (where the $AdS_5 \times S^5$ background was investigated). The resulting theory is described by an Hamiltonian with derivative interactions which was computed up to the sixth order in the fields and momenta. This is a necessary step to compute three-to-three amplitudes perturbatively at the tree-level. The interacting Lagrangian was also obtained and split into the relevant pieces describing the 4-point or 6-point Feynman vertices. The interacting Hamiltonian and the pieces composing the interacting Lagrangian are presented in appendices B and C respectively.

After deriving the Lagrangian of the theory we have obtained the Feynman rules for the four and six-leg vertices; the first kind are reported in appendix D in full. The others are not reported due to the length of the expressions but can be read off from the sextic Lagrangian. In the fourth chapter we computed all the possible two-to-two scattering processes at the tree level. We checked that only the processes allowed by integrability are non-vanishing; we reported all the amplitudes that contribute to the two-to-two S-matrix.

Finally in the last chapter we considered processes involving 6 external particles. The cancellation of all the amplitudes forbidden by integrability has been checked. We reported an example of a calculation explicitly showing this fact for a class of processes. Afterwards we successfully checked the tree-level factorisation of the allowed three-to-three amplitudes into products of two-to-two amplitudes and explained the mechanisms through which this happens.

The original results of this work are presented in [32].

Appendix A

G and B-fields in light-cone coordinates

Here we present the expression for the metric G and the B-field in light-cone coordinates. The metric can be written as spacetime interval:

$$
\begin{aligned}
ds^2 = &\Big[-\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + \Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2\Big]dx_+^2 \\
&+ \Big[-a^2\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + (1-a)^2\Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2\Big]dx_-^2 \\
&+ \Big[2a\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + 2(1-a)\Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2\Big]dx_+dx_- \\
&+ \frac{8}{(2-|z|^2)^2}dzd\bar{z} + \frac{8}{(2+|y|^2)^2}dyd\bar{y} + 2dud\bar{u} + 2dvd\bar{v}.
\end{aligned}
\tag{A.1}
$$

The B-field takes the form:

$$
\begin{aligned}
B = &\frac{8iq}{(2-|z|^2)^2}\left[\bar{z}dx_+ \wedge dz - zdx_+ \wedge d\bar{z} - a\bar{z}dx_- \wedge dz + azdx_- \wedge d\bar{z}\right] \\
&+ \frac{8iq}{(2+|y|^2)^2}\left[\bar{y}dx_+ \wedge dy - ydx_+ \wedge d\bar{y} + (1-a)\bar{y}dx_- \wedge dy - (1-a)ydx_- \wedge d\bar{y}\right].
\end{aligned}
\tag{A.2}
$$

From these we read the terms in the coefficients in equation (3.27):

$$
\begin{aligned}
B_{z+} &= \overline{B}_{\bar{z}+} = \frac{-4iq\bar{z}}{(2-|z|^2)^2}, \qquad B_{z-} = \overline{B}_{\bar{z}-} = \frac{4aiq\bar{z}}{(2-|z|^2)^2}, \\
B_{y+} &= \overline{B}_{\bar{y}+} = \frac{-4iq\bar{y}}{(2-|y|^2)^2}, \qquad B_{y-} = \overline{B}_{\bar{y}-} = \frac{-4(1-a)iq\bar{y}}{(2-|y|^2)^2}.
\end{aligned}
\tag{A.3}
$$

and

$$
\begin{aligned}
G_{++} &= -\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + \Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2, \\
G_{--} &= -a^2\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + (1-a)^2\Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2, \\
G_{+-} &= a\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + (1-a)\Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2, \\
G_{z\bar{z}} &= \frac{4}{(2-|z|^2)^2}, \qquad G_{y\bar{y}} = \frac{4}{(2+|y|^2)^2}, \qquad G_{u\bar{u}} = G_{v\bar{v}} = 1.
\end{aligned}
\tag{A.4}
$$

Appendix B

Free and interacting Hamiltonians

The free Hamiltonians of the non-linear sigma model is the following.

$$
\begin{aligned}
\mathcal{H}^{(2)} &= \left[p_z p_{\bar{z}} + p_y p_{\bar{y}} + p_u p_{\bar{u}} + p_v p_{\bar{v}} + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2\right] \\
&+ (|z|^2 + |y|^2) + iq(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}}).
\end{aligned} \tag{B.1}
$$

The quartic Hamiltonian is also reported. It has a a gauge dependence and is wrote in a way that it greatly simplifies in the $a = \frac{1}{2}$ gauge.

$$
\begin{aligned}
\mathcal{H}^{(4)} &= (|z|^2 - |y|^2)\left[p_z p_{\bar{z}} + p_y p_{\bar{y}} + p_u p_{\bar{u}} + p_v p_{\bar{v}} + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2\right] \\
&- p_z p_{\bar{z}}|z|^2 + p_y p_{\bar{y}}|y|^2 + |\acute{z}|^2|z|^2 - |\acute{y}|^2|y|^2 + \frac{iq}{2}(|z|^2 - |y|^2)(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}}) \\
&- \frac{iq}{2}(p_z\acute{\bar{z}} + p_{\bar{z}}\acute{z} + p_y\acute{\bar{y}} + p_{\bar{y}}\acute{y} + p_u\acute{\bar{u}} + p_{\bar{u}}\acute{u} + p_v\acute{\bar{v}} + p_{\bar{v}}\acute{v})(\bar{z}p_z - zp_{\bar{z}} - \bar{y}p_y + yp_{\bar{y}}) \\
&+ \frac{iq}{2}\left[p_z p_{\bar{z}} + p_y p_{\bar{y}} + p_u p_{\bar{u}} + p_v p_{\bar{v}} + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2\right](\bar{z}\acute{z} - z\acute{\bar{z}} - \bar{y}\acute{y} + y\acute{\bar{y}}) \\
&+ \frac{(2a-1)}{2}\left[(p_z p_{\bar{z}} + p_y p_{\bar{y}} + p_u p_{\bar{u}} + p_v p_{\bar{v}} + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2)^2\right] \\
&+ \frac{(2a-1)}{2}\left[-(|z|^2 + |y|^2)^2 - ((p_z\acute{\bar{z}} + p_{\bar{z}}\acute{z} + p_y\acute{\bar{y}} + p_{\bar{y}}\acute{y} + p_u\acute{\bar{u}} + p_{\bar{u}}\acute{u} + p_v\acute{\bar{v}} + p_{\bar{v}}\acute{v}))^2\right] \\
&+ \frac{(2a-1)iq}{2}\left[(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}})\left(p_z p_{\bar{z}} + p_y p_{\bar{y}} + p_u p_{\bar{u}} + p_v p_{\bar{v}} + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2 - (|z|^2 + |y|^2)\right)\right] \\
&- \frac{(2a-1)iq}{2}\left[(p_z\acute{\bar{z}} + p_{\bar{z}}\acute{z} + p_y\acute{\bar{y}} + p_{\bar{y}}\acute{y} + p_u\acute{\bar{u}} + p_{\bar{u}}\acute{u} + p_v\acute{\bar{v}} + p_{\bar{v}}\acute{v})(\bar{z}p_z - zp_{\bar{z}} + \bar{y}p_y - yp_{\bar{y}})\right].
\end{aligned} \tag{B.2}
$$

The part of the Hamiltonian containing 6 powers of the fields can be written as:

$$
\mathcal{H}^{(6)} = \mathcal{H}_0^{(6)} + \frac{(2a-1)}{2}\mathcal{H}_1^{(6)} + \left(\frac{2a-1}{2}\right)^2 \mathcal{H}_2^{(6)}. \tag{B.3}
$$

For completeness we also report the expressions of $\mathcal{H}_0^{(6)}$, $\mathcal{H}_1^{(6)}$ $\mathcal{H}_2^{(6)}$ obtained from Mathematica.

