UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

The polyharmonic Kuramoto model

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Academic Year 2025/2026

"Grau, theurer Freund, ist alle Theorie,
Und grün des Lebens goldner Baum."

— W. F. Goethe, *Faust*

Acknowledgements

First and foremost, I would like to express my deepest gratitude to Professor Antonio Ponno for his constant guidance and intellectual challenge throughout this work, and for being a daily source of academic inspiration since my bachelor's years.

I am sincerely grateful to my parents for allowing me to grow freely, to pursue my interests, and for always standing by me, even through my moments of difficulty. I would like to thank my brothers, whom I have always looked up to as models of integrity, and to whom I owe much of my curiosity and love of culture.

I am grateful to the friends I have had the fortune to meet over the years: in particular, Quizi and the LGF, who brighten my days and kindly endure my occasional impatience, as well as my dear university friends, who have filled the cold lecture halls with warmth and humanity.

Finally, I would like to express my deepest gratitude to Eleonora, who has always believed in me with a strength and constancy that I have often struggled to find in myself, and who has supported me far beyond what I have ever been able to deserve or fully recognize.

Padova, July 2026

Simone

Abstract

The Kuramoto model is the most well-established framework for describing the emergence of synchronization in large populations of coupled oscillators. Among its many generalizations, the introduction of higher-harmonic interactions extends its ability to capture a broader range of collective behaviors. This thesis investigates the role of higher harmonics in synchronization phenomena, first by analyzing the effect of individual harmonic contributions to the coupling function and subsequently by deriving several properties of multi-harmonic extensions, with particular emphasis on the Biharmonic model, i.e. the Kuramoto model augmented by a second-harmonic coupling term. The investigation has been carried out using standard techniques from dynamical systems, such as stability analysis and bifurcation theory, along with statistical-mechanics approaches, chiefly self-consistency. Numerical simulations based on Euler's algorithm have been employed to validate analytical predictions. The study clarifies the origin of clustering in single-harmonic models and proves their equivalence to the classical Kuramoto model. In the thermodynamic limit, the Biharmonic model is shown to display a discontinuous phase transition within a non trivial region of the coupling space. Furthermore, results concerning multi-harmonic models have been achieved, providing the groundwork for future investigations of higher-order synchronization phenomena.

Contents

1 Introduction

In 2025, Yoshiki Kuramoto was awarded the Boltzmann Medal, one of the most prestigious awards in Statistical Physics, for his pioneering work on synchronization as an emergent behavior of systems of limit-cycle oscillators. In the framework he first developed in 1975, synchronization, which may be broadly defined as the spontaneous tendency of a system toward an organized collective state, is the result of the elementary interaction between single units. The key features of this model are essentially two:

- The oscillating units are dynamical systems capable of maintaining stable oscillations through an internal energy source, without requiring a periodic, external forcing. (*Self-sustainment*)
- The oscillating units are coupled pairwise and the interaction is all-to-all, regardless of their spatial distribution. (*Long-range coupling*)

These properties draw the line between synchronization *à la* Kuramoto and synchronization in classical mechanical systems, such as Huygens' pendula[1][2]: for the latter class of models, synchronization is not an emergent behavior of a complex system, but is instead due to the interplay between dissipation and the normal modes of oscillations of the system, in the sense that the long-time dynamics is governed by the least-dissipating mode. We could say that in this case synchronization is due to spectral selection, while in the Kuramoto model it is due to collective properties which are realized in the thermodynamic limit.

The original formulation of the model is characterized by three essential ingredients: the angular variables $\{\theta_i\}$, which describe the phase of each oscillator, the frequencies $\{\omega_i\}$, which are essentially the frequencies associated to the limit cycle of each oscillator (the "self-sustainment" we have introduced before) and the parameter K, which is a positive real number describing the interaction strength among units. The success of the Kuramoto models stems directly from this simple reduction: by disregarding the details of each unit and reducing them to an elementary phase with a fixed velocity, the model has been able to describe many synchronization phenomena in vastly different research areas[3], such as fireflies flashing and crickets chirping in Ecology, neurons firing and pacemaker cells beating in Biophysics, power grids in Electrical Engineering, Josephson junctions arrays and lasers in Quantum sciences, chemical oscillators in Chemistry, walk and clapping synchronization in Social Sciences, and many more.

The huge success of the model has inevitably led to the quest for a more general description of these phenomena, which could recover the classical case as a particular configuration. This generalization can be achieved in mainly four ways, all of which twist the structure of the interacting part of the model. The first possibility is to modify the interaction range and, consequently, to take into account the spatial organization of the units: thus, the coupling becomes of the nearest-neighbor type and the system is realized on a network, whose topology is now relevant. This is the approach which essentially breaks the long-range requirement of the original model and departs from the mean field description; it is also the most naturally justified since actual systems usually have a finite correlation length. The second way is to add phase lag or time delay to the interaction, meaning that the standard argument of the coupling is changed as follows:

$$\sim \theta_j(t) - \theta_i(t) \qquad \rightarrow \qquad \sim \theta_j(t-\tau) - \theta_i(t) - \alpha$$

This generalization is reasonable since it models the finite nature of communications and information transmission. The third possibility consists in adding higher harmonics to the standard sine-coupling: this allows for different kinds of synchronization, such as antiphase configurations or more general multi-phase clustering. This is particularly useful when considering systems whose interaction is a generic periodic function. The last possibility is to add many-body couplings, such as three-body ones: this approach usually generates higher order mean-field interactions, which might even be frustrated ones, and higher harmonics as well, depending on the specific form of the many-body coupling.

In this work, we will investigate the third way we have illustrated, namely the introduction of higher harmonics into the pairwise coupling function. Among the possible generalization outlined above, this one represents one of the simplest modifications of the original model while simultaneously producing a remarkably rich phenomenology. In fact, the first harmonic coupling of the classical Kuramoto model naturally favors configurations in which all oscillators synchronize onto a common phase; however, the addition of further Fourier modes modifies this picture substantially, allowing for the emergence of clustered states in which oscillators organize into multiple synchronized groups separated by fixed phase differences.

The simplest realization of this idea is obtained by considering a coupling function made up of the first two harmonics:

$$\Gamma(x) = K_1 \sin x + K_2 \sin(2x)$$

commonly referred to as the Biharmonic Kuramoto model. Despite the apparent simplicity of this extension, the competition between the two harmonic contributions introduces several new features that are absent in the original model. Depending on the relative strength between the couplings, the system may support either in-phase or antiphase synchronization and develop non-trivial bifurcation structures. Furthermore, the presence of two order different parameters, each associated to one Fourier mode, enriches the mean field description and raises new questions regarding the critical behavior of the phase transition. This kind of coupling is actually meaningful for many systems, such as Huygens self-sustained pendula hanging from a beam which can move horizontally and vertically[4], φ-Josephon junctions arrays[5], certain electrochemical oscillators[6] and, in general, any system whose coupling function is shaped like a double well over one period.

The main purpose of this dissertation is to investigate the role of higher harmonics in the synchronization of coupled oscillators, with particular emphasis on the Biharmonic model. To do so, we will review the classical Kuramoto framework, the analytical techniques that will be employed throughout the work and the single-harmonic generalization. We will then study finite-dimensional realizations of the Biharmonic model in order to characterize their equilibrium structure and bifurcations. Subsequently, we will consider the thermodynamical version of the Biharmonic model and derive its self-consistent equations, phase diagram and critical properties. Finally, we will discuss some features of generalized Kuramoto models which include many harmonic contributions in their coupling function.

The dissertation is organized as follows. In Chapter 2 we review the classical Kuramoto model and its main analytical results, including the synchronization transition and its critical behavior; moreover, we will study the single-harmonic model and its equivalency to the classical Kuramoto model in this section. In Chapter 3 we will begin our study of the Biharmonic model by considering its finite-dimensional realizations: to do so, we will first state some results about the finite-dimensional Kuramoto model, then we will study the $N = 2, 3$ Biharmonic model in detail. The section ends with a theorem about the critical threshold for identical oscillators separating different harmonic phases. In Chapter 4 we study the thermodynamic limit of the Biharmonic model. In particular, we will discuss the relevance of the additional mean field parameters and we will derive a parametric form for the solutions of the self-consistent equations, which will prove crucial when studying the non-trivial phase diagram of the model. Lastly, we will apply the formalism we have developed to a certain class of three-body-interacting Kuramoto model. In Chapter 5, we will consider three properties of some multi-harmonic Kuramoto models, namely the linear stability of the incoherent state, a "triviality" feature of a Kuramoto model which contains exclusively odd harmonics only and the generic scaling of the order parameters when branching from the incoherent solution.

2 The Kuramoto model

The whole work is devoted to the study of a generalization of the Kuramoto model, therefore this is a good place to revise the main properties of the original model Kuramoto worked on and to revise the techniques used[3][7][1].
We start from the governing dynamical equations. Assuming a population of N oscillators, each of them running at a constant frequency ω_i drawn from a normalized probability distribution $g(\omega)$, the system of coupled ODEs is given by:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{2.1}$$

where we have introduced a real parameter $K > 0$ which represents the coupling strength between oscillators. We notice immediately that the model exhibits symmetry under a global phase translation $\theta_i \to \theta_i + \alpha$ with $\alpha \in \mathbb{R}$. Under the extra assumption $g(\omega) = g(-\omega)$, the model is also invariant under the symmetry $\theta_i \to -\theta_i$, provided that $\omega_i \to -\omega_i$ as well, which formally does not change the properties of the system due to the evenness of g. The last thing we notice is that the introduction of the random frequencies $\{\omega_i\}$ acts as a quenched disorder in the thermodynamic limit.

Kuramoto's calculations relied on the introduction of a complex order parameter Ψ which makes the mean-field nature of the model explicit. For a finite number of oscillators, Ψ can be written as:

$$\Psi = re^{i\varphi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{2.2}$$

Exploiting this definition, we can rewrite the coupling term in the ODE as follows:

$$\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) = \frac{1}{N}\Im\left(e^{-i\theta_i}\sum_{j=1}^{N}e^{i\theta_j}\right) = \Im\Big(re^{i(\varphi-\theta_i)}\Big) = r\sin(\varphi - \theta_i)$$

Therefore, dropping the indices, we obtain the phase velocity relation for a single angle θ coupled to the mean-field:

$$\dot{\theta} = \omega + Kr\sin(\varphi - \theta) \tag{2.3}$$

2.1 Qualitative analysis of the model

We now want to analyze the model in a qualitative way in order to get an intuition of its behavior and set some expectations for further analysis. We first simplify the model by taking the identical-oscillator limit, which means that all oscillators share the same frequency ω_0, i.e. $g(\omega) = \delta(\omega - \omega_0)$. In this regime, we can use translation invariance to eliminate the frequency term by sending $\theta \to \theta + \omega_0 t$. The equation simplifies:

$$\dot{\theta} = Kr\sin(\varphi - \theta)$$

By setting the right-hand side to zero, we obtain the fixed points of the phase θ:

$$\bar{\theta} = \varphi, \varphi + \pi$$

By linearization, it is also straightforward to verify that the $\bar{\theta} = \varphi$ equilibrium is attracting, while the other fixed point is repulsive. This shows that all angles converge to a common phase, which is φ. In order to investigate the meaning of r, we simply use the definition of the order parameter with the equilibrium phases:

$$\Psi = \frac{1}{N} \sum_{j=1}^{N} e^{i\bar{\theta}_j} = \frac{1}{N} \sum_{j=1}^{N} e^{i\varphi} = e^{i\varphi}$$

which means that $r = 1$, and as such can be regarded as a measure of synchronicity of the model.

