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"STRATEGIC ASSET ASSOCATION IN A LOW INTEREST RATE ENVIRONMENT"

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Abstract

This work examines asset allocation problems in the long run, with a focus on how different assumption on the allocation problem can change the resulting optimal portfolios. The starting point is to assume constant expected returns, this specification is then relaxed in order to allow for return predictability. For an investor whose objective is to maximize its terminal wealth with i.i.d. returns, when parameter uncertainty is not accounted for, the allocation toward the risky component of the portfolio is constant with respect to the investment horizon. Conversely, when the investor incorporates parameter uncertainty the allocation is decreasing as the time horizon increases, even in a constant expected returns environment. This confirms that not including parameter uncertainty in the analysis lead to a non negligible over-allocation. These findings are confirmed with different asset classes and are present even when a multiplicity of risky securities are involved in the analysis. Furthermore when the assumption on return predictability is included, optimal allocation toward the risky component of the portfolio become increasing as the investment horizon grows. This work is complemented with various sensitivity analysis, which allows us to conclude that this over/under allocation phenomena is strongly dependent also from other assumption like investment horizon, risk free rate and risk aversion coefficients

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Introduction

Long run asset allocation has always been a topic that got a lot of attention, both from practitioners as well as from the academic world. The purpose of this work is to study the determinants and this optimal investment strategy, rather than studying the allocation per se. The rationale of this comes from the fact that each investor will have its own tailored made strategy, that can be driven by some specific parameters. Studying how the generic allocation is changing with respect to such specification, can help us outline a generic framework, and draw broader conclusions. Much of our focus will be not only on the parameters underlying the problem (e.g: risk aversion, investment horizon, investment universe), but also on the generic assumption that the investor makes on returns.

Throughout our work, we distinguish between four possible combinations of assumption that the investor can make on the estimation of returns. First of all, the investor can decide whether to incorporate parameter uncertainty or not. Given this choice, for each of the two approach he can then choose whether to assume constant expected returns, or whether to account for return predictability. Comparing these four cases one with the other, enables us to draw some conclusion on what is really driving long run allocation choices. Now we briefly motivate why we chose to study these assumption, and how the thesis is structured in order to analyze the problem.

Accounting for parameter uncertainty is fundamental, especially for highly risk averse individuals. As outlined by Michaud (1989), the Markowitz's optimization algorithm tend to maximize estimation errors, because the investor is not able to estimate with certainty expected returns and variances, which will contain an intrinsic error term. With the Markowitz methodology, we tend to assign more weight to those securities and asset classes that have strong expected returns and high negative correlations, while assigning low weight to those assets with low expected returns and positive correlation.

According to Michaud (1989), the securities to whom we tend to give more importance, are precisely those more likely to be subject to large estimation errors. In light of this, it is very important to accept that we cannot estimate with certainty returns and variances of expected returns, thus it is very important to account for parameter uncertainty. The effects of parameter uncertainty on the long run asset allocation have been studied by both Barberis (2000) and Bawa et al. (1979) extensively. Barberis (2000) focused more on the horizon effect (defined as a decrease in risky asset allocation) induced by the length of the investment horizon itself, while holding sample size still. Bawa et al. (1979) instead dedicated more attention on how estimation risk is affected by the use of different samples, and in particular how the uncertainty is driven by the size of the sample.

Regardless of uncertainty, the investor can decide whether to maintain the assumption of constant expected returns or not. The latter specification is supported by the evidence, historically established in literature, of long run predictability of returns. The works of Campbell (1987), Campbell and Shiller (1988), and Fama and French (1998) have shown how, by using a predictor variable, expected returns show some degree of of predictability in the long run. This ability to forecast returns, seems to increase as the investment horizon of returns increase (see Valkanov (2003))), and this is suitable for our purpose, since are studying allocation at long investment horizon. As a such, it is important to incorporate predictability in returns, in order to better form the expectations on future returns.

The first chapter of the thesis is dedicated to the study the environment of constant expected returns, in which the investor can choose whether to account for parameter uncertainty or not. First of all we show that the optimal allocation toward stocks (ω), is constant with respect to the investment horizon in the case in which the investor has a power utility function, its objective is to maximize the terminal wealth, and assumes constant expected returns. Such that an investor with one month investment horizon and an investor with a 30 years investment horizon, will allocate the same proportion of initial wealth toward stock. This results has been widely acknowledged in literature and is ought to Samuelson (1969) and Merton (1969). Moreover, when the investor gives up the assumption on certainty of estimated parameters and he acknowledges that he is uncertain about them, the horizon irrelevance result is no longer valid even while keeping the constant expected returns assumption. This was first noticed by Barberis

(2000), and we show how the horizon effect is a sizable factor, that drives long-run optimal allocation to stocks to be lower than the short-run ones, by about 10percentage points (on average, depending on risk aversion and investment horizon). Moreover, this horizon effect is strongly driven by the sample used in order to estimate the initial parameters, by reducing the estimation sample to a most-recent one (10years sample), the horizon effect nearly doubles in size, becoming an even more important factor to incorporate in the analysis.

In the second chapter, we explore optimal investment strategies with a multiplicity of assets. While assumption on the utility function and constant expected returns are maintained, we enlarge the investment universe available to the investor. Initially, we simply add another security and we show how the result of horizon irrelevance is confirmed. The allocation mix between the three asset is the same, irrespectively of the investment horizon When we incorporate parameter uncertainty, we confirm the presence of the "horizon effect" also with multiple assets. This implies that the share invested by a short-run investor is higher than the one invested by a long-run investor, for all the asset tested. The analysis is repeated for an an investor who has 5 assets available in his universe, and the results found are somehow different. While the horizon irrelevance results is still present, the effect of uncertainty is mixed with diversification effect given by the combination of different assets. The allocations obtained are no more strictly decreasing along with the investment horizon. This does not imply that the underlying uncertainty effect is disappeared, it is just blurred by some other effects.

In the third chapter, we give up the assumption of constant expected return assuming them to be time-varying. Relaxing this assumption is necessary in order to account for potential predictability in returns. Along with Barberis (2000), we use as a predictor variable the dividend yield. First of all, we outline the theoretical relationship between dividend price ratio and future expected returns (according to Cochrane (2016) Campbell and Kandel and Stambaugh (1996)). Then we provide also econometric evidence of this relationship, and along with Fama and French (1998) andValkanov (2003), we justify that predictability increases as the investment horizon increase. Lower is the frequency of returns, higher is our forecasting ability. After giving the economic intuition and the econometric justification about predictability of returns using the dividend yield, we incorporate this concept in a portfolio choice environment. The results show that, due to mean reversion in returns that slows down the evolution of volatility of cumulative expected excess returns, stocks in the long run appear less risky. This imply that the optimal fraction of wealth that the investor finds optimal to invest in the risky component of the portfolio, is again sensitive to the investment horizon. But now it will increase along with $\hat{T}$, in the sense that a long-run investor is much likely to allocate more to stocks that its corresponding short-run peers. Nevertheless, the results we found seem to be strongly driven by the assumption that we make on the value of the predictor variable, and the values with which it enters the regression equation. As a such, it is important also in the predictability environment to take into account uncertainty in parameter, assuming that we are uncertain about the regression's outcome (both in term of α as well as in term of β). Accounting for uncertainty will make our assumption on the predictor variable's value to be less relevant in our analysis, and allows us to demonstrate the existence of a sort of "convergence-effect" in allocation that account for predictability.

Throughout the various chapters, the results obtained changing the assumption on the initial problem are compared with what is defined as baseline scenario, identified by an investor who assumes constant expected returns and does not account for parameter uncertainty. This comparison helps us to realize how important is to take this phenomenon into account, in order to avoid over-exposure to risks.

Strategic Asset Allocation

Strategic asset allocation can be defined as the way in which an investor decides to allocate its wealth between different asset classes, for a long horizon. The investment horizon is usually really long, ranging between 10 and 50 years, and the re-balancing frequency is low. The goal of strategic asset allocation is generally twofold: it should represent the objective of the organization, and should help to construct a portfolio able to represent the risk/return trade off of the investor. In the sense that, being such a long-horizon choice, should represent the desire of the investor to undertake risks, and the corresponding return that he would like to obtain given the risk assumed. The determination of the strategic asset allocation (S.A.A.) should reflect expectations on the structural behavior of economic variables, and should not be influenced by cyclical shocks. This long-run allocation can be compared with tactical asset allocation (T.A.A.), which instead is supposed to target short and medium run variations in the economic cycle and market sentiment. Nevertheless T.A.A. and S.A.A. should not be threated as two different blocks, but should interact one with the other. Strategic asset allocation provides the reference allocation and asset mix in the long run, which will be cyclically adjusted with tactical choices, that are driven by shorter-term phenomena. Exactly because these two allocation pillars should "cooperate" one with the other, they should not be confused. A recent example of this confusion comes from the widespread behavior of investor, to change its strategic allocation after some shocks occur. A recent example of this, is coming from the two recent financial shocks: the dot-com bubble and the great financial crisis. In the later 90s, investors (and institutional investors as well) were piling up stocks in their portfolio, due to the fact that the return of this asset class where outperforming all the others. This led to the burst of the dot-com bubble, which undermined trust of investors in the stock market, and led them to partly revise their portfolio. In the subsequent period, between 2003 and 2007, investors started again to heavily rely on portfolio largely composed by stocks, which exposed them to severe losses during the great financial crisis in 2008. After this shock, investors radically changed their strategic mix, shifting much of their wealth toward fixed income instruments. This signals a confusion between strategic asset allocation and tactical asset allocation. While these two shocks belong to the short-run market cycle thus should be addressed by tactical choices, the lack of trust and the rise of fear and uncertainty led many investor to revise their strategic asset mix. The magnitude of these shocks was so big, that they make investors change their structural expectations on the economy.

Looking at the historical time series of different asset classes, we can try to see how an investor can decide its strategic mix. In particular, we want to see whether the common practice of investing more to stocks if the investment horizon is long, is supported by the data. The practice to tilt a long horizon portfolio toward stocks rather than toward bonds, is usually done because in the long run stocks tend to outperform all the other asset classes. Stocks are, by nature, riskier than bonds. The owner of a fixed income instrument, has the contractual right to receive back the amount lent, and additionally he can receive periodical payments of interests. This guarantee is not present for owners of stock, who can benefit from the increase in value of the instrument or from the payment of dividends, but has no guarantee that the invested amount will be repaid. As a such, stock investment should somehow repay the investor for the greater risk suffered. If we look at historical annual data of returns for US stock,T-Bills and

T-Bonds, we can find evidence of this. In table 1, we compare the average returns that an investor could have obtained with the different asset classes, in nominal terms. As expected, stock return have the highest average returns, but this is also very much dependent by the evaluation period, and in order to check this, we test this result using four different periods. The first period contain all the observations from 1952 onward

Table 1: Summary of average returns

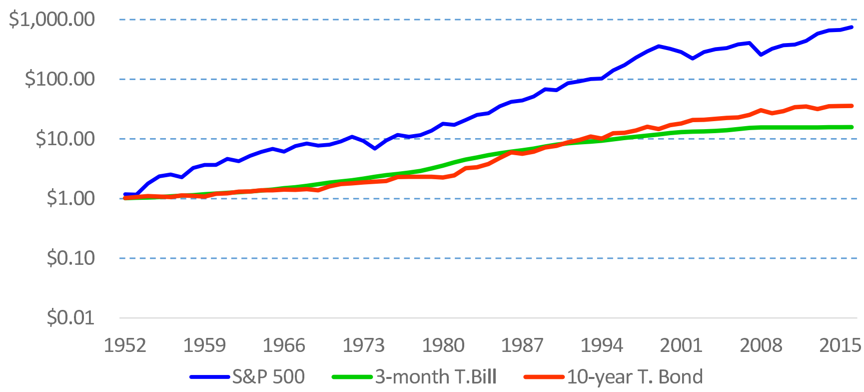
	S&P 500	*3-month T.Bill*	*10-year T. Bond*
1952-2016	12.01%	4.42%	6.07%
1984-2016	12.15%	3.64%	7.89%
2000-2016	6.07%	1.67%	5.72%
2007-2016	8.64%	0.74%	5.03%

Table 2: Risk premium of stocks

	Stocks - T.Bills	Stocks - T.Bonds
1952-2016	7.59%	5.94%
	(2.21%)	(2.49%)
1984-2016	8.51%	4.26%
	(2.86%)	(3.47%)
2000-2016	4.39%	0.34%
	(4.53%)	(6.12%)
2007-2016	7.90%	3.62%
	(6.06%)	(8.63)%

and will be used also throughout the thesis as the "baseline scenario" (instead of using annual data, we will use monthly continuously compounded returns). The other periods tested are shorter and more recent, and we used them in order to check the presence of the previously mentioned shocks. Across these four different periods, we can see that the return of the 10-year bond instrument is relatively stable, while the two other asset classes look more volatile. If we were to consider only the last 16 years of data, stock returns and bond returns were essentially equal, with a risk premium almost equal to zero. This can signal that an investor owning shares was suffering a higher risk, while receiving roughly the same return. The 2000-2016 can serve an example of possible risk incurred by owning a lot of risky asset in the portfolio. People who had a sizable amount of wealth invested in stocks suffered great losses, essentially due to the two recent great financial shocks above mentioned (the dot-com bubble and the great financial crisis). Such shocks can significantly undermine the wealth of investor over-exposed to stocks, and strongly decrease trust. Besides the shocks, we can still see how, for longer samples, both average yearly returns (table 1) and the respective risk premiums (table 2) are generally much higher for stock holdings. This can still suggest that, in the very long run, heavy investment in stock can be the way to go. This argument is also supported by the findings of figure 1, where we calculate the cumulative nominal returns since 1952. If in 1952 an investor would have allocate its wealth in stocks instead of bond, he would have obtained a clearly better performance.

Figure 1: Cumulative returns in nominal terms, 1952–2016



Cumulative nominal return, calculated from yearly returns. The initial point is 1951, which is assumed to be equal to 100. In the graph values are in hundreds of dollars, and the scale is in logarithm base 10.

Evidence from the data support the general guideline given by investment advisors, who suggest that investor with longer horizon should allocate more wealth to stocks, with respect to short run investors. Despite this idea is very widespread and backed by the data, it is against some historical findings in literature, which showed that an investor that estimates returns as i.i.d. should always allocate its wealth in the same way, regardless of the investment horizon.

In this work we try to investigate whether it is correct for a long-run investor to allocate more of its wealth to stock, and how the optimal strategy can change with respect to different assumption.

1

Allocation with parameter uncertainty

In this chapter we analyze investment strategies of an investor whose purpose is to maximize expected utility at the end of a pre-defined period of time (investment horizon). The main focus of the whole chapter is on the importance of the assumption that the investor makes on the first two moments of the returns distribution. In particular, we are interested in studying how to these assumption drive the percentage of wealth that an investor finds optimal to invest in the risky component of its portfolio. In order to do so, we dedicate section 1 to 4 to define the investment framework, specifying utility function, investment universe and the algorithm used to solve the problem. After, we explore the resulting optimal allocation that we find with two different assumptions on returns: whether the investor incorporates parameter uncertainty on its estimation or when he does not account for it. In section 1.5 we show the resulting optimal investment strategy for the latter case, confirming the result found by Samuelson (1969) of horizon irrelevance, such that the fraction of wealth invested is constant over time, regardless of the investment horizon. Then we show the resulting optimal investment strategy for the case in which the investor does incorporate parameter uncertainty, and we show that in this case the allocation is decreasing over time, and that this horizon effect can arise also in a context of independent and identically distributed returns. We then focus on analyzing the importance of the sample of data used in the problem, and we show how, by changing it, we end up with completely different optimal investment strategies. We use four different panel that we define ad-hoc and that help us to better understand the effect of different sample specification. In particular, much of our attention is dedicated to show that a reduction in sample size also cause an increase in uncertainty, which imply a greater horizon effect in the resulting allocation. Finally, we perform a sensitivity analysis on interest rates of the risk-less component of the portfolio, showing the sizable positive impact that a low interest environment has on the fraction of wealth that an investor finds optimal to invest in risky assets.

1.1 Investment universe

As a starting point, the investment universe that we set for our investor will be quiet simple, and identical to the one of Barberis (2000). His work will not only provide us the starting point and guidance, but will also be the benchmark against which we will compare some initial findings. The investor optimizes its allocation choices between a risky asset and a non-risky one. The risky asset is defined as the continuously compounded return on the SP500 index, in the period between 1952 and 18 April 2017. Returns are calculated on a monthly basis, obtaining a sample of 954 months.The risk-less asset is instead a rate held constant for the whole investment horizon. For the initial part of the section, we keep it constant at the exact same rate used by Barberis (2000), which is 0.34% monthly.

1.2 The asset allocation framework

For the purpose of our analysis, we consider a buy-hold investor, which is defined as an investor who, at a given starting point in time t, perform an allocation choice with the informations available at that time and, given its allocation decision, does not change it until the end of the investment period, regardless of the new information flow that he receives. For example, an investor who has a 30-year investment horizon, decides how to allocate at the starting point and then freezes its allocation until the defined period is over. This strategy is compared to the one of a myopic re balancing investor. An investor that myopically re balances is defined as someone who does not consider new informations coming after each period, and at each interval he comes back to the allocation of the initial period t. The re balancing is called myopic because does not take into consideration the additional flow of informations that he receives. Albeit this differentiation is relevant from a theoretical point of view, Barberis (2000) examines both of these approach and founds results essentially identical in term of optimal allocation and with respect to the factors that determines these choices. For these reason, we will restrict our analysis only to the buy-hold investor. As previously said, our investor has to make a decision of how to allocate its wealth at a starting point in time. Moreover, he also fixes an investment horizon $\hat{T}$, which is essentially the point in time until which he hold the allocation chosen. The ultimate goal is to maximize its final wealth (namely the $W_{t+\hat{T}}$), which is defined as follows:

$$W_{T+\hat{T}} = (1-\omega)exp\left(r_f\hat{T}\right) + \omega exp\left(r_f\hat{T} + r_{T+1} + r_{T+2} + ... + r_{T+\hat{T}}\right). \quad (1.1)$$

Of course the investor cannot estimate with certainty how will its wealth in $t+\hat{T}$ periods ahead, but he estimates an expected utility. We thus have to represent the investor's objective function in expected value terms $E_t(W_{T+\hat{T}})$. In order to calculate and optimize the expected wealth, we need to define how much satisfaction we can obtain from each allocation. Therefore, we need to define an utility function that allows us to rank the wealth obtained by the different allocations. As specified by Campbell and Viceira (2002), in order to obtain a tractable model of portfolio choice, we need to make some assumptions about the distribution of asset returns as well as the type of utility function. We choose a power utility function combined with log-normal asset returns. A power utility function tells us that relative risk aversion is constant, while absolute risk aversion is declining with respect to wealth (that in our case is fixed at 1 at investment time). The main motivation of this choice arises from the necessity to study long-term portfolio choice, and therefore the log-normality assumption of asset returns is particularly handy. This because log-normality can be extended easily to multi-period returns, since the product of two log-normal random variables is still log-normal. Having said that, we will define the power utility function as:

$$u\left(W\right) = \frac{W^{1-A}}{1-A}. \quad (1.2)$$

where A is the risk aversion coefficient that will shape our choices. Stating the utility function allows us to finally explicitly write the optimization problem that the investor faces:

$$max_t E_t \left(\frac{\{(1-\omega)\exp(r_f \hat{T}) + \omega \exp(r_f \hat{T} + R_{T+\hat{T}})\}^{1-A}}{1-A} \right) \tag{1.3}$$

with $R_{T+\hat{T}}$ defined as:

$$R_{T+\hat{T}} = r_{T+1} + r_{T+2} + \ldots + r_{T+\hat{T}} \tag{1.4}$$

Equation 1.3 is saying that we are looking for the optimal fraction of wealth to allocate to stocks, identified by the weight ω, that maximizes the terminal expected utility. The expected value in 1.3 not only indicates that the investor must estimated its expected utility, but it also signals that he must perform this calculation conditionally of the informations that he has at the starting point t. This is the assumption common to all the approaches that we will follow during this and the following chapter: the investor uses the past history as an input to perform its optimization. Given this initial assumption, we then give two possibilities to our investor for the estimation of future expected returns: either take into account parameter uncertainty or ignore it in full.

1.3 Including parameter uncertainty

In this first chapter, we assume that the investor does not reject the constant expected return null hypothesis, which is exactly equal to say that he does not allow for any predictability/time variation in expected return. Given this assumption, he will hypothesize future expected returns to be defined as realization of an independent and identically distributed random variable. This assumption will be kept whether we assume certainty in parameters, as well as when we do not. Keeping this assumption for both the cases, will not only enable us to isolate the "pure" effect of parameter uncertainty (without additional consequences of predictability, see chapter3), but will also allow us to demonstrate that parameter uncertainty can arise and have a sizable effect also in a constant expected returns environment. The choice that the investor faces when deciding whether to incorporate uncertainty or not, is whether he will estimate returns as $p(R_{T+\hat{T}}|r, \mu, \sigma^2)$, assuming μ and σ^2 as fixed and given (case 1) or either estimating $p(R_{T+\hat{T}}|r)$, therefore explicitly incorporating uncertainty in parameters.

1.3.1 Case 1: No uncertainty

We will consider first the situation in which the investor does not take into account uncertainty in parameters. In this case, the problem that the investor faces can be defined as follows:

$$max_{\omega} \int u(W_{T+\hat{T}})p(R_{T+\hat{T}}|r, \mu, \sigma^2)dR_{T+\hat{T}} \tag{1.5}$$

We can see how, in equation 1.4, the returns' distribution assumes no uncertainty on the estimation of the parameters. As previously mentioned, in this case the investor uses $p(R_{T+\hat{T}}|r, \mu, \sigma^2)$ in order to represent its belief in future returns. In order to generate returns then, still keeping the assumption of them being independent and identical realization of the same process, he computes them as follows:

$$\begin{aligned} r_{T+1} &= \mu + \epsilon_{t+1} \\ &\vdots \\ r_{T+\hat{T}} &= \mu + \epsilon_{t+\hat{T}} \end{aligned} \tag{1.6}$$

Thus, the sum of future expected returns $R_{T+\hat{T}} = r_{T+1} + r_{T+2} + \cdots + r_{T+\hat{T}}$ is normally distributed according to:

$$R_{T+\hat{T}} \sim N(\mu\hat{T}, \sigma^2\hat{T}) \tag{1.7}$$

The resulting allocation, are shown in the figures in the next section with a blue line.

