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Robust Factor Analysis of Moving Avarage processes

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Abstract

This thesis addresses the problem of learning dynamic factor models generated by zero-mean Gaussian moving avarage (MA) processes.

Factor models boast a long tradition and find natural application in many engineering and scientific disciplines, including, for example, psychology, econometrics, system engineering, machine learning and statistics. In general, the attention for this kind of models is motivated by their effectiveness in complex-data representation. Indeed they allow the compression of the information contained in a high dimensional data vector into a small number of common factors, based on the assumption of underlying latent non-observed variables influencing all the observations.

In this thesis, we propose an extension of factor analysis to MA processes in order to extract the compressible information from them. To robustly estimate the number of factors, we construct a confidence region centered in a finite sample estimate of the underlying model which contains the true model with a prescribed probability. In this confidence region, the problem, formulated as a rank minimization of a suitable spectral density, is efficiently approximated via a trace norm convex relaxation.

Contents

List of Symbols

Symbol	Description
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integer numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^{n\times m}$	Set of $n \times m$ matrices with real entries.
$\mathbb{C}^{n\times m}$	Set of $n \times m$ matrices with complex entries.
$\ker(\cdot)$	Kernel of a matrix
$\text{rank}(\cdot)$	Rank of a matrix
$(\cdot)^\top$	Transpose of a matrix or vector
$(\cdot)^*$	Conjugate transpose of a matrix or vector
$(\cdot)^{-1}$	Inverse of a square matrix
$\text{tr}(\cdot)$	Trace of a matrix
$\lvert\cdot\rvert$	Determinant of a square matrix
$\sigma(\cdot)$	Spectrum of a matrix
$A_{(i,j)}$	Element of matrix A in the i-th row and j-th column
I_n	Identity matrix of dimension $n \times n$ (the subscript is omitted if clear by the context)
$\langle\cdot,\cdot\rangle$	Frobenius inner product, $\langle A, B\rangle := \text{tr}(A^\top B)$ for $A, B \in \mathbb{R}^{n\times n}$
$\lVert\cdot\rVert$	Frobenius norm induced by the Frobenius inner product
$\lVert\cdot\rVert_2$	Matrix spectral norm
$\mathbf{Q}_n$	Vector space of real symmetric matrices of size $n \times n$
$\mathbf{Q}_n^+$	Cone of real symmetric positive semi-definite matrices of size $n \times n$
$\mathbf{M}_{m,n}$	Vector space of matrices in the form $[Y_0\ Y_1\ ...\ Y_n]$, $Y_0 \in \mathbf{Q}_m, \quad Y_1, ..., Y_n \in \mathbb{R}^{m\times m}$
$\mathbf{Q}_{m(n+1)}$	Vector space of real symmetric block-matrices with $(n+1) \times (n+1)$ square blocks of dimension $m \times m$

$\succeq$, $\succ$	Partial ordering induced by the cone of symmetric positive semi-definite matrices
$\mathrm{diag}(A)$	Operator mapping a matrix $A \in \mathbb{R}^{n\times n}$ into a n-dimensional vector containing the diagonal elements of A
$\mathrm{diag}(\mathbf{a})$	Operator mapping a n-dimensional vector $\mathbf{a} \in \mathbb{R}^n$ into a diagonal matrix A of dimension $n \times n$ such that $\mathrm{diag}(A) = \mathbf{a}$
$\mathrm{diag}^2(A)$	Operator mapping a square matrix A into a diagonal matrix of the same dimension having the same main diagonal
$\mathrm{ofd}(A)$	Operator mapping a matrix $A \in \mathbb{R}^{n\times n}$ into a matrix of the same dimension having the same off-diagonal elements and the elements of the main diagonal equal to zero.
$\mathrm{ofd}_B(A)$	"Block ofd " operator mapping a matrix $Y \in \mathbf{M}_{m,n}$ into the matrix $\mathrm{ofd}_B(Y) := [\,\mathrm{ofd}(Y_0)\ \mathrm{ofd}(Y_1)\ldots\mathrm{ofd}(Y_n)\,]$.
$\mathcal{S}_m^+$	Space of $m \times m$ matrix-valued coercive and bounded spectral densities defined on the unit circle $\{e^{i\theta};\ \theta \in [-\pi, +\pi]\}$.
$\mathcal{Q}_{m,n}$	Set of $m \times m$ Hermitian pseudo-polynomial matrices of order n
$\mathbb{E}\{\cdot\}$	Expected value of a random variable or vector
$\lvert\mathcal{A}\rvert$	Cardinality of a set $\mathcal{A}$

1 Introduction

In modern society, the great development of technology has led to a frenetic increase of the quantity of available data; the interpretation and understanding of the information contained in these data is, on the one hand, an extremely challenging task and, on the other, a crucial step for any engineering application. For this reason, the research literature has recently witnessed an increasing interest in developing efficient methods to organize the available data in suitable structured models and to provide a concise and parsimonious representation of them or, in other words, to *extract meaningful information from the data.*

One of the classical methods for this purpose is based on *factor models.* Factor models are a statistical tool used to describe the variability among observed, correlated variables in terms of a potentially lower number of unobserved variables. Even though, in principle, they are a pure mathematical construction, the attention for this kind of models is motivated by their effectiveness in complex-data representation. Indeed, if high-dimensional vectors of observed variables can be explained in terms of a smaller number of common relevant factors, the statistical description of the phenomenon significantly simplifies. It is therefore clear that factor analysis provides a powerful tool for data dimensionality reduction, with attractive applications in many disciplines.

In its simplest form, the factor analysis problem may be formulated in mathematical terms as follows. A covariance matrix Σ must be additively decomposed as the sum of two positive semi-definite matrices: D and L, where D is diagonal, and L has the lowest possible rank. The matrix D models the covariance of the *idiosyncratic noise* (i.e. a noisy component affecting each channel of the observations independently of the others), while L accounts for the common latent

factors that explain the available data. Clearly, this problem can be generalized to the dynamic case by considering a "low-rank plus diagonal decomposition" of spectral densities of stationary stochastic processes.

In this thesis the attention is focused on moving average (MA) Gaussian processes; although the estimation of a MA process is relatively simple, it would be interesting to extract the compressible information from it. To this purpose, this thesis proposes an extension of factor analysis to moving average processes. In particular, considerable effort is devoted to deriving an identification procedure which guarantees robustness on the complexity of the estimated model (especially in terms of the number of factors) with respect to the finiteness of the data. This has a strong motivation on the fact that, in realistic situations, only a finite sample estimate of the power spectral density to be decomposed is actually available. In these cases, even if the underlying process is truly low rank, the minimum rank solution of the factor analysis problem rapidly degrades when a certain degree of uncertainty affects the estimation.

The contribution of this thesis can be summarized as follows. Given a finite sample estimate $\hat{\Phi}$ of the power spectral density of a MA process, the proposed procedure takes into account the uncertainty in the estimation by computing a "neighborhood" of $\hat{\Phi}$ containing the true model with a prescribed probability. In this neighborhood, we search for the factor model that provides the description of the phenomenon in terms of the minimum number of factors. To do that, we formulate the problem as a trace minimization problem, which represents a convex relaxation of the corresponding minimum rank problem. The designed objective induces a low rank plus diagonal decomposition of the power spectral density, thus ensuring a small number of latent variables and hence a parsimonious representation of the observed data. In order to obtain a numerically viable procedure for solving the problem, as a first step, we provide a finite dimensional matrix parametrization of it; the latter is then analyzed by resorting to the Lagrange duality theory. The dual analysis is delicate to carry over, but it allows to prove the existence of a solution to the problem and it is efficiently exploited to show the equivalence of the original problem, stated in terms of power spectral densities, and its matricial formulation. Because of the zero duality gap between the primal and the dual problem, it is also possible to easily recover a solution to the primal optimization problem from a dual optimal value. We hasten to remark that solving the dual problem is not an easy task. We try to exploit an alternating direction method of multiplier (ADMM) algorithm which, however, does not produce the desired result.

1.1 Outline of the Thesis

This thesis is organized as follows.

Chapter 2 provides an introductory survey on factor models: it starts by motivating their importance and then illustrates the classical mathematical formulation of the factor analysis problem, as well as some of the numerous variants in which it is proposed in literature.

In Chapter 3 we review the Lagrangian duality theory, which plays a central role in convex optimization and finds sensible applications in the proposed identification procedure.

Chapter 4 serves as a background on the alternating direction method of multipliers, an iterative algorithm which solves convex optimization problems by "breaking" them into smaller subproblems, easier to handle.

In Chapter 5 we consider the robust dynamic factor analysis problem of MA Gaussian processes.

Finally, Chapter 6 draws some conclusions and suggests possible future lines of study.

At the end of this thesis, Appendix A covers basic definitions, concepts and results concerning stationary stochastic processes that are omitted from the main text, while Appendix B presents a review of the necessary mathematical background on convex analysis and convex optimization problems. To avoid interruption in the flow, many of the proofs of Section 5.3.3 are deferred to Appendix C.

2

Factor Analysis

Factor models are a statistical tool used to compress the information contained in a high dimensional data vector into a much smaller number of non-observed variables, called common factors, that influence all the observations.
Such models were initially developed at the beginning of the last century by Spearman [1] in the framework of the so-called *mental tests* as an attempt to explain the relations among a group of test scores. From this first seed, the interest for this kind of models has grown rapidly, also outside the psychology community, and analysis for factor models, or *factor analysis*, has become an important tool in several engineering and science disciplines, among which systems and control theory, computer science, statistics, econometry and biology.

The main idea in factor analysis is that in several practical situations we have a large number of observed variables (or processes) and each of these can be modelled as the sum of a component depending on a small number of common factors and of an idiosyncratic noise, i.e. a noisy component affecting each of the variable independently of the others. The typical visual representation of this situation is that of a flock of birds where the trajectory of each single bird is determined by the "average" trajectory of the flock and by a variation proper to the individual bird.

2.1 Static factor analysis

In its simplest form, a *factor model* is a m-dimensional static linear model

$$y = W_L u + W_D w \tag{2.1}$$

where $W_L \in \mathbb{R}^{m\times r}$, with $r \ll m$, and $W_D \in \mathbb{R}^{m\times m}$ diagonal; $u := \begin{bmatrix} u_1 \dots u_r \end{bmatrix}^\top$ and $w := \begin{bmatrix} w_1 \dots w_m \end{bmatrix}^\top$ are Gaussian random vectors of dimension r and m, respectively, with zero mean and covariance equal to the identity. Moreover, u and w are independent, that is

$$\mathbb{E}\{uw^\top\} = 0. \tag{2.2}$$

In (2.1), the m-dimensional random vector y is called *observed vector*; W_L is the *factor loading matrix*, u represents the (independent) *latent factors* and $W_L u$ is the *latent variable*; finally w corresponds to the *idiosyncratic component*.

From (2.1), it is easy to see that

$$y_i = \sum_{j=1}^{r} [W_L]_{(ij)} u_j + [W_D]_{(ii)} w_i \tag{2.3}$$

which explains the reason why (2.1) is referred to as a factor model: the i-th observed variable y_i is generated by the r independent common factors $u_1, \dots, u_r$ and by the specific factor w_i.

In view of (2.1) and (2.2), the covariance matrix Σ of y is given by

$$\Sigma = L + D \tag{2.4}$$

where $L := W_L W_L^\top$ is a low-rank matrix - with rank equal to r - and $D := W_D W_D^\top$ is diagonal. The matrix D models the covariance of the idiosyncratic noise affecting each channel of the observations, while L accounts for the common latent factors that explain the available data.

The purpose of factor analysis consists in characterizing the common factors, representing the compressed information, and the idiosyncratic noise. In mathematical terms, this problem may be formulated as that of additively decomposing a covariance matrix Σ as in (2.4). Clearly, the rank of L must be as small as possible in order to provide a description of the phenomenon in terms of the minimum number of factors, thus leading to the *minimum rank factor analysis* (MRFA) problem:

$$\begin{aligned} \min_{L,D\in\mathbf{Q}_m,} \quad & \operatorname{rank}(L) \\ \text{subject to} \quad & L, D \succeq 0 \\ & D \text{ diagonal} \\ & L + D = \Sigma. \end{aligned} \tag{2.5}$$

Starting from this basic formulation, countless variations of factor analysis problems have been studied over the years; see for example [2] and [3] for an extensive

discussion on the numerous variants in which the problem may be formulated.

It is important to remark that, in general, minimizing the rank of L turns out to be extremely hard because of the non-convexity of the rank function, so that a widely used alternative considers the convex relaxation of (2.5). The latter, known as *minimum trace factor analysis* (MTFA) problem, implies the minimization of the trace norm of L in place of the rank:

$$\begin{aligned} \min_{L,D\in\mathbf{Q}_m,} \quad & \operatorname{tr}(L) \\ \text{subject to} \quad & L, D \succeq 0 \\ & D \text{ diagonal} \\ & L + D = \Sigma. \end{aligned} \tag{2.6}$$

The relation between the minimum trace and the classic minimum rank factor analysis problem has been first studied in [4]; it turns out that, even though problem (2.5) and (2.6) are in general *not* equivalent, (2.6) is a very good approximation that often returns, with a reasonable computation burden, a decomposition (2.4) where L has indeed the minimum possible rank.

One substantial difficulty in deriving a good "low-rank plus diagonal decomposition" is that, in practice, we only see data and we do not actually know the covariance matrix Σ that needs to be decomposed. The first idea to address this issue is to estimate the covariance matrix from the available data and then perform the required decomposition. However, this procedure has poor performances because the decomposition problem is ill-posed, so that a small error in the estimated spectral density $\hat{\Sigma}$ may have disastrous consequences in the number of factors.

A viable strategy to deal with the problem of robustly estimating the number of factor is presented in [5]: the authors compute the covariance matrix $\Sigma = L+D$ in such a way that the trace of L is minimized under a constraint limiting the Kullback-Leibler divergence [1] between Σ and $\hat{\Sigma}$ to a prescribed tolerance. This

[1]The *Kullback-Leibler (KL) divergence* measures the deviance among probability densities. If p and q are two probability densities on $\mathbb{R}^m$, the KL divergence from q to p is defined to be

$$\mathcal{D}_{KL}(p||q) := \int_{\mathbb{R}^m} p \log(\frac{p}{q}).$$

When p and q are two zero-mean Gaussian densities with positive covariance matrices P and Q, respectively, then $\mathcal{D}_{KL}(p||q)$ can be easily computed as

$$\mathcal{D}_{KL}(p||q) = \frac{1}{2}(\log|P^{-1}Q| + \operatorname{tr}(Q^{-1}P) - m).$$

We should mention that $\mathcal{D}_{KL}$ is not a true metric because it does not obey the triangle inequality and the symmetric property.

tolerance must depend on the precision of the estimate $\hat{\Sigma}$ and it is reliably chosen on the basis of the data numerosity N and of the data dimension m by exploiting a scale invariance property of the Kullack-Leibler pseudo-distance.

2.2 Dynamic factor analysis

Whereas the initial approach to factor analysis was oriented to data originating from independent, identically distributed random variables, the idea has been further generalized to data originate from stochastic processes.
As shown in [6], the natural extension of (2.1) to the dynamic case is:

$$y_t = \Gamma_L u_t + \Gamma_D w_t, \qquad t \in \mathbb{Z}, \tag{2.7}$$

where Γ_L and Γ_D are transfer functions in the form

$$\Gamma_L(e^{i\vartheta}) = \sum_{k=-\infty}^{+\infty} W_{L,k} e^{-i\vartheta k}, \qquad W_{L,k} \in \mathbb{R}^{m\times r} \tag{2.8}$$

$$\Gamma_D(e^{i\vartheta}) = \sum_{k=-\infty}^{+\infty} W_{D,k} e^{-i\vartheta k}, \qquad W_{D,k} \in \mathbb{R}^{m\times m} \text{ diagonal} \tag{2.9}$$

and $u = \{u_t,\ t \in \mathbb{Z}\}$ and $w = \{w_t,\ t \in \mathbb{Z}\}$ are i.i.d. Gaussian processes of dimension r and m, respectively, with zero mean and covariance equal to the identity. Moreover, u and w are such that

$$\mathbb{E}\{u_t w_s^\top\} = 0 \qquad \forall t, s \in \mathbb{Z}. \tag{2.10}$$

From (2.10), it immediately follows that $y = \{y_t,\ t \in \mathbb{Z}\}$ is a zero mean Gaussian process with power spectral density

$$\Phi = \Phi_L + \Phi_D \tag{2.11}$$

where $\Phi_L = \Gamma_L \Gamma_L^* \succeq 0$ and $\Phi_D = \Gamma_D \Gamma_D^* \succeq 0$. By construction, $\text{rank}(\Phi_L) = r$, and Φ_D is diagonal. Therefore, y represents a factor model if its spectral density can be decomposed as "low rank plus diagonal" as in (2.11).

Different approaches have been considered to tackle the corresponding minimum-rank problem. For instance, in [7] the spectral density is approximated by a piecewise constant function and the factor analysis is then performed piecewise; in [8], [9] and [10] the authors consider the factor analysis problem in the special case in which the common (dynamic) factors are only combined in a static way.

It is worth remarking that, as in the static case, the minimum-rank factor

analysis problem is NP-hard, thus it is convenient to relax it in order to obtain a tractable convex-optimization problem. In [11] the function $\mathrm{tr}\int_{-\pi}^{+\pi}\Phi_L(e^{i\theta})\frac{d\theta}{2\pi}$ is proposed as an adequate approximation of $\mathrm{rank}(\Phi_L)$. The effectiveness of this approximation is justified by the following result:

Proposition 2.2.1. Let $\mathcal{A}_m$ be the linear space of $\mathbb{C}^{m\times m}$-valued analytic functions on the unit circle. Let $\Phi\in\mathcal{A}_m$ be an arbitrary analytic function such that $\Phi\succeq 0$. Define the following restricted rank function

$$\mathrm{rank}'(\Phi):=\begin{cases}\mathrm{rank}(\Phi), & \|\ \Phi\ \|\leq 1\\ +\infty, & \text{otherwise.}\end{cases}\tag{2.12}$$

Then, the convex hull of $\mathrm{rank}'(\Phi)$ is

$$\mathrm{co}\,\mathrm{rank}'(\Phi):=\begin{cases}\mathrm{tr}\int_{-\pi}^{+\pi}\Phi(e^{i\theta})\frac{d\theta}{2\pi}, & \|\ \Phi\ \|\leq 1\\ +\infty, & \text{otherwise.}\end{cases}\tag{2.13}$$

This result has been exploited in [12], where the factor analysis for moving average processes is formulated as a rank-minimization problem, and then approximated via the trace norm relaxation. The estimated decomposition, however, is good only provided that the number of data points is sufficiently large.

As already observed for the static case, in the realistic situation in which only a finite sample estimate $\hat{\Phi}$ of the spectral density Φ to be decomposed is available, the accuracy in the estimation may severely affect the goodness of the decomposition (2.11). An interesting approach to guarantee robustness of performances is presented in [13], where the authors face the problem of robustly identifying *latent variable auto-regressive (AR) dynamic graphical models* from a given finite sample estimate $\hat{\Phi}$ of the spectral density of the underlying process. In a similar spirit to [5], they introduce a confidence region for the spectral density which contains the true model with a prescribed probability; the "radius" of this set reflects the accuracy in the estimation, and it is computed by leveraging on a scale invariant property of the Itakura-Saito divergence. [2]

[2]The *Itakura-Saito divergence* measures the distance between two spectra $\Phi,\Psi\in\mathcal{S}_m^+$ [14]:

$$\mathcal{S}_{IS}(\Phi||\Psi):=\int_{-\pi}^{+\pi}\left\{\log|\Phi^{-1}(e^{i\theta})\Psi(e^{i\theta})|+\mathrm{tr}(\Psi^{-1}(e^{i\theta})\Phi(e^{i\theta}))-I_m\right\}\frac{d\theta}{2\pi}.$$

Notice that $\mathcal{S}_{IS}$ is not a true metric, since it is not symmetric and it does not fulfill the triangle inequality.

3

Duality Theory

This chapter covers Lagrangian duality, which plays a central role in constrained convex optimization.
After defining the Lagrange dual problem, whose solution provides a lower bound to the solution to the primal (minimization) problem, we illustrate the saddle point interpretation of duality and we state the classical Karush-Kuhn-Tucker conditions for optimality. For simplicity, throughout the chapter the attention is narrowed to real-valued functions defined on the Euclidean space $\mathbb{R}^n$.
Notice that the main part of the results reported below are taken from [15, chapter 5] and this reference will be omitted; we refer the reader to Appendix B for a review of the necessary background on convex analysis and mathematical optimization.

3.1 The Lagrange dual problem

Consider an optimization problem in the standard form (see Appendix B.3)

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f_0(x) \\ \text{subject to } &f_i(x) \leq 0, \quad i = 1, ..., m \\ &h_i(x) = 0, \quad i = 1, ..., p. \end{aligned} \tag{3.1}$$

with $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 0, ..., m$ and $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., p$.
We hasten to remark that we do not assume convexity of the problem.
Let

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\ h_i$$

be the domain of the optimization problem, which we supposed non-empty, and let p^* be the optimal value of (3.1).

The basic idea in Lagrangian duality is to take the constraints in (3.1) into account by augmenting the objective function with a weighted sum of the constraints functions. Accordingly, we define the *Langrangian function* as follows:

Definition 3.1.1 (Lagrangian function). Given the vectors $\lambda = [\lambda_1, \ldots, \lambda_m]$ and $\nu = [\nu_1 \ldots, \nu_p]$, called *dual variables* or *Lagrange multiplier vectors* associated with the problem (3.1), we define the *Lagrangian function* $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \tag{3.2}$$

with $\mathbf{dom}\ L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$.

The value of the Lagrangian function $L(x, \lambda, \nu)$ depends on the original variable x as well as the dual variables (λ, ν). By minimizing the Lagrangian over x, thus by solving un unconstrained optimization problem, we obtain the so-called *dual function*:

Definition 3.1.2 (Dual function). The *Lagrange dual function*, or just *dual function*, $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ associated with the problem (3.1) is defined as

$$g(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \Big(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \Big) \tag{3.3}$$

with domain

$$\mathbf{dom}\ g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}. \tag{3.4}$$

Proposition 3.1.1. The dual function g is concave.

Proof. The dual function (3.3) is the pointwise infimum of a family of affine (so concave) functions of (λ, ν). Consequently, it is concave (cf. Proposition B.2.4). □

It is important to remark that the dual function g is always concave, independently of the original problem (3.1).

Another important observation is that, by construction, the dual function yields lower bounds on the optimal values p^* of the original problem (3.1) since

for any $\lambda \geq 0$ and any ν it holds:

$$g(\lambda, \nu) \leq p^*. \tag{3.5}$$

This important property is easily verified. Let $\tilde{x}$ be any feasible point for the problem (3.1), i.e. $f_i(\tilde{x}) \leq 0$ and $h_i(\tilde{x}) = 0$; then

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x}) \tag{3.6}$$

since each term in the first sum is non-positive, and each term in the second sum is zero.
Hence

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq f_0(\tilde{x}); \tag{3.7}$$

since (3.7) holds for every feasible point $\tilde{x}$, (3.5) immediately follows.
By (3.5), for each pair (λ, ν) with $\lambda \geq 0$, the dual function $g(\lambda, \nu)$ provides a lower bound on the optimal value p^* which depends on the parameters λ and ν. Notice that such a bound is vacuous when $g(\lambda, \nu) = -\infty$: in order to have a non trivial lower bound on p^* we require a pair (λ, ν) with $\lambda \geq 0$ and $g(\lambda, \nu) > -\infty$, i.e. $(\lambda, \nu) \in \mathbf{dom}\ g$.

Now, a natural question is: what is the best lower bound that we can obtain from the dual function? The answer leads to the following optimization problem:

$$\begin{aligned} &\max_{\lambda, \nu} \ g(\lambda, \nu) \\ &\text{subject to} \ \ \lambda \geq 0. \end{aligned} \tag{3.8}$$

This problem is called *Lagrange dual problem* associated with the problem (3.1). In this context, the latter is usually referred to as *primal problem*. We say that a pair (λ, ν) with $\lambda \geq 0$ and $g(\lambda, \nu) > -\infty$ (i.e. $(\lambda, \nu) \in \mathbf{dom}\ g$) is *dual feasible*. We refer to (λ^*, ν^*) as *dual optimal* or *optimal Lagrange multipliers* if they are solution for problem (3.8), and we denote d^* the optimal value of the Lagrange dual problem.

It is important to remark that the Lagrange dual problem (3.8) is always a convex optimization problem, whether or not the primal problem (3.1) is convex. Indeed, problem (3.8) concerns the maximization of a concave function g over a convex set (that it equivalent to minimize the convex function $-g$ over a convex set).

The optimal value d^* of the Lagrange dual problem is, by definition, the best lower bound on p^* that we can obtain from the Lagrangian dual function g.

Clearly, by (3.5) we have

$$d^* \leq p^*, \tag{3.9}$$

which holds even when the original problem is not convex. This simple but important property is called *weak duality*.
The weak duality inequality (3.9) still holds when d^* and p^* are infinite: if the primal problem is infeasible, we have by definition $p^* = \infty$ and $d^* = \infty$, so that (3.9) is verified. Conversely, if the primal problem is unbounded below, i.e. $p^* = -\infty$, we get $d^* = -\infty$, and (3.9) is verified even in this situation.

The nonnegative difference $p^* - d^*$ is named *optimal duality gap* of the original problem. If the duality gap is zero, then

$$d^* = p^*, \tag{3.10}$$

and we say that *strong duality* holds.

