UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Master Degree in Physics

Final Dissertation

Renormalization group equations for theories with axion-like particles via on-shell amplitude methods

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Academic Year 2022/2023

Abstract

In this thesis work, we will carry out the renormalization group equations (RGEs) program in the context of theories with axion-like particles (ALPs). After specifying the ALP Effective Field Theory, in the first part of this thesis we will evaluate the relevant RGEs through standard one-loop calculations based on Feynman diagram techniques. Instead, the second part of the thesis, which represents an original contribution of the present work, is devoted to reproduce the ALP RGEs via on-shell amplitude methods, which have been proven to be quite powerful to study the ultraviolet properties of quantum field theories. The results of this thesis represent a crucial step towards the development of a two-loop RGEs program for ALP theories, which is missing so far in the literature.

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Chapter 1

Introduction

The Standard Model (SM) of particle physics has been validated down to scales close to 10^{-16} cm. Despite of its success, the SM fails to account for several observations, such as dark matter, neutrino masses and the cosmological matter-antimatter asymmetry, which are indications that the present theory requires further extensions. Additionally, the unnatural smallness of some parameters, as for example the CP violating angle of quantum chromodynamics (QCD) and the value of the cosmological constant cannot be explained in a satisfactory way within the SM framework.

Axion-like particles (ALPs) are light scalar or pseudoscalar spinless singlets of the SM gauge group that naturally arise in theories where an unspecified $U(1)$ global symmetry is both anomalous and spontaneously broken at high energy. Indeed, their lightness, relative to the scale Λ of new physics from which they stem, can be ascribed to their pseudo Nambu-Goldstone boson nature. They represent well motivated relics of new physics in a variety of explicit extensions of the SM. Their name is derived from the QCD axion, which was introduced by Peccei, Quinn *et al.* to address the strong CP problem. While the physical parameters of the QCD axion – its mass and decay constant – are closely related to each other, the ones associated to ALPs are arbitrary parameters to be determined or bounded by experiments. In this respect, ALPs can be regarded as a generalization of the QCD axion that are motivated by a wealth of other arguments, and they have often been suggested as possible particle physics solutions to the SM open problems briefly outlined before.

The phenomenology of ALPs at low energies can be described by effective operators. The leading-order interactions with SM fields can be parameterized in terms of the Wilson coefficients of dimension-five operators, suppressed by $1/\Lambda$. The ALP Effective Field Theory (EFT) considered in this thesis work has been introduced in Ref. [31] and it violates the CP symmetry, regardless of the scalar or pseudoscalar nature of the ALP. From the experimental point of view, these CP violating interactions have an impact on the permanent electric dipole moments (EDMs) of particles, nucleons, nuclei and molecules, with contributions that are expected to be by far dominant with respect to those predicted by the SM. Due to the lack of any significant SM background, flavor-diagonal CP violating observables, such as the permanent EDMs, represent indeed a very promising opportunity to probe these new physics scenarios. A fundamental point is that these observables are measured at very low energies, whereas the ALP EFT is defined around the electroweak scale. In this framework, the renormalization group equations (RGEs), which describe the evolution of the Wilson coefficients of the ALP EFT as functions of the energy scale, play a crucial role, and their calculation at one-loop order is the objective of this thesis.

This calculation can be performed in several ways. The standard one is based on Feynman diagrams and allows to extract the RGEs for any given operator from the divergences of the integrals associated to each loop diagram. However, this approach requires the computation of complicated mathematical objects which are not individually gauge invariant and involve off-shell intermediate states in internal propagators. Yet, other methods for computing the RGEs exist which allow to circumvent such unnecessary complications; among these are for instance those based on on-shell amplitudes. These are by construction gauge invariant quantities which only know about on-shell degrees of freedom. They are especially advantageous for dealing with massless particles with spin, such as gluons, where the focus on the two physical helicities eliminates the need to introduce gauge redundancies, removing at the same time intricate cancellations among large numbers of Feynman diagrams. The most natural language to deal with on-shell massless particles is provided by the spinor-helicity formalism [33], in which Poincaré covariance is efficiently implemented.

This thesis work is structured as follows. In Chapter 2, some of the most important open problems in particle physics are outlined, with a particular attention to the strong CP problem, which led to the formulation of the QCD axion solution. In Chapter 3 we review the renormalization program of EFTs, while in Chapter 4 we describe the specific CP violating ALP EFT considered in this thesis. Then, in Chapter 5 we perform the calculation at one-loop order of the RGEs corresponding to the Wilson coefficients of the theory with the standard Feynman diagrammatic approach, concluding the first part of the thesis. The objective of the second part is to reproduce the results of the RGEs through on-shell amplitude methods. In particular, in Chapter 6 we lay the foundations of such methods introducing the spinor-helicity formalism and analyzing the symmetry properties of scattering amplitudes. Additionally, the BCFW recursion relation is introduced and the relevant three-particle amplitudes involving the ALP and SM particles are reported. Finally, in Chapter 7, we discuss the on-shell method of form factors [22], which is based on the unitarity of the S-matrix. We then apply it to obtain the RGEs of the CP violating ALP EFT and we compare them to the ones computed with the standard Feynman diagrammatic approach.

Chapter 2

Motivations for axion physics

Our current understanding of elementary particle physics is described by the Standard Model (SM), a renormalizable quantum field theory (QFT) that has been corroborated by a number of astonishing accurate predictions, *e.g.* the measurement of the magnetic dipole moment of the electron, that culminated with the discovery of a scalar boson with a mass of 125 GeV in 2012 at LHC: the Higgs boson. However, despite these successes, there are extremely good reasons to believe it is not complete and that several efforts are needed in order to obtain a reasonable answer to some of the problems that are left unsolved. Finding a suitable extension of the SM is now the main challenge of particle physics.

In this Chapter we will outline the fundamental open questions that the SM is not able to answer, with a particular emphasis to the strong CP problem, reviewed in Section 2.2. The most compelling solution to this problem is represented by the QCD axion, a new hypothetical spinless elementary quantum field originally postulated by Peccei and Quinn in 1977, which is described in Section 2.3. Finally, in Section 2.4, we see how the QCD axion can be generalized through the introduction of axion-like particles, whose mass and decay constant are arbitrary parameters to be measured or bounded by experiments, which are able to solve other open problems.

2.1 Open problems in particle physics

2.1.1 Observational problems

The first class of open questions in particle physics arises from experimental observations, primarily in astrophysics and cosmology, which lack a coherent or satisfactory explanation within the SM. These "external" problems necessitate the presence of new physics beyond the SM to account for the observed phenomena.

Neutrino masses: experiments studying neutrino oscillations have demonstrated that at least two out of the three known left-handed neutrino families must have a non-vanishing mass. To explain this, several mechanisms have been proposed, and the two most popular ones involve the introduction of new sterile fermion fields. The first mechanism suggests that the left-handed SM neutrinos acquire their mass through a Dirac mass term by interacting with new light right-handed neutrino fields. Alternatively, the see-saw mechanism proposes that the left-handed SM neutrinos are coupled with new heavy neutrino fields, resulting in the SM neutrinos having a small mass while the sterile ones possess a much larger mass. In the latter case, neutrinos would be considered as Majorana particles.

Dark matter abundance: extensive evidence strongly suggests that more than 95% of the total matter in the Universe does not emit any sizeable radiation and is not composed of baryonic matter. This non-emitting and non-baryonic matter, known as dark matter, represents a perplexing mystery that drives the search for physics beyond the SM. Indeed, the cosmological abundance of dark matter cannot be accounted for by any particle within the SM framework. These new particles must be stable, very weakly interacting, and non-relativistic. Several candidates have been proposed, such as the axion, which can provide the observed amount of cold DM through the misalignment mechanism. Other candidates are present in supersymmetric extensions of the SM, such as gravitinos and neutralinos.

Matter-antimatter asymmetry: the matter-antimatter asymmetry observed in the Universe cannot be adequately explained within the framework of the SM. The level of CP violation introduced by SM interactions is insufficient to support an efficient baryogenesis mechanism. To address this issue, various proposals have been put forward to dynamically generate the matter-antimatter asymmetry from an initial symmetric state. Many of these solutions involve the introduction of new physical fields. Interestingly, the observed dominance of matter over antimatter could potentially be explained through a leptogenesis mechanism, driven by the decay of new heavy leptonic states such as the heavy neutrinos of the see-saw mechanism.

Cosmological inflation: in the field of cosmology, the inflationary epoch during the early stages of the Universe is widely regarded as essential for resolving fundamental issues within the standard cosmological model. These issues include the horizon problem, the flatness problem, and the relic problem. By introducing a non-SM scalar field that drives an accelerated period of inflationary expansion in the early Universe, it becomes possible to address all of these problems simultaneously.

Along with them, one should also take into account the recently confirmed discrepancy between the theoretically predicted and the observed value of the anomalous magnetic moment of the muon, as well as the so-called B flavor anomalies, from which we observed some hints of the breaking of lepton universality in the b quark decays.

2.1.2 Theoretical problems

This category of open questions in particle physics arises from theoretical concerns originating from the SM formulation itself, and thus can be viewed as "internal" problems. Typically, these issues revolve around the characteristics of coupling constants or the masses of SM fields.

Hierarchy problem: it revolves around the mass of the Higgs field. In absence of any protective symmetry, the Higgs boson mass is subject to quantum corrections that are proportional to the square of the next new physics scale beyond the electroweak scale. Therefore, the hierarchy problem involves explaining why the mass of the Higgs boson is significantly smaller than the Planck mass $M_{\rm Pl} \approx 10^{19}$ GeV, which is the next known new physics scale above which we cannot neglect the contribution of gravity and the predictions of SM are no longer valid.

Flavor problem: it pertains to the values attributed to masses and mixing parameters within the fermion sector of the SM. These parameters exhibit a wide range of values, spanning nearly five orders of magnitude. This is evident when considering the significant disparity in mass between the lightest measured fermion ($m_e = 511$ keV) and the heaviest one ($m_t \approx 173$ GeV). Accommodating such substantial differences poses challenges within the framework of the Higgs mechanism, which assigns fermions a mass determined by the product of the Higgs vacuum expectation value and the entries of the Yukawa matrices. Indeed, one would typically expect these entries to be of a similar order of magnitude.

Strong CP problem: the structure of the QCD vacuum brings a new term into the SM Lagrangian: the θ-term

$$\mathscr{L}_{\rm SM} \supset \theta \frac{g_s^2}{32\pi^2} G^{a\,\mu\nu} \widetilde{G}^a_{\mu\nu}\,, \tag{2.1}$$

which violates the combination of charge (C) and parity (P) symmetry, with θ being a dimensionless parameter. We then expect some CP violating phenomena in the QCD sector. However, the measurements of the electric dipole moment of the neutron lead to the bound $|\theta| \lesssim 10^{-10}$. This situation is analogous to the hierarchy problem, where again, the SM does not have any mechanism to explain why the θ parameter is so tiny. This problem is analyzed more in detail in the next Section.

2.2 Strong CP problem

With the development of quantum chromodynamics (QCD) in the 1970s, a puzzling problem arose. The QCD Lagrangian density for N quark flavors

$$\mathscr{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_{i=1}^{N} \bar{q}_i (i\not{D} - m_i) q_i \,, \tag{2.2}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{2.3}$$

is the QCD field strength tensor and the covariant derivative acting on the quark field is defined as

$$D_\mu q_i = (\partial_\mu + i g_s T^a A^a_\mu) q_i \,, \tag{2.4}$$

is invariant under the global symmetry

$$U(N)_V \times U(N)_A = SU(N)_L \times SU(N)_R \times U(1)_V \times U(1)_A \tag{2.5}$$

in the limit in which the quarks are massless. Since m_u and m_d are much lighter than the QCD scale $\Lambda_{\rm QCD} \approx 150$ MeV, this approximate symmetry is realized at least for $N = 2$. Indeed, the fact that the vector symmetry corresponding to isospin times baryon number $U(2)_V = SU(2)_I \times U(1)_B$ is a good approximate symmetry of nature is experimentally confirmed by the appearance of nucleon and pion multiplets in the spectrum of hadrons.

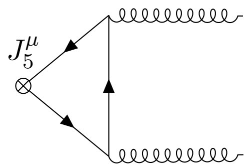
2.2.1 $U(1)$ problem

However, the axial symmetry is spontaneously broken by the dynamical formation of quark condensates $\langle \bar{q} q \rangle \neq 0$, and the four pseudo Nambu-Goldstone bosons associated with the spontaneous breakdown of $U(2)_A$ are expected to emerge in the hadronic spectrum. Despite the lightness of pions, there is no sign of another light state in the hadronic spectrum, as $m_\eta^2 \gg m_\pi^2$, which suggests that $U(1)_A$ might not be a symmetry of strong interactions. This is known as the $U(1)$ *problem* [69]. On the other hand, the absence of strong CP violation was believed to be one of the main successes of QCD.

After a few years, a new perspective emerged with the discovery of Yang-Mills instantons [10] and the non-trivial QCD vacuum structure [21, 46], which contradicted the previous viewpoint. The solution of the $U(1)$

problem resulted in the emergence of the *strong CP problem*, which was an unexpected outcome.

The chiral anomaly for axial currents [3, 11, 6] provides a possible solution to the $U(1)$ problem. Indeed, while at classical level and in the massless quark limit $\partial_\mu J_5^\mu = 0$, the divergence of the axial current J_5^μ gets a non-zero contribution at quantum level from the triangle diagrams $SU(3)_c^2 U(1)_A$



which connects it to two gluon fields with quarks running inside the loop:

$$\partial_\mu J_5^\mu = \frac{g_s^2 N}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}\,, \tag{2.6}$$

where

$$\widetilde{G}^{a\,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma} \tag{2.7}$$

is the QCD dual field strength tensor. Hence, the global $U(1)_A$ transformation acting on the quark wavefunctions as

$$q_i \longrightarrow e^{i\alpha\gamma_5/2} q_i\,, \tag{2.8}$$

with $i = 1, \ldots, N$, shifts the action by a quantity equal to

$$\delta S = \alpha \int \mathrm{d}^4x\, \partial_\mu J_5^\mu = \alpha \frac{g_s^2 N}{32\pi^2} \int \mathrm{d}^4x\, G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}\,. \tag{2.9}$$

This is a pure surface integral because the pseudoscalar density $G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ can be expressed as the divergence of a four-vector K^μ, known as Chern-Simons current

$$G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu} = \partial_\mu K^\mu = \partial_\mu[\epsilon^{\mu\nu\rho\sigma} A^a_\nu (G^a_{\rho\sigma} - \frac{g_s}{3} f^{abc} A^b_\rho A^c_\sigma)]\,, \tag{2.10}$$

and, as a consequence, it bears no effects in perturbation theory. However, there exist classical configurations, which are topologically non-trivial, such that the effects of this term cannot be ignored.

2.2.2 Yang-Mills instanton solutions

By going to Euclidean space $\mathbb{E}^4$ by means of a Wick rotation and using Gauss's theorem, we can write

$$\int \mathrm{d}^4x\, G^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} = \int \mathrm{d}^4x\, \partial_\mu K_\mu = \int_{S^3} \mathrm{d}\sigma_\mu\, K_\mu\,, \tag{2.11}$$

where S^3 is the three-sphere at infinity and $\mathrm{d}\sigma_\mu$ is an element of its hypersurface. If we use the naive boundary conditions for the gauge field $A^a_\mu|_{S^3} = 0$ at infinity, we realize that the variation of the action δS vanishes and $U(1)_A$ appears to be a symmetry again. However, if we instead apply a gauge transformation on the connection

$$A_\mu = A^a_\mu T^a \longrightarrow A'_\mu = A'^a_\mu T^a = S A_\mu S^{-1} + \frac{i}{g_s}(\partial_\mu S)S^{-1}\,, \tag{2.12}$$

resulting in

$$G_{\mu\nu} = G^a_{\mu\nu} T^a \longrightarrow G'_{\mu\nu} = G'^a_{\mu\nu} T^a = S G_{\mu\nu} S^{-1}\,, \tag{2.13}$$

we observe that, while $G'_{\mu\nu}|_{S^3} = 0$ so that the variation of the action is still finite, A'_μ is a pure gauge at infinity, *i.e.* $A'_\mu|_{S^3} = i g_s^{-1}(\partial_\mu S)S^{-1}$. With these modified boundary conditions, it turns out that there are gauge configurations for which $\delta S \neq 0$, and thus $U(1)_A$ is not a symmetry of QCD. In particular, this is achieved if S cannot be continuously deformed into the identity in group space. Considering the subgroup $SU(2)$ of $SU(3)$ and restricting the connection to this subgroup, the gauge potentials provide a mapping $S^3 \to S^3$, being S^3 the group space of $SU(2)$. Indeed, if $M \in SU(2)$, then it can be written as

$$M = a\mathbb{1} + i\vec{b}\cdot\vec{\sigma}\,, \tag{2.14}$$

with the real coefficients a and $\vec{b}$ satisfying $a^2 + |\vec{b}|^2 = 1$, which is the equation of a three-sphere $S^3 \subset \mathbb{E}^4$. It can be shown that, for mappings of non-trivial topology, the integral in Eq. (2.11) counts the number of times the hypersphere at infinity is wrapped around the group manifold S^3. More precisely,

$$\nu = \frac{g_s^2}{32\pi^2}\int \mathrm{d}^4x\, G^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} \in \mathbb{Z} \tag{2.15}$$

is called winding number or Pontryagin index. As a consequence, in Euclidean space $SU(2)$ field configurations of finite action fall in homotopy classes of different winding number, whose value cannot be modified by means of a deformation of the gauge configuration that maintains the action finite. Concerning $SU(3)$ gauge configurations, they can be classified in the same $SU(2)$ homotopy classes because any mapping from S^3 into any simple Lie group G can be continuously deformed into a mapping to a $SU(2)$ subgroup of G, hence with no change of homotopy class.

Solutions of the classical equations of motion in Euclidean space with non-trivial winding number are called *instantons*, because they are localized in all the four dimensions, and represent an interpolation between a vacuum state $|n\rangle$ (pure gauge) with homotopy class n at $t = -\infty$ and another vacuum state $|m\rangle$ with homotopy class m at $t = +\infty$. Indeed, returning to the physical Minkowski space and choosing the temporal gauge $A_0^a = 0$ so that $K_i = 0$, we can write the Pontryagin index in Eq. (2.15) as

$$\nu = \frac{g_s^2}{32\pi^2} \int \mathrm{d}^4x\, \partial_0 K^0 = \frac{g_s^2}{32\pi^2} \int \mathrm{d}^3x\, K^0\Big|_{t=-\infty}^{t=+\infty} = m - n\,, \tag{2.16}$$

and the tunnelling amplitude, in the semiclassical WKB approximation, is of the order of the exponential of the instanton action $\exp(-S_\nu)$, with $S_\nu = 8\pi^2|\nu|/g_s^2$: the transition amplitude between vacua belonging to different homotopy classes is not null.

2.2.3 QCD θ-vacua

Given that in the theory there are infinite minima in the potential, the vacuum is infinitely degenerate. This implies that the ground state can be written as the superposition of all the vacua:

$$|\theta\rangle = \sum_{n\in\mathbb{Z}} e^{in\theta}\, |n\rangle \ , \tag{2.17}$$

where θ is a global fundamental constant characterising the boundary condition on the wavefunction and that can be interpreted as an angle, because the previous expression is left invariant if $\theta \to \theta + 2\pi$. Vacua satisfying Eq. (2.17) are called *θ-vacua*, and the presence of the factor $\exp(in\theta)$ ensures the invariance of $|\theta\rangle$ under gauge transformations. Indeed, under the action of a large gauge transformation realized by the unitary operator $U(g_\nu) = U(g_1)^\nu$, which cannot be continuously deformed to the identity, the n-vacuum is shifted:

$$|n\rangle \longrightarrow U(g_\nu)\, |n\rangle = |n+\nu\rangle \ , \tag{2.18}$$

resulting in

$$|\theta\rangle \longrightarrow U(g_\nu)\, |\theta\rangle = e^{-i\nu\theta}\, |\theta\rangle \sim |\theta\rangle \ , \tag{2.19}$$

where the last step is the equivalence relation between vacua, which are equivalent if they differ by a phase. This means that $|\theta\rangle$ is an eigenstate of the unitary operator of the gauge transformation and is physically well-defined. Each $|\theta\rangle$ is the ground state of an independent sector of the Hilbert space. In fact, given the observable $\mathscr{O}$, it must be gauge invariant, and, in particular, it must commute with any gauge group element:

$$[\mathscr{O}, U(g_\nu)] = 0\,. \tag{2.20}$$

Taking two different vacua $\theta \neq \theta'$, this implies that

$$0 = \langle\theta|[\mathscr{O}, U(g_\nu)]|\theta'\rangle = (e^{-i\nu\theta'} - e^{i\nu\theta})\, \langle\theta|\mathscr{O}|\theta'\rangle \ , \tag{2.21}$$

so the only possibility is

$$\langle\theta|\mathscr{O}|\theta'\rangle = 0\,, \tag{2.22}$$

which is a superselection rule between vacua. This means that the Hilbert space is divided in different sectors which cannot communicate with each other: any observable connecting two different vacua would give zero.

The vacuum-to-vacuum transition amplitude can be computed using Eq. (2.17):

$$\langle\theta_+|\theta_-\rangle = \sum_{n,m\in\mathbb{Z}} e^{i(n-m)\theta}\,\langle m_+|n_-\rangle = \sum_{\nu\in\mathbb{Z}} e^{i\nu\theta} \sum_{m\in\mathbb{Z}} \langle m_+|(\nu+m)_-\rangle\ , \tag{2.23}$$

and, exploiting the definition of the winding number in Eq. (2.15), the phase factor $\exp(i\nu\theta)$ can be replaced by an effective contribution to the QCD action. Indeed, using the path integral formulation, we can write the above transition amplitude as

$$\langle\theta_+|\theta_-\rangle = \sum_{\nu\in\mathbb{Z}} \int \mathcal{D}[A,q,\bar{q}]\, e^{iS_{\text{eff}}[A,q,\bar{q}]}\delta\left(\nu - \frac{g_s^2}{32\pi^2}\int \mathrm{d}^4x\, G^{a\,\mu\nu}\widetilde{G}^a_{\mu\nu}\right)\,, \tag{2.24}$$

where

$$S_{\text{eff}}[A,q,\bar{q}] = S_{\text{QCD}}[A,q,\bar{q}] + \theta\frac{g_s^2}{32\pi^2}\int \mathrm{d}^4x\, G^{a\,\mu\nu}\widetilde{G}^a_{\mu\nu}\,. \tag{2.25}$$

The resolution of the $U(1)$ problem, by recognizing the complicated nature of the QCD vacuum, effectively adds an extra term to the QCD Lagrangian

$$\mathscr{L}_\theta = \theta\frac{g_s^2}{32\pi^2}G^{a\,\mu\nu}\widetilde{G}^a_{\mu\nu}\,, \tag{2.26}$$

which violates both parity (P) and time reversal (T) symmetries, and hence, because of CPT theorem, it violates CP. The most sensitive flavor-diagonal CP violating observable is the neutron electric dipole moment (EDM), which is defined in terms of the non-relativistic Hamiltonian

$$\mathcal{H}_{\text{NR}} = -d_n\vec{E}\cdot\hat{S} \tag{2.27}$$

and that can be written as the following Lorentz invariant dimension-five Lagrangian density

$$\mathscr{L} = -d_n\frac{i}{2}\bar{n}\sigma^{\mu\nu}\gamma_5 n F_{\mu\nu}\,. \tag{2.28}$$

Calculations based on QCD sum-rules [65] yield

$$|d_n| = 2.4(1.0)\times 10^{-16}|\theta|\,e\,\text{cm}\,, \tag{2.29}$$

which, compared to the current experimental limit [2]

$$|d_n^{\rm exp}| < 1.8 \times 10^{-26}\, e\,{\rm cm}\ (90\%\ {\rm CL})\,, \tag{2.30}$$

implies the bound

$$|\theta| \lesssim 10^{-10}\,. \tag{2.31}$$

Understanding the smallness of θ consists in the so-called strong CP problem.

Actually, if one considers the effect of chiral transformations on the θ-vacuum, the problem is even worse. In fact, because of the chiral anomaly in Eq. (2.6), the θ-vacuum is shifted under the application of a chiral transformation:

$$|\theta\rangle \longrightarrow e^{i\alpha Q_5}\,|\theta\rangle = |\theta+\alpha\rangle\ . \tag{2.32}$$

Then, if one considers a general complex quark mass matrix M, as in the case where weak interactions are included, so that the mass term is given by

$$\mathscr{L}_{\rm mass} = \sum_{i,j=1}^{N} \bar{q}_{iR} M_{ij} q_{jL} + {\rm h.c.}\,, \tag{2.33}$$

then only the combination

$$\bar{\theta} = \theta + \arg\det M \tag{2.34}$$

is left invariant under a global axial transformation, and hence physically observable. Now the problem is understanding the reason why the sum of these two uncorrelated coefficients should amount to such a small, fine-tuned value. Indeed, θ and M have completely different and independent origins, and one would expect, in absence of any symmetry, $\bar{\theta}$ to be of order one.

Additionally, if the value of $\bar{\theta}$ is not protected by any symmetry, it can receive radiative corrections from the interactions of quarks and gluons. However, it has been shown that in the SM the radiative corrections to $\bar{\theta}$ are extremely small, due to the fact that the leading contribution to the radiatively induced $\bar{\theta}$ arises at seven-loop level, and it is of the order of $10^{-33}\log\Lambda_{\rm UV}$ [32], where $\Lambda_{\rm UV}$ is an ultraviolet (UV) cutoff. Therefore, if the tree-level value of $\bar{\theta}$ is small at the UV scale, it will remain radiatively small when we run down to the QCD scale.

2.2.4 Possible solutions

Several solutions to the strong CP problem have been proposed. In the following, some of them are outlined.

Massless quark solution: if one of the quarks had no mass, then the contribution $\arg\det M$ would always be zero. This would mean that we could rotate the quark fields without any consequence, and the amount of CP violation, measured by the angle $\bar{\theta}$, would be unphysical and could be changed arbitrarily. However, nowadays we know that there are no massless quarks in nature by experimental matching with lattice calculations, so this option can be disregarded.

Soft P (CP) breaking: it is possible that either P or CP are symmetries of the high-energy theory, which would make the $\bar{\theta}$ term equal to zero. This idea was first suggested by Nelson [58] and Barr [7] and later discussed in the context of grand-unified models. Now, P has to be broken at some point to explain the chiral structure of the SM, and CP needs to be broken to generate the CKM phase. In these scenarios, the $\bar{\theta}$ term can be calculated, and the main challenge is to create the observed CP violation in the quark sector without causing the $\bar{\theta}$ term to be too large. This can be achieved, but it might require some fine-tuning or a somewhat unusual model building approach.

QCD solutions: the resolution to the strong CP problem could lie hidden in the infrared dynamics of QCD. However, efforts to explore this possibility by manipulating the topology of spacetime to trivialize the QCD vacuum [48] or through the use of confinement-induced screening [66] have often failed to provide a comprehensive solution to both the strong CP problem and the associated $U(1)$ problem.

Axion solutions: a new pseudoscalar degree of freedom – the QCD axion – is introduced in such a way that the $\bar{\theta}$ term is dynamically driven to zero [63, 62]. In the next Section we will outline this idea, which, among the other solutions, is the most compelling one, and can be experimentally tested at low energies.

2.3 QCD axion

The QCD axion has been proposed by Peccei and Quinn (PQ) [63, 62] in order to address the strong CP problem. The solution is based on the introduction of an approximate $U(1)_{\text{PQ}}$ global chiral symmetry, called PQ symmetry, realized at high energies. This symmetry necessarily has to possess the following characteristics:

- it is spontaneously broken at some high energy scale f_a;
- it is anomalous under strong interactions.

From Goldstone's theorem, for every spontaneously broken generator of a continuous internal symmetry, a new massless particle – a Goldstone boson – appears. Hence, the first property ensures the existence of a massless pseudoscalar field: the QCD axion $a(x)$. The action of the symmetry group $U(1)_{\mathrm{PQ}}$ effectively shifts the value of the axion field by a constant term

$$a(x) \longrightarrow a(x) + \kappa f_a \,, \tag{2.35}$$

where f_a is called axion decay constant, while the SM fields are left unchanged.

On the other hand, with the second property, which tells us that the PQ symmetry is not exact but is rather broken at quantum level, the QCD axion field ceases to be a pure Goldstone boson and instead becomes a pseudo Nambu-Goldstone boson (pNGB). This fact allows the axion to develop a non-null mass term and the interaction with the gluon field needed to cancel the $\bar{\theta}$ parameter. Indeed, if the shift symmetry transformation in Eq. (2.35) leaves the action invariant up to the term

$$\delta S = \frac{g_s^2 \kappa}{32\pi^2} \int \mathrm{d}^4x \, G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} \,, \tag{2.36}$$

the axion develops the anomalous coupling

$$\frac{a}{f_a} \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} \,. \tag{2.37}$$

The axion effective Lagrangian is then

$$\mathscr{L}_a = \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \mathscr{L}\left(\frac{\partial a}{f_a}, \psi\right) + \frac{a}{f_a} \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} \tag{2.38}$$

and, since the parameter κ is arbitrary, it can be chosen in such a way that the $\bar{\theta}$ term is driven to zero. This is indeed achieved simply by setting $\kappa = -\bar{\theta}$. Additionally, the Vafa-Witten theorem, which states that parity cannot be spontaneously broken in QCD, ensures that the axion potential has a CP preserving minimum for $\langle a \rangle = 0$. Therefore, the QCD axion field naturally evolves towards the CP preserving minimum of the scalar potential generated by the pseudoscalar density $G\widetilde{G}$, solving in a dynamical fashion the strong CP problem. This mechanism is known as *Peccei-Quinn mechanism*. Moreover, the same scalar potential provides a non-vanishing mass term that satisfies the relation

$$m_a f_a = \frac{\sqrt{m_u m_d}}{m_u + m_d} m_\pi f_\pi \,, \tag{2.39}$$

which implies

$$m_a \approx 5.7 \left(\frac{10^{12}\ \mathrm{GeV}}{f_a}\right) \mu\mathrm{eV} \tag{2.40}$$

and makes the axion a compelling DM candidate.

The QCD axion can can develop not only anomalous couplings to the pseudoscalar densities of vector bosons, but also derivative interactions with the fermion axial current. Its effective Lagrangian then reads

$$\mathscr{L}_a = \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \frac{\partial_\mu a}{2f_a} c_f^{ij} \bar{f}_i \gamma^\mu \gamma_5 f_j + \frac{a}{f_a}\frac{g_s^2}{32\pi^2} G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu} + \frac{1}{4} g_{a\gamma} a F_{\mu\nu}\widetilde{F}^{\mu\nu}\,, \tag{2.41}$$

where a sum over the fermions f is understood and i, j are flavor indices.

We can conclude this Section by highlighting the fact that the QCD axion solution to the strong CP problem does not depend on the specific UV completion of the chosen axion model. The crucial requirement is the existence of a spontaneously broken global $U(1)_{\mathrm{PQ}}$ symmetry at high energies, which is anomalous under strong interactions. This appealing property forms the foundation for various QCD axion models developed over the years, as well as the extensions of axions to ALPs.

2.4 Axion-like particles

Physicists have been driven to explore the potential of addressing additional unresolved issues within the SM by proposing the presence of axion-like particles (ALPs), thanks to the adaptability and straightforwardness of axion models.

ALPs are light scalar or pseudoscalar spinless bosons that originate from the spontaneous breaking of an unspecified $U(1)$ global symmetry at high energy that is both anomalous and spontaneously broken. As the QCD axion, ALPs are pNGBs associated with this symmetry, which implies the lightness of their mass compared to the energy scale at which the global symmetry gets broken. However, their physical quantities – the mass and the decay constant, as well as the couplings to other particles – are arbitrary parameters to be determined or bounded by experiments and that do not have to satisfy the strict relation in Eq. (2.39). Indeed, with this regard, ALPs can be understood as a generalization of the QCD axion.

Among the several ALP model that have been suggested to dynamically address some of the unresolved issues in particle physics, we can mention, excluding the QCD axion, the following two proposals.

Relaxion: this ALP is able to address the hierarchy problem by directly interacting with the Higgs field and influencing its evolution during the early stages of the Universe [42]. Specifically, the ALP potential enables the Higgs field to explore various values of its vacuum expectation value across a wide range of energy scales, eventually settling at the electroweak scale.

Flaxion: the objective of this ALP solution is to provide an explanation for the flavor structure of the SM. Alternatively, certain flaxion models [39] introduce a new complex field, where the angular component behaves as the axion, while the radial degree of freedom can serve as the inflaton during the early stages of the Universe.

Furthermore, various anomalies can be solved by the ALPs, for example the longstanding discrepancy of the anomalous magnetic moment of the muon [55] or the excess in excited Beryllium decays $^8\mathrm{Be}^* \to {}^8\mathrm{Be} + e^+e^-$ [50].

The explicit ALP effective Lagrangian considered in this thesis work is described in Chapter 4, only after having outlined the general concept of Effective Field Theories and their renormalization program.

Chapter 3

Renormalization of Effective Field Theories

In this Chapter we will introduce the concept of Effective Field Theories (EFTs), which represent a powerful tool to describe the behaviour of a system at a certain energy scale. In particular, they provide a general framework to study new physics effects in a systematic way.

An EFT is the simplest field theory characterized by a certain number of degrees of freedom, some symmetries and a set of expansion parameters, allowing computation of physical effects to a given precision in terms of a finite set of parameters directly deducible from experiments.

3.1 EFT expansion

If the scale separation between the SM and new physics is sufficiently large, it is then possible to consider the SM as a leading order approximation in the EFT expansion of a new fundamental theory. In the EFT framework, it is legitimate to parametrize new physics effects in terms of effective operators without referring to any UV model. Therefore, we can write the effective Lagrangian as

$$\mathscr{L}_{\mathrm{EFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_{n>4} \sum_{i} \frac{c_i^{(n)}}{\Lambda^{n-4}} \mathscr{O}_i^{(n)} \,, \tag{3.1}$$

where Λ is the energy scale above which the EFT ceases to be valid. The *effective operators* $\mathscr{O}_i^{(n)}$ have a mass dimension that is equal to n

$$[\mathscr{O}_i^{(n)}] = n \tag{3.2}$$

and must respect the postulated symmetries, such as Lorentz invariance and gauge invariance. An infinite set of such operators exist, but importantly there exists only a finite set of operators for each dimension n. On the other hand, $c_i^{(n)}$ are the corresponding dimensionless *Wilson coefficients*. The scale Λ has been factored out since $\mathscr{L}_{\text{EFT}}$ has to be treated as an expansion in powers of $1/\Lambda$.

The EFT Lagrangian represent a bridge that connects the UV theory at the high-energy scale with experimental measurements at low-energy scales. This connection can be exploited in both directions.

Bottom-up approach: the philosophy of this approach is to parametrize the EFT Lagrangian without any assumption of specific UV theory. By considering power counting arguments, described in the next Section, it is possible to truncate the EFT expansion, keeping only the relevant terms that are expected to give significant deviations from the SM. In this way, the truncated EFT will coincide with the low-energy limit of the UV model.

Top-down approach: the spirit of this approach is to study the implications at low-energy scales of the specific UV theory. The procedure to construct this connection consists of three steps.

1. Firstly, we can separate the light degrees of freedom of the system, denoted with φ_ℓ, from the heavy ones φ_h, namely those that cannot go on-shell at low-energies. Then the UV generating functional reads

$$Z_{\text{UV}}[J_\ell, J_h] = \int \mathcal{D}[\varphi_\ell, \varphi_h] \exp\left[i \int \mathrm{d}^4x (\mathscr{L}_{\text{UV}}(\varphi_\ell, \varphi_h) + J_\ell\varphi_\ell + J_h\varphi_h)\right] \tag{3.3}$$

and fully characterizes the UV theory. However, in the EFT we only need the correlators of φ_ℓ, hence we want to integrate out the heavy degrees of freedom. We can thus identify the effective generating functional as

$$Z_{\text{EFT}}[J_\ell] = Z_{\text{UV}}[J_\ell, J_h = 0]\,, \tag{3.4}$$

which is associated with an EFT Lagrangian, defined by

$$Z_{\text{EFT}}[J_\ell] = \int \mathcal{D}\varphi_\ell \exp\left[i \int \mathrm{d}^4x (\mathscr{L}_{\text{EFT}}(\varphi_\ell) + J_\ell\varphi_\ell)\right]. \tag{3.5}$$

In general, the resulting $\mathscr{L}_{\text{EFT}}$ is non-local, but can be approximated by a local Lagrangian exploiting the expansion in powers of $1/\Lambda$. This step is called *matching*.

2. The second step, named *running*, consists in evolving the Wilson coefficients from the UV scale down to lower energy scales where experimental measurements are performed, using the renormalization group equations (RGEs).

3. In the third step (*mapping*) we can finally compute physical quantities of interest considering the EFT Lagrangian at these low-energy scales.

These two approaches are complementary: with the bottom-up approach we can obtain model-independent constraints for the values of the Wilson coefficients from precision measurements, while the top-down approach can help us understand which EFT operators have significant contributions.

3.2 Power counting

The contributions of the effective operators to any observable are suppressed by powers of $(v/\Lambda)^{n-4}$ relative to the contributions of the SM operators, if the relevant energies in the process of interest are of order the electroweak scale v. In order to understand this point we can follow Ref. [54] and consider a scattering amplitude $\mathcal{M}$ in four spacetime dimensions. Working at a given energy scale E, the insertion of a single operator of dimension n gives a contribution to the amplitude of order

$$\mathcal{M} \sim \left(\frac{E}{\Lambda}\right)^{n-4} \tag{3.6}$$

by dimensional analysis, since the operator has a coefficient of mass dimension $1/\Lambda^{n-4}$. This means that the insertion of a set of higher dimension operators leads to an amplitude

$$\mathcal{M} \sim \left(\frac{E}{\Lambda}\right)^{N}, \tag{3.7}$$

where

$$N = \sum_i (n_i - 4) \tag{3.8}$$

and the sum is over all the inserted operators. This equation is known as *EFT power counting formula* and tells us how to organize the calculation: at leading order we can only use $\mathscr{L}_{\mathrm{SM}}$; corrections of order E/Λ are given by diagrams with a single insertion of a dimension-five operator; $(E/\Lambda)^2$ corrections are provided by diagrams with a single insertion of a dimension-six operator or two insertions of dimension-five operators, and so on.

This means that, at loop-level, if a diagram with two insertions of $\mathscr{O}^{(5)}$ operators is divergent, we must need a counterterm which is a $\mathscr{O}^{(6)}$ operator.

Its Wilson coefficient might vanish for a specific value of the renormalization scale μ, but in general it evolves with μ by the RGEs. Continuing in this way, we generate the infinite series of term of the EFT expansion of Eq. (3.1). Indeed, we can generate operators of arbitrarily high dimension by multiple insertions of operators with dimension $n > 4$. For this reason EFTs are referred to as non-renormalizable theories, because an infinite number of higher dimension operators are needed to renormalize the theory. However, as long as we are interested to corrections at a given accuracy, there are only a finite number of operators that contribute.

3.3 Anomalous dimension matrix

When we construct the effective Lagrangian, we dissociate the contributions stemming from virtual particles into short- and long-distance modes:

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega} = \int_\Lambda^\infty \frac{\mathrm{d}\omega}{\omega} + \int_0^\Lambda \frac{\mathrm{d}\omega}{\omega}\,, \tag{3.9}$$

where the Wilson coefficients $c_i^{(n)}(\Lambda)$ absorb the contributions of the first term since they are sensitive to the UV physics, while the second term is sensitive to IR physics and is absorbed into the matrix elements of the effective operators $\langle \mathscr{O}_i^{(n)}(\Lambda)\rangle$ [59]. If we perform a measurement at a characteristic energy scale E (with $v \ll E \leq \Lambda$), then we can integrate out the high-energy fluctuations of the light SM fields, namely those with frequencies $\omega > E$, from the generating functional, as shown in Eq. (3.4). The effective operators $\mathscr{O}_i^{(n)}$ are the same of before since the degrees of freedom are not changed, but in general the resulting effective Lagrangian is different. Indeed, now the Wilson coefficients $c_i^{(n)}(E)$ absorb the contributions of the high-energy modes $\int_E^\infty \mathrm{d}\omega/\omega$ and correspondingly the low-energy modes $\int_0^E \mathrm{d}\omega/\omega$ are absorbed into the operator matrix elements $\langle \mathscr{O}_i^{(n)}(E)\rangle$. The key point is that the matrix elements of the effective Lagrangian

$$\mathscr{L}_{\mathrm{EFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_{n>4}\sum_i \frac{c_i^{(n)}(\mu)}{\Lambda^{n-4}}\mathscr{O}_i^{(n)}(\mu) \tag{3.10}$$

are, by construction, independent of the arbitrary factorization scale μ, with $v \leq \mu \leq \Lambda$, which separates high-energy from low-energy contributions. Here $c_i^{(n)}(\mu)$ are the renormalized Wilson coefficients and $\mathscr{O}_i^{(n)}(\mu)$ the corresponding renormalized composite operators, while μ is at the same time the renormalization scale of these quantities.

In analogy with the standard renormalization procedure for renormalizable theories, we can write the bare operators of dimension n in terms of the renormalized ones as

$$\mathscr{O}_{i,0}^{(n)} = Z_{ij}^{(n)}(\mu)\mathscr{O}_j^{(n)}(\mu)\,, \tag{3.11}$$

where $Z^{(n)}$ is a matrix in the space of the operators that takes into account operator mixing effects. It contains the wavefunction renormalization parameters $Z_\Phi^{1/2}$ associated with each component field Φ in $\mathscr{O}_i^{(n)}$, as well as the renormalization parameters absorbing the UV divergences of the loop corrections to the operator matrix elements. From the fact that the bare operators are scale independent, we can write

$$0 = \frac{\mathrm{d}\mathscr{O}_{i,0}^{(n)}}{\mathrm{d}\log\mu} = \frac{\mathrm{d}Z_{ij}^{(n)}(\mu)}{\mathrm{d}\log\mu}\mathscr{O}_j^{(n)}(\mu) + Z_{ij}^{(n)}(\mu)\frac{\mathrm{d}\mathscr{O}_j^{(n)}(\mu)}{\mathrm{d}\log\mu}\,, \tag{3.12}$$

which can be solved setting

$$\frac{\mathrm{d}\mathscr{O}_j^{(n)}(\mu)}{\mathrm{d}\log\mu} = -\gamma_{ij}^{(n)}(\mu)\mathscr{O}_i^{(n)}(\mu)\,, \tag{3.13}$$

where

$$\gamma_{ij}^{(n)}(\mu) = (Z^{-1})_{jk}^{(n)}(\mu)\frac{\mathrm{d}Z_{ki}^{(n)}(\mu)}{\mathrm{d}\log\mu}\,. \tag{3.14}$$

On the other hand, if we exploit the scale independence of the effective Lagrangian, we find

$$0 = \frac{\mathrm{d}}{\mathrm{d}\log\mu}\left(c_i^{(n)}(\mu)\mathscr{O}_i^{(n)}(\mu)\right) = \left(\frac{\mathrm{d}c_i^{(n)}(\mu)}{\mathrm{d}\log\mu}\delta_{ij} - c_i^{(n)}(\mu)\gamma_{ji}^{(n)}(\mu)\right)\mathscr{O}_j^{(n)}(\mu)\,, \tag{3.15}$$

namely

$$\frac{\mathrm{d}c_i^{(n)}(\mu)}{\mathrm{d}\log\mu} = \gamma_{ij}^{(n)}(\mu)c_j^{(n)}(\mu)\,. \tag{3.16}$$

These are the RGEs for the renormalized Wilson coefficients, which describe their evolution as the energy scale changes. The matrix $\gamma_{ij}^{(n)}$ is called *anomalous dimension matrix* and implicitly depends on the renormalization scale μ through the renormalized marginal couplings.

Chapter 4

Effective Field Theory for CP violating axion-like particles

In Chapter 2 we have outlined the main open problems in particle physics. In particular, we have analyzed the strong CP problem. Its most compelling solution, the QCD axion, is extremely simple and elegant since it only relies on the existence of a $U(1)$ global symmetry at high energies that is both anomalous and spontaneously broken at the energy scale f_a. When have outlined in Section 2.4 the fact that the QCD axion has been generalized in order to address additional unresolved problems in particle physics, such as the hierarchy problem or the flavor problem, with the introduction of ALPs.

ALPs can display a limited range of interactions, which include anomalous couplings to gauge boson pseudoscalar densities, derivative interactions with matter fields, and non-derivative interactions with SM fields that slightly break their shift symmetry. Additionally, ALPs do not need to adhere to any accidental symmetries of the SM, making them capable of generating tree-level flavor-changing neutral currents effects. This characteristic makes ALPs particularly intriguing for experimental investigations aimed at detecting them.

A new line of research studies the possibility to probe ALPs through CP violating effects that would translate in observable permanent electric dipole moments (EDMs) of molecules, atoms, nuclei and nucleons. The ALP EFT considered in this thesis work has been introduced by Di Luzio, Gröber and

Paradisi [31] and reads

$$
\begin{aligned}
\mathscr{L}_\phi = e^2 \frac{C_\gamma}{\Lambda} \phi F^{\mu\nu} F_{\mu\nu} + g_s^2 \frac{C_g}{\Lambda} \phi G^{a\,\mu\nu} G^a_{\mu\nu} + \frac{v}{\Lambda} y_S^{ij} \phi \bar{f}_i f_j \\
+ e^2 \frac{\widetilde{C}_\gamma}{\Lambda} \phi F^{\mu\nu} \widetilde{F}_{\mu\nu} + g_s^2 \frac{\widetilde{C}_g}{\Lambda} \phi G^{a\,\mu\nu} \widetilde{G}^a_{\mu\nu} + i \frac{v}{\Lambda} y_P^{ij} \phi \bar{f}_i \gamma_5 f_j \,,
\end{aligned}
\tag{4.1}
$$

which is the most general $SU(3)_c \times U(1)_{\rm em}$ gauge invariant effective Lagrangian containing operators up to dimension-five. Here

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad \widetilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \,, \tag{4.2}
$$

$$
G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \,, \qquad \widetilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{a\,\rho\sigma} \,, \tag{4.3}
$$

are the EM and QCD field strengths and their respective dual tensors ($\epsilon^{0123} = +1$), while $f \in \{e, u, d\}$ denotes the SM fermions in the mass basis and i, j are flavor indices that appear in the Hermitian matrices y_S and y_P.

The second line of this Lagrangian correspond to the shift symmetry invariant sector, since the pseudoscalar Yukawa interactions between the ALP and fermions can be written as $\frac{\partial_\mu \phi}{\Lambda} \bar{f}_i \gamma^\mu \gamma_5 f_j$ by integrating by parts and exploiting the equations of motion. This is a dimension-five operator that is explicitly shift invariant.

On the other hand, the first line correspond to the sector that breaks explicitly the shift symmetry. The normalization factor v/Λ associated with the scalar Yukawa interaction is justified by the fact that, in the unbroken phase of the SM, scalar interactions can be written as $\phi H \bar{f}_{i\,,L(R)} f_{j\,,R(L)}$.

This effective Lagrangian is assumed to arise from integrating out some some new heavy particles at a scale Λ of global symmetry breaking, which is far above the electroweak scale $v = 246$ GeV and is typically assumed to be greater than 1 TeV. The Wilson coefficients, starting from this scale Λ, evolve at lower-energy scales according to their RGEs, whose derivation at one-loop level is the goal of this thesis work. Their renormalization is mediated by QED and QCD interactions, while weak interactions are neglected in a first approximation.

Moreover, the mass of the ALP ϕ is assumed to be $m_\phi \gtrsim$ few GeV, which means that, in the following, QCD can be treated perturbatively. The reason why we have factored out the gauge couplings e^2 and g_s^2 in front of the interactions between the ALP and the gauge bosons is given by the fact that, as we will see extensively, the couplings C_a and $\widetilde{C}_a$, with $a = \gamma, g$, turn out to be scale invariant at one-loop order.

As previously mentioned, this effective Lagrangian violates the CP symmetry. This is evident if we specialize to the interaction with photons:

$F_{\mu\nu}F^{\mu\nu} = 2|\vec{B}|^2 - 2|\vec{E}|^2$ is CP even, while $F_{\mu\nu}\widetilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$ is CP odd. Therefore, we cannot assign any CP transformation prescription to the ALP which is able to preserve the CP symmetry in both sectors of the Lagrangian. However, it is important to highlight that the two sectors do not violate CP individually, but is only their copresence that is CP violating. Indeed, the set of the Jarlskog invariants of the ALP EFT, namely those rephasing-invariant parameters that provide a measure of the CP violation, are given by products of the Wilson coefficients belonging to the two sectors:

$$C_a\widetilde{C}_b\,, \qquad y_S^{ii}\widetilde{C}_a\,, \qquad y_P^{ii}C_a\,, \qquad y_S^{ii}y_P^{jj}\,, \qquad y_S^{ik}y_{\rm SM}^{kk}y_P^{kj}\,, \tag{4.4}$$

where $a, b = \gamma, g$ and $y_{\rm SM}^{kk}$ are the SM Yukawa couplings in the diagonal basis.

From the phenomenological point of view, the most important signatures of a CP violating ALP are provided by EDMs. In general, the intrinsic angular momentum of a particle couples to external electric and magnetic fields, with strengths characterized by the electric and magnetic dipole moments, respectively. For a spin-1/2 fermion f, the non-relativistic Hamiltonian describing these interactions reads

$$\mathcal{H}_{\rm NR} = \frac{a_f e Q_f}{2m_f}\vec{\sigma}\cdot\vec{B} - d_f\vec{\sigma}\cdot\vec{E}\,, \tag{4.5}$$

where $\vec{\sigma}$ is the vector of Pauli matrices and a_f and d_f are the magnetic and EDMs of the fermion, while Q_f and m_f are its charge and mass. From this classical expression we can already deduce the transformation properties of the magnetic and EDMs, respectively, under CP: if the theory is invariant under CP, the only term which is allowed is the coupling to the magnetic field. The corresponding relativistic Lagrangian is

$$\mathscr{L} = -\frac{a_f e Q_f}{4m_f}\bar{f}\sigma^{\mu\nu}fF_{\mu\nu} - \frac{i}{2}d_f\bar{f}\sigma^{\mu\nu}\gamma_5 fF_{\mu\nu}\,, \tag{4.6}$$

where the first term is CP even and the second one is CP odd due to the presence of the γ_5 matrix. The SM predictions for the EDMs of the electron and the neutron are loop suppressed are respectively given by $|d_e^{\rm SM}| \lesssim 10^{-44}\,e$ cm and $|d_n^{\rm SM}| \lesssim 10^{-34}\,e$ cm, while their experimental bounds are $|d_e^{\rm exp}| \lesssim 10^{-29}\,e$ cm and $|d_n^{\rm exp}| \lesssim 10^{-26}\,e$ cm. We can now understand the reason why the permanent EDMs of molecules, atoms, nuclei and nucleons could represent CP violating signatures of ALPs. Examples of diagrams contributing to the fermion EDMs induced by ALP interactions are represented in Figure 4.1.

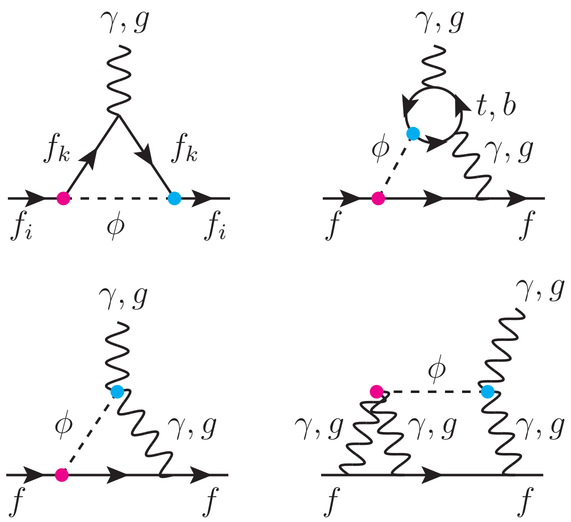


Figure 4.1: Examples of loop diagrams contributing to the EDMs of fermions induced by the ALP interactions. From Ref. [31].

Chapter 5

Renormalization of ALP EFT via Feynman diagrams

This Chapter is devoted to the computation at one-loop level of the anomalous dimension matrix of the ALP EFT previously defined in Chapter 4, through the standard techniques based on Feynman diagrams.

5.1 Dimensional regularization

The UV divergences that are present when computing loop diagrams only appear in the intermediate steps of the calculations. Indeed, when the counterterms are added to the Lagrangian density order by order in perturbation theory, these divergences cancel in all predictions for physical observables. In order to isolate the UV divergences, a regularization scheme is required. Ideally, it should respect all the symmetries of the theory, such as Lorentz and gauge invariance, and should not spoil the analytic structure of scattering amplitudes. The *dimensional regularization* scheme [1] preserves all these properties. Additionally and most importantly, it does not spoil the EFT expansion in powers of $1/\Lambda$ at loop-level. It consists in replacing the four-dimensional loop integrals by d-dimensional ones

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \longrightarrow \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,, \tag{5.1}$$

where $d = 4 - \varepsilon$ and ε is an infinitesimal parameter. In this way, as we will see, UV singularities show up as $1/\varepsilon^\ell$ pole terms, and, when the counterterms are added, these pole terms cancel and we can take the limit $\varepsilon \to 0$.

5.1.1 The issue of γ_5 and the BMHV scheme

It was soon realized that the three common properties valid in $d = 4$

$$\{\gamma_5, \gamma^\mu\} = 0\,, \tag{5.2}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma}\,, \tag{5.3}$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)\,, \tag{5.4}$$

are inconsistent in $d \neq 4$. Indeed, this statement follows from the fact that the combination of these three properties leads to

$$d\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = (8-d)\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5)\,, \tag{5.5}$$

heance, either $d = 4$ or the trace must vanish. This means that, in $d = 4 - \varepsilon$ dimensions, we must renounce at least one of them.

The *Breitenlohner-Maison-'t Hooft-Veltman* (BMHV) *scheme* [1, 18] is mathematically well-defined and consistent. This scheme gives up the anti-commutation property of γ_5. In particular, the d-dimensional Minkowski spacetime is regarded as a direct sum of two orthogonal subspaces: one that is four-dimensional and the other of dimension $d - 4$. Concerning the Dirac gamma matrices, they can thus be decomposed as

$$\underbrace{\gamma^\mu}_{d\text{-dim.}} = \underbrace{\bar{\gamma}^\mu}_{4\text{-dim.}} + \underbrace{\hat{\gamma}^\mu}_{(d-4)\text{-dim.}}\,, \tag{5.6}$$

which satisfy

$$\{\hat{\gamma}^\mu, \bar{\gamma}^\nu\} = 0\,, \quad \gamma_\mu\bar{\gamma}^\mu = 4\mathbb{1}\,, \quad \gamma_\mu\hat{\gamma}^\mu = (d-4)\mathbb{1}\,, \quad \bar{\gamma}_\mu\hat{\gamma}^\mu = 0\,. \tag{5.7}$$

On the other hand, the definition of γ_5 is intrinsically four-dimensional:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\bar{\gamma}^\mu\bar{\gamma}^\nu\bar{\gamma}^\rho\bar{\gamma}^\sigma\,. \tag{5.8}$$

In this way, the modified anti-commutation relations read

$$\{\gamma_5, \gamma^\mu\} = \{\gamma_5, \hat{\gamma}^\mu\} = 2\gamma_5\hat{\gamma}^\mu\,, \quad \{\gamma_5, \bar{\gamma}^\mu\} = 0\,, \quad [\gamma_5, \hat{\gamma}^\mu] = 0\,. \tag{5.9}$$

Only the original four-dimensional $\bar{\gamma}^\mu$ fully anti-commute with γ_5.

5.2 Dimensional analysis

The starting point of the renormalization procedure consists in observing that, if we use dimensional regularization, which lowers the number of spacetime dimensions from 4 to $d = 4 - \varepsilon$, then the mass dimensions of the fields change. Indeed, requiring the action to remain dimensionless, the Lagrangian density must have a mass dimension equal to d:

$$0 = [S] = \left[\int \mathrm{d}^d x\, \mathscr{L}\right] = \left[\int \mathrm{d}^d x\right] + [\mathscr{L}] = -d + [\mathscr{L}]\,. \tag{5.10}$$

Therefore, from the kinetic term of each field species, we can obtain the new mass dimensions. In particular

- for spin-0 fields

  $$d = [(\partial_\mu \phi)(\partial^\mu \phi)] = 2[\partial] + 2[\phi] = 2 + 2[\phi] \tag{5.11}$$

  implies

  $$[\phi] = \frac{d-2}{2} = \frac{2-\varepsilon}{2}\,; \tag{5.12}$$

- for spin-1/2 fields

  $$d = [\bar{\psi}\partial\!\!\!/\psi] = [\partial] + 2[\psi] = 1 + 2[\psi] \tag{5.13}$$

  leads to

  $$[\psi] = \frac{d-1}{2} = \frac{3-\varepsilon}{2}\,; \tag{5.14}$$

- for spin-1 fields

  $$d = [(\partial_\nu V_\mu)(\partial^\nu V^\mu)] = 2[\partial] + 2[V_\mu] = 2 + 2[V_\mu] \tag{5.15}$$

  implies as for the scalar case

  $$[V_\mu] = \frac{d-2}{2} = \frac{2-\varepsilon}{2}\,. \tag{5.16}$$

Accordingly, the mass dimensions of the ALP EFT composite operators are modified.

Regarding the bare ALP effective Lagrangian, we can write it as

$$\begin{aligned}\mathscr{L}_\phi^0 = {} & e_0^2 \frac{C_{\gamma,0}}{\Lambda}\phi_0 F_0^{\mu\nu} F_{0,\mu\nu} + g_{s,0}^2 \frac{C_{g,0}}{\Lambda}\phi_0 G_0^{a\,\mu\nu} G_{0,\mu\nu}^a + \frac{v_0}{\Lambda} y_{S,0}^{ij}\phi_0 \bar{f}_{i,0} f_{j,0} \\ & + e_0^2 \frac{\widetilde{C}_{\gamma,0}}{\Lambda}\phi_0 F_0^{\mu\nu}\widetilde{F}_{0,\mu\nu} + g_{s,0}^2 \frac{\widetilde{C}_{g,0}}{\Lambda}\phi_0 G_0^{a\,\mu\nu}\widetilde{G}_{0,\mu\nu}^a + i\frac{v_0}{\Lambda} y_{P,0}^{ij}\phi_0 \bar{f}_{i,0}\gamma_5 f_{j,0}\,,\end{aligned} \tag{5.17}$$

where the bare Wilson coefficients have a mass dimension that explicitly depends on ε. Indeed, from

$$[\phi_0 F_0^{\mu\nu} F_{0,\mu\nu}] = [\phi_0 F_0^{\mu\nu}\widetilde{F}_{0,\mu\nu}] = 5 - \frac{3}{2}\varepsilon\,, \tag{5.18}$$

$$[\phi_0 G_0^{a\,\mu\nu} G^a_{0,\mu\nu}] = [\phi_0 G_0^{a\,\mu\nu} \widetilde{G}^a_{0,\mu\nu}] = 5 - \frac{3}{2}\varepsilon \,, \tag{5.19}$$

$$[\phi_0 \bar{f}_{i,0} f_{j,0}] = [i\phi_0 \bar{f}_{i,0}\gamma_5 f_{j,0}] = 4 - \frac{3}{2}\varepsilon \,, \tag{5.20}$$

it follows that

$$[e_0^2 C_{\gamma,0}] = [e_0^2 \widetilde{C}_{\gamma,0}] = \frac{\varepsilon}{2} \,, \tag{5.21}$$

$$[g_{s,0}^2 C_{g,0}] = [g_{s,0}^2 \widetilde{C}_{g,0}] = \frac{\varepsilon}{2} \,, \tag{5.22}$$

$$[y^{ij}_{S,0}] = [y^{ij}_{P,0}] = \frac{\varepsilon}{2} \,. \tag{5.23}$$

5.3 Renormalized effective Lagrangian

The next step consists in writing the bare effective Lagrangian in Eq. (5.17) in terms of renormalized fields and coefficients. This is achieved with the introduction of the renormalization parameters: regarding the fields we can set

$$\phi_0 = Z_\phi^{1/2}\phi \,, \qquad A_{\mu,0} = Z_\gamma^{1/2} A_\mu \,, \tag{5.24}$$

$$A^a_{\mu,0} = Z_g^{1/2} A^a_\mu \,, \qquad f_{i,0} = Z_f^{1/2} f_i \,, \tag{5.25}$$

while the bare Wilson coefficients can be written in terms of the renormalized ones through

$$e_0^2 C_{\gamma,0} = Z_{C_\gamma} e^2 C_\gamma \mu^{\varepsilon/2} \,, \qquad e_0^2 \widetilde{C}_{\gamma,0} = Z_{\widetilde{C}_\gamma} e^2 \widetilde{C}_\gamma \mu^{\varepsilon/2} \,, \tag{5.26}$$

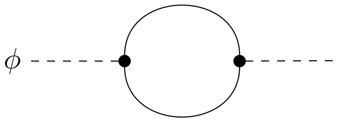
$$g_{s,0}^2 C_{g,0} = Z_{C_g} g_s^2 C_g \mu^{\varepsilon/2} \,, \qquad g_{s,0}^2 \widetilde{C}_{g,0} = Z_{\widetilde{C}_g} g_s^2 \widetilde{C}_g \mu^{\varepsilon/2} \,, \tag{5.27}$$

$$v_0 y^{ij}_{S,0} = Z_S^{ik} v y_S^{kj} \mu^{\varepsilon/2} \,, \qquad v_0 y^{ij}_{P,0} = Z_P^{ik} v y_P^{kj} \mu^{\varepsilon/2} \,, \tag{5.28}$$

where the renormalization scale μ, with $[\mu] = 1$, has been introduced to keep the renormalized Wilson coefficients dimensionless. Once we have defined the renormalization parameters, we can write the bare Lagrangian in Eq. (5.17) as

$$\begin{aligned} \mathscr{L}^0_\phi = {} & Z_{C_\gamma} Z_\gamma e^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2} \phi F^{\mu\nu} F_{\mu\nu} + Z_{C_g} Z_g g_s^2 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} \phi G^{a\,\mu\nu} G^a_{\mu\nu} \\ & + Z_S^{ik} Z_f \frac{v}{\Lambda} y_S^{kj} \mu^{\varepsilon/2} \phi \bar{f}_i f_j + Z_{\widetilde{C}_\gamma} Z_\gamma e^2 \frac{\widetilde{C}_\gamma}{\Lambda} \mu^{\varepsilon/2} \phi F^{\mu\nu} \widetilde{F}_{\mu\nu} \\ & + Z_{\widetilde{C}_g} Z_g g_s^2 \frac{\widetilde{C}_g}{\Lambda} \mu^{\varepsilon/2} \phi G^{a\,\mu\nu} \widetilde{G}^a_{\mu\nu} + i Z_P^{ik} Z_f \frac{v}{\Lambda} y_P^{kj} \mu^{\varepsilon/2} \phi \bar{f}_i \gamma_5 f_j \,. \end{aligned} \tag{5.29}$$

Note that we have neglected the renormalization parameter Z_ϕ associated with the ALP wavefunction. This is justified by the fact that the ALP propagator can receive loop corrections from diagrams containing at least two effective vertices, each of which is of order $1/\Lambda$



$$= \mathcal{O}(\Lambda^{-2}) \tag{5.30}$$

and consequently $Z_\phi = 1 + \mathcal{O}(\Lambda^{-2})$. Thus, it becomes relevant only when operators of dimension greater than five are taken into account. For the same reason, the renormalization parameters associated with the wavefunctions of fermion, photon and gluon fields are computed within the SM, namely without considering loop corrections mediated by ALP interactions.