1.3.2 Case 2: Uncertainty

In contrast to the approach just outlined, the investor could opt for an estimation process in which he does take into account uncertainty. It has been well acknowledged in literature that the degree of confidence that an investor should give to μ and σ, is far from 100%. For this reason, much of the past research tried to show how an investor can still base its observation on historical moments of the returns' distribution, but in parallel he can take into consideration that these estimates are neither fixed and neither certain. Keeping the assumption for which stock dynamics evolves as $r_t = \mu + \epsilon_t$ a Brownian motion, the investor could forecast his expected future returns as $p(r_{T+\hat{T}}|R_{1,j}, R_{1,j+1} \dots R_{1,T})$ thus without making assumptions on μ and σ. The uncertainty on them is explicitly accounted for, by constructing $p(r_{T+\hat{T}}|r)$ (where

r=$R_{1,j+1} \ldots R_{1,T}$), as a predictive distribution. The problem that the investor faces can be therefore represented by the following equation:

$$p(R_{T+\hat{T}}|r) = \int p(R_{T+\hat{T}}, \phi|r)d\phi = \int p(R_{T+\hat{T}}, \phi|r)p(\phi|r)d\phi \tag{1.8}$$

where ϕ=$[\mu, \sigma]$. In equation 1.8 we can see how, in contrast with 1.4, the investor forecasts expected returns only conditionally on historical returns, without making any further assumptions on the parameters μ and σ which are explicitly integrated out. The problem that the investor faces, can finally be expressed as:

$$\int u(W_{T+\hat{T}})p(R_{T+\hat{T}}|\phi, z)p(\phi|r)d\phi dR_{T+\hat{T}} \tag{1.9}$$

In order to integrate out uncertainty, the investor should create its forecast using a bayesian environment. The following steps are borrowed from Barberis (2000) who himself relies on the methodology outlined by Zellner (1971). The starting point is the specification of a prior distribution, that will be the baseline for the calculation of the posterior. We can express it as:

$$p(\mu, \sigma^2) \propto \frac{1}{\sigma^2} \tag{1.10}$$

with 1.10 being an uninformative prior, that summarizes our prior knowledge about parameters distribution. Then, following the approach of Zellner (1971) and using its conversion tables, we have that:

$$\begin{aligned} &\sigma^2|r \sim IG\left(\frac{T-1}{2}, \frac{1}{2}\sum_{t=1}^{T}(r_t - \bar{r})^2\right) \\ &\mu|\sigma^2, r \sim N\left(\bar{r}, \frac{\sigma^2}{T}\right) \end{aligned} \tag{1.11}$$

For the sake of notation, $\overline{r}$ indicates the average return of the stock over the estimation sample. T acts as a discount factor, playing the role of the uncertainty about the distribution of average historical excess returns, and T itself is defined as the size of the estimation sample. As we can see, T also enters into both the scale and the shape parameters of the inverse gamma distribution in 1.11. The steps to be followed in order to perform the integration over the uncertainty consists in:

- First we take a reasonably large number of draws (we use 1000000) from $p(\sigma^2|r)$, which is an Inverse Gamma distribution.

- For each of the simulation of σ^2 previously drawn, we sample another time from $p(\mu|r,\sigma^2)$, which is a normal distribution centered around the mean and with a dispersion which depends from the σ^2 we generated, and from the size of the sample T.

- After step one and two, we end up having a matrix with dimensions $nsim \times 2$, that can be described as

$$\begin{bmatrix} \mu_1, \sigma_1 \\ \mu_2, \sigma_2 \\ \vdots \\ \mu_{nsim}, \sigma_{nsim} \end{bmatrix}$$

  For each of these columns, we sample the last time from a normal distribution defined as:

$$R_{T+\hat{T}} \sim N(\mu\hat{T}, \sigma^2\hat{T}) \tag{1.12}$$

  where this time μ and σ are generated through the first two steps.

The intrinsic difference between accounting or not for parameter uncertainty, is given by step 1 and 2. While if we estimate $p(R_{T+\hat{T}}|r)$, we take μ and σ^2 as fixed and given, if instead we use $p(R_{T+\hat{T}}|r,\mu,\sigma^2)$ these parameter are described a distribution themselves.

The way in which we can compare the two approach is twofold: we can take a look at how different are the expected returns forecast with the two methods, or we look at how these two methodologies affect the asset allocation choices, with a particular focus on the long-run.

Before looking directly at the consequences that taking into account for uncertainty has on the optimal fraction of wealth to allocate to the risky component of the portfolio, we briefly describe the algorithm used to solve the problem, and the data that has been used to produce the graphs.

1.4 A sketch of the algorithm

In order to perform the maximization of the utility function described in 1.3, we solve the problem numerically since there is not a closed form solution of our utility function. Regardless of the way in which returns expectations are formed, the integral in equation and , are approximated by taking the average, over the number of simulations, of the wealth calculated using each of the simulated return. The formula used can be written as follows:

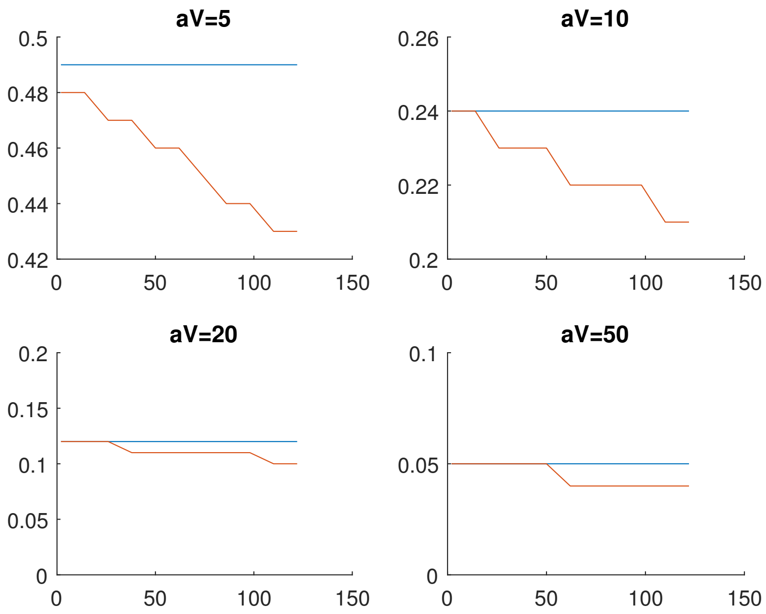
$$\frac{1}{NSim}\sum_{i=1}^{NSim}\left(\frac{\{(1-\omega)\exp(r_f\hat{T})+\omega\exp(r_f\hat{T}+R^i_{T+\hat{T}})\}^{1-A}}{1-A}\right) \tag{1.13}$$

In order to calculate end-of-period utility, the algorithm constructs a grid of possible values that the utility function can take given the weight of the risky and risk free component. We allow our weights to range between 0 and 100, such that we'll have 100 possible combinations (in the case of only 2 assets, one risky and one risk less). Then the algorithm calculates the terminal utility and finds the maximum one. The weight that ensure the maximum utility are the optimal ones, since they give the investor the maximum terminal utility for that period $\hat{T}$. Then the procedure is repeated for each investment horizon, such that we find a vector of optimal weights for each possible $T+\hat{T}$. In the base case, we end up with a grid that has a dimension of 100*$\hat{T}$ (such that in our baseline scenario, with a maximum horizon of 120months, the size will be 100x120), but the complexity of the problem increases quickly when we perform our sensitivity analysis or when we increase the number of assets between which the investor can choose. The number of simulation used to compute posterior distributions is 10000000, which should ensure reliability and stability of results.

1.5 Allocation results

In order to see the results, we plot the different optimal allocation choices for an investor that does take into account parameter uncertainty, and we compare these findings with the case in which he does not account for it. These results can be seen in figure 1.1 in which,first of all, we focus our attention to the blue line. It shows the optimal fraction of wealth that the investor would allocate to the risky component of the portfolio, for a

Figure 1.1: Optimal asset allocation - No-Uncertainty vs Uncertainty



The graphs displays on the y-axis the ω^* allocated to stocks, and on the x-axis the time horizon $\hat{T}$. The blue line represents the optimal allocation without uncertainty, while the red line displays the allocation in which the investor incorporates for it. aV is the risk aversion coefficient time horizon $\hat{T}$, when the investor does not take into account uncertainty in parameters. The line is constructed by connecting the allocations in each point in time, so it is a discrete vector of allocation, in which each entry corresponds to a time horizon. As we can see from the figure, the blue line is constant across the investment period, reflecting the famous result of "horizon-irrelevance" shown by Samuelson (1969) and Merton (1969). Their work showed how variations in the time horizon of the investment does not cause changes to the optimal investment strategy, given that asset returns are i.i.d., the utility function is CRRA and there are no transaction costs. The optimal resulting strategy would be a constant allocation, regardless of the $\hat{T}$. Given this results, it is interesting to compare it with the allocation for an investor that instead accounts for parameter uncertainty.. First of all, we can have a look at the table 1.5 which show the results of the

Table 1.1: Descriptive statistics of the posterior distribution generated using the full sample of informations available

μ	σ
0.0058	0.0012
(0.0013)	(6.36E-05)

posterior generated according to the methodology explained in subsection 1.3.2. Here we can see the mean and volatility of the posteriors' distribution parameters, and the respective standard deviations in parenthesis. As a first step we compare them with the historical parameters, with are respectively 0.58% for μ, and 0.0012 for the volatility. What we can see from table 1.5 is that the posterior distribution is fairly in line with the values we estimated from the historical observations. Nevertheless, since in this case the parameters are generated from a posterior sampling, then they have are also characterized by a standard deviation (value in brackets in table 1.5). The standard deviation is particularly important, since we can interpret it as a measure of confidence and certainty that the investor can give to the generated posterior. Higher is the standard deviation of the parameters, lower will be the confidence that we put on them. As we could expect, the main driver of allocations in the long run will be the standard deviation of the posterior distribution μ. Higher the standard deviation, higher we expect will be the impact of incorporating uncertainty in the problem. Moving to analyze the resulting optimal allocation, we can see from the red line in figure 1.1 that, even in a context of i.i.d. returns, relaxing the assumption of no-uncertainty causes the optimal investment strategy to be no more independent with respect to time. The magnitude of taking into account this uncertainty is substantial, in fact it causes allocation to decrease of a sizable amount, regardless of the risk aversion of the investor. As shown in the recap table 1.2, the

Table 1.2: Allocation results for different risk aversion, at different investment horizon expressed in number of months (e.g: $\hat{T}$=120 imply 10years). Δ is calculated as the difference between $\hat{T}$=120 and $\hat{T}$=1. Results identify the % of wealth invested in the risky component of the portfolio.

	$\hat{T}$=1	$\hat{T}$=36	$\hat{T}$=84	$\hat{T}$=120	Δ
aV=2	1.00	1.00	1.00	1.00	0%
aV=5	0.48	0.47	0.44	0.43	-10%
aV=10	0.24	0.23	0.23	0.21	-13%
aV=20	0.12	0.11	0.11	0.1	-17%
aV=50	0.05	0.05	0.04	0.04	-20%
aV=100	0.02	0.02	0.02	0.02	0%

decrease is essentially uniform in percentage regardless of the risk aversion coefficient used.This is a conclusion that could sometimes be a little bit difficult to get from the graphs, because when the aversion is higher, ω is much lower. In such cases, when we incorporate uncertainty the allocation changes only of some small amount of percentage wealth. Nevertheless, it is fundamental to acknowledge that, even though the decrease is much lower in absolute term, in relative term the effect is sizable and even higher. This conclusion can be taken from the most right column of table 1.2, that shows the change in percentage terms between the shortest and the longest horizons. This findings supports the argument that it is particularly important, especially for strongly risk averse individual, to take into account estimation risk. Looking at figure 1.1, we can now focus on the red line, showing the optimal strategy when we incorporate uncertainty. At very short investment horizon, we can confirm that the allocation is very similar with and without uncertainty. We can see how in this case ω is strongly decreasing through time, causing the allocation for a 120months investor to be radically different from the strategy of a 1month investor. In some cases, differences are greater that 10% in absolute terms. Again we stress the fact that differences in absolute term are only one side of the results, and that we should focus more in relative term variation (shown in table 1.2). The difference of the results obtained in the two allocation problems (with and without uncertainty), is due to the pace at which variance increases through time. In the case in which we do not incorporate uncertainty, the variance is defined according to equation 1.7, where we can see that $\hat{T}$ linearly scales the variance. On the other hand, when the investor incorporates uncertainty, then its forecasts are driven by a higher degree of uncertainty and, additionally, expected returns are autocorrelated. This because when an investor expect one-period ahead returns to be high, then he will also forecast the two period-ahead returns to be high as well. This autocorrelation component causes cumulative variances to increase more than proportionally, and is the main driver of the decrease in the optimal allocation toward stocks through time.

1.6 The importance of estimation sample

The sample that we use to estimate the parameters, plays a key role in our analysis. The consequences of using different sample are also much more sizable in the case in which we incorporate parameter uncertainty, with respect to when we do not. The drivers through which the sample influences the results, are the following:

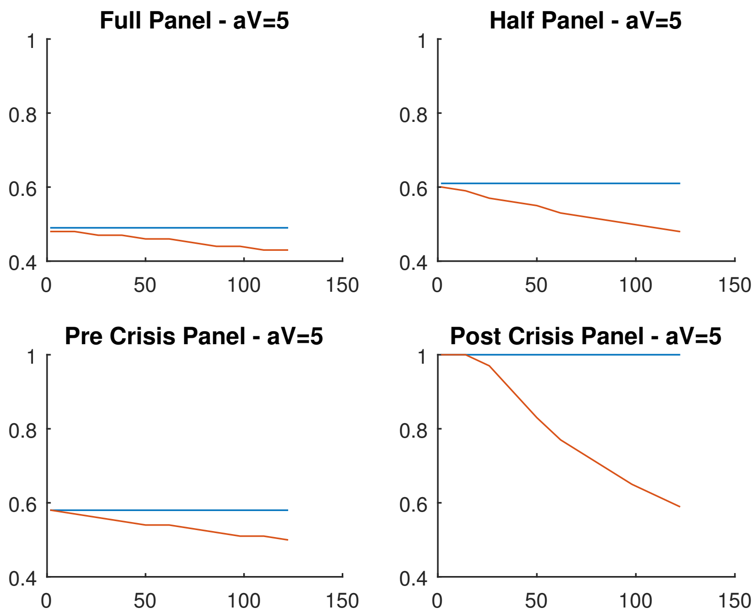
- First of all, it influences the historical μ and σ that we calculate from our sample. If we consider only a period of historically high returns, then we are likely to be biased upwards in our allocation.Since these two parameter are the base of our return forecast, then they play a key role in the resulting optimal allocations.
- The sample matters also in term of size, because shorter is the sample size, greater will be the parameter uncertainty faced by the investor.

In order to prove the importance of the initial sample, we repeat the analysis of the previous section, using four different possible sample compositions:

- The first choice we call it "Full sample", and is when we use the whole dataset from 1952 up until now, of continuously compounded monthly returns. This dataset would be the input for an investor who believes that using the whole history provides a better estimate.
- "Half sample" is instead defined as the full sample broken in half, out of which only the most recent half is considered in our estimation. An investor could prefer this sample because he might find not representative the observations too far in the past.

- A sample that we call "Pre crisis sample" considers only the historical values between 1952 and 2007. Therefore it does not include the great crisis and the most recent observations. This sample can be used in order the assess the impact of the financial crisis.
- Finally, we want to use a "Post crisis" sample. Which is a very-short estimation sample, defined as the observation from 2008 up until today. Such a dataset could be used by an investor who firmly believes that the underlying variable is strongly time-varying, thus prefers to use only very recent informations.

Figure 1.2: Optimal allocation with different samples



The blue line represents the case in which we ignore parameter uncertainty, while the red line is the case in which we incorporate it. All the four graphs assumes the same level of risk aversion α=5, and differ with respect to the sample considered in the estimation.

In the following section, we hold the "full-sample" case as our baseline scenario, and we use it as a benchmark. By comparing the allocation obtained varying the initial sample with respect to the baseline scenario, we can see the impact that the sample itself has on the investor's choices. We start by repeating the exercise done in previous section, and we first take a look at the descriptive statistics of the posterior distributions generated using different estimation samples. In table 1.3 we compare the results using the four

Table 1.3: Recap table of posterior distribution

FullSample		HalfSample	
μ	σ	μ	σ
0.0058	0.0012	0.0067	0.0013
(0.0013)	(0.0001)	(0.0019)	(0.0001)

PreCrisis		PostCrisis	
μ	σ	μ	σ
0.0062	0.0012	0.0098	0.0011
(0.0013)	(0.0001)	(0.0035)	(0.0002)

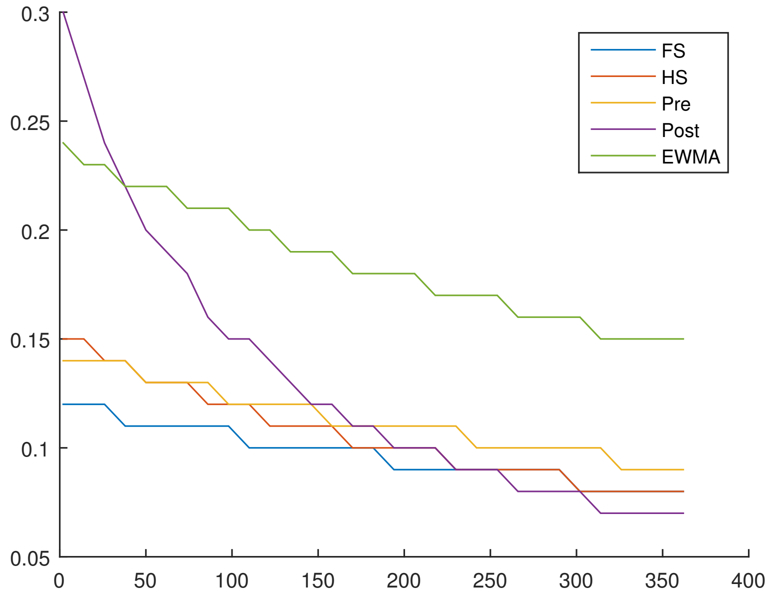
different specifications described above. As we can see, when we change the sample all the parameter estimates change a lot as well. The most important parameter for us is the standard deviation of the posterior μ. As previously mentioned, this parameters signals the degree of uncertainty that the investor does have on the posterior. Higher is the standard deviation of μ, higher will be the difference between the ω of an investor with $\hat{T}$ small and of an investor with $\hat{T}$ high. Additionally, another driver of the investor's optimal allocations in table 1.3 is the posterior' s distribution mean. Higher the mean of the posterior μ, drives up or down the fraction ω of wealth that the investor finds optimal to invest, at the beginning of its investment period. As a such, higher is this value, higher will be the allocation to stock for an investor with short-horizon $\hat{T}$. The reason why an investor could prefer to use a more recent subsample instead of a full one arise from the fact that he might believe that the parameters are described better using more recent data. In fact, he might flag 50-years old observations as unreliable or, at least, less reliable than more recent ones. In figure 1.2 we plot the optimal investment strategy for an investor who has the same risk aversion coefficient (held at α=5), but uses the four different samples. By doing so, we can isolate the effect of the different sample, since we hold all the other assumptions equal in the 4 plots. If we analyze the results, we can see that both the "horizon irrelevance" and the "horizon effect" described in the previous sections, are held regardless of the initial estimation sample. Nevertheless, the impact of these effects on the optimal investment strategy is very different across the four different sample specifications. In order to have an idea of the magnitude that this parameter uncertainty have on the ω, we can compare

Table 1.4: Allocation results for different risk aversion, at different investment horizon expressed in number of months (e.g: $\hat{T}$=360 imply 30years), using the four estimation samples. Δ is calculated as the difference between $\hat{T}$=360 and $\hat{T}$=1. Results identify the % of wealth invested in the risky component of the portfolio.

