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Dynamics of pendula hanging from a string

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a mia sorella e ai cielli

Abstract

This thesis aims to explain the phenomenon of the synchronization in mechanical – discrete and continuous – systems as an outcome of damping contributions affecting the normal modes of oscillation. We present and prove a result for finite-dimensional linear mechanical systems asserting the existence – under suitable conditions – of an attractive invariant undamped subspace in the phase space, on which the dynamics consists of small oscillations. Thereafter, we construct a hybrid model composed of a homogeneous flexible and elastic string, with fixed extremities, on which are suspended first one, and then two, pendula. We include a dissipative contribution due to internal damping within the string, modelling it as visco-elastic friction. The study, making use of the Lagrangian formalism, provides a linear analysis based on the determination of the undamped and damped normal modes of oscillation. Hence, we investigate beats and synchronization by means of a numerical study as the parameters that characterise the system change.

Sommario

Lo scopo di questa tesi è spiegare il fenomeno della sincronizzazione in sistemi meccanici – discreti e continui – in termini di selezione di modi normali dovuta al contributo dell'attrito sul sistema. Viene dimostrato un teorema per sistemi lineari meccanici finito dimensionali che asserisce l'esistenza – sotto certe ipotesi – di un sottospazio invariante e attrattivo privo di smorzamento, sul quale la dinamica consiste di piccole oscillazioni. In seguito si costruisce un modello ibrido composto da una corda omogenea, flessibile ed elastica, con gli estremi fissati, sui quali sono appesi prima uno, poi due, pendoli. Si include un termine di smorzamento interno alla corda, uniformemente distribuito nella stessa e modellato come attrito visco-elastico. Utilizzando il formalismo Lagrangiano, se ne studiano i modi normali smorzati e non, facendo un'analisi lineare attorno alla configurazione di equilibrio. Quindi si ricercano fenomeni di battimenti e sincronizzazione, studiando numericamente la dipendenza dai parametri del sistema.

Contents

Introduction

In 1665 Christiaan Huygens noticed that two of his recently invented pendulum clocks, when suspended on a common wood beam, exhibited a synchronous motion, namely they swung with the same frequency and in opposite phase. Based on his observations, he identified the small movements of the common support as the responsible for the interaction between the two pendula and ultimately for the mutual adjustment of their motion. Similar phenomena have been observed in nature, in which complex systems might exhibit a tendency toward a collective coherent behaviour. Some well-known examples in biology are the synchronization of fireflies in South-East Asia flashing in unison (see [3]) and that of cardiac pacemaker cells (see [24]). An extended description with various examples can be found in [20].

With regard to synchronization in mechanics, a lot of studies – both theoretical and experimental – have attempted to give a description of this mechanism for pendulum clocks (see e.g. [11]) and metronomes (see e.g. [18]). Despite specific details, most of works on the subject present some common aspects. In particular:

1. a restoring force (e.g. the escapement mechanism of the clocks) and friction are always included (see e.g. [14]), but a clear identification of elementary and universal mechanisms that lead to synchronization is lacking;

2. the frame supporting the pendula is modelled as a one-degree-of-freedom rigid bar, elastically fixed (see e.g. [2], [4]). Some studies have attempted to take into account the flexibility of the support, discretising the rigid beam by means of a finite number of masses (see e.g. [5], [19]), nonetheless it has always been neglected the continuous nature of the coupling structure.

The aim of the dissertation is to take a first step in both directions. Specifically,

1. we investigate quantitatively the role of damping in the synchronization in finite-dimensional mechanical systems;

2. we propose and study a model that accounts for the continuous and elastic nature of the support that couples the pendula.

First, we consider a generic discrete linear damped mechanical system of the form $M\ddot{q} + \Gamma\dot{q} + Kq = 0$. We shall prove that, under suitable conditions, the system admits an invariant subspace in the phase space on which the system does not dissipate, motions are undamped small oscillations and which is attractive for the dynamics. The main concept is that in-phase and anti-phase synchronization are just particular manifestations of patterns within the small oscillations (e.g. beats), as friction damps out other normal modes.

We then study a hybrid mechanical system consisting of a heavy homogeneous flexible and elastic string, with extremities fixed, on which – halfway – is hanged a pendulum. In our model we do not include any external restoring forces. Through a linear analysis, it will be investigated how the coupling affects the frequencies of oscillation of the string and of the pendulum, and the normal modes of oscillation will be determined.

Thereafter, we add a dissipative contribution to the system, including an internal damping term uniformly distributed within the string, of the type u_{txx}. In presence of friction only a finite number of "damped normal modes" has eigenvalues with a non-zero imaginary part and is therefore oscillating, while the others extinguish exponentially. Moreover, the model chosen produces decay rates strongly dependent on the frequency of oscillation, which entails that the higher the frequency the quicker the decay, so that after an initial transient the dynamics results in a small number of damped normal modes only.

Finally, this analysis will be repeated numerically in the case of two pendula suspended on the string, demonstrating that our model is able to describe synchronization phenomena.

The thesis is divided into five chapters, organized as follows:

1. In Chapter 1 the mechanism of synchronization in discrete systems is investigated. We recall the fundamental notion of normal mode of oscillation and its extension to damped systems. A theorem will prove that, under suitable conditions, there exists an invariant subspace in the phase space in which the dynamics consists of small oscillations, and therefore the corresponding normal modes are not damped. Three examples will illustrate how this can naturally lead to synchronization.

2. In Chapter 2 we construct a infinite-degree-of-freedom system consisting of a pendulum hanging from a heavy elastic string, fixed at the extremities, and study the small oscillations about the equilibrium configuration, describing the spectrum of frequencies of the coupled system and the associated eigenfunctions.

3. In Chapter 3 the analysis made in the previous chapter is replicated taking into account a damping force acting on the string. The eigenvalues are computed in the regime of weak damping, at the first perturbative order with respect to the damping coefficient.

4. In Chapter 4 we study the system consisting of two identical pendula suspended at one-third and two-thirds along the string, respectively. The frequencies of oscillation are determined numerically and the phenomenon of beats is checked.

5. In Chapter 5 the same dissipative contribution introduced in Chapter 3 is included in the case of two pendula suspended on the string. Through a numerical analysis, it is investigated the dependency on the parameters of the system and the possibility of synchronization.

Chapter 1

Discrete systems: A first look at synchronization

In this chapter we study the dynamics of n-dimensional mechanical systems, in the approximation of small oscillations about a stable equilibrium. We shall show that in presence of damping, synchronous motion of parts of the system can occur as a consequence of the existence of invariant undamped subspaces in the phase space.

We first recall the linearisation of Lagrange equations about an equilibrium point and the spectral analysis of the linearised system, with the definition of normal modes of oscillation. We will follow the notation and contents of [6]. As an example of small oscillation the phenomenon of beats is presented. We then extend the analysis to damped mechanical systems and we show how synchronization is a manifestation of patterns within the small oscillations, namely in-phase and anti-phase oscillations and beats. Some simple examples will illustrate such mechanism.

1.1 Lagrangian formalism

A n-degree-of-freedom holonomic system is described by the configuration space Q and the Lagrangian function $L : TQ \to \mathbb{R}$, $(q, \dot{q}) \mapsto L(q, \dot{q})$. The dynamics is determined by the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \tag{1.1}$$

For conservative mechanical systems the Lagrangian is of the form

$$L(q, \dot{q}) := T(q, \dot{q}) - V(q), \tag{1.2}$$

where $T(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q}$ is the kinetic energy of the system, with $A(q)$ symmetric and positive definite, and $V(q)$ is the potential energy.

1.1.1 Linearisation

Lagrange equations (1.1) are second-order differential equations in $(q_1, \dots, q_n) \in Q$. Let us consider the linearisation of these equations about an equilibrium point.

Proposition 1.1. *Consider the Lagrangian function $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$. The point $(q^*, 0)$ is an equilibrium of Lagrange equations (1.1) if and only if q^* is a critical point of the potential energy, i.e.*

$$\frac{\partial V}{\partial q}(q^*) = 0.$$

Proof. Lagrange equations can be written as

$$\left(\frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}} \ddot{q}\right)_i = \frac{\partial L}{\partial q_i} - \sum_{j=1}^{n} \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}.$$

For $\dot{q} = 0$, the second term vanishes iff $\frac{\partial L}{\partial q_i}(q^*, 0) = \frac{\partial V}{\partial q}(q^*) = 0$. □

Proposition 1.2. *Consider the Lagrangian function $L(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q} - V(q)$, with equilibrium $(q^*, 0)$. The linearisation of Lagrange equations for L about $(q^*, 0)$ is*

$$A(q^*)\ddot{q} + V''(q^*)(q - q^*) = 0, \tag{1.3}$$

where $V''(q)$ reads $\frac{\partial^2 V}{\partial q \partial q}(q)$. (1.3) are the Lagrange equations for the "quadratised" Lagrangian

$$L^*(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q^*)\dot{q} - \frac{1}{2}(q - q^*) \cdot V''(q^*)(q - q^*). \tag{1.4}$$

Proof. Compute the Taylor series expansion of the Lagrange equations $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0$ about the equilibrium $(q^*, 0)$, keeping only the linear terms:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} &= A(q)\ddot{q} + \mathcal{O}_2 \\
\frac{\partial T}{\partial q} &= \mathcal{O}_2 \\
\frac{\partial V}{\partial q} &= V''(q - q^*) + \mathcal{O}_2,
\end{aligned}$$

denoting with $\mathcal{O}_k$ the terms of order at least k in $(q - q^*)$ and $\dot{q}$. Thus, $A(q)\ddot{q} = -V''(q^*)(q - q^*) + \mathcal{O}_2$, and, since $A^{-1}(q) = A^{-1}(q^*) + \mathcal{O}_1$, the linearised equations are (1.3). □

1.2 Undamped mechanical systems

1.2.1 Spectral analysis

Fixing $q^* = 0$ through a translation of the coordinates, and renaming $A(0) =: M$ and $V''(0) =: K$, the linearised Lagrange equations (1.3) can be written as

$$M\ddot{q} + Kq = 0 \tag{1.5}$$

with $q = (q_1, \dots, q_n) \in \mathbb{R}^n$, M such that $M = M^T$ and positive definite, and K such that $K = K^T$. In order for the equilibrium to be a minimum of the energy, we require that K is positive semi-definite. As a first-order differential equation, (1.5) becomes

$$\dot{x} = \Lambda x$$

where $x = (q, \dot{q}) \in \mathbb{R}^{2n}$ and

$$\Lambda = \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -M^{-1}K & \mathbf{0} \end{pmatrix}. \tag{1.6}$$

The eigenvalues of Λ can be determined in terms of the eigenvalues of $M^{-1}K$. Since $M^{-1}Ku = \mu u$ is equivalent to $Ku = \mu M u$, the eigenvalues of $M^{-1}K$ are the solutions $\mu_1, \dots, \mu_n$ of

$$\det(K - \mu M) = 0,$$

and the associated eigenvectors of $M^{-1}K$ satisfy

$$Ku = \mu M u.$$

In particular, the spectral theorem ensures that the n eigenvalues of $M^{-1}K$ are real and there exist n linearly independent eigenvectors $u_1, \dots, u_n$ (see [17]). This property is called "simultaneous diagonalization of two quadratic forms, one of which is positive definite" (see e.g. [9]).

Proposition 1.3. *If $\mu_1, \dots, \mu_n \in \mathbb{R}$ are the eigenvalues of $M^{-1}K$, each one repeated many times as its multiplicity, then $\pm\sqrt{-\mu_1}, \dots, \pm\sqrt{-\mu_n}$ are the eigenvalues of Λ, each one repeated many times as its multiplicity.*

Proof.

$$\begin{aligned} \det(\Lambda - r\mathbb{I}) &= \det \begin{pmatrix} -r\mathbb{I} & \mathbb{I} \\ -M^{-1}K & -r\mathbb{I} \end{pmatrix} \\ &= \det \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -M^{-1}K - r^2\mathbb{I} & -r\mathbb{I} \end{pmatrix} \\ &= \det(M^{-1}K + r^2\mathbb{I}) \\ &= (r^2 + \mu_1) \dots (r^2 + \mu_n) \end{aligned}$$

and, since $\det(M^{-1}K - \mu\mathbb{I}) = (\mu - \mu_1) \dots (\mu - \mu_n)$, then

$$\det(\Lambda - r\mathbb{I}) = (r - \sqrt{-\mu_1})(r + \sqrt{-\mu_1}) \dots (r - \sqrt{-\mu_n})(r + \sqrt{-\mu_n}).$$

□

Since M is positive definite and K is positive semi-definite, the eigenvalue $\mu_j = \frac{u_j^T K u_j}{u_j^T M u_j} \geq 0$. If K is positive definite, then the equilibrium is stable and Λ has $2n$ imaginary eigenvalues $\pm i\omega_1, \dots, \pm i\omega_n$, with $\omega_j := \sqrt{\mu_j} > 0$. $\omega_1, \dots, \omega_n$ are called *frequencies of the small oscillations* about the equilibrium.

1.2.2 Normal modes of oscillation

The frequencies of the small oscillations ω_j, $j = 1, \ldots, n$, are the positive solutions of

$$\det(K - \omega^2 M) = 0, \tag{1.7}$$

with associated eigenvectors $u_1, \ldots, u_n$ given by

$$(K - \omega_j^2 M)u_j = 0. \tag{1.8}$$

Definition 1.1. Assume that the eigenvalues are simple. The j-th *normal mode of oscillation* about the equilibrium configuration is the two-parameter family of periodic solutions of linear system (1.5)

$$q_j(t; a, b) = (a_j \cos(\omega_j t) + b_j \sin(\omega_j t))\, u_j \tag{1.9}$$

with $a, b \in \mathbb{R}$.

The normal mode (1.9) can also be written as

$$q_j(t; c, \eta) = c_j \cos(\omega_j t + \eta_j) u_j,$$

with parameters $c_j > 0$, the amplitude, and $\eta_j \in [0, 2\pi[$, the initial phase. Hence, the j-th normal mode represents a harmonic motion in which the n coordinates $(q_1, \ldots, q_n)$ oscillate all with the same frequency, with period $T_j = 2\pi/\omega_j$, with the same initial phase and with amplitudes determined by the components of the eigenvector u_j. Each normal mode spans the two-dimensional invariant subspace $\mathbb{E}_j^{NM} \subset \mathbb{R}^{2n}$

$$\mathbb{E}_j^{NM} = \operatorname*{span}_{t \in \mathbb{R}} \{u_j \cos(\omega_j t); u_j \sin(\omega_j t)\}. \tag{1.10}$$

If all frequencies $\omega_1, \ldots, \omega_n$ are distinct, then the eigenvectors $u_1, \ldots, u_n$ are unique; if the algebraic multiplicity of ω_i, $\mathrm{alg}(\omega_i)$, is greater than one for some i, then there is a $2 \times \mathrm{alg}(\omega_i)$-dimensional subspace associated to the repeated eigenvalue.

Proposition 1.4. *Each integral curve of the linearised Euler-Lagrange equation* (1.5)*, called* small oscillation*, is a linear combination of the n normal modes of oscillation of the system:*

$$q(t; c, \eta) = \sum_{j=1}^{n} c_j \cos(\omega_j t + \eta_j) u_j, \tag{1.11}$$

with $c_j \geq 0$ and $\eta_j \in [0, 2\pi[$.

Proof. To prove that (1.11) is a solution, it is sufficient to verify, through direct substitution, that it satisfies equation (1.5). Moreover, there always exist constants c, η such that $q(0; c, \eta) = q_0$ and $\dot{q}(0; c, \eta) = \dot{q}_0$ for any choice of the initial conditions $q_0, \dot{q}_0$. □

1.2.3 Beats

Consider now the simple case of a small oscillation which is the sum of two normal modes with different frequencies ω_1, ω_2. If the initial phases are $\eta_{1,2} = 0$, then

$$\begin{pmatrix} q_1(t; c_1, c_2) \\ q_2(t; c_1, c_2) \end{pmatrix} = c_1 \cos(\omega_1 t) \begin{pmatrix} u_{1,1} \\ u_{1,2} \end{pmatrix} + c_2 \cos(\omega_2 t) \begin{pmatrix} u_{2,1} \\ u_{2,2} \end{pmatrix}.$$

$q_i(t; c_1, c_2) = c_1 \cos(\omega_1 t) u_{1,i} + c_2 \cos(\omega_2 t) u_{2,i}$, $i = 1, 2$, can also be written in terms of exponentials as the real part of

$$c_1 e^{i\omega_1 t} u_{1,i} + c_2 e^{i\omega_2} u_{2,i}.$$

Factoring out the average frequency, we have

$$c_1 e^{i\omega_1 t} u_{1,i} + c_2 e^{i\omega_2} u_{2,i} = e^{i\frac{\omega_1+\omega_2}{2}t}(c_1 e^{i\frac{\omega_1-\omega_2}{2}t} u_{1,i} + c_2 e^{-i\frac{\omega_1-\omega_2}{2}t} u_{2,i}). \tag{1.12}$$

In particular, if the two frequencies $\omega_{1,2}$ are slightly different, the resulting small oscillation is an oscillation at the mean frequency with a modulation in the amplitude at the lower frequency $\frac{\omega_1-\omega_2}{2}$. This phenomenon is known as *beats*.

Example

As an example, consider the two-degree-of-freedom system consisting of two identical pendula with mass m and length l connected by an ideal spring, with elastic constant k (this model is studied, for example, in [1]). Let q_1 and q_2 be the angular displacements of the pendula from the equilibrium. The total kinetic energy and the potential energy are, respectively,

$$T = \frac{1}{2} m l^2 (\dot{q}_1^2 + \dot{q}_2^2),$$
$$V = -mgl(\cos q_1 + \cos q_2) + \frac{1}{2} k l^2 (\sin q_1 - \sin q_2)^2.$$

The equilibrium configuration (q_1^*, q_2^*) is $(0, 0)$. Therefore,

$$A(0,0) = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}, \quad V''(0,0) = \begin{pmatrix} mgl + kl^2 & -kl \\ -kl^2 & mgl + kl^2 \end{pmatrix}.$$

The frequencies of the small oscillations and the eigenvectors of $A^{-1}V''$ are the following:

- normal mode 1:

$$\omega_1 = \sqrt{\frac{g}{l}}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In this normal mode both pendula oscillate in-phase with proper frequency and the spring is not stretched;

- normal mode 2:

$$\omega_1 = \sqrt{\frac{g}{l} + 2\frac{k}{m}}, \quad u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

  In this normal mode the pendula oscillate in anti-phase. The frequency is higher than the previous one due to the action of the spring.

Hence, the small oscillation is

$$\begin{aligned} q_1 &= c_1 \cos(\omega_1 t + \eta_1) + c_2 \cos(\omega_2 t + \eta_2), \\ q_2 &= c_1 \cos(\omega_1 t + \eta_1) - c_2 \cos(\omega_2 t + \eta_2). \end{aligned}$$

Let now the coupling due to the spring be weak, i.e. $\frac{k}{m} \ll \frac{g}{l}$. Therefore, $\omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}} \approx \sqrt{\frac{g}{l}}\big(1 + \frac{kl}{mg}\big)$. Consider the initial conditions such that at $t = 0$ the first pendulum has amplitude A and zero velocity, while the second one is at rest, i.e. $q_1(0) = A$, $\dot{q}_1 = 0$, $q_2(0) = 0$, $\dot{q}_2 = 0$. Then, $c_1 = c_2 = \frac{A}{2}$, $\eta_1 = \eta_2 = 0$ and the resulting small oscillation is

$$\begin{aligned} q_1 &= A \cos\Big(\frac{\omega_2 - \omega_1}{2} t\Big) \cos\Big(\frac{\omega_2 + \omega_1}{2} t\Big), \\ q_2 &= A \sin\Big(\frac{\omega_2 - \omega_1}{2} t\Big) \sin\Big(\frac{\omega_2 + \omega_1}{2} t\Big). \end{aligned}$$

Hence, with this choice of initial conditions, the second pendulum, which was initially at rest, after a time $\pi/(\omega_2 - \omega_1)$ reaches its maximum amplitude A, while the first pendulum is at rest in its equilibrium position. After a time $2\pi/(\omega_2 - \omega_1)$, the system will return to the initial configuration.

1.3 Damped mechanical systems

We include now a dissipative contribution to the system, adding to equation (1.5) a viscous damping force which is proportional to the velocity:

$$M\ddot{q} + \Gamma\dot{q} + Kq = 0 \tag{1.13}$$

with Γ a symmetric and positive semi-definite matrix of order n.

Remark 1.1. More in general, $\Gamma = \Gamma^S + \Gamma^A$, where Γ^S is its symmetric non-negative part and Γ^A is its skew-symmetric one. Nonetheless, Γ^A, which represents a gyroscopic term, does not contribute to dissipation (see [15]). In the sequel, $\Gamma = \Gamma^S$ is assumed.

As a first-order differential equation, (1.13) becomes

$$\dot{x} = \Lambda x$$

where $x = (q, \dot{q}) \in \mathbb{R}^{2n}$ and

$$\Lambda = \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -M^{-1}K & -M^{-1}\Gamma \end{pmatrix}. \tag{1.14}$$

1.3.1 Spectral analysis

For the spectral description of (1.13) we follow [16], to which the reader is referred for greater details.

Let $\lambda_1^\pm, \dots, \lambda_n^\pm$ be the $2n$ eigenvalues of (1.14), with multiplicities counted and such that if $\lambda_j^+ \notin \mathbb{R}$, then $\lambda_j^- = \left(\lambda_j^+\right)^*$. Let $u_j \in \mathbb{C}^n$, $j = 1, \dots, n$, satisfy

$$(M\lambda_j^2 + \Gamma\lambda_j + K)u_j = 0,$$

and $m_j, \gamma_j, k_j \in \mathbb{R}$ be defined as

$$m_j = u_j^* \cdot M u_j > 0, \quad \gamma_j = u_j^* \cdot \Gamma u_j \geq 0, \quad k_j = u_j^* \cdot K u_j \geq 0.$$

Then, if the matrix $M\lambda_j^2 + \Gamma\lambda_j + K$ has rank $n - \mathrm{alg}(\lambda_j)$ $\forall j$, the characteristic polynomial factorises in the product of n polynomials $m_j\lambda_j^2 + \gamma_j\lambda_j + k_j = 0$, so that the eigenvalues can be written as

$$\lambda_j^\pm = \frac{-\gamma_j \pm \sqrt{\gamma_j^2 - 4m_jk_j}}{2m_j}, \quad j = 1, \dots, n, \tag{1.15}$$

where "$\pm\sqrt{\gamma^2 - 4mk}$" indicates the pair of complex square roots of the complex number $\gamma^2 - 4mk$.

Proposition 1.5. *Let $\beta_1, \dots, \beta_n$ be the eigenvalues of Γ with respect to M, i.e. the roots of* $\det(\Gamma - \beta M) = 0$*, ordered so that $\beta_1 \leq \dots \leq \beta_n$, and let $\omega_{01} \leq \dots \leq \omega_{0n}$ be the frequencies of the small oscillations for $\Gamma = 0$. The eigenvalues* (1.15) *satisfy the following properties.*

1. *If $\lambda_j^+ = \left(\lambda_j^-\right)^* \notin \mathbb{R}$:*

$$-\frac{\beta_n}{2} \leq Re\left[\lambda_j^\pm\right] \leq -\frac{\beta_1}{2}, \tag{1.16a}$$

$$\omega_{01} \leq |\lambda_j^\pm| \leq \omega_{0n}. \tag{1.16b}$$

In particular, $Re\left[\lambda_j^\pm\right] \leq 0$. Moreover, if $0 < \beta_1 < \beta_n$ and $0 < \omega_{01} < \omega_{0n}$:
if

$$\beta_1 > 2\omega_{0n},$$

then all the eigenvalues are real,
if

$$\beta_n < 2\omega_{01},$$

then all the eigenvalues are complex with $Re[\lambda] \leq 0$.

2. *If $\lambda_j^\pm \in \mathbb{R}$:*

$$-\frac{\beta_n}{2} - \sqrt{\left(\frac{\beta_n}{2}\right)^2 - \omega_{10}^2} \leq \lambda_j^\pm \leq -\frac{\beta_n}{2} + \sqrt{\left(\frac{\beta_n}{2}\right)^2 - \omega_{10}^2}. \tag{1.17}$$

Proof. The simultaneous diagonalization result ensures that there exists an invertible matrix R such that $R^T M R = \mathbb{I}$ and $R^T K R =: C = \mathrm{diag}(\omega_{01}^2, \dots, \omega_{0n}^2)$. In the new coordinates, $r = R^{-1} q$:

$$\ddot{r} + G\dot{r} + Cr = 0,$$

where $G := R^T \Gamma R$. The eigenvalues of G are $\beta_1, \dots, \beta_n$, in fact:

$$\begin{aligned} \det(\Gamma - \beta M) &= 0 \\ &= \det(\Gamma - \beta (RR^T)^{-1}) \\ &= \det(R^T \Gamma R - \beta \mathbb{I}) \\ &= \det(G - \beta \mathbb{I}), \end{aligned}$$

where the identity $M^{-1} = RR^T$ has been used.

Let $\hat{u} = R^{-1} u$ satisfy

$$(\lambda^2 + G\lambda + C)\hat{u} = 0$$

and be such that, without loss of generality, $\hat{u}^* \cdot \hat{u} = 1$. Then, set $g := \hat{u}^* \cdot G\hat{u}$, $c := \hat{u}^* \cdot C\hat{u}$, the eigenvalues are the roots of

$$\lambda^2 + g\lambda + c = 0. \tag{1.18}$$

Since K and Γ are symmetric, $C = C^T$ and $G = G^T$, and $c, g \in \mathbb{R}$. The symmetry of C and G allows to apply the following result, which we will not prove, known as "variational characterization of the eigenvalues":

Lemma 1.1. *Let A be a real symmetric matrix of order n, with eigenvalues $a_1 \leq \cdots \leq a_n$, with multiplicities counted. Then, for any vector $x \in \mathbb{C}^n$ for which $x^* \cdot x = 1$,*

$$a_1 \leq x^* \cdot Ax \leq a_n. \tag{1.19}$$

Proof. See [8]. □

1. If $\lambda^{\pm} = \mu \pm i\omega$, with $\mu \in \mathbb{R}$, $\omega \neq 0$: substituting into (1.18) and equating real and imaginary parts to zero, one gets

   $$\begin{cases} \mu = -\frac{g}{2} \\ \mu^2 + \omega^2 = |\lambda|^2 = c. \end{cases}$$

   Thus, inequality (1.19) implies

   $$\beta_1 \leq g \leq \beta_n \text{ and } \omega_{01}^2 \leq c \leq \omega_{0n}^2$$

   and ultimately (1.16).

2. If $\lambda^{\pm} = \mu^{\pm} \in \mathbb{R}$: substituting into (1.18) one gets

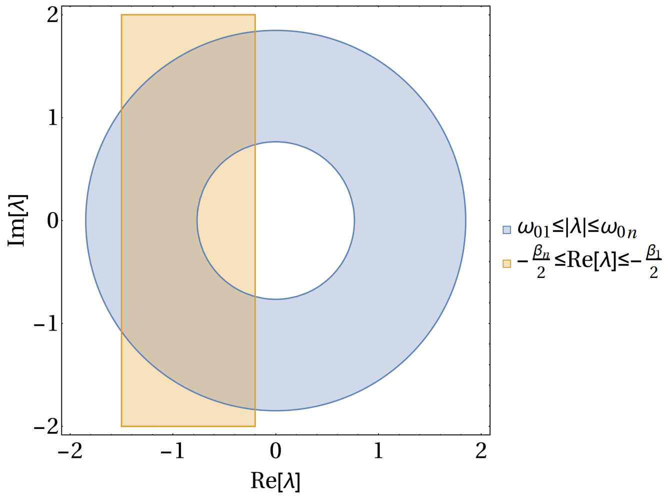
   $$\mu^{\pm} = -\frac{g}{2} \pm \sqrt{\left(\frac{g}{2}\right)^2 - k},$$

   thus, inequality (1.19) implies (1.17).

□

Figure 1.1 shows an example of region in the plane $(\mathrm{Re}[\lambda], \mathrm{Im}[\lambda])$ where the complex eigenvalues can be found, given by the intersection of the blue and orange portions.

Figure 1.1: Spectrum region $(\mathrm{Re}[\lambda], \mathrm{Im}[\lambda])$.



1.3.2 Damped normal modes

The eigenvalues $\lambda_1^{\pm}, \ldots, \lambda_n^{\pm}$, each one counted as many times as its multiplicity, are the solutions of

$$\det(M\lambda^2 + \Gamma\lambda + K) = 0, \tag{1.20}$$

with associated eigenvectors $u_1, \ldots, u_n$ given by

$$(M\lambda_j^2 + \Gamma\lambda_j + K)u_j = 0. \tag{1.21}$$

Definition 1.2. Assume the eigenvalues to be simple. The j-th *damped normal mode of oscillation* of system (1.13) is the two-parameter family of solutions of the form

$$q_j(t; c_1, c_2) = \mathrm{Re}\left[\left(c_{1j}e^{\lambda_j^+ t} + c_{2j}e^{\lambda_j^- t}\right) u_j\right] \tag{1.22}$$

with $c_1, c_2 \in \mathbb{R}$.

