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On the uniqueness of the solutions of a Stochastic Differential Equation with non-negativity constraint

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Chapter 0

Heuristic Prologue

0.1 Genesis of Stochastic Differential Equations

The starting point of stochastic processes in continuous time, is the study of dynamical systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^N \tag{0.1}$$

in which a perturbation, often called *noise* in the literature, is added to the equation, making f depending also on $\omega \in \Omega$, where $(\Omega, \mathscr{A}, \mathbf{P})$ is a probability space.

Equation (0.1) is an *ordinary differential equation* (ODE for short); these kind of mathematical objects turn out to be particularly fitting in the description of the evolution of certain situations. ODE are widely used in the more disparate settings, from physics to economics, from financial markets to demographical phenomena, from astronomy to different areas of engineering, from chemistry to biology.

One first example comes from Newton's works in classical physics: the second principle is usally written as

$$m\vec{a} = \vec{F},$$

which is an ODE: in fact, if $t \mapsto y(t) \in \mathbb{R}^3$ describes the position of a point (whose mass is denoted by m) in the space, then $\vec{v}(t) := y'(t)$ and $\vec{a}(t) := y''(t)$ are its velocity and acceleration respectively. In general, the force $\vec{F}$ depends on time, position and velocity of the considered point, thus the above equation can be rewritten as follows:

$$my''(t) = F(t, y(t), y'(t)).$$

We are dealing with an equation in which the unknown is a function of one variable (the function y), in which some of its derivatives are involved.

It is clear that the main aim is to "solve" these kind of equations, that is, find all possible functions make the identity true.

The second matter is study uniqueness of solutions; one can easily find ODEs which have not unique solution: take for istance $y'(t) = \sqrt{y(t)} \quad \forall t \in \mathbb{R}$.

Another classical example is the Malthus growth model: if $y(t)$ denotes the number of entities of a population, then the model assumes the growth speed of this population $y'(t)$ to be proportional with respect to the own entities mass, that is the following relation has to be verified:

$$y'(t) = \alpha y(t),$$

where α can be indended as $\nu - \mu$, where ν and μ are the natality rate and death rate respectively.

In this case it is readily checked that functions of the form $ce^{\alpha t}$ solves the equation; moreover, imposing a solution $\varphi : \mathbb{R} \to \mathbb{R}$ passes thru a fixed point y_0 at some starting time t_0, we can easily see that $t \mapsto \varphi(t)e^{-\alpha t}$ is constantly equal to y_0, from which one gets uniqueness of the solutions.

However let us come back to (0.1) and understand euristically what stochastical differential equations are and how do they born from this. We start by considering a discrete version of (0.1) with $N = 1$, that is a finite differences scalar equation: so for $0 < \Delta \ll 1$ we write

$$x_{(k+1)\Delta} = x_{k\Delta} + \Delta \cdot f(x_{k\Delta}) + g(x_{k\Delta}, \omega) \cdot W_k^\Delta(\omega) \tag{0.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is deterministic (the drift) while $g : \mathbb{R} \times \Omega \to \mathbb{R}$ brings into the equation some randomness (the diffusion coefficient) and $\{W_k^\Delta\}_{k \geq 0}$ is a sequence of i.i.d. random variables such that, without loss of generality $\mathbf{E}[W_k^\Delta] = 0$.

The next step is understand how W_k^Δ depend on Δ: what is a suitable order of magnitude with respect to Δ, such that the passage to the limit in (0.2) for $k \to +\infty$, gives back something reasonable?
So, let us focus on $\mathrm{Var}[W_k^\Delta]$; setting $t = n\Delta$, we can rewrite (0.2) as follows:

$$x_t = x_0 + \Delta \sum_{k=0}^{n-1} f(x_{k\Delta}) + \sum_{k=0}^{n-1} g(x_{k\Delta}, \omega) \cdot W_k^\Delta(\omega). \tag{0.3}$$

Now, fixing t, it is clear that $\Delta \to 0$ implies $n \to +\infty$, thus it follows that

$$\Delta \sum_{k=0}^{n-1} f(x_{k\Delta}) \overset{\Delta \to 0}{\longrightarrow} \int_0^t f(x_s)\, ds. \tag{0.4}$$

Next, suppose

- $g \equiv c$, with $c \in \mathbb{R} \setminus \{0\}$ constant; wlog $c = 1$
- $f \equiv 0$
- x_0 deterministic;

then, from (0.3) we would have

$$\mathrm{Var}[x_t] = \mathrm{Var}\left[\sum_{k=0}^{n-1} W_k^\Delta\right] = n\,\mathrm{Var}[W_0^\Delta],$$

from which we get that

$$\mathrm{Var}[W_0^\Delta] \approx \frac{1}{n} \approx \Delta$$

that is a kind of information about the behavior of W_k^Δ with respect to Δ we were searching for.

Let us then set

$$\mathrm{Var}[W_k^\Delta] := \Delta = \left(\sqrt{\frac{t}{n}}\right)^2.$$

At this point from the Central Limit Theorem one has that

$$\frac{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_k^\Delta}{\sqrt{\frac{t}{n}}} = \frac{1}{\sqrt{t}}\sum_{k=0}^{n-1} W_k^\Delta \xrightarrow[D]{\Delta\to 0} N(0,1)$$

from which

$$\sum_{k=0}^{n-1} W_k^\Delta \xrightarrow[D]{\Delta\to 0} B_t \sim N(0,t).$$

This holds for every $t \geq 0$, thus accepting all the B_t live in the same probability space, a stochastic process $B = \{B_t\}_{t\geq 0}$ is defined.

Next, we observe, that, fixed $0 \leq s < t$, the random variables $B_t - B_s$ and B_s are indipendent since they are limit in distribution of

$$\sum_{k=\lfloor \frac{s}{\Delta} \rfloor}^{\lfloor \frac{t}{\Delta} \rfloor -1} W_k^\Delta \quad \text{and} \quad \sum_{k=0}^{\lfloor \frac{s}{\Delta} \rfloor -1} W_k^\Delta$$

respectively.

It is now quite natural to expect that, fixed $0 \leq t_1 < \cdots < t_N$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}$ are indipendent, from which $(B_{t_1}, \ldots, B_{t_N})$ is a gaussian random vector.

Observing now that when $k \to +\infty$ one has $W_k^\Delta \sim B_{(k+1)\Delta} - B_{k\Delta}$ and considering now a non-trivial g, we can write

$$\sum_{k=0}^{n-1} g(x_{k\Delta}) W_k^\Delta \overset{\Delta \to 0}{\longrightarrow} \int_0^t g(x_s)\, dB_s \ . \tag{0.5}$$

where the limit is intended in probability and, if the process B has a.s. trajectories with bounded variation then the integral is a Riemann-Stieltjes integral; nevertheless a stochastic process having the properties B has, with moreover a.s. continuous trajectories, is called *Brownian Motion* and a.s. have not bounded variation: in this case the integral becomes a *Stochastic Integral*, which was introduced by the japanese mathematician Itô.

Putting togheter (0.3), (0.4) and (0.5) we get

$$x_t = x_0 + \int_0^t f(x_s)\, ds + \int_0^t g(x_s)\, dB_s \ , \tag{0.6}$$

which is a *Stochastic Differential Equation* driven by the Brownian Motion B.

Consider $x : [0, +\infty[\to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, both in $\mathscr{C}^1$; then we know that $\frac{d}{dx} f(x_t) = f'(x_t) \cdot x_t'$, which can be rewritten in integral form as $f(x_t) = f(x_0) + \int_0^t f'(x_s) \underbrace{dx_s}_{x_s'\, ds}$.
Then one can asks: is this rule true if $x = B$ (whose paths a.s. are not in $\mathscr{C}^1$)? That is: does $f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s$ hold true? The answer is no, and this is the motivation for the need of the celebrated *Itô Formula* which is basically a generalisation of the well known chain rule, and when $x = B$ and $f \in \mathscr{C}^2$, it reads as

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, ds.$$

Let us note that stochastic differential equations can be represented in two equivalent ways: the integral representation, which is the way (0.6) is written, and the differential representation, that is

$$dx_t = f(x_t)\, dt + g(x_t)\, dB_t$$

togheter with the initial value x_0 given. The differential representation is widely used, although it is only a formal writing, since involved paths are typically irregular.

0.2 More recent developments

Recently Stochastic Differential Equations driven by a more general process became object of interest in Stochastic Analysis research; such a process is called Fractional Brownian Motion; it is a generalisation of the standard Brownian Motion and it depends on an index $H \in]0,1[$, called *Hurst index*.

This process turns out to be useful in building many models in finance: in the classical Black & Scholes pricing model, the randomness of the stock price S is due to a Brownian Motion Z:

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dZ_t), \quad S_0 > 0 \\ dB_t &= rB_t \end{aligned}$$

where B is the bond price and $\mu \in \mathbb{R}$ and $r, \sigma \in \mathbb{R}_{>0}$. In particular the first equation is a Stochastic Differential Equation.

It was observed that interesting financial models can be derived from this last one by replacing the Brownian Motion Z with a fractional Brownian Motion Z^H; the solution of this new model is called *Geometric Fractional Brownian Motion*.

Many examples of situations in which the considered quantities are naturally positive can be reported. For istance the study of the motion of a particle on the plane with no constraints when its y-coordinate is ≥ 0 and "stopping" it when it goes below x-axis and is decreasing, removing this constraint once it becomes to increase.

These kind of models created the necessity to study SDEs with non-negativity constraints; such a constraint can be imposed in many ways; for this last model, for example, the Skorokhod problem (see chapter 2) turns out to be appropriate.

Studying these situations, the necessity to consider Skorokhod problem with SDEs driven by fBM arose; this argument is debated in the literature only a few, this is the reason why M. Ferrante and C. Rovira wrote their paper [2].

0.3 Last considerations

The stochastic integration with respect to the fBM has been studied by several authors; nevertheless throughout this thesis we will consider $H > \frac{1}{2}$, case in which the integral can be defined by a pathwise approach allowing a little regularity to the integrand function: the results discussed by Young [1] ensure the existence of the Riemann-Stieltjes integral $\int_0^T u_s \, dB_s^H$ for every stochastic process $\{u_t\}_{t\geq 0}$ whose trajectories are λ-Hölder continous, when $\lambda > 1 - H$; this result holds for every trajectory in the sense that $\int_0^T u_s(\omega) \, dB_s^H(\omega)$ is a Riemann-Stiltjes integral for every fixed $\omega \in \Omega$.

The starting point of this thesis is a paper by M. Ferrante and C. Rovira [2] in which the problem of the existence and uniqueness of the solutions of such differential equations was discussed at first exploiting the λ-Hölder norm methods; in this paper existence was proved, but some problem came out when the authors looked at uniqueness.

A different approach was held by A. Falkowski and L. Słomiński: in their paper [3] they treated a more general Stochastic Differential Equation but with some additional assumptions on the drift and diffusion coefficients, using the p-variation norm techniques, proving both existence and uniqueness.

We will analyze the first approach, suggesting a possible path to prove uniqueness with sup-norm approach.
In particular, this thesis is organized as follows: in the first chapter we will give the Stochastic Analysis tools required in the sequel, in the second one we will introduce our SDE with non-negativity constraints and the Young Integral, which is the one used to work with such an equation. Finally in the last chapter we will discuss about uniqueness.

Chapter 1

Stochastic Analysis Tools

1.1 The Fractional Brownian Motion

Let us fix a complete probability space $(\Omega, \mathscr{A}, \mathbf{P})$.

Definition 1.1.1. An N-dimensional *Gaussian Vector* is a random vector

$$X = (X_1, \ldots, X_N)$$

such that, $X \cdot a = \sum_{j=1}^{N} X_j a_j : \Omega \to \mathbb{R}$ is a normal random variable for every fixed $a \in \mathbb{R}^N$.

Definition 1.1.2. Let I be a total ordered set. A stochastic process

$$X : \Omega \to \mathbb{R}^I$$

is called *Gaussian Process* if for every $J \subseteq I,\ |J| < +\infty$, $X_{|J}$ is a gaussian vector.

Definition 1.1.3. A *scalar fractional Brownian Motion with Hurst parameter* $H \in]0, 1[$ is a stochastic process

$$B^H : \Omega \to \mathbb{R}^I$$

where $I = \mathbb{R}_{\geq 0}$, which is a centered (i.e. $\mathbb{E}[B_t^H] = 0 \ \ \forall t \geq 0$) Gaussian Process with covariance function given by

$$(s, t) \mapsto \mathrm{Cov}[B_s^H, B_t^H] = \mathbf{E}[B_s^H B_t^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right) \tag{1.1}$$

and a.s. with continous trajectories such that $B_0^H = 0$.

It's clear that when $H = \frac{1}{2}$ we recover the standard Brownian Motion; in particular the increments (i.e. the random variables $B_t - B_s$ for $0 \leq s < t$) are indipendent

(we have denoted $B := B^{1/2}$).
Instead, in the case $H \neq \frac{1}{2}$, the independency property of the increments does not hold true any longer.

The theory developed in this chapter deals with the above introduced scalar fractional Brownian Motion; nevertheless in the next chapters, working on the Stochastic Differential Equation, an m-dimensional version of this object is used:

Definition 1.1.4. A m-dimensional fractional Brownian Motion with Hurst parameter H is a m-dimensional random vector

$$W^H := (B^{H,1}, \dots, B^{H,m})$$

where the $B^{H,j}$ are indipendent fBMs of the same parameter H.

The fractional Brownian Motion is invariant in law under certain transformations:

Proposition 1.1.1. If B^H is an H-fractional Brownian Motion, then also the following stochastic processes are such:

1. $\{a^{-H} B^H_{at}\}_{t \geq 0}$ for every fixed $a > 0$, that is, the fBM is *self-similar*.

2. $\{B^H_{t_0+t} - B^H_{t_0}\}_{t \geq 0}$ for every fixed $t_0 \geq 0$, that is, the fBM has *stationary increments*.

3. $\{t^{2H} B^H_{1/t}\}_{t>0}$, that is, the law of the fBM is invariant under *time inversion*.

Conversely, any Gaussian process B^H a.s with continuous trajectories and $B^H_0 = 0$, such that $\mathrm{Var}[B^H_1] = 1$ for which 1. and 2. hold, then it is a fBM of index H.

Proof. The three invariace properties can be easily get recalling that two stochastic processes have the same law if and only if they have the same finite dimensional distributions; in particular for the second one, it can be useful to observe that from (1.1) it follows that

$$\mathbf{E}[(B^H_t - B^H_s)^2] = |t-s|^{2H}, \tag{1.2}$$

from which in particular we get that

$$B^H_{t_0+t} - B^H_{t_0} \sim N(0, t^{2H}) \sim B^H_t \quad \forall t \geq 0. \tag{1.3}$$

The converse part follows by direct computations. □

1.1.1 Hölder regolarity of the trajectories

Let us now prove the following important result:

Theorem 1.1.1. An H-fractional Brownian Motion $B^H = \{B^H_t\}_{t\in[0,1]}$ has a.s. α-Hölder continous paths for all $\alpha \in]0, H[$.

Proof. Let us define, for $n \geq 1$

$$\mathscr{D}_n := \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n - 1 \right\} \text{ and } \mathscr{D} := \bigcup_{n\geq 1} \mathscr{D}_n .$$

Step 1: Let us fix $0 < \alpha < H$ and prove that $\exists c, \rho > 0$ such that

$$\mathbf{P}\left(\exists s \in \mathscr{D}_n \ : \ |B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n}\right) \leq c2^{-\rho n}; \tag{1.4}$$

indeed we have

$$\begin{aligned}
\mathbf{P}\left(\exists s \in \mathscr{D}_n \ : \ |B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n}\right) &= \mathbf{P}\left(\bigcup_{s\in\mathscr{D}_n} \left\{|B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n}\right\}\right) \\
&\leq \sum_{s\in\mathscr{D}_n} \mathbf{P}\left(|B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n}\right) \\
&\overset{(1.3)}{=} 2^n \mathbf{P}\left(|B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n}\right) \\
&= 2^n \mathbf{P}\left(|B^H_{s+2^{-n}} - B^H_s|^{\frac{1}{H}m} > 2^{-\frac{1}{H}\alpha n m}\right) \\
&\overset{\text{Markov}}{\leq} 2^n \frac{\mathbf{E}\left[|B^H_{s+2^{-n}} - B^H_s|^{\frac{1}{H}m}\right]}{2^{-\frac{1}{H}\alpha n m}} \\
&= 2^{n\left(1+\frac{1}{H}\alpha m\right)} \mathbf{E}\left[\left|\frac{B^H_{s+2^{-n}} - B^H_s}{2^{-nH}}\right|^{\frac{1}{H}m}\right] 2^{-nm}
\end{aligned}$$

and observing that $\frac{B^H_{s+2^{-n}} - B^H_s}{2^{-nH}} \sim N(0,1)$, its expected value doesn't depend on n, thus setting $c_m := \mathbf{E}\left[\left|\frac{B^H_{s+2^{-n}} - B^H_s}{2^{-nH}}\right|^{\frac{1}{H}m}\right]$ and $\rho_m := -\left(1 + m\left(\frac{1}{H}\alpha - 1\right)\right)$ we have that

$$\mathbf{P}\left(\exists s \in \mathscr{D}_n \ : \ |B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n}\right) \leq c_m 2^{-n\rho_m}.$$

Observing that, since $0 < \alpha < H$, there exists $\overline{m}$ such that $\rho := \rho_{\overline{m}} > 0$, and setting $c := c_{\overline{m}}$, we get (1.4).

