**UNIVERSITÀ DEGLI STUDI DI PADOVA**

DIPARTIMENTO DI FISICA E ASTRONOMIA GALILEO GALILEI.

CORSO DI LAUREA MAGISTRALE IN FISICA

# *Moduli Spaces of $\mathcal{N} = 4$, $d = 3$ Quiver Gauge Theories and Mirror Symmetry*

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$16^{th}$ of September, 2014.
Academic Year 2013-2014.

*Folks given up*
*Under the quivers and lines,*
*You do the whirlwind,*
*Don't abandon,*
*Get a handle of yourself, son.*

Architecture in Helsinki.

**Abstract**

In this thesis we study the structure of moduli spaces for $\mathcal{N} = 4$ supersymmetric quiver gauge theories in $d = 2 + 1$ spacetime dimensions, which consist of Hyperkähler cones. Such moduli spaces have two different branches, named Higgs Branch and Coulomb Branch, joined at the origin, which in turn corresponds to a superconformal fixed point of the renormalization group flow. In this thesis, the standard procedure for computing the Higgs Branch via the Hyperkähler quotient is reviewed. Furthermore, a novel approach (introduced for the first time in [17]) to compute the Coulomb Branch is explained carefully. For this class of gauge theories there exist a conjectured *Mirror Symmetry*: a duality which swaps the moduli spaces' branches of dual theories. Along the way we provide some tests of such a symmetry. Applying this procedure, some new computation of such spaces are made, and some new mirror couples are conjectured.

# Contents

# 1 Introduction

One of the most common areas of research in Physics is that of elementary particles. These particles, such as the photon, the quarks or the electron, are the basic building blocks for all things that exist: all matter, including eventually bigger and bigger composite objects as protons, atoms, molecules, eventually cells and living organism, and also all radiation, including light as the most important example.

In order to understand how elementary particles behave and interact, back in the 50's the mathematical formalism of quantum field theory was invented. The idea underlying such a class of physical theories is indeed very simple: the universe is pervaded by some elementary fields that may experience little quantum fluctuations, which we call particles. By using this formalism, one can study what happens when two given particles interact, and compute very specific experimental predictions for the probability of this interaction to happen. Studies in this area culminated in the 70's with the establishment of the Standard Model of Particle Physics (from now on SM). This latter theory explains the three ways in which the fundamental interactions between the quantum fields can take place: namely the strong nuclear force and the electro-weak force. The most important feature such a Model is the role of symmetries within it. Infact, we believe that each field in SM is subject to a particular kind of symmetry, called *gauge symmetry*, which is local: the values of the fields at different spacetime points are affected by the same symmetry in a different way. Different fields however realize the gauge symmetry in different ways.

These symmetries are realized in physics by some well defined mathematical objects called *Lie Groups*: in few words they are groups which also carry the structure of a manifold, i.e. can be thought as some kind of surfaces. The symmetries fix the way in which the particles interact, by dictating the presence of some mediators for the interactions: some special particles called *gauge bosons*, being equal in number to the dimension of the symmetry group itself, thought as a manifold.

A theory that, similarly to the Standard Model, enjoys a gauge symmetry will be called a *gauge theory* and all the quantum field theories that we will study in this master thesis are of this kind. Historically, after some technical work on the standard model, amounting to prove its renormalizability and making it anomaly free, different predictions for scattering processes were made, and the Model was experimentally tested innumerable times at the particle accelerators. The success of this theory was immediate, and culminated recently in 2012, with the discovery of the last missing particles that was needed to make the Model self-consistent: the famous Higgs Boson. However, despite all the accomplishments and the success, the Standard Model is far from being considered a complete final theory for particle physics. Recent experiments show different phenomena which are not fully explained by the Model, such as neutrino masses and flavour oscillations, or the presence of Dark Matter in the universe: some kind of matter which can not be made out of any of the particles known and studied in the Model. Most importantly, the Standard Model does not include gravity. Among these problems, there is also another one which is particularly interesting and still unsolved.

In general, in physics and mathematics, there are some problems, like the ones stated above, which are universally considered so difficult that no one expects to be able to solve them completely, at once. Nevertheless, they act as guiding lampposts in the sense that they shed light onto which direction the research work should proceed, and this is the case: the quest for a solution of this last problem implicitly guides all the following technical work of this thesis. Such a problem consists in the understanding the dynamics of quarks and gluons in the strong coupling regime. Given the importance of this problem, it is worth to spend some words in explaining it in more details, and also explaining which had been the past attempts in attacking it, and how this thesis fits in all of this.

Out of the whole symmetry group of the Standard Model, the one that has a leading role in the dynamics of quarks and gluons is called $SU(3)$, which we now recall to be the group of all $3 \times 3$ unitary matrices, whose determinant is equal to 1. The strength of the interaction of these particles can be roughly quantified by a real parameter $\alpha_s$, which is called *coupling constant*: the bigger it is, the stronger the interaction. However, this parameter is not a mere number, but rather a function of the energy scale at which the theory itself is probed via a scattering process. This means the compling $\alpha_s$ evolves with energy: it runs. The reason for this feature relies in the mathematical procedure of *renormalization* of quantum field theories. In few words, most QFT predictions give some unphysical infinite result. This clearly makes no sense, for which a probability of a process must stay limited between 0, corresponding to the certainty of the fact that the process will not happen, and 1, corresponding to the certainty of the fact that the process will happen. It makes no sense to have an infinite probability, and to cure this problem renormalization was invented. Because of this procedure, the renormalized coupling constants of a generic theory run. One finds that the way those couplings run is dictated by the beta-function ordinary differential equation

$$\frac{dg_R}{d\Lambda} = \beta(g_R(\Lambda)) \tag{1}$$

where $g_R$ is the renormalized coupling constant, and $\Lambda$ is the energy scale, a parameter appearing as an integration constant in the most used renormalization scheme, called the $\overline{MS}$. Apart from sporadic special cases[1] the beta function does not vanish identically, and therefore solving equation (1) one finds a nontrivial dependence of $g_R$ on energy. Going back to chromodynamics, the $SU(3)$ gauge theory describing the dynamics of quark, gluons, and their binding states, the one loop beta function was computed to be

$$\beta(\alpha_s) = -\frac{9}{4\pi}\alpha_s^2. \tag{2}$$

From the minus sign, we see that the coupling constant decreases with increasing energy. Such a dependence was experimentally tested in details, as it can be seen from the following picture, taken from experimental data and showing a plot of $\alpha_s$ against energy.

[1] The most important of which is the supersymmetric $\mathcal{N} = 4$, $d = 4$ Yang-Mills theory.

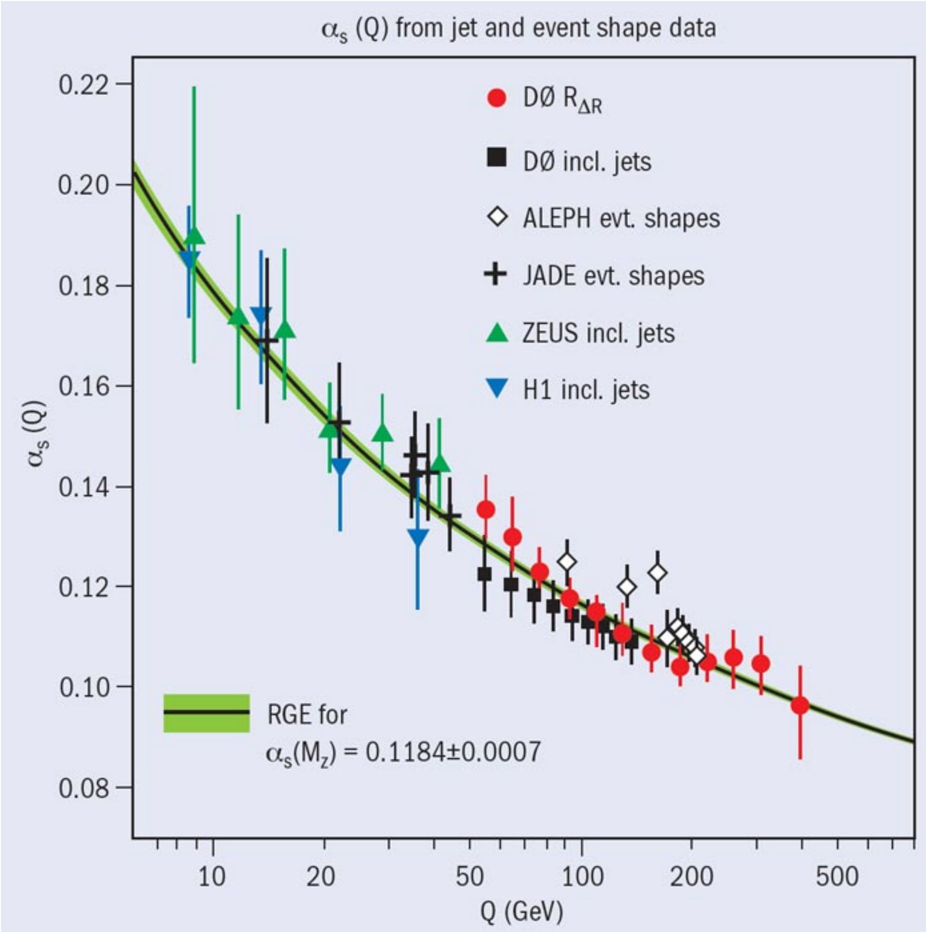



Figure 1: Running of the $\alpha_S$ coupling constant.

It can be clearly seen from this graph that at low energies the coupling constant grows larger and larger. In particular, below a given energy scale which is roughly $\Lambda_{max} \approx 300$ MeV it will grow larger than 1. We call *weak coupling regime* the energy range in which $\alpha_s < 1$ and *strong coupling regime* the energy range in which $\alpha_s > 1$.

Our understanding of quantum field theories allows us to know basically everything, within the weak coupling regime. Indeed, in this regime all the probabilities for particle interactions are known, and there exist a systematic and simple way to compute them, namely by using the Feynman diagrams expansion. In few words, this expansion consists in an infinite series of terms such that the first is proportional to $\alpha_s^2$, the second to $\alpha_s^4$, the third to $\alpha_s^6$, the n-th to $\alpha_s^{2n}$ and so on. Therefore, one can consider higher terms in this series to be just a small correction to the previous lower order terms, and stop the computation to whatever order one likes (or is able to compute). However, very few things are known in the strong coupling regime. The Feynman diagrams expansion does not work, because since now $\alpha_s^4 > \alpha_s^2$ and therefore the higher order terms in the series are not negligible when compared to the lower order terms, but in fact they are more important! So, we do not have a systematic way to proceed, in order to study the particle processes in a strong coupling limit. This fact is rather annoying since most of the interesting features of quantum chromodynamics, like the fact that quarks are confined inside protons and other heavier particles, arise exactly in the strong coupling limit. Since the beginning of the studies in QFT, this problem has prevent us to know in details the dynamics of chromodynamics, and other gauge theories, in the strong coupling regime.

An insight on how this problem could be addressed was first remarkably discovered by Sidney Coleman in [15], who found what we call *a duality* between two different theories, called the Sine-Gordon model and the Thirring model. He noted that for every process in the weak coupling regime of one of these theories, it corresponded exactly another process in the strong coupling regime of the other theory. Such a duality exchanges then non-perturbative, strongly coupled states of a theory with perturbative, weakly coupled states of the dual theory, and viceversa. The striking feature of such a duality relies on the fact that one could then employ perturbative methods like the Feynman diagrams expansion in the dual theory in order to study the dynamics of nonperturbative states in the original theory.

This idea generated a tremendous amount of excitement back in Coleman's days, since it was hoped (and it still is nowadays) that chromodynamics had a dual theory. If such dual existed, one could study the dual theory weak coupling regime, and then gain all the informations on the strong coupling one. Therefore virtually everything that can be asked about chromodynamics will be finally understood. However, although being extremely interesting, the Sine-Gordon $\simeq$ Thirring duality that Coleman discovered is merely an accademic example, rather far from being useful in describing real-world phenomena. Indeed, the Sine-Gordon model is a theory of one self-interacting spin 0 scalar field in two dimensions, and the Thirring model is a theory of just one self-interating spin 1 fermion in two dimension. The theories of phenomenological interest, however, contain both scalars, fermions and gauge fields interacting with each other in four spacetime dimesions. Eventually, the long standing goal is to study the strong coupling regime of quantum chromodynamics. The main idea is therefore to look for more complicated analogues of this duality among theories which resemble more to reality.

In particular we would like to consider a spacetime of more then two dimensions. We can not reach four in one jump, for it would be way too hard for anyone to do that, and therefore we would like to focus on the $d = 3$ case. Sadly, finding dualities in three dimension is still a strenuous problem. A mathematical result named Derrick's theorem [18] prevents the existence of dualities among theories containing only scalar and fermion fields. Therefore, to avoid Derrick's no-go, one must consider the case of gauge theories. This in principle is a good thing, since chromodynamics is indeed a $SU(3)$ gauge theory. However, the difficulty of finding strong-week dualities between different quantum field theories increases exponentially with the complexity of the gauge group and the number of dimensions of spacetime.

Following this "search for dualities paradigma", in the following we would like to simplify the problem as much as possible, and therefore we will focus on the easiest case after the one Coleman discovered: $\mathcal{N} = 4$ supersymmetric theories in $d = 2 + 1$ spacetime dimensions. One may wonder why we want to consider why do we consider theories enjoying a high amount of Supersymmetry, since Chromodynamics is not a supersymmetric theory. The reason for this is not casual, for one must realise that if a system under study enjoys many symmetries, its dynamic will be more constrained, and therefore easier to study. This fact is quite elementary but profound, and encompasses all the areas of Physics. Therefore, the assumptions of an extended supersymmetry and of a rather small number of dimensions, are just simplifying tools that we use in order to study other kind of dualities. We make no claim at all that the theories under studies have any resemblance to real world theories, they are just another (rather important) step towards the solution of a really arduous problem.

In particular in the following we study the geometric structure of the moduli spaces of such theories, defined as the set of all gauge inequivalent vacua (i.e states that minimize the potential). Let us call such set $\mathcal{M}$. One finds that $\mathcal{M}$ is composed of two branches, respectively called *The Higgs Branch* and *The Coulomb Branch*. In most of the cases, such a space is smooth, and it can be given the structure of a differentiable manifold. However, to simplify even more the problem we face, we wish to study the case of a theory in which all the gauge couplings have not just greater than one, but actually have run to infinity. In this case one finds that the theory enjoys even another symmetry, which is scale invariance. In jargon, we say that we have followed the renormalization group flow all the way down to the fixed point. One can prove that this theory is not simply scale invariant, but also invariant under a bigger set of symmetries which is called the whole conformal group. This extended supersymmetry, conformal symmetry, and low dimensions for spacetime, is really the maximum amount of simplification of the problem that we could resonably employ. However, it turns out that we are lucky, and indeed this is enough to find some more mirror couples.

The interest for such a class of gauge theories is then triple: First, it has been conjectured by Seiberg and Intriligator [26] that those superconformal field theories enjoy a duality called $3d$ *Mirror Symmetry* which exchanges the two branches of the moduli space. Since the Higgs Branch is protected against quantum correction (which it means that the moduli space for the classical theory is equal to the moduli space of the quantum theory) and on the other hand Coulomb Branch receives numerous quantum corrections (both from loop and instanton contributions) and is lifted by them, $3d$ Mirror Symmetry is a classical-quantum duality, much similar to the strong-weak duality found by Coleman in two dimensions. Secondly, there is a way to think of such a class of theories that is very fertile and allowed many new results to came to light for the first time. This way of thinking consists in picturing these theories as being realized within the framework of Type IIB string theory, as SuperYangMills theories arising on the worldvolume of certain systems of $D3$, $D5$ and $NS5$ branes ending on each other, as was shown by E. Witten and A. Hanany in [29]. This allows one to study the Mirror Symmetry duality stated above employing tecniques and ideas from the rich string theory scenario. In this thesis we will not discuss this "Brane Picture", for self consistency necessities of the thesis itself. However, it was at least fair to mention this interesting perspective. Thirdly, unexpected and not fully understood connections between the branches of the moduli spaces and other interesting physical and geometrical objects arise naturally in this context. For example, the Higgs Branch of certain theories is isomorphic (has the same properties) to the moduli space of certain instanton[2] configurations of other theories[3]. For all such reasons it is interesting to study supersymmetric gauge theories in $d = 3$, and this is what we will do in the following pages.

The outline of the thesis will be the following: In section 2 we introduce supersymmetry and review its basic aspects. This supersymmetry is a very famous conjectural symmetry exchanging integer spin bosons with semi-integer spin fermions. In section 3 we restrict our attention to the particular class of gauge theories that we will study in the following sections, since up to date mirror couples are known only within this very particular class of theories, which are called *Quiver Gauge Theories.* In section 4 we will discuss which new and interesting feature arise from the fact that we consider three-dimensional theories, instead of ordinarily four dimensional ones. In section 5 we will discuss which mathematical properties the moduli space of vacua of these theories has. Section 6 will be very technical, and here we will finally reach the heart of this thesis, explaining which is the main idea that allows us to compute the moduli spaces of vacua of quiver gauge theories, or at least gain as much information as possible about them. In section 7 we will employ the techniques of section 6 in order to develop a systematic way to attack the Higgs Branch of a quiver gauge theory. In section 8 we will do some explicit computation of those Higgs Branches, using the methods of section 6 and 7. In section 9 we will stop considering the Higgs Branch, and start turning our attention to the more difficult Coulomb Branch. In order to use the ideas of section 6, we would have first to make an aside and introduce some now concepts, as the monopole operators, and some new tecniques: here we will do that. In section 10 we will give the general prescription to compute Coulomb Branches, or at least to gain as much information as possible about them, using the methods of sections 6 and 9. In section 11 we will work out explicitly come computations of Coulomb Branches. Finally, in the last section we draw conclusions and summarize the meaning of the work we have done so far. The thesis is also equipped of an appendix, in which all the conventions about formulas, signs, and notations are carefully reported.

[2]We will not discuss instantons in this thesis, but for a sake of completeness we recall that an instanton is a "special" solution of the equation of motion of a gauge field theory, which is non-dissipative and has a finite energy.

[3]Here with moduli space of instantons we mean the space of parameters that enter in an instanton solution. This, in principle, has nothing to do with the moduli space of vacua $\mathcal{M}$ of a quantum field theory, which was described above.

# 2 Basic notions of Supersymmetry

In order to fix notation and introduce the subject, let us start by revising some basic notions on the famous topic of supersymmetry. Supersymmetry (or Susy for short) is a conjectural physical symmetry, extension of Poincarè spacetime symmetry, which “rotates fermions and bosons into each other”.

The phenomenological reasons behind Susy are numerous: it solves the hierarchy problem in the Standard Model of particle physics [21], it provides suitable candidates for Dark Matter [22] [9], and it improves the behaviour of the running of the SM coupling constants, in particular allowing the three couplings to meet at a single specific point in their running, opening thus the new possibility of a Grand Unification Theory, in which all the fundamental forces (apart from gravity) are unified. [35]. However, in our case, the interest for supersymmetry is not led by phenomenological reasons, but by the fact that the subset of quantum field theries which enjoy such a symmetry are particularly easy to study. With this we mean that supersymmetry, for our concerns, is merely a simplifying assumption.

In particular we will be eventually interested in studying $\mathcal{N} = 2$ Yang–Mills theories coupled to matter, in $d = 2 + 1$ spacetime dimension, but as for the moment, and for pedagogical reasons, we start with the much more familiar $\mathcal{N} = 1$ supersymmetry in 4 flat dimensions. The exposition will inevitably be contained and sketchy. For more details on supersymmetry, we refer the reader to standard texts on the topic, such as for example [41] [8] [10]. The interest for studying this basic case is that there is always a way to recast the $\mathcal{N} = 4$ $d = 3$ case we would like to focus on, into a particular example of this $\mathcal{N} = 1$ case that we approach in these preliminary pages.

## 2.1 The Poincaré supersymmetry algebra in $d = 3 + 1$

The supersymmetry algebra in $d = 3+1$ flat dimensions is an extension of the spacetime Poincaré algebra. It is a $\mathbb{Z}_2$ graded algebra $V = V_0 \oplus V_1$ where generators on $V_0$ are the usual generators of the Poincaré group ($P_\mu$ for the translations and $M_{\mu\nu}$ for boosts and rotations), and are called *bosonic generators*, while generators in $V_1$ are called *supercharges* and denoted with $Q^I_\alpha$. These latter ones are called *fermionic generators*. In general there can be $N$ fermionic generators. We call $N$ the *number of supercharges*. We can also define $\mathcal{N} = \dfrac{N}{d_S}$ which is the number of supercharges divided by the dimension of the smallest irreducible spinorial representation of $SO(1, d-1)$. We talk of a *minimal supersymmetric theory* if $\mathcal{N} = 1$ and of an *extended superymmetric theory* if $\mathcal{N} > 1$.

The Haag–Lopuszanski–Sohnius theorem [27] guarantees that any non trivial (interacting) quantum field theory which is also causal and possesses a positive definite energy functional, must have a (super)group of symmetries $G$ which factorizes in $G = G_{int,glob} \times G_{gauge} \times G_{Susy}$, forcing also the supercharges to transform themselves in a spin-$\frac{1}{2}$ representation of the Lorentz Algebra: i.e. the supercharges must be arranged into spinors trasforming under the Lorentz Group.

We can define the grading of an operator as the function

$$\eta : V \to \mathbb{Z}_2$$

$$\eta(O) = \begin{cases} +1 & \text{if } O \in V_0 \\ +1 & \text{if } O \in V_1 \end{cases} .$$

The bilinear, associative product which makes $V$ an algebra is given by

$$[O_a, O_b\} = O_a O_b + (-1)^{\eta(O_a)\eta(O_b)} O_b O_a.$$

In particular $[\cdot,\cdot\}$ reduces to the commutator if at least one of the two entries belongs to $V_0$.

The Susy algebra reads:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, P_\rho] &= i\eta_{\rho\mu} P_\nu - i\eta_{\rho\nu} P_\mu, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}\right), \\
[B_l, B_m] &= i f^n_{lm} B_n, \\
[P_\mu B_l] &= 0, \\
[M_{\mu\nu}, B_l] &= 0, \\
\left[P_\mu, Q^I_\alpha\right] &= 0, \\
\left[P_\mu, \bar{Q}^I_{\dot\alpha}\right] &= 0, \\
\left[M_{\mu\nu}, Q^I_\alpha\right] &= i(\sigma_{\mu\nu})_\alpha^{\ \beta} q^I_\beta, \\
\left[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}\right] &= i(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta} \bar{Q}^{I\dot\beta}, \\
\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ}, \\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI}, \\
\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta} (Z^{IJ})^*.
\end{aligned} \tag{3}$$

In the equations above, $P_\mu$ and $M_{\mu\nu}$ are the Poincaré generators. $P_\mu$ is the generator of spacetime translations, while $M_\mu$ is an antisymmetric matrix containing the three generators of spacelike rotations, and the three generators of boosts. $B_l$ are the generators of the internal symmetry group (which can be global + gauge). The epsilon symbol is the invariant tensor of $SU(2)$, defined in the appendix. The central charges $Z_{IJ}$ are antisymmetric quantities which commute with all the other generators of the algebra (i.e indeed they belong to the center of the Susy algebra.)

Three basic properties, which are common to all supersymmetric theories, follow immediately from the algebra written above.

1. The energy of any state of the Fock space is always greater or equal to zero. I.e, calling the hamiltonian operator $H$, $\forall\ |\psi\rangle, \langle\psi| H |\psi\rangle \geq 0$.
2. In the same representation of the Susy algebra there is always an equal number of bosonic and fermionic degrees of freedom.
3. All states in the same representations have the same mass, but they need not to have the same spin.

Given the Susy algebra, one can look for representations of it into the Fock space of states. This is what we will do in the following. Irreducible representations of the $\mathcal{N} = 1$ Susy algebra will be indeed the basic building blocks which we will use later, in order to build lagrangian densities for theories with an extended $\mathcal{N} = 4$ supersymmmetry in three dimensions.

## 2.2 Supermultiplets in $\mathcal{N} = 1$ and $\mathcal{N} = 2$, $d = 4$

In the following we want to look for irreducible finite dimensional representations of the Supersymmetry Algebra defined above. We are interested in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, in $d = 3 + 1$. These are the representations under which (on shell) particles will transform. We will only cover massless representations, since in all the theories considered, we study massless particles which only afterwards gain masses via a Higgs mechanism.