If $\lambda_j^{\pm} = \mu_j \pm i\omega_j$ not real, the damped normal mode (1.22) can also be written as

$$q_j(t; a, b) = a_j e^{\mu_j t}\cos(\omega_j t)\mathrm{Re}[u_j] + b_j e^{\mu_j t}\sin(\omega_j t)\mathrm{Im}[u_j],$$

with $a_j, b_j \in \mathbb{R}$. Hence, each damped normal mode spans in the phase space the two-dimensional invariant subspace $\mathbb{E}_j^{DNM} \subset \mathbb{R}^{2n}$

$$\mathbb{E}_j^{DNM} = \operatorname*{span}_{t\in\mathbb{R}}\left\{\mathrm{Re}[u_j]e^{\mathrm{Re}[\lambda_j]t}\cos\left(\mathrm{Im}[\lambda_j]t\right); \mathrm{Im}[u_j]e^{\mathrm{Re}[\lambda_j]t}\sin\left(\mathrm{Im}[\lambda_j]t\right)\right\}. \tag{1.23}$$

Definition 1.3. The *centre subspace* $\mathbb{E}^c$ is the direct sum of all the n_c two-dimensional invariant subspaces $\mathbb{E}_j^{DNM}$ associated to the purely imaginary eigenvalues:

$$\mathbb{E}^c = \bigoplus_{\mathrm{Re}[\lambda_j]=0} \mathbb{E}_j^{DNM}. \tag{1.24}$$

The centre subspace is invariant under the dynamics of the linear system, since it is a combination of invariant subspaces. Similarly, one can define the stable and unstable subspaces, as direct sum of invariant subspaces associated to eigenvalues with negative – respectively positive – real parts (see e.g. [10]).

1.4 Invariant undamped subspaces: theorem

In presence of dissipation, in order for the small oscillations not to be soon damped out, it is in general needed an external driving force to compensate the energy loss due to friction. However, in some cases the system may have some normal modes which are not damped. Therefore, whenever it oscillates in these normal modes, the system does not dissipate. In the following sections we will investigate the conditions under which there exist undamped invariant subspaces in the phase space, and how their presence can lead to synchronization. We will then apply the results found to three samples, which allow to understand intuitively this mechanism.

As above, we consider the n-degree-of-freedom linear mechanical system

$$M\ddot{q} + \Gamma\dot{q} + Kq = 0, \quad q \in \mathbb{R}^n, \tag{1.25}$$

with

M symmetric and positive definite,
Γ symmetric and positive semi-definite,
K symmetric.

Theorem 1.1. *Let $S \subseteq \ker(M^{-1}\Gamma)$ be a subspace and $S^\perp$ its orthogonal complement in $\mathbb{R}^n$ with respect to the ordinary scalar product. Consider the following conditions:*

C1: $(M^{-1}K)S \subseteq S$,

C2: $q \cdot Kq|_S$ *is positive definite,*

C3: S *is maximal among the subspaces of* $\ker(M^{-1}\Gamma)$ *which satisfy condition C1,*

C4: $q \cdot Kq|_{S^\perp}$ *is positive definite.*

Then,

1. *C1 implies that the tangent bundle TS is invariant under the flow of* (1.25)*;*

2. *C1 and C2 imply that TS is a subspace of the centre subspace and the dynamics on it consists of small oscillations;*

3. *C1 - C4 imply that TS is attractive.*

Proof. The result known as simultaneous diagonalization ensures the existence of an invertible matrix P such that $P^T M P = \mathbb{I}$ and $P^T \Gamma P =: \widetilde{\Gamma}$ is diagonal. Reordering rows and columns of P, $\widetilde{\Gamma}$ can be written as

$$\widetilde{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_p, 0, \ldots, 0) = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix},$$

with all $\gamma_i > 0$ and R a diagonal and positive definite matrix of order p.

In the new coordinates $Q = P^{-1}q$, equation (1.25) becomes

$$\ddot{Q} + \widetilde{\Gamma}\dot{Q} + \widetilde{K}Q = 0, \tag{1.26}$$

with $\widetilde{K} = P^T K P$ symmetric. Define the subspace $\widetilde{S} = P^{-1}S$. Since $S \subseteq \ker(M^{-1}\Gamma) = \ker(\Gamma)$ because M is invertible, and, because $\det(P) \neq 0$, $\widetilde{S} \subseteq P^{-1}\ker(\Gamma) = \ker(\widetilde{\Gamma})$. Let us work with the following splitting, omitting to indicate the matrix of change of basis:

$$\mathbb{R}^n = \mathrm{Im}(\widetilde{\Gamma}) \oplus \widetilde{U} \oplus \widetilde{S} \ni (x, u, s),$$

where $\widetilde{U}$ is the orthogonal complement of $\widetilde{S}$ in $\ker(\widetilde{\Gamma})$, with respect to the scalar product associated to M, i.e. $y \cdot Mz$, so that

$$\widetilde{\Gamma} = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{K} = \begin{pmatrix} A & B_u & B_s \\ B_u^T & D_u & D_{us} \\ B_s^T & D_{us}^T & D_s \end{pmatrix},$$

with certain matrices $A, \ldots, D_s$ such that $A = A^T$, $D_u = D_u^T$ and $D_s = D_s^T$.

1. Note that $M = (PP^T)^{-1}$. Thus, condition C1 becomes $PP^T KP\widetilde{S} \subseteq P\widetilde{S}$. Since $\det(P) \neq 0$ and $P^T KP = \widetilde{K}$, this condition is

$$\widetilde{K}\widetilde{S} \subseteq \widetilde{S}. \tag{1.27}$$

Therefore,

$$\widetilde{K}\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} B_s s \\ D_{us} s \\ D_s s \end{pmatrix} \in \widetilde{S} \quad \forall s \in \mathbb{R}^{n_s},$$

with $n_s = \dim \widetilde{S}$, namely $B_s = 0$, $D_{us} = 0$. Hence,

$$\widetilde{K} = \begin{pmatrix} A & B_u & 0 \\ B_u^T & D_u & 0 \\ 0 & 0 & D_s \end{pmatrix},$$

and equation (1.26) becomes

$$\begin{cases} \ddot{x} + R\dot{x} + Ax + B_u u = 0 \\ \ddot{u} + B_u^T x + D_u u = 0 \\ \ddot{s} + D_s s = 0. \end{cases} \tag{1.28}$$

The tangent bundle $T\widetilde{S}$, in which $x = \dot{x} = 0$ and $u = \dot{u} = 0$, is therefore invariant.

2. In the new coordinates, condition C2 is

$$Q \cdot \widetilde{K} Q|_{\widetilde{S}} > 0, \tag{1.29}$$

namely

$$\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} \cdot \widetilde{K} \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = s \cdot D_s s > 0 \quad \forall s \in \mathbb{R}^{n_s}.$$

Thus, D_s is positive definite and, as a consequence, on $T\widetilde{S}$ the dynamics consists of small oscillations.

3. If S satisfies condition C3, then $\widetilde{S}$ is largest subspace of $\ker(\widetilde{\Gamma})$ satisfying (1.27). Since

$$\begin{pmatrix} A & B_u & 0 \\ B_u^T & D_u & 0 \\ 0 & 0 & D_s \end{pmatrix} \begin{pmatrix} 0 \\ u \\ s \end{pmatrix} = \begin{pmatrix} B_u u \\ D_u u \\ D_s s \end{pmatrix},$$

it follows that $\ker(B_u) \oplus \widetilde{S}$ is a subspace of $\ker(\widetilde{\Gamma})$ which satisfies (1.27) and contains $\widetilde{S}$. Hence $\ker(B_u) = \{0\}$.

4. The dynamics on $\mathrm{Im}(\widetilde{\Gamma}) \oplus \widetilde{U}$ is described by

$$\begin{cases} \ddot{x} + R\dot{x} + Ax + B_u u = 0 \\ \ddot{u} + B_u^T x + D_u u = 0, \end{cases} \tag{1.30}$$

which can be written as

$$\begin{pmatrix} \ddot{x} \\ \ddot{u} \end{pmatrix} + \begin{pmatrix} R\dot{x} \\ 0 \end{pmatrix} + \hat{K} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with

$$\hat{K} = \begin{pmatrix} A & B_u \\ B_u^T & D_u \end{pmatrix}.$$

The equilibrium configurations of (1.30) are zeros of $\hat{K} \begin{pmatrix} x \\ u \end{pmatrix}$, namely they are the points in the kernel of $\hat{K}$.

If C3 holds, then $\hat{K}$ is positive definite. Therefore, the Lyapunov function $\mathcal{W}(x, u, \dot{x}, \dot{u}) = \frac{1}{2}\left(\dot{x}\cdot\dot{x} + \dot{u}\cdot\dot{u} + \begin{pmatrix} x \\ u \end{pmatrix}\cdot\hat{K}\begin{pmatrix} x \\ u \end{pmatrix}\right)$ defines a distance from $T\widetilde{S}$. Hence,

$$\frac{d\mathcal{W}}{dt} = -\dot{x}\cdot R\dot{x} \le 0,$$

with the equality holding iff $\dot{x} = 0$, because R is positive definite. Since $\det\hat{K} \ne 0$, $(\bar{x}, \bar{u}) = (0, 0)$ is the only equilibrium configuration. By LaSalle-Krasovskii principle (see e.g. [6]), if $(0, 0)$ is the only complete orbit in the set such that $\frac{d\mathcal{W}}{dt} = 0$, then it is asymptotically stable. Let us thus show that the set $\{(x, u, 0, \dot{u})\}$ does not contain any complete orbit different from $(0, 0)$. For $\dot{x} = 0$, from the first equation of (1.30), $u = -B_u^{-1}Ax$, because $\ker B_u = \{0\}$ for the maximality of $\widetilde{S}$. Substituting, the second equation of (1.30) becomes $\left(B_u^T - D_u B_u^{-1}A\right)x = 0$, and since $\det(B_u^T - D_u B_u^{-1}A) = \det\hat{K} \ne 0$, one gets $x = x_0 = 0$ and $u = u_0 = 0$. Therefore, $(\bar{x}, \bar{u}) = (0, 0)$ is the only complete orbit in $\{(x, u, 0, \dot{u})\}$ and $T\widetilde{S}$ is attractive.

□

This theorem provides sufficient conditions for the existence of an attractive invariant centre space TS in the phase space, and indicates how to compute it. The dimension of the undamped bundle TS coincides with the number of purely imaginary eigenvalues of the matrix $\Lambda = \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -M^{-1}K & -M^{-1}\Gamma \end{pmatrix}$ associated to (1.25). In particular, if S is one-dimensional, the motion in TS is a normal mode, therefore all the parts of the system oscillate, without being damped, at the same frequency. This mechanism is referred to as synchronization. More in general, S might be higher-dimensional, thus the motion consists of a linear superposition of normal modes, leading for example to beats.

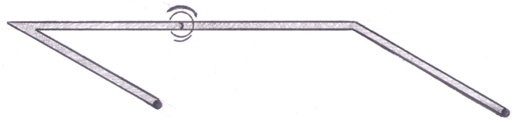
1.5 Examples

We apply now the results found to three simple mechanical systems, studying their spectrum and determining the invariant undamped subspace. This analysis will show that synchronization is not an unusual mechanism and that whether it occurs in anti-phase or in phase depends on how friction acts on the system.

1.5.1 Example 1

Consider the model consisting of two identical heavy L-shaped rigid bars. They are constrained so that one leg of each bar lays horizontally and with the extremities in contact, while the other leg is free to rotate in a perpendicular plane. Assume that there is friction in correspondence of the contact point between the two bars. Let m be the mass of each hanging leg, l be their length and I their moment of inertia with respect to the horizontal bars. Let then θ_1 and θ_2 be the angular displacements of the two hanging legs.

Figure 1.2: Example 1. In-phase synchronization model.



The undamped system is described by the Lagrangian

$$L = \frac{1}{2} I(\dot\theta_1^2 + \dot\theta_2^2) + mgl(\cos\theta_1 + \cos\theta_2).$$

For small oscillations, $\sin\theta \simeq \theta$, and rescaling time $t \to \sqrt{\frac{I}{mgl}}t$, the Lagrangian can be written as

$$L = \frac{1}{2}(\dot\theta_1^2 + \dot\theta_2^2) - \frac{1}{2}(\theta_1^2 + \theta_2^2). \tag{1.31}$$

It leads to the following Lagrange equations:

$$\begin{cases} \ddot\theta_1 + \theta_1 + \gamma(\dot\theta_1 - \dot\theta_2) = 0 \\ \ddot\theta_2 + \theta_2 + \gamma(\dot\theta_2 - \dot\theta_1) = 0, \end{cases} \tag{1.32}$$

where it has been added the contribution of the viscous damping, proportional to the difference between the angular velocities of the two bars, with $\gamma > 0$.

Invariant subspace

Equations (1.32) are of the form $M\ddot q + \Gamma\dot q + Kq = 0$ with

$$M = K = \mathbb{I}_2, \quad \Gamma = \begin{pmatrix} \gamma & -\gamma \\ -\gamma & \gamma \end{pmatrix}.$$

The kernel of Γ is

$$\ker(\Gamma) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle,$$

and it is invariant under the action of $M^{-1}K$, since $M^{-1}K \ker(M^{-1}\Gamma) = \ker(M^{-1}\Gamma)$. Thus, $S \equiv \ker(M^{-1}\Gamma) = \ker(\Gamma)$ and $S^\perp \equiv \mathrm{Im}(M^{-1}\Gamma) = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$. The hypotheses of theorem 1.1 are satisfied, therefore $T\ker\Gamma$ is invariant and attractive, and on it the motion is given by the normal mode corresponding to the two bars swinging in phase.

Sprectrum

- Without damping: $\gamma = 0$.
  The frequencies of the small oscillations are

$$\omega_{0k} = 1, \quad k = 1, 2.$$

- With damping: $\gamma \neq 0$.
  The eigenvalues satisfy the characteristic equation

$$(\lambda^2 + 1)(\lambda^2 + 2\gamma\lambda + 1) = 0.$$

  For $\gamma < 1$, the eigenvalues are

$$\begin{aligned} \lambda_1^{\pm} &= \pm i, \\ \lambda_2^{\pm} &= -\gamma \pm i\sqrt{1-\gamma^2}. \end{aligned}$$

  For $\gamma > 1$, the eigenvalues are

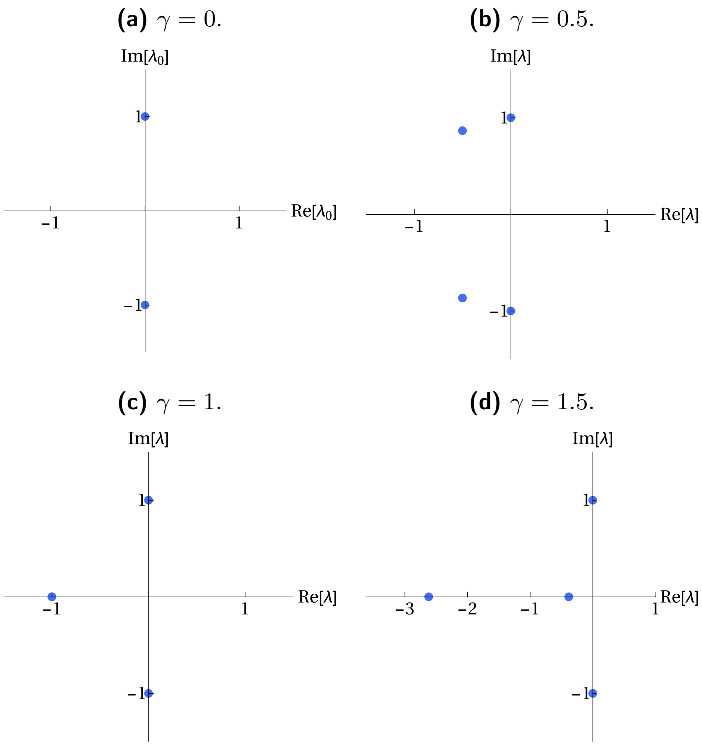
$$\begin{aligned} \lambda_1^{\pm} &= \pm i, \\ \lambda_2^{\pm} &= -\gamma \pm \sqrt{\gamma^2 - 1}. \end{aligned}$$

  For $\gamma = 1$, the eigenvalues are

$$\begin{aligned} \lambda_1^{\pm} &= \pm i, \\ \lambda_2^{\pm} &= -\gamma. \end{aligned}$$

Figure 1.3 shows the spectrum for the following values of γ: $\gamma = 0,\ 0.5,\ 1,\ 1.5$.

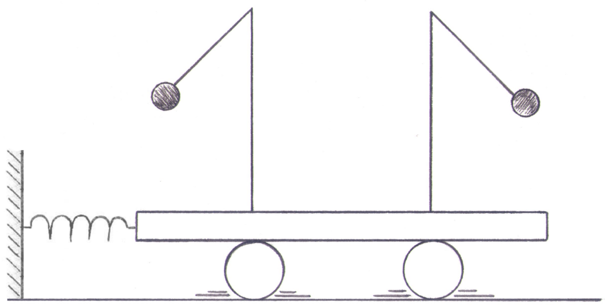
Figure 1.3: Example 1: Spectrum $(\mathrm{Re}[\lambda], \mathrm{Im}[\lambda])$.



1.5.2 Example 2

Consider the system consisting of two identical pendula on a common rigid support, which is constrained to move in one dimension horizontally and is connected to a spring. Assume that there is friction in correspondence of the wheels of the base support. Let m and l be the mass of the pendula and their length respectively, M the mass of the support and k the elastic constant of the spring. Let ϕ_1 and ϕ_2 be the angular displacements of the two pendula about their pivot points and X be the linear displacement of the support.

Figure 1.4: Example 2. Anti–phase synchronization model.



The undamped system is described by the Lagrangian

$$L=\frac{1}{2}(M+2m)\dot{X}^2+\frac{m}{2}l^2(\dot{\phi}_1^2+\dot{\phi}_2^2)+ml\dot{X}(\dot{\phi}_1\cos\phi_1+\dot{\phi}_2\cos\phi_2)+mgl(\cos\phi_1+\cos\phi_2)-\frac{k}{2}X^2.$$

For small oscillations, $\sin\phi \simeq \phi$, rescaling $X \to lX$, $t \to \sqrt{\frac{l}{g}}\,t$ and defining $\mu = \frac{M+2m}{m}$ and $\alpha = \frac{kl}{mg}$, the Lagrangian becomes

$$L = \frac{1}{2}\mu\dot{X}^2 + \frac{1}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2) + \dot{X}(\dot{\phi}_1 + \dot{\phi}_2) - \frac{1}{2}(\phi_1^2 + \phi_2^2) - \frac{1}{2}\alpha X^2. \tag{1.33}$$

It leads to the following Lagrange equations:

$$\begin{cases} \mu\ddot{X} + \gamma\dot{X} + \ddot{\phi}_1 + \ddot{\phi}_2 + \alpha X = 0 \\ \ddot{\phi}_1 + \ddot{X} + \phi_1 = 0 \\ \ddot{\phi}_2 + \ddot{X} + \phi_2 = 0, \end{cases} \tag{1.34}$$

where it has been added the contribution of the viscous damping to the common support, proportional to its velocity, with $\gamma > 0$.

Invariant subspace

Equations (1.34) are of the form $M\ddot{q} + \Gamma\dot{q} + Kq = 0$ with

$$M = \begin{pmatrix} \mu & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The kernel of Γ is

$$\ker(\Gamma) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle,$$

the invariant subspace $S \subseteq \ker(\Gamma)$ and its orthogonal are

$$S = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle, \qquad S^{\perp} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

The matrix P which diagonalizes simultaneously M and Γ is

$$P = \begin{pmatrix} \frac{-1}{\sqrt{\mu-2}} & 0 & 0 \\ \frac{1}{\sqrt{\mu-2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{\mu-2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore, $P^T M P = \mathbb{I}$, and $\tilde{\Gamma} = P^T \Gamma P$ and $\tilde{K} = P^T K P$ are

$$\tilde{\Gamma} = \frac{\gamma}{\mu - 2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} \frac{2+\alpha}{\mu-2} & \sqrt{\frac{2}{\mu-2}} & 0 \\ \sqrt{\frac{2}{\mu-2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The equations of motion, written in the form $\ddot{Q}+\tilde{\Gamma}\dot{Q}+\tilde{K}Q = 0$, with $Q = P^{-1}q = (x, u, s)$, become

$$\begin{cases} \ddot{x} + \frac{\gamma}{\mu-2}\dot{x} + \frac{2+\alpha}{\mu-2}x + \sqrt{\frac{2}{\mu-2}}u = 0 \\ \ddot{u} + \sqrt{\frac{2}{\mu-2}}x + u = 0 \\ \ddot{s} + s = 0. \end{cases}$$

On the invariant subspace $\widetilde{S} = P^{-1}S = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \subseteq \ker\Gamma$ the motion is described by $\ddot{s} + s = 0$, which corresponds to the normal mode in which the two pendula swing in anti-phase and the support is at rest.

Sprectrum

- Without damping: $\gamma = 0$.
  The frequencies of the small oscillations are

$$\omega_{01} = \left(\frac{(\alpha+\mu) - \sqrt{(\alpha+\mu)^2 - 4\alpha(\mu-2)}}{2(\mu-2)} \right)^{\frac{1}{2}},$$
$$\omega_{02} = 1,$$
$$\omega_{03} = \left(\frac{(\alpha+\mu) + \sqrt{(\alpha+\mu)^2 - 4\alpha(\mu-2)}}{2(\mu-2)} \right)^{\frac{1}{2}}.$$

- With damping: $\gamma \neq 0$.
  The eigenvalues satisfy the characteristic equation

$$(\lambda^2+1)\left[(\lambda^2+1)(\mu\lambda^2+\gamma\lambda+\alpha) - 2\lambda^4\right] = 0.$$

  The eigenvalues of Γ with respect to M, i.e. roots of $\det(\Gamma - \beta M) = 0$, are $\beta_1 = \beta_2 = 0$ and $\beta_3 = \frac{\gamma}{\mu+2}$. Therefore,
  if $\lambda_j^- = (\lambda_j^+)^* \notin \mathbb{R}$, then

$$-\frac{1}{2}\frac{\gamma}{\mu+2} \leq \mathrm{Re}[\lambda_j^\pm] \leq 0,$$

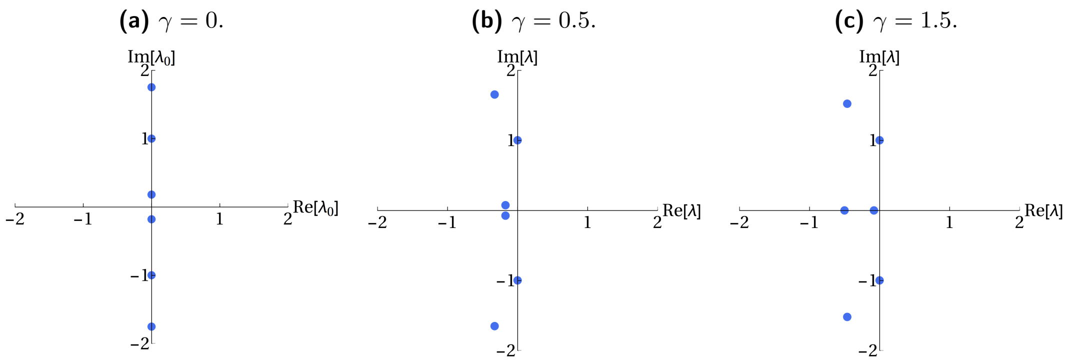
$$\left(\frac{(\alpha+\mu) - \sqrt{(\alpha+\mu)^2 - 4\alpha(\mu-2)}}{2(\mu-2)} \right)^{\frac{1}{2}} \leq |\lambda_j^\pm| \leq \left(\frac{(\alpha+\mu) + \sqrt{(\alpha+\mu)^2 - 4\alpha(\mu-2)}}{2(\mu-2)} \right)^{\frac{1}{2}};$$

  if $\lambda_j^\pm = \in \mathbb{R}$, then

$$-\frac{\gamma}{\mu+2} \leq \lambda_j^\pm \leq 0.$$

Figure 1.5 shows the spectrum for the following values of γ: $\gamma = 0,\ 1,\ 1.5$.

Figure 1.5: Example 2: Spectrum $(\mathrm{Re}[\lambda], \mathrm{Im}[\lambda])$ with $\mu = 3$, $\alpha = 0.1$.



1.5.3 Example 3

Consider the system consisting of a chain of three identical masses, with mass m, and four identical ideal springs, with elastic constant k. Assume there is friction in correspondence of the mass in the middle. Let x_i, $i = 1, 2, 3$, be the displacement of the i-th mass.

Figure 1.6: Example 3. Anti-phase synchronization model.

The undamped system is described by the Lagrangian

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2 - \frac{1}{2}kx_3^2.$$

Rescaling $t \to \sqrt{\frac{k}{m}}\, t$, the Lagrangian becomes

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}x_1^2 - \frac{1}{2}(x_2 - x_1)^2 - \frac{1}{2}(x_3 - x_2)^2 - \frac{1}{2}x_3^2. \tag{1.35}$$

It leads to the following Lagrange equations:

$$\begin{cases} \ddot{x}_1 + 2x_1 - x_2 = 0 \\ \ddot{x}_2 - x_1 + 2x_2 - x_3 + \gamma\dot{x}_2 = 0 \\ \ddot{x}_3 - x_2 + 2x_3 = 0, \end{cases} \tag{1.36}$$

where it has been added a viscous damping term to mass 2 proportional to its velocity, with $\gamma > 0$.

Invariant subspace

Equations (1.36) are of the form $M\ddot{q} + \Gamma\dot{q} + Kq = 0$ with

$$M = \mathbb{I}_3, \quad \Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The kernel of the matrix Γ is the two-dimensional subspace

$$\ker\Gamma = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle,$$

the invariant subspace $S \subseteq \ker(\Gamma)$ and its orthogonal are

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle, \qquad S^{\perp} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

The hypotheses of theorem 1.1 are satisfied only by the one-dimensional subspace S, and therefore TS is invariant and attractive. The motion on TS is given by the normal mode corresponding to the external masses oscillating in opposite phase and the mass in the middle at rest.

Sprectrum

- Without damping: $\gamma = 0$.
  The frequencies of the small oscillations are

$$\omega_{01} = \sqrt{2-\sqrt{2}},$$
$$\omega_{02} = \sqrt{2},$$
$$\omega_{03} = \sqrt{2+\sqrt{2}}.$$

- With damping: $\gamma \neq 0$.
  The eigenvalues satisfy the characteristic equation

$$(\lambda^2+2)\left[(\lambda^2+2)(\lambda^2+\gamma\lambda+2)-2\right] = 0.$$

  The eigenvalues of Γ with respect to M, i.e. roots of $\det(\Gamma - \beta M) = 0$, are $\beta_1 = \beta_2 = 0$ and $\beta_3 = \gamma$. Therefore,
  if $\lambda_j^- = (\lambda_j^+)^* \notin \mathbb{R}$, then

$$-\frac{\gamma}{2} \leq \mathrm{Re}[\lambda_j^\pm] \leq 0,$$
$$\sqrt{2-\sqrt{2}} \leq |\lambda_j^\pm| \leq \sqrt{2+\sqrt{2}};$$

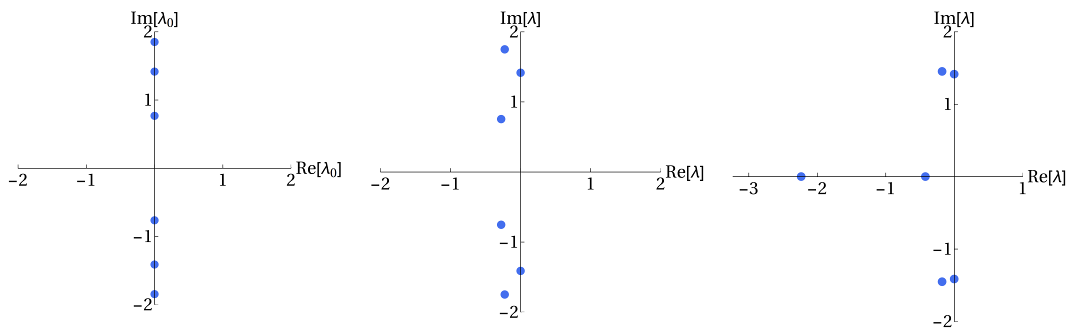
  if $\lambda_j^\pm \in \mathbb{R}$, then

$$-\gamma \leq \lambda_j^\pm \leq 0.$$

Figure 1.7 shows the spectrum for the following values of γ: $\gamma = 0,\ 1,\ 3$.

Figure 1.7: Example 3: Spectrum $(\mathrm{Re}[\lambda], \mathrm{Im}[\lambda])$.