We now relax the hypothesis on $g(\omega)$. In this case, the fixed point equation takes the form:

$$\frac{\omega}{Kr} = \sin\left(\theta - \varphi\right)$$

which exhibits a bifurcation following the ratio $\frac{\omega}{Kr}$:

$$\begin{cases} |\frac{\omega}{Kr}| < 1: & \bar{\theta} = \varphi + \arcsin\left(\frac{Kr}{\omega}\right) \\ |\frac{\omega}{Kr}| < 1: & \bar{\theta} = \varphi + \pi - \arcsin\left(\frac{Kr}{\omega}\right) \\ |\frac{\omega}{Kr}| > 1: & \nexists\ \bar{\theta} \end{cases}$$

Notice that now the positions of the equilibria depend on the intrisic frequencies of the oscillators: this fact calls for a statistical treatment of the problem.

2.2 The critical coupling

We now take the thermodynamic limit and describe the system in the continuum. To do so, we introduce the oscillator density $\rho(\theta, t, \omega)$, that is the density of oscillators with frequency ω whose phase is θ at time t. This must be normalized with respect to θ for every ω. We can therefore write down the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho v) = 0 \tag{2.4}$$

where v is the phase velocity, i.e. $v = \omega + Kr\sin\left(\varphi - \theta\right)$. This equation has an obvious stationary solution which is given by a constant:

$$\rho_0(\theta, \omega) = \frac{1}{2\pi} \tag{2.5}$$

We now generalize the equation for the order parameter in the continuum:

$$\Psi = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega e^{i\theta} \rho(\theta, t, \omega) g(\omega) \tag{2.6}$$

To characterize the solution ρ_0, we evaluate the order parameter using $\rho = \rho_0$:

$$\Psi = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega e^{i\theta} \frac{1}{2\pi} g(\omega) = 0$$

which shows that ρ_0 can be regarded as the incoherent solution since $r = 0$.

In order to solve the model, we employ the definition of the order parameter along with the solutions to the equilibria equation. For simplicity, we will shift to a rotating frame such that the equations are simplified to $\varphi = 0$: this only makes sense if we regard $g(\omega)$ as describing frequency deviations from the intrinsic frequency of the common phase φ. We must distinguish two cases corresponding to the bifurcation structure of the solutions:

$$\begin{cases} Kr > |\omega| : & \rho(\theta, \omega) = \rho_{lock}(\theta, \omega) = \delta\left(\theta - \arcsin\left(\frac{Kr}{\omega}\right)\right) \\ Kr < |\omega| : & \rho(\theta, \omega) = \rho_{drift}(\theta, \omega) = \frac{C(\omega)}{|\omega - Kr\sin\theta|} \end{cases}$$

The first θ distribution is due to the attracting solution of the equilibria equation, while the second distribution is due to a *Vlasov*-like argument: considering the phase velocity equation, the solution to the ODE $\hat{\theta}(t)$ must be a periodic function at stationarity with period T. Now, considering a generic function $f(\theta)$, if we are to average about one period at stationarity, we get:

$$\bar{f} = \frac{1}{T}\int_0^T dt f(\hat{\theta}(t)) = \frac{1}{T}\int_0^{2\pi} d\hat{\theta}\frac{dt}{d\hat{\theta}} f(\hat{\theta}) = \frac{1}{T}\int_0^{2\pi} d\hat{\theta}\frac{f(\hat{\theta})}{|\omega - Kr\sin\hat{\theta}|} = \int_0^{2\pi} d\hat{\theta} f(\hat{\theta})\rho_{drift}(\hat{\theta}, \omega) = \langle f \rangle_{drift}$$

which reveals the ergodic hypothesis underlying the argument. We can also identify $C(\omega) = \frac{1}{T}$. In particular:

$$C(\omega) = (T)^{-1} = \left(\int_0^T dt\right)^{-1} = \left(\int_0^{2\pi} d\theta \frac{dt}{d\theta}\right)^{-1} = \left(\int_0^{2\pi} \frac{d\theta}{|\omega - Kr\sin\theta|}\right)^{-1} = \frac{1}{2\pi}\sqrt{\omega^2 - (Kr)^2}$$

We can now solve the model by using the self-consistency of the order parameter. In particular, we are going to use its definition to write the average phase $\langle e^{i\theta}\rangle$ as an average over both population (θ) and disorder (ω): considering the bifurcation structure of the stationary problem, we also need to split the average into two contribution, namely the "locked" contribution, which follows the statistics ρ_{lock}, and the "unlocked" or "drifting" contribution, which follows the statistics ρ_{drift}. Therefore, remembering we have put $\varphi = 0$, we can rewrite (2.6) as:

$$\Psi = r = \langle e^{i\theta}\rangle_{lock} + \langle e^{i\theta}\rangle_{drift} = \int_0^{2\pi} d\theta \int_{-Kr}^{Kr} d\omega g(\omega)\rho_{lock}(\theta, \omega)e^{i\theta} + \int_0^{2\pi} d\theta \int_{|\omega|>Kr} d\omega g(\omega)\rho_{drift}(\theta, \omega)e^{i\theta}$$

The first thing we notice is that, given that $g(\omega)$ is an even function, the complex phase $e^{i\theta}$ can be simplified to its real part $\cos\theta$. Next, we can evaluate the drifting contribution explicitly:

$$\langle e^{i\theta}\rangle_{drift} = \int_0^{2\pi} \int_{|\omega|>Kr} d\omega g(\omega) C(\omega) \frac{\cos\theta}{|\omega - Kr\sin\theta|}$$

Though, remembering that $C(\omega)$ is an even function of ω as well, we find that the whole integral is vanishing due to the change of sign of the integrand under the transformations $\theta \to \theta + \pi$, $\omega \to -\omega$ and the symmetry of the integration domain.

Let us now evaluate the locked contribution:

$$\langle e^{i\theta}\rangle_{lock} = \int_0^{2\pi} d\theta \int_{-Kr}^{+Kr} d\omega g(\omega) \delta\left(\theta - \arcsin\left(\frac{Kr}{\omega}\right)\right) \cos\theta = \int_{-Kr}^{+Kr} d\omega g(\omega) \cos\bar{\theta}(\omega)$$

where $\bar{\theta}$ is the stable solution of the equilibria equation. We can evaluate this integral by performing a change of variable induced by the equilibrium condition:

$$\begin{cases} \omega & = Kr\sin\theta \\ d\omega & = Kr\cos\theta d\theta \end{cases}$$

which yields:

$$\langle e^{i\theta}\rangle_{lock} = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta \cos\theta g(Kr\sin\theta) \left(Kr\cos\theta\right)$$

Therefore, we can write the whole self-consistent equation as:

$$r = Kr \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta \cos^2\theta g(Kr\sin\theta) \tag{2.7}$$

As we can see, the equation always admits the solution $r = 0$, which, as we have seen, corresponds to the incoherent state of phases distributed uniformly along the unit circle. Simplifying r on both sides, we now take the $r \to 0^+$ limit in order to investigate the presence of a bifurcation from the incoherent solution. Doing so, we obtain the relation:

$$1 = Kg(0) \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta \cos^2\theta = Kg(0)\frac{\pi}{2}$$

This relation means that there exists a critical coupling K_c

$$K_c = \frac{2}{\pi g(0)} \tag{2.8}$$

such that for $K > K_c$ a new branch of solutions $r = r^{(\infty)}(K)$ bifurcates from the incoherent solution $r = 0$ (see Fig.(1) for an example of non-vanishing equilibrium order parameter and Fig.(2) for a numerical solution of the self-consistent equation).

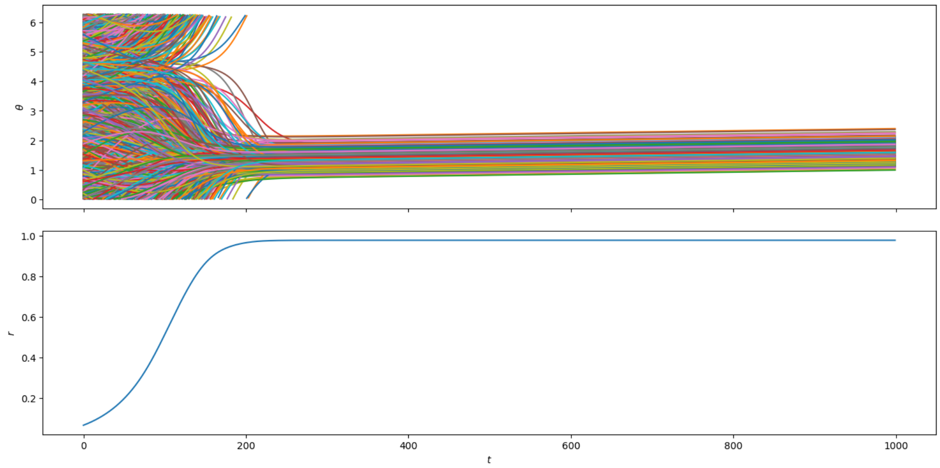


Figure 1: Example of the transition to equilibrium from the incoherent state in the Kuramoto model. The graph shows the dynamics of $N = 1000$ oscillators coupled through $K = 5$ with time step $\Delta t = 0.01$ repeated 1000 times.

We can also prove by implicit differentiation of the previous relation that the new branch is monotonically increasing with respect to K. In particular:

$$F(Kr) = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta \cos^2\theta g(Kr\sin\theta)$$

$$\frac{dr^{(\infty)}}{dK} = -\frac{F(Kr) + KrF'(Kr)}{K^2F'(Kr)}$$

where the numerator is strictly positive and the denominator is strictly negative since $g'(\omega)$ is an odd, negative function for $\omega > 0$. The exact shape of $r^{(\infty)}(K)$ is dependent on the specific distribution $g(\omega)$ and in general there is no closed form for it: a common method to investigate it is to treat the equation as a parametric equation by varying the parameter $t = Kr$ in the interval $[0, +\infty)$.

The significance of $g(0)$ in the critical coupling formula (2.8) can be explained as follows[7]: for an even, unimodal probability distribution, $g(0)$ measures the height of its only peak. By normalization, we can expect that the higher $g(0)$, the sharper the distribution is about 0, or, in other words, the smaller we expect the distribution's standard deviation to be. As a matter of fact, the quantity $\frac{1}{g(0)}$ can be related to the standard deviation for many distributions and we can interpret it as a measure of the "width" of the distribution. However, for heavy-tailed distribution, such as a Lorentzian, the standard deviation is ill-defined, while $g(0)$ is not: in this sense, its reciprocal still constitues a measure of the "width".

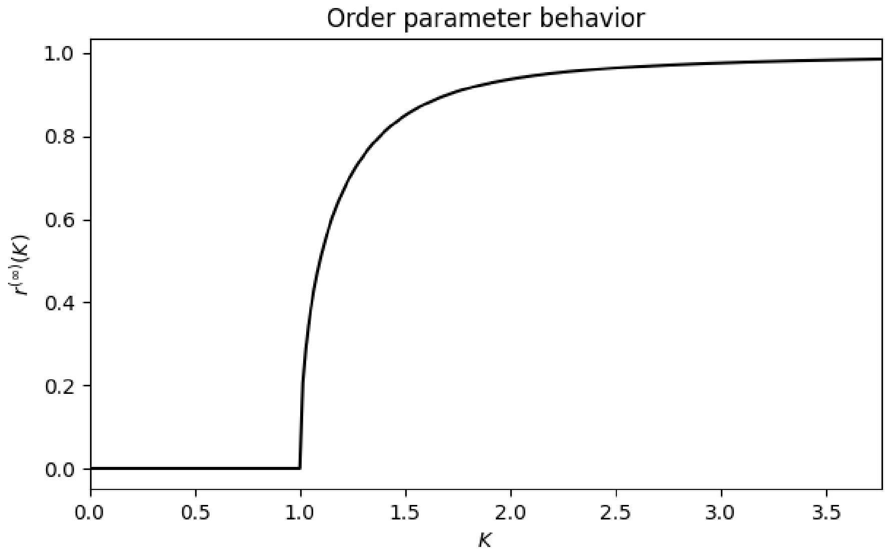


Figure 2: Numerical solution of the self-consistent equation for r where a normal Gaussian distribution has been used. The K-axis is rescaled by K_c and only the stable solution is marked.