Conditions for strong duality to hold are very important for the purpose of solving (3.1). In general, strong duality is not satisfied; however if the primal problem (3.1) is convex, we usually (not always) have strong duality. *Slater's condition* is a sufficient condition for strong duality to hold for a convex optimization problem:

Proposition 3.1.2 (Slater's condition). If (3.1) is a convex optimization problem and there exists $x \in \mathbf{relint}\ \mathcal{D}$ [1] such that

$$f_i(x) < 0, \quad i = 1, \ldots, m, \qquad h_i(x) = 0, \quad i = 1, \ldots, p, \tag{3.11}$$

then strong duality hold. (3.11) is called *Slater's condition*. Moreover, if the inequality constraint functions f_i, $i = 1, \ldots, k$, $k \leq m$ are affine functions, Slater's condition can be refined as follows: strong duality holds if there exists an $x \in \mathbf{relint}\ \mathcal{D}$ such that

$$\begin{aligned} f_i(x) &\leq 0, \quad i = 1, \ldots, k \\ f_i(x) &< 0, \quad i = k+1, \ldots, m \\ h_i(x) &= 0, \quad i = 1, \ldots, p. \end{aligned} \tag{3.12}$$

[1] The *relative interior* of a set C, denoted as $\mathbf{relint}\ C$, is the interior relative to the *affine hull* of C $\mathbf{aff}\ C := \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid n \in \mathbb{N}, x_i \in C, \alpha_i \in \mathbb{R} \text{ s.t.} \sum_{i=1}^n \alpha_i = 1\}$, i.e.

$$\mathbf{relint}\ C := \{x \in C \mid B(x, r) \cap \mathbf{aff}\ C \subseteq C \text{ for some } r > 0\}$$

where $B(x, r)$ is the ball of radius r and center x in any norm (all norms define the same relative interior).

In other words, Slater's condition for convex programming states that strong duality holds if there exists an x such that x is *strictly feasible*, namely it satisfies all the constraints of the problem (3.1), and in particular the nonlinear inequality constraints with strict sign.

Slater's condition (and its refinement) not only implies strong duality for convex problems, it also implies that there exists a dual feasible (λ^*, ν^*) with $g(\lambda^*, \nu^*) = d^* = p^*$ (and in particular $d^* > -\infty$).

3.2 Saddle-point interpretation

In this section we provide a *saddle point* formulation of the optimality conditions for the primal and the dual optimization problems. For the sake of simplicity, we assume that there are no equality constraints in the primal problem (3.1). It is easy to extend the results to cover them.

Consider the following problem:

$$\sup_{\lambda \geq 0} L(x, \lambda) = \sup_{\lambda \geq 0} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right)$$

Clearly, if x is not feasible, $f_i(x) > 0$ for some i, so that $\sup_{\lambda \geq 0} L(x, \lambda) = +\infty$, as can be seen by choosing $\lambda_j = 0$, $j \neq i$ and $\lambda_i \to \infty$. On the other hand, if x is feasible, i.e. $f_i(x) \leq 0$ for $i = 1, \dots m$, then the optimal choice of λ is $\lambda = 0$ and $\sup_{\lambda \geq 0} L(x, \lambda) = f_0(x)$. In view of this consideration,

$$\sup_{\lambda \geq 0} L(x, \lambda) = \begin{cases} f_0(x) & f_i(x) \leq 0,\ i = 1, \dots, m \\ +\infty & \text{otherwise.} \end{cases}$$

This means that we can express the optimal value of the primal problem as

$$p^* = \inf_x \sup_{\lambda \geq 0} L(x, \lambda).$$

Moreover, by definition of dual function, we have

$$d^* = \sup_{\lambda \geq 0} \inf_x L(x, \lambda).$$

As a consequence, weak duality can be expressed as

$$\sup_{\lambda \geq 0} \inf_x L(x, \lambda) \leq \inf_x \sup_{\lambda \geq 0} L(x, \lambda) \tag{3.13}$$

and strong duality becomes

$$\sup_{\lambda \geq 0} \inf_{x} L(x, \lambda) = \inf_{x} \sup_{\lambda \geq 0} L(x, \lambda). \tag{3.14}$$

Actually (3.13) does not depend on any properties of the Lagrangian function; it is a general result that holds for any function f:

Theorem 3.2.1 (max-min inequality). Let $W \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^m$. For any $f : W \times Z \to \mathbb{R}$ the *max-min inequality* holds:

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z). \tag{3.15}$$

When equality holds, i.e.

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) = \inf_{w \in W} \sup_{z \in Z} f(w, z), \tag{3.16}$$

one says that f (and W and Z) satisfies the *strong max–min property* or the *saddle-point property.*

We provide the following definition of *saddle point*:

Definition 3.2.1 (Saddle point). A pair $(\tilde{w}, \tilde{z}) \in W \times Z$ is called a *saddle point* for f if

$$f(\tilde{w}, z) \leq f(\tilde{w}, \tilde{z}) \leq f(w, \tilde{z}) \qquad \forall w \in W, \ \forall z \in Z. \tag{3.17}$$

In other words, $(\tilde{w}, \tilde{z})$ is a saddle point if $\tilde{w}$ minimizes $f(w, \tilde{z})$ over W and $\tilde{z}$ minimizes $f(\tilde{w}, z)$ over Z. The following result can be proved [16]:

Proposition 3.2.1. The pair $(\tilde{w}, \tilde{z})$ is a saddle point for $f : W \times Z \to \mathbb{R}$ if and only if the max-min equality (3.16) holds and

$$\tilde{w} \in \arg\min_{w \in W} \sup_{z \in Z} f(w, z), \qquad \tilde{z} \in \arg\max_{z \in Z} \inf_{w \in W} f(w, z). \tag{3.18}$$

Returning to our discussion of Lagrange duality, we see that if x^* and λ^* are primal and dual optimal points for a problem in which strong duality holds, they form a saddle point for the Lagrangian. The converse is also true: if (x, λ) is a saddle point for the Lagrangian, then x is primal optimal, λ is dual optimal and

the duality gap is zero.

3.3 Optimality conditions

Suppose that the primal and dual optimal values are attained, x^* be a primal optimal and (λ^*, ν^*) be a dual optimal point. Assume that strong duality holds. This means that

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\leq f_0(x*) \end{aligned}$$

where the first line states the zero duality gap and the second line is the definition of the dual function. The third line follows from the observation that the infimum of the Lagrangian over x is less or equal to its value at $x = x^*$. Finally the last inequality follows from $\lambda_i^* \geq 0$, $f_i(x^*) \leq 0$, $i = 1, \ldots, m$ and $h_i(x^*) = 0$, $i = 1, \ldots, p$. We conclude that the two inequalities in this chain hold with equality, from which we can draw some interesting conclusions.

For example, since the inequality in the last line is an equality, i.e.

$$f_0(x^*) = f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*),$$

we can conclude that

$$\sum_{i=1}^m \lambda_i^* f_i(x^*) = 0.$$

Then, since each term in this sum is nonpositive, it follows that

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m. \tag{3.19}$$

This condition is known as *complementary slackness condition* and it holds for any primal optimal x^* and any dual optimal (λ^*, ν^*) as soon as strong duality holds. The complementary slackness conditions means that

$$\lambda_i^* > 0 \implies f_i(x^*) = 0$$

or, equivalently,

$$f_i(x^*) < 0 \implies \lambda_i^* = 0.$$

In other words, the ith optimal Lagrange multiplier is zero unless the ith constraint is active at the optimum, i.e. $f_i(x^*) = 0$.

Moreover, since the inequality in the third line is an equality

$$\inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right) = f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*)$$

we conclude that x^* minimizes $L(x, \lambda^*, \nu^*)$ over x (the Lagrangian can have other minimizers, x^* is simply *a* minimizer).

Assume now that the functions $f_0, \dots f_m$ and $h_1, \dots, h_p$ are differentiable (and therefore they have open domains), so that the Lagrangian is a differentiable function. Since x^* minimizes $L(x, \lambda^*, \nu^*)$ over x, its gradient must vanish at x^*, i.e.

$$\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) = 0.$$

We introduce the following definition::

Definition 3.3.1 (Karush-Kuhn-Tucker conditions). We say that the points x^* and (λ^*, ν^*) satisfy the so-called *Karush-Kuhn-Tucker* (KKT) conditions for the problem (3.1) with f_i, $i = 1, \dots, m$ and h_i, $i = 1, \dots, p$ differentiable if

$$f_i(x^*) \leq 0, \qquad i = 1, \dots, m \tag{3.20}$$

$$h_i(x^*) \leq 0, \qquad i = 1, \dots, p \tag{3.21}$$

$$\lambda_i^* \geq 0, \qquad i = 1, \dots, m \tag{3.22}$$

$$\lambda_i^* f_i(x^*) = 0, \qquad i = 1, \dots, m \tag{3.23}$$

$$\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) = 0. \tag{3.24}$$

The first two conditions assures the feasibility of point x^*, while the third condition expresses the nonnegativity of the Lagrange multipliers associated with the inequality constraints. The equation (3.23) is the complementary slackness condition and finally (3.24) says that x^* is a stationary point for the Lagrangian at (λ^*, ν^*).

The previous observations lead to the following result:

Proposition 3.3.1. For *any* optimization problem (3.1) with differentiable objective and constraint functions for which strong duality obtains, any pair of primal and dual optimal points must satisfy the KKT conditions (3.20) - (3.24).

When the primal problem is convex, the KKT conditions are also sufficient for the points to be primal and dual optimal:

Proposition 3.3.2. If (3.1) is a convex optimization problem with differentiable objective and constraint functions, if x^*, λ^* and ν^* are any points that satisfy the KKT conditions (3.20) - (3.24), then x^* and (λ^*, ν^*) are primal and dual optimal, respectively, with zero duality gap.

Proof. (3.20) - (3.21) state that x^* is primal feasible. Since $\lambda_i^* \geq 0$, f_i, $i = 1, \ldots, m$ are convex and h_i, $i = 1, \ldots, p$ affine, the function $L(x, \lambda^*, \nu^*) = f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x)$ is convex in x. The last KKT condition states that the gradient of $L(x, \lambda^*, \nu^*)$ vanishes at x^*, so that, by convexity, x^* minimizes $L(x, \lambda^*, \nu^*)$ over x. We can conclude

$$\begin{aligned} g(\lambda^*, \nu^*) &= \inf_x L(x, \lambda^*, \nu^*) \\ &= L(x^*, \lambda^*, \nu^*) \\ &= f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &= f_0(x^*) \end{aligned}$$

where in the last equality we have used (3.21) and (3.23). This shows that x^* and (λ^*, ν^*) have zero duality gap, and therefore they are primal and dual optimal. □

Remark 1. If (3.1) is a convex optimization problem with differentiable objective and constraint functions satisfying the Slater's condition, then the KKT conditions become necessary and sufficient optimality conditions. Indeed, Slater's condition implies that the optimal duality gap is zero and the dual optimum is attained, so x is optimal if and only if there exist (λ, ν) that, together with x, satisfy the KKT conditions.

4

Alternating Direction Method of Multipliers

This chapter serves as a background on the Alternating Direction Method of Multipliers (ADMM), a simple but powerful iterative algorithm well-suited to distributed convex optimization, especially to some large-scale problems arising in statistics, machine learning and related area. It takes the form of a *decomposition-coordination* procedure, in which the solutions to small local sub-problems are coordinated to find a solution to a large global problem.
The method was developed in the 1970s, with roots in the 1950s, and then popularized by the work of Stephen Boyd [17], to which we refer throughout the chapter; this reference will be omitted.

4.1 Algorithm

In its most general form, ADMM can solve convex optimization problem in the form

$$\begin{aligned} \min_{x,z} \quad & f(x) + g(z) \\ \text{subject to} \quad & Ax + Bz = c \end{aligned} \tag{4.1}$$

where the optimization variables are $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ are convex. Here $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{p \times m}$ and $c \in \mathbb{R}^p$.
We suppose that problem (4.1) has at least one solution and p^* is the optimal value. In addition, we assume that f and g are differentiable, with gradients $\nabla f(x)$ and $\nabla g(z)$, respectively.

It is important to remark that in problem (4.1) we have two sets of optimization variables, x and z, and the objective function is separable across the splitting.

In order to introduce the ADMM algorithm, we have to define the so-called *augmented Lagrangian* for (4.1), which is

$$L_\rho(x, z, y) := f(x) + g(z) + y^\top (Ax + Bz - c) + \frac{\rho}{2} \parallel Ax + Bz - c \parallel_2^2 \tag{4.2}$$

where y is the *dual variable* and $\rho \geq 0$ is the so-called *penalty term*.
Note that (4.2) differs from the commonly used Lagrangian function (3.2) for the addition of the *penalty term* $\frac{\rho}{2} \parallel Ax + Bz - c \parallel_2^2$. The purpose of the penalty term is to ensure that the optimization algorithm is, at all times, close to satisfying the linear constraint $Ax + By = c$.
It is worth remarking that the augmented Lagrangian can be viewed as the (unaugmented) Lagrangian associated with the problem

$$\begin{aligned} \min_{x,z} \quad & f(x) + g(z) + \frac{\rho}{2} \parallel Ax + Bz - c \parallel_2^2 \\ \text{subject to} \quad & Ax + Bz = c, \end{aligned} \tag{4.3}$$

which is clearly equivalent to the original problem (4.1), since for any feasible (x, z) the term added to the objective is zero.

The ADMM algorithm is defined by the iteration of the following three steps:

$$x^{k+1} := \arg\min_x L_\rho(x, z^k, y^k) \tag{4.4}$$

$$z^{k+1} := \arg\min_z L_\rho(x^{k+1}, z, y^k) \tag{4.5}$$

$$y^{k+1} := y^k + \rho(Ax^{k+1} + Bz^{k+1} - c). \tag{4.6}$$

The equation (4.4) is an x-minimization step, (4.5) is a z-minimization step, and finally (4.6) is a dual variable update with step size equal to the augmented Lagrangian parameter ρ.
Note that, instead of jointly solving for x and z, ADMM alternates the update of x and z, which accounts for the term *alternating direction*.

4.2 Convergence

There are many convergence results for ADMM discussed in the literature. Boyd [17] shows that if the following two assumptions hold:

1. the (extended-real-valued) functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ are closed, convex and proper;

2. the unaugmented Lagrangian L_0 (that is the augmented Lagrangian with $\rho = 0$) has a saddle point, namely there exists (x^*, z^*, y^*), not necessarily unique, for which

$$L_0(x^*, z^*, y) \leq L_0(x^*, z^*, y^*) \leq L_0(x, z, y^*)$$

holds for all x, z and y;

then the ADMM iterates satisfy the following properties:

- *Residual convergence*: the residual $r^k := Ax^k + Bz^k - c \to 0$ as $k \to \infty$. This means that the iterates approach feasibility.
- *Objective convergence* $f(x^k) + g(z^k) \to p^*$ as $k \to \infty$, i.e. the objective function approaches the optimal value.
- *Dual variable convergence*: $y^k \to y^*$ as $k \to \infty$, where y^* is a dual optimal point.

Note that x^k and z^k need not to converge to optimal values, although such results can be shown under additional assumptions.

In literature there are also other less restrictive convergence proofs, with different penalty terms or inexact minimization.

4.3 Optimality conditions ans stopping criterion

In order to define a stopping criterion for the algorithm, we start analyzing optimality conditions for (4.1). Necessary and sufficient optimality conditions are primal feasibility,

$$Ax^* + Bz^* - c = 0 \tag{4.7}$$

and dual feasibility

$$0 = \nabla f(x^*) + A^\top y^* \tag{4.8}$$
$$0 = \nabla g(z^*) + B^\top y^*. \tag{4.9}$$

The last condition (4.9) always holds for $(x^{k+1}, z^{k+1}, y^{k+1})$: since z^{k+1} minimizes $L_\rho(x^{k+1}, z, y^k)$ by definition, we have

$$\begin{aligned} 0 &= \nabla g(z^{k+1}) + B^\top y^k + \rho B^\top (Ax^{k+1} + Bz^{k+1} - c) \\ &= \nabla g(z^{k+1}) + B^\top \big(y^k + \rho(Ax^{k+1} + Bz^{k+1} - c)\big) \\ &= \nabla g(z^{k+1}) + B^\top y^{k+1}, \end{aligned}$$

where in the last equality we have exploited the definition of the dual update (4.6). This means that z^{k+1} and y^{k+1} always satisfy (4.9), hence optimality boils down to satisfying (4.7) and (4.8).

The residuals for the conditions (4.7) and (4.8) can be defined as follows:

$$r^{k+1} := Ax^{k+1} + Bz^{k+1} - c \tag{4.10}$$

is the *primal residual* at the iteration $k+1$, while the quantity

$$s^{k+1} := \rho A^\top B(z^{k+1} - z^k) \tag{4.11}$$

is the *dual residual* at iteration $k+1$.
While the choice of the primal residual r^{k+1} is intuitive, it is necessary to motivate the choice of s^{k+1} as the residual for the dual feasibility condition (4.8). To this aim, we immediately notice that, since x^{k+1} minimizes $L_\rho(x, z^k, y^k)$ by definition, we have

$$\begin{aligned}
0 &= \nabla f(x^{k+1}) + A^\top y^k + \rho A^\top (Ax^{k+1} + Bz^k - c) \\
&= \nabla f(x^{k+1}) + A^\top \big(y^k + \rho(Ax^{k+1} + Bz^k - c)\big) \\
&= \nabla f(x^{k+1}) + A^\top \big(y^k + \rho(Ax^{k+1} + Bz^{k+1} - c + Bz^k - Bz^{k+1})\big) \\
&= \nabla f(x^{k+1}) + A^\top \big(y^k + \rho(Ax^{k+1} + Bz^{k+1} - c) + \rho(Bz^k - Bz^{k+1}))\big) \\
&= \nabla f(x^{k+1}) + A^\top y^{k+1} + \rho A^\top B(z^k - Bz^{k+1})
\end{aligned}$$

where the last equality follows from (4.6). Equivalently,

$$\nabla f(x^{k+1}) + A^\top y^{k+1} = \rho A^\top B(z^{k+1} - z^k) \tag{4.12}$$

and, by comparing (4.12) with (4.8), it is clear that the quantity $s^{k+1} = \rho A^\top B(z^{k+1} - z^k)$ can be viewed as a residual for the optimality condition (4.8).

Now, in order to define the stopping criterion for the ADMM, we can relate the residuals r^{k+1} and s^{k+1} to a bound on the objective suboptimality of the current point, i.e. to the quantity $f(x^k) + g(z^k) - p^*$. In particular, it is possible to show that if the residuals r^k and s^k are small, then the objective suboptimality must be small as well. This suggests that a reasonable termination criterion is

$$\| r^k \| \leq \epsilon^{pri}$$

and

$$\| s^k \| \leq \epsilon^{dual}$$

where $\epsilon^{pri} > 0$ and $\epsilon^{dual} > 0$ are the feasibility tolerances for optimality conditions (4.7) and (4.8), respectively. These tolerances can be chosen using an absolute and relative criterion, such as

$$\epsilon^{pri} = \sqrt{p}\epsilon^{abs} + \epsilon^{rel} \max\left\{\| Ax^k \|, \| Bz^k \|, \| c \|\right\} \tag{4.13}$$

$$\epsilon^{dual} = \sqrt{n}\epsilon^{abs} + \epsilon^{rel} \| A^\top y^k \| \tag{4.14}$$

where $\epsilon^{abs} > 0$ is an absolute tolerance and $\epsilon^{rel} > 0$ is a relative tolerance.
The factors $\sqrt{p}$ and $\sqrt{n}$ account for the fact that the norms are taken in $\mathbb{R}^p$ and $\mathbb{R}^n$, respectively.
A reasonable value for ϵ^{rel} might be 10^{-3} or 10^{-4}, depending on the application, while the choice of an absolute stopping criterion ϵ^{abs} depends on the scale of the typical variable values.

4.4 Extension and variations

Many variations of the classic ADMM algorithm have been explored in literature. Here, we briefly discuss two interesting variants that we will exploit in our factor analysis problem.

A first standard extension concerns the penalty parameters ρ. Experience on applications has shown that the number of ADMM iterations depends significantly on the value of the penalty parameter as it influences the decreasing speed of r^k and s^k to zero. As a matter of fact, from the definition of the augmented Lagrangian (4.2) and the ADMM update equations (4.4)-(4.6), it is clear that large values of ρ place a large penalty on violations of primal feasibility and so tend to produce small primal residuals. On the contrary, the definition (4.11) of s^k suggests that small values of ρ tend to reduce the dual residual, but at the expense of reducing the penalty on primal feasibility, which may result in a larger primal residual.
These observations originate the idea of using a variable penalty parameter, with the goal of improving the convergence in practice, as well as making performance less dependent on the initial choice of the penalty parameter.
Though it can be difficult to prove the convergence of ADMM when ρ varies by iteration, the standard ADMM theory still applies if we assumes that ρ becomes fixed after a finite number of iterations.

Another extension of the ADMM is related to the x and z-minimization steps. It is possible to prove that ADMM will converge even when the x- and z-minimization steps are not carried out exactly, provided certain suboptimality measures in the minimization satisfy appropriate conditions. This result is ex-

tremely important in the situations in which it is not possible to solve problems (4.4) in closed form: it allows to use iterative methods and solve the minimizations only approximately at first, and then more accurately as the iterations progress.

5

Robust Factor Analysis of MA Processes

In this chapter we face the problem of developing a robust identification procedure for the dynamic factor analysis of moving average (MA) processes.
First, we formulate the problem as a trace norm minimization of a suitable spectral density. Then, in order to provide a numerically viable procedure for solving it, we exploit a finite dimensional matrix parametrization of the problem. The latter is analysed by resorting to the Lagrange duality theory, which allows to prove the existence of a solution. In addition, we show how to recover the solution to the primal problem provided that a solution to the dual one is available. Finally, the last part of the chapter is devoted to developing a numerical algorithm to solve the dual problem by employing the alternating direction method of multipliers.

5.1 Problem formulation

Consider the moving average (MA) factor model:

$$y_t = \Gamma_L u_t + \Gamma_D w_t \tag{5.1}$$

where

$$\begin{aligned} \Gamma_L(e^{i\vartheta}) &= \sum_{k=0}^{n} W_{L,k} e^{-i\vartheta k}, \qquad W_{L,k} \in \mathbb{R}^{m\times r} \\ \Gamma_D(e^{i\vartheta}) &= \sum_{k=0}^{n} W_{D,k} e^{-i\vartheta k}, \qquad W_{D,k} \in \mathbb{R}^{m\times m} \text{ diagonal,} \end{aligned} \tag{5.2}$$

and $u = \{u_t,\ t \in \mathbb{Z}\}$ and $w = \{w_t,\ t \in \mathbb{Z}\}$ are normalized white Gaussian noises of dimension r and m, respectively, such that

$$\mathbb{E}\{u_t w_s^\top\} = 0 \quad \forall t, s. \tag{5.3}$$

The aforementioned model has the following interpretation: u is the process which describes the r factors, with $r \ll m$, not accessible to observation; Γ_L is the factor loading transfer matrix; $\Gamma_L u_t$ is the latent variable; $\Gamma_D w_t$ is idiosyncratic noise. By rewriting the m-dimensional Gaussian stationary stochastic process $y = \{y_t,\ t \in \mathbb{Z}\}$ as

$$y_t = \begin{bmatrix} \Gamma_L & \Gamma_D \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix},$$

and noticing that, in view of (5.3), $[u_t \quad w_t]^\top$ has covariance equal to the identity, from the Wiener-Kintchine formula it immediately follows that y_t has power spectral density

$$\Phi = \Phi_L + \Phi_D \tag{5.4}$$

with $\Phi_L := \Gamma_L \Gamma_L^* \succeq 0$ and $\Phi_D := \Gamma_D \Gamma_D^* \succeq 0$. Moreover, from (5.2), we have

$$\Phi_L = \left(\sum_{k=0}^{n} W_{L,k} e^{-i\vartheta k}\right)\left(\sum_{k=0}^{n} W_{L,k} e^{-i\vartheta k}\right)^*,$$
$$\Phi_D = \left(\sum_{k=0}^{n} W_{D,k} e^{-i\vartheta k}\right)\left(\sum_{k=0}^{n} W_{D,k} e^{-i\vartheta k}\right)^*;$$

thus, Φ_L, Φ_D and Φ belong to the finite dimensional space:

$$\mathcal{Q}_{m,n} = \left\{ \sum_{k=-n}^{n} R_k e^{-i\vartheta k},\ \ R_k = R_{-k}^T \in \mathbb{R}^{m\times m} \right\}. \tag{5.5}$$

By construction, $\mathrm{rank}(\Phi_L) = r$, where rank denotes the normal rank, and Φ_D is diagonal. Therefore, y represents a factor model if its spectral density can be decomposed as "low rank plus diagonal" as in (5.4).

Assume to collect a finite length realization of y, say $\mathrm{y}^N = \{\,\mathrm{y}_1 \ldots \mathrm{y}_N\,\}$. We want to to estimate the corresponding factor model, that is the decomposition in (5.4), as well as the number of factors r.