The bare Lagrangian in Eq. (5.29) can now be written as a sum of two structurally identical Lagrangians

$$\mathscr{L}^0_\phi = \mathscr{L}^{\text{ren.}}_\phi + \mathscr{L}^{\text{ct}}_\phi\,, \tag{5.31}$$

where

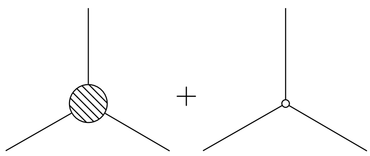
$$\begin{aligned}\mathscr{L}^{\text{ren.}}_\phi &= e^2\frac{C_\gamma}{\Lambda}\mu^{\varepsilon/2}\phi F^{\mu\nu}F_{\mu\nu} + g_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2}\phi G^{a\,\mu\nu}G^a_{\mu\nu} + \frac{v}{\Lambda}y_S^{ij}\mu^{\varepsilon/2}\phi\bar{f}_i f_j \\ &\quad + e^2\frac{\widetilde{C}_\gamma}{\Lambda}\mu^{\varepsilon/2}\phi F^{\mu\nu}\widetilde{F}_{\mu\nu} + g_s^2\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\phi G^{a\,\mu\nu}\widetilde{G}^a_{\mu\nu} + i\frac{v}{\Lambda}y_P^{ij}\mu^{\varepsilon/2}\phi\bar{f}_i\gamma_5 f_j\end{aligned} \tag{5.32}$$

denotes the renormalized Lagrangian, and

$$\begin{aligned}\mathscr{L}^{\text{ct}}_\phi &= (Z_{C_\gamma}Z_\gamma - 1)e^2\frac{C_\gamma}{\Lambda}\mu^{\varepsilon/2}\phi F^{\mu\nu}F_{\mu\nu} + (Z_{C_g}Z_g - 1)g_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2}\phi G^{a\,\mu\nu}G^a_{\mu\nu} \\ &\quad + (Z_S^{ik}Z_f - \delta^{ik})\frac{v}{\Lambda}y_S^{kj}\mu^{\varepsilon/2}\phi\bar{f}_i f_j + (Z_{\widetilde{C}_\gamma}Z_\gamma - 1)e^2\frac{\widetilde{C}_\gamma}{\Lambda}\mu^{\varepsilon/2}\phi F^{\mu\nu}\widetilde{F}_{\mu\nu} \\ &\quad + (Z_{\widetilde{C}_g}Z_g - 1)g_s^2\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\phi G^{a\,\mu\nu}\widetilde{G}^a_{\mu\nu} + i(Z_P^{ik}Z_f - \delta^{ik})\frac{v}{\Lambda}y_P^{kj}\mu^{\varepsilon/2}\phi\bar{f}_i\gamma_5 f_j\end{aligned} \tag{5.33}$$

is the counterterm one. As we will see, order by order in the perturbative expansion, these counterterms give rise to additional Feynman rules, which have the effects of cancelling the UV divergences stemming from Feynman diagrams constructed with the renormalized vertices. In the minimal subtraction scheme, the requirement

loop amplitude + counterterm =



= finite

(5.34)

fixes the counterterms $Z_i - 1$ to be proportional to the divergent terms $1/\varepsilon^\ell$ as $\varepsilon \to 0$

$$Z_i - 1 = \sum_{\ell=1}^{\infty} \frac{Z_i^{(\ell)}}{\varepsilon^\ell}\,, \tag{5.35}$$

where the coefficients $Z_i^{(\ell)}$ do not depend on ε and can be written as a perturbative expansion in the couplings. At one-loop level ($\ell = 1$), we will actually use the modified minimal subtraction ($\overline{\text{MS}}$) scheme, which exploit the freedom of adding the constant term $\log(4\pi) - \gamma_\text{E}$ to the counterterms

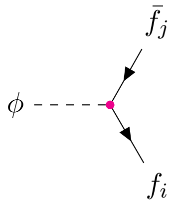
$$Z_i - 1 \propto \Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma_\text{E} + \log(4\pi)\,, \tag{5.36}$$

where $\gamma_\text{E} \approx 0.57721$ is the Euler-Mascheroni constant.

5.3.1 d-dimensional Feynman rules

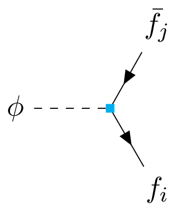
The d-dimensional Feynman rules associated with the renormalized effective operators in $\mathscr{L}_\phi^{\text{ren.}}$ read as follows

$\phi \bar{f} f$ operator:



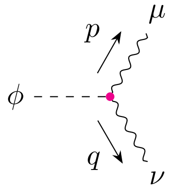
$$= i\frac{v}{\Lambda} y_S^{ij} \mu^{\varepsilon/2}\,; \tag{5.37}$$

$i\phi \bar{f} \gamma_5 f$ operator:



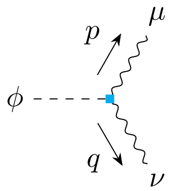
$$= -\frac{v}{\Lambda} y_P^{ij} \mu^{\varepsilon/2} \gamma_5\,; \tag{5.38}$$

$\phi F F$ operator:



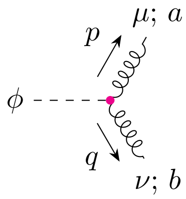
$$= 4ie^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2} (q^\mu p^\nu - p \cdot q g^{\mu\nu})\,; \tag{5.39}$$

$\phi F\widetilde{F}$ operator:

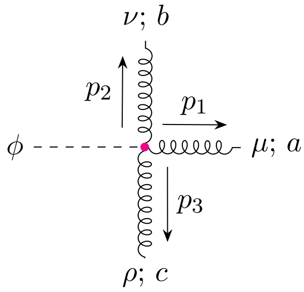


$$= 4ie^2\frac{\widetilde{C}_\gamma}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta\,; \tag{5.40}$$

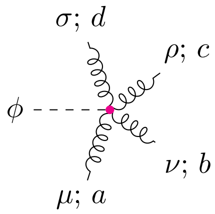
ϕGG operator:



$$= 4ig_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2}(q^\mu p^\nu - p\cdot q g^{\mu\nu})\delta^{ab}\,; \tag{5.41}$$

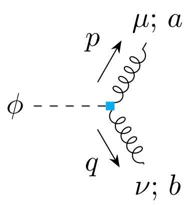


$$\begin{aligned} &= 4g_s^3\frac{C_g}{\Lambda}\mu^{\varepsilon}f^{abc}[g^{\mu\nu}(p_1-p_2)^\rho \\ &\quad + g^{\nu\rho}(p_2-p_3)^\mu + g^{\rho\mu}(p_3-p_1)^\nu]\,; \end{aligned} \tag{5.42}$$

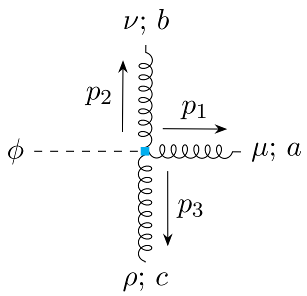


$$\begin{aligned} &= 4ig_s^4\frac{C_g}{\Lambda}\mu^{3\varepsilon/2}[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]\,; \end{aligned} \tag{5.43}$$

$\phi G\widetilde{G}$ operator:



$$= 4ig_s^2\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta\delta^{ab}\,; \tag{5.44}$$



$$= 4g_s^3 \frac{\widetilde{C}_g}{\Lambda} \mu^{\varepsilon} f^{abc} \epsilon^{\mu\nu\rho\alpha} (p_1 + p_2 + p_3)_\alpha \, . \tag{5.45}$$

The interaction vertex involving four gluons and the ALP induced by the operator $\phi G\widetilde{G}$ has a Feynman rule that is proportional to $f^{abe}f^{cde} + f^{cae}f^{bde} + f^{bce}f^{ade}$, which is zero due to the Jacobi identity.

5.4 One-loop 1PI Feynman diagrams

The general prescription to calculate d-dimensional one-loop integrals consists of five main steps.

1. Exploit the Feynman parametrization to write the denominator of the loop integral, which is given by the product of the internal propagators, as a sum

$$\frac{1}{D_1 \cdots D_n} = \Gamma(n) \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_n \frac{\delta(x_1 + \cdots + x_n - 1)}{(x_1 D_1 + \cdots + x_n D_n)^n} \, , \tag{5.46}$$

   where x_i are the Feynman parameters.

2. Since the denominators D_i are at most quadratic polynomials in the loop momentum k, it is useful to complete the square of $x_1 D_1 + \cdots + x_n D_n$ by introducing a shifted loop momentum ℓ

$$x_1 D_1 + \cdots + x_n D_n = \ell^2 - C + i\epsilon \, , \tag{5.47}$$

   where

$$\ell^\mu = k^\mu + \sum_i c_i(x_1, \ldots, x_n) p_i^\mu \, , \tag{5.48}$$

   p_i are the momenta of the external particles of the diagram and c_i are linear functions of the Feynman parameters. After performing the shift also in the numerator of the integral, it can be generically written as

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{N_0 + N_1 \ell^\mu + N_2 \ell^\mu \ell^\nu + \cdots}{(\ell^2 - C + i\epsilon)^n} \, , \tag{5.49}$$

   where the coefficients N_i do not depend on ℓ.

3. We can then use Lorentz invariance in order to reduce tensor integrals to scalar ones. Indeed the following identities

$$\int \mathrm{d}^d\ell\, f(\ell^2)\ell^{\mu_1}\cdots\ell^{\mu_{2n+1}} = 0\,, \tag{5.50}$$

$$\int \mathrm{d}^d\ell\, f(\ell^2)\ell^{\mu}\ell^{\nu} = \frac{g^{\mu\nu}}{d}\int \mathrm{d}^d\ell\, f(\ell^2)\ell^2 \tag{5.51}$$

   hold for every scalar function $f(\ell^2)$. We can then directly substitute in the numerator of the integral $\ell^\mu\ell^\nu$ with $g^{\mu\nu}\ell^2/d$ and neglect tensors constructed with an odd number of insertions of ℓ^μ.

4. The remaining scalar integrals can be computed according to the *master formula*

$$\begin{aligned} I_{m,n} &= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{(\ell^2)^m}{(\ell^2 - C + i\epsilon)^n} \\ &= i\frac{(-1)^{m-n}}{(4\pi)^{\frac{d}{2}}}C^{m-n+\frac{d}{2}}\frac{\Gamma(m+\frac{d}{2})\Gamma(n-m-\frac{d}{2})}{\Gamma(n)\Gamma(\frac{d}{2})}\,, \end{aligned} \tag{5.52}$$

   which makes use of a Wick rotation and of the residue theorem.

5. The final step consists in performing the integrals over the Feynman parameters x_i, either before or after a Laurent expansion about $\varepsilon = 0$. For this purpose, the following formula

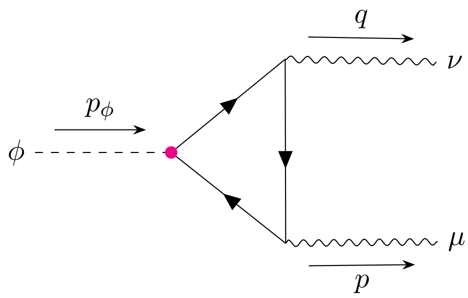
$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \sum_{k=1}^{n}\frac{1}{k} - \gamma_{\mathrm{E}}\right] + \mathcal{O}(\varepsilon)\,, \tag{5.53}$$

   valid for $n = 0, 1, 2, \ldots$, can be useful to expand the gamma function around negative integer values.

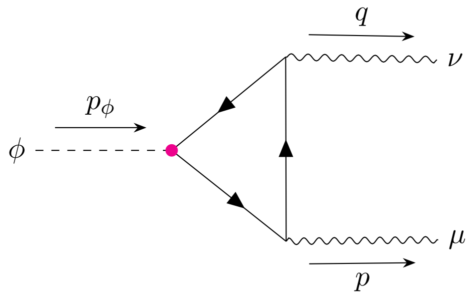
In order to renormalize the theory, it is sufficient to consider only the *one-particle irreducible* (1PI) diagrams, namely the connected Feynman diagrams that cannot be disconnected by cutting an internal line, since reducible diagrams are products of the integrals corresponding to their irreducible parts. Accordingly, the renormalization of the external legs is conducted separately in Appendix A.

5.4.1 ϕFF anomalous dimension

The only two 1PI diagrams of order $1/\Lambda$ contributing to the one-loop correction of the $\phi F_{\mu\nu}F^{\mu\nu}$ vertex are mediated by the operator $\phi\bar{f}_i f_j$ and are related by inverting the loop fermion line direction



$$= i\mathcal{M}^{(1)}_{\gamma S}(p_\phi, p, q)\,, \tag{5.54}$$



$$= i\mathcal{M}^{(2)}_{\gamma S}(p_\phi, p, q)\,. \tag{5.55}$$

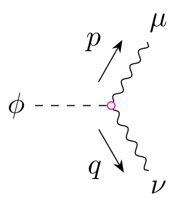
As shown in Appendix B, both diagrams do not develop an UV divergence

$$i\mathcal{M}^{(1)}_{\gamma S}(p_\phi, p, q)|_{\text{div.}} = i\mathcal{M}^{(2)}_{\gamma S}(p_\phi, p, q)|_{\text{div.}} = 0\,, \tag{5.56}$$

and, as a consequence, the relevant counterterm

$$\mathscr{L}^{\text{ct}}_{\phi} \supset (Z_{C_\gamma} Z_\gamma - 1) e^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2} \phi F^{\mu\nu} F_{\mu\nu}\,, \tag{5.57}$$

whose Feynman rule is



$$= 4i(Z_{C_\gamma} Z_\gamma - 1) e^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2} (q^\mu p^\nu - p \cdot q g^{\mu\nu})\,, \tag{5.58}$$

is fixed by requiring

$$\begin{aligned} 0 = {} & [i\mathcal{M}^{(1)}_{\gamma S}(p_\phi, p, q) + i\mathcal{M}^{(2)}_{\gamma S}(p_\phi, p, q)]|_{\text{div.}} \\ & + 4i(Z_{C_\gamma} Z_\gamma - 1) e^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2} (q^\mu p^\nu - p \cdot q g^{\mu\nu}) \epsilon^*_\mu(p) \epsilon^*_\nu(q) \end{aligned} \tag{5.59}$$

and does not need to cancel any divergence

$$Z_{C_\gamma} Z_\gamma = 1\,. \tag{5.60}$$

This implies that $Z_{C_\gamma} = Z_\gamma^{-1}$ and thus the bare Wilson coefficient from Eq. (5.26) reads

$$e_0^2 C_{\gamma,0} = Z_\gamma^{-1} e^2 C_\gamma \mu^{\varepsilon/2} \,. \tag{5.61}$$

On the other hand, we know from Appendix A that the bare electric charge satisfies

$$e_0 = Z_\gamma^{-1/2} e \mu^{\varepsilon/2} \,, \tag{5.62}$$

which leads to

$$e_0^2 C_{\gamma,0} = e_0^2 C_\gamma \mu^{-\varepsilon/2} \,. \tag{5.63}$$

We can now differentiate this expression with respect to $\log\mu$

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\log\mu}(e_0^2 C_{\gamma,0}) = \frac{\mathrm{d}}{\mathrm{d}\log\mu}(e_0^2 C_\gamma \mu^{-\varepsilon/2}) \\ &= e_0^2 \frac{\mathrm{d}C_\gamma}{\mathrm{d}\log\mu}\mu^{-\varepsilon/2} - \frac{\varepsilon}{2} e_0^2 C_\gamma \mu^{-\varepsilon/2} \,, \end{aligned} \tag{5.64}$$

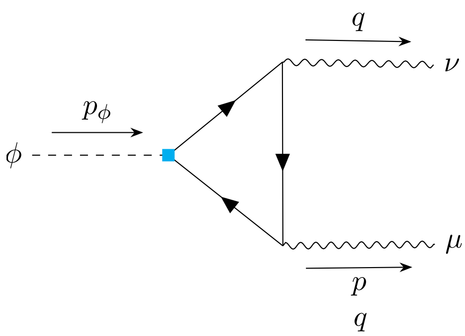
where the μ-independence of e_0 has been used, to find that

$$\frac{\mathrm{d}C_\gamma}{\mathrm{d}\log\mu} = \frac{\varepsilon}{2} C_\gamma \xrightarrow{\varepsilon\to 0} 0 \,. \tag{5.65}$$

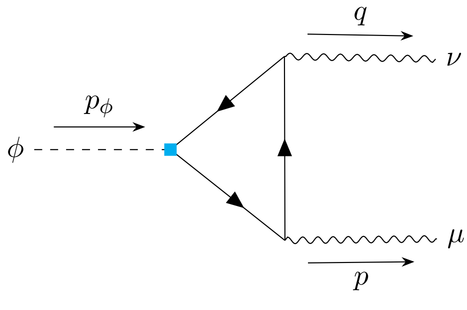
This proves that the Wilson coefficient $e^2 C_\gamma$ scales exactly as e^2 at one-loop order.

5.4.2 $\phi F\widetilde{F}$ anomalous dimension

The only two 1PI diagrams of order $1/\Lambda$ contributing to the one-loop correction of the $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ vertex are mediated by the operator $i\phi \bar{f}_i \gamma_5 f_j$ and are related by inverting the loop fermion line direction



$$= i\mathcal{M}^{(1)}_{\tilde{\gamma}P}(p_\phi, p, q) \,, \tag{5.66}$$



$$= i\mathcal{M}^{(2)}_{\tilde{\gamma}P}(p_\phi, p, q) \,. \tag{5.67}$$

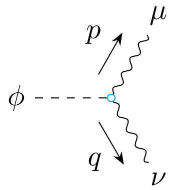
As in the previous case, both diagrams are UV finite

$$i\mathcal{M}^{(1)}_{\tilde{\gamma}P}(p_\phi, p, q)|_{\text{div.}} = i\mathcal{M}^{(2)}_{\tilde{\gamma}P}(p_\phi, p, q)|_{\text{div.}} = 0\,, \tag{5.68}$$

and, as a consequence, the relevant counterterm

$$\mathscr{L}^{\text{ct}}_\phi \supset (Z_{\widetilde{C}_\gamma} Z_\gamma - 1) e^2 \frac{\widetilde{C}_\gamma}{\Lambda} \mu^{\varepsilon/2} \phi F^{\mu\nu} \widetilde{F}_{\mu\nu}\,, \tag{5.69}$$

whose Feynman rule is



$$= 4i(Z_{C_\gamma} Z_\gamma - 1) e^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta\,, \tag{5.70}$$

is fixed by requiring

$$\begin{aligned} 0 &= [i\mathcal{M}^{(1)}_{\tilde{\gamma}P}(p_\phi, p, q) + i\mathcal{M}^{(2)}_{\tilde{\gamma}P}(p_\phi, p, q)]|_{\text{div.}} \\ &\quad + 4i(Z_{\widetilde{C}_\gamma} Z_\gamma - 1) e^2 \frac{\widetilde{C}_\gamma}{\Lambda} \mu^{\varepsilon/2} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \epsilon^*_\mu(p) \epsilon^*_\nu(q) \end{aligned} \tag{5.71}$$

and does not need to cancel any divergence

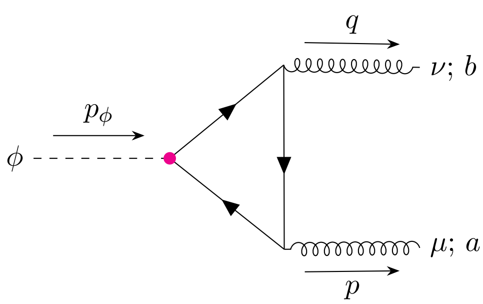
$$Z_{\widetilde{C}_\gamma} Z_\gamma = 1\,. \tag{5.72}$$

Following the same steps as before, we can conclude that also the Wilson coefficient $e^2\widetilde{C}_\gamma$ scales exactly as e^2 at one-loop order:

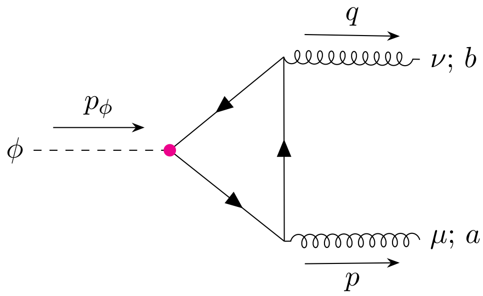
$$\frac{\mathrm{d}\widetilde{C}_\gamma}{\mathrm{d}\log\mu} = \frac{\varepsilon}{2}\widetilde{C}_\gamma \xrightarrow{\varepsilon\to 0} 0\,. \tag{5.73}$$

5.4.3 ϕGG anomalous dimension

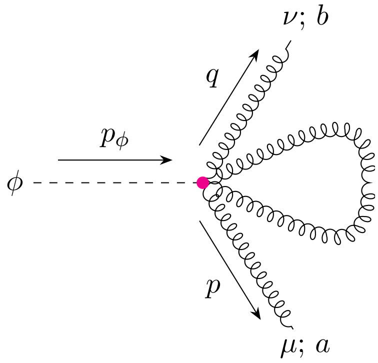
Concerning the interaction vertex between the ALP and two gluons corresponding to $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$, the one-loop 1PI diagrams that contribute to its correction at order $1/\Lambda$ are mediated by the operators $\phi \bar{f}_i f_j$ and $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ and read



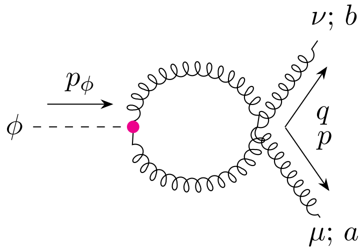
$$= i\mathcal{M}^{(1)}_{gS}(p_\phi, p, q)\,, \tag{5.74}$$



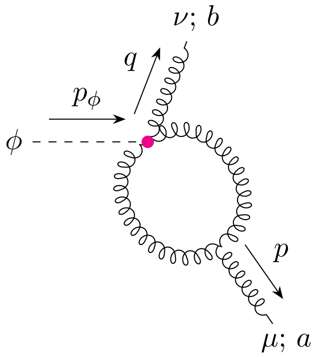
$$= i\mathcal{M}^{(2)}_{gS}(p_\phi, p, q)\,, \tag{5.75}$$



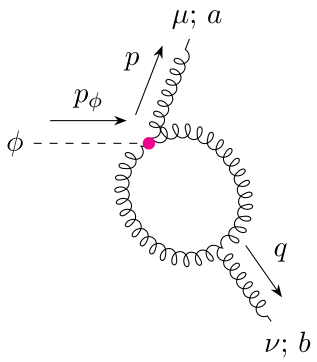
$$= i\mathcal{M}^{(1)}_{gg}(p_\phi, p, q)\,, \tag{5.76}$$



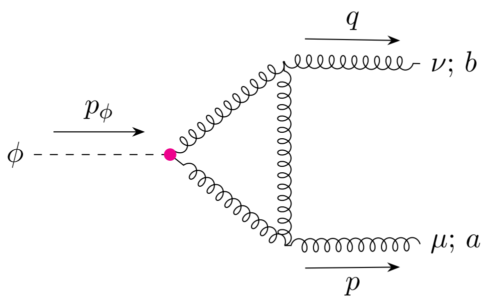
$$= i\mathcal{M}^{(2)}_{gg}(p_\phi, p, q)\,, \tag{5.77}$$



$$= i\mathcal{M}^{(3)}_{gg}(p_\phi, p, q)\,, \tag{5.78}$$



$$= i\mathcal{M}_{gg}^{(4)}(p_\phi, p, q)\,, \tag{5.79}$$



$$= i\mathcal{M}_{gg}^{(5)}(p_\phi, p, q)\,. \tag{5.80}$$

In the Feynman-'t Hooft gauge, their divergent parts, as explicitly computed in Appendix B, are respectively given by

$$i\mathcal{M}_{gS}^{(1)}(p_\phi, p, q)|_{\text{div.}} = i\mathcal{M}_{gS}^{(2)}(p_\phi, p, q)|_{\text{div.}} = 0\,, \tag{5.81}$$

$$i\mathcal{M}_{gg}^{(1)}(p_\phi, p, q)|_{\text{div.}} = 0\,, \tag{5.82}$$

$$\begin{aligned} i\mathcal{M}_{gg}^{(2)}(p_\phi, p, q)|_{\text{div.}} &= \frac{i}{24\pi^2} C_A g_s^4 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} (p^\nu q^\mu + 13 p\cdot q g^{\mu\nu}) \delta^{ab} \\ &\quad \times \epsilon_\mu^{a*}(p) \epsilon_\nu^{b*}(q) \Delta_\varepsilon\,, \end{aligned} \tag{5.83}$$

$$\begin{aligned} i\mathcal{M}_{gg}^{(3)}(p_\phi, p, q)|_{\text{div.}} &= i\mathcal{M}_{gg}^{(4)}(p_\phi, p, q)|_{\text{div.}} \\ &= \frac{3i}{8\pi^2} C_A g_s^4 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} (-p^\nu q^\mu + p\cdot q g^{\mu\nu}) \delta^{ab} \\ &\quad \times \epsilon_\mu^{a*}(p) \epsilon_\nu^{b*}(q) \Delta_\varepsilon\,, \end{aligned} \tag{5.84}$$

$$\begin{aligned} i\mathcal{M}_{gg}^{(5)}(p_\phi, p, q)|_{\text{div.}} &= \frac{i}{24\pi^2} C_A g_s^4 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} (29 p^\nu q^\mu - 43 p\cdot q g^{\mu\nu}) \delta^{ab} \\ &\quad \times \epsilon_\mu^{a*}(p) \epsilon_\nu^{b*}(q) \Delta_\varepsilon\,, \end{aligned} \tag{5.85}$$

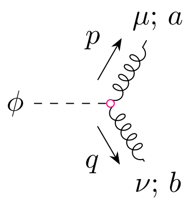
so that, when summed together, they provide

$$i\mathcal{M}_g(p_\phi, p, q)|_{\text{div.}} = \frac{i}{2\pi^2} C_A g_s^4 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} (p^\nu q^\mu - p\cdot q g^{\mu\nu}) \delta^{ab} \epsilon_\mu^{a*}(p) \epsilon_\nu^{b*}(q) \Delta_\varepsilon\,, \tag{5.86}$$

where $C_A = N_c = 3$ is the Casimir of the adjoint representation of $SU(N_c)$. It is important to notice that both diagrams $i\mathcal{M}_{gg}^{(3)}$ and $i\mathcal{M}_{gg}^{(4)}$ are also infrared divergent. The sum $i\mathcal{M}_g$ has the correct Lorentz structure that can be subtracted by the counterterm

$$\mathscr{L}_\phi^{\text{ct}} \supset (Z_{C_g} Z_g - 1) g_s^2 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} \phi G^{a\,\mu\nu} G^a_{\mu\nu} \,, \tag{5.87}$$

whose Feynman rule is



$$= 4i(Z_{C_g} Z_g - 1) g_s^2 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} (q^\mu p^\nu - p \cdot q g^{\mu\nu}) \delta^{ab} \,, \tag{5.88}$$

despite the fact that the individual diagrams $i\mathcal{M}_{gg}^{(2)}$ and $i\mathcal{M}_{gg}^{(5)}$ have a different Lorentz structure. From the condition

$$0 = i\mathcal{M}_g(p_\phi, p, q)|_{\text{div.}} + 4i(Z_{C_g} Z_g - 1) g_s^2 \frac{C_g}{\Lambda} \mu^{\varepsilon/2} (q^\mu p^\nu - p \cdot q g^{\mu\nu}) \delta^{ab} \epsilon_\mu^{a*}(p) \epsilon_\nu^{b*}(q) \tag{5.89}$$

we then obtain

$$Z_{C_g} Z_g = 1 - \frac{g_s^2}{8\pi^2} C_A \Delta_\varepsilon \,. \tag{5.90}$$

Therefore, from Eq. (5.27), the bare Wilson coefficient is given by

$$g_{s,0}^2 C_{g,0} = \left(1 - \frac{g_s^2}{8\pi^2} C_A \Delta_\varepsilon\right) Z_g^{-1} g_s^2 C_g \mu^{\varepsilon/2} \,. \tag{5.91}$$

On the other hand, the bare gauge coupling $g_{s,0}$, as shown in Appendix A, satisfies

$$g_{s,0} = Z_{g_s} Z_g^{-1/2} Z_f^{-1} g_s \mu^{\varepsilon/2} \,, \tag{5.92}$$

where the renormalization parameters read

$$Z_{g_s} = 1 - \frac{1}{16\pi^2} [g_s^2 (C_A + C_F c_f^2) + e^2 Q_f^2] \Delta_\varepsilon \,, \tag{5.93}$$

$$Z_f = 1 - \frac{1}{16\pi^2} (e^2 Q_f^2 + C_F g_s^2 c_f^2) \Delta_\varepsilon \,, \tag{5.94}$$

and $C_F = (N_c^2 - 1)/(2N_c) = 4/3$ is the Casimir of the fundamental representation of $SU(N_c)$. We can then isolate

$$Z_g^{-1} g_s^2 = Z_{g_s}^{-2} Z_f^2 g_{s,0}^2 \mu^{-\varepsilon} \tag{5.95}$$

and substitute it in Eq. (5.91) to obtain

$$g_{s,0}^2 C_{g,0} = \left(1 - \frac{g_s^2}{8\pi^2} C_A \Delta_\varepsilon\right) Z_{g_s}^{-2} Z_f^2 g_{s,0}^2 C_g \mu^{-\varepsilon/2} \,. \tag{5.96}$$

Remarkably, these renormalization parameters cancel each other at lowest order in the gauge couplings

$$\begin{aligned}\left(1 - \frac{g_s^2}{8\pi^2} C_A \Delta_\varepsilon\right) Z_{g_s}^{-2} Z_f^2 &= 1 - \frac{g_s^2}{8\pi^2} C_A \Delta_\varepsilon + \frac{1}{8\pi^2}[g_s^2(C_A + C_F c_f^2) + e^2 Q_f^2]\Delta_\varepsilon \\ &\quad - \frac{1}{8\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon + \mathcal{O}(e^4, g_s^4) \\ &= 1 + \mathcal{O}(e^4, g_s^4) \,,\end{aligned} \tag{5.97}$$

so that we are left with

$$g_{s,0}^2 C_{g,0} = g_{s,0}^2 C_g \mu^{-\varepsilon/2} \,. \tag{5.98}$$

We can now differentiate this expression with respect to $\log \mu$ and exploit the μ-independence of $g_{s,0}$ to find

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\log\mu}(g_{s,0}^2 C_{g,0}) = \frac{\mathrm{d}}{\mathrm{d}\log\mu}(g_{s,0}^2 C_g \mu^{-\varepsilon/2}) \\ &= g_{s,0}^2 \frac{\mathrm{d}C_g}{\mathrm{d}\log\mu}\mu^{-\varepsilon/2} - \frac{\varepsilon}{2} g_{s,0}^2 C_g \mu^{-\varepsilon/2} \,,\end{aligned} \tag{5.99}$$

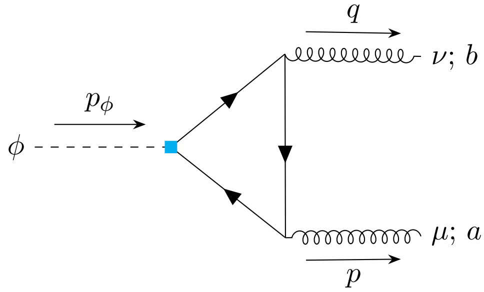
namely

$$\frac{\mathrm{d}C_g}{\mathrm{d}\log\mu} = \frac{\varepsilon}{2} C_g \xrightarrow{\varepsilon\to 0} 0 \,. \tag{5.100}$$

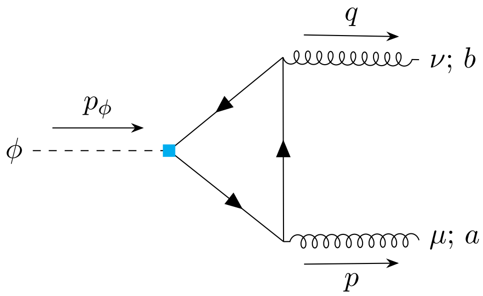
This proves that the Wilson coefficient $g_s^2 C_g$ scales exactly as g_s^2 at one-loop order.

5.4.4 $\phi G\widetilde{G}$ anomalous dimension

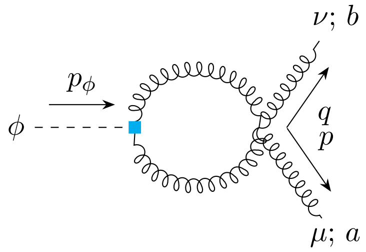
Concerning the interaction vertex between the ALP and two gluons corresponding to $\phi G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu}$, the one-loop 1PI diagrams that contribute to its correction at order $1/\Lambda$ are mediated by the operators $i\phi \bar{f}_i \gamma_5 f_j$ and $\phi G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu}$ and read



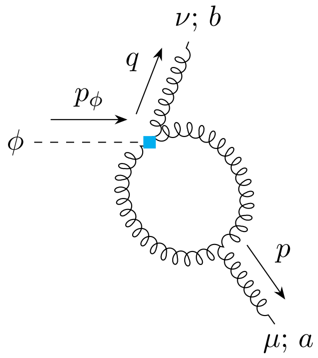
$$= i\mathcal{M}_{\tilde{g}P}^{(1)}(p_\phi, p, q) \,, \tag{5.101}$$



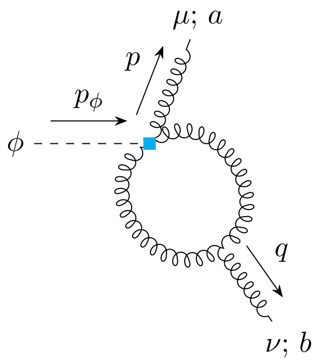
$$= i\mathcal{M}^{(2)}_{\tilde{g}P}(p_\phi, p, q)\,, \tag{5.102}$$



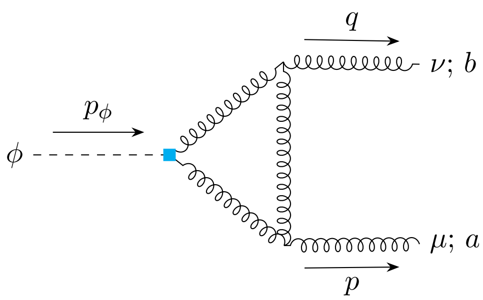
$$= i\mathcal{M}^{(1)}_{\tilde{g}\tilde{g}}(p_\phi, p, q)\,, \tag{5.103}$$



$$= i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}}(p_\phi, p, q)\,, \tag{5.104}$$



$$= i\mathcal{M}^{(3)}_{\tilde{g}\tilde{g}}(p_\phi, p, q)\,, \tag{5.105}$$



$$= i\mathcal{M}^{(4)}_{\tilde{g}\tilde{g}}(p_\phi, p, q)\,. \tag{5.106}$$

In the Feynman-'t Hooft gauge, their divergent parts, as explicitly computed in Appendix B, are respectively given by

$$i\mathcal{M}^{(1)}_{\tilde{g}P}(p_\phi,p,q)|_{\text{div.}} = i\mathcal{M}^{(2)}_{\tilde{g}P}(p_\phi,p,q)|_{\text{div.}} = 0\,, \tag{5.107}$$

$$i\mathcal{M}^{(1)}_{\tilde{g}\tilde{g}}(p_\phi,p,q)|_{\text{div.}} = 0\,, \tag{5.108}$$

$$\begin{aligned} i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}}(p_\phi,p,q)|_{\text{div.}} &= i\mathcal{M}^{(3)}_{\tilde{g}\tilde{g}}(p_\phi,p,q)|_{\text{div.}} \\ &= -\frac{3i}{8\pi^2}C_A g_s^4\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\delta^{ab}\epsilon^{a*}_\mu(p)\epsilon^{b*}_\nu(q)\Delta_\varepsilon\,, \end{aligned} \tag{5.109}$$

$$i\mathcal{M}^{(4)}_{\tilde{g}\tilde{g}}(p_\phi,p,q)|_{\text{div.}} = \frac{5i}{4\pi^2}C_A g_s^4\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\delta^{ab}\epsilon^{a*}_\mu(p)\epsilon^{b*}_\nu(q)\Delta_\varepsilon\,, \tag{5.110}$$

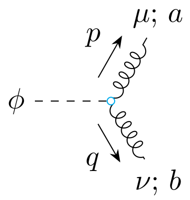
where $i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}}$ and $i\mathcal{M}^{(3)}_{\tilde{g}\tilde{g}}$ are also infrared divergent. The UV divergence of the sum

$$i\mathcal{M}_{\tilde{g}}(p_\phi,p,q)|_{\text{div.}} = \frac{i}{2\pi^2}N_c g_s^4\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\delta^{ab}\epsilon^{a*}_\mu(p)\epsilon^{b*}_\nu(q)\Delta_\varepsilon \tag{5.111}$$

is absorbed by the counterterm

$$\mathscr{L}^{\text{ct}}_\phi \supset (Z_{\widetilde{C}_g}Z_g - 1)g_s^2\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\phi G^{a\,\mu\nu}\widetilde{G}^a_{\mu\nu}\,, \tag{5.112}$$

whose Feynman rule is



$$= 4i(Z_{C_g}Z_g - 1)g_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\delta^{ab}\,. \tag{5.113}$$

From the condition

$$0 = i\mathcal{M}_{\tilde{g}}(p_\phi,p,q)|_{\text{div.}} + 4i(Z_{C_g}Z_g - 1)g_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\delta^{ab}\epsilon^{a*}_\mu(p)\epsilon^{b*}_\nu(q) \tag{5.114}$$

we then obtain

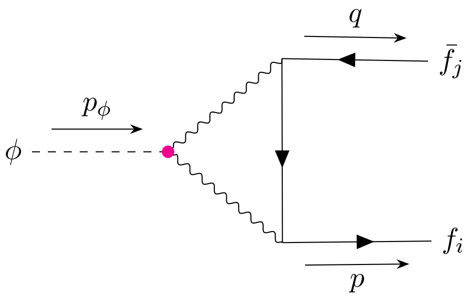
$$Z_{\widetilde{C}_g}Z_g = 1 - \frac{g_s^2}{8\pi^2}C_A\Delta_\varepsilon\,, \tag{5.115}$$

and following the exact same steps of the previous case, namely from Eq. (5.90) to (5.100), we can note that the same cancellation occurs, since $Z_{C_g} = Z_{\widetilde{C}_g}$, and conclude that also the Wilson coefficient $g_s^2\widetilde{C}_g$ scales exactly as g_s^2 at one-loop order:

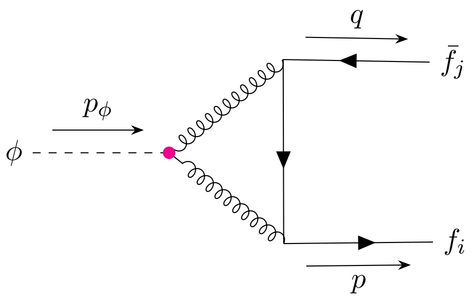
$$\frac{\mathrm{d}\widetilde{C}_g}{\mathrm{d}\log\mu} = \frac{\varepsilon}{2}\widetilde{C}_g \xrightarrow{\varepsilon\to 0} 0\,. \tag{5.116}$$

5.4.5 $\phi \bar{f} f$ anomalous dimension

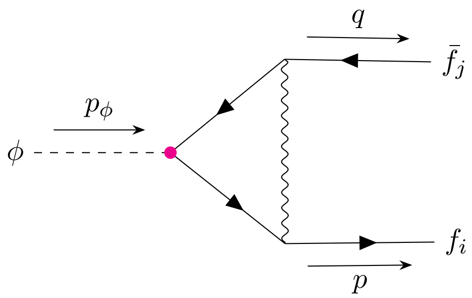
Regarding the interaction vertex between the ALP and two fermions corresponding to the Yukawa operator $\phi \bar{f}_i f_j$, the one-loop diagrams that contribute to its correction at order $1/\Lambda$ are mediated by the operators $\phi F_{\mu\nu} F^{\mu\nu}$, $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ and $\phi \bar{f}_i f_j$ itself and read



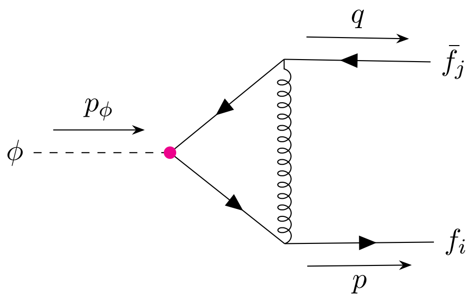
$$= i\mathcal{M}_{S\gamma}(p_\phi, p, q)\,, \tag{5.117}$$



$$= i\mathcal{M}_{Sg}(p_\phi, p, q)\,, \tag{5.118}$$



$$= i\mathcal{M}^{(\gamma)}_{SS}(p_\phi, p, q)\,, \tag{5.119}$$



$$= i\mathcal{M}^{(g)}_{SS}(p_\phi, p, q)\,. \tag{5.120}$$

In the Feynman-'t Hooft gauge, their divergent parts, as explicitly computed in Appendix B, are respectively given by

$$i\mathcal{M}_{S\gamma}(p_\phi, p, q)|_{\text{div.}} = -\frac{3i}{4\pi^2} e^4 Q_f^2 \frac{C_\gamma}{\Lambda} m_i \delta^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) v_j(q) \Delta_\varepsilon\,, \tag{5.121}$$

$$
\begin{aligned}
i\mathcal{M}_{Sg}(p_\phi, p, q))|_{\text{div.}} &= -\frac{3i}{4\pi^2} C_F g_s^4 c_f^2 \frac{C_g}{\Lambda} m_i \delta^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) v_j(q) \Delta_\varepsilon \,, && (5.122)\\
i\mathcal{M}_{SS}^{(\gamma)}(p_\phi, p, q))|_{\text{div.}} &= \frac{i}{4\pi^2} e^2 Q_f^2 \frac{v}{\Lambda} y_S^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) v_j(q) \Delta_\varepsilon \,, && (5.123)\\
i\mathcal{M}_{SS}^{(g)}(p_\phi, p, q))|_{\text{div.}} &= \frac{i}{4\pi^2} C_F g_s^2 c_f^2 \frac{v}{\Lambda} y_S^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) v_j(q) \Delta_\varepsilon \,. && (5.124)
\end{aligned}
$$

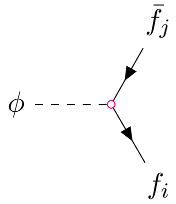
The divergence of their sum

$$
\begin{aligned}
i\mathcal{M}_S(p_\phi, p, q))|_{\text{div.}} = \frac{i}{4\pi^2\Lambda} \Big[& v y_S^{ij} (e^2 Q_f^2 + C_F g_s^2 c_f^2) \\
& - 3 m_i \delta^{ij} (e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \Big] \mu^{\varepsilon/2} \bar{u}_i(p) v_j(q) \Delta_\varepsilon
\end{aligned}
\tag{5.125}
$$

is absorbed by the counterterm

$$
\mathscr{L}_\phi^{\text{ct}} \supset (Z_S^{ik} Z_f - \delta^{ik}) \frac{v}{\Lambda} y_S^{kj} \mu^{\varepsilon/2} \phi \bar{f}_i f_j \,, \tag{5.126}
$$

whose Feynman rule is



$$
= i(Z_S^{ik} Z_f - \delta^{ik}) \frac{v}{\Lambda} y_S^{kj} \mu^{\varepsilon/2} \,. \tag{5.127}
$$

From the condition

$$
0 = i\mathcal{M}_S(p_\phi, p, q))|_{\text{div.}} + i(Z_S^{ik} Z_f - \delta^{ik}) \frac{v}{\Lambda} y_S^{kj} \mu^{\varepsilon/2} \bar{u}_i(p) v_j(q) \tag{5.128}
$$

we then obtain

$$
\begin{aligned}
Z_S^{ik} Z_f v y_S^{kj} = v y_S^{ij} - \frac{1}{4\pi^2} \Big[& v y_S^{ij} (e^2 Q_f^2 + C_F g_s^2 c_f^2) \\
& - 3 m_i \delta^{ij} (e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \Big] \Delta_\varepsilon \,,
\end{aligned}
\tag{5.129}
$$

and, recalling the expression for Z_f in Eq. (5.94), we can expand at lowest order in the Wilson coefficients and gauge couplings to find

$$
\begin{aligned}
Z_S^{ik} v y_S^{kj} &= v y_S^{ij} + \frac{v y_S^{ij}}{16\pi^2} (e^2 Q_f^2 + C_F g_s^2 c_f^2) \Delta_\varepsilon - \frac{1}{4\pi^2} \Big[v y_S^{ij} (e^2 Q_f^2 + C_F g_s^2 c_f^2) \\
&\quad - 3 m_i \delta^{ij} (e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \Big] \Delta_\varepsilon \\
&= v y_S^{ij} - \frac{1}{16\pi^2} \Big[3 v y_S^{ij} (e^2 Q_f^2 + C_F g_s^2 c_f^2) \\
&\quad - 12 m_i \delta^{ij} (e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \Big] \Delta_\varepsilon \,.
\end{aligned}
$$

$$(5.130)$$

At this point we can exploit the μ-independence of the bare Wilson coefficient to write

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\log\mu}(v_0 y_{S,0}^{ij}) = \frac{\mathrm{d}}{\mathrm{d}\log\mu}(Z_S^{ik} v y_S^{kj} \mu^{\varepsilon/2}) \\ &= \frac{\mathrm{d}}{\mathrm{d}\log\mu}(Z_S^{ik} v y_S^{kj})\mu^{\varepsilon/2} + \frac{\varepsilon}{2} Z_S^{ik} v y_S^{kj} \mu^{\varepsilon/2} \,, \end{aligned} \tag{5.131}$$

where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\log\mu}(Z_S^{ik} v y_S^{kj}) = \frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu} - \frac{1}{16\pi^2}\Bigg[& 3\frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ & + 3 v y_S^{ij}\left(\frac{\mathrm{d}e^2}{\mathrm{d}\log\mu} Q_f^2 + C_F \frac{\mathrm{d}g_s^2}{\mathrm{d}\log\mu} c_f^2\right) \\ & - 12\frac{\mathrm{d}m_i}{\mathrm{d}\log\mu}\delta^{ij}(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \\ & - 12 m_i \delta^{ij}\left(\frac{\mathrm{d}(e^4 C_\gamma)}{\mathrm{d}\log\mu} Q_f^2 + C_F \frac{\mathrm{d}(g_s^4 C_g)}{\mathrm{d}\log\mu} c_f^2\right)\Bigg]\Delta_\varepsilon \,. \end{aligned} \tag{5.132}$$

From the calculations carried out in Appendix A, and in particular from Eqs. (A.40), (A.72) and (A.89), we know that

$$\frac{1}{m_i}\frac{\mathrm{d}m_i}{\mathrm{d}\log\mu} = \mathcal{O}(e^2, g_s^2) \,, \tag{5.133}$$

$$\frac{1}{g_s^2}\frac{\mathrm{d}g_s^2}{\mathrm{d}\log\mu} = -\varepsilon + \mathcal{O}(g_s^2) \,, \tag{5.134}$$

$$\frac{1}{e^2}\frac{\mathrm{d}e^2}{\mathrm{d}\log\mu} = -\varepsilon + \mathcal{O}(e^2) \,, \tag{5.135}$$

which, combined with Eqs. (5.100) and (5.116), allow us to compute the following quantities

$$\begin{aligned} \frac{\mathrm{d}(e^4 C_\gamma)}{\mathrm{d}\log\mu} &= 2e^2 \frac{\mathrm{d}e^2}{\mathrm{d}\log\mu} C_\gamma + e^4 \frac{\mathrm{d}C_\gamma}{\mathrm{d}\log\mu} \\ &= 2e^2[-\varepsilon e^2 + \mathcal{O}(e^4)]C_\gamma + e^4 \frac{\varepsilon}{2} C_\gamma \\ &= -\frac{3}{2}\varepsilon e^4 C_\gamma + \mathcal{O}(e^6 C_\gamma) \,, \end{aligned} \tag{5.136}$$

$$\begin{aligned} \frac{\mathrm{d}(g_s^4 C_g)}{\mathrm{d}\log\mu} &= 2g_s^2 \frac{\mathrm{d}g_s^2}{\mathrm{d}\log\mu} C_g + g_s^4 \frac{\mathrm{d}C_g}{\mathrm{d}\log\mu} \\ &= 2g_s^2[-\varepsilon g_s^2 + \mathcal{O}(g_s^4)]C_g + g_s^4 \frac{\varepsilon}{2} C_g \\ &= -\frac{3}{2}\varepsilon g_s^4 C_g + \mathcal{O}(g_s^6 C_g) \,. \end{aligned} \tag{5.137}$$

Inserting these expression in Eq. (5.132) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\log\mu}(Z_S^{ik} v y_S^{kj}) &= \frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu} - \frac{1}{16\pi^2}\bigg[3\frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ &\quad - 3 v y_S^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\varepsilon \\ &\quad + 18 m_i \delta^{ij}(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g)\varepsilon\bigg]\Delta_\varepsilon \\ &= \frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu}\left[1 - \frac{3}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon\right] \\ &\quad + \frac{3}{8\pi^2} v y_S^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ &\quad - \frac{9}{4\pi^2} m_i \delta^{ij}(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g)\,.\end{aligned} \tag{5.138}$$

On the other hand, from Eqs. (5.130) and (5.131) we know that this quantity is also equal to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\log\mu}(Z_S^{ik} v y_S^{kj}) &= -\frac{\varepsilon}{2} Z_S^{ik} v y_S^{kj} \\ &= -\frac{\varepsilon}{2} v y_S^{ij} + \frac{3}{16\pi^2} v y_S^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ &\quad - \frac{3}{4\pi^2} m_i \delta^{ij}(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g)\,,\end{aligned} \tag{5.139}$$

so that, comparing Eqs. (5.138) and (5.139), we can write

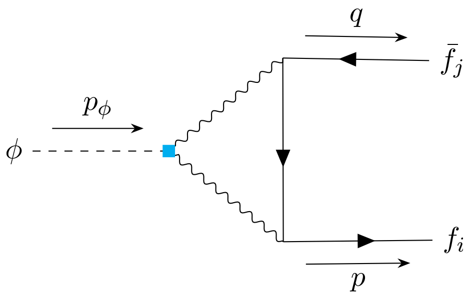
$$\begin{aligned}\frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu}\left[1 - \frac{3}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon\right] &= -\frac{\varepsilon}{2} v y_S^{ij} \\ - \frac{3}{16\pi^2} v y_S^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) + \frac{3}{2\pi^2} m_i \delta^{ij}&(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g)\end{aligned} \tag{5.140}$$

and isolate the beta function of the Wilson coefficient $v y_S^{ij}$ by expanding at lowest order in the gauge couplings and finally taking the limit $\varepsilon \to 0$:

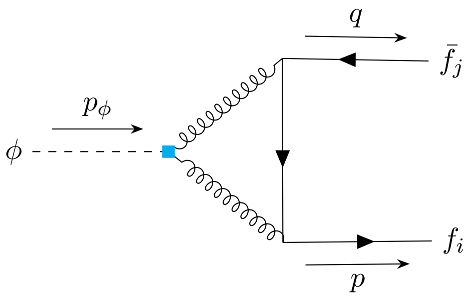
$$\begin{aligned}\frac{\mathrm{d}(v y_S^{ij})}{\mathrm{d}\log\mu} &= -\frac{\varepsilon}{2} v y_S^{ij} - \frac{3}{8\pi^2} v y_S^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ &\quad + \frac{3}{2\pi^2} m_i \delta^{ij}(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \\ &\xrightarrow{\varepsilon\to 0} -\frac{3}{8\pi^2} v y_S^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) + \frac{3}{2\pi^2} m_i \delta^{ij}(e^4 Q_f^2 C_\gamma + C_F g_s^4 c_f^2 C_g) \\ &= -\frac{3}{8\pi^2} v y_S^{ij}\left(e^2 Q_f^2 + \frac{4}{3} g_s^2 c_f^2\right) + \frac{3}{2\pi^2} m_i \delta^{ij}\left(e^4 Q_f^2 C_\gamma + \frac{4}{3} g_s^4 c_f^2 C_g\right).\end{aligned} \tag{5.141}$$

5.4.6 $i\phi\bar{f}\gamma_5 f$ anomalous dimension

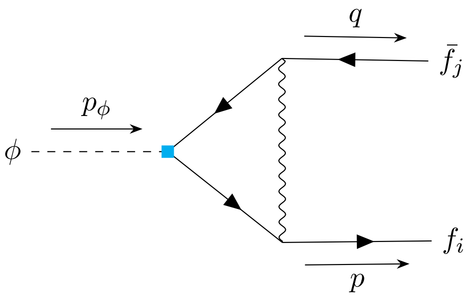
Regarding the interaction vertex between the ALP and two fermions corresponding to the Yukawa operator $i\phi\bar{f}_i\gamma_5 f_j$, the one-loop diagrams that contribute to its correction at order $1/\Lambda$ are mediated by the operators $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$, $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ and $i\phi\bar{f}_i\gamma_5 f_j$ itself and read



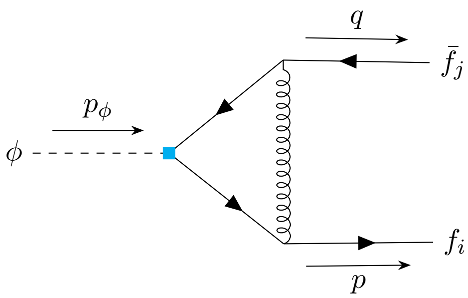
$$= i\mathcal{M}_{P\tilde{\gamma}}(p_\phi, p, q)\,, \tag{5.142}$$



$$= i\mathcal{M}_{P\tilde{g}}(p_\phi, p, q)\,, \tag{5.143}$$



$$= i\mathcal{M}^{(\gamma)}_{PP}(p_\phi, p, q)\,, \tag{5.144}$$



$$= i\mathcal{M}^{(g)}_{PP}(p_\phi, p, q)\,. \tag{5.145}$$

In the Feynman-'t Hooft gauge, their divergent parts, as explicitly computed in Appendix B, are respectively given by

$$i\mathcal{M}_{P\tilde{\gamma}}(p_\phi, p, q)|_{\text{div.}} = -\frac{3}{4\pi^2} e^4 Q_f^2 \frac{\widetilde{C}_\gamma}{\Lambda} m_i \delta^{ij} \mu^{\varepsilon/2} \bar{u}_i(p)\gamma_5 v_j(q)\Delta_\varepsilon\,, \tag{5.146}$$

$$i\mathcal{M}_{P\tilde{g}}(p_\phi, p, q)|_{\text{div.}} = -\frac{3}{4\pi^2} C_F g_s^4 c_f^2 \frac{\widetilde{C}_g}{\Lambda} m_i \delta^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) \gamma_5 v_j(q) \Delta_\varepsilon \,, \tag{5.147}$$

$$i\mathcal{M}_{PP}^{(\gamma)}(p_\phi, p, q)|_{\text{div.}} = -\frac{1}{4\pi^2} e^2 Q_f^2 \frac{v}{\Lambda} y_P^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) \gamma_5 v_j(q) \Delta_\varepsilon \,, \tag{5.148}$$

$$i\mathcal{M}_{PP}^{(g)}(p_\phi, p, q)|_{\text{div.}} = -\frac{1}{4\pi^2} C_F g_s^2 c_f^2 \frac{v}{\Lambda} y_P^{ij} \mu^{\varepsilon/2} \bar{u}_i(p) \gamma_5 v_j(q) \Delta_\varepsilon \,. \tag{5.149}$$

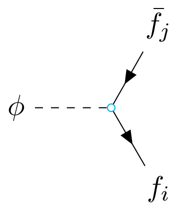
The divergence of their sum

$$\begin{aligned} i\mathcal{M}_P(p_\phi, p, q)|_{\text{div.}} = -\frac{1}{4\pi^2\Lambda}\Big[& v y_P^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ &+ 3 m_i \delta^{ij}(e^4 Q_f^2 \widetilde{C}_\gamma + C_F g_s^4 c_f^2 \widetilde{C}_g)\Big] \mu^{\varepsilon/2} \bar{u}_i(p) \gamma_5 v_j(q) \Delta_\varepsilon \end{aligned} \tag{5.150}$$

is absorbed by the counterterm

$$\mathscr{L}_\phi^{\text{ct}} \supset i(Z_P^{ik} Z_f - \delta^{ik}) \frac{v}{\Lambda} y_P^{kj} \mu^{\varepsilon/2} \phi \bar{f}_i \gamma_5 f_j \,, \tag{5.151}$$

whose Feynman rule is



$$= -(Z_P^{ik} Z_f - \delta^{ik}) \frac{v}{\Lambda} y_P^{kj} \mu^{\varepsilon/2} \gamma_5 \,. \tag{5.152}$$

From the condition

$$0 = i\mathcal{M}_P(p_\phi, p, q))|_{\text{div.}} - (Z_P^{ik} Z_f - \delta^{ik}) \frac{v}{\Lambda} y_P^{kj} \mu^{\varepsilon/2} \bar{u}_i(p) \gamma_5 v_j(q) \tag{5.153}$$

we then obtain

$$\begin{aligned} Z_P^{ik} Z_f v y_P^{kj} = v y_P^{ij} - \frac{1}{4\pi^2}\Big[& v y_P^{ij}(e^2 Q_f^2 + C_F g_s^2 c_f^2) \\ &+ 3 m_i \delta^{ij}(e^4 Q_f^2 \widetilde{C}_\gamma + C_F g_s^4 c_f^2 \widetilde{C}_g)\Big] \Delta_\varepsilon \,. \end{aligned} \tag{5.154}$$

At this point we can follow the exact same steps as the previous case, namely from Eq. (5.129) to (5.141), keeping in mind that the relative sign of the term proportional to the mass m_i is changed. Therefore, we can conclude that the beta function of the Wilson coefficient vy_P^{ij} is given by

$$\begin{aligned}\frac{\mathrm{d}(vy_P^{ij})}{\mathrm{d}\log\mu} &= -\frac{\varepsilon}{2}vy_P^{ij} - \frac{3}{8\pi^2}vy_P^{ij}(e^2Q_f^2 + C_F g_s^2 c_f^2)\\ &\quad - \frac{3}{2\pi^2}m_i\delta^{ij}(e^4Q_f^2\widetilde{C}_\gamma + C_F g_s^4 c_f^2\widetilde{C}_g)\\ &\xrightarrow{\varepsilon\to 0} -\frac{3}{8\pi^2}vy_P^{ij}(e^2Q_f^2 + C_F g_s^2 c_f^2) - \frac{3}{2\pi^2}m_i\delta^{ij}(e^4Q_f^2\widetilde{C}_\gamma + C_F g_s^4 c_f^2\widetilde{C}_g)\\ &= -\frac{3}{8\pi^2}vy_P^{ij}\left(e^2Q_f^2 + \frac{4}{3}g_s^2c_f^2\right) - \frac{3}{2\pi^2}m_i\delta^{ij}\left(e^4Q_f^2\widetilde{C}_\gamma + \frac{4}{3}g_s^4c_f^2\widetilde{C}_g\right).\end{aligned} \tag{5.155}$$

We can then summarize the results of the renormalization group running at one-loop order of the Wilson coefficients of the ALP EFT as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\begin{pmatrix}C_\gamma\\ C_g\\ vy_S^{ij}\end{pmatrix} = \gamma\begin{pmatrix}C_\gamma\\ C_g\\ vy_S^{ij}\end{pmatrix}, \quad \frac{\mathrm{d}}{\mathrm{d}\log\mu}\begin{pmatrix}\widetilde{C}_\gamma\\ \widetilde{C}_g\\ vy_P^{ij}\end{pmatrix} = \tilde{\gamma}\begin{pmatrix}\widetilde{C}_\gamma\\ \widetilde{C}_g\\ vy_P^{ij}\end{pmatrix}, \tag{5.156}$$

where the anomalous dimension matrices corresponding to the shift symmetry breaking and shift symmetry invariant sectors of the ALP EFT are respectively given by

$$\gamma = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \frac{3e^4}{2\pi^2}m_i\delta^{ij}Q_f^2 & \frac{2g_s^4}{\pi^2}m_i\delta^{ij}c_f^2 & -\frac{3}{8\pi^2}(e^2Q_f^2 + \frac{4}{3}g_s^2c_f^2)\end{pmatrix}, \tag{5.157}$$

$$\tilde{\gamma} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\frac{3e^4}{2\pi^2}m_i\delta^{ij}Q_f^2 & -\frac{2g_s^4}{\pi^2}m_i\delta^{ij}c_f^2 & -\frac{3}{8\pi^2}(e^2Q_f^2 + \frac{4}{3}g_s^2c_f^2)\end{pmatrix}. \tag{5.158}$$

Chapter 6

On-shell amplitudes

In this Chapter we will lay the foundations for the computation of the ALP RGEs through on-shell amplitude methods, which will be conducted in Chapter 7. Firstly, in Section 6.1 we introduce the spinor-helicity formalism [33, 43], which provides the natural language to deal with amplitudes of on-shell massless particles. In Section 6.3 we show how three-particle amplitudes can be constructed without any reference to the Lagrangian density and how their form is completely fixed by their symmetry properties, described in Section 6.2. Finally, in Section 6.5 we derive the BCFW recursion formula, which allows us to construct any tree-level multi-particle scattering amplitude starting from lower-point ones, bypassing the expansions in terms of Feynman diagrams.

6.1 Spinor-helicity formalism

The spinor-helicity formalism is a powerful tool for calculating scattering amplitudes in gauge theories. It is based on the idea of representing the momenta of particles in terms of spinors, which are mathematical objects that describe the intrinsic angular momentum of particles: the spin $\vec{S}$.

6.1.1 Helicity of massless particles

The projection of the spin onto the axis of the three-momentum of the particle is a quantity known as *helicity*

$$h = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|} \,. \tag{6.1}$$

If the particle is massless, then h is a Lorentz invariant quantity because the particle travels at the speed of light and no Lorentz boost can invert the direction of propagation. Additionally, for a massless spin-s particle, the helicity can only take the extremal values $h = \pm s$.

According to Wigner's classification, the scattering state of a particle is characterized by the on-shell momentum p and its helicity h: $|p, h\rangle$. In this thesis, we will only focus on particles with spin lower or equal to 1.

Spin-1/2: fermions

The Dirac equations in momentum space for the positive and negative energy bispinors are respectively given by

$$(\not{p} - m)u(p) = 0\,, \qquad\qquad (\not{p} + m)v(p) = 0\,. \tag{6.2}$$

In the massless case they coincide, since $\not{p}u(p) = \not{p}v(p) = 0$. By applying the projectors $(\mathbb{1} \pm \gamma_5)/2$, one can obtain states with definite helicity:

$$u_\pm(p) = \frac{1}{2}(\mathbb{1} \pm \gamma_5)u(p)\,, \qquad\qquad v_\mp(p) = \frac{1}{2}(\mathbb{1} \pm \gamma_5)v(p)\,, \tag{6.3}$$

and if $m = 0$ we can identify $u_\pm(p) = v_\mp(p)$. Spin-1/2 on-shell states are labeled by $|p, \pm 1/2\rangle$.

Spin-1: gauge bosons

Gauge fields carry helicities $h = \pm 1$ and are described by the polarization four-vectors $\epsilon_\mu^{(\pm)}(p)$, which obey the transversality condition

$$p \cdot \epsilon^{(\pm)}(p) = 0\,, \tag{6.4}$$

as well as the following relations

$$\epsilon^{(\pm)}(p) \cdot \epsilon^{(\pm)}(p) = 0\,, \tag{6.5}$$

$$\epsilon^{(+)}(p) \cdot \epsilon^{(-)}(p) = -1\,, \tag{6.6}$$

$$(\epsilon_\mu^{(\pm)}(p))^* = \epsilon_\mu^{(\mp)}(p)\,. \tag{6.7}$$

Spin-1 on-shell states are labeled by $|p, \pm 1\rangle$.

6.1.2 Helicity spinors

Now we can introduce the spinor-helicity formalism for the description of scattering amplitudes for massless particles. It provides an uniform description of the on-shell degrees of freedom for the scattering states of all helicities of massless particles and renders the analytic expressions of scattering amplitudes in a more compact form compared to the standard four-vector notation.

The first step consists in mapping the four-momentum of an on-shell state onto a 2×2 matrix in spinor indices

$$p^\mu = (p^0, \vec{p}) \longrightarrow p^{\dot{\alpha}\alpha} = \bar{\sigma}^{\dot{\alpha}\alpha}_\mu p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} , \tag{6.8}$$

where $\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = (\mathbb{1}, -\vec{\sigma})^{\dot{\alpha}\alpha}$ and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{6.9}$$

The transformation in Eq. (6.8) explicitly implements the local isomorphism between the complexified four-dimensional Lorentz group $SO(1,3)$ and $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})^*$. The mass-shell condition can be rewritten as a determinant condition on the matrix $p^{\dot{\alpha}\alpha}$:

$$m^2 = p^\mu p_\mu = (p^0)^2 - |\vec{p}|^2 = \det\bigl(p^{\dot{\alpha}\alpha}\bigr) . \tag{6.10}$$

Any 2×2 matrix has at most rank two and therefore can be written as $p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha + \tilde{\mu}^{\dot{\alpha}}\mu^\alpha$, where λ, μ and $\tilde{\lambda}, \tilde{\mu}$ are commuting Weyl spinors in the $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group respectively. If we now specialize to massless particles, we have

$$\det\bigl(p^{\dot{\alpha}\alpha}\bigr) = 0 , \tag{6.11}$$

meaning that $p^{\dot{\alpha}\alpha}$ has a vanishing eigenvalue (the other one is $2p^0$), thus its rank is one and its decomposition over the eigenspectrum looks as

$$p^{\dot{\alpha}\alpha} = 2p^0 \tilde{\psi}^{\dot{\alpha}}\psi^\alpha = \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha , \tag{6.12}$$

where

$$\lambda^\alpha = \sqrt{2p^0}\psi^\alpha \qquad\qquad \tilde{\lambda}^{\dot{\alpha}} = \sqrt{2p^0}\tilde{\psi}^{\dot{\alpha}} \tag{6.13}$$

are known as *helicity spinors*.