Step 2: $\exists \Omega^* \in \mathscr{A}$ with $\mathbf{P}(\Omega^*) = 1$ such that

$$\forall \omega \in \Omega^* \ \exists n(\omega) \geq 1 \ : \ |B^H_{s+2^{-n}} - B^H_s| \leq 2^{-\alpha n} \quad \forall s \in \mathscr{D}, \ \forall n \geq n(\omega). \tag{1.5}$$

Let us set

$$A_n := \left\{ \exists s \in \mathscr{D}_n \ : \ |B^H_{s+2^{-n}} - B^H_s| > 2^{-\alpha n} \right\} \in \mathscr{A};$$

then, by Step 1 we get $\sum_{n \geq 1} \mathbf{P}(A_n) < +\infty$, hence, once we have set

$$A := \limsup_{n \to +\infty} A_n =: \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k$$

the first Borel-Cantelli Lemma allows to deduce that $P(A) = 0$; clearly $A^c \in \mathscr{A}$ and $\mathbf{P}(A^c) = 1$.
Now, $\omega \in A \Leftrightarrow \omega \in A_n$ for countable many $n \geq 1$, thus $\omega \in A^c \Leftrightarrow \exists n(\omega) \geq 1$ such that $\omega \in A_n^c \ \forall n \geq n(\omega)$ and thus we get

$$\begin{aligned} A^c &= \{\omega \in \Omega \ : \ \exists n(\omega) \geq 1 \text{ s.t. } |B^H_{s+2^{-n}} - B^H_s| \leq 2^{-\alpha n} \ \ \forall s \in \mathscr{D}_n, \ \forall n \geq n(\omega)\} \\ &\subseteq \{\omega \in \Omega \ : \ \exists n(\omega) \geq 1 \text{ s.t. } |B^H_{s+2^{-n}} - B^H_s| \leq 2^{-\alpha n} \ \ \forall s \in \mathscr{D}, \ \forall n \geq n(\omega)\} =: \Omega^* \end{aligned}$$

from which $\Omega^* \in \mathscr{A}$ (by completeness) and clearly $\mathbf{P}(\Omega^*) = 1$ thus we have proved (1.5).

Step 3: Let us prove that for $m > n$ big enough, and for $s, t \in \mathscr{D}_m$ such that $|t - s| < 2^{-n}$, then a.s.

$$\left| B^H_t - B^H_s \right| \leq \sum_{j=n+1}^{m} 2^{-\alpha j}. \tag{1.6}$$

So, let us take $\omega \in \Omega^*$, fix $n(\omega)$ as in Step 2 and consider $m > n \geq n(\omega)$.
We will use induction on $m - n$. Let us suppose without loss of generality $0 \leq s < t$.

- Suppose $m - n = 1$. Then $s = \frac{k}{2^{n+1}}$ and $t = \frac{k'}{2^{n+1}}$, where $0 \leq k < k' \leq 2^{n+1} - 1$; hence

$$\begin{aligned} |t - s| < 2^{-n} &\Longleftrightarrow \frac{k' - k}{2^{n+1}} < 2^{-n} \\ &\Longleftrightarrow k' - k < 2 \\ &\Longleftrightarrow k' = k + 1. \end{aligned}$$

Thus $t = s + 2^{-(n+1)}$, from which we get $\left| B^H_{s+2^{-(n+1)}} - B^H_s \right| \leq 2^{-\alpha(n+1)}$ by (1.5).

- Suppose now (1.6) holds for every pair $m > n$ such that $m - n \leq h$ (fix one of these pairs) for some $h \geq 1$; let us show (1.6) holds for suitable $m' > n'$ such that $m' - n' = h + 1$.

Pick $s, t \in \mathscr{D}_{m+1}$ such that $|t - s| < 2^{-n}$ and define

$$\begin{aligned} s_0 &:= s \\ s_j &:= s_{j-1} + 2^{-(n+j)}, \quad j = 1, \ldots, m + 1 - n \ . \end{aligned}$$

Observe that

$$s_{m+1-n} = s + \sum_{r=n+1}^{m+1} 2^{-r} = s - 2^{-(m+1)} + 2^{-n} \in \mathscr{D}_{m+1} \;;$$

if by contradiction $t > s_{m+1-n}$, say $t = s_{m+1-n} + 2^{-(m+1)} \in \mathscr{D}_{m+1}$, then $|t-s| = 2^{-n} \not< 2^{-n}$, absurd. Thus we have

$$0 \le s < t \le s_{m+1-n}.$$

So we define

$$u := \max\{j = 0, 1, \dots, m+1-n \;:\; s_j \le t\} \ge 0$$

from which we have two cases: if $s_u = t$ we have

$$\begin{aligned}
\left|B_t^H - B_s^H\right| &\le \sum_{j=0}^{m-n} \left|B_{s_{j+1}}^H - B_{s_j}^H\right| \\
&\le \sum_{j=0}^{m-n} \left|B_{s_j+2^{-(n+j+1)}}^H - B_{s_j}^H\right| \\
&\overset{(1.5)}{\le} \sum_{j=0}^{m-n} 2^{-\alpha(n+j+1)} \\
&= \sum_{j=n+1}^{m+1} 2^{-\alpha j}
\end{aligned}$$

which is (1.6) with $(m+1) - n = h+1$, thus the inductive step is proved.

The other possibility is $s_u < t$. In this case clearly the following relations hold:

$$\begin{aligned}
&u \le m-n \\
&t < s_{u+1}.
\end{aligned}$$

But, noticing that $|s_{m-n+1} - s_{m-n}| = 2^{-(m+1)}$, since $t \in \mathscr{D}_{m+1}$, it cannot be $s_u < t < s_{u+1}$ for $u = m-n$, thus $u \le m-n-1$.

Now we have to proceed as follows

$$\begin{aligned}
\left|B_t^H - B_s^H\right| &\le \left|B_t^H - B_{s_u}^H\right| + \sum_{j=0}^{u-1} \left|B_{s_{j+1}}^H - B_{s_j}^H\right| \\
&\le \left|B_t^H - B_{s_u}^H\right| + \sum_{j=0}^{u-1} \left|B_{s_j+2^{-(n+j+1)}}^H - B_{s_j}^H\right| \\
&\overset{(1.5)}{\le} \left|B_t^H - B_{s_u}^H\right| + \sum_{j=0}^{u-1} 2^{-\alpha(n+j+1)} \\
&= \left|B_t^H - B_{s_u}^H\right| + \sum_{j=n+1}^{n+u} 2^{-\alpha j} \;;
\end{aligned}$$

then observing that $t, s_u \in \mathscr{D}_{m+1}$ and noticing that $s_u < t$ implies $t < s_{u+1}$, we get

$$|t - s_u| < |s_{u+1} - s_u| = 2^{-(n+u+1)}$$

and since

$$(m+1) - (n+u+1) = h - u \leq h \ ,$$

by inductive hypotesis we obtain

$$\left|B_t^H - B_{s_u}^H\right| \leq \sum_{j=n+u+2}^{m+1} 2^{-\alpha j} \leq \sum_{j=n+u+1}^{m+1} 2^{-\alpha j}$$

which leads to the conclusion.

Let us note that the sum $\sum_{j=n+u+2}^{m+1} 2^{-\alpha j}$ makes sense since $m+1 \geq n+u+2$ which is equivalent to $u \leq m-n-1$.

Step 4: Conclusion.
Let us fix $\omega \in \Omega^*$ and take $n(\omega)$ as in Step 2; then consider $n \geq n(\omega)$ and observe that $\mathscr{D} = \bigcup_{m>n} \mathscr{D}_m$. Hence, for $s, t \in \mathscr{D}$ such that $|t-s| < 2^{-n}$, when m tends to infinity in (1.6), we get

$$\left|B_t^H - B_s^H\right| \leq \sum_{j=n+1}^{+\infty} 2^{-\alpha j} = 2^{-\alpha n} \underbrace{\left(\frac{1}{2^\alpha + 1}\right)}_{=:c_\alpha} .$$

Now, we are dealing with $t, s \in \mathscr{D}$ such that $|t-s| < 2^{-n(\omega)}$; thus, there exists a unique $n \geq n(\omega)$ such that

$$2^{-(n+1)} \leq |t-s| < 2^{-n}$$

from which we find out that

$$\left|B_t^H - B_s^H\right| \leq c_\alpha 2^{-\alpha n} = 2^\alpha c_\alpha 2^{-\alpha(n+1)} \leq 2^\alpha c_\alpha |t-s|^\alpha. \tag{1.7}$$

Then, let us take $\widetilde{\Omega} \in \mathscr{A}$ with $\mathbf{P}(\widetilde{\Omega}) = 1$ such that $t \mapsto B_t^H(\omega)$ is continuous for every $\omega \in \widetilde{\Omega}$.
Next, clearly $\widetilde{\Omega} \cap \Omega^* \in \mathscr{A}$ and $\mathbf{P}(\widetilde{\Omega} \cap \Omega^*) = 1$. So let us fix $\omega \in \widetilde{\Omega} \cap \Omega^*$.
We get

$$\sup_{\substack{t,s\in[0,1] \\ s\neq t}} \frac{\left|B_t^H - B_s^H\right|}{|t-s|^\alpha} < +\infty \iff \sup_{\substack{t,s\in\mathscr{D} \\ s\neq t}} \frac{\left|B_t^H - B_s^H\right|}{|t-s|^\alpha} < +\infty$$

$$\iff \sup_{\substack{t,s\in\mathscr{D},\ t\neq s \\ |t-s|<2^{-n(\omega)}}} \frac{\left|B_t^H - B_s^H\right|}{|t-s|^\alpha} < +\infty.$$

Now, the first equivalence is true because of the density of $\mathscr{D}$ in $[0,1]$ and the continuity of the chosen trajectory $B^H(\omega)$. The second one is obvious and the latter side is true by (1.7). Thus we have proved that a.s. the fractional brownian motion has continuous α-Hölder trajectories, for every $0 < \alpha < H$, as wanted. $\square$

Now, in order to prove that the H-fractional Brownian Motion a.s. has *not* H-Hölder continuous trajectories, let us state a suitable *law of the iterated logarithm*, as reported in [8]:

Theorem 1.1.2 (Law of the Iterated Logarithm for continuous Gaussian Processes)**.** Let $\{Y_t\}_{t\geq 0}$ be a gaussian process a.s. with continuous trajectories, $\mathbf{E}[Y_t] \equiv 0$. Then set

$$w(s,t) := \mathbf{E}[Y_s Y_t] \qquad Q(t) := \frac{1}{2} w(t,t) \geq 0 \; .$$

Then define the process

$$X_t := \frac{Y_t}{\sqrt{2Q(t)}}.$$

Suppose there exists a monotone, non-decreasing function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $v(0) = 0$, and there exists $s_0, \beta_1, \beta_2, \beta_3 > 0$ with $\beta_3 < \frac{1}{2}\beta_1 + 1$ such that the following relations are satisfied:

- $\displaystyle\lim_{t\to+\infty} \frac{Q(s+t) - Q(s)}{v(s+t) - v(s)} = 1 \quad \text{uniformly in } \; s \geq 0;$
- $v(t) \geq \left(\dfrac{t}{s}\right)^{\beta_1} \cdot v(s) > 0, \quad \forall\, t \geq s > s_0 \;;$
- $v(t) \leq \left(\dfrac{t}{s}\right)^{\beta_3} \cdot v(s), \quad \forall\, t \geq s > s_0 \;;$
- $Q(t) = O_{0+}(t^{\beta_2})$.

Then, a.s. the following laws of the iterated logarithm hold:

$$\lim_{T\to+\infty} \left(\sup_{0\leq t\leq T} X_t - \sqrt{2\log\log T} \right) = 0$$

from which

$$\limsup_{t\to+\infty} \left(X_t - \sqrt{2\log\log t} \right) = 0$$

and finally

$$\limsup_{t\to+\infty} \frac{X_t}{\sqrt{2\log\log t}} = 1 \; . \tag{1.8}$$

Now, if $Y = B^H$, it turns to be

$$\begin{aligned} w(s,t) &= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \\ Q(t) &= \frac{1}{2}t^{2H} \\ X_t &= \frac{B_t^H}{t^H} \end{aligned}$$

and taking $v := Q$, $\beta_1 := 2H - \varepsilon$, $\beta_2 := 2H$ and $\beta_3 := 2H + \frac{\varepsilon}{2}$, where $\varepsilon := \frac{1-H}{2} > 0$, the hypotesis of the theorem are easily fulfilled, thus, by (1.8) we get that a.s.

$$\limsup_{t\to+\infty} \frac{B_t^H}{t^H\sqrt{2\log\log t}} = 1 \ . \tag{1.9}$$

Then fixing $t_0 \geq 0$ and considering the process $\left\{t^{2H}\left(B_{t_0+\frac{1}{t}} - B_{t_0}\right)\right\}_{t>0}$, which is a realisation of the fractional Brownian Motion, since it is obtained by composing a stationary increment of an H-fBM together with a time inversion, from (1.9) we get that a.s.

$$\limsup_{t\to+\infty} \frac{t^H\left(B^H_{t_0+\frac{1}{t}} - B^H_{t_0}\right)}{\sqrt{2\log\log t}} = 1$$

and using the substitution $t \to \frac{1}{t}$ this last lim sup reads as

$$\limsup_{t\to0+} \frac{B^H_{t_0+t} - B^H_{t_0}}{t^H\sqrt{2\log\log(1/t)}} = 1 \ ,$$

from which we immediately get, a.s. for any $t_0 \geq 0$

$$\limsup_{t\to0+} \frac{B^H_{t_0+t} - B^H_{t_0}}{t^H} = +\infty \ ,$$

which implies that, for a set of probability one, the trajectories of the fractional Brownian Motion of Hurst index H are not H-Hölder continuous around any point $t_0 \geq 0$, as claimed.

Finally we want to spend a couple of words about the modulus of continuity of a function.
Given a function $f : (X, d_X) \to (Y, d_Y)$ defined between metric spaces, a *global modulus of continuity* for f, is a function $\omega_f : [0 + \infty] \to [0, +\infty]$ which vanishes at 0 and is continuous at 0 such that

$$d_Y(f(x), f(x')) \leq \omega_f(d_X(x, x')) \qquad \forall x, x' \in X.$$

Just for completeness, we recall the local notion: f admits a *modulus of continuity at the point* $x \in X$ if there exists $\omega_x : [0+\infty] \to [0,+\infty]$ (again zero at 0 and continuous there) such that

$$d_Y(f(x), f(x')) \leq \omega_x(d_X(x,x')) \qquad \forall x' \in X.$$

At this point, we report from [9] that a.s. the fractional Brownian Motion of Hurst index $0 < H < 1$ admits a global modulus of continuity given by

$$\omega_{B^H}(\delta) = \delta^H |\log \delta|^{\frac{1}{2}}.$$

1.1.2 On the p-variation of the fractional Brownian Motion

Let us consider a function $f : [a,b] \to \mathbb{R}$ and a partition of the interval $[a,b]$, say $\Pi := \{a = t_0 < t_1 < \cdots < t_{k_n} < t_{k_n+1} = b\}$; we define the *mesh* of the partition Π as $|\Pi| := \max_{0 \leq j \leq k_n} |t_{j+1} - t_j|$. Let us fix $p > 0$.