$$\mathcal{H}_0^{(6)} = \frac{1}{4}\Big(-\Big((y\bar{y}+z\bar{z})\left(\bar{y}y'+\bar{z}z'-y\bar{y}'-z\bar{z}'\right)^2 q^2\Big)-(y\bar{y}+z\bar{z})\left(-\bar{y}y'+\bar{z}z'+y\bar{y}'-z\bar{z}'\right)^2 q^2+$$
$$+(y\bar{y}+z\bar{z})\left(p_u u'+p_v v'+p_y y'+p_z z'+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\right)^2 q^2+2(y\bar{y}+z\bar{z})$$
$$\Big(\bar{z}^2 z'^2-2z\bar{z}\bar{z}'z'+\left(\bar{y}y'-y\bar{y}'\right)^2+z^2\left(\bar{z}'\right)^2\Big)q^2-i\Big(3\bar{z}^2\bar{z}'z^3+\bar{z}\left(-3z'\bar{z}^2-4\bar{y}y'\bar{z}+4y\bar{y}'\bar{z}+4y\bar{y}\bar{z}'\right)z^2+$$
$$+4y\bar{y}\left(-z'\bar{z}^2-\bar{y}y'\bar{z}+y\bar{y}'\bar{z}+y\bar{y}\bar{z}'\right)z+y^2\bar{y}^2\left(-3\bar{y}y'-4\bar{z}z'+3y\bar{y}'\right)\Big)q+4i(y\bar{y}+z\bar{z})$$
$$\Big(y\left(-\bar{y}\bar{z}z'+z\bar{z}\bar{y}'+z\bar{y}\bar{z}'\right)-z\bar{y}\bar{z}y'\Big)q-(y\bar{y}+z\bar{z})\Big(p_u u'+p_v v'+p_y y'+p_z z'+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+$$
$$+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)^2+(y\bar{y}+z\bar{z})\Big(y\bar{y}+iqy'\bar{y}+z\bar{z}+p_u p_{\bar{u}}+p_v p_{\bar{v}}+p_y p_{\bar{y}}+p_z p_{\bar{z}}+iq\bar{z}z'+u'\bar{u}'+v'\bar{v}'-$$
$$-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)^2+3\left(y^3\bar{y}^3+z^3\bar{z}^3\right)+y^2\bar{y}^2 p_y p_{\bar{y}}+z^2\bar{z}^2 p_z p_{\bar{z}}+3y^2\bar{y}^2 y'\bar{y}'+3z^2\bar{z}^2 z'\bar{z}'+$$
$$+i\Big(2iy\bar{y}-qy'\bar{y}-2iz\bar{z}+q\bar{z}z'+qy\bar{y}'-qz\bar{z}'\Big)\Big(\left(\bar{y}y'+\bar{z}z'-y\bar{y}'-z\bar{z}'\right)\left(-\bar{y}y'+\bar{z}z'+y\bar{y}'-z\bar{z}'\right)q^2+$$
$$+2i(y\bar{y}+z\bar{z})\left(\bar{y}y'-\bar{z}z'-y\bar{y}'+z\bar{z}'\right)q+2i(z\bar{z}-y\bar{y})\left(-\bar{y}y'-\bar{z}z'+y\bar{y}'+z\bar{z}'\right)q+2i\Big(-\bar{z}\bar{z}'z^2+$$
$$+\bar{z}^2 z'z+y\bar{y}\left(y\bar{y}'-\bar{y}y'\right)\Big)q-i\left(yp_y-zp_z-\bar{y}p_{\bar{y}}+\bar{z}p_{\bar{z}}\right)\Big(p_u u'+p_v v'+p_y y'+p_z z'++p_{\bar{u}}\bar{u}'+$$
$$+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)q+2y^2\bar{y}^2-2z^2\bar{z}^2+2y\bar{y}p_y p_{\bar{y}}-2z\bar{z}p_z p_{\bar{z}}-2y\bar{y}y'\bar{y}'+2z\bar{z}z'\bar{z}'\Big)+$$
$$+\Big(y\bar{y}+iqy'\bar{y}+z\bar{z}+p_u p_{\bar{u}}+p_v p_{\bar{v}}+p_y p_{\bar{y}}+p_z p_{\bar{z}}+iq\bar{z}z'+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-$$
$$-iqz\bar{z}'+z'\bar{z}'\Big)\Big(-q^2\left(-\bar{y}y'+\bar{z}z'+y\bar{y}'-z\bar{z}'\right)^2-8yz\bar{y}\bar{z}-2iq\Big(\bar{z}\bar{z}'z^2-\Big(z'\bar{z}^2-2\bar{y}y'\bar{z}+$$
$$+2y\bar{y}'\bar{z}+2y\bar{y}\bar{z}'\Big)z+y\bar{y}\left(-\bar{y}y'+2\bar{z}z'+y\bar{y}'\right)\Big)\Big)\Big). \tag{B.4}$$

$$\mathcal{H}_1^{(6)} = \Big((z\bar{z}-y\bar{y})\left(\bar{y}y'+\bar{z}z'-y\bar{y}'-z\bar{z}'\right)^2 q^2\Big)-(y\bar{y}-z\bar{z})\Big(p_u u'+p_v v'+p_y y'+p_z z'+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+$$
$$+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)^2 q^2-(y\bar{y}+z\bar{z})\left(\bar{y}y'-\bar{z}z'-y\bar{y}'+z\bar{z}'\right)\left(-\bar{y}y'-\bar{z}z'+y\bar{y}'+z\bar{z}'\right)q^2-\Big[2\Big(\bar{y}y'+\bar{z}z'-$$
$$-y\bar{y}'-z\bar{z}'\Big)\left(-\bar{z}\bar{z}'z^2+\bar{z}^2 z'z+y\bar{y}\left(y\bar{y}'-\bar{y}y'\right)\right)q^2-2(y\bar{y}+z\bar{z})\Big(\bar{z}^2 z'^2-2z\bar{z}\bar{z}'z'-\left(\bar{y}y'-y\bar{y}'\right)^2+$$
$$+z^2\bar{z}'^2\Big)q^2+\frac{1}{2}i\left(\bar{y}y'-\bar{z}z'-y\bar{y}'+z\bar{z}'\right)\Big(y\bar{y}+iqy'\bar{y}+z\bar{z}+p_u p_{\bar{u}}+p_v p_{\bar{v}}+p_y p_{\bar{y}}+p_z p_{\bar{z}}+iq\bar{z}z'+$$
$$+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)^2 q+(z\bar{z}-y\bar{y})\Big(p_u u'+p_v v'+p_y y'+p_z z'+p_{\bar{u}}\bar{u}'+$$
$$+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)^2+\frac{1}{2}i\left(2iy\bar{y}-qy'\bar{y}-2iz\bar{z}+q\bar{z}z'+qy\bar{y}'-qz\bar{z}'\right)\Big(p_u u'+p_v v'+p_y y'+p_z z'+$$
$$+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)\Big(-iqyp_y+y'p_y-iqzp_z+iq\bar{y}p_{\bar{y}}+iq\bar{z}p_{\bar{z}}+p_u u'+p_v v'+p_z z'+$$
$$+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)+\Big(y\bar{y}+iqy'\bar{y}+z\bar{z}+p_u p_{\bar{u}}+p_v p_{\bar{v}}+p_y p_{\bar{y}}+p_z p_{\bar{z}}+iq\bar{z}z'+u'\bar{u}'+$$
$$+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)\Big(\Big(\left(\bar{y}y'-y\bar{y}'\right)^2-\bar{z}^2 z'^2\Big)q^2+3iy\bar{y}\left(y\bar{y}'-\bar{y}y'\right)q+$$
$$+z\bar{z}z'\left(3i\bar{z}+2q\bar{z}'\right)q-2y^2\bar{y}^2+z^2\left(2\bar{z}^2-3iq\bar{z}'\bar{z}-q^2\bar{z}'^2\right)\Big)- \tag{B.5}$$