The way one constructs the massless supermultiplets is the following. Consider $\mathcal{N} = 1$ in $d = 4$. Now boost to a frame in which $P^\mu = (E, 0, 0, E)$. From the Susy algebra we have

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu, \tag{4}$$

which becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2E\sigma^0_{\alpha\dot{\beta}} - 2E\sigma^3_{\alpha\dot{\beta}} = 4E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

Therefore, we can define operators $a := \dfrac{Q_2}{2\sqrt{E}}$ and $a^\dagger := \dfrac{\bar{Q}_{\dot{2}}}{2\sqrt{E}}$ which satisfy the fermionic harmonic oscillator algebra

$$\begin{aligned} \{a, a^\dagger\} &= 1, \\ \{a, a\} &= 0, \\ \{a^\dagger, a^\dagger\} &= 0. \end{aligned} \tag{6}$$

We can now build the supermultiplets by using the operators defined above. Start with a state $|\Omega\rangle$ called *Clifford Vacuum*[4], defined to be the state of the supermultiplets which

[4]This "vacuum state" should not be confused with the vaccum of the theory, which in turn is the state of minimal energy. This vacuum $|\Omega\rangle$ is simply the lowest weight state of an irreducible infinitedimesional Susy representation.

is annihilated by $a$.
From $|\Omega\rangle$ we can build another state, $a^\dagger |\Omega\rangle$.
We see immediately that we cannot build another state after $a^\dagger |\Omega\rangle$ since $a^\dagger a^\dagger = 0$.

Furthermore one can show that is the state $|\Omega\rangle$ corresponds to a particle of helicity $\lambda_0$ then the state $a^\dagger |\Omega\rangle$ corresponds to a particle of helicity $\lambda_0 + \frac{1}{2}$, which means that $a^\dagger$ raises the helicity of a state by a half. Therefore all supermultiplets of $\mathcal{N} = 1$ Susy in $d = 4$ are given by a couple of particles $\{|\lambda_0\rangle, |\lambda_0 + \frac{1}{2}\rangle\}$ The most important ones for the following are:

1. Chiral Multiplet in $\mathcal{N} = 1$.
   The degrees of freedom (d.o.f.) of this supermultiplet consist in a scalar one and a spinorial one. This supermultiplet is clearly not self-conjugate under CPT.
   Therefore we complete it, adding its conjugate[5]. This gives
   $$\left(-\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).$$
   The Chiral supermultiplet is where the matter sits in a $\mathcal{N} = 1$ theory. One can see that the CPT complection of this multiplet consist in a scalar field $\varphi$ and a spinor field $\psi$.
   Calling the whole supermultiplet $\Phi$ we write
   $$\Phi = (\varphi, \psi).$$
2. Vector Multiplet in $\mathcal{N} = 1$.
   The degrees of freedom of this supermultiplet consist in a spinorial one and in a vectorial one, and we see that also this supermultiplet is not self-conjugate under CPT.
   Therefore we complete it, adding the conjugate degrees of freedom. This gives
   $$\left(-1, -\frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right).$$
   One can see that the CPT complection of this multiplet consist in a spinor field $\chi$ and a vector field $A$.
   Calling the whole supermultiplet $V$ we write
   $$V = (\chi, A).$$

By applying a similar procedure to the one detailed above, one works out the supermultiplets of $\mathcal{N} = 2$ Susy, finding

1. Hypermultiplet in $\mathcal{N} = 2$.
   The degrees of freedom of this supermultiplet consist of $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$. This supermultiplet looks self-conjugate under CPT at a first sight, but really it is not for

[5] Remember that CPT symmetry flips the sign of helicity.

a subtle reason.[6]
We complete it, adding its CPT conjugate. This gives

$$\left(-\frac{1}{2},0,0,\frac{1}{2}\right)\oplus\left(-\frac{1}{2},0,0,\frac{1}{2}\right).$$

One can recognize two $\mathcal{N}=1$ chiral multiplets together.
This supermultiplet is where the matter sits in a $\mathcal{N}=2$ theory. Calling the hypermultiplet $H$ and the two $\mathcal{N}=1$ supermultiplets which compose it $X_1$ and $X_2$, we have

$$H=(X_1,X_2).$$

2. Vector Multiplet in $\mathcal{N}=2$.
The degrees of freedom of this supermultiplet consist in a scalar one, two spinorial ones, and a vectorial one, which is $\left(0,\frac{1}{2},\frac{1}{2},1\right)$.
Again. the supermultiplet is clearly not self-conjugate under CPT.
Upon completing it, adding its CPT conjugate we find

$$\left(-1,-\frac{1}{2},-\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2},\frac{1}{2},1\right).$$

One can recognize a $\mathcal{N}=1$ vector multiplet and a $\mathcal{N}=1$ chiral multiplet together.
Calling the $\mathcal{N}=2$ vector multiplet $W$ and the two $\mathcal{N}=1$ supermultiplets which compose it $V$ and $\tilde{X}$, we have

$$Z=(\Phi,V).$$

In particular, in the following we will work with the latter two supermultiplets defined above, which will be fundamental for our purpose.

### 2.3 $\mathcal{N}=1$ Superspace and Superfields

In order to be able to write a manifestly supersymmetry lagrangian density $\mathcal{L}$ it is convenient to introduce the ($\mathcal{N}=1$) superspace formalism. Introduce a set of four new grassman variables $\theta_\alpha$, $\overline{\theta}_{\dot{\alpha}}$ and define the superspace to be the supermanifold $\mathcal{S}$ parametrized by coordinates $\{x^\mu,\theta_\alpha,\overline{\theta}_{\dot{\alpha}}\}$ (where $\mu=0,\cdots 3$ and $\alpha=1,2$, $\dot{\alpha}=\dot{1},\dot{2}$). For a rigorous definition of a supermanifold, a supergroup, and supergeometry in general we refer the reader to [5][19].

We define a superfield $X:\mathcal{S}\to\mathbb{C}$ as a generic function of superspace.
Since any square of fermionic coordinates vanish, the most general superfield has a Taylor

[6] The reason for this is the following: The way the various states are constructed out of the clifford vacuum shows that under $SU(2)$ R-symmetry the helicity 0 state behaves like a doublet while the fermionic states are singlets. Now, suppose by absurdum that the supermultiplet is self-CPT conjugate. Then the two scalars degree of freedom must both be real. However, the fundamental representation of $SU(2)$ is not real, and therefore the supermultiplet is not self-CPT conjugate.

expansion which actually terminates after a finite number of terms.
We can write a superfield as

$$Y(x,\theta_\alpha,\bar\theta_{\dot\alpha}) = f(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \\ + \theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta d(x). \tag{7}$$

We recall now the basic rules for differentiation and integration on superspace. The differential operators of partial derivative with respect of the fermionic coordinates are defined as

$$\begin{aligned} \partial_\alpha &= \frac{\partial}{\partial\theta^\alpha}, \\ \partial^\alpha &= -\epsilon^{\alpha\beta}\partial_\beta, \\ \bar\partial_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \\ \bar\partial^{\dot\alpha} &= -\epsilon^{\dot\alpha\dot\beta}\partial_\beta\bar\partial_{\dot\alpha}, \end{aligned} \tag{8}$$

where it holds that

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta, \qquad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}, \qquad \partial_\alpha\bar\theta_{\dot\beta} = 0, \qquad \bar\partial^{\dot\alpha}\theta^\beta = 0. \tag{9}$$

As for integration, we define the integral over a single fermionic variable to be a linear map which satisfies the following two conditions

$$\int d\theta = 0, \tag{10a}$$

$$\int d\theta\ \theta = 0. \tag{10b}$$

We can extend this last definition for integrals over two different fermionic variables $\theta_1$ and $\theta_2$, and therefore defining an integral over superspace, by requiring the integration measure to satisfy

$$d^2\theta = \frac{1}{2}d\theta^1\ d\theta^2 \qquad d^2\bar\theta = d\bar\theta^{\dot 2}\ d\bar\theta^{\dot 1}. \tag{11}$$

With this conventions, one can see that

$$\int d^2\theta\ \theta\theta = \int d^2\bar\theta\ \bar\theta\bar\theta = 1. \tag{12}$$

Now, one can prove that a Susy transformation acts on a generic superfield just as a translation in superspace. In particular, under a Susy transformation, the last (the highest) component of a generic superfield will transform into itself plus a total derivative.
Therefore, the spacetime integral of the last component of a superfield is manifestly supersymmetry invariant, and thus the general procedure to build susy invariant lagrangians is to use arbitrary products of superfields, and afterwards projecting only on

the highest component.

In formulae,

$$\int d^4x d^2\theta d^2\overline{\theta}\ Y(x,\theta,\overline{\theta}) \tag{13}$$

is manifestly Susy invariant.

However, those superfields we defined don't carry an irreducible representation of the susy algebra on the superfield space. Therefore, even if possible in principle, a theory built with generic superfields only would be extremely cumbersome to work with.
Therefore we look for irreducible representation onto the superfield space.
We find them by imposing restrictions (which of course suold not spoil susy invariance) on a generic superfields.
Doing this, we find two different kinds of superfields, the *Chiral Superfield* and the *Vector Superfield*.
We will see that each one of these, after being sent on shell, encodes exactly the same degrees of freedom of the chiral supermultiplet and the vector supermultiplet which we discussed before.

**The Chiral Superfield**

In order to start imposing conditions on a generic superfield, we define two differential operators called *Superspace Covariant Derivatives* in the following way:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu, \tag{14a}$$

$$\overline{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{14b}$$

and we give the following definition

**Def 1.** *A Chiral superfield* $\Phi$ *is a superfield such that*

$$\overline{D}_{\dot{\alpha}}\Phi = 0. \tag{15}$$

Defining a shifted spacetime coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ one can express a chiral superfield as

$$\Phi(y) = \varphi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y) \tag{16}$$

From which is clear that it encoloses almost the same degrees of freedom of a chiral supermultiplet: a scalar field $\varphi$, a spinor field $\psi$, but also another scalar field $F$.

One can prove that on-shell (imposing the equations of motion), $F$ does not propagate, and is fixed by the other fields of the theory.
Therefore, on-shell, the degrees of freedom carried by the chiral supermultiplet are exaclty the same of those of a chiral supermultiplet.

### The Vector Superfield

We give the following definition: A Vector superfield is a superfield which is real. This means it satisfies the condition

$$V^\dagger = V \tag{17}$$

A generic vector superfield can be written in the form

$$\begin{aligned} V(x,\theta,\overline{\theta}) = c(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi} + \theta\sigma^\mu\overline{\theta}A_\mu + \frac{i}{2}\theta\theta\left(M(x) + iN(x)\right) + \\ - \frac{i}{2}\overline{\theta\theta}\left(M(x) - iN(x)\right) + i\theta\theta\overline{\theta}\left(\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)\right) + \\ - i\overline{\theta\theta}\theta\left(\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\right) + \frac{1}{2}\left(d(x) - \frac{1}{2}\partial\partial c(x)\right). \end{aligned} \tag{18}$$

However, one can perform a supergauge transformation which brings this superfield in the so called *Wess-Zumino Gauge*, in which it takes the easier form

$$V_{WZ}(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta\lambda}(x) - i\overline{\theta\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}d(x). \tag{19}$$

We will always work with vector superfields in the Wess-Zumino gauge.
From this expression for the vector superfield it is clear that it encloses almost the same degrees of freedom of a vector supermultiplet: here we have a vector field $A_\mu$, a spinor gaugino field $\lambda$, its complex conjugate field $\bar{\lambda}$, but also a scalar field $d$.
One can prove that on-shell (imposing the equations of motion), $d$ does not propagate, and is fixed by the other fields of the theory.
Furthermore, on shell, the dynamics of $\bar{\lambda}$ is fixed by the dynamics of $\lambda$ i.e. they are no longer independent fields.
This leaves us with only a vector field and a weyl spinor on-shell, and therefore the on-shell degrees of freedom carried by the chiral supermultiplet are exactly the same of those of a chiral supermultiplet.

### 2.4 $\mathcal{N}=1$ Matter coupled Super Yang–Mills.

In this section we wish to build the most general supersymmetric lagrangian of a vector superfield interacting with a chiral and an anti-chiral superfield.
As a first thing fix a (non necessarily semisimple) gauge group $G$. Which are all the supersymmetric invariant terms which we can build out of a chiral superfield $\Phi$, an antichiral superfield $\overline{\Phi}$ and a vector superfield $V$?
Suppose that $V$ is in the adjoint representation of $G$.
Then we can define a *Superfield Strength* as

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \tag{20a}$$

$$\overline{W}_{\dot\alpha} = -\frac{1}{4}DD\bar{D}_{\dot\alpha}V \tag{20b}$$

We see immediately that $W_\alpha$ is a chiral superfield and therefore to project on its highest component is sufficient to integrate over half superspace.
One finds that

$$\mathcal{L}_{YM} = \int d^2\theta \frac{1}{32\pi} Im\left(\tau \int d^2\theta \ \mathrm{tr} W^\alpha W_\alpha\right) \tag{21}$$

The most general $\mathcal{N}=1$ supersymmetric lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{FI} + \mathcal{L}_{matter} =$$

$$= \frac{1}{32\pi} Im\left(\tau \int d^2\theta \ \mathrm{tr} W^\alpha W_\alpha\right) + \tag{22}$$
$$+ 2g\sum_A \xi_A \int d^2\theta d^2\bar\theta V^A + \int d^2\theta d^2\bar\theta \bar\Phi e^{2gV}\Psi + \int d^2\theta W(\Psi) + \int d^2\bar\theta \bar{W}(\bar\Psi)$$

where

- $\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}$ is the complex coupling constant. Its real part is the topological theta angle, and the immaginary part is the stangard YangMills gauge coupling.
- $\mathcal{W}$ is the superpotential, a holomorphic function of its arguments.
- $W_\alpha$ is the superfield strenght defined above and trasforming in the adjoint representation of $G$.
- $\xi_i \quad i = i...N_{U(1)}$ are the Fayet-Iliopulos Parameters. There can be one for every $U(1)$ factor of the gauge group.

## 2.5 $\mathcal{N}=2$ Matter coupled Super Yang-Mills

Although we want to build a $\mathcal{N}=2$ superinvariant lagrangian, we stick with the $\mathcal{N}=1$ superspace formalism, where only $\mathcal{N}=1$ supersymmetry is manifest.
The reason for this seemingly inconvenient choice relies on the fact that a construction of an $\mathcal{N}=2$ superspace, called generally the *Harmonic Superspace* (see [25]) is not yet well known.
Therefore in writing a lagrangian we must use the $\mathcal{N}=2$ supermultiplets decomposed into $\mathcal{N}=1$ ones, and all the restrictions coming from the fact that now the $R$-symmetry is $U(2)\simeq U(1)\times SU(2)$.
For simplicity, let us build the lagrangian for a theory of a single hypermultiplet $H$ (and its conjugate) interacting with a gauge superfield $Z$. Recall that bot the hypermultilet and the vector supermultiplet split into a direct sum of $\mathcal{N}=1$ multiplets, namely $H=(X_1,X_2)$ and $Z=(\Phi,V)$.
The $R$-symmetry constraints strongly the form of the superpotential term $\mathcal{W}$. This indeed can not contain terms composed solely of $X_1$ and $X_2$. I.e. a term such as $X_1X_1X_2$, would break explicitly $\mathcal{N}=2$ supersymmetry.
Therefore, the superpotential term is basically fixed to be of the form $X_1\Phi X_2$.
Furthermore, picking a canonical Kähler potential, in order to avoid theories containing more than two derivatives in the equation of motion of the fields[7], one has finds that the most general $\mathcal{N}=2$ supersymmetric lagrangian is given by

$$\begin{aligned} L &= L^{\mathcal{N}=2}_{SYM}+L^{\mathcal{N}=2}_{matter}+L^{\mathcal{N}=2}_{FI}= \\ &=\frac{1}{32\pi}Im\left(\tau\int d^2\theta\ \mathrm{tr}W^\alpha W_\alpha\right)+\int d^2\theta d^2\bar\theta\ \mathrm{tr}\bar\Phi e^{2gV}\Phi+ \\ &+\int d^2\theta d^2\overline{\theta}\left(\overline{X_1}e^{2gV}X_1+\overline{X}_2e^{2gV}X_2\right)+\int d^2\theta\sqrt{2}X_1\Phi X_2+h.c+ \\ &+2g\sum_A\xi^A\int d^2\theta d^2\bar\theta V^A \end{aligned} \tag{23}$$

Notice that a $\mathcal{N}=2$ pure Super Yang–Mills lagrangian is just a special kind of a $\mathcal{N}=1$ lagrangian.
A few comments are due at this stage.

- As said before, $\mathcal{W}$ is now fixed from the assignment of the gauge group and the representations in which the chral fields $X_{1,\ i}$ and $X_{2,\ i}$. This will be crucial in the following.
- $\xi_i\quad i=i...N_{U(1)}$ are the Fayet-Iliopulos Parameters. Again there can be one for every $U(1)$ factor of the gauge group.
- In principle one could also add mass terms of the form $mX_1X_2$ but we would not do that.

[7]In principle one could relax this condition, but we would not do it here, and stick with a canonical Kähler term

### 2.6 Dimensional reduction. $\mathcal{N}=2,\ d=4\ \mapsto \mathcal{N}=4,\ d=3$

In the introduction of this thesis we stated that in the end we would like to work on $\mathcal{N}=4$ theories defined on a $d=2+1$ dimensional Minkowski spacetime. However, up to now we have only considered the case of supersymmetry in $d=3+1$ dimensions. We will now solve such a problem.
The way to get down from a higher dimensional theory to a lower dimensional one is called *dimensional reduction.*
Recall that $\mathcal{N}=\dfrac{N}{d_s}$. This means that for $\mathcal{N}=2$ and $d=4$, given that the real dimension of the minimal spinor for this dimension is $d_s=4$, then we have a theory of 8 supercharges.
But we immediately notice that also for $\mathcal{N}=4$ and $d=3$ we have 8 supercharges, since in that case the real dimension of the minimal spinorial representation of $\mathfrak{so}(3)$ is 4.
Given that these two assignments of $D$ and $\mathcal{N}$ correspond to the same number of supercharges, the $V_1$ part of our superalgebra must be the same in the two cases, and the only possible differences arise from the $V_0$ part, which is the usual Poincaré Group.
The problem is thus simpler then expected: it is sufficient to find the branching rules for the Lorentz group, as one reduces the dimensions.
As usual we work with the complexification of their algebras:

$$\mathfrak{so}(3)_{\mathbb{C}} \hookrightarrow \mathfrak{so}(4)_{\mathbb{C}}. \tag{24}$$

Here and in the following, the highest weight convention is used in order to denote representations of Lie Algebras. Furthermore, in the case of $\mathfrak{so}(4)_{\mathbb{C}}$ we have the accidental isomorphism $\mathfrak{so}(4)_{\mathbb{C}} \simeq \mathfrak{so}(2)_{\mathbb{C}} \times \mathfrak{so}(2)_{\mathbb{C}}$, and it is convenient to use a double weight notation. We refer the reader to the appendix, where the highest weight notation and such a convention is defined. We denote with $[1,0]_4$ and $[0,1]_4$ are the two spinorial representations of $\mathfrak{so}(4)_{\mathcal{C}}$ and $[1,1]_4$ is the vectorial.
On the other hand $[n]$ is the representation of dimension $n+1$ of $\mathfrak{su}(3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}}$.
Finally $[0,0]_4$ and $[0]$ denote the trivial representations of the two algebras.
The branching rules are known to be:

$$[0,0]_{\mathfrak{su}(4)} \mapsto [0]_{\mathfrak{su}(2)}, \tag{25a}$$
$$[1,0]_{\mathfrak{su}(4)} \mapsto [1]_{\mathfrak{su}(2)}, \tag{25b}$$
$$[0,1]_{\mathfrak{su}(4)} \mapsto [1]_{\mathfrak{su}(2)}, \tag{25c}$$
$$[1,1]_{\mathfrak{su}(4)} \mapsto [2]_{\mathfrak{su}(2)} + [0]_{\mathfrak{su}(2)}. \tag{25d}$$

Therefore the field components of the supermultiplet decompose as follows:

- $\psi_4 \mapsto \psi_3$: the spinor field in $d=4$ dimension goes to the spinor field in $d=3$.
- $\overline{\psi}_4 \mapsto \psi_3$: also the spinor of the other chirality goes to the spinor field in $d=3$
- $A_\mu(x) = A_0(x) + A_i(x)$: the vector field goes to a scalar field and a threedimensional vector.

## 2.7 Moduli Spaces

Consider a generic classical field theory. Given the Lagrangian density $\mathcal{L}$, one can define the energy momentum tensor $T^{\mu\nu}$ as the Noether current associated with translational invariance of the theory. The 00 component of such a tensor is the energy density, and is an explicit function of the fields. One can define $E[\Phi] = \int d^3x T^{00}[\Phi]$ to be the energy functional. This map takes a field configuration $\Phi$ (i.e an assignment of a specific functional form for all the fields of the theory) and gives a real number $E[\Phi]$, which is the energy associated with that particular field configuration. We can define an equivalence relation on the set of all vacua $\mathcal{V}$ in the following way: two vacua $\Phi_1$ and $\Phi_2$ are equivalent if there exists a gauge transformation which sends $\Phi_1$ into $\Phi_2$. The moduli space of a generic classical field theory is therefore defined to be the set $\mathcal{M}$ of all inequivalent states of minimal energy. Such states are called *vacua*.

After quantization, the situation is quite different from a mathematical prospective, although the physical interpretation is the same. Now $E[\Phi] = \int d^3x T^{00}[\Phi]$ is a functional which takes an operator $\Phi$ and gives another operator $E[\Phi]$ acting on the Fock space of states of the quantum field theory. Therefore, we say that a state is a vacuum state is the expectation value of $E[\Phi]$ on such state is minimal.

Since the kinetic terms in the Hamiltonian are quadratic in the derivatives of the fields, in a vacuum field configuration, all the fields must be constants over spacetime. This is $\partial_\mu \phi_{vac} = 0 \ \forall \phi$. Furthermore, Lorentz invariance of the vacuum forces all fields to be not only constant, but identically zero in any vacuum, apart from the scalar fields. Therefore, only scalar fields can assume non-vanishing vacuum expectation values. Furthermore, such scalar vevs provide coordinate on the moduli space, turning it into a differentiable manifold.

**An explicit example.**

Let us give now an explicit example of a computation of a classical moduli space. Consider a $\mathcal{N} = 1$ theory in $4d$. Suppose that the theory is only a theory of matter: i.e. no vector superfields are present, nor a gauge group. Suppose there are three chiral superfields $X, Y, Z$, take the Kähler potential to be canonical and that the superpotential $W(X, Y, Z)$ given by $W(X, Y, Z) = 0$.

The lagrangian for this theory is given by

$$\begin{aligned} \mathcal{L} &= \int d^2\theta d^2\bar{\theta}\ K(X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}) + \int d^2\theta\ W(X, Y, Z) + h.c. = \\ &= \int d^2\theta d^2\bar{\theta}\left(\bar{X}X + \bar{Y}Y + \bar{Z}Z\right) + \int d^2\theta\ XYZ + h.c. \end{aligned} \tag{26}$$

Let us now look for the scalar potential.

$$\begin{aligned} V(x, y, z) &= \left|\frac{\partial W}{\partial X}\right|^2_{X\to x} + \left|\frac{\partial W}{\partial Y}\right|^2_{Y\to y} + \left|\frac{\partial W}{\partial Z}\right|^2_{Z\to z} = \\ &= y^2z^2 + x^2z^2 + y^2z^2. \end{aligned} \tag{27}$$

It is immediate to see that $V(x, y, z) \geq \ \forall x, y, z$ and that its minimum value is obtained when $V = 0$.

$$V(x, y, z) = 0 \implies \begin{cases} yz = 0, \\ xz = 0, \\ yx = 0. \end{cases} \tag{28}$$

However we see that this system is overdetermined, which means that the solution is not unique. There are indeed three different regions of solutions, where the scalar fields vanish in pairs:

$$\langle x \rangle \neq 0 \ y = z = 0, \tag{29a}$$
$$\langle y \rangle \neq 0 \ x = z = 0, \tag{29b}$$
$$\langle z \rangle \neq 0 \ x = y = 0. \tag{29c}$$

Pictorically we can represent this as three different algebraic varieties which meet at a single point.

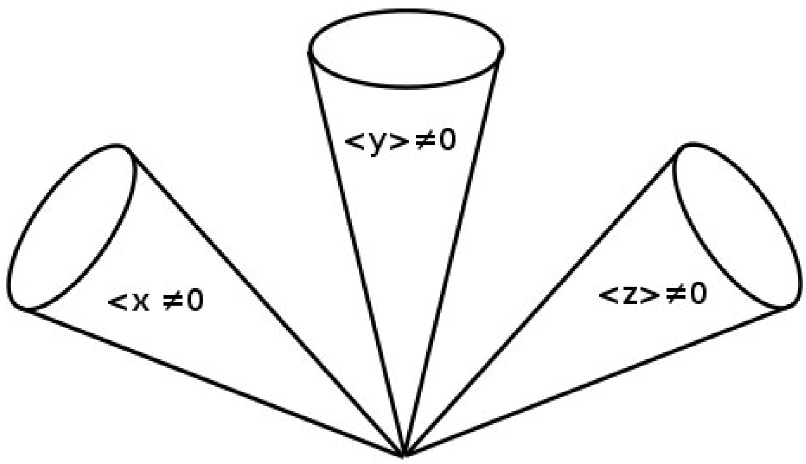



Figure 2: A pictorial representation of the moduli space for the $XYZ$ model.

This very simple model exhibits two peculiar aspects that will be encountered again for other spaces: as a first thing this moduli space is composed of pieces that join at the origin. Furthermore, it is not strictly a manifold, for which it has a singularity. These two elements will be recurring in the following, for which are common to all the moduli spaces that we will encounter in this thesis.