Then we define the *p-variation of f on the partition* Π as

$$\mathrm{V}^{(p)}_{a,b}(f,\Pi) := \sum_{j=0}^{k_n} |f(t_{j+1}) - f(t_j)|^p$$

and the *p-variation of f* is thus given by

$$\mathrm{V}^{(p)}_{a,b}(f) := \sup_{\Pi} \mathrm{V}^{(p)}_{a,b}(f,\Pi).$$

Let us take $T > 0$ and consider $t_j := \frac{T}{n} j, \;\; j = 0, \dots, n$, thus, we have $\Pi_n := \{t_j\}_{j=0}^n$, which is a sequence of partitions of $[0,T]$ whose mesh goes to 0 as $n \to +\infty$. Then as stated in [10], we get

$$\exists \lim_{n \to +\infty} \mathrm{V}^{(p)}_{0,T}(B^H, \Pi_n) = \begin{cases} +\infty & p < 1/H \\ T \cdot \mathbf{E}\left[\left|B_1^H\right|\right] & p = 1/H \\ 0 & p > 1/H \end{cases} \tag{1.10}$$

where the limit is intended to be in $L^1(\Omega, \mathscr{A}, \mathbf{P})$ and thus, up to passing a subsequence, it holds a.s..

Since it is clear that surely

$$\mathrm{V}^{(p)}_{0,T}(B^H) \geq \lim_{n \to +\infty} \mathrm{V}^{(p)}_{0,T}(B^H, \Pi_n) \ ,$$

then, it is clear too that, for every $0 < H < 1$, a.s.

$$\mathrm{V}^{(1)}_{0,T}(B^H) = +\infty$$

that is, a.s. the trajectories of the fractional Brownian Motion, have *not* bounded variation.

Let us then note that the position $\|f\|_{p\text{-var},[a,b]} := \left(\mathrm{V}_{a,b}^{(p)}(f)\right)^{1/p}$ defines a seminorm, called the *p-variation seminorm* which is linked to the α-Hölder seminorm defined by (2.1): if f is α-Hölder continuous, then it has finite $1/\alpha$-variation; to be precise the following inequality holds:

$$\|f\|_{\frac{1}{\alpha}\text{-var},[a,b]} \leq \|f\|_{\alpha,[a,b]}(b-a)^{\alpha}.$$

Let us finally highlight the above p-variation and α-Hölder seminorms become norms by setting

$$\Theta(f)_{p\text{-var},[a,b]} := \|f\|_{\infty,[a,b]} + \|f\|_{p\text{-var},[a,b]}$$
$$\Theta(f)_{\alpha,[a,b]} := \|f\|_{\infty,[a,b]} + \|f\|_{\alpha,[a,b]}$$

called the *p-variation norm* and the *α-Hölder norm* respectively.

Chapter 2

Young Integration Theory

2.1 Foreword on Young Integral and overview on the SDE

Consider $0 < \lambda \leq 1$ and $s, t \in \mathbb{R},\ s < t$; we will denote with $\mathscr{C}^\lambda([s,t];\mathbb{R}^d)$ the space of measurable functions $f : [s,t] \to \mathbb{R}^d$ such that

$$\|f\|_{\lambda,[s,t]} := \sup_{s \leq u < v \leq t} \frac{|f(v) - f(u)|}{|v-u|^\lambda} < +\infty \tag{2.1}$$

which is the usual λ-Hölder space on $[s,t]$ of $\mathbb{R}^d$-valued functions (with $|\cdot|$ a norm on $\mathbb{R}^d$).

Notation: throughout this thesis, when there will be not ambiguities on the domain we are working on, we could omit it, e.g. writing $\|f\|_\lambda$ and $\mathscr{C}^\lambda(\mathbb{R}^d)$ instead of $\|f\|_{\lambda,[s,t]}$ and $\mathscr{C}^\lambda([s,t];\mathbb{R}^d)$ respectively.

The object of our interest is the following Stochastic Differential Equation

$$x_t = \xi_0 + \int_0^t b(s, x_s)\, ds + \int_0^t \sigma(s, x_s)\, dW_s^H + y_t \tag{2.2}$$

which we will now describe precisely.

(2.2) is an equation in $\mathbb{R}^d$, where $\xi_0 \in \mathbb{R}^d_+ := \{(x_1, \dots, x_d) : x_i > 0\ \forall i = 1, \cdots, d\}$ is fixed and $t \in [0,T]$ for some fixed $T > 0$.

Next,

$$b : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$$

and

$$\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathscr{M}_{d,m}(\mathbb{R})$$

are bounded measurable functions, called *drift* and *diffusion coefficient*, respectively. We will assume the following hypotesis on them:

$$|b(t,x) - b(t,y)| \leq K_0|x-y| \ \ \forall x,y \in \mathbb{R}^d,\ \forall t \in [0,T] \tag{2.3}$$

$$|\sigma(t,x) - \sigma(t,y)| \leq K_0|x-y| \ \ \forall x,y \in \mathbb{R}^d,\ \forall t \in [0,T] \tag{2.4}$$

$$|\sigma(t,x) - \sigma(s,x)| \leq K_0|t-s|^\nu \ \ \forall x \in \mathbb{R}^d,\ \forall s,t \in [0,T] \tag{2.5}$$

where $\nu \in]\frac{1}{2}, 1]$, $K_0 > 0$ and $|\cdot|$ denotes every time a suitable norm, conveniently chosen, which will be specified in the next chapter. Hence, both b and σ are Lipschitz in space and furthermore σ is ν-Hölder continous in time.

The term $y = (y_t)_{t\geq 0}$ is a vector valued, non-decreasing (the sup, the negative part, and in general all tools which takes into account the total order of $\mathbb{R}$, when used for $\mathbb{R}^d$ elements, are clearly intended componentwise) process, whose role is to mantain x non-negative; thus it allows the non-negativity constraint to be satisfied. We want to describe explicitly this process: a suitable tool to tackle this problem is the Skorokhod Problem, which we will now recall.

First of all let us set

$$\mathscr{C}_+(\mathbb{R}_+;\mathbb{R}^d) := \{x \in \mathscr{C}(\mathbb{R}_+;\mathbb{R}^d) : x_0 \in \mathbb{R}^d_+\} \ .$$

Then the *Skorokhod Problem with normal reflection on* $\mathbb{R}^d$ (SP for short) is the following:

Definition 2.1.1. Given a path $z \in \mathscr{C}_+(\mathbb{R}_+;\mathbb{R}^d)$, a solution to the SP related to z is a pair of functions (x,y), both in $\mathscr{C}_+(\mathbb{R}_+;\mathbb{R}^d)$, such that

1. $x_t = z_t + y_t$ and $x_t \in \mathbb{R}^d_+$ for all $t \geq 0$;
2. $y^i_0 = 0$ and $t \mapsto y^i_t$ is non-decreasing for each $i = 1, \dots, d$;
3. $\int_0^t x^i_s\, dy^i_s = 0$ for each $i = 1, \dots, d$ and for all $t \geq 0$.

Let us observe that since $t \mapsto y^i_t$ is non-decreasing, it belongs to $\mathrm{BV}[0,T]$ for any fixed $T > 0$, thus the integral above is a Riemann-Stiltjes one, hence expressing it as Riemann sums

$$\int_0^t x^i_s\, dy^i_s = \lim_{|\Pi_{0t}|\to 0} \sum_{j=0}^{n-1} x^i_{s_j}(y^i_{s_{j+1}} - y^i_{s_j})$$

where the limit is taken over any partition $\Pi_{0,t} = \{0 = s_0 < s_1 < \cdots < s_n = t\}$ of $[0,t]$ whose mesh tends to 0, it can be seen that y^i can increases only when x^i is zero.

Then it is well known (see [4] or [5]) how to write a solution to SP_z explicitly for each component of y (and thus of x):

$$y_t^i = \sup_{s\in[0,t]} \left(z_s^i\right)^- . \tag{2.6}$$

The path x is called *reflector of* z, while the path y is called *regulator of* z.

We will apply this result to obtain the non-negativity constraint for each path in (2.2): namely, setting

$$z_t := \xi_0 + \int_0^t b(s, x_s)\, ds + \int_0^t \sigma(s, x_s)\, dW_s^H$$

and solving for SP_z, we can rewrite (2.2) as $x_t = z_t + y_t$ with x as the only unknown, which satisfies a non-negativity constraint forced as wanted.
In the case of equation (2.2), x and y are called *reflector term* and *regulator term*, respectively.

At this point, we need to explain how the stochastic integral

$$\int_0^t \sigma(s, x_s)\, dW_s^H$$

has to be intended.

The classical stochastic integral (the one defined by Itô) is defined with respect to the standard Brownian Motion; one of the central point in the construction of this integral, is the fact that the increments of a Brownian Motion are indipendent. In the case we consider ($1/2 < H < 1$), independence of the increments is lost, thus, we cannot use the classical stochastic integral.

The strategy we will apply, is to use a pathwise approach, that is, working with a fixed $\omega \in \Omega$: in this way we will both eliminate the indipendence problem and take advantage from the regularity of the trajectories of the fractional Brownian Motion. The right tool for our purposes is a generalization of the Riemann-Stiltjes integral, which allows to perform integration with respect to functions (paths, in our case) which has not necessarely bounded variation.

This is the Young integration, whose theory allows to define the integral

$$\int_s^t f_u\, dg_u$$

where f is λ-Hölder continous and g is γ-Hölder continous, where $\lambda + \gamma > 1$. Young integral will be introduced in the following sections, for which we will refer to [6] for the first subsection and to [7] for the second one.

2.2 Increments, the map δ and the Sewing Map Λ

The Young Integral is defined constructing an algebraic structure based on the notion of *increment* togheter with an elementary operator δ acting on them. Let us describe it.

First of all, for an arbitrary $T > 0$, a finite dimensional normed vector space $(V, |\cdot|)$ on $\mathbb{R}$ (where $|\cdot| : V \to \mathbb{R}_{\geq 0}$ is a suitable norm on V) and an integer $k \geq 1$, we denote by $\mathscr{C}_k(V)$ the set of all continous functions $f : [0,T]^k \to V$ such that $f_{t_1 \cdots t_k} = 0$ if $t_i = t_{i+1}$ for some $1 \leq i \leq k-1$.
Such functions are called $(k-1)$-*increments*.
Moreover we set $\mathscr{C}_0(V) := V$.

Then endowing $\mathscr{C}_k(V)$ with the pointwise sum (i.e. given $f, h \in \mathscr{C}_k(V)$ we define $(f+h)_{t_1 \cdots t_k} := f_{t_1 \cdots t_k} + h_{t_1 \cdots t_k}$ where this last sum is the one defined on the vector space V), and defining a scalar product as $(\alpha f)_{t_1 \cdots t_k} := \alpha f_{t_1 \cdots t_k}$ (where $\alpha \in \mathbb{R}$ and $f \in \mathscr{C}_k(V)$ and this last product is the product of an element of V with a scalar) we have given $\mathscr{C}_k(V)$ an $\mathbb{R}$-vector space structure; in particular $(\mathscr{C}_k(V), +)$ becomes an abelian group.

Next we define the operator δ_k (for any $k \geq 1$) as follows:

$$\delta_k : \mathscr{C}_k(V) \to \mathscr{C}_{k+1}(V) \ , \ (\delta_k f)_{t_1 \cdots t_{k+1}} := \sum_{j=1}^{k+1} (-1)^j f_{t_1 \cdots \widehat{t_j} \cdots t_{k+1}}$$

where the hat $\widehat{\cdot}$ means the corresponding argument is omitted; define then $\delta_0 : V \to \mathscr{C}_1(V)$ that maps $v \in V$ into the constant function $f_t \equiv v$.

These maps δ_k are clearly $\mathbb{R}$-linear maps and in particular are homomorphisms of abelian groups.

Hence we can write

$$0 \xrightarrow{0} \mathscr{C}_0(V) \xrightarrow{\delta_0} \mathscr{C}_1(V) \xrightarrow{\delta_1} \mathscr{C}_2(V) \xrightarrow{\delta_2} \cdots \tag{2.7}$$

where $\delta_{k+1} \circ \delta_k = 0$ (equivalently $\operatorname{Im} \delta_k \subseteq \ker \delta_{k+1}$): indeed, take $f \in \mathscr{C}_k(V)$ and consider

$$\delta_{k+1}(\delta_k f_{s_1 \cdots s_{k+1}})_{t_1 \cdots t_{k+2}} = \delta_{k+1} \left(\sum_{j=1}^{k+1} (-1)^j f_{s_1 \cdots \widehat{s_j} \cdots s_{k+1}} \right)_{t_1 \cdots t_{k+2}} =$$

$$\begin{aligned}
&= -\Bigg(- \underbrace{f_{t_3t_4\cdots t_{k+2}}}_{=:a_{1,2}} + \underbrace{f_{t_2t_4\cdots t_{k+2}}}_{=:a_{1,3}} - \underbrace{f_{t_2t_3t_5\cdots t_{k+2}}}_{=:a_{1,4}} + \cdots (-1)^{k+1} \underbrace{f_{t_2t_3t_4\cdots t_{k+1}}}_{=:a_{1,k+2}} \Bigg) \\
&\quad + \Bigg(- \underbrace{f_{t_3t_4\cdots t_{k+2}}}_{=:a_{2,1}} + \underbrace{f_{t_1t_4\cdots t_{k+2}}}_{=:a_{2,3}} - \underbrace{f_{t_1t_3t_5\cdots t_{k+2}}}_{=:a_{2,4}} + \cdots (-1)^{k+1} \underbrace{f_{t_1t_3t_4\cdots t_{k+1}}}_{=:a_{2,k+2}} \Bigg) \\
&\quad - \Bigg(- \underbrace{f_{t_2t_4\cdots t_{k+2}}}_{=:a_{3,1}} + \underbrace{f_{t_1t_4\cdots t_{k+2}}}_{=:a_{3,2}} - \underbrace{f_{t_1t_2t_5\cdots t_{k+2}}}_{=:a_{3,4}} + \cdots (-1)^{k+1} \underbrace{f_{t_1t_2t_5\cdots t_{k+1}}}_{=:a_{3,k+2}} \Bigg) \\
&\quad + \cdots + \\
&\quad (-1)^{k+2} \Bigg(- \underbrace{f_{t_2t_3\cdots t_{k+1}}}_{=:a_{k+2,1}} + \underbrace{f_{t_1t_3\cdots t_{k+1}}}_{=:a_{k+2,2}} - \underbrace{f_{t_1t_2t_4\cdots t_{k+1}}}_{=:a_{k+2,3}} + \cdots (-1)^{k+1} \underbrace{f_{t_1t_2t_3\cdots t_k}}_{=:a_{k+2,k+1}} \Bigg) \\
&= - \sum_{1\le i<j\le k+2} (-1)^{i+j} a_{i,j} + \sum_{1\le j<i\le k+2} (-1)^{i+j} a_{i,j} = 0
\end{aligned}$$

since clearly $a_{i,j} = a_{j,i}$.

Thus, the sequence (2.7) of abelian groups connected with such homomorphisms is a cochain complex, denoted with $(\mathscr{C}_*(V), \delta)$, where $\mathscr{C}_*(V) := \bigcup_{k\ge 0} \mathscr{C}_k(V)$, and the above mentioned operator $\delta : \mathscr{C}_*(V) \to \mathscr{C}_*(V)$ is called *coboundary operator* and is defined acting on the increments as δ_k do, i.e. $\delta_{|_{\mathscr{C}_k(V)}} := \delta_k$.
We can rewrite the above property as $\delta \circ \delta \equiv 0$.

