Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Corso di Laurea Triennale in Matematica

Hausdorff dimension and Iterated Function Systems

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Anno Accademico 2022/2023

22/09/2023

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Introduction

Prior to the XX century, mathematicians were mostly concerned with studying sets, functions and methods related to classical calculus and its results. Mathematics, and in particular Analysis and Geometry, was largely focused on the concepts of smoothness and regularity at the time.
Anything which did not fall under these categories was not considered worthy of study, or seen as pathological. A curious example of this enmity towards these irregularities is the debate which shook the mathematical world after Karl Weierstrass's discovery in 1895 of a continuous functions which is nowhere differentiable, nowadays known as the Weierstrass function.
Renowned French mathematician Charles Hermite, after learning about the Weierstrass function, wrote to his doctoral student Thomas Stieltjes: "I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives."

During the first half of the last century, interest in these pathologies started to rise, although without a formal nor general theory behind it. In 1904 the Swedish mathematician Helge von Koch constructed a curve which, at any point, is impossible to draw a tangent line to. Unlike Weierstrass' analytical proof, hard to visualize, von Koch's construction was purely geometrical and thus its peculiar property could be more intuitively understood at the time.
Later on in 1951, English researcher Lewis Fry Richardson noticed that the coastline of Great Britain had no "definitive" length: in trying to measure it as one would measure a rectifiable curve, through smaller and smaller subdivisions into line segments, he found that this process would yield an infinitely long coastline, due to its "roughness" at every point or scale of observation.

In 1975 the mathematician Benoit B. Mandelbrot, while studying invariant sets under transformations of the complex plane, previously studied by Gaston Julia and Pierre Fatou, found yet another example of these "rough" and irregular sets, now known as the Mandelbrot set. What piqued his interest was the local structures of these sets, which he was studying with the aid of one of the first computer programs in the field to plot his images.
Studying all examples cited above, he thought to group all objects that had some "roughness" under one name: *fractals* (derived from the Latin *fractus*, meaning "broken" or "fractured"). Mandelbrot also expanded on this, observing that many objects in nature are described much more appropriately with these concepts in mind.

Although the definition of fractal is not particularly pedantic or agreed upon, it usually translates to "objects that have non-integer dimension", where the dimension in question is often the *Hausdorff dimension.* This concept was first introduced in 1918 by German mathematician Felix Hausdorff, and in this work our first goal is to present its definition and properties, through the study of the *Hausdorff measure.*
Briefly, fractal dimension describes how efficiently a subset of $\mathbb{R}^n$ fills space: as mentioned earlier, it serves as an excellent quantifier of "roughness", that is the prominence of the irregularities of a set when viewing it at very small scales. Related to this, we will also cover the definition of the so called *Box-counting dimension*, an alternative definition similar to its counterpart.

Lastly, our survey will cover a class of objects known as *Iterated Function Systems*, i.e. families of contracting transformations, which are the main focus of this thesis. They were first introduced and studied by Hutchinson in 1981 (see [5]), which lay the foundations for mathematicians from the 1980s onwards.
In particular, we are interested in the study of their *attractor*, the invariant set under these contractions (which we will prove to exist and be unique), and the calculation of its dimensions. A particular sub-family of attractors is that of *self-similar sets*, objects that are geometrically similar to smaller com-

ponents of their own structure, i.e. attractors of contractions which are also similarities. In conclusion, the last section covers a number of examples of classic fractal constructions.
Note that we will follow more or less in detail Falconer's book on these topics, see [3], in particular Chapters 1, 2, 3, 4 and 9.

Chapter 1

Hausdorff measure and dimension

1.1 Measure Theory background

Definition 1.1. Let $\mu : M \to [0, +\infty]$ be a function defined on a σ-algebra M of $X \neq \emptyset$, and consider the following properties:

(a) $\mu(\emptyset) = 0$

(b) $\mu(A) \leq \mu(B)$ for all $A \subseteq B$ in M

(c) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

(c') $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ if the A_i are disjoint sets in M.

We call μ an *outer measure* on X if it satisfies (a), (b) and (c) and if $M = \mathcal{P}(X)$, or *measure* if it satisfies all four properties (actually, (a) and (c') suffice).

For the purposes of this thesis, we will be working with Borel measures on $X = \mathbb{R}^n$, that is measures defined on the σ-algebra generated by the open subsets of $\mathbb{R}^n$, also called Borel sets. In particular, the Hausdorff measure:

Definition 1.2. Let F be a subset of $\mathbb{R}^n$ and $s \geq 0$. For any $\delta > 0$, we call $\{U_i\}_{i\geq 1}$ a δ-*cover* of F if for all U_i we have $|U_i| = \sup\Big\{|x-y| : x, y \in U_i\Big\} \leq \delta$ and $F \subseteq \bigcup_{i=1}^{\infty} U_i$. Then let us define

$$\mathcal{H}^s_\delta(F) = \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ is a } \delta\text{-cover of } F\Big\}. \tag{1.1}$$

Note that as δ decreases, the class of admissible δ-covers gets smaller, so $\mathcal{H}^s_\delta$ increases. Thus we may define

$$\mathcal{H}^s(F) = \sup_{\delta>0} \mathcal{H}^s_\delta(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F). \tag{1.2}$$

We call $\mathcal{H}^s(F)$ the *s-dimensional Hausdorff measure* of F.

We have that $\mathcal{H}^s$ is indeed an outer measure:

- $\mathcal{H}^s(\emptyset) = 0$ obviously.
- $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ if $A \subseteq B$, since $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s_\delta(B)$ for all $\delta > 0$ (every δ-cover of B is also a δ-cover of A)

- $\mathcal{H}^s\Big(\bigcup\limits_{i=1}^{\infty} A_i\Big) \leq \sum\limits_{i=1}^{\infty} \mathcal{H}^s(A_i)$, since $\mathcal{H}^s_\delta\Big(\bigcup\limits_{i=1}^{\infty} A_i\Big) \leq \sum\limits_{i=1}^{\infty} \mathcal{H}^s_\delta(A_i)$ for all $\delta > 0$:
  if for some A_i it occurs that $\mathcal{H}^s_\delta(A_i) = \infty$, the inequality is trivial. Otherwise, let $\varepsilon > 0$: every A_i has a δ-cover $\{U^i_k\}_{k\geq 1}$ such that

$$\sum_{k=1}^{\infty} |U^i_k|^s < \mathcal{H}^s_\delta(A_i) + \frac{\varepsilon}{2^i}.$$

  Thus, since $\{U^i_k\}_{k,i\geq 1}$ is a δ-cover of $\bigcup\limits_{i=1}^{\infty} A_i$,

$$\mathcal{H}^s_\delta\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |U^i_k|^s < \sum_{i=1}^{\infty}\Big(\mathcal{H}^s_\delta(A_i) + \frac{\varepsilon}{2^i}\Big) = \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(A_i) + \varepsilon$$

  which yields the desired result as ε approaches zero.

Furthermore, $\mathcal{H}^s$ is a Borel measure: this is a direct consequence of the (non-trivial) fact that Borel sets are $\mathcal{H}^s$-measurable:

Definition 1.3. An outer measure μ on a metric space (X, d) is called a *metric outer measure* if μ is additive on any pair of positively separated sets E and F, which means that if $d(E, F) = \inf\{d(x, y) : x \in E, y \in F\} > 0$, then $\mu(E \cup F) = \mu(E) + \mu(F)$.

Theorem 1.4. Let μ be a metric outer measure on a metric space X. Then all Borel subsets of X are μ-measurable

Cf. [2, pages 5-6] for more details.

This result, along with Caratheodory's extension theorem (Cf. [4, Theorem 1.11]), proves that $\mathcal{H}^s$ is a Borel measure, provided $\mathcal{H}^s$ is a metric outer measure, which can be easily verified: let $E, F \subseteq \mathbb{R}^n$ and $\mathcal{F}$ be a δ-cover of $E \cup F$ with $0 < \delta < d(E, F)$, and

$$\mathcal{F}_E = \{U \in \mathcal{F} : U \cap F = \emptyset\} \qquad \mathcal{F}_F = \{U \in \mathcal{F} : U \cap E = \emptyset\}.$$

Then no set in $\mathcal{F}$ intersects both E and F, and thus $\mathcal{F}_E$ and $\mathcal{F}_F$ are disjoint and form δ-covers of E and F respectively. Then,

$$\mathcal{H}^s_\delta(E) \leq \sum_{U\in\mathcal{F}_E} |U|^s \qquad \text{and} \qquad \mathcal{H}^s_\delta(F) \leq \sum_{U\in\mathcal{F}_F} |U|^s,$$

and we get

$$\mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) \leq \sum_{U\in\mathcal{F}_E} |U|^s + \sum_{U\in\mathcal{F}_F} |U|^s \leq \sum_{U\in\mathcal{F}} |U|^s.$$

Taking infimum over such covers as $\mathcal{F}$, we get $\mathcal{H}^s_\delta(E \cup F) \leq \mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) \leq \mathcal{H}^s_\delta(E \cup F)$, where the first inequality is property (c) in Definition (1.1).

Another fundamental fact to consider is that the n-dimensional Hausdorff measure is, up to a constant multiple, the n-dimensional Lebesgue measure $\mathcal{L}^n$. More precisely, if F is a Borel subset of $\mathbb{R}^n$, $\mathcal{H}^n(F) = c_n^{-1}\mathcal{L}^n(F)$ where

$$c_n = \frac{\pi^{\frac{n}{2}}}{2^n\Gamma(\frac{n}{2} + 1)}$$

is the volume of an n-dimensional ball of diameter 1 (Cf. [1, section 2.2]).
Not only that, $\mathcal{H}^m$ measures the m-dimensional area of "nice" subsets of $\mathbb{R}^n$, such as m-dimensional smooth submanifolds: $\mathcal{H}^0$ counts the number of points, $\mathcal{H}^1$ measures the length of smooth curves, $\mathcal{H}^2$ the area of smooth surfaces, etc...
This is a direct consequence of the famous (but hard to prove) Area Formula (Cf. [1, section 3.3]):

Theorem 1.5 (Area Formula)**.** Consider an injective Lipschitz map $f : \mathbb{R}^m \to \mathbb{R}^n$, where $m \leq n$. By Rademacher's theorem f is differentiable almost everywhere, and at any point of differentiability $y \in \mathbb{R}^m$ we may define $Jf(y) = \sqrt{\det\left(Df|_y^t \cdot Df|_y\right)}$. Then the following identity holds:

$$c_m^{-1} \int_A Jf(y)\; d\mathcal{L}^m(y) = \int_{f(A)} d\mathcal{H}^m(x) = \mathcal{H}^m(f(A)). \tag{1.3}$$

In particular, this applies locally for diffeomorphisms that define our smooth m-submanifolds of $\mathbb{R}^n$.

We now see a couple of basic results that determine how $\mathcal{H}^s$ behaves under certain mappings.

Proposition 1.6 (Scaling property)**.** Let $S : \mathbb{R}^n \to \mathbb{R}^m$ be a similarity transformation of scale $\lambda > 0$. Then for all $F \subseteq \mathbb{R}^n$,

$$\mathcal{H}^s(S(F)) = \lambda^s \mathcal{H}^s(F).$$

Proof. Suppose $\{U_i\}_{i\geq 1}$ is a δ-cover of F. Then $\{S(U_i)\}_{i\geq 1}$ is a $\lambda\delta$-cover of $S(F)$ and

$$\sum_{i\geq 1} |S(U_i)|^s = \lambda^s \sum_{i\geq 1} |U_i|^s,$$

so $\mathcal{H}^s_{\lambda\delta}(S(F)) \leq \lambda^s \mathcal{H}^s_\delta(F)$ which gives $\mathcal{H}^s(S(F)) \leq \lambda^s \mathcal{H}^s(F)$ letting $\delta \to 0$. By replacing S with S^{-1}, similarity of scale $1/\lambda$, and F with $S(F)$, we have the opposite inequality, since $\mathcal{H}^s(F) \leq \lambda^{-s}\mathcal{H}^s(S(F))$. □

Proposition 1.7. Let $f : F \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a Hölder transformation of exponent $\alpha > 0$, that is

$$|f(x) - f(y)| \leq c|x-y|^\alpha \qquad \forall x, y \in F$$

for some constant $c > 0$. Then for each $s \geq 0$,

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F).$$

Proof. Suppose $\{U_i\}_{i\geq 1}$ is a δ-cover of F. Since

$$|f(F \cap U_i)| \leq c|F \cap U_i|^\alpha \leq c|U_i|^\alpha,$$

$\{f(F \cap U_i)\}_{i\geq 1}$ is a $c\delta^\alpha$-cover of $f(F)$. Then

$$\sum_{i\geq 1} |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum_{i\geq 1} |U_i|^s$$

gives that $\mathcal{H}^{s/\alpha}_{c\delta^\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s_\delta(F)$, which, letting $\delta \to 0$, yields the desired inequality. □

As a remark, $\mathcal{H}^s$ is invariant under isometries because of Proposition 1.6 (with $\lambda = 1$) and in particular rotation and translation invariant, as could be expected.
Furthermore, Proposition 1.7 tell us that a Lipschitz mapping f (Hölder mapping of exponent $\alpha = 1$) satisfies

$$\mathcal{H}^s(f(F)) \leq c^s \mathcal{H}^s(F).$$

1.2 Hausdorff dimension

We are now interested in the behaviour of $\mathcal{H}^s$ as s changes. Consider a δ-cover $\{U_i\}_{i\geq 1}$ of $F \subseteq \mathbb{R}^n$ and $0 \leq s < t$. We have

$$\sum_{i\geq 1} |U_i|^t \leq \sum_{i\geq 1} |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_{i\geq 1} |U_i|^s$$

and taking infima yields $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. As $\delta \to 0$, $\delta^{t-s} \to 0$ and $\delta^{s-t} \to \infty$, so it is clear that

$$\mathcal{H}^s(F) < \infty \Rightarrow \mathcal{H}^t(F) = 0$$

$$\mathcal{H}^s(F) = \infty \Leftarrow \mathcal{H}^t(F) > 0$$

This tell us that $\mathcal{H}^s(F)$ jumps from ∞ to 0 as s increases, which leads us to the following definition:

Definition 1.8. There is a critical value of s called *Hausdorff dimension of F* at which the jump occurs: we denote it as $\dim_{\mathrm{H}} F$. More precisely

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\} \tag{1.4}$$

Where we agree that the supremum of the empty set is 0.
Note that $\dim_{\mathrm{H}} F$ is also the only value of s for which $0 < \mathcal{H}^s(F) < \infty$ could occur: if this happens for some Borel set F, F is called an s-set.