If we require the four-momentum to be real, the matrix $p^{\dot{\alpha}\alpha}$ is Hermitian and its eigenvectors satisfy

$$(\lambda^\alpha)^* = \pm\tilde{\lambda}^{\dot{\alpha}} , \tag{6.14}$$

where the sign is the same as that of the energy p^0, namely $p^0 > 0$ for outgoing particles and $p^0 < 0$ for incoming ones. An explicit realization of the helicity spinors is given by

$$\lambda^\alpha = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} p^0 + p^3 \\ p^1 + ip^2 \end{pmatrix} , \quad \tilde{\lambda}^{\dot{\alpha}} = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} p^0 + p^3 \\ p^1 - ip^2 \end{pmatrix} . \tag{6.15}$$

If instead we extend the definition of the four-momentum into the complex plane, then the spinors λ and $\tilde{\lambda}$ are independent. An important point is that the momentum bispinor in Eq. (6.12) is invariant under a little group transformation

$$\lambda \longrightarrow \rho^{-1}\lambda\,, \qquad \tilde{\lambda} \longrightarrow \rho\tilde{\lambda}\,, \tag{6.16}$$

where $\rho \in \mathbb{C}$. For real four-momentum, this transformation reduces to a phase redefinition, or a $U(1)$ transformation, since $|\rho| = 1$.

For negative momenta, we will define

$$\bar{\lambda} = \lambda\,, \qquad \bar{\tilde{\lambda}} = -\tilde{\lambda}\,, \tag{6.17}$$

in such a way that $\bar{p}^{\dot{\alpha}\alpha} = (-p)^{\dot{\alpha}\alpha} = \bar{\tilde{\lambda}}^{\dot{\alpha}}\bar{\lambda}^{\alpha}$.

Spinor indices are raised and lowered with the invariant anti-symmetric spinor metric tensors, defined as

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.18}$$

The epsilon symbols with undotted and dotted indices respectively satisfy

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = -\delta_{\alpha}^{\gamma}\delta_{\beta}^{\delta} + \delta_{\alpha}^{\delta}\delta_{\beta}^{\gamma}\,, \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\gamma}\dot{\delta}} = -\delta_{\dot{\alpha}}^{\dot{\gamma}}\delta_{\dot{\beta}}^{\dot{\delta}} + \delta_{\dot{\alpha}}^{\dot{\delta}}\delta_{\dot{\beta}}^{\dot{\gamma}}\,, \tag{6.19}$$

from which it respectively follows that

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta_{\alpha}^{\gamma}\,, \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \epsilon^{\dot{\gamma}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}\,. \tag{6.20}$$

With these conventions, in order to raise or lower an index of a spinor quantity, adjacent spinor indices are summed over when multiplied on the left by the appropriate epsilon symbol:

$$\lambda_{\alpha} = \epsilon_{\alpha\beta}\lambda^{\beta}\,, \qquad \lambda^{\alpha} = \epsilon^{\alpha\beta}\lambda_{\beta}\,, \tag{6.21}$$

$$\tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}\,, \qquad \tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}\,. \tag{6.22}$$

Moreover, if we define

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\beta} = (\mathbb{1}, \vec{\sigma})_{\alpha\dot{\alpha}}\,, \tag{6.23}$$

we can write the completeness relations as

$$\sigma_{\mu\,\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\bar{\sigma}^{\mu\,\dot{\beta}\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\,. \tag{6.24}$$

Additionally, the identity

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_{\mu} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{6.25}$$

holds and the normalization condition is given by

$$\mathrm{Tr}(\bar{\sigma}_{\mu}\sigma_{\nu}) = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\bar{\sigma}^{\dot{\beta}\beta}_{\nu} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma_{\mu\,\alpha\dot{\alpha}}\sigma_{\nu\,\beta\dot{\beta}} = 2g_{\mu\nu}\,. \tag{6.26}$$

Using these relations, it is easy to show that, given the map between Lorentz four-vectors V^{μ} and bispinors $V^{\dot{\alpha}\alpha}$

$$V^{\mu} \longrightarrow V^{\dot{\alpha}\alpha} = \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}V^{\mu} \tag{6.27}$$

as the one introduced in Eq. (6.8), its inverse is provided by

$$V^{\dot{\alpha}\alpha} \longrightarrow V^{\mu} = \frac{1}{2}\sigma^{\mu}_{\alpha\dot{\alpha}}V^{\dot{\alpha}\alpha}\,. \tag{6.28}$$

Similarly, the inverse of

$$V^{\mu} \longrightarrow V_{\alpha\dot{\alpha}} = \sigma_{\mu\,\alpha\dot{\alpha}}V^{\mu} \tag{6.29}$$

is given by

$$V_{\alpha\dot{\alpha}} \longrightarrow V^{\mu} = \frac{1}{2}\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}}\,. \tag{6.30}$$

6.1.3 Angle and square inner products

In the spinor-helicity formalism, a general scattering amplitude involving massless particles is a function of the set of helicity spinors $\{\lambda_i, \tilde{\lambda}_i\}$, with the index i running all ingoing and outgoing external legs. From now on, in order to uniformize the description, we will take all particles as outgoing, namely $p^0_i > 0$ for $i = 1, \ldots, n$.

Starting from the helicity spinors, we can construct Lorentz invariant quantities through the angle $\langle\cdot\,\cdot\rangle$ and square $[\cdot\,\cdot]$ inner products, defined as

$$\langle i\, j\rangle = \langle\lambda_i\,\lambda_j\rangle = \lambda^{\alpha}_i\lambda_{j\,\alpha} = \epsilon_{\alpha\beta}\lambda^{\alpha}_i\lambda^{\beta}_j\,, \tag{6.31}$$

$$[i\, j] = [\lambda_i\,\lambda_j] = \tilde{\lambda}_{i\,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_j = -\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\alpha}}_i\tilde{\lambda}^{\dot{\beta}}_j\,, \tag{6.32}$$

with $i, j = 1, \ldots, n$, where we have introduced the NW-SE $\searrow$ (SW-NE $\nearrow$) contraction rule for the undotted (dotted) Weyl indices. Since the helicity spinors are bosonic and the Levi-Civita tensors are antisymmetric, the angular and square brackets are antisymmetric as well under the exchange of their entries:

$$\langle i\, j\rangle = -\langle j\, i\rangle\,, \qquad\qquad [i\, j] = -[j\, i]\,. \tag{6.33}$$

As a consequence, we have that $\langle i\,i\rangle = 0 = [i\,i]$.

The Mandelstam invariants can be written in terms of these brackets in a simple way

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \lambda_i^{\alpha}\lambda_{j\,\alpha}\tilde{\lambda}_{j\,\dot{\alpha}}\tilde{\lambda}_i^{\dot{\alpha}} = \langle i\,j\rangle[j\,i] \tag{6.34}$$

and this expression can be inverted for real momenta as

$$\langle i\,j\rangle = \sqrt{|s_{ij}|}e^{i\phi_{ij}} = [j\,i]^*\,, \qquad [j\,i] = \sqrt{|s_{ij}|}e^{-i\phi_{ij}} = \langle i\,j\rangle^*\,, \tag{6.35}$$

where the phases $\phi_{ij} \in \mathbb{R}$ follow from the momenta p_i and p_j as

$$\cos\phi_{ij} = \frac{p_i^1(p_j^0 + p_j^3) - p_j^1(p_i^0 + p_i^3)}{\sqrt{|s_{ij}|(p_i^0 + p_i^3)(p_j^0 + p_j^3)}}\,, \tag{6.36}$$

$$\sin\phi_{ij} = \frac{p_i^2(p_j^0 + p_j^3) - p_j^2(p_i^0 + p_i^3)}{\sqrt{|s_{ij}|(p_i^0 + p_i^3)(p_j^0 + p_j^3)}}\,. \tag{6.37}$$

6.1.4 Useful formulae

Schouten identities

Any three two-component spinors have to be linearly dependent, so they must satisfy

$$c_1\lambda_1^{\alpha} + c_2\lambda_2^{\alpha} + c_3\lambda_3^{\alpha} = 0\,. \tag{6.38}$$

where at least one coefficient c_i is different from zero. Projecting both sides onto $\lambda_{3\,\alpha}$ and taking into account that $\langle\lambda_3\,\lambda_3\rangle = 0$ we obtain

$$c_1\,\langle\lambda_1\,\lambda_3\rangle + c_2\,\langle\lambda_2\,\lambda_3\rangle = 0\,, \tag{6.39}$$

which leads to

$$c_2 = -\frac{\langle\lambda_1\,\lambda_3\rangle}{\langle\lambda_2\,\lambda_3\rangle}c_1 = \frac{\langle\lambda_3\,\lambda_1\rangle}{\langle\lambda_2\,\lambda_3\rangle}c_1\,. \tag{6.40}$$

In a similar manner, the projection of Eq. (6.38) onto $\lambda_{2\,\alpha}$ gives

$$c_1\,\langle\lambda_1\,\lambda_2\rangle + c_3\,\langle\lambda_3\,\lambda_2\rangle = 0\,, \tag{6.41}$$

meaning that

$$c_3 = -\frac{\langle\lambda_1\,\lambda_2\rangle}{\langle\lambda_3\,\lambda_2\rangle}c_1 = \frac{\langle\lambda_1\,\lambda_2\rangle}{\langle\lambda_2\,\lambda_3\rangle}c_1\,. \tag{6.42}$$

Substituting Eqs. (6.40) and (6.42) inside Eq. (6.38) and dividing by $c_1 \neq 0$ we obtain

$$\lambda_1^\alpha \langle \lambda_2 \, \lambda_3 \rangle + \lambda_2^\alpha \langle \lambda_3 \, \lambda_1 \rangle + \lambda_3^\alpha \langle \lambda_1 \, \lambda_2 \rangle = 0 \, , \tag{6.43}$$

known as Schouten identity. The contraction with an arbitrary spinor $\lambda_{a\,\alpha}$ leads to

$$\langle 1\, 2 \rangle \langle 3\, a \rangle + \langle 2\, 3 \rangle \langle 1\, a \rangle + \langle 3\, 1 \rangle \langle 2\, a \rangle = 0 \, . \tag{6.44}$$

Clearly, the same steps can be repeated for the conjugate spinors, and the identities in Eqs. (6.43) and (6.44) respectively become

$$\tilde{\lambda}_1^{\dot\alpha} [\lambda_2 \, \lambda_3] + \tilde{\lambda}_2^{\dot\alpha} [\lambda_3 \, \lambda_1] + \tilde{\lambda}_3^{\dot\alpha} [\lambda_1 \, \lambda_2] = 0 \, , \tag{6.45}$$

$$[1\, 2][3\, a] + [2\, 3][1\, a] + [3\, 1][2\, a] = 0 \, . \tag{6.46}$$

Momentum conservation

This identity makes use of the total momentum conservation in scattering amplitudes. If all the particles involved are considered as outgoing, the total momentum conservation reads

$$\sum_{i=1}^{n} p_i^\mu = 0 \, , \tag{6.47}$$

which can be translated in terms of the helicity spinors:

$$\sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha} = 0 \, . \tag{6.48}$$

Then, if we contract the previous identity with two arbitrary spinors $\lambda_{a\,\alpha}$ and $\tilde{\lambda}_{b\,\dot\alpha}$ we obtain

$$\sum_{i=1}^{n} \langle a\, i \rangle [i\, b] = 0 \, . \tag{6.49}$$

Levi-Civita contraction

We finally quote two identities [34] that will be very useful in order to express the contraction of the Levi-Civita tensor with four null four-vectors in terms of angle and square inner products:

$$\begin{aligned} \langle a\, b \rangle [b\, c] \langle c\, d \rangle [d\, a] &= \mathrm{Tr} \left(\frac{\mathbb{1} - \gamma_5}{2} \not{p}_a \not{p}_b \not{p}_c \not{p}_d \right) \\ &= \frac{1}{2} \left(s_{ab} s_{cd} - s_{ac} s_{bd} + s_{ad} s_{bc} - 4 i \epsilon(a,b,c,d) \right) \end{aligned} \tag{6.50}$$

and similarly

$$
\begin{aligned}
[a\,b]\langle b\,c\rangle[c\,d]\langle d\,a\rangle &= \mathrm{Tr}\left(\frac{\mathbb{1}+\gamma_5}{2}\not{p}_a\not{p}_b\not{p}_c\not{p}_d\right) \\
&= \frac{1}{2}\left(s_{ab}s_{cd} - s_{ac}s_{bd} + s_{ad}s_{bc} + 4i\epsilon(a,b,c,d)\right),
\end{aligned}
\tag{6.51}
$$

where $\epsilon(a,b,c,d) = \epsilon_{\mu\nu\rho\sigma}p_a^\mu p_b^\nu p_c^\rho p_d^\sigma$.

6.1.5 Fermion polarization states

Helicity spinors solve the Dirac equation in the massless limit. If we exploit the chiral or Weyl representation of the Dirac gamma matrices, in which they take the form

$$
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\alpha} \\ \bar\sigma^{\mu\,\dot\beta\beta} & 0 \end{pmatrix}, \tag{6.52}
$$

we can write $\not{p}$ as

$$
\not{p} = p_\mu\gamma^\mu = \begin{pmatrix} 0 & p_{\alpha\dot\alpha} \\ p^{\dot\beta\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_\alpha\tilde\lambda_{\dot\alpha} \\ \tilde\lambda^{\dot\beta}\lambda^\beta & 0 \end{pmatrix}, \tag{6.53}
$$

where $p_{\alpha\dot\alpha}$ is given by the contraction of p^μ with $\sigma_{\mu\,\alpha\dot\alpha}$:

$$
p_{\alpha\dot\alpha} = p^\mu\sigma_{\mu\,\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}\,. \tag{6.54}
$$

Now, setting

$$
u_+(p) = v_-(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} = |p\rangle\,, \tag{6.55}
$$

$$
u_-(p) = v_+(p) = \begin{pmatrix} 0 \\ \tilde\lambda^{\dot\alpha} \end{pmatrix} = |p]\,, \tag{6.56}
$$

using the convenient bra-ket notation with angle $|\cdot\rangle$ and square $|\cdot]$ brackets, we note that the Dirac equations in the massless limit

$$
\not{p}\,|p\rangle = 0\,, \qquad\qquad \not{p}|p] = 0 \tag{6.57}
$$

are automatically satisfied, since $\langle\lambda\,\lambda\rangle = 0 = [\lambda\,\lambda]$.

On the other hand, concerning the Dirac adjoint of the spinor field ψ, defined as $\bar\psi = \psi^\dagger\gamma^0$, we can write

$$
\bar u_+(p) = \bar v_-(p) = \begin{pmatrix} (\lambda_\alpha)^* & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde\lambda_{\dot\alpha} \end{pmatrix} = [p|\,, \tag{6.58}
$$

$$
\bar u_-(p) = \bar v_+(p) = \begin{pmatrix} 0 & (\tilde\lambda^{\dot\alpha})^* \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} \lambda^\alpha & 0 \end{pmatrix} = \langle p|\,, \tag{6.59}
$$

where we have exploited Eq. (6.14) with the plus sign since the particles are considered as outgoing.

Using this notation, we can immediately see that

$$\langle k|\gamma^\mu|p\rangle = 0 = [k|\gamma^\mu|p]\,, \tag{6.60}$$

while

$$\langle p|\gamma^\mu|p] = \lambda^\alpha \sigma^\mu_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha} = 2p^\mu \tag{6.61}$$

and similarly

$$[p|\gamma^\mu|p\rangle = \tilde\lambda_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha}\lambda_\alpha = 2p^\mu\,. \tag{6.62}$$

More generically, for an arbitrary four-vector q^μ, we have the following relations

$$[i|\not{q}|j\rangle = \tilde\lambda_{i\,\dot\alpha}q^{\dot\alpha\alpha}\lambda_{j\,\alpha}\,, \qquad\qquad \langle j|\not{q}|i] = \lambda_j^\alpha q_{\alpha\dot\alpha}\tilde\lambda_i^{\dot\alpha}\,, \tag{6.63}$$

which imply the identity

$$[i|\gamma^\mu|j\rangle = \langle j|\gamma^\mu|i]\,. \tag{6.64}$$

Additionally, by exploiting the completeness relations of sigma matrices in Eq. (6.24), the Fierz identity takes the form

$$[i|\gamma^\mu|j\rangle\langle l|\gamma_\mu|k] = 2[i\,k]\langle l\,j\rangle\,. \tag{6.65}$$

In this way we have expressed the polarization degrees of freedom of external massless fermion states by means of the helicity spinors λ and $\tilde\lambda$.

6.1.6 Gauge boson polarization states

The gauge boson polarization vectors $\epsilon_\mu^{(\pm)}(p)$ can be expressed as bispinors through

$$\epsilon^{(\pm)}_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\epsilon^{(\pm)}_\mu\,, \tag{6.66}$$

with the negative helicity bispinor given by

$$\epsilon^{(-)}_{\alpha\dot\alpha} = \sqrt{2}\frac{\lambda_\alpha\tilde\xi_{\dot\alpha}}{[\lambda\,\xi]} \tag{6.67}$$

and the positive helicity one by

$$\epsilon^{(+)}_{\alpha\dot\alpha} = \sqrt{2}\frac{\xi_\alpha\tilde\lambda_{\dot\alpha}}{\langle\xi\,\lambda\rangle}\,. \tag{6.68}$$

Here ξ and $\tilde{\xi}$ are arbitrary reference spinors, with the only condition that they are not parallel to λ and $\tilde{\lambda}$, so that $\langle\xi\,\lambda\rangle$ and $[\xi\,\lambda]$ are different from zero. Their nature corresponds to the freedom of performing gauge transformations and, as a consequence, they will drop out of any final expression for a scattering amplitude. Indeed, if we consider an infinitesimal shift of the reference spinor $\xi \to \xi + \delta\xi$, the corresponding variation of $\epsilon^{(+)}_{\alpha\dot{\alpha}}$ is proportional to the momentum bispinor $p_{\alpha\dot{\alpha}}$:

$$\begin{aligned}\delta\epsilon^{(+)}_{\alpha\dot{\alpha}} &= \sqrt{2}\frac{\delta\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}}{\langle\xi\,\lambda\rangle} - \sqrt{2}\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}\frac{\langle\delta\xi\,\lambda\rangle}{\langle\xi\,\lambda\rangle^2}\\ &= \sqrt{2}\frac{\langle\xi\,\lambda\rangle\delta\xi_\alpha - \langle\delta\xi\,\lambda\rangle\xi_\alpha}{\langle\xi\,\lambda\rangle^2}\tilde{\lambda}_{\dot{\alpha}}\\ &= \sqrt{2}\frac{\langle\delta\xi\,\xi\rangle\lambda_\alpha}{\langle\xi\,\lambda\rangle^2}\tilde{\lambda}_{\dot{\alpha}}\\ &= \sqrt{2}\frac{\langle\delta\xi\,\xi\rangle}{\langle\xi\,\lambda\rangle^2}p_{\alpha\dot{\alpha}}\,,\end{aligned} \tag{6.69}$$

where in the second step we have used the Schouten identity in Eq. (6.43) to write $\langle\xi\,\lambda\rangle\delta\xi_\alpha - \langle\delta\xi\,\lambda\rangle\xi_\alpha$ as $\langle\delta\xi\,\xi\rangle\lambda_\alpha$. We note that the variation of $\epsilon^{(+)}_{\alpha\dot{\alpha}}$ is proportional to the associated momentum, and thus this can be understood as a gauge transformation.

Exploiting the inverse relation of Eq. (6.66)

$$\epsilon^{(\pm)}_\mu = \frac{1}{2}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\epsilon^{(\pm)}_{\alpha\dot{\alpha}}\,, \tag{6.70}$$

we can show that the properties in Eqs. (6.4), (6.5), (6.6) and (6.7) that must be satisfied by the polarization vectors are actually fulfilled. Indeed

$$\begin{aligned}p\cdot\epsilon^{(h)}(p) &= \frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha\epsilon^{(h)}_{\alpha\dot{\alpha}}\\ &= \begin{cases}\langle\lambda\,\lambda\rangle/\sqrt{2} & \text{if } h=-\\ [\lambda\,\lambda]/\sqrt{2} & \text{if } h=+\end{cases}\\ &= 0\end{aligned} \tag{6.71}$$

as a consequence of the anti-symmetry of the angular and square spinor products, while

$$\begin{aligned}\epsilon^{(+)}(p)\cdot\epsilon^{(+)}(p) &= \frac{1}{4}\bar{\sigma}_\mu^{\dot\alpha\alpha}\bar{\sigma}^{\mu\,\dot\beta\beta}\epsilon^{(+)}_{\alpha\dot\alpha}\epsilon^{(+)}_{\beta\dot\beta}\\ &= \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\epsilon^{(+)}_{\alpha\dot\alpha}\epsilon^{(+)}_{\beta\dot\beta}\\ &= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\frac{\xi_\alpha\xi_\beta\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}}{\langle\xi\,\lambda\rangle^2}\\ &= \frac{[\lambda\,\lambda]\langle\xi\,\xi\rangle}{\langle\xi\,\lambda\rangle^2}\\ &= 0\end{aligned} \tag{6.72}$$

and analogously

$$\epsilon^{(-)}(p)\cdot\epsilon^{(-)}(p) = \frac{[\xi\,\xi]\langle\lambda\,\lambda\rangle}{[\xi\,\lambda]^2} = 0 \tag{6.73}$$

follow for the same reason. On the other hand, the scalar product between polarization vectors of different helicities gives

$$\begin{aligned}\epsilon^{(+)}(p)\cdot\epsilon^{(-)}(p) &= \frac{1}{4}\bar{\sigma}_\mu^{\dot\alpha\alpha}\bar{\sigma}^{\mu\,\dot\beta\beta}\epsilon^{(+)}_{\alpha\dot\alpha}\epsilon^{(-)}_{\beta\dot\beta}\\ &= \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\epsilon^{(+)}_{\alpha\dot\alpha}\epsilon^{(-)}_{\beta\dot\beta}\\ &= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\frac{\xi_\alpha\tilde\lambda_{\dot\alpha}\lambda_\beta\tilde\xi_{\dot\beta}}{\langle\xi\,\lambda\rangle[\lambda\,\xi]}\\ &= \frac{\langle\xi\,\lambda\rangle[\xi\,\lambda]}{\langle\xi\,\lambda\rangle[\lambda\,\xi]}\\ &= -1\end{aligned} \tag{6.74}$$

as required.

For the following calculations, it is helpful to write the polarization four-vectors in terms of the Dirac matrices in the chiral representation: the positive helicity one takes the form

$$\begin{aligned}\epsilon^{(+)}_\mu(p) &= \frac{1}{2}\bar{\sigma}_\mu^{\dot\alpha\alpha}\epsilon^{(+)}_{\alpha\dot\alpha}\\ &= \frac{1}{2}\bar{\sigma}_\mu^{\dot\alpha\alpha}\sqrt{2}\frac{\xi_\alpha\tilde\lambda_{\dot\alpha}}{\langle\xi\,\lambda\rangle}\\ &= \frac{1}{\sqrt{2}}\frac{[\lambda|\gamma_\mu|\xi\rangle}{\langle\xi\,\lambda\rangle}\\ &= \frac{1}{\sqrt{2}}\frac{\langle\xi|\gamma_\mu|\lambda]}{\langle\xi\,\lambda\rangle}\,,\end{aligned} \tag{6.75}$$

while the negative helicity one can be written as

$$\begin{aligned}\epsilon_\mu^{(-)}(p) &= \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\epsilon_{\alpha\dot{\alpha}}^{(-)} \\ &= \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\sqrt{2}\frac{\lambda_\alpha\tilde{\xi}_{\dot{\alpha}}}{[\lambda\,\xi]} \\ &= -\frac{1}{\sqrt{2}}\frac{\langle\lambda|\gamma_\mu|\xi]}{[\xi\,\lambda]} \\ &= -\frac{1}{\sqrt{2}}\frac{[\xi|\gamma_\mu|\lambda\rangle}{[\xi\,\lambda]}\,,\end{aligned} \tag{6.76}$$

where he have exploited the identity in Eq. (6.64).

In this way we have expressed the polarization degrees of freedom of external massless gauge boson states by means of the helicity spinors λ and $\tilde{\lambda}$.

6.2 Symmetries of scattering amplitudes

The utility of spinor-helicity variables flows from the fact that they linearly realize the symmetries of the system, and, in particular, of scattering amplitudes.

Scattering amplitudes are defined as on-shell matrix elements of the S-matrix

$$\begin{aligned}S_{nm}(p_1,\ldots,p_n;p_1',\ldots,p_m') &= \langle p_1,\ldots,p_n|S|p_1',\ldots,p_m'\rangle \\ &= {}_{\text{out}}\langle p_1,\ldots,p_n|p_1',\ldots,p_m'\rangle_{\text{in}}\end{aligned} \tag{6.77}$$

where $|p_1',\ldots,p_m'\rangle$ and $\langle p_1,\ldots,p_n|$ are asymptotic multi-particle on-shell states, obtained by applying creation operators to the vacuum

$$|p_1',\ldots,p_m'\rangle = a_\Phi^\dagger(p_1')\ldots a_\Phi^\dagger(p_m')\,|0\rangle\;, \tag{6.78}$$

$$\langle p_1,\ldots,p_n| = \langle 0|\,a_\Phi(p_1)\ldots a_\Phi(p_n)\,, \tag{6.79}$$

and Φ denotes all fields. This $m\to n$ scattering process, illustrated in Figure 6.1, can be traded for a $0\to n+m$ process, where all the particles are regarded as outgoing, by making use of crossing symmetry, which consists of changing the sign of the momenta of the ingoing particles, and allowing them to have negative energy. Therefore, in the following, we will focus on n-particle scattering amplitudes of the form

$$i\mathscr{M}_n(1^{h_1},\ldots,n^{h_n}) = \langle 1^{h_1},\ldots,n^{h_n}|S|0\rangle \tag{6.80}$$

without loss of generality, where we have specified the helicity h_i for each particle and omitted the label p of the momenta.

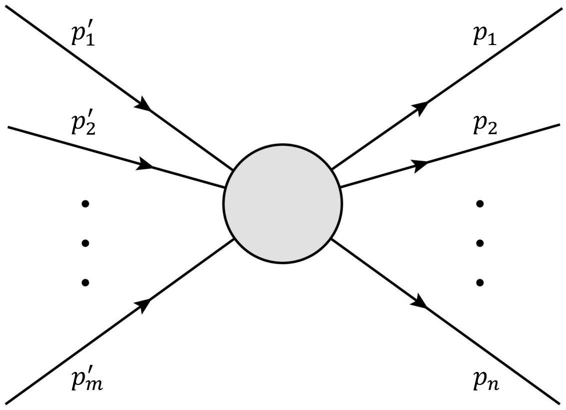


Figure 6.1: Scheme of a scattering process.

6.2.1 Helicity operator

From the Dirac equations

$$0 = \not{p}|p\rangle = p^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \lambda_\alpha = (p^0 + \vec{p} \cdot \vec{\sigma})\lambda \,, \tag{6.81}$$
$$0 = \not{p}|p] = p^\mu \sigma_{\mu\,\alpha\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} = (p^0 - \vec{p} \cdot \vec{\sigma})\tilde{\lambda} \,, \tag{6.82}$$

where we have replaced $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$, we can actually show that the helicity spinors λ and $\tilde{\lambda}$ are associated with half-integer values of the helicity. Indeed, for massless particles we can write the helicity as

$$h = \frac{\vec{p} \cdot \vec{\sigma}}{2p^0} \,, \tag{6.83}$$

from which it follows that

$$h\lambda = -\frac{1}{2}\lambda \,, \qquad\qquad h\tilde{\lambda} = +\frac{1}{2}\tilde{\lambda} \,, \tag{6.84}$$

or equivalently $h_\lambda = -1/2$ and $h_{\tilde{\lambda}} = +1/2$. Thus, the fermion states $u_+(p) = v_-(p) = |p\rangle$ and $\bar{u}_-(p) = \bar{v}_+(p) = \langle p|$ carry helicity $(-1/2)$, while $u_-(p) = v_+(p) = |p]$ and $\bar{u}_+(p) = \bar{v}_-(p) = [p|$ carry helicity $(+1/2)$.

The dependence of scattering amplitudes on λ and $\tilde{\lambda}$ enters through the wavefunctions of particles, and, as a consequence, amplitudes should be polynomials in the helicity spinors. This property can be expressed in a compact form by introducing the *helicity operator*

$$H_i = -\frac{1}{2}\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \frac{1}{2}\tilde{\lambda}_{i\,\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{i\,\dot{\alpha}}} \,, \tag{6.85}$$

which gives back the helicity of the particle i when it acts on an arbitrary polynomial in λ and $\tilde{\lambda}$. As an example, we can verify that the polarization bispinors of the gauge bosons given by Eqs. (6.67) and (6.68) are eigenvectors of the helicity operator associated with the correct eigenvalues:

$$H\epsilon^{(+)}_{\beta\dot\beta} = \left(-\frac{1}{2}\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha} + \frac{1}{2}\tilde\lambda_{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_{\dot\alpha}}\right)\sqrt{2}\frac{\xi_\beta\tilde\lambda_{\dot\beta}}{\langle\xi\,\lambda\rangle} = (+1)\epsilon^{(+)}_{\beta\dot\beta}\,, \tag{6.86}$$

$$H\epsilon^{(-)}_{\beta\dot\beta} = \left(-\frac{1}{2}\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha} + \frac{1}{2}\tilde\lambda_{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_{\dot\alpha}}\right)\sqrt{2}\frac{\lambda_\beta\tilde\xi_{\dot\beta}}{[\lambda\,\xi]} = (-1)\epsilon^{(-)}_{\beta\dot\beta}\,. \tag{6.87}$$

Therefore, the scattering amplitude of particles with momenta p_i and helicities h_i has to satisfy the homogeneity condition

$$H_i\mathscr{M}_n(1^{h_1},\ldots,n^{h_n}) = h_i\mathscr{M}_n(1^{h_1},\ldots,n^{h_n}) \tag{6.88}$$

for each $i = 1,\ldots,n$, or equivalently, under local little group transformations acting of the helicity spinors as

$$\lambda_i \longrightarrow \rho_i^{-1}\lambda_i\,, \qquad\qquad \tilde\lambda_i \longrightarrow \rho_i\tilde\lambda_i\,, \tag{6.89}$$

the amplitude should transform as

$$\mathscr{M}_n(1^{h_1},\ldots,n^{h_n}) \longrightarrow \rho_i^{2h_i}\mathscr{M}_n(1^{h_1},\ldots,n^{h_n})\,. \tag{6.90}$$

6.2.2 Angular momentum operators

The invariance under translations implies the conservation of the total momentum, so that we can write

$$i\mathscr{M}_n(1^{h_1},\ldots,n^{h_n}) = (2\pi)^4\delta^{(4)}\left(\sum_{i=1}^n p_i\right)i\mathcal{M}_n(1^{h_1},\ldots,n^{h_n})\,, \tag{6.91}$$

where $\mathcal{M}_n$ is the reduced amplitude.

On the other hand, scattering amplitudes should be invariant also under Lorentz transformations. Their action on the helicity spinors looks as

$$\delta\lambda^\alpha = \omega^{\alpha\beta}\lambda_\beta\,, \qquad\qquad \delta\tilde\lambda_{\dot\alpha} = \bar\omega_{\dot\alpha\dot\beta}\tilde\lambda^{\dot\beta}\,, \tag{6.92}$$

where ω and $\bar\omega$ are symmetric matrices. As an example, we can easily verify that the inner products $\langle i\,j\rangle$ and $[i\,j]$ are actually Lorentz invariant:

$$\delta\langle i\,j\rangle = \langle\delta i\,j\rangle + \langle i\,\delta j\rangle = \langle\delta i\,j\rangle - \langle\delta j\,i\rangle = \omega^{\alpha\beta}\lambda_{i\,\beta}\lambda_{j\,\alpha} - \omega^{\alpha\beta}\lambda_{j\,\beta}\lambda_{i\,\alpha} = 0 \tag{6.93}$$

since $\omega^{\alpha\beta} = \omega^{\beta\alpha}$, and similarly

$$\delta[i\,j] = [\delta i\,j] + [i\,\delta j] = [\delta i\,j] - [\delta j\,i] = \bar\omega_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\beta}\tilde\lambda_j^{\dot\alpha} - \bar\omega_{\dot\alpha\dot\beta}\tilde\lambda_j^{\dot\beta}\tilde\lambda_i^{\dot\alpha} = 0 \tag{6.94}$$

due to $\bar\omega_{\dot\alpha\dot\beta} = \bar\omega_{\dot\beta\dot\alpha}$.

The transformations in Eq. (6.92) are respectively generated by the differential operators $\omega^{\alpha\beta}M_{\alpha\beta}/2$ and $\bar{\omega}_{\dot{\alpha}\dot{\beta}}\bar{M}^{\bar{\alpha}\bar{\beta}}/2$, where he have defined the *angular momentum operators* as

$$M_{\alpha\beta} = \sum_{i=1}^{n}\left(\lambda_{i\,\alpha}\frac{\partial}{\partial\lambda_i^\beta} + \lambda_{i\,\beta}\frac{\partial}{\partial\lambda_i^\alpha}\right), \tag{6.95}$$

$$\bar{M}^{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n}\left(\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_{i\,\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\beta}}\frac{\partial}{\partial\tilde{\lambda}_{i\,\dot{\alpha}}}\right). \tag{6.96}$$

The invariance of scattering amplitudes under Lorentz transformations translates to the property that the reduced amplitude depends on the momenta only through the brackets $\langle i\,j\rangle$ and $[i\,j]$

$$\mathcal{M}_n = \mathcal{M}_n(\{\langle i\,j\rangle, [i\,j]\}) \tag{6.97}$$

and the condition for the particle i to have helicity h_i becomes

$$\mathcal{M}_n(\{(\rho_i\rho_j)^{-1}\langle i\,j\rangle, \rho_i\rho_j[i\,j]\}) = \left(\prod_{i=1}^{n}\rho_i^{2h_i}\right)\mathcal{M}_n(\{\langle i\,j\rangle, [i\,j]\})\,. \tag{6.98}$$

6.2.3 Dilatation operator

From the definition of the scattering amplitude

$$i\mathscr{M}_n = \langle p_1,\ldots,p_n|S|0\rangle = (2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{n}p_i\right)i\mathcal{M}_n \tag{6.99}$$

we can compute its mass dimension. Indeed, since an asymptotic one-particle on-shell state $\langle p_i| = \langle 0|\,a(p_i)$ has dimension (-1) (which can be seen from the commutation relations, *e.g.* for scalars $[a(p), a^\dagger(k)] = (2\pi)^3 2p^0\delta^{(3)}(\vec{p}-\vec{k})$), we deduce that

$$[\mathscr{M}_n] = -n\,, \tag{6.100}$$

and, as a consequence, the reduced amplitude has dimension

$$[\mathcal{M}_n] = 4 - n\,. \tag{6.101}$$

On the other hand, recalling that $p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha$, the helicity spinors have dimension

$$[\lambda] = [\tilde{\lambda}] = \frac{1}{2}[p] = \frac{1}{2}\,. \tag{6.102}$$

Therefore, under a rescaling of the spinors associated with each particle

$$\lambda_i \longrightarrow \rho^{1/2}\lambda_i\,, \qquad \tilde{\lambda}_i \longrightarrow \rho^{1/2}\tilde{\lambda}_i\,, \tag{6.103}$$

the amplitude should transform as

$$\mathcal{M}_n \longrightarrow \rho^{4-n}\mathcal{M}_n \, . \tag{6.104}$$

The *dilatation operator* generating these transformations is

$$D = \frac{1}{2}\sum_{i=1}^{n}\left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tilde{\lambda}_{i\,\dot{\alpha}}\frac{\partial}{\partial \tilde{\lambda}_{i\,\dot{\alpha}}}\right) \tag{6.105}$$

and the corresponding restriction on the amplitude takes the form

$$D\mathcal{M}_n = (4-n)\mathcal{M}_n \, . \tag{6.106}$$

6.3 Three-particle amplitudes

In this Section we show, following Ref. [13], how the constraints coming from locality, Poincaré invariance and dimensional analysis uniquely fix the form of three-particle on-shell scattering amplitudes.

We start noticing that, according to momentum conservation,

$$p_1 + p_2 + p_3 = 0 \, , \tag{6.107}$$

which implies for massless on-shell particles

$$0 = p_1^2 = s_{23} = \langle 2\,3\rangle[3\,2] \, , \tag{6.108}$$
$$0 = p_2^2 = s_{31} = \langle 3\,1\rangle[1\,3] \, , \tag{6.109}$$
$$0 = p_3^2 = s_{12} = \langle 1\,2\rangle[2\,1] \, . \tag{6.110}$$

From the last line, we deduce that if $\langle 1\,2\rangle \neq 0$, then $[1\,2] = 0$. Furthermore, applying Eq. (6.49) with $a = 1$ and $b = 3$, we find that $\langle 1\,2\rangle[2\,3] = -\langle 1\,1\rangle[1\,3] - \langle 1\,3\rangle[3\,3] = 0$, so $[2\,3] = 0$. Repeating this procedure cyclically, we conclude that $[1\,2] = [2\,3] = [3\,1] = 0$. On the other hand, assuming $[1\,2] \neq 0$ and following the same line of reasoning, we would find $\langle 1\,2\rangle = \langle 2\,3\rangle = \langle 3\,1\rangle = 0$.

To summarize, three-particle amplitudes have support on two possible kinematic configurations:

- the *holomorphic* configuration, which corresponds to the case in which all square brackets are vanishing

  $$[1\,2] = [2\,3] = [3\,1] = 0 \tag{6.111}$$

  implying $\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$;

- the *anti-holomorphic* configuration, which corresponds to the case in which all angle brackets are vanishing

$$\langle 1\,2\rangle = \langle 2\,3\rangle = \langle 3\,1\rangle = 0 \tag{6.112}$$

  implying $\lambda_1 \propto \lambda_2 \propto \lambda_3$.

Both of them require λ_i and $\tilde{\lambda}_i$ to be independent variables, meaning that $\langle i\,j\rangle \neq [j\,i]^*$ and the Mandelstam invariants are not real.

Recalling that amplitudes are polynomials in the helicity spinors, we can consider the following ansatz for the three-particle amplitude of particles with helicities h_i:

$$\mathcal{M}_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = \mathcal{M}_3^H(\langle 1\,2\rangle, \langle 2\,3\rangle, \langle 3\,1\rangle) + \mathcal{M}_3^A([1\,2], [2\,3], [3\,1])\,, \tag{6.113}$$

where

$$\mathcal{M}_3^H(\langle 1\,2\rangle, \langle 2\,3\rangle, \langle 3\,1\rangle) = g_H \langle 1\,2\rangle^a \langle 2\,3\rangle^b \langle 3\,1\rangle^c \tag{6.114}$$

$$\mathcal{M}_3^A([1\,2], [2\,3], [3\,1]) = g_A [1\,2]^{-a} [2\,3]^{-b} [3\,1]^{-c} \tag{6.115}$$

are the holomorphic and anti-holomorphic parts respectively and g_H and g_A are the corresponding coupling constants. Matching the helicity of each particle with the exponents a, b and c by means of the homogeneity condition in Eq. (6.98), we obtain a simple system of three equations in three variables

$$a + c = -2h_1\,, \tag{6.116}$$

$$a + b = -2h_2\,, \tag{6.117}$$

$$b + c = -2h_3\,, \tag{6.118}$$

whose solution is given by

$$a = h_3 - h_1 - h_2\,, \tag{6.119}$$

$$b = h_1 - h_2 - h_3\,, \tag{6.120}$$

$$c = h_2 - h_3 - h_1\,. \tag{6.121}$$

These exponents are integers, since fermions always appear in pairs. Imposing the amplitude to have the right mass dimension $[\mathcal{M}_3] = 4 - 3 = 1$ we obtain

$$1 = [g_H] + a + b + c = [g_H] - h_1 - h_2 - h_3\,, \tag{6.122}$$

$$1 = [g_A] - a - b - c = [g_H] + h_1 + h_2 + h_3\,, \tag{6.123}$$

leading to

$$[g_H] = 1 + h_1 + h_2 + h_3 = 1 + h\,, \tag{6.124}$$

$$[g_A] = 1 - h_1 - h_2 - h_3 = 1 - h\,, \tag{6.125}$$

where $h = h_1 + h_2 + h_3$. The final step consists in requiring that the three-particle amplitude has the correct physical behavior in the limit of real momenta, namely $\mathcal{M}_3$ should vanish when $\langle i\,j\rangle$ and $[i\,j]$ go to zero. It is straightforward to verify that if $a+b+c = -h > 0$, then we must set $g_A = 0$ in order to avoid an infinity, and similarly, if $a+b+c = -h < 0$, then $g_H = 0$ follows for the same reason.

To summarize, we can distinguish two cases depending on the sign of the total helicity h:

- if $h < 0$, then the three-particle amplitude is given by its holomorphic configuration

$$\mathcal{M}_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = g_H \langle 1\,2\rangle^{h_3 - h_1 - h_2} \langle 2\,3\rangle^{h_1 - h_2 - h_3} \langle 3\,1\rangle^{h_2 - h_3 - h_1}\,, \tag{6.126}$$

  where the coupling constant g_H has dimension $[g_H] = 1 + h < 1$;

- if $h > 0$, then the three-particle amplitude is given by its anti-holomorphic configuration

$$\mathcal{M}_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = g_A [1\,2]^{-h_3 + h_1 + h_2} [2\,3]^{-h_1 + h_2 + h_3} [3\,1]^{-h_2 + h_3 + h_1}\,, \tag{6.127}$$

  where the coupling constant g_A has dimension $[g_A] = 1 - h < 1$.

By noticing that the coupling constants must have dimension lower than 1 in both cases, we can ascertain the fact that the three-particle amplitude has the correct physical behaviour in the limit of real momenta is equivalent to require the theory to be local. Indeed, a three-particle amplitude with $[g] > 1$ would originate from an operator $\mathscr{O}$ with mass dimension $[\mathscr{O}] = 4 - [g] < 3$ containing three fields. If this were the case, negative powers of derivatives would be involved, ensuring the non-locality of the operator.

In the following Subsection we will explore some relevant examples, including the interactions of an ALP with SM fields.

6.3.1 Gauge interactions

The interaction between a gauge boson and two fermions is generated at the Lagrangian level by the promotion of a partial derivative ∂_μ to a covariant one D_μ inside the fermion kinetic term:

$$i\bar{f}\gamma^\mu \partial_\mu f \longrightarrow i\bar{f}\gamma^\mu D_\mu f = i\bar{f}\gamma^\mu(\partial_\mu + ieQ_f A_\mu + ig_s c_f T^a A^a_\mu) f\,. \tag{6.128}$$

Since the gauge couplings are dimensionless, the three-particle amplitudes with one gauge boson $v = \gamma, g$ and two fermions

$$i\mathcal{M}_3(1_v^{h_1}, 2_f^{h_2}, 3_{\bar{f}}^{h_3}) \tag{6.129}$$

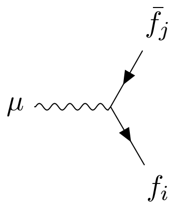
are non-vanishing if and only if $[g] = 1 \mp h = 0$, namely the sum of the three helicities h is ± 1. This, combined with the fact that $|h_1| = 1$, implies that f and $\bar{f}$ must have opposite helicities $h_2 = -h_3 = \pm 1/2$. From Eqs. (6.126) and (6.127) we then expect to find the following structures

$$i\mathcal{M}_3(1_v^{\mp}, 2_f^-, 3_{\bar{f}}^+) \propto \begin{cases} \langle 1\,2\rangle^2 \langle 2\,3\rangle^{-1} \\ [1\,3]^2 [2\,3]^{-1} \end{cases} . \tag{6.130}$$

In order to evaluate the coefficients multiplying the spinor products, we can rely on Feynman rules.

QED interaction

The Feynman rule for the QED interaction vertex is given by



$$= -ieQ_f \delta^{ij} \gamma^\mu , \tag{6.131}$$

and, in order to obtain the corresponding three-particle amplitudes, we can contract it with the photon polarization vector and the fermion spinors. Therefore, considering the holomorphic configuration, in which the photon has negative helicity, we have

$$\begin{aligned} i\mathcal{M}_3(1_\gamma^-, 2_{f_i}^-, 3_{\bar{f}_j}^+) &= -ieQ_f \delta^{ij} \bar{u}_-(p_2) \gamma^\mu v_+(p_3) \epsilon_\mu^{(-)}(p_1) \\ &= ieQ_f \delta^{ij} \langle 2|\gamma^\mu|3] \frac{1}{\sqrt{2}} \frac{[\xi|\gamma_\mu|1\rangle}{[\xi\,1]} \\ &= \sqrt{2} ieQ_f \delta^{ij} \langle 2\,1\rangle \frac{[\xi\,3]}{[\xi\,1]} . \end{aligned} \tag{6.132}$$

From the total momentum conservation we can write $[\xi\,1]\langle 1\,2\rangle + [\xi\,3]\langle 3\,2\rangle = 0$, from which follows

$$\frac{[\xi\,3]}{[\xi\,1]} = \frac{\langle 1\,2\rangle}{\langle 2\,3\rangle} , \tag{6.133}$$

leading to

$$i\mathcal{M}_3(1_\gamma^-, 2_{f_i}^-, 3_{\bar{f}_j}^+) = -\sqrt{2} ieQ_f \delta^{ij} \frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle} . \tag{6.134}$$

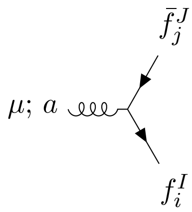
Similarly, the three-particle amplitude in the anti-holomorphic configuration, namely with positive photon helicity, reads

$$\begin{aligned} i\mathcal{M}_3(1^+_\gamma, 2^-_{f_i}, 3^+_{\bar{f}_j}) &= -ieQ_f\delta^{ij}\bar{u}_-(p_2)\gamma^\mu v_+(p_3)\epsilon^{(+)}_\mu(p_1) \\ &= -ieQ_f\delta^{ij}\langle 2|\gamma^\mu|3]\frac{1}{\sqrt{2}}\frac{\langle\xi|\gamma_\mu|1]}{\langle\xi\,1\rangle} \\ &= -\sqrt{2}ieQ_f\delta^{ij}[3\,1]\frac{\langle\xi\,2\rangle}{\langle\xi\,1\rangle} \\ &= -\sqrt{2}ieQ_f\delta^{ij}\frac{[1\,3]^2}{[2\,3]}\,, \end{aligned} \tag{6.135}$$

where in the last line $[3\,1]\langle 1\,\xi\rangle + [3\,2]\langle 2\,\xi\rangle = 0$ has been used. Clearly, both results agree with Eq. (6.130).

QCD interaction

The Feynman rule for the QCD interaction vertex is given by



$$= -ig_s c_f\delta^{ij}T^a_{IJ}\gamma^\mu\,, \tag{6.136}$$

and, in order to obtain the corresponding three-particle amplitudes, we can follow the same steps of the QED case: the spinorial structure does not change and the only difference is provided by the prefactor. Therefore, the three-particle amplitudes in the holomorphic and anti-holomorphic configurations are respectively given by

$$i\mathcal{M}_3(1^-_{g^a}, 2^-_{f^I_i}, 3^+_{\bar{f}^J_j}) = -\sqrt{2}ig_s c_f\delta^{ij}T^a_{IJ}\frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle}\,, \tag{6.137}$$

$$i\mathcal{M}_3(1^+_{g^a}, 2^-_{f^I_i}, 3^+_{\bar{f}^J_j}) = -\sqrt{2}ig_s c_f\delta^{ij}T^a_{IJ}\frac{[1\,3]^2}{[2\,3]}\,. \tag{6.138}$$

Three-gluon amplitudes

For the case of identical vectors the helicities are $h_i = \pm 1$, which imply that the exponents a, b and c are odd integers. As a consequence, the three-vector amplitude is odd under the exchange of any two external particles, and must identically vanish in order to preserve Bose symmetry. This is indeed what happens for photons: any three-photon amplitude is vanishing by charge conjugation symmetry. However, this argument does not forbid an interaction between vectors carrying different colors, as in the case of gluons.

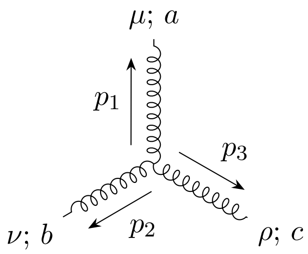
We expect the three-gluon amplitude to be proportional to g_s, since it is the only coupling constant in QCD, and, from the fact that it is dimensionless $[g_s] = 0$, we can infer that the amplitude is non-vanishing if and only if the sum of the three helicities h is equal to ± 1. Concerning the holomorphic configuration ($h = -1$), from the general formula in Eq. (6.126) we then expect the three-gluon amplitude to take the form

$$i\mathcal{M}_3(1^-_{g^a}, 2^-_{g^b}, 3^+_{g^c}) \propto g_s f^{abc} \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle}\,, \tag{6.139}$$

where the insertion of the structure constants f^{abc} is essential in order to restore Bose symmetry. Indeed, without them the amplitude would be antisymmetric under the exchange of particle 1 with particle 2. Similarly, for the anti-holomorphic configuration ($h = 1$), we expect from Eq. (6.127)

$$i\mathcal{M}_3(1^+_{g^a}, 2^+_{g^b}, 3^-_{g^c}) \propto g_s f^{abc} \frac{[1\,2]^3}{[2\,3][3\,1]}\,. \tag{6.140}$$

In order to find the exact values of the coefficients in front of the angular and square brackets, we can rely on Feynman rules. The three-gluon Feynman rule reads



$$= -g_s f^{abc}[g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu]\,, \tag{6.141}$$

which stems from the operator $-G^a_{\mu\nu}G^{a\,\mu\nu}/4$, where the convention used is $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$. In order to obtain the amplitude, we can then contract this Feynman rule with the polarization vectors $\epsilon^{(\pm)}_\mu(p_i) =$

$\epsilon^{(\pm)}_{i\,\mu}$. Therefore, for the holomorphic configuration we have

$$
\begin{aligned}
i\mathcal{M}_3(1^-_{g^a}, 2^-_{g^b}, 3^+_{g^c}) &= -g_s f^{abc}[\epsilon^{(-)}_1 \cdot \epsilon^{(-)}_2 (p_1 - p_2) \cdot \epsilon^{(+)}_3 \\
&\quad + \epsilon^{(-)}_2 \cdot \epsilon^{(+)}_3 (p_2 - p_3) \cdot \epsilon^{(-)}_1 \\
&\quad + \epsilon^{(+)}_3 \cdot \epsilon^{(-)}_1 (p_3 - p_1) \cdot \epsilon^{(-)}_2] \\
&= -g_s f^{abc}[\epsilon^{(-)}_1 \cdot \epsilon^{(-)}_2 (p_1 - p_2) \cdot \epsilon^{(+)}_3 \\
&\quad + \epsilon^{(-)}_2 \cdot \epsilon^{(+)}_3 (p_1 + 2p_2) \cdot \epsilon^{(-)}_1 \\
&\quad - \epsilon^{(+)}_3 \cdot \epsilon^{(-)}_1 (2p_1 + p_2) \cdot \epsilon^{(-)}_2] \\
&= -g_s f^{abc}[\epsilon^{(-)}_1 \cdot \epsilon^{(-)}_2 (p_1 - p_2) \cdot \epsilon^{(+)}_3 \\
&\quad + 2\epsilon^{(-)}_2 \cdot \epsilon^{(+)}_3 p_2 \cdot \epsilon^{(-)}_1 - 2\epsilon^{(+)}_3 \cdot \epsilon^{(-)}_1 p_1 \cdot \epsilon^{(-)}_2] \,,
\end{aligned}
\tag{6.142}
$$

where the momentum conservation $p_3 = -p_1 - p_2$ and the transversality conditions $p_i \cdot \epsilon^{(\pm)}_i = 0$ have been used. The calculation is more manageable if we choose the same reference momentum ξ for the three polarization vectors:

$$
\epsilon^{(-)}_1 \cdot \epsilon^{(-)}_2 = \frac{1}{2} \frac{[\xi|\gamma^\mu|1\rangle}{\langle \xi\, 1\rangle} \frac{[\xi|\gamma_\mu|2\rangle}{\langle \xi\, 2\rangle} = \frac{\langle 1\, 2\rangle [\xi\, \xi]}{\langle \xi\, 1\rangle \langle \xi\, 2\rangle} = 0\,; \tag{6.143}
$$

$$
\epsilon^{(-)}_1 \cdot \epsilon^{(+)}_3 = -\frac{1}{2} \frac{[\xi|\gamma^\mu|1\rangle}{[\xi\, 1]} \frac{\langle \xi|\gamma_\mu|3]}{\langle \xi\, 3\rangle} = -\frac{\langle 1\, \xi\rangle [3\, \xi]}{[1\, \xi] \langle 3\, \xi\rangle}\,; \tag{6.144}
$$

$$
\epsilon^{(-)}_2 \cdot \epsilon^{(+)}_3 = -\frac{1}{2} \frac{[\xi|\gamma^\mu|2\rangle}{[\xi\, 2]} \frac{\langle \xi|\gamma_\mu|3]}{\langle \xi\, 3\rangle} = -\frac{\langle 2\, \xi\rangle [3\, \xi]}{[2\, \xi] \langle 3\, \xi\rangle}\,; \tag{6.145}
$$

$$
p_2 \cdot \epsilon^{(-)}_1 = -\frac{1}{2\sqrt{2}} \langle 2|\gamma^\mu|2] \frac{[\xi|\gamma_\mu|1\rangle}{[\xi\, 1]} = \frac{1}{\sqrt{2}} \frac{\langle 1\, 2\rangle [2\, \xi]}{[1\, \xi]}\,; \tag{6.146}
$$

$$
p_1 \cdot \epsilon^{(-)}_2 = -\frac{1}{2\sqrt{2}} \langle 1|\gamma^\mu|1] \frac{[\xi|\gamma_\mu|2\rangle}{[\xi\, 2]} = -\frac{1}{\sqrt{2}} \frac{\langle 1\, 2\rangle [1\, \xi]}{[2\, \xi]}\,. \tag{6.147}
$$

Inserting this contractions in the expression for the amplitude we obtain

$$
\begin{aligned}
i\mathcal{M}_3(1^-_{g^a}, 2^-_{g^b}, 3^+_{g^c}) &= \sqrt{2} g_s f^{abc} \frac{\langle 1\, 2\rangle [3\, \xi]}{\langle 3\, \xi\rangle} \left(\frac{\langle 2\, \xi\rangle}{[1\, \xi]} + \frac{\langle 1\, \xi\rangle}{[2\, \xi]} \right) \\
&= -\sqrt{2} g_s f^{abc} \frac{\langle 1\, 2\rangle [3\, \xi]}{\langle 3\, \xi\rangle} \frac{\langle \xi\, 2\rangle [2\, \xi] + \langle \xi\, 1\rangle [1\, \xi]}{[1\, \xi][2\, \xi]} \\
&= -\sqrt{2} g_s f^{abc} \frac{\langle 1\, 2\rangle [3\, \xi]}{\langle 3\, \xi\rangle} \frac{\langle \xi|p_1 + p_2|\xi]}{[1\, \xi][2\, \xi]} \\
&= \sqrt{2} g_s f^{abc} \frac{\langle 1\, 2\rangle [3\, \xi]}{\langle 3\, \xi\rangle} \frac{\langle \xi\, 3\rangle [3\, \xi]}{[1\, \xi][2\, \xi]} \\
&= -\sqrt{2} g_s f^{abc} \langle 1\, 2\rangle \frac{[3\, \xi]}{[1\, \xi]} \frac{[3\, \xi]}{[2\, \xi]}\,.
\end{aligned}
\tag{6.148}
$$

In order to eliminate ξ, we can exploit once again the total momentum conservation and write the identities

$$[\xi\, 3]\langle 3\, 2\rangle + [\xi\, 1]\langle 1\, 2\rangle = 0\,, \qquad [\xi\, 3]\langle 3\, 1\rangle + [\xi\, 2]\langle 2\, 1\rangle = 0\,, \tag{6.149}$$

which respectively lead to

$$\frac{[3\,\xi]}{[1\,\xi]} = \frac{\langle 1\, 2\rangle}{\langle 2\, 3\rangle}\,, \qquad \frac{[3\,\xi]}{[2\,\xi]} = \frac{\langle 1\, 2\rangle}{\langle 3\, 1\rangle}\,. \tag{6.150}$$

Finally we can write the three-gluon amplitude in the holomorphic configuration as

$$i\mathcal{M}_3(1^-_{g^a}, 2^-_{g^b}, 3^+_{g^c}) = -\sqrt{2} g_s f^{abc} \frac{\langle 1\, 2\rangle^3}{\langle 2\, 3\rangle\langle 3\, 1\rangle}\,. \tag{6.151}$$

The calculation of the anti-holomorphic three-gluon amplitude is completely analogous and we omit it. It follows that

$$i\mathcal{M}_3(1^+_{g^a}, 2^+_{g^b}, 3^-_{g^c}) = \sqrt{2} g_s f^{abc} \frac{[1\, 2]^3}{[2\, 3][3\, 1]}\,. \tag{6.152}$$

These results are clearly in perfect agreement with our expectations based on symmetry arguments.

If we instead take all-plus or all-minus helicities, from Eqs. (6.126) and (6.127) we can write

$$i\mathcal{M}_3(1^-_{g^a}, 2^-_{g^b}, 3^-_{g^c}) \propto f^{abc}\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 1\rangle \tag{6.153}$$

$$i\mathcal{M}_3(1^+_{g^a}, 2^+_{g^b}, 3^+_{g^c}) \propto f^{abc}[1\, 2][2\, 3][3\, 1] \tag{6.154}$$

and by power counting we can see that these amplitudes can be solely generated by dimension-six irrelevant operators of the form $\mathrm{Tr}\, G^3$.

6.3.2 ALP interactions

Concerning the interactions between an ALP ϕ and the SM particles, it is more useful to compute form factors instead of scattering amplitudes. The reason for that will be unfolded in the next Chapter. The *form factor* associated with a local and gauge invariant operator $\mathscr{O}(x)$ is defined as the matrix element between the operator evaluated at $x = 0$ (this does not lose generality by translational invariance) and the multi-particle asymptotic on-shell state

$$F_{\mathscr{O}}(1^{h_1}, \ldots, n^{h_n}) = {}_{\text{out}}\langle 1^{h_1}, \ldots, n^{h_n}|\mathscr{O}(0)|0\rangle\,. \tag{6.155}$$

We will specialize to the case in which $F_{\mathscr{O}}$ is a *minimal* form factor, namely the lowest order form factor that does not vanish in the free theory limit.

Minimal form factors share with n-particle amplitudes the property of being polynomials in the kinematic variables, and as a consequence, for the case $n = 3$, the general formulae in Eqs. (6.126) and (6.127) are valid also for them. We then expect to find

$$F_{\mathscr{O}}(1^{h_1}, 2^{h_2}, 3^{h_3}) = \begin{cases} \kappa_H \langle 1\,2\rangle^{h_3-h_1-h_2} \langle 2\,3\rangle^{h_1-h_2-h_3} \langle 3\,1\rangle^{h_2-h_3-h_1} & \text{if } h < 0 \\ \kappa_A [1\,2]^{h_1+h_2-h_3} [2\,3]^{h_2+h_3-h_1} [3\,1]^{h_3+h_1-h_2} & \text{if } h > 0 \end{cases} \tag{6.156}$$

with $\kappa_H, \kappa_A \in \mathbb{C}$.

ϕVV operators

The operators ϕVV and $\phi V\widetilde{V}$ – with $V = F, G$ – that mediate the interaction between the ALP and the gauge bosons have mass dimension equal to 5. This means that the three-particle amplitudes

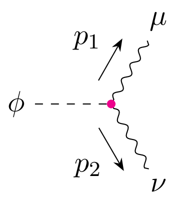
$$i\mathcal{M}_3(1_v^{h_1}, 2_v^{h_2}, 3_\phi) \tag{6.157}$$

are proportional to a Wilson coefficient $g_{\phi vv}$ with dimension $[g_{\phi vv}] = -1 = 1 \mp h$, implying $h = h_1 + h_2 = \pm 2$, since $h_\phi = 0$. By symmetry arguments and power counting, we have found that the ALP couples with two gauge bosons having the same helicities $h_1 = h_2 = \pm 1$. According to Eqs. (6.126) and (6.127), for the three-particle amplitudes the two possible cases are

$$i\mathcal{M}_3(1_v^{\mp}, 2_v^{\mp}, 3_\phi) \propto \begin{cases} g_{\phi vv} \langle 1\,2\rangle^2 \\ g_{\phi vv} [1\,2]^2 \end{cases} . \tag{6.158}$$

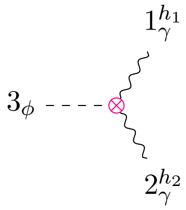
In order compute the minimal form factors associated with each operator, we can start from the Feynman rules, divide them by i times the corresponding Wilson coefficient and contract with the polarization vectors.

Regarding the operator $\phi F_{\mu\nu} F^{\mu\nu}$, its Feynman rule is provided by



$$= 4ie^2 \frac{C_\gamma}{\Lambda} (p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu}) , \tag{6.159}$$

which, once it is divided by $ie^2 C_\gamma / \Lambda$ and contracted with the photon polarization vectors, can lead to the form factor F_γ, graphically denoted as



$$= F_\gamma(1_\gamma^{h_1}, 2_\gamma^{h_2}, 3_\phi)\,. \tag{6.160}$$

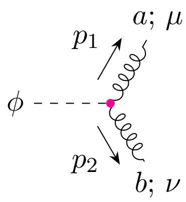
Considering the configuration with $h_1 = h_2 = -1$, we have

$$\begin{aligned} F_\gamma(1_\gamma^-, 2_\gamma^-, 3_\phi) &= \langle 1_\gamma^-, 2_\gamma^-, 3_\phi | \phi F_{\mu\nu} F^{\mu\nu} | 0\rangle \\ &= 4(p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu}) \epsilon_\mu^{(-)}(p_1) \epsilon_\nu^{(-)}(p_2) \\ &= 2(p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu}) \frac{[\xi_1|\gamma_\mu|1\rangle}{[\xi_1\, 1]} \frac{[\xi_2|\gamma_\nu|2\rangle}{[\xi_2\, 2]} \\ &= 2\frac{[\xi_1\, 2]\langle 2\, 1\rangle[\xi_2\, 1]\langle 1\, 2\rangle - \langle 1\, 2\rangle[2\, 1]\langle 1\, 2\rangle[\xi_2\, \xi_1]}{[\xi_1\, 1][\xi_2\, 2]} \\ &= -2\frac{\langle 1\, 2\rangle^2}{[\xi_1\, 1][\xi_2\, 2]}([\xi_1\, 2][\xi_2\, 1] + [2\, 1][\xi_2\, \xi_1]) \end{aligned} \tag{6.161}$$

and if we take the same reference momentum $\xi_1 = \xi_2$ the last expression reduces to:

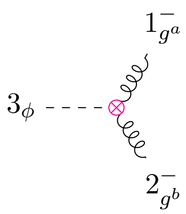
$$F_\gamma(1_{\gamma_-}, 2_{\gamma_-}, 3_\phi) = -2\langle 1\, 2\rangle^2\,. \tag{6.162}$$

We can repeat the same steps for the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$, since it is sufficient to consider the form factors with the least number of particles. The Feynman rule with two gluons reads



$$= 4 i g_s^2 \frac{C_g}{\Lambda} \delta^{ab} (p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu})\,, \tag{6.163}$$

which leads to the corresponding form factor F_g:

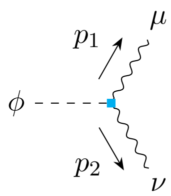


$$\begin{aligned} &= F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) \\ &= \langle 1^-_{g^a}, 2^-_{g^b}, 3_\phi | \phi G^c_{\mu\nu} G^{c\,\mu\nu} | 0\rangle \\ &= -2\delta^{ab}\langle 1\, 2\rangle^2\,. \end{aligned} \tag{6.164}$$

$\phi V\widetilde{V}$ operators

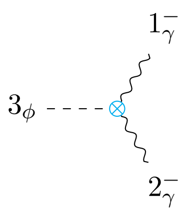
The calculation of the form factors associated with the $\phi V\widetilde{V}$ operators is not as straightforward as it was for ϕVV. The reason is that it involves the contraction between four null vectors with the Levi-Civita tensor.