To this aim, given our data y^N, we first compute the sample covariance lags

$\hat{R}_j$ as

$$\hat{R}_j = \frac{1}{N}\sum_{t=1}^{N-j} \mathrm{y}(t+j)\mathrm{y}(t)^\top, \quad j = 0 \dots n \tag{5.6}$$

where n is fixed by the user. Then, the estimate $\hat{\Phi}$ of Φ is obtained by the truncated periodogram:

$$\hat{\Phi} = \sum_{k=-n}^{n} \hat{R}_k e^{i\vartheta k}. \tag{5.7}$$

Notice that $\hat{\Phi}$ could be not positive definite for all ϑ. In that case, it is possible to use the periodogram properly smoothed using a windowing method [18].
On the other hand, $\hat{\Phi}$ may not admit a low rank plus diagonal decomposition. Thus, we estimate directly the two terms Φ_L and Φ_D of the decomposition (5.4) by solving the following optimization problem

$$\begin{aligned} \min_{\Phi,\Phi_L,\Phi_D \in \mathcal{Q}_{m,n}} \ & \mathrm{tr} \int \Phi_L \\ \text{subject to } & \Phi_L + \Phi_D = \Phi, \\ & \Phi \succ 0 \text{ a.e.}, \ \ \Phi_L, \Phi_D \succeq 0, \\ & \Phi_D \text{ diagonal}, \\ & \mathcal{S}_{IS}(\Phi||\hat{\Phi}) \leq \delta. \end{aligned} \tag{5.8}$$

Notice that, when omitted, the integrals are always defined from $-\pi$ to π with respect to the normalized *Lebesgue* measure $d\theta/2\pi$.
Here, the objective function induces low-rankness on Φ_L, see [11]. The first three constraints impose that Φ_L and Φ_D provide a genuine spectral density decomposition of type (5.4). The last constraint, in which $\mathcal{S}_{IS}(\Phi||\hat{\Phi})$ is the Itakura-Saito divergence defined by

$$\mathcal{S}_{IS}(\Phi||\hat{\Phi}) := \int \log|\hat{\Phi}\Phi^{-1}| + \mathrm{tr}[\hat{\Phi}^{-1}\Phi - I_m], \tag{5.9}$$

imposes that this spectral density belongs to a set "centered" in the nominal spectral density $\hat{\Phi}$ and with prescribed tolerance δ.

Notice that Φ_D is univocally determined by Φ and Φ_L. Thus, problem (5.8)

can be rewritten by removing Φ_D:

$$(\hat{\Phi}_L, \hat{\Phi}_D) = \underset{\Phi, \Phi_L \in \mathcal{Q}_{m,n}}{\arg\min} \ \mathrm{tr} \int \Phi_L$$
$$\text{subject to } \Phi \succ 0 \text{ a.e.}, \ \ \Phi_L, \Phi - \Phi_L \succeq 0, \tag{5.10}$$
$$\Phi - \Phi_L \text{ diagonal},$$
$$\mathcal{S}_{IS}(\Phi || \hat{\Phi}) \leq \delta.$$

5.2 The Choice of δ

Before solving our problem, two important considerations are in order on the choice of the tolerance parameter δ appearing in the constraint of (5.10). This choice should reflect the accuracy of the estimation $\hat{\Phi}$ of Φ. This can be accomplished by choosing a desired probability $\alpha \in (0, 1)$ and considering a ball of radius δ_α (in the Itakura-Saito topology) centered in $\hat{\Phi}$ and containing the true spectrum Φ with probability α. To estimate δ_α we proceed in two steps.

First, we consider the periodogram in (5.7) of Φ and we rely on a scale invariance property of the Itakura-Saito divergence. To introduce this property we define $\hat{\mathbf{R}}_j$ as the *estimator* corresponding to $\hat{R}_j$, namely the *random matrix* defined analogously to (5.6), but taking the Gaussian random variables in place of the corresponding realization y_t. In a similar way, $\hat{\mathbf{\Phi}}$ and $\hat{\Phi}$ denotes the periodogram, understood as estimator and estimate, respectively.

Lemma 5.2.1. Let $y = \{y(t) : t \in \mathbb{Z}\}$ be a zero mean, stationary, full rank, Gaussian process with spectral density Φ. Let $\hat{\mathbf{\Phi}}$ be the truncated periodogram based on a sample of y of length N. Then, $\mathcal{S}_{IS}(\Phi || \hat{\mathbf{\Phi}})$ is a random variable whose distribution depends only on the numerosity N of the sample, on the dimension m and the order n of the process.

Proof. We use arguments similar to the ones in proof of [13, Lemma 4.1].
Let $W(e^{i\vartheta})$ be the minimum phase spectral factor of Φ; define the process $\tilde{y} = \{\tilde{y}(t), \ t \in \mathbb{Z}\}$ as $\tilde{y}(t) := W(e^{i\vartheta})^{-1} y(t)$. Clearly, $\tilde{y}(t)$ is the normalized white

Gaussian noise process. Then we have

$$\begin{aligned}\hat{\mathbf{\Phi}}(e^{i\vartheta}) &= \sum_{k=-n}^{k=n}\left(\frac{1}{N}\sum_{t=1}^{N-k} y(t+k)y(t)^\top\right)e^{-ik\vartheta}\\ &= \sum_{k=-n}^{k=n}\left(\frac{1}{N}\sum_{t=1}^{N-k} W(e^{i\vartheta})\tilde{y}(t+k)\tilde{y}(t)^\top W(e^{i\vartheta})^*\right)e^{-ik\vartheta}\\ &= W(e^{i\vartheta})\sum_{k=-n}^{k=n}\left(\frac{1}{N}\sum_{t=1}^{N-k} \tilde{y}(t+k)\tilde{y}(t)^\top\right)e^{-ik\vartheta}W(e^{i\vartheta})^*\\ &= W(e^{i\vartheta})\hat{\mathbf{\Omega}}_N(e^{i\vartheta})W(e^{i\vartheta})^*\end{aligned}$$

where $\hat{\mathbf{\Omega}}_N(e^{i\vartheta}) := \sum_{k=-n}^{k=n}\left(\frac{1}{N}\sum_{t=1}^{N-k}\tilde{y}(t+k)\tilde{y}(t)^\top\right)e^{-ik\vartheta}$ is the truncated periodogram (understood as estimator) based on a sample of the normalized white Gaussian noise $\tilde{y}$ of length N. Hence, the Itakura-Saito divergence between Φ and $\hat{\mathbf{\Phi}}$ is

$$\begin{aligned}\mathcal{S}_{IS}(\Phi||\hat{\mathbf{\Phi}}) &= \int \log|\hat{\mathbf{\Phi}}\Phi^{-1}| + \mathrm{tr}[\hat{\mathbf{\Phi}}^{-1}\Phi - I_m]\\ &= \int \log|W\hat{\mathbf{\Omega}}_N W^* W^{-*}W^{-1}| + \mathrm{tr}[W^{-*}\hat{\mathbf{\Omega}}_N^{-1}W^{-1}WW^*] - m\\ &= \int \log\det(\hat{\mathbf{\Omega}}_N) + \mathrm{tr}(\hat{\mathbf{\Omega}}_N^{-1}) - m \qquad (5.11)\end{aligned}$$

where we have exploited the fact that $\Phi = WW^*$.
(5.11) results also in the case we consider a smoothed version of the periodogram obtained by a windowing method, [19]. In this case, $\hat{\mathbf{\Omega}}_N$ must be understood as the smoothed periodogram of the normalized white noise. □

In view of this result, we can easily generate a realization of the random variable $\mathcal{S}_{IS}(\Phi||\hat{\mathbf{\Phi}})$ from a realization of the normalized white Gaussian noise process. Accordingly, we can compute numerically δ_α such that $\Pr(\mathcal{S}_{IS}(\Phi||\hat{\mathbf{\Phi}}) \leq \delta_\alpha) = \alpha$ by a standard Monte Carlo procedure.

The second consideration is the following. If the chosen α is too large with respect to the data length N, the resulting δ_α may be too generous yielding to a diagonal Φ obeying $\mathcal{S}_{IS}(\Phi||\hat{\Phi}) \leq \delta_\alpha$. In this case problem (5.10) admits the trivial solution $\Phi_L = 0$ and $\Phi_D = \Phi$ diagonal. To rule out this trivial case, δ in (5.10) must be be strictly smaller than the upper bound

$$\delta_{\max} := \min_{\substack{\Phi\in\mathcal{S}_m^+\\ \Phi \text{ diagonal}}} \mathcal{S}_{IS}(\Phi||\hat{\Phi})$$

where $\mathcal{S}_m^+$ denotes the family of bounded and coercive functions defined on the unit circle and taking values in the cone of positive definite $m \times m$ Hermitian matrices. This problem can be easily solved as follows. Since Φ must be diagonal, by denoting as ϕ_i and by $\hat{\gamma}_i$ the i-th element in the diagonal of Φ and of $\hat{\Phi}^{-1}$, respectively, we have

$$
\begin{aligned}
\delta_{\max} &= \min_{\substack{\Phi \in \mathcal{S}_m^+ \\ \Phi \text{ diagonal}}} \int \log|\hat{\Phi}\Phi^{-1}| + \operatorname{tr}[\hat{\Phi}^{-1}\Phi - I_m] \\
&= \min_{\substack{\Phi \in \mathcal{S}_m^+ \\ \Phi \text{ diagonal}}} \int \log|\hat{\Phi}\big(\operatorname{diag}^2(\hat{\Phi}^{-1})\big)\big(\operatorname{diag}^2(\hat{\Phi}^{-1})\big)^{-1}\Phi^{-1}| + \operatorname{tr}[\hat{\Phi}^{-1}\Phi - I_m] \\
&= \min_{\substack{\Phi \in \mathcal{S}_m^+ \\ \Phi \text{ diagonal}}} \int \log|\big(\operatorname{diag}^2(\hat{\Phi}^{-1})\big)^{-1}\Phi^{-1}| + \operatorname{tr}[\hat{\Phi}^{-1}\Phi - I_m] + \int \log|\hat{\Phi}\operatorname{diag}^2(\hat{\Phi}^{-1})| \\
&= \min_{\substack{\Phi \in \mathcal{S}_m^+ \\ \Phi \text{ diagonal}}} \int \log(\prod_{i=1}^{m} \hat{\gamma}_i^{-1}\phi_i^{-1}) + \sum_{i=1}^{m} \hat{\gamma}_i\phi_i - m + \int \log|\hat{\Phi}\operatorname{diag}^2(\hat{\Phi}^{-1})| \\
&= \left[\sum_{i=1}^{m} \min_{\phi_i \in \mathcal{S}_1^+} \int \log(\hat{\gamma}_i^{-1}\phi_i^{-1}) + \hat{\gamma}_i\phi_i - 1\right] + \int \log|\hat{\Phi}\operatorname{diag}^2(\hat{\Phi}^{-1})| \\
&= \left[\sum_{i=1}^{m} \min_{\phi_i \in \mathcal{S}_1^+} \mathcal{S}_{IS}(\phi_i||\hat{\gamma}_i^{-1})\right] + \int \log|\hat{\Phi}\operatorname{diag}^2(\hat{\Phi}^{-1})|
\end{aligned}
$$

where $\operatorname{diag}^2(\cdot)$ is the (orthogonal projection) operator mapping a square matrix M into a diagonal matrix of the same size having the same main diagonal of M. Therefore, since the Itakura-Saito divergence is non-negative, the solution corresponds to $\phi_i^{opt}(e^{i\vartheta}) = (\hat{\gamma}_i(e^{i\vartheta}))^{-1}$, $i = 1, ..., m$ for which $\mathcal{S}_{IS}(\phi_i^{opt}||\hat{\gamma}_i^{-1}) = 0$. Accordingly,

$$\delta_{max} = \int \log|\hat{\Phi}\operatorname{diag}^2(\hat{\Phi}^{-1})|. \tag{5.12}$$

A more generous upper bound can be derived by assuming that Φ is the spectrum of an MA process of order n. However, numerical experiments showed that $\delta_{max} \gg \delta_\alpha$ even in the case that N is relatively small.

5.3 Problem solution

5.3.1 Matricial parametrization of the problem

To study problem (5.10) it is convenient to introduce a matricial parametrization for Φ, Φ_L and $\Phi - \Phi_L$. To this end, we first introduce the so-called *shift operator*

$\Delta(e^{i\theta})$, defined as

$$\Delta(e^{i\vartheta}) := [I_m \quad e^{i\vartheta} I_m \quad \dots \quad e^{in\vartheta} I_m], \tag{5.13}$$

and the space $\mathbf{Q}_{m(n+1)}$ of symmetric block-matrices with $(n+1) \times (n+1)$ square blocks of dimension $m \times m$; if $X \in \mathbf{Q}_{m(n+1)}$, X_{ij} is the block in position i, j with $i, j = 0, \dots, n$, so that

$$X = \begin{bmatrix} X_{00} & X_{01} & \dots & X_{0n} \\ X_{01}^\top & X_{11} & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ X_{0n}^\top & X_{1n}^\top & \dots & X_{nn} \end{bmatrix}.$$

Moreover, we will use $\mathbf{M}_{m,n}$ to denote the vector space of matrices of the form

$$Y := [Y_0 \quad Y_1 \quad ... \quad Y_n], \qquad Y_0 \in \mathbf{Q}_m, \quad Y_1, ..., Y_n \in \mathbb{R}^{m \times m}, \tag{5.14}$$

and the linear mapping $T : \mathbf{M}_{m,n} \to \mathbf{Q}_{m(n+1)}$ which constructs a symmetric block-Toeplitz matrix from its first block row so that if Y is given by (5.14),

$$T(Y) = \begin{bmatrix} Y_0 & Y_1 & \dots & Y_n \\ Y_1^\top & Y_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & Y_1 \\ Y_n^\top & \dots & Y_1^\top & Y_0 \end{bmatrix}.$$

The adjoint of T is a mapping $D : \mathbf{Q}_{m(n+1)} \to \mathbf{M}_{m,n}$ defined by $D(X) = [[D(X)]_0 \quad \dots \quad [D(X)]_n]$ with

$$[D(X)]_0 = \sum_{h=0}^{n} X_{hh}, \quad [D(X)]_j = 2 \sum_{h=0}^{n-j} X_{h\, h+j}, \ j = 1, ..., n.$$

Given $X \in \mathbf{Q}_{m(n+1)}$, by direct computation we obtain

$$\Delta X \Delta^* = [D(X)]_0 + \frac{1}{2} \sum_{j=1}^{n} e^{-ij\vartheta} [D(X)]_j + e^{ij\vartheta} [D(X)]_j^\top, \tag{5.15}$$

thus $\Delta X \Delta^* \in \mathcal{Q}_{m,n}$. Conversely, since D is a surjective map, any element in $\mathcal{Q}_{m,n}$ may be parametrized as (5.15). We conclude that

$$\mathcal{Q}_{m,n} = \{\Delta X \Delta^* \text{ s.t. } X \in \mathbf{Q}_{m(n+1)}\} \tag{5.16}$$

and we introduce the following matrix parametrization for Φ, Φ_L and $\Phi - \Phi_L$:

$$\begin{aligned} \Phi &= \Delta X \Delta^* \in \mathcal{Q}_{m,n} \\ \Phi_L &= \Delta L \Delta^* \in \mathcal{Q}_{m,n} \\ \Phi - \Phi_L &= \Delta (X - L) \Delta^* \in \mathcal{Q}_{m,n} \end{aligned} \tag{5.17}$$

with X and L matrices in $\mathbf{Q}_{m(n+1)}$.

Next, the objective is to provide a more convenient formulation of Problem (5.10) in terms of X and L. To this end, we have to take into account the following points.

1. *Positivity Constraints $\Phi \succ 0$ a.e. and $\Phi_L, \Phi - \Phi_D \succeq 0$:*
   It can been shown (see, for example, [11, Appendix A]) that, for any $\Psi \in \mathcal{Q}_{m,n}$, $\Psi \succeq 0$ if and only if there exists a matrix $P \in \mathbf{Q}_{m(n+1)}$ such that $\Psi = \Delta P \Delta^*$ and $P \succeq 0$. Therefore, we replace the conditions $\Phi_L \succeq 0$ with $L \succeq 0$ and the condition $\Phi - \Phi_L \succeq 0$ with $X - L \succeq 0$. Note that the latter only guarantees $X \succeq 0$ and thus Φ to be positive semidefinite, however we will show that this is sufficient to guarantee that $\Phi \succ 0$ a.e. at the optimum.

2. *Constraint $\Phi - \Phi_L$ diagonal:*
   Let ofd : $\mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m}$ denote the linear operator such that, given $A \in \mathbb{R}^{m \times m}$, ofd$(A)$ is the matrix in which each off-diagonal element is equal to the corresponding element of A and each diagonal element is zero. We define the "block ofd" linear operator ofd$_B$: $\mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ as follows. Given $Z = [\, Z_0 \; Z_1 \ldots Z_n \,] \in \mathbf{M}_{m,n}$, then

   $$\text{ofd}_B(Z) = [\, \text{ofd}(Z_0) \; \text{ofd}(Z_1) \ldots \text{ofd}(Z_n) \,]. \tag{5.18}$$

   It is not difficult that ofd$_B$ is a self-adjoint operator, since ofd is self-adjoint as well. Then, it is easy to see that, in view of (5.15), the condition $\Phi - \Phi_L$ diagonal is equivalent to the condition $[D(X-L)]_j$ diagonal for $j = 0, \ldots, n$, that is ofd$_B(D(X - L)) = 0$.

3. *The Low Rank Regularizer:*

We have

$$\mathrm{tr}\int \Phi_L = \mathrm{tr}\int \Delta L \Delta^* = \mathrm{tr}\left(L\int \Delta^*\Delta\right) = \mathrm{tr}(L)$$

where we exploited the cyclic property of the trace and the fact that $\int e^{ij\vartheta} = 1$ if $j = 0$, and $\int e^{ij\vartheta} = 0$ otherwise.

4. *The Divergence Constraint:*
A convenient matrix parameterization of the Itakura-Saito divergence $\mathcal{S}_{IS}(\Phi||\hat{\Phi})$ can be obtained by making use of the following facts.

First, since $\Phi = \Delta X \Delta^*$ with $X \succeq 0$, there exists $A \in \mathbb{R}^{m\times m(n+1)}$ such that $X = A^\top A$. This can be easily seen by noticing that $\Phi \in \mathcal{Q}_{m,n}$ such that $\Phi \succeq 0$ admits the spectral factorization $\Phi = \Gamma\Gamma^*$ where $\Gamma := \Delta A^\top$ and $A \in \mathbf{R}^{m\times m(n+1)}$. Hence, from $\Phi = \Delta X \Delta^* = \Delta A^\top A \Delta^*$ we conclude that $X = A^\top A$.
Then, by using the Jensen-Kolmogorov formula we obtain

$$\int \log|\Phi| = \int \log|\Delta A^\top A \Delta^*| = \log|A_0^\top A_0| = \log|X_{00}|, \tag{5.19}$$

which holds provided that $X_{00} \succ 0$.
Strictly speaking, (5.19) is valid when Φ is free from zeros over the unit circle. However, as we will show in Lemma 5.3.1, (5.19) can be extended by continuity to the case in which $\Phi \succeq 0$ has some zeros on the unit circle. The proof of the Lemma exploits the celebrated Beppo Levi's monotone convergence theorem which is reported in the following:

Theorem 5.3.1 (Beppo Levi's monotone convergence theorem for Lebesgue integral). Let (X, Σ, μ) be a measure space, and $X \in \Sigma$. Consider a pointwise non-decreasing sequence $\{f_k\}_{k=1}^{\infty}$ of measurable non-negative functions $f_k : X \to [0, +\infty]$, i.e. for every $k \geq 1$ and every $x \in X$

$$0 \leq f_k(x) \leq f_{k+1}(x) \leq +\infty.$$

Set the pointwise limit of sequence $\{f_k\}$ to be f. That is, for every $x \in X$,

$$f(x) := \lim_{k\to\infty} f_k(x).$$

Then, f is measurable and

$$\lim_{k\to\infty}\int_X f_k d\mu = \int_X f d\mu.$$

Lemma 5.3.1. Consider a power spectral density $\Phi \in \mathcal{Q}_{m,n}$, such that $\Phi \succeq 0$ with Φ singular for some $\vartheta \in [-\pi, +\pi]$. Let $X \in \mathbf{Q}_{m(n+1)}$, $X \succeq 0$ such that $\Phi = \Delta X \Delta^*$. Then

$$\int \log|\Phi| = \log|X_{00}|.$$

Proof. As previously notice, since $\Phi = \Delta X \Delta^*$ with $X \succeq 0$, there exists $A \in \mathbb{R}^{m\times m(n+1)}$ such that $X = A^\top A$. The matrix A is such that $\Phi \succeq 0$ admits the spectral factorization $\Phi = \Gamma\Gamma^*$ where $\Gamma := \Delta A^\top$.
Introduce $\Phi_n := \Phi + \frac{1}{n}I$ with $n \in \mathbb{N}$. Let $\Gamma_n := \Delta A_n$ with $A_n \in \mathbb{R}^{m\times m(n+1)}$ be a spectral factor of Φ_n, i.e. $\Phi_n = \Gamma_n\Gamma_n^*$.
Clearly, $\lim_{n\to+\infty}\Phi_n = \Phi$; accordingly $\lim_{n\to+\infty}\Gamma_n = \Gamma$ and $\lim_{n\to+\infty}A_n = A$.
Moreover, $\Phi_n \succ 0\ \forall\vartheta$; therefore, we can exploit (5.19) to obtain

$$\int \log|\Phi_n| = \log|A_{n_0}^\top A_{n_0}|.$$

Then, applying the limit operator to both sides, we have

$$\begin{aligned}\lim_{n\to+\infty}\int \log|\Phi_n| &= \lim_{n\to+\infty}\log|A_{n_0}^\top A_{n_0}| \\ &= \log|A_0^\top A_0| = \log|X_{00}|.\end{aligned}$$

To conclude the proof, it remains to show that in the left side of the previous equation it is possible to interchange the limit and the integral operators, so that

$$\lim_{n\to+\infty}\int \log|\Phi_n| = \int \lim_{n\to+\infty}\log|\Phi_n| = \int \log|\Phi|.$$

To this aim, we introduce the sequence of functions $\{f_n\}_{n=1}^{+\infty}$ where

$$f_n(\theta) := \log|\Phi_n(\vartheta)| = \log|\Phi(\vartheta) + \frac{1}{n}I|$$

with limit $f(\vartheta) := \lim_{n\to+\infty} f_n(t) = \log|\Phi(\vartheta)|$.
Observe that, since the interval of integration $[-\pi, \pi]$ is bounded and $f_1(\vartheta) <$

$+\infty$ for any $\vartheta \in [-\pi, \pi]$, we have

$$\int f_1(\vartheta) d\vartheta < +\infty.$$

Now, consider a new sequence $\{g_n\}_{n=1}^{+\infty}$ with $g_n(\vartheta) := f_n(\vartheta) - f_1(\vartheta)$ and let $g(\vartheta) := \lim_{n\to+\infty} g_n(\vartheta)$. $\{g_n\}$ is a a pointwise non-increasing sequence of measurable non-positive functions,

$$\cdots \leq g_2(\vartheta) \leq g_1(\vartheta) \leq 0, \quad \text{for every } \vartheta \in [-\pi, +\pi]\,.$$

converging to $g(\vartheta)$ from above. Clearly, $\{g_n\}$ satisfies all the hypotheses of Beppo-Levi's monotone convergence theorem 5.3.1 (applied with opposite signs), from which it immediately follows that

$$\lim_{n\to+\infty} \int g_n(\vartheta) = \int g(\vartheta). \tag{5.20}$$

Therefore,

$$\begin{aligned} \lim_{n\to+\infty} \int f_n(\vartheta) &= \lim_{n\to+\infty} \int f_n(\vartheta) - f_1(\vartheta) + \int f_1(\vartheta) \\ &= \lim_{n\to+\infty} \int g_n(\vartheta) + \int f_1(\vartheta) \\ &= \int g(\vartheta) dt + \int f_1(\vartheta). \end{aligned} \tag{5.21}$$

But now, since $f_1(\vartheta) < +\infty$ for all ϑ,

$$g(\vartheta) := \lim_{n\to+\infty} g_n(\vartheta) = f(\vartheta) - f_1(\vartheta). \tag{5.22}$$

Finally, plugging (5.22) into (5.21), we obtain

$$\lim_{n\to+\infty} \int f_n(\vartheta) = \int f(\vartheta), \tag{5.23}$$

concluding the proof. □

A second observation in order to conveniently parametrize the Itakuro-Saito divergence constraint is that, by exploiting the cyclic property of the trace,

$$\begin{aligned} \int \mathrm{tr}(\hat{\Phi}^{-1}\Phi) &= \int \mathrm{tr}(\hat{\Phi}^{-1}\Delta X \Delta^*) \\ &= \mathrm{tr}\left(X \int \Delta^* \hat{\Phi}^{-1} \Delta\right) = \langle X, T(\hat{P})\rangle, \end{aligned}$$

where $T(\hat{P})$ is defined as follows. Consider the following expansion of $\hat{\Phi}^{-1}$:

$$\hat{\Phi}^{-1} = \sum_{k=-\infty}^{\infty} \hat{P}_k e^{-i\vartheta k}; \tag{5.24}$$

by construction, since $\int e^{ij\vartheta} = 1$ if $j = 0$ and $\int e^{ij\vartheta} = 0$ otherwise, we have

$$\begin{aligned}
\int \Delta^* \hat{\Phi}^{-1} \Delta &= \int \begin{bmatrix} \Phi^{-1} & e^{i\theta}\Phi^{-1} & \dots & e^{in\theta}\Phi^{-1} \\ e^{-i\theta}\Phi^{-1} & \Phi^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{i\theta}\Phi^{-1} \\ e^{-in\theta}\Phi^{-1} & \dots & e^{-i\theta}\Phi^{-1} & \Phi^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \hat{P}_0 & \hat{P}_1 & \dots & \hat{P}_n \\ \hat{P}_{-1} & \hat{P}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{P}_1 \\ \hat{P}_{-n} & \dots & \hat{P}_{-1} & \hat{P}_0 \end{bmatrix} = \begin{bmatrix} \hat{P}_0 & \hat{P}_1 & \dots & \hat{P}_n \\ \hat{P}_1^\top & \hat{P}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{P}_1 \\ \hat{P}_n^\top & \dots & \hat{P}_1^\top & \hat{P}_0 \end{bmatrix} \\
&= T(\hat{P})
\end{aligned}$$

where $\hat{P} := [\hat{P}_0 \dots \hat{P}_n]$.