Now, we have proved that $\operatorname{Im} \delta_k \subseteq \ker \delta_{k+1}$, but something stronger holds: the inclusion is indeed an equality, as we now prove.
Let $f \in \mathscr{C}_{k+1}(V)$ such that

$$\begin{aligned}
\delta_{k+1} f_{t_1\cdots t_{k+1}\tilde{t}} = &- f_{t_2t_3t_4\cdots t_{k+1}\tilde{t}} + f_{t_1t_3t_4\cdots t_{k+1}\tilde{t}} - f_{t_1t_2t_4\cdots t_{k+1}\tilde{t}} + \cdots \\
&+ (-1)^{k+1} f_{t_1\cdots t_k\tilde{t}} + (-1)^{k+2} f_{t_1\cdots t_k t_{k+1}} = 0
\end{aligned}$$

from which we have that

$$(-1)^{k+1} f_{t_1\cdots t_k t_{k+1}} = \underbrace{-f_{t_2t_3t_4\cdots t_{k+1}\tilde{t}} + f_{t_1t_3t_4\cdots t_{k+1}\tilde{t}} - f_{t_1t_2t_4\cdots t_{k+1}\tilde{t}} + \cdots + (-1)^{k+1} f_{t_1\cdots t_k\tilde{t}}}_{=:F_{t_1\cdots t_{k+1}\tilde{t}}}$$

and thus F does not depend on $\tilde{t}$; in particular we have that

$$F_{t_1\cdots t_{k+1}\tilde{t}} \equiv F_{t_1\cdots t_{k+1}0}$$

from which, setting $g_{s_1\cdots s_k} := (-1)^{k+1} f_{s_1\cdots s_k 0}$, we obtain that $g \in \mathscr{C}_k(V)$ and $\delta_k g = f$, as wanted.

Thus the sequence (2.7) is exact, which is another way to say the cochain complex $(\mathscr{C}_*(V), \delta)$ is *acyclic*, that is, having denoted $\mathscr{ZC}_k(V) := \mathscr{C}_k(V) \cap \ker \delta_k$ and $\mathscr{BC}_k(V) := \mathscr{C}_k(V) \cap \operatorname{Im} \delta_{k-1}$ the spaces of *k-cocycles* and *k-coboundaries* respectively, $\mathscr{ZC}_k(V) = \mathscr{BC}_k(V)$ for any $k \geq 1$.

Even if we have introduced the subject in a general setting, the cases we will investigate the most are $k = 1, 2$, for which we will write explicitly how δ_k works: taking $f \in \mathscr{C}_1(V)$, $h \in \mathscr{C}_2(V)$ and $s, t, u \in [0, T]$, we have that

$$(\delta_1 f)_{st} = f_t - f_s \ , \quad (\delta_2 h)_{stu} = -h_{tu} + h_{su} - h_{st} \ .$$

We resume now the fact above proved in the following

Lemma 2.2.1. Let $k \geq 1$ and $h \in \mathscr{ZC}_{k+1}(V)$. Then there exists a (nonunique) $f \in \mathscr{C}_k(V)$ such that $\delta_k f = h$.

In particular, taking a 1-increment $h \in \mathscr{C}_2(V)$ such that $\delta_2 h = 0$, Lemma (2.2.1) ensures the existence of a (nonunique) $f \in \mathscr{C}_1(V)$ such that $h_{ts} = f_t - f_s$; this provides an heuristic interpretation on how δ works on $\mathscr{C}_2(V)$: it measures how much a 1-increment is far from being an exact increment of a function, i.e. a finite difference.

We have already pointed out that we will deal mainly with k-increments, with $k \leq 2$. We now need a tool which will allow us to handle these objects properly: namely, we measure the size of the increments with suitable Hölder-type norms, which we are going to describe.

Let us fix $\lambda > 0$; if $h \in \mathscr{C}_2(V)$, we set

$$\|h\|_\lambda := \sup_{\substack{s,t \in [0,T] \\ s \neq t}} \frac{|h_{ts}|}{|t - s|^\lambda}$$

and

$$\mathscr{C}_2^\lambda(V) := \{h \in \mathscr{C}_2(V) \ : \ \|h\|_\lambda < +\infty\} \ .$$

Note that we have not specified the domain we are working on is $[0, T]$, since from the context it is clear and this will not lead to any ambiguity.

Let us remark that the usual λ-Hölder space on $[0, T]$ of V-valued functions as defined at the beginning of the section, can be recovered in the k-increments framework: if $f \in \mathscr{C}_1(V)$ we have

$$\|f\|_\lambda = \|\delta_1 f\|_\lambda$$

and thus, setting

$$\mathscr{C}_1^\lambda(V) := \{f \in \mathscr{C}_1(V) \ : \ \|\delta_1 f\|_\lambda < +\infty\}$$

we trivially have $\mathscr{C}^\lambda(V) = \mathscr{C}_1^\lambda(V)$.

Finally, given $g \in \mathscr{C}_3(V)$ and $\gamma, \rho > 0$, we set

$$\|g\|_{\rho,\gamma} := \sup_{\substack{s,t,u\in[0,T] \\ t\neq u\neq s}} \frac{|g_{sut}|}{|u-s|^\rho |t-u|^\gamma}$$

from which we define

$$\|g\|_\lambda := \inf \left\{ \sum_j \|g^j\|_{\rho_j,\lambda-\rho_j} \ : \ g = \sum_j g^j, \ 0 < \rho_j < \lambda \right\} ,$$

where this last infimum is taken over all sequences $\{g^j\}_j \subset \mathscr{C}_3(V)$ such that $g = \sum_j g^j$ (when the sequence is not finite, we intend the sum converging pointwise, with respect to the norm considered on V - notice that being V finite dimensional $\mathbb{R}$-linear space, all the norms on it are equivalent) and all choices of the numbers $0 < \rho_j < \lambda$.
Let us note that, given $g \in \mathscr{C}_3^\lambda$, we have

$$\|g\|_\lambda \leq \|g\|_{\rho,\lambda-\rho} \quad \forall\, 0 < \rho < \lambda. \tag{2.8}$$

Then, $\|\cdot\|_\lambda$ is a norm on $\mathscr{C}_3(V)$ and we can consider

$$\mathscr{C}_3^\lambda(V) := \{g \in \mathscr{C}_3(V) \ : \ \|g\|_\lambda < +\infty\} ,$$

which is clearly an $\mathbb{R}$-vector space.
Note that the same kind of norms can be considered on the space $\mathscr{Z}\mathscr{C}_3(V)$, leading to the definition of the spaces $\mathscr{Z}\mathscr{C}_3^\lambda(V)$ and thus $\mathscr{Z}\mathscr{C}_3^{1+}(V)$, where we have set $\mathscr{C}_3^{1+}(V) := \bigcup_{\lambda>1} \mathscr{C}_3^\lambda(V)$.

The crucial point in this approach to pathwise integration of irregular processes is that, under suitable conditions, the operator δ admits a right inverse. We call this operator the *sewing map*, and we denote it by Λ; we prove its existence in the following

Theorem 2.2.1 (The Sewing Map)**.** There exists a unique map

$$\Lambda : \mathscr{Z}\mathscr{C}_3^{1+}(V) \to \mathscr{C}_2^{1+}(V)$$

such that

$$\delta \circ \Lambda = \mathrm{Id}_{\mathscr{Z}\mathscr{C}_3^{1+}(V)} .$$

Moreover, we have that

$$\|\Lambda h\|_\lambda \leq \frac{1}{2^\mu - 2}\|h\|_\lambda \quad \forall h \in \mathscr{Z}\mathscr{C}_3^\lambda(V), \tag{2.9}$$

from which $\Lambda : \mathscr{Z}\mathscr{C}_3^\lambda(V) \to \mathscr{C}_2^\lambda(V)$ is lipschitz and thus continous for every fixed $\lambda > 1$, since Λ is linear when seen as an operator $\mathscr{Z}\mathscr{C}_3^\lambda(V) \to \mathscr{C}_2^\lambda(V)$.

Proof. Let us fix $h \in \mathscr{Z}\mathscr{C}_3^{1+}$; hence $h \in \mathscr{Z}\mathscr{C}_3^\lambda$ for some $\lambda > 1$.

Step 1: Let us start by proving the uniqueness of the sewing map.
In the next Steps, we will prove the existence of a 1-increment $M \in \mathscr{C}_2^\lambda(V)$ such that $\delta M = h$, that is $\delta : \mathscr{C}_2^\lambda(V) \to \mathscr{Z}\mathscr{C}_3^\lambda(V)$ is surjective, which is equivalent to say that it is right invertible; this right inverse is denoted by Λ and it is clearly defined as $\Lambda h := M$.
Now, the existence of another 1-increment $M' \in \mathscr{C}_2^\lambda(V)$ such that $\delta M' = h$ is equivalent to the existence of another right inverse Λ', which is defined as $\Lambda' M' := h$, as above.
Thus, proving that $M' = M$ (for every fixed h) will imply that $\Lambda' = \Lambda$, that is the sewing map is unique, as claimed.
So, setting $Q := M - M'$, it is clear that $\delta Q = 0$ by linearity.
Hence $Q \in \mathscr{Z}\mathscr{C}_2^\lambda(V)$ and thus, by Lemma (2.2.1), there exists $q \in \mathscr{C}_1^\lambda(V)$ such that $Q_{ts} = (\delta q)_{ts} = q_t - q_s$ for every $t, s \in [0, T]$, from which we have that

$$\sup_{0 \leq s < t \leq T} \frac{|q_t - q_s|}{|t - s|^\lambda} = \sup_{0 \leq s < t \leq T} \frac{|Q_{ts}|}{|t - s|^\lambda} = \|Q\|_\lambda .$$

So for every $0 \leq s < t \leq T$ we have that

$$0 \leq \frac{|q_t - q_s|}{|t - s|} \leq \|Q\|_\lambda |t - s|^{\lambda - 1}$$

hence, taking s fixed and passing to the limit for $t \to s$ in the above expression, since $\lambda - 1 > 0$, we get $|q'_s| = 0$ (notice that both the value and the existence of this limit are guaranteed by two policemen theorem). Since this holds for every s, we have that $q' \equiv 0$ on $[0, T]$, from which we deduce q is constant on $[0, T]$, which implies $Q = 0$, i.e. $M' = M$, as wanted.

Step 2: Let us now construct a 1-increment $M \in \mathscr{C}_2^\lambda(V)$ satisfying $\delta M = h$. Since $\delta h = 0$, invoking again Lemma (2.2.1) we know that there exists a $B \in \mathscr{C}_2(V)$ such that $\delta B = h$. We will build our M starting from B.
Pick $s, t \in [0, T]$ with $s < t$ and, for $n \geq 0$, consider the dyadic partition $\{r_i^n : 0 \leq i \leq 2^n\}$ of the interval $[s, t]$, that is

$$r_i^n := s + \frac{t-s}{2^n} i \ , \quad 0 \leq i \leq 2^n.$$

Then, we can define

$$M_{ts}^n := B_{ts} - \sum_{i=0}^{2^n-1} B_{r_{i+1}^n r_i^n} \tag{2.10}$$

which is a continous function and vanishes whenever $s = t$, thus $M^n \in \mathscr{C}_2(V)$.

Now we have

$$\begin{aligned} M_{ts}^{n+1} - M_{ts}^n &= - \sum_{i=0}^{2^{n+1}-1} B_{r_{i+1}^{n+1} r_i^{n+1}} + \sum_{i=0}^{2^n-1} B_{r_{i+1}^n r_i^n} \\ &= - \sum_{i=0}^{2^n-1} B_{r_{2i+1}^{n+1} r_{2i}^{n+1}} - \sum_{i=0}^{2^n-1} B_{r_{2i+2}^{n+1} r_{2i+1}^{n+1}} + \sum_{i=0}^{2^n-1} B_{r_{i+1}^n r_i^n} \\ &= \sum_{i=0}^{2^n-1} B_{r_{2i+2}^{n+1} r_{2i}^{n+1}} - B_{r_{2i+1}^{n+1} r_{2i}^{n+1}} - B_{r_{2i+2}^{n+1} r_{2i+1}^{n+1}} \\ &= \sum_{i=0}^{2^n-1} (\delta B)_{r_{2i+2}^{n+1} r_{2i+1}^{n+1} r_{2i}^{n+1}} \\ &= \sum_{i=0}^{2^n-1} h_{r_{2i+2}^{n+1} r_{2i+1}^{n+1} r_{2i}^{n+1}} \end{aligned}$$

where the second equality is obtained by splitting odd and even terms, while the third one is got by observing that $r_i^n = r_{2i}^{n+1}$ and $r_{i+1}^n = r_{2i+2}^{n+1}$.
Considering then some $\{h^j\}_j \subset \mathscr{C}_3(V)$ such that $h = \sum_j h^j$, some $0 < \rho_j < \lambda$ and

setting $a_s := r^{n+1}_{2i+s}$ for $s = 0, 1, 2$, one has that

$$
\begin{aligned}
|M^{n+1}_{ts} - M^n_{ts}| &\leq \sum_{i=0}^{2^n-1} |h_{a_2a_1a_0}| \\
&\leq \sum_{i=0}^{2^n-1} \sum_j |h^j_{a_2a_1a_0}| \\
&= \sum_{i=0}^{2^n-1} \sum_j \frac{|h^j_{a_2a_1a_0}|}{|a_2-a_1|^{\lambda-\rho_j}|a_1-a_0|^{\rho_j}} \underbrace{|a_2-a_1|}_{=\frac{|t-s|}{2^{n+1}}}{}^{\lambda-\rho_j} \underbrace{|a_1-a_0|}_{=\frac{|t-s|}{2^{n+1}}}{}^{\rho_j} \\
&\leq \frac{|t-s|^\lambda}{2^{\lambda(n+1)}} \sum_{i=0}^{2^n-1} \sum_j \|h^j\|_{\rho_j,\lambda-\rho_j} \\
&= \frac{|t-s|^\lambda}{2^\lambda 2^{n(\lambda-1)}} \sum_j \|h^j\|_{\rho_j,\lambda-\rho_j}
\end{aligned}
$$

thus, passing to the inf on the sequences $\{h^j\}_j \subset \mathscr{C}_3(V)$ such that $h = \sum_j h^j$ and on the $\rho_j \in]0, \lambda[$ we get

$$
|M^{n+1}_{ts} - M^n_{ts}| \leq \frac{|t-s|^\lambda \|h\|_\lambda}{2^\lambda 2^{n(\lambda-1)}} \ . \tag{2.11}
$$

Since $M^0_{ts} = 0$, we have that $M^n_{ts} = \sum_{j=0}^{n-1} M^{j+1}_{ts} - M^j_{ts}$, from which

$$
|M^n_{ts}| \leq \sum_{j=0}^{n-1} |M^{j+1}_{ts} - M^j_{ts}| \leq \frac{|t-s|^\lambda \|h\|_\lambda}{2^\lambda} \sum_{j=0}^{n-1} \frac{1}{2^{j(\lambda-1)}} \ ,
$$

and thus $M_{ts} := \lim_{n\to+\infty} M^n_{ts}$ exists and satisfies

$$
\frac{|M_{ts}|}{|t-s|^\lambda} \leq \frac{\|h\|_\lambda}{2^\lambda} \sum_{j=0}^{+\infty} \frac{1}{2^{j(\lambda-1)}} = \frac{1}{2^\lambda - 2} \|h\|_\lambda \ . \tag{2.12}
$$

From this we deduce that M_{ts} vanishes whenever $t = s$; moreover in step 4 we will the continuity of M and thus it belongs to $\mathscr{C}_2(V)$.
Finally, since (2.12) holds for all $s, t \in [0, T]$ with $s \neq t$, passing to the $\sup_{s,t\in[0,T],s\neq t}$ we get

$$
\|M\|_\lambda \leq \frac{1}{2^\lambda - 2} \|h\|_\lambda
$$

from which $M \in \mathscr{C}_2^\lambda(V)$. In step 6 it will be shown that $\delta M = h$, from which the right inverse of δ is defined as $\Lambda h := M$ and thus (2.9) is satisfied.