The Feynman rule for the operator $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ is



$$= 4ie^2\frac{\widetilde{C}_\gamma}{\Lambda}\epsilon^{\mu\nu\rho\sigma}p_{1\,\rho}p_{2\,\sigma}\,, \tag{6.165}$$

from which the form factor $F_{\tilde{\gamma}}$ evaluated for negative photon helicities can be computed as



$$\begin{aligned} &= F_{\tilde{\gamma}}(1^-_\gamma, 2^-_\gamma, 3_\phi) \\ &= \langle 1^-_\gamma, 2^-_\gamma, 3_\phi|\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}|0\rangle \\ &= 4\epsilon^{\mu\nu\rho\sigma}\epsilon^{(-)}_\mu(p_1)\epsilon^{(-)}_\nu(p_2)p_{1\,\rho}p_{2\,\sigma}\,. \end{aligned} \tag{6.166}$$

Since the polarization vectors and the momenta are null, we can apply the identity in Eq. (6.50), which can be rewritten isolating the Levi-Civita contraction as

$$\begin{aligned} 4\epsilon(a,b,c,d) = {} & 2i\langle a\,b\rangle[b\,c]\langle c\,d\rangle[d\,a] - i\langle a\,b\rangle[b\,a]\langle c\,d\rangle[d\,c] + i\langle a\,c\rangle[c\,a]\langle b\,d\rangle[d\,b] \\ & - i\langle a\,d\rangle[d\,a]\langle b\,c\rangle[c\,b]\,, \end{aligned} \tag{6.167}$$

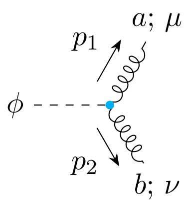
so that we can write Eq. (6.166) as

$$\begin{aligned} F_{\tilde{\gamma}}(1^-_\gamma, 2^-_\gamma, 3_\phi) = {} & 2i\frac{2\langle 1\,2\rangle[\xi_2\,1]\langle 1\,2\rangle[2\,\xi_1]}{[1\,\xi_1][2\,\xi_2]} \\ & - i\frac{2\langle 1\,2\rangle[\xi_2\,\xi_1]\langle 1\,2\rangle[2\,1]}{[1\,\xi_1][2\,\xi_2]} \\ & - i\frac{2\langle 1\,2\rangle[2\,\xi_1]\langle 2\,1\rangle[1\,\xi_2]}{[1\,\xi_1][2\,\xi_2]} \\ = {} & \frac{2i\langle 1\,2\rangle^2(2[\xi_2\,1][2\,\xi_1] - [\xi_2\,\xi_1][2\,1] + [2\,\xi_1][1\,\xi_2])}{[1\,\xi_1][2\,\xi_2]} \\ = {} & -\frac{2i\langle 1\,2\rangle^2}{[1\,\xi_1][2\,\xi_2]}([1\,\xi_2][2\,\xi_1] + [\xi_2\,\xi_1][2\,1])\,. \end{aligned} \tag{6.168}$$

Then, if we take $\xi_1 = \xi_2$ the last expression reduces to

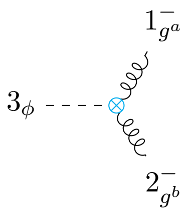
$$F_{\tilde{\gamma}}(1^-_\gamma, 2^-_\gamma, 3_\phi) = -2i\langle 1\, 2\rangle^2\,. \tag{6.169}$$

Concerning the operator $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$, we recall that its Feynman rule for the interaction with two gluons is given by



$$= 4ig_s^2\frac{\widetilde{C}_g}{\Lambda}\delta^{ab}\epsilon^{\mu\nu\rho\sigma}p_{1\,\rho}p_{2\,\sigma}\,, \tag{6.170}$$

and repeating the same steps of Eq. (6.168) we can write the corresponding form factor $F_{\tilde{g}}$ as



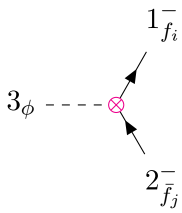
$$\begin{aligned} &= F_{\tilde{g}}(1^-_{g^a}, 2^-_{g^b}, 3_\phi) \\ &= \langle 1^-_{g^a}, 2^-_{g^b}, 3_\phi|\phi G^c_{\mu\nu}\widetilde{G}^{c\,\mu\nu}|0\rangle \\ &= -2i\delta^{ab}\langle 1\, 2\rangle^2\,. \end{aligned} \tag{6.171}$$

Yukawa operators

The Yukawa operators $\phi\bar{f}_i f_j$ and $i\phi\bar{f}_i\gamma_5 f_j$ effectively mediate the interaction between the ALP and SM fermions. They have mass dimension equal to 4, implying the corresponding three-particle amplitudes

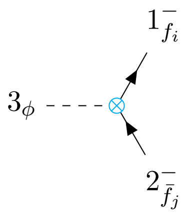
$$i\mathcal{M}_3(1^{h_1}_f, 2^{h_2}_{\bar{f}}, 3_\phi) \tag{6.172}$$

and form factors to be non-vanishing if and only if $h = h_1 + h_2 = \pm 1$, namely $h_1 = h_2 = \pm 1/2$. Not only gauge bosons, but also fermions, in order to interact with the ALP at the level of the EFT considered, must have the same helicities. The form factors can be straightforwardly computed contracting the fields with the corresponding external states. Regarding the $\phi\bar{f}_i f_j$ operator and negative helicity fermion states, we can write its form factor F_S as



$$
\begin{aligned}
&= F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) \\
&= \langle 1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi | \phi \bar{f}_i f_j | 0 \rangle \\
&= \bar{u}_-(p_1) v_-(p_2) = \begin{pmatrix} \lambda_1^\alpha & 0 \end{pmatrix} \begin{pmatrix} \lambda_{2\,\alpha} \\ 0 \end{pmatrix} \\
&= \langle 1\, 2 \rangle \, .
\end{aligned} \tag{6.173}
$$

On the other hand, the form factor F_P, associated with the operator $i\phi \bar{f}_i \gamma_5 f_j$ and computed for negative helicity fermion states, is given by



$$
\begin{aligned}
&= F_P(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) \\
&= \langle 1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi | i\phi \bar{f}_i \gamma_5 f_j | 0 \rangle \\
&= i\bar{u}_-(p_1) \gamma_5 v_-(p_2) = i \begin{pmatrix} \lambda_1^\alpha & 0 \end{pmatrix} \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \lambda_{2\,\alpha} \\ 0 \end{pmatrix} \\
&= -i \langle 1\, 2 \rangle \, ,
\end{aligned} \tag{6.174}
$$

since, in the Weyl basis, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \mathrm{diag}(-1, -1, 1, 1)$.

All the minimal form factors involving the ALP ϕ found in Eqs. (6.162), (6.164), (6.169), (6.171), (6.173) and (6.174) are in agreement with Eq. (6.156) and are summarized in Table 6.1.

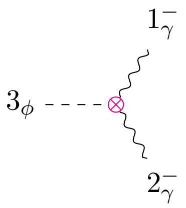
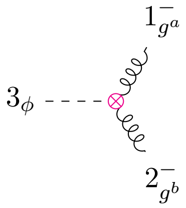
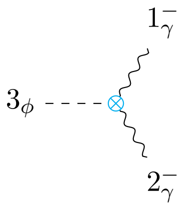
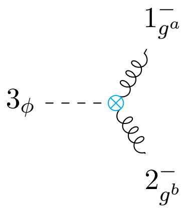
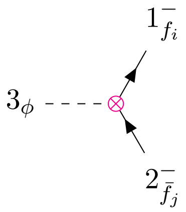
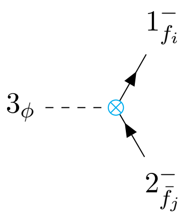
Operator	**Minimal form factor**		
$\phi F_{\mu\nu}F^{\mu\nu}$	$F_\gamma(1^-_\gamma, 2^-_\gamma, 3_\phi)$	$-2\langle 1\,2\rangle^2$	
$\phi G^c_{\mu\nu}G^{c\,\mu\nu}$	$F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi)$	$-2\langle 1\,2\rangle^2\delta^{ab}$	
$\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$	$F_{\tilde{\gamma}}(1^-_\gamma, 2^-_\gamma, 3_\phi)$	$-2i\langle 1\,2\rangle^2$	
$\phi G^c_{\mu\nu}\widetilde{G}^{c\,\mu\nu}$	$F_{\tilde{g}}(1^-_{g^a}, 2^-_{g^b}, 3_\phi)$	$-2i\langle 1\,2\rangle^2\delta^{ab}$	
$\phi\bar{f}_i f_j$	$F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi)$	$\langle 1\,2\rangle$	
$i\phi\bar{f}_i\gamma_5 f_j$	$F_P(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi)$	$-i\langle 1\,2\rangle$	

Table 6.1: Minimal form factors – with the least number of particles, taken with negative helicities – corresponding to the effective operators mediating the interaction between the ALP ϕ and the SM particles.

6.4 Color decomposition

Concerning gauge field theories, there exists a very useful technique that is able to disentangle the color and kinematical degrees of freedom in a scattering amplitude. In order apply it, it is convenient to rescale the generators T^a of the gauge group $SU(N_c)$ as

$$t^a = \sqrt{2} T^a \,. \tag{6.175}$$

In this way, these rescaled generators are normalized as

$$\mathrm{Tr}(t^a t^b) = \delta^{ab} \tag{6.176}$$

and satisfy the commutation relation

$$[t^a, t^b] = i\sqrt{2} f^{abc} t^c \,. \tag{6.177}$$

With these conventions the structure constants take the form

$$f^{abc} = -\frac{i}{\sqrt{2}} \mathrm{Tr}(t^a [t^b, t^c]) \tag{6.178}$$

and the Fierz-type identity reads

$$t^a_{IJ} t^a_{KL} = \delta_{IL}\delta_{KJ} - \frac{1}{N_c}\delta_{IJ}\delta_{KL} \,. \tag{6.179}$$

A key observation is that the color dependence of a generic diagram is given by terms involving contractions of the generator matrices t^a and the structure constants f^{abc}. Indeed, the three-gluon vertex carries one structure constant f^{abc}, the four-gluon interaction a product of two f^{abc} and the interaction between a gluon and a quark anti-quark pair comes with a generator t^a_{IJ}. Thus, we can exploit the trace formula in Eq. (6.178) to replace the dependence on the structure constants f^{abc} of the generic graph by products of the generators t^a, with open and contracted indices. Open fundamental indices correspond to quark lines in the diagram, while open adjoint indices to the external gluon states. Additionally, applying Eq. (6.179), we can perform the contractions over the adjoint indices. As a consequence, we can infer that the color dependence of any diagram can be reduced entirely to traces and strings of the generators t^a_{IJ}.

Concerning a pure-gluon tree-level amplitude, its color degrees of freedom reduce to a single-trace structure for the generator matrices and can be brought into the color-decomposed form

$$i\mathcal{M}_n(1^{h_1}_{g^{a_1}}, \dots, n^{h_n}_{g^{a_n}}) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(t^{a_{\sigma_1}} \cdots t^{a_{\sigma_n}}) iM_n((\sigma_1)^{h_{\sigma_1}}_g, \dots, (\sigma_n)^{h_{\sigma_n}}_g) \,,$$

(6.180)

where $S_n/\mathbb{Z}_n$ is the set of all non-cyclic permutations of n elements and is equivalent to S_{n-1}, while M_n is called *partial* or *color-ordered* amplitude and carry all kinematic information that is now separated from the colour degrees of freedom. It is simpler than the full amplitude, since it only receive contributions from a fixed cyclic ordering of the gluons: its poles can indeed appear only in channels of adjacent momenta $(p_i + \dots + p_{i+s})^2 \to 0$. We can consider as an example the case $n = 4$:

$$\begin{aligned} i\mathcal{M}_4^{abcd} &= i\mathcal{M}_4(1_{g^a}, 2_{g^b}, 3_{g^c}, 4_{g^d}) \\ &= iM_4(1_g, 2_g, 3_g, 4_g)\mathrm{Tr}(t^a t^b t^c t^d) + iM_4(1_g, 2_g, 4_g, 3_g)\mathrm{Tr}(t^a t^b t^d t^c) \\ &\quad + iM_4(1_g, 3_g, 2_g, 4_g)\mathrm{Tr}(t^a t^c t^b t^d) + iM_4(1_g, 3_g, 4_g, 2_g)\mathrm{Tr}(t^a t^c t^d t^b) \\ &\quad + iM_4(1_g, 4_g, 2_g, 3_g)\mathrm{Tr}(t^a t^d t^b t^c) + iM_4(1_g, 4_g, 3_g, 2_g)\mathrm{Tr}(t^a t^d t^c t^b)\,. \end{aligned} \tag{6.181}$$

Regarding QCD amplitudes with one fermion line at tree-level, they take the form

$$\begin{aligned} i\mathcal{M}_{n+2}(1^{h_1}_{g^{a_1}}, \dots, n^{h_n}_{g^{a_n}}, (n+1)^{h_{n+1}}_{fI}, (n+2)^{h_{n+2}}_{\bar{f}J}) &= \sum_{\sigma \in S_n} (t^{a_{\sigma_1}} \cdots t^{a_{\sigma_n}})_{IJ} \\ \times\, iM_{n+2}((\sigma_1)^{h_{\sigma_1}}_g, \dots, (\sigma_n)^{h_{\sigma_n}}_g, (n+1)^{h_{n+1}}_f, (n+2)^{h_{n+2}}_{\bar{f}})\,, \end{aligned} \tag{6.182}$$

which, for the case $n = 2$, reads

$$\begin{aligned} i\mathcal{M}_4(1_{g^a}, 2_{g^b}, 3_{f^I}, 4_{\bar{f}^J}) = iM_4(1_g, 2_g, 3_f, 4_{\bar{f}})(t^a t^b)_{IJ} \\ + iM_4(2_g, 1_g, 3_f, 4_{\bar{f}})(t^b t^a)_{IJ}\,. \end{aligned} \tag{6.183}$$

Color-ordered amplitudes are gauge invariant and enjoy general properties which considerably reduce the number of independent amplitudes.

Cyclicity:

$$iM_n(1_g, 2_g, \dots, n_g) = iM_n(2_g, \dots, n_g, 1_g) \tag{6.184}$$

directly follows from the definition of the color-ordered amplitude in Eq. (6.180).

Parity:

$$[iM_n(1^{h_1}_g, \dots, n^{h_n}_g)]^* = iM_n(1^{-h_1}_g, \dots, n^{-h_n}_g)\,. \tag{6.185}$$

Charge conjugation:

$$iM_n(1_f, 2_{\bar{f}}, 3_g, \dots, n_g) = -iM_n(1_{\bar{f}}, 2_f, 3_g, \dots, n_g)\,, \tag{6.186}$$

namely flipping the helicity on a fermion line changes the amplitude up to a sign.

Reflection:

$$iM_n(1_g, 2_g, \dots, n_g) = (-1)^n iM_n(n_g, (n-1)_g, \dots, 1_g) \qquad (6.187)$$

follows from the anti-symmetry of color-ordered gluon vertices.

Photon decoupling:

$$\sum_{\sigma \in \mathbb{Z}_{n-1}} iM_n((\sigma_1)_g, \dots, (\sigma_{n-1})_g, n_g) = 0\,, \qquad (6.188)$$

where the elements $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ are cyclic permutations of the set $\{1, 2, \dots, n-1\}$. It follows from the fact that pure-gluon amplitudes in the $U(N_c)$ theory containing a $U(1)$ photon must vanish since $f^{0ab} = 0$, where we have defined the $U(1)$ generator $t^0_{IJ} = \delta_{IJ}/\sqrt{N_c}$.

6.5 BCFW recursion relation

The Britto-Cachazo-Feng-Witten (BCFW) recursion relation [19, 20] provides a systematic way to compute scattering amplitudes by recursively relating them to simpler ones. Its key idea is to use the power of complex analysis, in particular Cauchy's integral theorem and residue theorem, and exploit the analytic properties of on-shell scattering amplitudes. In particular, they are analytic functions of the kinematic variables and it should be possible to reconstruct them knowing their behavior in singular limiting kinematics.

We start considering a tree-level and color-ordered n-particle amplitude $iM_n(1, \dots, n)$ and deforming the helicity spinors of the particles i and n:

$$\lambda_1 \longrightarrow \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n\,, \qquad \tilde{\lambda}_1 \longrightarrow \tilde{\lambda}_1\,, \qquad (6.189)$$

$$\lambda_n \longrightarrow \lambda_n\,, \qquad \tilde{\lambda}_n \longrightarrow \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1\,, \qquad (6.190)$$

where $z \in \mathbb{C}$. This transformation is known as $[n, 1\rangle$ shift. Correspondingly, the two momenta are deformed as

$$p_1^{\dot{\alpha}\alpha} \longrightarrow \hat{p}_1^{\dot{\alpha}\alpha}(z) = \tilde{\lambda}_1^{\dot{\alpha}}(\lambda_1 - z\lambda_n)^{\alpha}\,, \qquad (6.191)$$

$$p_n^{\dot{\alpha}\alpha} \longrightarrow \hat{p}_n^{\dot{\alpha}\alpha}(z) = (\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}}\lambda_n^{\alpha}\,, \qquad (6.192)$$

but the total momentum conservation is not spoiled since $\hat{p}_1^{\dot{\alpha}\alpha}(z) + \hat{p}_n^{\dot{\alpha}\alpha}(z) = p_1^{\dot{\alpha}\alpha} + p_n^{\dot{\alpha}\alpha}$, and the on-shell conditions still hold: $\hat{p}_1^2(z) = \hat{p}_n^2(z) = 0$. The amplitude that is function of these shifted momenta is called deformed amplitude

$$i\hat{M}_n(z) = iM_n(\hat{1}(z), 2, \dots, n-1, \hat{n}(z)) \qquad (6.193)$$

and it is the analytic continuation of the original one, which can be found setting $z = 0$: $iM_n = i\hat{M}_n(0)$.

Cauchy's integral theorem states that if we consider a function $f = f(x)$ defined on the real axis that admits an analytical continuation to the complex plane and that is holomorphic in x, then $f(x)$ can be computed as

$$f(x) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} \frac{f(z)}{z - x} , \qquad (6.194)$$

where γ is an infinitesimal circumference around x. Blowing up the integration contour, as illustrated in Figure 6.2, we can write

$$f(x) = -\sum_k \operatorname*{Res}_{z=z_k} \frac{f(z)}{z - x} + \oint_{|z|=\infty} \frac{\mathrm{d}z}{2\pi i} \frac{f(z)}{z - x} \qquad (6.195)$$

due to residue theorem, where z_k are the poles of $f(z)$. Additionally, if $f(z)$

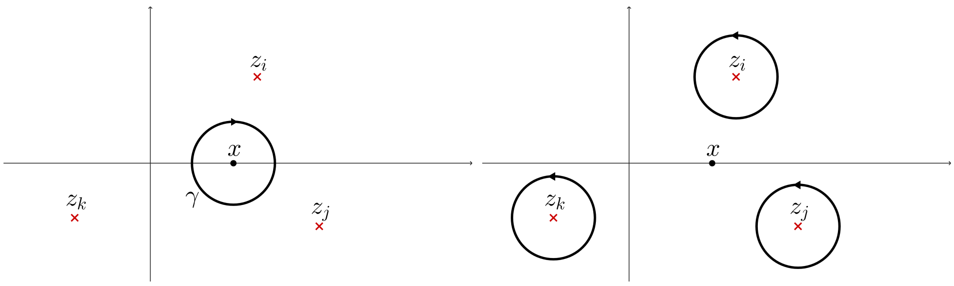


Figure 6.2: Deformation of the integration contour γ.

goes to zero fast enough for $|z| \to \infty$ so that the residue of $f(z)$ at infinity is null, $f(x)$ can be evaluated just by knowing the behaviour of its analytical continuation around the singularities:

$$f(x) = -\sum_k \operatorname*{Res}_{z=z_k} \frac{f(z)}{z - x} . \qquad (6.196)$$

This result can be applied to the n-particle amplitude, provided that it is computed at tree-level. Indeed, at tree-level the amplitude iM_n is singular when a multi-particle channel corresponding to the Feynman propagator i/P_i^2 goes on-shell, namely when $P_i^2 = 0$, where $P_i = p_1 + \cdots + p_i$ is the sum of adjacent momenta, since the amplitude is color-ordered, and $i = 2, \ldots, n-2$. Therefore, concerning the deformed amplitude $i\hat{M}_n(z)$, it has $n - 3$ isolated poles located at $z = z_i$, which can be found imposing $\hat{P}_i^2(z_i) = 0$, where

$$\hat{P}_i(z) = \hat{p}_1(z) + p_2 + \cdots + p_i , \qquad (6.197)$$

$$\hat{P}_i^2(z) = P_i^2 - z\langle n|P_i|1] = -\langle n|P_i|1](z - z_i) . \qquad (6.198)$$

Thus, the $n-3$ poles of $i\hat{M}_n(z)$ are located at

$$z_i = \frac{P_i^2}{\langle n|P_i|1]} \,. \tag{6.199}$$

We also need to know what the residues at the poles are. As we approach a pole z_i, which is simple since $\hat{P}_i^2(z)$ is linear in z, the deformed amplitude $i\hat{M}_n(z)$ factorizes into a product of two causally disconnected amplitudes with fewer legs, since the intermediate state that connects them is on its mass shell and can therefore propagate an arbitrary distance:

$$\begin{aligned} i\hat{M}_n(z) \sim \frac{1}{z-z_i}\frac{-i}{\langle n|P_i|1]} \sum_h iM_{i+1}(\hat{1}(z_i), 2, \ldots, i, -\hat{P}_i^{-h}(z_i)) \\ \times\, iM_{n-i+1}(\hat{P}_i^h(z_i), i+1, \ldots, n-1, \hat{n}(z_i))\,, \end{aligned} \tag{6.200}$$

where h runs over all possible helicity states propagating between the amplitude on the left side $iM_{i+1}(\hat{1}(z_i), 2, \ldots, i, -\hat{P}_i^{-h}(z_i))$ and the one on the right side $iM_{n-i+1}(\hat{P}_i^h(z_i), i+1, \ldots, n-1, \hat{n}(z_i))$. This is schematically represented in Figure 6.3. Therefore, the residues can be computed as

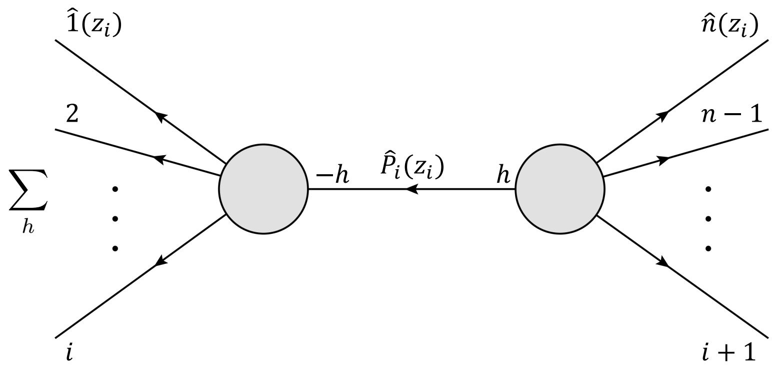


Figure 6.3: Factorization of the deformed color-ordered amplitude $i\hat{M}_n(z)$ near the pole z_i.

$$\begin{aligned} \operatorname*{Res}_{z=z_i} i\hat{M}_n(z) &= \lim_{z\to z_i}(z-z_i) i\hat{M}_n(z) \\ &= \frac{-i}{\langle n|P_i|1]} \sum_h iM_{i+1}(\hat{1}(z_i), 2, \ldots, i, -\hat{P}_i^{-h}(z_i)) \\ &\quad \times\, iM_{n-i+1}(\hat{P}_i^h(z_i), i+1, \ldots, n-1, \hat{n}(z_i)) \end{aligned} \tag{6.201}$$

and, if we assume that $i\hat{M}_n(z) \to 0$ as $|z| \to \infty$ so that the residue at infinity is zero

$$\oint_{|z|=\infty} \frac{\mathrm{d}z}{2\pi i} \frac{i\hat{M}_n(z)}{z} = 0 \tag{6.202}$$

(in which case the shift is said to be valid or good), we can apply Eq. (6.196) to reconstruct the original amplitude as

$$iM_n(1,\ldots,n) = i\hat{M}_n(0) = -\sum_{i=2}^{n-2} \frac{1}{z_i} \operatorname*{Res}_{z=z_i} i\hat{M}_n(z)\,. \tag{6.203}$$

Finally, plugging in Eqs. (6.199) and (6.201), we can conclude that

$$\begin{aligned} iM_n(1,\ldots,n) = \sum_{i=2}^{n-2}\sum_h & iM_{i+1}(\hat{1}(z_i), 2, \ldots, i, -\hat{P}_i^{-h}(z_i)) \frac{i}{P_i^2} \\ & \times iM_{n-i+1}(\hat{P}_i^h(z_i), i+1, \ldots, n-1, \hat{n}(z_i))\,. \end{aligned} \tag{6.204}$$

This is the *BCFW recursion formula.*

This relation is valid in any spacetime dimension d and is constructive, since the amplitudes appearing on the right hand side have lower multiplicity than iM_n. Hence, with the seed three-particle amplitudes we can use this formula to construct all n-particle amplitudes at tree-level without using Feynman diagrams.

6.5.1 Example: Parke-Taylor formula

A straightforward application of the BCFW recursion formula concerns the calculation of the tree-level scattering amplitude involving n gluons. The three-gluon color-ordered amplitudes in the holomorphic and anti-holomorphic configuration can be respectively derived from Eqs. (6.151) and (6.152)

$$iM_3(1_g^-, 2_g^-, 3_g^+) = ig_s \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle}\,, \tag{6.205}$$

$$iM_3(1_g^+, 2_g^+, 3_g^-) = -ig_s \frac{[1\,2]^3}{[2\,3][3\,1]}\,. \tag{6.206}$$

These two are the only non-vanishing amplitudes at tree-level for $n = 3$.

A general n-gluon tree amplitude, with $n > 3$, must depend on the n polarization vectors involved $\epsilon_{i\,\mu} = \epsilon_\mu(p_i)$, which have to be contracted with themselves $(\epsilon_i \cdot \epsilon_j)$ or with the external momenta $(\epsilon_i \cdot p_j)$. Moreover, in general it can be written as a sum of terms, each of which contains at least one one polarization contraction $\epsilon_i \cdot \epsilon_j$. This implies that if the gluons have all the same helicity, all the terms are proportional to $\epsilon_i^{(\pm)} \cdot \epsilon_j^{(\pm)}$, which is zero if they share the same reference momentum ($\xi_i = \xi_j$)

$$\epsilon_i^{(\pm)} \cdot \epsilon_j^{(\pm)} = \begin{cases} \langle \xi_i\, \xi_j \rangle [j\, i] \langle i\, \xi_i \rangle^{-1} \langle j\, \xi_j \rangle^{-1} \\ \langle i\, j \rangle [\xi_j\, \xi_i][i\, \xi_i]^{-1} [j\, \xi_j]^{-1} \end{cases} = 0\,, \tag{6.207}$$

leading to

$$iM_n(1_g^{\pm}, \ldots, n_g^{\pm}) = 0\,. \tag{6.208}$$

Similarly, the gluon tree-level amplitude with one flipped helicity state vanishes

$$iM_n(1_g^{\mp}, 2_g^{\pm}, \ldots, n_g^{\pm}) = 0 \tag{6.209}$$

by choosing $\xi_1 \neq p_1$ and $\xi_2 = \cdots = \xi_n = p_1$. Indeed, in this way all the terms containing a contraction $\epsilon_i^{(\pm)} \cdot \epsilon_j^{(\pm)} = 0$, with $i, j = 2, \ldots, n$, vanish, and at the same time

$$\epsilon_i^{(\pm)} \cdot \epsilon_1^{(\mp)} = \begin{cases} \langle 1\, \xi_i \rangle [\xi_1\, i] \langle i\, \xi_i \rangle^{-1} [1\, \xi_1]^{-1} \\ \langle i\, \xi_1 \rangle [\xi_i\, 1] \langle 1\, \xi_1 \rangle^{-1} [i\, \xi_i]^{-1} \end{cases} = 0 \tag{6.210}$$

for each $i = 2, \ldots, n$, since $\xi_i = p_1$.

Therefore, the first non-vanishing pure-gluon tree-level amplitudes are the ones with two flipped helicities, of the form

$$iM_n(1_g^-, 2_g^+, \ldots, (n-1)_g^+, n_g^-)\,, \tag{6.211}$$

which are known as *maximally helicity violating* (MHV) gluon amplitudes. Eq. (6.205) is the first of this family of amplitudes, and, for any n, they are expressed through the *Parke-Taylor formula* [60]

$$iM_n(1_g^-, 2_g^+, \ldots, (n-1)_g^+, n_g^-) = ig^{n-2} \frac{\langle 1\, n \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle \langle n\, 1 \rangle}\,. \tag{6.212}$$

We can prove it inductively applying the BCFW recursion formula, since it is already valid for $n = 3$. The $[n, 1\rangle$ shift, which reads

$$\lambda_1 \longrightarrow \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n\,, \qquad \tilde{\lambda}_1 \longrightarrow \tilde{\lambda}_1\,, \tag{6.213}$$

$$\lambda_n \longrightarrow \lambda_n\,, \qquad \tilde{\lambda}_n \longrightarrow \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1\,, \tag{6.214}$$

can be proven to yield a deformed color-ordered amplitude $i\hat{M}_n(z)$ that has a large-z falloff as z^{-1}, thus the residue at infinity is zero and the shift is valid. This holds since the gluons 1 and n are in the same helicity configuration ($h_1 = h_n$). If the two helicities were different ($h_1 \neq h_n$), we would have

$$i\hat{M}_n(z) \sim \begin{cases} z^{-1} & \text{if } h_1 = +1,\, h_n = -1 \\ z^3 & \text{if } h_1 = -1,\, h_n = +1 \end{cases} \tag{6.215}$$

as $|z| \to \infty$, so that for the second case the shift would be forbidden. The MHV amplitude has only one factorization channel, which is given, keeping in mind Figure 6.3, by a left hand side amplitude

$$iM_3(\hat{1}_g^-(z), 2_g^+, -\hat{P}_g^+(z)) \tag{6.216}$$

and a right hand side amplitude

$$iM_{n-1}(\hat{P}_g^-(z), 3_g^+, \ldots, (n-1)_g^+, \hat{n}_g^-(z)), \tag{6.217}$$

where $\hat{P}(z) = \hat{p}_1(z) + p_2$. The first one is provided by Eq. (6.206) by applying the cyclicity property

$$\begin{aligned} iM_3(\hat{1}_g^-(z), 2_g^+, -\hat{P}_g^+(z)) &= iM_3(2_g^+, -\hat{P}_g^+(z), \hat{1}_g^-(z)) \\ &= -ig_s \frac{[2\,(-\hat{P}(z))]^3}{[\hat{1}(z)\,2][(-\hat{P}(z))\,\hat{1}(z)]}, \end{aligned} \tag{6.218}$$

while the second one is a $(n-1)$-gluon MHV amplitude, which, by induction hypothesis, reads

$$\begin{aligned} iM_{n-1}(\hat{P}_g^-(z), 3_g^+, \ldots, (n-1)_g^+, \hat{n}_g^-(z)) &= ig_s^{n-3} \\ &\times \frac{\langle \hat{n}(z)\,\hat{P}(z)\rangle^3}{\langle \hat{P}(z)\,3\rangle\langle 3\,4\rangle \cdots \langle (n-1)\,\hat{n}(z)\rangle}. \end{aligned} \tag{6.219}$$

The pole associated with this factorization is located at

$$z_P = \frac{(p_1+p_2)^2}{\langle n|(p_1+p_2)|1]} = \frac{\langle 1\,2\rangle[2\,1]}{\langle n\,2\rangle[2\,1]} = \frac{\langle 1\,2\rangle}{\langle n\,2\rangle}, \tag{6.220}$$

and the following relations hold

$$[\hat{1}(z)\,*] = [1\,*], \tag{6.221}$$

$$\langle \hat{n}(z)\,*\rangle = \langle n\,*\rangle, \tag{6.222}$$

$$\langle n\,\hat{P}(z)\rangle[\hat{P}(z)\,2] = \langle n\,\hat{1}(z)\rangle[1\,2] = \langle n\,1\rangle[1\,2], \tag{6.223}$$

$$\langle 3\,\hat{P}(z)\rangle[\hat{P}(z)\,1] = \langle 3\,2\rangle[2\,1]. \tag{6.224}$$

Therefore, exploiting the BCFW recursion formula, we obtain

$$\begin{aligned}
&iM_n(1_g^-, 2_g^+, \ldots, (n-1)_g^+, n_g^-) \\
&= iM_3(\hat{1}_g^-(z_P), 2_g^+, -\hat{P}_g^+(z_P))\frac{i}{s_{12}} iM_{n-1}(\hat{P}_g^-(z_P), 3_g^+, \ldots, (n-1)_g^+, \hat{n}_g^-(z_P)) \\
&= -ig_s^{n-2}\frac{[2\,\hat{P}(z_P)]^3}{[\hat{1}(z_P)\,2][\hat{P}(z_P)\,\hat{1}(z_P)]}\frac{1}{\langle 1\,2\rangle[2\,1]}\frac{\langle \hat{n}(z_P)\,\hat{P}(z_P)\rangle^3}{\langle \hat{P}(z_P)\,3\rangle\langle 3\,4\rangle\cdots\langle (n-1)\,\hat{n}(z_P)\rangle} \\
&= ig_s^{n-2}\frac{\langle n\,1\rangle^3[1\,2]^3}{[1\,2]\langle 3\,2\rangle[1\,2]\langle 1\,2\rangle[2\,1]\langle 3\,4\rangle\cdots\langle (n-1)\,n\rangle} \\
&= -ig_s^{n-2}\frac{\langle n\,1\rangle^3}{\langle 3\,2\rangle\langle 1\,2\rangle\langle 3\,4\rangle\cdots\langle (n-1)\,n\rangle} \\
&= ig_s^{n-2}\frac{\langle n\,1\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\langle n\,1\rangle},
\end{aligned} \tag{6.225}$$

which is exactly the Parke-Taylor formula. It can be straightforwardly generalized to the case in which the only two gluons having negative helicities are labelled by i and j simply by changing the numerator as

$$iM_n(1_g^+, \ldots, i_g^-, \ldots, j_g^-, \ldots, n_g^+) = ig_s^{n-2}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\langle n\,1\rangle}. \tag{6.226}$$

However, for the following calculations we will only need the four-gluon MHV amplitude, which reads

$$iM_4(1_g^-, 2_g^-, 3_g^+, 4_g^+) = ig_s^2\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} \tag{6.227}$$

and can be used to derive the full colored amplitude through Eq. (6.181):

$$\begin{aligned}
i\mathcal{M}_4(1_{g^a}^-, 2_{g^b}^-, 3_{g^c}^+, 4_{g^d}^+) = -2ig_s^2\langle 1\,2\rangle^4\Bigg[&\frac{f^{abe}f^{cde}}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} \\
&+\frac{f^{ace}f^{bde}}{\langle 1\,3\rangle\langle 3\,2\rangle\langle 2\,4\rangle\langle 4\,1\rangle}\Bigg].
\end{aligned} \tag{6.228}$$

Chapter 7

Renormalization of ALP EFT via on-shell amplitudes

Arrived at this point, we have introduced the spinor-helicity formalism, described the symmetries of scattering amplitudes and exploited them to compute the relevant three-particle amplitudes, as well as the minimal form factors involving the ALP. Additionally, we have glimpsed the true power of on-shell methods, which allow us to recursively obtain tree-level amplitudes starting from lower-point ones. We now want to know how to calculate the anomalous dimension matrix of composite operators within this framework without performing loop integrals, and ultimately reproduce the RGEs of the ALP EFT obtained with the standard Feynman diagrammatic approach in Chapter 5.

The anomalous dimension of an operator can be computed using a number of different methods. The method outlined in this Chapter has been proposed by Caron-Huot and Wilhelm [22] and adopted by Miró, Ingoldby and Riembau [36] for the calculation of the anomalous dimensions of SM EFT dimension-six operators.

The central role is played by the form factors $F_{\mathscr{O}}$ previously defined in Eq. (6.155) and that can be written in a more compact notation as

$$F_{\mathscr{O}}(\vec{n}) = {}_{\text{out}}\langle \vec{n}|\mathscr{O}(0)|0\rangle \,, \tag{7.1}$$

where $\langle \vec{n}| = \langle 1, \ldots, n|$ denotes a n-particle asymptotic on-shell state. Their calculation in perturbation theory involves the regularization of both ultraviolet (UV) and infrared (IR) divergences. Thus, in dimensional regularization they acquire a dependence on the 't Hooft scale μ and satisfy the Callan-Symanzik equation

$$\left(\frac{\partial}{\partial \log \mu} + \gamma - \gamma_{\text{IR}} + \beta_g \frac{\partial}{\partial g}\right) F_{\mathscr{O}}(\vec{n}; \mu) = 0 \,. \tag{7.2}$$

Here, γ is the anomalous dimension matrix, $\gamma_{\rm IR}$ is the IR anomalous dimension and β_g collectively denotes the beta functions of the couplings in the theory. Additionally, form factors are closely related to the S-matrix elements

$$S_{nm} = \langle \vec{n}|S|\vec{m}\rangle = {}_{\rm out}\langle \vec{n}|\vec{m}\rangle_{\rm in} \tag{7.3}$$

through an equation that will be derived following Ref. [36].

7.1 S-matrix and dilatation operator

By Lorentz invariance, form factors must depend on the Mandelstam invariants $s_{ij} = 2p_i \cdot p_j = \langle i\, j\rangle[j\, i]$, *i.e.* $F_{\mathcal{O}} = F_{\mathcal{O}}(\{s_{ij} + i\epsilon\})$, and are not real because the Feynman prescription adds the small positive imaginary part given by $i\epsilon$. However, they can be related to their complex conjugates in two different ways.

- The first one is based on the analyticity of form factors: the complex conjugation of $F_{\mathcal{O}}$ amounts to complex conjugating the time-ordered propagators, with denominators of the form $s_{ij} + i\epsilon$, into anti-time-ordered propagators, whose denominators are given by $s_{ij} - i\epsilon$. Thus, under complex conjugation, $\{s_{ij} + i\epsilon\}$ are effectively replaced by $\{s_{ij} - i\epsilon\}$, and the analyticity relation reads

$$F^*_{\mathcal{O}}(\{s_{ij} - i\epsilon\}) = F_{\mathcal{O}}(\{s_{ij} + i\epsilon\})\,. \tag{7.4}$$

- The second one consists in analytically continuing all the momenta p_i to the complex plane and rotate them along a counter-clockwise circle with a common phase α:

$$p_i \longrightarrow e^{i\alpha} p_i\,. \tag{7.5}$$

  Correspondingly, the Mandelstam invariants are rotated by 2α and the form factors are transformed as

$$F_{\mathcal{O}} \longrightarrow e^{i\alpha D} F_{\mathcal{O}}\,, \tag{7.6}$$

  where D is the dilatation operator, which is the generator of the transformation and is defined as

$$D = \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu}\,. \tag{7.7}$$

  For $\alpha = \pi$, as illustrated in Figure 7.1, the Mandelstam invariants are back to their original values, but with the opposite imaginary part. Therefore, the change of sign of all outgoing momenta leads to the

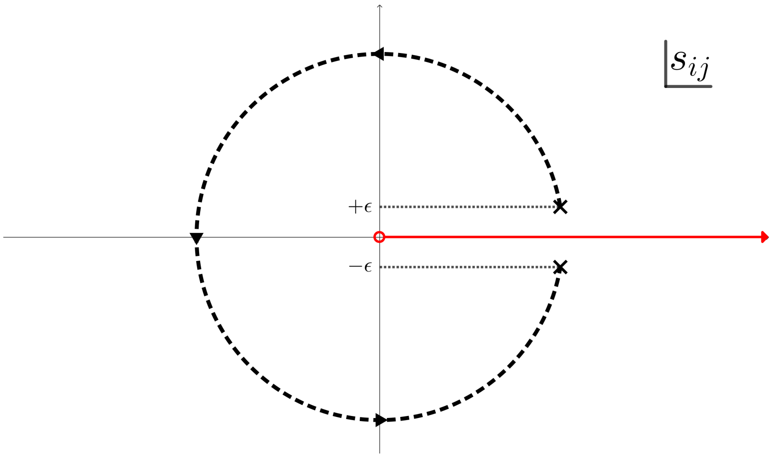


Figure 7.1: Transformation of the analytically continued Mandelstam invariants s_{ij} under the action of the dilatation operator D. For $\alpha = \pi$ their infinitesimal imaginary part ϵ changes sign.

following relation between the form factors evaluated at $\{s_{ij} + i\epsilon\}$ and at $\{s_{ij} - i\epsilon\}$:

$$F_{\mathscr{O}}(\{s_{ij} - i\epsilon\}) = e^{i\pi D} F_{\mathscr{O}}(\{s_{ij} + i\epsilon\}) , \tag{7.8}$$

which, combined with the analyticity relation in Eq. (7.4), implies

$$e^{-i\pi D} F^*_{\mathscr{O}}(\{s_{ij} + i\epsilon\}) = F_{\mathscr{O}}(\{s_{ij} + i\epsilon\}) . \tag{7.9}$$

Next, we can exploit unitarity in the form of the completeness relation satisfied by multi-particle asymptotic states $|\vec{n}\rangle$

$$\mathbb{1} = \sum_{\vec{n}} \int \mathrm{d}\Pi_{\vec{n}} \; |\vec{n}\rangle \langle \vec{n}| \; , \tag{7.10}$$

where

$$d\Pi_{\vec{n}} = \frac{1}{k!} \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{2E_i} \tag{7.11}$$

is their Lorentz invariant phase space element, and the factor $1/k!$ accounts for k identical particles. We can exploit this relation to rewrite a generic form factor $F_{\mathscr{O}}$ as

$$\begin{aligned} F_{\mathscr{O}}(\vec{n}) &= {}_{\text{out}} \langle \vec{n}|\mathscr{O}(0)|0\rangle \\ &= \sum_{\vec{m}} \int \mathrm{d}\Pi_{\vec{m}} \, {}_{\text{out}} \langle \vec{n}|\vec{m}\rangle_{\text{ in}} \, {}_{\text{in}} \langle \vec{m}|\mathscr{O}(0)|0\rangle \; . \end{aligned} \tag{7.12}$$

The final ingredient is provided by CPT theorem. CPT is a discrete symmetry transformation that is represented in the Hilbert space by an anti-linear and anti-unitary operator. It relates ingoing and outgoing states when inserted in the inner product and acts on local operators $\mathcal{O}(x)$ as

$$\text{CPT}\mathcal{O}(x)[\text{CPT}]^{-1} = \mathcal{O}^\dagger(-x)\,. \tag{7.13}$$

Therefore, it follows that

$$\begin{aligned} {}_{\text{out}}\langle \vec{m}|\mathcal{O}(0)|0\rangle &= \langle 0|[\text{CPT}]^{-1}\text{CPT}\mathcal{O}^\dagger(x)[\text{CPT}]^{-1}\text{CPT}|\vec{m}\rangle_{\text{in}} \\ &= \langle 0|\mathcal{O}^\dagger(0)|\vec{m}\rangle_{\text{in}}\,, \end{aligned} \tag{7.14}$$

which implies for a Hermitian operator $\mathcal{O}^\dagger(x) = \mathcal{O}(x)$

$${}_{\text{in}}\langle \vec{m}|\mathcal{O}(0)|0\rangle = \langle 0|\mathcal{O}(0)|\vec{m}\rangle_{\text{out}} = \left({}_{\text{out}}\langle \vec{m}|\mathcal{O}(0)|0\rangle\right)^* = F^*_{\mathcal{O}}(\vec{m})\,. \tag{7.15}$$

Thus, we can rewrite Eq. (7.12) as

$$F_{\mathcal{O}}(\vec{n}) = \sum_{\vec{m}} \int \mathrm{d}\Pi_{\vec{m}}\, S_{nm} F^*_{\mathcal{O}}(\vec{m}) \tag{7.16}$$

and combine this result with Eq. (7.9) to obtain

$$e^{-i\pi D} F^*_{\mathcal{O}}(\vec{n}) = \sum_{\vec{m}} \int \mathrm{d}\Pi_{\vec{m}}\, S_{nm} F^*_{\mathcal{O}}(\vec{m})\,. \tag{7.17}$$

This is the central equation we were seeking that relates form factors and S-matrix elements. Its interpretation consists in identifying the dilatation operator as minus the phase of the S-matrix, divided by π.

On the other hand, the dilatation operator is also closely related to the renormalization group evolution. Indeed, at energies much higher than any mass or equivalently if all the particles are massless, we can infer by dimensional analysis that $F_{\mathcal{O}}$ can depend only on dimensionless ratios s_{ij}/μ^2, and consequently $D = \sum_i p_i \cdot \partial/\partial p_i$ can be traded by $-\mu\partial/\partial\mu$. Thus, the Callan-Symanzik equation yields

$$DF_{\mathcal{O}} = -\frac{\partial}{\partial \log \mu} F^{(1)}_{\mathcal{O}} = \left(\gamma - \gamma_{\text{IR}} + \beta_g \frac{\partial}{\partial g}\right)^{(1)} F^{(0)}_{\mathcal{O}}\,, \tag{7.18}$$

where the super-index (1) on the right hand side denotes the coefficients of the leading single $\log \mu$ that typically arises at one-loop order, while $F^{(0)}_{\mathcal{O}}$ are the minimal form factors, which have been defined in Section 6.3. Expanding Eq. (7.17) in powers of D at first non-trivial order we obtain

$$-i\pi DF_{\mathcal{O}}(\vec{n}) = \sum_{\vec{m}} \int \mathrm{d}\Pi_{\vec{m}} \left(S_{nm} F_{\mathcal{O}}(\vec{m})\right)^{(0)}, \tag{7.19}$$

which, combined with Eq. (7.18), gives

$$-i\pi(\gamma - \gamma_{\text{IR}})^{(1)} F_{\mathscr{O}}^{(0)}(\vec{n}) = \sum_{\vec{m}} \int \mathrm{d}\Pi_{\vec{m}} \, (S_{nm} F_{\mathscr{O}}(\vec{m}))^{(0)} \,, \tag{7.20}$$

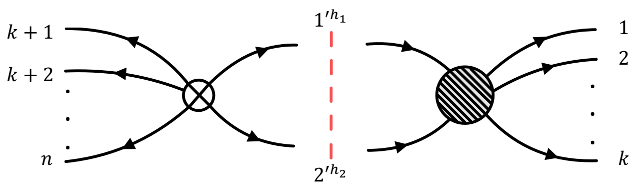
where we have neglected the action of $\partial/\partial g$ on $F_{\mathscr{O}}^{(0)}$ since minimal form factors do not depend on the couplings of the theory as shown previously. From now on we will omit the super-indices (0) and (1). This is a matrix equation in the space of all local operators accounting for all possible RG mixing effects; we can then explicitly write their indices and note that the IR anomalous dimension is diagonal if the operators are kinematically independent, since it is due to soft and collinear emission of particles:

$$-i\pi(\gamma_{ij} - \gamma_{\text{IR}}^{i}\delta_{ij}) \langle \vec{n}|\mathscr{O}_i|0\rangle = \langle \vec{n}|S \otimes \mathscr{O}_j|0\rangle \,, \tag{7.21}$$

where the symbol "$\otimes$" denotes the convolution operation, *i.e.* the insertion of the completeness relation in Eq. (7.10).

At one-loop level, the right hand side of this equation involves a tree-level form factor and a tree-level scattering amplitude, contracted with a two-particle phase space integral. This is effectively illustrated by

$$\langle \vec{n}|S \otimes \mathscr{O}|0\rangle = \sum_{k=2}^{n} \sum_{h_1,h_2}$$



$$+ \text{ permutations of external particles}$$

$$= \sum_{k=2}^{n} \sum_{h_1,h_2} \frac{1}{2} \int \prod_{i=1}^{2} \frac{\mathrm{d}^3 p_i'}{(2\pi)^3 2E_i'} (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_k - p_1' - p_2')$$
$$\times \, i\mathcal{M}_{k+2}(1, \ldots, k; 1'^{h_1}, 2'^{h_2}) F_{\mathscr{O}}(1'^{h_1}, 2'^{h_2}, k+1, \ldots, n)$$
$$+ \text{ permutations of external particles} \,, \tag{7.22}$$

where the arrows denote the direction of the momenta.

Our computations will involve only $2 \to 2$ scattering amplitudes of the form $i\mathcal{M}_4(1, 2; 1'^{h_1}, 2'^{h_2})$. In this case, the integrals in Eq. (7.22) can be performed expressing the amplitudes and the form factors in terms of the helicity spinors and exploiting the following parametrization [70] of the internal spinor variables λ_1', λ_2' in terms of the external ones λ_1, λ_2

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \tag{7.23}$$

and similarly for the complex conjugate spinors $\tilde{\lambda}'_1, \tilde{\lambda}'_2$

$$\begin{pmatrix} \tilde{\lambda}'_1 \\ \tilde{\lambda}'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} . \tag{7.24}$$

We can easily check that this parametrization satisfies the total momentum conservation $p_1 + p_2 = p'_1 + p'_2$:

$$\begin{aligned} (p'_1 + p'_2)^{\dot{\alpha}\alpha} &= (\cos\theta\tilde{\lambda}_1 - \sin\theta e^{-i\phi}\tilde{\lambda}_2)^{\dot{\alpha}}(\cos\theta\lambda_1 - \sin\theta e^{i\phi}\lambda_2)^{\alpha} \\ &\quad + (\sin\theta e^{i\phi}\tilde{\lambda}_1 + \cos\theta\tilde{\lambda}_2)^{\dot{\alpha}}(\sin\theta e^{-i\phi}\lambda_1 + \cos\theta\lambda_2)^{\alpha} \\ &= \cos^2\theta\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^{\alpha} - \sin\theta\cos\theta(e^{i\phi}\tilde{\lambda}_1^{\dot{\alpha}}\lambda_2^{\alpha} + e^{-i\phi}\tilde{\lambda}_2^{\dot{\alpha}}\lambda_1^{\alpha}) + \sin^2\theta\tilde{\lambda}_2^{\dot{\alpha}}\lambda_2^{\alpha} \\ &\quad + \sin^2\theta\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^{\alpha} + \sin\theta\cos\theta(e^{i\phi}\tilde{\lambda}_1^{\dot{\alpha}}\lambda_2^{\alpha} + e^{-i\phi}\tilde{\lambda}_2^{\dot{\alpha}}\lambda_1^{\alpha}) + \cos^2\theta\tilde{\lambda}_2^{\dot{\alpha}}\lambda_2^{\alpha} \\ &= (\cos^2\theta + \sin^2\theta)(\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^{\alpha} + \tilde{\lambda}_2^{\dot{\alpha}}\lambda_2^{\alpha}) \\ &= (p_1 + p_2)^{\dot{\alpha}\alpha} . \end{aligned} \tag{7.25}$$

Additionally, the two-particle Lorentz invariant phase space integral becomes an integral over the solid angle $\mathrm{d}\Omega_2$ and reduces to

$$\frac{1}{2}\int\prod_{i=1}^{2}\frac{\mathrm{d}^3p'_i}{(2\pi)^3 2E'_i}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p'_1 - p'_2) = \frac{1}{16\pi}\int\frac{\mathrm{d}\Omega_2}{4\pi} , \tag{7.26}$$

as explicitly derived in Ref. [36], where the angular integration measure has been defined as

$$\int\frac{\mathrm{d}\Omega_2}{4\pi} = \int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta . \tag{7.27}$$

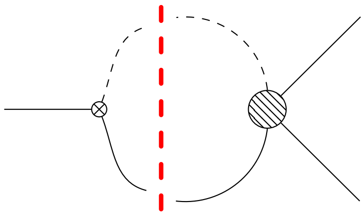
Therefore, we can express Eq. (7.21) at one-loop level and for four-particle scattering amplitudes as

$$\begin{aligned} -i\pi(\gamma_{ij} - \gamma^i_{\mathrm{IR}}\delta_{ij})F_{\mathscr{O}_i}(1, \ldots, n) = &\frac{1}{16\pi}\sum_{h_1,h_2}\int\frac{\mathrm{d}\Omega_2}{4\pi} i\mathcal{M}_4(1, 2; 1'^{h_1}, 2'^{h_2}) \\ &\times F_{\mathscr{O}_j}(1'^{h_1}, 2'^{h_2}, 3, \ldots, n) \\ &+ \text{permutations of external particles} . \end{aligned} \tag{7.28}$$

7.2 Anomalous dimensions of ALP EFT

In order to evaluate the anomalous dimensions associated with the ALP EFT operators, it is sufficient to consider the minimal form factors with the least number of particles, which is equal to three for all operators.

Additionally, since we are working at order $1/\Lambda$, we can discard the contributions on the right hand side of Eq. (7.28) stemming from the convolution of four-particle amplitudes and minimal form factors that share an internal ALP state. Indeed, these convolutions would contribute at order $1/\Lambda^2$, since both $i\mathcal{M}_n(\phi, \{\text{SM particles}\})$ and $F_{\mathcal{O}}(\phi, \{\text{SM particles}\})$ are of order $1/\Lambda$:



$$= \mathcal{O}(\Lambda^{-2})\,. \tag{7.29}$$

Therefore, concerning this specific case, Eq. (7.28) can be rewritten as

$$\begin{aligned}-i\pi(\gamma_{ij} - \gamma^{i}_{\text{IR}}\delta_{ij})F_{\mathcal{O}_i}(1,2,3_\phi) = \frac{1}{16\pi}\sum_{h_1,h_2}\int \frac{d\Omega_2}{4\pi} i\mathcal{M}_4(1,2;1'^{h_1},2'^{h_2}) \\ \times F_{\mathcal{O}_j}(1'^{h_1},2'^{h_2},3_\phi)\,,\end{aligned} \tag{7.30}$$

where the amplitudes $i\mathcal{M}_4$ are built of SM interactions. This is the central formula we will exploit to compute the anomalous dimension matrix elements for the ALP EFT, defined by

$$\frac{d}{d\log\mu}\begin{pmatrix} e^2 C_\gamma \\ g_s^2 C_g \\ v y_S^{ij}\end{pmatrix} = \begin{pmatrix}\gamma_{\gamma\gamma} & \gamma_{\gamma g} & \gamma_{\gamma S} \\ \gamma_{g\gamma} & \gamma_{gg} & \gamma_{gS} \\ \gamma_{S\gamma} & \gamma_{Sg} & \gamma_{SS}\end{pmatrix}\begin{pmatrix} e^2 C_\gamma \\ g_s^2 C_g \\ v y_S^{ij}\end{pmatrix}, \tag{7.31}$$

$$\frac{d}{d\log\mu}\begin{pmatrix} e^2 \widetilde{C}_\gamma \\ g_s^2 \widetilde{C}_g \\ v y_P^{ij}\end{pmatrix} = \begin{pmatrix}\gamma_{\tilde\gamma\tilde\gamma} & \gamma_{\tilde\gamma\tilde g} & \gamma_{\tilde\gamma P} \\ \gamma_{\tilde g\tilde\gamma} & \gamma_{\tilde g\tilde g} & \gamma_{\tilde g P} \\ \gamma_{P\tilde\gamma} & \gamma_{P\tilde g} & \gamma_{PP}\end{pmatrix}\begin{pmatrix} e^2 \widetilde{C}_\gamma \\ g_s^2 \widetilde{C}_g \\ v y_P^{ij}\end{pmatrix}. \tag{7.32}$$

Moreover, throughout the text we will choose to perform the calculations with the minimal form factors on the left hand side of Eq. (7.30) evaluated for negative helicity configurations. This yields the same results that could be obtained considering positive helicity configurations due to CPT theorem.

The final observation before the computation of the anomalous dimensions is that, as can be seen from Table 6.1, minimal form factors associated with Hermitian operators that transform in an opposite way under a CP transformation differ in a phase i, and, since the anomalous dimension matrix elements must be real, we can ignore the mixing of such operators at lowest order in $1/\Lambda$.

7.2.1 ϕFF anomalous dimension

Considering the operator $\phi F_{\mu\nu}F^{\mu\nu}$, its anomalous dimension can receive contributions from the operators $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$, $\phi \bar{f}_i f_j$ and $\phi F_{\mu\nu}F^{\mu\nu}$ itself.

$\gamma_{\gamma g}$

The anomalous dimension matrix element $\gamma_{\gamma g}$ can be computed starting from Eq. (7.30) with $i=\gamma$ and $j=g$

$$-i\pi\gamma_{\gamma g}F_\gamma(1^-_\gamma,2^-_\gamma,3_\phi) = \frac{1}{16\pi}\sum_{h_1,h_2}\int\frac{\mathrm{d}\Omega_2}{4\pi}i\mathcal{M}_4(1^-_\gamma,2^-_\gamma;1'^{h_1}_{g^a},2'^{h_2}_{g^b}) \times F_g(1'^{h_1}_{g^a},2'^{h_2}_{g^b},3_\phi)\,, \tag{7.33}$$

and, since $F_g(1'^{h_1}_{g^a},2'^{h_2}_{g^b},3_\phi) = F_g(1'^{\pm}_{g^a},2'^{\pm}_{g^b},3_\phi)$ holds, we can write

$$-i\pi\gamma_{\gamma g}F_\gamma(1^-_\gamma,2^-_\gamma,3_\phi) = \frac{1}{16\pi}\int\frac{\mathrm{d}\Omega_2}{4\pi}\Big[i\mathcal{M}_4(1^-_\gamma,2^-_\gamma;1'^{+}_{g^a},2'^{+}_{g^b})F_g(1'^{+}_{g^a},2'^{+}_{g^b},3_\phi) + i\mathcal{M}_4(1^-_\gamma,2^-_\gamma;1'^{-}_{g^a},2'^{-}_{g^b})F_g(1'^{-}_{g^a},2'^{-}_{g^b},3_\phi)\Big]\,, \tag{7.34}$$

which can be schematically represented as in Figure 7.2.

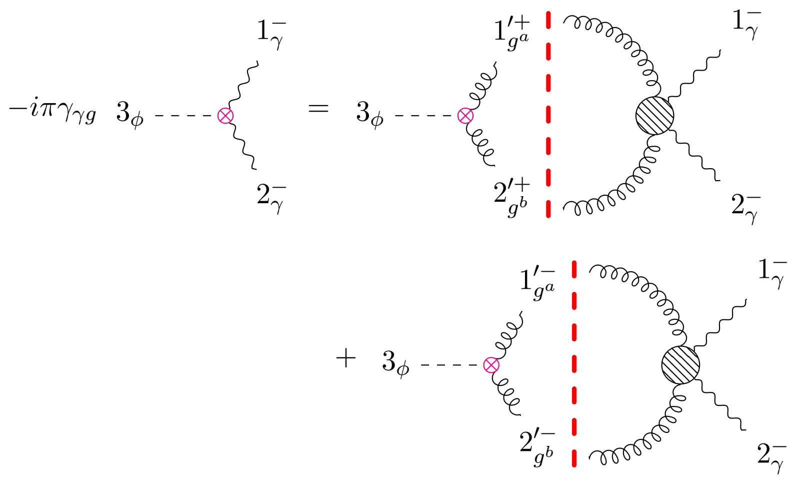


Figure 7.2: Contributions to the anomalous dimension matrix element $\gamma_{\gamma g}$.

This case is particularly simple because, within the SM, it is impossible to construct a tree-level four-particle amplitude involving two gluons and two photons with any helicity configuration

$$i\mathcal{M}_4(1_{g^a},2_{g^b},3_\gamma,4_\gamma) = 0\,. \tag{7.35}$$

Thus $\gamma_{\gamma g} = 0$.

$\gamma_{\gamma S}$

The anomalous dimension matrix element $\gamma_{\gamma S}$ can be computed starting from Eq. (7.30) with $i = \gamma$ and $j = S$

$$\begin{aligned} -i\pi\gamma_{\gamma S} F_\gamma(1_\gamma^-, 2_\gamma^-, 3_\phi) = &\sum_f \frac{1}{16\pi} \sum_{h_1,h_2} \int \frac{\mathrm{d}\Omega_2}{4\pi} \\ &\times \Big[i\mathcal{M}_4(1_\gamma^-, 2_\gamma^-; 1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}) F_S(1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}, 3_\phi) \\ &+ i\mathcal{M}_4(1_\gamma^-, 2_\gamma^-; 2'^{h_2}_{f_i}, 1'^{h_1}_{\bar{f}_j}) F_S(2'^{h_2}_{f_i}, 1'^{h_1}_{\bar{f}_j}, 3_\phi) \Big], \end{aligned} \tag{7.36}$$

where he have symmetrized over the internal fermions, and, since $F_S(1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}, 3_\phi) = F_S(1'^{\pm}_{f_i}, 2'^{\pm}_{\bar{f}_j}, 3_\phi)$ holds, we can write

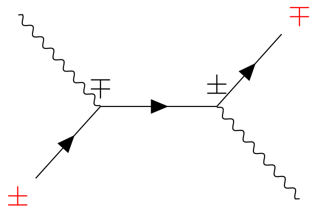
$$\begin{aligned} -i\pi\gamma_{\gamma S} F_\gamma(1_\gamma^-, 2_\gamma^-, 3_\phi) = &\sum_f \frac{1}{16\pi} \int \frac{\mathrm{d}\Omega_2}{4\pi} \\ &\times \Big[i\mathcal{M}_4(1_\gamma^-, 2_\gamma^-; 1'^{+}_{f_i}, 2'^{+}_{\bar{f}_j}) F_S(1'^{+}_{f_i}, 2'^{+}_{\bar{f}_j}, 3_\phi) \\ &+ i\mathcal{M}_4(1_\gamma^-, 2_\gamma^-; 1'^{-}_{f_i}, 2'^{-}_{\bar{f}_j}) F_S(1'^{-}_{f_i}, 2'^{-}_{\bar{f}_j}, 3_\phi) \\ &+ (1' \longleftrightarrow 2') \Big], \end{aligned} \tag{7.37}$$

which can be schematically represented as in Figure 7.3.

Also this case is particularly simple because, within the SM, it is impossible to construct a tree-level four-particle amplitude involving two fermions with the same helicity and two photons with any helicity

$$i\mathcal{M}_4(1^{\pm}_{\bar{f}_i}, 2^{\pm}_{f_j}, 3_\gamma, 4_\gamma) = 0\,. \tag{7.38}$$

This can be understood from the fact that the QED interaction couples two fermions with opposite helicities, and if we try to construct the four-particle amplitude



we can see that we are forced to assign opposite helicities to the external outgoing fermions. Thus $\gamma_{\gamma S} = 0$.

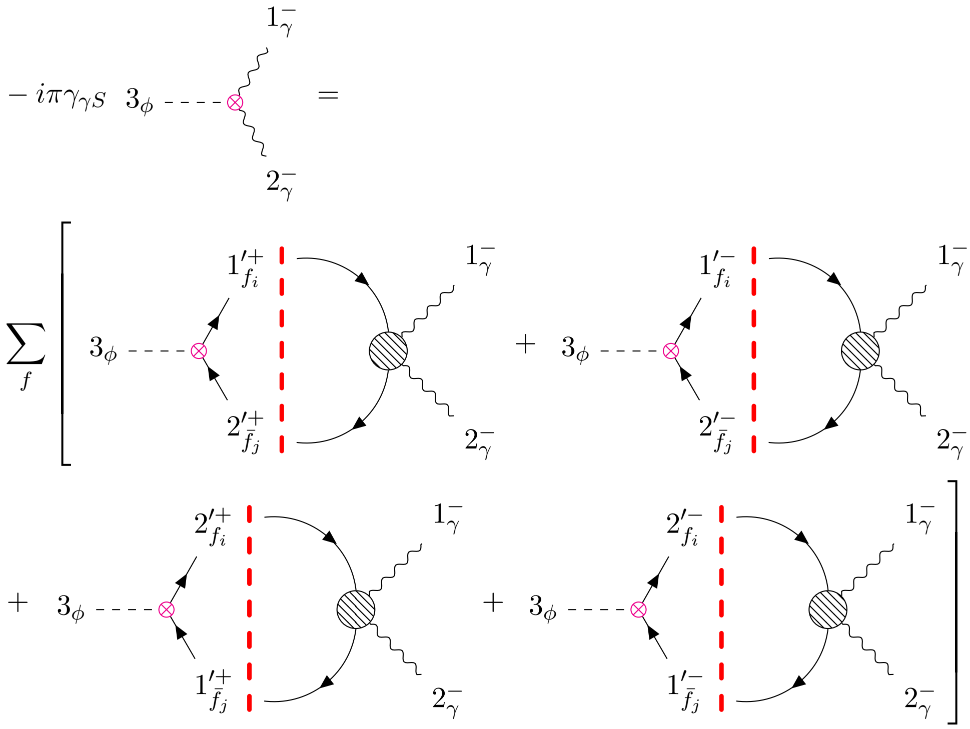


Figure 7.3: Contributions to the anomalous dimension matrix element $\gamma_{\gamma S}$.