2.3 The critical exponents

As we have proved, the system undergoes a phase transition corresponding to the arising of a non-zero solution of the self-consistent equation from the branch $r = 0$. Moreover, we can see that as $K \to K_c^+$:

$$\frac{dr^{(\infty)}}{dK} \to +\infty$$

This is exactly the phenomenology of a continuous, second-order phase transition. Therefore, we are motivated to investigate its critical behavior and to calculate its critical exponents in order to identify a universality class.

Let us start with the critical exponent β[3]. To calculate it, we simply expand the integrand in (2.7) equation up to second order in r (the first order contribution is vanishing due to the evenness of $g(\omega)$):

$$r = Kr \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta \cos^2\theta \left(g(0) + \frac{1}{2} g''(0)(Kr\sin\theta)^2 \right) = \frac{Kr}{K_c} + \frac{(Kr)^3}{2} g''(0) \left(\frac{\pi}{8}\right)$$

simplifying r on both sides and introducing the deviation from criticality $K = K_c(1+\mu)$, under the assumption of small μ, we obtain:

$$1 = \mu + \frac{\pi K_c^3}{16} g''(0) r^2$$

which reveals the scaling relation:

$$r \sim \mu^{\frac{1}{2}} \tag{2.9}$$

So we have found that the critical exponent β takes the value $\beta = \frac{1}{2}$, which is the standard mean-field exponent.

We can also calculate the correlation exponent $\bar{\nu}$ exploiting the Finite Size Scaling ansatz[8]. We start from the definition of the finite-size order parameter:

$$r = \frac{1}{N} \sum_{|\omega_j|<Kr} \sqrt{1 - \left(\frac{\omega_j}{Kr}\right)^2} = \tilde{\Psi}(r)$$

By definition, we know that $\langle \tilde{\Psi}(r) \rangle = \Psi(r)$ which is simply the locked contribution in the mean-field self-consistent equation. We define the finite-size fluctuations as:

$$\delta\tilde{\Psi}(r) = \tilde{\Psi}(r) - \Psi(r) \propto \frac{\sqrt{N_{locked}}}{N}$$

In order to get an estimate of $\delta\tilde{\Psi}(r)$, we calculate its variance:

$$\langle \delta\tilde{\Psi}^2(r) \rangle = \frac{N_{locked}}{N^2} \int_{-Kr}^{Kr} d\omega g(\omega) \left(1 - \left(\frac{\omega}{Kr}\right)^2\right) \approx \frac{N_{locked}}{N^2} Krg(0) \int_{-1}^{+1} dx(1-x^2) \propto \frac{4(Kr)}{3N} g(0)$$

which is valid near the onset of synchronicity. Therefore, we can argue that in this regime $\delta\tilde{\Psi}(r) \sim \sqrt{\frac{r}{N}}$.

We can now write the self-consistent equation (2.7) near criticality including finite-size corrections:

$$r = \frac{K}{K_c} r + \frac{\pi g''(0)}{16} K^3 r^3 + \delta\tilde{\Psi}(r)$$

Taking the $K \to K_c$ limit and using the deviation from criticality μ, we obtain:

$$0 \approx \mu r + \frac{\pi g''(0)}{16} K_c^3 r^3 + \delta\tilde{\Psi}(r)$$

In order to employ the FSS ansatz in the mean-field case, we need to calculate the scaling relation for the order parameter at criticality. To do so, we set $\mu = 0$, use the ansatz $r \sim N^{-\alpha}$ and consider competition between the third order term and the fluctuation term:

$$r \sim N^{-3\alpha} \sim N^{-\frac{\alpha+1}{2}} \sim \sqrt{\frac{r}{N}}$$

By comparing exponents, we obtain $\alpha = \frac{1}{5}$. Next, we consider the FSS ansatz:

$$r(N, \mu) = b^{-\frac{\beta}{\bar{\nu}}} r_b(b^{-1}N, b^{\frac{1}{\bar{\nu}}}\mu)$$

Taking $b = N$, we can rewrite the scaling relation as:

$$r = N^{-\frac{\beta}{\bar{\nu}}} f(\mu N^{\frac{1}{\bar{\nu}}}) \tag{2.10}$$

for some scaling function f. By substituting $\beta = \frac{1}{2}$ and comparing the expression with the scaling relation we have obtained previously, we obtain $\bar{\nu} = \frac{5}{2}$. The scale free parameter inside the scaling function must then be $\mu N^{\frac{2}{5}}$, which is compatible with the result we obtain by evaluating competition between the linear and the cubic terms:

$$\mu r \sim \mu N^{-\frac{1}{5}} \sim N^{\frac{3}{5}} \sim r^3$$

which means that $\mu \sim N^{-\frac{2}{5}}$ as predicted. In Fig.(3) it is shown the validity of this result using FSS data collapse.

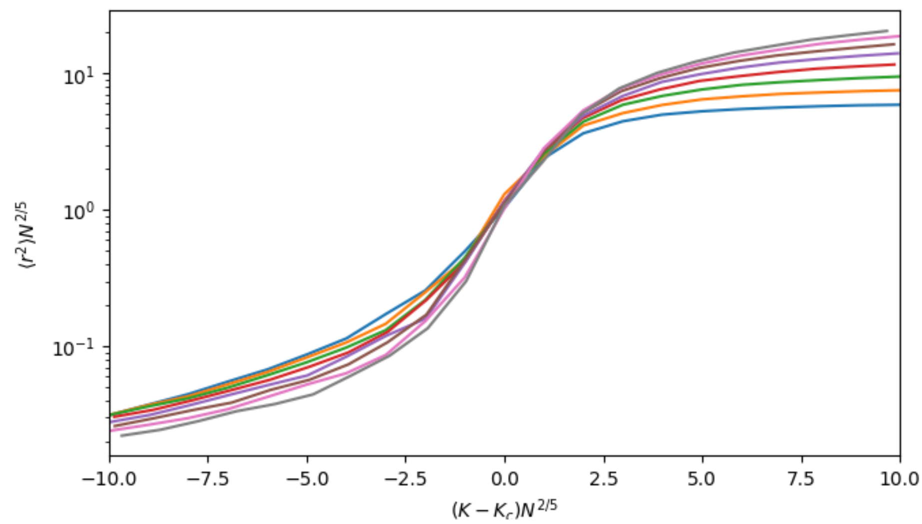


Figure 3: Data collapse for the Kuramoto model. Each point has been obtained by averaging 100 order parameters at equilibrium in independent simulations of $N = 100 \cdot 2^k$ oscillators with $k = 0, 1, \ldots, 7$.

To obtain the time correlation exponent, we postulate the existence of a mean synchronization time τ_s and consider the following: since the dynamics is ruled by an ODE in which the coupling K appears at linear order, we can always rescale time by K. This intuitively gives rise to a scale-free combination $\mu\tau_s$, which can be inverted to argue that $\tau_s \sim \mu^{-1}$, so that we can extract the time correlation exponent $\nu_{\parallel} = 1$. A numerical proof of this conjecture can be found in [9].

2.4 The single-harmonic model

We now introduce one possible generalization of the Kuramoto model, namely the one in which we change the functional form of the coupling $\sin(x)$ to a more general $\sin(mx)$, with $m \in \mathbb{N}$. We start from the ODE for the i-th oscillator:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin\left(m(\theta_j - \theta_i)\right) \tag{2.11}$$

If we perform the substitution $\psi_i = m\theta_i$ and we rescale time as $\tau = mt$, we obtain the following equation:

$$\dot{\psi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\psi_j - \psi_i\right)$$

which is exactly the usual Kuramoto model. This relation means that the single-harmonic generalization is equivalent to the original problem and, as such, is characterized by the same critical coupling $K_c = \frac{2}{\pi g(0)}$ and belongs to the same universality class[10][11]. Another way to see this is to solve the model using the same methods we have previously employed: this method is instructive because it lets us get an insight on the peculiarity of the m-th-harmonic model phenomenology. We introduce the generalized Daido's order parameter:

$$\Psi_m = r_m e^{i\varphi_m} = \frac{1}{N}\sum_{j=1}^{N} e^{im\theta_j} \tag{2.12}$$

By the same imaginary-part argument of the Kuramoto model, we are able to rewrite the ODEs for the N oscillators as an ODE for the phase θ coupled to the mean-field:

$$\dot{\theta} = \omega + Kr_m \sin\left(\varphi_m - m\theta\right) \tag{2.13}$$

Now, setting the mean-field phase $\varphi_m = 0$, we need to consider the fixed-point equation:

$$0 = \omega - Kr\sin m\theta$$

As before, this equation has no solution for $|\omega| > Kr$; on the other hand, it has $2m$ solutions for $|\omega| < Kr$, namely:

$$\begin{cases} \bar{\theta} &= \frac{1}{m}\left(\arcsin\left(\frac{\omega}{Kr}\right) + 2k\pi\right) \\ \bar{\theta} &= \frac{1}{m}\left(\pi - \arcsin\left(\frac{\omega}{Kr}\right) + 2k\pi\right) \end{cases}$$

with $k \in \{0, 1, \dots, m-1\}$. Linearization shows that the first family of solutions is attracting while the second one is repulsive. In principle, this multiplicity of stable equilibria for the same frequency ω forces us to introduce $m-1$ indipendent indicator (or redistribution) functions $S_i(\omega) \in [0, 1]$ in the locked contribution to the self-consistent equation: each one of these indicator functions is associated to a particular branch of stable solutions (i.e. a fixed k) and describes the fraction of oscillators with frequency ω which lie on the i-th branch. Notice that there is no need for an m-th indicator function since they are constrained to live on the $(m-1)$-dimensional simplex and, as such, $S_m(\omega)$ can be expressed as:

$$S_m(\omega) = 1 - \sum_{i=1}^{m-1} S_i(\omega)$$

We can now consider the locked contribution to the self-consistent equation. After changing integration variable from ω to θ, we can write it as a sum over stable branches:

$$\langle e^{im\theta}\rangle_{lock} = \sum_{k=1}^{m} \int_{\frac{1}{m}\left(-\frac{\pi}{2}+2k\pi\right)}^{\frac{1}{m}\left(+\frac{\pi}{2}+2k\pi\right)} d\theta \cos{(m\theta)} g(Kr_m \sin{(m\theta)})(mKr_m \cos{(m\theta)}) S_k(Kr_m \sin{(m\theta)})$$

This complicated sum can be radically simplified by virtue the periodicity of the integrand: exploiting it, all pairs of integration extrema can be substituted by the pair $\left(-\frac{\pi}{2m}, +\frac{\pi}{2m}\right)$. This allows us to bring the sum inside the integral, which is now applied only to the indicator functions; though, by definition of indicator function, the sum simplifies to 1. Therefore, the whole locked contribution simply amounts to:

$$\langle e^{im\theta}\rangle_{lock} = \int_{-\frac{\pi}{2m}}^{+\frac{\pi}{2m}} d\theta (mKr_m) \cos^2{(m\theta)} g(Kr_m \sin{(m\theta)}) = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta K r_m \cos^2{(\theta)} g(Kr_m \sin{(\theta)})$$

where in the last step we perfomed the change of variable $\theta' = m\theta$. This shows that the locked contribution is the same of the usual Kuramoto model, though referred to a different order parameter.