Step 3: We want to do the same as in step 2, but now with a general sequence of partitions of $[s,t]$.
Consider thus $\{\Pi_n\}_{n\geq 1}$, say $\Pi_n := \{s = r_0^n < r_1^n < \cdots < r_{k_n}^n < r_{k_n+1}^n = t\}$, such that its mesh $|\Pi_n| := \max_{0\leq i\leq k_n} |r_{i+1}^n - r_i^n|$ tends to 0 as $n \to +\infty$ and, without loss of generality, $\Pi_n \subsetneq \Pi_{n+1}$ $\forall n \geq 1$, from which $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}$ is a strictly increasing sequence.
Set then

$$M_{ts}^{\Pi_n} := B_{ts} - \sum_{l=0}^{k_n} B_{r_{l+1}^n r_l^n} \, .$$

Now, if $\frac{2|t-s|}{k_n} \lneq |r_{l+1}^n - r_{l-1}^n|$ for all $1 \leq l \leq k_n$, supposing k_n odd, say $k_n = 2m-1$, we have that

$$|t-s| = \sum_{j=1}^{m} |r_{2j}^n - r_{2(j-1)}^n| \gneq m\frac{2|t-s|}{k_n} \gneq \frac{k_n}{2}\frac{2|t-s|}{k_n} = |t-s|$$

which is a contradiction. If k_n was even, once set $k_n = 2m$, the argument is the same.
Thus there exists some $1 \leq l_1 \leq k_n$ such that

$$|r_{l_1+1}^n - r_{l_1-1}^n| \leq \frac{2|t-s|}{k_n} \tag{2.13}$$

Set then

$$\Pi_n^{(1)} := \Pi_n \setminus \{r_{l_1}^n\}$$

from which it is clear that

$$\begin{aligned} M_{ts}^{\Pi_n^{(1)}} &= \underbrace{B_{ts} - \sum_{l=0}^{k_n} B_{r_{l+1}^n r_l^n}}_{=M_{ts}^{\Pi_n}} + \underbrace{\left(B_{r_{l_1+1}^n r_{l_1}^n} + B_{r_{l_1}^n r_{l_1-1}^n} - B_{r_{l_1+1}^n r_{l_1-1}^n}\right)}_{=-(\delta B)_{r_{l_1+1}^n r_{l_1}^n r_{l_1-1}^n}} \\ &= M_{ts}^{\Pi_n} - h_{r_{l_1+1}^n r_{l_1}^n r_{l_1-1}^n} \end{aligned}$$

and thus, as before, we consider some $\{h^j\}_j \subset \mathscr{C}_3(V)$ such that $h = \sum_j h^j$, some $0 < \rho_j < \lambda$ and setting $a_s := r_{l_1+s}^{n+1}$ for $s = -1, 0, 1$ we have that

$$
\begin{aligned}
\left|M_{ts}^{\Pi_n} - M_{ts}^{\Pi_n^{(1)}}\right| &= \left|h_{a_1 a_0 a_{-1}}\right| \\
&\le \sum_j \left|h^j_{a_1 a_0 a_{-1}}\right| \\
&= \sum_j \frac{\left|h^j_{a_1 a_0 a_{-1}}\right|}{|a_1-a_0|^{\lambda-\rho_j}\,|a_0-a_{-1}|^{\rho_j}} \underbrace{|a_1-a_0|}_{\le |a_1-a_{-1}|}{}^{\lambda-\rho_j} \underbrace{|a_0-a_{-1}|}_{\le |a_1-a_{-1}|}{}^{\rho_j} \\
&\overset{(2.13)}{\le} \left(\frac{2|t-s|}{k_n}\right)^{\lambda} \sum_j \|h^j\|_{\rho_j,\lambda-\rho_j} \ ,
\end{aligned}
$$

from which, passing to the inf on $\{h^j\}_j$ and ρ_j as above, we get

$$\left|M_{ts}^{\Pi_n} - M_{ts}^{\Pi_n^{(1)}}\right| \le 2^\lambda \|h\|_\lambda \left(\frac{|t-s|}{k_n}\right)^{\lambda} . \tag{2.14}$$

We can now repeat the argument on $\Pi_n^{(1)}$, finding thus an index l_2 for which (2.13) holds and allows hence to define $\Pi_n^{(2)} := \Pi_n^{(1)} \backslash \{r^n_{l_2}\} = \Pi_n \backslash \{r^n_{l_1}, r^n_{l_2}\}$ and, as did above, after having observed that $M_{ts}^{\Pi_n^{(2)}} = M_{ts}^{\Pi_n^{(1)}} - h_{r^n_{l_2+1} r^n_{l_2} r^n_{l_2-1}}$, to obtain the following esimate

$$\left|M_{ts}^{\Pi_n^{(1)}} - M_{ts}^{\Pi_n^{(2)}}\right| \le 2^\lambda \|h\|_\lambda \left(\frac{|t-s|}{k_n-1}\right)^{\lambda} .$$

We proceed inductively by defining

$$\Pi_n^{(j)} := \Pi_n \setminus \{r^n_{l_1}, \dots, r^n_{l_j}\}, \quad j = 1, \dots, k_n$$

and getting

$$\left|M_{ts}^{\Pi_n^{(j)}} - M_{ts}^{\Pi_n^{(j-1)}}\right| \le 2^\lambda \|h\|_\lambda \left(\frac{|t-s|}{k_n-j+1}\right)^{\lambda} . \tag{2.15}$$

for $j = 2, \dots, k_n$.

Thus, observing that $M_{ts}^{\Pi_n^{(k_n)}} = 0$ (since $\Pi_n^{(k_n)} = \{s,t\}$ is the trivial partition), one gets immediately $M_{ts}^{\Pi_n} = M_{ts}^{\Pi_n} - M_{ts}^{\Pi_n^{(1)}} + \sum_{j=2}^{k_n} \left(M_{ts}^{\Pi_n^{(j-1)}} - M_{ts}^{\Pi_n^{(j)}}\right)$, from which, exploiting (2.14) and (2.15) we have that

$$
\begin{aligned}
\left|M_{ts}^{\Pi_n}\right| &\le \left|M_{ts}^{\Pi_n} - M_{ts}^{\Pi_n^{(1)}}\right| + \sum_{j=2}^{k_n} \left|M_{ts}^{\Pi_n^{(j-1)}} - M_{ts}^{\Pi_n^{(j)}}\right| \\
&\le 2^\lambda \|h\|_\lambda |t-s|^\lambda \sum_{j=1}^{k_n} \frac{1}{j^\lambda} \\
&\le \underbrace{2^\lambda \|h\|_\lambda \zeta(\lambda)}_{=:c_{h,\lambda}} |t-s|^\lambda
\end{aligned}
$$

from which the sequence $\left\{M_{ts}^{\Pi_n}\right\}_{n\geq 1} \subset V$ is bounded and thus, by Bolzano-Weierstrass (in fact being V a finite dimensional $\mathbb{R}$-vector space, it is isomorphic to $\mathbb{R}^N$, for some N) admits a converging subsequence $\left\{M_{ts}^{\Pi_{n_j}}\right\}_{j\geq 1} \subset V$ to an element we denote by M_{ts} which satisfy $|M_{ts}| \leq c_{h,\lambda}|t-s|^{\lambda}$.

Step 4: Let us see that M is continous in a fixed point (t_0, s_0).
Let us fix $\varepsilon > 0$ and search for some $\delta > 0$ such that

$$|M_{ts} - M_{t_0 s_0}| < \varepsilon \quad \forall (t,s) \in \mathrm{B}_{(t_0,s_0)}(\delta)$$

where $\mathrm{B}_{(x_0,y_0)}(r)$ is the open ball centered in (x_0, y_0) and radius $r > 0$.
Denote the n-th dyadic partition of $[s,t]$ already used in Step 2 with $\Pi_n^{[s,t]}$, specifying now the interval $[s,t]$.
So, calling $S^{\Pi_n^{[s,t]}} := \sum_{i=0}^{2^n-1} B_{r_{i+1}^n r_i^n}$, where clearly $r_i^n = s + \frac{t-s}{2^n} i$, let us rewrite (2.10) with this notation:

$$M_{ts}^{\Pi_n^{[s,t]}} = B_{ts} - S^{\Pi_n^{[s,t]}}.$$

Moreover we have seen $\left\{M_{ts}^{\Pi_n^{[s,t]}}\right\}_{n\geq 1}$ admits limit M_{ts}, so does $\left\{S^{\Pi_n^{[s,t]}}\right\}_{n\geq 1}$; we denote this last limit by S_{ts}.

We will consider different cases according on all the possible relations between the two intervals $[s,t]$ and $[s_0,t_0]$, supposing always, without loss of generality, that $[s,t] \cap [s_0,t_0] \neq \emptyset$ and $[s,t] \neq [s_0,t_0]$.

$$\text{CASE A:} \qquad [s,t] \setminus [s_0,t_0] \neq \emptyset \neq [s_0,t_0] \setminus [s,t]. \tag{2.16}$$

We now split S in a suitable way, writing

$$M_{ts}^{\Pi_n^{[s,t]}} = B_{ts} + \left(B_{b_n a_n} - S^{\Pi_n^{[s,t]} \cap [s_0,t_0]}\right) - B_{b_n a_n} - S^{\Pi_n^{[s,t]} \setminus [s_0,t_0]} \tag{2.17}$$

and

$$M_{t_0 s_0}^{\Pi_n^{[s_0,t_0]}} = B_{t_0 s_0} + \left(B_{d_n c_n} - S^{\Pi_n^{[s_0,t_0]} \cap [s,t]}\right) - B_{d_n c_n} - S^{\Pi_n^{[s_0,t_0]} \setminus [s,t]} \tag{2.18}$$

where

$$a_n := \min\left\{\Pi_n^{[s,t]} \cap [s_0,t_0]\right\}, \quad b_n := \max\left\{\Pi_n^{[s,t]} \cap [s_0,t_0]\right\}$$
$$c_n := \min\left\{\Pi_n^{[s_0,t_0]} \cap [s,t]\right\}, \quad d_n := \max\left\{\Pi_n^{[s_0,t_0]} \cap [s,t]\right\} .$$

Now, as $n \to +\infty$, supposing without loss of generality that the intersections are not empty, it is clear that,

$$a_n \to \inf\left\{\bigcup_{n\geq 1} \Pi_n^{[s,t]} \cap [s_0,t_0]\right\} = \min\{[s,t] \cap [s_0,t_0]\} =: p$$

where the last equality follows since $\bigcup_{n\geq 1}\Pi_n^{[s,t]}$ is dense in $[s,t]$.
Similarly we have that $b_n \to \max\{[s,t]\cap[s_0,t_0]\} =: q$ and $c_n \to p$, $d_n \to q$.
Hence it should be clear that

$$B_{b_n a_n} - S^{\Pi_n^{[s,t]}\cap[s_0,t_0]} \longrightarrow M_{qp}\ ,\quad n\to+\infty$$

and

$$B_{d_n c_n} - S^{\Pi_n^{[s_0,t_0]}\cap[s,t]} \longrightarrow M_{qp}\ ,\quad n\to+\infty.$$

Moreover, setting

$$\xi := \inf\left\{\bigcup_{n\geq 1}\Pi_n^{[s,t]}\setminus[s_0,t_0]\right\} = \inf\{[s,t]\setminus[s_0,t_0]\}\,,\quad \eta := \sup\{[s,t]\setminus[s_0,t_0]\}$$

and

$$\xi^* := \inf\{[s_0,t_0]\setminus[s,t]\}\,,\quad \eta^* := \sup\{[s_0,t_0]\setminus[s,t]\}$$

and passing to the limit in (2.17) and (2.18) for $n\to+\infty$, they turn into

$$M_{ts} = B_{ts} + M_{qp} - B_{qp} - S_{\eta\xi}$$

and

$$M_{t_0 s_0} = B_{t_0 s_0} + M_{qp} - B_{qp} - S_{\eta^*\xi^*}$$

respectively, from which we immediately have

$$|M_{ts} - M_{t_0 s_0}| \leq |B_{ts} - B_{t_0 s_0}| + |S_{\eta\xi}| + |S_{\eta^*\xi^*}|\ .$$

Recalling then that B is continous, we find $\delta_1 > 0$ such that $|B_{ts} - B_{t_0 s_0}| < \frac{\varepsilon}{3}$ for every $(t,s)\in \mathrm{B}_{(t_0,s_0)}(\delta_1)$.

Next, since $\|M\|_\lambda < +\infty$ (Step 2), we have that $|M_{ts}| \leq \|M\|_\lambda |t-s|^\lambda\quad \forall s,t\in[0,T]$, from which M is continuous on the diagonal of $[0,T]^2$. Now, since $S_{ts} = B_{ts} - M_{ts}$ and B is continuous on the whole $[0,T]^2$, it follows that S is continuous on the diagonal too.

Now, observing the pair (s,t) respects the restriction (2.16) if and only if it belongs to

$$X_{(s_0,t_0)} := \big\{(s,t)\in[0,T]^2 : (0\leq s < s_0 < t < t_0 \leq T)\vee(0\leq s_0 < s < t_0 < t\leq T)\big\}\,,$$

we have that

$$\lim_{\substack{(s,t)\to(s_0,t_0)\\ (s,t)\in X_{(s_0,t_0)}}} |\eta-\xi| = 0$$

and thus

$$\lim_{\substack{(s,t)\to(s_0,t_0)\\(s,t)\in X_{(s_0,t_0)}}} S_{\eta\xi} = 0 \ .$$

Clearly we have identical results when dealing with (ξ^*, η^*).
Hence we can find $\delta_2 > 0$ such that $|S_{\eta\xi}|, |S_{\eta^*\xi^*}| < \frac{\varepsilon}{3}$ for every $(t,s) \in \mathrm{B}_{(t_0,s_0)}(\delta_2) \cap X_{(s_0,t_0)}$ and thus, putting all togheter and setting $\delta_A := \min\{\delta_1, \delta_2\}$, we finally get

$$|M_{ts} - M_{t_0 s_0}| < \varepsilon \quad \forall (t,s) \in \mathrm{B}_{(t_0,s_0)}(\delta_A) \cap X_{(s_0,t_0)}.$$

$$\text{Case B:} \qquad [s,t] \setminus [s_0,t_0] = \emptyset. \tag{2.19}$$

Now a pair (s,t) satisfy this condition if and only if it belongs to

$$Y_{(s_0,t_0)} := \left\{(s,t) \in [0,T]^2 : 0 \le s_0 \le s < t \le t_0 \le T\right\}.$$

In particular in this case we have $[s_0,t_0] \setminus [s,t] = [s_0, s[\cup]t, t_0]$; as did before we split S conveniently (but now the equation for $M_{ts}^{\Pi_n^{[s_0,t_0]}}$ is enough):

$$M_{t_0 s_0}^{\Pi_n^{[s_0,t_0]}} = B_{t_0 s_0} + \left(B_{ts} - S^{\Pi_n^{[s_0,t_0]} \cap [s,t]}\right) - B_{ts} - S^{\Pi_n^{[s_0,t_0]} \cap [s_0,s[} - S^{\Pi_n^{[s_0,t_0]} \cap]t,t_0]}$$

and letting $n \to +\infty$ this last equation turns into

$$M_{t_0 s_0} = B_{t_0 s_0} + M_{ts} - B_{ts} - S_{s s_0} - S_{t_0 t}; ,$$

from which we immediately get

$$|M_{ts} - M_{t_0 s_0}| \le |B_{ts} - B_{t_0 s_0}| + |S_{s s_0}| + |S_{t_0 t}| \ .$$

and with the same arguments used in Case A, we find a $\delta_B > 0$ such that

$$|M_{ts} - M_{t_0 s_0}| < \varepsilon \quad \forall (t,s) \in \mathrm{B}_{(t_0,s_0)}(\delta_B) \cap Y_{(s_0,t_0)}.$$

$$\text{Case C:} \qquad [s_0,t_0] \setminus [s,t] = \emptyset. \tag{2.20}$$

Now a pair (s,t) satisfy this condition if and only if it belongs to

$$Z_{(s_0,t_0)} := \left\{(s,t) \in [0,T]^2 : 0 \le s \le s_0 < t_0 \le t \le T\right\}.$$

It is identical to Case B; thus there exists $\delta_C > 0$ such that

$$|M_{ts} - M_{t_0 s_0}| < \varepsilon \quad \forall (t,s) \in \mathrm{B}_{(t_0,s_0)}(\delta_C) \cap Z_{(s_0,t_0)}.$$

Since $X_{(s_0,t_0)} \cup Y_{(s_0,t_0)} \cup Z_{(s_0,t_0)} = [0,T]^2$, setting $\delta := \min\{\delta_A, \delta_B, \delta_C\}$, we can finally conclude that

$$|M_{ts} - M_{t_0 s_0}| < \varepsilon \quad \forall (t,s) \in \mathrm{B}_{(t_0,s_0)}(\delta),$$

obtaining then the continuity of M at the point (s_0, t_0).