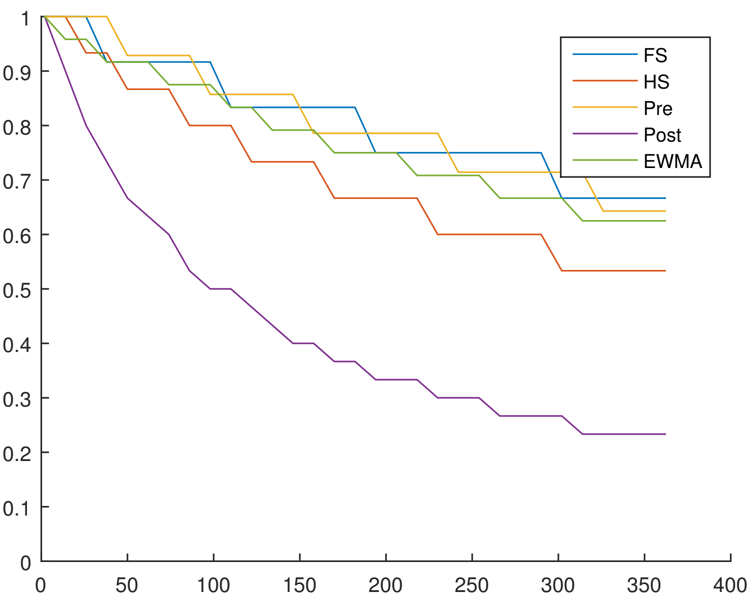
	$\hat{T}$=1	$\hat{T}$=60	$\hat{T}$=120	$\hat{T}$=180	$\hat{T}$=240	$\hat{T}$=360	Δ
			Full sample				
aV=2	1.00	1.00	1.00	1.00	0.98	0.92	-8%
aV=5	0.48	0.46	0.43	0.41	0.39	0.35	-27%
aV=10	0.24	0.22	0.21	0.2	0.18	0.17	-29%
aV=20	0.12	0.11	0.1	0.1	0.09	0.08	-33%
aV=50	0.05	0.04	0.04	0.04	0.04	0.03	-40%
aV=100	0.02	0.02	0.02	0.02	0.02	0.02	0%
			Half sample				
aV=2	1.00	1.00	1.00	1.00	1.00	0.92	-8%
aV=5	0.6	0.53	0.48	0.44	0.4	0.34	-43%
aV=10	0.3	0.26	0.23	0.21	0.19	0.16	-47%
aV=20	0.15	0.13	0.11	0.1	0.09	0.08	-47%
aV=50	0.06	0.05	0.05	0.04	0.04	0.03	-50%
aV=100	0.03	0.03	0.02	0.02	0.02	0.02	-33%
			Pre-crisis sample				
aV=2	1.00	1.00	1.00	1.00	1.00	1.00	0%
aV=5	0.58	0.54	0.5	0.47	0.44	0.4	-31%
aV=10	0.29	0.26	0.24	0.23	0.21	0.19	-34%
aV=20	0.14	0.13	0.12	0.11	0.1	0.09	-36%
aV=50	0.06	0.05	0.05	0.04	0.04	0.04	-33%
aV=100	0.03	0.03	0.02	0.02	0.02	0.02	-33%
			Post-crisis sample				
aV=2	1.00	1.00	1.00	1.00	1.00	0.87	-13%
aV=5	1.00	0.77	0.59	0.47	0.4	0.3	-70%
aV=10	0.6	0.38	0.28	0.22	0.19	0.14	-77%
aV=20	0.3	0.19	0.14	0.11	0.09	0.07	-77%
aV=50	0.12	0.07	0.05	0.04	0.04	0.03	-75%
aV=100	0.06	0.04	0.03	0.02	0.02	0.01	-83%

Figure 1.3

(a) Optimal allocation with different samples



(b) Indexed optimal allocation with different samples. Values displayed are indexed, such that at $\hat{T}$=1 we have y=100.



In both figure a and b, the risk aversion coefficient is held at α=20, in order to isolate the effect of a reduction in the estimation sample size on parameter uncertainty the top-left and bottom right plot in figure 1.2, which show the "Full-sample" and the "Post-crisis" samples respectively. Even though the starting point is different, we can see how using the full sample specification, the horizon effect is around 10% with a 10year horizon. On the other hand, when the investor uses the most recent (and shorter in term of size) subsample, then this effect is around 25%, almost three times higher that the previous specification. The number just quoted can give us an idea about the decrease in the allocation in absolute term of initial wealth. Nevertheless, if we look at the variations in relative terms, then the difference that arise changing the panel is even more striking. If we look at figure 1.4b, we can see the indexed evolution of the variation in the allocation. In order to construct this graph, we divide each ω at each $\hat{T}$, for the ω at $\hat{T}$=1. Using this methodology, we can see how different is the allocation at the end of the horizon with respect to the allocation at the beginning of the horizon. The difference between the four panels that we are using is even more clear here. If we consider the baseline scenario (blue line on the graph), we can see that the difference, in 30 years time, is around 30%. Instead using the "post-crisis" sample, then the difference is around 70%, meaning that, for an investor with risk aversion of 20, $\omega_{\hat{T}=360}$ is 30% of the $\omega_{\hat{T}=1}$. The driver of this difference is again the standard deviation of the posterior distribution's μ. Looking at table 1.3, we can see how the standard deviation using the "Post-Crisis" sample is 0.0035, nearly three times greater than the standard deviation obtained using the entire sample (0.0013). In order to further investigate the impact of parameter uncertainty, we decided to extend the investment horizon, from 120months to 360months (representing 30years ahead). From table 1.4, we can see that not only the decreasing path continues after 10 years, but that it continues in an uniform or, in some case, more than uniform pace. The most-right column of table 1.4, displays the delta in ω, that we calculated as difference between the allocation at $\hat{T}$=30years and $\hat{T}$=1month. As we can see, the greater changes are shown in the bottom panel of table 1.4, that displays the allocation when we use the "Post-Crisis" sample. In relative percentage terms, the horizon effect changes the allocation of the huge amount of 65-75% when we use the most recent sample. As explained in the previous section, the difference in the two allocation problems (with and without uncertainty), is due to the speed at which variance increases through time, and the fact that with uncertainty this pace is more than linear. This higher uncertainty is driven by the size of the sample used in the estimation, as it is shown in the recap table. Lower is the size of the sample, higher is the standard deviation surrounding the posterior distribution's μ. The way in which sample size affect the posterior, can be seen in equation 1.11, where T exactly identify the size of the sample. The dimension of the sample induces the higher uncertainty because the standard deviation of the posterior distribution of returns that the investor generates is higher. Having this higher dispersion in its distribution, makes the investor slightly more uncertain and since, as we said before, when we account for uncertainty variance grows more than linearly with $\hat{T}$, then this difference gets bigger and bigger as the investment horizon increases.

1.6.1 EWMA

In both figure 1.4a and 1.4b, in addition to our four samples we also plotted an allocation line called "EWMA". This line identifies the resulting optimal investment strategy for an investor that uses the full sample as input, but instead of calculating μ and σ as the arithmetic average, calculates them as an exponentially moving average with rolling windows. More in detail, in order to obtain these results, we used a rolling window of 180months, and a time-decay factor λ=0.96. The purpose of this additional methodology is to solve the trade-off that the investor has, between the willingness to use more recent samples, and the fact that by reducing the sample its uncertainty increase sharply. In order to prove that the EWMA methodology can alleviate this problem, we repeat the optimization procedure performed in the previous section, considering EWMA as a new input sample. From table 1.5, we can look at the descriptive statistics of the posterior distribution generated with this methodology. We can see how we obtain a posterior

Table 1.5: Descriptive statistics of the posterior distribution generated using the full sample of informations available, and estimating moments with a rolling EWMA methodology, with a window size of 180months, and λ=0.96

μ	σ
0.0078	0.00092
(0.0013)	(5.40E-05)

returns mean of 0.0078, which is quiet close to the estimate that we had in the "post-crisis" specification (see bottom-right panel of table 1.3). This suggests that the EWMA is successful in giving more weight to the most recent data point. Additionally, we can see that the standard deviation of the returns posterior distribution's mean is much lower using the EWMA, being equal to the full-sample figure. This allows us to conclude that EWMA, both do not induce higher uncertainty in the estimation, and fulfill the desire to give more importance to recent observations. The results obtained using this method are very interesting and can be see in figure 1.4a and 1.4b. The green line plot the optimal strategy obtained, and we can see how the degree of uncertainty that we have is much lower. If we focus on the bottom plot, we can see how the indexed allocation line shows us that the ω is decreasing over time, but at a much lower pace with respect to the case in which we use the most-recent sample. The optimal share of wealth invested in the risky asset at the very end of the investment horizon is 70% of the optimal one at the starting $\hat{T}$, signaling that the horizon effect in 30years time is roughly 30%. Recalling that, when the investor was using the shortest estimation sample, this share was only 30% (so a decrease of 70%), it is quiet a big improvement. The Exponentially weighted moving average, allow us to keep the horizon effect to a reasonable level while assigning more weight to recent observation. Thus, it look like a good strategy to represent investor's preferences.

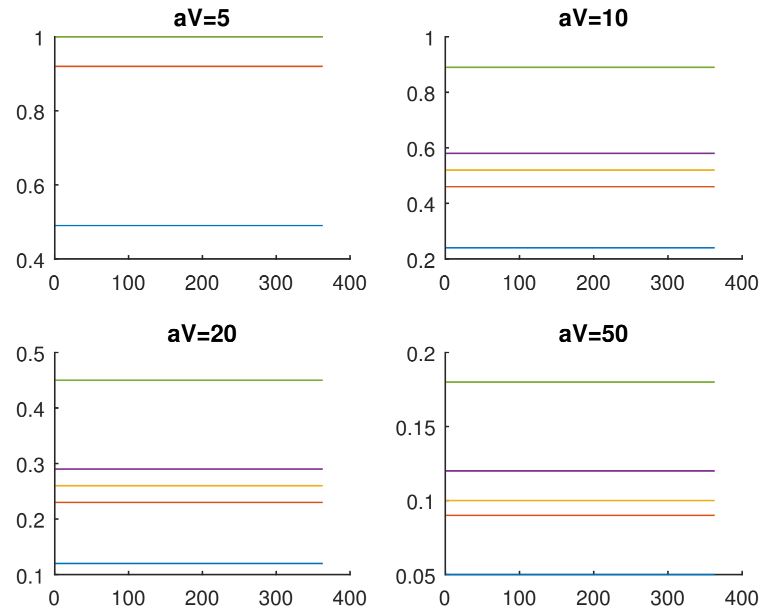
1.7 Low interest rate environment

Up until now, we have assumed that the investor was allowed to choose between two different asset classes: a risky and a risk less one. While the risky asset performance was measured according to its past performance, the risk free asset was assumed constant. Throughout the previous section, our approach followed the one of Barberis (2000), such that we held the risk free rate constant for the whole investment period (even for the 30years!) at a rate of 0.36%. This rate was the real return on T-bills over December 1995, and might have been possible in the past, but currently is no more realistic, especially in light of the fact that an hypothetical risk free security does not exist per se. As a such, the allocations obtained in the previous section, do not really reflect current market conditions and we try to examine what happens when we push rates downward. We perform a sensitivity analysis on the risk free rate, maintaining the assumption that they are constant through the whole horizon, but now we use the following rates as input: [0.34%,0.074%,0,-0.074%,-0.46%]. This sensitivity analysis is carried out simply by repeating the optimization algorithm/problem that we had before, but trying different inputs. Recall that the risk free interest rate enters into our problem in equation 1.3. As of now, we are not interested in analyzing the value of the new allocation per se, but we are interested in the general trend and how changing the risk free rate impacts the optimal resulting allocation overall. For this reason, we analyze the allocations obtained using different risk less rates altogether, focusing on what we could call the "range" of allocation, identified by the difference in ω between the highest and lowest value of the risk free rate. In order to express this "range" concept in term of allocation, we can define this difference as a Δ, calculated as follows:

$$\Delta = \left[\frac{\omega_{(r_f=0.35\%)} - \omega_{(r_f=-0.46\%)}}{\omega_{(r_f=-0.46\%)}}\right] \tag{1.14}$$

Thus, using this Δ we can easily compare and see the effect that a move in the r_f rate cause. In order to see how the risk-less component impacts our results, we can look at figure 1.5, where we display the spectrum of allocations using different rates.

Figure 1.5: Low rates- allocation with different samples - IID



In each graph we plot the ω for an investor that does not account for parameter uncertainty and uses the full estimation sample. Each line corresponds to a different value of the r_f component, and each plot corresponds to a different risk aversion coefficient. Please note that in some case there might be displayed less than 5 lines, when two or more allocations are overlapping (displaying the same values).

In figure 1.5, we can see the allocation range for different input rates, for an investor that uses the full sample and that does not take into account parameter uncertainty. In each of the four graphs, every line correspond to a different risk-free rate, and each graph itself corresponds to a different risk aversion coefficient. Here we can see that the results of the sensitivity is a shift in implied optimal allocation. We can see how, if we consider an investor who has a risk aversion coefficient of α=10, the optimal allocation is 24% in the case with the highest risk-free rate, and 89% when instead we use the lowest r_f. Solely by changing the rate, we obtained completely different optimal investment choices. This results is straightforward, since when we decrease the return of one component of our portfolio, the attractiveness of the other assets(that are not affected by the change) increases.This is clearly shown by the fact that the lines go up as the risk free rate goes down, and the movement in ω is sizable. Provided that the first round effect is the one above described, then we would like to see whether this finding is uniform with respect to different allocations. In particular, we want to investigate whether the risk aversion coefficient plays a role in the impact that the variation of the deterministic rate have. In fact, looking at figure 1.5 and comparing the four plots, we can see how, in absolute terms of ω, it looks like high risk aversion partly neutralize the impact of the sensitivity analysis. Comparing the two figures on the right, with aversion equal to 10 and 50 respectively, we can see how the range between the lowest and the highest allocation is 0.65 in the former case, and 0.13 in the latter. This could potentially lead us to say that the "neutralizing" effect of risk aversion is present and sizable. This finding could also be backed by an economic intuition that, as the investor is more and more risk averse, the optimal share that he is willing to invest in the risky component is less determined by the rate but it is almost solely determined by its riskiness. Albeit this strong argument, the empirical evidence is not that simple. If instead of displaying the allocations in absolute term of wealth, we compare their Δ in relative terms, then the picture changes. In table 1.6 we display the allocations values as well as the Δ for each value of risk aversion. Looking at the results in table 1.6 we can see that, contrary to what expected, a really high averse individual is as sensitive as a low risk averse to changes in the risk free rate. In order to come to this conclusion, we compare the allocation for each different risk free rate, for the four risk aversion coefficients that we are considering. As we can see, the ω of an investor with an α=10 is 3.5 times higher when the risk free rate is at -0.46%, than when the rate is 0.36%. And this

Table 1.6: Low interest rates vs Risk aversion - No Uncertainty

r_f	av=5	av=10	av=20	av=50
0.34%	0.49	0.24	0.12	0.05
0.08%	0.92	0.46	0.23	0.09
0.00%	1.00	0.52	0.26	0.10
-0.08%	1.00	0.58	0.29	0.12
-0.46%	1.00	0.89	0.45	0.18
Δ	-52%	-72%	-72%	-71%

is the same relative difference that we can see with the higher value of risk aversion. Please note that we dropped α=5 from the comparison because the allocation (even though it could potentially go higher) reaches one, which is the maximum possible weight according to our initial constraint, therefore it is not comparable with the others which are below the threshold. The next step is to see the effect of this low interest rate environment in the case in which the investor does take into account predictability. Figure 1.6 shows, for various risk aversion coefficient (each graph) and risk free rate (each colored line correspond to a different rate), the optimal investment choice for an investor that incorporates predictability. We split the that analysis in two, such that all the figures on the right use the full sample as input, while plots on the left use only the shorter and most recent subsample. In order to better compare the results, we indexed the series, such that they all start from one. In particular, for each line the value of every is point in time is expressed as:

$$\left[\omega_{T+\hat{T}}^{r_f,\alpha}/\omega_{T+1}^{r_f,\alpha}\right] \tag{1.15}$$

Thus, the lines displayed do not display the allocation results ω, but they show how different is the optimal allocation in a given $\hat{T}$, with respect to the $\hat{T}$=1. Again, we stress that the message we want to deliver now is no more the optimal allocation itself, but the trend in allocation with respect to: risk aversion, horizon, estimation sample and risk-less rate. In these plots, we can see how the allocations are decresing with $\hat{T}$, because of the horizon effect caused by incorporating parameter uncertainty. The main takeaway is that the horizon effect itself does not influence the impact that variations in the risk free rate have. This is clear when we compare the Δ, shown in the bottom row of table 1.6 for the case without uncertainty, and in the left-panel of table 1.7 for the case that incorporates uncertainty. Values as fairly in line, going against what one could conclude just by looking at the graphs 1.6, in which allocation lines seems to converge. While the convergence is present in absolute terms of ω, in relative term of percentage

Table 1.7: Low interest rates vs Risk aversion - Uncertainty

$\hat{T}$=2

	Full-sample				Reduced-sample			
r_f	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
0.34%	0.48	0.24	0.12	0.05	1.00	0.54	0.27	0.11
0.08%	0.9	0.45	0.22	0.09	1.00	0.74	0.37	0.15
0.00%	1.00	0.51	0.25	0.10	1.00	0.79	0.40	0.16
-0.08%	1.00	0.57	0.28	0.11	1.00	0.85	0.42	0.17
-0.46%	1.00	0.87	0.43	0.17	1.00	1.00	0.57	0.23
Δ	-52%	-72%	-72%	-71%	0%	-46%	-53%	-52%

$\hat{T}$=120

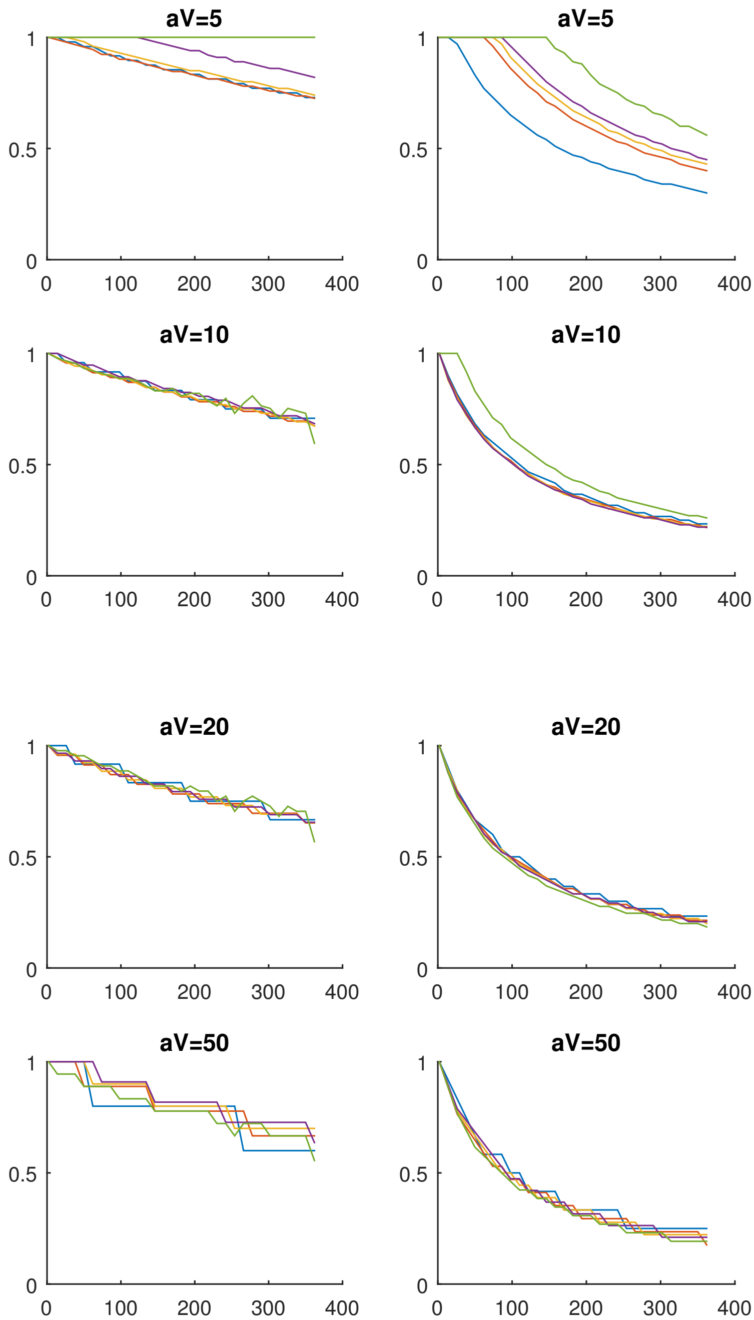
	Full-sample				Reduced-sample			
r_f	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
0.34%	0.49	0.24	0.12	0.05	0.56	0.27	0.13	0.05
0.08%	0.92	0.46	0.23	0.09	0.75	0.36	0.18	0.07
0.00%	1.00	0.52	0.26	0.10	0.79	0.39	0.19	0.07
-0.08%	1.00	0.58	0.29	0.12	0.84	0.41	0.20	0.08
-0.46%	1.00	0.89	0.45	0.18	1.00	0.53	0.26	0.10
Δ	-51%	-73%	-73%	-72%	-44%	-49%	-50%	-50%

$\hat{T}$=360

	Full-sample				Reduced-sample			
r_f	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
0.34%	0.35	0.17	0.08	0.03	0.30	0.14	0.07	0.03
0.08%	0.66	0.31	0.15	0.06	0.40	0.18	0.09	0.03
0.00%	0.74	0.35	0.17	0.07	0.43	0.20	0.09	0.04
-0.08%	0.82	0.39	0.19	0.07	0.45	0.21	0.10	0.04
-0.46%	1.00	0.53	0.25	0.1	0.56	0.26	0.12	0.05
delta	-65%	-68%	-68%	-70%	-46%	-46%	-42%	-40%

variation the proportions are kept as in the no-uncertainty case. As a consequence, we can confirm the sizable effect that the r_f rate has, and that this results is held also when the investor incorporates uncertainty. Moreover, since now that we include uncertainty, the allocation is no more fixed along with the time horizon, we can also see how the horizon effect itself does not change the relative impact that the risk-less rate has. If we focus again on the left part of table 1.7, we can see the Δ for three different snapshot dates. They aim at capture the size of the allocation spectrum at some time horizon. By doing so, we can see whether the sensitivity to the risk-free rate increases or decreases as the horizon effect grows. From the table, we can see that the Δ is more or less equal for the three snapshot dates that we pick when the investor uses the full sample for its estimation. An investor with a risk aversion α=20, shows a range of 72% at the beginning of the horizon, and a range of 73% when the horizon is 120month or 360months. Interestingly enough, we do not find that this finding is kept, when we change the initial estimation sample. As shown in the previous section, the sample used for the estimation of the parameters plays a key role since using a shorter sample induces a greater uncertainty in the returns' posterior distribution and thus results in a stronger horizon effect. If we repeat the sensitivity analysis to the r_f with the "reduced-sample" (identified as the 10 most recent years of observations), then results are radically different and are shown in the right part of table 1.7. Comparing the results with the "full-sample" at the beginning of the investment horizon, we can immediately see that the range of allocation, identified by the Δ, is about 30% lower. Nevertheless, the most interesting result is that not only the allocation spectrum is smaller, but it is also decreasing along with the horizon. If we compare at the three snapshot dates the values of Δ for a given allocation, we can see a sizable decrease, of around 10%, that was not present in the case in which we were using the full estimation sample. The graphical proof of these findings can be seen in figure 1.6, where we show the indexed allocation for the investor using the two different samples (all the graphs on the left refer to the full sample, while graphs on the right to the reduced-sample). These plots help us to again see how the horizon effect is much higher using the smaller sample, and that the allocation are more converging in this case, with respect to the case in which we use the full sample where they are more disperse.