## 3 Quiver Gauge Theories

In this section we will define a class of graphs called *quivers*. Those graphs are directed graphs, in which arrows starting and ending at the same node are also allowed. From a physical point of view, quivers are extremely interesting since they provide a very compact way for writing a whole Super Yang–Mills lagrangian density. Let us start with giving a formal definition of a quiver graph. Subsequently we will explain the rule to associate a lagrangian to a quiver. In order to avoid confusion, we will then split the discussion in two parts: one for a subset of quivers which we will call $\mathcal{N} = 1$ quivers, and one for another subset which we will call $\mathcal{N} = 2$ quivers.

**Def 2.** *A quiver* $\Gamma$ *is a mathematical object which consists of:*

- *The set* $V$ *of vertices (or nodes) of* $\Gamma$*.*
- *The set* $E$ *of edges (or arrows) of* $\Gamma$*.*
- *A function:* $s : E \to V$ *giving the source of the arrow, and another function,* $t : E \to V$ *giving the target of the arrow.*

An example of a quiver is the following:

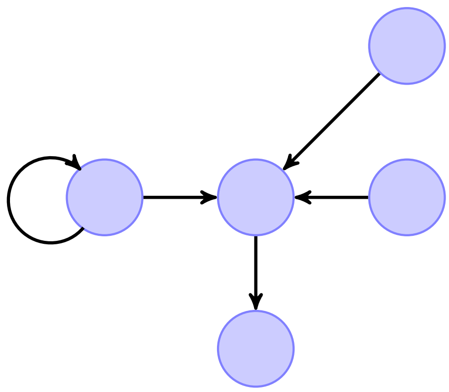


Figure 3: A generic quiver graph.

There is also a second, more abstract definition of a quiver, which may be interesting to the more mathematically oriented reader.

We define the *Walking Quiver* as the category $\mathcal{Q}$ consisting of two objects $\{E, V\}$ and four morphisms: the identities of $V$ and $E$, namely $id_v$ and $id_E$, and two more morphisms $s : E \to V$ and $t : E \to V$.

A quiver graph is then simply a covariant functor $\Gamma : \mathcal{Q} \to \mathbf{Set}$, from the Walking Quiver category to the category of sets and functions between sets.

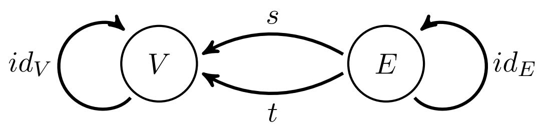



Figure 4: Pictorical representation of the Walking Quiver category.

### 3.1 $\mathcal{N} = 1$ quivers

Given those mathematical definitions, in the following we explain how quivers turn useful in physics, giving explicitly the rule to *read off* the lagrangian from a quiver diagram. We will also support this rule by showing a couple of examples. However before giving this rule, we need to enlarge the definition given above, allowing also for a second type of node.

Let us introduce another set $\tilde{V}$ of vertices, which we will represent pictorically with squares. Arrows between any kind of nodes are allowed: namely an arrow $e$ can have as a source any element $v \in V$ or $\tilde{v} \in \tilde{V}$ and as a target any element $w \in V$ or $\tilde{w} \in \tilde{W}$. We will call such a quiver a $\mathcal{N} = 1$ quiver. For example, the graph showed in the following picture is a $\mathcal{N} = 1$ quiver.

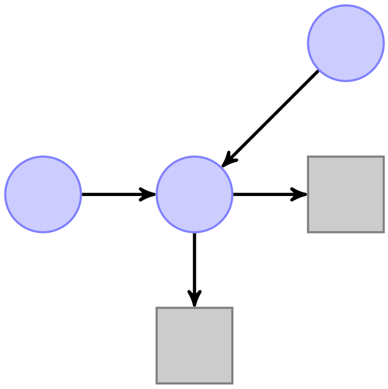


Figure 5: A generic $\mathcal{N} = 1$ quiver.

The rule to read off the gauge group and the matter content of a Super Yang–Mills theory from a quiver graph is the following. Suppose the cardinality of the set $V$ is given by $|V| = n$, and the cardinality of the set of arrows is given by $|E| = m$.

- Each node $v_i \in V$ $i = 1 \ldots n$ of the quiver diagram corresponds to a factor $G_i$ of the gauge group $G = G_1 \times G_2 \times \ldots G_n$.
- Each arrow $e_i \in E$ $i = i \ldots m$ corresponds to a $\mathcal{N} = 1$ chiral superfield transforming in the fundamental representation of the target of $e_i$ and the antifuntamental[8]

[8]Sometimes known as the conjugate fundamental.

representation of the source of $e_1$.

Note that these rules do not fix the lagrangian completely. One needs to specify the superpotential, the mass terms and the Fayet-Iliopuos terms by hand, since they cannot be read off the quiver.
In our case we will always assume, unless explicitly stated otherwise, that

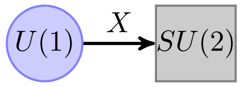
1. The Kähler potential is canonical.

2. Mass terms are zero.

3. Fayet-Iliopulos' terms are zero.

Therefore, limiting ourself to the class of lagrangians which satisfy the three requirements stated above, a generic $\mathcal{N} = 1$ quiver fixes the Lagrangian up to the choice of the superpotential.

A few examples will show how this rule works. Consider first the following quiver graph.




$$W = 0$$

Figure 6: Quiver graph for electrodynamics with 2 flavours.

This quiver correspond to the Super QED lagrangian with two flavours and zero superpotential. The matter content is explicitly stated in the following table:

| Superfield | Rep of $U(1)$ | Rep of $SU(2)$ |
|---|---|---|
| $X$ | $q^{-1}$ | $[1]_2$ |

Table 1: Matter content for the $\mathcal{N} = 1$ QED with 2 flavours.

In the $\mathcal{N} = 1$ superspace formalism, the lagrangian is given by

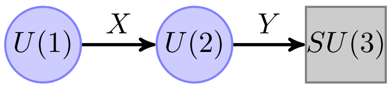
$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{matter} = \frac{1}{32\pi} Im \left( \tau \int d^2\theta tr W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} \bar{X}^a e^{2gV} X_a. \qquad (30)$$

Expanding this into the usual spacetime notation, and going on shell (substituting the auxiliary fields in terms of their equations of motion) we quickly find that the quiver above is associated to the lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^{\mu}D_{\mu}\bar{\lambda} + \sum_{j=1}^{2}\left(\overline{D_{\mu}\varphi_j}D^{\mu}\varphi^{j} - i\overline{\psi_j}\sigma^{\mu}D_{\mu}\psi_j + i\sqrt{2}e\bar{\varphi}_j\lambda\psi - i\sqrt{2}p\bar{s}i_j\bar{\lambda}\varphi_j\right), \tag{31}$$

Where $A_\mu$ is the usual gauge field, $D_{mu} = \partial_\mu - ieA_\mu$ is the usual $U(1)$ covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength associated to the gauge field $A_\mu$. The other ordinary fields appearing are $\psi_1$ and $\psi_2$, the spinor fields associated to the massless electron and the muon respectively. Furthermore, we see that there are obviously all the superpartners of the fields stated above: the photino $\lambda$, a fermion superpartner of $A_\mu$, and the selectron $\varphi_1$ and the smuon $\varphi_2$, the two scalar superpartners of $\psi_1$ and $\psi_2$. Also, from this lagrangian one can immediately see manifest the global $SU(2)$ symmetry which rotates the fermions.

Another example of a $\mathcal{N} = 1$ is the one corresponding to the $(1) - (2) - [3]$ theory, written below.




$$W = 0$$

Figure 7: Quiver graph for $(1) - (2) - [3]$ theory.

### 3.2 $\mathcal{N} = 2$ quivers

We can consider a $\mathcal{N} = 2$ lagrangian just as a special case of a $\mathcal{N} = 1$ lagrangian. Therefore, a $\mathcal{N} = 2$ quiver is just a special type of a $\mathcal{N} = 1$ quiver. We define a $\mathcal{N} = 2$ quiver as a quiver such that

1. For each arrow $X_i \in E$ having as a source $a_i \in V$ and as a target $b_i \in V$, there is also an arrow $\tilde{X}_i \in E$ having as a source $b_i \in V$ and as a target $a_i \in V$.

2. For each node $v_i \in V$ there is an arrow $\Phi_i \in E$ having as a source and as a target $v_i \in V$.

For example, the quiver in the following picture is a $\mathcal{N} = 2$ quiver.

The rule for associating a lagrangian to the quiver is the same as the one given above, for $\mathcal{N} = 1$ quivers.

### 3.3 A shorthand notation for $\mathcal{N} = 2$ quivers

In order to avoid writing numerous lines in a single quiver, we use a shorthand notation to write down $\mathcal{N} = 2$ quivers. The prescription is the following:

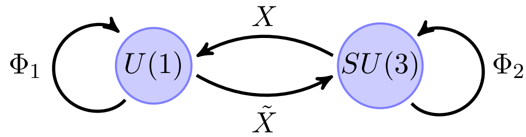



Figure 8: An example of a $\mathcal{N} = 2$ quiver.

- Instead of two opposite directed arrows between two nodes, draw just an unoriented line between the two nodes.
- For each node, avoid writing down the line that starts and end on that node.

For example, the quiver

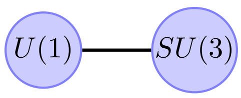



Figure 9: Shorthand notation for a $\mathcal{N} = 2$ quiver.

# 4 Features of gauge theories in $d = 3$

In this section we will carefully explain all the features that belong to a generic $\mathcal{N} = 4$ Yang–Mills gauge theory in $d = 2 + 1$ spacetime dimensions. As a first thing we will see that due to the small number of dimensions, there exist a new type of symmetries which are called *Hidden Symmmetries.* These symmetries are topological in nature: they depend explicitly on the dimension of spacetime, and are of fundamental importance for the following developments of this thesis.

Indeed, in the following sections we will look for operators which are non trivially charged under these hidden symmetries, for which they will play a leading role in the study of the Coulomb Branch. Subsequently, we will discuss the dualization of the photon, which is the fact that in three dimension a gauge field is dual to a scalar field, and one could in principle replace all gauge fields with scalars and obtain a completely equivalent physical theory. Finally, we will discuss in more details the structure of the moduli space for $\mathcal{N} = 4$ supersymmetric theories in $d = 3$, in particular, we will .

## 4.1 Hidden Symmetries

Suppose that we are studying the quantum electrodynamics in three dimensions: the gauge group is $U(1)$. The field strength 2-form $F$ satisfies both the equations of motion

$$dF = 0, \tag{32}$$

and the Bianchi identity

$$d^*F = 0. \tag{33}$$

Where the $*$ operation is the Hodge dual of $k$-forms.

In $d = 3$, taking a coordinate chart, the Hodge dual of $F^{\mu\nu}$ is

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\rho\sigma}, \tag{34}$$

and therefore a current. This current is conserved in virtue of Bianchi identity, giving

$$dJ = 0. \tag{35}$$

This implies that the physical theory under study enjoys a symmetry not explicitly readable from the lagrangian. This current can not be thought of as a Noether current arising from imposing a vanishing variation of the action $S = \int d^3x\, \mathcal{L}$ with respect of an action of a Lie group $G$, for which the fields which carry a non-zero charge associated with this symmetry are *not* explicitly present in the lagrangian.

Now consider the more interesting case of a gauge group $G$ which is not semisimple, but actually contains an $U(1)$ factor. Also in this case there exists a topologically conserved current, in three dimensions:

$$J^\mu = \frac{1}{12\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}F_{\nu\lambda}. \tag{36}$$

Furthermore, if the gauge group contains more than a $U(1)$ factor, then there will be as many currents as the number of $U(1)$ factors. Because of the fact that this current does not arise from an explicit invariance of the lagrangian, we call such a symmetry an *Hidden Symmetry.* It is also called a *Topological Symmetry* for the fact that it depends explicitly on the fact we are in $d = 3$. Also, it is called an ANO symmetry, because the states of the Fock space which carry a non-zero topological charge, in the Higgs Phase, are a certain class of solitons called Abrikosov–Nielsen–Olsen vortices.

In one of the following chapters, we will define a certain class of operators which carry a non-zero ANO charge, and that in the Higgs Phase create such ANO vortex states. Those operators are called *Monopole Operators*, but for the moment we will postpone their discussion, and focus our attention on other peculiar aspects of the dynamics of $\mathcal{N} = 4$ and $d = 3$.

## 4.2 The dualization of the photon

As we have seen from the branching rules of the previous section, in three dimensions the photon has only one polarization. Therefore, as firstly pointed out by Polyakov in [37], the action of a free photon can be explicitly written in terms of a free scalar field, which is called *The Dual Photon*, and usually denoted with $\gamma$. How does one proceed in defining such a dual photon?

Consider the equation of motion of $F$, namely $d^*F = 0$, and regard it as a Bianchi identity for the 1-form $J =^* F$, the hodge dual of $F$, defined above. Notice that $J$ is a closed form. We use this equation to infer that, due to Poincarè Lemma, in a contractible[9] topological space such as $\mathbb{R}$, $J$ is also an exact form, and therefore it exists a 0-form $\gamma$ such that $d\gamma = J$. Therefore, we see that the Bianchi identity and the equation of motion are swapped, when one interprets them for the photon $A_i$ and for the dual photon $\gamma$. In particular now, the bianchi identity for $F$, which is $dF = 0$, will be the equation of motion $d^*J = 0$ which is $d^*d\gamma = 0$, written in terms of the dual photon $\gamma$.

Notice also that under a gauge transformation of $A_i$, the dual photon does not change, therefore the $U(1)$ gauge group acts trivially on the dual photon $\gamma$. Let us now go back to the topological current defined above. When written in terms of the dual photon, this is

$$J_\mu = \frac{1}{2\pi}\partial_\mu \gamma \tag{37}$$

From this, one sees immediately that this current is generated by translation on the target space of $\varphi$.

$$\gamma(x) \mapsto \gamma(x) + \alpha \tag{38}$$

[9]Recall that a topological space $(X, \tau)$ is called contractible if it has the same homotopy type of a point, or equivalently if all its homotopy groups are trivial $\pi_k(X) = \{e\}\ \forall k \in \mathbb{N}$

### 4.3 Structure of the Moduli Space

We want to discuss here some of the main properties that moduli spaces of vacua have, for theories with a different amount of supersymmetry.

As usual, we start considering the simplest $\mathcal{N}=1$ case in four spacetime dimensions. In a generic $\mathcal{N}=1$ supersymmetric theory, the moduli space of vacua is a Kähler manifold. This is true since the lagrangian term for a nonlinear sigma-model can be written, after sending the auxiliary fields $F_i$ on shell, is

$$\begin{aligned}&\int d^2\theta d^2\bar{\theta} K(\Phi,\bar{\Phi}) = \\ &= K_i^j\left(\partial_\mu\varphi_i\partial^\mu\bar{\varphi}^j + \frac{i}{2}D_\mu\psi_i\sigma^\mu\bar{\psi}^j - \frac{i}{2}\psi^i\sigma^\mu D_\mu\bar{\psi}_j)\right) - \left(K^{-1}\right)_j^i W_i W^j = \\ &= -\frac{1}{2}\left(W_{ij} - \Gamma_{ij}^k W_k\right)\psi^i\psi^j - \frac{1}{2}\left(W^{ij} - \Gamma_k^{ij} W^k\right)\bar{\psi}_i\bar{\psi}_j + \frac{1}{4}R_{ij}^{kl}\psi^i\psi^j\bar{\psi}_k\bar{\psi}_l\end{aligned}, \tag{39}$$

where

$$K_i^j = \frac{\partial^2}{\partial\varphi^i\partial\bar{\varphi}_j}K(\varphi,\bar{\varphi}) \tag{40}$$

is the called the Kähler metric, a symmetric second rank hermitean tensor locally built out of the partial derivatives of the Kähler potential $K(\Phi,\bar{\Phi})$ with respect to the scalar fields of the theory, which by definition of a sigma model are the coordinates on the target space. The remaining quantities appearing above are the Christoffel symbols $\Gamma_{ij}^k$ and the Riemann curvature tensor $R_{jkl}^i$ associated with this metric. Therefore, all the quantities of interest can be defined as geometrical objects which take values on the set of scalar fields of the theory, which is on the other hand the moduli space.

In particular, the moduli space $\mathcal{M}$ is a complex manifold which admits a locally hermitean metric $K_{ij}$ which satisfies certain specific properties. In the following section we will define rigorously which are these properties. However, we refrain from doing that at this moment, but we anticipate that such a geometric object is what we will call a *Kähler manifold.*

Let us now turn to the more restricted $\mathcal{N}=2$ case, in $d=4$. In this case, the Kähler potential is not simply a real function of $\Phi$ and $\bar{\Phi}$, but can actually be written in terms of a single holomorphic function $\mathcal{F}(\Phi)$, which we call *The Prepotential.* Indeed it holds that

$$K(\Phi,\bar{\Phi}) = -\frac{i}{32\pi}\bar{\Phi}_a\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^a} + \frac{i}{32\pi}\frac{\partial\overline{\mathcal{F}}(\Phi)}{\partial\bar{\Phi}_a}\Phi^a \tag{41}$$

A Kähler manifold which enjoys a Kähler potential that can be derived from a prepotential is called *Special Kähler Manifold.* Notice that, quite trivially, a Special Kähler manifolds are Kähler manifolds, but the other implication is not true.

However, in the $\mathcal{N}=2$ case this is not the end of the story, since one can add different hypermultiplets to this sigma-model. Each hypermultiplet contains two complex scalars. The result is that the moduli space is given in few words by an extension of the Kähler case, in which the manifold has three complex structures all of them compatible with the hermitean metric. Again, in the following section we will be more precise in what we mean with this statement, but we anticipate now that this is what we will call an Hyperkähler manifold.

The existence of two different sets of scalars, the ones in the hypermultiplets, and the ones in the $\mathcal{N} = 2$ vector supermultiplets in $d = 4$, implies that the moduli space of vacua is composed of two different "parts" (as we will explain later, they are two different irreducible branches of a reducible algebraic variety), which we call *The Higgs Branch*, in case the hypermultiplets' scalars assume vev, and the *Coulomb Branch*, in case the vector supermultiplets' scalars assume vev.

$$\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H \tag{42}$$

Here, the Coulomb branch $\mathcal{M}_C$ is a Special Kähler manifold, as discussed before, while the Higgs branch $\mathcal{M}_H$ is a Hyperkähler manifold.

Now, performing a dimensional reduction in order to go down to the $\mathcal{N} = 4$ case in $d = 2 + 1$, one finds that, because of the branching rules of the previous section and of the dualization of the photon discussed above, one has 4 real scalars both for each hypermultiplet and each vector supermultiplet.
In the geometry of the problem this is reflected promoting the Special Kähler manifold to another Hyperkähler maniofold.
Therefore, in the end, the moduli space of the theories of our interest is given by

$$\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H \tag{43}$$

where now both the Higgs and the Coulomb branches are HyperKähler manifolds.

Motivated by these physical reasonings, in the following section we will define in a mathematically more rigorous way what such Hyperkähler manifolds really are.

# 5 Hyperkähler Geometry.

As we have seen in the previous sections, the moduli space of supersymmetric gauge theories is in general a Kähler manifold. Furthermore, the more the supersymmetry of the theory is extended, the more the dynamics is constrained, and also the more the geometric structure of the moduli space is rigid, and richer.

In this section we will define in a rigorous mathematical way what these Kähler manifolds are, and in particular we will pay attention to a subset of Kähler manifolds which are actually Hyperkähler. For more details, we refer the reader to [36].

## 5.1 Kähler manifolds

**Def 3.** *A complex differentiable manifold of dimension n is a topological manifold* $(X, \tau)$ *with an open covering of charts* $\{(U_i, \varphi_i)\}_{i \in I}$ *such that* $\forall i$ $\varphi_i$ *is a homeomorphism from* $U_i$ *to* $\mathbb{C}^n$*, and the transition functions* $\varphi_i \circ \varphi_j^{-1}$ *are holomorphic.*

On a complex manifold it exists a well defined tensor field $J$, section of $TM \otimes T^*M$ such that $J^2 = -1$. We call such a tensor field the *complex structure* of $M$. It is worth to notice that viceversa, if a real $2n$-dimensional manifold admits a tensor field $J$, section of $TM \otimes T^*M$ such that $J^2 = -1$, the manifold itself needs not to be complex (i.e. the complex atlas might fail to exist due to other topological obstructions), and is called *quasicomplex* in this case.

Now consider a riemmanian complex manifold $(M, g)$. If the metric tensor field $g$ satisfies

$$g_p(J_p X, J_p Y) = g_p(X, Y) \tag{44}$$

at each point $p \in M$ and for any $X, Y \in T_p M$, $g$ is said to be a *hermitean metric.* The couple $(M, g)$ is then called an *hermitean manifold.*

Using this hermitian metric g, we can define a two-form $\omega$ on $M$ called the Hermitian form by $\omega(v, w) = g(Jv, w)$ for all vector fields $v, w$ sections of $TM$.

We are now ready to give the following definition:

**Def 4.** *A Kähler manifold is a hermitean manifold* $(M, g)$ *whose Kähler form* $\omega$ *is closed:* $d\omega = 0$*. In this case the metric* $g$ *is called the Kähler metric of* $M$*.*

Recall that the physical interest for Kähler manifolds arises since in any $\mathcal{N} = 1$ supersymmmetric theory, the moduli space of vacua is indeed Kähler.

#### An example: $\mathbb{C}^m$ is Kähler.

Consider $M = \mathbb{C}^m = \{(z^1, z^2 ... z^m)\}$, the $m$ dimensional complex space. As a real differential manifold $\mathbb{C}^m$ is isomoprphic to $\mathbb{R}^m$, via the identification

$$z^\mu = x^\mu + iy^\mu \tag{45}$$

Regarding $\mathbb{C}^m$ as a riemannian manifold, endowed with the usual euclidean metric of $\mathbb{R}^{2m}$, namely

$$\delta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \delta\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right) = \delta_{\mu\nu}, \tag{46a}$$

$$\delta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\right) = 0. \tag{46b}$$

The action of the complex structure $J$ on the tangent plane in given by

$$J\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu} \tag{47a}$$

$$J\frac{\partial}{\partial y^\mu} = -\frac{\partial}{\partial x^\mu} \tag{47b}$$

And by using this we can find the complex metric, which action on two tangent vectors of the complexified tangent plane is

$$\delta\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = \delta\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) = 0 \tag{48a}$$

$$\delta\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) = \delta\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\right) = \frac{1}{2}\delta_{\mu\nu} \tag{48b}$$

From this we see that the Kähler form is given by

$$\Omega = \frac{i}{2}\sum_{\mu=1}^{m} dz^\mu \wedge d\bar{z}^\mu = \frac{i}{2}\sum_{\mu=1}^{m} dx^\mu \wedge dy^\mu \tag{49}$$

It is immediate to check that $d\Omega = 0$ and therefore $\mathbb{C}^m$ is Kähler. The Kähler potential is given globally by

$$\mathcal{K} = \frac{1}{2}\sum_{\mu=1}^{m} z^\mu \bar{z}^\mu \tag{50}$$

**An example: The complex projective space is Kähler**

More important than the plane, for our purposes, is the fact that the complex projective space $\mathbb{C}P^m$ is a Kähler manifold. We will prove this in the following.

The interest for this result, in connection to this thesis, relies in the fact that all the moduli spaces we will consider will be algebraic cones, and therefore algebraic projective varieties, or schemes. Now, an algebraic projective variety is a subset of $\mathbb{C}P^m$ which satisfies some properties[10]. Therefore, it enjoys the Kähler structure just because of the fact that the ambient space in which it is immersed is $\mathbb{C}P^m$, which is kähler. We then conclude that by proving that $\mathbb{C}P^m$ is kähler, what we actually prove is that all the moduli spaces that we will find later are (at least) kähler.

The proof we present in the following lines is a standard result, and in particular we borrowed it entirely from [36]. To start, take on $\mathbb{C}P^m$ a chart $(U_\alpha, \varphi_\alpha)$ of usual inhomogeneous coordinates such that $\varphi_\alpha(p) = \xi^\nu_{(\alpha)}$, with $\nu \neq \alpha$. It is convenient to define a slightly tidier notation, renaming $\xi_\alpha$ such as

$$\xi^\nu_{(\alpha)} = \zeta^\nu_\alpha \quad (\nu \leq \alpha - 1) \tag{51a}$$

$$\xi^{\nu+1}_{(\alpha)} = \zeta^\nu_\alpha \quad (\nu \geq \alpha) \tag{51b}$$

Within the chart $(U_\alpha, \varphi_\alpha)$ we can define a positive function

$$\mathcal{K}_\alpha(p) = \sum_{\nu=1}^{m} \left| \zeta^\nu_{(\alpha)(p)} \right|^2 + 1 = \sum_{\nu=1}^{m+1} \left| \frac{z^\nu}{z^\alpha} \right|^2 \tag{52}$$

If now we consider two charts $(U_\alpha, \varphi_\alpha)$ and $(U_\beta, \varphi_\beta)$, in a point $p \in U_\alpha \cap U_\beta$ the two functions $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are related by

$$\mathcal{K}_\alpha(p) = \left| \frac{z^\beta}{z^\alpha} \right| \tag{53}$$

From this it follows almost immediately that

$$\partial\bar{\partial} \log \mathcal{K}_\alpha = \partial\bar{\partial} \log \mathcal{K}_\beta \tag{54}$$

Therefore we define a closed two form $\Omega$ by

$$\Omega := i\partial\bar{\partial} \log \mathcal{K}_\alpha \tag{55}$$

This will be our Kähler form.