Similarly, we can work out that the drifting contribution is vanishing in this case as well:

$$\langle e^{im\theta}\rangle_{drift} = \int_{|\omega|>Kr_m} d\omega g(\omega) \frac{\int_0^{2\pi} d\theta \frac{\cos{(m\theta)}}{|\omega - Kr_m \sin{(m\theta)}|}}{\int_0^{2\pi} d\theta \frac{1}{|\omega - Kr_m \sin{(m\theta)}|}} = \int_{|\omega|>Kr_m} d\omega g(\omega) \frac{\int_0^{2\pi} d\theta \frac{\cos\theta}{|\omega - Kr_m \sin\theta|}}{\int_0^{2\pi} d\theta \frac{1}{|\omega - Kr_m \sin\theta|}} = 0$$

where we used the same change of variable and the result from the analysis of the drifting contribution in the Kuramoto model.

To complete the analysis, we now look for solutions to the self-consistent equation in the $r_m \to 0^+$ limit. This is completely equivalent to the Kuramoto model, since we always have the incoherent branch $r_m = 0$ and the new solution $r_m^{(\infty)}$ bifurcating from the previous one for $K > K_c = \frac{2}{\pi g(0)}$. By the same methods, we can prove that the critical exponents are the same of the original model. See Fig.(4) for two examples of clustering synchronization and Fig.(5) for a numerical validation of the equivalence claim.

Lastly, we present a heuristic way to extract information for the model. Let us consider the case of N identical oscillators and let us rewrite the family of ODEs in terms of the complex phases $z_i = e^{i\theta_i}$:

$$\dot{\theta}_i = \frac{K}{2i}(\Psi_m z_i^{-m} - z_i^m \Psi_{-m})$$

Notice that $\Psi_{-m} = \overline{\Psi_m}$ and $\Psi_0 = 1$. Now, we take the time derivative of the n-th order parameter:

$$\dot{\Psi}_n = \frac{in}{N}\sum_i \dot{\theta}_i z_i^n = \frac{nK}{2}(\Psi_m \Psi_{n-m} - \Psi_{n+m}\Psi_{-m})$$

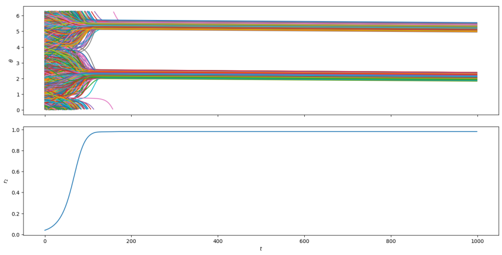
This derivative is linear in the complex order parameter only when $n = m$, which shows that Ψ_m is the correct order parameter to describe the system:

$$\dot{\Psi}_m = \frac{mK}{2}(\Psi_m - \Psi_{2m}\overline{\Psi_m})$$

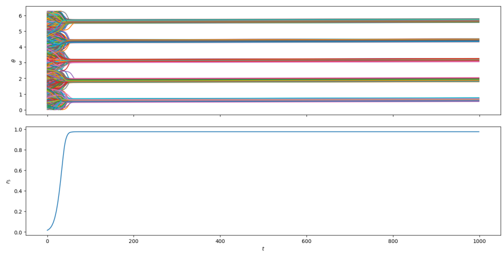
The stationarity condition of the order parameter is always satisfied by the obvious solution $\Psi_m = 0$, i.e. the disordered state. Assuming now that the order parameter has non-negative magnitude instead, we get the conditions:

$$\begin{cases} r_{2m} &= r_m = 1 \\ \varphi_{2m} &= 2\varphi_m \end{cases}$$

which are only satisfied when the oscillators are spread around the unit circle with a spacing between them which is a multiple of $\frac{\pi}{m}$. Due to the periodic nature of the problem, we are forced to conclude that these fixed points must alternate stability between attracting and repulsive.

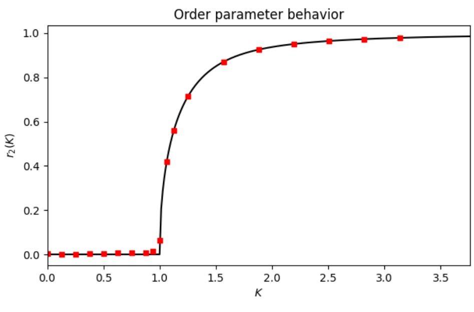


(a) $m = 2$

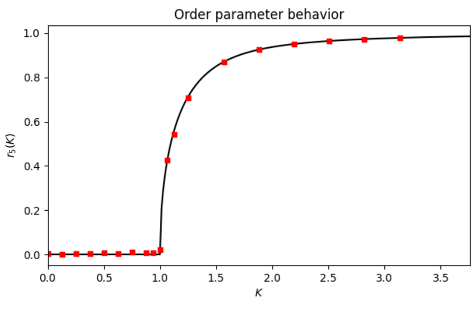


(b) $m = 5$

Figure 4: Example of synchronous states for two values of m. The equilibrium configuration is given by clusters of oscillators separated by $\frac{2\pi}{m}$ steps. Each run represents the dynamics of $N = 1000$ oscillators initialized at uniformly random phases with coupling strength $K = 5$; the time step is $\Delta t = 0.01$ and the total number of steps is 1000.



(a) $m = 2$



(b) $m = 5$

Figure 5: Average value of the m-th order parameter obtained from independent runs of the corresponding model. The data is plotted over the stable numerical solution of the self-consistent equation of the Kuramoto model, showing that the single-harmonic models belongs to the same universality class. The K-axis is normalized by K_c.

2.5 Euler's algorithm

Let us conclude this section by briefly discussing the numerical integration method we have employed and that will be employed in future chapters to simulate Kuramoto models.

Euler's method for ODEs integration is given by the well-known recursion relation:

$$\theta_i^{n+1} = \theta_i^n + \Delta t f_i(\theta^n)$$

where n denotes the time step, i.e. $\theta_i^n = \theta_i(t_0 + n\Delta t)$, Δt is the time interval and $f_i(\theta)$ is the vector field of the associated ODE. As we know, this algorithm has global accuracy $\mathcal{O}(\Delta t)$, which is one of the lowest among ODE-integration algorithms. Nonetheless, Euler's algorithm is still enough to capture the essence of the dynamics of Kuramoto models for sufficiently small time steps. There are a few reasons why this is the case:

- The phase variables $\{\theta_i\}$ are naturally periodic, which means that after each step one can simply mod out the coordinates back to the fundamental interval. This prevents the sort of unbounded growth errors that sometimes plagues Euler's algorithm and other explicit methods.
- The dynamics are usually dissipative, meaning that the system tends toward attractors. Therefore, in the long-time limit, the system reaches a synchronized state nonetheless, which is the correct one for a good choice of the time step.
- The computational bottleneck is given by the pairwise coupling $\sim \sin(m(\theta_j - \theta_i))$, which costs $\mathcal{O}(N^2)$ per iteration, accounting for the sum over oscillators. Other methods (e.g. Runge-Kutta methods) require many evaluations, rendering this computation heavy; on the other hand, Euler's algorithm requires only one evaluation of this kind per timestep, which is a good compromise if we are interested in statistical properties rather than precise dynamics.
- For finite systems, the relevant time scales in the problem are given by $\hat{\omega}^{-1}$ and K^{-1}, where $\hat{\omega}$ is the mean frequency of the distribution $g(\omega)$ and K is the largest coupling. Therefore, since we can always find a reference frame where $\hat{\omega} = 0$, the problem is not plagued by stiffness unless K gets very large. Thus, the appropriate time step is selected by the magnitude of K.

We can prove the last step by using the fact that Euler's algorithm is stable if and only if the time step satisfies:

$$|1 + \lambda \Delta t| < 1$$

with λ the largest eigenvalue of the Jacobian of the vector field. For the synchronized state, we have $\lambda = -K$, therefore a good time satisfies the following inequality:

$$\Delta t < \frac{2}{K}$$

3 Finite N analysis of the Biharmonic model

In the upcoming sections, we are going to analyze one of the simplest, non-trivial extensions of the Kuramoto model by introducing a second coupling into the model which is proportional to a second harmonic. Our discussion will be divided in two parts: we are going to discuss some aspects of the low-dimensional realizations of the model first and then we will solve the model in the thermodynamic limit. Let us remark that studying finite-N models is not redundant, but rather strategic to get an insight about some aspects of the theory, such as synchronization mechanisms or exact trajectories (an example is the recent result on the $N = 3$ Kuramoto model[12]), and to understand the foundations of reduction techniques (like the Ott-Antonsen ansatz[13] and the Watanabe-Strogatz approach[14]).

Let us start from the family of ODEs:

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\left(K_1 \sum_{j=1}^{N} \sin\left(\theta_j - \theta_i\right) + K_2 \sum_{j=1}^{N} \sin\left(2(\theta_j - \theta_i)\right)\right) \tag{3.1}$$

where $K_1, K_2 > 0$ are the couplings and the frequencies $\{\omega_i\}$ are drawn from an even probability distribution $g(\omega)$, though they are simply fixed real parameters when studying finite N realizations.

3.1 Review of the low-dimensional Kuramoto model

Before proceeding, it might be useful to get familiar with the low-dimensional solutions of the ordinary Kuramoto model and the techniques used first. For this reason, we present the simple cases of $N = 2, 3$ oscillators subject to Kuramoto's coupling and discuss their behavior as a dynamical system.

3.1.1 $N = 2$

For a couple of oscillators, we can explicitly write the system of ODEs governing their dynamics as:

$$\begin{cases} \dot{\theta}_1 &= \omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= \omega_2 - \frac{K}{2}\sin(\theta_2 - \theta_1) \end{cases}$$

The system appears to be a planar one, however we can reduce its dimensionality by exploiting the model's symmetry under phase translations: this means that the physical interesting quantities are not the phases themselves but rather their distance from one another. Therefore, we will as a paradigm always want to rewrite the system using the new variables

$$\Delta_i = \theta_N - \theta_i \tag{3.2}$$

where the reference phase has been arbitrarily picked to be θ_N. This means that the physical dimensionality of the system is always $N - 1$.

Applying what we have discussed to the $N = 2$ case, we obtain a one-dimensional dynamical system described by the ODE:

$$\dot{\Delta} = \Omega - K \sin \Delta \tag{3.3}$$

where we have introduced the parameter $\Omega = \omega_2 - \omega_1$. The fixed points of the system are given by the condition

$$\frac{\Omega}{K} = \sin \Delta$$

which induces a bifurcation structure ruled by the threshold $K_c = |\Omega|$:

$$\begin{cases} K > K_c : \bar{\Delta} = \arcsin\left(\frac{\Omega}{K}\right) \\ K > K_c : \bar{\Delta} = \pi - \arcsin\left(\frac{\Omega}{K}\right) \\ K < K_c : \nexists\ \bar{\Delta} \end{cases}$$

where the first equilibrium is attracting and the second one is repulsive, by linearization. We recognize that the overall phenomenology resembles our previous analysis of the Kuramoto model: the two oscillators tend to synchronize and their phases tend to cluster.