Figure 1.6: Low interest rate environment - Uncertainty



In each graph we plot the ω for an investor that does include parameter uncertainty. All the four plots on the left side use the full sample, while the four grpahs on the right use the "Post-Crisis" sample. Each line corresponds to a different value of the r_f component, and each plot corresponds to a different risk aversion coefficient. Please note that in some case there might be displayed less than 5 lines, when two or more allocations are overlapping (displaying the same values).

2

Extending the investment universe

Throughout the whole first chapter, we analyzed how the long term asset allocation of an investor that chooses between two assets classes, varies when we account or not for uncertainty in estimation of parameters. Given the interesting findings obtained, we try now to extend this environment and get rid of some assumptions that were limiting its potential application to real cases. In particular, we would like to modify the some of the assumption that we used in chapter 1 on the asset classes, and that were adopted in order to follow Barberis (2000).

So far, the investor was given the possibility to choose only between a risky asset (stock-index), and a risk-less asset. The latter was held constant throughout the whole investment period, ad its return was fixed a priori at the value of 0.034% monthly. This value was the real return on a 3M t-bill in December 1995, and we used it in order to be able to compare the results obtained with the literature we were using. Nevertheless, this assumption seems no more realistic in actual market conditions, first of all due to the high return used. The return on the risk-free rate was largely addressed in the last section of Chapter 1, where we performed a sensitivity analysis on its value in order to adapt it to the current low and negative interest rate environment. Additionally, the assumption of a bond as a proxy of a risk free security, does not seems reasonable anymore. As a such, we want to stop treating the fixed income component of the investment universe as a risk-free, and start considering it as a part of the risky share. In order to do so, we now assume that the bond is proxied by a bond index, but whose future expected returns are no more constant but are realization of an i.i.d. process, and the risk-less component of the portfolio is now assumed to be cash, which has a return equal to zero in any period and will be constant. As a result of these changes in the assumption, we end-up having a lager investment horizon, that will be composed by three different securities: cash, stock-index and a bond-index. The last two will form the risky component of the allocation, while cash is composing the remaining risk-less part.

In this extended investment universe, the investor's choice is now no more the optimal fraction of wealth to allocate to one risky asset, but now he can choose how to allocate its wealth between two securities. We can rewrite the investor's objective (previously defined in equation 1.3) in light of the new assumptions, as:

$$max_t E_t \left(\frac{\{(1-\omega_1-\omega_2)\exp(r_f\hat{T}) + \omega_1 \exp(+R^i_{T+\hat{T}}) + \omega_2 \exp(R^j_{T+\hat{T}})\}^{1-A}}{1-A} \right) \tag{2.1}$$

where ω_1 and ω_2 identify each of the two security, and the allocation toward stocks is then defined by difference with respect to the previous two $(1-\omega_1-\omega_2)$. Also in this case, each weight ω ranges between zero and one, such that there is no short selling and no borrowing. As we saw in the previous chapter, the optimization algorithm that we used is based on a grid, that calculates the wealth and the expected utility for each combination of the weights. Now that the number of securities is increased, the grid is increasing in dimension, thus each additional security represent a computational burden for the algorithm. Keeping the number of securities to three, allow us to preserve some speed in the optimization algorithm, that is useful especially when we perform some sensitivity analysis. Nevertheless, we keep in mind that it is relatively straightforward to increase the number of securities.

2.1 The data

The risk less component is defined as cash, held constant at a rate of return of zero. The risky component of the portfolio is represented by the S&P500 index and the Merrill Lynch U.S. Treasuries Index (5-7 Yrs), the former being the "Stock index" and the latter the "Bond index". In the last section, we further extend the investment universe, adding two assets which are:MSCI EMU index, and Merrill Lynch EMU Direct Government Index (7-10 Yrs). All these four assets are considered in the period between December 1990 and October2015, for which we calculate the continuously compounded monthly returns. The European indexes are converted in dollars.

2.2 Methodology

When we extend the investment universe the number of risky securities is no more equal to one, thus the methodology used in previous chapter to estimate future expected returns is no more viable. While in chapter 1 we were focusing only on one risky asset, now we need to capture joint variations of all the assets that we are considering. As a such, if before we could hypothesize the returns to be evolving as $R_t \sim N(\mu, \sigma^2)$, now we need to assume

$$R_t \sim N(\mu, \Sigma) \tag{2.2}$$

with μ and Σ which are respectively the mean vector and the variance covariance matrix. The difference is given by the usage of a variance covariance matrix, that allows us to capture co-movements of the various securities in our portfolio. The case in which expected returns are generated according to 2.2 will form our baseline scenario, namely when an investor with constant expected returns does not account for uncertainty.

Also when we assume that the investor acknowledges his uncertainty in the parameter, we need to revise the methodology used in chapter 1. This because we were using an inverse-gamma distributions in order to model the variance of the posterior distribution, now that we have multiple securities we need to use its multivariate peer, which is the inverse Wishart. The steps for the derivation of the posterior distribution with a multiplicity of assets are borrowed from Meucci (2009) and Meucci (2011), where robust Bayesian portfolio allocations are derived. We use this methodology with some adaptations, in order to be consistent with the environment set out in the previous chapter. The starting point in order to build the predictive distribution, is to assume that the investor summarizes its knowledge about the securities using the historical observations:

$$\hat{\mu} \equiv \frac{1}{T}\sum_{t=1}^{T} r_{t,\tau},\ \hat{\Sigma} \equiv \frac{1}{T}\sum_{t=1}^{T}(r_{t,\tau} - \hat{\mu})(r_{t,\tau} - \hat{\mu}) \tag{2.3}$$

These estimates constitute what we can define as the market knowledge of the investor. It identify the informations that an investor has, based on the sample of data observed. This set of informations, will be then combined with a prior-knowledge on the parameters, in order to construct a posterior distribution that combines these two sets of knowledge. In order to generate the posterior distribution of returns, we need to first specificity our prior distribution. The prior should contain additional information and insights that we have about the market, and that should allow us to improve the precision of our posterior by putting additional information to the simple observation that we can have from historical realization of some assets. The prior that we use is defined, as a Normal Inverse Wishart:

$$\begin{aligned} \Sigma^{-1} &\sim Wishart\left(v_0, \frac{\Sigma_0^{-1}}{v_0}\right) \\ \mu|\Sigma &\sim N\left(\mu_0, \frac{\Sigma}{T_0}\right) \end{aligned} \tag{2.4}$$

with (μ_0, Σ_0) that represent the investor's experience on the parameters, and (v_0, T_0) represent its confidence. In practical terms, the prior distribution is defined in the following manner: Σ_0 is a matrix that is composed by zero everywhere apart from the main diagonal, where we have the sample covariance between the different securities; the prior mean is instead set as:

$$\mu_0 \equiv 0.5 \times \Sigma_0 \frac{1}{N}$$

Albeit the prior is defined as described in equation 2.4, we would like it to be as close as possible to the environment used in chapter 1, in order to make our results comparable. There we were using a non-informative prior, which allowed us to avoid making assumption on prior knowledge on parameters. In order to be as close as possible to such a specification, we must act on how the informations coming from the prior and from the market-observations are combined, in order to merge these two informations set. The update of the market-knowledge with the prior, is done firstly calculating the following parameters:

$$\begin{aligned} T_1 &\equiv T_0 + T \\ \mu_{sum} &\equiv \frac{1}{T_1}\left[T_0\mu_0 + T\hat{\mu}\right] \\ v_1 &= v_0 + T \\ \Sigma_1 &\equiv \frac{1}{v_1}\left[v_0\Sigma_0 + T\hat{\Sigma} + \frac{(\mu_0 - \hat{\mu})(\mu_0 - \hat{\mu})'}{\frac{1}{T} + \frac{1}{T_0}}\right] \end{aligned} \tag{2.5}$$

with T being the observed sample size, and (v_0, T_0) bein the confidence we put on the prior. In order to be as close as possible to a non-informative prior, we can recall the concept of a sample based posterior. In particular we know that, in an allocation model, when we have T that is large with respect to T_0 and v_0, we obtained what is defined as a sample based efficient frontier. In the sense that the results are based almost solely on the sample estimates, and the prior is not adding any relevant information. In order to obtain a sample-based posterior, then we have to assign very low confidence to the prior with respect to the confidence to the sample. In practical term, we set $T_0 \equiv v_0 \equiv \frac{T}{100}$, where T is the number of monthly returns that we have in our sample, which in the case in which we consider data from 1990 up until today, is equal to 318. By doing this, the confidence assigned to the prior is negligible with respect to the confidence assigned to the market observations, such that the assumption made on the distribution of the prior resemble the ones of an uninformative prior. Additionally, this specification has the advantage that, in the case in which the investor will have additional informations about the market, he could simply revise its confidence on the prior in order to obtain prior based efficient frontiers.

The last step is to use the parameters defined according to 2.5, in order to draw from our posterior distribution. As previously mentioned, we used "conjugate" assumptions for the prior dynamics, such that we end up with a posterior which is of the same family of distribution of the prior itself. As a such, the posterior used is the following:

$$\begin{aligned} \Sigma^{-1} \sim & Wishart\left(v_1, \frac{\Sigma_p^{-1}}{v_1}\right), \\ \mu|\Sigma \sim & N\left(\mu_{sum}, \frac{\Sigma}{T_1}\right) \end{aligned} \tag{2.6}$$

where the parameters used, are given by 2.5.

The final step is to draw from the predictive distribution based on the posterior that we just outlined, such that expected returns are simulated as:

$$R_{T+\hat{T}} \sim N(\mu\hat{T}, \sigma^2\hat{T}) \tag{2.7}$$

from which we obtain a vector of simulated expected returns for the different assets. Then in order to find the optimal solution to the optimization problem, the procedure is the same described in section 1.4.

2.3 Portfolio allocation results

First of all, we want to check whether the results found in the previous chapter, namely when the investor could choose only between a stock index and a r_f, are still held. As a such, we should expect our new allocations to be:

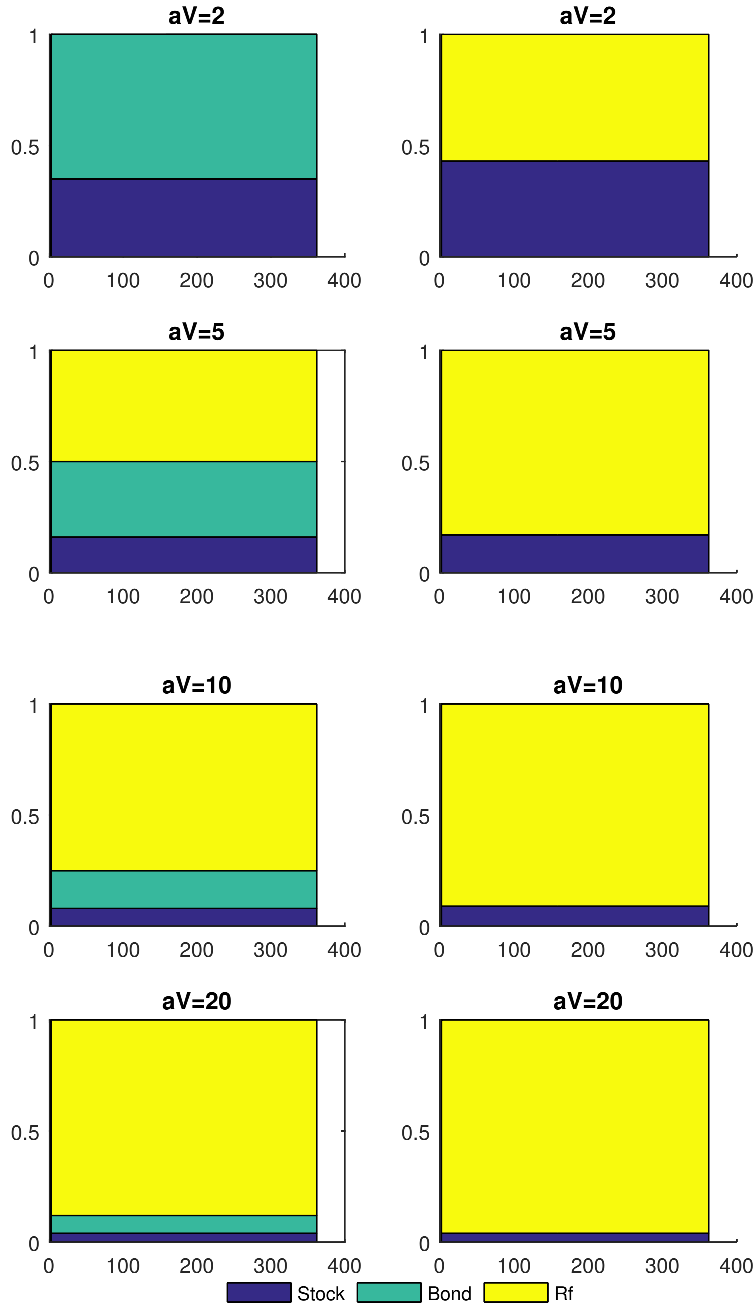
- Constant through time in the baseline scenario (no uncertainty)
- Decreasing as $\hat{T}$ grows, if the investor acknowledges uncertainty

Figure 2.1 shows the optimal investment strategies for an investor that assumes i.i.d. returns, and that does not account for parameter uncertainty. The two plots on the left, correspond to the case in which the investment universe is composed by three asset, whereas the rights plots use the same specifications of chapter 1 (one risky asset and one risk-less). Please note that in all graphs r_f is cash, such that it is assumed constant and with a return of 0. These plots helps us to confirm that the well established horizon irrelevance result is kept, also when multiple assets are available. As we can see the optimal weights of the different securities are not sensitive to the investment horizon since all the areas are constant.

Having established this, then we want to compare the two specification. Firstly, we dedicate our attention to compare the "blue area" for different values of risk aversion for the case in which a bond index is included (all the left plots), with respect to when it is not (all the right plots). As we can see, a slight difference is found when the risk aversion is at aV=2, having that in the left plot ω is 0.35, about 20% lower than 0.43 that we can see in right plot. Despite this difference with the low values of risk aversion, the allocation are fairly similar and in particular we can see that, as the risk aversion increases, the two results in the two specifications become equal. Given that the optimal allocation toward the stock index is fairly stable between the two different specifications, we expect the allocation to the bond index to make the whole difference between these two cases. Recall that the bond index is assumed to be an i.i.d. process, and as a such should be considered as a part of the risky component of the portfolio, rather than part of the risk-less. As it could have been foreseen, given its low volatility (0.0036) the bond index is perceived, when the risk aversion coefficient is low, as a substitute of cash (and in general of the risk free investment). This can be seen looking at the top-left plot in figure 2.1, where we can see that the investor does not allocate any percentage of its wealth in the risk-less component. Of course, as the investor becomes more and more risk averse, despite its low volatility also the bond is perceived as risky and the investor increases its allocation toward cash. In relative terms, we can see how the ω of the bond index is the double (or more) than the ω of the stock index, across all the different risk aversion levels. If we sum both of them, in order to represent the allocation risk appetite of the investor (defined as the overall share of wealth invested in risky assets), we can see how this value is almost 2.5 times higher than the case in which he only has two assets available. The motivation for this difference can be explained using Markowitz portfolio theory and the concept of diversification effect. When the number of available assets increase, such that we move from N to $N + X$ assets, in the worst-case scenario the possible efficient combination will be the same, but for sure will not be less. Generally speaking, the addition of new assets can potentially move the efficient frontier to the left on the risk-return plane, but surely will not make it move to the right. Since a movement in the left of the efficient frontier represents an improvement for the investor, we are generally better off when we add assets in our investment universe. In particular, this improvement is driven by the correlation that the new asset has with the existing one, in particular if it is less that linearly correlated with it. In our simple case, the correlation between the stock and the bond index is only 0.0925, far below 1, and this explains the improvement (defined in term of share invested in the risky component) that we obtain in our portfolio. This improvement can be interpreted again from the efficient frontiers, in particular from the fact that this expansion in the investment universe drags our efficient frontier in the top left of the risk-return space. Such that, with respect to the case in which the investor does not consider the bond index, we now have a portfolio composition that is able to give us a greater return at the same level of risk, or the same return at a lower level of risk. From table 2.1 we can see that risk aversion does not play a role in this, improvement, such that the increase in allocation is stable across the 5 different risk coefficients that we show.

If instead we allow our investor to incorporate parameter uncertainty, then we can see how the allocations are no more stable through time. In figure 2.3, we plot the resulting strategy for the investor that acknowledge his uncertainty, in the two specifica-

Figure 2.1: Increase in the investment universe - Stocks and bonds

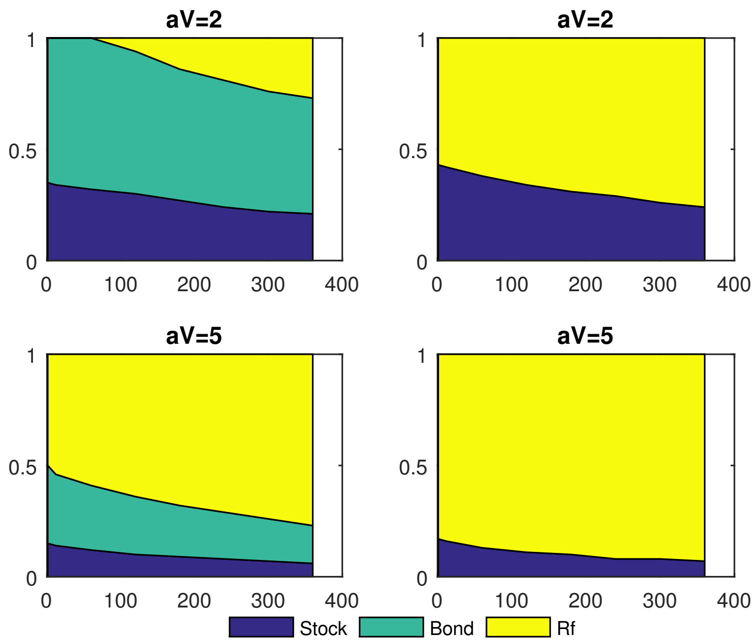


All the figures on the left use the "new" investment universe, assuming cash as risk free and introducing a bond index in the investment universe. The right figures instead use the same assumption of chapter 1. All the figures use daily data since 1990.

Table 2.1: Optimal investment strategy - Portfolio vs Baseline scenario

	Baseline scenario				
	aV=2	aV=5	aV=10	aV=20	aV=50
Stock index	0.43	0.17	0.09	0.04	0.02
R_f*	0.57	0.83	0.91	0.96	0.98
	Portfolio				
	aV=2	aV=5	aV=10	aV=20	aV=50
Stock index	0.35	0.16	0.08	0.04	0.02
Bond index	0.65	0.34	0.17	0.08	0.03
R_f (Cash)	0.00	0.50	0.75	0.88	0.95

Figure 2.3: Increase in the investment universe - Uncertainty



Optimal investment strategy for an investor who does incorporate uncertainty, and has an extended investment universe. "Stock" is the S&P 500, "Bond" is the Merrill Lynch U.S. Treasuries Index (7-10 Yrs) tions. In both plots the r_f rate is held constant at a rate of zero, and the sample of data used is since 1990. As we can see, regardless of the number of securities that form the investment universe, the horizon effect (identified by the decrease of allocation as the horizon increases) is present and sizable. Looking at the left plots, we can see how it is present for all the asset classes that we considered, regardless of their riskiness. Interestingly enough, incorporating or not parameter uncertainty can even change the number of securities that the investor considers. If we look at the top-left figure where we show the allocation with a risk aversion coefficient equal to 2, we can see how an investor is investing only in the bond index and the stock index when the horizon is small while, as T increase (and uncertainty increases along with it), some weight is assigned also to the the risk free security. Another interesting point is that the horizon effect seems to be more sizable for some securities than for others. Looking at the two left plots in figure 2.3, then we can see how the decrease in the bond index (light blue area) is more pronounced than the one of the stock index (blue area) in absolute terms of optimal allocation shares. Despite the fact that the former is decreasing more than the latter in absolute terms, when we look at the relative changes, the situation is reverse. In table 2.3, we express the variation in percentage term, calculated taking ω at the longest and at the shortest horizon. As we can see, the first column (stock index), shows a higher variation with respect to the second one (bond index), and this finding is stable across all the different risk aversion coefficients. In order to explain the higher horizon effect that one security has, we need to recall what causes this effect to arise. As explained in chapter 1, the main drivers of uncertainty through time is the standard deviation of the μ of the posterior distribution. In table 2.2 we show these statistics, and we can see how the uncertainty on this parameter is almost the double for the stock index than for the bond index. This makes the investor more uncertain about future expected returns of that securities, which in turns triggers a higher horizon effect. In particular, we recall that when we account for parameter uncertainty, the variances of cumulative expected returns grow more than linearly, such that as $\hat{T}$ increases, a given security is perceived more and more risky.