Now we only need to prove that there really exists an hermitean metric which has $\Omega$ as a Kähler form. The existence of such a metric, called *The Fubini–Study metric* for the complex projective space is a classical result. In local affine coordinates, the metric is given by

$$g_{i\bar{j}} = g\left(\partial_i, \bar{\partial}_j\right) = \frac{\left(1 + |\mathbf{z}|^2\right)\delta_{i\bar{j}} - \bar{z}_i z_j}{\left(1 + |\mathbf{z}|^2\right)^2} \tag{56}$$

[10] It should be a closed set in the Zariski topology of $\mathbb{C}P^m$

where $|\mathbf{z}|^2 = z_1^2 + z_2^2 + \cdots z_m^2$.

It is a straightforward computation to show that $g(X,Y) = \Omega(X,JY)$, finishing therefore the proof.

## 5.2 Hyperkähler manifolds

We wish now to discuss the Hyperkähler case, since we have seen that in the $mathcalN = 4$ $d = 3$, the two branches of the moduli space are indeed hyperkähler. An Hyperkähler manifold is nothing more than the quaternionic analogue of a Kähler manifold. Consider a Riemmannian Manifold $(M, g)$ for which exists not only one complex structure, but three of them. We call them $I,\ J,\ K \in \Gamma(T^*M \otimes T^*M)$, and they satisfy the quaternion algebra, namely

$$I^2 = J^2 = K^2 = -1 \quad IJK = -1 \tag{57}$$

For any of these three complex structures, the riemannian metric could be hermitian. Suppose it is hermitian for all three of them. Then we could build three Hermitean forms $\omega_I$, $\omega_J$, $\omega_K$ just as we did before. If all three of them are closed, the manifold is said to be HyperKaher.

**Def 5.** *An Hyperkähler manifold is a riemannian manifold $(M, g)$ equipped with three complex structures $I, J, K$ which follow the quaternion algebra, and such that the metric tensor field $g$ is hermitean with respect to all the three complex structures, and the three hermitean forms associated with $I, J, K$ are closed.*

# 6 The Chiral Ring, Algebraic Geometry and the Plethystic Program.

In this rather technical section we introduce some algebraic and combinatorial techniques that will prove themselves useful in the systematic study of the Moduli Spaces. Along the way we state the fundamental result on which this thesis relies: roughly speaking the fact that a geometrical object can be defined by the set of all "well behaved" functions over it. Indeed this is the procedure we will adopt to study the moduli spaces, shifting the problem from a geometrical approach to a (hopefully easier) algebraic approach.

## 6.1 The Chiral Ring

Following [3] and [14] we define now a special class of gauge invariant operators, which will have a leading role in all the following.

In section 2 of this thesis we have seen that a chiral superfield is defined to be a special type of superfield which is defined by

$$\overline{D}_{\dot{\alpha}} X(x, \theta, \bar{\theta}) = 0,$$

where $\overline{D}_{\dot{\alpha}}$ is the covariant superspace derivative. This implies that the lowest component of this superfield is annihilated by $\overline{\mathcal{Q}}_{\dot{\alpha}}$, namely

$$[\overline{\mathcal{Q}}_{\dot{\alpha}}, \varphi(x)] = 0.$$

A similar property also holds true for the lowest component of the chiral superfield strength $W_\alpha$, which is the gluino $\lambda_\alpha$. This fermionic field satisfies

$$\{\overline{\mathcal{Q}}_{\dot{\alpha}}, \lambda_\alpha(x)\}$$

Starting from these two examples, we give a general definition of a class of operators, which we would like to call *Chiral Operators*.

**Def 6.** *A chiral operator $O(x)$ is a* gauge invariant *operator such that it is annihilated by all the supercharges of one chirality.*

$$[Q_\alpha, O(x)\} = 0$$

Let us call the set of all chiral operators $\mathcal{C}_o$. Notice for example that the scalar field $\varphi(x)$ and the gluino $\lambda_\alpha(x)$ considered above are not chiral operators. This is because they are not gauge invariant in general: $\lambda_\alpha(x)$ sits in the same representation of the superfield $W_\alpha$ it belongs to, and that representation is the adjoint, and $\varphi(x)$ sits in a generic representation $R$ of the gauge group, which is whatever representation the matter carries (usually the fundamental representation of $G$). Therefore, in order to build gauge invariant operators, one should use more than a single field, and build combinations of traces, such as $\mathrm{tr}\lambda^2(x)$ for example.

Now that we have defined those chiral operators, we can immediately derive the important properties for the following: the v.e.v of any time ordered product of chiral operators is independent on their spacetime position.

*Proof.* Consider for instance the product of two bosonic chiral operators at different spacetime points $O_1(x)$ and $O_2(y)$ and take a derivative with respect of the spacetime coordinates $x^\mu$.

$$
\begin{aligned}
&\frac{\partial}{\partial x^\mu}\langle 0|T\left(O_1(x)O_2(y)\right)|0\rangle = \\
&= \left\langle 0|T\left(\frac{\partial}{\partial x^\mu}O_1(x)O_2(y)\right)|0\right\rangle + \delta^0_\mu\langle 0|[O_1(x),O_2(y)]|0\rangle\,\delta(x^0-y^0)
\end{aligned}
\tag{58}
$$

Now, both terms vanish separately. The first one because

$$
\begin{aligned}
\left\langle 0|\frac{\partial}{\partial x^\mu}O_1(x)O_2(y)|0\right\rangle &= -i\langle 0|[\mathcal{P}_\mu,O_1(x)]O_2(y)|0\rangle = \\
&= \frac{i}{2}\overline{\sigma}^{\dot\alpha\alpha}_\mu\langle 0|[\{\mathcal{Q}_\alpha\overline{\mathcal{Q}}_{\dot\alpha}\}O_1(x)]O_2(y)|0\rangle = \\
&= \frac{i}{2}\overline{\sigma}^{\dot\alpha\alpha}_\mu\langle 0|\{\overline{\mathcal{Q}}_{\dot\alpha}[\mathcal{Q}_\alpha,O_1(x)]\}O_2(y)|0\rangle = \\
&= \frac{i}{2}\overline{\sigma}^{\dot\alpha\alpha}_\mu\langle 0|\{\overline{\mathcal{Q}}_{\dot\alpha}[\mathcal{Q}_\alpha,O_1(x)]O_2(y)\}|0\rangle = \\
&= 0
\end{aligned}
\tag{59}
$$

where we used the SUSY algebra, the Jacobi identity and the chirality of the operators in order to bring $\overline{\mathcal{Q}}_{\dot\alpha}$ to act on the vacuum which is assumed to be supersymmetric, and therefore annihilated by the supercharges.

The second term in (58) is also zero because the equal time commutator vanishes since the same arguments used above to show that the first term vanishes apply more generally to the OPE of the two operators.

□

Therefore we have found that the v.e.v of a generic chiral operator $O(x)$ does not depend on $x$.

Suppose the vacuum is supersymmetric, then objects of the type $\overline{Q}_{\dot\alpha}, ...$ do not contribute to the expectation values. Therefore one can define an equivalence relation between chiral operators. Two chiral operators $O_1(x)$ and $O_2(x)$ are equivalent if there exist a gauge invariant operator $X_{\dot\alpha}(x)$ such that

$$
O_1(x) = O_2(x) + [\overline{Q}^{\dot\alpha}, X_{\dot\alpha}\}
$$

**Def 7.** *The chiral ring is defined to be the quotient ring of the ring of chiral operators, over its ideal defined by the equivalence relation defined above. This is*

$$
\mathcal{CR} := \mathcal{C}_o/\sim . \tag{60}
$$

From this construction, we see immediately a very important consequence: the fact that there is a map from the chiral ring, to the ring of holomorphic functions over the moduli space.

$$
\begin{aligned}
\mathcal{CR} &\to O[x] \\
O(x) &\mapsto f(z_1\cdots z_n)
\end{aligned}
\tag{61}
$$

It is immediate to see that this map is injective, which means that each chiral operator defines a different holomorphic function over the Moduli Space. However, it has not been proven yet that this map is surjective, although it is generally believed so. There might exist holomorphic functions which are not associated with any chiral operator. As a working assumption, one conjectures that this map is indeed bijective and therefore invertible, and it is also a isomorphism of rings.

$$\mathcal{CH} \simeq \mathcal{O}[X] \tag{62}$$

## 6.2 Some notions in Algebraic Geometry.

Following [30] we give now some basic definitions in algebraic geometry. For our purpose, we are interested only on the main result for which, given a commutative ring, one can always associate a geometric object to it, which is called the affine algebraic scheme modelled over that ring.

In particular, in our case, the ring we study is the chiral ring which is isomorphic to the ring of holomorphic functions over the moduli space. However, we will see that the spaces that we consider are cones, and in this particular case, the ring of holomorphic function is also isomorphic to the ring of algebraic functions over the moduli space.

It turns out, and it is indeed the main result of modern (post Groethendieck) algebraic geometry, the remarkable fact that such a scheme associated to the ring of algebraic functions over a algebraic variety is indeed the variety itself[11]. This allows us to study a geometric object: the moduli space of a supersymmetric theory, in an indirect and easier algebraic way. Our strategy consists then in the following steps:

- Start with studying the chiral ring $\mathcal{CR}$, which is a well defined physical object.
- Associate the ring $O[X]$, of holomorphic functions over $X$, to $\mathcal{CR}$.
- Associate the ring $A[X]$, of algebraic functions over $X$, to $O[X]$.
- Associate $X$ to $A[X]$, via the construction of the affine scheme over $A[X]$.

This line of reasoning can be represented pictorically in the following way

$$\mathcal{CR} \implies O[X] \implies A[X] \implies X$$

### A quick review of Commutative Algebra

To make sense of the following definitions, we need to review some basic facts of commutative algebra. Here we are only interested of getting to the result we want with as little machinery as possible. Therefore, for a more complete exposition to the subject, we refer the reader to [4].

[11] Augumented with a very special "point at infinity" which we will discuss later

Recall that given a ring $A$, a subset $\mathfrak{a}$ of $A$ is called an ideal if it is an abelian addiction subgroup of $A$, and furthermore for all $x \in \mathfrak{a}$ and forall $y$ in $A$, then $xy \in \mathfrak{a}$. Intuitively, an ideal is an absorbent subset of a ring. As an example, consider $\mathfrak{a} \subset \mathbb{Z}$ generated by the number 3. This is $(3) = \{\cdots - 6, -3, 0, 3, 6, 9, \cdots\}$ and is indeed clearly an ideal.

Furthermore, an ideal $\mathfrak{p} \subset A$ is called prime if the following property holds: Given any $x, y \in A$ such that $x \cdot y \in \mathfrak{p}$ then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Again, as an example consider $(3) = \{\cdots - 6, -3, 0, 3, 6, 9, \cdots\}$. This ideal is obviously prime, since if $xy \in (3)$ then $xy$ is a multiple of 3 and therefore either $x$ or $y$ must be multiple of 3. As a counterexample, consider $(4) = \{\cdots - 6, -3, 0, 3, 6, 9, \cdots\}$. This is clearly not prime, since for example $60 = 6 * 10 \in (4)$ but nor 6 nor 10 are in (4).

**Def 8.** *The spectrum of a ring $A$, denoted by* Spec$A$ *is the set which has as elements all the prime ideals of $A$.*

For example, take $A = \mathbb{Z}$ then $\text{Spec}\mathbb{Z} = \{(0), (p)\ \forall p \in \mathbb{Z}\}$. Also, an ideal $\mathfrak{m} \subset A$ is called maximal if it is a proper ideal[12] and the smallest ideal containing $\mathfrak{m}$ is the ring $A$ itself. Again, consider $(3) \subset \mathbb{Z}$. This is maximal. Notice that (4) is not maximal, because $(4) \subset (2)$.

In this easy case of $\mathbb{Z}$, the two notions of a maximal and a prime ideal coincide, but one should not confuse them, because in other contexts they differ. In particular, we are interested in the prime and maximal ideals of the ring $\mathbf{C}[x_1, \cdots, x_n]$ of polynomials in $n$ variables, over the field $\mathbb{C}$.

**Def 9.** *A ring $A$ is said to be local if it has just one maximal ideal.*

For example, we have just seen that $\mathbb{Z}$ is not a local ring.

### Sheaves over topological spaces

Let $(X, \tau)$ a topological space. Now we want to define a mathematical object called a *Sheaf of rings* over $X$. The main idea is that for every open set $U \in \tau$ one associates a ring $R_U$. We would like to call 'Sheaf" the map $\mathcal{R}$ which does exactly this: $\mathcal{R}$ sends $U$ to its ring $R_U$. However, this map has to satisfy a certain number of properties, i.e. one can not associate random rings to open sets, but has to do it "nicely".

**Def 10.** *Given a topological space $(X, \tau)$, a presheaf of rings over $X$ is the assignment of the following two things*

- *For every open set $U \in \tau$, a ring $\mathcal{R}(U)$ is defined,*
- *For every inclusion $V \subseteq U$ it is given a morphism of rings $\rho_{UV} : \mathcal{R}(U) \to \mathcal{R}(V)$,*

*such that the folowing conditions are satisfied:*

1. $\mathcal{R}(\emptyset) = \{0\}$,
2. *For every open set $U$, $\rho_{UU} = id$,*

[12] Remember that an ideal is proper if it is different than 0 and than the whole ring.

3. *For every inclusion of sets $W \subseteq V \subset U$, the morphism maps must satisfy $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.*

**Def 11.** *A presheaf $\mathcal{R}$ over $X$ is called a sheaf if it satisfies the following two additional conditions:*

- *Let $U$ be a open subset of $X$ and $\{V_i\}_{i \in I}$ an open cover of $U$. If $s \in \mathcal{R}(U)$ is such that $s|_{V_i} = \rho_{UV_i}(s) = 0$ for all $i \in I$, then it must be $s = 0$.*
- *Let $U$ be an open set of $X$ and $\{V_i\}_{i \in I}$ an open cover of $U$. If are given sections $s_i \in \mathcal{R}(V_i)$for all $i \in I$, and those sections are given such that for all $i \in I$ it holds $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then it must exist a unique $s \in \mathcal{R}(U)$ such that $s_i = s|_{V_i}$ for all $i \in I$.*

**Def 12.** *Consider two presheaves of rings $\mathcal{F}$ and $\mathcal{R}$ over a topological space $X$. A morphism of presheaves $\varphi : \mathcal{F} to \mathcal{R}$ is given by the assignment of a ring homomorphism $\phi(U) : \mathcal{F}(U) \to \mathcal{R}(U)$ for each open subset $U \subset X$, with the condition that for each inclusion $V \subseteq U$ of open subsets, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{R}(U) \\ {\scriptstyle \rho_{UV}} \downarrow & & \downarrow {\scriptstyle \rho'_{UV}} \\ \mathcal{G}(V) & \xrightarrow[\phi_V]{} & \mathcal{F}(V) \end{array}$$

*If the presheaf is indeed a sheaf, the same definition can be used for a morphism of sheaves.*

Sheaves are particularly important since they encode the idea of describing global properties by means of local ones. When this can be done, we have a sheaf. Otherwise we only have a presheaf.

### Ringed spaces and the Zariski Topology over *SpecA*

Now we would like to build a sheaf of rings over the very particular topological space given by the spectrum of another ring. In order to do this, we must first of all define a topology on Spec$A$, by declaring which sets of prime ideals of $A$ are open.

However, before going into the discussion of the topology of Spec$A$, we give two easy definitions which are completely empty of new mathematical content. Simply, they allow us to adopt shorter names for the same things we have been discussing so far.

**Def 13.** *A Ringed Space is a couple $(X, \mathcal{R}_X)$ where $X$ is a topological space and $\mathcal{A}$ is a sheaf of rings over $X$.*

In case the sheaf is a sheaf of commutative rings, we call the couple $(X, \mathcal{R}_X)$ a commutative ringed space.

**Def 14.** *A morphism of ringed spaced* $(X, \mathcal{R}_X)$ *to* $(Y, \mathcal{R}_Y)$ *is a pair* $(f, f^\sharp)$*, where* $f : X \to Y$ *is a continuous function and* $f^\sharp$ *is a morphism of sheaves of rings over* $Y$

$$f^\sharp : \mathcal{R}_Y \to f_* \mathcal{R}_X$$

Furthermore, by using the concept of the stalks of a sheaf, we can give the following definition.

**Def 15.** *A Locally Ringed Space is a commutative ringed space* $(X, \mathcal{R}_X)$ *such that for every* $p \in X$ *the stalk* $\mathcal{R}_{X,p}$ *is a local ring.*

Having defined a locally ringed space, now we want to define the concept of a map between two locally ringed spaces. Again, this is exactly what we expect to be, and the definition we are about to give does not add any new key concept.

**Def 16.** *An affine scheme is a locally ringed space* $(X; \mathcal{R}_X)$ *which is isomorphic, as a locally ringed space, to the spectrum of a ring* $A$*.*

Given these definitions, we would like to discuss the topology on Spec$A$. Such a topology is called the *Zariski Topology.*

Given a ring $A$, consider a generic ideal (not necessarily prime) $\mathfrak{a}$ of $A$. For each $\mathfrak{a}$ we can define a subset of Spec$A$ in the following way

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}A | \mathfrak{a} \subseteq \mathfrak{p}\}$$

Now, the following result holds true:

**Prop 1.** *1. If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals of* $A$*, then*

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$$

*2. For each family of ideals* $\{\mathfrak{a}_i\}^{i \in I}$ *of* $A$*, it holds*

$$V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$$

*3. If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals of* $A$*, then*

$$V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \leftrightarrow \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$$

*4.* $V((0)) =$ *Spec* $A$ *and* $V(A) = \emptyset$*.*

We omit the proof, referring the interested reader to [30].

From this result we can see that the subsets of Spec$A$ of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ satisfy the axioms of the closed sets of a topology. Therefore one could define a topology on Spec$A$, by declaring closed all the sets of the form $V(\mathfrak{a})$.

### 6.2.1 Schemes

We are now ready to give the most important definition of this section: the one of a scheme. In few words, the scheme modelled over a ring is a locally ringed space that is isomorphic to the spectrum of a ring (thought as a topological space equipped with a sheaf of rings.) For our physical concern, the ring we are given is the ring of algebraic functions over an unknown algebraic variety, which is the moduli space.

Before defining a scheme, we should augument the spectrum of a ring, with a sheaf of rings. We can define it in the following way: For every open set $U \in \text{Spec}A$, we can define $\mathcal{O}_X(U)$ as the set of all functions

$$s : U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$.

It can be proven that $\mathcal{O}_X$ is a sheaf of rings over $X = \text{Spec}A$, and that therefore the couple $(X, \mathcal{O}_X)$ is a locally ringed space.

We are now ready to give the following definition:

**Def 17.** *Am affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic, as a locally ringed space, to the spectrum of a ring. A morphism of affine schemes is simply a morphism of locally ringed spaces.*

Let us give an easy example of this construction, in order to shed some light and explain how it works in a easy, pedagogical case. Consider the field $k = \mathbb{C}$ and consider the ring of polynomials in one unknown, over $\mathbb{C}$, namely $A = \mathbb{C}[z]$. We first look for the spectrum of $A$. The prime ideals of $A = \mathbb{C}[z]$ are $(0)$ and all ideals of the kind $(f(z))$ where $f(z)$ is an irreducible polynomial. The points of $X = \text{Spec}A$ of the kind $(f(z))$ are closed. However, notice that $(0)$ is not closed, since its closure is $\text{Spec}A$ itself. We call this point *The generic point.* Since $\mathbb{C}$ is algebraically closed, irreducible polynomials in $\mathbb{C}[z]$ are of the kind $f(z) = z - a$ for some $a \in \mathbb{C}$. Therefore, the closed points of $\text{Spec}A$ are the prime ideals of $A$ which are of the form $(x - a)$, for all $a \in \mathbb{C}$.

Therefore, it can be easily seen that the affine scheme over $A = \mathbb{C}[z]$ is exactly the complex line $X = \mathbb{C}$, with the extra presence of the generic point.

## 6.3 The Plethystic Exponential

In the previous subsection we have seen how to associate a geometric object (the affine scheme) to a generic commutative ring $A$. In case that $A$ is the ring of algebraic functions over some unknown algebraic variety, the scheme associated to $A$ will be the algebraic variety itself.

From a physical point of view, the ring of holomorphic functions over the moduli space is given by the chiral ring. Restricting our discussion to the case of cones, i.e. projective varieties, the ring of holomorphic function coincides with the ring of algebraic function. However, it is often too hard to compute the whole chiral ring. Therefore, our approach will be more modest: we will limit ourselves to *count* the different chiral operators, and count them in a graded way.

In order to do that, we would like to define now a function that "counts" the symmetrized products of a given set of objects. The interest for such a function is obvious: since the ring of affine coordinates over an algebraic variety is a quotient of $\mathbb{C}[x_1...x, n]$, it is commutative noetherian ring, an therefore we know it must be generated by a finite number of elements, subject to a certain number of relations. This means that all elements of $A[\mathcal{M}]$ can be constructed by sums of products of the generators, and furthermore those products are symmetric in the exchange of their entries, since the ring itself is commutative. Therefore we would like to dispose of a function which generates the whole ring by knowing the generators and relations of it and, most importantly, we would like to invert this function, in order to get the generators and relation from the whole ring.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, which maps $\mathbb{R}^n \ni (t_1...t_n) \mapsto f(t_1, ..., t_n) \in \mathbb{R}$ such that $f(0, 0, ..., 0) = 0$, we define the *Plethystic Exponential* of $f$ to be the function

$$PE[f(t_1...t_n)] = \exp\left(\frac{\sum_{k=1}^{\infty} f(t_1^k, ...t_n^k)}{f}\right) \tag{63}$$

To get a feel of how the plethystic exponential of some easy function looks like, let us compute $PE[f(t)]$ where $f(t) = t$.

Given $f(t) = t$ it holds that $PE[f(t)] = \dfrac{1}{1-t}$, for one has

$$PE[t] = \exp\left(\frac{\sum_{k=1}^{\infty} t^k}{k}\right) = \exp\left(-\ln(1-t)\right) = \exp\left(\ln\left(\frac{1}{1-t}\right)\right) = \frac{1}{1-t} \tag{64}$$

Now notice that the plethystic exponential enjoys the usual sum to product property of the ordinary exponential, which is

$$PE[f(t) + g(t)] = PE[f(t)]PE[g(t)]$$

for any couple of functions $f$ and $g$ which vanish at the origin.

Now let us discuss why the plethystic exponential is exactly the "counting function" we desired, and how it proves useful. The main idea is that this function keeps track of the cardinality of the set of all symmetric momonials at generic degree. More precisely, given $n$ basic monomials, consider the set $S_{n,k}$ whose elements are all the symmetric monomials of degree $k$. In general, the coefficient of the $k$-th power of $t$ in the Taylor expansion of $PE[nt]$ gives the cardinality of $S_{(n,k)}$.

Let us clarify this point with a straightforward example. Suppose we are given a set $S_1$ of 3 base monomials to start with, and take each of these monomial to be of degree one by definition.

$$S_{(3,1)} = \{a, b, c\}$$

Then we can build the set of all symmetric monomials of degree two. This is

$$S_{(3,2)} = \{a^2, b^2, c^2, ab, ac, bc\}$$

The cardinality of $S_2$ is 6. The set of all symmetric monomials of degree three, on the other hand, is given by

$$S_{(3,3)} = \{a^3, b^3, c^3, a^2b, ab^2, a^2c, ac^2, b^2c, bc^2, abc\}$$

This set $S_3$ has cardinality 10.

It is not wrong to guess that in general $S_n$ will have a cardinality given by the $n+1$-esimal triangular number[13]

$$T_{n+1} = \frac{(n+1)(n+2)}{2} \tag{65}$$

Now, how the plethystic exponential encodes this informations? Let us compute $PE[3t] = \dfrac{1}{(1-t)^3}$ and expand it in Maclaurin series.

$$PE[3t] = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \cdots \tag{66}$$

We can see that the coefficient in front of $t^k$ gives us the cardinality of the set $S_{(3,k)}$

As we have seen in the example above, $PE[\dot{t}]$ is the generating funcion for products of $n$ "base objects".

We can generalize this.

Notice that here we graded the set of all monomials, via a grading given by the ordinary degree.

However, the same procedure works for any grading of any ring.

In particular we wish to grade the ring of chiral operators, with a grading given by physical properties of the operators themselves. This grading is given by the charges these operators carry with respect of physical symmetries, i.e. the operators quantum numbers such as the conformal dimension, the topological ANO charge, etc.