Summing up, we get the following matrix re-parametrization of Problem (5.10):

$$\begin{aligned}
(\hat{X}, \hat{L}) = \underset{X,L \in \mathbf{Q}_{m(n+1)}}{\arg\min} \quad & \operatorname{tr}(L) \\
\text{subject to} \quad & X_{00} \succ 0,\ L \succeq 0,\ X - L \succeq 0, \\
& \operatorname{ofd}_B(D[X - L]) = 0, \\
& -\log|X_{00}| + \int \log|\hat{\Phi}| + \langle X, T(\hat{P}) \rangle - m \leq \delta.
\end{aligned} \tag{5.25}$$

We remark once again that to prove the equivalence between (5.10) and (5.25) we still need to show that $\Phi \succ 0$ a.e. at the optimum: this fact will be established after the variational analysis.

5.3.2 The dual problem

We reformulate the constrained minimization problem in (5.25) as an unconstrained problem by means of Duality Theory.
If we use $V, U \in \mathbf{Q}_{m(n+1)},\ V, U \succeq 0$ as the multipliers associated with the constraints on the positive semi-definiteness of $X - L$ and L, respectively; $Z \in \mathbf{M}_{m,n}$

as the multiplier associated with the constraint $\mathrm{ofd}_B(D(X-L))=0$ and $\lambda\in\mathbb{R}, \lambda\geq 0$, as the multiplier associated with the Itakura-Saito divergence, then the Lagrangian of problem (5.25) is

$$\begin{aligned}
\mathcal{L}(X,L,\lambda,U,V,Z) &= \mathrm{tr}(L) - \langle V, X-L\rangle - \langle U, L\rangle + \langle Z, \mathrm{ofd}_B(D(X-L))\rangle \\
&\quad + \lambda\Big(-\log|X_{00}| + \int \log|\hat{\Phi}| + \langle X, T(\hat{P})\rangle - m - \delta\Big) \\
&= \langle L, I\rangle - \langle V, X-L\rangle - \langle U, L\rangle + \langle T(\mathrm{ofd}_B(Z)), X-L\rangle + \langle X, \lambda T(\hat{P})\rangle \\
&\quad - \lambda\Big(\log|X_{00}| - \int \log|\hat{\Phi}| + m + \delta\Big) \\
&= \langle L, I\rangle - \langle V, X\rangle + \langle V, L\rangle - \langle U, L\rangle + \langle T(\mathrm{ofd}_B(Z)), X\rangle - \langle T(\mathrm{ofd}_B(Z)), L\rangle \\
&\quad + \langle X, \lambda T(\hat{P})\rangle - \lambda\Big(\log|X_{00}| - \int \log|\hat{\Phi}| + m + \delta\Big) \\
&= \langle L, I - U + V - T(\mathrm{ofd}_B(Z))\rangle + \langle X, T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\rangle \\
&\quad - \lambda\Big(\log|X_{00}| - \int \log|\hat{\Phi}| + m + \delta\Big), \qquad (5.26)
\end{aligned}$$

where in the first equality we exploited the fact that the operator ofd_B is self-adjoint and the mappings T and D are adjoints, so that $\langle Z, \mathrm{ofd}_B(D(X-L))\rangle = \langle \mathrm{ofd}_B(Z), D(X-L))\rangle = \langle T(\mathrm{ofd}_B(Z)), X-L\rangle$.
Note that we have not included the constraint $X_{00}\succ 0$ because, as we will show later on, this condition is automatically met by the solution of the dual problem.

The dual function is defined as the infimum of $\mathcal{L}(X,L,\lambda,U,V,Z)$ over X and L. Thanks to the convexity of the Lagrangian, we rely on standard variational methods to characterize the minimum.

- *Partial minimization with respect to L:* $\mathcal{L}$ depends on L only through $\langle L, I-U+V-T(\mathrm{ofd}_B(Z))\rangle$ which is bounded below only if

$$I - U + V - T(\mathrm{ofd}_B(Z)) = 0. \qquad (5.27)$$

  Thus, we get that

$$\inf_L \mathcal{L} = \begin{cases} \langle X, T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\rangle + \\ -\lambda\big(\log|X_{00}| - \int\log|\hat{\Phi}| + m + \delta\big) & \text{if (5.27)} \\ -\infty & \text{otherwise.} \end{cases}$$

- *Partial minimization with respect to X:* $\mathcal{L}$ depends on X_{00} through the terms

$$\langle X_{00}, \Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{00}\rangle - \lambda\big(\log|X_{00}|\big).$$

which are unbounded below unless

$$\Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{00} \succ 0. \tag{5.28}$$

If (5.28) holds, then, by taking convexity into account, the matrix X_{00} achieving the minimum is easily obtained by imposing the optimality condition $\delta\mathcal{L}(X_{00}; \delta X_{00}) = 0 \ \ \forall \delta X_{00} \in \mathbf{Q}_m$. This is equivalent to

$$\begin{aligned}\delta\mathcal{L}(X_{00}; \delta X_{00}) &= \mathrm{tr}\,\Big(\Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{00} \delta X_{00}\Big) - \lambda\,\mathrm{tr}\,\big(X_{00}^{-1}\delta X_{00}\big) \\ &= \langle \Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{00} - \lambda X_{00}^{-1}, \delta X_{00}\rangle = 0\end{aligned}$$

for any δX_{00}, from which it follows that

$$\Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{00} - \lambda X_{00}^{-1} = 0$$

and so

$$X_{00} = \Big(\Big[T(\hat{P}) + \lambda^{-1}(T(\mathrm{ofd}_B(Z)) - V)\Big]_{00}\Big)^{-1}. \tag{5.29}$$

provided that $\lambda > 0$.
The Lagrangian is linear in the remaining variables X_{lh}, for $(l,h) \neq (0,0)$, and therefore bounded below only if

$$\Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{lh} = 0 \qquad \forall (l,h) \neq (0,0). \tag{5.30}$$

Summarizing, the minimization of the Lagrangian with respect to X and L is finite if and only if (5.27), (5.28), and (5.30) hold in which case

$$\begin{aligned}\min_{X,L} \mathcal{L} &= \langle \Big(\Big[\lambda^{-1}(T(\mathrm{ofd}_B(Z)) - V) + T(\hat{P})\Big]_{00}\Big)^{-1}, \Big[T(\mathrm{ofd}_B(Z)) - V + \lambda T(\hat{P})\Big]_{00}\rangle \\ &\quad - \lambda\big(- \log\big|\big[T(\hat{P}) + \lambda^{-1}(T(\mathrm{ofd}_B(Z)) - V)\big]_{00}\big| - \int \log|\hat{\Phi}| + m + \delta\big) \\ &= \lambda m - \lambda\big(- \log\big|\big[T(\hat{P}) + \lambda^{-1}(T(\mathrm{ofd}_B(Z)) - V)\big]_{00}\big| - \int \log|\hat{\Phi}| + m + \delta\big) \\ &= -\lambda\big(- \log\big|\big[T(\hat{P}) + \lambda^{-1}(T(\mathrm{ofd}_B(Z)) - V)\big]_{00}\big| - \int \log\big|\hat{\Phi}\big| + \delta\big).\end{aligned}$$

Otherwise the Lagrangian has no minimum and its infimum is $-\infty$.

In order to simplify the notation, let us define the vector space $\mathcal{O}$ as:

$$\mathcal{O} := \{Z \in \mathbf{M}_{m,n} : \mathrm{ofd}_B(Z) = Z\}.$$

Since Z always appears in the form $\mathrm{ofd}_B(Z)$, we can replace it with $Z \in \mathcal{O}$.

We can now formulate the dual problem for the Lagrangian (5.26) as

$$\sup_{(\lambda,U,V,Z)\in\tilde{\mathcal{C}}} \tilde{J} \tag{5.31}$$

where

$$\tilde{J} := \lambda\Big(\log\big|\big[T(\hat{P}) + \lambda^{-1}(T(\ Z) - V)\big]_{00}\big| + \int \log|\hat{\Phi}| - \delta\Big)$$

and the feasible set $\tilde{\mathcal{C}}$ is given by:

$$\begin{aligned}\tilde{\mathcal{C}} := \{(\lambda, U, V, Z) : U, V \in \mathbf{Q}_{m(n+1)}, U, V \succeq 0, Z \in \mathcal{O}, \lambda \in \mathbb{R}, \lambda > 0,\\ I - U + V - T(Z) = 0, [\lambda T(\hat{P}) + T(Z) - V]_{00} \succ 0,\\ [\lambda T(\hat{P}) + T(Z) - V]_{lh} = 0,\ \forall (l,h) \neq (0,0)\}.\end{aligned}$$

Note that, constraints $I - U + V - T(Z) = 0$ and $U \succeq 0$ are equivalent to the constraint $I + V - T(Z) \succeq 0$. Thus, we can eliminate the redundant variable U; moreover, by changing the sign to the objective function $\tilde{J}$ and observing that $\big[T(\hat{P}) + \lambda^{-1}(T(Z) - V)]_{00} = \hat{P}_0 + \lambda^{-1}(Z_0 - V_{00})$, we can rewrite (5.31) as:

$$\min_{(\lambda,V,Z)\in\mathcal{C}} J \tag{5.32}$$

where

$$J := \lambda\Big(- \log\big|\hat{P}_0 + \lambda^{-1}(Z_0 - V_{00})\big| - \int \log|\hat{\Phi}| + \delta\Big).$$

and the corresponding feasible set $\mathcal{C}$ is:

$$\begin{aligned}\mathcal{C} := \{(\lambda, V, Z) : V \in \mathbf{Q}_{m(n+1)}, V \succeq 0, Z \in \mathcal{O}, I + V - T(Z) \succeq 0, \lambda \in \mathbb{R}, \lambda > 0,\\ [\lambda \hat{P}_0 + Z_0 - V_{00}] \succ 0, [\lambda(T(\hat{P})) + T(Z) - V]_{lh} = 0, \forall (l,h) \neq (0,0)\}.\end{aligned}$$

Next we address existence of solutions to (5.32).

5.3.3 Existence of solutions

The aim of this subsection is to show that (5.32) admits solution. The set $\mathcal{C}$ is not compact, as it is neither closed nor bounded. Our strategy consists in showing that we can restrict the set $\mathcal{C}$ to a compact set over which the minimization is equivalent. Then, since the objective function is continuous over $\mathcal{C}$ (and hence over the restricted compact set), we can use Weierstrass's Theorem to conclude that the problem does admit a minimum.

The first step consists in showing that, similarly to [13, Proposition 6.1], we can restrict $\mathcal{C}$ to a subset where $\lambda \geq \varepsilon$ with $\varepsilon > 0$ a positive constant.

Proposition 5.3.1. Let $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{C}$ such that

$$\lim_{k\to\infty} \lambda^{(k)} = 0. \tag{5.33}$$

Then, such a sequence cannot be an infimizing sequence.

The proof can be found in Appendix C.

As a consequence, minimizing the dual functional over the set $\mathcal{C}$ is equivalent to minimize it over the set:

$$\begin{aligned}\mathcal{C}_1 :=&\{(\lambda, V, Z) : V \in \mathbf{Q}_{m(n+1)} V \succeq 0, Z \in \mathcal{O}, I + V - T(Z) \succeq 0, \lambda \in \mathbb{R}, \lambda \geq \varepsilon,\\ &[\lambda \hat{P}_0 + Z_0 - V_{00}] \succ 0, [\lambda(T(\hat{P})) + T(Z) - V]_{lh} = 0, \forall (l,h) \neq (0,0)\}.\end{aligned}$$

Next we show that we can restrict the search for the optimal solution to a subset of $\mathcal{C}_1$ in which both $(T(Z) - V)$ and λ cannot diverge.

Proposition 5.3.2. Let $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{C}_1$ such that either

$$\lim_{k\to\infty} \parallel T(Z^{(k)}) - V^{(k)} \parallel = +\infty \tag{5.34}$$

or

$$\lim_{k\to\infty} \lambda^{(k)} = +\infty \tag{5.35}$$

or both. Then, such a sequence cannot be an infimizing sequence.

The proof can be found in Appendix C.

It follows from the previous result that there exists $\beta \in \mathbb{R}$ with $\mid \beta \mid < \infty$ such that

$$T(Z) - V \succeq \beta I$$

and $0 < \gamma < \infty$ such that $\lambda \leq \gamma$. Therefore, the set $\mathcal{C}_1$ can be further restricted to the set:

$$\begin{aligned}\mathcal{C}_2 :=& \{(\lambda, V, Z) : V \in \mathbf{Q}_{m(n+1)}, V \succeq 0, Z \in \mathcal{O}, \beta I \preceq T(Z) - V \preceq I, \lambda \in \mathbb{R},\\ &\gamma \geq \lambda \geq \varepsilon, [\lambda \hat{P}_0 + Z_0 - V_{00}] \succ 0, [\lambda(T(\hat{P})) + T(Z) - V]_{lh} = 0, \forall (l,h) \neq (0,0)\}.\end{aligned}$$

In addition, we show that it is not possible for V and Z to diverge while keeping finite the difference $T(Z) - V$. Accordingly, we can further restrict $\mathcal{C}_2$ to a subset $\mathcal{C}_3$ in which neither V nor Z can diverge:

Proposition 5.3.3. Let $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{C}_2$ such that

$$\lim_{k\to\infty} \| V^{(k)} \| = +\infty \tag{5.36}$$

or

$$\lim_{k\to\infty} \| Z^{(k)} \| = +\infty \tag{5.37}$$

or both. Then, such a sequence cannot be an infimizing sequence.

The proof can be found in Appendix C.

Thus, the minimization over $\mathcal{C}_2$ is equivalent to the minimization over the subset:

$$\begin{aligned}\mathcal{C}_3 := \{(\lambda, V, Z) : V \in \mathbf{Q}_{m(n+1)}, \alpha I \succeq V \succeq 0, Z \in \mathcal{O}, \beta I \preceq T(Z) - V \preceq I, \lambda \in \mathbb{R},\\ \gamma \geq \lambda \geq \varepsilon, [\lambda \hat{P}_0 + Z_0 - V_{00}] \succ 0, [\lambda(T(\hat{P})) + T(Z) - V]_{lh} = 0, \forall (l,h) \neq (0,0)\}.\end{aligned}$$

for certain $\alpha > 0$ positive constant.

Finally, we consider a sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{Z}} \in \mathcal{C}_3$ such that $[(\lambda^{(k)})^{-1}\big([Z^{(k)}]_0 - [V^{(k)}]_{00}\big) + \hat{P}_0]$ tends to be singular as $k \to \infty$. This implies that $|(\lambda^{(k)})^{-1}\big([Z^{(k)}]_0 - [V^{(k)}]_{00}\big) + \hat{P}_0|$ tends to zero and hence $J \to +\infty$. Thus, such a sequence cannot be an infimizing sequence.
Therefore, the final set $\mathcal{C}_C$ is :

$$\begin{aligned}\mathcal{C}_C := \{(\lambda, V, Z) : V \in \mathbf{Q}_{m(n+1)}, \alpha I \succeq V \succeq 0, Z \in \mathcal{O}, \beta I \preceq T(Z) - V \preceq I, \lambda \in \mathbb{R},\\ \gamma \geq \lambda \geq \varepsilon, [\lambda \hat{P}_0 + Z_0 - V_{00}] \succeq \mu I, [\lambda(T(\hat{P})) + T(Z) - V]_{lh} = 0, \forall (l,h) \neq (0,0)\}\end{aligned}$$

where $\alpha, \beta, \gamma, \varepsilon$ and μ such that $|\alpha|, |\beta|, |\gamma|, |\varepsilon|$ and $|\mu| < +\infty$.

Theorem 5.3.2. Problem (5.32) is equivalent to

$$\min_{(\lambda,V,Z)\in\mathcal{C}_C} J(\lambda, V, Z)$$

and it admits solution.

Proof. Equivalence of the two problems has already been proven by the previous arguments.
Since $\mathcal{C}_C$ is clearly closed and bounded, hence compact, and J is continuous over $\mathcal{C}_C$, by the Weierstrass's Theorem the minimum exists. □

5.4 Solution to the original problem

In this section we prove that the primal problem (5.10) and its matrix reformulation (5.25) are equivalent and we show how to recover the solution of the original problem.

Let $(\lambda^\circ, V^\circ, Z^\circ)$ be a solution of (5.32) and (X°, L°) be the corresponding solution of (5.25).
First observe that the primal problem (5.25) satisfies the *refined Slater's conditions*. Indeed, pick for instance $X = L = T(\hat{R})$: (X, L) is feasible, and the divergence constraint holds with strict inequality. From convex duality, *strong duality* holds, namely the value of the primal objective at (X°, L°) is equal to the value of the dual objective evaluated at $(\lambda^\circ, V^\circ, Z^\circ)$. By strong duality and the existence of a dual optimal solution $(\lambda^\circ, V^\circ, Z^\circ)$, it also follows that the primal optimal point (X°, L°) is a minimizer of $\mathcal{L}(X, L; \lambda^\circ, V^\circ, Z^\circ)$. In particular, reviewing the derivation of the dual problem, we see that X° minimizes the Lagrangian only if

$$X_{00}^\circ = \left(\hat{P}_0 + (\lambda^\circ)^{-1}(Z_0^\circ - V_{00}^\circ)\right)^{-1} \succ 0. \tag{5.38}$$

We are now ready to prove that (5.10) and (5.25) are equivalent. Since X_{00}° is positive definite, $\log|X_{00}^\circ|$ is finite. By Lemma 5.3.1, at the optimum $\int \log|\Phi|$ must be finite as well; this implies that at most $\Phi(e^{i\vartheta})$, $\vartheta \in [-\pi, +\pi]$ may be singular on a set of zero measure, or, in other terms, $\Delta X^\circ \Delta^* \succ 0$ a.e.. This observation leads to the following proposition:

Proposition 5.4.1. Let (X°, L°) be a solution of (5.25). Then $\Delta X^\circ \Delta^* \succ 0$ a.e.. Accordingly, (5.10)) and (5.25) are equivalent.

In the following we show how to recover the solution of the primal problem from an optimal solution $(\lambda^\circ, V^\circ, Z^\circ)$ of the dual problem; to this aim we need the following result, see [20].

Lemma 5.4.1. Let $Z \in \mathbf{M}_{m,n}$ and $W \in \mathbf{Q}_m$. If $W \succ 0$ is such that

$$T(Z) \succeq \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \tag{5.39}$$

then $T(Z) \succ 0$.

Exploiting the constraints $[\lambda(T(\hat{P})) + T(Z) - V]_{lh} = 0, \forall (l,h) \neq (0,0)$ and $[\lambda \hat{P}_0 + Z_0 - V_{00}] \succ 0$, it is not difficult to see that

$$V^\circ = \lambda^\circ T(\hat{P}) + T(Z^\circ) - \begin{bmatrix} W^\circ & 0 \\ 0 & 0 \end{bmatrix} \tag{5.40}$$

where

$$W^\circ := Z_0^\circ - V_{00}^\circ + \lambda^\circ \hat{P}_0 \succ 0. \tag{5.41}$$

Since $V^\circ \succeq 0$ and in view of Lemma 5.4.1, we have that $\lambda^\circ T(\hat{P}) + T(Z^\circ) \succ 0$. Hence, V° has rank at least equal to mn.

Since the duality gap between (5.25) and (5.32) is equal to zero, from convex duality we know that the primal and dual optimal solutions are related by the *complementary slackness conditions*. In particular, for the optimal Lagrange multiplier V° associated with the inequality $X - L \succeq 0$, we have that

$$\langle V^\circ, X^\circ - L^\circ \rangle = 0. \tag{5.42}$$

Since $V^\circ, X^\circ - L^\circ \succeq 0$, (5.42) implies

$$V^\circ (X^\circ - L^\circ) = 0. \tag{5.43}$$

Indeed, by the positive semi-definiteness of the two matrices, there exist $V^{1/2}$ and $D^{1/2} \in \mathbf{Q}_{m(n+1)}$ such that $V^\circ = V^{1/2} V^{1/2}$ and $X^\circ - L^\circ = D^{1/2} D^{1/2}$. Thus, by exploiting the cyclic property of the trace, (5.42) can be rewritten as

$$\begin{aligned} 0 &= \mathrm{tr}\left(V^\circ (X^\circ - L^\circ) \right) \\ &= \mathrm{tr}(V^{1/2} V^{1/2} D^{1/2} D^{1/2}) \\ &= \mathrm{tr}(V^{1/2} D^{1/2} D^{1/2} V^{1/2}) \\ &= \mathrm{tr}(V^{1/2} D^{1/2} (V^{1/2} D^{1/2})^\top) \end{aligned}$$

which implies

$$V^{1/2} D^{1/2} = 0.$$

Finally, premultiplying by $V^{1/2}$ and postmultiplying by $D^{1/2}$ the last equality, we get (5.43).

Recalling that $\mathrm{rank}(V^\circ) \geq mn$, in view of (5.43) we have that $X^\circ - L^\circ$ has rank at most equal to m. Let $\mathrm{rank}(X^\circ - L^\circ) = \tilde{m}$ with $\tilde{m} \leq m$. There exists

a full-row rank matrix $A \in \mathbb{R}^{\tilde{m} \times m(n+1)}$ such that $X^\circ - L^\circ = A^\top A$. By (5.43), it follows that

$$V^\circ A^\top = 0. \tag{5.44}$$

This means that $\mathrm{Im}(A^\top) \subseteq \ker(V^\circ)$. Let us denote by $Y_D := [v_o \quad v_1 \quad ... \quad v_l] \in \mathbb{R}^{m(n+1) \times l}$ the matrix whose columns $v_o, v_1, ..., v_l$ form a basis of $\ker V^\circ$. Note that the dimension l of the null space of V° is al least $\tilde{m}$ because $\mathrm{Im}(A^\top) \subseteq \ker(V^\circ)$ and $\mathrm{rank}(A^\top) = \tilde{m}$, and also $l \leq m$ because $\mathrm{rank}(V^\circ) \geq mn$.
Writing now the columns of $A^\top$ as a linear combination of the vectors $v_o, v_1, ..., v_l$, we obtain:

$$A^\top = Y_D S \tag{5.45}$$

where $S \in \mathbf{R}^{l \times \tilde{m}}$. Accordingly,

$$X^\circ - L^\circ = A^\top A = Y_D S S^\top Y_D^\top,$$

thus

$$X^\circ - L^\circ = Y_D Q_D Y_D^\top, \tag{5.46}$$

where the matrix $Q_D := SS^\top \in \mathbf{Q}_l$ is unknown.

In a similar fashion, by the zero duality gap between (5.25) and (5.32), the *complementary slackness condition* for the multiplier associated to the positive semi-definiteness of L reads as

$$\langle U^\circ, L^\circ \rangle = 0 \tag{5.47}$$

which in turn implies

$$U^\circ L^\circ = 0. \tag{5.48}$$

In the previous expressions, we remember that from (5.27) it follows

$$U^\circ = I + V^\circ - T(Z^\circ).$$

Repeating the same reasoning as before, it can be seen that, if the dimension of the null space of U° is $\tilde{r}$ with $\tilde{r} \geq r$ and $Y_L := [u_o \quad u_1 \quad ... \quad u_{\tilde{r}}] \in \mathbb{R}^{m(n+1) \times \tilde{r}}$ is a

matrix whose columns form a basis of $\ker U^\circ$, then L° can be written as

$$L^\circ = Y_L Q_L Y_L^\top \tag{5.49}$$

with $Q_L \in \mathbf{Q}_{\tilde{r}}$ unknown. Plugging (5.49) into (5.46), we then obtain

$$X^\circ - Y_L Q_L Y_L^\top = Y_D Q_D Y_D^\top. \tag{5.50}$$

Assume now that each block of $X^\circ - L^\circ$ is diagonal, namely

$$\mathrm{ofd}(\left[Y_D Q_D Y_D^\top\right]_{hk}) = 0 \quad h,k = 0,...,n. \tag{5.51}$$

Remark 2. We can make the previous assumption without loss of generality. Indeed, let $\hat{\Phi}$, $\hat{\Phi}_L$ be the solutions of the original problem (5.10) and $\hat{\Phi}_D = \hat{\Phi} - \hat{\Phi}_L$; let X, L and $D := X - L$ in $\mathbf{Q}_{m(n+1)}$ be any matrices such that $\hat{\Phi} = \Delta X \Delta^*$, $\hat{\Phi}_L = \Delta L \Delta^*$ and $\Phi_D = \Delta D \Delta^*$. We can always consider a different matrix parametrization $(\tilde{X}, \tilde{L}, \tilde{D})$ for $\hat{\Phi}$, $\hat{\Phi}_L$ and $\hat{\Phi}_D$ as follows. Let $\delta X \in \mathbf{Q}_{m(n+1)}$ such that $\Delta \delta X \Delta^* = 0$ and $\tilde{X} := X + \delta X$ satisfies (5.29). Moreover, there always exists a matrix $\tilde{D}$ with all diagonal blocks such that $\hat{\Phi}_D = \Delta \tilde{D} \Delta^*$; in other words, there always exists $\delta D \in \mathbf{Q}_{m(n+1)}$ such that $\Delta \delta D \Delta^* = 0$ and $\tilde{D} := D + \delta D$ satisfies $\mathrm{ofd}(\left[\tilde{D}\right]_{hk}) = 0$ for $h,k = 0,...,n.$. Let $\tilde{L} := \tilde{X} - \tilde{D} = X - D + \delta L$ where $\delta L := \delta X - \delta D$. It is easy to see that $\hat{\Phi} = \Delta \tilde{X} \Delta^*$ and $\hat{\Phi}_L = \Delta \tilde{L} \Delta^*$, thus $(\tilde{X}, \tilde{L})$ represents a solution of (5.25). This allows us to restrict to solutions (X°, L°) of problem (5.25) for which (5.51) holds.