(a) $\gamma = 0$. **(b)** $\gamma = 1$. **(c)** $\gamma = 3$.



Chapter 2

Pendulum hanging from a string

In this chapter we study the small oscillations about the equilibrium configuration of a system composed of a pendulum suspended on a string. The Euler-Lagrange equations for infinite-degree-of-freedom systems are first recalled, thus the Lagrangian for the string-pendulum system is derived and the linearised system is analysed. Finally, the spectrum and the eigenfunctions are computed in the case in which the pendulum is suspended in the middle point of the string.

2.1 Infinite dimensional systems

The configuration space Q is the space of square-integrable real functions u on $D \subseteq \mathbb{R}^3$. Let L be the Lagrangian functional $L : TQ \to \mathbb{R}$, $(u, u_t) \mapsto L(u, u_t)$, which can be expressed in terms of a Lagrangian density $\mathcal{L}$ as

$$L(u, u_t) = \int_D \mathcal{L}(u, u_t, u_x)\, dx,$$

denoting the partial derivatives with the correspondent subscript. Analogously to the finite-dimensional case, the equations of motion can be derived through the principle of least action. Let

$$S(u) := \int_{t_1}^{t_2} L(u, u_t)\, dt$$

be the action functional. Then its critical points, solutions of $dS(u) = 0$, determine the Euler-Lagrange equations for the Lagrangian L, with the notation of functional derivatives

$$\begin{cases} \frac{\delta L}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u_x} \\ \frac{\delta L}{\delta u_t} = \frac{\partial \mathcal{L}}{\partial u_t}. \end{cases}$$

Hence, the Euler-Lagrange equations for an infinite-dimensional system are

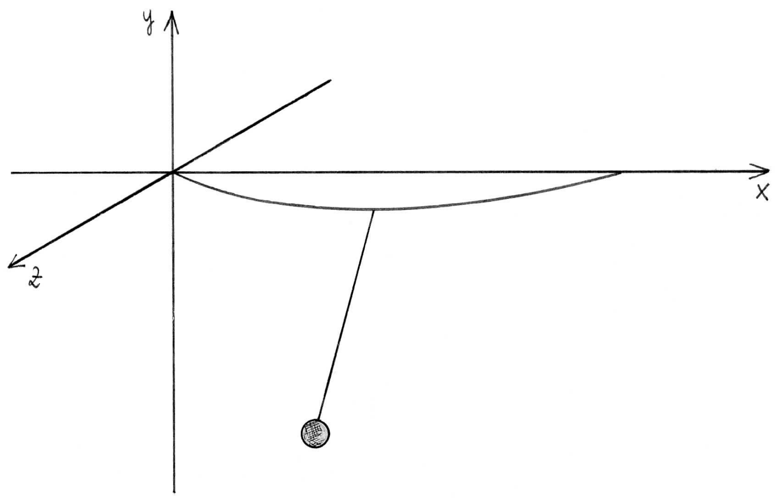
$$\frac{d}{dt}\frac{\delta L}{\delta u_t} - \frac{\delta L}{\delta u} = 0. \tag{2.1}$$

2.2 Description of the system

The system consists of a heavy homogeneous flexible and elastic string with fixed extremities and a pendulum hanging from a point O_1 of the string. We shall consider the transverse displacements of the string (see [23]) and assume that the pendulum swings in the vertical plane transverse to the string.

Let us choose a coordinate system with the origin on the left extremity of the string, the x-axis horizontal and directed along the resting string and the y-axis directed as the ascendant vertical.

Figure 2.1: Model: pendulum hanging from a string.



Let ψ be the horizontal transverse displacement of a point of the string from the x-axis on the xz-plane and χ the vertical displacement from the x-axis on the xy-plane. The string is thus described by the following embedding:

$$[0,\Lambda] \ni x \mapsto \begin{pmatrix} x \\ y(x,t) = \chi(x,t) \\ z(x,t) = \psi(x,t) \end{pmatrix} \in \mathbb{R}^3$$

with functions $\psi, \chi : [0,\Lambda] \times \mathbb{R} \to \mathbb{R}$ such that $\psi(0,t) = 0 = \psi(\Lambda,t)$ and $\chi(0,t) = 0 = \chi(\Lambda,t)$ $\forall t$, and Λ the length of the string.

Let ϕ be the angular displacement of the pendulum measured from the descendant y-axis on the yz-plane, let l be its length and let us denote with x_1 the x-coordinate of the suspension point O_1. Therefore, the pendulum has coordinates

$$\begin{cases} x_p = x_1 \\ y_p(t) = \chi(x_1,t) - l\cos\phi(t) \\ z_p(t) = \psi(x_1,t) + l\sin\phi(t). \end{cases}$$

Hence, this model describes a hybrid dynamical system consisting of a one-degree-of-freedom system (the pendulum) and of a continuous system (the string).

2.3 Lagrangian

We now derive the Lagrangian of the coupled system. Since the system is conservative, the Lagrangian is the difference between the kinetic energy and the potential energy of the system:

$$L(\phi, \dot{\phi}, \psi, \psi_t, \chi, \chi_t) = T(\phi, \dot{\phi}, \psi, \psi_t, \chi, \chi_t) - V(\phi, \psi, \chi),$$

where each term accounts both for the string and for the pendulum. Thus,

$$T = \frac{m}{2}\Big[l^2\dot{\phi}^2 + \psi_t^2(x_1) + \chi_t^2(x_1) + 2l\dot{\phi}\big(\psi_t(x_1)\cos(\phi) + \chi_t(x_1)\sin(\phi)\big)\Big] + \int_0^\Lambda \Big[\frac{\rho}{2}\left(\psi_t^2 + \chi_t^2\right)\Big]\,dx,$$

$$V = mg\big(\chi(x_1) - l\cos(\phi)\big) + \int_0^\Lambda \left[\frac{\tau}{2}\left(\psi_x^2 + \chi_x^2\right) + \rho g\chi\right]\,dx,$$

where ρ is the linear mass density of the string, τ its tension, l is the length of the pendulum and m its mass and g indicates the standard acceleration of gravity.

In terms of a Lagrangian density $\mathcal{L}$, the Lagrangian is

$$L(\phi, \dot{\phi}, \psi, \psi_t, \chi, \chi_t) = \int_0^\Lambda \mathcal{L}(\phi, \dot{\phi}, \psi, \psi_t, \psi_x, \chi, \chi_t, \chi_x)\,dx.$$

Hence,

$$\begin{aligned} L = \int_0^\Lambda \Bigg\{ & \left[\frac{1}{2}\rho\left(\psi_t^2 + \chi_t^2\right) - \frac{1}{2}\tau\left(\psi_x^2 + \chi_x^2\right) - \rho g\chi\right] \\ & + \left[\frac{1}{2}m\Big(\psi_t^2 + \chi_t^2 + 2l\dot{\phi}\big(\psi_t\cos(\phi) + \chi_t\sin(\phi)\big)\Big) - mg\chi\right]\delta(x - x_1) \\ & + \left[\frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos(\phi)\right]\delta(x - x_1)\Bigg\}\,dx. \end{aligned} \tag{2.2}$$

This Lagrangian can be written as sum of the following terms:

$$\begin{aligned} L_p &= \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi, \\ L_s &= \int_0^\Lambda \left[\frac{1}{2}\rho\left(\psi_t^2 + \chi_t^2\right) - \frac{1}{2}\tau\left(\psi_x^2 + \chi_x^2\right) - \rho g\chi\right]\,dx, \\ L_i &= \int_0^\Lambda \left[\frac{1}{2}m\left(\psi_t^2 + \chi_t^2\right) + ml\dot{\phi}\left(\psi_t\cos\phi + \chi_t\sin\phi\right) - mg\chi\right]\delta(x - x_1)\,dx, \end{aligned}$$

where L_p describes a pendulum with fixed suspension point, L_s describes the string and L_i contains the terms due to the interaction between the pendulum and the string.

Applying (2.1), the Lagrangian (2.2) gives the following Euler-Lagrange equations:

$$\begin{cases} ml^2\ddot{\phi}(t) + ml\psi_{tt}(x_1,t)\cos(\phi(t)) + ml\chi_{tt}(x_1,t)\sin(\phi(t)) + mgl\sin(\phi(t)) = 0 \\ \rho\psi_{tt}(x,t) + \Big(m\psi_{tt}(x,t) + ml\ddot{\phi}(t)\cos(\phi(t)) - ml\dot{\phi}^2(t)\sin(\phi(t))\Big)\,\delta(x-x_1) - \tau\psi_{xx}(x,t) = 0 \\ \rho\chi_{tt}(x,t) + \Big(m\chi_{tt}(x,t) + ml\ddot{\phi}(t)\sin(\phi(t)) + ml\dot{\phi}^2(t)\cos(\phi(t))\Big)\,\delta(x-x_1) - \tau\chi_{xx}(x,t) \\ \qquad\qquad + \rho g + mg\delta(x-x_1) = 0. \end{cases}$$

Let us look for the equilibrium state $(\phi_{eq}, \psi_{eq}(x), \chi_{eq}(x))$. The equilibrium coordinates satisfy the conditions of extremum

$$\frac{\delta L}{\delta\phi} = 0, \quad \frac{\delta L}{\delta\psi} = 0, \quad \frac{\delta L}{\delta\chi} = 0,$$

and the boundary conditions

$$\psi(0,t) = \psi(\Lambda,t) = 0, \quad \chi(0,t) = \chi(\Lambda,t) = 0, \quad \forall t.$$

The computation of ϕ_{eq} and $\psi_{eq}(x)$ is straightforward, while in order to determine $\chi_{eq}(x)$ one has to solve the equation $\tau\chi''(x) - \rho g = mg\delta(x - x_1)$, which descends from the condition $\frac{\delta L}{\delta\chi} = 0$. The Dirac delta function can be handled in the following way: one solves the equation independently on the left-hand side and on the right-hand side of the discontinuity x_1, so that $\chi_{eq}(x) = \chi_-(x)\Theta(x_1 - x) + \chi_+(x)\Theta(x - x_1)$; then one imposes, together with the boundary conditions, the following interface conditions:

$$\begin{cases} \chi_-(x_1) = \chi_+(x_1) \\ \chi'_+(x_1) - \chi'_-(x_1) = \frac{mg}{\tau}. \end{cases}$$

Thus, the equilibrium configuration, which is stable since it is a strict minimum of the potential energy, is

$$\begin{cases} \phi_{eq} = 0 \\ \psi_{eq}(x) = 0 \\ \chi_{eq}(x) = \frac{\rho g}{2\tau}x^2 + x\big[\frac{mg}{\tau}\big(\frac{x_1}{\Lambda} - 1\big) - \frac{\rho g\Lambda}{2\tau}\big] + \big[\frac{mg}{\tau}(x - x_1)\big]\Theta(x - x_1). \end{cases} \tag{2.3}$$

In order to make a linear analysis, we consider small oscillations about the stable equilibrium state, namely

$$\begin{cases} \phi(t) = \phi_{eq} + \phi'(t) \\ \psi(x,t) = \psi_{eq}(x) + \psi'(x,t) \\ \chi(x,t) = \chi_{eq}(x) + \chi'(x,t). \end{cases}$$

In general, in doing so, the Lagrangian becomes quadratic in its variables. In fact, from the expansion about the equilibrium, the Lagrangian can be written as

$$L = L_0 + L_1 + L_2,$$

where L_0 is constant, thus does not influence the dynamics, L_1 contains linear terms but vanishes because of the extremum condition and L_2 contains quadratic terms.

We relabel for convenience the displacements from the equilibrium, ϕ', ψ', χ', omitting the primes, i.e. $\phi' \mapsto \phi$, $\psi' \mapsto \psi$, $\chi' \mapsto \chi$. Therefore, to second order in the displacements, the Lagrangian is

$$L = \int_0^\Lambda \left\{ \left[\frac{1}{2}\rho\left(\psi_t^2 + \chi_t^2\right) - \frac{1}{2}\tau\left(\psi_x^2 + \chi_x^2\right) \right] \right. \\ \left. + \left[\frac{1}{2}m\left(l^2\dot{\phi}^2 + \psi_t^2 + \chi_t^2 + 2l\dot{\phi}\psi_t\right) - \frac{1}{2}mgl\phi^2 \right] \delta(x - x_1) \right\} dx.$$

Let us now rescale the variables in order to obtain an adimensional Lagrangian, as follows:

$$x \mapsto \Lambda x, \quad \phi \mapsto \frac{\Lambda}{l}\phi, \quad \psi \mapsto \Lambda\psi, \quad \chi \mapsto \Lambda\chi, \quad t \mapsto \sqrt{\frac{\rho\Lambda^2}{\tau}}\,t,$$

and let us define the following dimensionless parameters $\alpha, \beta > 0$, which characterise the geometry of the system:

$$\alpha^2 = \frac{g\rho\Lambda^2}{l\tau}, \qquad \beta = \frac{m}{\rho\Lambda}. \tag{2.4}$$

The definitive adimensional Lagrangian is

$$L = \int_0^1 \left\{ \left[\frac{1}{2}\left(\psi_t^2 + \chi_t^2\right) - \frac{1}{2}\left(\psi_x^2 + \chi_x^2\right) \right] \right. \\ \left. + \beta\left[\frac{1}{2}\left(\dot{\phi}^2 + \psi_t^2 + \chi_t^2 + 2\dot{\phi}\psi_t\right) - \frac{1}{2}\alpha^2\phi^2 \right] \delta(x - x_1) \right\} dx. \tag{2.5}$$

Remark 2.1. In the quadratic Lagrangian (2.5), the only coupling between the pendulum and the string is $\dot{\phi}\psi_t$, while there is not any terms proportional to $\dot{\phi}\chi_t$. In fact, expanding about the equilibrium the interaction terms in (2.2), $\dot{\phi}\psi_t\cos\phi = \dot{\phi}\psi_t + o(\phi^2)$ while $\dot{\phi}\chi_t\sin\phi = \dot{\phi}\chi_t\phi + o(\phi^3)$, i.e. the second term is one order higher in ϕ with respect to the first one and can therefore be neglected.

2.4 Equations of motion

The equations of motion are the Euler-Lagrange equations for infinite-dimensional systems:

$$\begin{cases} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\psi_t} - \frac{\partial\mathcal{L}}{\partial\psi} + \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\psi_x} = 0 \\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\chi_t} - \frac{\partial\mathcal{L}}{\partial\chi} + \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\chi_x} = 0. \end{cases}$$

Since Lagrangian (2.5) contains a discontinuity in x_1, detectable from the presence of the Dirac delta function $\delta(x - x_1)$, in order to be able to derivate $\mathcal{L}$, a weak formulation is required. Nonetheless, we neglect in this treatise that study, and we proceed deriving like for usual functions, keeping in mind to define the interface conditions in x_1 when needed.

The equations of motion of the linearised coupled system are

$$\begin{cases} \beta\big(\ddot{\phi}(t) + \psi_{tt}(x_1, t) + \alpha^2\phi(t)\big) = 0 \\ \psi_{tt}(x, t) + \beta\big(\psi_{tt}(x, t) + \ddot{\phi}(t)\big)\delta(x - x_1) - \psi_{xx}(x, t) = 0 \\ \chi_{tt}(x, t) + \beta\chi_{tt}(x, t)\delta(x - x_1) - \chi_{xx}(x, t) = 0, \end{cases} \tag{2.6}$$

with the boundary conditions

$$\begin{cases} \psi(0, t) = 0 = \psi(1, t) \\ \chi(0, t) = 0 = \chi(1, t) \end{cases} \tag{2.7}$$

for all $t \in \mathbb{R}$.

2.5 Normal modes of oscillation

The equations of motion (2.6) are a linear system, therefore every solution, called small oscillation, can be written as a linear superposition of solutions. In particular, we are interested in periodic solutions in which every part of the system oscillates with the same frequency, namely normal modes of oscillation. We will not be able to prove the unicity of the solutions found, since this would require a weak formulation of the problem, with the use of test functions. Nonetheless, we shall characterise solutions which are $\mathcal{C}^0$.

2.5.1 Spectrum of the vibrating string

Let us start considering the decoupled system. The first equation of (2.6), which describes the motion of the pendulum, decouples from the others if $\psi(x_1, t) = 0\ \forall t$, that is when its suspension point stands still. In this case, the pendulum swings periodically with proper frequency $\omega = \alpha$ (in rescaled time units).

The equation of the vibrating string is restored in the limit of $\beta = 0$. The equations of motion (2.6) for the string, with the boundary conditions, become

$$\begin{cases} \psi_{tt}(x, t) - \psi_{xx}(x, t) = 0 \\ \chi_{tt}(x, t) - \chi_{xx}(x, t) = 0 \\ \psi(0, t) = 0 = \psi(1, t) \\ \chi(0, t) = 0 = \chi(1, t). \end{cases} \tag{2.8}$$

Definition 2.1. The *normal modes of oscillation* of frequency ω of system (2.8) are periodic solutions of the form:

$$\Psi(x, t) = \begin{pmatrix} \psi(x, t) \\ \chi(x, t) \end{pmatrix} = \widehat{\Psi}_\omega(x)\cos(\omega t + \eta), \tag{2.9}$$

with $\omega \in \mathbb{R}$, $\eta \in [0, 2\pi[$ and $\widehat{\Psi}_\omega : [0, 1] \to \mathbb{R}^2$, called eigenfunction associated to ω.

Proposition 2.1. *The frequencies of the normal modes of oscillation* (2.9) *are a countable family* $\{\omega_n^0 : n \in \mathbb{N}_+\}$ *of degeneracy* $\deg(\omega_n^0) = 2$*, with*

$$\omega_n^0 = n\pi. \tag{2.10}$$

Proof. The equations in ψ and in χ are decoupled. Let us seek solutions of system (2.8) of the form

$$\Psi(x,t) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \cos(\omega t + \eta).$$

Then, $f : [0,1] \to \mathbb{R}$ has to satisfy

$$\begin{cases} f''(x) + \omega^2 f(x) = 0 \\ f(0) = 0 \\ f(1) = 0. \end{cases} \tag{2.11}$$

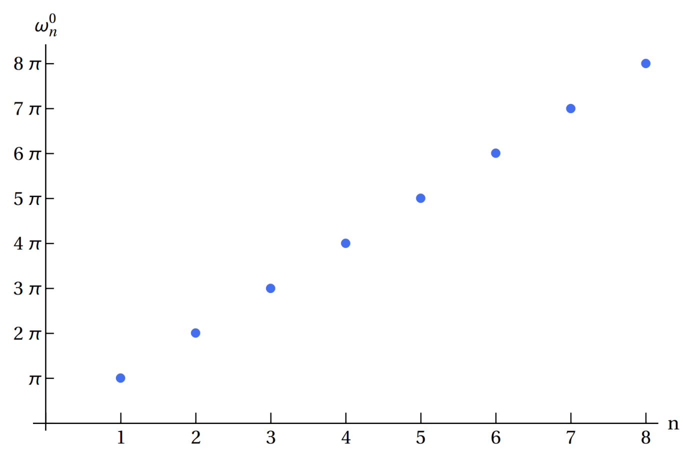
The first equation of system (2.11) has solution $f(x) = a\cos(\omega x) + b\sin(\omega x)$, with $a, b, \omega \in \mathbb{R}$; the boundary conditions give $a = 0$ and $\sin(\omega) = 0$, which implies solutions (2.10). Analogously, $g : [0,1] \to \mathbb{R}$ has to satisfy

$$\begin{cases} g''(x) + \omega^2 g(x) = 0 \\ g(0) = 0 \\ g(1) = 0 \end{cases} \tag{2.12}$$

which yields solution (2.10) as well. Therefore the frequency ω_n^0 has degeneracy two, for every n. □

The frequencies (2.10) are proportional to n, which labels the wave number. Hence, the vibrating string has a linear dispersion relation (Figure 2.2).

Figure 2.2: Dispersion relation of the string: ω_n^0.



Let us now compute the associated eigenfunctions.

Proposition 2.2. *For every n, the eigenfunction associated to ω_n^0 is*

$$\widehat{\Psi}_{\omega_n^0}(x) = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \sin(\omega_n^0 x), \tag{2.13}$$

with $a_n, b_n \in \mathbb{R}$.

Proof. System (2.11) implies $f_n(x) = a_n \sin(\omega_n^0 x)$ and system (2.12) implies $g_n(x) = b_n \sin(\omega_n^0 x)$ with $a_n, b_n \in \mathbb{R}$, for every n. □

Since ω_n^0 is degenerate, for every n, to each frequency is associated a plane of normal modes $\begin{pmatrix} a_n \\ b_n \end{pmatrix}$, with a_n, b_n fixed by the initial conditions.

2.5.2 Spectrum of the string with pendulum

Let us now consider the coupled system. The presence of the pendulum hanging from the string modifies the spectrum of frequencies and breaks the degeneracy; such spectrum will consist of "infinity-plus-one" eigenvalues. For simplicity, let us assume the suspension point of the pendulum to be lying halfway on the string, i.e. $x_1 = \frac{1}{2}$. For $\beta \neq 0$, the equations of motion, with the boundary conditions, are

$$\begin{cases} \ddot{\phi}(t) + \psi_{tt}(x_1, t) + \alpha^2 \phi(t) = 0 \\ \psi_{tt}(x,t) + \beta\big(\psi_{tt}(x,t) + \ddot{\phi}(t)\big)\delta(x - x_1) - \psi_{xx}(x,t) = 0 \\ \chi_{tt}(x,t) + \beta \chi_{tt}(x,t)\delta(x - x_1) - \chi_{xx}(x,t) = 0 \\ \psi(0,t) = 0 = \psi(1,t) \\ \chi(0,t) = 0 = \chi(1,t). \end{cases} \tag{2.14}$$

Definition 2.2. The *normal modes of oscillation* of frequency ω of system (2.14) are periodic solutions of the form:

$$\Phi(x,t) = \begin{pmatrix} \phi(t) \\ \psi(x,t) \\ \chi(x,t) \end{pmatrix} = \widehat{\Phi}_\omega(x) \cos(\omega t + \eta), \tag{2.15}$$

with $\omega \in \mathbb{R}$, $\eta \in [0, 2\pi[$ and $\widehat{\Phi}_\omega : [0,1] \to \mathbb{R}^3$, called eigenfunction associated to ω.

Let us consider first the general case in which the parameter α is not a multiple of 2π. For these special values of the parameter, in fact, one frequency of the coupled system coincides with the one of the pendulum alone, leading to a different dynamics.

Proposition 2.3. *For every $\beta > 0$ and $\alpha \neq 2\pi r$ with $r \in \mathbb{N}_+$, the frequencies of the normal modes of oscillation* (2.15) *are three countable families and a special frequency: $\{\omega_{2j}^{\alpha,\beta} : j \in \mathbb{N}_+\}$, $\{\omega_{2j-1}^{\alpha,\beta} : j \in \mathbb{N}_+\}$, $\{\omega_{\chi,2j-1}^{\beta} : j \in \mathbb{N}_+\}$ and $\widehat{\omega}^{\alpha,\beta}$, with the following properties. Let s be an index, dependent on α, such that*

$$s(\alpha) = \left\lceil \frac{\alpha}{2\pi} \right\rceil, \tag{2.16}$$

where $\lceil \cdot \rceil$ indicates the ceiling function: $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$. Then, *1.* $\omega_{2j}^{\alpha,\beta} = \omega_{2j}^{0}$

*1*a) $\deg(\omega_{2j}^{\alpha,\beta}) = 2$,

2. $\omega_{2j-2}^{0} < \omega_{2j-1}^{\alpha,\beta} < \omega_{2j-1}^{0}$ *for* $j < s$

*2*a) $|\omega_{2j-1}^{\alpha,\beta} - \omega_{2j-1}^{0}| < |\omega_{2j+1}^{\alpha,\beta} - \omega_{2j+1}^{0}|$,

3. $\omega_{2j-1}^{0} < \omega_{2j-1}^{\alpha,\beta} < \omega_{2j}^{0}$ *for* $j > s$

*3*a) $\omega_{2j-1}^{\alpha,\beta} - \omega_{2j-1}^{0} > \omega_{2j+1}^{\alpha,\beta} - \omega_{2j+1}^{0}$,

4. $\omega_{2s-1}^{0} < \omega_{2s-1}^{\alpha,\beta} < \omega_{2s}^{0}$ *for* $(2s-2)\pi < \alpha < (2s-1)\pi$

*4*a) $\omega_{2s-1}^{\alpha,\beta} - \omega_{2s-1}^{0} > \omega_{2s+1}^{\alpha,\beta} - \omega_{2s+1}^{0}$,

5. $\omega_{2s-2}^{0} < \widehat{\omega}^{\alpha,\beta} < \omega_{2s-1}^{0}$ *for* $(2s-2)\pi < \alpha < (2s-1)\pi$,

6. $\omega_{2s-2}^{0} < \omega_{2s-1}^{\alpha,\beta} < \omega_{2s-1}^{0}$ *for* $(2s-1)\pi \leq \alpha < (2s)\pi$

*6*a) $|\omega_{2s-1}^{\alpha,\beta} - \omega_{2s-1}^{0}| > |\omega_{2s-2}^{\alpha,\beta} - \omega_{2j-2}^{0}|$,

7. $\omega_{2s-1}^{0} < \widehat{\omega}^{\alpha,\beta} < \omega_{2s}^{0}$ *for* $(2s-1)\pi \leq \alpha < (2s)\pi$,

8. $\omega_{2j-2}^{0} < \omega_{\chi,2j-1}^{\beta} < \omega_{2j-1}^{0}$

*8*a) $|\omega_{\chi,2j-1}^{\alpha,\beta} - \omega_{\chi,2j-1}^{0}| < |\omega_{\chi,2j+1}^{\alpha,\beta} - \omega_{\chi,2j+1}^{0}|$.

Moreover, asymptotically

$$\omega_{2j-1}^{\alpha,\beta} \xrightarrow[j\to\infty]{} \omega_{2j-1}^{0}$$
$$\omega_{\chi,2j-1}^{\alpha,\beta} \xrightarrow[j\to\infty]{} \omega_{2j-2}^{0},$$

and for $\beta \to 0$

$$\omega_{n}^{\alpha,\beta} \xrightarrow[\beta\to 0]{} \omega_{n}^{0}$$
$$\widehat{\omega}^{\alpha,\beta} \xrightarrow[\beta\to 0]{} \alpha$$
$$\omega_{\chi,n}^{\beta} \xrightarrow[\beta\to 0]{} \omega_{n}^{0}.$$

Proof. Let us seek solutions of system (2.14) of the form

$$\Phi(x,t) = \begin{pmatrix} A \\ f(x) \\ g(x) \end{pmatrix} \cos(\omega t + \eta),$$

with $A \in \mathbb{R}$ and $f, g : [0,1] \to \mathbb{R}$. In order to handle the Dirac delta function, we determine independently the solutions on the left-hand side, f_-, g_-, and on the right-hand side, f_+, g_+, of the discontinuity x_1 and impose the interface conditions. Then, $f(x) = f_-(x)\Theta(x_1 - x) + f_+(x)\Theta(x - x_1)$ has to satisfy

$$\begin{cases} (\alpha^2 - \omega^2)A - \omega^2 f(x_1) = 0 \\ f''(x) + \omega^2 f(x) = -\omega^2\beta(f(x) + A)\delta(x - x_1) \\ f_-(0) = 0 \\ f_+(1) = 0 \\ f_-(x_1) = f_+(x_1) \\ f'_+(x_1) - f'_-(x_1) = -\omega^2\beta(f(x_1) + A) \end{cases} \tag{2.17}$$

and $g(x) = g_-(x)\Theta(x_1 - x) + g_+(x)\Theta(x - x_1)$ has to satisfy

$$\begin{cases} g''(x) + \omega^2 g(x) = -\beta\omega^2 g(x)\delta(x - x_1) \\ g_-(0) = 0 \\ g_+(1) = 0 \\ g_-(x_1) = g_+(x_1) \\ g'_+(x_1) - g'_-(x_1) = -\beta\omega^2 g(x_1). \end{cases} \tag{2.18}$$

Let us start considering the first system. The second equation of (2.17) has solution $f_\pm(x) = a_\pm \cos(\omega x) + b_\pm \sin(\omega x)$, $a_\pm, b_\pm, \omega \in \mathbb{R}$; the other five equations correspond to the condition

$$\det \begin{pmatrix} \alpha^2 - \omega^2 & -\omega^2\cos(\omega x_1) & -\omega^2\sin(\omega x_1) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\omega) & \sin(\omega) \\ 0 & \cos(\omega x_1) & \sin(\omega x_1) & -\cos(\omega x_1) & -\sin(\omega x_1) \\ \beta\alpha^2 & \omega\sin(\omega x_1) & -\omega\cos(\omega x_1) & -\omega\sin(\omega x_1) & \omega\cos(\omega x_1) \end{pmatrix} = 0,$$

where in the last raw it has already been substituted $\omega^2(f(x_1) + A)$ with $\alpha^2 A$, from the first equation in (2.17). The vanishing of this determinant, for $x_1 = \frac{1}{2}$, leads to the characteristic equation

$$F(\omega) := -\omega \sin\left(\frac{\omega}{2}\right)\left[(\alpha^2 - \omega^2)2\cos\left(\frac{\omega}{2}\right) - \alpha^2\beta\omega\sin\left(\frac{\omega}{2}\right)\right] = 0. \tag{2.19}$$

It has two families of solutions, which are the roots of

$$F_1(\omega) := \sin\left(\frac{\omega}{2}\right) = 0, \tag{2.20a}$$

$$F_2(\omega) := (\alpha^2 - \omega^2)2\cos\left(\frac{\omega}{2}\right) - \alpha^2\beta\omega\sin\left(\frac{\omega}{2}\right) = 0, \tag{2.20b}$$

and can be written in the form

$$\omega_n^{\alpha,\beta} = \omega_n^0 + \delta_n^{\alpha,\beta} \tag{2.21a}$$

$$\widehat{\omega}^{\alpha,\beta} = \alpha + \widehat{\delta}^{\alpha,\beta} \tag{2.21b}$$

with $\delta_{2j-1}^{\alpha,\beta}$ and $\widehat{\delta}^{\alpha,\beta}$ satisfying, respectively,

$$\left[\alpha^2-\left((2j-1)\pi+\delta_{2j-1}^{\alpha,\beta}\right)^2\right]2\sin\left(\frac{\delta_{2j-1}^{\alpha,\beta}}{2}\right)+\alpha^2\beta\left((2j-1)\pi+\delta_{2j-1}^{\alpha,\beta}\right)\cos\left(\frac{\delta_{2j-1}^{\alpha,\beta}}{2}\right)=0 \tag{2.22}$$

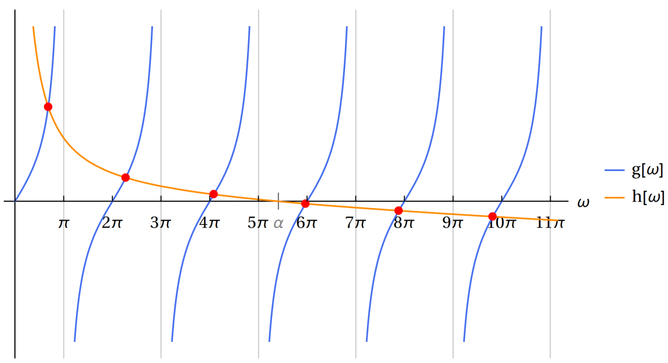
and

$$\left(2\alpha+\widehat{\delta}^{\alpha,\beta}\right)2\widehat{\delta}^{\alpha,\beta}\cos\left(\frac{\alpha+\widehat{\delta}^{\alpha,\beta}}{2}\right)+\alpha^2\beta\left(\alpha+\widehat{\delta}^{\alpha,\beta}\right)\sin\left(\frac{\alpha+\widehat{\delta}^{\alpha,\beta}}{2}\right)=0. \tag{2.23}$$

- Equation (2.20a) has roots $\omega_{2j}^{\alpha,\beta}=(2j)\pi$, $j=1,2,\dots$, which coincide with the even roots of the string without pendulum (this proves the first part of 1.).