$\gamma_{\gamma\gamma}$

The anomalous dimension matrix element $\gamma_{\gamma\gamma}$ can be computed starting from Eq. (7.30) with $i = j = \gamma$

$$-i\pi(\gamma_{\gamma\gamma} - \gamma^{\gamma}_{\text{IR}})F_\gamma(1^-_\gamma, 2^-_\gamma, 3_\phi) = \frac{1}{16\pi}\sum_{h_1,h_2}\int \frac{d\Omega_2}{4\pi} i\mathcal{M}_4(1^-_\gamma, 2^-_\gamma; 1'^{h_1}_\gamma, 2'^{h_2}_\gamma) \times F_\gamma(1'^{h_1}_\gamma, 2'^{h_2}_\gamma, 3_\phi)\,, \tag{7.39}$$

and, since $F_\gamma(1'^{h_1}_\gamma, 2'^{h_2}_\gamma, 3_\phi) = F_\gamma(1'^{\pm}_\gamma, 2'^{\pm}_\gamma, 3_\phi)$ holds, we can write

$$\begin{aligned}-i\pi(\gamma_{\gamma\gamma} - \gamma^{\gamma}_{\text{IR}})F_\gamma(1^-_\gamma, 2^-_\gamma, 3_\phi) = \frac{1}{16\pi}\int \frac{d\Omega_2}{4\pi} \\ \times \Big[i\mathcal{M}_4(1^-_\gamma, 2^-_\gamma; 1'^+_\gamma, 2'^+_\gamma)F_\gamma(1'^+_\gamma, 2'^+_\gamma, 3_\phi) \\ + i\mathcal{M}_4(1^-_\gamma, 2^-_\gamma; 1'^-_\gamma, 2'^-_\gamma)F_\gamma(1'^-_\gamma, 2'^-_\gamma, 3_\phi)\Big]\,,\end{aligned} \tag{7.40}$$

which can be schematically represented as in Figure 7.4.

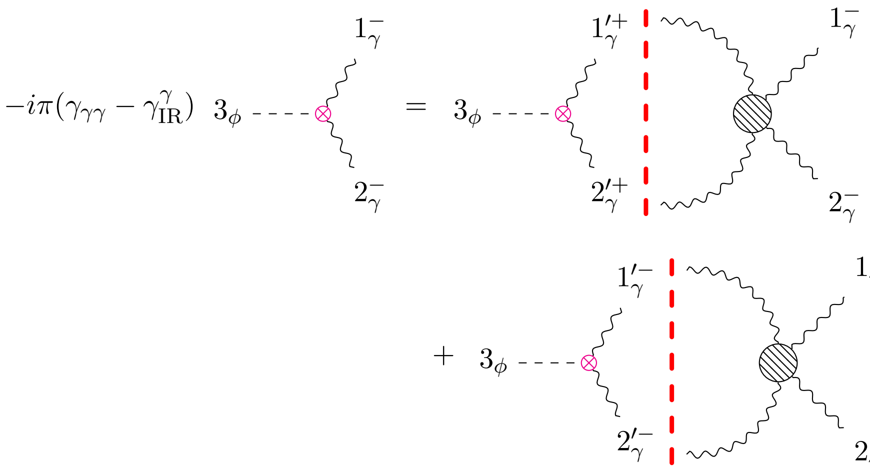


Figure 7.4: Contributions to the anomalous dimension matrix element $\gamma_{\gamma\gamma}$.

Also this case is trivial because, within the SM, it is impossible to construct a tree-level four-photon amplitude with any helicity configuration

$$i\mathcal{M}_4(1_\gamma, 2_\gamma, 3_\gamma, 4_\gamma) = 0\,. \tag{7.41}$$

Thus $\gamma_{\gamma\gamma} = \gamma_{\rm IR}^{\gamma}$, where the IR anomalous dimension $\gamma_{\rm IR}^{\gamma}$ corresponding to the operator $\phi F_{\mu\nu}F^{\mu\nu}$ will be computed in Section 7.3.

Therefore, we can summarize the results obtained for the $\phi F_{\mu\nu}F^{\mu\nu}$ operator as

$$\gamma_{\gamma g} = 0\,, \tag{7.42}$$

$$\gamma_{\gamma S} = 0\,, \tag{7.43}$$

$$\gamma_{\gamma\gamma} = \gamma_{\rm IR}^{\gamma}\,. \tag{7.44}$$

7.2.2 $\phi F\widetilde{F}$ anomalous dimension

The anomalous dimensions associated with the $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ operator can receive contributions from the operators $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$, $i\phi\bar{f}_i\gamma_5 f_j$ and $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ itself. Their calculation is completely analogous to the one we have just trivially performed for $\phi F_{\mu\nu}F^{\mu\nu}$, thus we can just report the schemes in Figures 7.5, 7.6 and 7.7.

Since all these four-particle amplitudes are vanishing, we can summarize the anomalous dimensions of the operator $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ as

$$\gamma_{\tilde{\gamma}\tilde{g}} = 0\,, \tag{7.45}$$

$$\gamma_{\tilde{\gamma}P} = 0\,, \tag{7.46}$$

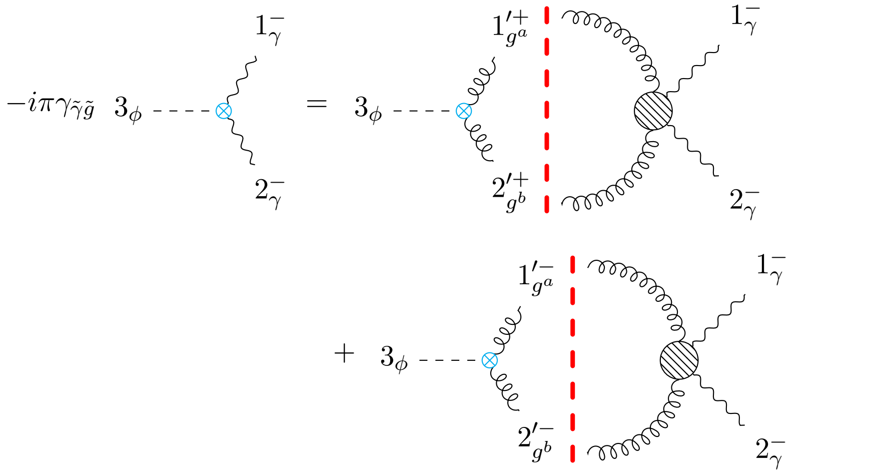


Figure 7.5: Contributions to the anomalous dimension matrix element $\gamma_{\tilde{\gamma}\tilde{g}}$.

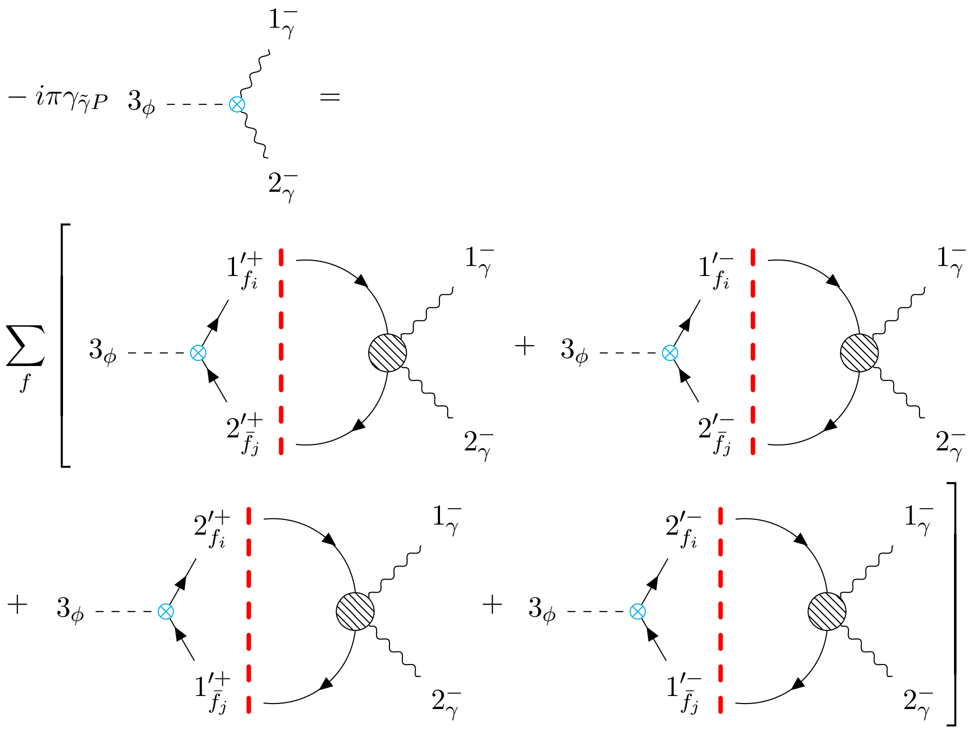


Figure 7.6: Contributions to the anomalous dimension matrix element $\gamma_{\tilde{\gamma}P}$.

$$\gamma_{\tilde{\gamma}\tilde{\gamma}} = \gamma_{\text{IR}}^{\tilde{\gamma}}\,, \tag{7.47}$$

where the IR anomalous dimension $\gamma_{\text{IR}}^{\tilde{\gamma}}$ corresponding to the operator $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ will be computed in Section 7.3.

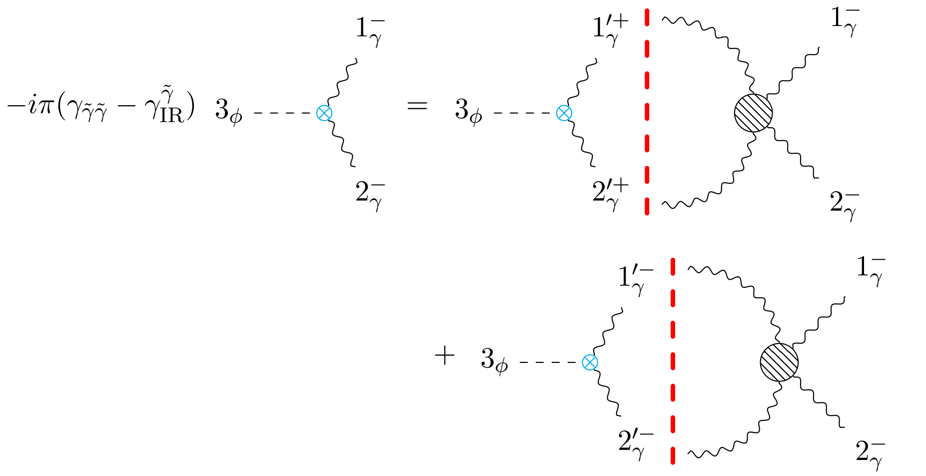


Figure 7.7: Contributions to the anomalous dimension matrix element $\gamma_{\tilde{\gamma}\tilde{\gamma}}$.

7.2.3 ϕGG anomalous dimension

Considering the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$, its anomalous dimension can receive contributions from the operators $\phi F_{\mu\nu}F^{\mu\nu}$, $\phi \bar{f}_i f_j$ and $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ itself.

$\gamma_{g\gamma}$

The anomalous dimension matrix element $\gamma_{g\gamma}$ can be computed starting from Eq. (7.30) with $i = g$ and $j = \gamma$

$$\begin{aligned} -i\pi\gamma_{g\gamma}F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) = \frac{1}{16\pi}\sum_{h_1,h_2}\int \frac{\mathrm{d}\Omega_2}{4\pi} i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^{h_1}_\gamma, 2'^{h_2}_\gamma) \\ \times F_\gamma(1'^{h_1}_\gamma, 2'^{h_2}_\gamma, 3_\phi)\,, \end{aligned} \tag{7.48}$$

and, since $F_\gamma(1'^{h_1}_\gamma, 2'^{h_2}_\gamma, 3_\phi) = F_\gamma(1'^{\pm}_\gamma, 2'^{\pm}_\gamma, 3_\phi)$ holds, we can write

$$\begin{aligned} -i\pi\gamma_{g\gamma}F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) = \frac{1}{16\pi}\int \frac{\mathrm{d}\Omega_2}{4\pi}\Big[i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^+_\gamma, 2'^+_\gamma) F_\gamma(1'^+_\gamma, 2'^+_\gamma, 3_\phi) \\ + i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^-_\gamma, 2'^-_\gamma) F_\gamma(1'^-_\gamma, 2'^-_\gamma, 3_\phi)\Big]\,, \end{aligned} \tag{7.49}$$

which can be schematically represented as in Figure 7.8.

These amplitudes, as mentioned previously, are vanishing, and consequently $\gamma_{g\gamma} = 0$.

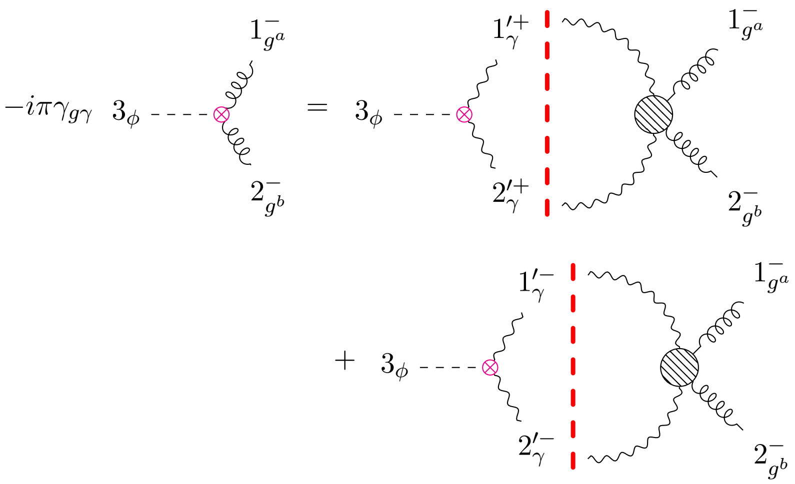


Figure 7.8: Contributions to the anomalous dimension matrix element $\gamma_{g\gamma}$.

γ_{gS}

The anomalous dimension matrix element $\gamma_{g\gamma}$ can be computed starting from Eq. (7.30) with $i = g$ and $j = S$

$$
\begin{aligned}
-i\pi\gamma_{gS}F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) = \sum_f \frac{1}{16\pi} \sum_{h_1,h_2} \int \frac{\mathrm{d}\Omega_2}{4\pi} \\
\times \Big[i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}) F_S(1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}, 3_\phi) \\
+ i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 2'^{h_2}_{f_i}, 1'^{h_1}_{\bar{f}_j}) F_S(2'^{h_2}_{f_i}, 1'^{h_1}_{\bar{f}_j}, 3_\phi) \Big],
\end{aligned}
\tag{7.50}
$$

and, since $F_S(1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}, 3_\phi) = F_S(1'^{\pm}_{f_i}, 2'^{\pm}_{\bar{f}_j}, 3_\phi)$ holds, we can write

$$
\begin{aligned}
-i\pi\gamma_{gS}F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) = \sum_f \frac{1}{16\pi} \int \frac{\mathrm{d}\Omega_2}{4\pi} \\
\times \Big[i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^{+}_{f_i}, 2'^{+}_{\bar{f}_j}) F_S(1'^{+}_{f_i}, 2'^{+}_{\bar{f}_j}, 3_\phi) \\
+ i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^{-}_{f_i}, 2'^{-}_{\bar{f}_j}) F_S(1'^{-}_{f_i}, 2'^{-}_{\bar{f}_j}, 3_\phi) \\
+ (1' \longleftrightarrow 2') \Big],
\end{aligned}
\tag{7.51}
$$

which can be schematically represented as in Figure 7.9.

Also these amplitudes are vanishing, as in the QED case. Indeed, the seed QCD three-particle amplitude involving two fermions and one gluon vanishes if the two fermions have the same helicity, leading to

$$
i\mathcal{M}_4(1^{\pm}_{f_i}, 2^{\pm}_{\bar{f}_j}, 3_{g^a}, 4_{g^b}) = 0 \tag{7.52}
$$

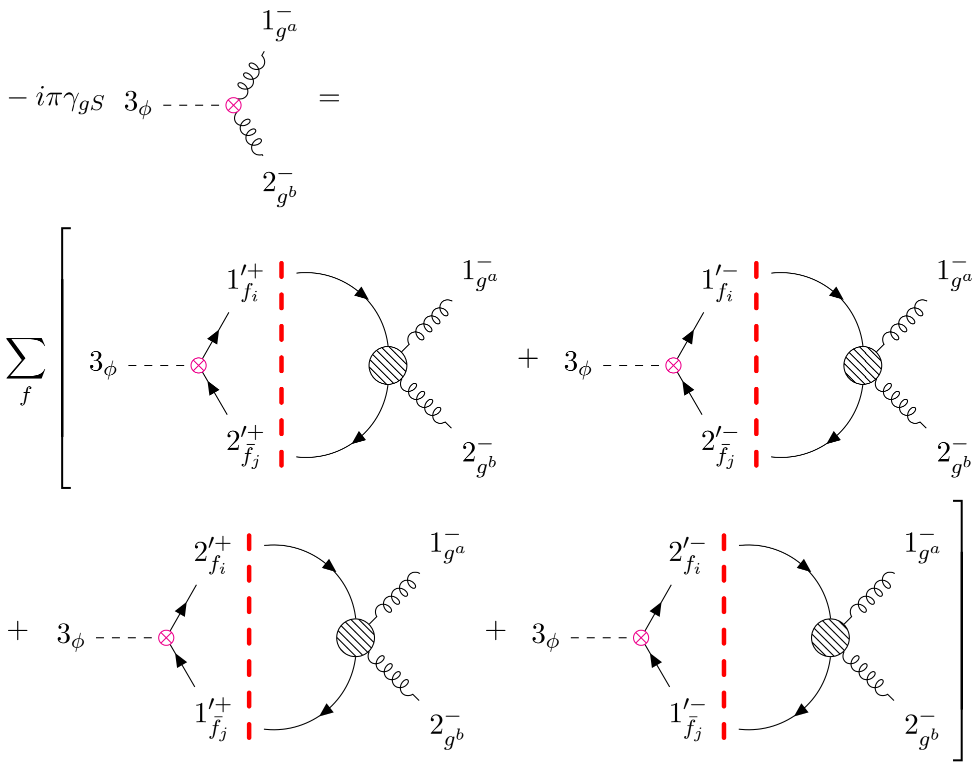


Figure 7.9: Contributions to the anomalous dimension matrix element γ_{gS}.

and consequently $\gamma_{gS} = 0$.

γ_{gg}

The anomalous dimension matrix element γ_{gg} can be computed starting from Eq. (7.30) with $i = j = g$

$$-i\pi(\gamma_{gg} - \gamma_{\text{IR}}^{g}) F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) = \frac{1}{16\pi} \sum_{h_1,h_2} \int \frac{\mathrm{d}\Omega_2}{4\pi} i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^{h_1}_{g^c}, 2'^{h_2}_{g^d}) \times F_g(1'^{h_1}_{g^c}, 2'^{h_2}_{g^d}, 3_\phi)\,, \tag{7.53}$$

and, since $F_g(1'^{h_1}_{g^c}, 2'^{h_2}_{g^d}, 3_\phi) = F_g(1'^{\pm}_{g^c}, 2'^{\pm}_{g^d}, 3_\phi)$ holds, we can write

$$-i\pi(\gamma_{gg} - \gamma_{\text{IR}}^{g}) F_g(1^-_{g^a}, 2^-_{g^b}, 3_\phi) = \frac{1}{16\pi} \int \frac{\mathrm{d}\Omega_2}{4\pi} \times \Big[i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^+_{g^c}, 2'^+_{g^d}) F_g(1'^+_{g^c}, 2'^+_{g^d}, 3_\phi) + i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^-_{g^c}, 2'^-_{g^d}) F_g(1'^-_{g^c}, 2'^-_{g^d}, 3_\phi) \Big]\,,$$

(7.54)

which can be schematically represented as in Figure 7.10.

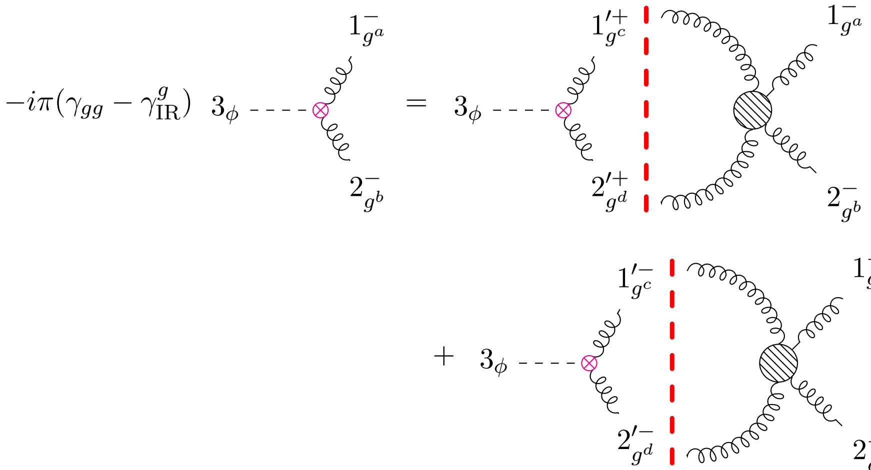


Figure 7.10: Contributions to the anomalous dimension matrix element γ_{gg}.

The first amplitude can be written in terms of outgoing gluons as

$$i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^+_{g^c}, 2'^+_{g^d}) = i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}, \bar{1}'^-_{g^c}, \bar{2}'^-_{g^d}) \tag{7.55}$$

and it can be proven to vanish, as shown in Subsection 6.5.1, since the outgoing gluons are all in the same helicity configuration. On the other hand, the second amplitude is provided by the Parke-Taylor formula

$$\begin{aligned} i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^-_{g^c}, 2'^-_{g^d}) &= i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}, \bar{1}'^+_{g^c}, \bar{2}'^+_{g^d}) \\ &= -2ig_s^2 \langle 1\,2\rangle^4 \left[\frac{f^{abe} f^{cde}}{\langle 1\,2\rangle \langle 2\,\bar{1}'\rangle \langle \bar{1}'\,\bar{2}'\rangle \langle \bar{2}'\,1\rangle} \right. \\ &\quad \left. + \frac{f^{ace} f^{bde}}{\langle 1\,\bar{1}'\rangle \langle \bar{1}'\,2\rangle \langle 2\,\bar{2}'\rangle \langle \bar{2}'\,1\rangle} \right], \end{aligned} \tag{7.56}$$

which, multiplied by $F_g(1'^-_{g^c}, 2'^-_{g^d}, 3_\phi) = -2\delta^{cd}\langle 1'\,2'\rangle^2$ and recalling that we adopt the convention $\bar{\lambda}_i = \lambda_i$, yields

$$4iC_A g_s^2 \delta^{ab} \langle 1'\,2'\rangle^2 \frac{\langle 1\,2\rangle^4}{\langle 1\,1'\rangle \langle 1'\,2\rangle \langle 2\,2'\rangle \langle 2'\,1\rangle}\,, \tag{7.57}$$

where $f^{ace} f^{bde} \delta^{cd} = C_A \delta^{ab}$ has been exploited and $C_A = N_c$ denotes the Casimir of the adjoint representation of $SU(N_c)$. From the parametrization of the spinor variables in Eq. (7.23), it follows that

$$\langle 1'\,2'\rangle = \langle 1\,2\rangle \cos^2\theta - \langle 2\,1\rangle \sin^2\theta = \langle 1\,2\rangle\,, \tag{7.58}$$

$$\langle 1\,1'\rangle = -\langle 1\,2\rangle \sin\theta e^{i\phi}\,, \tag{7.59}$$

$$\langle 1\,2'\rangle = \langle 1'\,2\rangle = \langle 1\,2\rangle \cos\theta\,, \tag{7.60}$$

$$\langle 2\,2'\rangle = -\langle 1\,2\rangle \sin\theta e^{-i\phi}\,, \tag{7.61}$$

leading to

$$i\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}; 1'^-_{g^c}, 2'^-_{g^d}) F_g(1'^-_{g^c}, 2'^-_{g^d}, 3_\phi) = -4ig_s^2 C_A \delta^{ab} \frac{\langle 1\,2\rangle^2}{\cos^2\theta \sin^2\theta}\,. \tag{7.62}$$

We can insert this expression inside Eq. (7.54) and obtain

$$2i\pi(\gamma_{gg} - \gamma^g_{\mathrm{IR}})\delta^{ab}\langle 1\,2\rangle^2 = -\frac{1}{16\pi}\int \frac{\mathrm{d}\Omega_2}{4\pi} 4ig_s^2 C_A \delta^{ab} \frac{\langle 1\,2\rangle^2}{\cos^2\theta\sin^2\theta}\,, \tag{7.63}$$

namely

$$\gamma_{gg} = \gamma^g_{\mathrm{IR}} - \frac{1}{8\pi^2} g_s^2 C_A \int \frac{\mathrm{d}\Omega_2}{4\pi} \frac{1}{\cos^2\theta\sin^2\theta}\,. \tag{7.64}$$

This integral is divergent, but it is cured by the IR anomalous dimension γ^g_{IR} associated with the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$, which will be calculated in Section 7.3.

Therefore, we can summarize the results obtained for the $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ operator as

$$\gamma_{g\gamma} = 0\,, \tag{7.65}$$

$$\gamma_{gS} = 0\,, \tag{7.66}$$

$$\gamma_{gg} = \gamma^g_{\mathrm{IR}} - \frac{1}{8\pi^2} g_s^2 C_A \int \frac{\mathrm{d}\Omega_2}{4\pi} \frac{1}{\cos^2\theta\sin^2\theta}\,. \tag{7.67}$$

7.2.4 $\phi G\widetilde{G}$ anomalous dimension

The anomalous dimensions associated with the $\phi G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}$ operator can receive contributions from the operators $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$, $i\phi \bar{f}_i \gamma_5 f_j$ and $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ itself. Their calculation is completely analogous to the one we have just performed for $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$, thus we can just report the schemes in Figures 7.11, 7.12 and 7.13 and the results.

The results for these anomalous dimension matrix elements are respectively given by

$$\gamma_{\tilde{g}\tilde{\gamma}} = 0\,, \tag{7.68}$$

$$\gamma_{\tilde{g}P} = 0\,, \tag{7.69}$$

$$\gamma_{\tilde{g}\tilde{g}} = \gamma^{\tilde{g}}_{\mathrm{IR}} - \frac{1}{8\pi^2} g_s^2 C_A \int \frac{\mathrm{d}\Omega_2}{4\pi} \frac{1}{\cos^2\theta\sin^2\theta}\,. \tag{7.70}$$

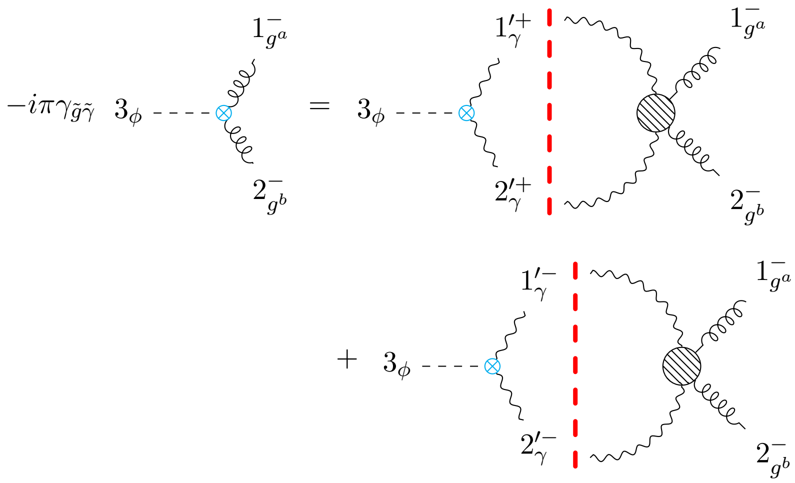


Figure 7.11: Contributions to the anomalous dimension matrix element $\gamma_{\tilde{g}\tilde{\gamma}}$.

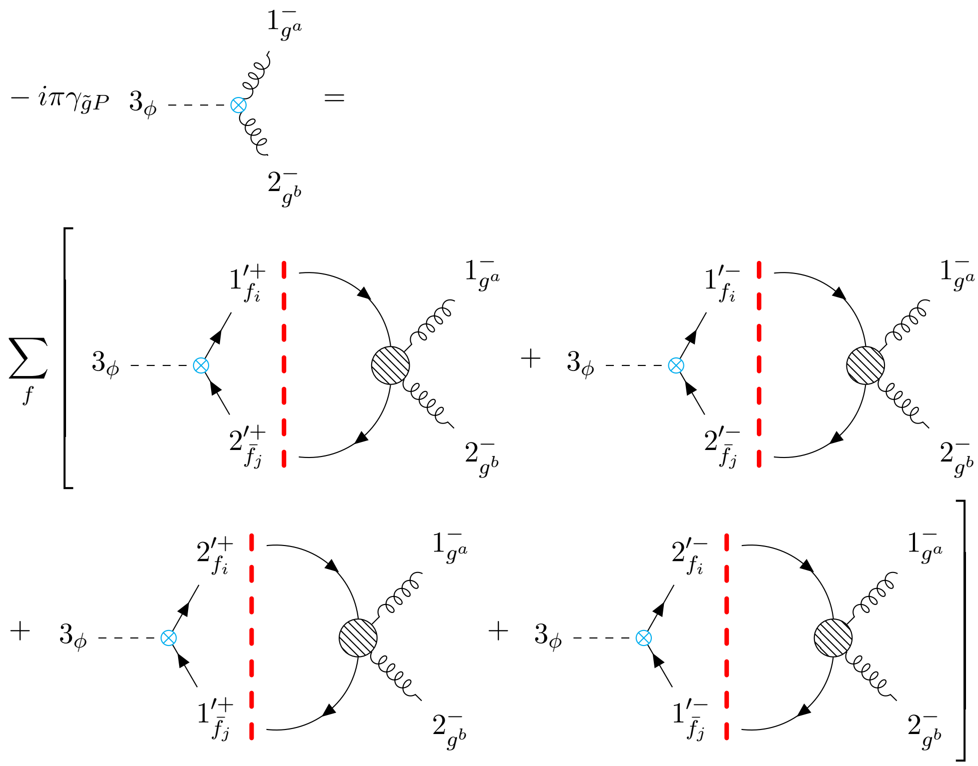


Figure 7.12: Contributions to the anomalous dimension matrix element $\gamma_{\tilde{g}P}$.

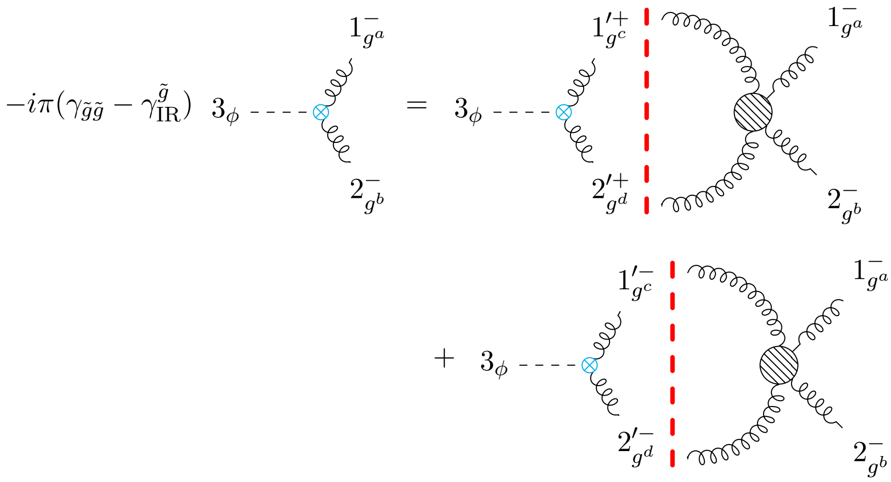


Figure 7.13: Contributions to the anomalous dimension matrix element $\gamma_{\tilde{g}\tilde{g}}$.

7.2.5 $\phi \bar{f} f$ anomalous dimension

Considering the operator $\phi \bar{f}_i f_j$, its anomalous dimension can receive contributions from the operators $\phi F_{\mu\nu}F^{\mu\nu}$, $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ and $\phi \bar{f}_i f_j$ itself.

$\gamma_{S\gamma}$

The anomalous dimension matrix element $\gamma_{S\gamma}$ can be computed starting from Eq. (7.30) with $i = S$ and $j = \gamma$

$$-i\pi\gamma_{S\gamma}F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) = \frac{1}{16\pi}\sum_{h_1,h_2}\int \frac{\mathrm{d}\Omega_2}{4\pi} i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^{h_1}_\gamma, 2'^{h_2}_\gamma) \times F_\gamma(1'^{h_1}_\gamma, 2'^{h_2}_\gamma, 3_\phi)\,, \tag{7.71}$$

and, since $F_\gamma(1'^{h_1}_\gamma, 2'^{h_2}_\gamma, 3_\phi) = F_\gamma(1'^{\pm}_\gamma, 2'^{\pm}_\gamma, 3_\phi)$ holds, we can write

$$-i\pi\gamma_{S\gamma}F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) = \frac{1}{16\pi}\int \frac{\mathrm{d}\Omega_2}{4\pi}\Big[i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^+_\gamma, 2'^+_\gamma)F_\gamma(1'^+_\gamma, 2'^+_\gamma, 3_\phi) + i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^-_\gamma, 2'^-_\gamma)F_\gamma(1'^-_\gamma, 2'^-_\gamma, 3_\phi)\Big]\,, \tag{7.72}$$

which can be schematically represented as in Figure 7.14.

As discussed previously for the case of $\gamma_{\gamma S}$, these amplitudes are vanishing since the external fermions are in the same helicity configuration. Therefore $\gamma_{S\gamma} = 0$.

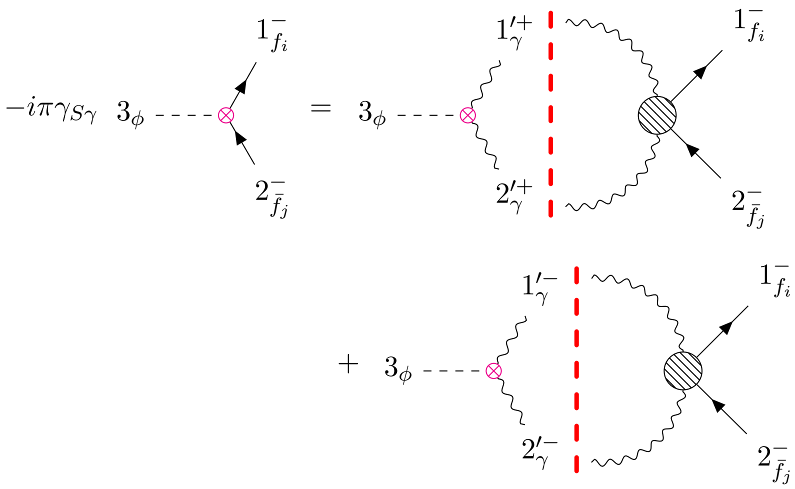


Figure 7.14: Contributions to the anomalous dimension matrix element $\gamma_{S\gamma}$.

γ_{Sg}

The anomalous dimension matrix element γ_{Sg} can be computed starting from Eq. (7.30) with $i = S$ and $j = g$

$$-i\pi\gamma_{Sg}F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) = \frac{1}{16\pi}\sum_{h_1,h_2}\int \frac{\mathrm{d}\Omega_2}{4\pi} i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^{h_1}_{g^a}, 2'^{h_2}_{g^b}) \times F_g(1'^{h_1}_{g^a}, 2'^{h_2}_{g^b}, 3_\phi)\,, \tag{7.73}$$

and, since $F_g(1'^{h_1}_{g^a}, 2'^{h_2}_{g^b}, 3_\phi) = F_g(1'^{\pm}_{g^a}, 2'^{\pm}_{g^b}, 3_\phi)$ holds, we can write

$$-i\pi\gamma_{Sg}F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) = \frac{1}{16\pi}\int \frac{\mathrm{d}\Omega_2}{4\pi}\Big[i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^+_{g^a}, 2'^+_{g^b})F_g(1'^+_{g^a}, 2'^+_{g^b}, 3_\phi) + i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^-_{g^a}, 2'^-_{g^b})F_g(1'^-_{g^a}, 2'^-_{g^b}, 3_\phi)\Big]\,, \tag{7.74}$$

which can be schematically represented as in Figure 7.15.

As discussed previously for the case of γ_{gS}, these amplitudes are vanishing since the external fermions are in the same helicity configuration. Therefore $\gamma_{Sg} = 0$.

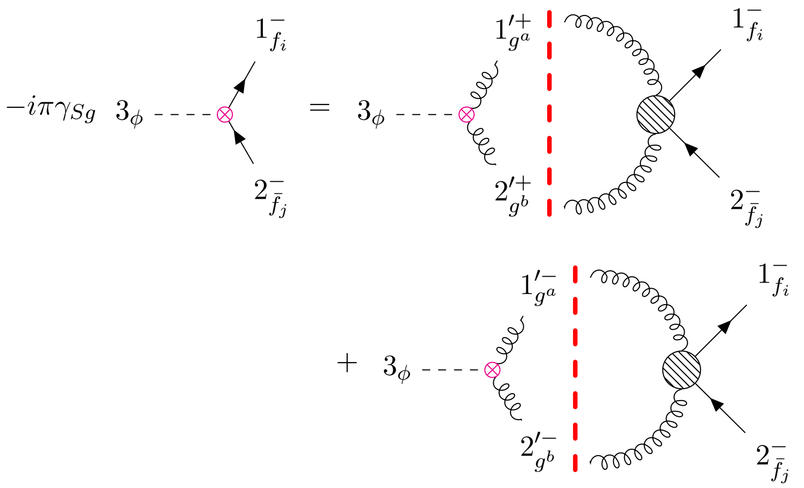


Figure 7.15: Contributions to the anomalous dimension matrix element γ_{Sg}.

γ_{SS}

The anomalous dimension matrix element γ_{Sg} can be computed starting from Eq. (7.30) with $i = j = S$

$$
\begin{aligned}
-i\pi(\gamma_{SS} - \gamma^S_{\mathrm{IR}})F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) = \frac{1}{16\pi} \sum_{h_1,h_2} \int \frac{\mathrm{d}\Omega_2}{4\pi} \\
\times \Big[i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}) F_S(1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}, 3_\phi) \\
+ i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 2'^{h_2}_{f_i}, 1'^{h_1}_{\bar{f}_j}) F_S(2'^{h_2}_{f_i}, 1'^{h_1}_{\bar{f}_j}, 3_\phi) \Big] ,
\end{aligned}
\tag{7.75}
$$

and, since $F_S(1'^{h_1}_{f_i}, 2'^{h_2}_{\bar{f}_j}, 3_\phi) = F_S(1'^{\pm}_{f_i}, 2'^{\pm}_{\bar{f}_j}, 3_\phi)$ holds, we can write

$$
\begin{aligned}
-i\pi(\gamma_{SS} - \gamma^S_{\mathrm{IR}})F_S(1^-_{f_i}, 2^-_{\bar{f}_j}, 3_\phi) = \frac{1}{16\pi} \int \frac{\mathrm{d}\Omega_2}{4\pi} \\
\times \Big[i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^+_{f_i}, 2'^+_{\bar{f}_j}) F_S(1'^+_{f_i}, 2'^+_{\bar{f}_j}, 3_\phi) \\
+ i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}; 1'^-_{f_i}, 2'^-_{\bar{f}_j}) F_S(1'^-_{f_i}, 2'^-_{\bar{f}_j}, 3_\phi) \\
+ (1' \longleftrightarrow 2') \Big] ,
\end{aligned}
\tag{7.76}
$$

which can be schematically represented as in Figure 7.16.

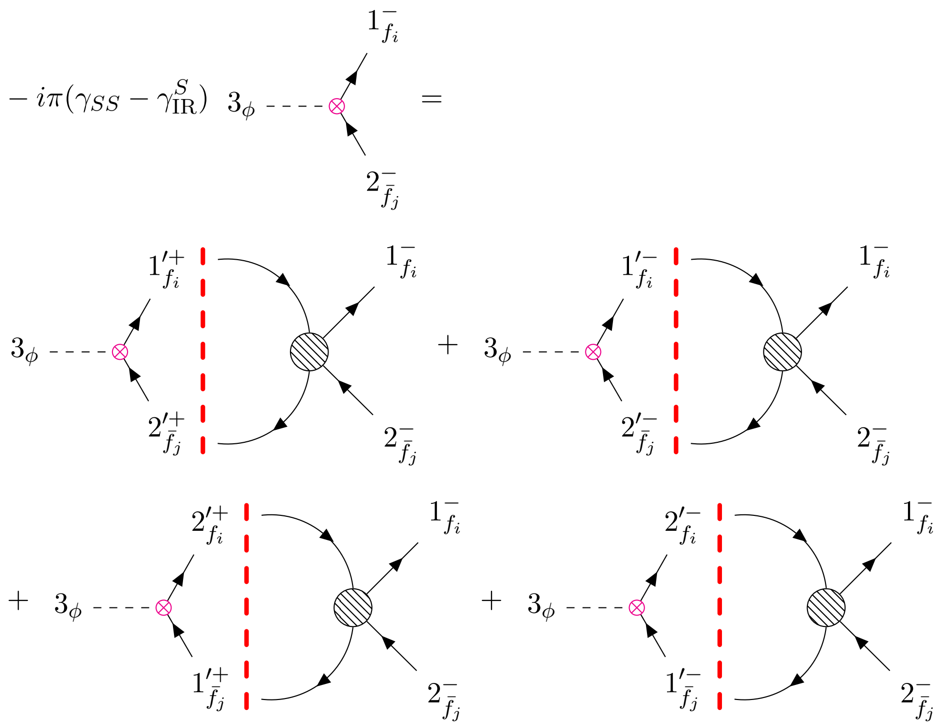


Figure 7.16: Contributions to the anomalous dimension matrix element γ_{SS}.

The amplitudes on the left hand side are vanishing since they correspond to four-fermion amplitudes with the outgoing fermions in the same helicity configuration

$$i\mathcal{M}_4(1^\pm_{f_i}, 2^\pm_{\bar f_j}, 3^\pm_{\bar f_i}, 4^\pm_{f_j}) = 0\,. \tag{7.77}$$

Indeed, as already mentioned, this is a consequence of the fact that the seed QED and QCD three-particle amplitudes couple fermions having different helicities. On the other hand, the amplitudes on the right hand side are non-vanishing and, for a color-singlet in the initial state, are derived from the ones reported in Appendix C through crossing symmetry and read

$$\begin{aligned} i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar f_j}; 1'^-_{f_i}, 2'^-_{\bar f_j}) &= -2i(e^2 Q_f^2 + C_F g_s^2 c_f^2)\frac{\langle 1\,2\rangle[1'\,2']}{\langle 1\,1'\rangle[1'\,1]} \\ &= 2i(e^2 Q_f^2 + C_F g_s^2 c_f^2)\frac{1}{\sin^2\theta}\,, \end{aligned} \tag{7.78}$$

$$\begin{aligned} i\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar f_j}; 2'^-_{f_i}, 1'^-_{\bar f_j}) &= -2i(e^2 Q_f^2 + C_F g_s^2 c_f^2)\frac{\langle 1\,2\rangle[2'\,1']}{\langle 1\,2'\rangle[2'\,1]} \\ &= -2i(e^2 Q_f^2 + C_F g_s^2 c_f^2)\frac{1}{\cos^2\theta}\,, \end{aligned} \tag{7.79}$$

where we have parametrized the internal helicity spinors as in Eqs. (7.23) and (7.24), and $C_F = (N_c^2-1)/(2N_c)$ is the Casimir of the fundamental representation of $SU(N_c)$: $T^a_{IK}T^a_{KJ} = C_F\delta_{IJ}$. We can insert these expression inside Eq. (7.76) and exploit

$$F_S(1'^{-}_{f_i}, 2'^{-}_{\bar{f}_j}, 3_\phi) = -F_S(2'^{-}_{f_i}, 1'^{-}_{\bar{f}_j}, 3_\phi) = \langle 1'\,2'\rangle = \langle 1\,2\rangle = F_S(1^{-}_{f_i}, 2^{-}_{\bar{f}_j}, 3_\phi) \tag{7.80}$$

to find

$$-i\pi(\gamma_{SS}-\gamma^S_{\text{IR}})\langle 1\,2\rangle = \frac{1}{16\pi}\int\frac{\mathrm{d}\Omega_2}{4\pi}2i(e^2Q_f^2+C_Fg_s^2c_f^2)\langle 1\,2\rangle\left(\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right), \tag{7.81}$$

which leads to

$$\gamma_{SS} = \gamma^S_{\text{IR}} - \frac{1}{16\pi^2}(e^2Q_f^2 + C_Fg_s^2c_f^2)\int\frac{\mathrm{d}\Omega_2}{4\pi}\frac{2}{\cos^2\theta\sin^2\theta}\,. \tag{7.82}$$

Again, this is a divergent integral, but it is cured by the IR anomalous dimension γ^S_{IR} associated with the operator $\phi\bar{f}_if_j$, which will be calculated in Section 7.3.

Therefore, we can summarize the results obtained for the $\phi\bar{f}_if_j$ operator as

$$\gamma_{S\gamma} = 0\,, \tag{7.83}$$
$$\gamma_{Sg} = 0\,, \tag{7.84}$$
$$\gamma_{SS} = \gamma^S_{\text{IR}} - \frac{1}{16\pi^2}(e^2Q_f^2 + C_Fg_s^2c_f^2)\int\frac{\mathrm{d}\Omega_2}{4\pi}\frac{2}{\cos^2\theta\sin^2\theta}\,. \tag{7.85}$$

7.2.6 $i\phi\bar{f}\gamma_5 f$ anomalous dimension

The anomalous dimensions associated with the $i\phi\bar{f}_i\gamma_5 f_j$ operator can receive contributions from the operators $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$, $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ and $i\phi\bar{f}_i\gamma_5 f_j$ itself. Their calculation is completely analogous to the one we have just performed for $\phi\bar{f}_if_j$, thus we can just report the schemes in Figures 7.17, 7.18 and 7.19 and the results.

The results for these anomalous dimension matrix elements are respectively given by

$$\gamma_{P\tilde{\gamma}} = 0\,, \tag{7.86}$$
$$\gamma_{P\tilde{g}} = 0\,, \tag{7.87}$$
$$\gamma_{PP} = \gamma^P_{\text{IR}} - \frac{1}{16\pi^2}(e^2Q_f^2 + C_Fg_s^2c_f^2)\int\frac{\mathrm{d}\Omega_2}{4\pi}\frac{2}{\cos^2\theta\sin^2\theta}\,. \tag{7.88}$$

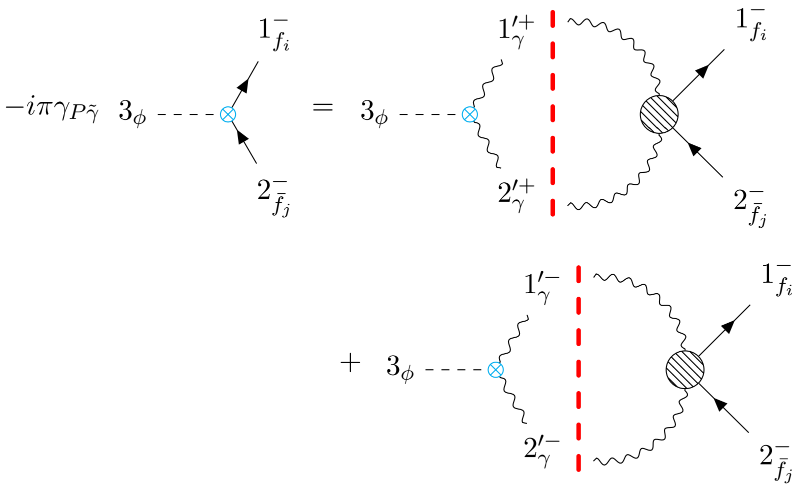


Figure 7.17: Contributions to the anomalous dimension matrix element $\gamma_{P\tilde{\gamma}}$.

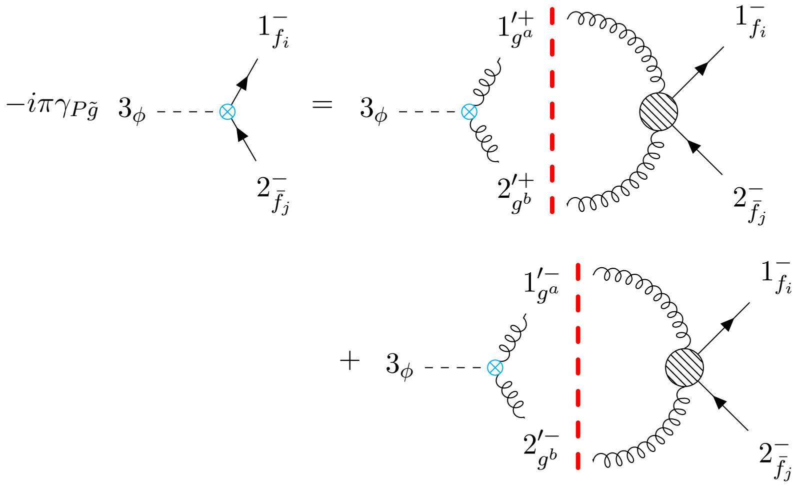


Figure 7.18: Contributions to the anomalous dimension matrix element $\gamma_{P\tilde{g}}$.

7.3 Infrared structure

The final task consists in the evaluation of the IR anomalous dimensions. Indeed, not only UV singularities, but also IR singularities emerge in perturbative results for on-shell scattering amplitudes of theories with massless fields. This implies that form factors need to be renormalized also in the IR limit and that the IR anomalous dimension must be taken into account in the Callan-Symanzik equation.

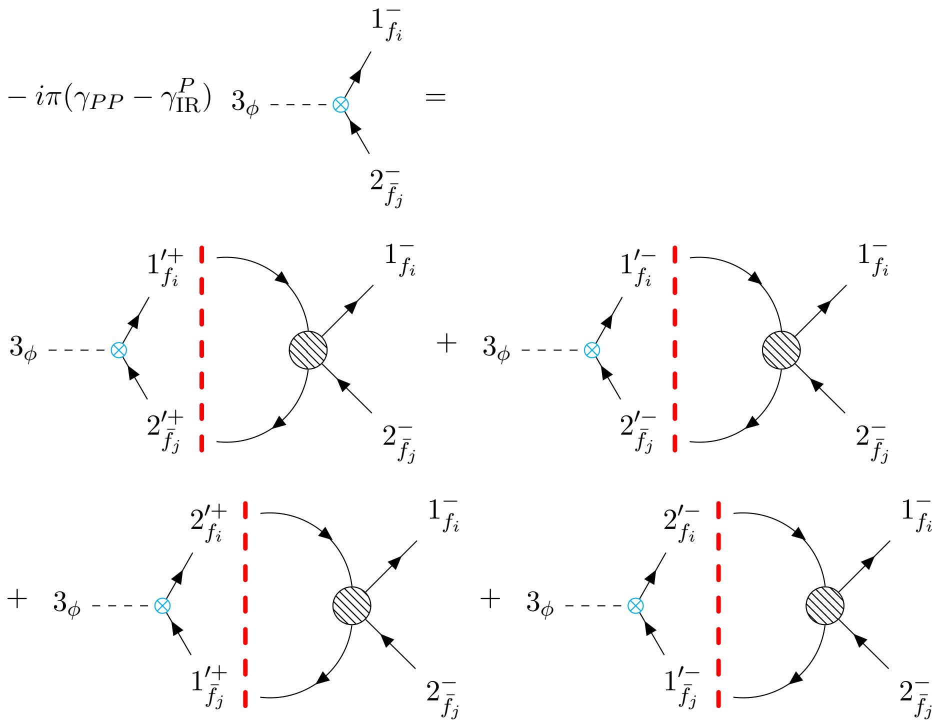


Figure 7.19: Contributions to the anomalous dimension matrix element γ_{PP}.

IR singularities originate from loop-momentum configurations where particle momenta become either soft or collinear, and, as stated by the Kinoshita-Lee-Nauenberg theorem [49, 52], physical observables are insensitive to these, since they cancel against the singularities stemming from the real emission of soft and collinear particles. They are universal and depend only on the particles that are involved in the process.

A key result [9] is that in any gauge theory the IR anomalous dimension is a function of the external states that – at least at one-loop level – takes the form

$$\gamma_{\rm IR}(\{s_{ij}\};\mu) = \frac{g^2}{4\pi^2}\sum_{i<j} T_i^a T_j^a \log\frac{\mu^2}{-s_{ij}} + \sum_i \gamma_i^{\rm coll.}\,, \tag{7.89}$$

where T_i^a is the generator of the gauge group acting on particle i and $\gamma_i^{\rm coll.}$ is its the collinear dimension. In general, this equation can be exploited in Eq. (7.28) in order to extract the anomalous dimension matrix element γ_{ij}

as

$$\begin{aligned}\gamma_{ij} F_{\mathscr{O}_i}(1,\dots,n) = -\frac{1}{16\pi^2}\sum_{h_1,h_2}\int\frac{\mathrm{d}\Omega_2}{4\pi}\Big[&\mathcal{M}_4(1,2;1'^{h_1},2'^{h_2})F_{\mathscr{O}_j}(1'^{h_1},2'^{h_2},3,\dots,n)\\ &+\text{permutations of external particles}\\ &+\frac{2g^2T_1^aT_2^a}{\cos^2\theta\sin^2\theta}F_{\mathscr{O}_j}(1,\dots,n)\Big]\\ &+F_{\mathscr{O}_j}(1,\dots,n)\sum_k\gamma_k^{\text{coll.}}\,,\end{aligned} \tag{7.90}$$

provided that the collinear dimensions of each particle are known.

However, we can obtain the IR anomalous dimensions in a different and more practical way, without knowing a priori the values of the collinear dimensions of the particles. This method relies on the fact that the conservation of the stress-energy tensor

$$T^{\mu\nu}(x) = \sum_\Phi \frac{\partial\mathscr{L}}{\partial(\partial_\mu\Phi(x))}\partial^\nu\Phi(x) - g^{\mu\nu}\mathscr{L}(x)\,, \tag{7.91}$$

where the sum runs over all the fields Φ of the theory, holds at quantum level and that it does not develop an UV anomalous dimension. We can thus substitute the generic operator $\mathscr{O}$ in Eq. (7.21) with the stress-energy tensor in spinor indices $T^{\alpha\beta\,\dot\alpha\dot\beta}$, defined as

$$T^{\alpha\beta\,\dot\alpha\dot\beta} = \sigma_\mu^{\alpha\dot\alpha}\sigma_\nu^{\beta\dot\beta}T^{\mu\nu}\,, \tag{7.92}$$

obtaining that the IR anomalous dimension associated with the multi-particle external state $\langle\vec{n}|$ can be computed through

$$i\pi\gamma_{\text{IR}}\,\langle\vec{n}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle = \langle\vec{n}|S\otimes T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\;. \tag{7.93}$$

This equation is particularly simple for $n=2$, since in this case we can avoid the explicit calculation of the stress-energy tensor and obtain its minimal form factor exploiting the symmetries of the problem. In particular, we require $\langle 1_\Phi|T^{\alpha\beta\,\dot\alpha\dot\beta}|1_\Phi\rangle$ to return the momentum of the particle Φ

$$\langle 1_\Phi|T^{\alpha\beta\,\dot\alpha\dot\beta}|1_\Phi\rangle = 2p_1^{\alpha\dot\alpha}p_1^{\beta\dot\beta} \tag{7.94}$$

and we impose $\langle 1_\Phi^{h_1},2_\Phi^{h_2}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle$ to be orthogonal to (p_1+p_2), since the stress-energy tensor is conserved, and to obey the homogeneity conditions

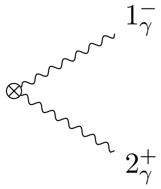
$$\langle 1_\Phi^{h_1},2_\Phi^{h_2}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle \longrightarrow \rho_i^{2h_i}\,\langle 1_\Phi^{h_1},2_\Phi^{h_2}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle \tag{7.95}$$

under the action of the little group transformations acting of the helicity spinors as

$$\lambda_i \longrightarrow \rho_i^{-1}\lambda_i\,, \tag{7.96}$$
$$\tilde{\lambda}_i \longrightarrow \rho_i\tilde{\lambda}_i\,, \tag{7.97}$$

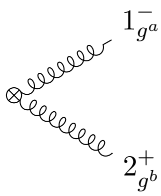
for each $i = 1, 2$. In this way, the minimal form factors associated with the stress-energy tensor are uniquely fixed and take the following forms depending on the particle species.

Photons:



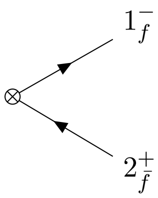
$$= \langle 1^-_\gamma, 2^+_\gamma | T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle = 2\lambda_1^\alpha\lambda_1^\beta\tilde{\lambda}_2^{\dot\alpha}\tilde{\lambda}_2^{\dot\beta}\,. \tag{7.98}$$

Gluons:



$$= \langle 1^-_{g^a}, 2^+_{g^b} | T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle = 2\delta^{ab}\lambda_1^\alpha\lambda_1^\beta\tilde{\lambda}_2^{\dot\alpha}\tilde{\lambda}_2^{\dot\beta}\,. \tag{7.99}$$

Fermions:



$$\begin{aligned} &= \langle 1^-_f, 2^+_{\bar f} | T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle \\ &= \frac{1}{2}(\lambda_1^\alpha\lambda_1^\beta\tilde{\lambda}_1^{\dot\alpha}\tilde{\lambda}_2^{\dot\beta} + \lambda_1^\alpha\lambda_1^\beta\tilde{\lambda}_1^{\dot\beta}\tilde{\lambda}_2^{\dot\alpha} - \lambda_1^\alpha\lambda_2^\beta\tilde{\lambda}_2^{\dot\alpha}\tilde{\lambda}_2^{\dot\beta} - \lambda_1^\beta\lambda_2^\alpha\tilde{\lambda}_2^{\dot\alpha}\tilde{\lambda}_2^{\dot\beta})\,. \end{aligned} \tag{7.100}$$

Now, we can proceed to the calculation of the IR anomalous dimensions of the ALP EFT. Since they depend only on the particles in the final state, we have that the operators in the shift symmetry invariant and shift symmetry breaking sectors of the ALP Lagrangian share the same $\gamma_{\rm IR}$'s, which can be conveniently renamed as Γ_γ, Γ_g and Γ_f:

$$\gamma_{\rm IR}^{\gamma} = \gamma_{\rm IR}^{\tilde\gamma} = \Gamma_\gamma\,, \tag{7.101}$$

$$\gamma^{g}_{\rm IR} = \gamma^{\tilde{g}}_{\rm IR} = \Gamma_g\,, \tag{7.102}$$

$$\gamma^{S}_{\rm IR} = \gamma^{P}_{\rm IR} = \Gamma_f\,. \tag{7.103}$$

Moreover, the ALP is a gauge-singlet and therefore can only develop a collinear anomalous dimension, which is of order $1/\Lambda^2$. Consequently we can ignore its contribution to the IR anomalous dimensions and restrict ourselves to the case $n = 2$ discussed above.

7.3.1 ϕFF & $\phi F\widetilde{F}$ IR anomalous dimension

The IR anomalous dimension Γ_γ can be computed through Eq. (7.93) with $\langle\vec{n}| = \langle 1^-_\gamma, 2^+_\gamma|$:

$$i\pi\Gamma_\gamma\,\langle 1^-_\gamma, 2^+_\gamma|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle = \langle 1^-_\gamma, 2^+_\gamma|S\otimes T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ , \tag{7.104}$$

where the convolution must be expanded allowing for all possible intermediate states. The non-vanishing contributions are schematically illustrated in Figure 7.20, which can be read as

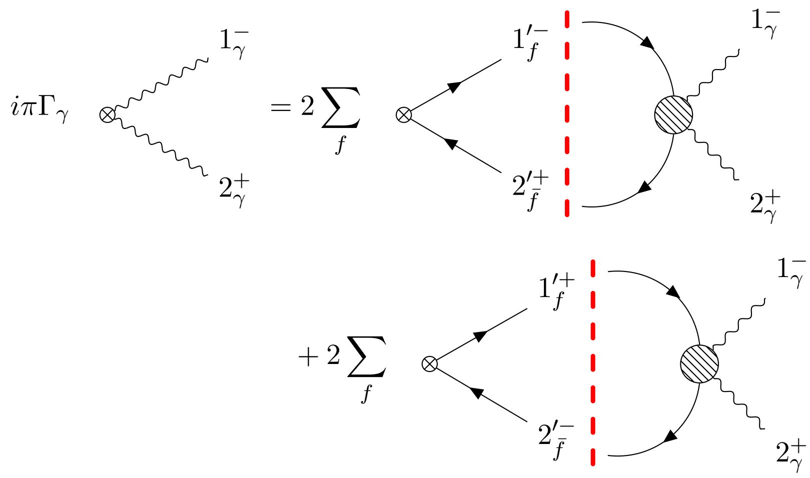


Figure 7.20: Non-vanishing contributions to the IR anomalous dimension Γ_γ associated with the operators ϕFF and $\phi F\widetilde{F}$.

$$\begin{aligned} i\pi\Gamma_\gamma\,\langle 1^-_\gamma, 2^+_\gamma|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle = 2\sum_f \frac{1}{16\pi}\int\frac{{\rm d}\Omega_2}{4\pi} \\ \times\Big[i\mathcal{M}_4(1^-_\gamma, 2^+_\gamma; 1'^-_f, 2'^+_{\bar f})\,\langle 1'^-_f, 2'^+_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle \\ + i\mathcal{M}_4(1^-_\gamma, 2^+_\gamma; 1'^+_f, 2'^-_{\bar f})\,\langle 1'^+_f, 2'^-_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\Big]\,, \end{aligned} \tag{7.105}$$

where the factor 2 has been introduced to run the sum $\sum_f$ over all Dirac fermions (instead of Weyl fermions)[1].

The first four-particle amplitude reads

$$i\mathcal{M}_4(1_\gamma^-, 2_\gamma^+; 1_f'^-, 2_{\bar{f}}'^+) = -2ie^2 Q_f^2 \frac{\langle 1\,2'\rangle[2\,1']}{\langle 2\,2'\rangle[1\,2']}\,, \tag{7.106}$$

while the second one can be obtained through the substitution $1' \leftrightarrow 2'$:

$$i\mathcal{M}_4(1_\gamma^-, 2_\gamma^+; 1_f'^+, 2_{\bar{f}}'^-) = -2ie^2 Q_f^2 \frac{\langle 1\,1'\rangle[2\,2']}{\langle 2\,1'\rangle[1\,1']}\,. \tag{7.107}$$

We can write them as functions of the angular variables θ, ϕ parametrizing the two-particle phase space as follows

$$i\mathcal{M}_4(1_\gamma^-, 2_\gamma^+; 1_f'^-, 2_{\bar{f}}'^+) = -2ie^2 Q_f^2 \frac{\cos\theta}{\sin\theta} e^{i\phi}\,, \tag{7.108}$$

$$i\mathcal{M}_4(1_\gamma^-, 2_\gamma^+; 1_f'^+, 2_{\bar{f}}'^-) = -2ie^2 Q_f^2 \frac{\sin\theta}{\cos\theta} e^{3i\phi}\,, \tag{7.109}$$

where

$$\langle 1\,2'\rangle = \langle 1\,2\rangle \cos\theta\,, \qquad \langle 1\,1'\rangle = -\langle 1\,2\rangle \sin\theta e^{i\phi}\,, \tag{7.110}$$

$$\langle 2\,2'\rangle = -\langle 1\,2\rangle \sin\theta e^{-i\phi}\,, \qquad \langle 2\,1'\rangle = -\langle 1\,2\rangle \cos\theta\,, \tag{7.111}$$

$$[2\,1'] = -[1\,2] \cos\theta\,, \qquad [2\,2'] = -[1\,2] \sin\theta e^{i\phi}\,, \tag{7.112}$$

$$[1\,2'] = [1\,2] \cos\theta\,, \qquad [1\,1'] = -[1\,2] \sin\theta e^{-i\phi} \tag{7.113}$$

have been exploited.