3.1.2 $N = 3$

For 3 oscillators, the system of ODEs is given by:

$$\begin{cases} \dot{\theta}_1 &= \omega_1 + \frac{K}{3}\sin(\theta_2 - \theta_1) + \frac{K}{3}\sin(\theta_3 - \theta_1) \\ \dot{\theta}_2 &= \omega_2 - \frac{K}{3}\sin(\theta_2 - \theta_1) + \frac{K}{3}\sin(\theta_3 - \theta_2) \\ \dot{\theta}_3 &= \omega_3 - \frac{K}{3}\sin(\theta_3 - \theta_1) - \frac{K}{3}\sin(\theta_3 - \theta_2) \end{cases}$$

Once again, we remove the redundant degree of freedom by rewriting the system as a planar one in the phase differences:

$$\begin{cases} \dot{\Delta}_1 &= \Omega_1 - \frac{K}{3}\left(2\sin(\Delta_1) + \sin(\Delta_2) + \sin(\Delta_1 - \Delta_2)\right) \\ \dot{\Delta}_2 &= \Omega_2 - \frac{K}{3}\left(\sin(\Delta_1) + 2\sin(\Delta_2) - \sin(\Delta_1 - \Delta_2)\right) \end{cases} \tag{3.4}$$

where $\Delta_i = \theta_3 - \theta_i$ and $\Omega_i = \omega_3 - \omega_i$.

Let us consider the identical-oscillators case first, that is we take $\Omega_1 = \Omega_2 = 0$. The fixed points are determined by the system of equations:

$$\begin{cases} 0 &= -\frac{K}{3}\left(2\sin(\Delta_1) + \sin(\Delta_2) + \sin(\Delta_1 - \Delta_2)\right) \\ 0 &= -\frac{K}{3}\left(\sin(\Delta_1) + 2\sin(\Delta_2) - \sin(\Delta_1 - \Delta_2)\right) \end{cases}$$

which has six solutions on the 2-torus. We report the equilibria along with their stability properties obtained by linearization in the following table:

We recognize three types of states: first, we have the synchronized state, which is the only attracting equilibrium for the system, characterized by all phases collapsing onto a single one. Next, we have three partially synchronized states, corresponding to the saddles: in this case only two phases are locked while the remaining one is antipodal to them. Finally,

Equilibrium	Eigenvalues	Stability
$(0,0)$	$(-3,-3)$	attracting
(π,π)	$(-1,+3)$	saddle
$(0,\pi)$	$(-1,+3)$	saddle
$(\pi,0)$	$(-1,+3)$	saddle
$(\pm\frac{2\pi}{3},\mp\frac{2\pi}{3})$	$(+3,+3)$	repulsive

Table 1: Fixed points of the 3-identical-oscillators Kuramoto model

we have two incoherent, repulsive states, which are given by the equal partition of phases along the unit circle.

We now relax the hypothesis on the differences Ω_i. We want to study the bifurcation threshold $K_c = K_c(\Omega_1, \Omega_2)$, which means we are looking for the value of K that rules the appearance of attracting fixed points in the system. We already know that $K_c(0,0) = 0$ and we also expect that when two oscillators are identical (i.e. when either $\Omega_1 = 0$, $\Omega_2 = 0$ or $\Omega_1 = \Omega_2$) the critical behavior reduces to the $N = 2$ case, so $K_c(\Omega, 0) = K_c(0, \Omega) = K_c(\Omega, \Omega) = |\Omega|$. Unfortunately, no analytical approach yields a closed form expression for the critical curve in parameter space, therefore we have to resort to a numerical approach. We know that that the bifurcation we are looking for is a saddle-node one, so we look for solutions to the fixed point equation along with the condition of vanishing Jacobian; we also need to verify that the emerging equilibria are attracting. To ease the analysis, we reduce the number of parameters by one introducing the reduced frequencies $a = \frac{\Omega_1}{K}$ and $b = \frac{\Omega_2}{K}$. The resulting bifurcation diagram is presented in the following figure:

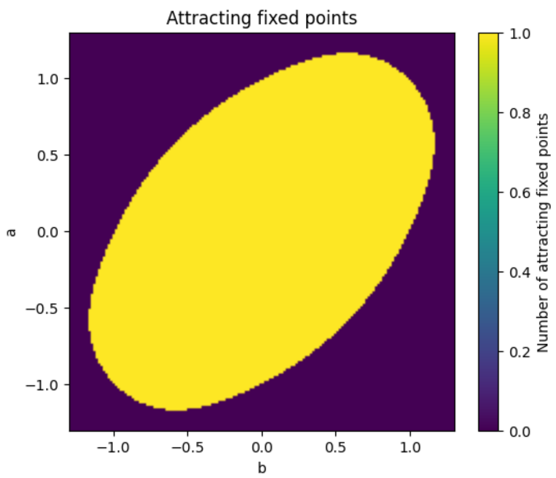


Figure 6: Bifurcation diagram for the 3-oscillators Kuramoto model. The yellow area looks like an ellipse but is actually slightly wider than the ellipse interpolating the obvious points discussed in the text.

3.2 Low-dimensional Biharmonic model

We are finally ready to analyze some low-dimensional realizations of the biharmonic extension of the Kuramoto model. We have already defined the system, but we can reduce the number of effective parameters by rescaling time as $\tau = K_1 t$:

$$\dot{\theta}_i = \frac{\omega_i}{K_1} + \frac{1}{N}\left(\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + c\sum_{j=1}^{N}\sin(2(\theta_j - \theta_i))\right) \tag{3.5}$$

where $c = \frac{K_2}{K_1} > 0$ is the relative strength between the couplings.

3.2.1 $N = 2$

We start from the pair of oscillators subject to biharmonic coupling. We can write the reduced system as a single ODE for the phase difference $\Delta = \theta_2 - \theta_1$:

$$\dot{\Delta} = \frac{\Omega}{K_1} - \sin\Delta - c\sin(2\Delta) \tag{3.6}$$

Let us consider the identical-oscillators case first, setting $\Omega = \omega_2 - \omega_1 = 0$. The fixed points of the system are given by:

$$0 = \sin\Delta + 2c\sin\Delta\cos\Delta$$

We can immediately recognize that $\bar{\Delta} = 0, \pi$ are always solutions to the equation, while the solutions $\bar{\Delta} = \pm\arccos\left(-\frac{1}{2c}\right)$ are real only for $c > \frac{1}{2}$. By studying the linearized system, we determine the following bifurcation structure:

- $0 < c < \frac{1}{2}$: The solution $\bar{\Delta} = 0$ is an attracting fixed point for the system, while $\bar{\Delta} = \pi$ is repulsive. The system behaves like the usual Kuramoto model, therefore we can call this the 1^{st} harmonic regime.
- $c > \frac{1}{2}$: Both solutions $\bar{\Delta} = 0, \pi$ are attracting fixed points for the system and the emerging $\bar{\Delta} = \pm\arccos\left(-\frac{1}{2c}\right)$ are repulsive. The system behaves like a Kuramoto model with second harmonic coupling, therefore we will call this the 2^{nd} harmonic regime.

Notice that, in the second case, the presence of the attracting equilibrium at $\Delta = \pi$ does not imply that the system is going to collapse towards it: in fact, for an initial condition $|\Delta_0| < \arccos\left(-\frac{1}{2c}\right)$ the system will collapse to the $\bar{\Delta} = 0$ equilibrium, making it impossible to verify "phenomenologically" the presence of the second harmonic term. This problem is, in fact, analogous to the introduction of the redistribution functions in the single-harmonic model.

We not turn to the $\Omega \neq 0$ problem. We know that the onset of synchronicity is strictly related to a tangential bifurcation, which means we need to investigate the shape of the coupling term first. In particular, we are interested in studying the extrema of the following function:

$$f(x) = \sin x + c\sin(2x) \tag{3.7}$$

We expect that, for low values of c, f resembles a sine function with period 2π, while, for high values of c, f should resemble a sine function with period π. To formalize this, we look for the points where the first derivative vanishes:

$$f'(x) = \cos x + 2c\cos(2x) = \cos x + 2c(2\cos^2 x - 1) = 0$$

where in the last step we have used the standard trigonometric identity for $\cos(2x)$. This quadratic equation in $\cos x$ has two solutions:

$$\begin{cases} \cos x_1^{(c)} & = \frac{1}{8c}(-1 + \sqrt{1+32c^2}) \\ \cos x_2^{(c)} & = \frac{1}{8c}(-1 - \sqrt{1+32c^2}) \end{cases} \tag{3.8}$$

However, requiring that each solution is contained within the interval $[0,1]$, we find that $x_1^{(c)}$ is real for every choice of c, while $x_2^{(c)}$ is physical only for $c \geq \frac{1}{2}$. Now, let us denote by $M(c)$ and $m(c)$ the global and secondary maximum of f respectively. By direct calculation, one can verify that, for our choice of c, $M(c) = f(x_1^{(c)})$ and $m(c) = f(x_2^{(c)})$. We also notice that f is an odd function and as such $M(c)$ and $m(c)$ are also the absolute values of the global and secondary minima. Explicitly:

$$\begin{cases} M(c) & = +(1 + 2c\cos x_1)\sqrt{1 - \cos^2 x_1} \\ m(c) & = -(1 + 2c\cos x_2)\sqrt{1 - \cos^2 x_2} \end{cases} \tag{3.9}$$

One final observation is the following: in both c regimes, the explicit solution to the equation $f(x) = k$ with $k \in \mathbb{R}$ is in general convoluted due to the non-trivial dependence on both harmonics and on c (it can be reduced to a quartic polynomial equation). In order to avoid this complication, we will resort to a qualitative description given by the bifurcation structure of f alone.

As we have already said, the bifurcation to synchronicity is ruled by the extrema of the coupling function f: this means that the value $c = \frac{1}{2}$ plays a key role in the transition between 1st and 2nd harmonic regimes. In particular, we obtain the following structure:

- $\frac{|\Omega|}{K_1} > M(c)$: There are no intersections between the horizontal line at $\frac{|\Omega|}{K_1}$ and the coupling function $f(\Delta)$, so there are no fixed points for the system. The oscillators never lock.
- $0 < c < \frac{1}{2}$ and $\frac{|\Omega|}{K_1} < M(c)$: The coupling function has only two extrema, therefore the system has two fixed points, which have opposite stability properties. Phenomenologically, we observe a 1st harmonic behavior.
- $c > \frac{1}{2}$ and $m(c) < \frac{|\Omega|}{K_1} < M(c)$: The coupling function has four extrema, however the horizontal line at $\frac{|\Omega|}{K_1}$ intersects the function in only two points (that is, in the region on the y-axis between the two extrema). Therefore, we observe oscillators locking like in a 1st harmonic regime, even though the system allows for 2nd harmonic entrainment.
- $c > \frac{1}{2}$ and $\frac{|\Omega|}{K_1} < m(c)$: The horizontal line now intersects the coupling function at four distinct values of Δ, creating two pairs of fixed points with opposite stability. The phenomenology depends on the initial condition, since the oscillators can either lock in phase (near $\Delta = 0$) or in antiphase (near $\Delta = \pi$): this is analogous to the introduction of indicator functions in the single-harmonic model.

We present the phase diagram for this model in the following picture, where we have used $\frac{K_{1,2}}{|\Omega|}$ as bifurcation parameters for clarity.