### 6.4 The Plethystic Logarithm

More important than the Plethystic exponential is its inverse function, which we will now define. We will use the Plethysitc logarithm in order to find out the number and degree of generators and relations of a given ring, knowing the Plethystic exponential. This is indeed the situation that we will face from a physical point of view: we can count how many operators there are with a given charge (relative of a given symmetry), and out of all this "list" of operators, we would like to isolate just the generaotrs and the relations defining the chiral ring itself. Such generators, seen as holomorphic functions, can be thought as coordinates on the moduli space, subject to certain relations.

[13] We remember that the $n$-esimal triangular number, which we call $T_n$ is given by the sum of the first $n$ integers. I.e

$$T_n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Therefore let us now introduce the inverse function. The Plethystic Logarithm of a multivariable function $g(t_1, \cdots t_n)$ that equals 1 at the origin, $g(0, \cdots, 0) = 1$ is defined as

$$PL[g(t_1, t_2, \cdots, t_n)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g(t_1^k, \cdots, t_n^k)) \tag{67}$$

where $\mu(k)$ is the Möbius function defined as

$$\mu(k) = \begin{cases} 0 & k \text{ has one or more repeted prime factors} \\ 1 & k = 1 \\ (-1)^n & k \text{ is the product of } n \text{ distinct primes.} \end{cases} \tag{68}$$

The Plethystic Logarithm is the inverse function of the plethystic exponential.

*Proof.* Call

$$g(t) = PE[f(t)] = PE\left[\sum_{k=1}^{\infty} a_k t^k\right] = \exp\left[\sum_{p=1}^{\infty} \frac{1}{p} f(t^p)\right] = \prod_{m=1}^{\infty} \frac{1}{(1-t^m)^{a_m}} \tag{69}$$

Taking the logarithm of this, and expanding in series we have

$$\begin{aligned} \log(g(t)) = \log\left(\prod_{m=1}^{\infty} \frac{1}{(1-t^m)^{a_m}}\right) &= -\sum_{k=1}^{\infty} \log\left((1-t^m)^{a_m}\right) = \\ = -\sum_{k=1} a_m \log\left(1-t^m\right) &= \sum_{k=1}^{\infty} (-a_k) \sum_{m=1}^{\infty} -\frac{1}{m} (t^k)^m \end{aligned} \tag{70}$$

Therefore we have

$$\begin{aligned} PL[g(t)] &= \sum_{l=1}^{\infty} \frac{\mu(l)}{l} \log(g(t^l)) = \sum_{l=1}^{\infty} \frac{\mu(l)}{l} \left(\sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} \frac{1}{m} (t^{lk})^m\right) = \\ &= \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \sum_{l|n} \mu(l) \frac{1}{n} (t^k)^n \end{aligned} \tag{71}$$

where we have re-written the double sum on $m$ and $l$ as the alternative sum on $n = ml$ and its divisors $l$.
Now we make use of a fundamental theorem of analytic number theory, namely the Möbius inversion formula

$$\sum_{d|n} \mu(d) = \delta_{n,1} \tag{72}$$

We find

$$PL[g(t)] = \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \sum_{l|n} \mu(l) \frac{1}{n} (t^k)^n = \\ = \sum_{k=1} a_k \sum_{n=1}^{\infty} \frac{1}{n} (t^k)^n \delta_{n,1} = \sum_{k=1}^{\infty} a_k t_k = f(t) \tag{73}$$

Which closes the proof.

# 7 Higgs Branch

In this section we will outline the method to explicitly compute the Hilbert Series of the Higgs Branch of a $\mathcal{N} = 4$ $d = 3$ quiver gauge theory.
In particular, we will first look at three equivalent procedures to compute the classical moduli space of a generic supersymmetric gauge theory.
Since the Higgs branch is classically exact, we could in principle employ any of these methods to compute it. However, we choose to use just a particular one out of those.
Our strategy will be the one of computing the Hibert series for all the operators which annihilate the supercharges of one kind, but are not necessarily gauge invariant and therefore are not chiral.

Subsequently, we will restrict the sum over *only* the strictly chiral operators, namely the gauge invariant ones.
The systematic way with which this operation is performed is called the *Molyen Weyl Projection*, and in a few words consists in integrating the Hilbert series over the whole gauge group, in a certain sense averaging away all the non-gauge invariant operators in the sum.
In the following section, we will give a few worked examples on how to apply this method in real computations: in particular we will find the generators and relations for the Higgs branch of the $U(1)$ with $n$ flavours and $SU(2)$ with $n$ flavours.

To start, consider a generic supersymmetric field theory.
Remember that the moduli space is defined to be the set of minima of the scalar potential of the theory, quotiented over the gauge group. With this we mean that the vacua $|\Omega_1\rangle$ and $|\Omega_2\rangle$ which are related by a gauge transformation (i.e there exists a unitary infinite-dimensional representation $U(\cdot)$ of $G$ onto the Fock space, such that there exists a group element $g \in G$ for which $|\Omega_1\rangle = U(g)\,|\Omega_2\rangle$ is true.)

We also recall that the scalar potential of a generic supersymmetry theory is given by

$$V(\varphi) = \sum_i |F_i(\varphi)|^2 + \sum_a |D_a(\varphi)|^2 \tag{74}$$

where the $F_i$ are called the $F$-terms and the $D_a$ are called the $D$ terms. It is clear that the minima of $V$ are realized for the vacuum expectation values of the scalar fields $\varphi$ such that the $F$-terms and the $D$-terms are zero.
Therefore we could simply solve these equations and then take a quotient over the gauge group.

However, it turns out that imposing the $D$-terms equations and quotienting over the gauge group is equivalent to taking a quotient over the complexified gauge group.
Furthermore, by exploiting the correspondence between chiral operators and holomorphic functions stated in the previous chapter, we could simply use the generators of the chiral ring as coordinates on the moduli space.

In order to find its moduli space we have then three equivalent procedures:

1. By Imposing the $F$ terms equations and the $D$ terms equations, and later taking a quotient with respect to the gauge group $G_{\mathbb{C}}$
2. Impose the $F$ terms equations and later to take a quotient with respect of the complexified gauge group $G$.
3. Parametrize the moduli space with coordinates given by the generators of the ring of all the gauge invariant chiral operators, and later taking a quotient with respect of the set of classical relations among them.

Our strategy to compute the Higgs Branch is the third one.
The first and the second way, despite seemingly more direct, become quite cumbersome to be carried out for theories with a high rank of the gauge group, or a high number of matter fields. Furthermore, the system of $F$-terms equations and $D$-terms equations is not necessarily solvable in a closed for.
In particular, it really is convenient and rather simple to parametrize the moduli space with the generators of the chiral ring.

In general it could be hard to figure out which is the chiral ring, starting from the matter field assignment. However, if we now restrict ourselves to the $\mathcal{N} = 4$ $d = 3$ case, and we focus only on the Higgs branch, we see that the generators of the chiral ring are simply given by the scalar components of the matter hymermultiplets of the theory.

We grade the chiral ring according to the global symmetries of the thoery.
Therefore, the strategy we adopt is the following:

1. Draw the quiver diagram for a given theory.
2. Read out of the diagram the assignment of the representations in which the hyermultiplets transform. It is very important in this step to count $ALL$ the symmetries involved, both global and gauge ones.
3. Write a generic Hilbert Series counting the operators made out of symmetric products of the hypermultiples, and grade them with respect to the symmetries found in the previous step.
4. Project the Hilbert Series into the sector of gauge invariant chiral operators, finding in this way a new Hilbert Series which only counts elements of the chiral ring.
5. Read the number and the gradings of generators and the relations of the Chiral ring of the Higgs Branch out of this last Hilbert Series.
6. Is the informations collected performing point 5 are enough, use them to build explicitly the Higgs branch, by the construction of the scheme associated to the Chiral ring.

In the following section we will carry out explicitly some examples, applying this method a couple of specific cases.

However, before that, we still need to clarify the point 4 stated above.
How does one perform this projection on the Hilbert Series?
In the next few pages we will answer such a question.

## 7.1 The Haar measure

In order to retain only the chiral operators, we would like to project the Hilbert Series counting all the operators $\mathcal{O}$ such that $[Q_\alpha, \mathcal{O}\}$, onto the sector of the gauge invariant operators.
The way to perform such a task consists in performing the integral of the Hilbert series over the whole gauge group itself.
To be able to do this, and understand how this method works, we must first define a theory of integration for Lie groups.

This will be the task of this section. We will define here a measure of integration, which is left and right invariant under the group multiplication, and is unique up to a scaling factor.
Such a measure is called the *Haar Measure* of a compact Lie group.

We begin giving two basic definition in point set topology, which are useful since they come as necessary conditions for the existence of a Haar measure. In a way, they restrict the class of groups for which a Haar measure can be defined. For more details on these topological properties and conditions, we refer the reader to the excellent books [32] [13].

**Def 18.** *A topological space $(X, \tau)$ is said to be* Hausdorff *if for every couple of points $p, q \in X$ there exist a open neighborhoods $U_p$ of $p$ and $U_q$ of $q$ such that $U_p \cap U_q \neq \emptyset$.*

As an easy example of a topological space which is Hausdorff, we consider $\mathbb{R}$ endowed with the usual topology. Given two points $p$ and $q$ in $\mathbb{R}$, and supposing without loss of generality that $p > q$ one can find open sets $U_p = ]p-\epsilon, p+\epsilon[$ and $U_q = ]q-\epsilon, q+\epsilon[$ which respectively contain $p$ and $q$.
Clearly, by choosing $\epsilon = (p-q)/3$, the two open sets do not intersect.

On the other hand, let us give an example of a space which is not Hausdorff.
Consider again $X = \mathbb{R}$, but now give it the following topology: the open sets are exactly all the intervals of the form $]a, +\infty[$, and no others.
This is a topology, since every union of half lines is a half line, and an intersection of two half lines is a half line.
However, if now we take $p, q \in \mathbb{R}$, and without loss of generality we assume $p > q$, we see that every open neighborhood of $q$ inevitably contains $p$.
Therefore $\mathbb{R}$ endowed with this topology is not Hausdorff.

For our physical concerns, all the gauge groups used in particle physics are indeed Hausdorff topological spaces.
This is true because they are not simply topological groups, but Lie groups. This means that any of such groups $G$ would be locally homeomorphic to $\mathbb{R}^n$ and therefore, in virtue of Whitney embedding theorem there exists a $N \in \mathbb{N}$ such that they can be embedded in $\mathbb{R}^N$. Therefore they can be thought as topological subspaces of $\mathbb{R}^N$ endowed with the

usual topology, which is Hausdorff. Then, since the Hausdorff condition is hereditary, all the topological subspaces of $\mathbb{R}^N$, including $G$, are themselves Hausdorff.

**Def 19.** *A topological space* $(X, \tau)$ *is said to be* locally compact *if for every point* $p \in X$ *there exists a open neighborhood* $U_p$ *of* $p$ *such that* $U_p$ *is compact, regarded as a topological space with the subset topology.*

Now, consider a locally compact topological group $G$.
The $\sigma$-algebra generated by all the open subsets of $G$ is called the *Borel algebra*, and it's element are called Borel sets.

A measure on the Borel algebra is said to be left invariant if, for every $S$ in the Borel Algebra it holds that

$$\mu(gS) = S. \tag{75}$$

Now, the fact that one such measure exists is the content of Haar's theorem which we will now state without giving a proof.

**Prop 2.** *Given a Hausdorff locally compact topological group* $G$*, there exist (up to a positive multiplicative constant) a unique countably additive nontrivial measure* $\mu$ *on the Borel subsets of* $G$*, satisfying the following properties*

- *The measure* $\mu$ *is left invariant.*
- *If* $K \subset G$ *is compact, then* $\mu(K) < \infty$
- *The measure is outer regular on Borel sets* $E$*:*
  $\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U\, open\}$
- *The measure is inner regular on open sets* $E$*:*
  $\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K\, compact\}$

As stated before, we will not prove here the uniqueness of this measure, referring the interested reader to [28] [33] for such a proof.

Usually, one chooses the Haar measure to be normalized in such a way that the volume of the topological group, which is defined as

$$\mathrm{Vol}(G) := \int d\mu_G 1 \tag{76}$$

is evaluated to 1.

Such a convention is particularly useful for our purposes.

Let us see now how to use this Haar measure in order to reduce the Hilbert Series counting all operators such that $[Q_\alpha, \mathcal{O}\}$, to the Hilbert Series counting just the gauge invariant ones.

### 7.2 The Molien-Weyl projection

When dealing with a set of polynomials of formal series expansions, on which a compact lie group acts, the Haar measure defined in the previous section can be computed explicitly by using the Weyl integral formula.
We avoid a complete derivation of this lengthy result, referring the interested reader to [23].

The main point of such result is that one can write:

$$\int d\mu_G = \frac{k_G}{(2\pi i)^n} \oint_{|z_1|=1} \cdots \oint_{z_n=n} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \prod_{\alpha\in\Delta} \left(1 - \prod_{l=1}^{n} z_n^{\alpha_l}\right), \tag{77}$$

where $\alpha \in \Delta$ are the roots of $G$, and $\alpha_l$ is the l-esim component of the vector $\alpha$.

As an example of the application of this formula, we compute now the Haar measure for $U(1)$ and for $SU(2)$ because we will use them in the following section to explicitly compute the Hilbert Series for the Higgs branch of theories with gauge group $U(1)$ or $SU(2)$.

Since $U(1)$ is abelian, there are no roots at all.
Therefore the Haar measure is given by

$$\int d\mu_{U(1)} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z}. \tag{78}$$

On the other hand, $SU(2)$ has a non empty set of roots $\Delta_{SU(2)} = \{\alpha, -\alpha\}$. Furthermore, $\alpha$ is a one-dimensional vector, which we can assume to be equal to the number 1. This allows us to write

$$\int d\mu_{SU(2)} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1-z)\left(1 - \frac{1}{z}\right). \tag{79}$$

In our specifc case, the variable $z$ appearing on this expression of the Haar measure is the fugacity associated to the gauge symmetry, readable from the quiver diagram.
From an intuitive point of view, it is easy to believe that integrating an Hilbert series over the fugacity that counts the grading of the operators with respect of the gauge symmetry will give something that does not carry a nontrivial representation of the gauge group.

Let us try to explicate this idea a little more formally.
To start, recall that since $G$ is a group, for every element $g \in G$ there exists the inverse element $g^{-1} \in G$.
Suppose now that, $G$ acts nontrivially on an addendum $a(t, z)$ of the Hilbert series $HS(t, z)$.
This is equivalent to say that $a(t, z)$ lies in a nontrivial representation of $G$. When $g \in G$ acts on $a(t, z)$ we find $a(t, gz)$.
Also, when $g^{-1} \in G$ acts on $a(t, z)$, we find $a(t, g^{-1}z)$.
Now if we sum over all possible $g \in G$ (actually if we integrate over all $g \in G$, since $G$

has a continuous infinity of elements), the contribution of the action of $g$ will cancel out with the contribution of the action of $g^{-1}$.

$$\int d\mu_G \; a(t, gz) = 0 \tag{80}$$

Suppose now, on the other hand, that $G$ acts trivially on another addendum $b(t, z)$ of the Hilbert Series.
This is equivalent of saying that $b(t, z)$ lies in the trivial representation: the singlet of the gauge group.
Now, all $g \in G$ are represented by the identity 1.
Therefore when $g \in G$ acts on $b(t, z)$ we find $b(t, gz) = b(t, z)$.
Also when $g^{-1} \in G$ acts on $b(t, z)$, we find $b(t, g^{-1}z) = b(t, z)$.
Therefore, integrating over the gauge group now gives

$$\int d\mu_G \; b(t, gz) = \mathrm{Vol}(G) \cdot b(t, z), \tag{81}$$

where $\mathrm{Vol}(G)$ is the volume of the gauge group.
If the Haar measure is normalized such that $\mathrm{Vol}(G)1$, then we see that integrating the Hilbert Series over the whole gauge group will set to zero all the terms which contain factors transforming nontrivially under $G$, and therefore leaving only the terms which count singlets of the gauge group.

From the explicit expression of the Haar measure for $U(1)$ and $SU(2)$ given above, we see that this *Weyl Projection* technique amounts to compute some contour integrals over the gauge fugacity $z$.
This is task is usually easy to perform, despite being sometimes tedious.

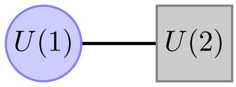
For a more formal explanation of the Weyl Projection, and a proof of the fact that indeed it works, we refer the reader to [23].

# 8 Some Computations of Higgs Branches.

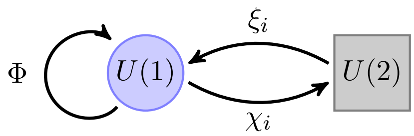
In this section we compute the Higgs Branches of the moduli space of two different quiver gauge theories, namely the $\mathcal{N} = 4$ supersymmetric three-dimensional $QED$ with $n$ flavours, and the $SU(2)$ theory with $n$ flavour.

## 8.1 $U(1)$ with 2 flavours

Let us compute in detail the Higgs Branch of the moduli space for the theory associated to the following quiver diagram.




As a first thing, we will write this $\mathcal{N} = 2$ quiver diagram in the $\mathcal{N} = 1$ notation.




The gauge and global symmetries acting on the Higgs branch are encoded in the following table:

Field/Dynkin Label assignment.

| | | $U(2)_f$ | |
|---|---|---|---|
| | $U(1)$ | $SU(2)_f$ | $U(1)_f$ |
| | $z$ | $x$ | $q$ |
| $\Phi$ | 0 | [0] | 0 |
| $\xi$ | 1 | [1] | $-1$ |
| $\chi$ | $-1$ | [1] | 1 |

Table 2: Representation assigment for the different matter fields in the $U(1)$ theory with 2 flavours.

The $F$- terms are obtained by letting the derivative of the superpotential with respect to the supermultiples vanish. We find

$$\begin{cases} \dfrac{\partial \mathcal{W}}{\partial \chi_i} = \Phi \xi^i = 0 \\ \dfrac{\partial \mathcal{W}}{\partial \xi_i} = \chi_i \Phi = 0 \\ \dfrac{\partial \mathcal{W}}{\partial \Phi} = \chi_i \xi^i = 0 \end{cases} \tag{82}$$

So we have two different possibilities:

1. $\chi_i$ , $\xi^i = 0$ and $\langle \Phi \rangle = any$ would correspond to the Coulomb Branch, where the expectation value of the scalars in the vector multiplet take non-zero value.
2. $\Phi = 0$, $\chi_i = any$, $\xi^i = any$ with $\chi_i \xi^i = 0$, corresponds to the Higgs Branch, where scalars in the hypermultiplet take non-zero expectation value.

However we recall notice that, as explained before, one cannot calculate the Coulomb branch in this manner since it receives numerous quantum corrections, which means that the superpotential would need loop renormalisation and instanton corrections.
On the other hand the Higgs Branch is not renormalised, hence this classical computation is valid and gives an exact result.

Here we focus only on the Higgs Branch, leaving the computation for the Coulomb Branch for later.
The space of solutions $\Phi = 0$, $\chi_i$ , $\xi^i = any$ with $\chi_i \xi^i = 0$ is called the *F-flat space*, or sometimes the *Master Space* and it is usually denoted by $\mathcal{F}^\flat$

$$\mathcal{F}^\flat = \{\Phi = 0, \langle \chi_i \rangle = any, \langle \xi^i \rangle = any | \chi_i \xi^i = 0\} \tag{83}$$

Let us compute the generating function encoding the $\mathcal{F}^\flat$ space, namely its Hilbert Series. We will call this $g_{1,N}^{\mathcal{F}^\flat}$.

$$g_{1,N}^{\mathcal{F}^\flat} = \left(1 - t^2\right) PE \left( \chi([1]) \omega t + \chi([1]) \frac{t}{\omega} \right), \tag{84}$$

where

- $w = \dfrac{z}{q}$ is a redefined fugacity, to absorb the $U(1)$ global symmetry, with fugacity $q$ into the gauge local $U(1)$ with fugacity $z$.
- $PE\left([1]wt\right)$ counts the symmetric products of $\xi$ and $PE\left([1]wt\right)$ the symmetric products of $\chi$.
- The $(1 - t^2)$ prefactor in front takes care of the relation occurring at degree two in the generators.

We find

$$
\begin{aligned}
g^{\mathcal{F}^\flat}_{1,N} &= \left(1-t^2\right) PE\left(\chi([1]\omega t)\right) PE\left(\chi([1])\frac{t}{\omega}\right) = \\
&= \left(1-t^2\right) PE\left(\left(x+x^{-1}\right)\omega t\right) PE\left(\left(x+x^{-1}\right)\frac{t}{\omega}\right) = \\
&= \left(1-t^2\right) \frac{1}{1-x\omega t}\frac{1}{1-x^{-1}\omega t}\frac{1}{1-x\omega^{-1}t}\frac{1}{1-x^{-1}\omega^{-1}t}.
\end{aligned}
\tag{85}
$$

**The Weyl-Molien Projection**

Now we should perform the integral over the gauge group, in order to project onto the gauge invariant operators. The Haar measure for $U(1)$ is simply $\frac{dw}{w}$.

In the end we have

$$
HS_{U(1),N}(t,x) = \frac{1}{2\pi i}\int_{|w|=1} g^{\mathcal{F}^\flat}_{1,N}(t,x,w). \tag{86}
$$

The tecnique to do the integration is the standard use of the Residue Theorem [2], and therefore

$$
HS_{U(1),N}(t,x) = \sum_i \mathrm{Res}\left(g^{\mathcal{F}^\flat}_{1,N}(t,x,w)\right). \tag{87}
$$

The function

$$
g^{\mathcal{F}^\flat}_{1,N}(t,x,w) = \left(1-t^2\right) \frac{1}{1-x\omega t}\frac{1}{1-x^{-1}\omega t}\frac{1}{1-x\omega^{-1}t}\frac{1}{1-x^{-1}\omega^{-1}t} \tag{88}
$$

has four poles:

- $\omega = x^{-1}t^{-1}$,
- $\omega = xt^{-1}$,
- $\omega = xt$,
- $\omega = x^{-1}t$.

Given that $|t|<1$ and $|x|=1$ we find that the only poles for which $|w|<1$ are

$$
\omega_1 = tx^{-1} \quad \text{and} \quad \omega_2 = tx. \tag{89}
$$

After computing the residuals one finds

$$
HS_{U(1),N}(t,z) = \frac{1}{(1-t^2x^{-2})(1-x^2)} + \frac{1}{(1-t^2x^2)(1-x^{-2})} \tag{90}
$$

which, after easy algebraic manipulations can be recast into the form

$$
\frac{1-t^4}{(1-t^2)(1-t^2x^2)(1-t^2x^{-2})}. \tag{91}
$$

Let us now redefine the fugacities, such as

$$t^2 \mapsto t \quad \text{and} \quad x^2 \mapsto z. \tag{92}$$

Upon doing this we find

$$HS_{U(1),N}(t,z) = PE[\left(1 + z + z^{-1}\right)t - t^2] \tag{93}$$

**Application of the PL.**

By a direct application of the Plethysitc Logarithm we find that the Higgs Branch is a complete intersection algebraic variety, defined by three generators in the adjoint of $SU(2)$, of order one, and one relation in the singlet of $SU(2)$ and of order two.

Therefore the Higgs Branch of the moduli space of $U(1)$ with 2 flavours is given by

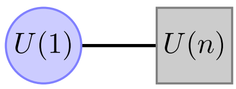
$$\mathcal{M}_H = \{X, Y, Z \in \mathbb{C}^3 | XY = Z^2\} \tag{94}$$

It is not hard to prove that the algebraic variety defined in the equation above is simply

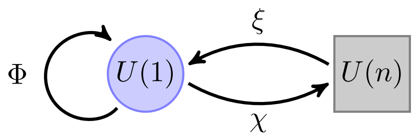
$$\mathcal{M}_H = \mathbb{C}^2/\mathbb{Z}_2 \tag{95}$$

## 8.2 $U(1)$ with $n$ flavours.

We now repeat the same procedure of the previous pages to compute the Higgs Branch for the slightly more general theory given by the following quiver.




This is a theory of supersymmetric electrodynamics, with $n$ flavours. The corresponding $\mathcal{N} = 2$ quiver is given by




The gauge and global symmetries acting on the Higgs branch are encoded in the following table:

Field/Dynkin Label assignment.

| | | $U(n)_f$ | |
|---|---|---|---|
| | $U(1)$ | $SU(n)_f$ | $U(1)_f$ |
| | $z$ | $x_1, \cdots x_{n-1}$ | $q$ |
| $\Phi$ | 0 | $[0, \cdots, 0]$ | 0 |
| $\xi$ | 1 | $[0, \cdots, 1$ | $-1$ |
| $\chi$ | $-1$ | $[1, \cdots, 0]$ | 1 |

Table 3: Representation assigment for the different matter fields in the $U(1)$ theory with $n$ flavours.