Equation (2.20b) can not be solved analytically, nonetheless the existence of its roots can be determined through continuity and monotony arguments. Graphically, the roots of (2.20b) can be visualized as intersection points between $g(\omega):=\tan\left(\frac{\omega}{2}\right)$ and $h(\omega):=\frac{2}{\alpha^2\beta}\frac{\alpha^2-\omega^2}{\omega}$ (Figure 2.3). Let us define the index s such that $\omega_{2s-2}^{\alpha,\beta}<\alpha\leq\omega_{2s}^{\alpha,\beta}$, from which (2.16).

Figure 2.3: Plot $g(\omega),h(\omega)$ with $\alpha=17$, $\beta=0.5$; $s=3$.



- For $j<s$, in every interval $[(2j-2)\pi,(2j-1)\pi]$: $F_2((2j-2)\pi)=(\alpha^2-(2j-2)^2\pi^2)2(-1)^{j-1}$ is positive for every odd j and negative for every even j, and never vanishing for any j; vice versa, $F_2((2j-1)\pi)=-\alpha^2\beta(2j-1)\pi(-1)^{j-1}$ is negative for every odd j and positive for every even j, and never vanishing for any j. Moreover, F_2 is monotone, since $F_2'(\omega)=-4\omega\cos\left(\frac{\omega}{2}\right)-(\alpha^2-\omega^2)\sin\left(\frac{\omega}{2}\right)-\alpha^2\beta\sin\left(\frac{\omega}{2}\right)-\alpha^2\beta\frac{\omega}{2}\cos\left(\frac{\omega}{2}\right)$, and $F_2'<0$ for j odd and $F_2'>0$ for j even. Bolzano's theorem ensures that there exists a unique root of F_2 in $](2j-2)\pi,(2j-1)\pi[$, $j=1,2,\dots,s-1$. It can be written in the form (2.21a) with $\delta_{2j-1}^{\beta}<0$ and $|\delta_{2j-1}^{\beta}|<\pi$. Moreover, since $g(\omega)$ is increasing and 2π-periodic, and $h(\omega)$ is decreasing and positive for $j<s$, then $|\delta_{2j-1}^{\alpha,\beta}|<|\delta_{2j+1}^{\alpha,\beta}|$ for every $j<s$. This proves 2.

- For $j > s$, in every interval $[(2j-1)\pi, (2j)\pi]$:
  $F_2((2j-1)\pi) = -\alpha^2\beta(2j-1)\pi(-1)^{j-1}$ is negative for any odd j and positive for any even j, and never vanishing for any j; vice versa, $F_2((2j)\pi) = (\alpha^2-(2j)^2\pi^2)2(-1)^j$ is positive for any odd j and negative for any even j, and never vanishing for any j. Moreover, F_2 is monotone, since $F_2' < 0$ for j even and $F_2' > 0$ for j odd. Bolzano's theorem ensures that there exists a unique root of F_2 in $](2j-1)\pi, (2j)\pi[$, $j = s+1, s+2, \dots$. It can be written in the form (2.21a) with $\delta^{\beta}_{2j-1} > 0$ and $|\delta^{\beta}_{2j-1}| < \pi$. Moreover, since $g(\omega)$ is increasing and 2π-periodic, and $h(\omega)$ is decreasing and negative for $j > s$, then $\delta^{\alpha,\beta}_{2j-1} > \delta^{\alpha,\beta}_{2j+1}$ for every $j > s$. This proves 3.
- For $j = s$, if $\alpha \in](2s-2)\pi, (2s-1)\pi]$, in $[(2s-2)\pi, (2s-1)\pi]$ and in $[(2s-1)\pi, (2s)\pi]$:
  $F_2((2s-2)\pi) = (\alpha^2 - (2s-2)^2\pi^2)2(-1)^{s-1}$ is positive if s is odd and negative if s is even, and never vanishing; vice versa, $F_2((2s-1)\pi) = -\alpha^2\beta(2s-1)\pi(-1)^{s-1}$ is negative if s is odd and positive if s is even, and never vanishing. Moreover, F_2 is monotone in $[(2s-2)\pi, (2s-1)\pi]$, therefore Bolzano's theorem ensures that there exists a unique root of F_2 in $](2s-2)\pi, (2s-1)\pi[$. It can be labelled $\widehat{\omega}^{\alpha,\beta}$, of the form (2.21b), with $\widehat{\delta}^{\alpha,\beta} < 0$ and $|\widehat{\delta}^{\alpha,\beta}| < \pi$. Analogously, in $](2s-1)\pi, (2s)\pi[$ there is a unique root of F_2 of the form (2.21a) with $\delta^{\alpha,\beta}_{2s-1} > 0$ and $\delta^{\alpha,\beta}_{2s-1} < \pi$. This proves 4. and 5.
- For $j = s$, if $\alpha \in [(2s-1)\pi, (2s)\pi[$, in $[(2s-2)\pi, (2s-1)\pi]$ and in $[(2s-1)\pi, (2s)\pi]$:
  $F_2((2s-2)\pi) = (\alpha^2 - (2s-2)^2\pi^2)2(-1)^{s-1}$ is positive if s is odd and negative if s is even, and never vanishing; vice versa, $F_2((2s-1)\pi) = -\alpha^2\beta(2s-1)\pi(-1)^{s-1}$ is negative if s is odd and positive if s is even, and never vanishing. Moreover, F_2 is monotone in $[(2s-2)\pi, (2s-1)\pi]$, therefore Bolzano's theorem ensures that there exists a unique root of F_2 in $](2s-2)\pi, (2s-1)\pi[$. It can be written in the form (2.21a), with $\delta^{\alpha,\beta}_{2s-1} < 0$ and $|\delta^{\alpha,\beta}_{2s-1}| < \pi$. Analogously, in $](2s-1)\pi, (2s)\pi[$ there is a unique root of F_2, labelled $\widehat{\omega}^{\alpha,\beta}$, of the form (2.21b) with $\widehat{\delta}^{\alpha,\beta} > 0$ and $\widehat{\delta}^{\alpha,\beta} < \pi$. This proves 6. and 7. Note that if $\alpha = (2s-1)\pi$, since there is no unique choice, either one of the two roots can be labelled as $\widehat{\omega}^{\alpha,\beta}$.

An estimate of the modification of the odd spectrum of the string due to the presence of the pendulum, $\delta^{\alpha,\beta}_{2j-1}$, is given by the expansion in $\delta^{\alpha,\beta}_{2j-1}$ of (2.22). Keeping only the linear terms, one gets

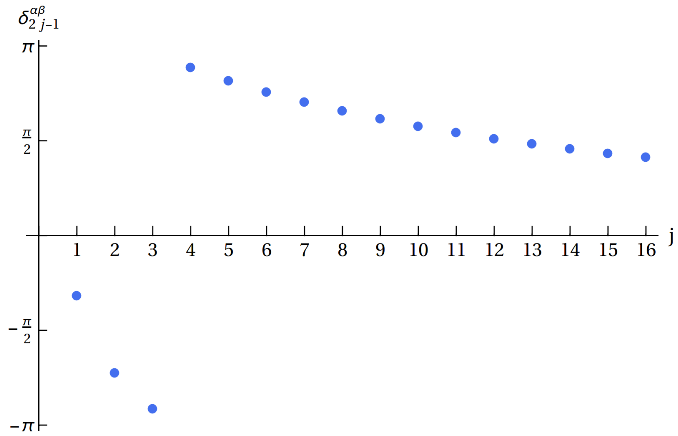
$$\delta^{\alpha,\beta(1)}_{2j-1} = \frac{(2j-1)\pi\alpha^2\beta}{-\alpha^2 - \alpha^2\beta + (2j-1)^2\pi^2}. \tag{2.24}$$

In particular, $\delta^{\alpha,\beta(1)}_{2j-1} \xrightarrow[j\to\infty]{} 0$, therefore $\omega^{\alpha,\beta}_{2j-1}$ tend asymptotically to the odd roots of the string without pendulum (Figure 2.4).

Let us consider now system (2.18). The first equation of (2.18) has solution $g_\pm(x) = a_{\chi,\pm}\cos(\omega x) + b_{\chi,\pm}\sin(\omega x)$, $a_{\chi,\pm}, b_{\chi,\pm}, \omega \in \mathbb{R}$; the other four equations correspond to the condition

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(\omega) & \sin(\omega) \\ 0 & \sin(\omega x_1) & -\cos(\omega x_1) & -\sin(\omega x_1) \\ 0 & -\omega\cos(\omega x_1) + \beta\omega^2\sin(\omega x_1) & -\omega\sin(\omega x_1) & \omega\cos(\omega x_1) \end{pmatrix} = 0.$$

Figure 2.4: $\delta_{2j-1}^{\alpha,\beta}$, with $\alpha = 17$, $\beta = 0.5$; $s = 3$.



For $x_1 = \frac{1}{2}$, it gives the following characteristic equation

$$G(\omega) := -\omega \sin\left(\frac{\omega}{2}\right)\left[2\cos\left(\frac{\omega}{2}\right) - \beta\omega\sin\left(\frac{\omega}{2}\right)\right] = 0. \tag{2.25}$$

It has two families of solutions, which are the roots of

$$G_1(\omega) := \sin\left(\frac{\omega}{2}\right) = 0, \tag{2.26a}$$

$$G_2(\omega) := 2\cos\left(\frac{\omega}{2}\right) - \beta\omega\sin\left(\frac{\omega}{2}\right) = 0, \tag{2.26b}$$

and can be written as

$$\omega_{\chi,n}^{\beta} = \omega_n^0 + \delta_{\chi,n}^{\beta}, \quad n = 1, 2, 3, \ldots \tag{2.27}$$

with $\delta_{\chi,2j-1}^{\beta}$ satisfying

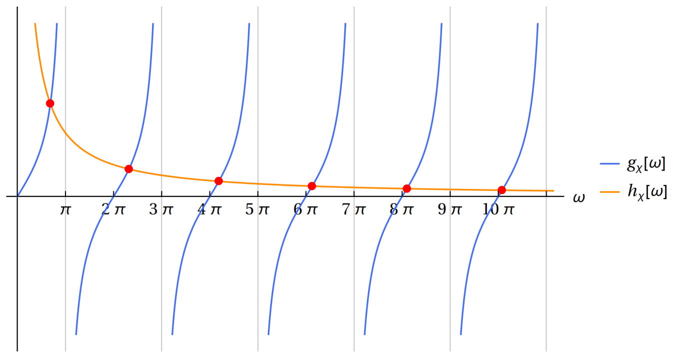
$$2\sin\left(\frac{\delta_{\chi,2j-1}^{\beta}}{2}\right) + \beta\left[(2j-1)\pi + \delta_{\chi,2j-1}^{\beta}\right]\cos\left(\frac{\delta_{\chi,2j-1}^{\beta}}{2}\right) = 0. \tag{2.28}$$

- Equation (2.26a) has roots $\omega_{\chi,2j}^{\beta} = (2j)\pi$, $j = 1, 2, \ldots$, which coincide with the even roots of the string without pendulum, hence $\deg(\omega_{2j}^{\alpha,\beta}) = 2$, which proves $1a$).

Equation (2.26b) can not be solved analytically, nonetheless the existence of its roots can be determined through continuity and monotony arguments. Graphically, the roots of (2.26b) can be visualized as intersection points between $g_\chi(\omega) := \tan\left(\frac{\omega}{2}\right)$ and $h_\chi(\omega) := \frac{2}{\beta\omega}$ (Figure 2.5).

- In every interval $[(2j-2)\pi, (2j-1)\pi]$:
  $F_2((2j-2)\pi) = 2(-1)^{j-1}$ is positive for any odd j and negative for any even j, and

Figure 2.5: Plot $g_\chi(\omega), h_\chi(\omega)$ with $\beta = 0.5$.



never vanishing; vice versa, $F_2((2j-1)\pi) = -\beta(2j-1)\pi(-1)^{j-1}$ is negative for any odd j and positive for any even j, and never vanishing. Moreover, F_2 is monotone in $[(2j-2)\pi,(2j-1)\pi]$, since $F_2'(\omega) = -\sin\left(\frac{\omega}{2}\right)(1+\beta) - \beta\frac{\omega}{2}\cos\left(\frac{\omega}{2}\right)$, and $F_2' < 0$ for j odd and $F_2' > 0$ for j even. Bolzano's theorem ensures that there exists a unique root of F_2 in $](2j-2)\pi,(2j-1)\pi[$, $j = 1,2,\ldots$. It can be written in the form (2.27) with $\delta^\beta_{\chi,2j-1} < 0$ and $|\delta^\beta_{\chi,2j-1}| < \pi$. Moreover, since $g_\chi(\omega)$ is increasing and 2π-periodic, and $h_\chi(\omega)$ is decreasing and positive, then $|\delta^{\alpha,\beta}_{2j-1}| < |\delta^{\alpha,\beta}_{2j+1}|$ for every $j \in \mathbb{N}_+$. This proves 8.

An estimate of the modification of the odd spectrum of the string due to the dishomogeneity in x_1 is given by the expansion of (2.28) in $\delta^\beta_{\chi,2j-1}$. Keeping only the linear terms, one gets:

$$\delta^{\beta(1)}_{\chi,2j-1} = \frac{4-\beta(2j-1)\pi^2}{\beta(2j)\pi} \xrightarrow[j\to\infty]{} -\pi.$$

Finally, in order to determine the behaviour of the frequencies for small values of the parameter β, let us prove the following result:

Lemma 2.1. *For every j, the displacement $\delta^{\alpha,\beta}_{2j-1}$ of the eigenvalue* (2.21a) *from ω^0_{2j-1}, the displacement $\delta^\beta_{\chi,2j-1}$ of the eigenvalue* (2.27) *from ω^0_{2j-1} and the displacement $\widehat{\delta}^{\alpha,\beta}$ of the eigenvalue* (2.21b) *from α depend differentially on the parameter β, if $\alpha \neq 2\pi r$ with $r \in \mathbb{N}_+$.*

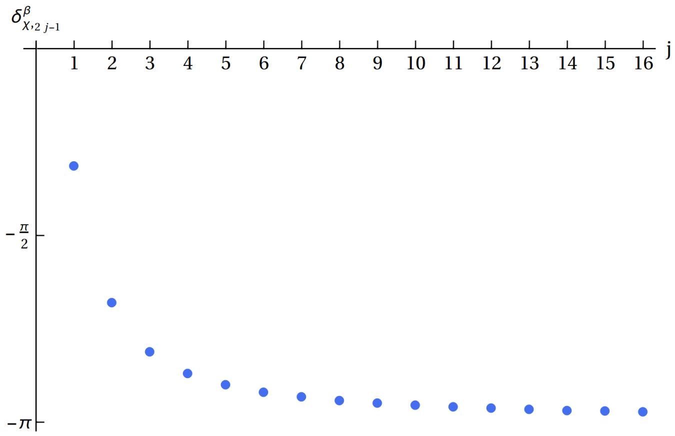
Proof. For $n = 2j$, $\delta^{\alpha,\beta}_{2j} = 0$, $j = 1,2,\ldots$, for any value of β. For $n = 2j-1$, $\delta^{\alpha,\beta}_{2j-1}$ are the roots of the function g_n, defined as

$$g_n(\beta,\delta^{\alpha,\beta}) := \left(\alpha^2 - (\omega^0_n + \delta^{\alpha,\beta}_n)^2\right) 2\sin\left(\frac{\delta^{\alpha,\beta}_n}{2}\right) + \alpha^2\beta(\omega^0_n + \delta^{\alpha,\beta}_n)\cos\left(\frac{\delta^{\alpha,\beta}_n}{2}\right).$$

For every $n = 1,2,\ldots$, it satisfies

$$g_n(0,0) = 0, \quad \partial_\delta g_n(0,0) = (\alpha^2 - (\omega^0_n)^2) \neq 0;$$

Figure 2.6: $\delta^{\beta}_{\chi,2j-1}$ with $\beta = 0.5$; $s = 3$.



Analogously, $\delta^{\beta}_{\chi,2j} = 0$, for any value of β, and $\delta^{\beta}_{\chi,2j-1}$ are the roots of the function $g_{\chi,n}$, defined as

$$g_{\chi,n}(\beta, \delta^{\beta}_{\chi}) := 2\sin\left(\frac{\delta^{\beta}_{\chi,n}}{2}\right) + \beta(\omega^0_n + \delta^{\beta}_{\chi,n})\cos\left(\frac{\delta^{\beta}_{\chi,n}}{2}\right).$$

For every $n = 1, 2, \ldots$, it satisfies

$$g_{\chi,n}(0,0) = 0, \quad \partial_\delta g_{\chi,n}(0,0) = 1 \neq 0;$$

Dini's theorem ensures that in a neighbour of $\beta = 0$, for $\alpha \neq 2\pi r$ with $r \in \mathbb{N}_+$, $\delta^{\alpha,\beta}$ and δ^{β}_{χ} are differentiable functions of β implicitly defined by $g\left(\beta, \delta^{\alpha,\beta}(\beta)\right) = 0$ and $g_\chi\big(\beta, \delta^{\beta}_{\chi}(\beta)\big) = 0$ respectively. Similarly, $\widehat{\delta}^{\alpha,\beta}$ is root of the function h, defined as

$$h(\beta, \widehat{\delta}^{\alpha,\beta}) := \left(2\alpha + \widehat{\delta}^{\alpha,\beta}\right) 2\widehat{\delta}^{\alpha,\beta}\cos\left(\frac{\alpha + \widehat{\delta}^{\alpha,\beta}}{2}\right) + \alpha^2\beta\left(\alpha + \widehat{\delta}^{\alpha,\beta}\right)\cos\left(\frac{\alpha + \widehat{\delta}^{\alpha,\beta}}{2}\right),$$

which satisfies

$$h(0,0) = 0, \quad \partial_{\widehat{\delta}} h(0,0) = 4\alpha\cos\left(\frac{\alpha}{2}\right) \neq 0 \text{ for } \alpha \neq 2\pi r;$$

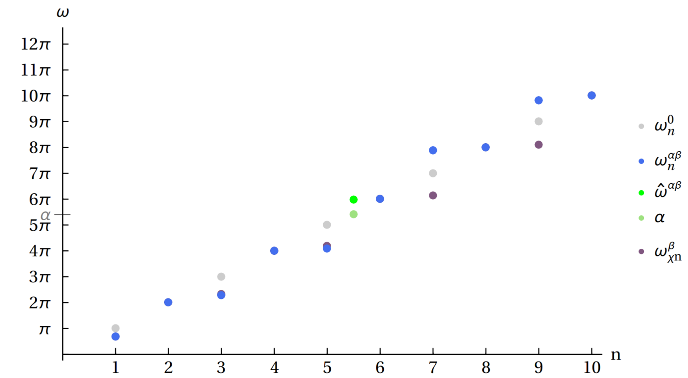
Dini's theorem ensures that in a neighbour of $\beta = 0$, for $\alpha \neq 2\pi r$ with $r \in \mathbb{N}_+$, $\widehat{\delta}^{\alpha,\beta}$ is a differentiable function of β implicitly defined by $h(\beta, \widehat{\delta}^{\alpha,\beta}(\beta)) = 0$. $\square$

Therefore, for $\beta \to 0$ the frequencies of the coupled system tend to those of the unperturbed string and to the proper frequency of the pendulum. $\square$

The presence of the pendulum modifies the odd frequencies of the string, while it leaves the even ones unchanged since the pendulum hangs from a node of the string. In Figure 2.7 is plotted the dispersion relation compared with the one without pendulum;

to the frequency $\widehat{\omega}^{\alpha,\beta}$ it has been conventionally assigned the index $2s-\frac{1}{2}$. In general, $\omega_n^{\alpha,\beta} \leq \omega_n^0$ for $n \leq 2s-2$ and $\omega_n^{\alpha,\beta} \geq \omega_n^0$ for $n \geq 2s$. In particular, the frequencies closer in value to the proper frequency of the pendulum are more strongly modified with respect to the others. $\omega_{\chi,n}^{\alpha,\beta} \leq \omega_n^0$ for every n, and $\widehat{\omega}^{\alpha,\beta} < \alpha$ for $(2s-2)\pi < \alpha < (2s-1)\pi$ while $\widehat{\omega}^{\alpha,\beta} > \alpha$ for $(2s-1)\pi < \alpha < (2s)\pi$. For small values of the parameter β, which physically correspond to a light pendulum with respect to the string, the spectrum of the unperturbed string is slightly modified by the presence of the hanging pendulum and $\widehat{\omega}^{\alpha,\beta}$ is close in value to the proper frequency of the pendulum.

Figure 2.7: Dispersion relation for the string with pendulum: $\omega_n^{\alpha,\beta}$, $\omega_{\chi,n}^{\beta}$ and $\widehat{\omega}^{\alpha,\beta}$ with $\alpha = 17$ and $\beta = 0.5$; $s = 3$.



Let us now compute the eigenfunctions of the normal modes.

Proposition 2.4. *For every j, for $\alpha \neq 2\pi r$ with $r \in \mathbb{N}_+$, the eigenfunctions are the following:*

- *the eigenfunction associated to $\omega_{2j}^{\alpha,\beta}$ is*

$$\widehat{\Phi}_{\omega_{2j}^{\alpha,\beta}}(x) = \begin{pmatrix} 0 \\ a_{2j} \\ b_{2j} \end{pmatrix} \sin\left(\omega_{2j}^{\alpha,\beta} x\right), \tag{2.29}$$

- *the eigenfunction associated to $\omega_{2j-1}^{\alpha,\beta}$ is*

$$\widehat{\Phi}_{\omega_{2j-1}^{\alpha,\beta}}(x) = a_{2j-1} \begin{pmatrix} \frac{(\omega_{2j-1}^{\alpha,\beta})^2}{\alpha^2-(\omega_{2j-1}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{2j-1}^{\alpha,\beta}}{2}\right) \\ \sin\left(\omega_{2j-1}^{\alpha,\beta} x\right)\Theta(x_1-x) + \sin\left(\omega_{2j-1}^{\alpha,\beta}(1-x)\right)\Theta(x-x_1) \\ 0 \end{pmatrix}, \tag{2.30}$$

- *the eigenfunction associated to $\widehat{\omega}^{\alpha,\beta}$ is*

$$\widehat{\Phi}_{\widehat{\omega}^{\alpha,\beta}}(x) = \hat{a}_{2j-1} \begin{pmatrix} \frac{(\widehat{\omega}^{\alpha,\beta})^2}{\alpha^2-(\widehat{\omega}^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}^{\alpha,\beta}}{2}\right) \\ \sin\left(\widehat{\omega}^{\alpha,\beta} x\right)\Theta(x_1-x) + \sin\left(\widehat{\omega}^{\alpha,\beta}(1-x)\right)\Theta(x-x_1) \\ 0 \end{pmatrix}, \tag{2.31}$$

- *the eigenfunction associated to* $\omega^{\beta}_{\chi,2j-1}$ *is*

$$\widehat{\Phi}_{\omega^{\beta}_{\chi,2j-1}}(x) = b_{\chi,2j-1} \begin{pmatrix} 0 \\ 0 \\ \sin\left(\omega^{\beta}_{\chi,2j-1} x\right)\Theta(x_1-x) + \sin\left(\omega^{\beta}_{\chi,2j-1}(1-x)\right)\Theta(x-x_1) \end{pmatrix}, \tag{2.32}$$

with $a_n, b_n, \hat{a}, b_{\chi,n} \in \mathbb{R}$.

Proof. From (2.17) and (2.18), in order to determine $A, a_\pm, b_\pm$ and $a_{\chi,\pm}, b_{\chi,\pm}$, the following eigenvector problems have to be satisfied:

$$\begin{pmatrix} \alpha^2-\omega^2 & -\omega^2\cos\left(\frac{\omega}{2}\right) & -\omega^2\sin\left(\frac{\omega}{2}\right) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\omega) & \sin(\omega) \\ 0 & \cos\left(\frac{\omega}{2}\right) & \sin\left(\frac{\omega}{2}\right) & -\cos\left(\frac{\omega}{2}\right) & -\sin\left(\frac{\omega}{2}\right) \\ \beta\alpha^2 & \omega\sin\left(\frac{\omega}{2}\right) & -\omega\cos\left(\frac{\omega}{2}\right) & -\omega\sin\left(\frac{\omega}{2}\right) & \omega\cos\left(\frac{\omega}{2}\right) \end{pmatrix} \begin{pmatrix} A \\ a_- \\ b_- \\ a_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with ω solution of (2.19), and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(\omega) & \sin(\omega) \\ 0 & \sin\left(\frac{\omega}{2}\right) & -\cos\left(\frac{\omega}{2}\right) & -\sin\left(\frac{\omega}{2}\right) \\ 0 & -\omega\cos\left(\frac{\omega}{2}\right)+\beta\omega^2\sin\left(\frac{\omega}{2}\right) & -\omega\sin\left(\frac{\omega}{2}\right) & \omega\cos\left(\frac{\omega}{2}\right) \end{pmatrix} \begin{pmatrix} a_{\chi,-} \\ b_{\chi,-} \\ a_{\chi,+} \\ b_{\chi,+} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with ω solution of (2.25).

- For $\omega^{\alpha,\beta}_{2j}$:

$$\begin{cases} A = 0 \\ a_- = 0 \\ a_+ = 0 \\ b_+ = b_- \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = 0 \\ b_{\chi,+} = b_{\chi,-} \end{cases}$$

which yields eigenfunction (2.29).

- For $\omega^{\alpha,\beta}_{2j-1}$:

$$\begin{cases} A = \frac{(\omega^{\alpha,\beta}_{2j-1})^2}{\alpha^2-(\omega^{\alpha,\beta}_{2j-1})^2}\sin\left(\frac{\omega^{\alpha,\beta}_{2j-1}}{2}\right) b_- \\ a_- = 0 \\ a_+ = \sin\left(\omega^{\alpha,\beta}_{2j-1}\right) b_- \\ b_+ = -\cos\left(\omega^{\alpha,\beta}_{2j-1}\right) b_-, \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = 0 \\ b_{\chi,-} = 0 \\ b_{\chi,+} = 0 \end{cases}$$

which yields eigenfunction (2.30).