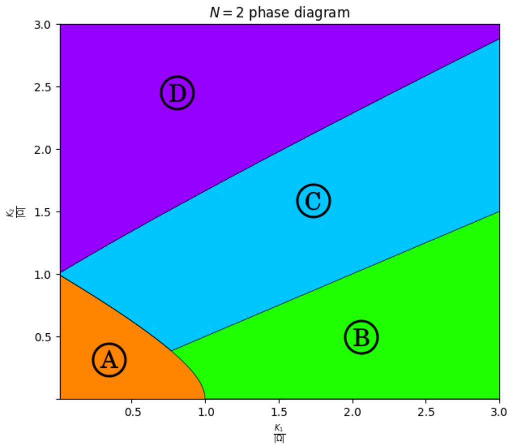


Figure 7: Phase diagram for two oscillators with biharmonic coupling function. Each coloured region represents one regime described in thet text. A: Disordered state, no entrainment is possible. B: 1^{st} harmonic regime. C: 1^{st} harmonic phenomenology even though the coupling function has four extrema. D: 2^{nd} harmonic regime. As both $K_{1,2}$ grow, the blue (C) region gets narrower, as predicted by the strong coupling limit.

3.2.2 $N = 3$

We now study the system of three oscillators subject to biharmonic coupling. The reduced system of ODEs is a planar one:

$$\begin{cases} \dot{\Delta}_1 &= a - \frac{1}{3}(2\sin\Delta_1 + \sin\Delta_2 + \sin(\Delta_1 - \Delta_2) + c(2\sin(2\Delta_1) + \sin(2\Delta_2) + \sin(2(\Delta_1 - \Delta_2)))) \\ \dot{\Delta}_2 &= b - \frac{1}{3}(\sin\Delta_1 + 2\sin\Delta_2 - \sin(\Delta_1 - \Delta_2) + c(\sin(2\Delta_1) + 2\sin(2\Delta_2) - \sin(2(\Delta_1 - \Delta_2)))) \end{cases}$$

where $a = \frac{\Omega_1}{K_1}$ and $b = \frac{\Omega_2}{K_1}$.

As before, we investigate the identical-case first, setting $a = b = 0$. In this case, the system undergoes two distinct bifurcations. In order to verify this, we need to solve the fixed-point equations:

$$\begin{cases} 0 &= -\frac{1}{3}(2\sin\Delta_1 + \sin\Delta_2 + \sin(\Delta_1 - \Delta_2) + c(2\sin(2\Delta_1) + \sin(2\Delta_2) + \sin(2(\Delta_1 - \Delta_2)))) \\ 0 &= -\frac{1}{3}(\sin\Delta_1 + 2\sin\Delta_2 - \sin(\Delta_1 - \Delta_2) + c(\sin(2\Delta_1) + 2\sin(2\Delta_2) - \sin(2(\Delta_1 - \Delta_2)))) \end{cases}$$

Due to the permutation symmetry of the model (exchanging oscillators does not change the dynamics), we are motivated to look for solutions which exhibit symmetric properties, namely $\Delta_1 = \Delta_2$, $\Delta_1 = 0$, $\Delta_2 = 0$ and $\Delta_1 = -\Delta_2$.

Along the symmetric branch, we can formulate the equilibrium condition as:

$$\sin\Delta_1 + c\sin(2\Delta_1) = 0$$

We notice that this is the same equation as the $N = 2$ case, so the solutions retain the same stability properties along with the same existence conditions. This is also the case for the

$\Delta_i = 0$ branches, for which the constraint yields the same equations in the complementary variable $\Delta_{j \neq i}$.

Along the antisymmetric branch, the fixed-point equation takes the form:

$$\sin \Delta_1 + (1 + c) \sin(2\Delta_1) + c \sin(4\Delta_1) = 0$$

which is always solved by $\bar{\Delta_1} = 0, \pi$. Using the trigonometric expansions for $\sin(2x)$ and $\sin(4x)$, we find the additional condition:

$$1 + 2(1 - c) \cos \Delta_1 + 8c \cos^3 \Delta_1 = (2 \cos \Delta_1 + 1)(4c \cos^2 \Delta_1 - 2c \cos \Delta_1 + 1) = 0$$

where we can identify the expected solution $\Delta_1 = \pm \frac{2\pi}{3}$. The quadratic polynomial in $\cos \Delta_1$ has real solutions only for $c > 4$ and one can verify that these roots satisfy $\cos \bar{\Delta}_1 < 1$ when they exist: therefore, the quadratic factor contributes with four additional fixed points for $c > 4$.

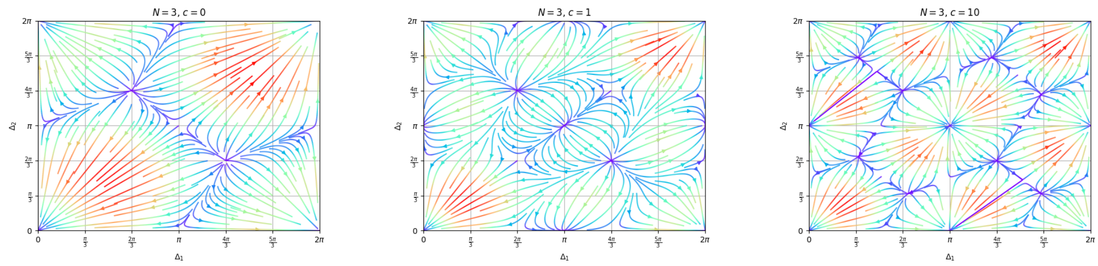
Counting all distinct solutions, we have found 16 equilibria. Via numerical analysis of the fixed-point equations, we have found 24 total equilibria, with the missing 8 having non-evident special properties (except in the limit of large c). The most relevant features of the phase portrait are the following:

- The point $(0, 0)$ is an attractor and never loses stability.
- The points $(0, \pi)$, $(\pi, 0)$ and (π, π) transition from being saddles to attracting nodes as c crosses $\frac{1}{2}$ from below. The 6 fixed points branching from these ones are always saddles.
- The special points arising from the quadratic polynomial in the antisymmetric branch (and their analogs in other sectors of the phase space) when $c > 4$ make the phase portrait in the interval $[0, \pi)^2$ topologically equivalent to the phase portrait in the interval $[0, 2\pi)^2$ when $c < \frac{1}{2}$, effectively creating four copies of the latter.

We can interpret all these features by defining three distinct regimes for the system:

- $0 < c < \frac{1}{2}$: The system is in a 1^{st} harmonic phase, all phases synchronize to $\Delta_1 = \Delta_2 = 0$.
- $\frac{1}{2} < c < 4$: The system undergoes the onset of the 2^{nd} harmonic behavior, phases either synchronize in phase or in antiphase.
- $c > 4$: The effect of the second harmonic is much stronger that the first harmonic coupling, which can be effectively neglected.

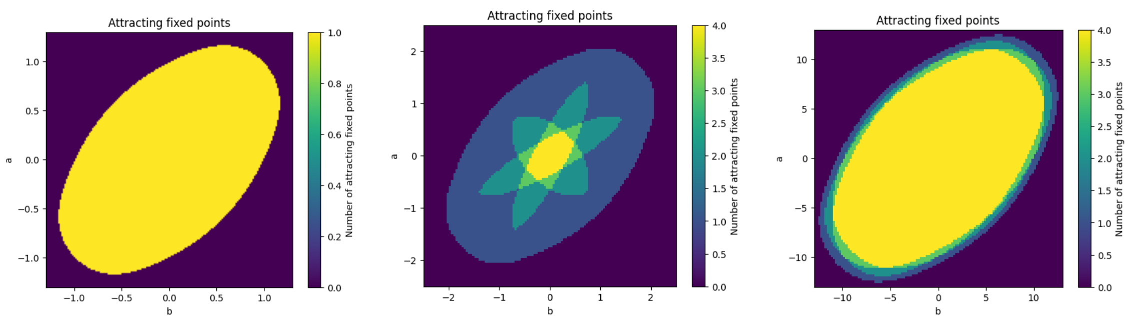
In the following figures we present the phase portrait of the system for different values of the relative strength c.



(a) $c = 0$ (b) $c = 1$ (c) $c = 10$

Figure 8: Comparison between phase portraits with different values of the relative strength c. Each plot is representative of one of the regimes defined in the text.

We now turn to the non-identical oscillators case: just as in the usual Kuramoto model, an analytical study of the model's bifurcations is difficult. We present some numerical result obtained by counting the number of attracting equilibria in the (a, b) plane in the following figure:



(a) $c = 0$ (b) $c = 1$ (c) $c = 10$

Figure 9: Number of attracting fixed points in the (a,b) plane for different values of c. Notice, in the second and third diagrams, the special dynamics along the lines $a = 0$, $b = 0$ and $a = b$ and the widening of the critical boundary induced by the growing K_2.

3.3 Critical threshold for identical oscillators

Let us now consider the problem of identical oscillators subject to biharmonic coupling for a generic number N of oscillators. Investigating numerically the cases $N = 4$ and $N = 5$ always yields a bifurcation when the relative strength c crosses the value $c_c = \frac{1}{2}$ (other bifurcations are present for different c values). We are thus tempted to conjecture that $c = c_c = \frac{1}{2}$ is the universal threshold for the onset of the 2^{nd} harmonic regime in ensembles of identical oscillators. In this section we will prove this claim.

The ruling equations are:

$$\dot{\theta}_i = \sum_{k \neq i}^{N} \left(\sin(\theta_k - \theta_i) + c \sin(2(\theta_k - \theta_i)) \right) \tag{3.10}$$

As before, we reduce the dimensionality by introducing the phase differences $\Delta_i = \theta_N - \theta_i$. We will denote by $\mathcal{N} = N - 1$ the reduced dimension.

$$\dot{\Delta}_i = -\left(2\sin\Delta_i + \sum_{k\neq i}^{\mathcal{N}}(\sin\Delta_k + \sin(\Delta_i - \Delta_k)) + \right.$$
$$\left. + \, c\left(2\sin(2\Delta_i) + \sum_{k\neq i}^{\mathcal{N}}(\sin(2\Delta_k) + \sin(2(\Delta_i - \Delta_k)))\right)\right) \tag{3.11}$$

The strategy to prove our conjecture is to study the stability of all the configurations for which Δ_i is either 0 or π, which are values that make the vector field vanish trivially. To do so, we first need to linearize the system, which is equivalent to calculating the Jacobian of the vector field. We can evaluate its entries via the definition $J_{ij} = \frac{\partial\dot{\Delta}_i}{\partial\Delta_j}$:

$$J_{ij} = \begin{cases} -\left(2\cos\Delta_i + \sum_{k\neq i}^{\mathcal{N}}\cos(\Delta_i - \Delta_k) + 2c\left(2\cos(2\Delta_i)) + \sum_{k\neq i}^{\mathcal{N}}\cos(2(\Delta_i - \Delta_k))\right)\right) & \text{if } i = j \\ -(\cos\Delta_j - \cos(\Delta_i - \Delta_j) + 2c(\cos(2\Delta_j) - \cos(2(\Delta_i - \Delta_j)))) & \text{if } i \neq j \end{cases}$$

The next step is to exploit the model's symmetry under permutations: since the dynamics is invariant under the exchange of phases, which is ruled by a permutation matrix P, the Jacobian also maintains all of its interesting properties under the similarity transformation $J' = PJP^{-1}$, namely its eigenvalues. This allows us to restrict the analysis to the following family of microstates, that is the vectors $\bar{\boldsymbol{\Delta}}$ whose entries are:

$$\bar{\Delta}_i = \begin{cases} 0 & \text{if } 1 \leq i \leq \mathcal{M} \\ \pi & \text{if } \mathcal{M}+1 \leq i \leq \mathcal{N} \end{cases} \tag{3.12}$$

where $0 \leq \mathcal{M} \leq \mathcal{N}$. For this choice, the Jacobian takes the form of a block matrix, which we can write as follows:

$$J = \left(\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right)$$

where A is a $\mathcal{M} \times \mathcal{M}$ matrix and D is a $(\mathcal{N}-\mathcal{M}) \times (\mathcal{N}-\mathcal{M})$ matrix. In particular, the entries for each matrix are given by:

$$\begin{cases} A_{ij} & = ((\mathcal{N}+1)(1-2c) - 2(\mathcal{M}+1))\delta_{ij} \\ B_{ij} & = 0 \\ C_{ij} & = -2 \\ D_{ij} & = 2 + ((\mathcal{N}+1)(1-2c) - 2(\mathcal{N}-\mathcal{M}))\delta_{ij} \end{cases}$$

We need to investigate the stability of the microstate by analyzing the signs of the eigenvalues of J; since B is the null matrix, J has the structure of a block-triangular matrix and, as such, by Schur's decomposition lemma, the set of its eigenvalues are given by the union of the sets of the eigenvalues of the matrices along the diagonal, which are A and D. A is already in diagonal form, while D is given by the sum of an "all-ones" matrix and a diagonal matrix, therefore we can diagonalize both simultaneously and sum the eigenvalues. We find:

$$\begin{cases} \lambda_1 = (\mathcal{N}+1)(1-2c) & \textit{multiplicity: } 1 \\ \lambda_2 = \lambda_1 - 2(\mathcal{M}+1) & \textit{multiplicity: } \mathcal{M} \\ \lambda_3 = \lambda_1 - 2(\mathcal{N}-\mathcal{M}) & \textit{multiplicity: } \mathcal{N}-\mathcal{M}-1 \end{cases} \tag{3.13}$$

where λ_2 is the eigenvalue of the matrix A while λ_1 and λ_3 are the eigenvalues of the matrix D.