Applying the ofd operator to both sides of (5.50) and exploiting (5.51), it is not difficult to derive:

$$\mathrm{ofd}(\left[Y_L Q_L Y_L^\top\right]_{00}) = \mathrm{ofd}(X^\circ_{00}) \tag{5.52}$$

which is a system of $m(m-1)/2$ linear equations in the $\tilde{r}(\tilde{r}+1)/2$ unknowns Q_L. Finally, once L° is computed, in order to retrieve Q_D, we first exploit (5.51) to reduce its dimension; then the remaining unknowns of Q_D are obtained from the following system of $m(m+1)/2$ linear equations:

$$\left[Y_D Q_D Y_D^\top\right]_{00} = X^\circ_{00} - L^\circ_{00}. \tag{5.53}$$

In virtue of the fact that both the dual and the primal problem admit solution, the resulting systems of equations (5.52) and (5.53) admit solutions.

5.5 The proposed algorithm

In this section we propose an algorithm to solve numerically the dual problem.

To start with, as observed in Section 5.4, we rewrite (5.32) in a different fashion by getting rid of the slack variable $V \in \mathbf{Q}_{m(n+1)}$. This is done by introducing a new variable $W \in \mathbf{Q}_m$ defined, similarly to (5.41), as

$$W := Z_0 - V_{00} + \lambda \hat{P}_0 \succ 0 \tag{5.54}$$

such that, equally to (5.40), the variable V can be expressed as

$$V = \lambda T(\hat{P}) + T(Z) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.55}$$

Therefore, the constraint $V \succeq 0$ of problem (5.32) reads as

$$\lambda T(\hat{P}) + T(Z) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \succeq 0,$$

and the constraint $I + V - T(Z) \succeq 0$ as

$$I + \lambda T(\hat{P}) + T(Z) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} - T(Z) = I + \lambda T(\hat{P}) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \succeq 0.$$

Consequently, the dual problem (5.32) can be stated in terms of the variables λ, W and Z as follows:

$$\min_{(\lambda, W, Z) \in \mathcal{C}} J \tag{5.56}$$

where

$$J := \lambda \Big(- \log \big| \lambda^{-1} W \big| - \int \log |\hat{\Phi}| + \delta \Big).$$

and the corresponding feasible set $\mathcal{C}$ is:

$$\begin{aligned} \mathcal{C} := \{ (\lambda, W, Z) : W \in \mathbf{Q}_m, W \succ 0, Z \in \mathcal{O}, \lambda \in \mathbb{R}, \lambda > 0, \\ \lambda T(\hat{P}) + T(Z) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \succeq 0, I + \lambda T(\hat{P}) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \succeq 0 \}. \end{aligned}$$

We can further simplify our problem as follows.

First, we observe that the constraint

$$V = \lambda T(\hat{P}) + T(Z) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \succeq 0 \tag{5.57}$$

implies

$$\lambda T(\hat{P}) + T(Z) \succeq \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}$$

and then, by Lemma 5.4.1, $\lambda T(\hat{P}) + T(Z) \succ 0$.

Now, recalling the characterization of a symmetric positive semi-definite matrix using the Schur complement (see Proposition C.0.3), we can easily rewrite the constraint $V \succeq 0$ by computing the Schur complement of the south-east block of V. To this aim, it is convenient to introduce the linear operators $T_{0,0} : \mathbf{M}_{m,n} \to \mathbf{Q}_m$, $T_{0,1:n} : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n-1}$ and $T_{1:n,1:n} : \mathbf{M}_{m,n} \to \mathbf{Q}_{mn}$ that, for a given matrix $H \in \mathbf{M}_{m,n}$ construct a symmetric block-Toeplitz matrix and extract the blocks in position $(0,0)$, $(0,1:n)$ and $(1:n,1:n)$, respectively. With this notation, we have

$$T(Z + \lambda\hat{P}) = \begin{bmatrix} T_{0,0}(Z + \lambda\hat{P}) & T_{0,1:n}(Z + \lambda\hat{P}) \\ T_{0,1:n}^{\top}(Z + \lambda\hat{P}) & T_{1:n,1:n}(Z + \lambda\hat{P}) \end{bmatrix}$$

and the constraint (5.57) is equivalent to require:

$$W \preceq T_{0,0}(Z + \lambda\hat{P}) - T_{0,1:n}(Z + \lambda\hat{P})T_{1:n,1:n}^{-1}(Z + \lambda\hat{P})T_{0,1:n}^{\top}(Z + \lambda\hat{P}). \tag{5.58}$$

Denoting by $Q(\lambda, Z)$ the function of the dual variables (λ, Z)

$$Q(\lambda, Z) := T_{0,0}(Z + \lambda\hat{P}) - T_{0,1:n}(Z + \lambda\hat{P})T_{1:n,1:n}^{-1}(Z + \lambda\hat{P})T_{0,1:n}^{\top}(Z + \lambda\hat{P}),$$

(5.58) becomes

$$W \preceq Q(\lambda, Z). \tag{5.59}$$

Notice that, since $Q(\lambda, Z)$ is the Schur complement of the positive definite matrix $T(Z) + \lambda T(\hat{P})$, it takes values in the cone of positive definite $m \times m$ symmetric matrices.

Consider now the constraint

$$I + \lambda T(\hat{P}) - \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \succeq 0; \tag{5.60}$$

by noticing that $I + \lambda T(\hat{P}) \succ 0$ and computing the Schur complement of the

south-east block, the latter can be equivalently expressed as

$$W \preceq I + T_{0,0}(\lambda\hat{P}) - T_{0,1:n}(\lambda\hat{P})\big(I + T_{1:n,1:n}(\lambda\hat{P})\big)^{-1}T_{0,1:n}^{\top}(\lambda\hat{P}). \tag{5.61}$$

To simplify the notation, let us define the function of the variable λ

$$R(\lambda) := I + T_{0,0}(\lambda\hat{P}) - T_{0,1:n}(\lambda\hat{P})\big(I + T_{1:n,1:n}(\lambda\hat{P})\big)^{-1}T_{0,1:n}^{\top}(\lambda\hat{P})$$

which, being the Schur complement of the positive definite matrix $I+\lambda T(\hat{P})$, takes values in the cone of positive definite $m \times m$ symmetric matrices; the condition (5.60) reads as

$$W \preceq R(\lambda). \tag{5.62}$$

Summing up, problem (5.32) can be stated as

$$\min_{(\lambda,W,Z)\in\mathcal{C}} J = \lambda\Big(- \log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\Big). \tag{5.63}$$

where

$$\begin{aligned}\mathcal{C} := \{(\lambda, W, Z) :\ & Z \in \mathcal{O},\ \lambda \in \mathbb{R},\ \lambda > 0, W \in \mathbf{Q}_m,\\ & W \succ 0, W \preceq Q(\lambda, Z),\ W \preceq R(\lambda)\}.\end{aligned}$$

Finding a descending direction (λ, W, Z) for $J(\lambda, W, Z)$ satisfying simultaneously the constraints $W \preceq Q(\lambda, Z)$ and $W \preceq R(\lambda)$ may be a difficult task. Therefore, the idea is to implement an Alternating Direction Method of Multipliers (ADMM) algorithm to decouple such constraints.

Actually, three different algorithms based on the ADMM are presented in the following: *Algorithm 1* and *Algorithm 2* represent the first failed attempts at solving problem (5.63); as we will observe, they contain wrong considerations and implications. Then, the procedure has been reexamined and corrected in *Algorithm 3*. However, as we will see, in this third formulation of the problem we are not able to numerically solve the first minimization step required by the ADMM.

Before proceeding, we state the following result that we will exploit in the three algorithms:

Proposition 5.5.1. Let $A = UDU^{\top}$ be the eigenvalue decomposition of $A \in \mathbf{Q}_m$ with $D = \mathrm{diag}(d_{11}, \dots d_{mm})$. Let $\Pi_{\mathbf{Q}_m^+}$ be the projection operator onto the cone

of symmetric positive semi-definite $m \times m$ matrices $\mathbf{Q}_m^+$, defined by

$$\begin{aligned}\Pi_{\mathbf{Q}_m^+} &: \mathbf{Q}_m \to \mathbf{Q}_m^+ \\ \Pi_{\mathbf{Q}_m^+}(A) &= \arg\min_{S \in \mathbf{Q}_m^+} \| S - A \|^2 .\end{aligned}$$

Then

$$\Pi_{\mathbf{Q}_m^+}(A) = U \operatorname{diag}(f(d_{11}), \dots f(d_{mm}))U^\top$$

where

$$f(d_i) := \begin{cases} d_i & \text{if } d_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $S \in \mathbf{Q}_m^+$ be a $m \times m$ symmetric positive semi-definite matrix. Let d_{ij} denote the element of D in position (i, j), with $d_{ij} = 0$ for $i \neq j$, and let u_i denote the i-th column of the orthogonal matrix U.
Since the Frobenius norm of a matrix is invariant to the change of basis we have,

$$\begin{aligned}\| A - S \|^2 =& \| U^\top (A - S) U \|^2 \\ =& \| U^\top A U - U^\top S U \|^2 \\ =& \| D - U^\top S U \|^2 \\ =& \sum_{i=1}^m \sum_{j=1}^m |d_{ij} - u_i^\top S u_j|^2 \\ =& \sum_{j=1}^m |d_{jj} - u_j^\top S u_j|^2 + \sum_{i=1}^m \sum_{j=1}^m |u_i^\top S u_j|^2.\end{aligned}$$

Since $S \succeq 0$, then necessarily $u_j^\top S u_j \geq 0$; therefore, the matrix S minimizing the Frobenius distance from A is such that

$$u_j^\top S u_j = f(d_{jj})$$

and

$$u_i^\top S u_j = 0 \quad \text{for } i \neq j.$$

In a compact way, at the optimum

$$U^\top S U = \operatorname{diag}(f(d_{11}), \dots f(d_{mm})),$$

from which it follows

$$S = U \operatorname{diag}(f(d_{11}), \dots f(d_{mm}))U^\top.$$

□

5.5.1 Algorithm 1

To begin with, we reformulate problem (5.63) in a suitable format for ADMM implementation. This is achieved by introducing a new variable $Y \in \mathbf{Q}_m$ defined as

$$Y = W$$

subject to

$$Y \preceq R(\lambda).$$

The reformulated problem is

$$\begin{aligned} \min_{(\lambda,W,Z)\in\mathcal{C}_{\lambda,W,Z}, Y\in\mathcal{C}_Y} \; & J = \lambda\big(-\log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\big) \\ \text{subject to } & Y = W \end{aligned} \tag{5.64}$$

where

$$\begin{aligned} \mathcal{C}_{\lambda,W,Z} &:= \{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0,\ W \preceq Q(\lambda, Z)\} \\ \mathcal{C}_Y &:= \{Y : Y \in \mathbf{Q}_m : Y \preceq R(\lambda)\} \end{aligned}$$

Notice now that, since $J = \lambda(-\log|\lambda^{-1}W| - \int \log|\hat{\Phi}| + \delta)$, at the optimum W is necessarily equal to $Q(\lambda, Z)$. Indeed, given that the logarithm is a monotonic function and $\lambda > 0$, in order to minimize $J(\lambda, W, Z)$, W must have the largest possible determinant. In view of this observation, we can eliminate the reduntant variable W, obtaining:

$$\begin{aligned} \min_{(\lambda,Z)\in\mathcal{C}_{\lambda,Z}, Y\in\mathcal{C}_Y} \; & F = \lambda\big(-\log\big|\lambda^{-1}Q(\lambda,Z)\big| - \int \log|\hat{\Phi}| + \delta\big) \\ \text{subject to } & Y = Q(\lambda, Z) \end{aligned} \tag{5.65}$$

where

$$\begin{aligned} \mathcal{C}_{\lambda,Z} &:= \{(\lambda, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ Q(\lambda, Z) \succ 0\} \\ \mathcal{C}_Y &:= \{Y \in \mathbf{Q}_m : Y \preceq R(\lambda)\} \end{aligned}$$

The *augmented Lagrangian* for (5.65) is:

$$\mathcal{L}_\rho(\lambda, Z, Y, M) := \lambda\Big(- \log\left|\lambda^{-1}Q\right| - \int \log|\hat{\Phi}| + \delta\Big) + \langle M, Y - Q\rangle + \frac{\rho}{2} \parallel Y - Q \parallel^2$$

where $M \in \mathbf{Q}_m$ is the Lagrange multiplier, and $\rho > 0$ is the *penalty parameter*.

Accordingly, given the initial guesses $\lambda^{(0)}, Z^{(0)}, Y^{(0)}$ and $M^{(0)}$, the ADMM updates are:

$$(\lambda^{(k+1)}, Z^{(k+1)}) = \underset{(\lambda,Z)\in\mathcal{C}_{\lambda,Z}}{\arg\min}\ \mathcal{L}_\rho(\lambda, Z, Y^{(k)}, M^{(k)}) \tag{5.66}$$

$$Y^{(k+1)} = \underset{Y\in\mathcal{C}_Y}{\arg\min}\, \mathcal{L}_\rho(\lambda^{(k+1)}, Z^{(k+1)}, Y, M^{(k)}) \tag{5.67}$$

$$M^{(k+1)} = M^{(k)} + \rho\Big(Y^{(k+1)} - Q(\lambda^{(k+1)}, Z^{(k+1)})\Big) \tag{5.68}$$

Problem (5.66) does not admit a closed form solution, therefore we approximate the optimal solution by a gradient projection step:

$$\begin{aligned}
\lambda^{(k+1)} &= \lambda^{(k)} - t_k \nabla_\lambda \mathcal{L}_\rho(\lambda^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)}) \\
Z^{(k+1)} &= \Pi_{\mathcal{O}}\left(Z^{(k)} - t_k \nabla_Z \mathcal{L}_\rho(\lambda^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)})\right)
\end{aligned}$$

where:

- $\nabla_\lambda \mathcal{L}_\rho(\lambda, Z, Y, M)$ denotes the gradient of the augmented Lagrangian with respect to λ.
- $\nabla_Z \mathcal{L}_\rho(\lambda, Z, Y, M)$ denotes the gradient of the augmented Lagrangian with respect to Z.
- $\Pi_{\mathcal{O}}$ denotes the projector onto $\mathcal{O}$.
  We immediately see that $\Pi_{\mathcal{O}} : \mathbf{M}_{mn} \to \mathcal{O}$ is given by

  $$\Pi_{\mathcal{O}}(A) = \mathrm{ofd}_B(A)$$

- the step-size t_k is determined at each step k in an iterative fashion: we start by setting $t_k = 1$ and we decrease it progressively of a factor β with $0 < \beta < 1$ until the conditions $\lambda^{(k+1)} > 0$ and $Q(\lambda^{(k+1)}, Z^{(k+1)}) \succ 0$ are met and the so-called Armijo condition [15] is satisfied.

On the other hand, Problem (5.67) admits a closed form solution which can be easily computed as follows.

First notice that, by performing the change of variable

$$\tilde{Y} := R(\lambda^{(k+1)}) - Y, \tag{5.69}$$

the constraint $Y \succeq R(\lambda)$ is equivalent to $\tilde{Y} \succeq 0$ (i.e. $\tilde{Y} \in \mathbf{Q}_m^+$). Rewriting the augmented Lagrangian as a function of $\tilde{Y}$, the problem (5.67) can be stated in terms of $\tilde{Y}$ as

$$\begin{aligned}\tilde{Y}^{(k+1)} = \underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \lambda^{(k+1)}\Big(- \log\big|(\lambda^{(k+1)})^{-1}Q(\lambda^{(k+1)}, Z^{(k+1)})\big| - \int \log|\hat{\Phi}| + \delta\Big) + \\ + \langle M^{(k)}, R(\lambda^{(k+1)}) - \tilde{Y} - Q(\lambda^{(k+1)}, Z^{(k+1)})\rangle + \\ + \frac{\rho}{2} \parallel R(\lambda^{(k+1)}) - \tilde{Y} - Q(\lambda^{(k+1)}, Z^{(k+1)}) \parallel^2 .\end{aligned} \tag{5.70}$$

Since the arg min operator is invariant under addition of a constant and multiplication by a positive constant, it holds:

$$\begin{aligned}&\underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \lambda\Big(- \log\big|\lambda^{-1}Q\big| - \int \log|\hat{\Phi}| + \delta\Big) + \langle M, R - \tilde{Y} - Q\rangle + \\ &\qquad + \frac{\rho}{2} \parallel R - \tilde{Y} - Q \parallel^2 = \\ &\underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \langle M, -\tilde{Y}\rangle + \frac{\rho}{2} \parallel R - Q - \tilde{Y} \parallel^2 = \\ &\underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \frac{2}{\rho}\langle M, -\tilde{Y}\rangle + \parallel R - Q - \tilde{Y} \parallel^2 = \\ &\underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \frac{2}{\rho}\langle M, -\tilde{Y}\rangle + \parallel R - Q - \tilde{Y} \parallel^2 + \frac{1}{\rho^2} \parallel M \parallel^2 + \frac{2}{\rho}\langle M, R - Q\rangle = \\ &\underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \parallel R - Q - \tilde{Y} \parallel^2 + \frac{1}{\rho^2} \parallel M \parallel^2 + \frac{2}{\rho}\langle M, R - Q - \tilde{Y}\rangle .\end{aligned}$$

Therefore, problem (5.70) is equivalent to

$$\tilde{Y}^{(k+1)} = \underset{\tilde{Y}\in\mathbf{Q}_m^+}{\arg\min}\ \parallel R(\lambda^{(k+1)}) - Q(\lambda^{(k+1)}, Z^{(k+1)}) + \frac{1}{\rho}M^{(k)} - \tilde{Y} \parallel^2 . \tag{5.71}$$

If $\Pi_{\mathbf{Q}_m^+}$ denotes the projection operator onto the cone $\mathbf{Q}_m^+$ of symmetric positive semi-definite matrices of size $m \times m$,

$$\Pi_{\mathcal{C}_{\tilde{Y}}} : \mathbf{Q}_m \to \mathbf{Q}_m^+$$

defined as

$$\Pi_{\mathbf{Q}_m^+}(A) := \underset{\in\mathbf{Q}_m^+}{\arg\min}\ \parallel A - S \parallel^2, \tag{5.72}$$

from (5.71) it immediately follows that the update $\tilde{Y}^{(k+1)}$ is given by

$$\tilde{Y}^{(k+1)} = \Pi_{\mathbf{Q}_m^+}\Big(R(\lambda^{k+1}) - Q(\lambda^{(k+1)}, Z^{(k+1)}) + \frac{1}{\rho}M^{(k)}\Big).$$

Accordingly, the solution to problem (5.67) is

$$\begin{aligned} Y^{(k+1)} &= R(\lambda^{k+1}) - \tilde{Y}^{(k+1)} \\ &= R(\lambda^{k+1}) - \Pi_{\mathbf{Q}_m^+}\Big(R(\lambda^{k+1}) - Q(\lambda^{(k+1)}, Z^{(k+1)}) + \frac{1}{\rho}M^{(k)}\Big). \end{aligned}$$

We remind that we can compute $\Pi_{\mathbf{Q}_m^+}(\cdot)$ by exploiting Proposition 5.5.1.

Remark 3. As already mentioned, algorithm 1 is incorrect: problem (5.63) and (5.65) are not equivalent. As a matter of fact, in (5.65) we are searching the solution over

$$\tilde{\mathcal{C}} := \{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0,\ W = Q(\lambda, Z) \preceq R(\lambda)\},$$

that is a subset of the feasible set

$$\mathcal{C} = \{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0,\ W \preceq Q(\lambda, Z),\ W \preceq R(\lambda)\}$$

of problem (5.63); however it is not true that the optimal solution of (5.63) necessarily belong to $\tilde{\mathcal{C}}$. The error is in the transition from (5.64) to (5.65), where we have erroneously carried out the optimization of the objective function J with respect to W without taking into account the constraint $Y = W$, and, in doing so, we have eliminated the dual variable W by setting $W = Q(\lambda, Z)$.
The first idea to solve the issue is to maintain the dual variable W and apply the ADMM directly to (5.64). However, this approach would lead to the augmented Lagrangian

$$\mathcal{L}_\rho(\lambda, W, Y, M) := \lambda\Big(-\log\left|\lambda^{-1}W\right| - \int \log|\hat{\Phi}| + \delta\Big) + \langle M, Y - W\rangle + \frac{\rho}{2} \parallel Y - W \parallel^2$$

which does not depend on Z, so that it would be not possible to update the variable Z through a gradient projection step. In algorithm 2 we then propose a different formulation of the dual problem (5.63) with the aim of obtaining an augmented Lagrangian which is also function of the variable Z.

There is also another issue in the above procedure: it is not correct to apply the ADMM to solve problem (5.65) because it is not appropriately formulated in a suitable fashion for the ADMM implementation. As a matter of fact, the feasible sets $\mathcal{C}_{\lambda,Z}$ and $\mathcal{C}_Y$ are coupled since the dual variable λ appears in both of them. Hence, it is not possible to split the minimization over (λ, Z) and Y as requested by the ADMM algorithm. We reveal in advance that an analogous problem will arise in Algorithm 2.

5.5.2 Algorithm 2

Consider again Problem (5.63):

$$\min_{(\lambda,W,Z)\in\mathcal{C}} J = \lambda\Big(-\log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\Big)$$

where

$$\begin{aligned}\mathcal{C} := \{(\lambda, W, Z) :\ & Z \in \mathcal{O},\ \lambda \in \mathbb{R},\ \lambda > 0, W \in \mathbf{Q}_m,\\ & W \succ 0, W \preceq Q(\lambda, Z),\ W \preceq R(\lambda)\}.\end{aligned}$$

We recall that

$$Q(\lambda, Z) := T_{0,0}(Z + \lambda\hat{P}) - T_{0,1:n}(Z + \lambda\hat{P})T^{-1}_{1:n,1:n}(Z + \lambda\hat{P})T^{\top}_{0,1:n}(Z + \lambda\hat{P})$$

and

$$R(\lambda) := I + T_{0,0}(\lambda\hat{P}) - T_{0,1:n}(\lambda\hat{P})\big(I + T_{1:n,1:n}(\lambda\hat{P})\big)^{-1}T^{\top}_{0,1:n}(\lambda\hat{P}).$$

The aim is to formulate Problem (5.63) in a suitable format for ADMM implementation to decouple the constraint $W \preceq R(\lambda)$ and $W \preceq Q(\lambda, Z)$. This is achieved by introducing a new variable $Y \in \mathbf{Q}_m$ defined as

$$Y = Q(\lambda, Z)$$

subject to

$$Y \succeq W.$$

Then, Problem (5.63) can be stated as

$$\begin{aligned}\min_{(\lambda,W,Z)\in\mathcal{C}_{\lambda,W,Z}, Y\in\mathcal{C}_Y} \ & J = \lambda\Big(-\log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\Big)\\ \text{subject to } & Y = Q(\lambda, Z)\end{aligned} \tag{5.73}$$

where

$$\begin{aligned}\mathcal{C}_{\lambda,W,Z} &:= \{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0,\ W \preceq R(\lambda)\}\\ \mathcal{C}_Y &:= \{Y : Y \in \mathbf{Q}_m : Y \succeq W\}.\end{aligned}$$

We attack (5.73) by implementing an ADMM algorithm.

The *augmented Lagrangian* for (5.73) is:

$$\mathcal{L}_\rho(\lambda, W, Z, Y, M) := \lambda\Big(- \log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\Big) + \langle M, Y - Q(\lambda, Z)\rangle + \\ + \frac{\rho}{2} \parallel Y - Q(\lambda, Z) \parallel^2$$

where $M \in \mathbf{Q}_m$ is the Lagrange multiplier, and $\rho > 0$ is the *penalty parameter*.