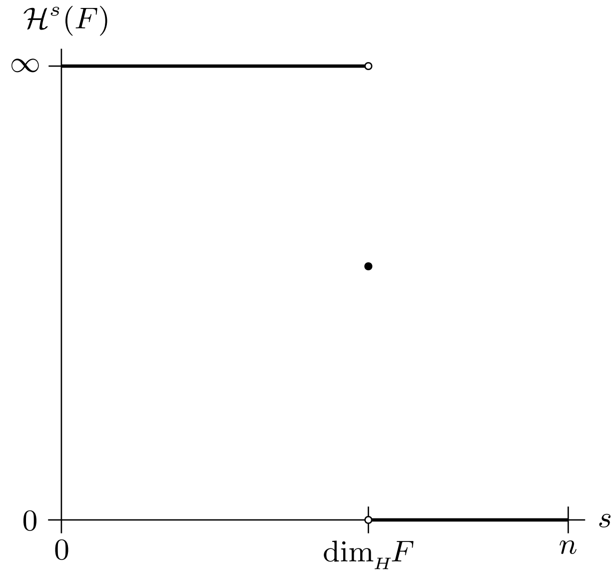


Figure 1.1: $\mathcal{H}^s(F)$ plotted against s for some $F \subseteq \mathbb{R}^n$ which is also a $\dim_{\mathrm{H}} F$-set

What follows are some basic properties of the Hausdorff dimension:

- *Monotonicity*: if $E \subseteq F \subseteq \mathbb{R}^n$ then $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$.
  This is immediate from the monotonicity of $\mathcal{H}^s$ for all $s \geq 0$, since
  $$\forall s > \dim_{\mathrm{H}} F \quad \mathcal{H}^s(E) \leq \mathcal{H}^s(F) = 0$$
  gives that $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$ by (1.4).

- *Countable Stability*: supposing $\{F_i\}_{i \geq 1}$ are countably many subsets of $\mathbb{R}^n$, then
  $$\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i = \sup_{i \geq 1}\{\dim_{\mathrm{H}} F_i\}.$$
  Clearly $\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i \geq \dim_{\mathrm{H}} F_i$ for all F_i by monotonicity. On the other hand, if $s > \dim_{\mathrm{H}} F_i$ for all F_i, we have $\mathcal{H}^s(F_i) = 0$ and so $\mathcal{H}^s\Big(\bigcup_{i=1}^{\infty} F_i\Big) = 0$, thus $s > \dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i$. By choosing $s = \sup_{i \geq 1}\{\dim_{\mathrm{H}} F_i\} + \varepsilon$, with $\varepsilon > 0$, and letting $\varepsilon \to 0$, it is clear that $\sup_{i \geq 1}\{\dim_{\mathrm{H}} F_i\} \geq \dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i$.

- *$\dim_H$ on countable sets*: if $F \subseteq \mathbb{R}^n$ is countable, $\dim_{\mathrm{H}} F = 0$.
  This is immediate from the countable stability property, since $F = \bigcup_{i=1}^{\infty}\{f_i\}$ and clearly, f_i being a point in F, $\mathcal{H}^0(\{f_i\}) = 1$, which means $\dim_{\mathrm{H}}\{f_i\} = 0$ and so $\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty}\{f_i\} = 0$.

- *$\dim_H$ on open sets*: if $F \subseteq \mathbb{R}^n$ is open, $\dim_{\mathrm{H}} F = n$.
  Recall that, given any ball $B \subseteq \mathbb{R}^n$, $\mathcal{H}^n(B) = c_n^{-1}\mathcal{L}^n(B)$ and thus B is clearly an n-set. Now, since

F is open, it contains an n-ball of some radius, thus $\dim_{\mathrm{H}} F \geq n$. Then, F is contained in a countable union of n-balls (since $\mathbb{R}^n$ is contained in the union of the countably many balls $\{B_i\}_i$ centered in $\mathbb{Z}^n$ each of radius 1), and so by countable stability $\dim_{\mathrm{H}} F \leq \dim_{\mathrm{H}} \bigcup_i B_i = \sup_i \dim_{\mathrm{H}} B_i = n$.

- *$\dim_H$ on smooth sets*: if F is a smooth m-dimensional submanifold of $\mathbb{R}^n$, $\dim_{\mathrm{H}} F = m$.
  This follows from Corollary 1.10 which we will prove in a moment:
  Let $M \subseteq \mathbb{R}^n$ be the m-dimensional smooth submanifold. By countable stability, it is enough to show that locally, that is for $N_r = B(\overline{x}, r) \cap M$ for some $r > 0$ and $\overline{x} \in M$, we have dimension m. Without loss of generality there is an orthogonal projection $\pi : N_r \to \mathbb{R}^m$ on some m-plane of $\mathbb{R}^n$ (e.g. its tangent m-plane) such that $\pi(N_r) \subseteq \mathbb{R}^m$ is open and thus $\dim_{\mathrm{H}} \pi(N_r) = m$, and since $|\pi(x) - \pi(y)| \leq |x - y|$ for all $x, y \in N_r$, $m = \dim_{\mathrm{H}} \pi(N_r) \leq \dim_{\mathrm{H}} N_r$.
  On the other hand, M is locally diffeomorphic to some open subset $U \subseteq \mathbb{R}^m$, which means that without loss of generality there exists a diffeomorphism $\phi \in C^1(U, N_r)$. By Lagrange's mean value theorem, $|\phi(x) - \phi(y)| \leq \Big(\sup_{z \in K} J\phi(z)\Big) \cdot |x - y|$ for all $x, y \in K$ and some compact subset $K \subseteq U$, so that $\sup_{z \in K} J\phi(z) < +\infty$ by continuity: thus $\phi_{|K}$ is Lipschitz, and $\dim_{\mathrm{H}} \phi(K) \leq \dim_{\mathrm{H}} K \leq m$.
  By choosing some $r' < r$ sufficiently small, $N_{r'} \subseteq \phi(K)$ which gives $\dim_{\mathrm{H}} N_{r'} \leq \dim_{\mathrm{H}} \phi(K) \leq m$.

Other important transformation properties of $\dim_{\mathrm{H}}$ follow from the corresponding ones for $\mathcal{H}^s$ given in Propositions 1.6 and 1.7:

Proposition 1.9. Let $f : F \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a Hölder transformation of exponent $\alpha > 0$ and constant $c > 0$. Then

$$\dim_{\mathrm{H}} f(F) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} F.$$

Proof. Let $s > \dim_{\mathrm{H}} F$. Then by Proposition 1.7

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F) = 0,$$

thus $\dim_{\mathrm{H}} f(F) \leq s/\alpha$ for all $s > \dim_{\mathrm{H}} F$, which means $\dim_{\mathrm{H}} f(F) \leq (\dim_{\mathrm{H}} F)/\alpha$. □

Corollary 1.10. Let $f : F \to \mathbb{R}^n$ be a function. Then:

(a) If f is a Lipschitz transformation then $\dim_{\mathrm{H}} f(F) \leq \dim_{\mathrm{H}} F$.

(b) If f is a bi-Lipschitz transformation then $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$, that is if

$$c_1 |x - y| \leq |f(x) - f(y)| \leq c_2 |x - y| \quad \forall x, y \in F$$

where $0 < c_1 \leq c_2 < \infty$.

Proof.

(a) Directly from the last result by choosing $\alpha = 1$.

(b) Obviously by (a) we have $\dim_{\mathrm{H}} f(F) \leq \dim_{\mathrm{H}} F$.
Moreover, since $c_1 |x - y| \leq |f(x) - f(y)|$, f is injective:

$$x \neq y \quad \Leftrightarrow \quad |x - y| > 0 \quad \Rightarrow \quad |f(x) - f(y)| \geq c_1 |x - y| > 0 \quad \Leftrightarrow \quad f(x) \neq f(y).$$

Thus $f : F \to f(F)$ is invertible and so $f^{-1} : f(F) \to F$ is also Lipschitz of constant $1/c_1$, since $c_1 |f^{-1}(x) - f^{-1}(y)| \leq |x - y|$. Thus by (a) we have $\dim_{\mathrm{H}} F \leq \dim_{\mathrm{H}} f(F)$.

□

The last result is particularly important. *Hausdorff dimension is invariant under bi-Lipschitz transformations*: since bi-Lipschitz transformations are necessarily homeomorphisms, this corollary tell us that the Hausdorff dimension is a "finer" invariant than topological invariants, that is, we can further subdivide a class of homeomorphic topological spaces in classes of those who have the same Hausdorff dimension. However, dimension alone tells us little about the topological properties of a set, but something can be said for dimension less than 1:

Proposition 1.11. A set $F \subseteq \mathbb{R}^n$ with $\dim_{\mathrm{H}} F < 1$ is totally disconnected, that is, its connected components are exactly its points.

Proof. Let $x, y \in F$ be distinct points and define $f : \mathbb{R}^n \to [0, \infty)$ as $f(z) = |z - x|$.
Since

$$|f(z) - f(w)| = \big||z - x| - |w - x|\big| \leq |(z - x) - (w - x)| = |z - w|,$$

from part (a) of the last result we have $\dim_{\mathrm{H}} f(F) \leq \dim_{\mathrm{H}} F < 1$, thus $f(F)$ is a subset of $\mathbb{R}$ of $\mathcal{H}^1$-measure zero, or equivalently Lebesgue measure zero.

By the following Lemma 1.12, $\mathbb{R} \backslash f(F)$ is dense in $\mathbb{R}$: then there must exist an r both in the open set $\{s \in \mathbb{R} : 0 < s < f(y)\}$ (which is not empty since $f(y) \neq 0$) and in $\mathbb{R} \backslash f(F)$, so that $F = A \cup B$ can be divided in the two connected components

$$A = \{z \in F : |z - x| < r\} \ \text{ and } \ B = \{z \in F : |z - x| > r\}.$$

Since $r < f(y)$, y lies in B, and clearly $x \in A$. Thus every point lies in a different connected component than all other points. □

Lemma 1.12. Let $E \subseteq \mathbb{R}$ be such that $\mathcal{L}(E) = 0$. Then $\mathbb{R} \backslash E$ is a dense subset of $\mathbb{R}$.

Proof. Suppose $\mathbb{R} \neq \overline{\mathbb{R} \backslash E} = \mathbb{R} \backslash \mathrm{int}(E)$, where $\mathrm{int}(E)$ denotes the interior of E. It follows that $\mathrm{int}(E) \neq \emptyset$, so there is a ball centered on $x \in E$ of radius $r > 0$ such that $B_{\mathbb{R}}(x, r) \subseteq \mathrm{int}(E) \subseteq E$, and so $0 < \mathcal{L}\Big(B_{\mathbb{R}}(x, r)\Big) \leq \mathcal{L}(E)$, which is a contradiction. □

1.3 Direct derivation for some examples

Example 1.13 (Cantor dust)**.** Let E_0 be the unit square in $\mathbb{R}^2$ and subdivide it in 16 sub-squares with a quarter of the side length. Now discard all of them except four so that the remaining squares are in different columns of the original square, and call E_1 this set. Now construct E_2 subdividing each square of E_1 as we just did with E_0 (with the same square pattern), and so forth for all $k \in \mathbb{N}$ construct E_k, as shown in Figure 1.2.
Now consider $F = \bigcap\limits_{k=0}^{\infty} E_k$, which is called a *Cantor dust*: we show that

$$1 \leq \mathcal{H}^1(F) \leq \sqrt{2}$$

which means that $\dim_{\mathrm{H}} F = 1$.

Observe that E_k consists of 4^k squares of side length 4^{-k}, thus of diameter $4^{-k}\sqrt{2}$. Taking the squares of E_k as a δ-cover, where $\delta = 4^{-k}\sqrt{2}$, we see that $\mathcal{H}^1_\delta(F) \leq 4^k \cdot 4^{-k}\sqrt{2}$. Letting $k \to \infty$, we get $\delta \to 0$ and so

$$\mathcal{H}^1(F) \leq \sqrt{2}.$$

For the lower bound, consider the orthogonal projection π on the x-axis, that is

$$\pi : \mathbb{R}^2 \to \mathbb{R}, \ (x_1, x_2) \longmapsto \pi(x_1, x_2) = x_1.$$

Clearly $|\pi(x) - \pi(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}^2$, thus π is a Lipschitz transformation, and observe that the way we constructed F tells us that $\pi(F) = [0, 1] \subseteq \mathbb{R}$: by Proposition 1.7,

$$1 = \mathcal{H}^1\big(\pi(F)\big) \leq \mathcal{H}^1(F).$$

Another, less formal, way of calculating the dimension of this Cantor dust is as follows: since F is made up of 4 smaller copies of itself scaled by a factor of $\frac{1}{4}$, by the scaling property we have that

$$\mathcal{H}^s(F) = 4 \cdot \Big(\frac{1}{4}\Big)^s \mathcal{H}^s(F).$$

If we assume that F is a s-set for $s = \dim_{\mathrm{H}} F$, that is $0 < \mathcal{H}^s(F) < \infty$, then it must be that $1 = 4\big(\frac{1}{4}\big)^s$, or $s = 1$.