$$
\begin{aligned}
&-\frac{1}{2}i\left(2iy\bar{y}-qy'\bar{y}+2iz\bar{z}-q\bar{z}z'+qy\bar{y}'+qz\bar{z}'\right)\Big[\left(\bar{y}y'+\bar{z}z'-y\bar{y}'-z\bar{z}'\right)\left(-\bar{y}y'+\bar{z}z'+y\bar{y}'-z\bar{z}'\right)q^2+\\
&+2i\left(y\bar{y}+z\bar{z}\right)\left(\bar{y}y'-\bar{z}z'-y\bar{y}'+z\bar{z}'\right)q+i\left(-yp_y+zp_z+\bar{y}p_{\bar{y}}-\bar{z}p_{\bar{z}}\right)\Big(p_uu'+p_vv'+p_yy'+p_zz'+\\
&+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)q+2i\left(y\left(\bar{y}\bar{z}z'+z\bar{z}\bar{y}'-z\bar{y}\bar{z}'\right)-z\bar{y}\bar{z}y'\right)q+2y^2\bar{y}^2-2z^2\bar{z}^2+2y\bar{y}p_yp_{\bar{y}}-\\
&-2z\bar{z}p_zp_{\bar{z}}-2y\bar{y}y'\bar{y}'+2z\bar{z}z'\bar{z}'\Big]-\Big(y\bar{y}+iqy'\bar{y}+z\bar{z}+p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+iq\bar{z}z'+u'\bar{u}'+\\
&+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)\Big[\left(\bar{y}y'+\bar{z}z'-y\bar{y}'-z\bar{z}'\right)\left(-\bar{y}y'+\bar{z}z'+y\bar{y}'-z\bar{z}'\right)q^2+\\
&+2i\left(y\bar{y}+z\bar{z}\right)\left(\bar{y}y'-\bar{z}z'-y\bar{y}'+z\bar{z}'\right)q+i\left(-yp_y+zp_z+\bar{y}p_{\bar{y}}-\bar{z}p_{\bar{z}}\right)\Big(p_uu'+p_vv'+p_yy'+p_zz'+\\
&+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)q+2i\left(y\left(\bar{y}\bar{z}z'+z\bar{z}\bar{y}'-z\bar{y}\bar{z}'\right)-z\bar{y}\bar{z}y'\right)q+2y^2\bar{y}^2-2z^2\bar{z}^2+2y\bar{y}p_yp_{\bar{y}}-\\
&-2z\bar{z}p_zp_{\bar{z}}-2y\bar{y}y'\bar{y}'+2z\bar{z}z'\bar{z}'+i\left(2iy\bar{y}-qy'\bar{y}-2iz\bar{z}+q\bar{z}z'+qy\bar{y}'-qz\bar{z}'\right)\Big(y\bar{y}+iqy'\bar{y}+z\bar{z}+\\
&+p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+iq\bar{z}z'+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)\Big]\Big].
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
\mathcal{H}_2^{(6)}=&-\Big(\Big(p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+\bar{y}\left(y+iqy'\right)+\bar{z}\left(z+iqz'\right)+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+\\
&+z'\bar{z}'\Big)\left(2iz\bar{z}-qz'\bar{z}+\bar{y}\left(2iy-qy'\right)+qy\bar{y}'+qz\bar{z}'\right)^2\Big)-i\Big(p_uu'+p_vv'+p_yy'+p_zz'+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+\\
&+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)\Big(iq\bar{y}p_{\bar{y}}+\bar{y}'p_{\bar{y}}+iq\bar{z}p_{\bar{z}}+p_uu'+p_vv'+p_y\left(y'-iqy\right)+p_z\left(z'-iqz\right)+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'\\
&+p_{\bar{z}}\bar{z}'\Big)\left(2iz\bar{z}-qz'\bar{z}+\bar{y}\left(2iy-qy'\right)+qy\bar{y}'+qz\bar{z}'\right)+q^2\left(y\bar{y}+z\bar{z}\right)\Big(p_uu'+p_vv'+p_yy'+p_zz'+\\
&+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)^2-\left(y\bar{y}+z\bar{z}\right)\left(p_uu'+p_vv'+p_yy'+p_zz'+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\right)^2-\\
&-\Big(p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+\bar{y}\left(y+iqy'\right)+\bar{z}\left(z+iqz'\right)+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+\\
&+z'\bar{z}'\Big)^2\left(y\bar{y}+iq\left(\bar{y}y'+\bar{z}z'-y\bar{y}'\right)+z\left(\bar{z}-iq\bar{z}'\right)\right)-2\Big(p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+\bar{y}\left(y+iqy'\right)+\\
&+\bar{z}\left(z+iqz'\right)+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)\Big(\Big(p_uu'+p_vv'+p_yy'+p_zz'+p_{\bar{u}}\bar{u}'+\\
&+p_{\bar{v}}\bar{v}'+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)^2+iq\left(-yp_y-zp_z+\bar{y}p_{\bar{y}}+\bar{z}p_{\bar{z}}\right)\Big(p_uu'+p_vv'+p_yy'+p_zz'+p_{\bar{u}}\bar{u}'+p_{\bar{v}}\bar{v}'+\\
&+p_{\bar{y}}\bar{y}'+p_{\bar{z}}\bar{z}'\Big)-\Big(p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+\bar{y}\left(y+iqy'\right)+\bar{z}\left(z+iqz'\right)+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+\\
&+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)^2+2iq\left(y\bar{y}+z\bar{z}\right)\left(\bar{y}y'+\bar{z}z'-y\bar{y}'-z\bar{z}'\right)+2iq\left(y\bar{y}+z\bar{z}\right)\Big(-\bar{y}y'-\bar{z}z'+y\bar{y}'+\\
&+z\bar{z}'\Big)-i\Big(2iz\bar{z}-qz'\bar{z}+\bar{y}\left(2iy-qy'\right)+qy\bar{y}'+qz\bar{z}'\Big)\Big(p_up_{\bar{u}}+p_vp_{\bar{v}}+p_yp_{\bar{y}}+p_zp_{\bar{z}}+\bar{y}\left(y+iqy'\right)+\\
&+\bar{z}\left(z+iqz'\right)+u'\bar{u}'+v'\bar{v}'-iqy\bar{y}'+y'\bar{y}'-iqz\bar{z}'+z'\bar{z}'\Big)\Big).
\end{aligned}
\tag{B.7}
$$

Appendix C

Pieces of the sextic Lagrangian

Here we report all the different interaction pieces of the sextic Lagrangian of the string sigma model. These contribute to $\mathcal{L}^{(6)}$ as illustrated in section 3.4.

$$\begin{aligned} A_X^{(6)} &= \frac{1}{4}|X|^2\left[|X|^2(|\dot{X}|^2 - 9|X'|^2) - (X'^2 - \dot{X}^2)(\bar{X}'^2 - \dot{\bar{X}}^2)\right] \\ &+ \frac{iq}{4}|X|^2\left[|X|^2(X\bar{X}' - \bar{X}X') + 2(\dot{X}^2\bar{X}\bar{X}' - \dot{\bar{X}}^2XX') + 6|X'|^2(X\bar{X}' - \bar{X}X')\right] \\ &+ \frac{q^2}{2}|X'|^2\left[|X|^2(|\dot{X}|^2 - |X'|^2) - \frac{1}{2}X^2(\dot{\bar{X}}^2 - \bar{X}'^2) - \frac{1}{2}\bar{X}^2(\dot{X}^2 - X'^2)\right]. \end{aligned} \tag{C.1}$$

$$\begin{aligned} B_X^{(6)} &= |X|^2\left[\dot{\bar{X}}^2\acute{X}^2 + \acute{\bar{X}}^2\dot{X}^2 - 2|\acute{X}|^4\right] \\ &+ iq\Big[-(X\acute{\bar{X}} - \bar{X}\acute{X})(-\frac{3}{2}|\acute{X}|^2|X|^2 + \frac{1}{2}|X|^4 + \frac{1}{2}|\acute{X}|^2|\dot{X}|^2 - \frac{3}{2}|\acute{X}|^4 - \frac{1}{2}|\dot{X}|^4) \\ &+ (\frac{1}{2}|\dot{X}|^2 - \frac{3}{2}|\acute{X}|^2 - |X|^2)(X\acute{X}\dot{\bar{X}}^2 - \bar{X}\acute{\bar{X}}\dot{X}^2) + \bar{X}\dot{\bar{X}}^2\acute{X}^3 - X\dot{X}^2\acute{\bar{X}}^3\Big] \\ &+ q^2|X'|^2\left[|X|^2(|\dot{X}|^2 - |X'|^2) + 2(\bar{X}^2\acute{X}^2 + \acute{\bar{X}}^2X^2) - 2(\bar{X}^2\dot{X}^2 + \dot{\bar{X}}^2X^2)\right]. \end{aligned} \tag{C.2}$$

$$\begin{aligned} C_X^{(6)} &= -\frac{1}{4}\Big[|X|^2(|X|^4 - \dot{\bar{X}}^2X'^2 - \dot{X}^2\bar{X}'^2 + |X'|^4 + |\dot{X}|^4) + 2|\dot{X}|^2(\dot{\bar{X}}^2X'^2 + \dot{X}^2\bar{X}'^2 - |X'|^4 - |\dot{X}|^4)\Big] \\ &- \frac{1}{2}\Big[|X'|^2(|X'|^4 - \dot{\bar{X}}^2X'^2 - \dot{X}^2\bar{X}'^2)\Big] - \frac{iq}{4}\Big[(X\bar{X}' - X'\bar{X})(-|X|^4 - |\dot{X}|^4 + 2|\dot{X}|^2|X'|^2 - 3|X'|^4) \\ &+ (XX'\dot{\bar{X}}^2 - \bar{X}\bar{X}'\dot{X}^2)(3|X'|^2 - 2|\dot{X}|^2) + \dot{X}^2X\bar{X}'^3 - \dot{\bar{X}}^2\bar{X}X'^3\Big] - \frac{q^2}{4}\Big[2|X|^2(|X'|^4 - +|\dot{X}|^2|X'|^2) \\ &+ (\dot{X}^2\bar{X}^2 + \dot{\bar{X}}^2X^2 - X'^2\bar{X}^2 - \bar{X}'^2X^2)|X'|^2\Big]. \end{aligned} \tag{C.3}$$

$$D_\mu^{(6)} = -\frac{1}{2}\Big[|\mu'|^2(|\dot{\mu}|^4 - |\dot{\mu}|^2|\mu'|^2 + |\mu'|^4 - \dot{\mu}^2\bar{\mu}'^2 - \dot{\bar{\mu}}^2\mu'^2) + |\dot{\mu}|^2(\dot{\mu}^2\bar{\mu}'^2 + \dot{\bar{\mu}}^2\mu'^2 - |\dot{\mu}|^4)\Big]. \tag{C.4}$$

$$E^{(6)}_{\mu\mu X} = -\frac{1}{4}|X|^2\Big[|\dot\mu|^4 - \dot\mu^2\bar\mu'^2 - \mu'^2\dot{\bar\mu}^2 + |\mu'|^4\Big]. \tag{C.5}$$