Accordingly, given the initial guesses $\lambda^{(0)}, W^{(0)}, Z^{(0)}, Y^{(0)}$ and $M^{(0)}$, the ADMM updates are:

$$(\lambda^{(k+1)}, W^{(k+1)}, Z^{(k+1)}) = \underset{(\lambda,W,Z)\in\mathcal{C}_{\lambda,W,Z}}{\arg\min} \mathcal{L}_\rho(\lambda, W, Z, Y^{(k)}, M^{(k)}) \tag{5.74}$$

$$Y^{(k+1)} = \underset{Y\in\mathcal{C}_Y}{\arg\min}\, \mathcal{L}_\rho(\lambda^{(k+1)}, W^{(k+1)}, Z^{(k+1)}, Y, M^{(k)}) \tag{5.75}$$

$$M^{(k+1)} = M^{(k)} + \rho\Big(Y^{(k+1)} - Q(\lambda^{(k+1)}, Z^{(k+1)})\Big). \tag{5.76}$$

We first consider Problem (5.74).
We immediately observe that W appears in the augmented Lagrangian $\mathcal{L}_\rho$ only in the logarithmic term. Then, since the logarithm is a monotonic function and $\lambda > 0$, in order to minimize $\mathcal{L}_\rho$ the variable W must assume the largest possible determinant in $\mathcal{C}_{\lambda,W,Z}$, so that W is necessary equal to $R(\lambda)$ at the optimum. In view of this observation, we can eliminate the variable W, and reformulate the augmented Lagrangian as:

$$\tilde{\mathcal{L}}_\rho(\lambda, Z, Y, M) := \lambda\Big(- \log\big|\lambda^{-1}R(\lambda)\big| - \int \log|\hat{\Phi}| + \delta\Big) + \langle M, Y - Q(\lambda, Z)\rangle + \\ + \frac{\rho}{2} \parallel Y - Q(\lambda, Z) \parallel^2$$

with

$$(\lambda, Z) \in \mathcal{C}_{\lambda,Z} := \{(\lambda, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O}\}$$
$$Y \in \mathcal{C}_Y := \{Y : Y \in \mathbf{Q}_m, Y \succeq R(\lambda)\}.$$

The ADMM updates become:

$$(\lambda^{(k+1)}, Z^{(k+1)}) = \underset{(\lambda,Z)\in\mathcal{C}_{\lambda,Z}}{\arg\min}\, \tilde{\mathcal{L}}_\rho(\lambda, Z, Y^{(k)}, M^{(k)}) \tag{5.77}$$

$$Y^{(k+1)} = \underset{Y\in\mathcal{C}_Y}{\arg\min}\, \tilde{\mathcal{L}}_\rho(\lambda^{(k+1)}, Z^{(k+1)}, Y, M^{(k)}) \tag{5.78}$$

$$M^{(k+1)} = M^{(k)} + \rho\Big(Y^{(k+1)} - Q(\lambda^{(k+1)}, Z^{(k+1)})\Big). \tag{5.79}$$

Problem (5.77) does not admit a closed form solution, therefore we approximate the optimal solution by a gradient projection step:

$$\begin{aligned}\lambda^{(k+1)} &= \lambda^{(k)} - t_k \nabla_\lambda \tilde{\mathcal{L}}_\rho(\lambda^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)}) \\ Z^{(k+1)} &= \Pi_{\mathcal{O}} \left(Z^{(k)} - t_k \nabla_Z \tilde{\mathcal{L}}_\rho(\lambda^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)}) \right)\end{aligned}$$

where:

- $\nabla_\lambda \tilde{\mathcal{L}}_\rho(\lambda, Z, Y, M)$ denotes the gradient of the augmented Lagrangian $\tilde{\mathcal{L}}_\rho$ with respect to λ.
- $\nabla_Z \tilde{\mathcal{L}}_\rho(\lambda, Z, Y, M)$ denotes the gradient of the augmented Lagrangian $\tilde{\mathcal{L}}_\rho$ with respect to Z.
- $\Pi_{\mathcal{O}}$ denotes the projector onto $\mathcal{O}$, which, as already observed in Algorithm 1, is given by
$$\Pi_{\mathcal{O}}(A) = \mathrm{ofd}_B(A).$$
- the step-size t_k is determined at each step k in an iterative fashion: we start by setting $t_k = 1$ and we decrease it progressively of a factor β with $0 < \beta < 1$ until the condition $\lambda^{(k+1)} > 0$ is met and the Armijo condition [15] is satisfied.

Concerning (5.75), a closed form solution to the problem can be easily computed as follows. We first perform the change of variable

$$\tilde{Y} := Y - R(\lambda), \tag{5.80}$$

and notice that the constraint $Y \succeq R(\lambda)$ is equivalent to $\tilde{Y} \succeq 0$ (i.e. $\tilde{Y} \in \mathbf{Q}_m^+$). Then, we state Problem (5.75) in terms of the variable $\tilde{Y}$ as

$$\begin{aligned}\tilde{Y}^{(k+1)} = \underset{\tilde{Y} \in \mathbf{Q}_m^+}{\arg\min}\ \lambda^{(k+1)} \Big(&- \log \left| (\lambda^{(k+1)})^{-1} R(\lambda^{(k+1)}) \right| - \int \log |\hat{\Phi}| + \delta \Big) + \\ &+ \langle M^{(k)}, \tilde{Y} + R(\lambda^{(k+1)}) - Q(\lambda^{(k+1)}, Z^{(k+1)}) \rangle \\ &+ \frac{\rho}{2} \parallel \tilde{Y} + R(\lambda^{(k+1)}) - Q(\lambda^{(k+1)}, Z^{(k+1)}) \parallel^2 .\end{aligned} \tag{5.81}$$

Since the arg min operator is invariant under addition of a constant and multiplication by a positive constant, it holds:

$$\begin{aligned}\underset{\tilde{Y} \in \mathbf{Q}_m^+}{\arg\min}\ \lambda \Big(&- \log \left| \lambda^{-1} R \right| - \int \log |\hat{\Phi}| + \delta \Big) + \langle M, \tilde{Y} + R - Q \rangle + \\ &+ \frac{\rho}{2} \parallel \tilde{Y} + R - Q \parallel^2 =\end{aligned}$$

$$
\begin{aligned}
&\arg\min_{\tilde{Y}\in\mathbf{Q}_m^+} \ \langle M,\tilde{Y}\rangle + \frac{\rho}{2}\parallel \tilde{Y}+R-Q\parallel^2 = \\
&\arg\min_{\tilde{Y}\in\mathbf{Q}_m^+} \ \frac{2}{\rho}\langle M,\tilde{Y}\rangle + \parallel \tilde{Y}+R-Q\parallel^2 = \\
&\arg\min_{\tilde{Y}\in\mathbf{Q}_m^+} \ \frac{2}{\rho}\langle M,\tilde{Y}\rangle + \parallel \tilde{Y}+R-Q\parallel^2 + \frac{1}{\rho^2}\parallel M\parallel^2 + \frac{2}{\rho}\langle M,R-Q\rangle = \\
&\arg\min_{\tilde{Y}\in\mathbf{Q}_m^+} \ \parallel \tilde{Y}+R-Q\parallel^2 + \frac{1}{\rho^2}\parallel M\parallel^2 + \frac{2}{\rho}\langle M,\tilde{Y}+R-Q\rangle = \\
&\arg\min_{\tilde{Y}\in\mathbf{Q}_m^+} \ \parallel \tilde{Y}+R-Q+\frac{1}{\rho}M\parallel^2 .
\end{aligned}
$$

Therefore, Problem (5.81) is equivalent to

$$
\tilde{Y}^{(k+1)} = \arg\min_{\tilde{Y}\in\mathbf{Q}_m^+} \ \parallel \tilde{Y} - \big(-R(\lambda^{(k+1)}) + Q(\lambda^{(k+1)}, Z^{(k+1)}) - \frac{1}{\rho}M^{(k)}\big)\parallel^2 . \quad (5.82)
$$

Now, recalling that the projection operator $\Pi_{\mathbf{Q}_m^+}$ onto the cone $\mathbf{Q}_m^+$ of symmetric positive semi-definite matrices of size $m \times m$ is defined as

$$
\Pi_{\mathbf{Q}_m^+}(A) := \arg\min_{S\in\mathbf{Q}_m^+} \ \parallel A-S\parallel^2, \quad (5.83)
$$

from (5.82) it immediately follows that the update $\tilde{Y}^{(k+1)}$ is given by

$$
\tilde{Y}^{(k+1)} = \Pi_{\mathbf{Q}_m^+}\Big(Q(\lambda^{(k+1)}, Z^{(k+1)}) - R(\lambda^{(k+1)}) - \frac{1}{\rho}M^{(k)}\Big).
$$

Here, $\Pi_{\mathbf{Q}_m^+}$ can be computed by exploiting Proposition 5.5.1.
Then, the solution to Problem (5.75) is given by

$$
\begin{aligned}
Y^{(k+1)} &= R(\lambda^{(k+1)}) + \tilde{Y}^{(k+1)} \\
&= R(\lambda^{(k+1)}) + \Pi_{\mathbf{Q}_m^+}\Big(Q(\lambda^{(k+1)}, Z^{(k+1)}) - R(\lambda^{(k+1)}) - \frac{1}{\rho}M^{(k)}\Big).
\end{aligned}
$$

Remark 4. As previously said, Algorithm 2 does not represent a right strategy to attack Problem (5.63). This is because Problem (5.73) is not properly formulated in a suitable fashion for the ADMM implementation: the feasible sets $\mathcal{C}_{\lambda,W,Z}$ and $\mathcal{C}_Y$ are coupled because of the presence of the dual variable W in both of them. Consequently, it is not possible to split the minimization over (λ, W, Z) and Y, and thus to successfully apply the ADMM algorithm.
Algorithm 3 overcomes this problem by rewriting (5.63) in a different format in which both the objective function and the feasible sets are decoupled.

5.5.3 Algorithm 3

We recall that we want to solve Problem (5.63):

$$\min_{(\lambda,W,Z)\in\mathcal{C}} J = \lambda\Big(-\log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\Big).$$

where

$$\begin{aligned}\mathcal{C} := \{(\lambda, W, Z) :\ & Z \in \mathcal{O},\ \lambda \in \mathbb{R},\ \lambda > 0, W \in \mathbf{Q}_m,\\ & W \succ 0, W \preceq Q(\lambda, Z),\ W \preceq R(\lambda)\}\end{aligned}$$

and the functions $Q(\lambda, Z)$ and $R(\lambda)$ are defined as:

$$\begin{aligned}Q(\lambda, Z) &:= T_{0,0}(Z + \lambda\hat{P}) - T_{0,1:n}(Z + \lambda\hat{P})T^{-1}_{1:n,1:n}(Z + \lambda\hat{P})T^{\top}_{0,1:n}(Z + \lambda\hat{P})\\ R(\lambda) &:= I + T_{0,0}(\lambda\hat{P}) - T_{0,1:n}(\lambda\hat{P})\big(I + T_{1:n,1:n}(\lambda\hat{P})\big)^{-1}T^{\top}_{0,1:n}(\lambda\hat{P}).\end{aligned}$$

The aim is to formulate Problem (5.63) in a suitable format for ADMM implementation to decouple the constraint $W \preceq R(\lambda)$ and $W \preceq Q(\lambda, Z)$. This is achieved by introducing a new variable $Y \in \mathbf{Q}_m$ defined as

$$Y = Q(\lambda, Z) - W$$

subject to

$$Y \succeq 0.$$

Then, Problem (5.63) can be stated as

$$\begin{aligned}\min_{(\lambda,W,Z)\in\mathcal{C}_{\lambda,W,Z}, Y\in\mathbf{Q}_m^+} \ & J = \lambda\big(-\log\big|\lambda^{-1}W\big| - \int \log|\hat{\Phi}| + \delta\big)\\ \text{subject to } & Y = Q(\lambda, Z) - W\end{aligned} \tag{5.84}$$

where

$$\mathcal{C}_{\lambda,W,Z} := \{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0,\ W \preceq R(\lambda)\}.$$

Remark 5. Problem (5.84) is formulated in a suitable format for ADMM implementation; in particular, differently from algorithm 1 and 2, here the feasible sets for (λ, W, Z) and Y, i.e. $\mathcal{C}_{\lambda,W,Z}$ and $\mathbf{Q}_m^+$, are actually decoupled.

The *augmented Lagrangian* for (5.84) is:

$$\mathcal{L}_\rho(\lambda, W, Z, Y, M) := \lambda\Big(- \log \left|\lambda^{-1} W\right| - \int \log |\hat{\Phi}| + \delta\Big) + \langle M, Y - Q(\lambda, Z) + W\rangle + \frac{\rho}{2} \parallel Y - Q(\lambda, Z) + W \parallel^2$$

where $M \in \mathbf{Q}_m$ is the Lagrange multiplier, and $\rho > 0$ is the *penalty parameter*.

Accordingly, given the initial guesses $\lambda^{(0)}, W^{(0)}, Z^{(0)}, Y^{(0)}$ and $M^{(0)}$, the ADMM updates are:

$$(\lambda^{(k+1)}, W^{(k+1)}, Z^{(k+1)}) = \underset{(\lambda, W, Z) \in \mathcal{C}_{\lambda, W, Z}}{\arg\min} \mathcal{L}_\rho(\lambda, W, Z, Y^{(k)}, M^{(k)}) \tag{5.85}$$

$$Y^{(k+1)} = \underset{Y \in \mathbf{Q}_m^+}{\arg\min} \mathcal{L}_\rho(\lambda^{(k+1)}, W^{(k+1)}, Z^{(k+1)}, Y, M^{(k)}) \tag{5.86}$$

$$M^{(k+1)} = M^{(k)} + \rho \left(Y^{(k+1)} - Q(\lambda^{(k+1)}, Z^{(k+1)}) + W^{(k+1)}\right). \tag{5.87}$$

We start considering Problem (5.86), which can be easily solved as follows. First, by exploiting the invariance of the arg min operator under addition of a constant and multiplication by a positive constant, we have that

$$\begin{aligned}
&\underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \lambda(-\log \left|\lambda^{-1} W\right| - \int \log |\hat{\Phi}| + \delta) + \langle M, Y - Q + W\rangle + \frac{\rho}{2} \parallel Y - Q + W \parallel^2 \\
&= \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \langle M, Y\rangle + \frac{\rho}{2} \parallel Y - Q + W \parallel^2 \\
&= \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \frac{2}{\rho}\langle M, Y\rangle + \parallel Y - Q + W \parallel^2 \\
&= \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \frac{2}{\rho}\langle M, Y\rangle + \parallel Y - Q + W \parallel^2 + \frac{1}{\rho^2} \parallel M \parallel^2 + \frac{2}{\rho}\langle M, -Q + W\rangle \\
&= \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \parallel Y - Q + W \parallel^2 + \frac{1}{\rho^2} \parallel M \parallel^2 + \frac{2}{\rho}\langle M, Y - Q + W\rangle \\
&= \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \parallel Y - Q + W + \frac{1}{\rho} M \parallel^2 \\
&= \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \parallel Y - (Q - W - \frac{1}{\rho} M) \parallel^2 .
\end{aligned}$$

This allows to rewrite Problem (5.86) as

$$Y^{(k+1)} = \underset{Y \in \mathbf{Q}_m^+}{\arg\min}\, \parallel Y - \left(Q(\lambda^{(k+1)}, Z^{(k+1)}) - W^{(k+1)} - \frac{1}{\rho} M^{(k)}\right) \parallel^2 . \tag{5.88}$$

Now, recalling that the projection operator $\Pi_{\mathbf{Q}_m^+}$ onto the cone $\mathbf{Q}_m^+$ of symmetric

positive semi-definite matrices of size $m \times m$

$$\Pi_{\mathbf{Q}_m^+} : \mathbf{Q}_m \to \mathbf{Q}_m^+$$

is defined as

$$\Pi_{\mathbf{Q}_m^+}(A) := \underset{S=S^\top \succeq 0}{\arg\min} \quad \| S - A \|^2, \tag{5.89}$$

from (5.88) it immediately follows that the update $Y^{(k+1)}$ is given by

$$Y^{(k+1)} = \Pi_{\mathbf{Q}_m^+}\Big(Q(\lambda^{(k+1)}, Z^{(k+1)}) - W^{(k+1)} - \frac{1}{\rho}M^{(k)}\Big).$$

Here, in order to compute $\Pi_{\mathbf{Q}_m^+}$ we can exploit Proposition 5.5.1.

On the other hand, Problem (5.85) does not admit a closed form solution. It is also difficult to approximate the optimal solution by a gradient projection step because a closed-form expression of the projector onto the set $\mathcal{C}_{\lambda,W,Z}$ is not available. Consequently, we try to perform the (λ, W, Z)-update exploiting some heuristics.

The first idea is to initially perform a gradient projection step by considering that (λ, W, Z) belongs to the set $\{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0\}$ (that is, we do not take the constraint $W \preceq R(\lambda)$ into account for the moment). Then, in order to guarantee that the new point satisfies also the constraint $W \preceq R(\lambda)$, we consider λ fixed and we project W onto the convex cone $\{S : S \in \mathbf{Q}_m, S \preceq R(\lambda)\}$. More precisely, we perform the following operations:

$$\begin{aligned}
\lambda^{(k+1)} &= \lambda^{(k)} - t_k \nabla_\lambda \mathcal{L}_\rho(\lambda^{(k)}, W^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)}) \\
Z^{(k+1)} &= \Pi_{\mathcal{O}}\left(Z^{(k)} - t_k \nabla_Z \mathcal{L}_\rho(\lambda^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)})\right) \\
W^{(k+1)} &= \Pi\left(W^{(k)} - t_k \nabla_W \mathcal{L}_\rho(\lambda^{(k)}, W^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)})\right)
\end{aligned}$$

where:

- $\nabla_\lambda \mathcal{L}_\rho(\lambda, W, Z, Y, M)$ denotes the gradient of the augmented Lagrangian with respect to λ. By standard computation we get

$$\begin{aligned}
\nabla_\lambda \mathcal{L}_\rho &= -\log\left|\lambda^{-1}W\right| - \int \log|\hat{\Phi}| + \delta - \lambda \operatorname{tr}\left(\lambda W^{-1}(-\lambda^{-2}W)\right) \\
&\quad - \operatorname{tr}(M\nabla_\lambda Q) - \rho \operatorname{tr}\left((Y - Q + W)\nabla_\lambda Q\right) \\
&= -\log|\lambda^{-1}W| - \int \log|\hat{\Phi}| + \delta + \operatorname{tr}(I) - \operatorname{tr}(M\nabla_\lambda Q) \\
&\quad - \rho \operatorname{tr}\left((Y - Q + W)\nabla_\lambda Q\right) \\
&= -\log|\lambda^{-1}W| - \int \log|\hat{\Phi}| + \delta + m - \operatorname{tr}\left(\left(M + \rho(Y - Q + W)\right)\nabla_\lambda Q\right)
\end{aligned}$$

where $\nabla_\lambda Q$ denotes the gradient of the function $Q(\lambda, Z)$ with respect to λ. Introducing the convention that, whenever the argument of the operators $T_{0,1:n}$ and $T_{1:n,1:n}$ is omitted, it is intended to be equal to $(Z + \lambda\hat{P})$, and recalling the definitions $\mathcal{T} := T([\hat{P}_0, \dots, \hat{P}_{n-1}])$ and $\mathcal{K} := [\hat{P}_1, \dots, \hat{P}_n]$ given in Section 5.3.3, we get

$$\nabla_\lambda Q(\lambda, Z) = \hat{P}_0 - \mathcal{K} T_{1:n,1:n}^{-1} T_{0,1:n}^\top + T_{0,1:n} T_{1:n,1:n}^{-1} \mathcal{T} T_{1:n,1:n}^{-1} T_{0,1:n}^\top - T_{0,1:n} T_{1:n,1:n}^{-1} \mathcal{K}^\top.$$

- $\nabla_Z \mathcal{L}_\rho(\lambda, W, Z, Y, M)$ denotes the gradient of the augmented Lagrangian with respect to Z.
  Computing the first variation of $\mathcal{L}_\rho$ along the direction δZ we get:

$$\begin{aligned} \mathcal{L}_\rho(\delta Z) &:= \mathcal{L}_\rho(\lambda, W, Z, Y, M; \delta Z) \\ &= -\operatorname{tr}(M \delta Q) - \rho \operatorname{tr}\big((Y - Q + W)\delta Q\big) \\ &= \operatorname{tr}\Big(\big(-M - \rho(Y - Q + W)\big)\delta Q\Big). \end{aligned} \tag{5.90}$$

  Here, δQ denotes the first variation of $Q(\lambda, Z)$ along th direction δZ:

$$\begin{aligned} \delta Q := \delta Q(\lambda, Z; \delta Z) &= T_{0,0}(\delta Z) - T_{0,1:n}(\delta Z) T_{1:n,1:n}^{-1} T_{0,1:n}^\top + \\ &+ T_{0,1:n} T_{1:n,1:n}^{-1} T_{1:n,1:n}(\delta Z) T_{1:n,1:n}^{-1} T_{0,1:n}^\top - T_{0,1:n} T_{1:n,1:n}^{-1} T_{0,1:n}^\top(\delta Z) \end{aligned} \tag{5.91}$$

  with the usual convention for the arguments of the operators $T_{0,1:n}$ and $T_{1:n,1:n}$. Therefore, plugging (5.91) into (5.90) and exploiting the cyclic property of the trace, it is easy to get

$$\begin{aligned} \mathcal{L}_\rho(\delta Z) &= \operatorname{tr}\Bigg(A\Big(T_{0,0}(\delta Z) - T_{0,1:n}(\delta Z) T_{1:n,1:n}^{-1} T_{0,1:n}^\top + \\ &+ T_{0,1:n} T_{1:n,1:n}^{-1} T_{1:n,1:n}(\delta Z) T_{1:n,1:n}^{-1} T_{0,1:n}^\top - T_{0,1:n} T_{1:n,1:n}^{-1} T_{0,1:n}^\top(\delta Z)\Big)\Bigg) \\ &= \operatorname{tr}\Big(A T_{0,0}(\delta Z) - A T_{0,1:n} T_{1:n,1:n}^{-1} T_{0,1:n}^\top(\delta Z)\Big) + \\ &+ \operatorname{tr}\Big(-T_{1:n,1:n}^{-1} T_{0,1:n}^\top A T_{0,1:n}(\delta Z) + T_{1:n,1:n}^{-1} T_{0,1:n}^\top A T_{0,1:n} T_{1:n,1:n}^{-1} T_{1:n,1:n}(\delta Z)\Big) \\ &= \operatorname{tr}\Big(\begin{bmatrix} A & -A T_{0,1:n} T_{1:n,1:n}^{-1} \\ -T_{1:n,1:n}^{-1} T_{0,1:n}^\top A & T_{1:n,1:n}^{-1} T_{0,1:n}^\top A T_{0,1:n} T_{1:n,1:n}^{-1} \end{bmatrix} T(\delta Z)\Big) \\ &= \operatorname{tr}\Big(\begin{bmatrix} I_m \\ -T_{1:n,1:n}^{-1} T_{0,1:n}^\top \end{bmatrix} A \begin{bmatrix} I_m & -T_{0,1:n} T_{1:n,1:n}^{-1} \end{bmatrix} T(\delta Z)\Big) \end{aligned}$$

  where, for simplicity, we have denoted with A the expression $(-M - \rho(Y -$

$Q + W)$).
Then, since T and D are adjoint operators, we get

$$\mathcal{L}_\rho(\delta Z) = \mathrm{tr}\Big(D\big(\begin{bmatrix} I_m \\ -T^{-1}_{1:n,1:n}T^{\top}_{0,1:n} \end{bmatrix} A \begin{bmatrix} I_m & -T_{0,1:n}T^{-1}_{1:n,1:n} \end{bmatrix}\big)\delta Z\Big),$$

from which it immediately follows

$$\nabla_Z \mathcal{L}_\rho = D\big(\begin{bmatrix} I_m \\ -T^{-1}_{1:n,1:n}T^{\top}_{0,1:n} \end{bmatrix} (-M-\rho(Y-Q+W)) \begin{bmatrix} I_m & -T_{0,1:n}T^{-1}_{1:n,1:n} \end{bmatrix}\big)$$

- $\nabla_W \mathcal{L}_\rho(\lambda, W, Z, Y, M)$ denotes the gradient of the augmented Lagrangian with respect to W.
  By computing the first variation of $\mathcal{L}_\rho$ along the direction δW we get:

$$\begin{aligned}\mathcal{L}_\rho(\delta W) &:= \mathcal{L}_\rho(\lambda, W, Z, Y, M; \delta W)\\ &= -\lambda\,\mathrm{tr}\left(\lambda W^{-1}(\lambda^{-1}\delta W)\right) + \mathrm{tr}\left(M\delta W\right) + \rho\,\mathrm{tr}\left((Y - Q + W)\delta W\right)\\ &= \mathrm{tr}\left((-\lambda W^{-1} + M + \rho(Y - Q + W))\delta W\right).\end{aligned}$$

  Therefore,

$$\nabla_W \mathcal{L}_\rho = -\lambda W^{-1} + M + \rho(Y - Q + W).$$

- $\Pi_{\mathcal{O}}$ denotes the projector operator onto $\mathcal{O}$.
  We immediately see that $\Pi_{\mathcal{O}} : \mathbf{M}_{mn} \to \mathcal{O}$ is given by

$$\Pi_{\mathcal{O}}(A) = \mathrm{ofd}_B(A)$$

- Π denotes the projector operator onto the convex cone $\{S : S \in \mathbf{Q}_m, S \preceq R(\lambda^{k+1})\}$ which is defined as:

$$\Pi(A) = \underset{S\in\mathbf{Q}_m, S\preceq R(\lambda^{k+1})}{\arg\min} \parallel S - A \parallel^2 . \tag{5.92}$$

  In order to find an explicit expression for Π, we perform the change of variable

$$\tilde{S} := R(\lambda^{k+1}) - S \tag{5.93}$$

  and we notice that the condition $S \preceq R(\lambda^{k+1})$ reads as $\tilde{S} \succeq 0$. Then,

Problem (5.92) can be stated in terms of $\tilde{S}$ as

$$\arg\min_{\tilde{S}\in\mathbf{Q}_m^+} \| R(\lambda^{k+1}) - \tilde{S} - A \|^2 = \arg\min_{\tilde{S}\in\mathbf{Q}_m^+} \| \tilde{S} - (R(\lambda^{k+1}) - A) \|^2 . \quad (5.94)$$

By exploiting the definition of the projector $\Pi_{\mathbf{Q}_m^+}$ provided in Proposition 5.5.1, the optimal $\tilde{S}$ is given by

$$\tilde{S} = \Pi_{\mathbf{Q}_m^+}(R(\lambda^{k+1}) - A).$$

Finally, from (5.93) we get

$$\Pi(A) = R(\lambda^{k+1}) - \Pi_{\mathbf{Q}_m^+}(R(\lambda^{k+1}) - A). \quad (5.95)$$

- the step-size t_k is determined at each step k in an iterative fashion: we start by setting $t_k = 1$ and we decrease it progressively of a factor β with $0 < \beta < 1$ until the conditions $\lambda^{(k+1)} > 0$ and $W^{(k+1)} \succ 0$ are met and the so-called Armijo condition [15] is satisfied:

$$\begin{aligned}\mathcal{L}_\rho\big(\lambda^{(k+1)}, W^{(k+1)}, Z^{(k+1)}, Y^{(k)}, M^{(k)}\big) \leq \mathcal{L}_\rho(\lambda^{(k)}, W^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)}) + \\ + \alpha(\lambda^{(k+1)} - \lambda^{(k)})\nabla_\lambda \mathcal{L}_\rho(\lambda^{(k)}, W^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)}) + \\ + \alpha\langle W^{(k+1)} - W^{(k)}, \nabla_W \mathcal{L}_\rho(\lambda^{(k)}, W^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)})\rangle + \\ + \alpha\langle Z^{(k+1)} - Z^{(k)}, \nabla_Z \mathcal{L}_\rho(\lambda^{(k)}, W^{(k)}, Z^{(k)}, Y^{(k)}, M^{(k)})\rangle\end{aligned}$$

with $0 < \alpha < 1$.