- For $\widehat{\omega}^{\alpha,\beta}$:

$$\begin{cases} A = \frac{(\widehat{\omega}^{\alpha,\beta})^2}{\alpha^2-(\widehat{\omega}^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}^{\alpha,\beta}}{2}\right) b_- \\ a_- = 0 \\ a_+ = -\sin\left(\widehat{\omega}^{\alpha,\beta}\right) b_- \\ b_+ = -\cos\left(\widehat{\omega}^{\alpha,\beta}\right) b_-, \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = 0 \\ b_{\chi,-} = 0 \\ b_{\chi,+} = 0 \end{cases}$$

which yields eigenfunction (2.31).

- For $\omega^{\beta}_{\chi,2j-1}$:

$$\begin{cases} A = 0 \\ a_- = 0 \\ a_+ = 0 \\ b_- = 0 \\ b_+ = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = \sin\left(\omega^{\beta}_{\chi,2j-1}\right) b_{\chi,-} \\ b_{\chi,+} = -\cos\left(\omega^{\beta}_{\chi,2j-1}\right) b_{\chi,-} \end{cases}$$

which yields eigenfunction (2.32).

□

The normal modes found have the following physical interpretation. The family associated to the even frequencies leaves the pendulum still, hanging from a node of the wave described by the string and every combination of normal modes is still a periodic solution, due to the degeneracy of $\omega^{\alpha,\beta}_{2j}$. In the normal modes associated to $\omega^{\alpha,\beta}_{2j-1}$ and to $\widehat{\omega}^{\alpha,\beta}$ the pendulum and the transverse horizontal component of the string are in motion. Lastly, for $\omega^{\alpha,\beta}_{\chi,2j-1}$ the string oscillates only vertically and the pendulum is at rest. Note that for the odd frequencies the string swings with different amplitudes on the two sides with respect to the suspension point of the pendulum.

Let us now determine the normal modes of oscillation in the case in which the parameter α is a multiple of 2π. For such values, $\widehat{\omega}^{\alpha,\beta}$ coincides with an even frequency of the string, hence this configuration can be thought as a kind of resonance. Fixed $\alpha = 2\pi s$, the behaviour of the system is identical to the case considered above for all frequencies except for $\omega^{\alpha,\beta}_{2s}$, for which the pendulum instead of being at rest oscillates with the string.

Proposition 2.5. *For every $\beta > 0$ and $\alpha = 2\pi r$ with $r \in \mathbb{N}_+$, the frequencies of the normal modes of oscillation* (2.15) *are three countable families: $\{\omega^{\alpha,\beta}_{2j} : j \in \mathbb{N}_+\}$, $\{\omega^{\alpha,\beta}_{2j-1} : j \in \mathbb{N}_+\}$, $\{\omega^{\beta}_{\chi,2j-1} : j \in \mathbb{N}_+\}$, with the following properties. Let s be such that $s = \frac{\alpha}{2\pi}$, then*

1. $\omega^{\alpha,\beta}_{2j} = \omega^0_{2j}$

   *1*a) $\deg(\omega^{\alpha,\beta}_{2j}) = 2$ *if $j \neq s$*

   *1*b) $\deg(\omega^{\alpha,\beta}_{2j}) = 3$ *if $j = s$,*

2. $\omega^0_{2j-2} < \omega^{\alpha,\beta}_{2j-1} < \omega^0_{2j-1}$ *for $j \leq s$*

2a) $|\omega^{\alpha,\beta}_{2j-1} - \omega^0_{2j-1}| < |\omega^{\alpha,\beta}_{2j+1} - \omega^0_{2j+1}|$,

3. $\omega^0_{2j-1} < \omega^{\alpha,\beta}_{2j-1} < \omega^0_{2j}$ *for* $j > s$

3a) $\omega^{\alpha,\beta}_{2j-1} - \omega^0_{2j-1} > \omega^{\alpha,\beta}_{2j+1} - \omega^0_{2j+1}$,

4. $\omega^0_{2j-2} < \omega^{\beta}_{\chi,2j-1} < \omega^0_{2j-1}$

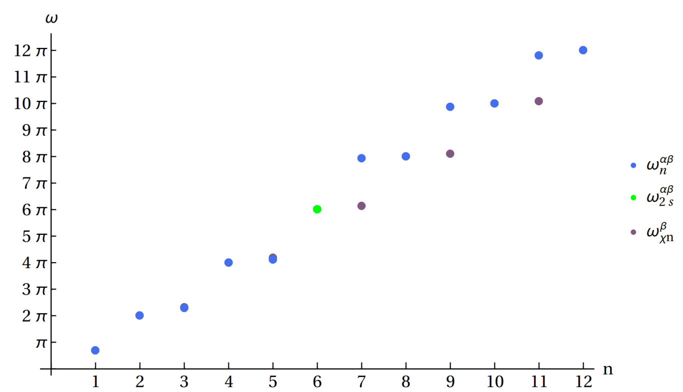
4a) $|\omega^{\alpha,\beta}_{\chi,2j-1} - \omega^0_{\chi,2j-1}| < |\omega^{\alpha,\beta}_{\chi,2j+1} - \omega^0_{\chi,2j+1}|$.

Moreover, asymptotically

$$\omega^{\alpha,\beta}_{2j-1} \xrightarrow[j\to\infty]{} \omega^0_{2j-1}$$
$$\omega^{\alpha,\beta}_{\chi,2j-1} \xrightarrow[j\to\infty]{} \omega^0_{2j-2}.$$

Proof. The proof is analogous to the one of Proposition 2.3. In this case, $\widehat{\omega}^{\alpha,\beta}$ coincides with $\omega^{\alpha,\beta}_{2s}$, therefore $\deg(\omega^{\alpha,\beta}_{2s}) = 3$. □

Figure 2.8: Dispersion relation for the string with pendulum with double root: $\omega^{\alpha,\beta}_n$ and $\omega^{\beta}_{\chi,n}$ with $\alpha = 6\pi$ and $\beta = 0.5$.



Finally, let us compute the corresponding eigenfunctions.

Proposition 2.6. *For every j, for $\alpha = 2\pi s$ with $s \in \mathbb{N}_+$, the eigenfunctions are the following:*

- *the eigenfunction associated to $\omega^{\alpha,\beta}_{2j}$, with $j \neq s$, is* (2.29),

- *the eigenfunction associated to $\omega^{\alpha,\beta}_{2s}$ is*

$$\widehat{\Phi}_{\omega_{2s}^{\alpha,\beta}}(x) = \begin{pmatrix} \frac{(-1)^s}{\beta(2s)\pi}(a-a') \\ a\sin\left(\omega_{2s}^{\alpha,\beta}x\right)\Theta(x_1-x) + a'\sin\left(\omega_{2s}^{\alpha,\beta}x\right)\Theta(x-x_1) \\ b\sin\left(\omega_{2s}^{\alpha,\beta}x\right) \end{pmatrix}, \tag{2.33}$$

- *the eigenfunction associated to* $\omega_{2j-1}^{\alpha,\beta}$ *is* (2.30),

- *the eigenfunction associated to* $\omega_{\chi,2j-1}^{\beta}$ *is* (2.32),

with $a, a', b \in \mathbb{R}$.

Proof. The proofs for $\omega_{2j}^{\alpha,\beta}$ with $j \neq s$, for $\omega_{2j-1}^{\alpha,\beta}$ and for $\omega_{\chi,2j-1}^{\beta}$ are analogous to those of Proposition 2.4.

- For $\omega_{2s}^{\alpha,\beta} = (2s)\pi$:

$$\begin{cases} A = \frac{(-1)^s}{\beta(2s)\pi}(b_- - b_+) \\ a_- = 0 \\ a_+ = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = 0 \\ b_{\chi,+} = b_{\chi,-} \end{cases}$$

which yields eigenfunction (2.33).

□

For $\omega_{2s}^{\alpha,\beta}$ the system exhibits a particular dynamics, in which both the pendulum and the string oscillate. In particular, for certain initial conditions, that is for $a = 0$ or $a' = 0$, one half of the string can stand still, or swings only vertically, while the other half and the pendulum are in motion.

Chapter 3

Pendulum hanging from a string with damping

The system considered up to now is conservative since the total energy is preserved during the motion. In this chapter we introduce a dissipative contribution to the string and repeat the analysis made in Chapter 2. This time, in order to determine the frequencies of oscillation, we will assume a weak damping and make an expansion of the eigenvalues in terms of the damping coefficient.

As shown through simple examples in Chapter 1, friction plays a crucial role in the selection of the normal modes of oscillation and, ultimately, in the evolution of the dynamics of the system. Hence, we first present the main damping models, in order to justify the choice of friction made.

3.1 Damping model

In analogy with n-dimensional systems, a damped infinite-dimensional system can be written in the form

$$u_{tt} + \Gamma u_t + K u = 0, \tag{3.1}$$

where $u \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, and Γ and K are positive self-adjoint operators.

Different choices of damping lead, in general, to different dynamics. A classification of the types of friction can be done distinguishing between internal and external damping, depending whether the dissipation of energy is due to elements within the system or not. A physical interpretation of various models can be found e.g. in [23].

External damping. It accounts for the energy loss due to the interaction of the system with external elements. The two main contributions are the following:

- Coulomb (or frictional) damping, for which the kinetic energy of the system is transformed into heat as the system slides against another dry surface. Such friction is proportional to the normal force and it opposes to the motion, but does not depend on the velocity of the system once in motion:

$$F_c = -\mu N.$$

- Viscous damping, which is due to the medium in which the system is immersed. It produces a force proportional to the relative velocity of the system with respect to the medium:

$$F_v = -\gamma u_t.$$

Internal damping. It accounts for the dissipation within the system, which might be due, for example, to the friction between particles and depends therefore on their relative displacements and speeds. In the case of a string or a beam it is connected to the rate of change of bending, and depends in general on the elastic properties of the material. Two main models should be mentioned, which differ from one another for the dependence of the damping rate on the frequency (see [22], [13]):

- Kelvin-Voigt (or visco-elastic) model, which introduces a damping term proportional to the linear operator which describes the elastic force acting on the velocity of the system:

$$F_{kv} = -\gamma K u_t.$$

- Structural (or hysteretic) damping, which produces a damping rate proportional to the frequency of oscillation of the system. The damping force is

$$F_s = -\gamma K^{\frac{1}{2}} u_t.$$

Let us compare the damping rates, namely the real part of the eigenvalues, for the different models. Let ω_j^2 be the eigenvalues of K with ϕ_j the associated eigenvectors, and let us seek solutions of the form $u_j = e^{\lambda_j t}\phi_j$.

- Viscous damping: the equation of motion is

$$u_{tt} + \gamma u_t + K u = 0.$$

  Inserting the expression of u_j and solving the characteristic equation, one gets the following eigenvalues:

$$\lambda_j = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_j^2}}{2}.$$

  The real part of λ_j for weak damping is constant, therefore viscous damping yields a uniform damping rate.

- Kelvin-Voigt damping: the equation of motion is

$$u_{tt} + \gamma K u_t + K u = 0,$$

whose eigenvalues are

$$\lambda_j = \frac{-\gamma\omega_j^2 \pm \sqrt{\gamma^2\omega_j^4 - 4\omega_j^2}}{2}.$$

The damping rate is proportional to the frequency squared, hence there is a dichotomy of decay rates, in particular higher frequencies decay much faster than the lower ones.

- Structural damping: the equation of motion is

$$u_{tt} + \gamma K^{\frac{1}{2}} u_t + Ku = 0,$$

whose eigenvalues are

$$\lambda_j = \frac{-\gamma\omega_j \pm \sqrt{\gamma^2\omega_j^2 - 4\omega_j^2}}{2}.$$

The real part decays proportionally to the frequency.

Experimental studies (see e.g. [12]) show that for various materials and in a wide range of frequencies, the damping rate depends linearly on the frequency of oscillation. The most fitting model would therefore be structural damping, nonetheless it might present some difficulties since, in general, if K is a differential operator, $K^{\frac{1}{2}}$ might not be such. For a detailed description of frequency-dependent damping refer to [22]. However, when the damping is light the predictions of the Kelvin-Voigt model and the structural one are comparable, therefore in this dissertation we adopt the visco-elastic damping model. For a string the elastic operator K is $-c^2\frac{\partial^2}{\partial x^2}$, therefore the dissipating force is given by

$$\gamma\frac{\partial^3 u}{\partial t \partial x^2}.$$

3.2 Equations of motion

Consider the system studied in Chapter 2, consisting of a pendulum hanging from a flexible and elastic string with extremities fixed. Let us introduce a dissipating contribution to the string, adding to the equations of motion of the string (2.6) a damping force proportional to ψ_{txx} and χ_{txx}, namely a Kelvin-Voigt damping term. Let ν be the dimensionless parameter that accounts for internal damping. In our model we neglected the friction acting on the pendulum, such as, for example, viscous damping of the air, since it is reasonable to assume that, in general, its contribution is secondary with respect to the internal friction of the string. The equations of motion become

$$\begin{cases} \beta\big(\ddot{\phi}(t) + \psi_{tt}(x_1,t) + \alpha^2\phi(t)\big) = 0 \\ \psi_{tt}(x,t) + \beta\big(\psi_{tt}(x,t) + \ddot{\phi}(t)\big)\delta(x-x_1) - \psi_{xx}(x,t) - \nu\psi_{txx}(x,t) = 0 \\ \chi_{tt}(x,t) + \beta\chi_{tt}(x,t)\delta(x-x_1) - \chi_{xx}(x,t) - \nu\chi_{txx}(x,t) = 0, \end{cases} \tag{3.2}$$

where

$$\alpha^2 = \frac{g\rho\Lambda^2}{l\tau}, \qquad \beta = \frac{m}{\rho\Lambda},$$

and with the boundary conditions

$$\begin{cases} \psi(0,t) = 0 = \psi(1,t) \\ \chi(0,t) = 0 = \chi(1,t) \end{cases}$$

for every $t \in \mathbb{R}$.

3.3 Damped normal modes

The equations of motion (3.2) are a linear system, therefore every solution, called damped small oscillation, is a linear combination of solutions. In particular, we look for harmonic solutions with an exponentially decreasing amplitude.

3.3.1 Spectrum of the damped vibrating string

Let us consider first the decoupled system. The pendulum does not have a dissipative component, therefore, when decoupled, its motion is unchanged. The eigenvalues of the string are, instead, modified.

For $\beta = 0$, the equations of motion of the damped string, and its boundary conditions, are

$$\begin{cases} \psi_{tt}(x,t) - \psi_{xx}(x,t) - \nu\psi_{txx}(x,t) = 0 \\ \chi_{tt}(x,t) - \chi_{xx}(x,t) - \nu\chi_{txx}(x,t) = 0 \\ \psi(0,t) = 0 = \psi(1,t) \\ \chi(0,t) = 0 = \chi(1,t). \end{cases} \tag{3.3}$$

Definition 3.1. The *damped normal modes* of eigenvalue λ of system (3.3) are solutions of the form:

$$\Psi(x,t) = \begin{pmatrix} \psi(x,t) \\ \chi(x,t) \end{pmatrix} = \operatorname{Re}\left[\widehat{\Psi}_\lambda(x)e^{\lambda t}\right], \tag{3.4}$$

with $\lambda \in \mathbb{C}$, $\operatorname{Re}[\lambda] < 0$ and $\widehat{\Psi}_\lambda : [0,1] \to \mathbb{C}^2$, called eigenfunction associated to λ.

The real part of the eigenvalues, $\mu := \operatorname{Re}[\lambda]$ is called decay rate, while its imaginary part, $\omega := \operatorname{Im}[\lambda]$ is the oscillation frequency. Let us denote "$\pm\sqrt{x+iy}$" the pair of complex squared roots of the complex number $x + iy$.

Proposition 3.1. *The eigenvalues of the damped normal modes* (3.4) *are a countable family* $\{\lambda_n^{0\pm} : n \in \mathbb{N}_+\}$ *of degeneracy* $\deg(\lambda_n^{0\pm}) = 2\ \forall n$*, with*

$$\lambda_n^{0\pm} = \frac{-n^2\pi^2\nu \pm n\pi\sqrt{n^2\pi^2\nu^2 - 4}}{2}. \tag{3.5}$$

Proof. The equations in ψ and in χ are decoupled. Let us seek solutions of system (3.3) of the form

$$\Psi(x,t) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} e^{\lambda t},$$

of which only the real part has to be considered. Then, $f : [0, 1] \to \mathbb{C}$ has to satisfy

$$\begin{cases} f''(x) - \xi^2 f(x) = 0 \\ f(0) = 0 \\ f(1) = 0 \end{cases} \tag{3.6}$$

where $\xi \in \mathbb{C}$, with $\mathrm{Im}[\xi] > 0$, is defined by the dispersion relation:

$$\xi(\lambda) := \pm \frac{\lambda}{\sqrt{1 + \nu\lambda}}. \tag{3.7}$$

The first equation of system (3.6) has solution $f(x) = ae^{\xi x} + be^{-\xi x}$, with $a, b, \xi \in \mathbb{C}$; the boundary conditions give $a + b = 0$ and $\sinh(\xi) = 0$, which implies

$$\xi_n^0 = in\pi. \tag{3.8}$$

From inversion of (3.7):

$$\lambda^\pm = \frac{\xi^2\nu \pm \sqrt{\xi^4\nu^2 + 4\xi^2}}{2}. \tag{3.9}$$

Substituting (3.8) into (3.9), one gets (3.5).
Analogously, $g : [0, 1] \to \mathbb{C}$ has to satisfy

$$\begin{cases} g''(x) - \xi^2 g(x) = 0 \\ g(0) = 0 \\ g(1) = 0 \end{cases} \tag{3.10}$$

which yields solution (3.5) as well. Therefore the eigenvalues $\lambda_n^{0\pm}$ have degeneracy two, for every n. □

The spectrum presents therefore the following properties (Figure 3.1):

- For $n \geq n_m := \left\lceil \frac{2}{\pi\nu} \right\rceil$, the eigenvalues $\lambda_n^{0\pm}$ are real. Therefore, for every $\nu \neq 0$, there is a finite number of oscillating solutions only. Moreover, λ_n^{0+} tends to $-1/\nu$, while λ_n^{0+} tends to $-\infty$.

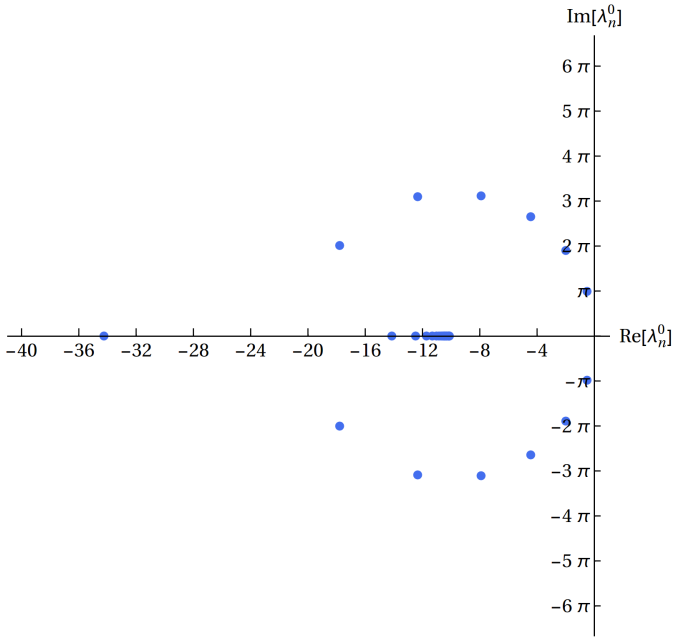
- For $n < n_m$, the eigenvalues λ_n^0 lie on a circle centred at $(-1/\nu, 0)$ of radius $1/\nu$ in the plane $(\mathrm{Re}[\lambda], \mathrm{Im}[\lambda])$. The frequencies of oscillation, $\mathrm{Im}[\lambda^{0\pm}] = n\pi\sqrt{4 - n^2\pi^2\nu^2}/2$, are lower with respect to those of the string without damping, $\omega_n^0 = n\pi$.

- λ_r^{0+} and λ_r^{0-} coincide for values of ν such that $\frac{2}{\pi\nu} = r \in \mathbb{N}^+$, which corresponds to the critical damping.

In Figure 3.2 are reported the real and imaginary parts of the dispersion relation. For weak damping, namely for $n < n_m$, the real part of the eigenvalues is negative and monotonically decreasing, and there is a quadratic dependence of the decay rate on the frequency of the undamped system, $\omega_n^0 = n\pi$.

Let us now compute the corresponding eigenfunctions.

Figure 3.1: Spectrum of the damped string: λ_n^0 with $\nu = 0.1$.



Proposition 3.2. *For every n, the eigenfunction associated to $\lambda_n^{0\pm}$ is*

$$\widehat{\Psi}_{\lambda_n^{0\pm}}(x) = \begin{pmatrix} a_n^{\pm} \\ b_n^{\pm} \end{pmatrix} \sin(n\pi x), \tag{3.11}$$

with $a_n^{\pm}, b_n^{\pm} \in \mathbb{C}$.

Proof. System (3.6) implies $f_n(x) = a_n \sinh(\xi_n^0 x)$ and system (3.10) implies $g_n(x) = b_n \sinh(\xi_n^0 x)$ with $a_n, b_n \in \mathbb{C}$, for every n. Substituting (3.8) within, one gets (3.11). □

Since $\lambda_n^{0\pm}$ are degenerate, for every n, to each eigenvalue is associated a plane of normal modes $\begin{pmatrix} a_n^{\pm} \\ b_n^{\pm} \end{pmatrix}$, with $a_n^{\pm}, b_n^{\pm}$ fixed by the initial conditions.

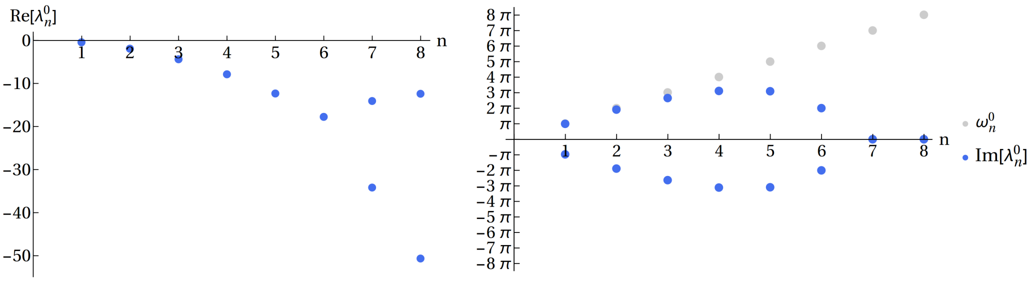
3.3.2 Spectrum of the damped string with pendulum

Let us now consider the coupled system, assuming $x_1 = \frac{1}{2}$. For $\beta \neq 0$, the equations of motion in presence of damping, with the boundary conditions, are

$$\begin{cases} \ddot{\phi}(t) + \psi_{tt}(x_1, t) + \alpha^2 \phi(t) = 0 \\ \psi_{tt}(x, t) + \beta\big(\psi_{tt}(x, t) + \ddot{\phi}(t)\big)\delta(x - x_1) - \psi_{xx}(x, t) - \nu\psi_{txx}(x, t) = 0 \\ \chi_{tt}(x, t) + \beta\chi_{tt}(x, t)\delta(x - x_1) - \chi_{xx}(x, t) - \nu\chi_{txx}(x, t) = 0 \\ \psi(0, t) = 0 = \psi(1, t) \\ \chi(0, t) = 0 = \chi(1, t). \end{cases} \tag{3.12}$$

Figure 3.2: Dispersion relation for the damped string: λ_n^0 with $\nu = 0.1$; $n_m = 7$.

(a) $\mathrm{Re}[\lambda_n^0]$ **(b)** $\mathrm{Im}[\lambda_n^0]$



Definition 3.2. The *damped normal modes* of eigenvalue λ of system (3.12) are solutions of the form:

$$\Phi(x,t) = \begin{pmatrix} \phi(t) \\ \psi(x,t) \\ \chi(x,t) \end{pmatrix} = \mathrm{Re}\left[\widehat{\Psi}_\lambda(x) e^{\lambda t}\right], \tag{3.13}$$

with $\lambda \in \mathbb{C}$, $\mathrm{Re}[\lambda] < 0$ and $\widehat{\Psi}_\lambda : [0,1] \to \mathbb{C}^3$, called eigenfunction associated to λ.

Proposition 3.3. *The eigenvalues of the damped normal modes* (3.13) *are three countable families and two special ones:* $\{\lambda_{2j}^{\alpha,\beta\pm} : j \in \mathbb{N}_+\}$, $\{\lambda_{2j-1}^{\alpha,\beta\pm} : j \in \mathbb{N}_+\}$, $\{\lambda_{\chi,2j-1}^{\beta\pm} : j \in \mathbb{N}_+\}$ *and* $\widehat{\lambda}^{\alpha,\beta\pm}$, *with the following properties.*

1. $\lambda_{2j}^{\alpha,\beta\pm} = \lambda_{2j}^{0\pm}$

   *1*a*)* $\deg(\lambda_{2j}^{\alpha,\beta\pm}) = 2$,

2. $\lambda_{2j-1}^{\alpha,\beta\pm} = \pm i\omega_{2j-1}^{\alpha,\beta} + \epsilon^{\alpha,\beta}(\omega_{2j-1}^{\alpha,\beta})\nu + \mathcal{O}(\nu^2)$,

3. $\widehat{\lambda}^{\alpha,\beta\pm} = \pm i\widehat{\omega}^{\alpha,\beta} + \epsilon^{\alpha,\beta}(\widehat{\omega}^{\alpha,\beta})\nu + \mathcal{O}(\nu^2)$,

4. $\lambda_{\chi,2j-1}^{\beta\pm} = \frac{-(\omega_{\chi,2j-1}^{\beta})^2\nu \pm \sqrt{(\omega_{\chi,2j-1}^{\beta})^4\nu^2 - 4(\omega_{\chi,2j-1}^{\beta})^2}}{2}$.

with $\epsilon^{\alpha,\beta}(\omega) = -\frac{\frac{1}{2}\alpha^2\beta\omega^3 + 2\omega(\alpha^2-\omega^2) + \frac{2\omega(\alpha^2-\omega^2)^2}{\alpha^2\beta}}{8\omega + \alpha^2\beta\omega + \frac{4(\alpha^2-\omega^2)}{\omega} + \frac{4(\alpha^2-\omega^2)^2}{\alpha^2\beta\omega}} < 0.$

Proof. Let us seek solutions of system (3.12) of the form

$$\Phi(x,t) = \begin{pmatrix} A \\ f(x) \\ g(x) \end{pmatrix} e^{\lambda t},$$

with $A \in \mathbb{C}$ and $f, g : [0,1] \to \mathbb{C}$, and the real part intended. In order to handle the Dirac delta function, we determine independently the solutions on the left-hand side, f_-, g_-, and on the right-hand side, f_+, g_+, of the discontinuity x_1 and impose the interface conditions. Then, defined ξ as

$$\xi(\lambda) := \pm\frac{\lambda}{\sqrt{1+\nu\lambda}}, \tag{3.14}$$

$f(x) = f_-(x)\Theta(x_1 - x) + f_+(x)\Theta(x - x_1)$ has to satisfy

$$\begin{cases} A(\alpha^2+\lambda^2) + \lambda^2 f(x_1) = 0 \\ f''(x) - \xi^2 f(x) = \xi^2\beta(f(x)+A)\delta(x-x_1) \\ f_-(0) = 0 \\ f_+(1) = 0 \\ f_-(x_1) = f_+(x_1) \\ f'_+(x_1) - f'_-(x_1) = \xi^2\beta(f(x_1)+A) \end{cases} \tag{3.15}$$

and $g(x) = g_-(x)\Theta(x_1 - x) + g_+(x)\Theta(x - x_1)$ has to satisfy

$$\begin{cases} g''(x) - \xi^2 g(x) = \beta\xi^2 g(x)\delta(x-x_1) \\ g_-(0) = 0 \\ g_+(1) = 0 \\ g_-(x_1) = g_+(x_1) \\ g'_+(x_1) - g'_-(x_1) = \beta\xi^2 g(x_1). \end{cases} \tag{3.16}$$

Let us start considering the first system. The second equation of (3.15) has solution $f_\pm(x) = a_\pm e^{\xi x} + b_\pm e^{-\xi x}$, $a_\pm, b_\pm, \xi \in \mathbb{C}$; the other five equations correspond to the condition

$$\det\begin{pmatrix} \alpha^2+\lambda^2 & \lambda^2 e^{\xi x_1} & \lambda^2 e^{-\xi x_1} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\xi} & e^{-\xi} \\ 0 & e^{\xi x_1} & e^{-\xi x_1} & -e^{\xi x_1} & -e^{-\xi x_1} \\ \frac{\alpha^2\beta}{1+\nu\lambda} & -\xi e^{\xi x_1} & \xi e^{-\xi x_1} & \xi e^{\xi x_1} & -\xi e^{-\xi x_1} \end{pmatrix} = 0,$$

where in the last raw it has already been substituted $\lambda^2(f(x_1)+A)$ with $-\alpha^2 A$, from the first equation in (3.15). The vanishing of this determinant, for $x_1 = \frac{1}{2}$, leads to the characteristic equation

$$F(\lambda) := \beta\xi(\lambda)\sinh\left(\frac{\xi(\lambda)}{2}\right)\left[\left(\alpha^2+\lambda^2\right)2\cosh\left(\frac{\xi(\lambda)}{2}\right) + \alpha^2\beta\xi(\lambda)\sinh\left(\frac{\xi(\lambda)}{2}\right)\right] = 0. \tag{3.17}$$

It has two families of solutions, which are the roots of

$$F_1(\lambda) := \sinh\left(\frac{\xi(\lambda)}{2}\right) = 0, \tag{3.18a}$$

$$F_2(\lambda) := \left(\alpha^2+\lambda^2\right)2\cosh\left(\frac{\xi(\lambda)}{2}\right) + \alpha^2\beta\xi(\lambda)\sinh\left(\frac{\xi(\lambda)}{2}\right) = 0. \tag{3.18b}$$

- Equation (3.18a) has roots $\xi_{2j}^{\alpha,\beta}(\lambda) = i(2j)\pi$, $j = 1, 2, \ldots$, from which, inverting (3.14), $\lambda_{2j}^{\alpha,\beta} = \frac{-n^2\pi^2\nu \pm n\pi\sqrt{n^2\pi^2\nu^2-4}}{2}$, which coincide with the even roots of the damped string without pendulum (this proves the first part of 1.).