The $F$- terms are obtained by letting the derivative of the superpotential with respect to the supermultiples vanish. We find

$$\begin{cases} \dfrac{\partial \mathcal{W}}{\partial \chi_i} = \Phi \xi^i = 0 \\ \dfrac{\partial \mathcal{W}}{\partial \xi_i} = \chi_i \Phi = 0 \\ \dfrac{\partial \mathcal{W}}{\partial \Phi} = \chi_i \xi^i = 0 \end{cases} . \tag{96}$$

So, just as in the previous case, we have two different possibilities:

1. $\chi_i$ , $\xi^i = 0$ and $\langle \Phi \rangle = any$ would correspond to the Coulomb Branch, where the expectation value of the scalars in the vector multiplet take non-zero value.

2. $\Phi = 0$, $\chi_i = any$, $\xi^i = any$ with $\chi_i \xi^i = 0$, corresponds to the Higgs Branch, where scalars in the hypermultiplet take non-zero expectation value.

However we recall notice that, as explained before, one cannot calculate the Coulomb branch in this manner since it receives numerous quantum corrections, which means that the superpotential would need loop renormalisation and instanton corrections. On the other hand the Higgs Branch is not renormalised, hence this classical computation is valid and gives an exact result.

Here we focus only on the Higgs Branch, leaving the computation for the Coulomb Branch for later. The space of solutions $\Phi = 0$, $\chi_i$ , $\xi^i = any$ with $\chi_i \xi^i = 0$ is called the *F-flat space*, or sometimes the *Master Space* and it is usually denoted by $\mathcal{F}^\flat$.

$$\mathcal{F}^\flat = \{\Phi = 0, \langle \chi_i \rangle = any, \langle \xi^i \rangle = \text{any} | \chi_i \xi^i = 0\} \tag{97}$$

Let us compute the generating function encoding the $\mathcal{F}^\flat$ space, namely its Hilbert Series. We will call this $g_{1,N}^{\mathcal{F}^\flat}$.

$$g_{1,N}^{\mathcal{F}^\flat} = \left(1 - t^2\right) PE \left( \text{car}([1, 0, \cdots, 0]) \omega t + \text{car}([0, 0, \cdots, 1]) \frac{t}{\omega} \right) \tag{98}$$

Up to now, this case is almost identical to the previous one.
However, now the global flavour group is different and therefore the characters of its representations are different.

Here

- $w = \frac{z}{q}$ is a redefined fugacity, to absorb the $U(1)$ factor of the $U(n)$ global symmetry, with fugacity $q$, into the gauge local $U(1)$ with fugacity $z$.
- $PE\left(\text{car}([1,\cdots,0])wt\right)$ counts the symmetric products of $\xi$ and $PE\left(\text{car}([0,\cdots,1])wt\right)$ the symmetric products of $\chi$.
- The $(1-t^2)$ prefactor in front takes care of the relation occurring at degree two in the generators.

Recall that the characters of $SU(n)$ are given by

$$\text{car}([1,\cdots,0]) = x_1 + \frac{1}{x_{N-1}} + \sum_{k=2}^{N-1} \frac{x_k}{x_{k-1}} \tag{99a}$$

$$\text{car}([0,\cdots,1]) = \frac{1}{x_1} + x_{N-1} + \sum_{k=2}^{N-1} \frac{x_{k-1}}{x_k} \tag{99b}$$

and therefore, using the properties of the plethystic exponential, we can write

$$\begin{aligned} &g^{\mathcal{F}^\flat}_{1,N}(t, x_1, \cdots, x_{N-1}, w) = \\ &= \frac{(1-t^2)}{(1-x_1wt)\left(1-\frac{1}{x_{N-1}}wt\right)} \prod_{k=2}^{N-1} \frac{1}{1-\frac{x_k}{x_{k-1}}wt} \cdot \\ &\cdot \frac{1}{\left(1-\frac{1}{x_1}\frac{t}{w}\right)\left(1-x_{N-1}\frac{t}{w}\right)} \prod_{k=2}^{N-1} \frac{1}{1-\frac{x_{k-1}}{x_k}\frac{t}{w}} \end{aligned} \tag{100}$$

Now we would like to perform the Molien-Weyl projection, in order to find the Hilbert Series which just counts the gauge invariant operators such that $[Q_\alpha, \mathcal{O}\}$, and not all of them.
The Haar measure for $SU(2)$ was given in the previous section.

$$HS_{SU(n)}(t, x_1, \cdots, x_{n-1}) = \frac{1}{2\pi i} \int_{|w|=1} \frac{dw}{w} g^{\mathcal{F}^\flat}_{1,N}(t, x_1, \cdots, x_n, w), \tag{101}$$

where we restrict to the unit circle, since the radius of convergence for $t$ is 1 and therefore only the poles within such circle must be considered. They are given by

$$w_1 = \frac{t}{x_1}, \quad w_2 = \frac{x_1}{x_2}t, \quad \cdots \quad w_{n-1} = \frac{x_{n-2}}{x_{n-1}}t, \quad w_n = tx_{n-1}. \tag{102}$$

After performing the contour integration, one finds

$$HS_{SU(n)}(t, x_1, \cdots, x_{n-1}) = \sum_{p=0}^{\infty} \text{car}([p, 0, \cdots, 0, p])_{SU(N)} t^{2p}. \tag{103}$$

We can now find the dimension of the $[p, 0, \cdots, 0, p]$ representation, by setting $x_i = 1$, in virtue of the Weyl dimension formula. Thus, the unrefined Hilbert series which counts the gauge invariant operators at a given degree is given by

$$HS_{1,SU(N)}(t, 1, \cdots, 1) = \frac{\sum_{p=0}^{N-1} \binom{N-1}{p}^2 t^{2p}}{(1-t^2)^{2(N-1)}} \tag{104}$$

From this we can infer the dimensionality of the Higgs Branch, since the pole at $t = 1$ is of order $2(N-1)$. Therefore

$$\dim_{\mathbb{C}} \mathcal{M}_H = 2\,(N-1)\,. \tag{105}$$

## 8.3 $SU(2)$ with $n$ flavours

Consider now the quiver gauge theory summarized in the following graph.

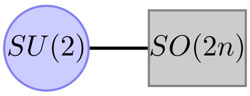



Figure 10: $\mathcal{N} = 2$ quiver for $SU(2)$ with $n$ flavours

As a first thing, we will write this $\mathcal{N} = 2$ quiver diagram in the $\mathcal{N} = 1$ notation.

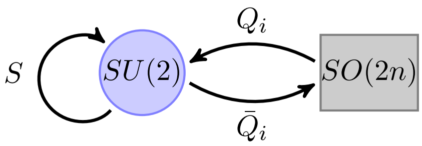



Figure 11: $\mathcal{N} = 1$ quiver for $SU(2)$ with $n$ flavours

The gauge and global symmetries acting on the Higgs branch can be read off the quiver.
The superpotential is given by

$$W = Q \cdot S \cdot Q = Q^i_a \epsilon^{ab} S_{bc} \epsilon^{cd} Q^i_d \tag{106}$$

where $\epsilon^{ab}$ is the $SU(2)$ invariant tensor, defined in the appendix.
The F-terms are obtained by taking derivatives with respect to the (scalars in the) multiplets.

The Higgs branch occurs when the scalars coming from the $\mathcal{N}=4$ vector multiplet vanish while the ones from the hypermultiplet take nonzero vev.

$$\frac{\partial W}{\partial Q^i_f} = 2\epsilon^{fb}\epsilon^{cd}S_{bc}Q^i_d \tag{107a}$$

$$\frac{\partial W}{\partial S_{bc}} = \epsilon^{ab}\epsilon^{cd}Q^i_aQ^i_d = Q^b_iQ^i_c + Q^i_cQ^i_b \tag{107b}$$

Then $\mathcal{F}^\flat_{1,SO(2N)} = \{S=0, \langle Q^i_a\rangle \neq 0 | Q^a_iQ^i_b + Q^i_bQ^i_a = 0\}$, and the condition on the $Q^i_a$ implies that the second symmmetric product of two of them has to vanish. This relation is of order squared in the fields, and trasforms as the $[2]_{SU(2)}$. This character will appear as the prefactor of the Plethystic exponential.

$$\begin{aligned} g^{\mathcal{F}^\flat}_{1,SO(2n)}(t,x_1,\cdots x_n,z) = \left(1-z^2t^2\right)\left(1-t^2\right)\left(1-z^{-2}t^2\right)\cdot \\ \cdot PE\left[\mathrm{car}([1,0,\cdots,0]_{SO(2n)})\left(z+z^{-1}\right)\right] \end{aligned} \tag{108}$$

Now recall that the character for the fundamental representation of $SO(2n)$ is given by

$$\mathrm{car}([1,\cdots,0]_{SO(2n)}) = \sum_{a=1}^{n}\left(x_a + x_a^{-1}\right) \tag{109}$$

Using this, we can rewrite the plethystic exponential as follows

$$\begin{aligned} g^{\mathcal{F}^\flat}_{1,SO(2n)}(t,x_1,\cdots x_n,z) = \left(1-z^2t^2\right)\left(1-t^2\right)\left(1-z^{-2}t^2\right)\cdot \\ \cdot\prod_{a=1}^{n}\left(\frac{1}{1-zx_at}\frac{1}{1-z^{-1}x_at}\frac{1}{1-zx_a^{-1}t}\frac{1}{1-z^{-1}x_a^{-1}t}\right) \end{aligned} \tag{110}$$

This Hilbert series counts all the operators, disregarding the fat that they are gauge invariant or not. In order to retain only the gauge invariant ones, we perform the Weyl-Molien projection. We will make use of equation (79)

$$\begin{aligned} HS(t,x_1,\cdot,x_n) &= \int d\mu_{SU(2)}g^{\mathcal{F}^\flat}_{1,SO(2n)}(t,x_1,\cdots x_n,z) = \\ &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1-z)\left(1-\frac{1}{z}\right)g^{\mathcal{F}^\flat}_{1,SO(2n)}(t,x_1,\cdots x_n,z) = \\ &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1-z)\left(1-\frac{1}{z}\right)\left(1-z^2t^2\right)\left(1-t^2\right)\left(1-z^{-2}t^2\right)\cdot \\ &\cdot\prod_{a=1}^{n}\left(\frac{1}{1-zx_at}\frac{1}{1-z^{-1}x_at}\frac{1}{1-zx_a^{-1}t}\frac{1}{1-z^{-1}x_a^{-1}t}\right) = \\ &= \sum_{p=0}^{\infty}\mathrm{car}([0,p,0,\cdots,0]_{SO(2n)})t^{2p} \end{aligned} \tag{111}$$

# 9 Monopole Operators

Having studied the Higgs branch of some theories, we would like now to move on to the Coulomb branch. However, in order to do that we must first of all discuss monopole operators. The reason for this is that the relevant chiral operators necessary to describe such a branch of the moduli space, are no longer simply hypermultiplets, but indeed monopole operators.

Therefore, in this chapter we define those peculiar operators, which are a certain class of local operators in three dimensional conformal field theories that are not polynomial in the fundamental fields and create topological disorder. The importance of those operators relies on the fact that they are "soliton creating operators" [40][15][39], for which they carry nonvanishing charge under the topological symmetry. In the exposition, we follow closely [12] [6].

## 9.1 The abelian, easy case

Let us start by considering the simple case of QED in three dimension. This theory is not supersymmetric, and the gauge group is abelian. However, this is the easiest possible scenario for introducing monopole operators. The action for three-dimensional QED with $N_f$ flavours in the Euclidean Space is given by

$$S_{QED} = \int d^3x \left( \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \psi_j^\dagger \left( \sigma \cdot i D_A \right) \psi^j \right) \tag{112}$$

where $A$ is the $U(1)$ gauge field, $F = dA$ is its field strength and $\psi$ is a two component complex Weyl fermion. $j$ runs from 1 to the number of flavours $N_f$.

Since the gauge coupling $e$ has mass dimension $\frac{1}{2}$, the theory is super-renormalizable and free in the UV.
On the other hand, this theory is strongly coupled in the infrared.

As we have seen in section 3, this theory possesses an interesting conserved current, the dual of the field strength:

$$J^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}$$

We want to look for operators possessing a nonvanishing charge under such a topological symmetry. In the Higgs phase, such operators are called vortex operators (or ANO) operators since they create Abrikosov-Nielsen-Olesen vortices when acting on a state, as it was shown in [1]. They are the 3d analogues of the twist and winding-state operators of $d = 2$ CFT. In that case, one sees that the winding state operator creates a kink. One can express this more precisely in the following way: consider a CFT of free bosons in $d = 2$ and add a perturbative periodic potential (such as in the Sine-Gordon model). The resulting theory, in which now the boson particles have mass, possesses various vacua and kinks interpolating among them.

As it was shown in the series of papers [12] [6], an operator which carries a nontrivial charge under the topological symmetry group can be defined by requiring that the gauge fields have a singularity at the insertion point in the path integral, and such singularity should be that of a Dirac monopole field.

$$A^{N,S}(\vec{r}) = \frac{m}{2}\left(\pm 1 - \cos\theta\right) d\varphi \tag{113}$$

Where the opposite signs corresponds to two different charts covering the two hemispheres of $S^2$ that surround the insertion point. The magnetic charge $m$ is subject to the usual Dirac quantisation condition, as we will prove in the next subsection. In order for the gauge fields to have such singularities, one must also define some operators $V_m$ which must be inserted exactly at the gauge field singularity, acting therefore as some "monopole creating operators".

Let us now restrict ourselves to the case in which the theory has followed the renormalization group flow all the way down to the infrared fixed point. Here, all the gauge couplings have run to infinity, and the theory is superconformal.

In a generic CFT there exists a concept called *Operator-State Correspondence*. This is a map that associated to each local operator, a state in the Fock space of the theory. In few words this works since a conformal theory on $\mathbb{R}^3$ can be "radially quantized" [20][38], i.e written as a theory on $R \times S^2$. Through this procedure local operators on $\mathbb{R}^3$ can be brought to infinity by a conformal transformation, but operators at infinity are incoming/outgoing states on $R \times S^2$. Thus monopole operators of the original theory carrying GNO charge $m$ are in a one-to-one mapping to states on the radially quantized theory with flux $m$ through the sphere. We call these states t'Hooft monopoles.

In the next subsection we will prove that the flux of a t'Hooft monopole is quantized by a generalization of the usual Dirac quantization for monopole charges. Later, we will exploit again the radial quantization and the operator-state correspondence, in order to compute the conformal dimension of these operators, which we will discover to be equal to the energy of the corresponding state in the redially quantized theory.

## 9.2 Generalized Dirac condition

We now prove that a t'Hooft-Polyakov monopole's magnetic charge satisfies a quantization condition which forces it to belong to the weight lattice of the dual group of the gauge group.

Consider the principal bundle $\mathbb{R}^3 \to G$. As a topological space, the base space of this bundle has the same homotopy type of the base space of the bundle $S^2 \to G$, meaning that there exist continuous functions

$$\begin{aligned} f &: \mathbb{R}^3 \to S^2 \\ g &: S^2 \to \mathbb{R}^3 \end{aligned} \tag{114}$$

such that $g \circ f \simeq 1_Y$ and $f \circ g \simeq 1_X$. Here $\simeq$ means being homotopic [31] as continuous maps.[14] Therefore, in virtue of the theorem of equivalence of homotopic bundles (see

[14]Recall that given two topological spaces $X$ and $Y$, two continuous maps $f, g : X \to Y$ are said *homotopic* if there exists a third continuous map $h : [0,1] \times X \to Y$ such that $h(0,\cdot) = f(\cdot)$ and $h(1,\cdot) = g(\cdot)$.

[36]) the two bundles given above define physically equivalent gauge theories, and we should then consider $S^2 \to U(1)$.

An open cover of charts for the manifold $S^2$ can be takes as the couple of open sets

$$U_N = \{\phi \in [0, 2\pi], \theta \in [0, \frac{\pi}{2} + \epsilon]\} \tag{115a}$$

$$U_S = \{\phi \in [0, 2\pi], \theta \in [\frac{\pi}{2} - \epsilon, \pi]\} \tag{115b}$$

The gauge connection is given, in the two patches, by

$$A_N \simeq \frac{m}{2}\,(1 - \cos\theta)\,d\phi, \tag{116a}$$

$$A_S \simeq \frac{m}{2}\,(-1 - \cos\theta)\,d\phi. \tag{116b}$$

Where $A_N$ is defined in patch covering the upper hemisphere surrounding the origin, and $A_S$ is defined in the lower one. Here $m$ belongs to the the Lie Algebra $\mathfrak{g}$ of $G$.

The transition function between the two patches of the bundle is given by

$$\begin{aligned} t_{NS} : U_N \cap U_S \to G \\ \phi \mapsto \exp(i\Phi(\phi)) \end{aligned} \tag{117}$$

The two fields $A_N$ and $A_S$ are related by a Yang-Mills gauge transformation:

$$A_N = t_{NS}^{-1} A_S t_{NS} - i t_{NS}^{-1} dt_{NS}. \tag{118}$$

Computing the exterior derivative of the transition function gives

$$dt_{NS} = i t_{NS} d\Phi \tag{119}$$

Putting the last two equations together implies

$$d\Phi = A_N - t_{NS}^{-1} A_S t_{NS} \tag{120}$$

Integrating this one finds

$$\Phi = \int_0^{2\pi} d\varphi A_N - t_{NS}^{-1} A_S t_{NS} = 2\pi m \tag{121}$$

Therefore, by requiring the transition function between the patches to be smooth and single-valued, one finds the Dirac quantization condition:

$$\exp(2\pi i m) = 1_G \tag{122}$$

This condition requires $m$ to belong to the weight lattice of $\hat{G}$, the Langland dual of the group $G$.

Here we recall, for the following, the definition of the Langland dual [34] of the group $G$.

**Def 20.** *Consider a Lie algebra $\mathfrak{g}$. The Lie algebra $\hat{\mathfrak{g}}$ having as a root system the set of coroots of $\mathfrak{g}$ is called the Langland dual of $\mathfrak{g}$.*
*In general, different Lie groups $\hat{G}_i \quad i = 1 \cdots n$, differing for their center $Z(G)$ and their fundamental group $\pi_1(G)$, have $\hat{\mathfrak{g}}$ as a Lie algebra.*
*However, Dirac quantization condition also singles out exactly one of these groups, by stating that $m$ belongs to the kernel of $exp(\cdot)$.*
*We will call such a group $\hat{G}$, the Langland dual of $G$.*[15]

As a reference, we will report now which are the GNO duals of some common groups:

- $U(1)$ is self dual, since there are no roots at all.
- $SU(2)$ is dual to $SO(3)$.
- $U(n)$ is self dual.
- $G_2$ is self dual.

### 9.3 BPS Monopole operators as chiral operators for the Coulomb Branch

We would like exploit the radial quantization and the operator state correspondence, in order to be able to compute the quantum numbers of these monopole operators: i.e. their mass dimension, or their charges with respect to the symmetries of the theory.

As it was shown in [12], the energy of the state in the radial quantized theory corresponds to the $R$-charge of the monopole operator in the superconformal theory. Also, we recall that the $R$-charge of an operator in a $\mathcal{N} = 4$, $d = 3$ superconformal theory coincides with its conformal dimension.

Now, the problem to compute the energy of a particular state in a SCFT on a curved spacetime is certainly a well posed problem, although a difficult one. We will refrain from entering in these details, and refer the reader to the cited works of Kapustin, Bashirov, Borokhov.

For example, they show that for the case of $U(1)$ with $N_f$ flavours, the energy of the t'Hooft monopole state corresponding to a monopole operator has energy

$$E = N_f \frac{|m|}{2} \tag{123}$$

and therefore the monopole operator $V_m$ will have conformal dimension

$$\Delta = N_f \frac{|m|}{2} \tag{124}$$

The procedure for the nonabelian case is similar to the abelian one. However, the details for the radial quantization of the CFT and in particular the computation of the

[15] In physics literature such a group is also called the GNO dual of $G$.

energy $E = \Delta$ are much more difficult. We omit the explicit computations, suggesting the reader to refer to [11] for further details.

The main result is the following: the conformal dimension of a monopole operator of GNO (magnetic) charge $m$ in the infrared CFT is given by

$$\Delta(m) = -\sum_{\alpha \in \Delta^+} |\alpha(m)| + \frac{1}{2} \sum_i \sum_{\rho_i \in R_i} |\rho_i(m)|. \tag{125}$$

where the first sum is over the set of all positive roots $\alpha \in \Delta^+$ of the gauge group represents the contribution arising from the $\mathcal{N} = 4$ vector multiplets, while the second sum is the contribution from the matter multiplets.

This formula (125) will be of fundamental importance in all the following computations.

Furthermore, it should be said that all the considerations we have made so far (in non supersymmetric cases) about the allowed values of the magnetic charge will also hold in the more specific scenario of $\mathcal{N} = 4$. See for example [7].

However, it happens that not all the monopole operators defined above will be such that they preserve supersymmetry (i.e sit in the lowest, "scalar" component of a whole chiral supermultiplet). We would like to look for monopole operators which preserve $\frac{1}{2}$ of the supercharges, and therefore are BPS operators. The interest these operators is completely obvious: they are annihilated by a half of the supercharges and they sit in the lowest component of a chiral supermultiplet. In case we could find a subset of them that are also gauge invariant, we will have found our *chiral operators* for the Coulomb branch, in the sense of the definition given in section 6.

In order to do this, and restrict the set of all monopole operators to that of the BPS ones, further constraints on the variation of the gauginos, and on the singularities of the matter fields configuration are to be imposed. For details on this, we refer to [17] and [12].

The crucial fact for our purposes is that in the $\mathcal{N} = 4$ $d = 3$ theory, there is the possibility of turning on a constant background for the complex scalar $\phi$ in the adjoint, other then the $\mathcal{N} = 2$ monopole background discussed above, and still preserving the fact that the monopole operator is $\frac{1}{2}$ BPS.

Indeed, the presence of a magnetic monopole operator of GNO charge $m$ will break the gauge group $G$ to a residual gauge group $H_m$, the commutant of $m$ in $G$. Most of the times $H_m$ is $U(1)^r$, where $r$ is the rank of the gauge group $G$. However, sometimes the gauge group is not broken to its maximal torus, and in the following we must keep particular attention to this aspect.

Now, if we take the complex scalar $\phi$ to assume values in the lie algebra $\mathfrak{h}_m$ of the residual gauge group, it can be proven that the monopole operator on top of this nonvanishing $\phi$ background still preserves $\frac{1}{2}$ of the supersymmetries.

In a sense, we have two different types of BPS $\mathcal{N} = 4$ monopole operators in $d = 3$. Some of them have $\phi = 0$, and we will call them *bare*, but others have $\phi \neq 0$, and we will call them *dressed by $\phi$*. Both of these types of monopoles operators are candidates to be the chiral operators of the chiral ring for the Coulomb branch of our theory.

We only need to find the subset of BPS monopoles operators which are also gauge invariant. A gauge transformation will act both on $m$ and on $\phi$ via an action of the Weyl group of the gauge group $G$.

Therefore we can conclude that the chiral operators for the chiral ring of the Coulomb branch, are those combinations of monopole operators which are invariants under an action of the Weyl group.

## 10 Coulomb Branch

Unlike the Higgs branch, the Coulomb branch is not protected against quantum corrections. The traditional way to describe this branch is to give give vacuum expectation value to the three scalars in the $\mathcal{N} = 4$ vector multiplets. For a generic vev, the gauge group $G$ is completely broken down to $U(1)^r$. In this way, all the W-bosons and the matter fields will be massive. The low energy dynamics on such a point of the Coulomb branch is described by an effective field theory of $r$ abelian vector multiplets.

By using the dualization of the photon we sae in section 3, we can dualize the vector supermultiplets into twisted hypermultiplets. The hyperkähler metric on the Coulomb branch can be computed semiclassically by integrating out the massive hypermultiplets and W-boson vector multiplets at one loop. This 1-loop description is only reliable in weakly coupled regions where the fields that have been integrated out are very massive, well above the renormalization cutoff scale $\Lambda$ which dictates the range of validity of the effective field theory. In particular, nowadays it is not known how to dualize a non-abelian vector multiplet.

Our strategy for the computation of the Coulomb branch dismisses this old procedure, and follows the new approach outlined in [17] and [16]. In particular, we will count the number of bare and dressed monopole operator, grading them with their conformal dimension[16] $\Delta$, and the topological symmetry $Z(\hat{G})$ in case this is non-trivial.

As we have seen before, these BPS monopole operators will be the chiral operators relevant for the description of the Coulomb branch of the moduli space. In particular, there will be a bijection between the chiral ring of bare and dressed BPS monopole operators, and the ring of holomorphic functions over the moduli space.

An important assumption, on which relies our good counting, is that for a given GNO charge $m$ there will be an unique BPS monopole operator. In few words, this allows us to count *one* for every different magnetic charge $m$. Up to our understanding of the subject, there is no proof of this fact. However, there are strong indications that this is indeed correct, since all the computations for the Coulomb branches we have made (and we report later) agree perfectly with the perturbative computations known in literature.

### 10.1 Dimension Formula for "good" and "bad" theories.

To start, we recall that the GNO magnetic charge of a (bare or dressed) monopole operator belongs to the weight lattice of $\hat{G}$ the GNO dual of the gauge group $G$. Furthermore, since we are only interested in gauge invariant monopole operators, and the gauge group $G$ acts on $m$ via the Weyl group $W_G$, taking $m$ from a weyl chamber to another, then we can safely restrict $m$ to take values in a single Weyl chamber of the weight lattice $\Gamma_{\hat{G}}$ of the GNO dual.