Table 2.2: Recap table of posterior distributions

μ		Σ	
Stock index	Bond index	Stock index	Bond index
0.0053	0.0048	0.0151	0.0007
(0.0069)	(0.0034)	(0.0012)	(0.0004)

Table 2.3: Optimal allocation for each risk aversion, evaluated every 10 years

	av=2		
$\hat{T}$	Stock index	Bond index	R_f
1	0.35	0.65	0.00
120	0.30	0.64	0.06
240	0.24	0.57	0.19
360	0.21	0.52	0.27
Δ	-40%	-20%	27%
	av=5		
$\hat{T}$	Stock index	Bond index	R_f
1	0.15	0.35	0.50
120	0.10	0.26	0.64
240	0.08	0.21	0.71
360	0.06	0.17	0.77
Δ	-60%	-51%	54%
	av=10		
$\hat{T}$	Stock index	Bond index	R_f
1	0.08	0.17	0.75
120	0.05	0.12	0.83
240	0.03	0.10	0.87
360	0.03	0.08	0.89
Δ	-63%	-53%	19%
	av=20		
$\hat{T}$	Stock index	Bond index	R_f
1	0.04	0.08	0.88
120	0.02	0.06	0.92
240	0.02	0.05	0.93
360	0.01	0.04	0.95
Δ	-75%	-50%	8%

2.4 Increasing the number of available assets

Having established that uncertainty in parameter is a key factor also when we have more than one asset in our portfolio, we try now to incorporate two additional asset to see how the optimal allocation can change. Up until now we considered a stock index and a bond index, both belonging to the US market. We increase the investment universe, adding two other index, representing the European segment. These two new securities are the MSCI EMU index, and Merrill Lynch EMU Direct Government Index (7-10 Yrs). As a such, our portfolio is now composed by five securities, of which four are considered risky. Two out of these five are representing the stock market (one US and the other EU), and two others are bond index (one US and the other EU). The 5th security is cash, which is considered risk free and has a constant return equal to zero. We stress the fact that all of the five securities were evaluated pairwise (either in a 3-asset portfolio, or as in previous section and chapter 1), and a sizable horizon effect was confirmed.

Figure 2.5 shows the resulting allocations for a portfolio of five securities, when the investor acknowledges its uncertainty and has a risk aversion coefficient equal to two. As we can see, the clear horizon effect that was driving down the allocation is no more so evident, now that we have an increased amount of securities, this effect seems rather mixed. As shown in figure 2.5, the allocations are still overall decreasing with respect to the risk free, but for each of the security we do not have anymore the monotonically decreasing path that we found so far with uncertainty. Interestingly enough, in some cases the ω seem to be increasing through time (i.e:SP500), such that the initial allocation in an asset is lower than the one at the end of the investment horizon.

More in detail, we can look at the allocation tables in tab2.4. While for the first two column the horizon effect is predominant, for the other two we can see how the path of the allocation is not monotonically decreasing. This signals that, apart from the underlying uncertainty, these allocations are driven by some other factors, that can even make the horizon effect less predominant.

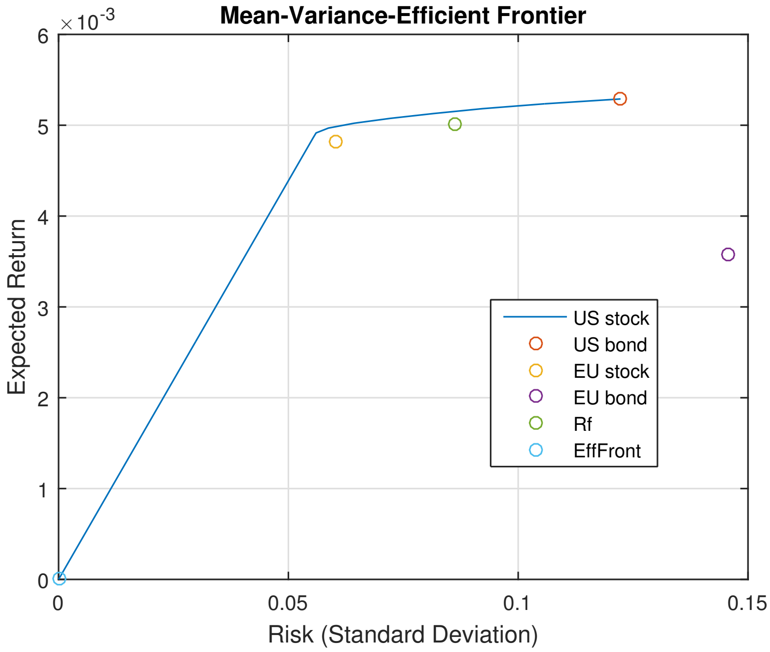
In order to explain this mixed effect on allocation, we need to investigate further how the combination of many assets can influence the optimal strategies. Given that uncertainty effect has been shown for all the securities that we include in our investment universe, we know that this underlying factor (uncertainty), is pushing down the allocations as the horizon increase. For each asset evaluated singularly, the allocation

Table 2.4: Portfolio allocation, evaluated every 5 yeas

T	US bond	EU stock	US stock	EU bond	R_f
1	0.68	0.32	0.00	0.00	0.00
12	0.67	0.28	0.05	0.00	0.00
60	0.65	0.19	0.11	0.05	0.00
120	0.59	0.13	0.13	0.11	0.04
180	0.54	0.11	0.13	0.10	0.12

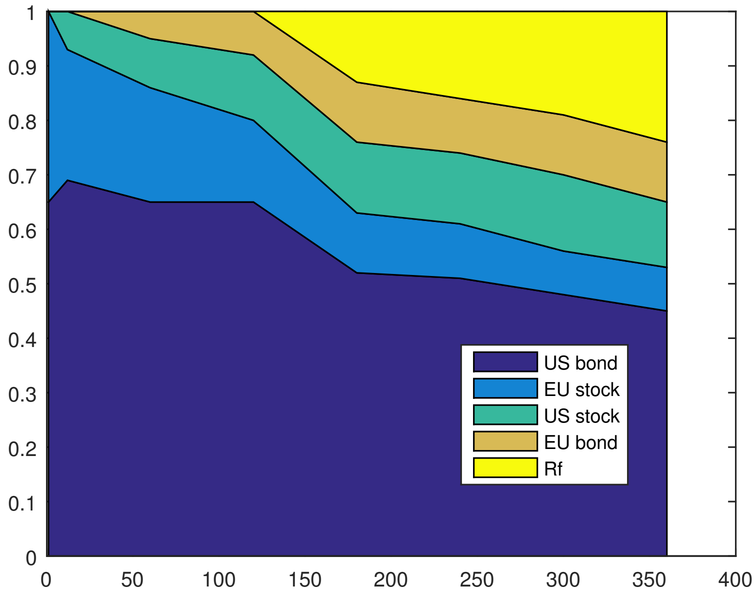
at the beginning of the investment horizon is higher than the one at the end of the horizon. Conversely, we expect the diversification effect not to contribute in an unique way: it might drag down some assets, while it might push upward some others. To better understand how the investor can choose between these portfolio of assets, we can plot on a risk return plane all our securities, as well as the resulting efficient frontier. As

Figure 2.4



we can see from figure 2.4, some assets might seem more appealing than others from a risk-return perspective, and generally speaking securities on the top-left of the plot are the most desirable one. When one asset is not clearly better than the other (i.e: has

Figure 2.5: Optimal allocation with sample data - Uncertainty



higher return but also greater variance), then the risk aversion coefficient helps us in our choice. Despite each security intrinsic characteristic, an important aspect is given by how the different assets can combine one with the other. In particular, if two assets are low or negatively correlated, an investor might find optimal to allocate some weight to them, even if this is not justified from a pure risk-returns analysis. In order to understand how the different securities interact with each other, we can have a look at the variance-covariance matrix estimated from the sample data (see table 2.5). From this,

Table 2.5: Variance-covariance matrix with 5 assets (data estimated from the sample 1990-2015)

	US stock	US bond	EU stock	EU bond	R_f
US stock	0.0150	0.0007	0.0153	0.0035	0.0000
US bond	0.0007	0.0036	-0.0008	0.0030	0.0000
EU stock	0.0153	-0.0008	0.0212	0.0048	0.0000
EU bond	0.0035	0.0030	0.0048	0.0075	0.0000
R_f	0.0000	0.0000	0.0000	0.0000	0.0000

we can understand better what is driving the overall allocation. In particular, the main diagonal is showing us the variance of the single assets, while all the other elements identify the respective covariances. Generally speaking two assets can provide an improvement in investment opportunities if they have negative or low correlation, because their combination can help to hedge some risk.

Given the characteristics of the securities that compose our portfolio, we want to see how are the optimal investment strategies. First of all, we analyze the result shown in figure 2.5, where most of the wealth is allocated to "US bond" and "EU stock". While the former is a very good security in risk returns terms (as can be see from 2.4), the latter is the worse among all the assets we have in our pocket. Despite its low attractiveness, we still allocate about 30% of our wealth to it, which makes it the second most important asset in ur portfolio. This is motivated by the slightly negative covariance (-0.00082) that this assets (EU stock) has with our main security (US bond).

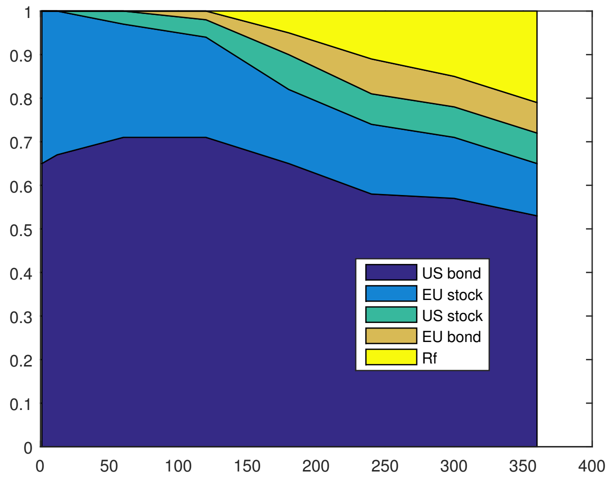
In the converse way, we can explain the low allocation toward the "EU bond index". From a risk-return perspective, this assets is a really good one, on the other hand we can see how its covariances is positive and relatively high with respect to all the other securities. This penalizes it, such that we assign only a weight of about 5-7% (with $aV = 2$) despite its high return and relatively low risk.

To better explain the diversification effect, we can artificially try to fix the correlation between two assets. In order to do so, we assume that the covariance between two assets is no more the one estimated from the real dataset, but that it is set arbitrarily. As a such, we end up having a new variance covariance matrix, that reflects the assumption we make on the correlation. Note that this will results also in different posterior distribution of the variance itself, which is estimated according to the methodology described in section 2.2. We impose this assumption on the correlation only on 2.3, while 2.4 is left untouched.

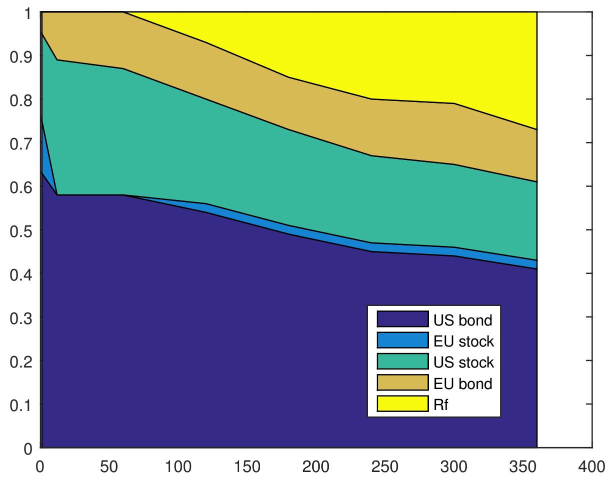
More in detail, in order to verify the effect of co-movements between assets on the allocation, we change the correlation coefficient between the "US bond" and "EU stock". We try two different cases, by imposing it equal to -0.5 in the first case and to 0.5 in the second specification. The choice of the assets is given by the fact that these are the two main assets in our portfolio (in terms of ω), and the allocation to the "EU stock" seems only correlation driver, rather than risk-return driven .The results for the first specification, are shown in figure 2.7a, for a risk aversion coefficient of aV=2. As we can see, a decrease in the correlation implies a even greater allocation toward the

Figure 2.6: Portfolio allocation - changing the correlation

(a) Optimal allocation with correlation at -0.5 - Uncertainty



(b) Optimal allocation with correlation at 0.5 - Uncertainty



In the upper figure, we impose the correlation coefficient between US bond and EU stock to be equal to -0.5. On the bottom plot, we impose the same coefficient to be equal to 0.5

"EU bond", and a lower allocation toward the other peers. This specification results in an optimal portfolio which is even more concentrated on two assets.

When we evaluate the second case, namely when the correlation coefficient is set at 0.5, we can see that the optimal strategy radically changes. Now that the security "EU stock" is no more uncorrelated with the others, it immediately loses appeal to the investor. Since the correlation coefficient is changed for that asset, its evaluation is based on the risk-return performance which is quiet bad compared with the other peers. As a such, the artificial increase in correlation cause the investor to stop considering it as a desirable asset. This result in a drop in allocation toward the "EU stock" to a negligible amount (around 3%). If we compare it with the previous specification, the decrease is in the order of 30% for short investment horizon, and around 15% for longer horizon.

Given that uncertainty effect makes stock appear riskier in the long run, driving allocation down as the horizon increases, the diversification effect seems to be rather mixed. It does not have a clear path with respect to time. In some cases it might cause allocation to decrease while in other cases it might push them upward. Including more assets in the analysis can provide some interesting results, but also makes uncertainty effect to be more blurred.

3

The predictability of asset returns

Predictability in stock returns has always been a trending topic in finance, and evidence of it has been widely provided in literature (see Keim and Stambaugh (1986), Fama and French (1998) Campbell and Shiller (1988)). The whole purpose of predictability is to try to infer some informations about the realizations of a process, using another variable. Of course the goal of predictability is not to forecast precisely the future returns, but to try to see whether expected risk premium vary over time (such that they are not constant in expected value), and if some of this variation can be somehow explained. Even the ability to explain a small part of this variation, can be a great advantage for an investor that has to decide how to build its portfolio. Much of the researches in the past used as a predictor variable the dividend yield of a stock, in order to predict future realizations of that stock itself. This concept and predictor variable are our starting point as well, and we use it in order to incorporate the concept of predictability into the portfolio optimization problem that we have seen in previous chapters. In order to do this, we need to re-consider some of the assumptions used up until now. While we will keep the characteristics of the investor, optimization algorithm and constraints on parameters exactly equal, in order to incorporate predictability we need to drop the constant expected returns assumption. It comes straightforward that, in order to try to predict future returns, we need to assume that they are time varying rather than constant. Additionally, some of the results shown in chapter 1, especially the evidence of mean reversion in expected returns as the horizon increases, provide us further motivation to relax the constant expected returns assumption. In this chapter, we firstly focus on understanding the issue of predictability from an economic point of view, then we justify the economic intuition providing econometric evidence, finally we implement the predictability environment in a strategic asset allocation model. We focus our analysis on the impact that including predictability have on the investment choice in the long run, showing how this can lead to strategies completely different with respect to the one we have seen in the previous chapter. Then we show the importance of the value of the predictor variable used in the estimation, and we perform a sensitivity analysis in order to show that, when the investor accounts for uncertainty, this importance decreases.

3.1 The relationship between dividends and expected returns

In order to understand the economic relationship between returns and dividend, we strongly rely on the work of Campbell and Shiller (1988) and Campbell (1991) and some of their conclusions are the starting point of our analysis. They showed how returns are predictable, especially in the long-run framework, while the forecasting variable (dividend yield) is not predictable itself. These statements are the basis of the whole topic of predictability, and we now justify them in details. To show the argumentation of these statements, we start by analyzing what is defined as the linearized present value identity, in the one-period securities case. We define dividend yield as the ratio between the dividend D_t paid a time t and the stock price at P_t. This will give us a number expressed in percentage, which is defined as the dividend yield or dividend price ratio. It is very important to understand both the time relationship between dividend and stocks as well as the causal relationship between dividend yield and stock prices. First of all, we need to clarify that an owner of a stock at time t has the right on the next period's dividends payments, such that there is a (t) to $(t+1)$ relationship. This allows us to introduce the concept of total return of a stock, that is defined as the return obtained by the combination of capital gains and of dividend payments. This relationship can be stated as 3.1:

$$R_{t+1} \equiv \frac{P_{t+1} + D_{t+1}}{P_t} - 1 \tag{3.1}$$

in which we can define R_{t+1} as the returns for holding a given stock between two periods t and $t+1$, D_{t+1} is the dividend per share paid in the next period, and the denominator

P_t signals the ex-dividend price which can also be represented at the price that must be paid in order to obtain the capital gain of that stocks and the right ot obtain the dividend. We stress again the fact that the dividend payment is forward looking $t+1$, because purchasing a stock at the end of the period t, gives right to next period's payments D_{t+1} but not to current ones D_t. Taking the $\log()$ to both sides, we can rearrange the equation as follows

$$r_{t+1} = \log(R_t) = d_{t+1} - p_t \tag{3.2}$$

which can be further turned into a present value identity, by moving p to the left hand side, and subtracting d_t to both sides, giving

$$p_t - d_t = pd_t = \Delta d_{t+1} - r_{t+1} \tag{3.3}$$

3.3 is defined as an ex-post equation, or identity relationship. This means that is ex-post verified. It tells us that the price dividend ratio is determined as the difference between tomorrow's dividend growth and tomorrow's returns growth. Expressing the same equation in expected value term, we can see that:

$$pd_t = E_t(\Delta d_{t+1}) - E_t(r_{t+1}) \tag{3.4}$$

which tells us that prices cannot vary, unless expected dividends or expected returns vary as well. Thus confirming that we cannot be in an a world with returns independent and identically distributed, because evidence from markets show that the left hand side variable is varying over time. Conditional expectations are therefore varying over time. From this linear one-period relationship, we can draw our conclusion in a more clear way, before extending the equations to a multi-period setting. We can say that

- High current prices P_t, must be followed by lower future expected returns $P_t + 1$, by higher dividends entitlement D_{t+1} or by some sort of combination of the two.
- High returns instead must be associated with high expected future dividends, downward revision in expected future returns, or some sort of combination of the two.

Moreover, being the above stated 3.4 an identity, we can conclude that expected returns and expected dividend growth, not only are strongly interconnected but knowing one of the two allows us to infer the other. If we try to represent these concepts using regression identities, we can also see some interesting facts about the relationship between these time varying variables. Starting from the

$$dp_t = r_{t+1} - \Delta d_{t+1} \tag{3.5}$$

we can define our regression identities as:

$$r_{t+1} = \beta^r dp_t + \varepsilon^r_{t+1} \tag{3.6}$$

$$\Delta d_{t+1} = \beta^d dp_t + \varepsilon^d_{t+1} \tag{3.7}$$

Then we can express our initial identity 3.5 as a function of 3.1 and 3.7, such that

$$dp_t = r_{t+1} - \Delta d_{t+1} = \beta^r dp_t + \varepsilon^r_{t+1} - \beta^d dp_t + \varepsilon^d_{t+1} \tag{3.8}$$

But given that these are identities, then the coefficient of the regression must, by definition, add up to one . Since expected returns and expected dividends are telling exactly the same thing, we can express their coefficient in relation one to the other:

$$\begin{aligned} \beta^r dp_t - \beta^d dp_t &= 1 \\ \varepsilon^r_{t+1} - \varepsilon^d_{t+1} &= 0 \end{aligned} \tag{3.9}$$

This results allow us to tie volatility and predictability:

$$\beta^r = \frac{cov(r_{t+1}, dp_t)}{Var(dp_t)} \tag{3.10}$$

$$Var(dp_t) = cov(r_{t+1}, dp_t) - cov(\Delta dp_{t+1}, dp_t) \tag{3.11}$$

Variance of dividend yield is covariance with return minus the covariance with future dividend growth. As a such, 3.9 can be viewed as a decomposition of variance, it tells us how much of the variance of prices is coming from its covariance with returns, and how much of that variance is coming from time varying dividend growth. This ties up and unites the observations about volatility with observations about predictability: volatility tells us about predictability. These are consistent with an environment in which expected returns are varying over time.

If instead we move to a multi-period setting, our simple equations get more complicated but underlying concepts and conclusion remain unchanged. Moving to a multi-period securities is necessary because we want to exploit allocation choices in the long horizon, and because in the long run we have evidence of return predictability. The first step is borrowed from Campbell and Shiller (1988), and their linearized return identity. Total return is defined as the ratio between price plus dividend over price, expressed in terms of price dividend ratios and dividend growth rates:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{(1 + \frac{P_{t+1}}{D_{t+1}})\frac{D_{t+1}}{D_t}}{\frac{P_t}{D_t}} \tag{3.12}$$

Taking logs we have:

$$r_{t+1} = \log(1 + \exp(pd_{t+1})) + \Delta dp_{t+1} - pd_t \tag{3.13}$$

In which we have pd_t that is defined as the price-dividend ratio, while dp_t is the dividend-price ratio. In order to further simplify the equation 3.13, since we have log(1+x), we need to take a first-order Taylor expansion over $PD = e^{pd}$, from which we obtain that:

$$r_{t+1} \approx \log(1 + \exp(pd_{t+1})) + \frac{\exp(pd)}{1 + \exp(pd)}(pd_{t+1} - pd_t) + \Delta d_{t+1} - pd_t$$

additionally, define

$$\rho = \frac{PD}{PD + 1}$$

such that

$$r_{t+1} \approx \log(1 + PD) + \frac{PD}{1 + PD}(p_{t+1} - d_{t+1} - pd_t)$$

We borrow the last step from Cochrane (2016), assuming that $\rho = \frac{1}{1+\frac{D}{P}} \approx 0.96$, which allows us to finally state the Campbell-Shiller return identity:

$$r_{t+1} \approx \rho(p_{t+1} - d_{t+1}) + \Delta d_{t+1} - (p_t - d_t) \tag{3.14}$$

3.14 which is a log-linearization of the definition of returns, and tells us that future returns are explained by tomorrow log price dividend ratio, the next period's dividend growth, and today price dividend ratio. Thus, to get future high returns we need to have high P_{t+1}, receive high d_{t+1} , or have a low price today. If we move the price to the left hand side of the equation, we obtain 3.15

$$pd_t \approx \rho \times pd_{t+1} + \Delta d_{t+1} - r_{t+1} \tag{3.15}$$

$$pd_t \approx \sum_{j=1}^{\infty} \rho^{j-1} \Delta d_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} r_{t+j} \tag{3.16}$$

Formula 3.16 states a linear present value formula, discounted with time-varying discount rates and time varying expected returns, but all terms enter linearly in the equation. This is just a definition of returns in the long-run (is an identity), and the message that it delivers is the following:

- long run returns are implied by low price or high future dividends. High dividends play a special role in the long run because they are not temporary or short lived streams as in a short horizon investment

- High prices come from either high expectations of future dividend growth, or from low expectations of future returns (low risk premium), that represents the willingness to hold assets besides the low returns.