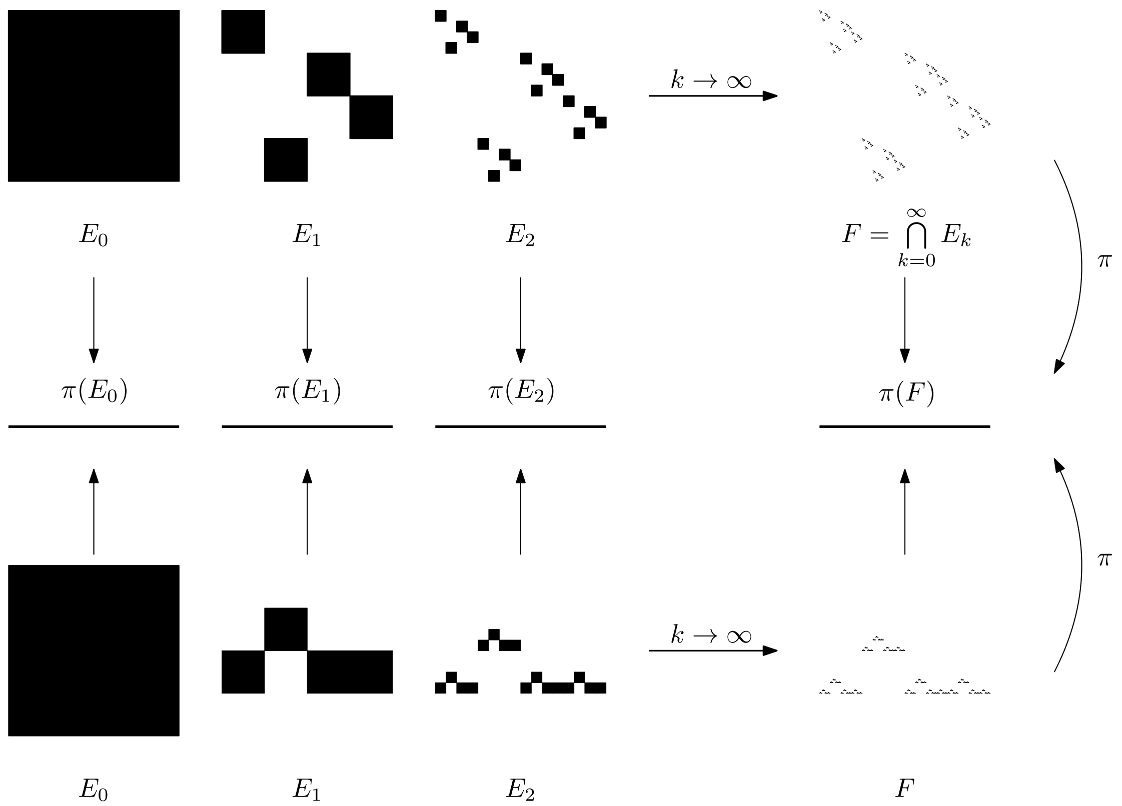


Figure 1.2: Two examples of Cantor dust and their orthogonal projections

Example 1.14 (Middle third Cantor set)**.** Let F be the *middle third Cantor set*, that is the "1-dimensional Cantor dust" obtained with repeated subdivisions of the intervals in thirds and taking out the middle one, starting from $[0, 1]$.
We now show that if $s = \log 2/ \log 3$,

$$\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$$

so that $\dim_{\mathrm{H}} F = s$.

Let us call *level-k intervals* all intervals of length 3^{-k} involved in the construction of F, which are 2^k (shown in Figure 1.3). Then all level-k intervals form a 3^{-k}-cover of F, and so

$$\mathcal{H}^s_{3^{-k}}(F) \leq 2^k 3^{-ks} = 1$$

if $s = \log 2/ \log 3$. As $k \to \infty$, we get

$$\mathcal{H}^s(F) \leq 1.$$

Conversely, to prove that $\frac{1}{2} \leq \mathcal{H}^s(F)$, since $3^s = 2$, we show that

$$\sum_{i=1}^{\infty} |U_i|^s \geq \frac{1}{2} = 3^{-s} \tag{1.5}$$

for any given cover $\{U_i\}_{i \geq 1}$ of F. In reality, it suffices to show this for finitely many closed subintervals of $[0, 1]$: without loss of generality all U_i may be intervals, and by expanding them slightly and using the compactness of $[0, 1]$, if we show (1.5) for the obtained finite subcover of $\{\overline{U_i}\}$, we are done.
Then consider a cover $\{U_i\}_{i=1}^N$, of closed subintervals of $[0, 1]$, of F. For any given U_i, let k be the integer such that

$$3^{-(k+1)} \leq |U_i| \leq 3^{-k}. \tag{1.6}$$

Then U_i can intersect at most one level-k interval, let us call it $E_{k,i}$, since all level-k intervals have at least a pairwise distance of 3^{-k}. Thus if $j \geq k$, by construction U_i intersects at most all 2^{j-k} level-j

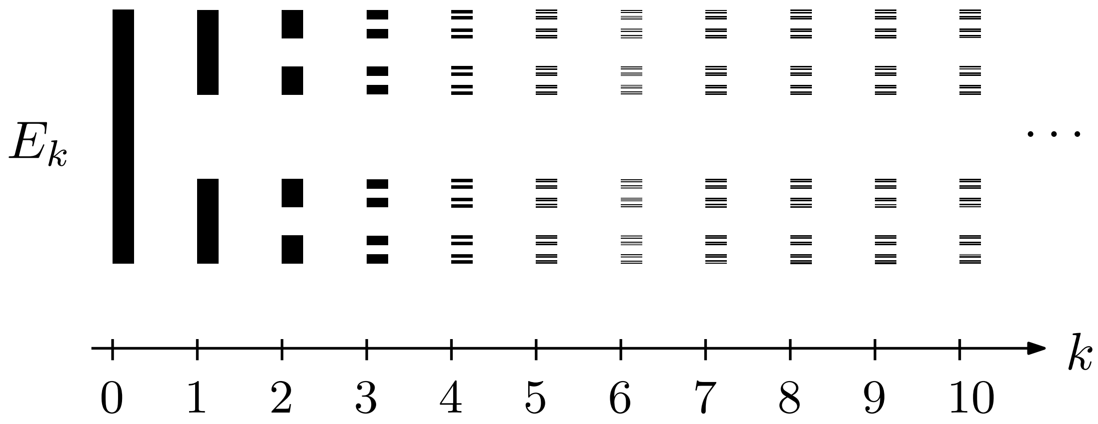


Figure 1.3: The construction of the middle third Cantor set. Here E_k is the union of all level-k intervals

subintervals of $E_{k,i}$. Now, since

$$2^{j-k} = 2^j 3^{-sk} = 2^j 3^{-s(k+1)} 3^s \leq 2^j 3^s |U_i|^s$$

by (1.6), if we choose j large enough so that $3^{-(j+1)} \leq |U_i|$ for all $i = 1, 2, \dots, N$, the last inequality applies for all U_i and so they all cumulatively intersect at most

$$\sum_{i=1}^{N} 2^{j-k} \leq 2^j 3^s \sum_{i=1}^{N} |U_i|^s$$

level-j intervals. If we recall that $\{U_i\}_{i=1}^N$ covers completely F, then it is clear that all 2^j level-j intervals intersect at least one U_i: counting intervals gives that

$$2^j \leq 2^j 3^s \sum_{i=1}^{N} |U_i|^s,$$

thus $3^{-s} \leq \sum_{i=1}^{N} |U_i|^s$ which is (1.5) and proves the lower bound.

Just as in the last example, we can obtain the same result in a more heuristic manner: since F is made up of two copies of itself (one contained in $[0, \frac{1}{3}]$, the other in $[\frac{2}{3}, 1]$) scaled by a factor $\frac{1}{3}$, we have that

$$\mathcal{H}^s(F) = 2\left(\frac{1}{3}\right)^s \mathcal{H}^s(F),$$

and if we assume F to be an s-set for $s = \dim_{\mathrm{H}} F$, we can simplify $\mathcal{H}^s(F)$ and get $3^s = 2$.

These examples and their “heuristic” solutions hint at a formula for calculating the dimension of self-similar sets, which we will prove later on and formalizes the same intuition.

Chapter 2

Other definitions of dimension

2.1 Box-counting dimensions

Box dimension is one of the most used dimensions, largely due to its applicability and easy empirical calculation. In this chapter we introduce its basic properties and relationships with the Hausdorff dimension.

Definition 2.1. Let $F \subseteq \mathbb{R}^n$ a non-empty bounded set, and $N_\delta(F)$ the smallest number of sets of diameter at most δ which can cover F, that is the smallest number of elements a δ cover of F can have. We define the *lower* and *upper box-counting dimensions* of F respectively as

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{2.1}$$

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{2.2}$$

If these have equal value, we refer to it as *box-counting dimension* or simply *box dimension* of F, and denote it as

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{2.3}$$

Here, and further on, we are assuming $0 < \delta < 1$ to ensure that $-\log \delta$ is strictly positive.

This definition entails a dimension under this intuition: a dimension s of F may be determined assuming that $N_\delta(F) \sim c\delta^{-s}$ as δ approaches zero, for some constant $c > 0$ which can be thought of as the s-dimensional volume of F. The idea is that in an s-dimensional volume c there ought to be at least approximately $c\delta^{-s}$ elements in every δ-cover, and we are searching for an s that makes this work for smaller and smaller scales of measurement δ (so that $N_\delta(F)$ satisfies a sort of scaling property). Taking logarithms, we have

$$s = \lim_{\delta \to 0} \frac{\log N_\delta(F) - \log c}{-\log \delta} = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

The following proposition reveals why it is called *box* dimension, and gives other equivalent definitions:

Proposition 2.2. The lower and upper box-counting dimensions of $F \subseteq \mathbb{R}^n$ are given by

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

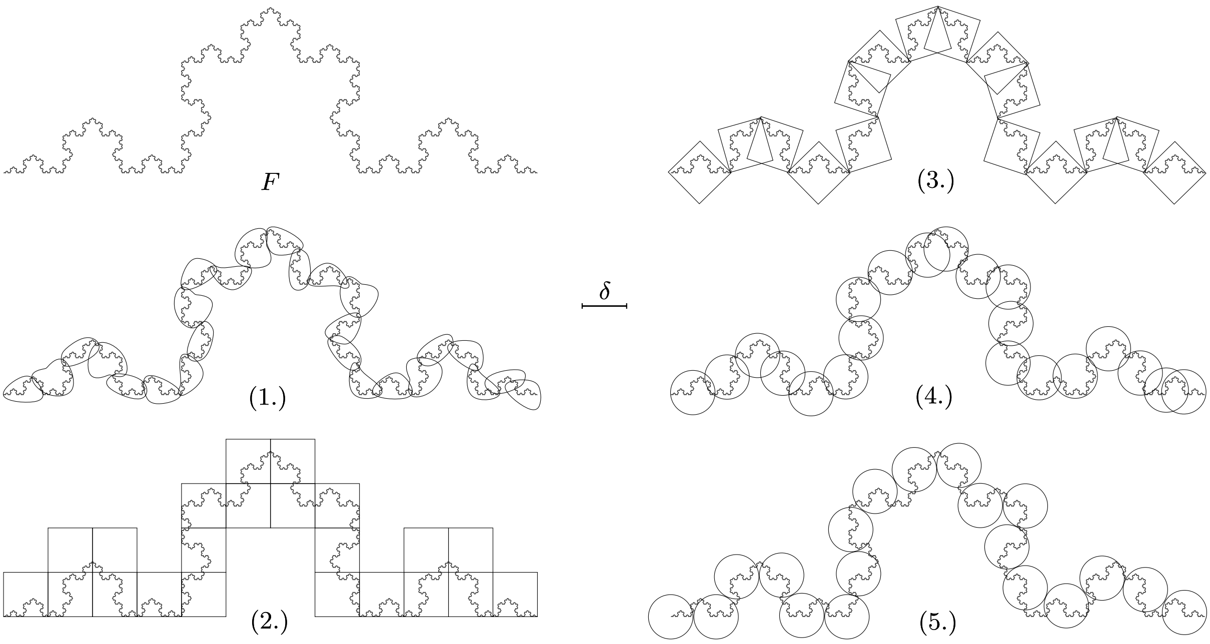
$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and the box-counting dimension of F by

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

if the limit exists, where $N_\delta(F)$ is any of the following:

1. the smallest number of sets of diameter at most δ that cover F
2. the number of δ-mesh cubes that intersect F
3. the smallest number of cubes of side δ that cover F
4. the smallest number of closed balls of radius δ that cover F
5. the largest number of disjoint balls of radius δ with centers in F



Proof.