Further, as we have seen before, the R-charge of a bare monopole operator of GNO charge $m$, in the infrared CFT, is given by

[16] We recall that *usually* (we will explain later when), for a superconformal theory such as this, the R-charge coincides with the conformal dimension

$$\Delta(m) = -\sum_{\alpha\in\Delta^+} |\alpha(m)| + \frac{1}{2}\sum_i \sum_{\rho_i\in R_i} |\rho_i(m)| \tag{126}$$

where the first sum is over the set of all positive roots $\alpha \in \Delta^+$ of the gauge group represents the contribution arising from the $\mathcal{N} = 4$ vector multiplets, while the second sum is the contribution from the matter multiplets. This second term is a sum over the weights of the matter representations.

This formula was first conjectured in [24] based on some weak coupling result from [11] and group theory results. Later, this formula was proven in [7] [6]. However, for our purpose, this formula can not be applied to any $\mathcal{N} = 4$, $d = 3$ theory, but just a subset of them. Using the nomenclature proposed by Gaiotto and Witten in [24], a theory is called

- *good* if all the BPS monopole operators have $\Delta > \frac{1}{2}$,
- *ugly* if all the BPS monopole operators have $\Delta \geq \frac{1}{2}$,
- *bad* if some of the BPS monopole operators have $\Delta < \frac{1}{2}$.

In case the bad case, it is no longer true that the R-charge of the monopole operator corresponds to its conformal dimension, since the superconformal $R$-symmetry mixes with other accidental symmetries. In other words, in the bad case the theory is no longer unitary. In the ugly case, the monopoles such that $\Delta = \frac{1}{2}$ will be free decoupled fields, while in the good case, all the monopole operators will be coupled.

## 10.2 The Hilbert Series

Knowing the formula for the dimension and the topological symmetry of a monopole operator in terms of the magnetic charge $\vec{m}$, we are now ready to write the Hilbert Series for the Coulomb branch. A first guess is

$$H_G(t) = \sum_{m\in\Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} t^{\Delta(m)} z^{J(m)}. \tag{127}$$

We immediately tell, and warn the reader, that this first guess is not correct although it is very close to the final result. It is instructive, from a pedagogical point of view, to first have a look at this formula and highlight some basic features of it, and only later explain what is the problem, and how it is avoided.

In the formula above, we see that we are counting the monopole operators of GNO charge $m$, grading them accordingly to the superconformal dimension $\Delta$ and the topological ANO charge $J(m)$. $t$ and $z$ are fugacities associated with $\Delta$ and $J(m)$.

The Hilbert Series written above is not correct since it does not take into account the fact that there exist some magnetic charges $m$ such that the gauge group is not fully broken to $H_m = U(1)^r$ and therefore the sum above overcounts the monopole operators.

We need to add a correction factor in order to avoid the overcounting of monopole operators which occurs on the borders of the Weyl chambers of $G$. Indeed, in this case the gauge group is not completely broken down to $U(1)^r$, but to a generic subgroup of $H < G$.

It was shown in [17] that this classical factor is given by

$$P_G(t, \vec{m}) = \prod_{i=1}^{r} \frac{1}{1 - t^{d_i(\vec{m})}} \tag{128}$$

where $d_i(\vec{m})$ is the degree of the $i$th casimir operator of the group $H_m$ which is unbroken by the flux of the magnetic operators.

We recall that there is always a semisimple lie group has a number of casimir operators which is equal to the rank of the group itself. As a reference, the degrees casimir operators of the classical groups are given in the following table:

| Simple Lie Algebra $\mathfrak{g}$ | Degrees |
|---|---|
| $a_l, \quad l \geq 1$ | $2, 3, \cdots, l+1$ |
| $b_l, \quad l \geq 2$ | $2, 4, \cdots, 2l$ |
| $c_l, \quad l \geq 3$ | $2, 4, \cdots, 2l$ |
| $d_l, \quad l \geq 4$ | $2, 4, \cdots, 2l-2, l$ |
| $e_6,$ | $2, 5, 6, 8, 9, 12$ |
| $e_7,$ | $2, 6, 8, 10, 12, 14, 18$ |
| $e_8,$ | $2, 8, 12, 14, 18, 20, 24, 30$ |
| $f_4,$ | $2, 6, 8, 12$ |
| $g_2,$ | $2, 6$ |

Table 4: Degrees of the casimir invariants of the simple Lie algebras.

The last ingredient we need to be able to write the Hilbert Series for a generic coulomb branch, is a rule which allows us to compute, in a simple way, the residual $H_m$ group, knowing the gauge group $G$ and the magnetic charge $m$. Let us now explain how to find out which is the residual $H_m$ gauge group, given the gauge group $G$ and a magnetic charge $m$.

Recall that $m$ can be thought as a matrix in the adjoint representation of $\hat{G}$. In order to make a concrete example, let the rank of the gauge group be $r = 3$. Then $m$ is a diagonal $3 \times 3$ matrix

$$m = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{129}$$

The gauge group acts on $m$ via the adjoint representation $m \mapsto m' = gmg^{-1}$.

By requiring that $m$ is left invariant by a gauge symmetry, namely $m' = m$ one sees that if $m_1 \neq m_2 \neq m_3$, then $g$ must take the form

$$g = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \tag{130}$$

with the $a_i \in \mathbb{R}$.
Therefore, wee see that $g$ is the adjoint representation of a generic element of $U(1) \times U(1) \times U(1)$ and therefore we can say that if the $m_i$ are all different, then the gauge group is completely broken.

On the other hand, consider now the case in which $m_1 = m_2 \neq m_3$. In this case

$$m = \begin{pmatrix} m_1 \cdot 1_2 & \\ 0 & m_3 \end{pmatrix}. \tag{131}$$

Therefore, looking for the most generic $g$ such that $gmg^{-1} = m$, we should look for a block diagonal $g$. The upper-right block is $2 \times 2$ and should satisfy $g_1 \cdot 1_2 \cdot g_1^{-1} = 1_2$, and (of course) should be an element of a subgroup of $U(3)$. On the other hand, the lower-right block gives a $U(1)$ factor, just as the previous case. We conclude that if $m_1 = m_2 \neq m_3$, the gauge group $G = U(3)$ is broken to $H_m = U(2) \times U(1)$.

It should be clear now how to proceed in a generic case, given the assignment of a gauge group $G$ and of a monopole charge $m$. Having defined the correction factor $P(t, m)$, and the rule for finding the residual gauge groups, we are now finally ready to state the correct Hilbert Series for the Coulomb branch.

$$H_G(t) = \sum_{m \in \Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} t^{\Delta(m)} P_G(t, m). \tag{132}$$

In the next section, we will perform explicitly numerous computation using this formula, hoping to shed some light on how this procedure works.

# 11 Some Computations of Coulomb Branches.

In the following we will reproduce some computations of the generators and the relations defining a Coulomb branch of a moduli space for a $\mathcal{N} = 4$ $d = 3$ quiver gauge theory. In in order to give detailed examples and show explicitly how the Hilbert Series Method works.

## 11.1 $U(1)$ with $n$ flavours

Let us compute in detail the Hilbert series for the Coulomb Branch of the theory associated to the following quiver diagram, corresponding to the supersymmetric QED with $n$ flavours.

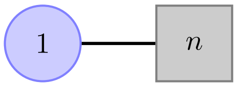



Figure 12: Quiver for $U(1)$ with $n$ flavours.

Recall the dimension formula for the monopole operators.

$$\Delta(m) = -\sum_{\alpha\in\Delta^+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)| \tag{133}$$

In this case, since $U(1)$ is abelian the set of positive roots is empty. $\Delta^+ = \emptyset$.
This leaves us with

$$\Delta(m) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)|. \tag{134}$$

Since we have $n$ hypermultiplets, all in the fundamental representation of the gauge group, we have

$$\Delta(m) = \frac{1}{2}n|\rho(m)|. \tag{135}$$

And the weight of the fundamental reprsentation of $U(1)$ is 1, which gives $|\rho(m)| = |1 \cdot m| = |m|$.
This allows us to write

$$\Delta(m) = \frac{n}{2}|m|. \tag{136}$$

The Hilbert Series is

$$HS_{U(1),n} = \frac{1}{1-t}\sum_{m\in\mathbb{Z}} z^m t^{\Delta(m)}. \tag{137}$$

With a direct computation one finds

$$HS_{U(1),n} = \frac{1-t^n}{(1-t)(1-zt^{\frac{n}{2}})(1-z^{-1}t^{\frac{n}{2}})}. \tag{138}$$

| | dim | charge |
|---|---|---|
| $a$ | 1 | 0 |
| $b$ | $\frac{n}{2}$ | 1 |
| $c$ | $\frac{n}{2}$ | -1 |
| $R_1$ | $n$ | 0 |

Table 5: Generators and Relations for the Coulomb branch of $U(1)$ with $n$ flavours.

Now, in order to look for the generators and relations of the Coulomb branch, we can apply the plethystic logarithm to the Hilbert Series given above.

$$PL\left(HS_{U(1),n}(t)\right) = 1 + t + \left(z + z^{-1}\right) t^{\frac{n}{2}} - t^n. \tag{139}$$

As it is possible to see from the PL above, we find that there are three generators $a, b, c$ of the chiral ring, and one relation $R_1$.
We collect in the following table the dimensions and topological charges of such operators.

From this table we can see immediately that $a = \varphi$, $b = V_1$, $c = V_{-1}$ and the relation is given by

$$a^n = bc \qquad \Longrightarrow \varphi^n = V_1 V_{-1} \tag{140}$$

Therefore the Coulomb Branch of the moduli space of $U(1)$ with $n$ flavours is given by

$$\mathcal{M}_C = \{X, Y, Z \in \mathbb{C}^3 | XY = Z^n\} \tag{141}$$

We can work explicitly on this algebraic variety, and show that for every $n$, there exists a finite group $G_n$ such that $\mathcal{M}_C \simeq \dfrac{\mathbb{C}^2}{G_n}$.

*Proof.* Such a group is $\mathbb{Z}_n$ and the action on $\mathbb{C}^2$ is given by

$$\begin{cases} z_1 \mapsto \omega z_1 \\ z_2 \mapsto \omega z_2 \end{cases}$$

where $\omega$ is the $n-$th root of 1.
We see immediately that we can form three monomials of degree two, which are invariant under such an action. They are $a = z_1 z_2$ $b = z_1^n$ and $c = z_2^n$. Furtheremore they are not independent, but subject to the relation $a^n = bc$.
We conclude that the coordinate ring[17] is given by $\mathbb{C}[a, b, c]/ \langle bc = a^n \rangle$.

Some remarks:

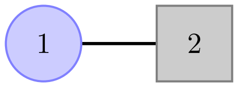
- The complex dimension of $\mathcal{M}_C$ is $\dim_{\mathbb{C}} \mathcal{M}_C = 2$ since there are three complex coordinates and one polynomial equation they must satisfy.

[17] Recall that the coordinate ring $A[k]$ of an algebraic variety $X$ is the set of all well-definite algebraic function over $X$

- Since there is a finite number of generators and a finite number of relations, the Coulomb branch is a complete intersection algebraic variety.
- For a generic number of flavours, there is no enhancement of the hidden symmetry, which remains $U(1)$

**Special case** $(1) - [2]$

Consider now the special case of $U(1)$ with 2 flavours.




As a special case of equation (138) have seen before, the Hilbert Series is given by

$$HS_{U(1),2} = \frac{1-t^2}{(1-t)(1-zt)(1-z^{-1}t)}. \tag{142}$$

Now we would like to make manifest the enhancement of the hidden symmetry. In order to do this we make a change of variables in the fugacity, redefining $z = w^2$.
For further reference, every time we will make a change of variables in the fugacities, we will call it "The fugacity map".
We find

$$\begin{aligned} HS_{U(1),2} &= \frac{1-t^2}{(1-t)(1-w^2t)(1-w^{-2}t)} = \\ &= (1-t^2)\frac{1}{1-t}\frac{1}{1-w^2t}\frac{1}{1-w^{-2}t} = \\ &= (1-t^2)\cdot PE[w^2t]\cdot PE[t]\cdot PE[w^{-2}t] = \\ &= PE\left[\left(w^2+1+w^{-2}\right)t\right] = \\ &= (1-t^2)PE[car([2]_2)_w t] \end{aligned} \tag{143}$$

Since we have been able to recast the Hilbert series into a plethystic exponential of the character of the adjoint of $SU(2)$, this means that the 3 generators of order 1 which we have found, will not transform into themselves according to that representation.
Therefore the hidden topological symmetry is no longer $U(1)$, but is enhanced to $SU(2)$.
This is even more explicit with an application of the plethystic logarithm to the Hilbert series, in order to isolate the generators and the relations defining this algebraic variety.

$$PL(HS_{U(1),2}) = \left(1 + w^2 + \frac{1}{w^2}\right)t - t^2. \tag{144}$$

Some remarks:

- Comparing the $\mathcal{M}_C$ with $\mathcal{M}_H$ (which was computed in section 7) we see that $U(1)$ with 2 flavours is self-mirror since the Coulomb and Higgs branches coincide.
- Quite trivially, since this is a subcase of $U(1)$ with $n$ flavours, the Coulomb (and also the Higgs) branch is a complete intersection algebraic variety.

- As discussed above, there is a symmetry enhancement: the topological hidden symmetry is $SU(2)$ now, and not simply $U(1)$.

### 11.2 $U(2)$ with $n$ flavours

Let us compute in detail the Hilbert series for the Coulomb Branch of the theory associated to the following quiver diagram, corresponding to a theory of gauge group $U(2)$ and with $n$ flavours.

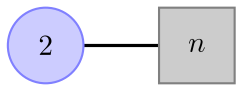



Figure 13: Quiver for $U(2)$ with $n$ flavours.

$U(2)$ has rank 2 and therefore the monopole operator will have as a magnetic charge a two-dimensional matrix $\vec{m} = (m_1, m_2)$.
Recall the dimension formula for the monopole operators.

$$\Delta(m) = -\sum_{\alpha\in\Delta^+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)| \tag{145}$$

In this case, $U(2)$ is not abelian and we must figure out its root system.
Write $U(2) = U(1) \times SU(2)$.
The root system factorizes as a disjoint union of vector spaces $\Delta = \Delta_{U(1)} \sqcup \Delta_{SU(2)}$. The one for $U(1)$ is empty. $\Delta_{U(1)} = \emptyset$.
This leaves us with the computation of the root system of $SU(2)$ and the identification of its positive roots.
It is well known that $SU(2)$ has two roots of length squared 2. $\Delta = \{\alpha, -\alpha\}$ and therefore $\Delta^+ = \{\alpha\}$ Furthermore, in a convenient basis $\alpha = (1, -1)$.
Therefore $|\alpha(m)| = |\alpha \cdot \vec{m}| = |m_1 - m_2|$.
This leaves us with

$$\Delta(m_1, m_2) = -|m_1 - m_2| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)|. \tag{146}$$

Since we have $n$ hypermultiplets, all in the fundamental representation of the gauge group, we have

$$|\rho(\vec{m})| = |m_1| + |m_2|\,. \tag{147}$$

This allows us to write

$$\Delta(m_1, m_2) = -\,|m_1 - m_2| + \frac{n}{2}\left(|m_1| + |m_2|\right). \tag{148}$$

The Hilbert Series is

$$HS_{U(2),n} = \sum_{m_1\geq m_2} t^{\Delta(m_1,m_2)} z^{(m1+m2)} P_{U(2)}(\vec{m}, t). \tag{149}$$

The classical factor $P_{U(2)}(\vec{m}, t)$ is given by

$$P_{U(2)}(\vec{m}, t) = \begin{cases} \dfrac{1}{(1-t)(1-t^2)} & m_1 = m_2 \implies H_{\vec{m}} = U(2) \\ \dfrac{1}{(1-t)^2} & m_1 \neq m_2 \implies H_{\vec{m}} = U(1)^2 \end{cases} . \quad (150)$$

With a direct computation one finds

$$HS_{U(2),n} = \frac{(1-t^n)(1-t^{n-1})}{(1-t)(1-t^2)(1-zt^{\frac{n}{2}})(1-zt^{\frac{n}{2}-1})(1-z^{-1}t^{\frac{n}{2}})(1-z^{-1}t^{\frac{n}{2}-1})}. \quad (151)$$

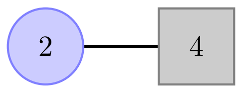
From the Hilbert Series we find that there are six generators $g_i$ $i = 1 \cdots 6$ of the chiral ring, and two relations $R_1, R_2$. We collect in the following table the dimensions and topological charges of such operators.

| | $\Delta$ | $z$ |
|---|---|---|
| $g_1$ | $1$ | $0$ |
| $g_2$ | $2$ | $0$ |
| $g_3$ | $\frac{n}{2}$ | $1$ |
| $g_4$ | $\frac{n}{2}-1$ | $1$ |
| $g_5$ | $\frac{n}{2}$ | $-1$ |
| $g_6$ | $\frac{n}{2}-1$ | $-1$ |
| $R_1$ | $n$ | $0$ |
| $R_2$ | $n-1$ | $0$ |

Table 6: Generators and Relations for the Coulomb branch of $U(2)$ with $n$ flavours.

**The special case** $(2) - [4]$

Consider now the special case of $U(2)$ with 4 flavours.
The quiver diagram is the following




Now we would like to make manifest the enhancement of the hidden symmetry. In order to do this we make a change of variables, with fugacity map given by $z = w^2$.

We find

$$
\begin{aligned}
HS_{U(2),4} &= \frac{\left(1-t^3\right)\left(1-t^4\right)}{(1-t)\left(1-t^2\right)\left(1-w^2t\right)\left(1-w^{-2}t\right)\left(1-w^2t^2\right)\left(1-w^{-2}t^2\right)} = \\
&= \left(1-t^3\right)\left(1-t^4\right)\frac{1}{1-t}\frac{1}{1-t^2}\frac{1}{1-w^2t}\frac{1}{1-w^{-2}t}\frac{1}{1-w^2t^2}\frac{1}{1-w^{-2}t^2} = \\
&= \left(1-t^3\right)\left(1-t^4\right)\cdot PE[w^2t]\cdot PE[t]\cdot PE[w^{-2}t]\cdot \\
&\cdot PE[w^2t^2]\cdot PE[t^2]\cdot PE[w^{-2}t^2] = \\
&= PE\left[\left(w^2+1+w^{-2}\right)t\right] PE\left[\left(w^2+1+w^{-2}\right)t^2\right] = \\
&= \left(1-t^3\right)\left(1-t^4\right) PE[car([2]_2)_w t] PE[car([2]_2)_w t^2].
\end{aligned}
\tag{152}
$$

Since we have been able to recast the Hilbert series into a plethystic exponential of the character of the adjoint of $SU(2)$, this means that the three generators of dimension 1, and the three of dimension 2 will transform into themselves according to that representation.
This is even more explicit with an application of the plethystic logarithm to the Hilbert series, in order to isolate the generators and the relations defining this algebraic variety.

$$
PL(HS_{U(2),4}) = 1 + \left(1 + w^2 + \frac{1}{w^2}\right)t - \left(1 + w^2 + \frac{1}{w^2}\right)t^2 - t^3 - t^4. \tag{153}
$$

In this case we see an enhancement of the topological symmetry, which is $SU(2)$ in this case, and not simply $U(1)$.
Furthermore, from the formula (151) we see that only for $n=4$ there can be symmetry enhancement, and in all the other cases the topological symmetry is $U(1)$.

## 11.3 $U(3)$ with $n$ flavours

Let us compute in detail the Hilbert series for the Coulomb Branch of the theory associated to the following quiver diagram, corresponding to a theory of gauge group $U(3)$ and with $n$ flavours.

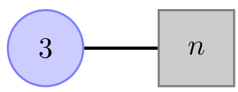



Figure 14: Quiver for $U(3)$ with $n$ flavours.

$U(3)$ has rank 3 and therefore the monopole operator will have as a magnetic charge a three-dimensional vector $\vec{m} = (m_1, m_2, m_3)$.

The classical factor $P_{U(3)}(\vec{m},t)$ is given by

$$P_{U(2)}(\vec{m},t) = \begin{cases} \dfrac{1}{(1-t)(1-t^2)(1-t^3)} & m_1 = m_2 = m_3 \\ \dfrac{1}{(1-t)^2(1-t^2)} & m_1 = m2 \neq m_3 \text{ or cyclic perm} \\ \dfrac{1}{(1-t)^3} & m_1 \neq m_2 \neq m_3 \end{cases} \tag{154}$$

The Hilbert Series is

$$HS_{U(3),n} = \sum_{m_1 \geq m_2 \geq m3} t^{\Delta(m_1,m_2,m3)} z^{(m1+m2+m3)} P_{U(3)}(\vec{m},t). \tag{155}$$

With a direct computation one finds

$$\begin{aligned} HS_{U(3),n} = & \frac{(1-t^n)(1-t^{n-1})(1-t^{n-2})}{(1-t)(1-t^2)(1-t^3)} \cdot \\ & \cdot \frac{1}{(1-zt^{\frac{n}{2}})(1-zt^{\frac{n}{2}-1})(1-zt^{\frac{n}{2}-2})(1-z^{-1}t^{\frac{n}{2}})} \cdot \\ & \cdot \frac{1}{(1-z^{-1}t^{\frac{n}{2}-1})(1-z^{-1}t^{\frac{n}{2}-2})}. \end{aligned} \tag{156}$$

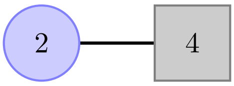
From the Hilbert Series we find that there are nine generators $g_i$ $i = 1 \cdots 9$ of the chiral ring, and three relations $R_1, R_2, R_3$. We collect in the following table the dimensions and topological charges of such operators.

| | $\Delta$ | $z$ |
|---|---|---|
| $g_1$ | $1$ | $0$ |
| $g_2$ | $2$ | $0$ |
| $g_2$ | $3$ | $0$ |
| $g_3$ | $\frac{n}{2}$ | $1$ |
| $g_4$ | $\frac{n}{2}-1$ | $1$ |
| $g_5$ | $\frac{n}{2}-2$ | $1$ |
| $g_6$ | $\frac{n}{2}$ | $-1$ |
| $g_7$ | $\frac{n}{2}-1$ | $-1$ |
| $g_8$ | $\frac{n}{2}-2$ | $-1$ |
| $R_1$ | $n$ | $0$ |
| $R_2$ | $n-1$ | $0$ |
| $R_2$ | $n-2$ | $0$ |

Table 7: Generators and Relations for the Coulomb branch of $U(3)$ with $n$ flavours.

**The special case** $(3)-[6]$

Consider now the special case of $U(2)$ with 4 flavours.
The quiver diagram is the following




As a particular case of the equation (156), we find

$$HS_{U(3),n} = \frac{(1-t^6)(1-t^5)(1-t^4)}{(1-t)(1-t^2)(1-t^3)(1-zt^3)} \cdot \\ \cdot \frac{1}{(1-zt^2)(1-zt)(1-z^{-1}t^3)(1-z^{-1}t^2)(1-z^{-1}t)} \tag{157}$$

Now we would like to make manifest the enhancement of the hidden symmetry. In order to do this we make a change of variables, with fugacity map given by $z = w^2$.

We find

$$\begin{aligned} HS_{U(3),4} &= \frac{\left(1-t^4\right)\left(1-t^5\right)\left(1-t^6\right)}{(1-t)\left(1-t^2\right)\left(1-w^2t\right)(1-w^{-2}t)\left(1-w^2t^2\right)\left(1-w^{-2}t^2\right)} = \\ &= \frac{\left(1-t^4\right)}{1-t}\frac{\left(1-t^5\right)}{1-t^2}\frac{\left(1-t^6\right)}{1-w^2t}\frac{1}{1-w^{-2}t}\frac{1}{1-w^2t^2}\frac{1}{1-w^{-2}t^2} = \\ &= \left(1-t^4\right)\left(1-t^5\right)\left(1-t^6\right)\cdot PE[w^2t]\cdot \\ &\cdot PE[t]\cdot PE[w^{-2}t]PE[w^2t^2]\cdot PE[t^2]\cdot PE[w^{-2}t^2] = \\ &= \left(1-t^4\right)\left(1-t^5\right)\left(1-t^6\right) PE\left[\left(w^2+1+w^{-2}\right)t\right]\cdot \\ &\cdot PE\left[\left(w^2+1+w^{-2}\right)t^2\right] PE\left[\left(w^2+1+w^{-2}\right)t^3\right] \\ &= \left(1-t^4\right)\left(1-t^5\right)\left(1-t^6\right) PE[car([2]_2)_w t]\cdot \\ &\cdot PE[car([2]_2)_w t^2] PE[car([2]_2)_w t^3] \end{aligned} \tag{158}$$

Since we have been able to recast the Hilbert series into a plethystic exponential of the character of the adjoint of $SU(2)$, this means that the three generators of dimension 1, the three of dimension 2, and the three of dimension 3 will transform into themselves according to that representation.
This is even more explicit with an application of the plethystic logarithm to the Hilbert series, in order to isolate the generators and the relations defining this algebraic variety.

$$PL(HS_{U(3),6}) = 1+\left(1+w^2+\frac{1}{w^2}\right)t+\left(1+w^2+\frac{1}{w^2}\right)t^{+}2\left(1+w^2+\frac{1}{w^2}\right)t^3-t^4-t^5-t^6 \tag{159}$$

In this case we see an enhancement of the topological symmetry, which is $SU(2)$ in this case, and not simply $U(1)$.
Furthermore, from the formula (156) we see that only for $n = 6$ there can be symmetry enhancement, and in all the other cases the topological symmetry is $U(1)$.