But now we can observe that, having the k-period ahead in time expectations discounted to form current prices, then we can say that even expectations far in the future, can raise prices today.

3.2 Econometric evidence of predictability

The economic and theoretical relationship between dividend price ratio and stock market returns has been deeply analyzed in the previous section, so now we want to provide some empirical evidence from the data. In particular, we follow the approach of Fama and French (1998), in order to prove that the power of the predictor variable increases as the forecasting horizon increases and we complement our analysis following Valkanov (2003), in order to better clarify that the lower is the frequency of returns used in the regression equation, higher is the explanatory and forecasting power of dividend price ratio. The starting point is to write down our regression model:

$$r_{t:t+\hat{T}} = \alpha_{\hat{T}} + \beta_{\hat{T}} log(\frac{D_t}{P_t}) + \epsilon_{t+\hat{T}},\ \epsilon_{t+\hat{T}} \sim N(0,1) \tag{3.17}$$

which, in compact form, can be re-wrote as:

$$r_{t+\hat{T}} = a_t + \beta_t dp_t + \epsilon_{t+\hat{T}} \tag{3.18}$$

where r_t is the continuously compounded return on a stock index (SP500 in our case), and dp_t is the dividend yield, that we took from the Shiller dataset (see appendix). The explanation for the increase in the predictive power of the dividend yield is given by the nature of the two processes that we are using. The dividend yield is a noisy process in the short run, such that its noise overcome the volatility of short period returns. However, as the returns horizon increases, the above mentioned noise tends to be weaker in term of return variance, and the portion of long run returns that can be explained through dividend price ratio increases sharply. Some summary statistics of the returns are computed in table 3.1 where wee can see how the mean and standard deviation of this process are more or less stable, even when changing the sample. An important thing to notice is that, using the most recent subsample (post-crisis), we can see how it has a higher volatility as well as a lower average return. This confirms the allocation shown in the previous chapter. The most important statistics displayed in 3.1 are the autocorrelation coefficients. We can see a quiet high autocorrelation of the process at the first lag, but this autocorrelation is not persistent, since it decays very fast, and the coefficients show that autocorrelation becomes negligible soon. The decaying characteristic of the return seems to confirm the intuition of strong mean reversion of the process, and this finding is strongly in line with what we concluded previously in chapter 1. On the other hand, the explanatory variable is a strongly persistent process (see table 3.2), suggesting that a shock to dividend yield is very persistent. This finding is very important for many reasons. First of all it allows us to "be safe" when using the VAR, even to forecast longer returns, since the low variability of the forecasting variable is a very important characteristic in order to iterate our model forward. In addition, as we have seen in the previous section when calculating the present value formula in the long run, even long run expectations on dividends can impact current prices, and this strong persistence is one of the key determinants of that (see the formula 3.16 which shows the Campbell-Shiller present value identity). The next step in assessing the predictability of returns

Table 3.1: Descriptive statistics of the stock return time series

Start	*End*	μ	σ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
Jan-53	Aug-16	0.006	0.035	0.243	0.015	0.043	0.071	0.100	-0.073
Nov-84	Aug-16	0.007	0.036	0.266	-0.005	0.025	0.059	0.075	-0.070
Jan-53	Dec-06	0.006	0.034	0.234	0.024	0.022	0.022	0.101	-0.036
Feb-09	Aug-16	0.005	0.036	0.249	0.000	0.039	0.064	0.095	-0.076

Table 3.2: Descriptive statistics of the dividend yield time series

Start	*End*	μ	σ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
Jan-53	Aug-16	0.031	0.012	0.991	0.980	0.968	0.956	0.942	0.927
Nov-84	Aug-16	0.023	0.008	0.983	0.962	0.940	0.919	0.899	0.879
Jan-53	Dec-06	0.033	0.012	0.990	0.978	0.965	0.952	0.937	0.921
Feb-09	Aug-16	0.030	0.011	0.994	0.984	0.974	0.964	0.953	0.941

from an econometric point of view, is to investigate the predictive power of the dividend yield using equation 3.17. In table 3.3 we performed a long run forecast of returns using lagged dividend price ratio, and from it we can see how the predictability of returns increases as the horizon increases as well. Both the β coefficient of the regression as well as its adjusted R^2, increase sharply as we stretch the forecasting horizon forward. We also displayed the coefficient $\frac{t}{\sqrt{T}}$, which is an adjusted t-stat borrowed from Valkanov (2003). Moreover, we performed this forecasting using different sample, in order to show how different are the estimated coefficients with respect to the sampling period used for the estimation. As we can see, the increase in both β and R^2 estimates along with the forecasting horizon remains also if we change the sampling period. On the other hand, the coefficients look quiet different in the 4 cases displayed in table 3.3. Performing this analysis using different samples is a key intuition in order to analyze the time varying relationship between our explanatory variables and returns, and additionally it tells us that, in order to assess the relationship between our forecasting variable and the one to be explained, is not always better or proper to use the largest sample. It is better for the investor to evaluate this relationship using one sample or the other, based on how much he believes that this relationship is time varying. The intuition behind the increase in forecasting power along with the investment horizon can be seen graphically

Table 3.3: Long run regression

	Full sample				*Half sample*		
Horizon k	β	$\frac{t}{\sqrt{T}}$	R^2	Horizon k	β	$\frac{t}{\sqrt{T}}$	R^2
1	0.838	0.103	0.009	1	1.172	0.131	0.014
4	3.897	0.206	0.039	4	5.589	0.265	0.063
8	8.267	0.293	0.078	8	12.204	0.398	0.135
16	16.983	0.436	0.159	16	23.215	0.547	0.229
24	24.632	0.550	0.232	24	32.707	0.655	0.299
60	52.117	0.892	0.443	60	72.878	1.240	0.606
	Post sample				*Pre sample*		
Horizon k	β	$\frac{t}{\sqrt{T}}$	R^2	Horizon k	β	$\frac{t}{\sqrt{T}}$	R^2
1	2.213	0.103	-0.033	1	0.948	0.114	0.011
4	21.878	0.757	0.357	4	4.408	0.232	0.050
8	38.213	1.119	0.558	8	8.841	0.317	0.090
16	32.385	1.650	0.737	16	17.315	0.443	0.163
24	54.099	1.877	0.784	24	24.570	0.545	0.229
60	43.031	2.610	0.876	60	51.069	0.837	0.412

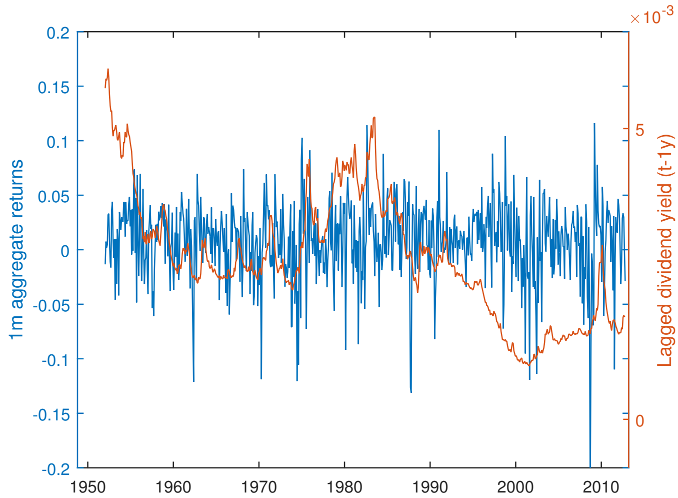
in figure 3.2a and 3.2b, in which we can see how, using a lower frequency of returns, the predictive power of the dividend yield increases as well. Coming back to the results in table 3.3, we want to try to motivate the reason behind the increase over time of both the regression coefficient β as well as of the R^2. We can see the point mathematically, if we write the cumulative returns as:

$$\begin{aligned} r_{t+1} + r_{t+2} &= \beta(1+\phi)dp_t + \epsilon_{t+1} \\ r_{t+1} + r_{t+2} + r_{t+3} &= \beta(1+\phi+\phi^2)dp_t + \epsilon_{t+2} \end{aligned} \tag{3.19}$$

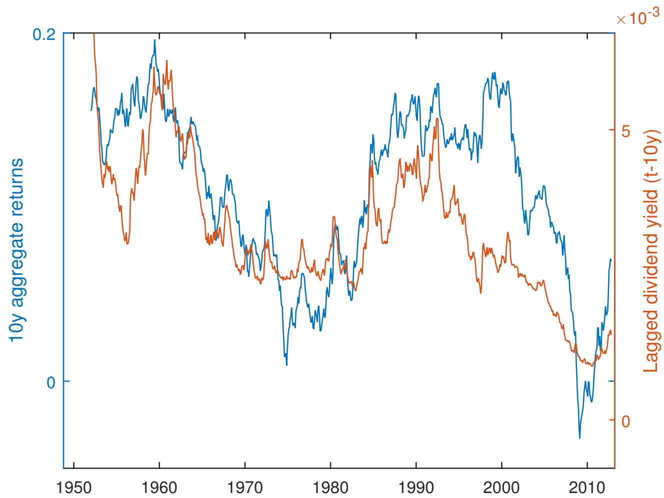
From equation 3.19,we can see how coefficients are rising along with the horizon. This happens because in our case the explanatory variable (dividend price ratio), is very persistent since, as we recall from table 3.2, its autocorrelation is nearly 1. This imply not only that coefficients grow along with the investment horizon, but also tells us that they increase almost linearly with T. A similar methodology can be shown in order to motivate the increase in R^2 coefficients. If we write out the one and two period ahead

Figure 3.1

(a) 1Month returns and dividend yield at t-1y



(b) 10y returns and dividend yield at t-10y



R^2 we can see that:

$$\begin{aligned} R^2_{t+1} &= \frac{\beta^2\sigma^2(dp_t)}{\sigma^2(r_{t+1}} \\ R^2_{t+2} &= \frac{\beta^2(1+\phi)^2\sigma^2(dp_t)}{\sigma^2(r_{t+1}+r_{t+2}} \\ &\approx \frac{\beta^2(1+\phi)^2\sigma^2(dp_t)}{2\sigma^2(r_{t+1})} = \frac{(1+\phi)^2}{2}R^2_{t+1} \end{aligned} \tag{3.20}$$

Again, if we consider that $\phi \approx 0.99$, then we can conclude that the two period R^2 is almost the double of the single-period one. Thus, R^2 coefficient grows almost linearly with T. This mathematical finding is in line with what we can see in table 3.3, if we multiply the one-period R^2 and the forecasting horizon, we can roughly replicate the coefficients implied by the predictive regression.

3.3 Predictability and asset allocation

Having established in the previous section the tight link between the dividend yield and the expected returns, we would like to see how an investor that takes into account this phenomenon changes its long run asset allocation accordingly. In order to incorporate the concept of predictability into our asset allocation framework, we will rely again on Barberis (2000), also in order to give continuity to the results obtained in the previous chapter. Nevertheless, in the forthcoming paragraphs we also analyze an extend the environment that he proposes, performing various sensitivity analysis to motivate the results in a clearer way. Again, the starting point of our problem is the one of a buy and hold investors (who build a portfolio at a given moment t and does not change it until a $t + \hat{T}$) who would like to maximize its expected wealth W_{t+T} at the end of a pre-determined investment period. Its investment universe can consist in a constant risk-free asset, and a stock component, thus its only relevant choice for now is the optimal percentage ω of its actual wealth (assumed to be 1) to allocate to the risky portion of its portfolio. Since we include a restriction on borrowing for the investor, as well as the constraint of no short selling, its ω can range in an interval that goes from 0 to 1.The returns that we obtain from ω are given by the sum of each period's realized returns, such that $R_{t+T} = R_{t+1} + R_{t+2} + \ldots + R_{t+T}$. The terminal wealth is the function that the investor would like to maximize, and is described as:

$$W_{t+T} = (1-\omega)exp(r_f T) + \omega exp(R_{t+T}) \tag{3.21}$$

Moreover, as well as in Chapter 1 and 2, the preference of the investor are described by a power utility function, driven by a risk aversion coefficient α that represent how much risk an investor is willing to take. As we will discuss later, this parameter plays a key role in shaping optimal asset allocation choices that we found. This kind of utility function is a CRRA (Constant Relative Risk Aversion), will imply that the problem faced by the investor can be stated as follows:

$$max_\omega E_t \left\{ \frac{(1-\omega)exp(r_f T) + \omega exp(R_{t+T})^{1-\alpha}}{1-\alpha} \right\} \tag{3.22}$$

As we highlighted in the previous chapter, the solution to 3.22 is not know in closed form, and therefore the integral characterizing the future expected wealth maximization problem, must be solved numerically. Nevertheless, the procedure to follow in order to solve the problem in an iterative way, works exactly in the same way describe in section 1.4, thus we will not describe it again in full. The key difference that we have when we incorporate predictability, is that now our expected returns distribution are generated from our regression framework. In order to describe the new sampling procedure, we recall first our regression:

$$z_t = a + Bx_{t-1} + \epsilon_t \tag{3.23}$$

In which we have $Z_t = (r_t, x_t')$ and $x_t = (x_{1,t}, x_{2,t}, \ldots, x_{n,t})$ and ϵ_t $i.i.d. \propto N(0, \Sigma)$. Equation 3.23 is an Ordinary Least Square regression,in which the dependent variable is a matrix composed by the continuously compounded return of an asset (r_t), and a vector of predictor variables. In our case, the predictor variable in only one, such that z_t will be only a matrix of continuously compounded returns of a security over its dividend yield (where the dividend yield is defined as the ratio between dividend paid in the last year over current price of the index). Unlike in previous chapter, now we will use the regression environment to forecast expected returns. We will see how including this predictor variable will change radically how the optimal portion of wealth to allocate to stocks will change. In order to compute the posterior distribution of returns $p(a, B, \Sigma | r)$, we can rewrite the model as follows:

$$Z = XC + E \tag{3.24}$$

Where Z is a matrix composed by $T-1$ continuously compounded returns, combined with the vectors of $T-1$ predictor variables. Since in our case we are only using one predictor variable in the regression, in our case Z is composed only by two columns in total. Generally speaking, if we define n as the amount of predictor variables that we include in our analysis (for our specification, n=1), the $Z = [(T-1) \times (n+1)]$ matrix. X is instead defined as the $T-1$ vector of dividend yields, combined with a vector of ones 1_{T-1}. The difference between Z and X is that both contains the predictor variables as a column, but one is the first lag of the other. E is a $[(T-1) \times (n+1)]$ matrix of identically and independently distributed realizations of the following a $N(0, \Sigma)$. In the specification in which we include predictability of returns in our problem, the procedure is fairly in line with what we did when we were only incorporating uncertainty, what changes is the way in which we forecast expected returns, and the shape of the distributions used (must be adapted because we are now in a multi-dimensional environment). In order to do so, we start by specifying a prior, that will be the starting point of our Bayesian vector autoregression:

$$p\,(C, \Sigma) \propto |\Sigma|^{-\frac{n+2}{2}} \tag{3.25}$$

Following the conversion tables of Zellner (1971), we can derive directly the posterior distribution $p\,(C, \Sigma^{-}1|z)$, which is given by

$$\begin{aligned} \Sigma^{-1}|z \propto Wishart(T-n-2, S^{-}1) \\ vec(C)|\Sigma, z \propto N(vec(\hat{C}), \Sigma \otimes (X'X)^{-1}) \end{aligned} \tag{3.26}$$

with $S = (Z - X\hat{C})'(Z - X\hat{C})$ and $\hat{C} = (X'X)^{-1}X'Z$. The steps to generate the posterior distribution then are the same described in section 1.4, such that we first draw the variance-covariance matrix, and the we draw the mean. Once we have generated our posterior distribution of the parameters from equation 3.26 , then we need to draw from the predictive distribution, in order to obtain our expected returns. In order to do so, equation 3.23 an be rewritten as:

$$Z_t = a + B_0 X_{t-1} + \epsilon_t \tag{3.27}$$

if we assume $B_0 = [0_{n+1} B]$, we can additionally rewrite the model as:

$$B_0 = [(0 \dots 0)')\, B] \tag{3.28}$$

$$Z_t = a + B_0 Z_{t-1} + \epsilon_t \tag{3.29}$$

which is a fully autoregressive equation. As a such, we can iterate forward the calculation, deriving that $Z_{T+\hat{T}} = Z_{T+1} + Z_{T+2} + \dots + Z_{T+\hat{T}}$ is distributed as a multivariate normal random variable, which first two moments are defined respectively as:

$$\begin{aligned}\mu_{sum} =& \hat{T}a + \\ & (\hat{T}-1)B_0 a + (\hat{T}-2)B_0^2 a + \dots + B_0^{\hat{T}-1} a + \\ & (B_0 + B_0^2 + \dots + B_0^{\hat{T}}) z_T\end{aligned} \tag{3.30}$$

$$\begin{aligned}\Sigma_{sum} =& \Sigma \\ & + (I + B_0)\Sigma(I + B_0)' + \\ & (I + B_0 + B_0^2)\Sigma(I + B_0 + B_0^2)' + \dots \\ & (I + B_0 + B_0^2 + \dots + B_0^{\hat{T}-1})\Sigma(I + B_0 + B_0^2 + \dots B_0^{\hat{T}-1})'\end{aligned} \tag{3.31}$$

Finally, the $\hat{T}$ period ahead forecast of returns can be obtained by drawing from a normal distribution, defined as

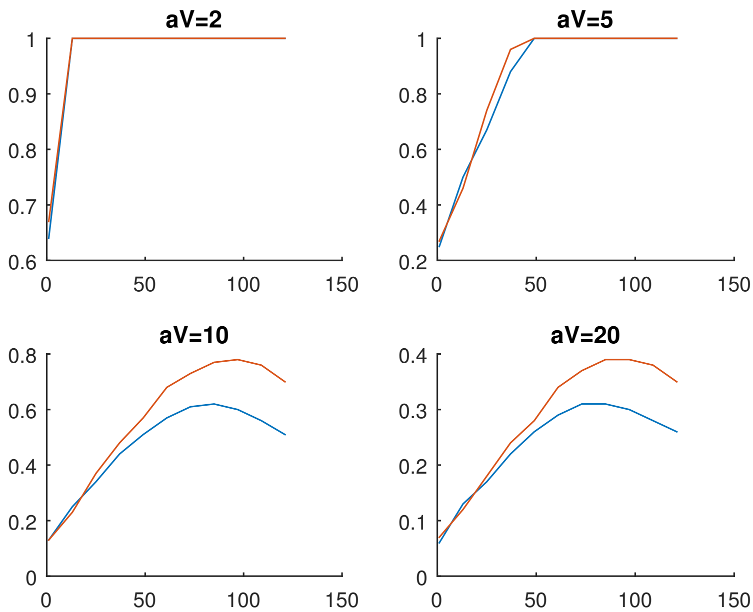
$$p(R_{T+\hat{T}} | \theta, z) \sim N(\mu_{sum}, \Sigma_{sum}) \tag{3.32}$$

with mean and covariance matrix defined from 3.5 and 3.31 respectively. At the end of this estimation process, we end up having a simulated matrix of 1000000 expected return, which are incorporated in the utility function and averaged out, according to formula 1.13 of chapter 1. All the steps above are for an investor who does include uncertainty and predictability all together. Nevertheless, in this chapter as well we let the investor choose whether to incorporate uncertainty in the estimation of parameters or not. In this case, he decides whether to be uncertain or not with respect to the regression coefficients, and when he does not, then he will simply use the sample estimates of the regression (without performing the steps in equation 3.26), and then draw from 3.32 in order to obtain the expected returns.