$$\begin{aligned} F^{(6)}_{\mu\mu X} = |X|^2\Big[|\dot\mu|^2 - |\mu'|^2\Big] - \frac{1}{4}iq\Big[&(X\bar X' - \bar X X')(|\dot\mu|^4 - 3|\mu'|^4 + \bar\mu'^2\dot\mu^2 + \mu'^2\dot{\bar\mu}^2) \\ &- (X\dot{\bar X} - \bar X\dot X)(2|\dot\mu|^2 - 2|\mu'|^2)(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')\Big]. \end{aligned} \tag{C.6}$$

$$\begin{aligned} G^{(6)}_{\mu\mu X} =& \frac{1}{4}\Big[(\dot X\bar X' + \dot{\bar X}X')(4(|\mu'|^2 - |\dot\mu|^2)(\dot\mu\bar\mu' + \dot{\bar\mu}\mu') + (|\dot X|^2 - |X'|^2)(-2\dot\mu^2\bar\mu'^2 - 2\dot{\bar\mu}^2\mu'^2 + 6|\dot\mu|^4 - 2|\mu'|^4) \\ &+ |X|^2(\dot\mu^2\bar\mu'^2 + \dot{\bar\mu}^2\mu'^2 - |\dot\mu|^4 - |\mu'|^4)\Big] + \frac{1}{4}iq\Big[-(X\bar X' - \bar X X')(|\dot\mu|^4 + 3|\mu'|^4 + \bar\mu'^2\dot\mu^2 + \mu'^2\dot{\bar\mu}^2) \\ &- (X\dot{\bar X} - \bar X\dot X)(2|\dot\mu|^2 - 2|\mu'|^2)(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')\Big]. \end{aligned} \tag{C.7}$$

$$\begin{aligned} H^{(6)}_{\mu XX} =& \frac{1}{2}|X|^2\Big[(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')(X'\dot{\bar X} + \bar X'\dot X) - (|\dot\mu|^2 + |\mu'|^2)(|X'|^2 + |\dot X|^2 + |X|^2)\Big] \\ &+ iq|X|^2\Big[|\mu'|^2(X\bar X' - \bar X X')\Big] + \frac{q^2}{2}|X|^2\Big[(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')(X'\dot{\bar X} + \bar X'\dot X) \\ &- (|\dot\mu|^2 + |\mu'|^2)(|X'|^2 + |\dot X|^2 + |X|^2)\Big]. \end{aligned} \tag{C.8}$$

$$\begin{aligned} I^{(6)}_{\mu XX} =& |X|^2\Big[|\dot\mu|^2(|\dot X|^2 - |X'|^2) + (\dot\mu\bar\mu' + \dot{\bar\mu}\mu')(X'\bar X + \bar X'X) + |\mu'|^2(|\dot X|^2 + 3|X'|^2)\Big] \\ &- \frac{iq}{2}\Big[2(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')(X\dot{\bar X} - \bar X\dot X)(|X'|^2 - |\dot X|^2) + (XX'\dot{\bar X}^2 - \bar X\bar X'\dot X^2)(|\mu'|^2 - |\dot\mu|^2) \\ &+ (\dot\mu\bar\mu' + \dot{\bar\mu}\mu')(X\dot X\bar X'^2 - \bar X\dot{\bar X}X'^2) - |\mu'|^2(X\bar X' - \bar X X')|X|^2\Big] \\ &- \frac{q^2}{2}\Big[(|\mu'|^2 - |\dot\mu|^2)(\bar X^2\dot X^2 + X^2\dot{\bar X}^2) - 2|X|^2|\mu'|^2(|\dot X|^2 - |X'|^2)\Big]. \end{aligned} \tag{C.9}$$

$$\begin{aligned} J^{(6)}_{\mu XX} =& \Big[\frac{1}{2}(|\mu'|^2(-|X|^2|\dot X|^2 + |\dot X|^4 - |X|^2|X'|^2 + (\bar X^2\dot X^2 + X^2\dot{\bar X}^2)) \\ &+ |\dot\mu|^2(-|X|^2|\dot X|^2 - |X'|^4 - |X|^2|X'|^2 - (\bar X^2\dot X^2 + X^2\dot{\bar X}^2)) \\ &+ (|X|^2 - 2|\dot X|^2 + 2|X'|^2)(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')(X'\dot{\bar X} + \bar X'\dot X) + 3|\dot\mu|^2|\dot X|^4 - 3|\mu'|^2|X'|^4)\Big] \\ &\frac{iq}{2}\Big[(\dot\mu\bar\mu' + \dot{\bar\mu}\mu')\Big((|\dot X|^2 - 2|X'|^2)(X\dot{\bar X} - \bar X\dot X) - (X\dot X\bar X'^2 - \bar X\dot{\bar X}X'^2)\Big) \\ &- (X\bar X' - \bar X X')(|\dot\mu|^2|X'|^2 + 3|\mu'|^2|X'|^2) + (XX'\dot{\bar X}^2 - \bar X\bar X'\dot X^2)(|\dot\mu|^2 - |\mu'|^2)\Big] \\ &\frac{q^2}{4}\Big[|\mu'|^2(\bar X^2X'^2 + X^2\bar X'^2 - \bar X^2\dot X^2 - X^2\dot{\bar X}^2) + 2|X|^2(|\dot X|^2 - |X'|^2)|\mu'|^2\Big]. \end{aligned} \tag{C.10}$$

$$\begin{aligned}K^{(6)}_{uvv} = \frac{1}{2}\Big[&2(|v'|^2-|\dot v|^2)(\dot u\bar u'+\dot{\bar u}u')(\dot v\bar v'+\dot{\bar v}v')+3|\dot u|^2|\dot v|^4-3|u'|^2|v'|^4\\ &-|u'|^2|\dot v|^4-|\dot u|^2|v'|^4+(|u'|^2-|\dot u|^2)(\dot{\bar v}^2v'^2+\dot v^2\bar v'^2)\Big].\end{aligned} \tag{C.11}$$

$$\begin{aligned}L^{(6)}_{yzz} = \frac{1}{4}\Big[&|z|^2\Big(|z|^2(|y|^2-|y'|^2+2|\dot y|^2)-2|\dot y|^2|\dot z|^2-2|y'|^2|\dot z|^2-2|z'|^2|\dot y|^2-2|y'|^2|z'|^2\\ &+2(\dot z\bar z'+\dot{\bar z}z')(\dot y\bar y'+\dot{\bar y}y')+8|y|^2|z'|^2\Big)-|\dot z|^4|\dot y|^2+|y|^2\Big(-|z'|^4+\dot{\bar z}^2z'^2+\dot z^2\bar z'^2\Big)\Big].\\ &\frac{iq}{2}\Big[(z\bar z'-\bar z z')(-|z|^2|y|^2-|y|^2|z'|^2+2|z|^2|y'|^2)-|y|^2(zz'\dot{\bar z}^2-\bar z\bar z'\dot z^2)-2(y\bar y'-\bar y y')|z|^2|z'|^2\Big]\\ &\frac{q^2}{2}\Big[|z|^2(|\dot z|^2|y'|^2-|y'|^2|z'|^2)+|z'|^2\Big((\bar z\dot z-\dot{\bar z}z)(\bar y\dot y-\dot{\bar y}y)-(\bar z z'-\bar z'z)(\bar y y'-\bar y'y)\ \Big)\Big].\end{aligned} \tag{C.12}$$

$$\begin{aligned}M^{(6)}_{yzz} = -\Big[&|z|^2\Big(-|\dot z|^2|\dot y|^2+|\dot z|^2|y'|^2+|\dot y|^2|z'|^2+3|z'|^2|y'|^2-(\dot z\bar z'+\dot{\bar z}z')(\dot y\bar y'+\dot{\bar y}y')\Big)\\ &+|y|^2(|\dot z|^4-|z'|^4)\Big]-\frac{iq}{2}\Big[(\dot{\bar y}y'+\dot y\bar y')\Big((\bar z\dot z-\dot{\bar z}z)(|\dot z|^2-2|z'|^2)-z'^2\bar z\dot{\bar z}+\bar z'^2z\dot z-2|z|^2(\bar z z'-\bar z'z)\Big)\\ &+(zz'\dot{\bar z}^2-\bar z\bar z'\dot z^2)\Big(|y|^2-|\dot y|^2+|y'|^2\Big)-|\dot z|^2(\dot{\bar z}z'-\bar z'\dot z)(\bar y\dot y+\dot{\bar y}y)\\ &+(\dot{\bar z}z'+\bar z'\dot z)(\bar y\dot y-\dot{\bar y}y)|z'|^2+(\bar z z'-\bar z'z)\Big(-|y|^2|z'|^2+|\dot y|^2|z'|^2+2|z|^2|y'|^2+3|z'|^2|y'|^2\Big)\\ &+(\bar y y'-\bar y'y)\Big(\frac{1}{2}(-|z|^4+|\dot z|^4+z'^2\dot{\bar z}^2+\bar z'^2\dot z^2)+2|z|^2|z'|^2+\frac{3}{2}|z'|^4\Big)\Big]\\ &-\frac{q^2}{2}\Big[|y'|^2\Big(-2|z|^2|\dot z|^2+|z|^2|z'|^2-z'^2\bar z^2-\bar z'^2z^2+z^2\dot{\bar z}^2+\bar z^2\dot z^2\Big)\Big].\end{aligned} \tag{C.13}$$