- ($1 \Leftrightarrow 2$) Let us recall that for a δ-mesh cube we mean $[m_1\delta, (m_1+1)\delta] \times \ldots \times [m_n\delta, (m_n+1)\delta] \subseteq \mathbb{R}^n$ for some integers $m_1, \ldots, m_n$. Now let $N'_\delta(F)$ be the number of δ-mesh cubes that intersect F: they provide a collection of sets of diameter $\delta\sqrt{n}$ that cover F, so $N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$. If δ is small enough so that $\delta\sqrt{n} < 1$, we get

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log \delta\sqrt{n}} \leq \frac{\log N'_\delta(F)}{-\log \delta\sqrt{n}},$$

so letting δ approach zero yields

$$\underline{\dim}_{\mathrm{B}} F \leq \liminf_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta - \log \sqrt{n}} = \liminf_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta},$$

$$\overline{\dim}_{\mathrm{B}} F \leq \limsup_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta - \log \sqrt{n}} = \limsup_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta}.$$

On the other hand, any set of diameter at most δ is surely contained in 3^n δ-mesh cubes, which are any δ-mesh cube containing some point of the set and all its neighbouring cubes. So $N'_\delta(F) \leq 3^n N_\delta(F)$, which means that, taking logarithms and limits like earlier,

$$\liminf_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta} = \liminf_{\delta \to 0} \frac{\log N'_\delta(F) - n\log 3}{-\log \delta} \leq \underline{\dim}_{\mathrm{B}} F,$$

$$\limsup_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta} = \limsup_{\delta \to 0} \frac{\log N'_\delta(F) - n\log 3}{-\log \delta} \leq \overline{\dim}_{\mathrm{B}} F.$$

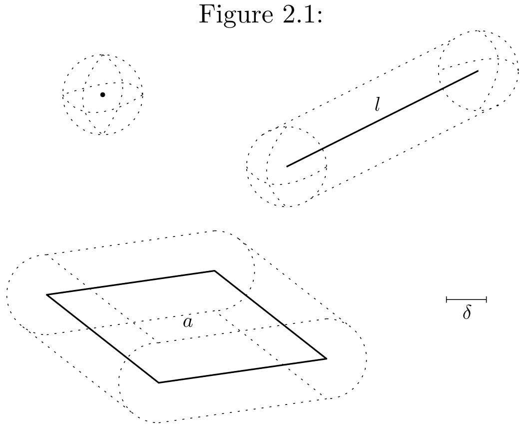
- ($1 \Leftrightarrow 3$) The equivalence follows as in the mesh cube case, since any cube of side δ has diameter $\delta\sqrt{n}$ and any set of diameter of at most δ is contained in a cube of side δ.
- ($1 \Leftrightarrow 4$) Precisely as the last equivalence, since any closed ball of radius δ has diameter 2δ and any set of diameter of at most δ is obviously contained in a closed ball of radius δ.
- ($1 \Leftrightarrow 5$) Let $N'_\delta(F)$ be the largest number of disjoint balls of radius δ centered in F, and $B_1, B_2, \ldots, B_{N'_\delta(F)}$ be disjoint balls centered in F and of radius δ. Let x be a point in F: x must be within distance δ of one of the B_i, otherwise the ball $B(x,\delta)$ could be added to form a larger collection of disjoint balls, which is a contradiction. Thus the balls $2B_1, 2B_2, \ldots, 2B_{N'_\delta(F)}$ (concentric with the B_i but with double radius, thus of diameter 4δ) cover F, so $N_{4\delta}(F) \leq N'_\delta(F)$. Conversely, let us again consider any balls $B_1, B_2, \ldots, B_{N'_\delta(F)}$ of radii δ centered in F and disjoint, and $U_1, U_2, \ldots, U_k$ any finite δ-cover of F: the U_j surely cover the centers of each B_i, so each B_i contains at least one of the U_j. Since the balls are disjoint, different B_i contain different U_j, thus $N'_\delta(F) \leq k$, or better $N'_\delta(F) \leq N_\delta(F)$.
  As for the last equivalences, taking logarithms and limits of these two inequalities yields the equivalence.

□

There is another intuition behind this definition which is worth mentioning:
Let $F_\delta = \left\{x \in \mathbb{R}^n : |x-y| \leq \delta \text{ for some } y \in F\right\}$ be the *δ-neighbourhood* of $F \subseteq \mathbb{R}^n$. We consider the rate at which the n-dimensional volume (that is, its n-dimensional Lebesgue measure) of F_δ decreases as $\delta \to 0$, for example in $\mathbb{R}^3$, as shown in Figure 2.1:

- If F is a point, $\mathrm{vol}(F_\delta) = \frac{4\pi}{3}\delta^3$
- If F is a segment of length l, $\mathrm{vol}(F_\delta) \sim \pi l \delta^2$
- If F is a flat surface of area a, $\mathrm{vol}(F_\delta) \sim 2a\delta$

Figure 2.1:



In general, this pattern more or less extends to fractional dimensions, that is to say, we expect there to be some $s > 0$ such that $\mathrm{vol}^n(F_\delta) \sim c\delta^{n-s}$ for some $c > 0$ called *Minkowski content* or *s-dimensional content* of F. In this sense, as δ approaches zero,

$$n - \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta} \sim n - \frac{\log \delta^{n-s}}{\log \delta} = s.$$

This definition of dimension, sometimes called *Minkowski* or *Minkowski-Bouligand dimension*, as it turns out, coincides with the box-counting dimension, even if the limit does not exist.

Proposition 2.3. If F is a subset of $\mathbb{R}^n$, then

$$\underline{\dim}_{\mathrm{B}} F = n - \liminf_{\delta \to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta}$$

$$\overline{\dim}_{\mathrm{B}} F = n - \limsup_{\delta \to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta},$$

where F_δ is the δ-neighbourhood of F.

Proof. If F can be covered by $N_\delta(F)$ balls of radius $\delta < 1$, F_δ can be covered by the corresponding concentric balls of radius 2δ, hence

$$\mathrm{vol}^n(F_\delta) \leq N_\delta(F) c_n (2\delta)^n$$

where $c_n = \mathrm{vol}^n\big(B_{\mathbb{R}^n}(0,1)\big)$. Taking logarithms and dividing by $-\log\delta$ yields

$$\frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \leq \frac{\log 2^n c_n + n\log\delta + \log N_\delta(F)}{-\log\delta},$$

so

$$\liminf_{\delta\to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \leq -n + \underline{\dim}_{\mathrm{B}} F, \qquad \text{or better} \qquad n - \liminf_{\delta\to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log\delta} \leq \underline{\dim}_{\mathrm{B}} F$$

with a similar inequality taking the upper limit.

On the other hand (using Proposition 2.2, point 5), if there are $N_\delta(F)$ disjoint balls of radius δ centered in F, summing up the volumes gives

$$N_\delta(F) c_n \delta^n \leq \mathrm{vol}^n(F_\delta).$$

Taking logarithms just as before, by letting δ approach zero we get

$$n - \liminf_{\delta\to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log\delta} \geq \underline{\dim}_{\mathrm{B}} F$$

and similarly for the upper limit. □

2.2 A comparison with Hausdorff dimension and some examples

Example 2.4. Let F be the middle third Cantor set. Then $\underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = \log 2/\log 3$.
(See Example 1.14 for terminology and notation)

The 2^k level-k intervals of length 3^{-k} form a δ-cover of F, if $3^{-k} < \delta \leq 3^{-k+1}$, so $N_\delta(F) \leq 2^k$ and

$$\overline{\dim}_{\mathrm{B}} = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \leq \limsup_{k\to+\infty} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}.$$

On the other hand, any interval (a box in $\mathbb{R}$) of length δ and part of a box covering of F, with $3^{-k-1} \leq \delta < 3^{-k}$, intersects exactly one level-k interval: there are 2^k such intervals, so at least 2^k intervals of length δ are required to cover F, thus $N_\delta(F) \geq 2^k$ which leads to $\underline{\dim}_{\mathrm{B}} F \geq \log 2/\log 3$.

Thus for the middle third Cantor set F, $\dim_{\mathrm{B}} F = \log 2/\log 3 = \dim_{\mathrm{H}} F$. This is in general false, but something can be said nonetheless:
Let $F \subseteq \mathbb{R}^n$ be covered by $N_\delta(F)$ sets of diameter δ. Then by definition of Hausdorff measure $\mathcal{H}^s_\delta(F) \leq N_\delta(F)\delta^s$.
If $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) > 1$, then $N_\delta(F)\delta^s \geq \mathcal{H}^s_\delta(F) > 1$ if δ is sufficiently small, and taking logarithms gives

$$\log N_\delta(F) + s\log\delta > 0 \qquad \text{or better} \qquad s \leq \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} = \underline{\dim}_{\mathrm{B}} F.$$

So if F has non-zero s-dimensional Hausdorff measure (since, by the scaling property, if $\mathcal{H}^s(F) > 0$ we can scale F by some factor so that $\mathcal{H}^s(F) > 1$) then $s \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F$, and by definition (1.4) we get

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F \tag{2.4}$$

We do not always get the equality here, as we shall see with this next example.

Example 2.5. Let $F = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. Then $\dim_{\mathrm{H}} F = 0 \neq \frac{1}{2} = \dim_{\mathrm{B}} F$.

As seen in Chapter 1, $\dim_{\mathrm{H}} F = 0$ since F is countable.
On the other hand, let $0 < \delta < \frac{1}{2}$ and k be the integer satisfying $\frac{1}{k(k+1)} \leq \delta < \frac{1}{(k-1)k}$. Suppose now U is a subset of $\mathbb{R}$ with $|U| \leq \delta$: U can cover at most one of the points in $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}\right\} \subseteq F$, since $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{(k-1)k} > \delta$.
Thus at least k sets of diameter at most δ are required to cover F, so $N_\delta(F) \geq k$, which gives

$$\frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log k}{-\log \delta} \geq \frac{\log k}{\log k(k+1)}.$$

Letting $\delta \to 0$ means letting $k \to +\infty$, and we get $\underline{\dim}_{\mathrm{B}} F \geq \frac{1}{2}$.
Conversely, take δ and k as before: $(k+1)$ intervals of length δ cover $[0, \frac{1}{k}]$, and other $k-1$ intervals cover $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k-1}\right\}$, one for each point. By definition $N_\delta(F) \leq 2k$, which yields

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log 2k}{-\log \delta} \leq \frac{\log 2k}{\log(k-1)k}.$$

Taking the upper limit as $\delta \to 0$, we get $\overline{\dim}_{\mathrm{B}} F \leq \frac{1}{2}$.

If we recall the definitions of Hausdorff and box dimension, we can see that

$$N_\delta(F)\delta^s = \inf\left\{\sum_i \delta^s : \{U_i\} \text{ is a (finite) } \delta\text{-cover of } F\right\}$$

and

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}.$$

are very similar quantities, from which we extrapolate a dimension s by studying how they behave as δ approaches zero.

This should shine a light on the qualitative differences between these two dimensions: in calculating Hausdorff dimension we assign a weight $|U_i|^s$ to the covering sets U_i, whereas for the box dimensions we use the same weight δ^s for each covering set. Roughly speaking, box dimensions indicate the efficiency with which a set may be covered by small sets of equal size, whereas Hausdorff dimension involves covering sets of small but varying size.

2.3 Techniques for computing dimensions

Here we introduce some basic techniques aimed at finding bounds for Hausdorff measures and dimensions: generally speaking, we get upper bounds by finding effective small sets coverings, and lower bounds by putting "sensible" measures on the set. For most fractals, natural coverings of the set arise in its construction, which give "obvious" upper bounds.

Proposition 2.6. Let $F \subseteq \mathbb{R}^n$.

1. Suppose F can be covered by n_k sets of diameter at most δ_k for each $k \in \mathbb{N}$, and such that $\delta_k \to 0$ as $k \to \infty$. Then
$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \liminf_{k\to\infty} \frac{\log n_k}{-\log \delta_k} \tag{2.5}$$
Moreover, if $n_k \delta_k^s$ remains bounded as $k \to \infty$, then $\mathcal{H}^s(F) < \infty$.

2. In all limits in definitions (2.1)-(2.3) it is sufficient to consider δ approaching zero through any decreasing sequence δ_k such that $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, in particular if $\delta_k = c^k$.
In other words, we have the following:
$$\underline{\dim}_{\mathrm{B}} F = \liminf_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}$$

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}$$

$$\dim_{\mathrm{B}} F = \lim_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}$$

for any decreasing infinitesimal sequence δ_k such that $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$. In addition, we have

$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \limsup_{k\to\infty} \frac{\log n_k}{-\log \delta_k}. \tag{2.6}$$

Proof.

1. By definition (2.1) and the observation resulting in (2.4), we get (2.5):

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \liminf_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \leq \liminf_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

Furthermore, if $n_k\delta_k^s$ is bounded by some constant $0 < M < \infty$, since $\mathcal{H}^s_{\delta_k}(F) \leq n_k\delta_k^s \leq M$ because the n_k sets form a δ_k-cover of F, letting $k \to \infty$ yields $\mathcal{H}^s(F) \leq M < \infty$.

2. Suppose δ_k is decreasing and $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, let $N_\delta(F)$ be the smallest number of sets in a δ-cover of F and choose $\delta_{k+1} \leq \delta < \delta_k$. Then, since $N_\delta(F) \leq N_{\delta_{k+1}}(F)$ because all δ_{k+1}-covers of F are also δ-covers of F,

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_k} = \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log(\delta_{k+1}/\delta_k)} \leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log c}.$$

Thus letting $\delta \to 0$ also means $k \to \infty$, and

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \leq \limsup_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}.$$

Similarly, since $N_{\delta_k}(F) \leq N_\delta(F)$ because $\delta < \delta_k$, we have

$$\frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log N_{\delta_k}(F)}{-\log \delta_{k+1}} = \frac{\log N_{\delta_k}(F)}{-\log \delta_k + \log(\delta_k/\delta_{k+1})} \geq \frac{\log N_{\delta_k}(F)}{-\log \delta_k - \log c}$$

which leads to the opposite inequality taking upper limits. The same statement for $\underline{\dim}_{\mathrm{B}}$ is obtained taking lower limits instead of upper limits.
Thus with the same assumptions we have (2.6):

$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F = \limsup_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \leq \limsup_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

□

Before stating the next definition, let us recall that the support $\mathrm{spt}(\mu)$ of a measure μ on $\mathbb{R}^n$ is the smallest closed set X such that $\mu(\mathbb{R}^n \backslash X) = 0$.