- In order to solve equation (3.18b), let us utilise Dini theorem to determine at the first order in ν the roots of $F_2(\nu,\lambda) = \left(\alpha^2+\lambda^2\right) 2\cosh\left(\frac{\lambda}{2\sqrt{1+\nu\lambda}}\right) + \alpha^2\beta\frac{\lambda}{\sqrt{1+\nu\lambda}}\sinh\left(\frac{\lambda}{2\sqrt{1+\nu\lambda}}\right) = 0$. Set $\lambda = i\omega$ for $\nu = 0$,

$$F_2(0,i\omega) = (\alpha^2-\omega^2)2\cos\left(\frac{\omega}{2}\right) - \alpha^2\beta\omega\sin\left(\frac{\omega}{2}\right) = 0$$
$$\partial_\lambda F_2(0,i\omega) = \frac{1}{2}i\cos\left(\frac{\omega}{2}\right)\left[8\omega+\alpha^2\beta\omega+\frac{4(\alpha^2-\omega^2)}{\omega}+\frac{4(\alpha^2-\omega^2)^2}{\alpha^2\beta\omega}\right] \neq 0,$$

  and
$$\partial_\lambda F_2(0,i\omega) = \frac{\omega}{2}i\cos\left(\frac{\omega}{2}\right)\left[\frac{1}{2}\alpha^2\beta\omega+2(\alpha^2-\omega^2)+\frac{2(\alpha^2-\omega^2)^2}{\alpha^2\beta}\right].$$

  Hence, set $\epsilon = -\frac{\partial_\nu F_2(0,i\omega)}{\partial_\lambda F_2(0,i\omega)}$, at the linear order in ν, $\lambda^{\alpha,\beta\pm}_{2j-1} = \pm i\omega^{\alpha,\beta}_{2j-1} + \epsilon^{\alpha,\beta}(\omega^{\alpha,\beta}_{2j-1})\nu + \mathcal{O}(\nu^2)$. Analogous proof holds for $\widehat{\lambda}^{\alpha,\beta\pm}$. This proves 2. and 3.

Let us consider now system (3.16). The first equation of (3.16) has solution $g_\pm(x) = a_{\chi,\pm}e^{\xi x} + b_{\chi,\pm}e^{-\xi x}$, $a_{\chi,\pm}, b_{\chi,\pm}, \omega \in \mathbb{C}$; the other four equations correspond to the condition

$$\det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{\xi} & e^{-\xi} \\ e^{\xi x_1} & e^{-\xi x_1} & -e^{\xi x_1} & -e^{-\xi x_1} \\ -\xi e^{\xi x_1}(1+\beta\xi) & \xi e^{-\xi x_1}(1-\beta\xi) & \xi e^{\xi x_1} & -\xi e^{-\xi x_1} \end{pmatrix} = 0.$$

For $x_1 = \frac{1}{2}$, it gives the following characteristic equation:

$$G(\lambda) := \xi(\lambda)\sinh\left(\frac{\xi(\lambda)}{2}\right)\left[2\cosh\left(\frac{\xi(\lambda)}{2}\right)+\beta\xi(\lambda)\sinh\left(\frac{\xi(\lambda)}{2}\right)\right] = 0. \tag{3.19}$$

It has two families of solutions which are the roots of

$$G_1(\lambda) := \sinh\left(\frac{\xi(\lambda)}{2}\right) = 0, \tag{3.20a}$$
$$G_2(\lambda) := 2\cosh\left(\frac{\xi(\lambda)}{2}\right)+\beta\xi(\lambda)\sinh\left(\frac{\xi(\lambda)}{2}\right) = 0. \tag{3.20b}$$

- Equation (3.20a) has roots $\xi^\beta_{\chi,2j}(\lambda) = i(2j)\pi$, $j = 1,2,\ldots$, from which $\lambda^\beta_{\chi,2j} = \lambda^0_{2j}$. Therefore, $\deg(\lambda^{\alpha,\beta}_{2j}) = 2\ \forall j\in\mathbb{N}_+$, which proves $1a$).

- Equation (3.20b) can not be solved analytically, nonetheless, noticing that ξ is purely imaginary, it can be written as $\xi = i\omega$, with $\omega\in\mathbb{R}$ satisfying

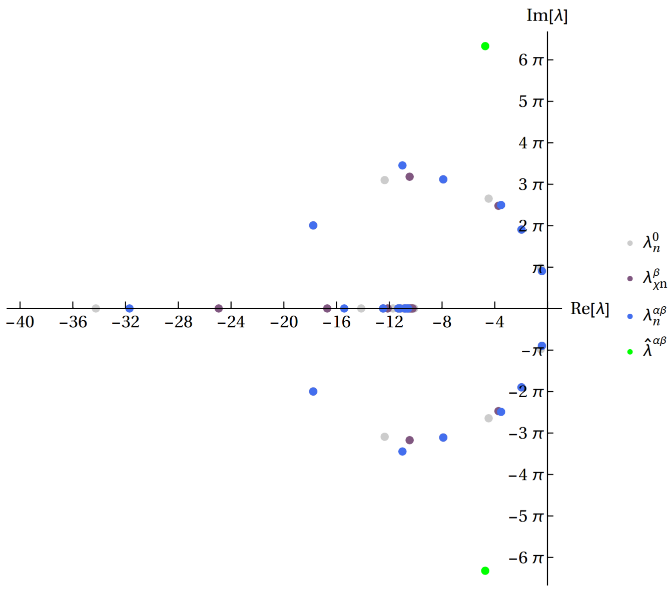
$$2\cos\left(\frac{\omega}{2}\right) - \beta\omega\sin\left(\frac{\omega}{2}\right) = 0.$$

  This equation corresponds to (2.26b), therefore its roots are $\omega^\beta_{\chi,2j-1}$. Hence, $\xi^\beta_{\chi,2j-1} = i\omega^\beta_{\chi,2j-1}$, which, from inversion of (3.14), yields 4.

□

For small values of the damping coefficient, at the linear order, the eigenvalues acquire a negative real part with respect to the frequencies of the undamped system. Figure 3.3 and Figure 3.4 show the spectrum of the system and the dispersion relation, respectively, compared with those of the unperturbed string. The even eigenvalues are unchanged, while the odd ones are modified by the presence of the pendulum. Note that while the eigenvalues $\lambda^{\beta}_{\chi,n}$ still lie on a circle, $\lambda^{\alpha,\beta}_n$ do not.

Figure 3.3: Spectrum of the damped string with pendulum: $\lambda^{\alpha,\beta}_n$, $\widehat{\lambda}^{\alpha,\beta}$, $\lambda^{\beta}_{\chi,n}$ with $\alpha = 17$, $\beta = 0.1$ and $\nu = 0.1$.



Remark 3.1. In presence of damping the eigenvalue $\widehat{\lambda}^{\alpha,\beta}$ does not ever coincide with an eigenvalue of the unperturbed string, unlike for the undamped case. In fact, the characteristic equation (3.17) does not admit double roots.

Remark 3.2. All the eigenvalues of the system have a non-vanishing real part, therefore, there are no invariant undamped subspaces in the phase space. This means that every small oscillation will be eventually damped out. Nonetheless, since we chose a model of friction which leads to decay rates proportional to the square of the frequency, we expect normal modes with high frequencies to vanish quickly and only a combination of a few damped normal modes with low frequencies to survive for longer.

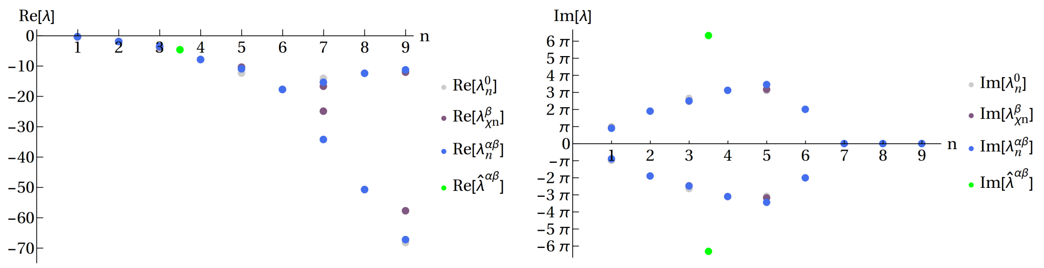
Finally, let us compute the associated eigenfunctions.

Proposition 3.4. *For every j, the eigenfunctions are the following:*

- *the eigenfunction associated to $\lambda^{\alpha,\beta\pm}_{2j}$ is*

Figure 3.4: Dispersion relation for the damped string with pendulum: $\lambda_n^{\alpha,\beta}, \widehat{\lambda}^{\alpha,\beta}, \lambda_{\chi,n}^{\beta}$ with $\alpha = 17$, $\beta = 0.1$ and $\nu = 0.1$. To the eigenvalue $\widehat{\lambda}^{\alpha,\beta}$ has been assigned the arbitrary index 3.5.

(a) Re[λ] **(b)** Im[λ]



$$\widehat{\Phi}_{\lambda_{2j}^{\alpha,\beta\pm}}(x) = \begin{pmatrix} 0 \\ a_{2j}^{\pm} \\ b_{2j}^{\pm} \end{pmatrix} \sinh(\xi(\lambda_{2j}^{\alpha,\beta})x), \tag{3.21}$$

- *the eigenfunction associated to* $\lambda_{2j-1}^{\alpha,\beta\pm}$ *is*

$$\widehat{\Phi}_{\lambda_{2j-1}^{\alpha,\beta\pm}}(x) = a_{2j-1}^{\pm} \left(\begin{array}{c} -\frac{(\lambda_{2j-1}^{\alpha,\beta})^2}{\alpha^2+(\lambda_{2j-1}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{2j-1}^{\alpha,\beta})}{2}\right) \\ \sinh\left(\xi(\lambda_{2j-1}^{\alpha,\beta})x\right) \Theta(x_1 - x) + \sinh\left(\xi(\lambda_{2j-1}^{\alpha,\beta})(1-x)\right) \Theta(x - x_1) \\ 0 \end{array} \right), \tag{3.22}$$

- *the eigenfunction associated to* $\widehat{\lambda}^{\alpha,\beta\pm}$ *is*

$$\widehat{\Phi}_{\widehat{\lambda}^{\alpha,\beta\pm}}(x) = \hat{a}^{\pm} \left(\begin{array}{c} -\frac{(\widehat{\lambda}^{\alpha,\beta})^2}{\alpha^2+(\widehat{\lambda}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\widehat{\lambda}^{\alpha,\beta})}{2}\right) \\ \sinh\left(\xi(\widehat{\lambda}^{\alpha,\beta})x\right) \Theta(x_1 - x) + \sinh\left(\xi(\widehat{\lambda}^{\alpha,\beta})(1-x)\right) \Theta(x - x_1) \\ 0 \end{array} \right), \tag{3.23}$$

- *the eigenfunction associated to* $\lambda_{\chi,2j-1}^{\beta\pm}$ *is*

$$\widehat{\Phi}_{\lambda_{\chi,2j-1}^{\beta\pm}}(x) = b_{\chi,2j-1}^{\pm} \left(\begin{array}{c} 0 \\ 0 \\ \sinh\left(\xi(\lambda_{\chi,2j-1}^{\beta})x\right) \Theta(x_1 - x) + \sinh\left(\xi(\lambda_{\chi,2j-1}^{\beta})(1-x)\right) \Theta(x - x_1) \end{array} \right), \tag{3.24}$$

with $a_n^{\pm}, b_n^{\pm}, \hat{a}^{\pm}, b_{\chi,n}^{\pm} \in \mathbb{C}$.

Proof. From (3.15) and (3.16), in order to determine $A, a_\pm, b_\pm$ and $a_{\chi,\pm}, b_{\chi,\pm}$, the following eigenvector problems have to be satisfied:

$$\begin{pmatrix} \alpha^2+\lambda^2 & \lambda^2 e^{\frac{\xi(\lambda)}{2}} & \lambda^2 e^{\frac{-\xi(\lambda)}{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\xi(\lambda)} & e^{-\xi(\lambda)} \\ 0 & e^{\frac{\xi(\lambda)}{2}} & e^{\frac{-\xi(\lambda)}{2}} & -e^{\frac{\xi(\lambda)}{2}} & -e^{\frac{-\xi(\lambda)}{2}} \\ \frac{\beta\alpha^2}{1+\nu\lambda} & -\xi(\lambda)e^{\frac{\xi(\lambda)}{2}} & \xi(\lambda)e^{\frac{-\xi(\lambda)}{2}} & \xi(\lambda)e^{\frac{\xi(\lambda)}{2}} & -\xi(\lambda)e^{-\frac{\xi(\lambda)}{2}} \end{pmatrix} \begin{pmatrix} A \\ a_- \\ b_- \\ a_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with λ solution of (3.17), and

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{\xi(\lambda)} & e^{-\xi(\lambda)} \\ e^{\frac{\xi(\lambda)}{2}} & e^{\frac{-\xi(\lambda)}{2}} & -e^{\frac{\xi(\lambda)}{2}} & -e^{\frac{-\xi(\lambda)}{2}} \\ \xi(\lambda)e^{\frac{\xi(\lambda)}{2}}(1+\beta\xi(\lambda)) & -\xi(\lambda)e^{\frac{-\xi(\lambda)}{2}}(1-\beta\xi(\lambda)) & \xi(\lambda)e^{\frac{\xi(\lambda)}{2}} & -\xi(\lambda)e^{\frac{-\xi(\lambda)}{2}} \end{pmatrix} \begin{pmatrix} a_{\chi,-} \\ b_{\chi,-} \\ a_{\chi,+} \\ b_{\chi,+} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with λ solution of (3.19).

- For $\lambda_{2j}^{\alpha,\beta}$:

$$\begin{cases} A = 0 \\ b_- = -a_- \\ b_+ = -a_+ \\ a_+ = a_- \end{cases} \quad \text{and} \quad \begin{cases} b_{\chi,-} = -a_{\chi,-} \\ b_{\chi,+} = -a_{\chi,+} \\ a_{\chi,+} = a_{\chi,-} \end{cases}$$

  which yields eigenfunction (3.21).

- For $\lambda_{2j-1}^{\alpha,\beta}$:

$$\begin{cases} A = -a_- \frac{(\lambda_{2j-1}^{\alpha,\beta})^2}{\alpha^2-(\lambda_{2j-1}^{\alpha,\beta})^2} \left(e^{\frac{\xi(\lambda_{2j-1}^{\alpha,\beta})}{2}} - e^{\frac{-\xi(\lambda_{2j-1}^{\alpha,\beta})}{2}} \right) \\ b_- = -a_- \\ b_+ = -a_+ e^{2\xi(\lambda_{2j-1}^{\alpha,\beta})} \\ a_+ = -a_- e^{-\xi(\lambda_{2j-1}^{\alpha,\beta})}, \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = 0 \\ b_{\chi,-} = 0 \\ b_{\chi,+} = 0 \end{cases}$$

  which yields eigenfunction (3.22).

- For $\widehat{\lambda}^{\alpha,\beta}$:

$$\begin{cases} A = -a_- \frac{(\widehat{\lambda}^{\alpha,\beta})^2}{\alpha^2-(\widehat{\lambda}^{\alpha,\beta})^2} \left(e^{\frac{\xi(\widehat{\lambda}^{\alpha,\beta})}{2}} - e^{\frac{-\xi(\widehat{\lambda}^{\alpha,\beta})}{2}} \right) \\ b_- = -a_- \\ b_+ = -a_+ e^{2\xi(\widehat{\lambda}^{\alpha,\beta})} \\ a_+ = -a_- e^{-\xi(\widehat{\lambda}^{\alpha,\beta})}, \end{cases} \quad \text{and} \quad \begin{cases} a_{\chi,-} = 0 \\ a_{\chi,+} = 0 \\ b_{\chi,-} = 0 \\ b_{\chi,+} = 0 \end{cases}$$

  which yields eigenfunction (3.23).

- For $\lambda^{\beta}_{\chi,2j-1}$:

$$\begin{cases} A = 0 \\ a_- = 0 \\ a_+ = 0 \\ b_- = 0 \\ b_+ = 0 \end{cases} \quad \text{and} \quad \begin{cases} b_- = -a_- \\ b_+ = -a_+ e^{2\xi(\lambda^{\beta}_{\chi,2j-1})} \\ a_+ = -a_- e^{-\xi(\lambda^{\beta}_{\chi,2j-1})} \end{cases}$$

which yields eigenfunction (3.24).

$\square$

Chapter 4

Two pendula hanging from a string

In this chapter we extend the study made so far to the case in which there are two pendula hanging from a string. The phenomenology in this configuration is ever richer; in particular, we shall analyse the mutual interaction between the two pendula through the string, which – under suitable values of the parameters – leads to beating phenomena. Nonetheless, in absence of friction synchronization can not occur, therefore we have to wait for the next chapter to study that mechanism.

The computation of the eigenvalues of the small oscillations is done here numerically, assuming the two pendula to be identical and fixing their suspension points at one-third and two-thirds along the string.

4.1 Description of the system

The model is the same as before, with the only difference that two pendula are hanging from the string. The system consists of a homogeneous flexible and elastic string with fixed extremities and two pendula hanging from points O_1 and O_2 of the string, respectively.

Let ρ be the linear density of the string, τ its tension and Λ its length; let then l_k be the length of the k-th pendulum and m_k its mass, with $k = 1, 2$.

Since in Chapter 2 it emerged that, in the regime of small oscillations, the vertical component of the string always decouples from the transverse horizontal one and the pendulum, we assume now, without much loss of generality, that the string is constrained to lay on a horizontal plane, and it is therefore not subjected to the action of its weight, unlike the pendula. Let ψ be the horizontal transverse displacement of a point of the string from the x-axis on the xz-plane. The cable is thus described now by the following embedding:

$$[0, \Lambda] \ni x \mapsto \begin{pmatrix} x \\ 0 \\ z(x,t) = \psi(x,t) \end{pmatrix} \in \mathbb{R}^3$$

with function $\psi : [0, \Lambda] \times \mathbb{R} \to \mathbb{R}$ such that $\psi(0,t) = 0 = \psi(\Lambda, t)$ $\forall t$.

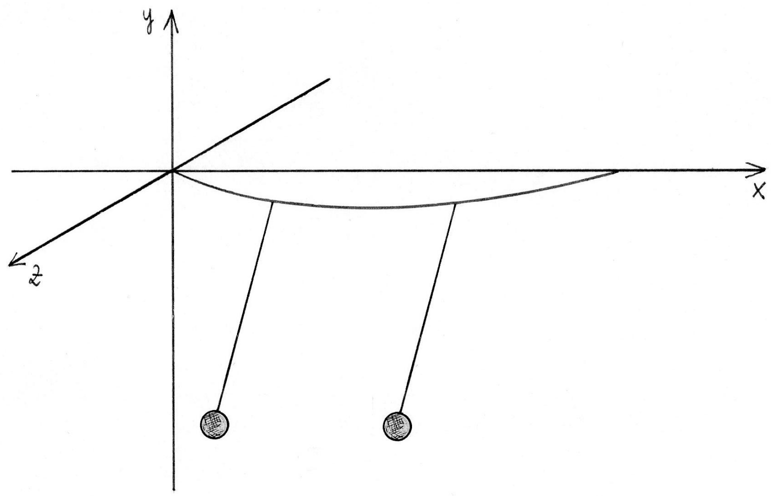
Let ϕ_k be the angular displacement of the k-th pendulum measured from the descendant y-axis on the yz-plane and let us denote with x_k the x-coordinate of the suspension point

O_k. The coordinates of the pendula are therefore

$$\begin{cases} x_{p_k} = x_k \\ y_{p_k}(t) = -l_k \cos\phi_k(t) \\ z_{p_k}(t) = \psi(x_k, t) + l_k \sin\phi_k(t), \end{cases}$$

with $k = 1, 2$.

Figure 4.1: Model: two pendula hanging from a string.



We initially consider the pendula having different masses and lengths and, for the sake of simplicity, constrain the study of the normal modes of oscillations to the case of two identical pendula.

4.2 Lagrangian

The system is described by a Lagrangian L, which is the difference between the kinetic energy and the potential energy of the system

$$L(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \psi, \psi_t) = T(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \psi, \psi_t) - V(\phi_1, \phi_2, \psi),$$

where each term is sum of the contributions of the two pendula and of the string:

$$T = \sum_{k=1}^{2} \frac{m_k}{2} \left[l_k^2 \dot{\phi}_k^2 + \psi_t^2(x_k) + 2 l_k \dot{\phi}_k \psi_t(x_k) \cos(\phi_k) \right] + \int_0^{\Lambda} \frac{\rho}{2} \psi_t^2 \, dx,$$

$$V = -\sum_{k=1}^{2} m_k g l_k \cos(\phi_k) + \int_0^{\Lambda} \frac{\tau}{2} \psi_x^2 \, dx.$$

As before, we work in the regime of small oscillations about the stable equilibrium configuration $(\phi_{1,eq}, \phi_{2,eq}, \psi_{eq}(x)) = (0,0,0)$. To second order in the displacements, the Lagrangian is therefore the following:

$$L = \int_0^\Lambda \left\{ \left[\frac{1}{2}\rho\psi_t^2 - \frac{1}{2}\tau\psi_x^2\right] + \sum_{k=1}^{2}\left[\frac{1}{2}m_k\left(l_k^2\dot{\phi}_k^2 + \psi_t^2 + 2l_k\dot{\phi}_k\psi_t\right) - \frac{1}{2}m_k g l_k \phi_k^2\right]\delta(x-x_k)\right\} dx.$$

Let us rescale the variables in order to obtain an adimensional Lagrangian, as follows:

$$x \mapsto \Lambda x, \quad \phi_k \mapsto \frac{\Lambda}{l_k}\phi, \quad \psi \mapsto \Lambda\psi, \quad t \mapsto \sqrt{\frac{\rho\Lambda^2}{\tau}}t.$$

and let us define the following dimensionless parameters $\alpha_k, \beta_k > 0$, $k = 1, 2$, which characterise the geometry of the system:

$$\alpha_k^2 = \frac{\rho g \Lambda^2}{l_k \tau}, \qquad \beta_k = \frac{m_k}{\Lambda\rho}. \tag{4.1}$$

The definitive adimensional Lagrangian is

$$L = \int_0^1 \left\{ \left[\frac{1}{2}\psi_t^2 - \frac{1}{2}\psi_x^2\right] + \sum_{k=1}^{2}\beta_k\left[\frac{1}{2}\left(\dot{\phi}_k^2 + \psi_t^2 + 2\dot{\phi}_k\psi_t\right) - \frac{1}{2}\alpha_k^2\phi_k^2\right]\delta(x-x_k)\right\} dx, \tag{4.2}$$

which can be written in terms of Lagrangian density $\mathcal{L}$:

$$L(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \psi, \psi_t) = \int_0^1 \mathcal{L}(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \psi, \psi_t, \psi_x)\, dx.$$

4.3 Equations of motion

The equations of motion of the coupled system, for $\beta_1, \beta_2 \neq 0$, are

$$\begin{cases} \ddot{\phi}_1(t) + \psi_{tt}(x_1, t) + \alpha_1^2\phi_1(t) = 0 \\ \ddot{\phi}_2(t) + \psi_{tt}(x_2, t) + \alpha_2^2\phi_2(t) = 0 \\ \psi_{tt}(x,t) + \sum_{k=1}^2 \beta_k\left(\psi_{tt}(x,t) + \ddot{\phi}_k(t)\right)\delta(x-x_k) - \psi_{xx}(x,t) = 0, \end{cases} \tag{4.3}$$

with the boundary conditions

$$\psi(0,t) = 0 = \psi(1,t) \tag{4.4}$$

for all $t \in \mathbb{R}$.

4.4 Normal modes of oscillation

The equations of motion (4.3) are a linear system, therefore every solution, called small oscillation, can be written as a linear superposition of solutions. In particular, we are interested in periodic solutions in which every part of the system oscillates with the same frequency, namely normal modes of oscillation. As before, we will not be able to prove the unicity of the solutions found, since this would require a weak formulation of the problem with use of test functions. Nonetheless, we shall characterise solutions which are $\mathcal{C}^0$.

In the computation of the normal modes we assume the two pendula to be identical, namely $\alpha_1 = \alpha_2 \equiv \alpha$ and $\beta_1 = \beta_2 \equiv \beta$, and we set $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$.

Definition 4.1. The *normal modes of oscillation* of frequency ω of system (4.3) are periodic solutions of the form:

$$\Phi(x,t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \psi(x,t) \end{pmatrix} = \widehat{\Phi}_\omega(x)\cos(\omega t + \eta), \tag{4.5}$$

with $\omega \in \mathbb{R}$, $\eta \in [0, 2\pi[$ and $\widehat{\Phi}_\omega : [0,1] \to \mathbb{R}^3$, called eigenfunction associated to ω.