3.4 Allocation results

From figure 3.3, we can see the resulting optimal allocation, for the case in which the investor includes return predictability in its optimization procedure. As in the previous section, we differentiate between two possible choices that the investor can make: whether he incorporates parameter uncertainty, or he assumes parameter are certain and fixed. As shown in figure 3.3, when the investor assumes return predictability, then the resulting investment strategies are completely different with respect from the case in which he does not (see section 1.5 of chapter 1). Up until now we saw that the allocation was either constant when there was no uncertainty, or decreasing when there was uncertainty. Now, the allocation with predictability are increasing, and in a strong way. In figure 3.3 the two lines both account for predictability, but one includes uncertainty (blue lune), while the other does not. At a first sight, the results that we obtain when we introduce predictability, seems to be completely opposite to previous conclusions. In fact, we can see from the two upper lines in figure 3.3, that when we account for predictability an investor with a longer horizon seems to allocate a sizeable larger portion of wealth to the risky component of his portfolio, compared to a short-term investor. This is clearly shown graphically because the both the lines are increasing (strongly increasing). This behaviour of the allocation could be explained in a converse way with respect to the previous chapter. In fact, as we shown in the previous chapter, when the investor estimated expected returns as realization of an independent and identically distributed process, the allocation should not change unless he accounts for parameter uncertainty. This because when the investor takes into account parameter uncertainty, expected returns have a strong positive autocorrelation, rational forecasts of one-year returns one to five years ahead are highly correlated. As a consequence, the variance of expected returns grows faster with the return horizon than the variance of unexpected returns, the variation of expected returns become a larger fraction of the variation of returns. When we consider predictability instead, we could said that, when the investor

Figure 3.3: Optimal allocation - Predictability



The blue line corresponds to the case in which the investor incorporates for both predictability ad uncertainty. The red line instead displays the allocation for the specification in which he accounts for predictability but not for uncertainty. The predictor variable is held at its mean value over the sample considered. Each graph plot the resulting optimal strategy for a different value of risk aversion. The results refer to a "full sample" used as input for the problem.

relaxes the assumption of constant expected returns and he accounts for predictability, the variance of cumulative expected stock returns might grow less than linearly with $\hat{T}$. As a consequence, attractiveness of stocks increase as the horizon of the investor increase since they are perceived as less risky. This leads an investor with a longer horizon be better off with a greater proportion of wealth allocated to the risky component of its portfolio. If we try to re-write and explain this concept mathematically, we come back to our regression equations:

$$r_{t+1} = \alpha + \beta x_{1,t} + \epsilon_{1,t+1} \tag{3.33}$$

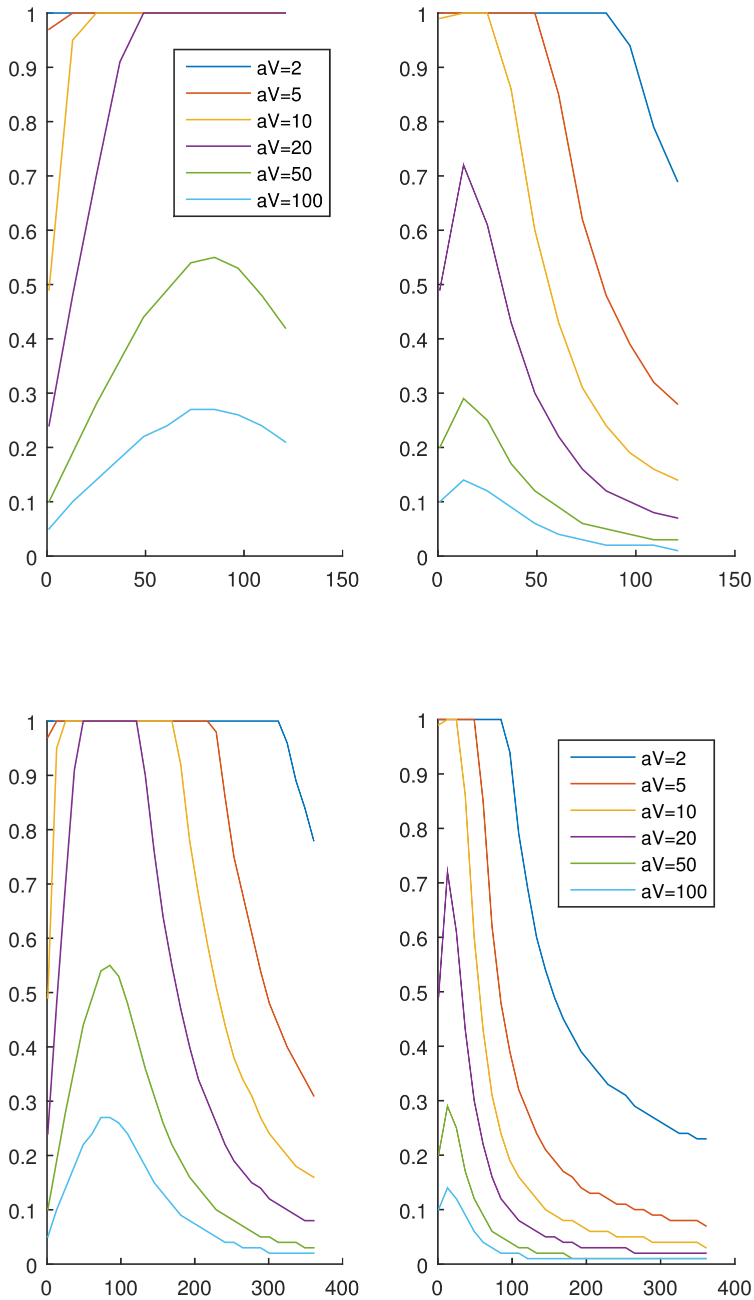
$$x_{1,t+1} = \gamma + \phi x_{1,t} + \epsilon_{2,t+1} \tag{3.34}$$

And if we calculate the one-step and two-step ahead conditional variances of cumulative returns, we can see that:

$$\begin{aligned} var_t(r_{t+1}) &= \sigma_1^2 \\ var_t(r_{t+1} + r_{t+2}) &= 2\sigma_1^2 + \beta^2\sigma_2^2 + 2\beta\sigma_{12} \end{aligned} \tag{3.35}$$

In order to compare $var_t(r_{t+1})$ and $var_t(r_{t+1} + r_{t+2})$ and show that they are decreasing over time, we recall an intuition that we gave in section 3.1, where we analyzed the relationship between shocks in dividend yields and returns. In particular, we recall that we have a negative correlation between shocks in dividend yield and subsequent shocks in stock returns, which in formula 3.35, is exactly represented by σ_{12}. This imply that a positive shocks in dividend yields makes current stock returns to be lower while, on the other hand, stock returns are expected to be higher in the future (since $\beta > 0$). This phenomena is the cause of mean reversion in returns, that makes stock more appealing to our investor as its horizon increases and is the reason why, incorporating the dividend yield in the analysis, induce this decreasing path of the variange of cumulative returns over time that makes our investor more willing to shift its wealth toward the risky component of the portfolio. Having said that, we can still notice how, in the two cases in which we introduce a predictor variable, one line ends up being consistently higher than the other one. Roughly speaking, this happens because in the lower line the investor is accounting for parameter uncertainty while in the line above he is not. This conclusion looks like an element of connection between what we shown in the previous chapter: accounting for parameter uncertainty reduce the optimal fraction of wealth to allocate to stocks, because of its effect on cumulative variances of expected returns. And we can now remark that the effect of parameter uncertainty is present regardless of the assumption that we make on whether expected returns are constant or a time varying/predictable. As we just mentioned, the decrease in the variance of cumulative expected returns causes the allocation toward stocks to increase as the investment horizon increases, regardless of how the investor estimates its expected returns and the assumption that he makes on them. Nevertheless, this effect appear to be non-monotonic with respect to the investment horizon. If we analyze how the optimal allocation percentage is varying over time, we could see how it is actually strictly increasing only up until a certain level of the investment horizon (around 8 years), while it start to decrease slowly afterwards. This becomes clearer when we further increase the investment horizon, moving from a 10year to a 30year time. This phenomenon is defined as the “Battling effect” of the two underlying concepts of predictability in returns and parameter uncertainty. In order to better visualize this "battling effects", and try to understand when one does prevail to the other, we can recall some of the conclusions that we draw in chapter 2, when we were discussing uncertainty. We saw how the uncertainty become much higher as we change the initial sample used to estimate the parameters. In particular, we showed how, using the most recent sub-sample (defined as "Post-Crisis"), the uncertainty that the investor had on the expected returns was three time higher than in the baseline case. Thus, in order to better understand the dynamics of predictability and uncertainty, we can try to repeat the same exercise and look at thee "battling effects". In figure 3.4, we plot the optimal allocation lines for an investor that incorporate for both predictability and uncertainty altogether. First of all, we can focus in the graphs on the left panel, in which the investor is using the “Full sample”. As we can see in the first graph, when the horizon goes until 10year, the investor always ends up with an end of horizon allocation that is greater than the allocation at $\hat{T}$=1. And this statement is true regardless of the level of risk aversion. Nevertheless, what can be seen clearly is that, after a first increasing phase, the allocation starts to decrease, and this appears only as the horizon stretches more forward, since in the initial part of the investment horizon, there is no sign of decrease whatsoever. The beginning of the decrease phase, is the signal that the effect of uncertainty is prevailing over the effect of predictability. In fact, as we have seen in the previous chapter, the uncertainty effect is increasing over time, causing allo-

Figure 3.4: Predictability and uncertainty – Full vs reduced sample



Each line displays the resulting optimal investment strategy for an investor with a different risk aversion coefficient. In the two left plot, the investor is using the full sample in order to estimate the parameters. The two graphs on the right, display the allocation in which he uses only the reduced one. The two graphs on the top go until a $\hat{T}$ of 10years, while the two bottom figures reach 30years. Predictor variable at 0.031%

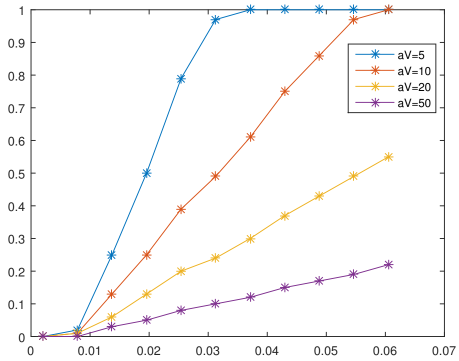
cation to decrease as the investment horizon increases, thus the "horizon effect" that it causes also becomes more sizable as $\hat{T}$ grows. This conclusion can be carried over also in the predictability case, and in order to do this, the plot on the bottom left is particularly helpful, since it show the case in which the horizon runs up until 30years. In this case, the difference between the optimal allocation at the end of the horizon and at the beginning is even more striking. As we stretch the horizon forward, the finding that the end-of period allocation was higher than the initial one, is no more held with certainty. This non-monotonicity of the increase in allocation is caused by the fact that, when we consider particularly long horizon, the negative effect of uncertainty more than offset the positive one of predictability. While the positive impact of predictability is greater, then the allocation is increasing, but when uncertainty gets bigger and bigger, then the path changes sign. In order to complement the analysis, we can look at the two subfigures of figure 3.4, in which we show the allocations for an investor that uses only the most recent sub-sample. In this case, as we recall from the previous chapter, the shorter sample induces greater uncertainty. As a result, the increasing path in which predictability dominates uncertainty is even shorter, and the allocation lines start to decrease even after a few years.

3.5 Playing with the value of the predictor variable

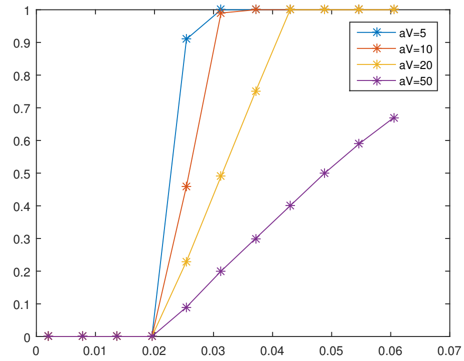
Thus far, we have held the predictor variable at its mean value over the sample used, which in the baseline scenario of an investor using the full estimation sample, was at 0.031%. Using only this value, we were able to isolate the effect of predictability and uncertainty themselves, and show the positive impact that predictability has on the allocation toward stocks. On the other hand, we want now to investigate the effect that the initial value of the predictor variable itself has on the optimal allocations. Recall that the predictor variable enters equation , which is one of the step necessary to build our predictive distribution, thus it is natural to hypothesize that this variable plays a big role in determining the expected future returns. The results of the sensitivity analysis conducted on the values of the predictor variable d/p, are shown in figure 3.6, in which we display the resulting ω corresponding to each value of the predictor variable, such that each line corresponds to a different risk aversion coefficient having on the x-axis the predictor variable and on the y-axis ω. Additionally, in order to see whether this

Figure 3.6: Predictability - Effect of the predictor variable

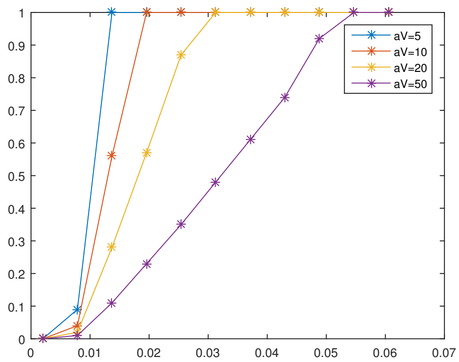
(a) $\hat{T}$=1Month



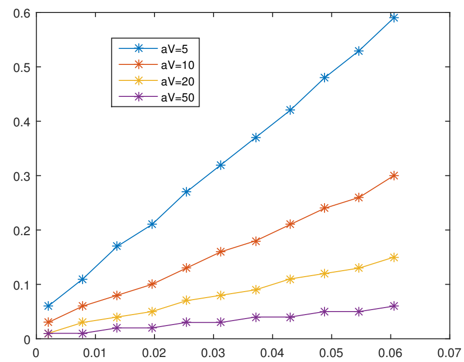
(b) $\hat{T}$=1Month



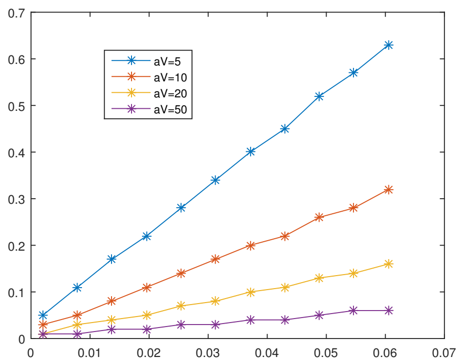
(c) $\hat{T}$=10Years



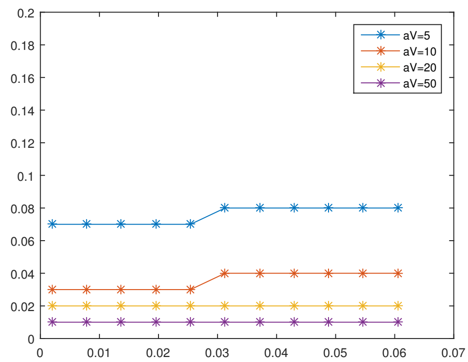
(d) $\hat{T}$=10Years



(e) $\hat{T}$=30Years



(f) $\hat{T}$=30Years



Each graph shows the optimal allocation for different values of the predictor variable (x-axis). Each line corresponds to a different level of risk aversion. All the plots on the left, use the "full-sample" in order to estimate the parameters, all the graphs on the right instead use the "Post-Crisis" sample. We consider three different snapshot dates, namely 1Month, 10Years and 30Years.

Table 3.4: Optimal fraction of wealth invested in the risky asset, for some values of the predictor variable and risk aversion. In each sub-table, we fix an investment horizon $\hat{T}$, at which we evaluate results.

T=1Month

	d/p=0.002	d/p=0.019	d/p=0.031	d/p=0.043	d/p=0.061
aV=2	0.00	1.00	1.00	1.00	1.00
aV=5	0.00	0.50	0.97	1.00	1.00
aV=10	0.00	0.25	0.49	0.75	1.00
aV=20	0.00	0.13	0.24	0.37	0.55
aV=50	0.00	0.05	0.10	0.15	0.22

T=10y

	d/p=0.002	d/p=0.019	d/p=0.031	d/p=0.043	d/p=0.061
aV=2	0.00	1.00	1.00	1.00	1.00
aV=5	0.00	1.00	1.00	1.00	1.00
aV=10	0.00	1.00	1.00	1.00	1.00
aV=20	0.00	0.50	1.00	1.00	1.00
aV=50	0.00	0.20	0.42	0.66	1.00

T=30y

	d/p=0.002	d/p=0.019	d/p=0.031	d/p=0.043	d/p=0.061
aV=2	0.15	0.54	0.78	0.98	1.00
aV=5	0.06	0.21	0.31	0.42	0.56
aV=10	0.03	0.10	0.16	0.21	0.28
aV=20	0.01	0.05	0.08	0.10	0.14
aV=50	0.01	0.02	0.03	0.04	0.06

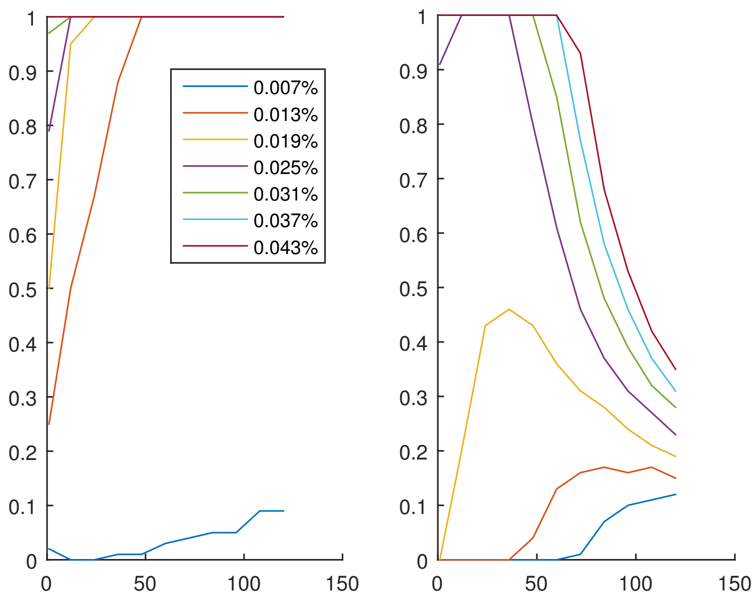
relationship varies as the investment horizon changes, we consider 3 snaphsot dates: 1month (3.7a and 3.7b),10years (3.7c and 3.7d) and 30years(3.7d and 3.7e). In order to study the effect of variation in the dividend yield, we can look at how each line is increasing as the dividend yield itself increases (recall that dividend yield and predictor variable in our case are synonymous, since we only have one predictor variable). Regardless of the effect of risk aversion, we can see clearly how, the higher is the dividend yield that we use as a starting point, higher is the resulting ω. This finding is quiet straightforward, and comes easily from the theory that we showed in the previous section. Higher current dividend yield lead the investor to forecast higher future expected return, on the other hand, lower expectations on returns are caused by below the average values of the predictor variable. As a such, when the predictor variable is above its average value, the attractiveness of our risky component of the portfolio increase since we expect them to have higher returns, thus we tend to allocate more toward stocks. As we can see in figure 3.6, this finding is uniform with respect to the various risk aversion coefficients that we tested. An interesting finding here comes from comparing what we can define as the slopes of each allocation line for a given risk aversion at a given snapshot date. We can define it as:

$$slope = \frac{\omega^{\alpha}_{max(d/p)} - \omega^{\alpha}_{min(d/p)}}{max(d/p) - min(d/p)} \tag{3.36}$$

Looking at this slope, we can see how the optimal fraction ω varies sharply along with d_t, such that changing the predictor variable can give us radically different optimal investment strategies. A positive slope tells us exactly that a higher value of the predictor variable imply a higher allocation toward stocks (for the reasons mentioned above). But a higher slope tells us also that the allocation is much more sensible to the value of the predictor variable. Such that, if the difference between the maximum ω that we can obtain (which is of course obtained using the highest d_t 0.061%) and the minimum one (obtained with the lowest d_t 0.002%), is high, then we are sensible to the initial value of the predictor variable, otherwise if the difference (or slope) is not so high, then we are more indifferent to it. Using this "slope" variable can help us in understanding how allocations change with respect to our initial parameters and assumptions. We can see how in 3.7a and 3.7b, which plot the allocation for the investor using the full sample and the reduced sample respectively (at a very short horizon of 1month), the slope is very high for all of the risk aversion coefficients. This tells us that the allocation that we end up having when we use different values of the predictor variable, will be much different one with respect to the other. In both graphs we are considering a very short horizon, such that we do not expect any uncertainty effect to be predominant here. Nevertheless, the same conclusion can be drawn at a 10years horizon, and can be seen from figure 3.7c. On the other hand, the message in graph 3.7d is a bit different, in particular looking at it we can see that the slope is not so high for most of the risk aversion coefficients, since only the least risk averse individual (aV=5) is changing a lot its ω as the predictor variable increase/decrease. This difference becomes even clearer when comparing 3.7e and 3.7f. While the former preserves relatively steep curves, in the latter the curves are almost flat, signaling an indifference of the optimal allocation with respect to the predictor variable. In order to explain this result, we must recall one of the main finding that we showed in chapter 1, namely that a reduction of the sample increases uncertainty and, in particular, increases the horizon effect (defined as the decrease of ω) as the $\hat{T}$ grew. As a such, we expect the three right plots in figure 3.6, to be subject to a much higher degree of uncertainty, since they are based on a reduced sample. Additionally, we showed above how this insensitivity of the allocation with respect to the value of the predictor variable is present and strong when using the reduced sample, and when the investment horizon is stretched forward. The explanation for this lack of sensitivity that we obtain in the long run and when uncertainty is high, comes from the intuition that, if the investor is uncertainty about the real forecasting power of the dividend yield, the allocation in the long run are less sensitive to its value.

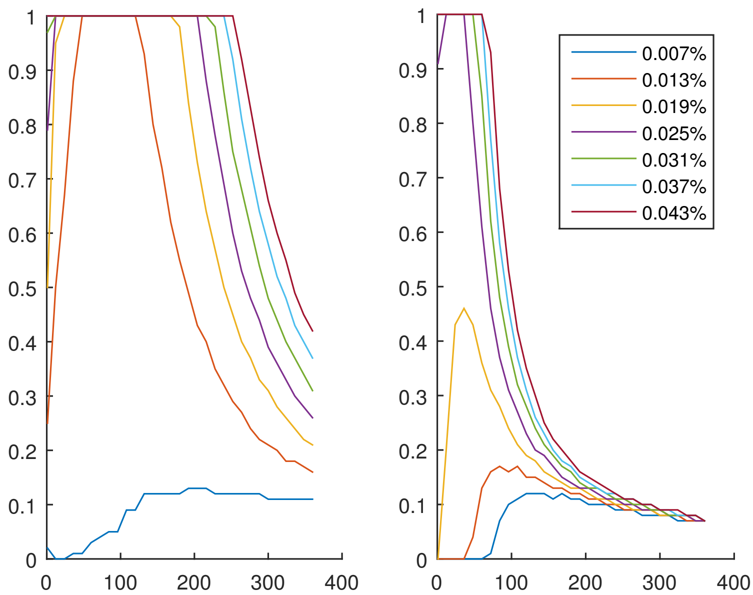
In light of this, if we plot altogether the allocation over different time horizon we would expect that, regardless of the initial value of the predictor variable, they will tend toward a common point. This is exactly what can be seen in figure 3.8. Again, when we use the full sample to estimate the parameter therefore we have a somehow low degree of uncertainty, the degree of convergence is not dramatic. Comparing the two different panels with the extended investment horizon (see figure 3.9), we can see that when with a reduction the sample size, the allocation lines fully overlap each other already at a 20 year horizon. Using the full sample instead, since the uncertainty is lower, this convergence is not complete even at 30 year horizon, and the difference between ω with the highest value of the predictor variable and the lowest, is still about 30%. This amount is quiet high, since it is telling us that an investor who is using the highest estimate of the dividend yield, is allocating toward stock an amount of wealth that is 3 times higher that an investor who uses the most cautious estimate about the predictor variable. Figure 3.9 helps us to conclude that the "converge effect" is full only when the uncertainty is really high, in normal conditions the assumption the initial value of the predictor variable still plays a major role, despite lowered by uncertainty.