On the other hand, the minimal form factors associated with the stress-energy tensor $\langle 1_f'^-, 2_{\bar{f}}'^+ | T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle$ and $\langle 1_f'^+, 2_{\bar{f}}'^- | T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle$ are provided by Eq. (7.100) and their expansion in terms of the helicity spinors λ_1, λ_2 and $\tilde\lambda_1, \tilde\lambda_2$ looks like is a complete mess. However, this is just an illusion. Indeed, all the terms having a non-zero integer phase give a vanishing contribution after the integration in the azimuthal angle ϕ:

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} e^{in\phi} = \delta_{0n}\,. \tag{7.114}$$

Since the amplitude in Eq. (7.108) has the phase $e^{i\phi}$, we can therefore select from $\langle 1_f'^-, 2_{\bar{f}}'^+ | T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle$ just the terms with phase $e^{-i\phi}$, denoted with the symbol $|_{-1}$. For example

$$\lambda_1'^\alpha \lambda_1'^\beta \tilde\lambda_1'^{\dot\alpha} \tilde\lambda_2'^{\dot\beta}|_{-1} = \lambda_1'^\alpha \lambda_1'^\beta \tilde\lambda_1'^{\dot\beta} \tilde\lambda_2'^{\dot\alpha}|_{-1} = -\lambda_1^\alpha \lambda_1^\beta \tilde\lambda_2^{\dot\alpha} \tilde\lambda_2^{\dot\beta} \cos^3\theta \sin\theta e^{-i\phi}\,, \tag{7.115}$$

[1] Equivalently, we could have symmetrized the internal fermion states mapping $1' \leftrightarrow 2'$ without considering the factor 2. The result cannot change since an outgoing Weyl fermion with helicity $\pm$ is formally equivalent to an ingoing Weyl anti-fermion with helicity $\mp$, namely $\bar{u}_\pm = \bar{v}_\mp$.

$$\lambda_1'^{\alpha}\lambda_2'^{\beta}\tilde{\lambda}_2'^{\dot{\alpha}}\tilde{\lambda}_2'^{\dot{\beta}}|_{-1} = \lambda_1'^{\beta}\lambda_2'^{\alpha}\tilde{\lambda}_2'^{\dot{\alpha}}\tilde{\lambda}_2'^{\dot{\beta}}|_{-1} = \lambda_1^{\alpha}\lambda_1^{\beta}\tilde{\lambda}_2^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}}\cos^3\theta\sin\theta e^{-i\phi}\,, \tag{7.116}$$

so that, when recombined together, they give

$$\langle 1_f'^-, 2_{\bar{f}}'^+|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle\,|_{-1} = -2\lambda_1^{\alpha}\lambda_1^{\beta}\tilde{\lambda}_2^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}}\cos^3\theta\sin\theta e^{-i\phi}\,. \tag{7.117}$$

Out of 64 terms, only one has survived, and it is precisely proportional to the form factor $\langle 1_\gamma^-, 2_\gamma^+|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle$ appearing on the left hand side of Eq. (7.105). Regarding $\langle 1_f'^+, 2_{\bar{f}}'^-|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle$, we have to isolate the terms with the phase $e^{-3i\phi}$ since it multiplies the amplitude in Eq. (7.109):

$$\langle 1_f'^+, 2_{\bar{f}}'^-|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle\,|_{-3} = -2\lambda_1^{\alpha}\lambda_1^{\beta}\tilde{\lambda}_2^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}}\sin^3\theta\cos\theta e^{-3i\phi}\,. \tag{7.118}$$

Again, we can note that the term that has survived has the same helicity structure of $\langle 1_\gamma^-, 2_\gamma^+|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle = 2\lambda_1^{\alpha}\lambda_1^{\beta}\tilde{\lambda}_2^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}}$, as it should.

Thus, we can rewrite Eq. (7.105) as

$$\begin{aligned} 2i\pi\Gamma_\gamma\lambda_1^{\alpha}\lambda_1^{\beta}\tilde{\lambda}_2^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}} = 2\sum_f \frac{1}{16\pi}\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta\, 4ie^2Q_f^2(\cos^4\theta+\sin^4\theta) \\ \times\,\lambda_1^{\alpha}\lambda_1^{\beta}\tilde{\lambda}_2^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}}\,, \end{aligned} \tag{7.119}$$

which leads to

$$\begin{aligned} \Gamma_\gamma &= 2\sum_f \frac{e^2}{4\pi^2}Q_f^2\int_0^{\pi/2}\mathrm{d}\theta\,\sin\theta\cos\theta(\cos^4\theta+\sin^4\theta) \\ &= \frac{e^2}{6\pi^2}\sum_f Q_f^2\,. \end{aligned} \tag{7.120}$$

Recalling Eqs. (7.44) and (7.47), this means that the value of the anomalous dimension matrix elements $\gamma_{\gamma\gamma}$ and $\gamma_{\tilde{\gamma}\tilde{\gamma}}$ is given by

$$\gamma_{\gamma\gamma} = \gamma_{\tilde{\gamma}\tilde{\gamma}} = \Gamma_\gamma = \frac{e^2}{6\pi^2}\sum_f Q_f^2 = \frac{1}{e^2}\frac{\mathrm{d}e^2}{\mathrm{d}\log\mu}\,, \tag{7.121}$$

which is precisely the anomalous dimension of e^2. This proves that the Wilson coefficients e^2C_γ and $e^2\widetilde{C}_\gamma$ of the respective operators $\phi F_{\mu\nu}F^{\mu\nu}$ and $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ scale exactly as e^2 at one-loop order:

$$\frac{\mathrm{d}C_\gamma}{\mathrm{d}\log\mu} = \frac{\mathrm{d}\widetilde{C}_\gamma}{\mathrm{d}\log\mu} = 0\,. \tag{7.122}$$

7.3.2 ϕGG & $\phi G\widetilde{G}$ IR anomalous dimension

The IR anomalous dimension Γ_g can be computed through Eq. (7.93) with $\langle \vec{n}| = \langle 1^-_{g^a}, 2^+_{g^b}|$:

$$i\pi\Gamma_g \, \langle 1^-_{g^a}, 2^+_{g^b}|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle = \langle 1^-_{g^a}, 2^+_{g^b}|S \otimes T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle \ , \tag{7.123}$$

where the convolution must be expanded allowing for all possible intermediate states. The non-vanishing contributions are schematically illustrated in Figure 7.21, and the factor 2 has been introduced for the same reason as the previous calculation of Γ_γ.

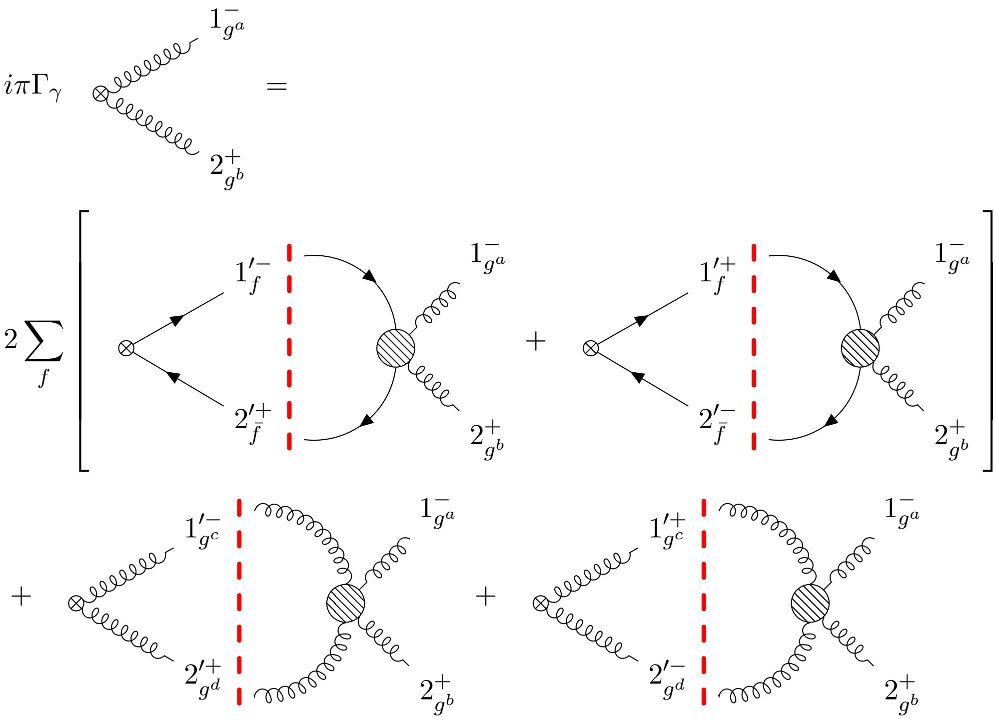


Figure 7.21: Non-vanishing contributions to the IR anomalous dimension Γ_g associated with the operators ϕGG and $\phi G\widetilde{G}$.

The four-particle amplitudes respectively read

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^-_f, 2'^+_{\bar{f}}) = -ig_s^2 c_f^2 \delta^{ab} \frac{\langle 1\,2'\rangle[2\,1']}{\langle 2\,2'\rangle[1\,2']} \,, \tag{7.124}$$

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^+_f, 2'^-_{\bar{f}}) = -ig_s^2 c_f^2 \delta^{ab} \frac{\langle 1\,1'\rangle[2\,2']}{\langle 2\,1'\rangle[1\,1']} \,, \tag{7.125}$$

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^-_{g^c}, 2'^+_{g^d})\delta^{cd} = -2iC_A g_s^2 \delta^{ab} \frac{\langle 1\,2'\rangle^4}{\langle 1\,1'\rangle\langle 1'\,2\rangle\langle 2\,2'\rangle\langle 2'\,1\rangle} \,, \tag{7.126}$$

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^+_{g^c}, 2'^-_{g^d})\delta^{cd} = -2iC_A g_s^2 \delta^{ab} \frac{\langle 1\,1'\rangle^4}{\langle 1\,1'\rangle\langle 1'\,2\rangle\langle 2\,2'\rangle\langle 2'\,1\rangle}\,, \quad (7.127)$$

and can be expressed in terms of θ and ϕ as

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^-_{f}, 2'^+_{\bar{f}}) = -ig_s^2 c_f^2 \delta^{ab} \frac{\cos\theta}{\sin\theta} e^{i\phi}\,, \quad (7.128)$$

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^+_{f}, 2'^-_{\bar{f}}) = -ig_s^2 c_f^2 \delta^{ab} \frac{\sin\theta}{\cos\theta} e^{3i\phi}\,, \quad (7.129)$$

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^-_{g^c}, 2'^+_{g^d})\delta^{cd} = 2iC_A g_s^2 \delta^{ab} \frac{\cos^2\theta}{\sin^2\theta}\,, \quad (7.130)$$

$$i\mathcal{M}_4(1^-_{g^a}, 2^+_{g^b}; 1'^+_{g^c}, 2'^-_{g^d})\delta^{cd} = 2iC_A g_s^2 \delta^{ab} \frac{\sin^2\theta}{\cos^2\theta} e^{4i\phi}\,. \quad (7.131)$$

As before, now we can isolate those terms of the stress-energy minimal form factors that have the appropriate phases, which compensate the amplitude phases:

$$\langle 1'^-_{f}, 2'^+_{\bar{f}} | T^{\alpha\beta\,\dot{\alpha}\dot{\beta}} | 0\rangle\,|_{-1} = -2\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} \cos^3\theta \sin\theta e^{-i\phi}\,, \quad (7.132)$$

$$\langle 1'^+_{f}, 2'^-_{\bar{f}} | T^{\alpha\beta\,\dot{\alpha}\dot{\beta}} | 0\rangle\,|_{-3} = -2\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} \sin^3\theta \cos\theta e^{-3i\phi}\,, \quad (7.133)$$

$$\langle 1'^-_{g^c}, 2'^+_{g^d} | T^{\alpha\beta\,\dot{\alpha}\dot{\beta}} | 0\rangle\,|_{0} = 2\delta^{cd}\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} \cos^4\theta\,, \quad (7.134)$$

$$\langle 1'^+_{g^c}, 2'^-_{g^d} | T^{\alpha\beta\,\dot{\alpha}\dot{\beta}} | 0\rangle\,|_{-4} = 2\delta^{cd}\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} \sin^4\theta e^{-4i\phi}\,. \quad (7.135)$$

As a check, we can verify that they all have the same helicity structure as $\langle 1^-_{g^a}, 2^+_{g^b} | T^{\alpha\beta\,\dot{\alpha}\dot{\beta}} | 0\rangle = 2\delta^{ab}\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}}$.

At this point, we can insert all these expressions inside Eq. (7.123) and obtain

$$\begin{aligned} 2i\pi\Gamma_g \delta^{ab}\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} = \frac{1}{16\pi}\int_0^{\pi/2} & 2\sin\theta\cos\theta\, \mathrm{d}\theta\, 2\delta^{ab}\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} \\ & \times \Bigg[2\sum_f i g_s^2 c_f^2 (\cos^4\theta + \sin^4\theta) \\ & + 2iC_A g_s^2 \left(\frac{\cos^6\theta}{\sin^2\theta} + \frac{\sin^6\theta}{\cos^2\theta} \right) \Bigg]\,, \end{aligned} \quad (7.136)$$

which can be simplified as

$$\Gamma_g = \frac{g_s^2}{8\pi^2}\int_0^{\pi/2} 2\sin\theta\cos\theta\, \mathrm{d}\theta \left[\sum_f c_f^2 (\cos^4\theta + \sin^4\theta) + C_A \frac{\cos^8\theta + \sin^8\theta}{\cos^2\theta \sin^2\theta} \right]. \quad (7.137)$$

Finally, we can recall Eqs. (7.67) and (7.70) to compute the value of the anomalous dimension matrix elements γ_{gg} and $\gamma_{\tilde{g}\tilde{g}}$ as

$$\begin{aligned}\gamma_{gg} = \gamma_{\tilde{g}\tilde{g}} &= \Gamma_g - \frac{g_s^2}{8\pi^2} C_A \int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta\,\frac{1}{\cos^2\theta\sin^2\theta} \\ &= \frac{g_s^2}{8\pi^2}\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta\left[\sum_f c_f^2(\cos^4\theta+\sin^4\theta)\right. \\ &\quad \left. + C_A\frac{\cos^8\theta+\sin^8\theta-1}{\cos^2\theta\sin^2\theta}\right] \\ &= \frac{g_s^2}{8\pi^2}\left(\frac{2}{3}\sum_f c_f^2 - \frac{11}{3}C_A\right) = -\frac{g_s^2}{8\pi^2}b_0 \\ &= \frac{1}{g_s^2}\frac{\mathrm{d}g_s^2}{\mathrm{d}\log\mu}\,,\end{aligned} \tag{7.138}$$

which is precisely the anomalous dimension of g_s^2, since $\sum_f c_f^2$ denotes the number of quarks and $C_A = 3$. This proves that the Wilson coefficients $g_s^2 C_g$ and $g_s^2 \widetilde{C}_g$ of the respective operators $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ and $\phi G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}$ scale exactly as g_s^2 at one-loop order:

$$\frac{\mathrm{d}C_g}{\mathrm{d}\log\mu} = \frac{\mathrm{d}\widetilde{C}_g}{\mathrm{d}\log\mu} = 0\,. \tag{7.139}$$

7.3.3 $\phi\bar{f}f$ & $i\phi\bar{f}\gamma_5 f$ IR anomalous dimension

The IR anomalous dimension Γ_f can be computed through Eq. (7.93) with $\langle\vec{n}| = \langle 1_f^-, 2_{\bar{f}}^+|$:

$$i\pi\Gamma_f\,\langle 1_f^-, 2_{\bar{f}}^+|T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle = \langle 1_f^-, 2_{\bar{f}}^+|S\otimes T^{\alpha\beta\,\dot{\alpha}\dot{\beta}}|0\rangle\,, \tag{7.140}$$

where the convolution must be expanded allowing for all possible intermediate states. The non-vanishing contributions are schematically illustrated in Figure 7.22.

The four-particle amplitudes are respectively given by

$$\begin{aligned} i\mathcal{M}_4(1_f^-, 2_{\bar{f}}^+; 1_f'^-, 2_{\bar{f}}'^+) &= 2i(e^2Q_f^2 + C_F g_s^2 c_f^2)\frac{\langle 1\,2'\rangle[2\,1']}{\langle 1\,1'\rangle[1'\,1]} \\ &= 2i(e^2Q_f^2 + C_F g_s^2 c_f^2)\frac{\cos^2\theta}{\sin^2\theta}\,, \end{aligned} \tag{7.141}$$

$$\begin{aligned} i\mathcal{M}_4(1_f^-, 2_{\bar{f}}^+; 2_f'^-, 1_{\bar{f}}'^+) &= 2i(e^2Q_f^2 + C_F g_s^2 c_f^2)\frac{\langle 1\,1'\rangle[2\,2']}{\langle 1\,2'\rangle[2'\,1]} \\ &= -2i(e^2Q_f^2 + C_F g_s^2 c_f^2)\frac{\sin^2\theta}{\cos^2\theta}e^{2i\phi}\,, \end{aligned} \tag{7.142}$$

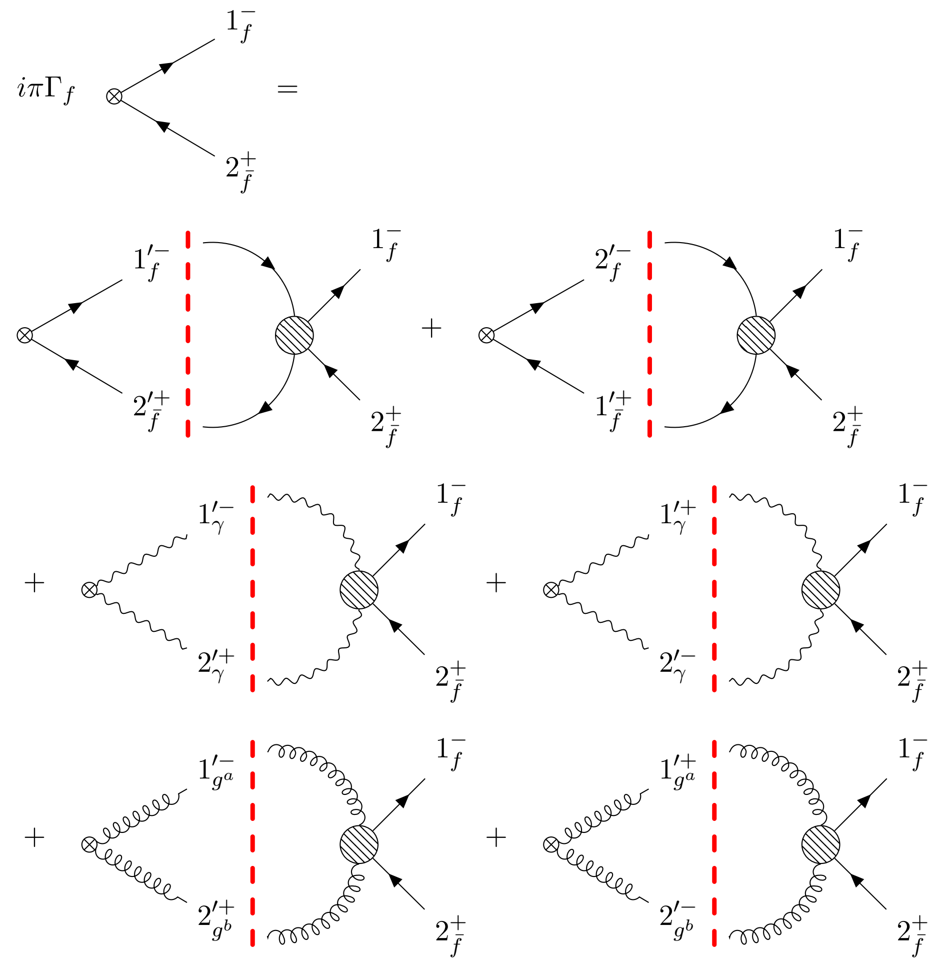


Figure 7.22: Non-vanishing contributions to the IR anomalous dimension Γ_f associated with the Yukawa operators $\phi\bar{f}f$ and $i\phi\bar{f}\gamma_5 f$.

$$i\mathcal{M}_4(1_f^-, 2_{\bar{f}}^+; 1_\gamma'^-, 2_\gamma'^+) = 2ie^2Q_f^2\frac{\langle 2'\,1\rangle[1'\,2]}{\langle 1'\,1\rangle[2'\,1]} = 2ie^2Q_f^2\frac{\cos\theta}{\sin\theta}e^{-i\phi}\,, \quad (7.143)$$

$$i\mathcal{M}_4(1_f^-, 2_{\bar{f}}^+; 1_\gamma'^+, 2_\gamma'^-) = 2ie^2Q_f^2\frac{\langle 1'\,1\rangle[2'\,2]}{\langle 2'\,1\rangle[1'\,1]} = -2ie^2Q_f^2\frac{\sin\theta}{\cos\theta}e^{3i\phi}\,, \quad (7.144)$$

$$i\mathcal{M}_4(1_f^-, 2_{\bar{f}}^+; 1_{g^a}'^-, 2_{g^b}'^+)\delta^{ab} = 2iC_F g_s^2 c_f^2\frac{\langle 2'\,1\rangle[1'\,2]}{\langle 1'\,1\rangle[2'\,1]} = 2iC_F g_s^2 c_f^2\frac{\cos\theta}{\sin\theta}e^{-i\phi}\,, \quad (7.145)$$

$$i\mathcal{M}_4(1_f^-, 2_{\bar{f}}^+; 1_{g^a}'^+, 2_{g^b}'^-)\delta^{ab} = 2iC_F g_s^2 c_f^2\frac{\langle 1'\,1\rangle[2'\,2]}{\langle 2'\,1\rangle[1'\,1]} = -2iC_F g_s^2 c_f^2\frac{\sin\theta}{\cos\theta}e^{3i\phi}\,, \quad (7.146)$$

where we exploited the usual parametrization of the internal helicity spinors.

These are then multiplied by the corresponding stress-energy form factors, out of which we can isolate the terms with the relevant phases:

$$\langle 1'^{-}_{f}, 2'^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\,|_0 = \cos^2\theta(\cos^2\theta - 3\sin^2\theta)\,\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ , \tag{7.147}$$

$$\langle 2'^{-}_{f}, 1'^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\,|_{-2} = \sin^2\theta(3\cos^2\theta - \sin^2\theta)e^{-2i\phi}\,\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ , \tag{7.148}$$

$$\langle 1'^{-}_{\gamma}, 2'^{+}_{\gamma}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\,|_1 = 4\cos^3\theta\sin\theta e^{i\phi}\,\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ , \tag{7.149}$$

$$\langle 1'^{+}_{\gamma}, 2'^{-}_{\gamma}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\,|_{-3} = -4\sin^3\theta\cos\theta e^{-3i\phi}\,\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ , \tag{7.150}$$

$$\langle 1'^{-}_{g^a}, 2'^{+}_{g^b}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\,|_1 = 4\cos^3\theta\sin\theta e^{i\phi}\delta^{ab}\,\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ , \tag{7.151}$$

$$\langle 1'^{+}_{g^a}, 2'^{-}_{g^b}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\,|_{-3} = -4\sin^3\theta\cos\theta e^{-3i\phi}\delta^{ab}\,\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle\ . \tag{7.152}$$

Finally, we can insert all these expressions in Eq. (7.140), in which the stress-energy form factor $\langle 1^{-}_{f}, 2^{+}_{\bar f}|T^{\alpha\beta\,\dot\alpha\dot\beta}|0\rangle$ correctly factorizes from both sides:

$$\begin{aligned} i\pi\Gamma_f = \frac{1}{16\pi}i(e^2Q_f^2 + C_F g_s^2 c_f^2)\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta \\ \times\left[2\frac{\cos^2\theta}{\sin^2\theta}\cos^2\theta(\cos^2\theta - 3\sin^2\theta) - 2\frac{\sin^2\theta}{\cos^2\theta}\sin^2\theta(3\cos^2\theta - \sin^2\theta)\right. \\ \left. + 8\frac{\cos\theta}{\sin\theta}\cos^3\theta\sin\theta + 8\frac{\sin\theta}{\cos\theta}\sin^3\theta\cos\theta\right], \end{aligned} \tag{7.153}$$

which can be rewritten as

$$\Gamma_f = \frac{1}{16\pi^2}(e^2Q_f^2 + C_F g_s^2 c_f^2)\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta\,\frac{2 - 6\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\ . \tag{7.154}$$

Finally, we can recall Eqs. (7.85) and (7.88) to compute the value of the anomalous dimension matrix elements γ_{SS} and γ_{PP} as

$$\begin{aligned} \gamma_{SS} = \gamma_{PP} &= \Gamma_f - \frac{1}{16\pi^2}(e^2Q_f^2 + C_F g_s^2 c_f^2)\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta\,\frac{2}{\sin^2\theta\cos^2\theta} \\ &= \frac{1}{16\pi^2}(e^2Q_f^2 + C_F g_s^2 c_f^2)\int_0^{\pi/2} 2\sin\theta\cos\theta\,\mathrm{d}\theta\,(-6) \\ &= -\frac{3}{8\pi^2}(e^2Q_f^2 + C_F g_s^2 c_f^2) \\ &= -\frac{3}{8\pi^2}\left(e^2Q_f^2 + \frac{4}{3}g_s^2 c_f^2\right)\ . \end{aligned}$$

(7.155)

Therefore, we can finally write the beta functions of the Wilson coefficients associated with the Yukawa operators $\phi\bar{f}_i f_j$ and $i\phi\bar{f}_i\gamma_5 f_j$ as

$$\frac{\mathrm{d}(vy_S^{ij})}{\mathrm{d}\log\mu} = -\frac{3}{8\pi^2}\left(e^2Q_f^2 + \frac{4}{3}g_s^2c_f^2\right)vy_S^{ij}\,, \tag{7.156}$$

$$\frac{\mathrm{d}(vy_P^{ij})}{\mathrm{d}\log\mu} = -\frac{3}{8\pi^2}\left(e^2Q_f^2 + \frac{4}{3}g_s^2c_f^2\right)vy_P^{ij}\,. \tag{7.157}$$

Chapter 8

Conclusions

ALPs are very compelling new physics candidates in extensions of the SM with a spontaneously broken global symmetry. The mass scale of the new physics sector is set by the scale Λ at which the global symmetry is broken, whereas the mass of the ALP, which is the pseudo Nambu-Goldstone boson associated with this symmetry, is significantly smaller. The phenomenology of ALPs at low energies is described at leading order in the EFT expansion by dimension-five effective operators, suppressed by $1/\Lambda$. The ALP EFT considered in this thesis is CP violating since is given by the sum of a shift symmetry invariant sector with a shift symmetry breaking one.

In this master thesis work, we successfully computed the RGEs for a CP violating ALP effective theory at one-loop level through two fundamentally different methods. The results obtained with the standard Feynman diagrammatic approach read

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\begin{pmatrix} C_\gamma \\ C_g \\ vy_S^{ij} \end{pmatrix} = \gamma \begin{pmatrix} C_\gamma \\ C_g \\ vy_S^{ij} \end{pmatrix} , \quad \frac{\mathrm{d}}{\mathrm{d}\log\mu}\begin{pmatrix} \widetilde{C}_\gamma \\ \widetilde{C}_g \\ vy_P^{ij} \end{pmatrix} = \tilde{\gamma} \begin{pmatrix} \widetilde{C}_\gamma \\ \widetilde{C}_g \\ vy_P^{ij} \end{pmatrix} , \tag{8.1}$$

where the anomalous dimension matrices corresponding to the shift symmetry breaking and shift symmetry invariant sectors of the ALP EFT are respectively given by

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3e^4}{2\pi^2} m_i \delta^{ij} Q_f^2 & \frac{2g_s^4}{\pi^2} m_i \delta^{ij} c_f^2 & -\frac{3}{8\pi^2}(e^2 Q_f^2 + \frac{4}{3} g_s^2 c_f^2) \end{pmatrix} , \tag{8.2}$$

$$\tilde{\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3e^4}{2\pi^2} m_i \delta^{ij} Q_f^2 & -\frac{2g_s^4}{\pi^2} m_i \delta^{ij} c_f^2 & -\frac{3}{8\pi^2}(e^2 Q_f^2 + \frac{4}{3} g_s^2 c_f^2) \end{pmatrix} . \tag{8.3}$$

The fact that the Wilson coefficients associated with the operators ϕVV and $\phi V\widetilde{V}$, with $V = F, G$, are scale independent at one-loop order once e^2 and g_s^2 are factored out, is in agreement with the results in the literature, in particular with Ref. [24] for the operator $\phi G\widetilde{G}$. Additionally, the result for the anomalous dimension matrix corresponding to the shift symmetry invariant sector agree with the expressions derived in Refs. [8, 23, 28].

The most important phenomenological implication of these results concerns the fact that γ and $\tilde{\gamma}$ contain non-vanishing non-diagonal elements, which are proportional to the fermion masses m_i and correspond to the flavor-diagonal diagrams in Eqs. (5.117), (5.118), (5.142) and (5.143). Indeed, these operator mixing effects induce at the quantum level an interaction between the ALP and fermions even if the tree-level Wilson coefficients y_S^{ii} and y_P^{ii} are vanishing. On the other hand, the diagrams in Eqs. (5.119), (5.120), (5.144) and (5.145) give rise to multiplicative renormalization effects and, in particular, to the matrix element $-\frac{3}{8\pi^2}(e^2 Q_f^2 + \frac{4}{3} g_s^2 c_f^2)$ shared by both the anomalous dimension matrices γ and $\tilde{\gamma}$.

The scale invariance of the Wilson coefficients C_a and $\widetilde{C}_a$, with $a = \gamma, g$, has been proven also via the on-shell method based on the form factors. The results obtained through this method agree with those computed with Feynman diagrams, except for the non-diagonal entries of γ and $\tilde{\gamma}$ that are proportional to the fermion masses m_i. This is a direct consequence of the fact that the method of form factors that we used relies on the fundamental assumption of all particles being massless.

The operator mixing effects that show such a mass dependence can nevertheless be approached in different ways. For example, one could extend the method of form factors to massive states, as it was recently developed in Ref. [30]. Alternatively, one could compute them by exploiting the fact that, in the SM, the masses of fermions are generated by the Higgs mechanism. Indeed, any amplitude proportional to the Higgs vacuum expectation value v can be directly connected to another amplitude where v is replaced by a physical Higgs boson h insertion. Indeed, v and h appear in the same component of the Higgs doublet, which in the unitary gauge read $H = \big(0,\ (v+h)/\sqrt{2}\big)^T$. As a consequence of the electroweak symmetry breaking pattern one can compute all the RGEs for a given operator that shows an implicit dependence on v, *e.g.* through fermion masses, by computing the RGEs relative to the very same operator but with an additional Higgs boson.

Additionally, we can mention the fact that the results we have obtained have been double-checked with a different parametrization of the internal helicity spinors, which makes use of the Stokes' integration technique developed in Ref. [57].

We can conclude that the results obtained via on-shell methods represent a crucial step towards the development of a two-loop RGEs program for ALP theories, which is missing so far in the literature. Indeed, the computational advantages of these unitarity-based on-shell methods over the traditional Feynman diagrammatic approach become apparent as the loop order increases.

Appendix A

Renormalization in QED & QCD

This Appendix is devoted to the renormalization program of the QED and QCD sectors of the SM. In particular, the renormalization parameters associated with the wavefunctions of fermion, photon and gluon fields are calculated, as well as the beta functions of the gauge couplings and masses. These results are exploited in Chapter 5 to obtain the values of the anomalous dimension matrix elements of the ALP EFT, defined in Chapter 4, through the standard Feynman diagrammatic approach.

The renormalized and bare Lagrangians associated with the $U(1)_{\rm em} \times SU(3)_c$ gauge invariant sector of the SM are respectively given by

$$
\begin{aligned}
\mathscr{L}_{\rm SM}^{\rm ren.} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - g_s\mu^{\varepsilon/2} f^{abc}(\partial_\mu A_\nu^a)A^{b\,\mu}A^{c\,\nu} \\
&- \frac{1}{4}g_s^2\mu^\varepsilon f^{eab}f^{ecd}A_\mu^a A_\nu^b A^{c\,\mu}A^{d\,\nu} - \bar{c}^a\Box c^a + g_s\mu^{\varepsilon/2}f^{abc}(\partial^\mu\bar{c}^a)A_\mu^b c^c \\
&+ i\bar{f}_i[\partial\!\!\!/ + i\mu^{\varepsilon/2}(eQ_f A\!\!\!/ + g_s c_f T^a A\!\!\!/^a)]f_i - m_i\bar{f}_i f_i\,,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\mathscr{L}_{\rm SM}^{0} = &-\frac{Z_\gamma}{4}F_{\mu\nu}F^{\mu\nu} - \frac{Z_g}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 \\
&- Z_g^{3/2}Z_{g_s}g_s\mu^{\varepsilon/2}f^{abc}(\partial_\mu A_\nu^a)A^{b\,\mu}A^{c\,\nu} \\
&- \frac{Z_g^2 Z_{g_s}^2}{4}g_s^2\mu^\varepsilon f^{eab}f^{ecd}A_\mu^a A_\nu^b A^{c\,\mu}A^{d\,\nu} - Z_c\bar{c}^a\Box c^a \\
&+ Z_c Z_g^{1/2}Z_{g_s}g_s\mu^{\varepsilon/2}f^{abc}(\partial^\mu\bar{c}^a)A_\mu^b c^c \\
&+ iZ_f\bar{f}_i[\partial\!\!\!/ + i\mu^{\varepsilon/2}(Z_\gamma^{1/2}Z_e eQ_f A\!\!\!/ + Z_g^{1/2}Z_{g_s}g_s c_f T^a A\!\!\!/^a)]f_i \\
&- Z_m Z_f m_i\bar{f}_i f_i\,,
\end{aligned}
\tag{A.2}
$$

where we have defined the additional renormalization parameters as

$$m_{i,0} = Z_m m_i \,, \qquad c_0^a = Z_c^{1/2} c^a \,, \tag{A.3}$$

$$e_0 = Z_e e \mu^{\varepsilon/2} \,, \qquad g_{s,0} = Z_{g_s} g_s \mu^{\varepsilon/2} \,. \tag{A.4}$$

Note that we have to consider the anti-commuting Faddeev-Popov ghost fields c^a to preserve unitarity. In particular, their interaction with gluons, as we will see, maintains the gluon propagator transverse with respect to the momentum also at loop-level, so that gauge invariance is not spoiled. The counterterm Lagrangian can be obtained as

$$\mathscr{L}_{\text{SM}}^{\text{ct}} = \mathscr{L}_{\text{SM}}^{0} - \mathscr{L}_{\text{SM}}^{\text{ren.}} \,. \tag{A.5}$$

A.1 Gauge group conventions

Concerning the gauge group $SU(N_c)$, we adopt the following conventions. The generators T^a, with $a = 1, \dots, N_c^2 - 1 = 8$ being an index of the adjoint representation, satisfy the commutation relation

$$[T^a, T^b] = i f^{abc} T^c \,, \tag{A.6}$$

where f^{abc} are the totally anti-symmetric structure constants, explicitly given by

$$f^{123} = 1 \,, \tag{A.7}$$

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \,, \tag{A.8}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2} \,, \tag{A.9}$$

while the ones that cannot be related to these by permuting indices are zero. In the fundamental representation, the generators T^a_{IJ} are Hermitian $N_c \times N_c$ traceless matrices, normalized as

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \tag{A.10}$$

and which can be written as $T^a = \lambda^a/2$, where

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{A.11}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{A.12}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{A.13}$$

are the Gell-Mann matrices. With this normalization, the structure constants can be traded with commutators and products of generators as

$$f^{abc} = -2i\mathrm{Tr}(T^a[T^b, T^c])\,, \tag{A.14}$$

which is a very useful relation in the context of gluon scatterings. Additionally, the $SU(N_c)$ Fierz-type identity reads

$$T^a_{IJ}T^a_{KL} = \frac{1}{2}\left(\delta_{IL}\delta_{KJ} - \frac{1}{N_c}\delta_{IJ}\delta_{KL}\right) \tag{A.15}$$

and can be understood as a completeness relation for a basis of Hermitian matrices spanned by $\{\mathbb{1}, T^a\}$. The Casimir operators in the fundamental and adjoint representations are respectively given by

$$T^a_{IK}T^a_{KJ} = C_F\delta_{IJ}\,, \qquad f^{acd}f^{bcd} = C_A\delta^{ab}\,, \tag{A.16}$$

where

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}\,, \qquad C_A = N_c = 3\,. \tag{A.17}$$

The gluon field strenght tensor is

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{A.18}$$

and the interaction between gluons and matter is mediated by the covariant derivative, defined as

$$\bar{f}^I i\not{D}_{IJ} f^J = \bar{f}^I i(\not{\partial}\delta_{IJ} + ig_s c_f T^a_{IJ}\not{A}^a) f^J\,, \tag{A.19}$$

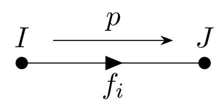
where c_f is a color number, which is 0 if f is a lepton (ℓ) and 1 if f is a quark (q):

$$c_f = \begin{cases} 0 & \text{if } f = \ell \\ 1 & \text{if } f = q \end{cases}\,. \tag{A.20}$$

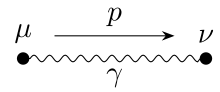
A.2 Feynman rules

Once fixed the conventions, we can report the Feynman rules associated with $\mathscr{L}_{\mathrm{SM}}^{\mathrm{ren.}}$.

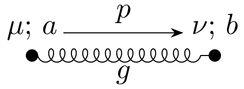
Propagators: in the Feynman-'t Hooft gauge the propagators read



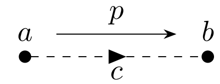
$$= \frac{i\delta_{IJ}}{\not{p} - m_i + i\epsilon}\,; \tag{A.21}$$



$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}\,; \tag{A.22}$$

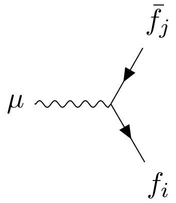


$$= \frac{-ig_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}\,; \tag{A.23}$$

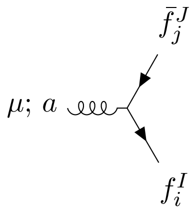


$$= \frac{i\delta^{ab}}{p^2 + i\epsilon}\,. \tag{A.24}$$

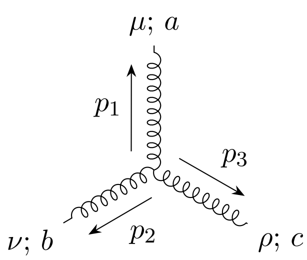
Interaction vertices:



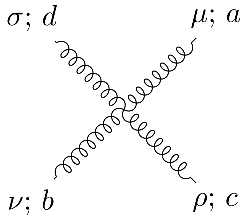
$$= -ieQ_f\delta^{ij}\mu^{\varepsilon/2}\gamma^{\mu}\,; \tag{A.25}$$



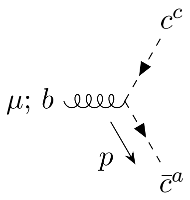
$$= -ig_s c_f T^a_{IJ}\delta^{ij}\mu^{\varepsilon/2}\gamma^{\mu}\,; \tag{A.26}$$



$$\begin{aligned} &= -g_s\mu^{\varepsilon/2}f^{abc}[g^{\mu\nu}(p_1 - p_2)^{\rho} \\ &\quad + g^{\nu\rho}(p_2 - p_3)^{\mu} + g^{\rho\mu}(p_3 - p_1)^{\nu}]\,; \end{aligned} \tag{A.27}$$



$$\begin{aligned} &= -ig_s^2\mu^{\varepsilon}[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]\,; \end{aligned} \tag{A.28}$$

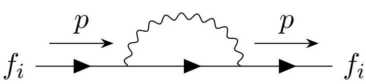


$$= -g_s f^{abc} \mu^{\varepsilon/2} p^\mu \,. \tag{A.29}$$

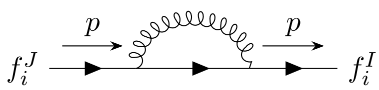
A.3 Propagator corrections

A.3.1 Fermion self-energy

The 1PI diagrams contributing to the fermion propagator correction at one-loop level are mediated by both QED and QCD:



$$= \frac{ie^2 Q_f^2}{16\pi^2}(\not{p} - 4m)\Delta_\varepsilon + \text{finite}\,, \tag{A.30}$$



$$= \frac{ig_s^2 c_f^2}{16\pi^2} C_F \delta_{IJ}(\not{p} - 4m)\Delta_\varepsilon + \text{finite}\,. \tag{A.31}$$

Thus, the sum of their divergent contributions is given by

$$i\Sigma(p)|_{\text{div.}} = \frac{i}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)(\not{p} - 4m)\Delta_\varepsilon \,, \tag{A.32}$$

which can be used to derive the renormalization parameters Z_f and Z_m. Indeed, the requirement that the relevant counterterms

$$\mathscr{L}_{\text{SM}}^{\text{ct}} \supset i(Z_f - 1)\bar{f}_i \not{\partial} f_i - (Z_f Z_m - 1) m_i \bar{f}_i f_i \tag{A.33}$$

cancel this divergence

$$i\Sigma(p)|_{\text{div.}} + i[(Z_f - 1)\not{p} - (Z_f Z_m - 1)m] = 0 \tag{A.34}$$

leads to

$$Z_f = 1 - \frac{1}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon \,, \tag{A.35}$$

$$Z_m = 1 - \frac{3}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon \tag{A.36}$$

at lowest order in the gauge couplings.

Running of the masses

The renormalization group evolution of the masses m_i can be computed imposing the μ-independence of the bare masses $m_{i,0}$:

$$\begin{aligned} 0 &= \frac{\mathrm{d}m_{i,0}}{\mathrm{d}\log\mu} = \frac{\mathrm{d}(Z_m m_i)}{\mathrm{d}\log\mu} = \frac{\mathrm{d}Z_m}{\mathrm{d}\log\mu} m_i + Z_m \frac{\mathrm{d}m_i}{\mathrm{d}\log\mu} \\ &= -\frac{3}{16\pi^2}\left(\frac{\mathrm{d}e^2}{\mathrm{d}\log\mu} Q_f^2 + C_F \frac{\mathrm{d}g_s^2}{\mathrm{d}\log\mu} c_f^2\right)\Delta_\varepsilon m_i \\ &\quad + \left[1 - \frac{3}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon\right]\frac{\mathrm{d}m_i}{\mathrm{d}\log\mu} . \end{aligned} \tag{A.37}$$

We can then exploit two results that will be derived in the following Section, namely

$$\frac{\mathrm{d}e^2}{\mathrm{d}\log\mu} = -\varepsilon e^2 + \mathcal{O}(e^4) , \qquad \frac{\mathrm{d}g_s^2}{\mathrm{d}\log\mu} = -\varepsilon g_s^2 + \mathcal{O}(g_s^4) , \tag{A.38}$$

to write

$$\left[1 - \frac{3}{16\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon\right]\frac{\mathrm{d}m_i}{\mathrm{d}\log\mu} = -\frac{3m_i}{8\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2) , \tag{A.39}$$

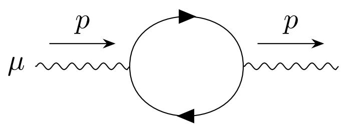
and, by expanding at lowest order in the gauge couplings, we can find the beta function for the masses:

$$\frac{\mathrm{d}m_i}{\mathrm{d}\log\mu} = -\frac{3m_i}{8\pi^2}(e^2 Q_f^2 + C_F g_s^2 c_f^2) . \tag{A.40}$$

In this case we do not have to take the limit $\varepsilon \to 0$, since the bare masses $m_{i,0}$ have a mass dimension equal to 1 also in $d = 4 - \varepsilon$ spacetime dimensions.

A.3.2 Photon vacuum polarization

The 1PI diagram that contributes to the photon propagator at one-loop is given by

$\sum_f$  $= i\Pi_{\mu\nu}(p) ,$ (A.41)

with

$$i\Pi_{\mu\nu}(p)|_{\text{div.}} = \frac{ie^2}{12\pi^2}(p_\mu p_\nu - g_{\mu\nu}p^2)\Delta_\varepsilon \sum_f Q_f^2 \tag{A.42}$$

and the sum runs over all the Dirac fermions. The relevant counterterm needed to cancel this divergence is

$$\mathscr{L}_{\text{SM}}^{\text{ct}} \supset -\frac{1}{4}(Z_\gamma - 1)F_{\mu\nu}F^{\mu\nu} , \tag{A.43}$$

from which we can obtain the renormalization parameter Z_γ imposing

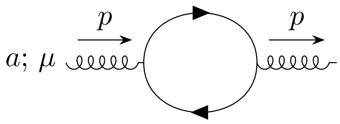
$$i\Pi_{\mu\nu}(p)|_{\text{div.}} + i(Z_\gamma - 1)(p_\mu p_\nu - g_{\mu\nu}p^2) = 0\,, \tag{A.44}$$

namely

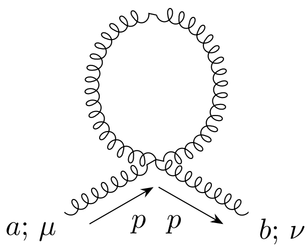
$$Z_\gamma = 1 - \frac{e^2}{12\pi^2}\Delta_\varepsilon \sum_f Q_f^2\,. \tag{A.45}$$

A.3.3 Gluon vacuum polarization

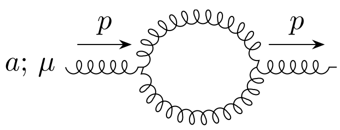
The 1PI diagrams contributing to the gluon propagator correction at one-loop level are given by

$\sum_f$ 

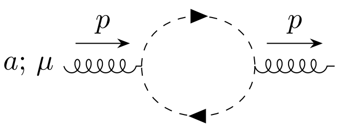
$= i\Pi^{ab(1)}_{\mu\nu}(p)\,,$ (A.46)



$= i\Pi^{ab(2)}_{\mu\nu}(p)\,,$ (A.47)



$= i\Pi^{ab(3)}_{\mu\nu}(p)\,,$ (A.48)



$= i\Pi^{ab(4)}_{\mu\nu}(p)\,,$ (A.49)

where their divergent parts are

$$i\Pi^{ab(1)}_{\mu\nu}(p)|_{\text{div.}} = \frac{ig_s^2}{24\pi^2}(p_\mu p_\nu - g_{\mu\nu}p^2)\delta^{ab}\Delta_\varepsilon \sum_f c_f^2\,, \tag{A.50}$$

$$i\Pi^{ab(3)}_{\mu\nu}(p)|_{\text{div.}} = \frac{ig_s^2}{32\pi^2}C_A\left(-\frac{11}{3}p_\mu p_\nu + \frac{19}{6}g_{\mu\nu}p^2\right)\delta^{ab}\Delta_\varepsilon\,, \tag{A.51}$$

$$i\Pi^{ab(4)}_{\mu\nu}(p)|_{\text{div.}} = \frac{ig_s^2}{32\pi^2}C_A\left(\frac{1}{3}p_\mu p_\nu + \frac{1}{6}g_{\mu\nu}p^2\right)\delta^{ab}\Delta_\varepsilon \tag{A.52}$$

and $i\Pi^{ab(2)}_{\mu\nu}(p)$ is identically vanishing since it is proportional to the scaleless integral

$$\int \frac{\mathrm{d}^d k}{k^2} = 0\,. \tag{A.53}$$

We can note that $i\Pi^{ab(3)}_{\mu\nu}(p)|_{\text{div.}}$ and $i\Pi^{ab(4)}_{\mu\nu}(p)|_{\text{div.}}$ are not separately transverse, but only their sum is:

$$i\Pi^{ab(3)}_{\mu\nu}(p)|_{\text{div.}} + i\Pi^{ab(4)}_{\mu\nu}(p)|_{\text{div.}} = \frac{5i}{48\pi^2} C_A g_s^2 (g_{\mu\nu}p^2 - p_\mu p_\nu)\delta^{ab}\Delta_\varepsilon\,, \tag{A.54}$$

$$p^\mu[i\Pi^{ab(3)}_{\mu\nu}(p)|_{\text{div.}} + i\Pi^{ab(4)}_{\mu\nu}(p)|_{\text{div.}}] = 0\,. \tag{A.55}$$

The total one-loop divergent correction to the gluon propagator

$$\begin{aligned} i\Pi^{ab}_{\mu\nu}(p)|_{\text{div.}} &= [i\Pi^{ab(1)}_{\mu\nu}(p) + i\Pi^{ab(2)}_{\mu\nu}(p) + i\Pi^{ab(3)}_{\mu\nu}(p) + i\Pi^{ab(4)}_{\mu\nu}(p)]|_{\text{div.}} \\ &= \frac{ig_s^2}{24\pi^2}\left(\frac{5}{2}C_A - \sum_f c_f^2\right)(g_{\mu\nu}p^2 - p_\mu p_\nu)\delta^{ab}\Delta_\varepsilon \end{aligned} \tag{A.56}$$

is then gauge invariant and is cured by the counterterm

$$\mathscr{L}^{\text{ct}}_{\text{SM}} \supset -\frac{1}{4}(Z_g - 1)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2\,. \tag{A.57}$$

From the condition

$$i\Pi^{ab}_{\mu\nu}(p)|_{\text{div.}} + i(Z_g - 1)(p_\mu p_\nu - g_{\mu\nu}p^2)\delta^{ab} = 0 \tag{A.58}$$

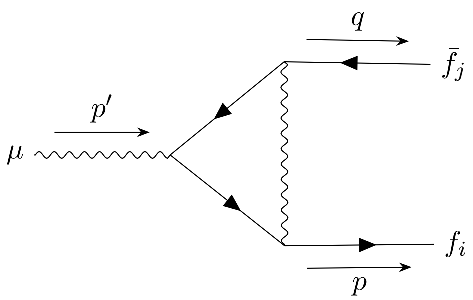
we can obtain the value of the renormalization parameter Z_g, namely

$$Z_g = 1 + \frac{g_s^2}{24\pi^2}\left(\frac{5}{2}C_A - \sum_f c_f^2\right)\Delta_\varepsilon\,. \tag{A.59}$$

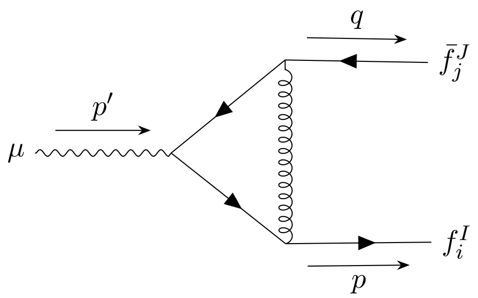
A.4 Vertex corrections

A.4.1 Running of the electric charge

The 1PI diagrams contributing to the QED vertex correction at one-loop level are given by



$$= -ieQ_f\mu^{\varepsilon/2}\Lambda^{(\gamma)}_\mu(p', p, q)\,, \tag{A.60}$$



$$= -ieQ_f\mu^{\varepsilon/2}\Lambda^{(g)}_{\mu\,IJ}(p',p,q)\,, \tag{A.61}$$

where their divergent contributions are respectively provided by

$$\Lambda^{(\gamma)}_{\mu}(p',p,q)|_{\text{div.}} = \frac{1}{16\pi^2}e^2Q_f^2\gamma_\mu\Delta_\varepsilon\,, \tag{A.62}$$

$$\Lambda^{(g)}_{\mu\,IJ}(p',p,q)|_{\text{div.}} = \frac{1}{16\pi^2}C_F g_s^2 c_f^2\delta_{IJ}\gamma_\mu\Delta_\varepsilon\,. \tag{A.63}$$

The divergence of their sum is absorbed by the counterterm

$$\mathscr{L}^{\text{ct}}_{\text{SM}} \supset -(Z_f Z_\gamma^{1/2} Z_e - 1)eQ_f\mu^{\varepsilon/2}\bar{f}_i \not{A} f_i \tag{A.64}$$

and from the condition

$$0 = -ieQ_f\mu^{\varepsilon/2}[\Lambda^{(\gamma)}_\mu(p',p,q)+\Lambda^{(g)}_\mu(p',p,q)]|_{\text{div.}} - i(Z_f Z_\gamma^{1/2} Z_e - 1)eQ_f\mu^{\varepsilon/2}\gamma_\mu \tag{A.65}$$

we obtain

$$Z_f Z_\gamma^{1/2} Z_e = 1 - \frac{1}{16\pi^2}(e^2Q_f^2 + C_F g_s^2 c_f^2)\Delta_\varepsilon\,, \tag{A.66}$$

which is precisely the value of Z_f. This is not a numerical coincidence, but is rather a well known result related to gauge invariance. Indeed, it follows from the Ward-Takahashi identity and it holds at all orders in perturbation theory. As a consequence, we find that the renormalization of the electric charge is determined completely by the renormalization of the photon field strength

$$e_0 = Z_\gamma^{-1/2} e\mu^{\varepsilon/2} = \left(1 - \frac{e^2}{12\pi^2}\Delta_\varepsilon \sum_f Q_f^2\right)^{-1/2} e\mu^{\varepsilon/2}\,. \tag{A.67}$$

We can expand this expression neglecting $\mathcal{O}(e^5)$ terms as

$$e_0 = \left(1 + \frac{e^2}{24\pi^2}\Delta_\varepsilon \sum_f Q_f^2\right) e\mu^{\varepsilon/2} \tag{A.68}$$

and from the μ-independence of the bare electric charge we can write

$$
\begin{aligned}
0 = \frac{\mathrm{d}e_0}{\mathrm{d}\log\mu} &= \frac{\mathrm{d}}{\mathrm{d}\log\mu}\left[\left(1+\frac{e^2}{24\pi^2}\Delta_\varepsilon\sum_f Q_f^2\right)e\mu^{\varepsilon/2}\right] \\
&= \frac{\mathrm{d}e}{\mathrm{d}\log\mu}\frac{e^2}{12\pi^2}\Delta_\varepsilon\sum_f Q_f^2\mu^{\varepsilon/2} + \left(1+\frac{e^2}{24\pi^2}\Delta_\varepsilon\sum_f Q_f^2\right)\frac{\mathrm{d}e}{\mathrm{d}\log\mu}\mu^{\varepsilon/2} \\
&+ \frac{\varepsilon}{2}\left(1+\frac{e^2}{24\pi^2}\Delta_\varepsilon\sum_f Q_f^2\right)e\mu^{\varepsilon/2}\,,
\end{aligned} \tag{A.69}
$$

namely

$$
\left(1+\frac{e^2}{8\pi^2}\Delta_\varepsilon\sum_f Q_f^2\right)\frac{\mathrm{d}e}{\mathrm{d}\log\mu} = -\frac{\varepsilon}{2}e - \frac{e^3}{24\pi^2}\sum_f Q_f^2\,. \tag{A.70}
$$

Expanding one more time in powers of e and taking the limit $\varepsilon \to 0$, we can finally obtain the beta function related to the electric charge at one-loop order

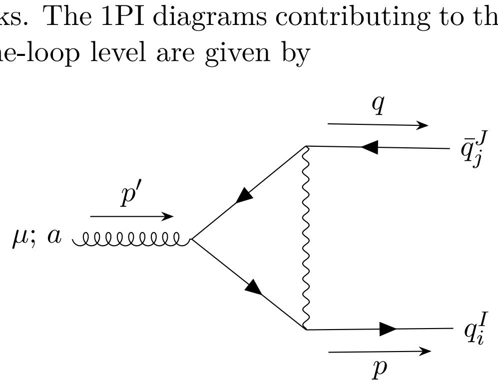
$$
\begin{aligned}
\frac{\mathrm{d}e}{\mathrm{d}\log\mu} &= -\frac{\varepsilon}{2}e - \frac{e^3}{24\pi^2}\sum_f Q_f^2 + \frac{e^3}{8\pi^2}\sum_f Q_f^2 \\
&= -\frac{\varepsilon}{2}e + \frac{e^3}{12\pi^2}\sum_f Q_f^2 \xrightarrow{\varepsilon\to 0} \frac{e^3}{12\pi^2}\sum_f Q_f^2\,.
\end{aligned} \tag{A.71}
$$

Equivalently, we can express the anomalous dimension of e^2 as

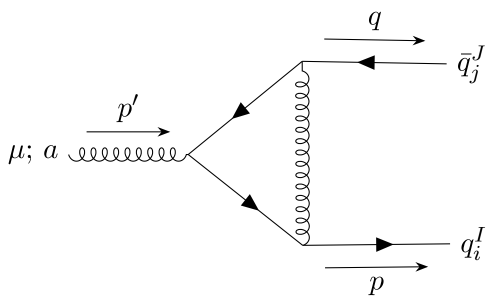
$$
\frac{1}{e^2}\frac{\mathrm{d}e^2}{\mathrm{d}\log\mu} = \frac{1}{e^2}2e\frac{\mathrm{d}e}{\mathrm{d}\log\mu} = -\varepsilon + \frac{e^2}{6\pi^2}\sum_f Q_f^2 \xrightarrow{\varepsilon\to 0} \frac{e^2}{6\pi^2}\sum_f Q_f^2\,. \tag{A.72}
$$

A.4.2 Running of the strong coupling constant

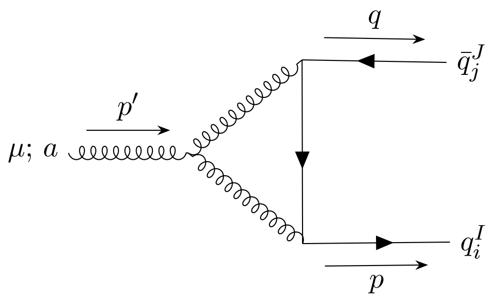
In order to obtain the beta function of the strong coupling constant, we can consider the renormalization of the interaction vertex between gluons and quarks. The 1PI diagrams contributing to the correction of this QCD vertex at one-loop level are given by



$$
= -ig_s\mu^{\varepsilon/2}\Lambda^{a(\gamma)}_{\mu\,IJ}(p',p,q)\,, \tag{A.73}
$$



$$= -i g_s \mu^{\varepsilon/2} \Lambda^{a(g1)}_{\mu\, IJ}(p', p, q)\,, \tag{A.74}$$



$$= -i g_s \mu^{\varepsilon/2} \Lambda^{a(g2)}_{\mu\, IJ}(p', p, q)\,, \tag{A.75}$$

where their divergent contributions are respectively provided by

$$\Lambda^{a(\gamma)}_{\mu\, IJ}(p', p, q)|_{\text{div.}} = \frac{e^2 Q_q^2}{16\pi^2} \gamma_\mu T^a_{IJ} \Delta_\varepsilon\,, \tag{A.76}$$

$$\Lambda^{a(g1)}_{\mu\, IJ}(p', p, q)|_{\text{div.}} = \frac{g_s^2}{16\pi^2} \gamma_\mu T^a_{IJ} \left(C_F - \frac{C_A}{2}\right) \Delta_\varepsilon\,, \tag{A.77}$$

$$\Lambda^{a(g2)}_{\mu\, IJ}(p', p, q)|_{\text{div.}} = \frac{g_s^2}{16\pi^2} \gamma_\mu T^a_{IJ} \frac{3C_A}{2} \Delta_\varepsilon\,, \tag{A.78}$$

where for the second and third diagrams the following identities have been exploited

$$T^b_{IK} T^a_{KL} T^b_{KJ} = \left(C_F - \frac{C_A}{2}\right) T^a_{IJ}\,, \tag{A.79}$$

$$f^{abc} T^c_{IK} T^b_{KJ} = -\frac{i}{2} C_A T^a_{IJ}\,. \tag{A.80}$$

The divergence of their sum is absorbed by the counterterm

$$\mathscr{L}^{\text{ct}}_{\text{SM}} \supset -(Z_q Z_g^{1/2} Z_{g_s} - 1) g_s \mu^{\varepsilon/2} \bar{q}_i \not{A} q_i \tag{A.81}$$

and from the condition

$$\begin{aligned} 0 = &-i g_s \mu^{\varepsilon/2} [\Lambda^{a(\gamma)}_{\mu\, IJ}(p', p, q) + \Lambda^{a(g1)}_{\mu\, IJ}(p', p, q) + \Lambda^{a(g2)}_{\mu\, IJ}(p', p, q)]|_{\text{div.}} \\ &- i(Z_q Z_g^{1/2} Z_{g_s} - 1) g_s \mu^{\varepsilon/2} \gamma_\mu T^a_{IJ} \end{aligned} \tag{A.82}$$

we obtain

$$Z_q Z_g^{1/2} Z_{g_s} = 1 - \frac{g_s^2}{16\pi^2}(C_F + C_A)\Delta_\varepsilon - \frac{e^2 Q_q^2}{16\pi^2}\Delta_\varepsilon\,. \tag{A.83}$$

Then, recalling from Eq. (A.35) the value of Z_q

$$Z_q = 1 - \frac{1}{16\pi^2}(e^2 Q_q^2 + C_F g_s^2)\Delta_\varepsilon \tag{A.84}$$

and Z_g from Eq. (A.59), we can isolate from Eq. (A.83) the renormalization parameter Z_{g_s} by expanding at lowest order in the gauge couplings

$$\begin{aligned} Z_{g_s} &= \left(1 - \frac{g_s^2}{16\pi^2}(C_F + C_A)\Delta_\varepsilon - \frac{e^2 Q_q^2}{16\pi^2}\Delta_\varepsilon\right) Z_q^{-1} Z_g^{-1/2} \\ &= 1 - \frac{g_s^2}{16\pi^2}(C_F + C_A)\Delta_\varepsilon - \frac{e^2 Q_q^2}{16\pi^2}\Delta_\varepsilon + \frac{1}{16\pi^2}(e^2 Q_q^2 + C_F g_s^2)\Delta_\varepsilon \\ &\quad - \frac{g_s^2}{48\pi^2}\left(\frac{5}{2}C_A - \sum_f c_f^2\right)\Delta_\varepsilon \\ &= 1 + \frac{g_s^2}{96\pi^2}(-11 C_A + 2 n_q)\Delta_\varepsilon\,, \end{aligned} \tag{A.85}$$

where we have denoted the sum $\sum_f c_f^2$ as the number of quarks n_q and observed that the contributions coming from QED have canceled each other, as a consequence of the Ward-Takahashi identity. Therefore, the bare strong coupling $g_{s,0}$ reads

$$g_{s,0} = \left[1 + \frac{g_s^2}{96\pi^2}(-11 C_A + 2 n_q)\Delta_\varepsilon\right] g_s \mu^{\varepsilon/2} \tag{A.86}$$

and following the exact same steps of the computation of the beta function of the electric charge, namely from Eq. (A.67) to (A.71), we can conclude that

$$\frac{\mathrm{d} g_s}{\mathrm{d}\log\mu} = -\frac{\varepsilon}{2} g_s - \frac{g_s^3}{16\pi^2} b_0 \xrightarrow{\varepsilon\to 0} -\frac{g_s^3}{16\pi^2} b_0\,, \tag{A.87}$$

where

$$b_0 = \frac{11}{3} C_A - \frac{2}{3} n_q = 11 - \frac{2}{3} n_q > 0\,. \tag{A.88}$$

Equivalently, we can express the anomalous dimension of g_s^2 as

$$\frac{1}{g_s^2}\frac{\mathrm{d} g_s^2}{\mathrm{d}\log\mu} = \frac{1}{g_s^2} 2 g_s \frac{\mathrm{d} g_s}{\mathrm{d}\log\mu} = -\varepsilon - \frac{g_s^2}{8\pi^2} b_0 \xrightarrow{\varepsilon\to 0} -\frac{g_s^2}{8\pi^2} b_0\,. \tag{A.89}$$

Appendix B

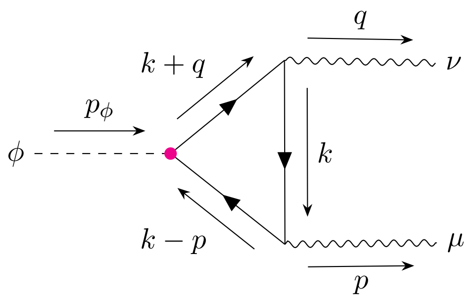
Calculation of one-loop Feynman integrals

In this Appendix, we will explicitly evaluate the divergent contributions of the one-loop 1PI Feynman integrals that are necessary to the calculation – conducted in Chapter 5 – of the anomalous dimension matrix of the ALP EFT defined in Chapter 4.