Proposition 4.1. *The frequencies of the normal modes of oscillation* (4.5)*, for* $x_1 = \frac{1}{3}$ *and* $x_2 = \frac{2}{3}$*, are the roots of the characteristic equation*

$$\begin{aligned} F(\omega) := -\omega^2 \sin\left(\frac{\omega}{3}\right) \Bigg[(\alpha^2-\omega^2)^2 2\cos^2\left(\frac{\omega}{3}\right) + \alpha^4\beta^2\omega^2\sin^2\left(\frac{\omega}{3}\right) \\ + (\alpha^2-\omega^2)^2\cos\left(\frac{2\omega}{3}\right) - 2\omega\alpha^2\beta(\alpha^2-\omega^2)\sin\left(\frac{2\omega}{3}\right)\Bigg] = 0. \end{aligned} \tag{4.6}$$

Proof. Seek solution of system of (4.3) of the form

$$\Phi(x,t) = \begin{pmatrix} A_1 \\ A_2 \\ f(x) \end{pmatrix} \cos(\omega t + \eta),$$

with $A_1, A_2 \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$. In order to handle the Dirac delta functions, we determine independently the solutions on the left-hand side and on the right-hand side of each discontinuity. Then, $f(x) = f_I(x)\Theta(x_1 - x) + f_{II}(x)\Theta(x - x_1)\Theta(x_2 - x) + f_{III}(x)\Theta(x - x_2)$ has to satisfy

$$\begin{cases} (\alpha^2-\omega^2)A_1 - \omega^2 f(x_1) = 0 \\ (\alpha^2-\omega^2)A_2 - \omega^2 f(x_2) = 0 \\ f''(x) + \omega^2 f(x) = -\beta\omega^2 \sum_{k=1}^2 (f(x) + A_k)\delta(x - x_k) \\ f_I(0) = 0 \\ f_{III}(1) = 0 \\ f_I(x_1) = f_{II}(x_1) \\ f_{II}(x_2) = f_{III}(x_2) \\ f'_{II}(x_1) - f'_I(x_1) = -\beta\omega(f(x_1) + A_1) \\ f'_{III}(x_2) - f'_{II}(x_2) = -\beta\omega(f(x_2) + A_2). \end{cases} \tag{4.7}$$

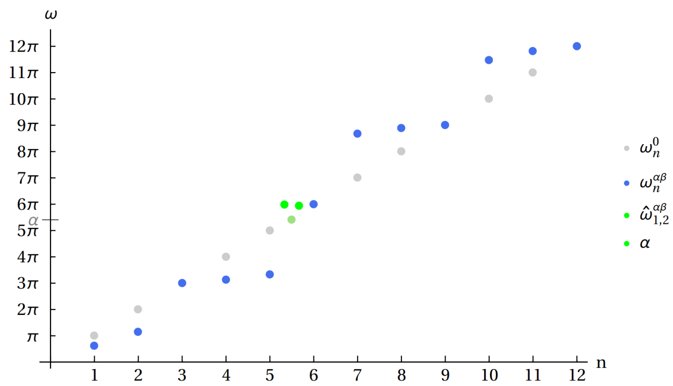
The third equation of (4.7) has solution $f_K(x) = a_K \cos(\omega x) + b_K \sin(\omega x)$, $a_K, b_K \in \mathbb{R}$, $K = I, II, III$, $\omega \in \mathbb{R}$; the other eight equations correspond to the condition

$$\det \begin{pmatrix} \alpha^2-\omega^2 & 0 & -\omega^2 c1 & -\omega^2 s1 & 0 & 0 & 0 & 0 \\ 0 & \alpha^2-\omega^2 & 0 & 0 & -\omega^2 c2 & -\omega^2 s2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & c1 & s1 & -c1 & -s1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c2 & s2 & -c2 & -s2 \\ \beta\alpha^2 & 0 & \omega s1 & -\omega c1 & -\omega s1 & \omega c1 & 0 & 0 \\ 0 & \beta\alpha^2 & 0 & 0 & \omega s2 & -\omega c2 & -\omega s2 & \omega c2 \end{pmatrix} = 0,$$

where $c = \cos(\omega)$, $s = \sin(\omega)$, $ck = \cos(\omega x_k)$, $sk = \sin(\omega x_k)$, $k = 1, 2$, and where in the last two raws it has already been substituted $\omega^2(f(x_k) + A_k)$ with $\alpha^2 A_k$. The vanishing of this determinant, for $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$, leads to the characteristic equation (4.6). $\square$

The frequencies can be determined solving numerically equation (4.6). For every $\beta > 0$ and $\alpha \neq 3\pi r$, with $r \in \mathbb{N}_+$, there are two countable families of frequencies, consisting respectively of those which are multiple of 3π and those which are not, and two special ones: $\{\omega_{3j}^{\alpha,\beta} : j \in \mathbb{N}_+\}$, $\{\omega_{3j-2}^{\alpha,\beta}, \omega_{3j-1}^{\alpha,\beta} : j \in \mathbb{N}_+\}$ and $\widehat{\omega}_1^{\alpha,\beta}$, $\widehat{\omega}_2^{\alpha,\beta}$. Analogously to the case of the single pendulum, $\omega_n^{\alpha,\beta} \leq \omega_n^0$ for $n \leq 3s - 1$ and $\omega_n^{\alpha,\beta} \geq \omega_n^0$ for $n \geq 3s$, with $s(\alpha) = \lceil \frac{\alpha}{3\pi} \rceil$ (Figure 4.2).

Figure 4.2: Dispersion relation for the string with two pendula: $\omega_n^{\alpha,\beta}$ and $\widehat{\omega}_{1,2}^{\alpha,\beta}$ with $\alpha = 17$ and $\beta = 0.5$; $s = 2$. To the frequencies $\widehat{\omega}_k^{\alpha,\beta}$ it has been assigned the arbitrary indices $3s - \frac{k}{3}$, with $k = 1, 2$.



Let us now compute the eigenfunctions of the normal modes.

Proposition 4.2. *For every j, $\beta > 0$ and $\alpha \neq 3\pi r$, with $r \in \mathbb{N}_+$, the eigenfunctions are the following:*

- *the eigenfunction associated to $\omega_{3j}^{\alpha,\beta}$ is*

$$\widehat{\Phi}_{\omega_{3j}^{\alpha,\beta}}(x) = \begin{pmatrix} 0 \\ 0 \\ a_{3j} \end{pmatrix} \sin(3j\pi x), \tag{4.8}$$

- *the eigenfunction associated to* $\omega_{3j-k}^{\alpha,\beta}$ *is*

$$\widehat{\Phi}_{\omega_{3j-k}^{\alpha,\beta}}(x) = a_{3j-k} \begin{pmatrix} \frac{(\omega_{3j-k}^{\alpha,\beta})^2}{\alpha^2-(\omega_{3j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right) \\ \frac{(\omega_{3j-k}^{\alpha,\beta})^2}{\alpha^2-(\omega_{3j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right)\left[2\cos\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\omega_{3j-k}^{\alpha,\beta}}{\alpha^2-(\omega_{3j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right)\right] \\ B\Theta(x_1-x) + C\Theta(x-x_1)\Theta(x_2-x) + D\Theta(x-x_2) \end{pmatrix}, \tag{4.9}$$

with $k = 1, 2$ *and*

$$B = \sin\left(\omega_{3j-k}^{\alpha,\beta} x\right)$$

$$C = \sin\left(\omega_{3j-k}^{\alpha,\beta} x\right) + \frac{\alpha^2\beta\omega_{3j-k}^{\alpha,\beta}}{\alpha^2 - (\omega_{3j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right) \sin\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3} - \omega_{3j-k}^{\alpha,\beta} x\right)$$

$$D = \left(2\cos\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\omega_{3j-k}^{\alpha,\beta}}{\alpha^2 - (\omega_{3j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{3j-k}^{\alpha,\beta}}{3}\right)\right) \sin\left(\omega_{3j-k}^{\alpha,\beta} - \omega_{3j-k}^{\alpha,\beta} x\right),$$

- *the eigenfunction associated to* $\widehat{\omega}_k^{\alpha,\beta}$, *is*

$$\widehat{\Phi}_{\widehat{\omega}_k^{\alpha,\beta}}(x) = \hat{a}_k \begin{pmatrix} \frac{(\widehat{\omega}_k^{\alpha,\beta})^2}{\alpha^2-(\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) \\ \frac{(\widehat{\omega}_k^{\alpha,\beta})^2}{\alpha^2-(\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\left[2\cos\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2-(\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\right] \\ \widehat{B}\Theta(x_1-x) + \widehat{C}\Theta(x-x_1)\Theta(x_2-x) + \widehat{D}\Theta(x-x_2) \end{pmatrix}, \tag{4.10}$$

with $k = 1, 2$ *and*

$$\widehat{B} = \sin\left(\widehat{\omega}_k^{\alpha,\beta} x\right)$$

$$\widehat{C} = \sin\left(\widehat{\omega}_k^{\alpha,\beta} x\right) + \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3} - \widehat{\omega}_k^{\alpha,\beta} x\right)$$

$$\widehat{D} = \left(2\cos\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\right) \sin\left(\widehat{\omega}_k^{\alpha,\beta} - \widehat{\omega}_k^{\alpha,\beta} x\right),$$

with $a_n, \hat{a}_k \in \mathbb{R}$.

Proof. From (4.7), in order to determine $A_k, a_K, b_K,\ k = 1, 2,\ K = I, II, III$, the following eigenvector problems have to be satisfied:

$$\begin{pmatrix} \alpha^2-\omega^2 & 0 & -\omega^2 c1 & -\omega^2 s1 & 0 & 0 & 0 & 0 \\ 0 & \alpha^2-\omega^2 & 0 & 0 & -\omega^2 c2 & -\omega^2 s2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & c1 & s1 & -c1 & -s1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c2 & s2 & -c2 & -s2 \\ \beta\alpha^2 & 0 & \omega s1 & -\omega c1 & -\omega s1 & \omega c1 & 0 & 0 \\ 0 & \beta\alpha^2 & 0 & 0 & \omega s2 & -\omega c2 & -\omega s2 & \omega c2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ a_I \\ b_I \\ a_{II} \\ b_{II} \\ a_{III} \\ b_{III} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with ω solution of (4.6).

- For $\omega_{3j}^{\alpha,\beta}$:

$$\begin{cases} A_1 = 0 \\ A_2 = 0 \\ a_I = a_{II} = a_{III} = 0 \\ b_I = b_{II} = b_{III} \end{cases}$$

  which yields eigenfunction (4.8).

- For $\omega_{2j-k}^{\alpha,\beta}$, $k = 1, 2$:

$$\begin{cases} A_1 = \frac{(\omega_{2j-k}^{\alpha,\beta})^2}{\alpha^2-(\omega_{2j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right) b_I \\ A_2 = A_1 \left[2\cos\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\omega_{2j-k}^{\alpha,\beta}}{\alpha^2-(\omega_{2j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right)\right] \\ a_I = 0 \\ a_{II} = \frac{\alpha^2\beta\omega_{2j-k}^{\alpha,\beta}}{\alpha^2-(\omega_{2j-k}^{\alpha,\beta})^2} \sin^2\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right) b_I \\ a_{III} = \sin\left(\omega_{2j-k}^{\alpha,\beta}\right) \left[2\cos\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\omega_{2j-k}^{\alpha,\beta}}{\alpha^2-(\omega_{2j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right)\right] b_I \\ b_{II} = \left[1 - \frac{\alpha^2\beta\omega_{2j-k}^{\alpha,\beta}}{\alpha^2-(\omega_{2j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right) \cos\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right)\right] b_I \\ b_{III} = \cos\left(\omega_{2j-k}^{\alpha,\beta}\right) \left[2\cos\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\omega_{2j-k}^{\alpha,\beta}}{\alpha^2-(\omega_{2j-k}^{\alpha,\beta})^2} \sin\left(\frac{\omega_{2j-k}^{\alpha,\beta}}{3}\right)\right] b_I \end{cases}$$

  which yields eigenfunction (4.9).

- For $\widehat{\omega}_k^{\alpha,\beta}$, $k = 1, 2$:

$$\begin{cases} A_1 = \frac{(\widehat{\omega}_k^{\alpha,\beta})^2}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) b_I \\ A_2 = A_1 \left[2\cos\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\right] \\ a_I = 0 \\ a_{II} = \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin^2\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) b_I \\ a_{III} = \sin\left(\widehat{\omega}_k^{\alpha,\beta}\right) \left[2\cos\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\right] b_I \\ b_{II} = \left[1 - \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) \cos\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\right] b_I \\ b_{III} = \cos\left(\widehat{\omega}_k^{\alpha,\beta}\right) \left[2\cos\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right) - \frac{\alpha^2\beta\widehat{\omega}_k^{\alpha,\beta}}{\alpha^2 - (\widehat{\omega}_k^{\alpha,\beta})^2} \sin\left(\frac{\widehat{\omega}_k^{\alpha,\beta}}{3}\right)\right] b_I \end{cases}$$

which yields eigenfunction (4.10).

□

The normal modes found have the following physical interpretation. The family associated to the frequencies multiple of 3π leaves the pendula still, hanging from two nodes of the wave described by the string. In the normal modes associated to $\omega_{2j-k}^{\alpha,\beta}$ and to $\widehat{\omega}_k^{\alpha,\beta}$, $k = 1, 2$, both pendula and the string are in motion.

Let us now determine the normal modes of oscillation in the case in which the parameter α is a multiple of 3π. For such values, $\widehat{\omega}_1^{\alpha,\beta} = \widehat{\omega}_2^{\alpha,\beta}$ and they coincide with a frequency of the string. Fixed $\alpha = 3\pi s$, the behaviour of the system is identical to the case considered above for all frequencies except for $\omega_{3s}^{\alpha,\beta}$, for which the pendula instead of being at rest oscillate with the string.

For every $\beta > 0$ and $\alpha = 3\pi r$, with $r \in \mathbb{N}_+$, there are two countable families of frequencies, consisting respectively of those which are multiple of 3π and those which are not: $\{\omega_{3j}^{\alpha,\beta} : j \in \mathbb{N}_+\}$, $\{\omega_{3j-2}^{\alpha,\beta}, \omega_{3j-1}^{\alpha,\beta} : j \in \mathbb{N}_+\}$, and, fixed $\alpha = 3\pi s$, $\deg(\omega_{3s}^{\alpha,\beta}) = 3$.

Finally, let us compute the associated eigenfunctions.

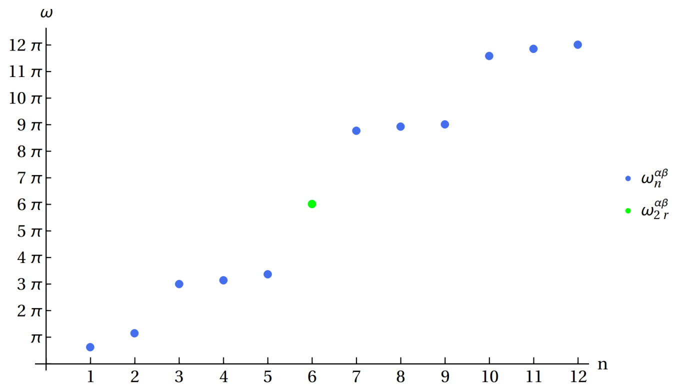
Proposition 4.3. *For every j, $\beta > 0$ and $\alpha = 3\pi s$, the eigenfunctions are the following:*

- *the eigenfunction associated to $\omega_{3j}^{\alpha,\beta}$, with $j \neq s$, is* (4.8),

- *the eigenfunction associated to $\omega_{3s}^{\alpha,\beta}$ is*

$$\widehat{\Phi}_{\omega_{3s}^{\alpha,\beta}}(x) = \begin{pmatrix} \frac{(-1)^s}{\beta(3s)\pi}(a - a') \\ \frac{1}{\beta(3s)\pi}(a' - a'') \\ \sin(3s\pi x)\big(a\Theta(x_1 - x) + a'\Theta(x - x_1)\Theta(x_2 - x) + a'''\Theta(x_1 - x)\big) \end{pmatrix}, \tag{4.11}$$

Figure 4.3: Dispersion relation for the string with two pendula with double root: $\omega_n^{\alpha,\beta}$ with $\alpha = 6\pi$ and $\beta = 0.5$.



- *the eigenfunction associated to* $\omega_{3j-k}^{\alpha,\beta}$ *is* (4.9),

with $a, a', a'' \in \mathbb{R}$.

Proof. The proofs for $\omega_{3j}^{\alpha,\beta}$ with $j \neq s$ and for $\omega_{3j-k}^{\alpha,\beta}$ are analogous to those of Proposition 4.2.

- For $\omega_{3s}^{\alpha,\beta} = (3s)\pi$:

$$\begin{cases} A_1 = \frac{(-1)^s}{\beta(3s)\pi}(b_I - b_{II}) \\ A_2 = \frac{1}{\beta(3s)\pi}(b_{II} - b_{III}) \\ a_I = a_{II} = a_{III} = 0 \end{cases}$$

  which yields eigenfunction (4.11).

□

When the system oscillates with a frequency equal to $\omega_{3s}^{\alpha,\beta}$, depending on the initial conditions, the following scenarios are possible:

- both pendula and the string are in motion,
- the string oscillates, while both pendula are at rest,
- one pendulum stand still, while the other one and the string oscillate,
- the two pendula swing, while segments of string are at rest,
- one pendulum and segments of string stand still, while the other one oscillates.

4.5 Beats

In the configurations in which two, or more, frequencies are close to each other one expects to find beating phenomena. Let us consider the situation in which the first two frequencies are close and – neglecting the contribution of higher frequencies – study the motion of the coupled system.

From a numerical computation of A_1 and A_2, it emerges that either $A_1 = A_2$ in the first normal mode and $A_1 = -A_2$ in the second one, or vice versa – depending on the parameters α and β.

Let us choose the initial conditions such that, at $t = 0$, one pendulum has amplitude A and zero velocity, while the other one is at rest in its equilibrium position:

$$\begin{cases} \phi_1(0) = A \\ \phi_2(0) = 0 \\ \dot{\phi}_1(0) = 0 \\ \dot{\phi}_1(0) = 0. \end{cases} \tag{4.12}$$

The small oscillation, sum of the first two normal modes, is therefore

$$\begin{cases} \phi_1(t) = c_1 \cos(\omega_1 + \eta_1) + c_2 \cos(\omega_2 + \eta_2) \\ \phi_2(t) = c_2 \cos(\omega_1 + \eta_1) - c_2 \cos(\omega_2 + \eta_2), \end{cases} \tag{4.13}$$

which satisfies the initial conditions chosen for $c_1 = c_2 = \frac{A}{2}$, $\eta_1 = \eta_2 = 0$. By substituting these values into (4.13), the small oscillation can be written as

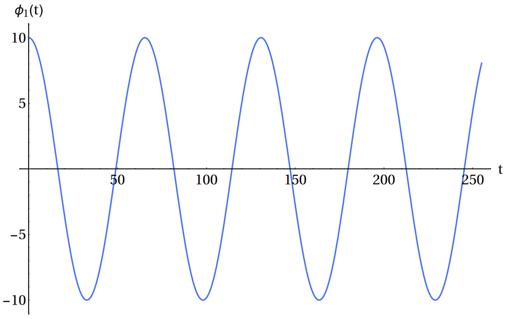
$$\begin{cases} \phi_1(t) = A \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_2 + \omega_1}{2} t\right) \\ \phi_2(t) = A \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_2 + \omega_1}{2} t\right), \end{cases} \tag{4.14}$$

which represent oscillations of the average frequency $\frac{\omega_2 + \omega_1}{2}$ and modulated amplitude (Figure 4.4).

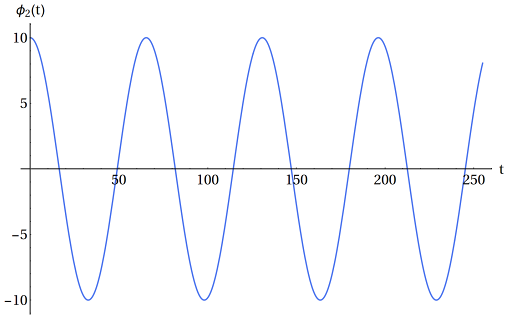
Remark 4.1. The choice of considering only the contribution of the first two normal modes will be justified with the introduction of an appropriate damping force, which produces damping rates dependent on the frequency of oscillation.

Figure 4.4: Beats: $\phi_1(t), \phi_2(t)$ with $\alpha = 0.1$ and $\beta = 25$.

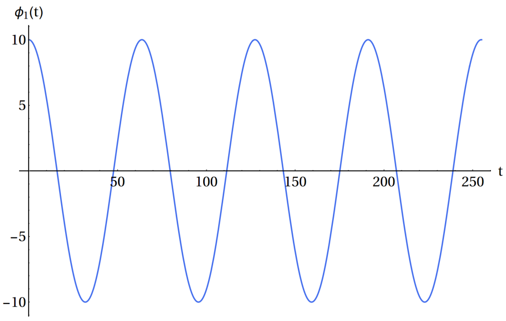
(a) $\phi_1(t)$ with ω_1.



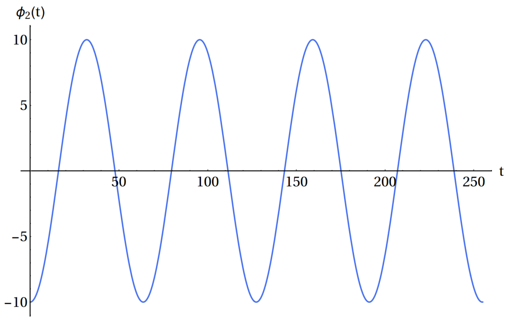
(b) $\phi_2(t)$ with ω_1.



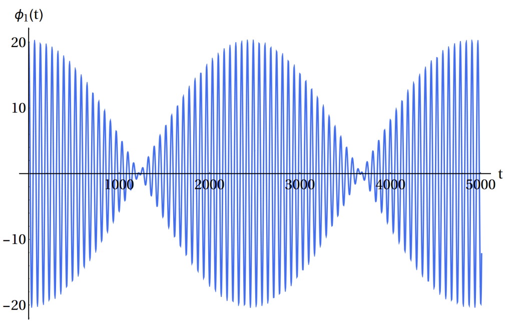
(c) $\phi_1(t)$ with ω_2.



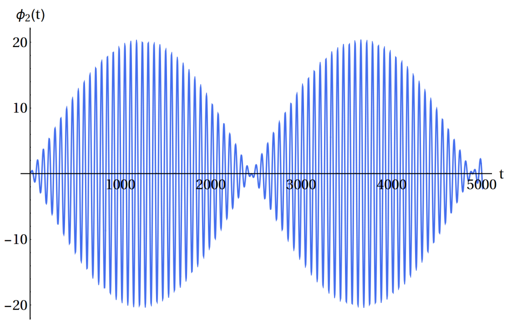
(d) $\phi_2(t)$ with ω_2.



(e) $\phi_1(t)$ with ω_1, ω_2.



(f) $\phi_2(t)$ with ω_1, ω_2.



Chapter 5

Two pendula hanging from a string with damping

In this chapter we add to the system, consisting of two pendula hanging from a flexible and elastic string, the same dissipative contribution as in Chapter 3 . We fix the suspension point of each pendula at one-third and two-thirds of the string, respectively, and study the small oscillations about the equilibrium configuration. The eigenvalues of the damped oscillations have been numerically determined and a qualitative analysis of the dependence on the parameters is presented. Thus, we investigate the role of damping in the possible synchronization of the two pendula.

5.1 Equations of motion

The equations of motion of the linearised damped coupled system, for $\beta_1, \beta_2 \neq 0$, are

$$\begin{cases} \ddot{\phi}_1(t) + \psi_{tt}(x_1,t) + \alpha_1^2\phi_1(t) = 0 \\ \ddot{\phi}_2(t) + \psi_{tt}(x_2,t) + \alpha_2^2\phi_2(t) = 0 \\ \psi_{tt}(x,t) + \sum_{k=1}^{2} \beta_k \left(\psi_{tt}(x,t) + \ddot{\phi}_k(t)\right) \delta(x-x_k) - \psi_{xx}(x,t) - \nu\,\psi_{txx}(x,t) = 0, \end{cases} \tag{5.1}$$

where

$$\alpha_k^2 = \frac{\rho g \Lambda^2}{l_k \tau}, \qquad \beta_k = \frac{m_k}{\Lambda\rho}, \qquad k = 1,2,$$

and with the boundary conditions

$$\psi(0,t) = 0 = \psi(1,t) \tag{5.2}$$

for all $t \in \mathbb{R}$.

5.2 Damped normal modes

In the computation of the damped normal modes we assume the two pendula to be identical, namely $\alpha_1 = \alpha_2 \equiv \alpha$ and $\beta_1 = \beta_2 \equiv \beta$, and we set $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$.

Definition 5.1. The *damped normal modes* of eigenvalue λ of system (5.1) are solutions of the form:

$$\Phi(x,t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \psi(x,t) \end{pmatrix} = \mathrm{Re}\left[\widehat{\Phi}_\lambda(x)e^{\lambda t}\right], \tag{5.3}$$

with $\lambda \in \mathbb{C}$, $\mathrm{Re}[\lambda] < 0$, and $\widehat{\Phi}_\lambda : [0,1] \to \mathbb{C}^3$, called eigenfunction associated to λ.

Proposition 5.1. *The eigenvalues of the damped normal modes* (5.3)*, for* $x_1 = \frac{1}{3}$ *and* $x_2 = \frac{2}{3}$*, are the roots of the characteristic equation*

$$\begin{aligned} F(\lambda) := \xi^2(\lambda)\sinh\left(\frac{\xi(\lambda)}{3}\right)\Bigg[(\alpha^2+\lambda^2)^2 2\cosh^2\left(\frac{\xi(\lambda)}{3}\right) + \alpha^4\beta^2\xi^2(\lambda)\sinh^2\left(\frac{\xi(\lambda)}{3}\right) \\ + (\alpha^2+\lambda^2)^2\cosh\left(\frac{2\xi(\lambda)}{3}\right) + 2\xi(\lambda)\alpha^2\beta(\alpha^2+\lambda^2)\sinh\left(\frac{2\xi(\lambda)}{3}\right)\Bigg] = 0, \end{aligned} \tag{5.4}$$

where $\xi \in \mathbb{C}$*, Im*$[\xi] > 0$*, is defined by the dispersion relation*

$$\xi(\lambda) := \pm\frac{\lambda}{\sqrt{1+\nu\lambda}}. \tag{5.5}$$

Proof. Seek solution of system of (5.1) of the form

$$\Phi(x,t) = \begin{pmatrix} A_1 \\ A_2 \\ f(x) \end{pmatrix} e^{\lambda t},$$

with $A_1, A_2 \in \mathbb{C}$ and $f : [0,1] \to \mathbb{C}$, and the real part intended. In order to handle the Dirac delta functions, we determine independently the solutions on the left-hand side and on the right-hand side of each discontinuity. Then, $f(x) = f_I(x)\Theta(x_1 - x) + f_{II}(x)\Theta(x - x_1)\Theta(x_2 - x) + f_{III}(x)\Theta(x - x_2)$ has to satisfy

$$\begin{cases} (\alpha^2+\lambda^2)A_1 + \lambda^2 f(x_1) = 0 \\ (\alpha^2+\lambda^2)A_2 + \lambda^2 f(x_2) = 0 \\ f''(x) - \xi^2 f(x) = \beta\xi^2\sum_{k=1}^2 (f(x)+A_k)\delta(x-x_k) \\ f_I(0) = 0 \\ f_{III}(1) = 0 \\ f_I(x_1) = f_{II}(x_1) \\ f_{II}(x_2) = f_{III}(x_2) \\ f'_{II}(x_1) - f'_I(x_1) = \xi^2\beta(f(x_1)+A_1) \\ f'_{III}(x_2) - f'_{II}(x_2) = \xi^2\beta(f(x_2)+A_2). \end{cases} \tag{5.6}$$

The third equation of (5.6) has solution $f_K(x) = a_K e^{\xi x} + b_K e^{-\xi x}$, with $a_K, b_K \in \mathbb{C}$,

$K = I, II, III$, $\xi \in \mathbb{C}$; the other eight equations correspond to the condition

$$\det\begin{pmatrix} \alpha^2+\lambda^2 & 0 & \lambda^2 e^{\xi x_1} & \lambda^2 e^{-\xi x_1} & 0 & 0 & 0 & 0 \\ 0 & \alpha^2+\lambda^2 & 0 & 0 & \lambda^2 e^{\xi x_2} & \lambda^2 e^{-\xi x_2} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\xi} & e^{-\xi} \\ 0 & 0 & e^{\xi x_1} & e^{-\xi x_1} & -e^{\xi x_1} & -e^{-\xi x_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\xi x_2} & e^{-\xi x_2} & -e^{\xi x_2} & -e^{-\xi x_2} \\ \frac{\beta\alpha^2}{1+\nu\lambda} & 0 & -\xi e^{\xi x_1} & \xi e^{-\xi x_1} & \xi e^{\xi x_1} & -\xi e^{-\xi x_1} & 0 & 0 \\ 0 & \frac{\beta\alpha^2}{1+\nu\lambda} & 0 & 0 & -\xi e^{\xi x_2} & \xi e^{\xi x_2} & \xi e^{\xi x_2} & -\xi e^{-\xi x_2} \end{pmatrix} = 0,$$

where in the last two raws it has already been substituted $\lambda^2(f(x_k) + A_k)$ with $-\alpha^2 A_k$. The vanishing of this determinant, for $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$, leads to the characteristic equation (5.4). □

The frequencies can be determined solving numerically equation (5.4). For every $\beta > 0$ and $\alpha > 0$, there are two countable families of frequencies, consisting respectively of those which are multiple of 3π and those which are not, and two couples of special ones: $\{\lambda_{3j}^{\alpha,\beta\pm} : j \in \mathbb{N}_+\}$, $\{\lambda_{3j-2}^{\alpha,\beta\pm}, \lambda_{3j-1}^{\alpha,\beta\pm} : j \in \mathbb{N}_+\}$ and $\widehat{\lambda}_1^{\alpha,\beta\pm}$, $\widehat{\lambda}_2^{\alpha,\beta\pm}$. Figure 5.1 and Figure 5.2 show the spectrum of the coupled system and the dispersion relation, respectively, compared with those of the unperturbed string. Frequencies which are multiple of 3π are unchanged, with the others are modified by the presence of the pendula.

Figure 5.1: Spectrum of the damped string with two pendula: $\lambda_n^{\alpha,\beta}, \widehat{\lambda}_{1,2}^{\alpha,\beta}$ with $\alpha = 17$, $\beta = 0.1$ and $\nu = 0.1$.