## 11.4 An educated guess for $U(k)$ with $N$ flavours.

Comparing equations (138), (151) and (156) one could see a recurring pattern.
From this pattern, it is possible to make a resonable conjecture about the formula for the summed Hilbert Series for $U(k)$ with $n$ flavours.
Indeed, we notice that

1. For the $(1)-[n]$ case, there is a generator of dimension 1 and vanishing topological charge, two generators of dimension $\frac{n}{2}$ and opposite topological charge, and a single relation of dimension $n$ and vanishing topological charge.
2. For the $(2)-[n]$ case, there are all the generators and relations of the $(1)-[n]$ case, plus a generator of dimension 2 and vanishing topological charge, two generators of dimension $\frac{n}{2}-1$ and opposite topological charges, and also another relation of dimension $n-1$ and vanishing topological charge.
3. For the $(3)-[n]$ case, there are all the generators and relations of the $(2)-[n]$ case, plus a generator of dimension 3 and vanishing topological charge, two generators of dimension $\frac{n}{2}-2$ and opposite topological charges, and also another relation of dimension $n-2$ and vanishing topological charge.

From this we can conjecture that for the $(k)-[n]$ case, there will all the generators and relations of the $(k-1)-[n]$ case, plus a generator of dimension $k$ and vanishing topological charge, two generators of dimension $\frac{n}{2}-(k-1)$ and opposite topological charges, and also another relation of dimension $n-k$ and vanishing topological charge.

The fact that this conjecture is infact true is not proven explicitly in this thesis.
As a reference, recently Hanany-Cremonesi-Zaffaroni showed perturbatively in [17] that this holds.

Assuming this conjecture true, one can guess the form of the summed Hilbert Series, finding

$$HS_{U(k),n} = \prod_{j=1}^{k} \frac{(1-t^{n+1-j})}{(1-t^j)\left(1-zt^{\frac{n}{2}+1-j}\right)\left(1-z^{-1}t^{\frac{n}{2}+1-j}\right)} \tag{160}$$

### 11.5 $SU(2)$ with $n$ flavours

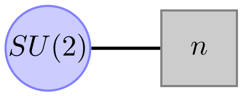



Figure 15: Quiver for $SU(2)$ with $n$ flavours.

It is well known that $SU(2)$ has two roots of the same length. $\Delta = \{\alpha, -\alpha\}$ and we can take[18] $\alpha$ to be of length squared 8. Therefore $\Delta = \{\alpha = 2\sqrt{2}, -\alpha = -2\sqrt{2}\}$ $\Delta^+ = \{\alpha\}$.
To find the GNO dual of $SU(2)$ let us compute the coroots.

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} = \frac{2\cdot 2\sqrt{2}}{8} = \frac{\alpha}{2}. \tag{161}$$

Therefore the GNO dual group of $SU(2)$ is $SO(3)$ which has the same algebra and therefore the same roots.
For this particular computation it is convenient to choose a basis of the weight space which is rotated by an angle $\theta = -\frac{\pi}{4}$ with respect from the original basis. The set of roots is now given by $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

We find $\vec{m} = \left(m, -\frac{1}{2}m\right)$.

Recall the dimension formula for the monopole operators.

$$\Delta(m) = -\sum_{\alpha\in\Delta^+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)| \tag{162}$$

We find $\alpha = (1, -1)$.
Therefore $|\alpha(m)| = |\alpha \cdot \vec{m}| = 2\,|m|$.

This leaves us with

$$\Delta(m) = -\left|m_1 - m_2\right| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in R_i} |\rho_i(m)|\,. \tag{163}$$

Since we have $n$ hypermultiplets, all in the fundamental representation of the gauge group, we have

$$\Delta(m) = \frac{1}{2} n\,|\rho(m)|\,. \tag{164}$$

[18]Remember that the length of a root is completely arbitrary, although by fixing it the lengths of all the other roots and of the weights must be fixed in an according way.

And the weights of the fundamental reprsentation of $SU(2)$ are $\left(\frac{1}{2}, -\frac{1}{2}\right)$, which gives $|\rho(m)| = \left|\frac{1}{2}m - \frac{1}{2}(-m)\right| = |m|$.
This allows us to write

$$\Delta(m) = (n-2)|m|. \tag{165}$$

Now we must figure out the classical corrective factor of $SU(2)$, which accounts for the overcounting of the dressed monopole operators.
$SU(2)$ can either be completely broken to $U(1) \times U(1)$, in the interior of a Weyl Chamber, or be unbroken, remaining just $SU(2)$ in the border.
Given that the Weyl group of $SU(2)$ is $\mathbb{Z}_2$, the border of the chamber occurs for $m = 0$. For all other $m \in \mathbb{N}^*$, the group is completely broken.

Therefore, the classical factor is

$$P_{SU(2)}(t,m) = \begin{cases} \dfrac{1}{(1-t^2)} & \text{if } m = 0 \\ \dfrac{1}{(1-t)} & \text{if } m \in \mathbb{N}^* \end{cases} . \tag{166}$$

The Hilbert Series is then

$$HS_{SU(2),n} = \sum_{m=0}^{\infty} t^{\Delta(m)} P_{SU(2)}(t,m). \tag{167}$$

With a direct computation one finds

$$\begin{aligned} HS_{SU(2),n} &= \frac{1}{(1-t^2)} t^{\Delta(0)} + \frac{1}{(1-t)} \sum_{m=1}^{\infty} t^{\Delta(m)} = \\ &= \frac{1}{(1-t^2)} + \frac{1}{(1-t)^2} \left( \sum_{m=0}^{\infty} t^{(n-2)m} - 1 \right) = \\ &= \frac{1}{(1-t^2)} + \frac{t^{n-2}}{(1-t)(1-t^{n-2})} = \\ &= \frac{1-t^{2n-2}}{(1-t^2)(1-t^{n-2})(1-t^{n-1})}. \end{aligned} \tag{168}$$

Now, in order to look for the generators and relations of the Coulomb branch, we can apply the plethystic logarithm to the Hilbert Series given above.

$$PL\left(HS_{SU(2),n}(t)\right) = 1 + t^2 + t^{n-1} + t^{n-2} - t^{2n-2} \tag{169}$$

As it is possible to see from the PL above, we find that there are three generators $a, b, c$ of the chiral ring, and one relation $R_1$.
We collect in the following table the dimensions and topological charges of such operators.

| | dim |
|---|---|
| $a$ | 2 |
| $b$ | $n-1$ |
| $c$ | $n-2$ |
| $R_1$ | $2n-2$ |

Table 8: Generators and Relations for the Coulomb branch of $SU(2)$ with $n$ flavours.

### 11.6 The $(1)-(2)-[3]$ quiver.

Consider the quiver gauge theory associated with the following graph.

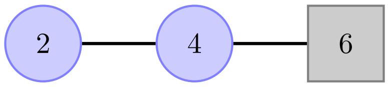



Figure 16: Quiver for the $U(1)\times U(2)$ theory with $n$ flavours.

The dimension formula is given by

$$\Delta(m_1,m_2,m_3) = -|m_2-m_3| + \frac{1}{2}\left(|m_1-m_2|+|m_1-m_3|\right) + \frac{3}{2}\left(|m_1|+|m_2|+|m_3|\right) \tag{170}$$

The center of $G=U(1)\times U(2)$ is given by $Z=U(1)\times U(1)$ and therefore we need two topological fugacities $z_1$ and $z_2$.

The Hilbert Series is given by

$$HS_{(1)-(2)-[3]} = \sum_{m1=-\infty}^{\infty}\sum_{m3=-\infty}^{\infty}\sum_{m2=m3}^{\infty} P_{U(1)}(m_1,t)P_{U(2)}(m2,m3,t)t^{\Delta(\vec{m})}z_1^{m1}z_2^{m2+m3} \tag{171}$$

This can be explicitly summed[19], giving

$$\begin{aligned} HS_{(1)-(2)-[3]}(t,z_1,z_2) = & \frac{(1-t)^2(1-t^3)}{(1-t)^2(1-z_1^{-1}t)(1-tz_1)}\cdot \\ & \cdot\frac{1}{(1-z_2^{-1}t)(1-tz_2)(1-z_1^{-1}z_2^{-1}t)(1-tz_1z_2)} \end{aligned} \tag{172}$$

Applying perturbatively the Plethystic Logarithm we find

$$PL\left(HS_{(1)-(2)-[3]}(t,z_1,z_2)\right) = \frac{(1+z_1)(1+z_2)(1+z_1z_2)}{z_1z_2}t - t^2 - t^3 + o[t^{1000}] \tag{173}$$

[19]For details on how the computation is explicitly performed, we refer the reader to the appendix B of this thesis. In such appendix there is attached the comprehensive code of the computer program that has been used to perform this task.

From which we can immediately infer that there is a relation of dimension 2 and one of dimension3.
Furthermore, by computing numerically the unrefined case, namely

$$\lim_{z_1,z_2\to 1} PL\left(HS_{(1)-(2)-[3]}(t,z_1,z_2)\right) = 8t - t^2 - t^3 + o[t^{1000}] \tag{174}$$

We see that there are 8 generators, of dimension 1.

**Fugacity Map**

Out of what we computed in this case, we could extract some more information about the Coulomb Branch.
Take a look at the term $\frac{(1+z_1)(1+z_2)(1+z_1z_2)}{z_1z_2}t$ in the plethystic logarithm.
We know that there is a hidden symmetry of $U(1)\times U(1)$. However, we could try to show that there is a symmetry enhancement, just as the one seen in the $(1)-[2]$ theory. In order to do this, let us try to perform a change of variables which recasts $(z_1,z_2)^t$ in terms of two new fugacities $(y_1,y_2)$.
The hope is that there exists a suitable change of variable such that the coefficient in front of the $t$ in the plethystic logarithm will take the form of a character of a Lie Group.

In order to find such a change of variable, we first need to define a product operation between a $n\times n$ matrix and a vector, which is different than the usual matrix multiplication.
Call a matrix $A=(a_{ij})$, and a vector $v=v_i$. Then the "star" product

$$A\star v$$

is defined as

$$(A\star v)_i = \prod_j v_j^{a_{ij}} \tag{175}$$

Let us give an explicit example, in order to make this point clear.
Consider a matrix

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

Take a vector

$$v=\begin{pmatrix}a\\ b\end{pmatrix}$$

Then the star product gives

$$A\star v = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}ab^2\\ a^3b^4\end{pmatrix} \tag{176}$$

Let us now go back to our former problem, which was finding a right change of variable between the old fugacities $z_1$, $z_2$, and the new ones $y_1$, $y_2$.

We will take as such change of variables the one given by the star product of the $\mathfrak{SU}(3)_{\mathbb{C}}$ Cartan matrix, with the vector of the new fugacities.

The $\mathfrak{SU}(3)_{\mathbb{C}}$ Cartan Matrix is given by

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{177}$$

We will use this to compute the change of variable discussed above.

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{178}$$

And therefore we find

$$\begin{cases} z_1 = y_1^2 y_2^{-1} \\ z_2 = y_1^{-2} y_2^2 \end{cases} \tag{179}$$

Rewriting the coefficient $\dfrac{(1+z_1)(1+z_2)(1+z_1z_2)}{z_1z_2}$ in terms of the new fugacities gives immediately

$$2 + \frac{y_1}{y_2^2} + \frac{1}{y_1y_2} + \frac{y_1^2}{y_2} + \frac{y_2}{y_1^2} + y_1y_2 + \frac{y_2^2}{y_1} \tag{180}$$

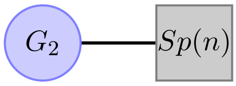
Which we recognize immediately to be the character of the 8-dimensional adjoint representation of $SU(3)$, namely the $[1,1]$ representation in Dynkin labels notation. Therefore, we showed that there is a symmetry enhancement in this case,

$$U(1) \times U(1) \rightarrow SU(3)$$

and the 8 generators of dimension 1 trasform into themselves according to this representation.

### 11.7 $G_2$ with $N$ fundamental flavours

Consider the quiver gauge theory associated with the following graph.




$G_2$ has rank 2 and therefore the monopole operator will have as a magnetic charge a two-dimensional vector $\vec{m} = (m, n)$

Recall the dimension formula for the monopole operators.

$$\Delta(m) = -\sum_{\alpha \in \Delta^+} |\alpha(m)| + \frac{1}{2} \sum_{i=1}^{n} \sum_{\rho_i \in R_i} |\rho_i(m)|$$

In this case, since $G_2$ is not abelian and we must figure out its root system.

This is given by the following diagram, in which $\alpha$ and $\beta$ are the fundamental roots.

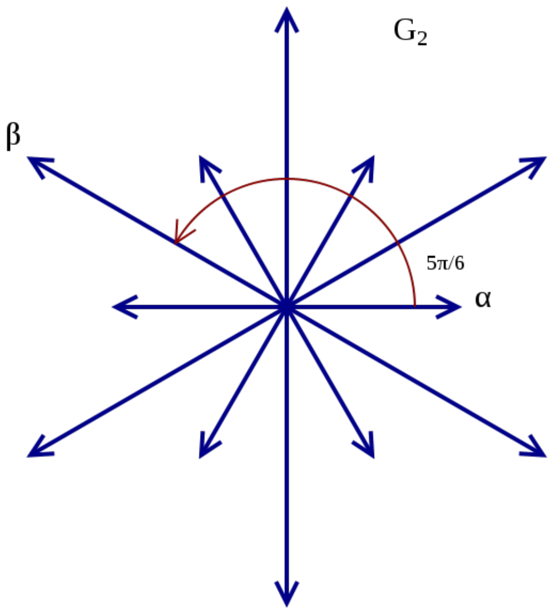



Figure 17: Root System for the complexified lie algebra $\mathfrak{g}(2)_{\mathbb{C}}$

We see immediately that the positive roots are given by

$$\alpha,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ 3\alpha+2\beta \tag{181}$$

Therefore, the negative contribution from the $\mathcal{N}=4$ vector multiplets reads

$$\Delta_{Vplet} = -\left(|m| + |n| + |m+n| + |2m+n| + |3m+n| + |3m+2n|\right) \tag{182}$$

Let us now compute the contribution from the matter fields. This depends on the representation of the fields, which we now assume to be the fundamental. Let us find the fundamental weights of $G_2$, as a first thing. The Cartan matrix reads

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{183}$$

and therefore its inverse is given by

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \tag{184}$$

Therefore we find

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{185}$$

$$\Lambda_1 = 2\alpha + \beta \tag{186a}$$
$$\Lambda_2 = 3\alpha + 2\beta \tag{186b}$$

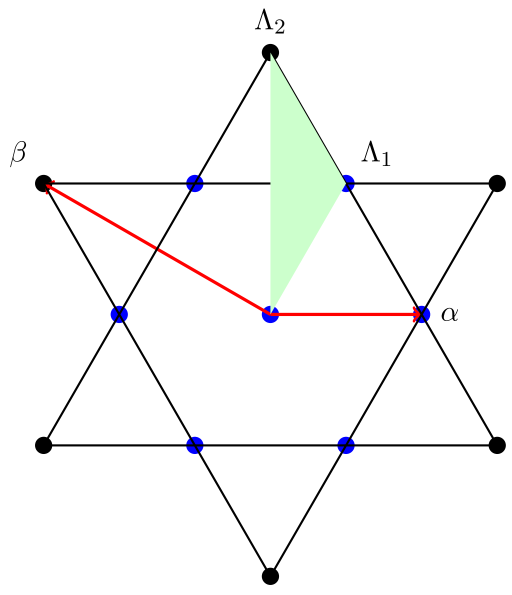



Figure 18: Weight diagram for $\mathfrak{g}(2)_{\mathbb{C}}$

**Classical dressing factor for $G_2$**

In order to find the classical factor, we need first of all to find the Weyl group and the Weyl chambers into which the weight diagram is partitioned. Regarding the Dynkin diagram of $G_2$ as a coxeter diagram, we read immediately the coxeter matrix associated to it. This is

$$C = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix} \tag{187}$$

Therefore, a presentation of the Weyl group $\mathcal{W}_{G_2}$ thought as a coxeter grup, is given by

$$\mathcal{W}_{G_2} = \langle p, q | p^2 = q^2 = 1, (pq)^6 = (qp)^6 = 1 \rangle \tag{188}$$

With some work one can show that this group has 12 elements and is isomorphic to the Dihedral group $D_6$.
Therefore we expect 12 Weyl chambers.

The fundamental weights delimitate the Weyl chamber, i.e the region the span the whole root system when acted upon by the Weyl group. The boundaries and the interior of the Weyl chamber are the "locations" of symmetry breaking/enhancement:

- In the interior of the Weyl chamber $G_2$ is maximally broken. The residual symmetry group is $U_{(1)}^2$ with two Casimir operators of degree $\{1, 1\}$.
- At the two boundaries of the Weyl chamber $G_2$. The residual symmetry group is $U_{(2)}$ with two Casimir operators of degree $\{1, 2\}$.
- At the boundary of the boundaries (the centre) of the Weyl chamber $G_2$ is unbroken. This is obvious, since there are no fluxes turned on! The residual symmetry group is $G_2$ with two Casimir operators of degree $\{2, 6\}$.

Therefore we find

$$P_{G_2}(t, m, n) = \begin{cases} \dfrac{1}{(1-t^2)(1-t^6)} \\ \dfrac{1}{(1-t)(1-t^2)} \\ \dfrac{1}{(1-t^2)} \end{cases} \tag{189}$$

**Hilbert Series and Chiral Ring**

The Hilbert Series is

$$HS_{G_2,N} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} t^{\Delta(m,n)} P_{G_2}(m, n, t). \tag{190}$$

With a direct computation one finds

$$HS_{G_2,N} = \frac{1 + t^{4N-5} + t^{2N-4} + t^{2N-3} + t^{2N-2} + t^{2N-1}}{(1-t^2)(1-t^6)(1-t^{2N-5})(1-t^{2N-6})}. \tag{191}$$

From this we can immediately see that the Coulomb Branch is not a complete intersection, for the fact that the numerator is not in the standard form $\prod_i 1 - t_i^{k_i}$, and it is impossible to put it in such standard form.

To find the lowest orders generators and relations we should compute the Plethystic Exponential. Let us do it in a specific case, $N = 4$. The Hilbert Series reads

$$HS_{G_2,N} = \frac{1 + t^1 1 + t^4 + t^5 + t^6 + t^7}{(1-t^2)(1-t^6)(1-t^3)(1-t^2)}. \tag{192}$$

## 12 Conclusions and Outlook.

As a conclusion of this work, we recall the fundamental steps we have taken, the results achieved, and possible lines of development for this really interesting subject. We have studied the structure of the moduli space of supersymmetric $\mathcal{N} = 4$, $d = 3$ gauge theories.

After a brief review on extended supersymmetry on the plane, we showed that the moduli space that we ought to study consists in two irreducible branches of an reducible algebraic veriety, which are named the *Higgs Branch* and the *Coulomb Branch*. The first of them is classically exact, therefore it does not gain corrections arising from loop or instantonic contributions. On the other hand, the Coulomb Branch is havily lifted by quantum corrections.

The physical interest for which this set of theories has been studied relies in the so called *3d Mirror Symmetry*, which is a duality that exchanges the two branches for a theory which has flown to the superconformal fixed point of the renormalization group. The hope is that a better understanding of this duality will shed light into discovering and understanding more complicated dualities for theories with less supersimmetry, and in four spacetimes dimensions, therefore getting one step closer to the final goal which is the study of the nonperturbative aspects of quantum chromodynamics.

After this introduction, we went over the standard way to compute the differential structure on the Higgs Branch (namely which kind of manifold it is) via the hyperkähler quotient and the Weyl-Molien projection. We gave some example of how this procedure works on simple cases, such as $U(1)$ with $n$ flavours and $SU(2)$ with n flavours.

Subsequently, we carefully explained a rather new procedure for computing the differential structure of the Coulomb Branch, introduced for the first time in [17]. This procedure bypasses other elder attempts of computing the differential structure and the metric, with a completely new approach to the problem. In few words, one exploits the conjectured bijection between "nonperturbative" chiral operators (which are the monopole operators) and holomorphic functions on the Coulomb Branch, in order to gain informations about the ring of holomorphic functions over the branch itself. Having this piece of information, encoded in a mathematical object called the *Hilbert Series* one is able to derive the number of generator and relations defining the fully corrected Coulomb Branch, and in some cases also the differentiable structure of the Coulomb Branch itself.

Some computations were carried out, writing an algorithm in Mathematica. All the computations in [17] have been redone and checked, and applying this method some new results were found: for example the computation of the number of generators and relations defining the Coulomb Branch of a $G_2$ gauge theory with $n$ hypemultiplets of matter sitting in the adjoint representation of the gauge group, or the computation of the number of generators and relationd for the Coulomb Branch of the $(1) - (2) - [3]$ quiver.

# 13 Appendix. Notation

In this appendix we recall the basic conventions on notation used troughout this paper. The first section regards conventions on the signature of the metric tensor and the spinors.
Subsequently, we review the highest weight notation for the irreducible representations of Lie Groups, which is very heavily used throughout this thesis.

## 13.1 Metric tensor and Spinors

The metric tensor field in $\mathbb{R}^{1,p}$ is always assumed to be $\eta_{\mu\nu} = diag(1, -1... - 1)$.
Next, we would like to discuss the spinorial representations in $d = 4$ and $d = 3$. The irreducible minimal spinorial representations of the (complexification of the) Lorentz Algebra in 4 spacetime dimensions, namely $\mathfrak{so}(1,3)^+_{\mathbb{C}} \simeq \mathfrak{su}(2)\times\mathfrak{su}(2)$ are two inequivalent representations of complex dimension 2.
Using a highest weight notation (for which we refer to the next section), we call these representations $[1,0]_{\mathfrak{su}(2)\times\mathfrak{su}(2)}$ and $[0,1]_{\mathfrak{su}(2)\times\mathfrak{su}(2)}$.
The vector space $\mathbb{C}^2$ onto which these representations act is called *the spinorial space.* Therefore, we have two different kind of spinors, which we call *left handed Weyl Spinor* and *right handed Weyl Spinor*, and denote them with

$$\psi_\alpha \qquad \psi_{\dot\alpha} \qquad \alpha, \dot\alpha = 1, 2 \tag{193}$$

Notice that both kind of Weyl spinors have 2 complex degrees of freedom, for a total of 4 real independent degrees of freedom.
In order to be able to consider spinorial fields $\psi_\alpha(x)$, and build a lagrangian density out of them, we must consider them taking values not in $\mathbb{R}$, but in a *Grassmann Algebra.* Such a Grassmann Algebra $\mathcal{G}$ is defined to be the $\mathbb{R}$-span of a set of $\dim\mathcal{G} = n$ anticummuting generators $\theta_i \quad i = 1...n$, which is

$$\{\theta_i, \theta_j\} = 0$$

From this we see, for example, that $\theta_i^2 = 0\ \forall i$ Notice also that the supercharges discussed in the first section of the thesis are arranged into Weyl spinors.

In order to raise or lower the spinorial indices we use the the $SU(2)$ invariant tensor, which is defined to be the 2x2 matrix

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{194}$$

and its inverse is given by

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{195}$$

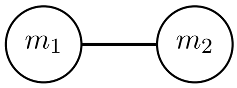
## 13.2 Dynkin Labels and the Dimension Formula

Given a Lie group $G$ we identify a $n$-dimensional representation $\rho : G \to GL(\mathbb{C}, n)$ with its highest weight $\Lambda$.
Expanding $\Lambda$ in a linear combination of the fundamental weight $\mu_i$ we have

$$\Lambda = \sum_i m_i \mu_i$$

We call the coefficients $m_i$ the *Dynkin Labels* of the representation. Therefore each representation is identified with a set of integer numbers $m_i \quad i = 1, \cdots, r$, where $r$ is equal to the rank of the group $G$.

As an example of this notation, consider the representations of $SU(3)$. The Dynkin Diagram is $A_2$




Therefore we label any representation with $[m_1, m_2]$.
For example, the trivial representation is given by $[0, 0]$, the fundamental is $[1, 0]$ and the antifundamental is given by $[0, 1]$.

It is possible to compute the dimension of an irreducible representation from its Dynkin Labels, by using the *Weyl Dimension Formula.* As a reference see [23]. In the case of $SU(3)$, the dimension formula reads

$$\dim[m_1, m_2] = (m_1 + 1)(m_2 + 1)\frac{(m_1 + m_2 + 2)}{2}$$

Therefore we see that the basic representation [1.0], i.e. the fundamental, has dimension 3.
Also the antifundamental, $[0, 1]$ has dimension 3, while the singlet $[0, 0]$ obviously has dimension 1.

# List of Figures



# List of Tables

## References

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