Definition 2.7. We will call *mass distribution* on a bounded set $A \subseteq \mathbb{R}^n$ any measure μ on $\mathbb{R}^n$ such that $\mathrm{spt}(\mu) \subseteq A$ and $0 < \mu(\mathbb{R}^n) < \infty$.

We may think of mass distributions as taking some finite mass and subdividing it on some bounded set, hence the name.

Proposition 2.8 (Mass distribution principle)**.** Let μ be a mass distribution on F and suppose that for some $s \geq 0$ there exist $c > 0$ and $\varepsilon > 0$ such that $\mu(U) \leq c|U|^s$ for all sets U with $|U| \leq \varepsilon$. Then

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{c} \qquad \text{and} \qquad s \leq \dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F.$$

Proof. If $\{U_i\}_i$ is any δ-cover of F with $\delta \leq \varepsilon$, then

$$0 < \mu(F) \leq \mu\Big(\bigcup_i U_i\Big) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s$$

by measure properties and the hypothesis on μ. Taking infima on both sides yields $\mathcal{H}^s_\delta(F) \geq \frac{\mu(F)}{c}$, so $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c} > 0$ letting $\delta \to 0$. Since $\mathcal{H}^s(F) > 0$, by definition (1.4) $\dim_{\mathrm{H}} F \geq s$. $\square$

Example 2.9. Let F be the middle third Cantor set. Let us apply the last two bounds on F as a demonstration:

Exactly as seen in Example 2.4, we can use Proposition 2.6 (which we now observe to be a generalization of that same idea) with the 2^k level-k intervals of length 3^{-k} to get $\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \log 2/\log 3$.
On the other hand, the lower bound is different this time. Let μ be the mass distribution on F defined as follows: if $E_{k,j}$ is the j-th level-k interval in the construction on F (ordered "from left to right"), $\mu(E_{k,j}) = 2^{-k}$, and for every other set $E \subseteq \mathbb{R}$ let $\mu(E) = \mu(F \cap E)$. Thus

$$\mu\Big(\bigcup_{j=1}^{2^k} E_{k,j}\Big) = 1 \quad \text{and so} \quad \mu(F) = \mu\Big(\bigcap_{k\in\mathbb{N}} \bigcup_{j=1}^{2^k} E_{k,j}\Big) = 1.$$

Now let U be a set with $|U| < 1$ and let k be the integer such that $3^{-(k+1)} \leq |U| < 3^{-k}$. U can intersect at most one among the level-k intervals, let that interval be $E_{k,j}$: we have that

$$\mu(U) = \mu(U \cap E_{k,j}) + \mu(U \backslash E_{k,j}) \leq \mu(E_{k,j}) = 2^{-k} = (3^{\log 2/\log 3})^{-k} = (3^{-k})^{\log 2/\log 3} \leq (3|U|)^{\log 2/\log 3}.$$

By Proposition 2.8 we get

$$\mathcal{H}^{\log 2/\log 3}(F) \geq \mu(F)/3^{\log 2/\log 3} = 3^{-\log 2/\log 3} = \frac{1}{2}$$

and $\dim_{\mathrm{H}} F \geq \log 2/\log 3$.

Chapter 3

Iterated Function Systems

3.1 Attractors and Hausdorff distance

Many fractals are made up of parts that are in some way similar to the whole object, as we have seen with the middle third Cantor set. These similarities are not only properties of these objects, but can be used to define them and find their dimension in a simple way: Iterated Function Systems are what we need to delve into these subjects appropriately.

Definition 3.1. Let D be a closed subset of $\mathbb{R}^n$ and consider a *contraction* $S : D \to D$, which means that there is a constant $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in D$ (in particular, any contraction is continuous). If the equality holds, we will call S a *contracting similarity.*
A finite family of contractions $\{S_1, S_2, \ldots, S_m\}$, usually with $m \geq 2$, is called an *iterated function system*, briefly *IFS*. An *attractor* or *invariant set* for the IFS is a non-empty compact subset F of D such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

For a quick and familiar example, let F be the middle third Cantor set and $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be the contracting similarities

$$S_1(x) = \frac{1}{3}x \qquad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

As seen in previous chapters, F is an attractor for the IFS given by $\{S_1, S_2\}$, which are the basic self-similarities of the Cantor set.

The fundamental property of any IFS is that it determines a unique attractor: to prove this we introduce a metric on the set $\mathcal{K}$ of non-empty compact subsets of D. Recall that by A_δ we denote the δ-neighbourhood of A, which is $A_\delta = \{x \in D : |x - a| \leq \delta \text{ for some } a \in A\}$. Then let us define the metric d known as the *Hausdorff metric* on $\mathcal{K}$

$$d(A, B) = \inf\{\delta > 0 : A \subseteq B_\delta \text{ and } B \subseteq A_\delta\},$$

which satisfies all the requirements:

1. $d(A, B) \geq 0$ and $d(A, B) = 0$ if and only if $A = B$:
   we only need to check that two sets $A, B \in \mathcal{K}$ at Hausdorff distance zero coincide. We know that $A \subseteq B_\delta$ and $B \subseteq A_\delta$ for all $\delta > 0$, which means that for every $n \in \mathbb{N}, a \in A, b \in B$ there are $a_n \in A$ and $b_n \in B$ such that $|a - b_n| \leq 1/n$ and $|b - a_n| \leq 1/n$ (using $\delta = 1/n$). Thus $\{a_n\}_n$ and $\{b_n\}_n$ are two sequences who respectively are in A and B and converge to b and a, but since A and B are closed sets, $b \in A$ and $a \in B$, which means $A \subseteq B$ and $B \subseteq A$.

2. $d(A, B) = d(B, A)$

3. $d(A, B) \leq d(A, C) + d(C, B)$:
   we need only verify that given $\delta_A, \delta_B > 0$ such that $A \subseteq C_{\delta_A}, C \subseteq A_{\delta_A}$ and $B \subseteq C_{\delta_B}, C \subseteq B_{\delta_B}$, then $A \subseteq B_{\delta_A+\delta_B}$ and $B \subseteq A_{\delta_A+\delta_B}$. This is true since $(A_{\delta_A})_{\delta_B} \subseteq A_{\delta_A+\delta_B}$, thus $B \subseteq C_{\delta_B} \subseteq (A_{\delta_A})_{\delta_B} \subseteq A_{\delta_A+\delta_B}$, and similarly $A \subseteq B_{\delta_A+\delta_B}$.

We will give two proofs of the fundamental result for IFSs, see Theorem 3.3 below. The first proof presented makes use of the Banach-Caccioppoli fixed-point theorem:

Theorem 3.2 (Banach-Caccioppoli)**.** Let (X, d) be a non-empty complete metric space and a contraction map $T : X \to X$. Then T admits a unique fixed point $\overline{x} \in X$, i.e. $T(\overline{x}) = \overline{x}$.
Furthermore, given any $x_0 \in X$ and the sequence $x_n = T(x_{n-1})$ for $n \geq 1$, then $\overline{x} = \lim_{n\to\infty} x_n$.

Theorem 3.3. Consider an iterated function system consisting of contractions $\{S_1, \dots, S_m\}$ on $D \subseteq \mathbb{R}^n$ of constants $c_1, \dots, c_m$ respectively. Then there is a unique attractor $F \subseteq D$ for it, i.e. such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Moreover, if we define $S : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K}$ is the family of non-empty compact subsets of D, by

$$S(E) = \bigcup_{i=1}^{m} S_i(E)$$

for $E \in K$ and denote the k-th iterate of S by S^k (so that S^0 is the identity on $\mathcal{K}$ and $S^k = S \circ S^{k-1}$ for $k \geq 1$), then

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

for every $E \in \mathcal{K}$ such that $S_i(E) \subseteq E$ for all $1 \leq i \leq m$.

First proof. Note that S, by the continuity of all S_i, does indeed map sets in $\mathcal{K}$ to sets in $\mathcal{K}$. If $A, B \in \mathcal{K}$ then

$$d(S(A), S(B)) = d\Big(\bigcup_{i=1}^{m} S_i(A), \bigcup_{i=1}^{m} S_i(B)\Big) \leq \max_{1\leq i\leq m} d(S_i(A), S_i(B)) \tag{3.1}$$

since taking some $\delta > \max_{1\leq i\leq m} d(S_i(A), S_i(B))$ means that $S_i(B) \subseteq (S_i(A))_\delta$ for all $1 \leq i \leq m$, thus we get $\bigcup_{i=1}^{m} S_i(B) \subseteq \Big(\bigcup_{i=1}^{m} S_i(A)\Big)_\delta$, and vice versa inverting the roles of $S_i(A)$ and $S_i(B)$.
Moreover, since S_i are contractions, $d(S_i(A), S_i(B)) \leq c_i \cdot d(A, B)$: letting $\delta > d(A, B)$ so that $A \subseteq B_\delta$ and $B \subseteq A_\delta$, we have that for every $a \in A$ we can pick some $b \in B$ such that $|a - b| \leq \delta$, thus $|S_i(a) - S_i(b)| \leq c_i\delta$, and so $S_i(A) \subseteq (S_i(B))_{c_i\delta}$, and similarly for the other inclusion. These last two inequalities together yield

$$d(S(A), S(B)) \leq (\max_{1\leq i\leq m} c_i) \cdot d(A, B). \tag{3.2}$$

This proves that S is a contraction on $\mathcal{K}$, since $0 < \max_{1\leq i\leq m} c_i < 1$. By Theorem 3.2, which we can apply since $(\mathcal{K}, d)$ is complete (see Lemma 3.4), we can conclude that there is a unique $F \in \mathcal{K}$ such that $S(F) = F$, or in other words F is the only attractor for $\{S_1, \dots, S_m\}$. Moreover, $S^k(E) \to F$ in $(\mathcal{K}, d)$ for any set $E \in \mathcal{K}$ as $k \to \infty$, and in particular if $S_i(E) \subseteq E$ for all $1 \leq i \leq m$, then $S(E) \subseteq E$ so that $S^k(E)$ is a decreasing sequence of non-empty compact sets whose limit, as will be seen during the proof of Lemma 3.4, is $\bigcap_{k=0}^{\infty} S^k(E)$ and must be equal to F. □

Second proof. Let $E \in \mathcal{K}$ be any set such that $S_i(E) \subseteq E$ for all $1 \leq i \leq m$, for example $E = D \cap B(0, r)$ will suffice provided r is large enough. Then $S^k(E)$ is a decreasing sequence of non-empty compact sets, thus

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

is also non-empty and compact, and furthermore

$$S(F) = \bigcap_{k=1}^{\infty} S^k(E) = \bigcap_{k=0}^{\infty} S^k(E) = F.$$

Thus F is an attractor, and to see its uniqueness we make use of (3.2), deriving it exactly as in the first proof. Suppose G is some other attractor for our IFS. Then

$$d(F,G) = d(S(F), S(G)) \leq (\max_{1\leq i\leq m} c_i) \cdot d(F,G),$$

and since $0 < \max_{1\leq i\leq m} c_i < 1$, $d(F,G)$ must equal zero and consequently $F = G$. □

Lemma 3.4. $\mathcal{K}$ is a complete metric space when endowed with the Hausdorff distance.

Proof. Let $\{E_n\}_n$ be any Cauchy sequence of sets in $\mathcal{K}$. Then, for every $\delta > 0$ one can find some $n_\delta \in \mathbb{N}$ such that $d(E_n, E_m) < \delta$ for all $n, m \geq n_\delta$, and in particular $E_n \subseteq (E_{n_\delta})_\delta$ and $E_{n_\delta} \subseteq (E_n)_\delta$. Now let

$$E = \bigcap_{n=0}^{\infty} \overline{\bigcup_{m=n}^{\infty} E_m},$$

which is a set in $\mathcal{K}$ since, choosing $\delta = 1$,

$$E \subseteq \overline{\bigcup_{m=n_1}^{\infty} E_m} \subseteq (E_{n_1})_1$$

and is thus closed and bounded, and non-empty since

$$A_n = \overline{\bigcup_{m=n}^{\infty} E_m}$$

is a decreasing sequence of non-empty closed sets. We now claim that $E_n \to E$ in $(\mathcal{K}, d)$: to prove this, we show that $d(E_n, A_n) \to 0$ and $d(A_n, E) \to 0$, so that $d(E_n, E) \leq d(E_n, A_n) + d(A_n, E) \to 0$, as $n \to \infty$.

Firstly, reasoning as in the derivation of (3.1), we have that

$$d(E_n, A_n) = d(E_n, \overline{\bigcup_{m=n}^{\infty} E_m}) \leq \sup_{m\geq n} d(E_n, E_m) \longrightarrow 0$$

as $n \to \infty$, since E_n is Cauchy.