B.1 ϕFF vertex corrections

B.1.1 $\phi \bar{f} f$ mediated diagrams

The first diagram contributing to the one-loop correction of the $\phi F_{\mu\nu}F^{\mu\nu}$ operator is mediated by the operator $\phi \bar{f}_i f_j$ and is given by



$$= i\mathcal{M}^{(1)}_{\gamma S}\,, \tag{B.1}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(1)}_{\gamma S} &= (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\left[(i\frac{v}{\Lambda}y_S^{ij}\mu^{\varepsilon/2})\frac{i}{\not{k}-\not{p}-m_j+i\epsilon}(-ieQ_f\gamma^\mu\delta^{jk}\mu^{\varepsilon/2})\right.\\ &\quad\times\left.\frac{i}{\not{k}-m_k+i\epsilon}(-ieQ_f\gamma^\nu\delta^{ik}\mu^{\varepsilon/2})\frac{i}{\not{k}+\not{q}-m_i+i\epsilon}\right]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\\ &= \sum_i A\mu^\varepsilon\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{N}{D_1D_2D_3}\,, \end{aligned} \tag{B.2}$$

where we have defined

$$A = \frac{v}{\Lambda}y_S^{ii}e^2Q_f^2\mu^{\varepsilon/2}\,, \tag{B.3}$$

$$N = \mathrm{Tr}\big[(\not{k}-\not{p}+m_i)\gamma^\mu(\not{k}+m_i)\gamma^\nu(\not{k}+\not{q}+m_i)\big]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,, \tag{B.4}$$

$$D_1 = k^2-m_i^2+i\epsilon\,, \tag{B.5}$$

$$D_2 = (k-p)^2-m_i^2+i\epsilon\,, \tag{B.6}$$

$$D_3 = (k+q)^2-m_i^2+i\epsilon\,. \tag{B.7}$$

Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned} \frac{1}{D_1D_2D_3} &= 2\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z\frac{\delta(x+y+z-1)}{[D_1x+D_2y+D_3z]^3}\\ &= 2\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z\frac{\delta(x+y+z-1)}{[(k-py+qz)^2-C+i\epsilon]^3}\,, \end{aligned} \tag{B.8}$$

with

$$C = m_i^2-2p\cdot qyz\,, \tag{B.9}$$

where the on-shellness of the external photons $p^2=q^2=0$ and $x+y+z=1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k=\ell+py-qz$, which implies $\mathrm{d}^dk=\mathrm{d}^d\ell$ and

$$\begin{aligned} \frac{1}{D_1D_2D_3} &= 2\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z\frac{\delta(x+y+z-1)}{(\ell^2-C+i\epsilon)^3}\\ &= 2\int_0^1\mathrm{d}y\int_0^{1-y}\mathrm{d}z\frac{1}{(\ell^2-C+i\epsilon)^3}\,. \end{aligned} \tag{B.10}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned} N &= \mathrm{Tr}[(\not{\ell}+\not{p}(y-1)-\not{q}z+m_i)\gamma^\mu(\not{\ell}+\not{p}y-\not{q}z+m_i)\gamma^\nu\\ &\quad\times(\not{\ell}+\not{p}y+\not{q}(1-z)+m_i)]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\\ &= f(d)m_i(4\ell^\mu\ell^\nu-\ell^2)\epsilon^*_\mu(p)\epsilon^*_\nu(q)+\mathcal{O}(\ell)\,, \end{aligned} \tag{B.11}$$

where the following identities involving the trace of gamma matrices in d-dimensions have been used

$$\mathrm{Tr}[\text{odd } \# \text{ of } \gamma_\mu\text{'s}] = 0 \tag{B.12}$$
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = f(d)(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,, \tag{B.13}$$
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = f(d)g^{\mu\nu}\,, \tag{B.14}$$

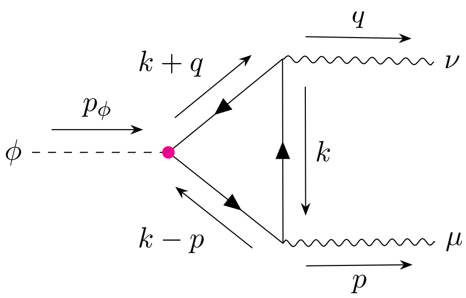
with the function $f(d)$ that has the property of approaching $f(d) \to 4$ as $d \to 4$. We can observe that, once we exploit Lorentz invariance by effectively replacing in the numerator $\ell^\mu\ell^\nu$ with $\ell^2 g^{\mu\nu}/d$ and sending to zero the terms that are proportional to the first power of ℓ, we obtain

$$\begin{aligned} N &= f(d)m_i\ell^2\left(\frac{4}{d} - 1\right) + \mathcal{O}(\ell^0) \\ &= f(4-\varepsilon)m_i\ell^2\left(\frac{4}{4-\varepsilon} - 1\right) + \mathcal{O}(\ell^0) \\ &= 4m_i\ell^2\mathcal{O}(\varepsilon) + \mathcal{O}(\ell^0)\,, \end{aligned} \tag{B.15}$$

namely the numerator is of order ℓ^0 once the $\varepsilon \to 0$ limit is performed. Thus, the degree of divergence of the integral is $d - 2\cdot 3 < 0$, meaning that the diagram converges:

$$i\mathcal{M}^{(1)}_{\gamma S} = \text{finite}\,. \tag{B.16}$$

The second diagram contributing to the one-loop correction of the $\phi F_{\mu\nu}F^{\mu\nu}$ operator is mediated by the operator $\phi\bar{f}_i f_j$ and is given by



$$= i\mathcal{M}^{(2)}_{\gamma S}\,, \tag{B.17}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(2)}_{\gamma S} = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\Bigg[&(i\frac{v}{\Lambda}y^{ij}_S\mu^{\varepsilon/2})\frac{i}{-\not{k} - \not{q} - m_j + i\epsilon}(-ieQ_f\gamma^\nu\delta^{ik}\mu^{\varepsilon/2}) \\ &\times \frac{i}{-\not{k} - m_k + i\epsilon}(-ieQ_f\gamma^\mu\delta^{jk}\mu^{\varepsilon/2})\frac{i}{-\not{k} + \not{p} - m_i + i\epsilon}\Bigg]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \end{aligned} \tag{B.18}$$

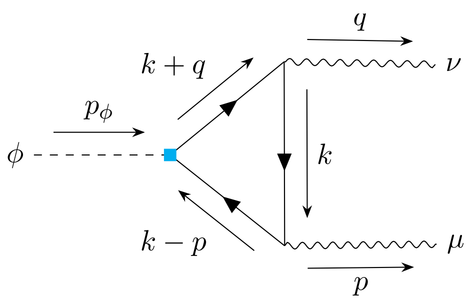
It can be related to the diagram $i\mathcal{M}^{(1)}_{\gamma S}$ via the substitutions $\mu \leftrightarrow \nu$ and $p \leftrightarrow q$, thus

$$i\mathcal{M}^{(2)}_{\gamma S} = \text{finite}\,. \tag{B.19}$$

B.2 $\phi F\widetilde{F}$ vertex corrections

B.2.1 $i\phi\bar{f}\gamma_5 f$ mediated diagrams

The first diagram contributing to the one-loop correction of the $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ operator is mediated by the operator $i\phi\bar{f}_i\gamma_5 f_j$ and is given by



$$= i\mathcal{M}^{(1)}_{\tilde{\gamma}P}\,, \tag{B.20}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(1)}_{\tilde{\gamma}P} &= (-1)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\Big[(-\frac{v}{\Lambda}y_P^{ij}\gamma_5\mu^{\varepsilon/2})\frac{i}{\not{k}-\not{p}-m_j+i\epsilon}(-ieQ_f\gamma^\mu\delta^{jk}\mu^{\varepsilon/2}) \\ &\quad\times\frac{i}{\not{k}-m_k+i\epsilon}(-ieQ_f\gamma^\nu\delta^{ik}\mu^{\varepsilon/2})\frac{i}{\not{k}+\not{q}-m_i+i\epsilon}\Big]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= \sum_i A\mu^\varepsilon\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{N}{D_1D_2D_3}\,, \end{aligned} \tag{B.21}$$

where we have defined

$$A = i\frac{v}{\Lambda}y_P^{ii}e^2Q_f^2\mu^{\varepsilon/2}\,, \tag{B.22}$$

$$N = \mathrm{Tr}\big[\gamma_5(\not{k}-\not{p}+m_i)\gamma^\mu(\not{k}+m_i)\gamma^\nu(\not{k}+\not{q}+m_i)\big]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,, \tag{B.23}$$

$$D_1 = k^2 - m_i^2 + i\epsilon\,, \tag{B.24}$$

$$D_2 = (k-p)^2 - m_i^2 + i\epsilon\,, \tag{B.25}$$

$$D_3 = (k+q)^2 - m_i^2 + i\epsilon\,. \tag{B.26}$$

Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned}\frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[D_1 x + D_2 y + D_3 z]^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[(k-py+qz)^2 - C + i\epsilon]^3}\,,\end{aligned} \tag{B.27}$$

with

$$C = m_i^2 - 2p\cdot qyz\,, \tag{B.28}$$

where the on-shellness of the external photons $p^2 = q^2 = 0$ and $x+y+z=1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = \ell + py - qz$, which implies $\mathrm{d}^d k = \mathrm{d}^d \ell$ and

$$\begin{aligned}\frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{(\ell^2 - C + i\epsilon)^3} \\ &= 2\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{1}{(\ell^2 - C + i\epsilon)^3}\,.\end{aligned} \tag{B.29}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N = \mathrm{Tr}[&\gamma_5(\not{\ell} + \not{p}(y-1) - \not{q}z + m_i)\gamma^\mu(\not{\ell} + \not{p}y - \not{q}z + m_i)\gamma^\nu \\ &\times (\not{\ell} + \not{p}y + \not{q}(1-z) + m_i)]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ = 4i&m_i(2y-1)\epsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \epsilon^*_\mu(p)\epsilon^*_\nu(q)\,,\end{aligned} \tag{B.30}$$

where the cyclicity of the trace and the following identities involving the trace of gamma matrices have been used

$$\mathrm{Tr}[\gamma_5] = 0\,, \tag{B.31}$$

$$\mathrm{Tr}[\gamma_5 \times (\text{odd \# of } \gamma_\mu\text{'s})] = 0\,, \tag{B.32}$$

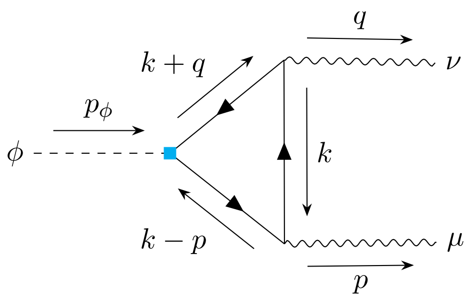
$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0\,, \tag{B.33}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma}\,. \tag{B.34}$$

This last identity indeed holds in the BMHV scheme. Thus, we have found that the numerator is ℓ-independent, and given the fact that the denominator scales as ℓ^6, we can conclude that the diagram is actually convergent. Therefore we can write

$$i\mathcal{M}^{(1)}_{\tilde{\gamma}P} = \text{finite}\,. \tag{B.35}$$

The second diagram contributing to the one-loop correction of the $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ operator is mediated by the operator $i\phi\bar{f}_i\gamma_5 f_j$ and is given by



$$= i\mathcal{M}^{(2)}_{\tilde{\gamma}P}\,, \tag{B.36}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(2)}_{\tilde{\gamma}P} = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\Big[(-\frac{v}{\Lambda}y^{ij}_P\gamma_5\mu^{\varepsilon/2})\frac{i}{-\not{k}-\not{q}-m_j+i\epsilon}(-ieQ_f\gamma^\nu\delta^{ik}\mu^{\varepsilon/2}) \\ \times\frac{i}{-\not{k}-m_k+i\epsilon}(-ieQ_f\gamma^\mu\delta^{jk}\mu^{\varepsilon/2})\frac{i}{-\not{k}+\not{p}-m_i+i\epsilon}\Big]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \end{aligned} \tag{B.37}$$

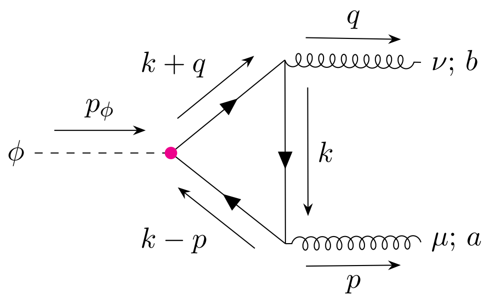
It can be related to the diagram $i\mathcal{M}^{(1)}_{\tilde{\gamma}P}$ via the substitutions $\mu \leftrightarrow \nu$ and $p \leftrightarrow q$, thus

$$i\mathcal{M}^{(2)}_{\tilde{\gamma}P} = \text{finite}\,. \tag{B.38}$$

B.3 ϕGG vertex corrections

B.3.1 $\phi\bar{f}f$ mediated diagrams

The first diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ operator is mediated by the operator $\phi\bar{f}_i f_j$ and is given by



$$= i\mathcal{M}^{(1)}_{gS}\,, \tag{B.39}$$

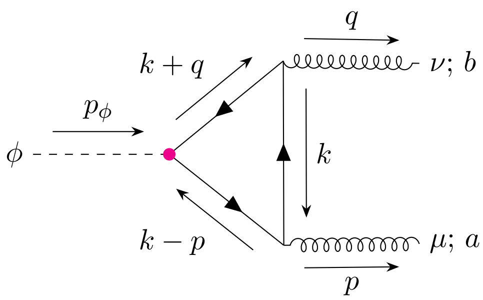
which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$i\mathcal{M}^{(1)}_{gS} = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\Big[(i\frac{v}{\Lambda}y^{ij}_S\mu^{\varepsilon/2})\frac{i}{\not{k}-\not{p}-m_j+i\epsilon}(-ig_s c_f T^a_{IJ}\gamma^\mu\delta^{jk}\mu^{\varepsilon/2}) \times \frac{i}{\not{k}-m_k+i\epsilon}(-ig_s c_f T^b_{JI}\gamma^\nu\delta^{ik}\mu^{\varepsilon/2})\frac{i}{\not{k}+\not{q}-m_i+i\epsilon}\Big]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \tag{B.40}$$

The structure of this integral is completely analogous to the one of the diagram $i\mathcal{M}^{(1)}_{\gamma S}$. Indeed they are equal except for their overall coefficient. Therefore

$$i\mathcal{M}^{(1)}_{gS} = \text{finite}\,. \tag{B.41}$$

The second diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ operator is mediated by the operator $\phi\bar{f}_i f_j$ and is given by



$$= i\mathcal{M}^{(2)}_{gS}\,, \tag{B.42}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

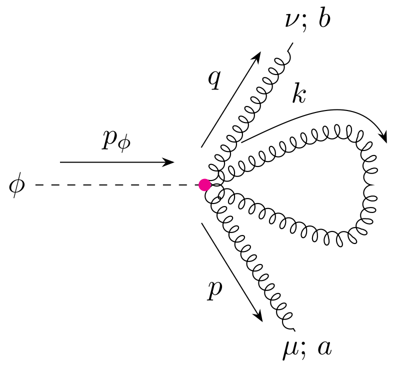
$$i\mathcal{M}^{(2)}_{gS} = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\Big[(i\frac{v}{\Lambda}y^{ij}_S\mu^{\varepsilon/2})\frac{i}{-\not{k}-\not{q}-m_j+i\epsilon}(-ig_s c_f T^b_{JI}\gamma^\nu\delta^{ik}\mu^{\varepsilon/2}) \times \frac{i}{-\not{k}-m_k+i\epsilon}(-ig_s c_f T^a_{IJ}\gamma^\mu\delta^{jk}\mu^{\varepsilon/2})\frac{i}{-\not{k}+\not{p}-m_i+i\epsilon}\Big]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \tag{B.43}$$

It can be related to the diagram $i\mathcal{M}^{(1)}_{gS}$ via the substitutions $\mu \leftrightarrow \nu$, $a \leftrightarrow b$ and $p \leftrightarrow q$, thus

$$i\mathcal{M}^{(2)}_{gS} = \text{finite}\,. \tag{B.44}$$

B.3.2 ϕGG mediated diagrams

The third diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(1)}_{gg}\,, \tag{B.45}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(1)}_{gg} = \frac{4\cdot 3}{4!}\int \frac{\mathrm{d}^d k}{(2\pi)^d} 4ig_s^4 \frac{C_g}{\Lambda}\mu^{3\varepsilon/2}\frac{-ig_{\rho\sigma}\delta^{cd}}{k^2+i\epsilon}&[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})] \\ &\times \epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \end{aligned} \tag{B.46}$$

This diagram is proportional to the scaleless integral

$$\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2+i\epsilon} = 0 \tag{B.47}$$

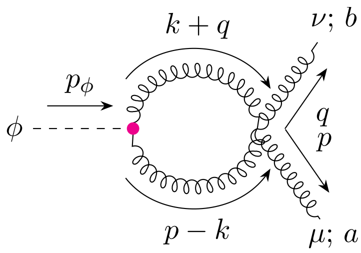
and is therefore vanishing. In order to see whether this is a consequence of a cancellation between UV and IR divergences (as in the case of diagrams $i\mathcal{M}^{(3)}_{gg}$ and $i\mathcal{M}^{(4)}_{gg}$), we can decouple them by giving the gluon a fictitious mass m_g and performing the limit $m_g \to 0$ at the end of the calculation. In this way, the integral can be computed according to the master formula in Eq. (**??**) as

$$\begin{aligned} \mu^\varepsilon I_{0,1} &= \mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-m_g^2+i\epsilon} \\ &= -\frac{i}{16\pi^2}m_g^2\left(\frac{4\pi\mu^2}{m_g^2}\right)^{\varepsilon/2}\frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)\Gamma(d/2)} \\ &= -\frac{i}{16\pi^2}m_g^2\left(\frac{4\pi\mu^2}{m_g^2}\right)^{\varepsilon/2}\Gamma(\varepsilon/2-1)\,. \end{aligned} \tag{B.48}$$

The pole of this integral is proportional to m_g^2 – as required by dimensional analysis – and we can therefore conclude that its divergent contribution vanishes as well

$$i\mathcal{M}_{gg}^{(1)}|_{\text{div.}} = 0\,. \tag{B.49}$$

The fourth diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}_{gg}^{(2)}\,, \tag{B.50}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned}
i\mathcal{M}_{gg}^{(2)} &= \frac{4\cdot 3\cdot 2}{2!\cdot 4!}\int \frac{\mathrm{d}^d k}{(2\pi)^d}(4ig_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2})[(p-k)^\alpha(k+q)^\beta - (p-k)\cdot(k+q)g^{\alpha\beta}] \\
&\quad \times \delta^{cd}\frac{-ig_{\alpha\gamma}\delta^{ce}}{(k+q)^2+i\epsilon}\frac{-ig_{\beta\delta}\delta^{df}}{(p-k)^2+i\epsilon}(-ig_s^2\mu^\varepsilon)[f^{abg}f^{efg}(g^{\mu\gamma}g^{\nu\delta}-g^{\mu\delta}g^{\nu\gamma}) \\
&\quad + f^{aeg}f^{bfg}(g^{\mu\nu}g^{\gamma\delta}-g^{\mu\delta}g^{\nu\gamma}) + f^{afg}f^{beg}(g^{\mu\nu}g^{\gamma\delta}-g^{\mu\gamma}g^{\nu\delta})]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\
&= A\mu^\varepsilon\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{N}{D_1 D_2}\,,
\end{aligned} \tag{B.51}$$

where we have defined

$$A = -2C_A g_s^4\frac{C_g}{\Lambda}\delta^{ab}\mu^{\varepsilon/2}\,, \tag{B.52}$$

$$\begin{aligned}
N &= [(p-k)^\alpha(k+q)^\beta - (p-k)\cdot(k+q)g^{\alpha\beta}]g_{\alpha\gamma}g_{\beta\delta} \\
&\quad \times (2g^{\mu\nu}g^{\gamma\delta} - g^{\mu\delta}g^{\nu\gamma} - g^{\mu\gamma}g^{\nu\delta})\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,,
\end{aligned} \tag{B.53}$$

$$D_1 = (k+q)^2 + i\epsilon\,, \tag{B.54}$$

$$D_2 = (p-k)^2 + i\epsilon\,, \tag{B.55}$$

and exploited the anti-symmetry of the structure constants $f^{abc} = f^{[abc]}$, as well as the fact that $f^{acd}f^{bcd}$ is the Casimir operator in the adjoint representation of $SU(N_c)$ and, by Schur's lemma, is proportional to the identity matrix

$$f^{acd}f^{bcd} = C_A\delta^{ab}\,, \tag{B.56}$$

with $C_A = N_c$. Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned}\frac{1}{D_1D_2} &= \int_0^1 \mathrm{d}x \frac{1}{[xD_1+(1-x)D_2]^2} \\ &= \int_0^1 \mathrm{d}x \frac{1}{\{[k+xq+(x-1)p]^2 - C + i\epsilon\}^2}\,,\end{aligned} \tag{B.57}$$

with

$$C = 2x(x-1)p\cdot q\,, \tag{B.58}$$

where the on-shellness of external gluons $p^2 = q^2 = 0$ has been used. In order to simplify the denominator, we can shift the integration variable k as $k = \ell - xq - (x-1)p$, which implies $\mathrm{d}^dk = \mathrm{d}^d\ell$ and

$$\frac{1}{D_1D_2} = \int_0^1 \mathrm{d}x \frac{1}{(\ell^2 - C + i\epsilon)^2}\,. \tag{B.59}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N &= [(-\ell + x(p+q))^\alpha(\ell-(x-1)(p+q))^\beta - (-\ell+x(p+q)) \\ &\quad \cdot(\ell-(x-1)(p+q))g^{\alpha\beta}](2g^{\mu\nu}g_{\alpha\beta} - \delta^\mu_\beta\delta^\nu_\alpha - \delta^\mu_\alpha\delta^\nu_\beta)\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= [(\ell - x(p+q))^\nu(\ell-(x-1)(p+q))^\mu + (\ell - x(p+q))^\mu \\ &\quad \times(\ell-(x-1)(p+q))^\nu - g^{\mu\nu}(\ell-x(p+q))\cdot(\ell-(x-1)(p+q)) \\ &\quad \times(4-2d)]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= \{(\ell-xp)^\nu(\ell-(x-1)q)^\mu + (\ell-xq)^\mu(\ell-(x-1)p)^\nu \\ &\quad - (4-2d)[\ell^2 + \ell\cdot(p+q) + 2x(x-1)p\cdot q]g^{\mu\nu}\}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,,\end{aligned} \tag{B.60}$$

where the transversality conditions $p\cdot\epsilon^*(p) = q\cdot\epsilon^*(q) = 0$ and $\delta^\mu_\mu = d$ have been used. We can observe that, once we exploit Lorentz invariance by effectively replacing in the numerator $\ell^\mu\ell^\nu$ with $\ell^2g^{\mu\nu}/d$ and sending to zero the terms that are proportional to the first power of ℓ, we obtain

$$\begin{aligned}N &= \ell^2 g^{\mu\nu}\left(\frac{2}{d} + 2d - 4\right)\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &\quad + 2x(x-1)[p^\nu q^\mu + (2d-4)p\cdot q g^{\mu\nu}]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,.\end{aligned} \tag{B.61}$$

We have to keep the terms that do not depend on ℓ since they yield a divergent integral too. Therefore, we have that

$$\begin{aligned}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{N}{D_1 D_2} &= \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 - C + i\epsilon)^2}\Bigg[\ell^2 g^{\mu\nu}\left(\frac{2}{d} + 2d - 4\right) \\ &\quad + 2x(x-1)[p^\nu q^\mu + (2d-4)p\cdot q g^{\mu\nu}]\Bigg]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= \epsilon^*_\mu(p)\epsilon^*_\nu(q)\int_0^1 \mathrm{d}x \Bigg[g^{\mu\nu}\left(\frac{2}{d}+2d-4\right)I_{1,2} \\ &\quad + 2x(x-1)[p^\nu q^\mu + (2d-4)p\cdot q g^{\mu\nu}]I_{0,2}\Bigg]\,. \end{aligned} \tag{B.62}$$

The divergent integrals $I_{1,2}$ and $I_{0,2}$ are computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_\mathrm{E} + \mathcal{O}(x)$ as $x \to 0$:

$$I_{0,2} = \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\Gamma(\varepsilon/2)\,, \tag{B.63}$$

$$I_{1,2} = -\frac{i}{16\pi^2}C\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\left(2-\frac{\varepsilon}{2}\right)\Gamma(\varepsilon/2-1)\,, \tag{B.64}$$

so that

$$\begin{aligned}\mu^\varepsilon g^{\mu\nu}\left(\frac{2}{d}+2d-4\right)I_{1,2} &= \frac{i}{16\pi^2}C\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2} 9\left[\frac{2}{\varepsilon} - \frac{1}{3} - \gamma_\mathrm{E} + \mathcal{O}(\varepsilon)\right] \\ &= \frac{i}{16\pi^2}9C\left[\Delta_\varepsilon - \frac{1}{3} + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right] \end{aligned} \tag{B.65}$$

and

$$\begin{aligned}\mu^\varepsilon I_{0,2} &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\Gamma(\varepsilon/2) \\ &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left[\frac{2}{\varepsilon} - \gamma_\mathrm{E} + \mathcal{O}(\varepsilon)\right] \\ &= \frac{i}{16\pi^2}\left[\Delta_\varepsilon + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right]\,, \end{aligned} \tag{B.66}$$

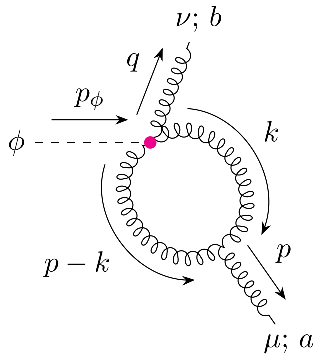
where $\Delta_\varepsilon = 2/\varepsilon - \gamma_\mathrm{E} + \log(4\pi)$. In this way, the divergent part of the diagram reads

$$\begin{aligned} i\mathcal{M}^{(2)}_{gg}|_{\text{div.}} &= \frac{i}{16\pi^2} A\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon \int_0^1 \mathrm{d}x\, \{g^{\mu\nu} 9C \\ &\quad + 2x(x-1)[p^\nu q^\mu + 4p\cdot q g^{\mu\nu}]\} \\ &= \frac{i}{16\pi^2} A\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon \int_0^1 \mathrm{d}x\, \{18x(x-1)p\cdot q g^{\mu\nu} \\ &\quad + 2x(x-1)[p^\nu q^\mu + 4p\cdot q g^{\mu\nu}]\} \\ &= \frac{i}{8\pi^2} A(p^\nu q^\mu + 13 p\cdot q g^{\mu\nu})\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon \times \int_0^1 \mathrm{d}x\, x(x-1) \\ &= \frac{i}{8\pi^2} A(p^\nu q^\mu + 13 p\cdot q g^{\mu\nu})\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon \times \left(-\frac{1}{6}\right) \end{aligned} \tag{B.67}$$

and we can finally write

$$i\mathcal{M}^{(2)}_{gg} = \frac{i}{24\pi^2} C_A g_s^4 \frac{C_g}{\Lambda} \mu^{\varepsilon/2}(p^\nu q^\mu + 13 p\cdot q g^{\mu\nu})\delta^{ab}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon + \text{finite}\,. \tag{B.68}$$

The fifth diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(3)}_{gg}\,, \tag{B.69}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(3)}_{gg} &= \frac{3\cdot 3\cdot 2}{3!\cdot 3!} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (4 g_s^3 \frac{C_g}{\Lambda}\mu^\varepsilon) f^{bcd}[g^{\nu\alpha}(q-k)^\beta + g^{\alpha\beta}(2k-p)^\nu \\ &\quad + g^{\beta\nu}(p-k-q)^\alpha] \frac{-ig_{\alpha\gamma}\delta^{ce}}{k^2+i\epsilon} \frac{-ig_{\beta\delta}\delta^{df}}{(p-k)^2+i\epsilon}(-g_s\mu^{\varepsilon/2}) f^{aef} \\ &\quad \times [g^{\mu\gamma}(p+k)^\delta + g^{\gamma\delta}(p-2k)^\mu + g^{\delta\mu}(k-2p)^\gamma]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= A\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N}{D_1 D_2}\,, \end{aligned} \tag{B.70}$$

where we have defined

$$A = 2 C_A g_s^4 \frac{C_g}{\Lambda} \delta^{ab} \mu^{\varepsilon/2}\,, \tag{B.71}$$

$$\begin{aligned} N = &[g^{\nu\alpha}(q-k)^{\beta} + g^{\alpha\beta}(2k-p)^{\nu} + g^{\beta\nu}(p-k-q)^{\alpha}] \\ &\times [g^{\mu\gamma}(p+k)^{\delta} + g^{\gamma\delta}(p-2k)^{\mu} + g^{\delta\mu}(k-2p)^{\gamma}] g_{\alpha\gamma} g_{\beta\delta} \epsilon^*_{\mu}(p) \epsilon^*_{\nu}(q)\,, \end{aligned} \tag{B.72}$$

$$D_1 = k^2 + i\epsilon\,, \tag{B.73}$$

$$D_2 = (p-k)^2 + i\epsilon\,, \tag{B.74}$$

and exploited the anti-symmetry of the structure constants $f^{abc} = f^{[abc]}$, as well as the fact that $f^{acd} f^{bcd}$ is the Casimir operator in the adjoint representation of $SU(N_c)$ and, by Schur's lemma, is proportional to the identity matrix

$$f^{acd} f^{bcd} = C_A \delta^{ab}\,, \tag{B.75}$$

with $C_A = N_c$. Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned} \frac{1}{D_1 D_2} &= \int_0^1 \mathrm{d}x \frac{1}{[(1-x)D_1 + xD_2]^2} \\ &= \int_0^1 \mathrm{d}x \frac{1}{\{[k-xp]^2 + i\epsilon\}^2}\,, \end{aligned} \tag{B.76}$$

where the on-shellness of an external gluon $p^2 = 0$ has been used. This is a scaleless integral ($C = 0$), thus it yields $i\mathcal{M}^{(3)}_{gg} = 0$. This zero is due to an exact cancellation between UV and IR divergences, and, in order to decouple the two contributions, we can give the gluon a small fictitious mass m_g in the denominator, and at the end of the calculation we can perform the limit $m_g \to 0$. In this way

$$\frac{1}{D_1 D_2} = \int_0^1 \mathrm{d}x \frac{1}{(\ell^2 - C + i\epsilon)^2}\,, \tag{B.77}$$

where

$$C = m_g^2 \tag{B.78}$$

and we have shifted the integration variable k as $k = \ell + xp$, which implies $\mathrm{d}^d k = \mathrm{d}^d \ell$. At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned} N = &[g^{\nu\alpha}(q-\ell-xp)^{\beta} + g^{\alpha\beta}(2\ell+2xp-p)^{\nu} + g^{\beta\nu}(p-\ell-xp-q)^{\alpha}] \\ &\times [g^{\mu\gamma}(p+\ell+xp)^{\delta} + g^{\gamma\delta}(p-2\ell-2xp)^{\mu} + g^{\delta\mu}(\ell+xp-2p)^{\gamma}] \\ &\times g_{\alpha\gamma} g_{\beta\delta} \epsilon^*_{\mu}(p) \epsilon^*_{\nu}(q)\,. \end{aligned} \tag{B.79}$$

Before expanding the numerator we can observe that the quadratic terms in ℓ do not contribute to the divergent part of the diagram, since they are associated to integral $I_{1,2}$, which, as we have seen in the calculation of $i\mathcal{M}_{gg}^{(2)}$, is proportional to C even after performing the limit $\varepsilon \to 0$. In this case $C \to 0$ and therefore $I_{1,2} \to 0$. The linear terms in ℓ do not contribute by Lorentz invariance, hence we can isolate those terms in N that do not depend on ℓ:

$$\begin{aligned} N &= [g^{\nu\alpha}(q-xp)^{\beta} + g^{\alpha\beta}(2xp-p)^{\nu} + g^{\beta\nu}(p-xp-q)^{\alpha}] \\ &\quad \times [g^{\mu\gamma}(p+xp)^{\delta} + g^{\gamma\delta}(p-2xp)^{\mu} + g^{\delta\mu}(xp-2p)^{\gamma}] \\ &\quad \times g_{\alpha\gamma}g_{\beta\delta}\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q) \\ &= 3(p\cdot q g^{\mu\nu} - p^{\nu}q^{\mu})\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\,, \end{aligned} \tag{B.80}$$

where $p^2 = 0$ and the transversality condition $p \cdot \epsilon^*(p) = 0$ have been used. Thus, the diagram reads

$$\begin{aligned} i\mathcal{M}_{gg}^{(3)} &= 3A(p\cdot q g^{\mu\nu} - p^{\nu}q^{\mu})\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\mu^{\varepsilon}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{1}{(\ell^2-C+i\epsilon)^2} \\ &= 3A(p\cdot q g^{\mu\nu} - p^{\nu}q^{\mu})\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\int_0^1\mathrm{d}x\,\mu^{\varepsilon}I_{0,2}\,. \end{aligned} \tag{B.81}$$

The integral $I_{0,2}$ is computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_{\mathrm{E}} + \mathcal{O}(x)$ as $x \to 0$:

$$I_{0,2} = \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\Gamma(\varepsilon/2)\,, \tag{B.82}$$

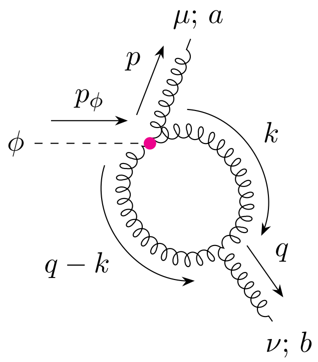
so that

$$\begin{aligned} \mu^{\varepsilon}I_{0,2} &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\Gamma(\varepsilon/2) \\ &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left[\frac{2}{\varepsilon} - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon)\right] \\ &= \frac{i}{16\pi^2}\left[\Delta_{\varepsilon} + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right], \end{aligned} \tag{B.83}$$

where $\Delta_{\varepsilon} = 2/\varepsilon - \gamma_{\mathrm{E}} + \log(4\pi)$. In this way, we can see that the divergent part of the diagram does not depend on C, and we can write it as

$$\begin{aligned} i\mathcal{M}_{gg}^{(3)}|_{\text{div.}} &= 3A(p\cdot q g^{\mu\nu} - p^{\nu}q^{\mu})\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\int_0^1\mathrm{d}x\,\frac{i}{16\pi^2}\Delta_{\varepsilon} \\ &= \frac{3i}{16\pi^2}A(p\cdot q g^{\mu\nu} - p^{\nu}q^{\mu})\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\times\int_0^1\mathrm{d}x \\ &= \frac{3i}{16\pi^2}A(p\cdot q g^{\mu\nu} - p^{\nu}q^{\mu})\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\times 1 \\ &= \frac{3i}{8\pi^2}C_A g_s^4\frac{C_g}{\Lambda}\mu^{\varepsilon/2}(-p^{\nu}q^{\mu} + p\cdot q g^{\mu\nu})\delta^{ab}\epsilon_{\mu}^{*}(p)\epsilon_{\nu}^{*}(q)\Delta_{\varepsilon}\,. \end{aligned} \tag{B.84}$$

The sixth diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(4)}_{gg}\,, \tag{B.85}$$

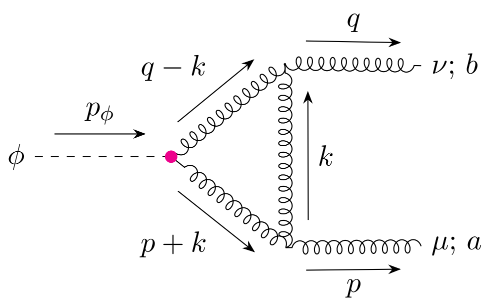
which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(4)}_{gg} = \frac{3\cdot 3\cdot 2}{3!\cdot 3!}\int \frac{\mathrm{d}^d k}{(2\pi)^d}(4g_s^3\frac{C_g}{\Lambda}\mu^{\varepsilon})f^{acd}[g^{\mu\alpha}(p-k)^{\beta} + g^{\alpha\beta}(2k-q)^{\mu} \\ + g^{\beta\mu}(q-k-p)^{\alpha}]\frac{-ig_{\alpha\gamma}\delta^{ce}}{k^2+i\epsilon}\frac{-ig_{\beta\delta}\delta^{df}}{(q-k)^2+i\epsilon}(-g_s\mu^{\varepsilon/2})f^{bef} \\ \times [g^{\nu\gamma}(q+k)^{\delta} + g^{\gamma\delta}(q-2k)^{\nu} + g^{\delta\nu}(k-2q)^{\gamma}]\epsilon^*_{\mu}(p)\epsilon^*_{\nu}(q)\,. \end{aligned} \tag{B.86}$$

It can be related to the diagram $i\mathcal{M}^{(3)}_{gg}$ via the substitutions $\mu \leftrightarrow \nu$, $a \leftrightarrow b$ and $p \leftrightarrow q$, thus

$$i\mathcal{M}^{(4)}_{gg}|_{\text{div.}} = \frac{3i}{8\pi^2}C_A g_s^4\frac{C_g}{\Lambda}\mu^{\varepsilon/2}(-p^{\nu}q^{\mu} + p\cdot q g^{\mu\nu})\delta^{ab}\epsilon^*_{\mu}(p)\epsilon^*_{\nu}(q)\Delta_{\varepsilon}\,. \tag{B.87}$$

The seventh diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu}G^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(5)}_{gg}\,, \tag{B.88}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$
\begin{aligned}
i\mathcal{M}_{gg}^{(5)} = & \frac{3\cdot 3\cdot 8}{2!\cdot 3!\cdot 3!}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\epsilon_\mu^*(p)\epsilon_\nu^*(q)(4ig_s^2\frac{C_g}{\Lambda}\mu^{\varepsilon/2})[(p+k)^\alpha(q-k)^\beta \\
& -(p+k)\cdot(q-k)g^{\alpha\beta}]\delta^{cd}\frac{-ig_{\alpha\gamma}\delta^{ce}}{(q-k)^2+i\epsilon}\frac{-ig_{\beta\delta}\delta^{df}}{(p+k)^2+i\epsilon}\frac{-ig_{\rho\sigma}\delta^{gh}}{k^2+i\epsilon} \\
& \times(-g_s\mu^{\varepsilon/2})f^{bge}[g^{\nu\rho}(q+k)^\gamma+g^{\rho\gamma}(q-2k)^\nu+g^{\gamma\nu}(k-2q)^\rho] \\
& \times(-g_s\mu^{\varepsilon/2})f^{ahf}[g^{\mu\sigma}(p-k)^\delta+g^{\sigma\delta}(2k+p)^\mu+g^{\delta\mu}(-k-2p)^\sigma] \\
= & A\mu^\varepsilon\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{N}{D_1D_2D_3}\,,
\end{aligned}
\tag{B.89}
$$

where we have defined

$$A = -4C_A g_s^4\frac{C_g}{\Lambda}\delta^{ab}\mu^{\varepsilon/2}\,, \tag{B.90}$$

$$
\begin{aligned}
N = & [(p+k)^\alpha(q-k)^\beta-(p+k)\cdot(q-k)g^{\alpha\beta}][g^{\nu\rho}(q+k)^\gamma \\
& +g^{\rho\gamma}(q-2k)^\nu+g^{\gamma\nu}(k-2q)^\rho][g^{\mu\sigma}(p-k)^\delta+g^{\sigma\delta}(2k+p)^\mu \\
& +g^{\delta\mu}(-k-2p)^\sigma]g_{\alpha\gamma}g_{\beta\delta}g_{\rho\sigma}\epsilon_\mu^*(p)\epsilon_\nu^*(q)\,,
\end{aligned}
\tag{B.91}
$$

$$D_1 = k^2+i\epsilon\,, \tag{B.92}$$

$$D_2 = (k+p)^2+i\epsilon\,, \tag{B.93}$$

$$D_3 = (k-q)^2+i\epsilon\,, \tag{B.94}$$

and exploited the anti-symmetry of the structure constants $f^{abc}=f^{[abc]}$, as well as the fact that $f^{acd}f^{bcd}$ is the Casimir operator in the adjoint representation of $SU(N_c)$ and, by Schur's lemma, is proportional to the identity matrix

$$f^{acd}f^{bcd} = C_A\delta^{ab}\,, \tag{B.95}$$

with $C_A = N_c$. Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$
\begin{aligned}
\frac{1}{D_1D_2D_3} &= 2\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z\frac{\delta(x+y+z-1)}{[D_1x+D_2y+D_3z]^3} \\
&= 2\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z\frac{\delta(x+y+z-1)}{[(k+yp-zq)^2-C+i\epsilon]^3}\,,
\end{aligned}
\tag{B.96}
$$

with

$$C = -2yzp\cdot q\,, \tag{B.97}$$

where the on-shellness of external gluons $p^2 = q^2 = 0$ and $x+y+z = 1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = \ell - yp + zq$, which implies $\mathrm{d}^d k = \mathrm{d}^d \ell$ and

$$\begin{aligned}\frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{(\ell^2 - C + i\epsilon)^3} \\ &= 2\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{1}{(\ell^2 - C + i\epsilon)^3}\,.\end{aligned} \tag{B.98}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N &= [(p+\ell-yp+zq)^\alpha (q-\ell+yp-zq)^\beta - (p+\ell-yp+zq)\cdot(q-\ell \\ &\quad + yp - zq) g^{\alpha\beta}][g^{\nu\rho}(q+\ell-yp+zq)^\gamma + g^{\rho\gamma}(q-2\ell+2yp-2zq)^\nu \\ &\quad + g^{\gamma\nu}(\ell-yp+zq-2q)^\rho][g^{\mu\sigma}(p-\ell+yp-zq)^\delta + g^{\sigma\delta}(2\ell-2yp \\ &\quad + 2zq+p)^\mu + g^{\delta\mu}(-\ell+yp-zq-2p)^\sigma] g_{\alpha\gamma} g_{\beta\delta} g_{\rho\sigma} \epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= (N_1+N_2+N_3+N_4+N_5)^{\mu\nu}\epsilon^*_\mu(p)\epsilon^*_\nu(q) + \mathcal{O}(\ell^0)\,,\end{aligned} \tag{B.99}$$

where we have defined

$$\begin{aligned}N_1^{\mu\nu} &= \ell^\mu \ell^\nu (5\ell^2 - 4d\ell^2 - 16p\cdot q + 6yp\cdot q + 6zp\cdot q - 10yzp\cdot q \\ &\quad + 4dp\cdot q - 4ydp\cdot q - 4zdp\cdot q + 8yzdp\cdot q)\,,\end{aligned} \tag{B.100}$$

$$\begin{aligned}N_2^{\mu\nu} &= \ell^\mu p^\nu (2\ell\cdot p - 8y\ell\cdot p + 10y^2\ell\cdot p + 4yd\ell\cdot p - 8y^2 d\ell\cdot p \\ &\quad + 6\ell\cdot q + 6y\ell\cdot q + 2z\ell\cdot q - 10yz\ell\cdot q - 4yd\ell\cdot q + 8yzd\ell\cdot q)\,,\end{aligned} \tag{B.101}$$

$$\begin{aligned}N_3^{\mu\nu} &= \ell^\nu q^\mu (6\ell\cdot p + 2y\ell\cdot p + 6z\ell\cdot p - 10yz\ell\cdot p - 4zd\ell\cdot p + 8yzd\ell\cdot p \\ &\quad + 2\ell\cdot q - 8z\ell\cdot q + 10z^2\ell\cdot q + 4zd\ell\cdot q - 8z^2 d\ell\cdot q)\,,\end{aligned} \tag{B.102}$$

$$N_4^{\mu\nu} = p^\nu q^\mu \ell^2 (-9 + y + z - 5yz + 4yzd)\,, \tag{B.103}$$

$$\begin{aligned}N_5^{\mu\nu} &= g^{\mu\nu}(-\ell^4 - 2(\ell\cdot p)^2 + 6y(\ell\cdot p)^2 - 4y^2(\ell\cdot p)^2 - 6y\ell\cdot p\ell\cdot q \\ &\quad - 6z\ell\cdot p\ell\cdot q + 8yz\ell\cdot p\ell\cdot q - 2(\ell\cdot q)^2 + 6z(\ell\cdot q)^2 - 4z^2(\ell\cdot q)^2 \\ &\quad + 9\ell^2 p\cdot q - 3y\ell^2 p\cdot q - 3z\ell^2 p\cdot q + 4yz\ell^2 p\cdot q)\,,\end{aligned} \tag{B.104}$$

and we have discarded all odd powers of ℓ by Lorentz invariance, as well as exploited $p^2 = q^2 = 0$, $\delta^\mu_\mu = d$ and the transversality conditions $p\cdot\epsilon^*(p) = q\cdot\epsilon^*(q) = 0$. Additionally, $\mathcal{O}(\ell^0)$ terms do not contribute to the divergent part of the diagram since the denominator of the integral scales as ℓ^6. Always by Lorentz invariance, we can substitute $\ell^\mu \ell^\nu$ with $\ell^2 g^{\mu\nu}/d$ in the numerator. With this substitution many terms vanish and others are simplified

$$\ell^\mu p^\nu \ell \cdot p \epsilon_\mu(p) = \frac{1}{d}\ell^2 p^\nu p^\mu \epsilon_\mu(p) = 0\,, \tag{B.105}$$

$$\ell^\nu q^\mu \ell \cdot q \epsilon_\nu(q) = \frac{1}{d}\ell^2 q^\nu q^\mu \epsilon_\nu(q) = 0\,, \tag{B.106}$$

$$\ell^\mu p^\nu \ell \cdot q = \ell^\nu q^\mu \ell \cdot p = \frac{1}{d}\ell^2 p^\nu q^\mu\,, \tag{B.107}$$

$$(\ell \cdot p)^2 = \frac{1}{d}\ell^2 p^2 = 0\,, \tag{B.108}$$

$$(\ell \cdot q)^2 = \frac{1}{d}\ell^2 q^2 = 0\,, \tag{B.109}$$

$$\ell \cdot p \ell \cdot q = \frac{1}{d} p \cdot q\,, \tag{B.110}$$

leading to

$$\begin{aligned} N_1^{\mu\nu} = \frac{1}{d}\ell^2 g^{\mu\nu}[\ell^2(5-4d) + p\cdot q(&-16+6y+6z-10yz+4d-4yd \\ &-4zd+8yzd)]\,, \end{aligned} \tag{B.111}$$

$$N_2^{\mu\nu} = \frac{1}{d}\ell^2 p^\nu q^\mu (6+6y+2z-10yz-4yd+8yzd)\,, \tag{B.112}$$

$$N_3^{\mu\nu} = \frac{1}{d}\ell^2 p^\nu q^\mu (6+2y+6z-10yz-4zd+8yzd)\,, \tag{B.113}$$

$$N_4^{\mu\nu} = p^\nu q^\mu \ell^2(-9+y+z-5yz+4yzd)\,, \tag{B.114}$$

$$N_5^{\mu\nu} = g^{\mu\nu}\ell^2\left[-\ell^2 + p\cdot q\left(\frac{-6y-6z+8yz}{d} + 9-3y-3z+4yz\right)\right], \tag{B.115}$$

so that they can be recombined to form the divergent part of the numerator as

$$\begin{aligned} N|_{\text{div.}} &= \Bigg\{\ell^4 g^{\mu\nu}\left(\frac{5-4d}{d}-1\right) + \ell^2\bigg[\frac{1}{d}p\cdot q g^{\mu\nu}(-16-2yz+4d-4yd \\ &\quad -4zd+8yzd) + p\cdot q g^{\mu\nu}(9-3y-3z+4yz) + \frac{1}{d}p^\nu q^\mu(12+8y+8z \\ &\quad -20yz-4yd-4zd+16yzd) + p^\nu q^\mu(-9+y+z-5yz+4yzd)\bigg]\Bigg\} \\ &\quad \times \epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= \Bigg\{-\frac{15}{4}\ell^4 g^{\mu\nu} + \ell^2\bigg[p\cdot q g^{\mu\nu}\left(9-7y-7z+\frac{23}{2}yz\right) \\ &\quad + p^\nu q^\mu(-6-y-z+22yz)\bigg]\Bigg\}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,, \end{aligned}$$

(B.116)

where in the last expression we have substituted $d = 4$[1]. The divergent integrals $I_{1,3}$ and $I_{2,3}$ are computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_{\mathrm{E}} + \mathcal{O}(x)$ as $x \to 0$:

$$\begin{aligned}
\mu^\varepsilon I_{1,3} &= \mu^\varepsilon \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - C + i\epsilon)^3} \\
&= \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2} \left(1 - \frac{\varepsilon}{4}\right) \Gamma(\varepsilon/2) \\
&= \frac{i}{16\pi^2} \left[1 + \frac{\varepsilon}{2} \log \frac{4\pi\mu^2}{C} + \mathcal{O}(\varepsilon^2)\right] \left[\frac{2}{\varepsilon} + 1 - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon)\right] \\
&= \frac{i}{16\pi^2} \left[\Delta_\varepsilon + 1 + \log \frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right],
\end{aligned} \tag{B.117}$$

$$\begin{aligned}
\mu^\varepsilon I_{2,3} &= \mu^\varepsilon \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{(\ell^2)^2}{(\ell^2 - C + i\epsilon)^3} \\
&= -\frac{i}{16\pi^2} C \left(\frac{4\pi\mu^2}{C}\right) \frac{\Gamma(2 + d/2)\Gamma(1 - d/2)}{\Gamma(3)\Gamma(d/2)} \\
&= -\frac{i}{16\pi^2} C \left(\frac{4\pi\mu^2}{C}\right) \frac{1}{8} (6 - \varepsilon)(4 - \varepsilon) \Gamma(\varepsilon/2 - 1) \\
&= -\frac{i}{16\pi^2} C \left[1 + \frac{\varepsilon}{2} \log \frac{4\pi\mu^2}{C} + \mathcal{O}(\varepsilon^2)\right] (-3) \left[\frac{2}{\varepsilon} - \gamma_{\mathrm{E}} + \frac{1}{6} + \mathcal{O}(\varepsilon)\right] \\
&= \frac{3i}{16\pi^2} C \left[\Delta_\varepsilon + \frac{1}{6} + \log \frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right],
\end{aligned} \tag{B.118}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_{\mathrm{E}} + \log(4\pi)$. The remaining integrals to solve are those over the Feynman parameters:

$$\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \, C = -\frac{1}{12} p \cdot q \,, \tag{B.119}$$

$$\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \left(9 - 7y - 7z + \frac{23}{2} yz\right) = \frac{127}{48} \,, \tag{B.120}$$

$$\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \, (-6 - y - z + 22yz) = -\frac{29}{12} \,. \tag{B.121}$$

[1] Although we should keep $d = 4 - \varepsilon$ and expand at the end around $\varepsilon = 0$, this works since we miss a term of order ε^0, which therefore does not contribute to the divergent part of the diagram.

Therefore, the divergent part of the diagram reads

$$
\begin{aligned}
i\mathcal{M}^{(5)}_{gg}|_{\text{div.}} &= 2A\epsilon^*_\mu(p)\epsilon^*_\nu(q)\frac{i}{16\pi^2}\bigg[-\frac{15}{4}\cdot 3\cdot\bigg(-\frac{1}{12}\bigg)p\cdot q g^{\mu\nu}+\frac{127}{48}p\cdot q g^{\mu\nu} \\
&\qquad -\frac{29}{12}p^\nu q^\mu\bigg]\Delta_\varepsilon \\
&= 2A\epsilon^*_\mu(p)\epsilon^*_\nu(q)\frac{i}{16\pi^2}\frac{1}{12}(43p\cdot q g^{\mu\nu}-29p^\nu q^\mu)\Delta_\varepsilon\,,
\end{aligned}
\tag{B.122}
$$

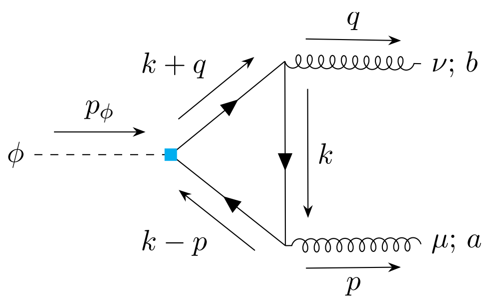
so that we can finally write

$$
i\mathcal{M}^{(5)}_{gg} = \frac{i}{24\pi^2}C_A g_s^4\frac{C_g}{\Lambda}\mu^{\varepsilon/2}(29p^\nu q^\mu - 43p\cdot q g^{\mu\nu})\delta^{ab}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon + \text{finite}\,. \tag{B.123}
$$

B.4 $\phi G\widetilde{G}$ vertex corrections

B.4.1 $i\phi\bar{f}\gamma_5 f$ mediated diagrams

The first diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ operator is mediated by the operator $i\phi\bar{f}_i\gamma_5 f_j$ and is given by



$$
= i\mathcal{M}^{(1)}_{\tilde{g}P}\,, \tag{B.124}
$$

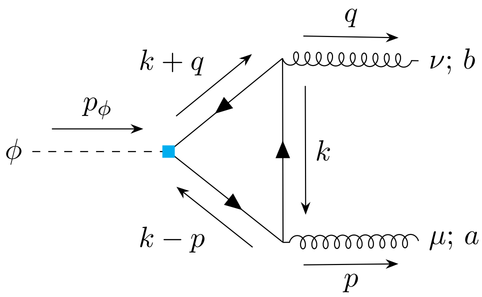
which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$
\begin{aligned}
i\mathcal{M}^{(1)}_{\tilde{g}P} = (-1)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{Tr}\bigg[&(-\frac{v}{\Lambda}y_P^{ij}\gamma_5\mu^{\varepsilon/2})\frac{i}{\not{k}-\not{p}-m_j+i\epsilon}(-ig_sc_fT^a_{IJ}\gamma^\mu\delta^{jk}\mu^{\varepsilon/2}) \\
\times\frac{i}{\not{k}-m_k+i\epsilon}&(-ig_sc_fT^b_{JI}\gamma^\nu\delta^{ik}\mu^{\varepsilon/2})\frac{i}{\not{k}+\not{q}-m_i+i\epsilon}\bigg]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,.
\end{aligned}
\tag{B.125}
$$

The structure of this integral is completely analogous to the one of the diagram $i\mathcal{M}^{(1)}_{\tilde{\gamma}P}$. Indeed they are equal except for their overall coefficient. Therefore

$$
i\mathcal{M}^{(1)}_{\tilde{g}P} = \text{finite}\,. \tag{B.126}
$$

The second diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ operator is mediated by the operator $i\phi \bar{f}_i \gamma_5 f_j$ and is given by



$$= i\mathcal{M}^{(2)}_{\tilde{g}P}\,, \tag{B.127}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

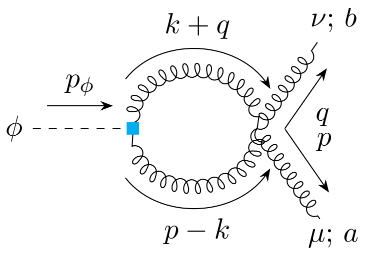
$$\begin{aligned} i\mathcal{M}^{(2)}_{\tilde{g}P} = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\Big[&(-\frac{v}{\Lambda}y^{ij}_P\gamma_5\mu^{\varepsilon/2})\frac{i}{-\not{k}+\not{p}-m_j+i\epsilon}(-ig_s c_f T^a_{IJ}\gamma^\mu\delta^{jk}\mu^{\varepsilon/2}) \\ &\times\frac{i}{-\not{k}-m_k+i\epsilon}(-ig_s c_f T^b_{JI}\gamma^\nu\delta^{ik}\mu^{\varepsilon/2})\frac{i}{-\not{k}-\not{q}-m_i+i\epsilon}\Big]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \end{aligned} \tag{B.128}$$

It can be related to the diagram $i\mathcal{M}^{(1)}_{\tilde{g}P}$ via the substitutions $\mu\leftrightarrow\nu$, $a\leftrightarrow b$ and $p\leftrightarrow q$, thus

$$i\mathcal{M}^{(2)}_{\tilde{g}P} = \text{finite}\,. \tag{B.129}$$

B.4.2 $\phi G\widetilde{G}$ mediated diagrams

The third diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(1)}_{\tilde{g}\tilde{g}}\,, \tag{B.130}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(1)}_{\tilde{g}\tilde{g}} = \frac{4\cdot 3\cdot 2}{2!\cdot 4!}\int\frac{\mathrm{d}^d k}{(2\pi)^d}&(4ig_s^2\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2})\epsilon^{\alpha\beta\zeta\eta}(p-k)_\zeta(q+k)_\eta\delta^{cd}\frac{-ig_{\alpha\gamma}\delta^{ce}}{(k+q)^2+i\epsilon} \\ &\times\frac{-ig_{\beta\delta}\delta^{df}}{(p-k)^2+i\epsilon}(-ig_s^2\mu^\varepsilon)[f^{abg}f^{efg}(g^{\mu\gamma}g^{\nu\delta}-g^{\mu\delta}g^{\nu\gamma}) \\ &+f^{aeg}f^{bfg}(g^{\mu\nu}g^{\gamma\delta}-g^{\mu\delta}g^{\nu\gamma})+f^{afg}f^{beg}(g^{\mu\nu}g^{\gamma\delta}-g^{\mu\gamma}g^{\nu\delta})]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,. \end{aligned}$$

(B.131)

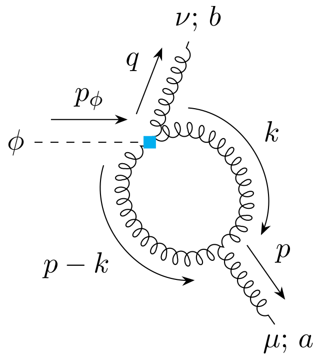
Since the Levi-Civita tensor $\epsilon^{\alpha\beta\zeta\eta}$ is contracted with a symmetric tensor in α, β

$$\epsilon^{\alpha\beta\zeta\eta} g_{\alpha\gamma} g_{\beta\delta} \delta^{ef} [f^{abg} f^{efg} (g^{\mu\gamma} g^{\nu\delta} - g^{\mu\delta} g^{\nu\gamma}) + f^{aeg} f^{bfg} (g^{\mu\nu} g^{\gamma\delta} - g^{\mu\delta} g^{\nu\gamma}) + f^{afg} f^{beg} (g^{\mu\nu} g^{\gamma\delta} - g^{\mu\gamma} g^{\nu\delta})] = 0 \,, \tag{B.132}$$

we can immediately conclude that this diagram is vanishing

$$i\mathcal{M}^{(1)}_{\tilde{g}\tilde{g}} = 0 \,. \tag{B.133}$$

The fourth diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}} \,, \tag{B.134}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}} &= \frac{3\cdot 3\cdot 2}{3!\cdot 3!} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (4 g_s^3 \frac{\widetilde{C}_g}{\Lambda} \mu^\varepsilon) f^{bcd} \epsilon^{\nu\alpha\beta\rho} (p+q)_\rho \frac{-i g_{\alpha\gamma} \delta^{ce}}{k^2 + i\epsilon} \frac{-i g_{\beta\delta} \delta^{df}}{(p-k)^2 + i\epsilon} \\ &\quad \times (-g_s \mu^{\varepsilon/2}) f^{aef} [g^{\mu\gamma} (p+k)^\delta + g^{\gamma\delta} (p-2k)^\mu + g^{\delta\mu} (k-2p)^\gamma] \epsilon^*_\mu(p) \epsilon^*_\nu(q) \\ &= A \mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N}{D_1 D_2} \,, \end{aligned} \tag{B.135}$$

where we have defined

$$A = 2 C_A g_s^4 \frac{\widetilde{C}_g}{\Lambda} \delta^{ab} \mu^{\varepsilon/2} \,, \tag{B.136}$$

$$\begin{aligned} N &= \epsilon^{\nu\alpha\beta\rho} (p+q)_\rho g_{\alpha\gamma} g_{\beta\delta} [g^{\mu\gamma} (p+k)^\delta + g^{\gamma\delta} (p-2k)^\mu + g^{\delta\mu} (k-2p)^\gamma] \\ &\quad \times \epsilon^*_\mu(p) \epsilon^*_\nu(q) \,, \end{aligned} \tag{B.137}$$

$$D_1 = k^2 + i\epsilon\,, \tag{B.138}$$

$$D_2 = (p-k)^2 + i\epsilon\,, \tag{B.139}$$

and exploited the anti-symmetry of the structure constants $f^{abc} = f^{[abc]}$, as well as the fact that $f^{acd}f^{bcd}$ is the Casimir operator in the adjoint representation of $SU(N_c)$ and, by Schur's lemma, is proportional to the identity matrix

$$f^{acd}f^{bcd} = C_A\delta^{ab}\,, \tag{B.140}$$

with $C_A = N_c$. Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned}\frac{1}{D_1D_2} &= \int_0^1 \mathrm{d}x \frac{1}{[(1-x)D_1 + xD_2]^2}\\ &= \int_0^1 \mathrm{d}x \frac{1}{\{[k-xp]^2 + i\epsilon\}^2}\,,\end{aligned} \tag{B.141}$$

where the on-shellness of an external gluon $p^2 = 0$ has been used. This is a scaleless integral ($C = 0$), thus it yields $i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}} = 0$. This zero is due to an exact cancellation between UV and IR divergences, and, in order to decouple the two contributions, we can give the gluon a small fictitious mass m_g in the denominator, and at the end of the calculation we can perform the limit $m_g \to 0$. In this way

$$\frac{1}{D_1D_2} = \int_0^1 \mathrm{d}x \frac{1}{(\ell^2 - C + i\epsilon)^2}\,, \tag{B.142}$$

where

$$C = m_g^2 \tag{B.143}$$

and we have shifted the integration variable k as $k = \ell + xp$, which implies $\mathrm{d}^dk = \mathrm{d}^d\ell$. At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N &= \epsilon^{\nu\alpha\beta\rho}(p+q)_\rho g_{\alpha\gamma}g_{\beta\delta}[g^{\mu\gamma}(p+\ell+xp)^\delta + g^{\gamma\delta}(p-2\ell-2xp)^\mu\\ &\quad + g^{\delta\mu}(\ell+xp-2p)^\gamma]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\\ &= -3\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\end{aligned} \tag{B.144}$$

and is ℓ-independent. Consequently, the diagram is simply given by

$$\begin{aligned}i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}} &= -3A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\mu^\varepsilon\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(\ell^2-C+i\epsilon)^2}\\ &= -3A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\int_0^1 \mathrm{d}x\,\mu^\varepsilon I_{0,2}\,.\end{aligned} \tag{B.145}$$

The integral $I_{0,2}$ is computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_{\mathrm{E}} + \mathcal{O}(x)$ as $x \to 0$:

$$I_{0,2} = \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\Gamma(\varepsilon/2)\,, \tag{B.146}$$

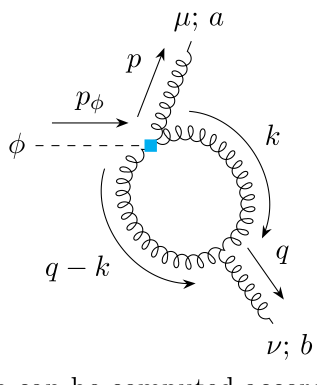
so that

$$\begin{aligned}\mu^\varepsilon I_{0,2} &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\Gamma(\varepsilon/2)\\ &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left[\frac{2}{\varepsilon} - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon)\right]\\ &= \frac{i}{16\pi^2}\left[\Delta_\varepsilon + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right],\end{aligned} \tag{B.147}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_{\mathrm{E}} + \log(4\pi)$. In this way, we can see that the divergent part of the diagram does not depend on C, and we can write it as

$$\begin{aligned}i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}}|_{\text{div.}} &= -3A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma \epsilon^*_\mu(p)\epsilon^*_\nu(q)\int_0^1 \mathrm{d}x\,\frac{i}{16\pi^2}\Delta_\varepsilon\\ &= -\frac{3i}{16\pi^2}A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\times\int_0^1\mathrm{d}x\\ &= -\frac{3i}{16\pi^2}A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\times 1\\ &= -\frac{3i}{8\pi^2}C_A g_s^4\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\delta^{ab}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon\,.\end{aligned} \tag{B.148}$$

The fifth diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(3)}_{\tilde{g}\tilde{g}}\,, \tag{B.149}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

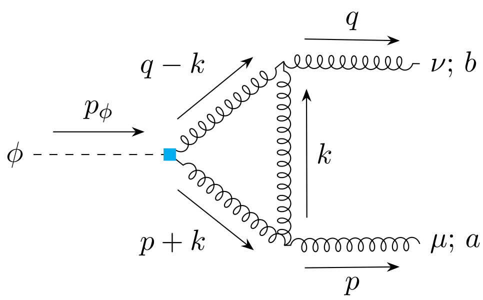
$$\begin{aligned}i\mathcal{M}^{(3)}_{\tilde{g}\tilde{g}} = \frac{3\cdot 3\cdot 2}{3!\cdot 3!}\int\frac{\mathrm{d}^d k}{(2\pi)^d}&(4g_s^3\frac{\widetilde{C}_g}{\Lambda}\mu^\varepsilon)f^{acd}\epsilon^{\mu\alpha\beta\rho}(p+q)_\rho\frac{-ig_{\alpha\gamma}\delta^{ce}}{k^2+i\epsilon}\frac{-ig_{\beta\delta}\delta^{df}}{(q-k)^2+i\epsilon}\\ &\times(-g_s\mu^{\varepsilon/2})f^{bef}[g^{\nu\gamma}(q+k)^\delta + g^{\gamma\delta}(q-2k)^\nu + g^{\delta\nu}(k-2q)^\gamma]\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,.\end{aligned}$$

(B.150)

It can be related to the diagram $i\mathcal{M}^{(2)}_{\tilde{g}\tilde{g}}$ through the substitutions $\mu \leftrightarrow \nu$, $a \leftrightarrow b$ and $p \leftrightarrow q$, thus

$$i\mathcal{M}^{(3)}_{\tilde{g}\tilde{g}}|_{\text{div.}} = -\frac{3i}{8\pi^2} C_A g_s^4 \frac{\widetilde{C}_g}{\Lambda} \mu^{\varepsilon/2} \epsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \delta^{ab} \epsilon^*_\mu(p) \epsilon^*_\nu(q) \Delta_\varepsilon\,. \tag{B.151}$$

The sixth diagram contributing to the one-loop correction of the $\phi G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}$ operator is mediated by the operator $\phi G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}$ itself and is given by



$$= i\mathcal{M}^{(4)}_{\tilde{g}\tilde{g}}\,, \tag{B.152}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}^{(4)}_{\tilde{g}\tilde{g}} &= \frac{3\cdot 3\cdot 8}{2!\cdot 3!\cdot 3!} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (4 i g_s^2 \frac{\widetilde{C}_g}{\Lambda} \mu^{\varepsilon/2}) \epsilon^{\alpha\beta\zeta\eta} (q-k)_\zeta (p+k)_\eta \delta^{cd} \\ &\quad \times \frac{-i g_{\alpha\gamma}\delta^{ce}}{(q-k)^2 + i\epsilon} \frac{-i g_{\beta\delta}\delta^{df}}{(p+k)^2 + i\epsilon} \frac{-i g_{\rho\sigma}\delta^{gh}}{k^2 + i\epsilon} (-g_s \mu^{\varepsilon/2}) f^{bge} \\ &\quad \times [g^{\nu\rho}(q+k)^\gamma + g^{\rho\gamma}(q-2k)^\nu + g^{\gamma\nu}(k-2q)^\rho](-g_s\mu^{\varepsilon/2}) f^{ahf} \\ &\quad \times [g^{\mu\sigma}(p-k)^\delta + g^{\sigma\delta}(2k+p)^\mu + g^{\delta\mu}(-k-2p)^\sigma] \epsilon^*_\mu(p)\epsilon^*_\nu(q) \\ &= A\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N}{D_1 D_2 D_3}\,, \end{aligned} \tag{B.153}$$

where we have defined

$$A = -4 C_A g_s^4 \frac{\widetilde{C}_g}{\Lambda} \delta^{ab} \mu^{\varepsilon/2}\,, \tag{B.154}$$

$$\begin{aligned} N &= \epsilon^{\alpha\beta\zeta\eta}(q-k)_\zeta (p+k)_\eta [g^{\nu\rho}(q+k)^\gamma + g^{\rho\gamma}(q-2k)^\nu + g^{\gamma\nu}(k-2q)^\rho] \\ &\quad \times [g^{\mu\sigma}(p-k)^\delta + g^{\sigma\delta}(2k+p)^\mu + g^{\delta\mu}(-k-2p)^\sigma] g_{\alpha\gamma} g_{\beta\delta} g_{\rho\sigma} \\ &\quad \times \epsilon^*_\mu(p)\epsilon^*_\nu(q)\,, \end{aligned} \tag{B.155}$$

$$D_1 = k^2 + i\epsilon\,, \tag{B.156}$$

$$D_2 = (k+p)^2 + i\epsilon\,, \tag{B.157}$$

$$D_3 = (k-q)^2 + i\epsilon\,, \tag{B.158}$$

and exploited the anti-symmetry of the structure constants $f^{abc} = f^{[abc]}$, as well as the fact that $f^{acd}f^{bcd}$ is the Casimir operator in the adjoint representation of $SU(N_c)$ and, by Schur's lemma, is proportional to the identity matrix

$$f^{acd}f^{bcd} = C_A\delta^{ab}\,, \tag{B.159}$$

with $C_A = N_c$. Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned}\frac{1}{D_1D_2D_3} &= 2\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z\frac{\delta(x+y+z-1)}{[D_1x+D_2y+D_3z]^3}\\ &= 2\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z\frac{\delta(x+y+z-1)}{[(k+yp-zq)^2-C+i\epsilon]^3}\,,\end{aligned} \tag{B.160}$$

with

$$C = -2yzp\cdot q\,, \tag{B.161}$$

where the on-shellness of external gluons $p^2 = q^2 = 0$ and $x+y+z = 1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = \ell - yp + zq$, which implies $\mathrm{d}^dk = \mathrm{d}^d\ell$ and

$$\begin{aligned}\frac{1}{D_1D_2D_3} &= 2\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z\frac{\delta(x+y+z-1)}{(\ell^2-C+i\epsilon)^3}\\ &= 2\int_0^1 \mathrm{d}y\int_0^{1-y} \mathrm{d}z\frac{1}{(\ell^2-C+i\epsilon)^3}\,.\end{aligned} \tag{B.162}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N = {}&\epsilon^{\alpha\beta\zeta\eta}(q-\ell+yp-zq)_\zeta(p+\ell-yp+zq)_\eta[g^{\nu\rho}(q+\ell-yp+zq)^\gamma\\ &+ g^{\rho\gamma}(q-2\ell+2yp-2zq)^\nu + g^{\gamma\nu}(\ell-yp+zq-2q)^\rho]\\ &\times[g^{\mu\sigma}(p-\ell+yp-zq)^\delta + g^{\sigma\delta}(2\ell-2yp+2zq+p)^\mu\\ &+ g^{\delta\mu}(-\ell+yp-zq-2p)^\sigma]g_{\alpha\gamma}g_{\beta\delta}g_{\rho\sigma}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\\ ={}&\{\epsilon^{\mu\nu\rho\sigma}[\ell_\rho p_\sigma N_1 + \ell_\rho q_\sigma N_1 + p_\rho q_\sigma N_2] + (\epsilon^{\mu\rho\sigma\eta}N_3^\nu + \epsilon^{\nu\rho\sigma\eta}N_4^\mu)\ell_\rho p_\sigma q_\eta\}\\ &\times\epsilon^*_\mu(p)\epsilon^*_\nu(q)\,,\end{aligned} \tag{B.163}$$

where we have defined

$$N_1 = -\ell^2 + 2\ell\cdot p(y-1) + 2\ell\cdot q(1-z) + 2p\cdot q(2-y-z+yz)\,, \tag{B.164}$$

$$
\begin{aligned}
N_2 &= \ell^2(y+z-1) + 2\ell\cdot p(-1+2y-y^2+z-yz) \\
&\quad + 2\ell\cdot q(1-y-2z+yz+z^2) \qquad \text{(B.165)} \\
&\quad + 2p\cdot q(2-3y+y^2-3z+3yz-y^2z+z^2-yz^2)\,, \\
N_3 &= -8\ell + 4p(2y-1)\,, \qquad \text{(B.166)} \\
N_4 &= 8\ell + 4q(2z-1)\,, \qquad \text{(B.167)}
\end{aligned}
$$

and exploited the the transversality conditions $p\cdot\epsilon^*(p) = q\cdot\epsilon^*(q) = 0$. Since the denominator scales as ℓ^6, we can ignore $\mathcal{O}(\ell^0)$ terms, and by Lorentz invariance we can effectively substitute $\ell^\mu\ell^\nu$ with $\ell^2 g^{\mu\nu}/d$ and send to zero the terms proportional to odd powers of ℓ. In this way many terms simplify

$$\epsilon^{\mu\nu\rho\sigma}\ell_\rho p_\sigma \ell\cdot p = \frac{\ell^2}{d}\epsilon^{\mu\nu\rho\sigma}p_\rho p_\sigma = 0\,, \qquad \text{(B.168)}$$

$$\epsilon^{\mu\nu\rho\sigma}\ell_\rho q_\sigma \ell\cdot q = \frac{\ell^2}{d}\epsilon^{\mu\nu\rho\sigma}q_\rho q_\sigma = 0\,, \qquad \text{(B.169)}$$

$$\epsilon^{\mu\nu\rho\sigma}\ell_\rho p_\sigma \ell\cdot q = -\frac{\ell^2}{d}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\,, \qquad \text{(B.170)}$$

$$\epsilon^{\mu\nu\rho\sigma}\ell_\rho q_\sigma \ell\cdot p = \frac{\ell^2}{d}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\,, \qquad \text{(B.171)}$$

$$\epsilon^{\mu\rho\sigma\eta}\ell^\nu\ell_\rho p_\sigma q_\eta = \frac{\ell^2}{d}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\,, \qquad \text{(B.172)}$$

$$\epsilon^{\nu\rho\sigma\eta}\ell^\mu\ell_\rho p_\sigma q_\eta = -\frac{\ell^2}{d}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\,, \qquad \text{(B.173)}$$

and the divergent contribution of the numerator takes the form

$$
\begin{aligned}
N|_{\text{div.}} &= \ell^2\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\left[-\frac{2}{d}(1-z)+\frac{2}{d}(y-1)+y+z-1-\frac{16}{d}\right]\epsilon^*_\mu(p)\epsilon^*_\nu(q) \\
&= \ell^2\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\frac{3}{2}(y+z-4)\,,
\end{aligned}
\qquad \text{(B.174)}
$$

where in the last expression we have substituted $d = 4$[2]. The divergent integral $I_{1,3}$ is computed according to master formula in Eq. (5.52), and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_{\text{E}} + \mathcal{O}(x)$ as

[2]Although we should keep $d = 4-\varepsilon$ and expand at the end around $\varepsilon = 0$, this works since we miss a term of order ε^0, which therefore does not contribute to the divergent part of the diagram.