$$\begin{aligned}N^{(6)}_{yzz} = \frac{1}{4}\Big[&|z|^2\Big(-3|z|^2|y|^2-2|z'|^2(|\dot y|^2+|y'|^2)-|\dot z|^2(|\dot y|^2+|y'|^2)\Big)\\ &+|z'|^4(-|y|^2-2|\dot y|^2-6|y'|^2)+|\dot z|^4(-|y|^2+2|y'|^2+6|\dot y|^2)\\ &+(\dot z\bar z'+\dot{\bar z}z')(\dot y\bar y'+\dot{\bar y}y')\Big(2|z|^2+4|z'|^2-4|\dot z|^2\Big)+(z'^2\dot{\bar z}^2+\bar z'^2\dot z^2)(|y|^2-2|\dot y|^2+2|y'|^2)\Big]\\ &+\frac{iq}{4}\Big[(\bar y y'-\bar y'y)\Big(-|z|^4-3|z'|^4+|\dot z|^4+z'^2\dot{\bar z}^2+\bar z'^2\dot z^2\Big)-(\bar z z'-\bar z'z)(|z|^2|y|^2+|z'|^2(2|\dot y|^2+6|y'|^2)\\ &+(\dot{\bar y}y'+\bar y'\dot y)\Big((\dot{\bar z}z-\bar z\dot z)(-4|z'|^2+2|\dot z|^2)-2z\dot z\bar z'^2+2\bar z\dot{\bar z}z'^2\Big)-(\dot{\bar z}z'+\bar z'\dot z)(\dot{\bar y}y-\bar y\dot y)(2|z'|^2-2|\dot z|^2)\\ &+2(|\dot y|^2-|y'|^2)(zz'\dot{\bar z}^2-\bar z\bar z'\dot z^2)\Big]+\frac{q^2}{4}\Big[2|z'|^2(\dot{\bar y}y-\bar y\dot y)(\dot{\bar z}z-\bar z\dot z)+|y'|^2\Big(z'^2\bar z^2+\bar z'^2z^2-z^2\dot{\bar z}^2-\bar z^2\dot z^2\\ &-2|z|^2|z'|^2+2|z|^2|\dot z|^2\Big)-2|z'|^2(\bar y'y-\bar y y')(\bar z'z-\bar z z')\Big].\end{aligned} \tag{C.14}$$

$$O^{(6)}_{yz\mu} = \frac{1}{2}(|\mu'|^2 + |\dot\mu|^2)(-|z|^2|\dot y|^2 - |y|^2|\dot z|^2 - |z|^2|y'|^2 - |y|^2|z'|^2) + (|\mu'|^2 - |\dot\mu|^2)|z|^2|y|^2$$
$$\frac{1}{2}\Big((\dot{\bar\mu}\mu' + \bar\mu'\dot\mu)(|z|^2(\dot{\bar y}y' + \bar y'\dot y) + |y|^2(\dot{\bar z}z' + \bar z'\dot z)\Big) + \frac{iq}{2}\Big[(|\mu'|^2 - |\dot\mu|^2)\Big(|z|^2(\bar y y' - \bar y' y) + |y|^2(\bar z z' - \bar z' z)\Big)$$
$$+ (\dot{\bar\mu}\mu' + \bar\mu'\dot\mu)(|z|^2(\bar y\dot y - \dot{\bar y}y) + |y|^2(\bar z\dot z - \dot{\bar z}z)\Big] + \frac{q^2}{2}\Big[|\mu'|^2\Big((\dot z\bar z - \dot{\bar z}z)(\dot y\bar y - \dot{\bar y}y) - (z\bar z' - \bar z z')(y\bar y' - \bar y y')\Big)\Big]. \tag{C.15}$$

$$P^{(6)}_{yz\mu} = -\frac{1}{2}\Big[4|\mu'|^2(|z|^2|y'|^2 - |y|^2|z'|^2) - 4|\dot\mu|^2(|z|^2|\dot y|^2 - |y|^2|\dot z|^2)\Big]$$
$$- \frac{iq}{2}\Big[(|\mu'|^2 - |\dot\mu|^2)\Big((\bar y y' - y\bar y')(|z|^2 - |\dot z|^2) - (\bar z z' - z\bar z')(|y|^2 - |\dot y|^2)$$
$$- (\dot z\bar z - \dot{\bar z}z)(\dot y\bar y' + \dot{\bar y}y') + (\dot y\bar y - \dot{\bar y}y)(\dot z\bar z' + \dot{\bar z}z')\Big) + (3|\mu'|^2 + |\dot\mu|^2)((\bar z z' - z\bar z')|y'|^2 - (\bar y y' - y\bar y')|z'|^2)$$
$$+ (\dot{\bar\mu}\mu' + \bar\mu'\dot\mu)\Big((\dot y\bar y - \dot{\bar y}y)(|z|^2 + |z'|^2 + |\dot z|^2) - (\dot z\bar z - \dot{\bar z}z)(|y|^2 + |y'|^2 + |\dot y|^2)$$
$$- (\dot y\bar y' + \dot{\bar y}y')(z'\bar z - \bar z' z) + (\dot z\bar z' + \dot{\bar z}z')(y'\bar y - \bar y' y)\Big)\Big]. \tag{C.16}$$

$$Q^{(6)}_{yz\mu} = \frac{1}{4}\Big[(|\mu'|^2 + |\dot\mu|^2)\Big(-2|z|^2(|y'|^2 + |\dot y|^2) - 2|y|^2(|z'|^2 + |\dot z|^2)\Big)$$
$$+ |\mu'|^2\Big(4(\dot y\bar y' + \dot{\bar y}y')(\dot z\bar z' + \dot{\bar z}z') - 4|\dot y|^2|z'|^2 - 12|y'|^2|z'|^2 + 4|\dot z|^2(|\dot y|^2 - |y'|^2)\Big)$$
$$- |\dot\mu|^2\Big(4(\dot y\bar y' + \dot{\bar y}y')(\dot z\bar z' + \dot{\bar z}z') - 4|\dot z|^2|y'|^2 - 12|\dot z|^2|\dot y|^2 - 4|z'|^2(|\dot y|^2 - |y'|^2)\Big)$$
$$+ (\dot{\bar\mu}\mu' + \bar\mu'\dot\mu)\Big((\dot{\bar y}y' + \bar y'\dot y)(2|z|^2 - 4|\dot z|^2 + 4|z'|^2) + (\dot{\bar z}z' + \bar z'\dot z)(2|y|^2 - 4|\dot y|^2 + 4|y'|^2)\Big)\Big]$$
$$+ \frac{iq}{4}\Big[(|\mu'|^2 - |\dot\mu|^2)\Big(-2(\dot{\bar z}z - \bar z\dot z)(\dot{\bar y}y' + \bar y'\dot y) - 2(\dot{\bar y}y - \bar y\dot y)(\dot{\bar z}z' + \bar z'\dot z) - 2(\bar y y' - \bar y' y)|\dot z|^2$$
$$- 2(\bar z z' - \bar z' z)|\dot y|^2\Big) + (3|\mu'|^2 + |\dot\mu|^2)\Big(-2(\bar y y' - \bar y' y)|z'|^2 - 2(\bar z z' - \bar z' z)|y'|^2\Big)$$
$$+ (\dot{\bar\mu}\mu' + \bar\mu'\dot\mu)\Big(2(\dot{\bar z}z - \bar z\dot z)(|\dot y|^2 + |y'|^2) + 2(\dot{\bar y}y - \bar y\dot y)(|\dot z|^2 + |z'|^2) - 2(\bar y' y - \bar y y')(\dot{\bar z}z' + \bar z'\dot z)$$
$$- 2(\bar z' z - \bar z z')(\dot{\bar y}y' + \bar y'\dot y)\Big)\Big] + \frac{q^2}{2}\Big[|\mu'|^2\Big((\bar y' y - \bar y y')(\bar z' z - \bar z z') - (\dot{\bar z}z - \bar z\dot z)(\dot{\bar y}y - \bar y\dot y)\Big)\Big]. \tag{C.17}$$

$$R^{(6)}_{Xuv} = \frac{1}{2}|X|^2\Big(-|u'|^2|v'|^2 - |\dot u|^2|\dot v|^2 - |\dot u|^2|v'|^2 - |u'|^2|\dot v|^2 + (\dot{\bar u}u' + \bar u'\dot u)(\dot{\bar v}v' + \bar v'\dot v)\Big). \tag{C.18}$$