Then A_n is Cauchy too, since $d(A_n, A_m) \leq d(A_n, E_n) + d(E_n, E_m) + d(E_m, A_m) \to 0$ as $m, n \to \infty$ for what we have shown so far, and this implies that $A_n \to E$: recall that

$$E = \bigcap_{n=0}^{\infty} A_n$$

and let $\varepsilon > 0$ and $\{n_k\}_k$ be a subsequence of the naturals such that $d(A_n, A_m) < \varepsilon/2^k$ for all $n, m \geq n_k$. Then for all $n \geq n_1$, $E \subseteq (A_n)_\varepsilon$ by construction, and $A_n \subseteq E_\varepsilon$: take any $x_1 \in A_n$, and consider $x_2 \in A_{n_2}$ such that $|x_1 - x_2| < \varepsilon/2$, and $x_3 \in A_{n_3}$ such that $|x_2 - x_3| < \varepsilon/2^2$, and so on for $x_k \in A_{n_k}$. This sequence is converging to some $x \in E$, and

$$|x_1 - x| \leq \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon,$$

which proves that $d(A_n, E) \to 0$. □

Let us now consider a way to "list" every point in F which will prove useful in the next section. For each $k \geq 1$ and $E \in \mathcal{K}$

$$S^k(E) = \bigcup_{i=1}^{m} S_i \Big(\bigcup_{i=1}^{m} S_i \Big(\cdots \bigcup_{i=1}^{m} S_i(E) \cdots \Big)\Big) = \bigcup_{(i_1,\ldots,i_k)\in\mathcal{I}_k} S_{i_1} \circ S_{i_2} \circ \ldots \circ S_{i_k}(E)$$

where $\mathcal{I}_k$ is the set of all k-term sequences $(i_1, \ldots, i_k)$ with $1 \le i_j \le m$. Then if $S_i(E) \subseteq E$ for all $1 \le i \le m$ it must be that any point $x \in F$ has a (not necessarily unique) corresponding sequence $(i_1, i_2, \ldots)$ such that $x \in S_{i_1} \circ \ldots \circ S_{i_k}(E)$ for all $k \ge 1$, since F is the intersection of all $S^k(E)$ by Theorem 3.3, in the sense that

$$\{x\} = \{x_{i_1,i_2,\ldots}\} = \bigcap_{k=1}^{\infty} S_{i_1} \circ \ldots \circ S_{i_k}(E), \tag{3.3}$$

where the intersection above consists of a single point since all S_i are contractions. Thus

$$F = \bigcup_{1 \le i_j \le m} \{x_{i_1,i_2,\ldots}\}. \tag{3.4}$$

Also note that the $S^k(E)$ are sometimes called *pre-fractals*, and the $S_{i_1} \circ \ldots \circ S_{i_k}(E)$ are the *level-k sets* (compare with the construction of the middle third Cantor set).

3.2 Open set condition: dimensions of self-similar attractors

Another significant advantage of working with iterated function systems is that their attractors have an easily deducible dimension, under some conditions. In this section we explore the calculation of the Hausdorff and Box dimensions of attractors of IFSs consisting of *contracting similarities*, also called *self-similar sets*, being the union of smaller similar copies of themselves.

In particular, let $S_i : \mathbb{R}^n \to \mathbb{R}^n$ and $|S_i(x) - S_i(y)| = c_i|x-y|$ for all $x, y \in \mathbb{R}^n$ and $0 < c_i < 1$. Similarly to the "heuristic calculation" introduced in the examples of Section 1.3, if $F = \bigcup_{i=1}^{m} S_i(F)$ is a "nearly disjoint" union, that is if we have that

$$\mathcal{H}^s(F) = \sum_{i=1}^{m} \mathcal{H}^s(S_i(F)) = \sum_{i=1}^{m} c_i^s \mathcal{H}^s(F)$$

then, under the assumption that $0 < \mathcal{H}^s(F) < \infty$, it follows that $s = \dim_{\mathrm{H}} F$ and

$$\sum_{i=1}^{m} c_i^s = 1.$$

We would like to put a similar condition on the IFS so that the $S_i(F)$ do not "overlap too much", in order to reach the same results.

Definition 3.5 (Open set condition)**.** We say that the similarities S_i satisfy the *open set condition* if there exists a non-empty bounded open set $V \subseteq \mathbb{R}^n$ such that

$$\bigcup_{i=1}^{m} S_i(V) \subseteq V$$

and the union is disjoint, that is $S_i(V)$ are pairwise-disjoint.

In our argument we will make use of the following geometrical lemma:

Lemma 3.6. Suppose $a, b, r > 0$ are fixed and let $\{V_i\}_i$ be a collection of pairwise-disjoint open subsets of $\mathbb{R}^n$ such that each V_i contains a ball of radius ar and is contained in a ball of radius br. Then any ball B of radius r intersects at most $(1+2b)^n a^{-n}$ sets among the closures $\overline{V_i}$.

Proof. If $\overline{V_i}$ intersects B, $\overline{V_i}$ is contained within the ball concentric with B of radius $(1+2b)r$, since $|\overline{V_i}| \le 2br$ and thus the center of B is at most distance $r + 2br$ from every point in $\overline{V_i}$.
Suppose q of the $\overline{V_i}$ intersect B: then, summing the (non-overlapping) volumes of the corresponding balls of radii ar contained in each of the q sets yields $q(ar)^n \le (1+2b)^n r^n$, i.e. $q \le (1+2b)^n a^{-n}$. □

Theorem 3.7. Suppose that the open set condition holds for the contracting similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \leq i \leq m$. If F is the attractor for the IFS $\{S_1, \dots, S_m\}$, then $s = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F$ where $s \geq 0$ is given by

$$\sum_{i=1}^{m} c_i^s = 1. \tag{3.5}$$

In addition, F is an s-set, i.e. $0 < \mathcal{H}^s(F) < \infty$.

Proof. Let s satisfy (3.5) and $\mathcal{I}_k$ be the set of all k-term sequences $(i_1, \dots, i_k)$ with $1 \leq i_j \leq m$. For any set $A \subseteq \mathbb{R}^n$ we will denote $A_{i_1,\dots,i_k} = S_{i_1} \circ \dots \circ S_{i_k}(A)$. Since

$$F = \bigcup_{i=1}^{m} S_i(F),$$

iterating this k times yields

$$F = \bigcup_{\mathcal{I}_k} F_{i_1,\dots,i_k}.$$

These are covers of F which provide an upper estimate for the Hausdorff measure: the composition of maps $S_{i_1} \circ \dots \circ S_{i_k}$ is a similarity of ratio $c_{i_1} \cdots c_{i_k}$, so

$$\sum_{\mathcal{I}_k} |F_{i_1,\dots,i_k}|^s = \sum_{\mathcal{I}_k} (c_{i_1} \cdots c_{i_k})^s |F|^s = \Big(\sum_{i_1=1}^{m} c_{i_1}^s\Big) \cdots \Big(\sum_{i_k=1}^{m} c_{i_k}^s\Big) |F|^s = |F|^s$$

by (3.5). Taking any $\delta > 0$, we may choose $k \geq 1$ such that

$$|F_{i_1,\dots,i_k}| \leq (\max_{1 \leq i \leq m} c_i)^k |F| \leq \delta \quad \text{since} \quad 0 < (\max_{1 \leq i \leq m} c_i) < 1,$$

so that $\{F_{i_1,\dots,i_k}\}_{\mathcal{I}_k}$ is a δ-cover of F, thus giving $\mathcal{H}^s_\delta(F) \leq |F|^s$ and hence $\mathcal{H}^s(F) \leq |F|^s$.

For a lower bound on the Hausdorff measure we need a lot more work:
Let $\mathcal{I} = \{(i_1, i_2, \dots) : 1 \leq i_j \leq m \text{ for all } j \geq 1\}$ and let $I_{i_1,\dots,i_k} = \{(i_1, i_2, \dots, i_k, q_{k+1}, \dots) : 1 \leq q_j \leq m \text{ for all } j > k\} \subseteq \mathcal{I}$ be the set off all sequences in $\mathcal{I}$ with given k initial terms.
Let also μ be a measure on $\mathcal{I}$ defined by setting $\mu(I_{i_1,\dots,i_k}) = (c_{i_1} \cdots c_{i_k})^s$. By (3.5) we have that

$$\mu(I_{i_1,\dots,i_k}) = (c_{i_1} \cdots c_{i_k})^s = \sum_{i=1}^{m} (c_{i_1} \cdots c_{i_k} c_i)^s = \sum_{i=1}^{m} \mu(I_{i_1,\dots,i_k,i}),$$

and also in particular

$$\mu(\mathcal{I}) = \sum_{i=1}^{m} \mu(I_i) = 1.$$

Then for any subset E of $\mathcal{I}$, from Caratheodory's extension theorem (Cf. [4, Theorem 1.11]) we can extend μ to a unique measure

$$\mu(E) = \inf\Big\{\sum_{i=1}^{\infty} \mu(U_i) : E \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } U_i = I_{i_1,\dots,i_k} \text{ for some } k\text{-term sequence } (i_1, \dots, i_k)\Big\}.$$

From μ we may induce a mass distribution ν on F in a natural manner, defining

$$\nu(A) = \mu(\{(i_1, i_2, \dots) \in \mathcal{I} : x_{i_1,i_2,\dots} \in A \cap F\})$$

for all subsets A of $\mathbb{R}^n$, where

$$\{x_{i_1,i_2,\dots}\} = \bigcap_{k=1}^{\infty} F_{i_1,\dots,i_k} \quad \text{as introduced in (3.3).}$$

Also by (3.4) we infer that $\mathcal{I} = \{(i_1, i_2, \dots) \in \mathcal{I} : x_{i_1,i_2,\dots} \in F\}$ and in turn $I_{i_1,\dots,i_k} = \{(i_1, i_2, \dots) \in \mathcal{I} : x_{i_1,i_2,\dots} \in F_{i_1,\dots,i_k}\}$, thus we get $\nu(F) = 1$ and $\nu(F_{i_1,\dots,i_k}) = (c_{i_1} \cdots c_{i_k})^s$.

We now show that this new mass distribution ν satisfies the conditions of the Mass distribution principle, Proposition 2.8.
Since the IFS satisfies the open set condition, let V be the non-empty bounded open set such that $S_i(V)$ are pairwise-disjoint and

$$\bigcup_{i=1}^{m} S_i(V) \subseteq V.$$

Defining

$$S(A) = \bigcup_{i=1}^{m} S_i(A)$$

for any subset A of $\mathbb{R}^n$, we get $S(\overline{V}) \subseteq \overline{V}$, so $S^k(\overline{V})$ converges to F by Theorem 3.3 in the sense that

$$F = \bigcap_{k=0}^{\infty} S^k(\overline{V}).$$

In particular $\overline{V} \supseteq F$ and in turn $\overline{V}_{i_1,\ldots,i_k} \supseteq F_{i_1,\ldots,i_k}$.
Let B be any ball of radius $0 < r < 1$ and $\mathcal{Q}$ as follows: for each sequence $(i_1, i_2, \ldots) \in \mathcal{I}$ we shorten it to $(i_1, \ldots, i_k)$ so that i_k is the first term for which

$$\Big(\min_{1\le i\le m} c_i\Big) r \le c_{i_1} \cdots c_{i_k} \le r. \tag{3.6}$$

Then $\mathcal{Q}$ is the set of all finite sequences obtained in this way. Clearly for any $(i_1, i_2, \ldots) \in \mathcal{I}$ there is exactly one value of k with $(i_1, \ldots, i_k) \in \mathcal{Q}$. Now, since $V_1, \ldots, V_m$ are disjoint subsets of V, then $V_{i_1,\ldots,i_k,1}, \ldots, V_{i_1,\ldots,i_k,m}$ are disjoint too for any choice of $(i_1, \ldots, i_k)$, and in particular $\{V_{i_1,\ldots,i_k} : (i_1, \ldots, i_k) \in \mathcal{Q}\}$ is a collection of disjoint sets. Also note that by (3.4) we have

$$F = \bigcup_{\mathcal{I}} \{x_{i_1,i_2,\ldots}\} \subseteq \bigcup_{\mathcal{Q}} F_{i_1,\ldots,i_k} \subseteq \bigcup_{\mathcal{Q}} \overline{V}_{i_1,\ldots,i_k}. \tag{3.7}$$

Now choose a and b such that V contains a ball of radius a and is contained in a ball of radius b (this is possible since V is open and bounded). Then for any $(i_1, \ldots, i_k) \in \mathcal{Q}$ we get that $V_{i_1,\ldots,i_k}$ contains a ball of radius $c_{i_1} \cdots c_{i_k} a$ and therefore one of radius

$$\Big(\min_{1\le i\le m} c_i\Big) ar,$$

and is contained in a ball of radius $c_{i_1} \cdots c_{i_k} b$ and therefore in one of radius br, because of the way we constructed $\mathcal{Q}$ by (3.6).
Let $\mathcal{Q}_B$ be the set of all sequences $(i_1, \ldots, i_k) \in \mathcal{Q}$ such that $\overline{V}_{i_1,\ldots,i_k}$ intersects B: by Lemma 3.6 there are at most

$$q = (1 + 2b)^n \Big(\min_{1\le i\le m} c_i\Big)^{-n} a^{-n}$$

sequences in $\mathcal{Q}_B$. We can conclude that

$$\nu(B) = \mu(\{(i_1, i_2, \ldots) \in \mathcal{I} : x_{i_1,i_2,\ldots} \in F \cap B\}) \le \mu\Big(\bigcup_{\mathcal{Q}_B} I_{i_1,\ldots,i_k}\Big)$$

since, if

$$x_{i_1,i_2,\ldots} \in F \cap B \subseteq \bigcup_{\mathcal{Q}_B} \overline{V}_{i_1,\ldots,i_k} \quad \text{by (3.7)},$$

then there must be some k such that $(i_1, \ldots, i_k) \in \mathcal{Q_B}$ and so $(i_1, i_2, \ldots) \in \bigcup_{\mathcal{Q}_B} I_{i_1,\ldots,i_k}$. Thus

$$\nu(B) \le \sum_{\mathcal{Q}_B} \mu(I_{i_1,\ldots,i_k}) = \sum_{\mathcal{Q}_B} (c_{i_1} \cdots c_{i_k})^s \le \sum_{\mathcal{Q}_B} r^s \le qr^s.$$

because of (3.6). Since any set U is contained in a ball of radius $|U|$, it follows that $\nu(U) \le q|U|^s$ for any set U such that $|U| < 1$ (recall that B is a ball of radius $r < 1$).