Step 5: Let us show that M does not depend on the particular sequence of partitions we use to define it, hence M is well defined.
Let us consider two arbitrary sequences of partitions of the interval $[s,t]$ such that their meshes go to 0.
Let us fix ideas: for $n \geq 1$ we write

$$\Pi_n := \{s = r_0^n < r_1^n < \cdots < r_{k_n}^n < r_{k_n+1}^n = t\}$$
$$\mathfrak{S}_n := \{s = u_0^n < u_1^n < \cdots < u_{h_n}^n < u_{h_n+1}^n = t\}$$

where we have $|\Pi_n|, |\mathfrak{S}_n| \to 0$ as $n \to +\infty$ and we suppose without loss of generality that $\Pi_n \subsetneq \Pi_{n+1}$ and $\mathfrak{S}_n \subsetneq \mathfrak{S}_{n+1}$, from which $\{h_n\}_{n\geq 1}, \{k_n\}_{n\geq 1} \subseteq \mathbb{N}$ are strictly increasing sequences.
Let us define then $\mathfrak{U}_n := \Pi_n \cup \mathfrak{S}_n$ and $u_n + 2$ as the cardinality of $\mathfrak{U}_n$; then, relabeling the elements of $\mathfrak{U}_n$, we can write $\mathfrak{U}_n = \{s = t_0^n < t_1^n < \cdots < t_{u_n}^n < t_{u_n+1}^n = t\}$

We recall that

$$M_{ts}^{\Pi_n} = B_{ts} - \sum_{i=0}^{k_n} B_{r_{i+1}^n r_i^n} \qquad \text{and} \qquad M_{ts}^{\mathfrak{S}_n} = B_{ts} - \sum_{i=0}^{h_n} B_{u_{i+1}^n u_i^n} \ .$$

We want to prove that

$$\left| M_{ts}^{\Pi_n} - M_{ts}^{\mathfrak{S}_n} \right| \overset{n \to +\infty}{\longrightarrow} 0.$$

So, let us consider

$$\left| M_{ts}^{\Pi_n} - M_{ts}^{\mathfrak{S}_n} \right| \leq \left| M_{ts}^{\Pi_n} - M_{ts}^{\mathfrak{U}_n} \right| + \left| M_{ts}^{\mathfrak{U}_n} - M_{ts}^{\mathfrak{S}_n} \right| .$$

Now

$$\begin{aligned}
\left|M_{ts}^{\Pi_n} - M_{ts}^{\mathfrak{U}_n}\right| &= \left|\sum_{i=0}^{k_n} B_{r_{i+1}^n r_i^n} - \sum_{j=0}^{u_n} B_{t_{j+1}^n t_j^n}\right| \\
&= \left|\sum_{i=0}^{k_n}\left(B_{r_{i+1}^n r_i^n} - \sum_{\substack{t_j^n, t_{j+1}^n \in \mathfrak{U}_n \\ r_i^n \le t_j^n < t_{j+1}^n \le r_{l+1}^n}} B_{t_{j+1}^n t_j^n}\right)\right| \\
&\le \sum_{i=0}^{k_n}\left|B_{r_{i+1}^n r_i^n} - \sum_{\substack{t_j^n, t_{j+1}^n \in \mathfrak{U}_n \\ r_i^n \le t_j^n < t_{j+1}^n \le r_{l+1}^n}} B_{t_{j+1}^n t_j^n}\right| \\
&= \sum_{i=0}^{k_n}\left|M_{r_{i+1}^n r_i^n}^{\mathfrak{U}_n \cap [r_i^n, r_{i+1}^n]}\right| \\
&\le c_{h,\lambda}\sum_{i=0}^{k_n}|r_{i+1}^n - r_i^n|^\lambda \\
&\le c_{h,\lambda}|\Pi_n|^{\lambda-1}\sum_{i=0}^{k_n}|r_{i+1}^n - r_i^n| \\
&= c_{h,\lambda}|\Pi_n|^{\lambda-1}|t-s| \overset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

The proof that even $\left|M_{ts}^{\mathfrak{U}_n} - M_{ts}^{\mathfrak{S}_n}\right|$ goes to 0 is the same because of the simmetry of Π_n and $\mathfrak{S}_n$ with respect to $\mathfrak{U}_n$.

Hence we have proved that $\lim_n M_{ts}^{\Pi_n}$ does not depend on the particular sequence of partitions used (provided its mesh goes to 0) and thus we can use the more useful one, depending on the situation.

Step 6: Let us finally prove that $\delta M = h$.
Pick $0 \le s < u < t \le T$, and consider two sequences of partitions of the intervals $[s,u]$ and $[u,t]$ whose meshes go to 0 as $n \to +\infty$, say $\{\Pi_n^{[s,u]}\}_{n\ge1}$ and $\{\Pi_n^{[u,t]}\}_{n\ge1}$ respectively.
Set then $\Pi_n^{[s,t]} := \Pi_n^{[s,u]} \cup \Pi_n^{[u,t]}$, which defines in turn a sequence of partitions of the interval $[s,t]$ whose mesh goes to 0.
As seen in Step 3, up to passing to a subsequence we have that

$$\lim_{n\to+\infty} M_{us}^{\Pi_n^{[s,u]}} = M_{us}\ , \quad \lim_{n\to+\infty} M_{tu}^{\Pi_n^{[u,t]}} = M_{tu}\ , \quad \lim_{n\to+\infty} M_{ts}^{\Pi_n^{[s,t]}} = M_{ts}\ .$$

Calling then $k_n^{[s,u]}+2$, $k_n^{[u,t]}+2$ and $k_n^{[s,t]}+2$ the number of points of the partitions

$\Pi_n^{[s,u]}$, $\Pi_n^{[u,t]}$ and $\Pi_n^{[s,t]}$ respectively, and setting $\Pi_n^{[s,t]} = \{r_i^n\}_{i=0}^{k_n^{[s,u]}+k_n^{[u,t]}+2}$ we have that

$$-M_{tu}^{\Pi_n^{[u,t]}} + M_{ts}^{\Pi_n^{[s,t]}} - M_{us}^{\Pi_n^{[s,u]}} = \overbrace{(-B_{tu} + B_{ts} - B_{us})}^{(\delta B)_{tus}} +$$
$$+ \underbrace{\left(\sum_{i=k_n^{[s,u]}+1}^{k_n^{[s,u]}+k_n^{[u,t]}+1} B_{r_{i+1}^n r_i^n} - \sum_{i=0}^{k_n^{[s,u]}+k_n^{[u,t]}+1} B_{r_{i+1}^n r_i^n} + \sum_{i=0}^{k_n^{[s,u]}} B_{r_{i+1}^n r_i^n} \right)}_{=0} = h_{tus}$$

from which, passing to the limit for $n \to +\infty$ we get

$$h_{tus} = -M_{tu} + M_{ts} - M_{us} = (\delta M)_{tus}$$

that is $\delta M = h$, as wanted. □

Remark: Given a linear map between vector spaces $\varphi : V \to W$, it is well known that if it is invertible on one side, a priori we cannot say anything nor on uniqueness of this one-side inverse, neither on the invertibility of φ on the other side.

Nevertheless it can be proved that if φ admits an inverse on one side, such a side-inverse is *unique* if and only if φ is invertible on the other side too.

In the previous theorem, for every $\lambda > 1$ fixed, we have proved the right invertibility of the linear map $\delta_2 : \mathscr{C}_2^\lambda(V) \to \mathscr{Z}\mathscr{C}_3^\lambda(V)$ and also that this right inverse is unique; hence this linear map is left invertible too.

What we will use in the sequel is indeed left invertibility of $\delta_2 : \mathscr{C}_2^\lambda(V) \to \mathscr{Z}\mathscr{C}_3^\lambda(V)$ and the efforts done in order to prove right invertibility are justified since right invertibility is equivalent to the uniqueness of the left inverse: in this way the left inverse is well defined and uniquely determined.

It is important to note that left invertibility is no more verified if we look at δ_2 as a map $\mathscr{C}_2(V) \to \mathscr{Z}\mathscr{C}_3(V)$, in fact it is equivalent to injectivity and $\ker \delta_2 \cap \mathscr{C}_2(V) = \operatorname{Im} \delta_1 \cap \mathscr{C}_2(V)$, but clearly this last one is not $\{0\}$.

Theorem 2.2.1, togheter with this last considerations, allows to get a canonical decomposition of the preimage of the space $\mathscr{Z}\mathscr{C}_3^{1+}(V)$ under δ, or, equivalently, of a function $h \in \mathscr{C}_2(V)$ whose increment δh is smooth enough:

Corollary 2.2.1. Given $h \in \mathscr{C}_2(V)$ such that $\delta_2 h \in \mathscr{C}_3^\lambda(V)$ for some $\lambda > 1$; then there exists, modulo a constant, a unique $f \in \mathscr{C}_1(V)$ such that h can be decomposed as

$$h = \delta_1 f + \Lambda \delta_2 h \tag{2.21}$$

Proof. First of all, let us note that since h is taken in $\mathscr{C}_2(V)$ (not in $\mathscr{C}_2^\lambda(V)$), as pointed out in the previous remark, $\Lambda\delta_2 h$ in general is not equal to h; thus $g :=$

$h - \Lambda\delta_2 h$ is not a trivial element of $\mathscr{C}_2(V)$.
On the other hand, having assumed $\delta_2 h \in \mathscr{C}_3^\lambda(V)$, we have that $\delta_2\Lambda\delta_2 h = \delta_2 h$ and thus $g \in \mathscr{Z}\mathscr{C}_2(V) = \mathscr{B}\mathscr{C}_2(V)$. Hence there exists an element $f \in \mathscr{C}_1(V)$ such that $\delta_1 f = g = h - \Lambda\delta_2 h$ that is (2.21).
If there was a different $\widetilde{f} \in \mathscr{C}_2(V)$ which satisfy (2.21), then we would have $\delta_1 f + \Lambda\delta_2 h = h = \delta_1\widetilde{f} + \Lambda\delta_2 h$ from which $\delta_1(f - \widetilde{f}) = 0$ that is, $f - \widetilde{f} \in \mathscr{Z}\mathscr{C}_1(V) = \mathscr{B}\mathscr{C}_1(V) = V$, i.e. $\exists v \in V$ such that $\widetilde{f} \equiv f + v$ as claimed. □

At this point the link between the structure we have introduced and the integration of irregular functions is not clear yet, but the next Corollary deals with a limit of a sum as the mesh of a sequence of partitions goes to 0, which begins to move closer to something that reminds integration territory:

Corollary 2.2.2 (Integration of small increments)**.** For any 1-increment $h \in \mathscr{C}_2(V)$ such that $\delta h \in \mathscr{C}_3^\lambda(V)$ for some $\lambda > 1$, set $\delta f = (\mathrm{Id} - \Lambda\delta)h$. Then

$$(\delta f)_{ts} = \lim_{n\to+\infty} \sum_{i=0}^{k_n} h_{t_{i+1}t_i} \tag{2.22}$$

where the limit is taken over any sequence of partitions $\Pi_n = \{s = t_0 < t_1 < \cdots < t_{k_n} < t_{k_n+1} = t\}$ of the interval $[s,t]$ whose mesh tends to 0.
The 1-increment δf is the *indefinite integral* of the 1-increment δh.

Proof. Let us define partial sums for h with respect to the sequence of partitions $\{\Pi_n\}_{n\geq 1}$:

$$S_{ts}^{\Pi_n} := \sum_{i=0}^{k_n} h_{t_{i+1}t_i}$$

Then, since $h = \delta f + \Lambda\delta h$ we have that

$$S_{ts}^{\Pi_n} = \sum_{i=0}^{k_n} (\delta f)_{t_{i+1}t_i} + \sum_{i=0}^{k_n} (\Lambda\delta h)_{t_{i+1}t_i} = (\delta f)_{ts} + \sum_{i=0}^{k_n} (\Lambda\delta h)_{t_{i+1}t_i}\ .$$

Now, being $\Lambda\delta h \in \mathscr{C}_2^\lambda(V)$, the latter series converges to 0, in fact for a generic element $\eta \in \mathscr{C}_2^\lambda(V)$ we have that

$$\begin{aligned}\left|\sum_{i=0}^{k_n} \eta_{t_{i+1}t_i}\right| &\leq \sum_{i=0}^{k_n} \left|\eta_{t_{i+1}t_i}\right| \\ &= \sum_{i=0}^{k_n} \frac{\left|\eta_{t_{i+1}t_i}\right|}{|t_{i+1}-t_i|^\lambda}|t_{i+1}-t_i|^\lambda \\ &\leq \|\eta\|_\lambda \sum_{i=0}^{k_n} |t_{i+1}-t_i|^\lambda \\ &\leq \|\eta\|_\lambda |\Pi_n|^{\lambda-1} \sum_{i=0}^{k_n} |t_{i+1}-t_i| \\ &= \|\eta\|_\lambda |\Pi_n|^{\lambda-1}|t-s| \overset{n\to+\infty}{\longrightarrow} 0\end{aligned}$$

This proves both the relation (2.22) and its well definition (the limit does not depend on the particular sequence of partitions, provided its mesh goes to 0). □

2.3 From Riemann Integral to Young Integral

Let us now fix an interval $[a, b] \subset \mathbb{R}$.
A *tagged partition* of $[a, b]$ is a pair (Π_n, τ_n), where $\Pi_n = \{a = t_0 < t_1 < \cdots < t_{k_n} < t_{k_n+1} = b\}$ is a usual partition of the interval $[a, b]$, while $\tau_n = \{a_i\}_i^{k_n}$ is a finite sequence of arbitrary numbers such that $t_i \leq a_i \leq t_{i+1}$.
The mesh of a tagged partition as above is defined as the mesh of Π_n.

The *Riemann Sums* of a real valued continuous function defined on $[a, b]$, say $f : [a, b] \to \mathbb{R}$, with respect to the tagged partition (Π_n, τ_n), is defined as:

$$S^{(\Pi_n,\tau_n)} := \sum_{i=0}^{k_n} f(a_i)(t_{i+1} - t_i) \ .$$

Then we can define the usual Riemann Integral of a continuous function on $[a, b]$ as the limit of its Riemann Sums over any sequence of tagged partitions whose mesh tends to 0:

$$\int_a^b f(x)\, dx := \lim_{n\to+\infty} \sum_{i=0}^{k_n} f(a_i)(t_{i+1} - t_i).$$

It can be proved that this object is well defined, that is, it does not depend on the particular sequence of partitions chosen (provided its mesh goes to 0).
Now we observe that the measure we integrate with respect to, is the one which measures the interval $[t_i, t_{i+1}]$ as $t_{i+1} - t_i = \mathrm{Id}(t_{i+1}) - \mathrm{Id}(t_i)$; it could be useful, in some contexts, to integrate a function over a path which is not a segment (or a line),

but a more irregular path; in order to perturb our segment we have to use a function g different from the identity.

The *Riemann-Stieltjes Integral* of a function f with respect to g, raises right from this consideration and, at this point, its definition is obvious:

$$\int_a^b f(x)\,dg(x) := \lim_{n\to+\infty} \sum_{i=0}^{k_n} f(a_{t_i})(g(t_{i+1}) - g(t_i)). \tag{2.23}$$

where the limit is over any sequence of tagged partitions such that its mesh tends to 0 (and, as before, this object is well defined).
One of the first problem that arises naturally is to detect all the possible pairs of functions (f, g) for which (2.23) make sense.
One possible class of such pairs is given by taking $f \in \mathscr{C}([a,b];\mathbb{R})$ and $g \in \mathrm{BV}([a,b];\mathbb{R})$, that is f continuous and g with bounded variation.

Let us consider now a trajectory of the fractional Brownian Motion of Hurst index $0 < H < 1$, say $t \mapsto W_t^H(\omega)$; we have already pointed out (see the end of the previous chapter) that a.s. it is not of bounded variation.

Thus we would have some problem to perform Riemann-Stieltjes integral of a continuous functions with respect to a trajectory of the fractional Brownian Motion of any Hurst parameter H and this is one of the reasons why we need a different integration tool.

A kind of integral that fits our problem is in fact the Young Integral, since it allows to integrate functions $f \in \mathscr{C}_1^\lambda(\mathscr{M}_{d,m}(\mathbb{R}))$ with respect to $g \in \mathscr{C}_1^\gamma(\mathbb{R}^d)$, when $\lambda + \gamma > 1$. Let us see how this can be performed.