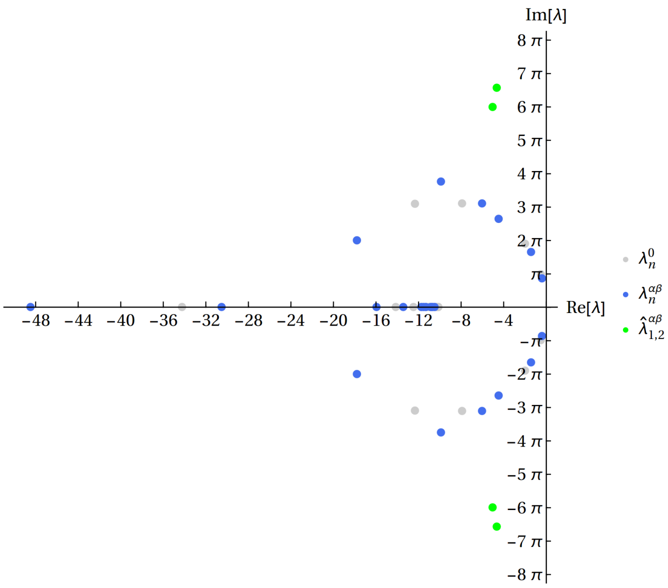
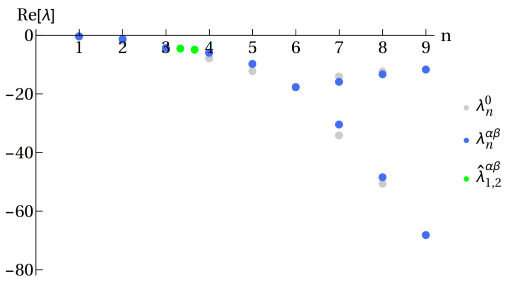
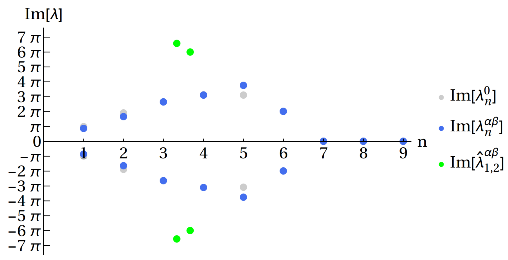


Figure 5.2: Dispersion relation for the damped string with two pendula: $\lambda_n^{\alpha,\beta}, \widehat{\lambda}_{1,2}^{\alpha,\beta}$ with $\alpha = 17$, $\beta = 0.1$ and $\nu = 0.1$. To the eigenvalues $\widehat{\lambda}_{1,2}^{\alpha,\beta}$ it has been assigned the arbitrary indices 3.3 and 3.7, respectively.

(a) Re[λ].



(b) Im[λ].



Let us finally compute the associated eigenfunctions.

Proposition 5.2. *For every j, the eigenfunctions are the following:*

- *the eigenfunction associated to $\lambda_{3j}^{\alpha,\beta\pm}$ is*

$$\widehat{\Phi}_{\lambda_{3j}^{\alpha,\beta\pm}}(x) = \begin{pmatrix} 0 \\ 0 \\ a_{3j}^{\pm} \end{pmatrix} \sin\left(3j\pi x\right), \tag{5.7}$$

- *the eigenfunction associated to $\lambda_{3j-k}^{\alpha,\beta\pm}$ is*

$$\widehat{\Phi}_{\lambda_{3j-k}^{\alpha,\beta\pm}}(x) = a_{3j-k}^{\pm} \begin{pmatrix} -\frac{(\lambda_{3j-k}^{\alpha,\beta})^2}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) \\ A_1 \left[\frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}\right] \\ B\Theta(x_1-x) + C\Theta(x-x_1)\Theta(x_2-x) + D\Theta(x-x_2) \end{pmatrix}, \tag{5.8}$$

with $k = 1, 2$ and

$$A_1 = -\frac{(\lambda_{3j-k}^{\alpha,\beta})^2}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right)$$

$$B = \sinh\left(\xi(\lambda_{3j-k}^{\alpha,\beta})x\right)$$

$$C = \sinh\left(\xi(\lambda_{3j-k}^{\alpha,\beta})x\right) + \frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) \sinh\left(\xi(\lambda_{3j-k}^{\alpha,\beta})\left(x-\frac{1}{3}\right)\right)$$

$$D = \left(\frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}\right) \sinh\left(\xi(\lambda_{3j-k}^{\alpha,\beta})(1-x)\right),$$

- *the eigenfunction associated to* $\widehat{\lambda}_k^{\alpha,\beta\pm}$ *is*

$$\widehat{\Phi}_{\widehat{\lambda}_k^{\alpha,\beta\pm}}(x) = \hat{a}_{3j-k}^{\pm} \begin{pmatrix} -\frac{(\widehat{\lambda}_k^{\alpha,\beta})^2}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2}\sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right) \\ \widehat{A}_1\left[\frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2}\sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right)+\frac{\sinh\left(\frac{2}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}\right] \\ \widehat{B}\Theta(x_1-x)+\widehat{C}\Theta(x-x_1)\Theta(x_2-x)+\widehat{D}\Theta(x-x_2) \end{pmatrix}, \tag{5.9}$$

with $k = 1, 2$ *and*

$$\begin{aligned}
\widehat{A}_1 &= -\frac{(\widehat{\lambda}_k^{\alpha,\beta})^2}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2}\sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right)\\
\widehat{B} &= \sinh\left(\xi(\widehat{\lambda}_k^{\alpha,\beta})x\right)\\
\widehat{C} &= \sinh\left(\xi(\widehat{\lambda}_k^{\alpha,\beta})x\right)+\frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2}\sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right)\sinh\left(\xi(\widehat{\lambda}_k^{\alpha,\beta})\left(x-\frac{1}{3}\right)\right)\\
\widehat{D} &= \left(\frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2}\sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right)+\frac{\sinh\left(\frac{2}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}\right)\sinh\left(\xi(\widehat{\lambda}_k^{\alpha,\beta})(1-x)\right),
\end{aligned}$$

with $a_n^{\pm}, \hat{a}_n^{\pm} \in \mathbb{C}$.

Proof. From (5.6), in order to determine A_k, a_K, b_K, $k = 1, 2$, $K = I, II, III$, the following eigenvector problems have to be satisfied:

$$\det\begin{pmatrix}
\alpha^2+\lambda^2 & 0 & \lambda^2 e^{\frac{\xi}{3}} & \lambda^2 e^{-\frac{\xi}{3}} & 0 & 0 & 0 & 0\\
0 & \alpha^2+\lambda^2 & 0 & 0 & \lambda^2 e^{\frac{2\xi}{3}} & \lambda^2 e^{-\frac{2\xi}{3}} & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & e^{\xi} & e^{-\xi}\\
0 & 0 & e^{\frac{\xi}{3}} & e^{-\frac{\xi}{3}} & -e^{\frac{\xi}{3}} & -e^{-\frac{\xi}{3}} & 0 & 0\\
0 & 0 & 0 & 0 & e^{\frac{2\xi}{3}} & e^{-\frac{2\xi}{3}} & -e^{\frac{2\xi}{3}} & -e^{-\frac{2\xi}{3}}\\
\frac{\beta\alpha^2}{1+\nu\lambda} & 0 & -\xi e^{\frac{\xi}{3}} & \xi e^{-\frac{\xi}{3}} & \xi e^{\frac{\xi}{3}} & -\xi e^{-\frac{\xi}{3}} & 0 & 0\\
0 & \frac{\beta\alpha^2}{1+\nu\lambda} & 0 & 0 & -\xi e^{\frac{2\xi}{3}} & \xi e^{\frac{2\xi}{3}} & \xi e^{\frac{2\xi}{3}} & -\xi e^{-\frac{2\xi}{3}}
\end{pmatrix}\begin{pmatrix} A_1\\ A_2\\ a_I\\ b_I\\ a_{II}\\ b_{II}\\ a_{III}\\ b_{III}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

with λ solution of (5.4).

- For $\lambda_{3j}^{\alpha,\beta\pm}$:

$$\begin{cases} A_1 = 0\\ A_2 = 0\\ b_I = -a_I\\ a_I = a_{II} = a_{III}\\ b_I = b_{II} = b_{III}\end{cases}$$

which yields eigenfunction (5.7).

- For $\lambda_{2j-k}^{\alpha,\beta\pm}$, $k=1,2$:

$$\begin{cases} A_1 = -a_I \frac{(\lambda_{3j-k}^{\alpha,\beta})^2}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} 2\sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) \\ A_2 = A_1 \left[\frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}\right] \\ b_I = -a_I \\ a_{II} = a_I \left[1 + \frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} e^{\frac{-\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right)\right] \\ b_{II} = -a_I \left[1 + \frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} e^{\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right)\right] \\ a_{III} = -a_I e^{-\xi(\lambda_{3j-k}^{\alpha,\beta})} \left[\frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}\right] \\ b_{III} = a_I e^{\xi(\lambda_{3j-k}^{\alpha,\beta})} \left[\frac{\alpha^2\beta\xi(\lambda_{3j-k}^{\alpha,\beta})}{\alpha^2+(\lambda_{3j-k}^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\lambda_{3j-k}^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\lambda_{3j-k}^{\alpha,\beta})\right)}\right] \end{cases}$$

which yields eigenfunction (5.8).

- For $\widehat{\lambda}_k^{\alpha,\beta\pm}$, $k=1,2$:

$$\begin{cases} A_1 = -a_I \frac{(\widehat{\lambda}_k^{\alpha,\beta})^2}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2} 2\sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right) \\ A_2 = A_1 \left[\frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}\right] \\ b_I = -a_I \\ a_{II} = a_I \left[1 + \frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2} e^{\frac{-\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}} \sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right)\right] \\ b_{II} = -a_I \left[1 + \frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2} e^{\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}} \sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right)\right] \\ a_{III} = -a_I e^{-\xi(\widehat{\lambda}_k^{\alpha,\beta})} \left[\frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}\right] \\ b_{III} = a_I e^{\xi(\widehat{\lambda}_k^{\alpha,\beta})} \left[\frac{\alpha^2\beta\xi(\widehat{\lambda}_k^{\alpha,\beta})}{\alpha^2+(\widehat{\lambda}_k^{\alpha,\beta})^2} \sinh\left(\frac{\xi(\widehat{\lambda}_k^{\alpha,\beta})}{3}\right) + \frac{\sinh\left(\frac{2}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}{\sinh\left(\frac{1}{3}\xi(\widehat{\lambda}_k^{\alpha,\beta})\right)}\right] \end{cases}$$

which yields eigenfunction (5.9).

□

The eigenfunctions found have the following interpretation: those associated to eigenvalues multiple of 3π correspond to the situation in which the pendula hang from nodes of the string, and are therefore at rest; in the other damped normal modes, instead, both pendula and the string are in motion.

5.3 Synchronization

In this final section we make a qualitative numerical study in order to investigate the possibility of synchronization between the two pendula.

The generic damped small oscillation is given by a linear superposition of the damped normal modes:

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \psi(x,t) \end{pmatrix} = \sum_{n=1}^{+\infty} \mathrm{Re}\left[\widehat{\Phi}_{\lambda_n^{\alpha,\beta+}}(x) e^{\lambda_n^{\alpha,\beta+} t} + \widehat{\Phi}_{\lambda_n^{\alpha,\beta-}}(x) e^{\lambda_n^{\alpha,\beta-} t} \right]$$
$$+ \sum_{k=1}^{2} \mathrm{Re}\left[\widehat{\Phi}_{\widehat{\lambda}_k^{\alpha,\beta+}}(x) e^{\widehat{\lambda}_k^{\alpha,\beta+} t} + \widehat{\Phi}_{\widehat{\lambda}_k^{\alpha,\beta-}}(x) e^{\widehat{\lambda}_k^{\alpha,\beta-} t} \right]. \quad (5.10)$$

The damping model chosen produces eigenvalues whose damping rates depend strongly on the frequency of oscillation, therefore every normal mode is damped with a different intensity and – after a transient – one expects that the motion is given by a combination of a few low-frequency damped normal modes only. Since the eigenvalues depend on the parameters α and β (from characteristic equation (5.4)), we confront now the damping rates for different values of the parameters.

5.3.1 Dependency on the parameters

Recalling that

$$\alpha^2 = \frac{\rho g \Lambda^2}{l\tau}, \quad \beta = \frac{m}{\Lambda\rho},$$

the physical meaning of the possible combinations of values of the parameters is the following:

a. low values of α and high values of β correspond to small ρ and Λ, and large m and l, namely, a short and light string with long and massive pendula;

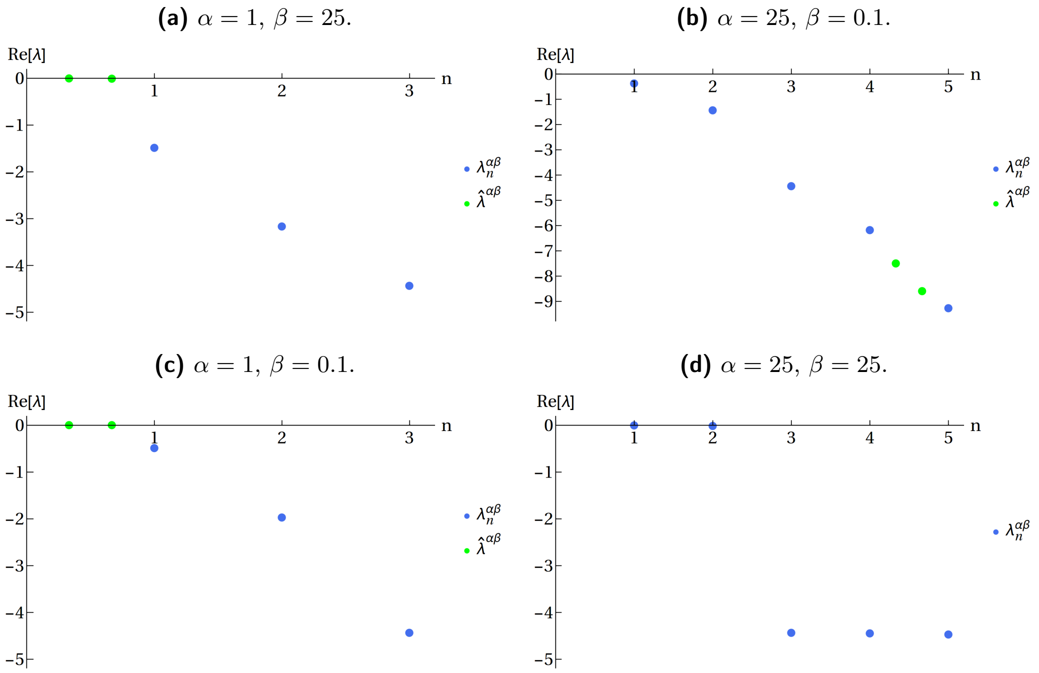
b. high values of α and low values of β correspond to large ρ and Λ, and small m and l, namely, a long massive string with short and light pendula;

c. low values of α and β correspond to large τ, and small m and large l, namely, a stretched string with light long pendula;

d. high values of α and β correspond to small τ, and large m and small l, namely a sagged string with massive and short pendula.

Qualitatively, the eigenvalues with frequency of oscillation closer to α are more strongly modified with respect to the values of the vibrating string and the perturbation increases as β increases. Figure 5.3 shows the damping rates of the first eigenvalues for the four different cases. The first two modes are always less damped with respect to the others, but the most interesting cases are those where β is large (a., d.), and the presence of the

Figure 5.3: Dispersion relation for the damped string with two pendula with $\nu = 0.1$ for different values of the parameters α, β.



(a) $\alpha = 1,\ \beta = 25.$ **(b)** $\alpha = 25,\ \beta = 0.1.$ **(c)** $\alpha = 1,\ \beta = 0.1.$ **(d)** $\alpha = 25,\ \beta = 25.$

pendula affects significantly the motion of the string. In these circumstances the splitting of the first two damping rates from the others is large.

In Figure 5.4 is presented the evolution of the angular displacements ϕ_1 and ϕ_2 of the two pendula for the first four damped normal modes of oscillation. While the first two normal modes are weakly damped, those with higher frequencies decay after a few oscillations. We are therefore justified to consider the damped small oscillation of ϕ_1 and ϕ_2 as the sum of just the first two damped normal modes.

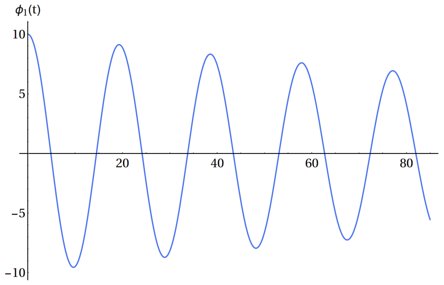
Below, Figures 5.5 - 5.8 illustrate – for different values of the parameters α, β – the temporal evolution of the displacements of the pendula as sum of the first two damped normal modes. The simulations have been made choosing the following initial conditions:

$$\begin{cases} \phi_1(0) = A \\ \phi_2(0) = 0 \\ \psi(x,0) = 0 \\ \dot{\phi}_1(0) = 0 \\ \dot{\phi}_2(0) = 0 \\ \psi_t(x,0) = 0. \end{cases}$$

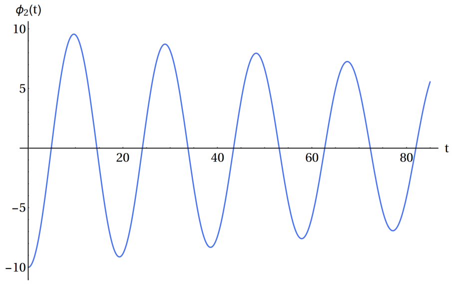
The initial evolution depends strongly on the parameters: small values of α lead to beats, while for large values, most of the initial impulse of the first pendulum is transferred to the second one within the first oscillation. However, in all cases, only the first damped normal mode survives at long times and eventually synchronization occurs.

Figure 5.4: $\phi_1(t)$ and $\phi_2(t)$ for the first four damped normal modes with $\alpha = 1$, $\beta = 25$.

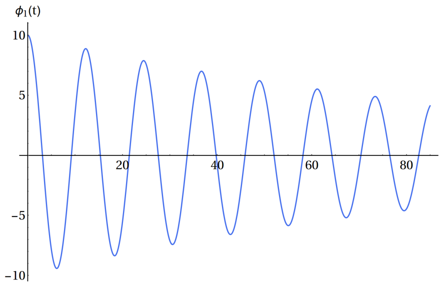
(a) $\phi_1(t)$ with λ_1.



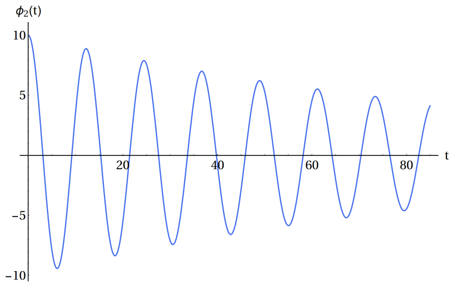
(b) $\phi_2(t)$ with λ_1.



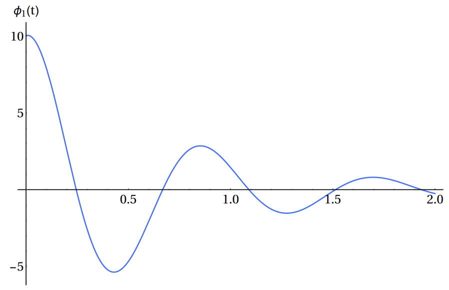
(c) $\phi_1(t)$ with λ_2.



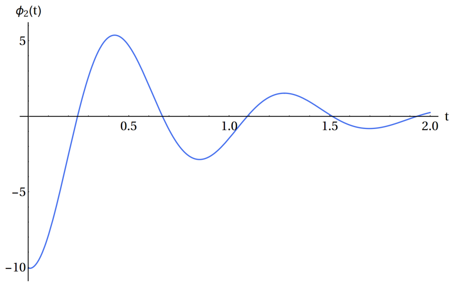
(d) $\phi_2(t)$ with λ_2.



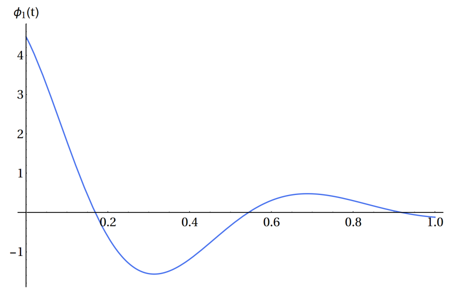
(e) $\phi_1(t)$ with λ_3.



(f) $\phi_2(t)$ with λ_3.



(g) $\phi_1(t)$ with λ_4.



(h) $\phi_2(t)$ with λ_4.

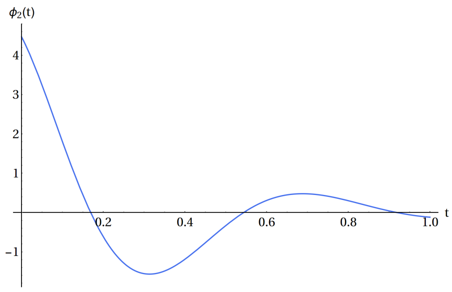
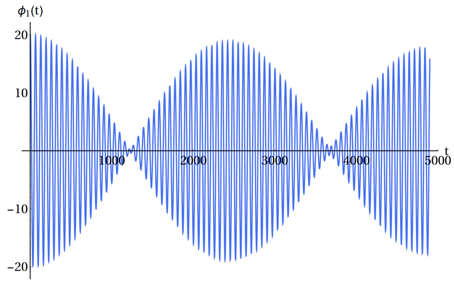
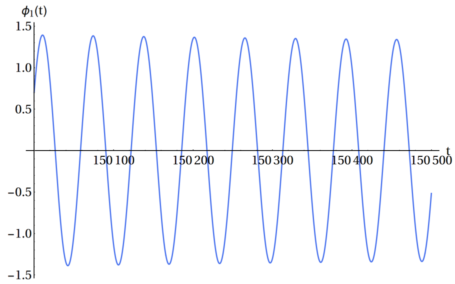


Figure 5.5: $\phi_1(t)$, $\phi_2(t)$ with λ_1, λ_2, with $\alpha = 0.1$, $\beta = 25$.

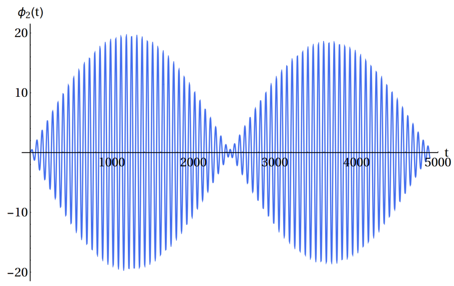
(a) $\phi_1(t)$ for $t \in (0, 5000)$.



(b) $\phi_1(t)$ for $t \in (150000, 200500)$.



(c) $\phi_2(t)$ for $t \in (0, 5000)$.



(d) $\phi_2(t)$ for $t \in (150000, 200500)$.

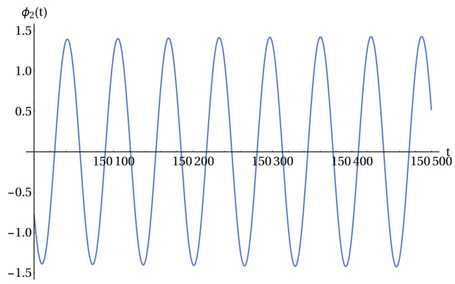
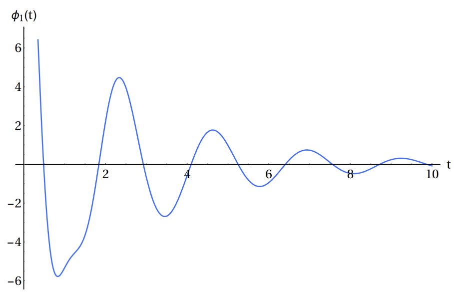


Figure 5.6: $\phi_1(t)$, $\phi_2(t)$ with λ_1, λ_2, with $\alpha = 25$, $\beta = 0.1$.

(a) $\phi_1(t)$.



(b) $\phi_2(t)$.

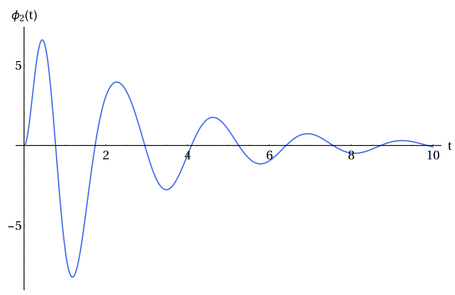


Figure 5.7: $\phi_1(t)$, $\phi_2(t)$ with λ_1, λ_2, with $\alpha = 1$, $\beta = 0.1$.

(a) $\phi_1(t)$. **(b)** $\phi_2(t)$.

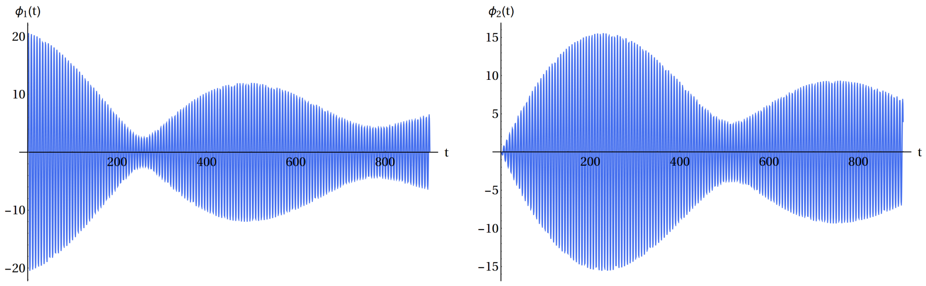
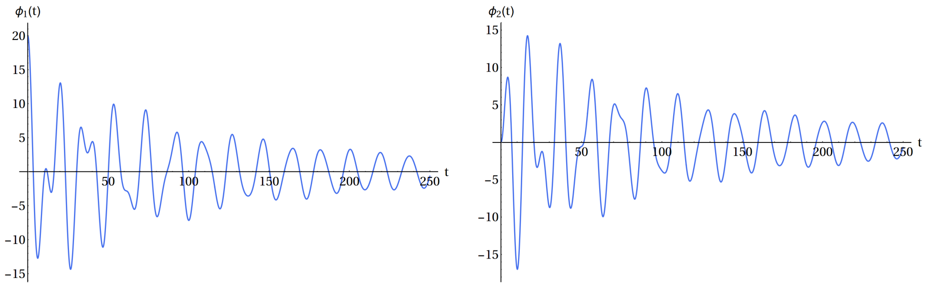


Figure 5.8: $\phi_1(t)$, $\phi_2(t)$ with λ_1, λ_2, with $\alpha = 25$, $\beta = 25$.

(a) $\phi_1(t)$. **(b)** $\phi_2(t)$.



Conclusions

In this thesis we have studied mechanisms of synchronization in finite-dimensional systems and in continuous systems coupled with discrete ones. In particular, it has been investigated the role of damping in the selection of some normal modes with respects to others and original results have been found.

We first considered a generic linear mechanical discrete system with a viscous damping term and proved that – under some hypotheses – the system admits an invariant subspace S in the configuration space such that its tangent bundle is the invariant centre space and it is attractive. On it the motion consists therefore of small oscillations, linear combinations of normal modes which do not dissipate, and the system tends asymptotically to this configuration. This property can lead to synchronization or beats, depending whether S is one-dimensional or higher-dimensional with frequencies close in values, respectively. The examples proposed showed this mechanism in the case of three simple models, which highlighted also that the type of synchronization, that is in-phase or anti-phase, depends on the particular system but it descends from the same mechanism.

As a natural continuation of this study, the case in which the damping matrix Γ has a trivial kernel but some of its eigenvalues are significantly smaller with respect to the others should be studied. In this configuration the system has a trivial centre space, nonetheless, one expects the existence of invariant subspaces in the phase space in which the damped normal modes are weakly damped.

Thereafter, we constructed a system consisting of a pendulum hanging from a heavy homogeneous flexible and elastic string, with extremities fixed. A linear analysis about the equilibrium configuration has been made and the study of the spectrum of frequencies, which is composed of a countable number of values plus an additional one, showed that the frequencies below the proper frequency of the pendulum are smaller with respect to the ones of the vibrating string, and, vice versa, are larger those above the proper frequency. Moreover, for certain values of the parameters, the special frequency might coincide with one of the unperturbed string, leading to normal modes in which half of the string is at rest, while the other half and the pendulum oscillate. We then added a dissipative contribution to the string, including an internal damping term, of Kelvin-Voigt type. This choice of friction, which studies showed to be in good agreement with experimental data, produces a splitting of the damping rates, which in the case of the vibrating string are proportional to the frequency squared. The eigenvalues of the coupled system have been computed in the limit of weak damping, through a linear series expansion with respect to the damping coefficient.

Since the spectrum of the system depends on the model of friction chosen, it might be interesting to employ alternative damping terms – for example, it may be realistic the inclusion of friction restricted to the extremities of the string – and to compare how different contributions affect the dynamics.

Finally, in the last two chapters we extended the study to the case in which there are two identical pendula suspended on the string. Such system accounts for the continuous nature of the support – unlike similar models in literature, which consider the pendula coupled through a rigid frame, elastically fixed, or discretise the beam by means of a finite number of masses. We determined the spectrum of the system numerically, both in the case of the undamped system and in the damped one, and we investigated qualitatively the dependence on the parameters which characterise the model. It emerged that the first two damped normal modes are significantly less damped with respect to those with higher frequencies, thus, after a transient, the motion is given by a linear superposition of the two of them only, and eventually the two pendula synchronise when only the less damped normal mode has not completely vanished.

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