Thus by the Mass distribution principle we get $\mathcal{H}^s(F) \geq \frac{\nu(F)}{q} = q^{-1} > 0$, and thus we have shown that $0 < q^{-1} \leq \mathcal{H}^s(F) \leq |F|^s < \infty$ and consequently $s = \dim_{\mathrm{H}} F$.

Now to complete the proof we just need to show that $\overline{\dim}_{\mathrm{B}} F \leq s$, since $s = \dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F$ by (2.4) (we have already shown that F has non-zero s-dimensional Hausdorff measure), which gives $s = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F$.
Note that if $\mathcal{R}$ is any set of finite sequences $(i_1, \ldots, i_k)$ such that for every $(i_1, i_2, \ldots) \in \mathcal{I}$ there is exactly one integer k such that $(i_1, \ldots, i_k) \in \mathcal{R}$ ($\mathcal{Q}$ satisfies this requirement), from (3.5) we get

$$\sum_{\mathcal{R}} (c_{i_1} \cdots c_{i_k})^s = 1$$

inductively. Choosing $\mathcal{R} = \mathcal{Q}$ and observing that

$$\sum_{\mathcal{Q}} \Big(\min_{1 \leq i \leq m} c_i \Big)^s r^s \leq \sum_{\mathcal{Q}} (c_{i_1} \cdots c_{i_k})^s = 1$$

by (3.6), it follows that $\mathcal{Q}$ has at most

$$p = \Big(\min_{1 \leq i \leq m} c_i \Big)^{-s} r^{-s}$$

elements. For each sequence $(i_1, \ldots, i_k) \in \mathcal{Q}$ we have $|\overline{V}_{i_1, \ldots, i_k}| = c_{i_1} \cdots c_{i_k} |\overline{V}| \leq r|\overline{V}|$, so F may be covered by at most p sets of diameter at most $r|\overline{V}|$ for each $r < 1$.
From Definition 2.1 of Box dimension, we have that $\overline{\dim}_{\mathrm{B}} F \leq s$ since

$$\frac{\log N_{r|\overline{V}|}(F)}{-\log(r|\overline{V}|)} \leq \frac{\log p}{-\log(r|\overline{V}|)} = \frac{-s \log \Big(\min_{1 \leq i \leq m} c_i \Big)}{-\log r - \log |\overline{V}|} + \frac{-s \log r}{-\log r - \log |\overline{V}|}$$

and taking upper limits on both sides as $r \to 0$. □

Note that, if the open set condition is not assumed, one can only assure that $\dim_{\mathrm{H}} F \leq s$. Even in this situation, it may be shown that $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F$.
Furthermore, there is a similar result if we do not require similarities but merely contractions:

Corollary 3.8. Let F be the attractor of an IFS consisting of contractions $\{S_1, \ldots, S_m\}$ on a closed subset D of $\mathbb{R}^n$ and of ratios $0 < c_1, \ldots, c_m < 1$ respectively. Then $\dim_{\mathrm{H}} F \leq s$ and $\overline{\dim}_{\mathrm{B}} F \leq s$, where

$$\sum_{i=1}^{m} c_i^s = 1.$$

Proof. These estimates can be obtained re-working through the proof of Theorem 3.7 (the first and last paragraphs, ignoring everything which strictly requires the open set condition), by observing that we have $|A_{i_1, \ldots, i_k}| \leq c_{i_1} \cdots c_{i_k} |A|$ instead of an equality for each set A. □

3.3 Applications to image encoding

Another fascinating property of IFSs is their ability to approximate, albeit a bit coarsely, any non-empty compact set of $\mathbb{R}^n$. This is widely used in image encoding and procedural generation of digital landscapes or plant-like structures (see Figures 3.4 and 3.6), since IFSs do not require a large amount of information to entail very complex and detailed attractors, often very similar to many structures found in nature.

Proposition 3.9. Let $\{S_1, \ldots, S_m\}$ be an IFS and let

$$c = \max_{1 \leq i \leq m} c_i$$

so that $|S_i(x) - S_i(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}^n$ and $1 \leq i \leq m$. Take any non-empty compact set $E \subset \mathbb{R}^n$ and let F be the attractor for the IFS and d the Hausdorff metric. Then

$$d(E, F) \leq \frac{1}{1 - c} d\Big(E, \bigcup_{i=1}^{m} S_i(E)\Big).$$

Proof. As seen in Theorem 3.3, by (3.2) we have

$$d\Big(\bigcup_{i=1}^{m} S_i(E), F\Big) = d\Big(\bigcup_{i=1}^{m} S_i(E), \bigcup_{i=1}^{m} S_i(F)\Big) \leq c \cdot d(E, F),$$

thus

$$d(E, F) \leq d\Big(E, \bigcup_{i=1}^{m} S_i(E)\Big) + d\Big(\bigcup_{i=1}^{m} S_i(E), F\Big) \leq d\Big(E, \bigcup_{i=1}^{m} S_i(E)\Big) + c \cdot d(E, F)$$

hence

$$(1-c) \cdot d(E, F) \leq d\Big(E, \bigcup_{i=1}^{m} S_i(E)\Big).$$

□

As a consequence, we get the result mentioned earlier:

Corollary 3.10. Let E be a non-empty compact subset of $\mathbb{R}^n$. Given $\delta > 0$ there exist a self-similar set F, attractor for some IFS consisting of the contracting similarities $\{S_1, \ldots, S_m\}$, such that $d(E, F) < \delta$.

Proof. Let $B_1, B_2, \ldots, B_m$ be a collection of balls that cover E whose centers are also in E and whose radii are at most $\delta/4$. This collection exists because of the compactness of E, since it is a finite sub-cover of any cover consisting of balls centered in E of radii at most $\delta/4$. Then

$$E \subseteq \bigcup_{i=1}^{m} B_i \subseteq E_{\delta/4},$$

where $E_{\delta/4}$ is the $\delta/4$-neighbourhood of E. For each i let S_i be any contracting similarity of ratio $c_i < 1/2$ that maps E into B_i, so that $c = \max_{1 \leq i \leq m} c_i < 1/2$. Then $S_i(E) \subseteq B_i \subseteq \Big(S_i(E)\Big)_{\delta/2}$ because $|B| \leq \delta/2$, so

$$\bigcup_{i=1}^{m} S_i(E) \subseteq E_{\delta/4} \qquad \text{and} \qquad E \subseteq \bigcup_{i=1}^{m} \Big(S_i(E)\Big)_{\delta/2}.$$

By definition of Hausdorff metric, $d\Big(E, \bigcup_{i=1}^{m} S_i(E)\Big) \leq \delta/2$, and by Proposition 3.9 we get

$$d(E, F) \leq \frac{1}{1-c} d\Big(E, \bigcup_{i=1}^{m} S_i(E)\Big) \leq \frac{\delta}{2(1-c)} < \delta$$

since $\frac{1}{1-c} < 2$. □

3.4 Classic examples of IFSs

Example 3.11 (Sierpiński triangle)**.** The *Sierpiński triangle* or *gasket* F is constructed from an equilateral triangle E_0 by repeatedly dividing it into four equal equilateral triangles with half of its side and removing inverted equilateral triangles (i.e. those that point down), forming E_1. Similarly from E_k we get E_{k+1}, see Figure 3.1. Its Hausdorff and Box dimension are both equal to $\log 3/\log 2$.

The set F is the attractor of the three similarities of ratios $1/2$ which map the starting triangle E_0 onto the three triangles of E_1. The open set condition for this IFS holds, taking V as the interior of E_0, and thus by Theorem 3.7, $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \log 3/\log 2$, the solution of $3/2^s = (1/2)^s + (1/2)^s + (1/2)^s = 1$ in terms of s.

Example 3.12 (Modified von Koch curve)**.** Fix $0 < a \leq \frac{1}{2}$ and construct a curve F by repeatedly replacing the middle portion a of each segment by the other two sides of an equilateral triangle. F is called a *modified von Koch curve*, as the original von Koch curve is obtained with $a = \frac{1}{3}$. Then $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F$ is the solution of

$$2a^s + 2\Big(\frac{1-a}{2}\Big)^s = 1 \tag{3.8}$$

in terms of s.

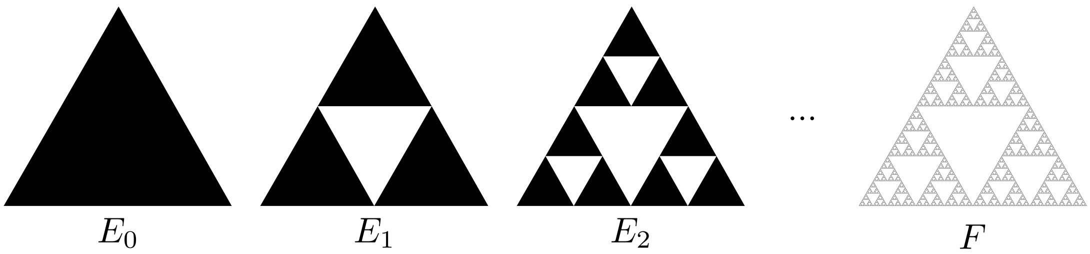


Figure 3.1: The construction of the Sierpiński triangle

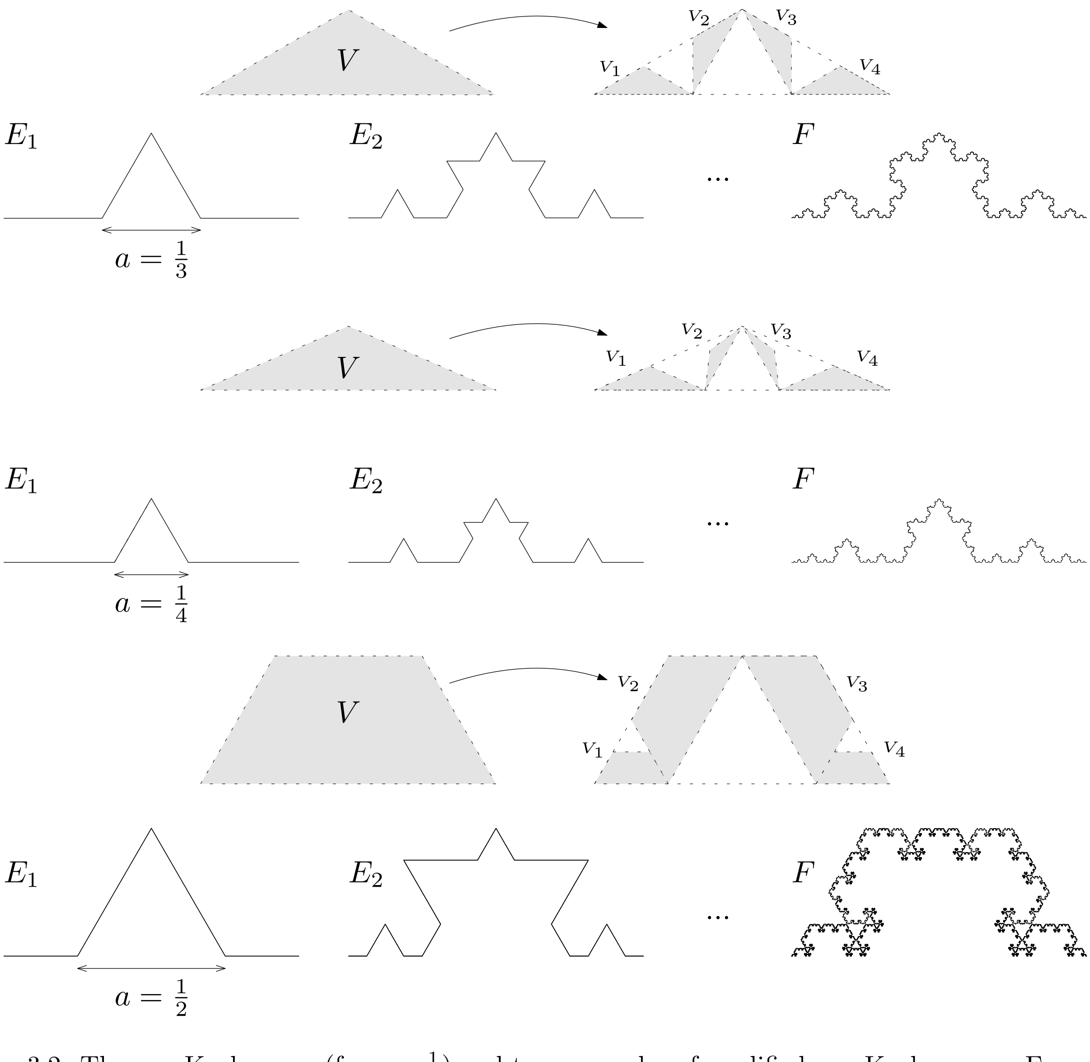


Figure 3.2: The von Koch curve (for $a = \frac{1}{3}$) and two examples of modified von Koch curves. For $a = \frac{1}{3}$, $a = \frac{1}{4}$ and $a = \frac{1}{2}$, (3.8) yields respectively $s = \frac{\log 4}{\log 3} \approx 1.262$, $s \approx 1.196$ and $s = \frac{\log(1+\sqrt{3})}{\log 2} \approx 1.45$

If $0 < a \leq \frac{1}{3}$, let V be the interior of the isosceles triangle of base 1 and height $a\frac{\sqrt{3}}{2}$ shown in Figure 3.2, and otherwise for $\frac{1}{3} < a \leq \frac{1}{2}$ let V be the interior of the isosceles trapezoid of identical larger base, identical height and lateral sides at 60 degrees angles with the larger base, also shown.
The curve F is the attractor of the IFS which maps the open set V into V_1, V_2, V_3, V_4 as displayed above, i.e. four similarities of which two have ratios a and the other ratios $\frac{1-a}{2}$. Note that V is also the set which satisfies the open set condition, hence (3.8) follows.