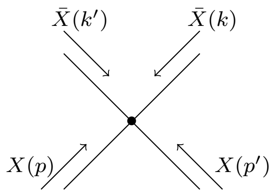
$$S^{(6)}_{Xuv} = -\frac{1}{2}\Big[|X|^2(-|\dot u|^2|\dot v|^2 + |u'|^2|v'|^2)\Big]$$
$$- \frac{iq}{2}\Big[(\bar z' z - \bar z z')\Big(-3|u'|^2|v'|^2 + |\dot u|^2|\dot v|^2 - |\dot u|^2|v'|^2 - |u'|^2|\dot v|^2 + (\dot{\bar u}u' + \bar u'\dot u)(\dot{\bar v}v' + \bar v'\dot v)\Big) \tag{C.19}$$
$$+ (\dot{\bar z}z - \bar z\dot z)\Big(-(\dot{\bar u}u' + \bar u'\dot u)|\dot v|^2 - (\dot{\bar v}v' + \bar v'\dot v)|\dot u|^2 + (\dot{\bar u}u' + \bar u'\dot u)|v'|^2 + (\dot{\bar v}v' + \bar v'\dot v)|u'|^2\Big)\Big)\Big].$$

$$
\begin{aligned}
T^{(6)}_{Xuv} = \frac{1}{4}\Big[&|X|^2\Big(-2|\dot{u}|^2|\dot{v}|^2 - 2|\dot{u}|^2|v'|^2 - 2|u'|^2|\dot{v}|^2 - 2|u'|^2|v'|^2 + 2(\dot{\bar{u}}u' + \bar{u}'\dot{u})(\dot{\bar{v}}v' + \bar{v}'\dot{v})\Big)\\
&|X'|^2\Big(+4|\dot{u}|^2|\dot{v}|^2 - 4|\dot{u}|^2|v'|^2 - 4|u'|^2|\dot{v}|^2 - 12|u'|^2|v'|^2 + 4(\dot{\bar{u}}u' + \bar{u}'\dot{u})(\dot{\bar{v}}v' + \bar{v}'\dot{v})\Big)\\
&|\dot{X}|^2\Big(12|\dot{u}|^2|\dot{v}|^2 + 4|\dot{u}|^2|v'|^2 + 4|u'|^2|\dot{v}|^2 - 4|u'|^2|v'|^2 - 4(\dot{\bar{u}}u' + \bar{u}'\dot{u})(\dot{\bar{v}}v' + \bar{v}'\dot{v})\Big)\\
&- 4(X\dot{\bar{X}} + \bar{X}\dot{X})\Big((|\dot{v}|^2 - |v'|^2)(\dot{\bar{u}}u' + \bar{u}'\dot{u}) + (|\dot{u}|^2 - |u'|^2)(\dot{\bar{v}}v' + \bar{v}'\dot{v})\Big)\Big]\\
&+ \frac{iq}{4}\Big[2(X\dot{\bar{X}} - \bar{X}\dot{X})\Big((|\dot{v}|^2 - |v'|^2)(\dot{\bar{u}}u' + \bar{u}'\dot{u}) + (|\dot{u}|^2 - |u'|^2)(\dot{\bar{v}}v' + \bar{v}'\dot{v})\Big)\\
&- (X\bar{X}' - X'\bar{X})\Big(2|\dot{u}|^2|\dot{v}|^2 - 2|u'|^2|\dot{v}|^2 - 2|\dot{u}|^2|v'|^2 - 6|u'|^2|v'|^2 + (\dot{\bar{u}}u' + \bar{u}'\dot{u})(\dot{\bar{v}}v' + \bar{v}'\dot{v})\Big)\Big].
\end{aligned}
\tag{C.20}
$$

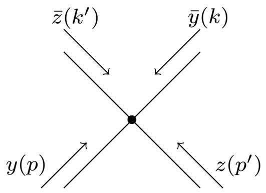
Appendix D

Feynman Rules

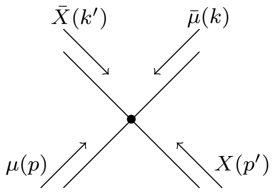
Here are shown the Feynman rules for the 4-point vertices of the string sigma model:



$$
\begin{aligned}
&= \pm 2i\Big[-\frac{1}{2}q\Big(\left(k_0k_0' - k_1k_1'\right)\left(p_1' + p_1\right) - \left(k_1' + k_1\right)\left(p_0p_0' - p_1p_1'\right) + k_1' + k_1 - p_1' - p_1\Big) \\
&+ p_1k_1' + k_1p_1' + k_1'p_1' + k_1p_1\Big] - \frac{1}{2}i(1-2a)\Big[q\big[2p_0\left(k_1' + k_1\right)p_0' - 2k_0k_0'\left(p_1' + p_1\right) \\
&- 2\left(p_1\left(k_1' + k_1\right)p_1' + p_1' + p_1\right) + 2\left(k_1k_1'\left(p_1' + p_1\right) + k_1' + k_1\right)\big] \\
&- 4k_0p_0k_0'p_0' - 4k_1p_1k_1'p_1' + 4\left(k_1p_0k_1'p_0' + k_0p_1k_0'p_1'\right) + 4\Big].
\end{aligned}
$$

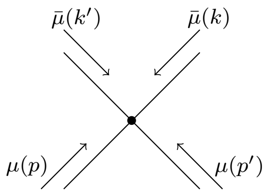


$$
\begin{aligned}
&= +i(p_0k_0 + p_1k_1 - p_0'k_0' - p_1'k_1') + \frac{iq}{2}\Big[(p_1'k_0' + k_1'p_0')(k_0 - p_0) \\
&+ (p_1k_0 + k_1p_0)(p_0' - k_0') + (1 + p_0'k_0' + p_1'k_1')(p_1 - k_1) \\
&- (1 + p_0k_0 + p_1k_1)(p_1' - k_1')\Big] + \frac{2a-1}{2}i\big[2(-(p_0'k_0' + p_1'k_1')(p_0k_0 + p_1k_1) \\
&+ 1 + (p_1'k_0' + p_0'k_1')(p_1k_0 + p_0k_1)) + q((k_1' - p_1')(1 + p_1k_1 + p_0k_0) \\
&+ (k_1 - p_1)(1 + p_1'k_1' + p_0'k_0') + (p_0 - k_0)(p_1'k_0' + k_1'p_0') + (p_0' - k_0')(p_1k_0 + k_1p_0))\big].
\end{aligned}
$$

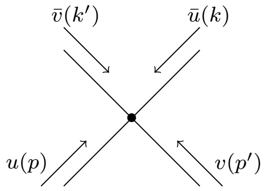


$$
\begin{aligned}
&= \pm i\left[p_0k_0 + p_1k_1 - \frac{q}{2}(p_0k_0 + p_1k_1)(p_1' - k_1') + \frac{q}{2}(p_1k_0 + k_1p_0)(p_0' - k_0')\right] \\
&+ \frac{2a-1}{2}i\big[2(-(p_0'k_0' + p_1'k_1')(p_0k_0 + p_1k_1) + (p_1'k_0' + p_0'k_1')(p_1k_0 + p_0k_1)) \\
&+ q((k_1' - p_1')(p_1k_1 + p_0k_0) + (p_0' - k_0')(p_1k_0 + k_1p_0))\big].
\end{aligned}
$$

Figure D.1: Massive and massless interaction vertices. $X = y, z$ and $\mu = u, v$. In the first and last vertices $+$ is for z and $-$ for y.



$$= -2i(2a-1)(p_0 p_0' - p_1 p_1')(k_0 k_0' - k_1 k_1').$$



$$= (2a-1)i\big[-(p_0' k_0' + p_1' k_1')(p_0 k_0 + p_1 k_1) + (p_1' k_0' + p_0' k_1')(p_1 k_0 + p_0 k_1)\big].$$

Figure D.2: Massless interaction vertices. Notice they all vanish in $a = \frac{1}{2}$ gauge.