Suppose at first $f : [0,T] \to \mathscr{M}_{d,m}(\mathbb{R})$ and $g : [0,T] \to \mathbb{R}^m$ be regular enough such that, for $0 \le s < t \le T$, the Riemann-Stieltjes integral $\int_s^t f_u\,dg_u$ is well defined (another case in which this happens is when f is Riemann-integrable and g is lipschitz). We will exploit the usual integration rules in order to manipulate $\int_s^t f_u\,dg_u$ until we express it in terms of Λ, fact that will allow us to extend the definition of integral to functions f and g with the regularity above promised.
This extended integral is the *Young Integral.*
Let us denote $I_{ab}(f\,dg) := \int_s^t f_u\,dg_u$, using, in the sequel, one or the other according to the convenience.

We have

$$
\begin{aligned}
I_{ab}(f\,dg) &= \int_s^t f_u\,dg_u \\
&= \lim_{n\to+\infty} \sum_{i=0}^{k_n} f_{a_{t_i}}(g_{t_{i+1}} - g_{t_i}) \\
&= \lim_{n\to+\infty} \left[\sum_{i=0}^{k_n} \underbrace{(f_{a_{t_i}} - f_s)}_{=(\delta f)_{sa_{t_i}}}(g_{t_{i+1}} - g_{t_i}) + f_s \sum_{i=0}^{k_n}(g_{t_{i+1}} - g_{t_i}) \right] \\
&= \int_s^t (\delta f)_{su}\,dg_u + f_s \int_s^t dg_u \\
&= I_{st}(\delta f\,dg) + f_s(\delta g)_{st}
\end{aligned}
$$

Let us concentrate on the term $I_{st}(\delta f\,dg)$; it clearly belongs to $\mathscr{C}_2(\mathbb{R}^d)$; then, given $s, u, t \in [0, T]$, we can define $h_{sut} := (\delta_2 I(\delta f\,dg))_{sut}$; let us work on it exploiting the usual integration rules. Without loss of generality let us consider $0 \le s < u < t \le T$.

$$
\begin{aligned}
h_{sut} &= (\delta_2 I(\delta f\,dg))_{sut} \\
&= -I_{ut}((f_\cdot - f_u)\,dg_\cdot) + I_{st}((f_\cdot - f_s)\,dg_\cdot) - I_{su}((f_\cdot - f_s)\,dg_\cdot) \\
&= -I_{ut}((f_\cdot - f_u)\,dg_\cdot) + I_{ut}((f_\cdot - f_s)\,dg_\cdot) \\
&= I_{ut}((f_\cdot - f_s - f_\cdot + f_u)\,dg_\cdot) \\
&= (f_u - f_s) I_{ut}(dg) \\
&= (\delta f)_{su}(\delta g)_{ut}.
\end{aligned}
$$

Remark: Since

$$
I_{ab}((f_\cdot - f_a)\,dg_\cdot) = \int_a^b \left(\int_a^u df_w \right) dg_u\ ,
$$

we read in the previous computations a nice property of the operator δ: it transforms iterated integrals into product of increments.

Now it is clear that $h \in \mathscr{C}_3(\mathbb{R}^d)$ such that $\delta h = 0$ (since we recall that $\delta\delta = 0$); moreover the previous computations highlights how the regularity of f and g affects the regularity of h:

$$
\begin{aligned}
\|h\|_{\lambda+\gamma} &\overset{(2.8)}{\leq} \|h\|_{\lambda,\gamma} \\
&= \sup_{\substack{s,u,t\in[0,T] \\ s<u<t}} \frac{|h_{sut}|}{|u-s|^{\lambda}|t-u|^{\gamma}} \\
&= \sup_{\substack{s,u,t\in[0,T] \\ s<u<t}} \frac{|f_u - f_s||g_t - g_u|}{|u-s|^{\lambda}|t-u|^{\gamma}} \\
&\leq \sup_{0\leq s<u\leq T} \frac{|f_u - f_s|}{|u-s|^{\lambda}} \cdot \sup_{0\leq u<t\leq T} \frac{|g_t - g_u|}{|t-u|^{\gamma}} \\
&= \|f\|_{\lambda}\|g\|_{\gamma} < +\infty
\end{aligned}
$$

and thus $h \in \mathscr{Z}\mathscr{C}_3^{\lambda+\gamma}(\mathbb{R}^d)$. So we can apply the sewing map to h, obtaining

$$
I_{st}(\delta f\, dg) = \Lambda_{st}(\delta f \delta g)
$$

which, plugged into the expression $I_{ab}(f\, dg) = I_{st}(\delta f\, dg) + f_s(\delta g)_{st}$ found before, gives back

$$
I_{st}(f\, dg) = f_s(\delta g)_{st} + \Lambda_{st}(\delta f \delta g) \ . \tag{2.24}
$$

We are arrived here supposing f and g regular enough to being legitimate in using Riemann-Stieltjes integration, but right hand side of (2.24) is defined whenever $f \in \mathscr{C}_1^{\lambda}(\mathscr{M}_{d,m}(\mathbb{R}))$ and $g \in \mathscr{C}_1^{\gamma}(\mathbb{R}^m)$, with $\lambda + \gamma > 1$, thus we have extended the R-S integral to a more general integral: (2.24) is in fact the definition of the Young integral.

The main theorem about Young integration follows:

Theorem 2.3.1 (Young Integral)**.** The Young Integral of a function $f \in \mathscr{C}_1^{\lambda}(\mathscr{M}_{d,m}(\mathbb{R}))$ with respect to a function $g \in \mathscr{C}_1^{\gamma}(\mathbb{R}^m)$ with $\lambda + \gamma > 1$ is defined as

$$
I_{st}(f\, dg) := f_s(\delta g)_{st} + \Lambda_{st}(\delta f \delta g) \ .
$$

Then we have:

1. Whenever f and g are regular enough to use R-S integral, then $I_{ab}(f\, dg)$ coincides with the R-S integral of f with respect to g.

2. For any $0 \leq \beta < 1$ such that $1 < \gamma + \lambda(1-\beta) =: \mu_\beta$ the Young integral satisfies the following inequality

$$
|I_{st}(f\, dg)| \leq \|f\|_{\infty}\|g\|_{\gamma}|t-s|^{\gamma} + c_{\gamma,\lambda,\beta}\|f\|_{\infty}^{\beta}\|f\|_{\lambda}^{1-\beta}\|g\|_{\gamma}|t-s|^{\mu_\beta} \tag{2.25}
$$

where $c_{\gamma,\lambda,\beta} = 2^{\beta}(2^{\mu_\beta} - 1)^{-1}$.

3. The Young Integral can be expressed as limit of Riemann Sums:

$$I_{st}(f\,dg) = \lim_{n\to+\infty} \sum_{i=0}^{k_n} f_{t_i}(\delta g)_{t_i t_{i+1}}$$

where the limit is taken over any sequence of partitions of $[s,t]$, say $\Pi_n = \{s = t_0 < t_1 < \cdots t_{k_n} < t_{k_n+1} = t\}$, whose mesh tends to 0. That is, the Young integral, coincides with the one defined by Young in [1].

Proof. The first claim is true by construction; just look at how we got (2.24).
Let us prove the second claim. Recalling that for $0 \le a < c < b \le T$ we have $(\delta f)_{ac}(\delta g)_{cb} = h_{acb}$ with $h \in \mathscr{Z}\mathscr{C}_3^{\lambda+\gamma}(\mathbb{R}^d)$ and, as seen in the previous page, $\|h\|_{\lambda+\gamma} \le \|f\|_\lambda \|g\|_\gamma$ it follows that

$$\begin{aligned}
|I_{st}(f\,dg)| &\le |f_s(g_t - g_s)| + |\Lambda_{st}(\delta f \delta g)| \\
&= |f_s| \frac{|g_t - g_s|}{|t-s|^\gamma}|t-s|^\gamma + \frac{|\Lambda_{st}(h)|}{|t-s|^{\lambda+\gamma}}|t-s|^{\lambda+\gamma} \\
&\le \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + \|\Lambda(h)\|_{\lambda+\gamma}|t-s|^{\lambda+\gamma} \\
&\overset{(2.9)}{\le} \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + (2^{\lambda+\gamma}-1)^{-1}\|h\|_{\lambda+\gamma}|t-s|^{\lambda+\gamma} \\
&\le \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + (2^{\lambda+\gamma}-1)^{-1}\|f\|_\lambda \|g\|_\gamma |t-s|^{\lambda+\gamma}
\end{aligned}$$

which is (2.25) with $\beta = 0$. Now, if in this one instead of λ we consider $\lambda(1-\beta)$ for $0 \le \beta < 1$ such that $\mu_\beta > 1$, then we get

$$|I_{st}(f\,dg)| \le \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + (2^{\mu_\beta}-1)^{-1}\|f\|_{\lambda(1-\beta)}\|g\|_\gamma |t-s|^{\mu_\beta}$$

and observing then

$$\begin{aligned}
\|f\|_{\lambda(1-\beta)} &= \sup_{0\le u<v\le T} \frac{|f_v - f_u|}{|v-u|^{\lambda(1-\beta)}} \\
&= \sup_{0\le u<v\le T} \frac{|f_v - f_u|^\beta |f_v - f_u|^{1-\beta}}{|v-u|^{\lambda(1-\beta)}} \\
&\le 2^\beta \|f\|_\infty^\beta \left(\sup_{0\le u<v\le T} \frac{|f_v - f_u|}{|v-u|^\lambda} \right)^{1-\beta} \\
&= 2^\beta \|f\|_\infty^\beta \|f\|_\lambda^{1-\beta}
\end{aligned}$$

from which claim 2 is proved.
Third claim: let us consider $p_{st} := f_s(\delta g)_{st} \in \mathscr{C}_2(\mathbb{R}^d)$; with a direct computation one shows that

$$(\delta_2 p)_{acb} = \delta_2(f_s(\delta g)_{st})_{acb} = -(\delta f)_{ac}(\delta g)_{cb} \tag{2.26}$$

from which clearly $\delta_2 p \in \mathscr{C}_3^{\lambda+\gamma}(\mathbb{R}^d)$.

Then using Corollary (2.2.2), we get

$$(\mathrm{Id} - \Lambda\delta) p_{st} = \lim_{n\to+\infty} \sum_{j=0}^{k_n} p_{t_j t_{j+1}},$$

that is

$$\begin{aligned}
\lim_{n\to+\infty} \sum_{j=0}^{k_n} f_{t_j} (\delta g)_{t_j t_{j+1}} &= f_s (\delta g)_{st} - \Lambda_{st} \left[\delta_2 (f_z (\delta g)_{zw})_{acb} \right] \\
&\overset{(2.26)}{=} f_s (\delta g)_{st} + \Lambda_{st} \left[(\delta f)_{ac} (\delta g)_{cb} \right] \\
&\overset{(2.24)}{=} I_{st}(f\, dg)
\end{aligned}$$

which concludes the proof. □

Chapter 3

A SDE driven by a Fractional Brownian Motion

3.1 A preliminary Lemma and some notation

Let us start with a lemma we will use in the sequel.

Lemma 3.1.1. If $K \Subset \mathbb{R}$ and $f, g : K \to \mathbb{R}$ are continuous, then

$$\left| \sup_K f - \sup_K g \right| \leq \sup_K |f - g| \,. \tag{3.1}$$

Proof. Let us take $x_0, x_1 \in K$ such that $\sup_K f = f(x_0)$ and $\sup_K g = g(x_1)$ assuming without loss of generality that $f(x_0) \geq g(x_1)$. Hence we have

$$\left| \sup_K f - \sup_K g \right| = f(x_0) - g(x_1).$$

Next consider $x_2 \in K$ which realizes $\sup_K |f - g| = |f(x_2) - g(x_2)|$; then it is clear that

$$|f(x_2) - g(x_2)| \geq |f(x_0) - g(x_0)| = f(x_0) - g(x_0)$$

and since

$$f(x_0) - g(x_0) \geq f(x_0) - g(x_1) \Longleftrightarrow g(x_0) \leq g(x_1)$$

we have proved (3.1). □

If $x \in \mathbb{R}^N$, we denote the usual euclidean norm in $\mathbb{R}^N$ with $\|x\|$: we omit the reference to the space $\mathbb{R}^N$ or to the dimension N since it should be clear from the context.

If $A = (a_{i,j}) \in \mathscr{M}_{d,m}(\mathbb{R})$, we define

$$\|A\|_1^{(d,m)} := \sup\{\|Av\| : v \in \mathbb{S}^{m-1}\}$$

$$\|A\|_2^{(d\times m)} := \sqrt{\sum_{i,j} a_{i,j}^2} \,.$$

We thus have two norms on $\mathscr{M}_{d,m}(\mathbb{R})$, which is a finite dimensional $\mathbb{R}$-vector space, hence they are equivalent, so there exist two absolute constants $C_{d,m}, c_{d,m} > 0$ such that

$$c_{d,m}\|\cdot\|_1^{(d,m)} \leq \|\cdot\|_2^{(d\times m)} \leq C_{d,m}\|\cdot\|_1^{(d,m)} . \tag{3.2}$$

Take now $f : [0,L] \to \mathscr{M}_{d,m}(\mathbb{R})$ and $h : [0,L] \to \mathbb{R}^N$; we define

$$\|f\|_{\infty,[0,L]} := \sup_{0\leq t\leq L} \|f(t)\|_1^{(d,m)} , \quad \|f\|_{\lambda,[0,L]} := \sup_{0\leq s<t\leq L} \frac{\|f(t)-f(s)\|_1^{(d,m)}}{|t-s|^\lambda}$$

$$\|h\|_{\infty,[0,L]} := \sup_{0\leq t\leq L} \|h(t)\| \quad , \quad \|h\|_{\lambda,[0,L]} := \sup_{0\leq s<t\leq L} \frac{\|h(t)-h(s)\|}{|t-s|^\lambda}.$$

3.2 A SDE driven by a fBM with $1/2 < H < 1$

Then let us recall quickly the problem: we want to prove uniqueness for the solutions of the following d-dimensional stochastic differential equation

$$x_t = \underbrace{\xi_0 + \int_0^t b(s,x_s)\,ds + \int_0^t \sigma(s,x_s)\,dg_s}_{=:z_t} + y_t \tag{3.3}$$

where $y_t = \sup_{0\leq s\leq t}(z_s)^-$, $\xi_0 \in \mathbb{R}_+^d$ is fixed and g is a fixed trajectory of an m-dimensional fractional Brownian Motion, that is, $g := W^H(\omega)$ for some $\omega \in \Omega^*$ fixed, where $\Omega^* \in \mathscr{A}$ is a set such that $\mathbf{P}(\Omega^*) = 1$ for which the regularity properties discussed in the first chapter are fulfilled.

Moreover

$$b : [0,L]\times\mathbb{R}^d \to \mathbb{R}^d \quad ,$$
$$\sigma : [0,L]\times\mathbb{R}^d \to \mathscr{M}_{d,m}(\mathbb{R})$$

are bounded measurable functions which satisfy (now we are able to be precise about the norms):

$$\|b(t,x)-b(t,y)\| \leq K_0\|x-y\| \;\; \forall x,y\in\mathbb{R}^d,\; \forall t\in[0,L] \tag{3.4}$$

$$\|\sigma(t,x)-\sigma(t,y)\|_1^{(d,m)} \leq K_0\|x-y\| \;\; \forall x,y\in\mathbb{R}^d,\; \forall t\in[0,L] \tag{3.5}$$

$$\|\sigma(t,x)-\sigma(s,x)\|_1^{(d,m)} \leq K_0|t-s|^\nu \;\; \forall x\in\mathbb{R}^d,\; \forall s,t\in[0,L] \tag{3.6}$$

where $\nu \in]\frac{1}{2},1]$ and $K_0 > 0$. Hence, both b and σ are Lipschitz in space, moreover σ is ν-Hölder continous in time.

Next we fix $H \in]\frac{1}{2},\nu]$ and observe that, $\forall\varepsilon \in]0,H[$ one has $g \in \mathscr{C}^{H-\varepsilon}([0,L],\mathbb{R}^m)$.

Then in [2] it is proved that for every fixed $\lambda \in]\frac{1}{2}, H[$, equation (3.3) admits a solution x such that, a.s. $x(\omega) \in \mathscr{C}^\lambda([0,L],\mathbb{R}^d)$. Hence, the notion of uniqueness has to be given on the space $\mathscr{C}^\lambda([0,L],\mathbb{R}^d)$.