We first analyze the degenerate state with $\mathcal{M} = \mathcal{N}$, that is the microstate with $\bar{\Delta}_i = 0$ for every i. In this case, the Jacobian is made up entirely by the matrix A and the only eigenvalue is λ_2, which takes the value $\lambda_2 = -(\mathcal{N}+1)(1+2c)$, meaning that the origin is always an attracting fixed point, as expected.

Now we consider the case $\mathcal{M} < \mathcal{N}$. The crux of the discussion is the sign of the eigenvalue λ_1:

$$\begin{cases} > 0 & \text{if } 0 < c < \frac{1}{2} \\ < 0 & \text{if } c > \frac{1}{2} \end{cases}$$

The dependence on λ_1 of $\lambda_{2,3}$ makes it so that for $0 < c < \frac{1}{2}$ the latter can be either positive or negative, however they can never be both positive simultaneously (see Fig.(10) for clarity). This means that for c in this interval the microstate is a saddle, as expected in the 1^{st} harmonic regime. In the case $c > \frac{1}{2}$, instead, both λ_2 and λ_3 are sums of two negative quantities, therefore all the eigenvalues share the same negative sign: this means that for these values of c the configuration is an attracting fixed point. We have thus proved the fact that $c_c = \frac{1}{2}$ separates the harmonic regimes.

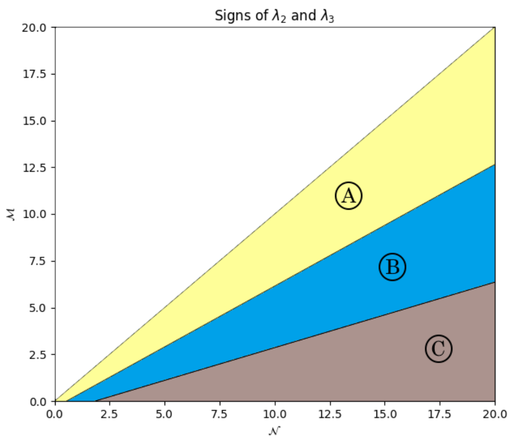


Figure 10: Sign diagram of the eigenvalues $\lambda_{2,3}$. A: $\lambda_2 < 0$ and $\lambda_3 > 0$. B: $\lambda_2 < 0$ and $\lambda_3 < 0$. C: $\lambda_2 > 0$ and $\lambda_3 < 0$.

As a last remark, let us prove that the incoherent state is given by the uniform distribution of phases along the unit circle and that this state is repulsive for the model we are considering. Using permutation symmetry, we can restrict the investigation to the fixed point given by:

$$\bar{\Delta}_k = \frac{2\pi}{\mathcal{N}+1}k = \alpha k \quad \text{with } k = 1, \ldots, \mathcal{N} \tag{3.14}$$

This configuration corresponds to setting each phase θ_i on one branch of the $N = \mathcal{N} + 1$ roots of unity in the complex plane. The main property we are going to exploit is the following:

$$\sum_{k=0}^{\mathcal{N}} e^{ik\alpha} = \sum_{k=0}^{\mathcal{N}} \sin(k\alpha) = \sum_{k=0}^{\mathcal{N}} \cos(k\alpha) = 0$$

In order to prove that $\bar{\boldsymbol{\Delta}}$ is an equilibrium of the system, we need to consider the sums in the vector field. Focusing on the ones in the first harmonic part of the field, they can be evaluated as follows:

$$S_1 = \sum_{k\neq i}^{\mathcal{N}} \sin(k\alpha) = 0 - \sin(i\alpha)$$

$$S_2 = \sum_{k\neq i}^{\mathcal{N}} \sin((i-k)\alpha) = \sum_{k=1}^{i-1} \sin((i-k)\alpha) + \sum_{k=i+1}^{\mathcal{N}} \sin((i-k)\alpha) = \sum_{m=1}^{i-1} \sin(m\alpha) - \sum_{m=1}^{\mathcal{N}-1} \sin(m\alpha)$$

Using the following reflection symmetry of the roots of unit:

$$\sin((\mathcal{N}+1-m)\alpha) = -\sin(m\alpha)$$

we can simplify the second sum:

$$S_2 = \sum_{m=1}^{i-1} \sin(m\alpha) - \sum_{m=1}^{i} \sin(m\alpha) = -\sin(i\alpha)$$

Therefore, we can easily see that the whole first-harmonic contribution vanishes:

$$2\sin(i\alpha) + S_1 + S_2 = 0$$

This argument still holds for the second-harmonic terms upon angle doubling. In the end, we have that every component of the vector field vanishes for our choice of $\bar{\Delta}_i$.

We will prove the instability of the solution by showing that there exists an unstable direction for the linearized system. We notice that the Jacobian acts as a convolution operator on the discretized unit circle, therefore its eigenvectors are given by Fourier modes. We are thus motivated to try as a candidate unstable direction the vector $\mathbf{v}$ whose entries are given by:

$$v_k = \sin(k\alpha)$$

We now work out the multiplication Jv, focusing only on the first-harmonic contribution:

$$(Jv)_i = J_{ii}\sin(i\alpha) + \sum_{k\neq i} J_{ik}\sin(k\alpha)$$

Using the properties of the roots of unity for $\cos(k\alpha)$, this calculation simplifies to $(Jv)_i = \sin(i\alpha)$, i.e. $\mathbf{v}$ is an eigenvector of the first-harmonic Jacobian with eigenvalue $\lambda = 1$. If we were to repeat the calculation for the second-harmonic Jacobian, we would find that its contribution vanishes (this is ultimately due to orthogonality of Fourier modes). The situation reverses for the eigenvector whose entries are:

$$w_k = \sin(2k\alpha)$$

In this case, the first-harmonic Jacobian brings no contribution while the second-harmonic one has an eigenvalue $\lambda = 2$. We have so proved that the incoherent state is a repulsive equilibrium.

4 Continuum analysis of the Biharmonic model

In this section we will fully carry out the discussion of the Biharmonic model in the thermodynamic limit. Let us recall the equations defining the system of N oscillators with frequencies $\{\omega_i\}$ drawn from a normalized even distribution $g(\omega)$:

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\left(K_1 \sum_{j=1}^{N} \sin\left(\theta_j - \theta_i\right) + K_2 \sum_{j=1}^{N} \sin\left(2(\theta_j - \theta_i)\right)\right) \tag{4.1}$$

with $K_1, K_2 > 0$. Due to the presence of two distinct harmonic couplings, in order to write the explict mean-field form of the equations we need to introduce two order parameters, which are the $m = 1, 2$ Daido's order parameters:

$$\Psi_1 = r_1 e^{i\varphi_1} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \qquad \Psi_2 = r_2 e^{i\varphi_2} = \frac{1}{N}\sum_{j=1}^{N} e^{2i\theta_j} \tag{4.2}$$

By the usual argument of equating imaginary parts, we can substitute the couplings with their mean-field counterparts, dropping the indices:

$$\dot{\theta} = \omega + K_1 r_1 \sin\left(\varphi_1 - \theta\right) + K_2 r_2 \sin\left(\varphi_2 - 2\theta\right) \tag{4.3}$$

Notice that we have two spurious phases φ_1 and φ_2 which cannot be simultaneously eliminated by phase-translation symmetry: in fact, if we are to send formally $\theta \to \theta + \varphi_1$, the second order parameter's phase still appears in the second harmonic coupling:

$$\dot{\theta} = \omega - K_1 r_1 \sin\theta + K_2 r_2 \sin\left(\Phi - 2\theta\right)$$

where we defined the phase difference $\Phi = \varphi_2 - 2\varphi_1$. We now propose an argument in favor of setting this difference to zero. Suppose we set $\varphi_1 = 0$ by Kuramoto's argument in the phase velocity equation. Then we can write schematically the self-consistency constraints as:

$$\begin{cases} \Psi_1 & = r_1 \\ \Psi_2 & = r_2 e^{i\Phi} \end{cases}$$

which means that the Ψ_2 order parameter has an imaginary part equal to:

$$\Im(\Psi_2) = r_2 \sin\Phi = \int_{-\infty}^{+\infty} d\omega \int_0^{2\pi} d\theta \sin\left(2\theta\right) f(\theta, \omega)$$

Therefore, in order for the phase difference Φ to vanish, we must show that the above integral is zero as well. We now consider reflection symmetry $(\theta, \omega) \to (-\theta, -\omega)$: this symmetry is realized in finite-N models and we require it holds in the continuum limit as well. We thus study the transformation of the velocity field under the transformation:

$$v(\theta,\omega) = \omega - K_1 r_1 \sin\theta + K_2 r_2 \sin(\Phi - 2\theta)$$
$$v(-\theta,-\omega) = -v(\theta,\omega) + 2K_2 r_2 \sin(\Phi)\cos(2\theta)$$

From the last equation, it is evident that in order for the reflection symmetry to be preserved we must set $\sin\Phi = 0$, which is true when $\Phi = 0$ (or when $\Phi = \pi$, but this essentially means changing the sign of the coupling strength K_2).

As we have stated above, this is only an argument for the choice $\Phi = 0$: we are basically selecting the stationary states which do not break the symmetry (which is what we usually do in simpler Kuramoto models). There is no counterargument against symmetry-broken synchronous states. In any case, it is worth noting that running simulations for large numbers of oscillators and long enough time intervals has never yielded a stationary state in which $\Phi \neq 0$ significantly.