Remark 6. Numerical simulations reveal that this approach is not able to guarantee that the ADMM iterates reach the optimal solution of Problem (5.84).

An alternative strategy is the following. As before, we perform a gradient projection step by considering that (λ, W, Z) belongs to the set $\{(\lambda, W, Z) : \lambda \in \mathbb{R},\ \lambda > 0,\ Z \in \mathcal{O},\ W \succ 0\}$. This time, however, in order to satisfy the constraint $W \preceq R(\lambda)$, we consider W fixed and we progressively increase λ until the condition considered condition is met.

Remark 7. Again, numerical simulations reveal that this heuristic does not achieve the desired outcome.

6
Conclusion

In this thesis we have faced the problem of developing a robust identification procedure for factor analysis of moving average processes.

In the first chapters we developed the necessary theoretical background on factor analysis and two important tools of convex mathematical optimization: the Lagrangian duality theory and the ADMM.

Then, the original contribution of this thesis was proposed in Chapter 5. Here, we first provided a formal mathematical statement of the MA factor analysis problem. The difficulty related to the non-convexity of the minimum-rank problem was overcome by considering the trace norm relaxation of the latter. Moreover, robustness on the complexity of the estimated model (especially in terms of the number of factors) was accomplished by searching the optimal solution on a confidence set about a finite sample estimate of the underlying spectral density. Such set contains the true model with a prescribed probability and its radius (in the Itakura-Saito topology) depends only on the numerosity of the sample and the order of the MA process. We then proposed a finite-dimensional matricial reformulation of the problem and, by means of the duality theory, we proved the equivalence between the original problem and its matricial parametrization, as well as the existence of a solution. Solving the dual problem turned out to be an extremely challenging task. The idea of exploiting the alternating direction method of multipliers to decouple its constraints and split the problem into smaller subproblems didn't return the desired result.

For future work, we plan to find an alternative strategy for numerically solving our MA factor analysis problem. Then, the efficiency of procedure will be tested in numerical simulations with synthetic data generated by a known "true" model:

after defining suitable metrics, a comparison between the true model and the model estimated from the data will be carried over. It would be also interesting to develop a second test with real data from a concrete example. The idea is to use data extracted from international daily stock markets in order to model the variability of some financial indicators commonly used in portfolio selection. This is indeed a case where the amount of data is enormous and there is evidence in the relevant literature supporting the assumption that a relatively small number of factors explains most of data variability. Moreover, there are many studies and analysis that propose models for these data so that this appears an ideal benchmark to assess the performance of the procedure and compare the results with competing methods.

A

Basics on Stochastic Processes

In this section, we provide miscellaneous information about stationary stochastic processes.

A.1 Stationary stochastic processes

The intuitive idea behind *stochastic processes* is the *randomness*. Briefly speaking, this means that we are not able to predict exactly the variation of such a signal outside the observed interval. More formally, a stochastic process can be defined as a collection of random variables that is indexed by a variable t, usually representing time.

The signals dealt with in the thesis are multidimensional, real-valued, discrete-time stochastic processes. Such signals are commonly obtained by sampling a continuous-time signal so that $t \in \mathbb{Z}$ is usually measured in units of the sampling interval.

Let $y = \{y(t),\ t \in \mathbb{Z}\}$ be a stochastic process. Throughout this work, we assume that y is *zero-mean*, i.e.

$$\mathbb{E}\{y(t)\} = 0 \qquad \forall t. \tag{A.1}$$

The *covariance function* of y is assumed to depend only on the lag k between any two samples so that it can be defined as

$$R(k) := \mathbb{E}\{y(t)y(t-k)^\top\} \qquad t, k \in \mathbb{Z}. \tag{A.2}$$

The two assumptions (A.1) and (A.2) imply that y is a *second-order stationary*

stochastic process [1].

Example 1. The stochastic process $y = \{y(t),\ t \in \mathbb{Z}\}$ is a *Gaussian* process if for every $t_1, \ldots, t_n \in \mathbb{Z}$ the collection of random variables $\{y(t_1), \ldots, y(t_n)\}$ has Gaussian distribution.

Example 2. The stochastic process $e = \{e(t),\ t \in \mathbb{Z}\}$ is a *white Gaussian noise* if it is a zero-mean Gaussian process and it is independent and identically distributed (i.i.d.), namely $e(t)$ and $e(s)$ have the same distribution and $\mathbb{E}\{e(t)e(s)^\top\} = 0$ for any $t, s \in \mathbb{Z}$ with $t \neq s$.

It is well-known (see, for example, [29, p.193-194]) that the function $R(\cdot)$ enjoys these simple but important properties:

$$R(k) = R(-k)^\top, \tag{A.3}$$

and the block-Toeplitz matrix

$$\mathbb{E}\left\{ \begin{bmatrix} y(0) \\ \vdots \\ y(N) \end{bmatrix} \begin{bmatrix} y(0)^\top & \ldots & y(N)^\top \end{bmatrix} \right\} = \begin{bmatrix} R(0) & R(-1) & \ldots & R(-N) \\ R(1) & R(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & R(-1) \\ R(N) & \ldots & R(1) & R(0) \end{bmatrix} \tag{A.4}$$

is symmetric, positive semi-definite for any $N \in \mathbb{N}$. In particular, $R(0) = R(0)^\top \succeq 0$.

A.2 Power Spectral Density

The *power spectral density* (PSD) of a zero-mean stationary stochastic process $y = \{y(t),\ t \in \mathbb{Z}\}$ is defined as the discrete-time Fourier transform (DTFT) of the covariance sequence:

$$\Phi(e^{i\theta}) := \sum_{k=-\infty}^{\infty} R(k) e^{-i\theta k}. \tag{A.5}$$

[1] The second-order stationarity will be simply called "stationarity"

We assume that the Fourier transform exists as an integrable function in $[-\pi, \pi]$, i.e.

$$\int_{-\pi}^{\pi} \Phi(e^{i\theta}) d\theta < \infty.$$

Example 3. The power spectral density of a white Gaussian noise with covariance equal to the identity I is $\Phi(e^{i\theta}) = I,\ \theta \in [-\pi, \pi]$.

We should also mention that we can recover $\{R(k)\}$ from Φ by exploiting the inverse DTFT:

$$R(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(\theta) e^{i\theta k} d\theta.$$

We recall that the power spectral density of a real process is a real-valued function satisfying the following properties (see for example [29, p.194-195]):

$$\Phi(e^{i\theta}) = \Phi(e^{-i\theta})^{\top} \qquad \forall \theta \in [-\pi, \pi]. \tag{A.6}$$

and

$$\Phi(e^{i\theta}) \succeq 0 \qquad \forall \theta \in [-\pi, \pi]. \tag{A.7}$$

For a power spectral density Φ, we define the norm as

$$\| \Phi \| := \max_{\theta \in [-\pi, \pi]} \sigma_1(\Phi(e^{i\theta})),$$

where $\sigma_1(\Phi(e^{i\theta}))$ denotes the largest singular value of $\Phi(e^{i\theta})$ at θ, and the (normal) rank as

$$\operatorname{rank} \Phi := \max_{\theta \in [-\pi, \pi]} \operatorname{rank}(\Phi(e^{i\theta})).$$

Next, we present a useful result which concerns the transformation of a PSD through a linear system [29, p.195]:

Proposition A.2.1 (Wiener-Kintchine formula). Consider a linear time-invariant

BIBO stable system [2] with $\mathcal{Z}$-transform

$$H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}.$$

Let x be the stationary input to the system with spectrum Φ_x, and let y be the corresponding output. Then, y is a stationary process with PSD

$$\Phi_y(e^{i\theta}) = H(e^{i\theta})\Phi_x(e^{i\theta})H(e^{i\theta})^* \qquad \theta \in [-\pi, \pi], \tag{A.8}$$

where $H(e^{i\theta}) = H(z)|_{z=e^{i\theta}}$. The relation (A.8) is known as *Wiener-Kintchine formula*.

A.3 Signals with rational spectra

An important class of signals is represented by the ones with rational spectral density (i.e. the PSD is expressed as the ratio of two (matricial) polynomials in $e^{i\theta}$). The following result holds [30, p.110]:

Theorem A.3.1. Let $y = \{y(t),\ t \in \mathbb{Z}\}$ be a m-dimensional process with rational power spectral density Φ. Suppose $\operatorname{rank} \Phi(e^{i\theta}) = p \leq m$. Then, Φ admits factorization

$$\Phi(e^{i\theta}) = W(e^{i\theta})W(e^{i\theta})^* \tag{A.9}$$

where $W(e^{i\theta})$ is a rational function of dimension $m \times p$ such that $W(z)$ - with $z = e^{i\theta}$ - has all the poles strictly inside the unit circle. The function $W(e^{i\theta})$ is called *spectral factor*.

Note that it is always possible to choose $W(e^{i\theta})$ in the decomposition (A.9) *minimum phase*, i.e. with all the zeros in the region $\{|z| \leq 1\}$ (see [30, p.92]).

In view of Theorem A.3.1, if y is a stochastic process with rational spectral

[2] We say that a linear time-invariant system with $\mathcal{Z}$-transform $H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}$ is BIBO-stable if its impulsive response is absolutely summable, i.e.

$$\sum_{k=-\infty}^{\infty} |h_k| < \infty.$$

This is equivalent to require that all the poles of $H(z)$ are strictly inside the unit circle, i.e. in the region $\{|z| < 1\}$.

density Φ, it admits the spectral factor $W(z) := \frac{B(z)}{A(z)}$ where

$$A(z) = I + \sum_{k=1}^{q} A_k z^{-k}, \qquad B(z) = \sum_{k=0}^{n} B_k z^{-k}$$

and $A(z)$ has all the zeros in $\{|z| < 1\}$. As a consequence, y can be represented as the output of a linear system with transfer function $W(z)$ driven by the normalized white noise $e(t)$:

$$y(t) = \frac{B(z)}{A(z)} e(t), \tag{A.10}$$

or equivalently,

$$y(t) + \sum_{k=1}^{q} A_k y(t-k) = B_0 e(t) + \sum_{k=1}^{n} B_k e(t-k). \tag{A.11}$$

A signal y satisfying equation (A.11) is called *autoregressive moving average* (ARMA) signal. In particular, if $n = 0$, then y is said to be an *autoregressive* (AR) signal, while, if $q = 0$, it is a *moving average* (MA) signal.

Notice that for a MA process of order n

$$y(t) = B_0 e(t) + \sum_{k=1}^{n} B_k e(t-k) \tag{A.12}$$

it holds that $R(k) = 0$ for $|k| > n$. In view of this simple observation, the PSD of a MA signal turns into the finite-dimensional summation:

$$\Phi = \sum_{k=-n}^{n} R(k) e^{-i\theta k}. \tag{A.13}$$

Hence, given a finite-length realization $\mathrm{y}^N = \{\, \mathrm{y}_1 \dots \mathrm{y}_N \,\}$ of the MA process y, a simple estimator of its spectrum is obtained by inserting the estimates of $\{R(k)\}$ in (A.13):

$$\hat{\Phi} = \sum_{k=-n}^{n} \hat{R}(k) e^{-i\theta k}. \tag{A.14}$$

In the previous formula, a standard way to compute the sample covariance is

$$\hat{R}(k) = \frac{1}{N} \sum_{t=k+1}^{N} y(t) y(t-k)^\top \qquad 0 \le k \le p. \tag{A.15}$$

B

Convex Analysis and Optimization

This Appendix contains different notions regarding convex analysis and convex optimization problems, focusing the attention on real-valued functions defined on the Euclidean space $\mathbb{R}^n$.
A valid reference for this part is [15]; this reference will be omitted.

B.1 Convex sets

Definition B.1.1 (Convex set). A set $C \subseteq \mathbb{R}^n$ is called convex if for all $x_1, x_2 \in \mathbb{R}^n$ the whole *segment* $[x_1, x_2] \subseteq C$, where

$$[x_1, x_2] := \{z = \lambda x_1 + (1-\lambda)x_2,\ 0 \leq \lambda \leq 1\}. \tag{B.1}$$

In other terms, a set C is convex if the line segment between any two points in C lies in C. We can easily prove the following characterization of convex sets:

Proposition B.1.1. Let $C \subseteq \mathbb{R}^n$ and $x_1, \ldots, x_n \in C$. A point of the form $\lambda_1 x_1 + \cdots + \lambda_n x_n$ where $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0,\ i = 1, \ldots, n$ is called *convex combination* of $x_1, \ldots, x_n$. The set C is convex if and only if it contains every convex combination of its points.

It is worth noticing that convexity is preserved under intersection:

Proposition B.1.2. Let $\{C_\alpha,\ \alpha \in I\}$ be a (possibly uncountable) collection of convex subsets of $\mathbb{R}^n$. Then $C := \bigcap_{\alpha \in I} C_\alpha$ is convex.

By proposition B.1.2, given any subset $C \subseteq \mathbb{R}^n$ it is possible to define the smallest convex set containing C as the intersection of all such sets; it is called *convex hull* and denoted $\mathbf{conv}C$.

Example 4. The empty set $\emptyset$, any singleton $\{x_0\}$ and the whole Euclidean space $\mathbb{R}^n$ are the simplest cases of convex subsets of $\mathbb{R}^n$. The *hyperplane*, defined as $H := \{x \in \mathbb{R}^n \mid a^\top x = b\}$ with $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$ is another convex set of $\mathbb{R}^n$.

An important class of convex sets is represented by the so-called *convex cones*:

Definition B.1.2 (Convex cone). A set C is called *cone* if for every $x \in C$ and $\lambda \geq 0$, $\lambda x \in C$. A set C is a *convex cone* if it is convex and a cone, which means that

$$\lambda_1 x_1 + \lambda_2 x_2 \in C \qquad \forall x_1, x_2 \in C \text{ and } \forall \lambda_1, \lambda_2 \geq 0. \tag{B.2}$$

Example 5. If we denote by $\mathbf{Q}_n$ the set of symmetric $n \times n$ matrices $\mathbf{Q}_n := \{X \in \mathbb{R}^{n \times n} \mid X = X^\top\}$, which is a vector space of dimension $n(n+1)/2$, then the set $\mathbf{Q}_n^+$ of symmetric positive semidefinite $n \times n$ matrices $\mathbf{Q}_n^+ := \{X \in \mathbf{Q}_n \mid X \succeq 0\}$ is a convex cone in $\mathbf{Q}_n$. This follows directly from the definition of positive semidefiniteness: for any $x \in \mathbb{R}^n$, $A, B \succeq 0$ and $\lambda_1, \lambda_2 \geq 0$, we have

$$x^\top(\lambda_1 A + \lambda_2 B)x = \lambda_1 x^\top A x + \lambda_2 x^\top B x \geq 0,$$

that is $(\lambda_1 A + \lambda_2 B) \in \mathbf{Q}_n^+$.
The interior of $\mathbf{Q}_n^+$ consists of the positive definite matrices: $\mathbf{Q}_n^{++} := \{X \in \mathbf{Q}_n \mid X \succ 0\}$.

B.2 Convex functions

Definition B.2.1 ((Strictly) Convex and Concave function). Let C be a convex subset of $\mathbb{R}^n$. A function $f : C \to (-\infty, +\infty]$ is called *convex* if it satisfies

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \qquad \forall x_1, x_2 \in C,\ \forall \lambda \in [0,1]. \tag{B.3}$$

It is called strictly convex if strict inequality holds in (B.3).
We say that f is concave if $-f$ is convex, and strictly concave if $-f$ is strictly convex.

Geometrically, (B.3) means that the secant line of a convex function through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ always lies above the graph of the function.

Observe that for an affine function [1] we always have equality in (B.3), so all affine (and therefore also linear) functions are both convex and concave. Conversely, it is possible to prove that any function that is both convex and concave is affine.

The basic inequality (B.3) is easily extended to convex combinations of more than two points, as shown in the following theorem:

Theorem B.2.1 (Jensen 1906). Let $C \subseteq \mathbb{R}^n$ convex and $f : C \to (-\infty, +\infty]$. f is convex if and only if for all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in C^n$, for all $\lambda_1, \ldots, \lambda_n \in [0,1]^n$ such that $\sum_{i=1}^n \lambda_i = 1$, it holds

$$f\big(\sum_{i=1}^{n} \lambda_i x_i\big) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \tag{B.4}$$

The inequality (B.4) is sometimes called *Jensen's inequality*, and it extends to infinite sums, integrals, and expected values.

Convex functions give rise to convex sets in an important way:

Proposition B.2.1. For any convex function $f : \mathbb{R}^n \to \mathbb{R}$ and any $\alpha \in \mathbb{R}$, the *α-sublevel set* $C_\alpha := \{x \in \mathbf{dom}\ f \mid f(x) \leq \alpha\}$ is a convex set.

Proof. The statement immediately follows from the definition of convexity: if $x, y \in C_\alpha$ then $f(x) \leq \alpha$ and $f(y) \leq \alpha$. Let $z = \lambda x + (1-\lambda)y$; by convexity of f

$$f(z) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda\alpha + (1-\lambda)\alpha \leq \alpha.$$

Thus, $z \in C_\alpha$. □

However, the converse is not always true: a function can have all its sublevel sets convex, even if it is not a convex function. For example, $f(x) = e^{-x}$ is not convex on $\mathbb{R}$ (indeed, it is strictly concave) but all its sublevel sets are convex.

The link between convex sets and convex functions is via the *epigraph* of a function. As the name suggests - 'epi' means 'above' - the epigraph is 'above the graph' of the function. More formally,

[1] A function $f : \mathcal{C} \to (-\infty, +\infty]$ is *affine* if $f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda) f(x_2)$ for any $x_1, x_2 \in \mathcal{C}$ and $\lambda \in \mathbb{R}$

Definition B.2.2 (Epigraph). Let $C \subseteq \mathbb{R}^n$ convex, and $f : C \to (-\infty, +\infty]$. Then the *epigraph* of f is defined as

$$\mathbf{epi} f = \{(x, \alpha) \in (C \times \mathbb{R}) \mid f(x) \leq \alpha\}.$$

Proposition B.2.2. Let $C \subseteq \mathbb{R}^n$ convex and $f : C \to (-\infty, +\infty]$. Then, f is convex if and only if $\mathbf{epi} f$ is convex.

Next, we provide a useful characterizations of convex function.

Theorem B.2.2. Let f be a twice differentiable function, that is its Hessian matrix (or second derivative) $H_{(i,j)}(x) := \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ exists at each point in $\mathbf{dom}\ f$, which is open. Then f is convex if and only if $\mathbf{dom}\ f$ is convex and its Hessian is positive semidefinite, i.e.

$$H(x) \succeq 0 \qquad \forall x \in \mathbf{dom}\ f. \tag{B.5}$$

From a geometric viewpoint, (B.5) can be interpreted as the requirement that the graph of f has positive curvature at x.
Strict convexity can be only partially characterized by second-order conditions: if $\nabla^2 f \succeq 0$ for all $x \in \mathbf{dom}\ f$, then f is strictly convex. The converse, however, is not true: for example the function $f(x) = x^2$ is strictly convex, but it has zero second derivative at $x = 0$.

Example 6. We have already mentioned that all linear and affine functions are convex (and concave). The exponential $f(x) = e^{ax}$ is convex on $\mathbb{R}$ for any $a \in \mathbb{R}$. The logarithm $f(x) = \log x$ is concave on $\mathbf{dom}\ f = \{x \in \mathbb{R} \mid x > 0\}$. The log-determinant function $f(X) = \log \det X$ is concave on $\mathbf{dom}\ f = \mathbf{Q}_n^{++}$.

It is important to establish what operations preserve the precious property of convexity, or allow us to construct new convex and concave functions.

Proposition B.2.3. The nonnegative weighted sum of convex functions $f_1, \ldots, f_n$

$$f = w_1 f_1 + \cdots + w_n f_n, \qquad w_i \geq 0,\ i = 1, \ldots, n$$

is convex.

Proof. The fact that convexity is preserved under nonnegative scaling and addition is easily verified directly, or can be stated in terms of the associated epigraphs. □

Proposition B.2.4. The pointwise supremum of an arbitrary collection of convex functions is convex. On the other hand, the pointwise infimum of an arbitrary collection of concave functions is a concave function.

Proof. Assume that for each $y \in \mathcal{I}$, $f(x, y)$ is convex in x, and consider $g(x) := \sup_{y \in \mathcal{I}} f(x, y)$. In terms of epigraph, the pointwise supremum of functions corresponds to the intersection of epigraph, i.e.

$$\mathbf{epi} g = \bigcap_{y \in \mathcal{I}} \mathbf{epi} f(\cdot, y),$$

so that the result immediately follows from Proposition B.1.2.
In analogous fashion, it is possible to prove the second part of the statement. □

We conclude this section providing some useful definitions:

Definition B.2.3 (Closed function). A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *closed* if for each $\alpha \in \mathbb{R}$, the α-sublevel set $\{x \in \mathbf{dom}\ f \mid f(x) \leq \alpha\}$ is a closed set.

This condition is equivalent to require that the epigraph of the function is closed. The definition of closed function is general, but it is usually only applied to convex functions.

Definition B.2.4 (Proper convex function). A *proper convex function* f is a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ taking value in the extended real number line such that $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x. [2]

Definition B.2.5 (Convex hull). Given $f : C \to [-\infty, +\infty]$, the *convex hull* $\operatorname{co} f$ is the greatest convex function such that $\operatorname{co} f(x) \leq f(x)\ \forall x \in \mathcal{C}$. [3]

It is clear that the convex hull provides the tightest convex lower approximation of a nonconvex function.

[2] [31, p.24].
[3] [31, p.36].

B.3 Convex optimization problems

An optimization problem concerns the minimization (or maximization) of a real-valued function f_0 over an admissible set C. Here, we concentrate on problems in finitely many variables, in which case C is a subspace of the vector space $\mathbb{R}^n$. Accordingly, an *optimization problem* can be formulated as

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f_0(x) \\ \text{subject to } &f_i(x) \leq 0, \quad i = 1, ..., m \\ &h_i(x) = 0, \quad i = 1, ..., p. \end{aligned} \tag{B.6}$$

The vector $x = [x_1, ..., x_n]^\top$ is the *optimization variable*, the function $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the *objective function* or *cost function*; the inequalities $f_i(x) \leq 0,\ i = 1, ..., m$ are called *inequality constraints* and $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, ..., m$ *inequality constraint functions*; the equations $h_i(x) = 0,\ i = 1, ..., p$ are called *equality constraints* and the functions $h_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, ..., p$ *equality constraint functions*. If there are no constraints, namely $m = p = 0$, then problem (B.6) is said to be *unconstrained*.

The *domain* of the optimization problem (B.6) is given by the set of points for which the objective and all the constraint functions are defined, namely

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\ h_i.$$

A point $x \in \mathcal{D}$ is *feasible* if it satisfies all the constraints $f_i(x) \leq 0,\ i = 1, ..., m$ and $h_i(x) = 0,\ i = 1, ..., p$. The problem is said to be *feasible* if there exists at least one feasible point, *infeasible* otherwise. The set of all feasible points is called *feasible set*.

The *optimal value* p^* of problem (B.6) is defined as

$$p^* = \inf_{x\in\mathcal{D}} \{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

We allow p^* to assume the extended values $\pm\infty$ with the following convention: if the problem is infeasible, we have $p^* = +\infty$, while if there are feasible points x_k for which $f_0(x_k) \to -\infty$, then $p^* = -\infty$. In the latter case we say that problem (B.6) is *unbounded below*. A vector x^* is called *optimal*, or a *solution* of the problem (B.6), if it has the smallest objective value, i.e. $f_0(x^*) = p^*$, among all the feasible points. The set of optimal points is the *optimal set* X_{opt}. If there exists at least one optimal point, we say the optimal value is *attained* and the problem (B.6) is *solvable*; on the contrary, if X_{opt} is empty, we say that the optimal value is not attained (notice that the latter situation always occur when the problem is

unbounded below).

If x is feasible and $f_i(x) = 0$ we say that the ith inequality constraint $f_i(x) \leq 0$ is *active* at x, while if $f_i(x) < 0$ then $f_i(x) \leq 0$ is *inactive*. Of course, the equality constraint are always active at all feasible points. We say that a constraint is *redundant* if deleting it doesn't change the feasible set.

A special class of mathematical optimization is *convex optimization*, characterized by the convexity of both the objective function and the feasible set. Formally, a *convex optimization problem* is formulated as

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \ f_0(x) \\ &\text{subject to } f_i(x) \leq 0, \quad i = 1, ..., m \\ &\qquad\qquad a_i^\top x = b_i, \quad i = 1, ..., p. \end{aligned} \tag{B.7}$$

where $f_0, \ldots, f_m$ are convex functions. Comparing (B.7) with the general standard form problem (B.6), it is easy to see that convex problem has three additional requirements:

- the objective function must be convex
- the inequality constraint functions must be convex
- the equality constraint functions $h_i(x) = a_i^\top x - b_i$ must be affine.

From (B.7) we immediately see that the feasible set of a convex optimization problem is indeed convex, since it is the intersection of the domain of the problem

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i$$

which is a convex set (f_i are convex), with m convex sublevel sets $\{x \mid f_i(x) \leq 0\}$ and p hyperplanes $\{x \mid a_i^\top x = b_i\}$.