Appendix E

Expression of $N(a_1, a_2, a_3)$

$$
\begin{aligned}
N(a_1, a_2, a_3) = i\Big[& -a_2^2a_3^3a_4a_5a_6^2a_1^6 + a_2^2a_3^3a_4a_5a_1^6 + a_2^2a_3^3a_5a_6a_1^6 - a_2^2a_3^3a_4^2a_5a_6a_1^6 + a_2^2a_3^3a_4a_5^2a_1^5 \\
& - a_2^2a_3^3a_4a_5^2a_6^2a_1^5 + a_2^2a_3^3a_4a_6^2a_1^5 - 3a_2^2a_3^4a_4a_5a_6^2a_1^5 - a_2^3a_3^3a_4a_5a_6^2a_1^5 + a_2a_3^3a_4a_5a_6^2a_1^5 \\
& + 3a_2^2a_3^2a_4a_5a_6^2a_1^5 - a_2^2a_3^3a_4a_1^5 + 3a_2^2a_3^4a_4a_5a_1^5 + a_2^3a_3^3a_4a_5a_1^5 - a_2a_3^3a_4a_5a_1^5 - 3a_2^2a_3^2a_4a_5a_1^5 \\
& - a_2^2a_3^3a_6a_1^5 + a_2^2a_3^3a_4^2a_6a_1^5 + a_2^2a_3^3a_5^2a_6a_1^5 - a_2^2a_3^3a_4^2a_5^2a_6a_1^5 + 3a_2^2a_3^4a_5a_6a_1^5 \\
& + a_2^3a_3^3a_5a_6a_1^5 - a_2a_3^3a_5a_6a_1^5 - 3a_2^2a_3^2a_5a_6a_1^5 - 3a_2^2a_3^4a_4^2a_5a_6a_1^5 - a_2^3a_3^3a_4^2a_5a_6a_1^5 \\
& + a_2a_3^3a_4^2a_5a_6a_1^5 + 3a_2^2a_3^2a_4^2a_5a_6a_1^5 + 2a_2^2a_3^4a_4a_5^2a_1^4 - 2a_2^2a_3^2a_4a_5^2a_1^4 - 2a_2^2a_3^4a_4a_5^2a_6^2a_1^4 \\
& + 2a_2^2a_3^2a_4a_5^2a_6^2a_1^4 + 2a_2^2a_3^4a_4a_6^2a_1^4 - 2a_2^2a_3^2a_4a_6^2a_1^4 - 3a_2^2a_3^5a_4a_5a_6^2a_1^4 - 2a_2^3a_3^4a_4a_5a_6^2a_1^4 \\
& + 2a_2a_3^4a_4a_5a_6^2a_1^4 + 9a_2^2a_3^3a_4a_5a_6^2a_1^4 + 2a_2^3a_3^2a_4a_5a_6^2a_1^4 - 2a_2a_3^2a_4a_5a_6^2a_1^4 - 3a_2^2a_3a_4a_5a_6^2a_1^4 \\
& - 2a_2^2a_3^4a_4a_1^4 + 2a_2^2a_3^2a_4a_1^4 + 3a_2^2a_3^5a_4a_5a_1^4 + 2a_2^3a_3^4a_4a_5a_1^4 - 2a_2a_3^4a_4a_5a_1^4 \\
& - 9a_2^2a_3^3a_4a_5a_1^4 - 2a_2^3a_3^2a_4a_5a_1^4 + 2a_2a_3^2a_4a_5a_1^4 + 3a_2^2a_3a_4a_5a_1^4 - 2a_2^2a_3^4a_6a_1^4 \\
& + 2a_2^2a_3^2a_6a_1^4 + 2a_2^2a_3^4a_4^2a_6a_1^4 - 2a_2^2a_3^2a_4^2a_6a_1^4 + 2a_2^2a_3^4a_5^2a_6a_1^4 - 2a_2^2a_3^2a_5^2a_6a_1^4 \\
& - 2a_2^2a_3^4a_4^2a_5^2a_6a_1^4 + 2a_2^2a_3^2a_4^2a_5^2a_6a_1^4 + 3a_2^2a_3^5a_5a_6a_1^4 + 2a_2^3a_3^4a_5a_6a_1^4 - 2a_2a_3^4a_5a_6a_1^4 \\
& - 9a_2^2a_3^3a_5a_6a_1^4 - 2a_2^3a_3^2a_5a_6a_1^4 + 2a_2a_3^2a_5a_6a_1^4 - 3a_2^2a_3^5a_4^2a_5a_6a_1^4 - 2a_2^3a_3^4a_4^2a_5a_6a_1^4 \\
& + 2a_2a_3^4a_4^2a_5a_6a_1^4 + 9a_2^2a_3^3a_4^2a_5a_6a_1^4 + 2a_2^3a_3^2a_4^2a_5a_6a_1^4 - 2a_2a_3^2a_4^2a_5a_6a_1^4 - 3a_2^2a_3a_4^2a_5a_6a_1^4 \\
& + 3a_2^2a_3a_5a_6a_1^4 + a_2^2a_3^5a_4a_5^2a_1^3 - 4a_2^2a_3^3a_4a_5^2a_1^3 + a_2^2a_3a_4a_5^2a_1^3 - a_2^2a_3^5a_4a_5^2a_6^2a_1^3 + 4a_2^2a_3^3a_4a_5^2a_6^2a_1^3 \\
& - a_2^2a_3a_4a_5^2a_6^2a_1^3 + a_2^2a_3^5a_4a_6^2a_1^3 - 4a_2^2a_3^3a_4a_6^2a_1^3 + a_2^2a_3a_4a_6^2a_1^3 - a_2^2a_3^6a_4a_5a_6^2a_1^3 \\
& - a_2^3a_3^5a_4a_5a_6^2a_1^3 + a_2a_3^5a_4a_5a_6^2a_1^3 + 9a_2^2a_3^4a_4a_5a_6^2a_1^3 + 4a_2^3a_3^3a_4a_5a_6^2a_1^3 - 4a_2a_3^3a_4a_5a_6^2a_1^3 \\
& + a_2^2a_4a_5a_6^2a_1^3 - 9a_2^2a_3^2a_4a_5a_6^2a_1^3 - a_2^3a_3a_4a_5a_6^2a_1^3 + a_2a_3a_4a_5a_6^2a_1^3 \\
& - a_2^2a_3^5a_4a_1^3 + 4a_2^2a_3^3a_4a_1^3 - a_2^2a_3a_4a_1^3 + a_2^2a_3^6a_4a_5a_1^3 + a_2^3a_3^5a_4a_5a_1^3 - a_2a_3^5a_4a_5a_1^3 \\
& - 9a_2^2a_3^4a_4a_5a_1^3 - 4a_2^3a_3^3a_4a_5a_1^3 + 4a_2a_3^3a_4a_5a_1^3 - a_2^2a_4a_5a_1^3 + 9a_2^2a_3^2a_4a_5a_1^3 + a_2^3a_3a_4a_5a_1^3 \\
& - a_2a_3a_4a_5a_1^3 - a_2^2a_3^5a_6a_1^3 + 4a_2^2a_3^3a_6a_1^3 + a_2^2a_3^5a_4^2a_6a_1^3 - 4a_2^2a_3^3a_4^2a_6a_1^3 \\
& + a_2^2a_3a_4^2a_6a_1^3 + a_2^2a_3^5a_5^2a_6a_1^3 - 4a_2^2a_3^3a_5^2a_6a_1^3 - a_2^2a_3^5a_4^2a_5^2a_6a_1^3 + 4a_2^2a_3^3a_4^2a_5^2a_6a_1^3 \\
& - a_2^2a_3a_4^2a_5^2a_6a_1^3 + a_2^2a_3a_5^2a_6a_1^3 - a_2^2a_3a_6a_1^3 + a_2^2a_3^6a_5a_6a_1^3 + a_2^3a_3^5a_5a_6a_1^3 \\
& - a_2a_3^5a_5a_6a_1^3 - 9a_2^2a_3^4a_5a_6a_1^3 - 4a_2^3a_3^3a_5a_6a_1^3 + 4a_2a_3^3a_5a_6a_1^3 - a_2^2a_5a_6a_1^3 \\
& + 9a_2^2a_3^2a_5a_6a_1^3 - a_2^2a_3^6a_4^2a_5a_6a_1^3 - a_2^3a_3^5a_4^2a_5a_6a_1^3 + a_2a_3^5a_4^2a_5a_6a_1^3 + 9a_2^2a_3^4a_4^2a_5a_6a_1^3 \\
& + 4a_2^3a_3^3a_4^2a_5a_6a_1^3 - 4a_2a_3^3a_4^2a_5a_6a_1^3 + a_2^2a_4^2a_5a_6a_1^3 - 9a_2^2a_3^2a_4^2a_5a_6a_1^3 - a_2^3a_3a_4^2a_5a_6a_1^3
\end{aligned}
\tag{E.1}
$$