Then, fixing $\frac{1}{2} < \lambda_1 < \lambda_2 < H$ and supposing uniqueness was proved, if $x^{(\lambda_1)}, x^{(\lambda_2)}$ are the solutions on $\mathscr{C}^{\lambda_1}$ and $\mathscr{C}^{\lambda_2}$ respectively, being $\mathscr{C}^{\lambda_2} \subseteq \mathscr{C}^{\lambda_1}$ it follows they must coincide; in particular $x^{(\lambda_1)} \in \mathscr{C}^{\lambda_2}$. From this we deduce that with uniqueness, we would have a solution x with a.s. trajectories belonging to $\mathscr{C}^\lambda$ for every $\lambda \in]\frac{1}{2}, H[$.

A different proof of existence was given by M. Gubinelli and can be found in [11].

We summarize here the relations between the parameters considered above, in order to be clear:

$$\frac{1}{2} < \lambda < H \leq \nu \leq 1 \ \ .$$

3.3 Uniqueness problem

For the sequel, let us fix $\omega \in \Omega^*$ and consider $x^{(1)}, x^{(2)} \in \mathscr{C}^\lambda([0,L],\mathbb{R}^d)$ two solutions of (3.3), writing $z^{(i)}, y^{(i)}$ with the obvious meaning: $x^{(i)} = z^{(i)} + y^{(i)} \ \ i = 1,2$.

We will take $0 \leq T \leq L$, which will be chosen conveniently later.

Let us prove first that the map $t \mapsto \sigma(t, x_t^{(1)}) - \sigma(t, x_t^{(2)}) =: \Delta_t$ is λ–Hölder continous on $[0,T]$; this will allow us to integrate this function with respect to g in Young sense.

Let us first take $0 \leq u < v \leq T$ and observe that

$$\begin{aligned}
\|\Delta_u - \Delta_v\|_1^{(d,m)} =& \|\sigma(u,x_u^{(1)}) - \sigma(u,x_u^{(2)}) - \sigma(v,x_v^{(1)}) + \sigma(v,x_v^{(2)})\|_1^{(d,m)} \\
\leq& \|\sigma(u,x_u^{(1)}) - \sigma(u,x_v^{(1)})\|_1^{(d,m)} + \|\sigma(u,x_v^{(1)}) - \sigma(v,x_v^{(1)})\|_1^{(d,m)} \\
&+ \|\sigma(v,x_v^{(2)}) - \sigma(u,x_v^{(2)})\|_1^{(d,m)} + \|\sigma(u,x_v^{(2)}) - \sigma(u,x_u^{(2)})\|_1^{(d,m)} \\
\overset{(3.5),(3.6)}{\leq}& K_0 \left[\|x_u^{(1)} - x_v^{(1)}\| + \|x_u^{(2)} - x_v^{(2)}\| + 2|u-v|^\nu\right]
\end{aligned}$$

from which we immediately get

$$\begin{aligned}
\|\Delta\|_{\lambda,[0,T]} &= \sup_{S \leq u < v \leq T} \frac{\|\Delta_v - \Delta_u\|_1^{(d,m)}}{|v-u|^\lambda} \\
&\leq K_0 \left[\|x^{(1)}\|_{\lambda,[0,T]} + \|x^{(2)}\|_{\lambda,[0,T]} + 2T^{\nu-\lambda}\right] \\
&\leq K_0 \left[\|x^{(1)}\|_{\lambda,[0,L]} + \|x^{(2)}\|_{\lambda,[0,L]} + 2L^{\nu-\lambda}\right] =: C_0
\end{aligned}$$

Hence we have proved both the λ-Hölder continuity of Δ (as claimed) and its λ-Hölder norm is bounded uniformely with respect to T.

We finally recall that for every $\eta \in \mathbb{R}$, the negative part can be expressed as

$$\eta^- = \frac{1}{2}(|\eta| - \eta) \ . \tag{3.7}$$

Fix then $t \in [0,T]$:

$$\begin{aligned}
\left\| y_t^{(1)} - y_t^{(2)} \right\|^2 &= \left\| \sup_{0\le s\le t} (z_s^{(1)})^- - \sup_{0\le s\le t} (z_s^{(2)})^- \right\|^2 \\
&= \sum_{j=1}^d \left(\sup_{0\le s\le t} (z_s^{(1),j})^- - \sup_{0\le s\le t} (z_s^{(2),j})^- \right)^2 \\
&\overset{(3.1)}{\le} \sum_{j=1}^d \left(\sup_{0\le s\le t} \left| (z_s^{(1),j})^- - (z_s^{(2),j})^- \right| \right)^2 \ .
\end{aligned}$$

Now in what follows we denote with $b^{(j)}$ and $(\Delta_t)_j$ (or $\sigma(t,x)_j$) the j-th component of b and the j-th row of Δ_t (or $\sigma(t,x)$) respectively.
We think to $(\Delta)_j$ as a $\mathscr{M}_{1,m}(\mathbb{R})$-valued function.
Before going on, we prove that $(\Delta)_j$ is λ-Hölder continous on $[0,T]$ as we did before for Δ and for the same reason: being allowed to integrate $(\Delta)_j$ with respect to g in Young sense.

$$\begin{aligned}
\|(\Delta)_j\|_{\lambda,[0,T]} &= \sup_{0\le s<t\le T} \frac{\|(\Delta_t)_j - (\Delta_s)_j\|_1^{(1,m)}}{|t-s|^\lambda} \\
&\le \frac{1}{c_{1,m}} \cdot \sup_{0\le s<t\le T} \frac{\|(\Delta_t)_j - (\Delta_s)_j\|_2^{(1\times m)}}{|t-s|^\lambda} \\
&\le \frac{1}{c_{1,m}} \cdot \sup_{0\le s<t\le T} \frac{\|\Delta_t - \Delta_s\|_2^{(d\times m)}}{|t-s|^\lambda} \\
&\le \frac{C_{d,m}}{c_{1,m}} \cdot \sup_{0\le s<t\le T} \frac{\|\Delta_t - \Delta_s\|_1^{(d,m)}}{|t-s|^\lambda} \\
&= \frac{C_{d,m}}{c_{1,m}} \|\Delta\|_{\lambda,[0,T]} \\
&\le \frac{C_{d,m}}{c_{1,m}} C_0
\end{aligned}$$

and we are done. In particular from this we deduce that

$$\|(\Delta)_j\|_{\infty,[0,T]} \leq \frac{C_{d,m}}{c_{1,m}} \|\Delta\|_{\infty,[0,T]} \ . \tag{3.8}$$

Working on the summands of the above sum, we have that

$$\begin{aligned}
\sup_{0\leq s\leq t} \left|(z_s^{(1),j})^- - (z_s^{(2),j})^-\right| &\overset{(3.7)}{=} \frac{1}{2} \sup_{0\leq s\leq t} \left|(|z_s^{(1),j}| - |z_s^{(2),j}|) + (z_s^{(2),j} - z_s^{(1),j})\right| \\
&\leq \frac{1}{2} \sup_{0\leq s\leq t} \left||z_s^{(1),j}| - |z_s^{(2),j}|\right| + \frac{1}{2} \sup_{0\leq s\leq t} \left|z_s^{(2),j} - z_s^{(1),j}\right| \\
&\leq \sup_{0\leq s\leq t} \left|z_s^{(1),j} - z_s^{(2),j}\right| \\
&= \sup_{0\leq s\leq t} \left|\int_0^s \left[b^{(j)}(u, x_u^{(1)}) - b^{(j)}(u, x_u^{(2)})\right] du + \int_0^s (\Delta_u)_j dg_u\right| \\
&\leq \sup_{0\leq s\leq t} \int_0^s \left\|b(u, x_u^{(1)}) - b(u, x_u^{(2)})\right\| du + \sup_{0\leq s\leq t} \left|\int_0^s (\Delta_u)_j \cdot dg_u\right| \\
&\overset{(2.24)}{\leq} K_0 \sup_{0\leq s\leq t} \|x^{(1)} - x^{(2)}\|_{\infty,[0,s]} \cdot s + \\
&\quad + \sup_{0\leq s\leq t} \left|(\Delta_0)_j \cdot (g_s - g_0) + \Lambda_{0s}(h_{acb}^{(j)})\right|
\end{aligned}$$

where we have set

$$h_{acb}^{(j)} := \delta\left((\Delta_\cdot)_j\right)_{ac} \cdot (\delta g_\cdot)_{cb} = \delta(\sigma(\cdot, x_\cdot^{(1)})_j - \sigma(\cdot, x_\cdot^{(2)})_j)_{ac} \cdot (\delta g_\cdot)_{cb} \ ;$$

moreover since $(\Delta_0)_j = 0$, the last term written turns into

$$K_0 \|x^{(1)} - x^{(2)}\|_{\infty,[0,t]} \cdot t + \sup_{0\leq s\leq t} \left|\Lambda_{0s}(h_{acb}^{(j)})\right| . \tag{3.9}$$

So it is now appropriate to study the term $\left|\Lambda_{0s}(h_{acb}^{(j)})\right|$:

$$\begin{aligned}
\left|\Lambda_{0s}(h_{acb}^{(j)})\right| &= \frac{\left|\Lambda_{0s}(h_{acb}^{(j)})\right|}{|s-0|^{\lambda+H-\varepsilon}} s^{\lambda+H-\varepsilon} \\
&\leq \|\Lambda(h^{(j)})\|_{\lambda+H-\varepsilon,[0,s]} \cdot s^{\lambda+H-\varepsilon} \\
&\overset{(2.9)}{\leq} (2^{\lambda+H-\varepsilon} - 2)^{-1} \|h^{(j)}\|_{\lambda+H-\varepsilon,[0,s]} \cdot s^{\lambda+H-\varepsilon} \\
&\overset{(2.8)}{\leq} (2^{\lambda+H-\varepsilon} - 2)^{-1} \|h^{(j)}\|_{\lambda,H-\varepsilon,[0,s]} \cdot s^{\lambda+H-\varepsilon} \ ,
\end{aligned}$$

and since

$$\begin{aligned}
\|h^{(j)}\|_{\lambda,H-\varepsilon,[0,s]} &= \sup_{\substack{a,c,b\in[0,s]\\ a\neq c\neq b}} \frac{|[(\Delta_c)_j-(\Delta_a)_j]\cdot(g_b-g_c)|}{|c-a|^{\lambda}|b-c|^{H-\varepsilon}} \\
\text{(C-S)} &\leq \sup_{\substack{a,c,b\in[0,s]\\ a\neq c\neq b}} \frac{\|(\Delta_c)_j-(\Delta_a)_j\|_2^{(1\times m)}}{|c-a|^{\lambda}} \frac{\|g_b-g_c\|}{|b-c|^{H-\varepsilon}} \\
&\leq \|g\|_{H-\varepsilon,[0,s]}\cdot \sup_{\substack{a,c\in[0,s]\\ a\neq c}} \frac{\|\Delta_c-\Delta_a\|_2^{(d\times m)}}{|c-a|^{\lambda}} \\
&\leq C_{d,m}\|g\|_{H-\varepsilon,[0,s]} \sup_{\substack{a,c\in[0,s]\\ a\neq c}} \frac{\|\Delta_c-\Delta_a\|_1^{(d,m)}}{|c-a|^{\lambda}} \\
&\leq \underbrace{K_0 C_{d,m}\|g\|_{H-\varepsilon,[0,s]}\left(\|x^{(1)}\|_{\lambda,[0,s]}+\|x^{(2)}\|_{\lambda,[0,s]}+2s^{\nu-\lambda}\right)}_{=:(2^{\lambda+H-\varepsilon}-2)\frac{1}{2\sqrt{d}}\eta_s}
\end{aligned}$$

where the last inequality was already seen. Thus, putting all togheter we get

$$\sup_{0\leq s\leq t}\left|\Lambda_{0s}(h^{(j)}_{acb})\right| \leq \frac{1}{2\sqrt{d}}\eta_t\cdot t^{\lambda+H-\varepsilon}\ , \tag{3.10}$$

hence, setting

$$H_t := \left\|x^{(1)}-x^{(2)}\right\|_{\infty,[0,t]}$$

we have that (3.9) is less or equal than

$$K_0 H_t\cdot t+\frac{1}{2\sqrt{d}}\eta_t\cdot t^{\lambda+H-\varepsilon} =: \gamma_t \tag{3.11}$$

and thus we can write

$$\left\|y_t^{(1)}-y_t^{(2)}\right\|^2 \leq \sum_{j=1}^{d}\gamma_t^2 = d\gamma_t^2$$

from which clearly

$$\left\|y_t^{(1)}-y_t^{(2)}\right\| \leq \sqrt{d}\gamma_t\ . \tag{3.12}$$

Next, since

$$\begin{aligned}
\left\|z_t^{(1)}-z_t^{(2)}\right\|^2 &= \sum_{j=1}^{d}\left(z_t^{(1),j}-z_t^{(2),j}\right)^2 \\
&\leq \sum_{j=1}^{d}\left(\sup_{0\leq s\leq t}\left|z_s^{(1),j}-z_s^{(2),j}\right|\right)^2,
\end{aligned}$$

repeating already done steps, we can easily get

$$\left\| z_t^{(1)} - z_t^{(2)} \right\| \leq \sqrt{d}\gamma_t \; . \tag{3.13}$$

Hence exploiting (3.12) and (3.13) we get, for all $t \in [0, T]$,

$$\left\| x_t^{(1)} - x_t^{(2)} \right\| \leq \left\| z_t^{(1)} - z_t^{(2)} \right\| + \left\| y_t^{(1)} - y_t^{(2)} \right\| \leq 2\sqrt{d}\gamma_t$$

hence, being $t \mapsto 2\sqrt{d}\gamma_t$ non-decreasing, we get

$$\left\| x^{(1)} - x^{(2)} \right\|_{\infty,[0,T]} \leq 2\sqrt{d}\gamma_T$$

that is, relabeling $K_1 := 2\sqrt{d}K_0$,

$$H_T \leq K_1 H_T \cdot T + \eta_T \cdot T^{\lambda+H-\varepsilon}$$

which is equivalent to (taking T small enough to get $1 - K_1 T > 0$)

$$\frac{H_T}{\eta_T T^{\lambda+H-\varepsilon}} \leq \frac{1}{1 - K_1 T}$$

and thus, setting $f_t := x_t^{(1)} - x_t^{(2)}$ (it is clear that $f_0 = 0$) and

$$\begin{aligned} 1 &= \limsup_{T \to 0^+} \frac{1}{1 - K_1 T} \\ &\geq \limsup_{T \to 0^+} \frac{H_T}{\eta_T T^{\lambda+H-\varepsilon}} \\ &\geq \limsup_{T \to 0^+} \frac{\left\| x_T^{(1)} - x_T^{(2)} \right\|}{\eta_T T^{\lambda+H-\varepsilon}} \\ &= \limsup_{T \to 0^+} \frac{\|f_T - f_0\|}{|T - 0|} \frac{1}{\eta_T T^{\alpha}} \end{aligned}$$

where $\alpha := \lambda + H - \varepsilon - 1$.

At this point we focus our attention on

$$\limsup_{T \to 0+} \frac{\|f_t - f_0\|}{|t - 0|} \geq 0 \; . \tag{3.14}$$

If (3.14) is > 0, then

$$\limsup_{T \to 0^+} \frac{\|f_T - f_0\|}{|T - 0|} \frac{1}{\eta_T T^{\alpha}} = +\infty$$

that is $1 \geq +\infty$, which is a contradiction. Thus there must exists a $T > 0$ such that two solutions agree on the time interval $[0, T]$.

If otherwise (3.14) is $= 0$, then

$$\exists \lim_{T \to 0^+} \frac{\|f_T - f_0\|}{|T - 0|} = 0.$$

This is where we have stopped the work. The obvious suggestion for future works, is to exclude (3.14) is $= 0$; we write some comments about this in the following concluding section.

3.4 Epilogue

Heuristically writing, the idea is that, the irregularity of the fractional BM affects the solutions in a way that allows to handle properly (3.14).

The problem is that we need some *local* informations about the behaviour of the solutions at 0 which does not seem trivial.

One last possibility, is expressing the solution of (3.3) as a limit of Riemann Sums, trying from this to extrapolate the needed informations.

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