One of the main reason why convex optimization plays a central role in mathematical optimization is that any locally optimal point is also globally optimal:

Proposition B.3.1. For a convex optimization problem (B.7) any locally optimal point is also globally optimal. Moreover, if the objective function f_0 is strictly convex, then the problem has at most one optimal point.

C

Proofs of Section 5.3.3

To establish the results in Section 5.3.3 we need the following preliminary results (see [13, Appendix C]).

Lemma C.0.1. Let $\Psi(z)$ be the spectral density of a second-order, zero-mean, purely non-deterministic process y and let $R_i := \mathbb{E}[y(t+i)y^\top(t)]$ be the i-th covariance lag of y. Finally, let $\mathcal{T}_n = T([R_0 \dots R_{n-1}])$ and $\mathcal{K}_n := [R_1 \dots R_n]$. Then,

$$\log|R_0 - \mathcal{K}_n \mathcal{T}_n^{-1} \mathcal{K}_n^\top| \geq \int \log|\Psi|.$$

Moreover, if y is AR of order n, then the previous formula holds with equality.

Lemma C.0.2. Let

$$M = \begin{bmatrix} M_{00} & M_{01} & \dots & M_{0n} \\ M_{01}^\top & M_{11} & \dots & M_{1n} \\ \vdots & \dots & \dots & \vdots \\ M_{0n}^\top & \dots & \dots & M_{nn} \end{bmatrix} \in \mathbf{Q}_{m(n+1)}$$

be a symmetric and positive (semi-) definite matrix partitioned in blocks M_{jl} of

dimension $m \times m$. Let $M_{jl,d} := \text{diag}^2(M_{jl})$ for all $j, l = 0, ..., n$ and

$$M_d := \begin{bmatrix} M_{00,d} & M_{01,d} & \dots & M_{0n,d} \\ M_{01,d}^\top & M_{11,d} & \dots & M_{1n,d} \\ \vdots & \dots & \dots & \vdots \\ M_{0n,d}^\top & \dots & \dots & M_{nn,d} \end{bmatrix}.$$

Then M_d is positive (semi-) definite.

In addition, it is useful to recall the Schur complement characterization for positive definiteness and positive semi-definiteness:

Lemma C.0.3. For any symmetric matrix M of the form

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

Then

- $M \succ 0$ if and only if $C \succ 0$ and the *Schur complement* of C in M is positive definite, i.e. $A - BC^{-1}B^\top \succ 0$
- if $C \succ 0$, then $M \succeq 0$ if and only if the *Schur complement* of C in M is positive semidefinite, i.e. $A - BC^{-1}B^\top \succeq 0$.

C.1 Proof of Proposition 5.3.1

The proof exploits arguments similar to [13, Proposition 6.1].

We consider two possible scenarios separately.

1) Let $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ be such that, besides (5.33), we have $\| (\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) \| \to +\infty$. Since we are dealing with symmetric matrices, this is equivalent to the fact that the maximum of the absolute values of the eigenvalues of $(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00})$ diverges:

$$\lim_{k\to\infty} \max_{\alpha^{(k)} \in \sigma\left((\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}\right)} |\alpha^{(k)}| = +\infty. \tag{C.1}$$

The next step is to show that (C.1) implies

$$\lim_{k\to\infty} \min_{\alpha^{(k)}\in\sigma\left((\lambda^{(k)})^{-1}([Z^{(k)}]_0-[V^{(k)}]_{00}\right)} \alpha^{(k)} = -\infty. \tag{C.2}$$

Indeed, (C.1) implies that at least one of the two conditions (C.2) and

$$\lim_{k\to\infty} \max_{\alpha^{(k)}\in\sigma\left((\lambda^{(k)})^{-1}([Z^{(k)}]_0-[V^{(k)}]_{00}\right)} \alpha^{(k)} = +\infty. \tag{C.3}$$

holds. To show now that (C.3) implies (C.2), notice that, since $V^{(k)} \succeq 0$ and $\lambda^{(k)} > 0\ \forall k$, $\max\{\alpha^{(k)} : \alpha^{(k)} \in \sigma\big((\lambda^{(k)})^{-1}[Z^{(k)}]_0\big)\}$ is non-smaller than the argument of the limit in the left-side of (C.3). Thus (C.3) implies

$$\lim_{k\to\infty} \max_{\alpha^{(k)}\in\sigma\left((\lambda^{(k)})^{-1}[Z^{(k)}]_0\right)} \alpha^{(k)} = +\infty. \tag{C.4}$$

But $(\lambda^{(k)})^{-1}[Z^{(k)}]_0$ is traceless $\forall k$, so the sum of its eigenvalues is zero, and thus so we have

$$\lim_{k\to\infty} \min_{\alpha^{(k)}\in\sigma\left((\lambda^{(k)})^{-1}[Z^{(k)}]_0\right)} \alpha^{(k)} = -\infty. \tag{C.5}$$

By the same argument as before, $V^{(k)} \succeq 0$ and $\lambda^{(k)} > 0$ imply that $\forall k$ $\min\{\alpha^{(k)} : \alpha^{(k)} \in \sigma\big((\lambda^{(k)})^{-1}[Z^{(k)}]_0\big)\}$ is non-smaller than the argument in the left side of (C.2). This fact, together with (C.5), leads to (C.2). By (C.2), we obtain that for k sufficiently large $[Z^{(k)}]_0-[V^{(k)}]_{00}+\lambda^{(k)}\hat{P}_0 = (\lambda^{(k)})\big((\lambda^{(k)})^{-1}([Z^{(k)}]_0-[V^{(k)}]_{00})+\hat{P}_0\big)$ has at least a negative eigenvalue, so the sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})$ is not in $\mathcal{C}$.

2) Consider now a sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ for which, besides (5.33), we have $\| (\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) \| \to c$ with $0 \le c < \infty$. Then, it can be seen that $\forall\varepsilon > 0$, $\exists\bar{k}$ such that $J(\lambda^{(k)}, V^{(k)}, Z^{(k)}) > -\varepsilon$ for all $k > \bar{k}$. Indeed, since $\| (\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) \|$ is bounded, there exists a real constant l such that for all k it holds

$$(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0 \preceq lI_m$$

Then, since for any two matrices $A, B \in \mathbf{Q}_m$, if $0 \preceq A \preceq B$ then $\det(A) < \det(B)$, we have:

$$\begin{gathered}
|(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0| \le l^m \\
\log|(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0| \le m\log l \\
-\log|(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0| \ge -m\log l \\
-\log|(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0| - \int\log|\hat{\Phi}| + \delta \ge -m\log l - \int\log|\hat{\Phi}| + \delta.
\end{gathered}$$

Therefore, we can define a second real constant $l_1 := -m \log l - \int \log |\hat{\Phi}| + \delta$, and $\forall k$ it holds $J(\lambda^{(k)}, V^{(k)}, Z^{(k)}) \geq \lambda^{(k)} l_1$. Since l_1 is constant, $\lambda^{(k)} l_1 \to 0$ so that, by definition, $\forall \varepsilon > 0$, $\exists \bar{k}$ such that

$$J(\lambda^{(k)}, V^{(k)}, Z^{(k)}) \geq \lambda^{(k)} l_1 > -\varepsilon, \quad \forall k > \bar{k}$$

Now, it is sufficient to find a triple $(\bar{\lambda}, \bar{V}, \bar{Z}) \in \mathcal{C}$ with $J(\bar{\lambda}, \bar{V}, \bar{Z})$ strictly negative to conclude that such a sequence is not an infimizing sequence.
To this purpose, let us consider $\bar{\lambda}$ sufficiently small, but striclty greater than zero. Moreover, let $\bar{Z}_j = -\bar{\lambda} \operatorname{ofd}(\hat{P}_j)$ for all $j = 0, \ldots, n$. Finally, we need to define $\bar{V}$. To this end, let $\hat{P}_{j,d} := \operatorname{diag}^2(\hat{P}_j)$, $j = 0, \ldots, n$, $\hat{P}_d := [\hat{P}_{0,d}| \ldots |\hat{P}_{n,d}]$ and $\mathcal{T}_{n+1,d} := T(\hat{P}_d)$. Observe that $\mathcal{T}_{n+1,d}$ is defined from $T(\hat{P})$ by the same "block by block diagonalization" procedure defined in Lemma C.0.2, so it is positive definite. Now we partition $\mathcal{T}_{n+1,d}$ as follows:

$$\mathcal{T}_{n+1,d} = \begin{bmatrix} \hat{P}_{0,d} & \mathcal{K}_d \\ \mathcal{K}_d^\top & \mathcal{T}_{n,d} \end{bmatrix}$$

which defines the matrices $\mathcal{K}_d$ and $\mathcal{T}_{n,d}$. We now set

$$\bar{V} := \bar{\lambda} \begin{bmatrix} \mathcal{K}_d \mathcal{T}_{n,d}^{-1} \mathcal{K}_d^\top & \mathcal{K}_d \\ \mathcal{K}_d^\top & \mathcal{T}_{n,d} \end{bmatrix}.$$

As already noticed, in view of Lemma C.0.2, $\mathcal{T}_{n,d}$ is positive definite (and hence invertible), so that $\bar{V}$ is positive semi-definite by Lemma C.0.3.
It is not difficult to check that the triple $(\bar{\lambda}, \bar{V}, \bar{Z})$ just defined is in $\mathcal{C}$ for $\bar{\lambda}$ sufficiently small. It remains to show that $J(\bar{\lambda}, \bar{V}, \bar{Z})$ is negative.
By linearity, $\operatorname{diag}^2(\hat{\Phi}^{-1})$ is the power spectral density of the process whose covariance lags are $\hat{P}_{d,j}$, so that, in view of Lemma C.0.1, we have

$$\begin{aligned} J(\bar{\lambda}, \bar{V}, \bar{Z}) &= -\bar{\lambda} \log |\hat{P}_{0,d} - \mathcal{K}_d \mathcal{T}_{n,d}^{-1} \mathcal{K}_d^\top| - \bar{\lambda} \int \log |\hat{\Phi}| + \bar{\lambda}\delta \\ &\leq -\bar{\lambda} \int \log |\operatorname{diag}^2(\hat{\Phi}^{-1})| - \bar{\lambda} \int \log |\hat{\Phi}| + \bar{\lambda}\delta \\ &= \bar{\lambda} \left(\delta - \int \log |\hat{\Phi} \operatorname{diag}^2(\hat{\Phi}^{-1})| \right) \\ &= \bar{\lambda}(\delta - \delta_{max}) \\ &< 0, \end{aligned}$$

where in the last equality we have taken into account the expression (5.12) while

the last inequality follows from the assumption $\delta_{max} > \delta$.

This suffices to conclude the proof. Indeed, the only possible remaining case is that for which $\lim_{k\to\infty} \parallel (\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) \parallel$ does not exist. However, in this case it is always possible to consider a subsequence $(\lambda^{(k_j)}, V^{(k_j)}, Z^{(k_j)})$ for which the limit exists (finite or infinite) and we can therefore reduce to one of the previous cases.

C.2 Proof of Proposition 5.3.2

The first part of the proof aims to show that (5.34) holds if and only if (5.35) holds as well. We are assuming that the estimated model has a non-trivial *dynamic*, i.e. there exists $i \neq 0$ such that $\hat{P}_i \neq 0$. Hence, there exists $(\bar{l}, \bar{h}) \neq (0, 0)$ such that $[T(\hat{P})]_{\bar{l}\bar{h}} \neq 0$. From the condition

$$[\lambda^{(k)} T(\hat{P}) + T(Z^{(k)}) - V^{(k)}]_{\bar{l}\bar{h}} = 0$$

(which is one of the conditions for the sequence to be in $\mathcal{C}_1$), the condition (5.35) holds if and only if $[T(Z^{(k)}) - V^{(k)}]_{\bar{l}\bar{h}}$ diverges.
Since (5.34) holds if and only if $[T(Z^{(k)}) - V^{(k)}]_{lh}$ diverges for some (l, h), it remains to show that if

$$\lim_{k\to\infty} \parallel [Z^{(k)}]_0 - [V^{(k)}]_{00} \parallel = +\infty, \tag{C.6}$$

then (5.35) holds. Notice that (C.6) implies that at least one eigenvalue of $([Z^{(k)}]_0 - [V^{(k)}]_{00})$ tends to $-\infty$ as $k \to \infty$. As a matter of fact, since we are dealing with symmetric matrices, (C.6) is equivalent to the fact that the maximum of the absolute values of the eigenvalues of $([Z^{(k)}]_0 - [V^{(k)}]_{00})$ diverges:

$$\lim_{k\to\infty} \max_{\alpha^{(k)} \in \sigma\left(([Z^{(k)}]_0 - [V^{(k)}]_{00}\right)} |\alpha^{(k)}| = +\infty. \tag{C.7}$$

(C.7) implies that at least one of the two conditions

$$\lim_{k\to\infty} \min_{\alpha^{(k)} \in \sigma\left(([Z^{(k)}]_0 - [V^{(k)}]_{00}\right)} \alpha^{(k)} = -\infty. \tag{C.8}$$

and

$$\lim_{k\to\infty} \max_{\alpha^{(k)} \in \sigma\left([Z^{(k)}]_0 - [V^{(k)}]_{00}\right)} \alpha^{(k)} = +\infty. \tag{C.9}$$

holds. To show now that (C.9) implies (C.8), notice that, since $V^{(k)} \succeq 0 \ \forall k$, $\max\{\alpha^{(k)} : \alpha^{(k)} \in \sigma\big([Z^{(k)}]_0\big)\}$ is non-smaller than the argument of the limit in the

left-side of (C.9). Thus (C.9) implies

$$\lim_{k\to\infty} \max_{\alpha^{(k)}\in\sigma\left([Z^{(k)}]_0\right)} \alpha^{(k)} = +\infty. \tag{C.10}$$

But $[Z^{(k)}]_0$ is traceless $\forall k$, so the sum of its eigenvalues is zero and thus we have

$$\lim_{k\to\infty} \min_{\alpha^{(k)}\in\sigma\left([Z^{(k)}]_0\right)} \alpha^{(k)} = -\infty. \tag{C.11}$$

By the same argument as before, $V^{(k)} \succeq 0$ implies that $\forall k$ $\min\{\alpha^{(k)} : \alpha^{(k)} \in \sigma\big((\lambda^{(k)})^{-1}[Z^{(k)}]_0\big)\}$ is non-smaller than the argument in the left side of (C.8). This fact, together with (C.11), leads to (C.8). As a consequence, since $\hat{P}_0$ is fixed, (C.6) implies that the constraint $(\lambda^{(k)}\hat{P}_0 + [Z^{(k)}]_0 - [V^{(k)}]_{00}) \succ 0$ can hold $\forall k$ only if (5.35) holds.

So far we have seen the equivalence between (5.34) and (5.35) and the fact that (C.6) implies (5.35). We now show that (5.35) implies not only (C.6), but the stronger condition

$$\lim_{k\to\infty} \frac{\| [Z^{(k)}]_0 - [V^{(k)}]_{00} \|}{\lambda^{(k)}} \neq 0. \tag{C.12}$$

In fact, we assume by contradiction that (C.12) does not hold, namely

$$\lim_{k\to\infty} \frac{\| [Z^{(k)}]_0 - [V^{(k)}]_{00} \|}{\lambda^{(k)}} = 0.$$

We show that the corresponding sequence does not belong to $\mathcal{C}_1$ as the constraint on the positive semi-definiteness of $(I + V^{(k)} - T(Z^{(k)}))$ fails for k sufficiently large. Indeed a symmetric matrix is positive semidefinite if and only if every principal minor is non-negative. Thus, let us consider the principal minor of order 2 obtained as follows. Select a block $\hat{P}_h$ with $h \neq 0$ and an element in position (p, q) such that $[\hat{P}_h]_{(p,q)} \neq 0$. Note that it is always possible to find such an element because we are assuming that the process has non-trivial dynamic. Then, consider the following 2×2 sub-matrix of $(I + V^{(k)} - T(Z^{(k)}))$:

$$\begin{aligned} &\begin{bmatrix} 1 + \left[[V^{(k)}]_{00} - [Z^{(k)}]_0\right]_{(p,p)} & \lambda^{(k)}[\hat{P}_h]_{(p,q)} \\ \lambda^{(k)}[\hat{P}_h^\top]_{(q,p)} & 1 + \lambda^{(k)}[\hat{P}_0]_{(q,q)} \end{bmatrix} \\ = &\begin{bmatrix} 1 + \left[[V^{(k)}]_{00} - [Z^{(k)}]_0\right]_{(p,p)} & \lambda^{(k)}[\hat{P}_h]_{(p,q)} \\ \lambda^{(k)}[\hat{P}_h]_{(p,q)} & 1 + \lambda^{(k)}[\hat{P}_0]_{(q,q)} \end{bmatrix} \end{aligned} \tag{C.13}$$

where $[\hat{P}_0]_{(q,q)} \geq 0$ and where we have expolited the constraint $[\lambda^{(k)}T(\hat{P}) + T(Z^{(k)}) - V^{(k)}]_{lh} = 0$ for $(l,h) \neq (0,0)$. Since we are assuming (by contradiction) that $\lim_{k\to\infty} \frac{\|[Z^{(k)}]_0 - [V^{(k)}]_{00}\|}{\lambda^{(k)}} = 0$, the determinant of (C.13) is negative for k sufficiently large. Indeed we immediately see that the determinant of (C.13) is

$$\begin{aligned}
&\big(1 + \big[[V^{(k)}]_{00} - [Z^{(k)}]_0\big]_{(p,p)}\big)\big(1 + \lambda^{(k)}[\hat{P}_0]_{(q,q)}\big) - \big(\lambda^{(k)}[\hat{P}_h]_{(p,q)}\big)^2 = \\
&(\lambda^{(k)})^2 \left\{ \big(\frac{1}{\lambda^{(k)}} + \frac{\big[[V^{(k)}]_{00} - [Z^{(k)}]_0\big]_{(p,p)}}{\lambda^{(k)}}\big)\big(\frac{1}{\lambda^{(k)}} + [\hat{P}_0]_{(q,q)}\big) - [\hat{P}_h]^2_{(p,q)} \right\}
\end{aligned}$$

and it diverges to $-\infty$ as $k \to +\infty$.
Therefore, the constraint on the positive semi-definiteness of $(I + V^{(k)} - T(Z^{(k)}))$ fails for k sufficiently large. Thus, the proof reduces to ruling out the following two possible cases:

1) Consider the case of a sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ such that, besides (5.35), we also have

$$\lim_{k\to\infty} \frac{\| [Z^{(k)}]_0 - [V^{(k)}]_{00} \|}{\lambda^{(k)}} = +\infty.$$

We can repeat the same reasoning of the Proof of Proposition 5.3.1 to conclude that at least one eigenvalue of $(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00})$ tends to $-\infty$ as $k \to \infty$. This implies that for k sufficiently large the positivity of $[Z^{(k)}]_0 - [V^{(k)}]_{00} + \lambda^{(k)}\hat{P}_0 = \lambda^{(k)}\big((\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0\big)$ fails, which rules out this case.

2) Finally, consider a sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ in $\mathcal{C}_1$ for which $\| [Z^{(k)}]_0 - [V^{(k)}]_{00} \| \to \infty$ at the same speed of $\lambda^{(k)}$ and $\| [T(Z^{(k)}) - V^{(k)}]_{lh} \|$. Note that, since

$$(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0 \succeq 0,$$

it holds that

$$\begin{aligned}
&(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) + \hat{P}_0 = C^{(k)} \\
&(\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) = C^{(k)} - \hat{P}_0 \\
&[V^{(k)}]_{00} - [Z^{(k)}]_0 = \lambda^{(k)}(\hat{P}_0 - C^{(k)})
\end{aligned} \tag{C.14}$$

for a certain $C^{(k)} \succeq 0$.
By (C.14) and $[\lambda^{(k)}T(\hat{P}) + T(Z^{(k)}) - V^{(k)}]_{lh} = 0$ for $(l,h) \neq (0,0)$, we get

$$(\lambda^{(k)})^{-1}(V^{(k)} - T(Z^{(k)})) = \begin{bmatrix} \hat{P}_0 - C^{(k)} & \mathcal{K} \\ \mathcal{K}^\top & \mathcal{T} \end{bmatrix}$$

with $\mathcal{K} := [\hat{P}_1 \dots \hat{P}_n]$ and $\mathcal{T} := T([\hat{P}_0 \dots \hat{P}_{n-1}])$; therefore, since $((\lambda^{(k)})^{-1}I)$ tends to zero as $1/\lambda^{(k)}$, we have that

$$\begin{aligned}(\lambda^{(k)})^{-1}(I + V^{(k)} - T(Z^{(k)})) &= \begin{bmatrix} \hat{P}_0 - C^{(k)} & \mathcal{K} \\ \mathcal{K}^\top & \mathcal{T} \end{bmatrix} + (\lambda^{(k)})^{-1}I \\ &= \begin{bmatrix} \hat{P}_0 - C^{(k)} & \mathcal{K} \\ \mathcal{K}^\top & \mathcal{T} \end{bmatrix} + \mathcal{O}(\frac{1}{\lambda^{(k)}}).\end{aligned}$$

Since $\mathcal{T}$ is positive definite and $(\lambda^{(k)})^{-1}(I + V^{(k)} - T(Z^{(k)}) \succeq 0$, by using the Schur complement, we get

$$\hat{P}_0 - C^{(k)} - \mathcal{K}\mathcal{T}^{-1}\mathcal{K}^\top + \mathcal{O}(\frac{1}{\lambda^{(k)}}) \succeq 0,$$

thus

$$C^{(k)} \preceq C^{(k)}_{max}$$

with

$$C^{(k)}_{max} := \hat{P}_0 - \mathcal{K}\mathcal{T}^{-1}\mathcal{K}^\top + \mathcal{O}(\frac{1}{\lambda^{(k)}}).$$

Therefore

$$\hat{P}_0 + (\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00}) \preceq C^{(k)}_{max}.$$

Hence,

$$\begin{aligned} J^{(k)} &:= J(\lambda^{(k)}, V^{(k)}, Z^{(k)}) \\ &= \lambda^{(k)}\Big(- \log\big|\hat{P}_0 + (\lambda^{(k)})^{-1}([Z^{(k)}]_0 - [V^{(k)}]_{00})\big| - \int \log|\hat{\Phi}| + \delta\Big) \\ &\geq \lambda^{(k)}\Big(- \log\big|C^{(k)}_{max}\big| - \int \log|\hat{\Phi}| + \delta\Big) \\ &= \lambda^{(k)}\Big(- \log\big|C^{(k)}_{max}\big| + \int \log|\hat{\Phi}^{-1}| + \delta\Big). \end{aligned} \tag{C.15}$$

Notice that $\hat{\Phi}^{-1} = \sum_{k=-\infty}^{\infty} \hat{P}_k e^{-i\vartheta k}$ is the power spectral density of an AR process of order n, then by using Lemma C.0.1 which holds with equality, from (C.15) we get

$$\begin{aligned} J^{(k)} &\geq \lambda^{(k)}\Big(- \log\big|C^{(k)}_{max}\big| + \log\big|\hat{P}_0 - \mathcal{K}\mathcal{T}^{-1}\mathcal{K}^\top\big| + \delta\Big) \\ &= \lambda^{(k)}\Big(\delta + \mathcal{O}(\frac{1}{\lambda^{(k)}}\Big) \to +\infty \qquad \text{as } k \to \infty. \end{aligned}$$

Thus $(\lambda^{(k)}, V^{(k)}, Z^{(k)})$ cannot be an infimizing sequence.

C.3 Proof of Proposition 5.3.3

Consider a sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})_{k\in\mathbb{N}}$ in $\mathcal{C}_2$.

We first show that $[Z^{(k)}]_0$ cannot diverge. Indeed, assume by contradiction that $\lim_{k\to\infty} \| [Z^{(k)}]_0 \| = +\infty$. Since it is a symmetric and traceless matrix, this implies

$$\lim_{k\to\infty} \min_{\alpha^{(k)}\in\sigma\left([Z^{(k)}]_0\right)} \alpha^{(k)} = -\infty. \tag{C.16}$$

In view of (C.16), since $\lambda^{(k)}\hat{P}_0$ is bounded and $V^{(k)}$ positive semidefinite $\forall k$, then $(\lambda^{(k)}\hat{P}_0 + [Z^{(k)}]_0 - [V^{(k)}]_{00})$ has at least a negative eigenvalue for k sufficiently large, so that the sequence $(\lambda^{(k)}, V^{(k)}, Z^{(k)})$ is not in $\mathcal{C}_2$. We conclude that

$$\lim_{k\to\infty} \| [Z^{(k)}]_0 \| < \infty. \tag{C.17}$$

As a consequence, since $\beta I \preceq T(Z^{(k)}) - V^{(k)} \preceq I$ (which is one of the condition for the sequence to be in $\mathcal{C}_2$), and $[T(Z^{(k)})]_{hh} = [Z^{(k)}]_0$ by construction, it holds that $\forall k$

$$\| [V^{(k)}]_{hh} \| < \infty, \quad h = 0, \dots, n. \tag{C.18}$$

Then, since $V^{(k)} \succeq 0$, it follows that also the off-diagonal blocks of $V^{(k)}$ must be bounded $\forall k$, i.e.

$$\| [V^{(k)}]_{hl} \| < \infty, \quad l \neq h, \;\; l, h = 0, \dots, n. \tag{C.19}$$

Finally, by the boundeness of $(T(Z^{(k)}) - V^{(k)})$ and (C.19), we obtain that $\forall k$

$$\| [Z^{(k)}]_h \| < \infty \quad h = 1, .., n, \tag{C.20}$$

concluding the proof.

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