$x \to 0$:

$$\begin{aligned}\mu^\varepsilon I_{1,3} &= \mu^\varepsilon \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2 - C + i\epsilon)^3}\\ &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left(1-\frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2)\\ &= \frac{i}{16\pi^2}\left[1+\frac{\varepsilon}{2}\log\frac{4\pi\mu^2}{C}+\mathcal{O}(\varepsilon^2)\right]\left[\frac{2}{\varepsilon}+1-\gamma_{\mathrm{E}}+\mathcal{O}(\varepsilon)\right]\\ &= \frac{i}{16\pi^2}\left[\Delta_\varepsilon + 1 + \log\frac{\mu^2}{C}+\mathcal{O}(\varepsilon)\right],\end{aligned} \tag{B.175}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_{\mathrm{E}} + \log(4\pi)$. Therefore, the divergent part of the diagram reads

$$\begin{aligned}i\mathcal{M}^{(4)}_{\tilde{g}\tilde{g}}|_{\mathrm{div.}} &= 2A\int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}z\frac{i}{16\pi^2}\Delta_\varepsilon \epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma \epsilon^*_\mu(p)\epsilon^*_\nu(q)\frac{3}{2}(y+z-4)\\ &= \frac{i}{16\pi^2}2A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon \times \int_0^1\mathrm{d}y\int_0^{1-y}\mathrm{d}z\frac{3}{2}(y+z-4)\\ &= \frac{i}{16\pi^2}2A\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon \times \left(-\frac{5}{2}\right)\end{aligned} \tag{B.176}$$

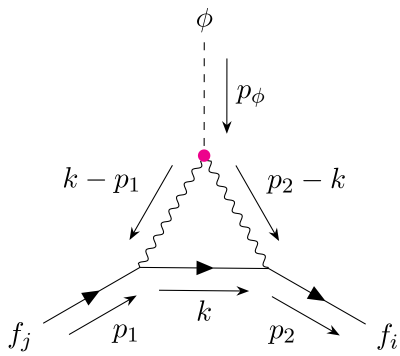
so that we can finally write

$$i\mathcal{M}^{(4)}_{\tilde{g}\tilde{g}} = \frac{5i}{4\pi^2}C_A g_s^4\frac{\widetilde{C}_g}{\Lambda}\mu^{\varepsilon/2}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma \delta^{ab}\epsilon^*_\mu(p)\epsilon^*_\nu(q)\Delta_\varepsilon + \text{finite}\,. \tag{B.177}$$

B.5 $\phi\bar{f}f$ vertex corrections

B.5.1 ϕFF mediated diagram

The first diagram contributing to the one-loop correction of the $\phi\bar{f}_i f_j$ operator is mediated by the operator $\phi F_{\mu\nu}F^{\mu\nu}$ and is given by



$$= i\mathcal{M}_{S\gamma}\,, \tag{B.178}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{S\gamma} = &\int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{u}(p_2)(-ie\gamma^\nu Q_f \mu^{\varepsilon/2}) \frac{i}{\not{k} - m_i + i\epsilon} \\ &\times (-ie\gamma^\mu Q_f \mu^{\varepsilon/2}) u(p_1) \delta^{ij} \frac{-ig_{\mu\rho}}{(k-p_1)^2 + i\epsilon} \frac{-ig_{\nu\sigma}}{(k-p_2)^2 + i\epsilon} \\ &\times (4ie^2 \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2})[(k-p_1)\cdot(k-p_2) g^{\rho\sigma} - (k-p_1)^\sigma (k-p_2)^\rho] \\ = &A\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N_1 + N_2}{D_1 D_2 D_3}\,, \end{aligned} \tag{B.179}$$

where we have defined

$$A = 4e^4 Q_f^2 \delta^{ij} \frac{C_\gamma}{\Lambda} \mu^{\varepsilon/2}\,, \tag{B.180}$$

$$N_1 = \bar{u}(p_2)(\not{k} - \not{p}_1)(\not{k} + m_i)(\not{k} - \not{p}_2) u(p_1)\,, \tag{B.181}$$

$$N_2 = -\bar{u}(p_2)\gamma^\mu(\not{k} + m_i)\gamma_\mu u(p_1)(k-p_1)\cdot(k-p_2)\,, \tag{B.182}$$

$$D_1 = k^2 - m_i^2 + i\epsilon\,, \tag{B.183}$$

$$D_2 = (k-p_1)^2 + i\epsilon\,, \tag{B.184}$$

$$D_3 = (k-p_2)^2 + i\epsilon\,. \tag{B.185}$$

Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned} \frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[D_1 x + D_2 y + D_3 z]^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[(k - p_1 y - p_2 z)^2 - C + i\epsilon]^3}\,, \end{aligned} \tag{B.186}$$

with

$$C = m_i^2(y^2 + z^2 - 2y - 2z + 1) + 2p_1\cdot p_2 yz\,, \tag{B.187}$$

where $p_1^2 = p_2^2 = m_i^2$, since this diagram is flavor diagonal, and $x+y+z=1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = p + p_1 y + p_2 z$, which implies $\mathrm{d}^d k = \mathrm{d}^d p$ and

$$\begin{aligned} \frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{(p^2 - C + i\epsilon)^3} \\ &= 2\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{1}{(p^2 - C + i\epsilon)^3}\,. \end{aligned} \tag{B.188}$$

At this point we have to perform the shift also in the numerator of the integral: N_1 becomes

$$\begin{aligned} N_1 &= \bar{u}(p_2)[\not{p} + \not{p}_1(y-1) + \not{p}_2 z][\not{p} + \not{p}_1 y + \not{p}_2 z + m_i] \\ &\quad \times [\not{p} + \not{p}_1 y + \not{p}_2(z-1)]u(p_1) \\ &= \bar{u}(p_2)\{\not{p}^3 + \not{p}^2[\not{p}_1 y + \not{p}_2(z-1)] + [\not{p}_1(y-1) + \not{p}_2 z]\not{p}^2 \\ &\quad + \not{p}[\not{p}_1 y + \not{p}_2 z + m_i]\not{p}\}u(p_1) + \mathcal{O}(p) \\ &= \bar{u}(p_1)\{\not{p}p^2 + p^2[\not{p}_1(2y-1) + \not{p}_2(2z-1)] \\ &\quad + \not{p}[\not{p}_1 y + \not{p}_2 z + m_i]\not{p}\}u(p_1) + \mathcal{O}(p) \\ &= \bar{u}(p_1)\{\not{p}p^2 + p^2[\not{p}_1(2y-1) + \not{p}_2(2z-1)] \\ &\quad + p_\mu p_\nu \gamma^\mu[\not{p}_1 y + \not{p}_2 z + m_i]\gamma^\nu\}u(p_1) + \mathcal{O}(p)\,, \end{aligned} \tag{B.189}$$

where we have used the identity $\not{p}^2 = p^2$, valid also in d dimensions. On the other hand, N_2 becomes

$$\begin{aligned} N_2 &= -\bar{u}(p_2)\gamma^\mu(\not{p} + \not{p}_1 y + \not{p}_2 z + m_i)\gamma_\mu[p + p_1(y-1) + p_2 z] \\ &\quad \cdot [p + p_1 y + p_2(z-1)]u(p_1) \\ &= -\bar{u}(p_2)\{\gamma^\mu \not{p}\gamma_\mu p^2 + \gamma^\mu\not{p}\gamma_\mu p\cdot[p_1(2y-1) + p_2(2z-1)] \\ &\quad + \gamma^\mu(\not{p}_1 y + \not{p}_2 z + m_i)\gamma_\mu p^2\}u(p_1) + \mathcal{O}(p) \\ &= \bar{u}(p_2)\{(d-2)\not{p}p^2 + (d-2)\not{p}p\cdot[p_1(2y-1) + p_2(2z-1)] \\ &\quad + p^2[(d-2)(\not{p}_1 y + \not{p}_2 z) - d m_i]\}u(p_1) + \mathcal{O}(p)\,. \end{aligned} \tag{B.190}$$

Now we can note that the terms proportional to $\not{p}p^2$, when integrated, give a vanishing contribution due to the fact that $p^\mu p^2$ is odd, and since we are interested in the divergent part of the integral, we can safely ignore $\mathcal{O}(p)$ terms in N_1 and N_2, keeping only the terms of order p^2: the divergent part of N_1 is

$$\begin{aligned} N_1|_{\text{div.}} &= \bar{u}(p_2)\{p^2[\not{p}_1(2y-1) + \not{p}_2(2z-1)] \\ &\quad + p_\mu p_\nu\gamma^\mu[\not{p}_1 y + \not{p}_2 z + m_i]\gamma^\nu\}u(p_1) \end{aligned} \tag{B.191}$$

and the divergent part of N_2 is

$$\begin{aligned} N_2|_{\text{div.}} &= \bar{u}(p_2)\{(d-2)\not{p}p\cdot[p_1(2y-1) + p_2(2z-1)] \\ &\quad + p^2[(d-2)(\not{p}_1 y + \not{p}_2 z) - d m_i]\}u(p_1)\,. \end{aligned} \tag{B.192}$$

The terms containing contractions between p and γ can be simplified by Lorentz invariance: $p^\mu p^\nu$ can effectively be replaced by $p^2 g^{\mu\nu}/d$ so that

$$\not{p}p\cdot p_i = p_\mu p_\nu p_i^\nu \gamma^\mu = \frac{1}{d}p^2 p_i^\nu\gamma^\mu g_{\mu\nu} = \frac{1}{d}p^2\not{p}_i\,, \tag{B.193}$$

$$p_\mu p_\nu \gamma^\mu \not{p}_i \gamma^\nu = \frac{1}{d}p^2\gamma^\mu\not{p}_i\gamma^\nu g_{\mu\nu} = \frac{1}{d}p^2\gamma^\mu\not{p}_i\gamma_\mu = \frac{2-d}{d}p^2\not{p}_i\,, \tag{B.194}$$

and the divergent parts of N_1 and N_2 become proportional to p^2:

$$N_1|_{\text{div.}} = p^2\bar{u}(p_2)\left[\frac{2-d}{d}(\not{p}_1 y+\not{p}_2 z)+m_i+\not{p}_1(2y-1)+\not{p}_2(2z-1)\right]u(p_1)\,, \tag{B.195}$$

while

$$\begin{aligned} N_2|_{\text{div.}} = p^2\bar{u}(p_2)\bigg\{&\frac{d-2}{d}[\not{p}_1(2y-1)+\not{p}_2(2z-1)] \\ &+(d-2)[\not{p}_1 y+\not{p}_2 z]-dm_i\bigg\}u(p_1)\,. \end{aligned} \tag{B.196}$$

Additionally, we can exploit the on-shellness of external states: from the Dirac equation follow $\bar{u}(p_2)\not{p}_2 = m_i\bar{u}(p_2)$ and $\not{p}_1 u(p_1) = m_j u(p_1)$, with $m_i = m_j$ since the diagram is flavor diagonal, and we can write

$$\begin{aligned} N_1|_{\text{div.}} &= p^2 m_i\left[\frac{2-d}{d}(y+z)+1+(2y-1)+(2z-1)\right]\bar{u}(p_2)u(p_1) \\ &= p^2 m_i\left[\frac{2}{d}(y+z)+y+z-1\right]\bar{u}(p_2)u(p_1) \\ &= p^2 m_i\left[\frac{1}{2}(3y+3z-2)+\frac{1}{8}\varepsilon(y+z)+\mathcal{O}(\varepsilon^2)\right]\bar{u}(p_2)u(p_1) \end{aligned} \tag{B.197}$$

and

$$\begin{aligned} N_2|_{\text{div.}} &= p^2 m_i\left[\frac{d-2}{d}(2y-1+2z-1)+(d-2)(y+z)-d\right]\bar{u}(p_2)u(p_1) \\ &= p^2 m_i\left[\left(d-\frac{4}{d}\right)(y+z-1)-2\right]\bar{u}(p_2)u(p_1) \\ &= p^2 m_i\left[3y+3z-5-\frac{5}{4}\varepsilon(y+z-1)+\mathcal{O}(\varepsilon^2)\right]\bar{u}(p_2)u(p_1)\,. \end{aligned} \tag{B.198}$$

Their sum is given by

$$(N_1+N_2)|_{\text{div.}} = p^2 m_i\left[\frac{3}{2}(3y+3z-4)+\frac{1}{8}\varepsilon(10-9y-9z)+\mathcal{O}(\varepsilon^2)\right]\bar{u}(p_2)u(p_1) \tag{B.199}$$

and the divergent part of the diagram reads

$$\begin{aligned}
i\mathcal{M}_{S\gamma}|_{\text{div.}} &= 2A\mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(N_1+N_2)|_{\text{div.}}}{(p^2 - C + i\epsilon)^3} \\
&= 2Am_i \bar{u}(p_2)u(p_1)\mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p^2}{(p^2 - C + i\epsilon)^3} \\
&\quad \times \left[\frac{3}{2}(3y+3z-4) + \frac{1}{8}\varepsilon(10-9y-9z) + \mathcal{O}(\varepsilon^2)\right] \\
&= Am_i \bar{u}(p_2)u(p_1) \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z\, \mu^\varepsilon I_{1,3} \\
&\quad \times \left[3(3y+3z-4) + \frac{1}{4}\varepsilon(10-9y-9z) + \mathcal{O}(\varepsilon^2)\right].
\end{aligned} \tag{B.200}$$

The integral $I_{1,3}$ is computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_\text{E} + \mathcal{O}(x)$ as $x \to 0$:

$$\begin{aligned}
I_{1,3} &= \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\left(1 - \frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2) \\
&= \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma_\text{E} + \mathcal{O}(\varepsilon)\right],
\end{aligned} \tag{B.201}$$

which, multiplied by μ^ε, gives

$$\begin{aligned}
\mu^\varepsilon I_{1,3} &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma_\text{E} + \mathcal{O}(\varepsilon)\right] \\
&= \frac{i}{16\pi^2}\left[1 + \frac{\varepsilon}{2}\log\frac{4\pi\mu^2}{C} + \mathcal{O}(\varepsilon^2)\right]\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma_\text{E} + \mathcal{O}(\varepsilon)\right] \\
&= \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma_\text{E} + \log(4\pi) + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right] \\
&= \frac{i}{16\pi^2}\left[\Delta_\varepsilon - \frac{1}{2} + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right],
\end{aligned} \tag{B.202}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_\text{E} + \log(4\pi)$. Therefore, the divergent contribution of the diagram is given by

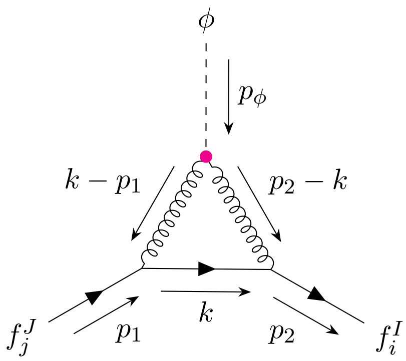
$$\begin{aligned}
i\mathcal{M}_{S\gamma}|_{\text{div.}} &= \frac{3i}{16\pi^2} Am_i \bar{u}(p_2)u(p_1)\Delta_\varepsilon \times \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z\,(3y+3z-4) \\
&= \frac{3i}{16\pi^2} Am_i \bar{u}(p_2)u(p_1)\Delta_\varepsilon \times (-1)
\end{aligned} \tag{B.203}$$

and we can finally write

$$i\mathcal{M}_{S\gamma} = -\frac{3i}{4\pi^2} e^4 Q_f^2 \frac{C_\gamma}{\Lambda} m_i \delta^{ij} \mu^{\varepsilon/2} \bar{u}(p_2)u(p_1)\Delta_\varepsilon + \text{finite}\,. \tag{B.204}$$

B.5.2 ϕGG mediated diagram

The second diagram contributing to the one-loop correction of the $\phi \bar{f}_i f_j$ operator is mediated by the operator $\phi G^a_{\mu\nu} G^{a\,\mu\nu}$ and is given by



$$= i\mathcal{M}_{Sg}\,, \tag{B.205}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{Sg} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{u}(p_2)(-ig_s\gamma^\nu c_f T^b_{IL}\mu^{\varepsilon/2}) \frac{i\delta_{LK}}{\not{k} - m_i + i\epsilon} \\ \times (-ig_s\gamma^\mu c_f T^a_{KJ}\mu^{\varepsilon/2}) u(p_1)\delta^{ij} \frac{-ig_{\mu\rho}\delta^{ac}}{(k-p_1)^2 + i\epsilon} \frac{-ig_{\nu\sigma}\delta^{bd}}{(k-p_2)^2 + i\epsilon} \\ \times (4ig_s^2 \frac{C_g}{\Lambda}\delta^{cd}\mu^{\varepsilon/2})[(k-p_1)\cdot(k-p_2)g^{\rho\sigma} - (k-p_1)^\sigma (k-p_2)^\rho]\,. \end{aligned} \tag{B.206}$$

This diagram is completely analogous to $i\mathcal{M}_{S\gamma}$, and its color structure is provided by

$$T^b_{IL} T^a_{KJ} \delta^{ab} \delta_{LK} = C_F \delta_{IJ} = \frac{N_c^2 - 1}{2N_c}\delta_{IJ}\,, \tag{B.207}$$

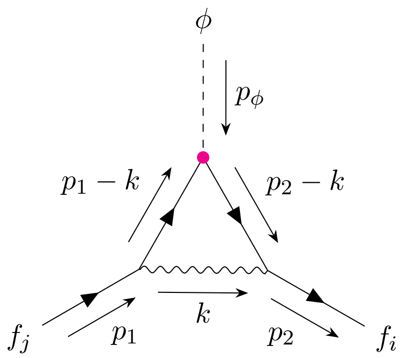
where C_F is the Casimir of the gauge group $SU(N_c)$ in the fundamental representation. Thus, we can immediately conclude that

$$i\mathcal{M}_{Sg} = -\frac{3i}{4\pi^2} C_F g_s^4 c_f^2 \frac{C_g}{\Lambda} m_i \delta^{ij} \delta_{IJ} \mu^{\varepsilon/2} \bar{u}(p_2) u(p_1) \Delta_\varepsilon + \text{finite}\,. \tag{B.208}$$

B.5.3 $\phi \bar{f} f$ mediated diagrams

EM-induced diagram

The third diagram contributing to the one-loop correction of the $\phi \bar{f}_i f_j$ operator is mediated by the operator $\phi \bar{f}_i f_j$ itself and is EM-induced. It is given by



$$= i\mathcal{M}_{SS}^{(\gamma)}\,, \tag{B.209}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{SS}^{(\gamma)} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{u}(p_2)(-ie\gamma^\nu Q_f \mu^{\varepsilon/2}) \frac{i}{\not{p}_2 - \not{k} - m_i + i\epsilon} \\ &\quad \times (i\frac{v}{\Lambda} y_S^{ij} \mu^{\varepsilon/2}) \frac{i}{\not{p}_1 - \not{k} - m_j + i\epsilon} (-ie\gamma^\mu Q_f \mu^{\varepsilon/2}) u(p_1) \frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \\ &= A\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N}{D_1 D_2 D_3}\,, \end{aligned} \tag{B.210}$$

where we have defined

$$A = e^2 Q_f^2 y_S^{ij} \frac{v}{\Lambda} \mu^{\varepsilon/2}\,, \tag{B.211}$$

$$N = \bar{u}(p_2)\gamma^\mu(\not{p}_2 - \not{k} + m_i)(\not{p}_1 - \not{k} + m_j)\gamma_\mu u(p_1)\,, \tag{B.212}$$

$$D_1 = k^2 + i\epsilon\,, \tag{B.213}$$

$$D_2 = (k - p_2)^2 - m_i^2 + i\epsilon\,, \tag{B.214}$$

$$D_3 = (k - p_1)^2 - m_j^2 + i\epsilon\,. \tag{B.215}$$

Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned} \frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[D_1 x + D_2 y + D_3 z]^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[(k - p_2 y - p_1 z)^2 - C + i\epsilon]^3}\,, \end{aligned} \tag{B.216}$$

with

$$C = m_i^2 y^2 + m_j^2 z^2 + 2p_1 \cdot p_2 yz\,, \tag{B.217}$$

where $p_1^2 = m_j^2$, $p_2^2 = m_i^2$ and $x + y + z = 1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = p + p_2 y + p_1 z$, which implies $\mathrm{d}^d k = \mathrm{d}^d p$ and

$$\begin{aligned}\frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{(p^2 - C + i\epsilon)^3} \\ &= 2\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{1}{(p^2 - C + i\epsilon)^3}\,.\end{aligned} \tag{B.218}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N &= \bar{u}(p_2)\gamma^\mu(\not{p} + \not{p}_2(y-1) + \not{p}_1 z - m_i)(\not{p} + \not{p}_2 y + \not{p}_1(z-1) - m_j)\gamma_\mu u(p_1) \\ &= \bar{u}(p_2)\gamma^\mu \not{p}^2 \gamma_\mu u(p_1) + \mathcal{O}(p) \\ &= p^2 \bar{u}(p_2)\gamma^\mu\gamma_\mu u(p_1) + \mathcal{O}(p) \\ &= dp^2 \bar{u}(p_2)u(p_1) + \mathcal{O}(p)\,.\end{aligned} \tag{B.219}$$

Since we are interested in the divergent part of the integral, we can safely ignore $\mathcal{O}(p)$ terms in N, keeping only the term of order p^2: the divergent part of N is

$$N|_{\text{div.}} = dp^2 \bar{u}(p_2)u(p_1) \tag{B.220}$$

and is already proportional to p^2. Thus, the divergent part of the diagram reads

$$\begin{aligned}i\mathcal{M}_{SS}^{(\gamma)}|_{\text{div.}} &= 2A\bar{u}(p_2)u(p_1)\mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{N|_{\text{div.}}}{(p^2 - C + i\epsilon)^3} \\ &= 2dA\bar{u}(p_2)u(p_1)\mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{p^2}{(p^2 - C + i\epsilon)^3} \\ &= 2A\bar{u}(p_2)u(p_1) \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z\, d\mu^\varepsilon I_{1,3}\,.\end{aligned} \tag{B.221}$$

The integral $I_{1,3}$ is computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_\mathrm{E} + \mathcal{O}(x)$ as $x \to 0$:

$$I_{1,3} = \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\left(1 - \frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2)\,, \tag{B.222}$$

which, multiplied by $d\mu^\varepsilon$, gives

$$\begin{aligned} d\mu^\varepsilon I_{1,3} &= (4-\varepsilon)\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left(1-\frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2) \\ &= (4-\varepsilon)\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left(1-\frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2) \\ &= \frac{i}{4\pi^2}\left[1+\frac{\varepsilon}{2}\log\frac{4\pi\mu^2}{C}+\mathcal{O}(\varepsilon^2)\right]\left[\frac{2}{\varepsilon}-1-\gamma_{\mathrm{E}}+\mathcal{O}(\varepsilon)\right] \\ &= \frac{i}{4\pi^2}\left[\Delta_\varepsilon - 1 + \log\frac{\mu^2}{C}+\mathcal{O}(\varepsilon)\right], \end{aligned} \tag{B.223}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_{\mathrm{E}} + \log(4\pi)$. Therefore, the divergent contribution of the diagram is given by

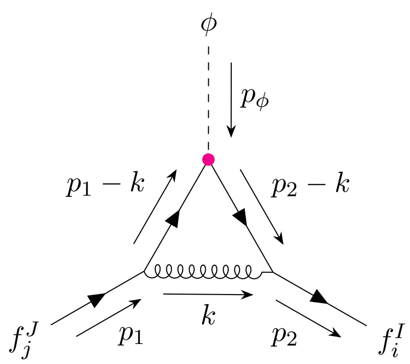
$$\begin{aligned} i\mathcal{M}_{SS}^{(\gamma)}|_{\text{div.}} &= 2A\bar{u}(p_2)u(p_1)\int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}z\,\frac{i}{4\pi^2}\Delta_\varepsilon \\ &= \frac{i}{2\pi^2}A\bar{u}(p_2)u(p_1)\Delta_\varepsilon \times \int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}z \\ &= \frac{i}{2\pi^2}A\bar{u}(p_2)u(p_1)\Delta_\varepsilon \times \frac{1}{2} \end{aligned} \tag{B.224}$$

and we can finally write

$$i\mathcal{M}_{SS}^{(\gamma)} = \frac{i}{4\pi^2}e^2Q_f^2\frac{v}{\Lambda}y_S^{ij}\mu^{\varepsilon/2}\bar{u}(p_2)u(p_1)\Delta_\varepsilon + \text{finite}\,. \tag{B.225}$$

QCD-induced diagram

The fourth diagram contributing to the one-loop correction of the $\phi\bar{f}_i f_j$ operator is mediated by the operator $\phi\bar{f}_i f_j$ itself and is QCD-induced. It is given by



$$= i\mathcal{M}_{SS}^{(g)}\,, \tag{B.226}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$i\mathcal{M}_{SS}^{(g)} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{u}(p_2)(-ig_s\gamma^\nu c_f T^b_{IM}\mu^{\varepsilon/2}) \frac{i\delta_{MK}}{\not{p}_2 - \not{k} - m_i + i\epsilon}(i\frac{v}{\Lambda}y_S^{ij}\mu^{\varepsilon/2}) \times \frac{i\delta_{KL}}{\not{p}_1 - \not{k} - m_j + i\epsilon}(-ig_s\gamma^\mu c_f T^a_{LJ}\mu^{\varepsilon/2})u(p_1)\frac{-ig_{\mu\nu}\delta^{ab}}{k^2 + i\epsilon}\,. \tag{B.227}$$

This diagram is completely analogous to $i\mathcal{M}_{SS}^{(\gamma)}$, and its color structure is provided by

$$T^b_{IM}T^a_{LJ}\delta^{ab}\delta_{ML} = C_F\delta_{IJ} = \frac{N_c^2 - 1}{2N_c}\delta_{IJ}\,, \tag{B.228}$$

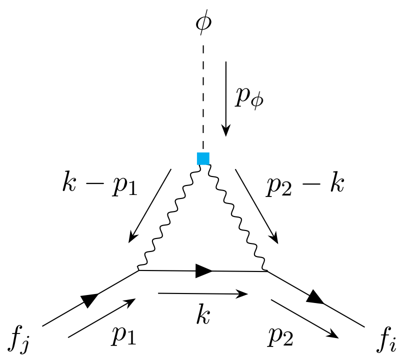
where C_F is the Casimir of the gauge group $SU(N_c)$ in the fundamental representation. Thus, we can immediately conclude that

$$i\mathcal{M}_{SS}^{(g)} = \frac{i}{4\pi^2}C_F g_s^2 c_f^2 \frac{v}{\Lambda} y_S^{ij}\delta_{IJ}\mu^{\varepsilon/2}\bar{u}(p_2)u(p_1)\Delta_\varepsilon + \text{finite} \tag{B.229}$$

B.6 $i\phi\bar{f}\gamma_5 f$ vertex corrections

B.6.1 $\phi F\widetilde{F}$ mediated diagram

The first diagram contributing to the one-loop correction of the $i\phi\bar{f}_i\gamma_5 f_j$ operator is mediated by the operator $\phi F_{\mu\nu}\widetilde{F}^{\mu\nu}$ and is given by



$$= i\mathcal{M}_{P\tilde{\gamma}}\,, \tag{B.230}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{P\tilde{\gamma}} = &\int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{u}(p_2)(-ie\gamma^\nu Q_f \mu^{\varepsilon/2}) \frac{i}{\not{k} - m_i + i\epsilon} \\ &\times (-ie\gamma^\mu Q_f \mu^{\varepsilon/2}) u(p_1) \delta^{ij} \frac{-ig_{\mu\rho}}{(k-p_1)^2 + i\epsilon} \frac{-ig_{\nu\sigma}}{(k-p_2)^2 + i\epsilon} \\ &\times (4ie^2 \frac{\widetilde{C}_\gamma}{\Lambda} \mu^{\varepsilon/2}) \epsilon^{\rho\sigma\alpha\beta} (k-p_1)_\alpha (p_2 - k)_\beta \\ = &A\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N}{D_1 D_2 D_3}, \end{aligned} \tag{B.231}$$

where we have defined

$$A = 4e^4 Q_f^2 \delta^{ij} \frac{\widetilde{C}_\gamma}{\Lambda} \mu^{\varepsilon/2}, \tag{B.232}$$

$$N = \epsilon^{\mu\nu\alpha\beta} \bar{u}(p_2) \gamma_\nu (\not{k} + m_i) \gamma_\mu u(p_1) (k-p_1)_\alpha (k-p_2)_\beta, \tag{B.233}$$

$$D_1 = k^2 - m_i^2 + i\epsilon, \tag{B.234}$$

$$D_2 = (k-p_1)^2 + i\epsilon, \tag{B.235}$$

$$D_3 = (k-p_2)^2 + i\epsilon. \tag{B.236}$$

Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned} \frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[D_1 x + D_2 y + D_3 z]^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[(k - p_1 y - p_2 z)^2 - C + i\epsilon]^3}, \end{aligned} \tag{B.237}$$

with

$$C = m_i^2 (y^2 + z^2 - 2y - 2z + 1) + 2p_1 \cdot p_2 yz, \tag{B.238}$$

where $p_1^2 = p_2^2 = m_i^2$, since this diagram is flavor diagonal, and $x + y + z = 1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = p + p_1 y + p_2 z$, which implies $\mathrm{d}^d k = \mathrm{d}^d p$ and

$$\begin{aligned} \frac{1}{D_1 D_2 D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{(p^2 - C + i\epsilon)^3} \\ &= 2\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{1}{(p^2 - C + i\epsilon)^3}. \end{aligned} \tag{B.239}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}
N &= \epsilon^{\mu\nu\alpha\beta}\bar{u}(p_2)\gamma_\nu(\not{p} + \not{p}_1 y + \not{p}_2 z + m_i)\gamma_\mu u(p_1)[p + p_1(y-1) + p_2 z]_\alpha \\
&\quad \times [p + p_1 y + p_2(z-1)]_\beta \\
&= \epsilon^{\mu\nu\alpha\beta} p_\alpha p_\beta \bar{u}(p_2)[\gamma_\nu \not{p}\gamma_\mu + \gamma_\nu(\not{p}_1 y + \not{p}_2 z + m_i)\gamma_\mu]u(p_1) \\
&\quad + \epsilon^{\mu\nu\alpha\beta}\bar{u}(p_2)\gamma_\nu\not{p}\gamma_\mu u(p_1)\{p_\alpha[p_1 y + p_2(z-1)]_\beta \\
&\quad + p_\beta[p_1(y-1) + p_2 z]_\alpha\} + \mathcal{O}(p) \\
&= \epsilon^{\mu\nu\alpha\beta}\bar{u}(p_2)\gamma_\nu\not{p}\gamma_\mu u(p_1)\{p_\alpha[p_1 y + p_2(z-1)]_\beta \\
&\quad + p_\beta[p_1(y-1) + p_2 z]_\alpha\} + \mathcal{O}(p) \\
&= \frac{1}{d}\epsilon^{\mu\nu\alpha\beta} p^2 \bar{u}(p_2)\gamma_\nu\gamma^\rho\gamma_\mu u(p_1)\{g_{\alpha\rho}[p_1 y + p_2(z-1)]_\beta \\
&\quad + g_{\beta\rho}[p_1(y-1) + p_2 z]_\alpha\} + \mathcal{O}(p)\,,
\end{aligned} \tag{B.240}$$

where in the last expression we have effectively substituted $p^\mu p^\nu$ with $p^2 g^{\mu\nu}/d$ by Lorentz invariance. Since we are interested in the divergent part of the integral, we can safely ignore $\mathcal{O}(p)$ terms in N, keeping only the term of order p^2: the divergent part of N is

$$\begin{aligned}
N|_{\text{div.}} = \frac{1}{d}\epsilon^{\mu\nu\alpha\beta} p^2 \bar{u}(p_2)\gamma_\nu\gamma^\rho\gamma_\mu u(p_1)\{g_{\alpha\rho}[p_1 y + p_2(z-1)]_\beta \\
+ g_{\beta\rho}[p_1(y-1) + p_2 z]_\alpha\}\,.
\end{aligned} \tag{B.241}$$

In order to simplify the numerator, we can exploit the following identity

$$\gamma_\mu\gamma_\nu\gamma_\rho = S_{\mu\nu\rho\sigma}\gamma^\sigma - i\epsilon_{\sigma\mu\nu\rho}\gamma^\sigma\gamma_5\,, \tag{B.242}$$

where

$$S_{\mu\nu\rho\sigma} = g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} \tag{B.243}$$

vanishes when contracted with the Levi-Civita tensor, and

$$\epsilon^{\mu\nu\rho\alpha}\epsilon_{\mu\nu\rho\beta} = -2(d-1)\delta^\alpha_\beta \tag{B.244}$$

is used, so that we obtain

$$\begin{aligned} N|_{\text{div.}} &= -\frac{i}{d} p^2 \epsilon^{\mu\nu\alpha\beta} \bar{u}(p_2)\gamma^\delta \gamma_5 u(p_1)\{\epsilon_{\delta\nu\alpha\mu}[p_1 y + p_2(z-1)]_\beta \\ &\quad + \epsilon_{\delta\nu\beta\mu}[p_1(y-1) + p_2 z]_\alpha\} \\ &= -2i\frac{d-1}{d} p^2 \bar{u}(p_2)\gamma^\mu \gamma_5 u(p_1)\{[p_1 y + p_2(z-1)]_\mu \\ &\quad - [p_1(y-1) + p_2 z]_\mu\} \\ &= -2i\frac{d-1}{d} p^2 \bar{u}(p_2)(\not{p}_1 - \not{p}_2)\gamma_5 u(p_1) \\ &= 2i\frac{d-1}{d} p^2 \bar{u}(p_2)(\gamma_5 \not{p}_1 + \not{p}_2 \gamma_5) u(p_1) \\ &= 4i\frac{d-1}{d} p^2 m_i \bar{u}(p_2)\gamma_5 u(p_1)\,, \end{aligned} \tag{B.245}$$

where $\bar{u}(p_2)\not{p}_2 = m_i \bar{u}(p_2)$ and $\not{p}_1 u(p_1) = m_j u(p_1)$ from the Dirac equation, with $m_i = m_j$ since the diagram is flavor diagonal. Thus, the divergent part of the diagram reads

$$\begin{aligned} i\mathcal{M}_{P\tilde{\gamma}}|_{\text{div.}} &= 2A\mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{N|_{\text{div.}}}{(p^2 - C + i\epsilon)^3} \\ &= 8i\frac{d-1}{d}\bar{u}(p_2)\gamma_5 u(p_1) A m_i \mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \\ &\quad \times \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p^2}{(p^2 - C + i\epsilon)^3} \\ &= 8i\bar{u}(p_2)\gamma_5 u(p_1) A m_i \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{d-1}{d}\mu^\varepsilon I_{1,3}\,. \end{aligned} \tag{B.246}$$

The integral $I_{1,3}$ is computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_{\mathrm{E}} + \mathcal{O}(x)$ as $x \to 0$:

$$I_{1,3} = \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\left(1 - \frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2)\,, \tag{B.247}$$

which, multiplied by $(d-1)\mu^\varepsilon/d$, gives

$$\begin{aligned} \frac{d-1}{d}\mu^\varepsilon I_{1,3} &= \frac{3-\varepsilon}{4-\varepsilon}\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left(1 - \frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2) \\ &= \frac{i}{16\pi^2}\frac{3}{4}\left[1 + \frac{\varepsilon}{2}\log\frac{4\pi\mu^2}{C} + \mathcal{O}(\varepsilon^2)\right]\left[\frac{2}{\varepsilon} - \frac{2}{3} - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon)\right] \\ &= \frac{3i}{64\pi^2}\left[\Delta_\varepsilon - \frac{2}{3} + \log\frac{\mu^2}{C} + \mathcal{O}(\varepsilon)\right], \end{aligned} \tag{B.248}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_{\rm E} + \log(4\pi)$. Therefore, the divergent contribution of the diagram is given by

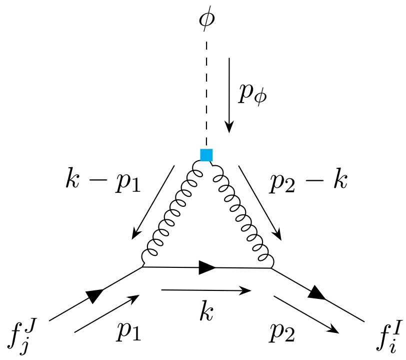
$$\begin{aligned} i\mathcal{M}_{P\tilde{\gamma}}|_{\rm div.} &= 8i\bar{u}(p_2)\gamma_5 u(p_1) A m_i \int_0^1 {\rm d}y \int_0^{1-y} {\rm d}z \frac{3i}{64\pi^2}\Delta_\varepsilon \\ &= -\frac{3}{8\pi^2}\bar{u}(p_2)\gamma_5 u(p_1) A m_i \Delta_\varepsilon \times \int_0^1 {\rm d}y \int_0^{1-y} {\rm d}z \\ &= -\frac{3}{8\pi^2}\bar{u}(p_2)\gamma_5 u(p_1) A m_i \Delta_\varepsilon \times \frac{1}{2} \end{aligned} \tag{B.249}$$

and we can finally write

$$i\mathcal{M}_{P\tilde{\gamma}} = -\frac{3}{4\pi^2} e^4 Q_f^2 \frac{\widetilde{C}_\gamma}{\Lambda} m_i \delta^{ij} \mu^{\varepsilon/2} \bar{u}(p_2)\gamma_5 u(p_1)\Delta_\varepsilon + \text{finite}\,. \tag{B.250}$$

B.6.2 $\phi G\widetilde{G}$ mediated diagram

The second diagram contributing to the one-loop correction of the $i\phi\bar{f}_i\gamma_5 f_j$ operator is mediated by the operator $\phi G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ and is given by



$$= i\mathcal{M}_{P\tilde{g}}\,, \tag{B.251}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{P\tilde{g}} = \int \frac{{\rm d}^d k}{(2\pi)^d} &\bar{u}(p_2)(-ig_s\gamma^\nu c_f T^b_{IL}\mu^{\varepsilon/2}) \frac{i\delta_{LK}}{\not{k} - m_i + i\epsilon} \\ &\times (-ig_s\gamma^\mu c_f T^a_{KJ}\mu^{\varepsilon/2}) u(p_1)\delta^{ij} \frac{-ig_{\mu\rho}\delta^{ac}}{(k-p_1)^2 + i\epsilon} \frac{-ig_{\nu\sigma}\delta^{bd}}{(k-p_2)^2 + i\epsilon} \\ &\times (4ig_s^2 \frac{\widetilde{C}_g}{\Lambda}\delta^{cd}\mu^{\varepsilon/2})\epsilon^{\rho\sigma\alpha\beta}(k-p_1)_\alpha (p_2-k)_\beta\,. \end{aligned} \tag{B.252}$$

This diagram is completely analogous to $i\mathcal{M}_{P\tilde{\gamma}}$, and its color structure is provided by

$$T^b_{IL} T^a_{KJ} \delta^{ab}\delta_{LK} = C_F \delta_{IJ} = \frac{N_c^2 - 1}{2N_c}\delta_{IJ}\,, \tag{B.253}$$

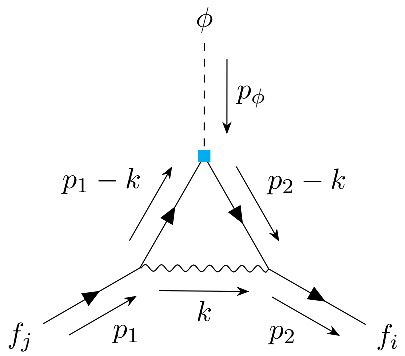
where C_F is the Casimir of the gauge group $SU(N_c)$ in the fundamental representation. Thus, we can immediately conclude that

$$i\mathcal{M}_{P\tilde{g}} = -\frac{3}{4\pi^2} C_F g_s^4 c_f^2 \frac{\widetilde{C}_g}{\Lambda} m_i \delta^{ij} \delta_{IJ} \mu^{\varepsilon/2} \bar{u}(p_2)\gamma_5 u(p_1)\Delta_\varepsilon + \text{finite}\,. \tag{B.254}$$

B.6.3 $i\phi\bar{f}\gamma_5 f$ mediated diagrams

EM-induced diagram

The third diagram contributing to the one-loop correction of the $i\phi\bar{f}_i\gamma_5 f_j$ operator is mediated by the operator $i\phi\bar{f}_i\gamma_5 f_j$ itself and is EM-induced. It is given by



$$= i\mathcal{M}_{PP}^{(\gamma)}\,, \tag{B.255}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{PP}^{(\gamma)} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{u}(p_2)(-ie\gamma^\nu Q_f \mu^{\varepsilon/2}) \frac{i}{\not{p}_2 - \not{k} - m_i + i\epsilon} \frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \\ &\quad \times (-\frac{v}{\Lambda} y_P^{ij} \gamma_5 \mu^{\varepsilon/2}) \frac{i}{\not{p}_1 - \not{k} - m_j + i\epsilon} (-ie\gamma^\mu Q_f \mu^{\varepsilon/2}) u(p_1) \\ &= A\mu^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N}{D_1 D_2 D_3}\,, \end{aligned} \tag{B.256}$$

where we have defined

$$A = ie^2 Q_f^2 \frac{v}{\Lambda} y_P^{ij} \mu^{\varepsilon/2}\,, \tag{B.257}$$

$$N = \bar{u}(p_2)\gamma^\mu(\not{p}_2 - \not{k} + m_i)\gamma_5(\not{p}_1 - \not{k} + m_j)\gamma_\mu u(p_1)\,, \tag{B.258}$$

$$D_1 = k^2 + i\epsilon\,, \tag{B.259}$$

$$D_2 = (k - p_2)^2 - m_i^2 + i\epsilon\,, \tag{B.260}$$

$$D_3 = (k - p_1)^2 - m_j^2 + i\epsilon\,. \tag{B.261}$$

Exploiting the Feynman parametrization, we can write the denominator in the integral as

$$\begin{aligned}\frac{1}{D_1D_2D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[D_1x+D_2y+D_3z]^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{[(k-p_2y-p_1z)^2-C+i\epsilon]^3}\,,\end{aligned} \tag{B.262}$$

with

$$C = m_i^2y^2 + m_j^2z^2 + 2p_1\cdot p_2yz\,, \tag{B.263}$$

where $p_2^2 = m_i^2$, $p_1^2 = m_j^2$ and $x+y+z=1$ have been used. In order to simplify the denominator, we can shift the integration variable k as $k = p + p_2y + p_1z$, which implies $\mathrm{d}^dk = \mathrm{d}^dp$ and

$$\begin{aligned}\frac{1}{D_1D_2D_3} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(x+y+z-1)}{(p^2-C+i\epsilon)^3} \\ &= 2\int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \frac{1}{(p^2-C+i\epsilon)^3}\,.\end{aligned} \tag{B.264}$$

At this point we have to perform the shift also in the numerator of the integral: N becomes

$$\begin{aligned}N &= \bar{u}(p_2)\gamma^\mu(\not{p} + \not{p}_1y + \not{p}_2(z-1) - m_i)\gamma_5 \\ &\quad \times (\not{p} + \not{p}_1(y-1) + \not{p}_2z - m_j)\gamma_\mu u(p_1) \\ &= \bar{u}(p_2)\gamma^\mu\not{p}\gamma_5\not{p}\gamma_\mu u(p_1) + \mathcal{O}(p) \\ &= p^\alpha p^\beta \bar{u}(p_2)\gamma^\mu\gamma_\alpha\gamma_5\gamma_\beta\gamma_\mu u(p_1) + \mathcal{O}(p) \\ &= \frac{1}{d}p^2 g^{\alpha\beta}\bar{u}(p_2)\gamma^\mu\gamma_\alpha\gamma_5\gamma_\beta\gamma_\mu u(p_1) + \mathcal{O}(p) \\ &= \frac{1}{d}p^2\bar{u}(p_2)\gamma^\beta\gamma^\alpha\gamma_5\gamma_\alpha\gamma_\beta u(p_1) + \mathcal{O}(p)\,,\end{aligned} \tag{B.265}$$

where we have effectively substituted $p^\mu p^\nu$ with $p^2g^{\mu\nu}/d$ by Lorentz invariance. Since we are interested in the divergent part of the integral, we can safely ignore $\mathcal{O}(p)$ terms in N, keeping only the term of order p^2: the divergent part of N is

$$N|_{\text{div.}} = \frac{1}{d}p^2\bar{u}(p_2)\gamma^\beta\gamma^\alpha\gamma_5\gamma_\alpha\gamma_\beta u(p_1)\,. \tag{B.266}$$

In order to compute $\gamma^\alpha\gamma_5\gamma_\alpha$, we can split γ^μ onto its 4-dimensional component $\bar{\gamma}^\mu$ and $(d-4)$-dimensional component $\hat{\gamma}^\mu$ as $\gamma^\mu = \bar{\gamma}^\mu + \hat{\gamma}^\mu$ according to the BMHV scheme introduced in Subsection 5.1.1

$$\begin{aligned}\gamma^\alpha\gamma_5\gamma_\alpha &= (\bar{\gamma}^\alpha + \hat{\gamma}^\alpha)\gamma_5(\bar{\gamma}_\alpha + \hat{\gamma}_\alpha) \\ &= \bar{\gamma}^\alpha\gamma_5\bar{\gamma}_\alpha + \bar{\gamma}^\alpha\gamma_5\hat{\gamma}_\alpha + \hat{\gamma}^\alpha\gamma_5\bar{\gamma}_\alpha + \hat{\gamma}^\alpha\gamma_5\hat{\gamma}_\alpha\end{aligned} \tag{B.267}$$

and exploit the following identities

$$\gamma_5\bar{\gamma}_\alpha = -\bar{\gamma}_\alpha\gamma_5\,, \tag{B.268}$$
$$\gamma_5\hat{\gamma}_\alpha = \hat{\gamma}_\alpha\gamma_5\,, \tag{B.269}$$
$$\bar{\gamma}^\alpha\bar{\gamma}_\alpha = 4\mathbb{1}\,, \tag{B.270}$$
$$\hat{\gamma}^\alpha\hat{\gamma}_\alpha = (d-4)\mathbb{1}\,, \tag{B.271}$$
$$\hat{\gamma}^\alpha\bar{\gamma}_\alpha = 0\,, \tag{B.272}$$

in such a way that

$$\gamma^\alpha\gamma_5\gamma_\alpha = (-\bar{\gamma}^\alpha\bar{\gamma}_\alpha + \bar{\gamma}^\alpha\hat{\gamma}_\alpha - \hat{\gamma}^\alpha\bar{\gamma}_\alpha + \hat{\gamma}^\alpha\hat{\gamma}_\alpha)\gamma_5 = -(8-d)\gamma_5\,. \tag{B.273}$$

Thus, we can write the divergent part of the numerator as

$$N|_{\text{div.}} = \frac{(8-d)^2}{d}p^2\bar{u}(p_2)\gamma_5 u(p_1)\,, \tag{B.274}$$

and the divergent part of the diagram reads

$$\begin{aligned}
i\mathcal{M}^{(\gamma)}_{PP}|_{\text{div.}} &= 2A\mu^\varepsilon \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{N|_{\text{div.}}}{(p^2-C+i\epsilon)^3} \\
&= 2A\mu^\varepsilon \frac{(8-d)^2}{d}\bar{u}(p_2)\gamma_5 u(p_1)\int_0^1 \mathrm{d}y \int_0^{1-y}\mathrm{d}z \\
&\quad\times \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{p^2}{(p^2-C+i\epsilon)^3} \\
&= 2A\bar{u}(p_2)\gamma_5 u(p_1)\int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}z\frac{(8-d)^2}{d}\mu^\varepsilon I_{1,3}\,.
\end{aligned} \tag{B.275}$$

The integral $I_{1,3}$ is computed according to master formula in Eq. (5.52) and then we can expand around $\varepsilon = 0$ knowing that $\Gamma(x) = 1/x - \gamma_{\text{E}} + \mathcal{O}(x)$ as $x \to 0$:

$$I_{1,3} = \frac{i}{16\pi^2}\left(\frac{4\pi}{C}\right)^{\varepsilon/2}\left(1-\frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2)\,, \tag{B.276}$$

which, multiplied by $(8-d)^2\mu^\varepsilon/d$, gives

$$\begin{aligned}
\frac{(8-d)^2}{d}\mu^\varepsilon I_{1,3} &= \frac{(4+\varepsilon)^2}{4-\varepsilon}\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\varepsilon/2}\left(1-\frac{\varepsilon}{4}\right)\Gamma(\varepsilon/2) \\
&= \frac{i}{16\pi^2}4\left[1+\frac{\varepsilon}{2}\log\frac{4\pi\mu^2}{C}+\mathcal{O}(\varepsilon^2)\right]\left[\frac{2}{\varepsilon}+1-\gamma_{\text{E}}+\mathcal{O}(\varepsilon)\right] \\
&= \frac{i}{4\pi^2}\left[\Delta_\varepsilon + 1 + \log\frac{\mu^2}{C}+\mathcal{O}(\varepsilon)\right],
\end{aligned} \tag{B.277}$$

where $\Delta_\varepsilon = 2/\varepsilon - \gamma_{\rm E} + \log(4\pi)$. Therefore, the divergent contribution of the diagram is given by

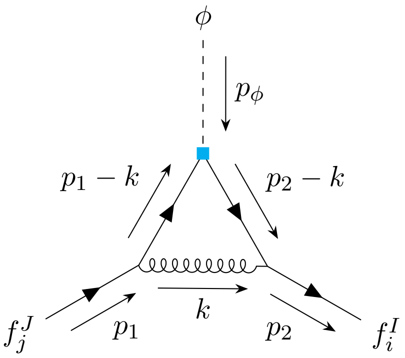
$$\begin{aligned} i\mathcal{M}_{PP}^{(\gamma)}|_{\rm div.} &= 2A\bar{u}(p_2)\gamma_5 u(p_1)\int_0^1 {\rm d}y \int_0^{1-y} {\rm d}z \frac{i}{4\pi^2}\Delta_\varepsilon \\ &= \frac{i}{2\pi^2}A\bar{u}(p_2)\gamma_5 u(p_1)\Delta_\varepsilon \times \int_0^1 {\rm d}y \int_0^{1-y} {\rm d}z \\ &= \frac{i}{2\pi^2}A\bar{u}(p_2)\gamma_5 u(p_1)\Delta_\varepsilon \times \frac{1}{2} \end{aligned} \tag{B.278}$$

and we can finally write

$$i\mathcal{M}_{PP}^{(\gamma)} = -\frac{1}{4\pi^2}e^2 Q_f^2 \frac{v}{\Lambda} y_P^{ij} \mu^{\varepsilon/2}\bar{u}(p_2)\gamma_5 u(p_1)\Delta_\varepsilon + \text{finite}\,. \tag{B.279}$$

QCD-induced diagram

The fourth diagram contributing to the one-loop correction of the $i\phi\bar{f}_i\gamma_5 f_j$ operator is mediated by the operator $i\phi\bar{f}_i\gamma_5 f_j$ itself and is QCD-induced. It is given by



$$= i\mathcal{M}_{PP}^{(g)}\,, \tag{B.280}$$

which can be computed according to the d-dimensional Feynman rules presented in Sections 5.3 and A.2 as

$$\begin{aligned} i\mathcal{M}_{PP}^{(g)} = \int \frac{{\rm d}^d k}{(2\pi)^d}\bar{u}(p_2)(-ig_s\gamma^\nu c_f T^b_{IM}\mu^{\varepsilon/2})\frac{i\delta_{MK}}{\not{p}_2 - \not{k} - m_i + i\epsilon}(-\frac{v}{\Lambda}y_P^{ij}\gamma_5\mu^{\varepsilon/2}) \\ \times \frac{i\delta_{KL}}{\not{p}_1 - \not{k} - m_j + i\epsilon}(-ig_s\gamma^\mu c_f T^a_{LJ}\mu^{\varepsilon/2})u(p_1)\frac{-ig_{\mu\nu}\delta^{ab}}{k^2 + i\epsilon}\,. \end{aligned} \tag{B.281}$$

This diagram is completely analogous to $i\mathcal{M}_{PP}^{(\gamma)}$, and its color structure is provided by

$$T^b_{IM}T^a_{LJ}\delta^{ab}\delta_{ML} = C_F\delta_{IJ} = \frac{N_c^2 - 1}{2N_c}\delta_{IJ}\,, \tag{B.282}$$

where C_F is the Casimir of the gauge group $SU(N_c)$ in the fundamental representation. Thus, we can immediately conclude that

$$i\mathcal{M}_{PP}^{(g)} = -\frac{1}{4\pi^2} C_F g_s^2 c_f^2 \frac{v}{\Lambda} y_P^{ij} \delta_{IJ} \mu^{\varepsilon/2} \bar{u}(p_2)\gamma_5 u(p_1)\Delta_\varepsilon + \text{finite}\,. \tag{B.283}$$

Appendix C

Four-particle tree-level amplitudes

In this Appendix we report the four-particle tree-level amplitudes composed of $SU(N_c)\times U(1)_{\rm em}$ gauge interactions. These results are exploited in Chapter 7 in order to compute the RGEs of the ALP EFT via on-shell amplitude methods.

Regarding all particles as outgoing and making explicit the indices I, J of the fundamental representation of $SU(N_c)$, the non-vanishing amplitudes read

$$\mathcal{M}_4(1^-_{f_i}, 2^-_{\bar{f}_j}, 3^+_{\bar{f}_i}, 4^+_{f_j}) = 2(e^2 Q_f^2 + C_F g_s^2 c_f^2)\frac{\langle 1\,2\rangle[4\,3]}{\langle 1\,3\rangle[3\,1]}\,, \tag{C.1}$$

$$\mathcal{M}_4(1^-_{f}, 2^+_{\bar{f}}, 3^+_{\bar{f}}, 4^-_{f}) = 2(e^2 Q_f^2 + C_F g_s^2 c_f^2)\frac{\langle 1\,4\rangle[2\,3]}{\langle 1\,3\rangle[3\,1]}\,, \tag{C.2}$$

$$\mathcal{M}_4(1^-_{f}, 2^+_{\bar{f}}, 3^-_{\gamma}, 4^+_{\gamma}) = 2e^2 Q_f^2\frac{\langle 3\,1\rangle[4\,2]}{\langle 1\,4\rangle[3\,1]}\,, \tag{C.3}$$

$$\mathcal{M}_4(1^-_{f^I}, 2^+_{\bar{f}^J}, 3^-_{g^a}, 4^+_{g^b})\delta^{ab} = 2C_F g_s^2 c_f^2 \delta_{IJ}\frac{\langle 3\,1\rangle[4\,2]}{\langle 1\,4\rangle[3\,1]}\,, \tag{C.4}$$

$$\mathcal{M}_4(1^-_{f^I}, 2^+_{\bar{f}^J}, 3^-_{g^a}, 4^+_{g^b})\delta_{IJ} = g_s^2 c_f^2 \delta^{ab}\frac{\langle 3\,1\rangle[4\,2]}{\langle 1\,4\rangle[3\,1]}\,, \tag{C.5}$$

$$\mathcal{M}_4(1^-_{g^a}, 2^-_{g^b}, 3^+_{g^c}, 4^+_{g^d})\delta^{cd} = -2C_A g_s^2 \delta^{ab}\frac{\langle 1\,2\rangle^4}{\langle 1\,3\rangle\langle 3\,2\rangle\langle 2\,4\rangle\langle 4\,1\rangle}\,, \tag{C.6}$$

where we recall that

$$C_A = N_c = 3\,, \qquad\qquad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}\,. \tag{C.7}$$

The adopted gauge group conventions are reported in Section A.1.

Crossing symmetry is extensively exploited to relate scattering matrix elements of ingoing and outgoing states with those computed with all the momenta and other quantum numbers considered as outgoing:

$$\mathcal{M}_4(1^{h_1}, 2^{h_2}; 3^{h_3}, 4^{h_4}) = (-1)^{n_f^-}\, \mathcal{M}_4(1^{h_1}, 2^{h_2}, \bar{4}^{-h_4}, \bar{3}^{-h_3}) \,. \tag{C.8}$$

Here n_f^- counts the number of (anti-)fermions with negative helicity in the initial state and the reverse order of fields upon crossing ensures the proper minus signs for fermion loops. Ref. [4] provides a detailed discussion about these phases. Moreover, the adopted convention for opposite momenta is

$$\bar{p}_i^{\dot{\alpha}\alpha} = (-p_i)^{\dot{\alpha}\alpha} = \bar{\tilde{\lambda}}_i^{\dot{\alpha}} \bar{\lambda}_i^{\alpha} \,, \tag{C.9}$$

with

$$\bar{\lambda}_i = \lambda_i \,, \qquad \bar{\tilde{\lambda}}_i = -\tilde{\lambda}_i \,, \tag{C.10}$$

as already specified in Section 6.1.

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