Figure 3.8: Predictability - Convergence in allocations



The plot on the left shows the resulting allocation for an investor that does include parameter uncertainty, and uses the full sample to estimate parameter. In the plot on the right, the investor uses only the most recent subsample. Aversion coefficient=20

Figure 3.9: Predictability Extended horizon - Convergence in allocations



The plot on the left shows the resulting allocation for an investor that does include parameter uncertainty, and uses the full sample to estimate parameter. In the plot on the right, the investor uses only the most recent subsample. Aversion coefficient=20

4

Concluding remarks

In this thesis we have analyzed strategic asset allocation choices, and we have shown how the results can vary in light of different assumption that can be done on the problem. Starting from a base case in which an investor estimates returns as i.i.d. realization and does not account for parameter uncertainty, we have seen how changing these assumption can lead to fairly different optimal strategies, which might signal over/under investment with respect to the baseline scenario.

An investor that accounts for parameter uncertainty tend to allocate about 10% less toward the risky component of the portfolio (in a 10year horizon), with respect to the case in which he does not account for uncertainty. Overall this signals that, when the investor does not acknowledge his uncertainty in the estimation of parameters, his allocation is biased toward the risky assets and he is taking too much unremunerated risk, which can potentially lead the great unexpected losses. High risk averse individual or investor that has constraints on losses are particularly sensible to such topic, and they should be therefore really careful on the assumption that they make on their returns.

Many factors can make the over-allocation problem even greater. Whilst in the base case of a 10-year horizon, the difference between uncertainty and certainty was about 10%, just by stretching forward the investment term the decrease is three times higher (around 30%) for a 30-year horizon. Uncertainty effect is therefore proportional with respect to $\hat{T}$, and does not slow down its pace after some periods. Additionally, a low risk averse individual seems to be less affect by the impact of uncertainty, since higher is the risk aversion higher is the difference between the allocation of a short horizon and a long horizon investor. Investor might find optimal to use a more recent (thus shorter) dataset in order to better capture the behavior of the assets that can be time varying. Changing the sample from a larger to a shorter one, can have dramatic effect on uncertainty. The horizon effect found with a 10 year sample is more than twice the one obtained using the full dataset, which imply that non incorporating uncertainty can lead to a 70% over-allocation toward the risky component in a 30-year horizon, and this amount is found irrespectively of the risk aversion of the individual. A potential solution that we propose is the use of Exponentially Weighted Moving Average techniques to calculate the first two moments of the return distribution. This allows the investor to assign greater weight to the most recent observations without reducing the estimation sample. The resulting horizon effect we found with this method is in line with the "full-sample" one, which tells us that the uncertainty with this model is about 40% lower than the case in which we were using the most recent sample.

We dedicated a lot of attention to study the value and the definition of the risk-free component of the portfolio. Many articles in literature that studied strategic asset allocation, assumed that the investment universe of the investor was formed by a risky asset and a risk-less one. With the latter being fixed income instrument, with some positive returns that were deterministic and constant throughout the whole investment period. Both the consideration of it having a positive return and being risk free are a bit too optimistic nowadays. In particular, current markets are characterized by the low interest rate environment for fixed income instruments. To explore this new low/negative interest rate environment, we perform a sensitivity analysis with respect to the value of the risk free, and we analyze how the share of wealth allocated to the risky asset varies. As expected, lower is the risk free rate of return, higher is the corresponding allocation to stocks. Intuitively, if we decrease the return of one component of the portfolio holding everything else equal, the other component will become more attractive. This is true both in the case in which the investor accounts or not for parameter uncertainty. The uncertainty effect is present regardless of the value of the risk-less asset, in the sense that also the assumptions on this rate are affected by uncertainty. This result is particularly intriguing, since it is telling us that there is some sort of convergence of allocations that use different R_f rates. This can be justified by noting that when we acknowledge our uncertainty on returns, this uncertainty is propagated also to the other assumptions underlying the problem. In order to show this, we used the concept of spectrum of allocation, that is defined ad the difference is allocation between the higher R_f and the lowest. The convergence is not so evident when we are using the full observation sample, but if we use the reduced one (which cause higher horizon effect), the spectrum of allocations is 20% lower. As a such, when account for uncertainty we know that potential differences in other assumptions will be reduced by uncertainty effect. This can be defined as "convergence effect", and is a recurring results in our work.

Some of the most interesting findings of this work come from what we called multi-asset environment. First of all, we showed how the result of horizon irrelevance ought to Samuelson (1969), is present also when a multiplicity of assets are included in our portfolio. On the other hand, when the investor acknowledges its uncertainty in parameters, the allocation becomes decreasing with respect to $\hat{T}$. Thus uncertainty effect is present regardless of the number of securities of which a portfolio is composed. Nevertheless we showed also that, when more and more assets are included in the investment universe, some other effects come into play and the horizon effect is no more as "pure" as with a small number of securities. With an increased number of assets in our portfolio, the main other driver is the diversification effect that we can obtain with the combination of different instruments. This effect is present and sizable, and in some cases more than offsets the horizon effect. As a such, the share of each asset is no more monotonically decreasing, but it is noisy (and in some cases even increasing). This should not be confused with the lack of presence of uncertainty effect, but should be interpreted as two effect competing one with the other, with none of the two clearly prevailing the other.

The evidence of predictability in asset returns can be used in our long run asset allocation problem. Using the dividend yield as predictor variable for stock returns, we can see how the investment strategy is increasing with respect to the investment horizon. Predictability makes risky assets more attractive in the long run, because its slows down the evolution of their variance making it grow less than linearly. This result is conflicting with the horizon effect induced by parameter uncertainty, such that these two effect battle each other, making the investment strategy hump-shaped with respect to time. The resulting optimal investment strategies are increasing when the investment horizon is small, while they start do decrease as $\hat{T}$ grows. This signals that predictability effect prevails at the beginning, namely when uncertainty is low, while horizon effect prevails in the long run. We showed how this even clearer when we reduce the estimation sample used in the problem. Since we knwe that a reduction in the sample size implied a greater uncertainty, we showed how in this specification the period in which predictability effect is greater than the horizon effect is much shorter. An important role is played by the value of the predictor variable:changing it can result in radically different allocation. When we account for parameter uncertainty, this value is still important but we found some convergence in allocation with respect to different values of the predictor variable. In the sense that, when the investor is uncertain about its return he is also uncertain about the initial value of the predictor variable. For example, we have seen how if the investor "wrongly" assumes the initial value of the predictor variable to be below the historical average instead of being above the average, thanks to uncertainty the allocation in the two cases will be converging and the error will be minimized.

Overall, we have seen how accounting for parameter uncertainty is important both when we keep the constant expected return assumption, or when we assume returns predictability. Without taking into account uncertainty, an investor can be pushed to over-allocate toward risky investment which might cause unexpected losses. Additionally, incorporating parameter uncertainty allows us to be better aware of the evolution of the variance of cumulative expected returns, that can be less than linearly of even more that linearly growing with the horizon. This effect can be mixed with diversification effect, or predictability effect which can in turn make the horizon effect look less evident. Moreover, uncertainty effect makes allocation more stable with respect to other assumptions (e.g: risk aversion, r_f rates, predictor variable). Being uncertain on return makes the investor uncertain also with respect to all his other assumptions, such that there is a convergence path caused by uncertainty that can even make other specification become less relevant. This propagation of uncertainty to the other assumption can partly offset small estimation errors in the other parameters.

4. CONCLUDING REMARKS

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Appendix

Dataset used

The dataset used for the analysis of Chapter 1 and 3 are provided by Robert Shiller's website, which contains the series used in the book "Irrational Exuberance". From this dataset, we extracted only the historical close prices of the S&P 500, and the respective dividend payments (D). The spreadsheet with the refreshed data, can be found here: `http://www.econ.yale.edu/~shiller/data.htm`.

For the introduction chapter, we used yearly returns of three different assets (S&P500,3M t-bill and 10year T-Bond) which can be found on the Damodaran website. The spreadsheet with the raw data is available here:`http://people.stern.nyu.edu/adamodar/pc/datasets/histretSP.xls`

Allocation tables for different inputs

The purpose of this sub-subsection is to show, more in details, the results of the sensitivity analysis with respect to the r_f. This is needed since, as previously mentioned, in the chapter we only showed the indexed version of the allocation. For each value of the risk free, we compare the full and the reduced sample, and we evaluate the allocation every 12 months.

Table 4.1: Allocation tables - r_f=0.34%

Rf=0.34%								
	FULL SAMPLE				REDUCED SAMPLE			
T	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
1	0.48	0.24	0.12	0.05	1.00	0.60	0.30	0.12
12	0.48	0.24	0.12	0.05	1.00	0.54	0.27	0.11
24	0.47	0.23	0.12	0.05	0.97	0.49	0.24	0.10
36	0.47	0.23	0.11	0.05	0.90	0.45	0.22	0.09
48	0.46	0.23	0.11	0.05	0.83	0.41	0.20	0.08
60	0.46	0.22	0.11	0.04	0.77	0.38	0.19	0.07
72	0.45	0.22	0.11	0.04	0.73	0.36	0.18	0.07
84	0.44	0.22	0.11	0.04	0.69	0.34	0.16	0.07
96	0.44	0.22	0.11	0.04	0.65	0.32	0.15	0.06
108	0.43	0.21	0.10	0.04	0.62	0.30	0.15	0.06
120	0.43	0.21	0.10	0.04	0.59	0.28	0.14	0.05
132	0.42	0.21	0.10	0.04	0.56	0.27	0.13	0.05
144	0.42	0.20	0.10	0.04	0.54	0.26	0.12	0.05
156	0.41	0.20	0.10	0.04	0.51	0.25	0.12	0.05
168	0.41	0.20	0.10	0.04	0.49	0.23	0.11	0.04
180	0.41	0.20	0.10	0.04	0.47	0.22	0.11	0.04
192	0.40	0.19	0.09	0.04	0.46	0.22	0.10	0.04
204	0.40	0.19	0.09	0.04	0.44	0.21	0.10	0.04
216	0.39	0.19	0.09	0.04	0.43	0.20	0.10	0.04
228	0.39	0.19	0.09	0.04	0.41	0.19	0.09	0.04
240	0.39	0.18	0.09	0.04	0.40	0.19	0.09	0.04
252	0.38	0.18	0.09	0.04	0.39	0.18	0.09	0.03
264	0.38	0.18	0.09	0.03	0.38	0.17	0.08	0.03
276	0.37	0.18	0.09	0.03	0.36	0.17	0.08	0.03
288	0.37	0.18	0.09	0.03	0.35	0.16	0.08	0.03
300	0.37	0.17	0.08	0.03	0.34	0.16	0.08	0.03
312	0.36	0.17	0.08	0.03	0.34	0.16	0.07	0.03
324	0.36	0.17	0.08	0.03	0.33	0.15	0.07	0.03
336	0.36	0.17	0.08	0.03	0.32	0.15	0.07	0.03
348	0.35	0.17	0.08	0.03	0.31	0.14	0.07	0.03
360	0.35	0.17	0.08	0.03	0.30	0.14	0.07	0.03

Table 4.2: Allocation tables - r_f=0.08%

Rf=0.08%								
	FULL SAMPLE				REDUCED SAMPLE			
T	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
1	0.91	0.46	0.23	0.09	1.00	0.83	0.42	0.17
12	0.90	0.45	0.22	0.09	1.00	0.74	0.37	0.15
24	0.89	0.44	0.22	0.09	1.00	0.67	0.33	0.13
36	0.88	0.44	0.22	0.09	1.00	0.61	0.30	0.12
48	0.87	0.43	0.21	0.08	1.00	0.56	0.28	0.11
60	0.86	0.42	0.21	0.08	1.00	0.52	0.26	0.10
72	0.84	0.42	0.21	0.08	0.96	0.48	0.24	0.09
84	0.84	0.41	0.20	0.08	0.91	0.45	0.22	0.09
96	0.82	0.41	0.20	0.08	0.86	0.43	0.21	0.08
108	0.82	0.40	0.20	0.08	0.82	0.40	0.20	0.08
120	0.81	0.40	0.19	0.08	0.78	0.38	0.19	0.07
132	0.80	0.39	0.19	0.08	0.75	0.36	0.18	0.07
144	0.79	0.39	0.19	0.07	0.71	0.34	0.17	0.07
156	0.78	0.38	0.19	0.07	0.69	0.33	0.16	0.06
168	0.77	0.38	0.18	0.07	0.66	0.31	0.15	0.06
180	0.77	0.37	0.18	0.07	0.63	0.30	0.15	0.06
192	0.76	0.37	0.18	0.07	0.61	0.29	0.14	0.05
204	0.75	0.36	0.18	0.07	0.59	0.28	0.13	0.05
216	0.74	0.36	0.17	0.07	0.57	0.27	0.13	0.05
228	0.74	0.36	0.17	0.07	0.55	0.26	0.12	0.05
240	0.73	0.35	0.17	0.07	0.53	0.25	0.12	0.05
252	0.72	0.35	0.17	0.07	0.52	0.24	0.12	0.05
264	0.71	0.34	0.17	0.07	0.50	0.23	0.11	0.04
276	0.71	0.34	0.16	0.06	0.48	0.22	0.11	0.04
288	0.70	0.34	0.16	0.06	0.47	0.22	0.10	0.04
300	0.69	0.33	0.16	0.06	0.46	0.21	0.10	0.04
312	0.69	0.33	0.16	0.06	0.45	0.21	0.10	0.04
324	0.68	0.32	0.16	0.06	0.43	0.20	0.10	0.04
336	0.67	0.32	0.16	0.06	0.42	0.19	0.09	0.04
348	0.67	0.32	0.15	0.06	0.41	0.19	0.09	0.04
360	0.66	0.31	0.15	0.06	0.40	0.18	0.09	0.03

Table 4.3: Allocation tables - r_f=0%

Rf=0%								
	FULL SAMPLE				REDUCED SAMPLE			
T	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
1	1.00	0.46	0.26	0.10	1.00	0.90	0.45	0.18
12	1.00	0.45	0.25	0.10	1.00	0.79	0.40	0.16
24	1.00	0.44	0.25	0.10	1.00	0.72	0.36	0.14
36	0.99	0.44	0.25	0.10	1.00	0.65	0.32	0.13
48	0.98	0.43	0.24	0.10	1.00	0.60	0.30	0.12
60	0.96	0.42	0.24	0.09	1.00	0.56	0.27	0.11
72	0.95	0.42	0.23	0.09	1.00	0.52	0.25	0.10
84	0.94	0.41	0.23	0.09	0.97	0.49	0.24	0.09
96	0.93	0.41	0.23	0.09	0.91	0.46	0.22	0.09
108	0.92	0.40	0.22	0.09	0.87	0.43	0.21	0.08
120	0.91	0.40	0.22	0.09	0.83	0.41	0.20	0.08
132	0.90	0.39	0.22	0.09	0.79	0.39	0.19	0.07
144	0.89	0.39	0.21	0.08	0.76	0.37	0.18	0.07
156	0.88	0.38	0.21	0.08	0.73	0.35	0.17	0.07
168	0.87	0.38	0.21	0.08	0.70	0.33	0.16	0.06
180	0.86	0.37	0.21	0.08	0.67	0.32	0.15	0.06
192	0.85	0.37	0.20	0.08	0.65	0.31	0.15	0.06
204	0.85	0.36	0.20	0.08	0.63	0.30	0.14	0.06
216	0.84	0.36	0.20	0.08	0.61	0.29	0.14	0.05
228	0.83	0.36	0.20	0.08	0.58	0.27	0.13	0.05
240	0.82	0.35	0.19	0.08	0.57	0.27	0.13	0.05
252	0.81	0.35	0.19	0.07	0.55	0.26	0.12	0.05
264	0.80	0.34	0.19	0.07	0.53	0.25	0.12	0.05
276	0.80	0.34	0.19	0.07	0.52	0.24	0.11	0.04
288	0.79	0.34	0.18	0.07	0.50	0.23	0.11	0.04
300	0.78	0.33	0.18	0.07	0.49	0.23	0.11	0.04
312	0.77	0.33	0.18	0.07	0.47	0.22	0.10	0.04
324	0.77	0.32	0.18	0.07	0.46	0.21	0.10	0.04
336	0.76	0.32	0.18	0.07	0.45	0.21	0.10	0.04
348	0.75	0.32	0.17	0.07	0.44	0.20	0.10	0.04
360	0.74	0.31	0.17	0.07	0.43	0.20	0.09	0.04

Table 4.4: Allocation tables - r_f=-0.08%

	Rf=-0.08%							
	FULL SAMPLE				REDUCED SAMPLE			
T	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
1	1.00	0.57	0.29	0.11	1.00	0.96	0.48	0.19
12	1.00	0.57	0.28	0.11	1.00	0.85	0.42	0.17
24	1.00	0.56	0.28	0.11	1.00	0.76	0.38	0.15
36	1.00	0.55	0.27	0.11	1.00	0.70	0.35	0.14
48	1.00	0.54	0.27	0.11	1.00	0.64	0.32	0.13
60	1.00	0.54	0.27	0.11	1.00	0.59	0.29	0.12
72	1.00	0.53	0.26	0.10	1.00	0.55	0.27	0.11
84	1.00	0.52	0.26	0.10	1.00	0.52	0.25	0.10
96	1.00	0.51	0.25	0.10	0.96	0.49	0.24	0.09
108	1.00	0.51	0.25	0.10	0.92	0.46	0.22	0.09
120	1.00	0.50	0.25	0.10	0.88	0.43	0.21	0.08
132	0.99	0.50	0.24	0.10	0.84	0.41	0.20	0.08
144	0.98	0.49	0.24	0.09	0.80	0.39	0.19	0.07
156	0.97	0.48	0.24	0.09	0.77	0.37	0.18	0.07
168	0.96	0.48	0.23	0.09	0.74	0.36	0.17	0.07
180	0.95	0.47	0.23	0.09	0.71	0.34	0.16	0.06
192	0.94	0.47	0.23	0.09	0.69	0.33	0.16	0.06
204	0.94	0.46	0.22	0.09	0.66	0.31	0.15	0.06
216	0.92	0.46	0.22	0.09	0.64	0.30	0.15	0.06
228	0.91	0.45	0.22	0.09	0.62	0.29	0.14	0.05
240	0.91	0.45	0.22	0.08	0.60	0.28	0.14	0.05
252	0.89	0.44	0.21	0.08	0.58	0.27	0.13	0.05
264	0.89	0.43	0.21	0.08	0.56	0.26	0.13	0.05
276	0.88	0.43	0.21	0.08	0.55	0.25	0.12	0.05
288	0.87	0.43	0.21	0.08	0.53	0.25	0.12	0.05
300	0.86	0.42	0.20	0.08	0.52	0.24	0.11	0.04
312	0.86	0.41	0.20	0.08	0.50	0.23	0.11	0.04
324	0.85	0.41	0.20	0.08	0.49	0.22	0.11	0.04
336	0.84	0.41	0.20	0.08	0.48	0.22	0.10	0.04
348	0.83	0.40	0.19	0.08	0.46	0.21	0.10	0.04
360	0.82	0.39	0.19	0.07	0.45	0.21	0.10	0.04

Table 4.5: Allocation tables - r_f=-0.34%

	Rf=-0.34%							
	FULL SAMPLE				REDUCED SAMPLE			
T	av=5	av=10	av=20	av=50	av=5	av=10	av=20	av=50
1	1.00	0.89	0.44	0.18	1.00	1.00	0.65	0.26
12	1.00	0.87	0.43	0.17	1.00	1.00	0.57	0.23
24	1.00	0.86	0.43	0.17	1.00	1.00	0.50	0.20
36	1.00	0.85	0.42	0.17	1.00	0.92	0.46	0.18
48	1.00	0.83	0.42	0.16	1.00	0.83	0.42	0.16
60	1.00	0.82	0.41	0.16	1.00	0.77	0.38	0.15
72	1.00	0.81	0.40	0.16	1.00	0.71	0.35	0.14
84	1.00	0.80	0.40	0.16	1.00	0.68	0.33	0.13
96	1.00	0.79	0.39	0.15	1.00	0.62	0.31	0.12
108	1.00	0.79	0.39	0.15	1.00	0.59	0.29	0.11
120	1.00	0.78	0.38	0.15	1.00	0.56	0.27	0.11
132	1.00	0.76	0.37	0.15	1.00	0.53	0.26	0.10
144	1.00	0.74	0.36	0.14	1.00	0.50	0.24	0.10
156	1.00	0.74	0.36	0.14	0.95	0.48	0.23	0.09
168	1.00	0.75	0.36	0.14	0.93	0.45	0.22	0.09
180	1.00	0.72	0.35	0.14	0.89	0.43	0.21	0.08
192	1.00	0.73	0.36	0.14	0.88	0.42	0.20	0.08
204	1.00	0.73	0.35	0.14	0.83	0.40	0.19	0.08
216	1.00	0.70	0.35	0.14	0.79	0.38	0.18	0.07
228	1.00	0.68	0.33	0.13	0.77	0.37	0.18	0.07
240	1.00	0.71	0.34	0.13	0.75	0.35	0.17	0.07
252	1.00	0.65	0.31	0.12	0.72	0.34	0.16	0.06
264	1.00	0.69	0.33	0.13	0.70	0.33	0.16	0.06
276	1.00	0.72	0.34	0.13	0.69	0.32	0.16	0.06
288	1.00	0.68	0.33	0.13	0.66	0.31	0.15	0.06
300	1.00	0.67	0.32	0.12	0.65	0.30	0.14	0.06
312	1.00	0.63	0.30	0.12	0.63	0.29	0.14	0.05
324	1.00	0.67	0.32	0.12	0.60	0.28	0.13	0.05
336	1.00	0.66	0.31	0.12	0.60	0.27	0.13	0.05
348	1.00	0.65	0.31	0.12	0.58	0.27	0.13	0.05
360	1.00	0.53	0.25	0.10	0.56	0.26	0.12	0.05