Self-similar curves such as the von Koch curve can be categorized in a convenient way through their *generator*, which is a number of straight line segments and two "special" points (in other words, E_1). We associate every line segment with a similarity that maps the special points to the endpoints of the segment: this defines the similarities up to reflections and 180 degrees rotations, but these are specified by the first step in the construction of the curve (i.e. E_2). Another way is to start from a line segment with a half-arrowhead at one end, and then display the generator by labeling each segment with a half-arrowhead like the segment we started with, so that there is no ambiguity in defining the similarities.

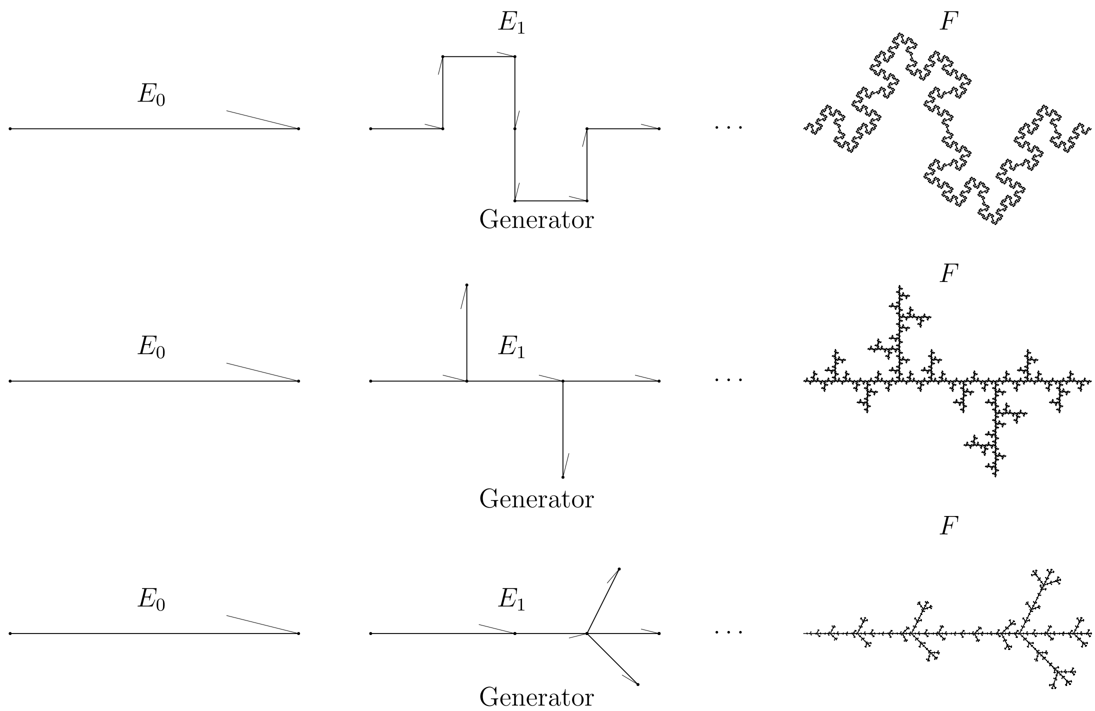


Figure 3.3: Some self-similar curves stemming from a generator. Their dimensions are, from top to bottom, $\frac{\log 8}{\log 4} = \frac{3}{2}$, $\frac{\log 5}{\log 3} \approx 1.465$ and $\frac{\log(1+\sqrt{17})}{\log 2} - 1 \approx 1.357$

In conclusion, we present an interesting class of sets, called *self-affine sets*, which are attractors of families of *affine* contractions, that is contracting transformations of the form

$$S(x) = T(x) + b,$$

where T is a linear transformation on $\mathbb{R}^n$ and b is a vector in $\mathbb{R}^n$. Unlike similarities, affine transformations can contract with differing ratios in different directions. Note that self-affine sets include self-similar sets as a particular case: it is then natural to look for analogies between self-affine and self-similar sets, such as a generalized dimension formula, or nice properties akin to those of self-similar sets. Unfortunately this is not the case, the situation is much more complicated and generally the information about the matrices and vectors that define an IFS of affine contractions is not enough to infer anything about its attractor. What follows is an example of how peculiar some of these sets can be:

Example 3.13. Let E_0 be the unit square and divide it into a $p \times q$ array of rectangles of sides $1/p$ and $1/q$, where $p < q$ and they are positive integers. Taking a subcollection of these rectangles to form E_1, let N_j be the number of rectangles in the j-th column, for $1 \leq j \leq p$. Iterating this construction yields

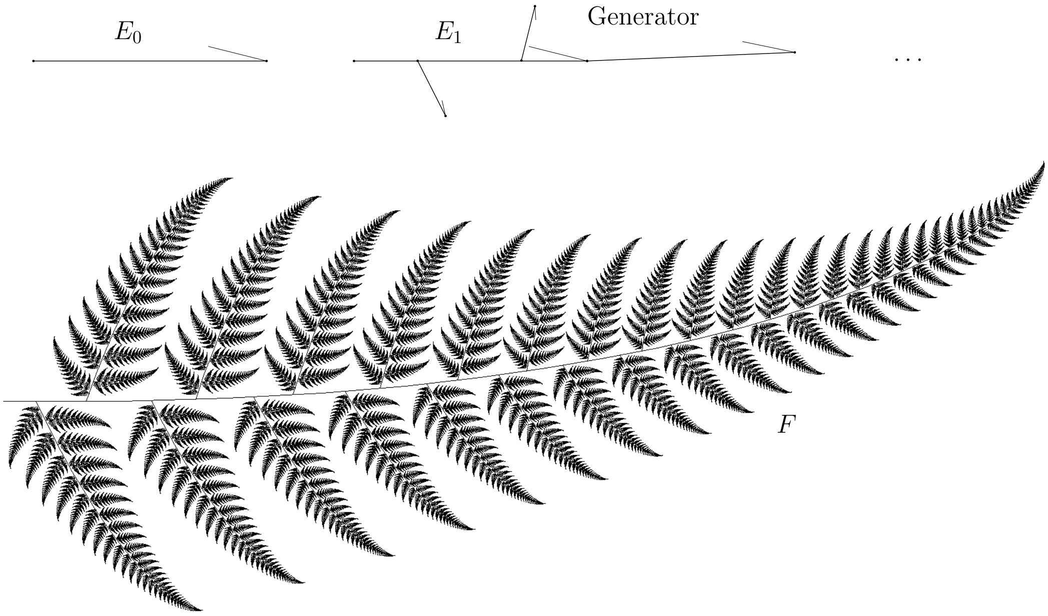


Figure 3.4: The Barnsley fern, a famous fractal introduced by Michael Barnsley

a self-affine set F, the limiting set obtained as the intersection of all E_k. Then

$$\dim_{\mathrm{H}} F = \log\Big(\sum_{j=1}^{p} N_j^{\log p/\log q}\Big)\frac{1}{\log p} \quad \text{and} \quad \dim_{\mathrm{B}} F = \frac{\log p_1}{\log p} + \log\Big(\frac{1}{p_1}\sum_{j=1}^{p} N_j\Big)\frac{1}{\log q}$$

where p_1 is the number of columns containing at least one rectangle of E_1.

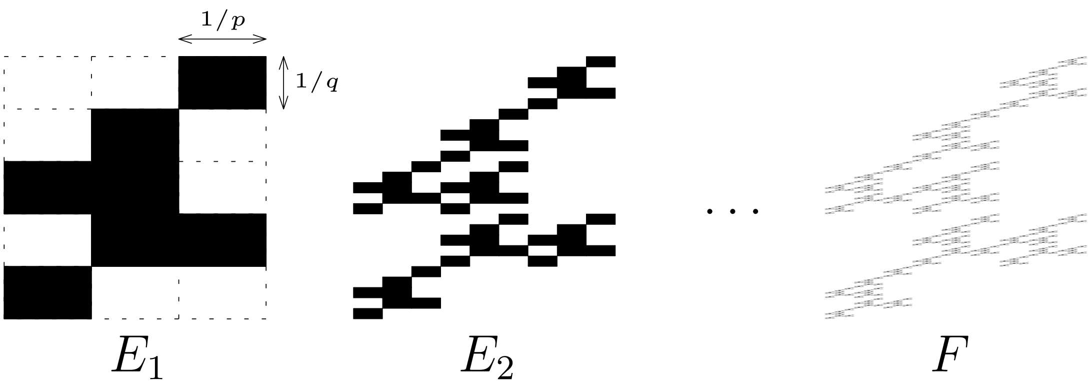


Figure 3.5: Construction of a self-affine set of the type considered in Example 3.13

Cf. [6] for the calculation. Note that the Hausdorff and Box dimensions are not always equal, and that their value depends not only on the number of rectangles, but also on their relative positions.

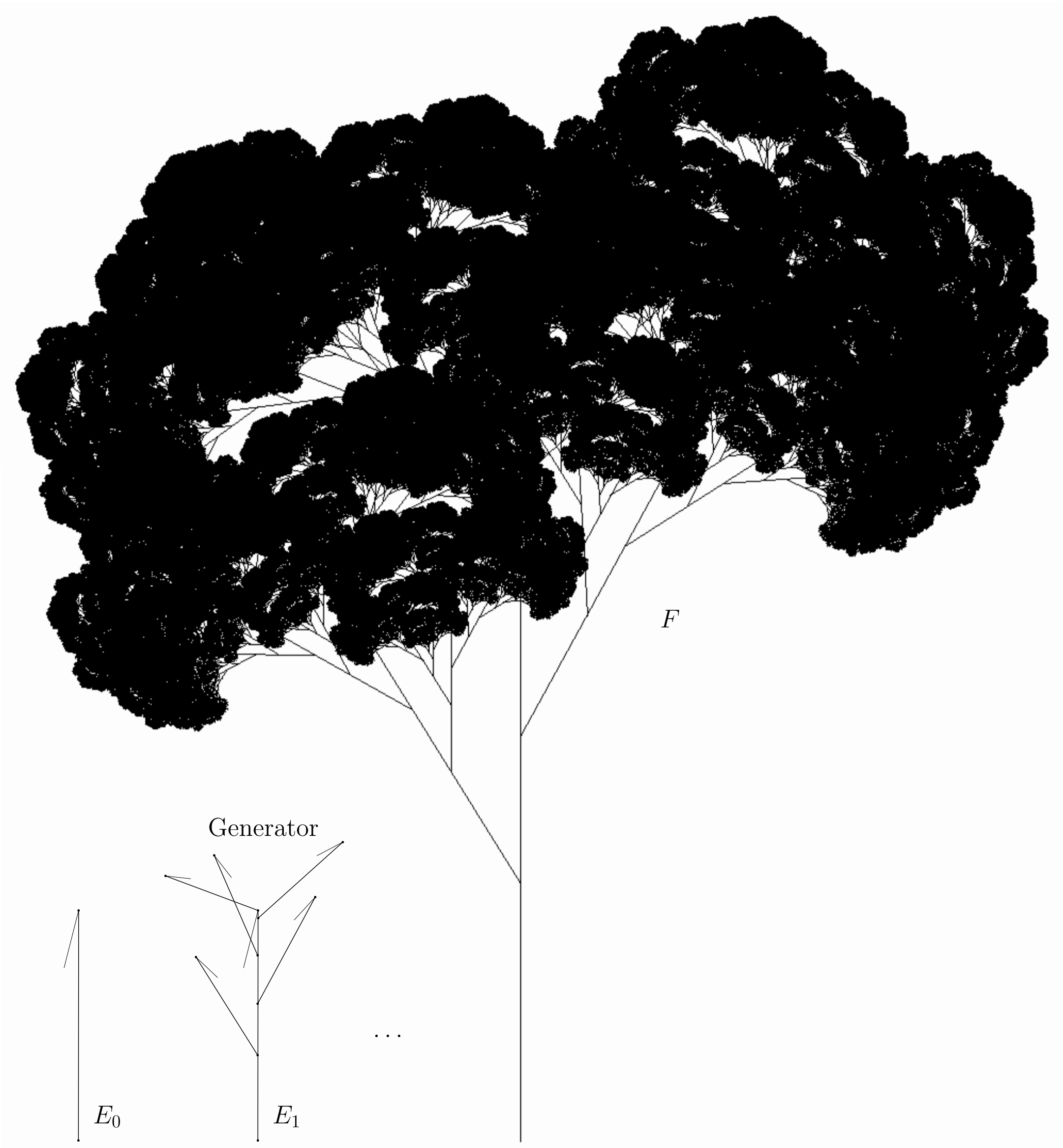


Figure 3.6: A fractal tree and its generator

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