$$
\begin{aligned}
&+ a_2a_3a_4^2a_5a_6a_1^3 + a_2^3a_3a_5a_6a_1^3 - a_2a_3a_5a_6a_1^3 - 2a_2^2a_3^4a_4a_5^2a_1^2 + 2a_2^2a_3^2a_4a_5^2a_1^2 + \\
&+ 2a_2^2a_3^4a_4a_5^2a_6^2a_1^2 - 2a_2^2a_3^2a_4a_5^2a_6^2a_1^2 - 2a_2^2a_3^4a_4a_6^2a_1^2 + 2a_2^2a_3^2a_4a_6^2a_1^2 + 3a_2^2a_3^5a_4a_5a_6^2a_1^2 \\
&+ 2a_2^3a_3^4a_4a_5a_6^2a_1^2 - 2a_2a_3^4a_4a_5a_6^2a_1^2 - 9a_2^2a_3^3a_4a_5a_6^2a_1^2 - 2a_2^3a_3^2a_4a_5a_6^2a_1^2 + 2a_2a_3^2a_4a_5a_6^2a_1^2 \\
&+ 3a_2^2a_3a_4a_5a_6^2a_1^2 + 2a_2^2a_3^4a_4a_1^2 - 2a_2^2a_3^2a_4a_1^2 - 3a_2^2a_3^5a_4a_5a_1^2 - 2a_2^3a_3^4a_4a_5a_1^2 \\
&+ 2a_2a_3^4a_4a_5a_1^2 + 9a_2^2a_3^3a_4a_5a_1^2 + 2a_2^3a_3^2a_4a_5a_1^2 - 2a_2a_3^2a_4a_5a_1^2 - 3a_2^2a_3a_4a_5a_1^2 \\
&+ 2a_2^2a_3^4a_6a_1^2 - 2a_2^2a_3^2a_6a_1^2 - 2a_2^2a_3^4a_4^2a_6a_1^2 + 2a_2^2a_3^2a_4^2a_6a_1^2 - 2a_2^2a_3^4a_5^2a_6a_1^2 + 2a_2^2a_3^2a_5^2a_6a_1^2 \\
&+ 2a_2^2a_3^4a_4^2a_5^2a_6a_1^2 - 2a_2^2a_3^2a_4^2a_5^2a_6a_1^2 - 3a_2^2a_3^5a_5a_6a_1^2 - 2a_2^3a_3^4a_5a_6a_1^2 + 2a_2a_3^4a_5a_6a_1^2 \\
&+ 9a_2^2a_3^3a_5a_6a_1^2 + 2a_2^3a_3^2a_5a_6a_1^2 - 2a_2a_3^2a_5a_6a_1^2 + 3a_2^2a_3^5a_4^2a_5a_6a_1^2 + 2a_2^3a_3^4a_4^2a_5a_6a_1^2 \\
&- 2a_2a_3^4a_4^2a_5a_6a_1^2 - 9a_2^2a_3^3a_4^2a_5a_6a_1^2 - 2a_2^3a_3^2a_4^2a_5a_6a_1^2 + 2a_2a_3^2a_4^2a_5a_6a_1^2 + 3a_2^2a_3a_4^2a_5a_6a_1^2 \\
&- 3a_2^2a_3a_5a_6a_1^2 + a_2^2a_3^3a_4a_5^2a_1 - a_2^2a_3^3a_4a_5^2a_6^2a_1 + a_2^2a_3^3a_4a_6^2a_1 - 3a_2^2a_3^4a_4a_5a_6^2a_1 \\
&- a_2^3a_3^3a_4a_5a_6^2a_1 + a_2a_3^3a_4a_5a_6^2a_1 + 3a_2^2a_3^2a_4a_5a_6^2a_1 - a_2^2a_3^3a_4a_1 + 3a_2^2a_3^4a_4a_5a_1 \\
&+ a_2^3a_3^3a_4a_5a_1 - a_2a_3^3a_4a_5a_1 - 3a_2^2a_3^2a_4a_5a_1 - a_2^2a_3^3a_6a_1 + a_2^2a_3^3a_4^2a_6a_1 + a_2^2a_3^3a_5^2a_6a_1 \\
&- a_2^2a_3^3a_4^2a_5^2a_6a_1 + 3a_2^2a_3^4a_5a_6a_1 + a_2^3a_3^3a_5a_6a_1 - a_2a_3^3a_5a_6a_1 - 3a_2^2a_3^2a_5a_6a_1 \\
&- 3a_2^2a_3^4a_4^2a_5a_6a_1 - a_2^3a_3^3a_4^2a_5a_6a_1 + a_2a_3^3a_4^2a_5a_6a_1 + 3a_2^2a_3^2a_4^2a_5a_6a_1 + a_2^2a_3^3a_4a_5a_6^2 \\
&- a_2^2a_3^3a_4a_5 - a_2^2a_3^3a_5a_6 + a_2^2a_3^3a_4^2a_5a_6\Big].
\end{aligned}
\tag{E.2}
$$

Bibliography

[1] Juan Martin Maldacena. "The Large N limit of superconformal field theories and supergravity". In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 231–252. DOI: 10.4310/ATMP.1998.v2.n2.a1. arXiv: hep-th/9711200.

[2] J. A. Minahan and K. Zarembo. "The Bethe ansatz for N=4 superYang-Mills". In: *JHEP* 03 (2003), p. 013. DOI: 10.1088/1126-6708/2003/03/013. arXiv: hep-th/0212208.

[3] Cagnazzo A. Zarembo K. "*B-field in AdS(3)/CFT(2) Correspondence and Integrability*". In: (2012). DOI: https://doi.org/10.48550/arXiv.1209.4049.

[4] Niklas Beisert et al. "Review of AdS/CFT Integrability: An Overview". In: *Lett. Math. Phys.* 99 (2012), pp. 3–32. DOI: 10.1007/s11005-011-0529-2. arXiv: 1012.3982 [hep-th].

[5] A. B. Zamolodchikov and A. B. Zamolodchikov. "*Factorized S matrices in two dimensions as the exact solutions of certain relativistic quantum field models*". In: *Annals Phys. 120, 253* (1979).

[6] V. I. Arnold. *"Mathematical Methods of Classical Mechanics"*. Springer. ISBN 9780387968902s, (1989).

[7] Bernard and M. Talon. *"Introduction to classical integrable systems"*. Cambridge University Press (2007).

[8] M. Gaudin. *"La fonction d'onde de Bethe"*. Masson (1983).

[9] L. D. Faddeev. "How algebraic Bethe ansatz works for integrable model". In: *Les Houches School of Physics: Astrophysical Sources of Gravitational Radiation*. May 1996, pp. 149–219. arXiv: hep-th/9605187.

[10] J.M. Evans M.Hassan N.J. MacKay A.J. Mountain. "*Local conserved charges in Principal Chiral models*". In: (1999). DOI: https://doi.org/10.48550/arXiv.hep-th/9902008.

[11] M. Luscher. "*Quantum non-local charges and absence of particle production in the two-dimensional non-linear σ-model*". In: Nucl. Phys. B135 (1978).

[12] Dorey P. "*Exact S-matrices*". In: (1998). DOI: https://doi.org/10.48550/arXiv.hep-th/9810026.

[13] Parke S.J. "*Absence of Particle Production and Factorization of the S Matrix in (1+1)-dimensional Models*". In: *Nucl.Phys.B 174* (1980).

[14] Bombardelli D. "*Lectures on S-matrices and Integrability*". In: (2016). DOI: https://doi.org/10.48550/arXiv.1606.02949.

[15] A. Doikou S. Evangelisti G. Feverati N. Karaiskos. "*Introduction to Quantum Integrability*". In: (2009). DOI: https://doi.org/10.48550/arXiv.0912.3350.

[16] S. Coleman and J. Mandula. "*All Possible Symmetries of the S Matrix*". In: *Phys. Rev. 159* (1967).

[17] A. Sfondrini. "*Towards integrability for AdS3/CFT2*". In: (2014). DOI: https://doi.org/10.48550/arXiv.1406.2971.

[18] M.D. Schwartz. *"Quantum Field Theory and the Standard Model"*. Cambridge University Press (2013).

[19] Rubinstein J. "*Sine-Gordon equation*". In: *Journal of Mathematical Physics. 11 (1)* (1970).

[20] R. K. Dodd R. K. Bullough. "*Polynomial Conserved Densities for the Sine-Gordon Equations*". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. The Royal Society.* (1971).

[21] Dorey P. Polvara D. "*Tree level integrability in 2d quantum field theories and affine Toda models*". In: (2021). DOI: https://doi.org/10.48550/arXiv.2111.02210.

[22] Davide Polvara. "Perturbative integrability in 1+1 dimensions and affine Toda theories". PhD thesis. Durham U., 2022.

[23] B. Hoare. "*Integrable Deformations of Sigma Models*". In: (2022). DOI: https://doi.org/10.48550/arXiv.2109.14284.

[24] Tong D. "*Lectures on String Theory*". In: (2009). DOI: https://doi.org/10.48550/arXiv.0908.0333.

[25] K. Becker Becker J. Schwarz. *"String Theory and M- Theory: A modern introduction"*. Cambridge University Press (2006).

[26] A. Babichenko B. Stefanski K. Zarembo. "*Integrability and the AdS(3)/CFT(2) correspondence*". In: (2009). DOI: https://doi.org/10.48550/arXiv.0912.1723.

[27] T. Lloyd O. Ohlsson Sax A. Sfondrini and jr. B. Stefanski. "*The complete worldsheet S matrix of superstrings on AdS3 × S3 × T4 with mixed three-form flux*". In: (2014). DOI: https://doi.org/10.48550/arXiv.1410.0866.

[28] G. Arutyunov S. Frolov. "*Foundations of the AdS5 × S5 Superstring Part I*". In: (2009). DOI: https://doi.org/10.48550/arXiv.0901.4937.

[29] S. Weinberg. *"The Quantum Theory of Fields"*. Volume 1, Cambridge University Press.

[30] B. Hoare and A.A. Tseytlin. "*On string theory on AdS3 × S3 × T4 with mixed 3-form flux: tree-level S-matrix*". In: (2018). DOI: https://doi.org/10.48550/arXiv.1303.1037.

[31] Wulff L. "*Classifying integrable symmetric space strings via factorized scattering*". In: (2018). DOI: https://doi.org/10.48550/arXiv.1711.00296.

[32] Nicola Baglioni et al. "On the worldsheet S matrix of the AdS3/CFT2 mixed-flux mirror model". In: (Aug. 2023). arXiv: 2308.15927 [hep-th].