The final mean field equation we are left with is:

$$\dot{\theta} = \omega - K_1 r_1 \sin\theta - K_2 r_2 \sin(2\theta) \tag{4.4}$$

4.1 Identical oscillators

We begin our study of the Biharmonic model with the simpler case of identical oscillators: this will prove useful to set expectations for the behavior of the full model. The reduced equation is the following:

$$\dot{\theta} = -K_1 r_1 \sin\theta - K_2 r_2 \sin(2\theta) \tag{4.5}$$

Let us briefly comment the structure of the coupling: we expect that for large values of K_1, the first harmonic contribution is much stronger than the second harmonic one and therefore the model should behave like the usual Kuramoto model; conversely, we expect a second-harmonic behavior for large K_2. This means that there is a phase transition between the two harmonic regimes ruled by the relative strength among K_2 and K_1. We investigate the nature of the transition by studying the fixed-point equation. At stationarity we have:

$$0 = -K_1 r_1 \sin\theta - K_2 r_2 \sin(2\theta) = -K_1 r_1 \sin\theta(1 + 2\gamma\cos\theta)$$

where we have defined the mean-field relative strength $\gamma = \frac{K_2 r_2}{K_1 r_1} > 0$. We notice that this equilibrium condition is the same of the $N = 2$ case we discussed in (3.2.1), provided that we send $c \to \gamma$. Applying those results, we have the following fixed points:

$$\begin{cases} \gamma > 0: & \bar{\theta} = 0 \\ \gamma > 0: & \bar{\theta} = \pi \\ \gamma > \frac{1}{2}: & \bar{\theta} = \pm\arccos\left(-\frac{1}{2\gamma}\right) \end{cases}$$

with the same stability properties we discussed in Section (3.2.1).

We can now write the self-consistent equation. Denoting by $H(x)$ the Heaviside step function:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

we can write:

$$r_m = H\left(\frac{1}{2} - \gamma\right) * \sum_{\bar{\theta}_{i,\text{stable}}} p_i^{(1)} \cos(m\bar{\theta}_i) + H\left(\gamma - \frac{1}{2}\right) * \sum_{\bar{\theta}_{i,\text{stable}}} p_i^{(2)} \cos(m\bar{\theta}_i) \tag{4.6}$$

where the sums run over the stable fixed points which exist in the corresponding γ interval (defined by the multiplied Heaviside function) and the parameters $p_i^{(k)}$ refer to the fraction of oscillators on the i-th stable branch in the k-th γ regime.
Let us briefly comment this equation. First, we notice that there is a hidden self-consistency requirement in the Heaviside functions since γ depends on both order parameters. Second, we notice that in the $\gamma < \frac{1}{2}$ regime there is only one stable fixed point, namely $\bar{\theta} = 0$, which conversely means that $p_1^{(1)} = 1$. Lastly, we can easily evaluate the self-consistent equation for r_2:

$$r_2 = H\left(\frac{1}{2} - \gamma\right) * (\cos 0) + H\left(\gamma - \frac{1}{2}\right) * \left(p_1^{(2)} \cos 0 + \left(1 - p_1^{(2)}\right) \cos 2\pi\right) = H\left(\frac{1}{2} - \gamma\right) + H\left(\gamma - \frac{1}{2}\right) = 1$$

which tells us that r_2 is always maximal for every value of the couplings K_1 and K_2.

We can now evaluate solve the self-consistency relation for r_1 as well. In order to ease the calculations, we will consider the different γ regimes separately. Let us start from the $\gamma < \frac{1}{2}$ regime first. In this case, we have:

$$r_1 = \cos 0 = 1$$

which allows us to solve self-consistently the γ relation as:

$$\gamma = \frac{K_2 r_2}{K_1 r_1} = \frac{K_2}{K_1} < \frac{1}{2}$$

which means that for $K_2 < \frac{K_1}{2}$ the system is in a 1$^{\text{st}}$ harmonic phase. Conversely, for $\gamma > \frac{1}{2}$ we have:

$$r_1 = p_1^{(2)} \cos 0 + \left(1 - p_1^{(2)}\right) \cos \pi = 2p_1^{(2)} - 1$$

Since $p_1^{(2)} \in [0, 1]$, r_1 can formally be negative valued, which does not makes sense for it: to solve this, we take the absolute values of the result, that is $r_1 = |2p_1^{(2)} - 1|$. This also allows us to restrict to the parameter values $p_1^{(2)} \in [0, \frac{1}{2}]$ without loss of generality. Solving for the couplings regime, we obtain:

$$\gamma = \frac{K_2 r_2}{K_1 r_1} = \frac{K_2}{K_1 |2p_1^{(2)} - 1|} > \frac{1}{2}$$

which means that this solution exists for $K_2 > \frac{K_1}{2}|2p_1^{(2)} - 1|$. What is evident is that for $p_1^{(2)} \neq 0$ there is a region in coupling space where the two solutions $r_1 = 1$ and $r_1 = |2p_1^{(2)} - 1|$ coexist. Also, by the fact that the parameter $p_1^{(2)}$ is arbitrary and does not provide any information about the dynamics that the system follows in order to synchronize, we deduce the fact that the solution $r_1 = |2p_1^{(2)} - 1|$ always exists for some $p_1^{(2)}$ since its domain is $K_2 > \inf \frac{K_1}{2}|2p_1^{(2)} - 1| = 0$.

In order to make sense of this we invoke the results we have achieved studying the finite N realization of this model. In fact, the solutions $r_1 = |2p_1^{(2)} - 1|$ can be thought of as the continuum limit of the particular microstate we have studied, meaning that essentially $p_1^{(2)} = \frac{\mathcal{M}}{\mathcal{N}}$. This allows us to conclude that the solution $r_1 = |2p_1^{(2)} - 1|$ is a saddle for $0 < K_2 < \frac{K_1}{2}$, while it is attracting for larger values of K_2. In Fig.(11) we show an example of the saddle-behavior we have mentioned.

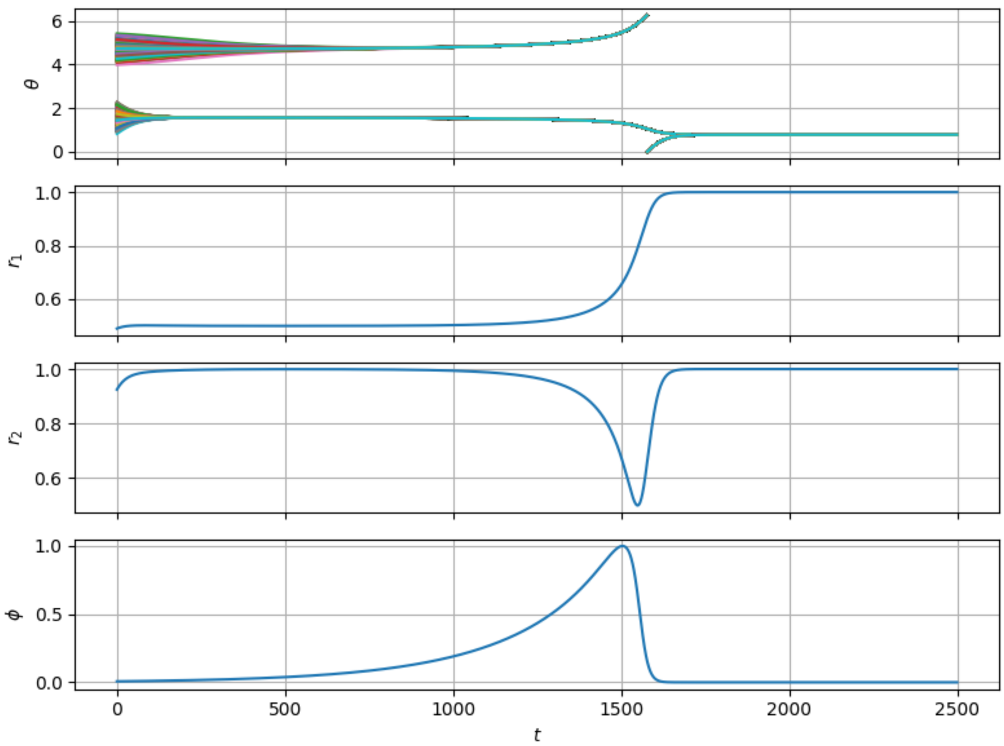


Figure 11: Example of the saddle-behavior in the identical-oscillators Biharmonic model. The system of $N = 10^4$ oscillators subject to the couplings $K_2 = 0.4K_1$ has been initialized near $r_1 = 0.5$, i.e. the partition is $p = 0.25$.

4.1.1 Relevance of identical-oscillators models

Let us now take a detour from the study of the Biharmonic model in order to discuss the physical relevance of identical-oscillators models. Suppose we have a generalized Kuramoto model with N coupled oscillators, where the coupling is through a generic periodic function $\Gamma(x)$. We also assign an intrinsic frequency ω_i drawn from a normalized probability distribution $g(\omega)$ to each oscillator. Then we can write the system of ODEs as:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} \Gamma(\theta_j - \theta_i)$$

If we rescale time as $\tau = Kt$, we obtain the system:

$$\dot{\theta}_i = \tilde{\Omega}_i + \sum_{j=1}^{N} \Gamma(\theta_j - \theta_i)$$

where $\tilde{\Omega}_i = \frac{\omega_i}{K}$ is a random variable extracted from the distribution $\tilde{g}(\tilde{\Omega}) = Kg(K\tilde{\Omega})$. If we now take the limit $K \to \infty$, naively we expect $\tilde{\Omega} \to 0$. More rigorously, we have that $\tilde{\Omega}$ converges to 0 in distribution, which means that $\tilde{g}(\tilde{\Omega})$ converges to $\delta(\tilde{\Omega})$ in the sense of distributions (weakly). This allows us to think of identical-oscillators models with coupling $\Gamma(x)$ as the large-coupling limit of the class of models with the same coupling function and a frequency distribution $g(\omega)$.

One last comment about this class of models is due to the fact that the synchronicity threshold is $K_c = 0$, i.e. synchronization always occurs. This can be seen in two ways: heuristically, using Kuramoto's formula $K_c = \frac{2}{\pi g(0)}$ and the fact that $g(0) = \delta(0) = +\infty$, we obtain $K_c = 0$; otherwise, since we are allowed to do the time-rescaling $\tau = Kt$ only for $K \neq 0$ and after the rescaling there is no movable physical parameter, the dynamics must be trivial only for $K = K_c = 0$. This fact can be generalized to the case of many couplings $\{K_i\}$ by noting that in the limiting cases of $K_i = K\delta_{i,i_0}$ for fixed i_0, the critical coupling must be $K_{i_0} = 0$; therefore, by the reasonable assumption that, in the non-identical case, the coherence threshold is a well-behaved hypersurface in parameter space, then it must collapse entirely to the origin by continuity.

4.2 The bifurcation in the (Φ, γ) plane

In order to monitor the transient from the inital time up until synchronization, we would like to study the bifurcation structure of the equilibrium equation:

$$\dot{\theta} = \omega - K_1 r_1 \sin\theta + K_2 r_2 \sin(\Phi - 2\theta)$$

As we have pointed out many times, the bifurcation is a saddle-node one, therefore the problem reduces to the study of the number of extrema of the coupling function $f(\theta)$ defined by:

$$f(\theta) = \sin\theta - \gamma \sin(\Phi - 2\theta)$$

This means that we need to discuss the number of solutions of the equation:

$$f'(\theta) = \cos\theta + 2\gamma \cos(\Phi - 2\theta) = 0$$

In the case $\Phi = 0$ the discussion is easy, however for $\Phi \neq 0$ the equation becomes much harder to solve. In particular, either by complexification or half-angle substitutions or expansion in elementary trigonometric functions, the equation becomes a quartic one, which means that it can be analytically solved through the quartic formula. However, due to its complexity, we will resort to other methods.
Due to the nature of the bifurcation, we require that the concavity of the function changes when the second pair of extrema appear (this can be also checked numerically), so we actually need to solve the system of equations:

$$\begin{cases} f'(\theta) & = \cos\theta + 2\gamma\cos(\Phi - 2\theta) = 0 \\ f''(\theta) & = -\sin\theta - 4\gamma\sin(\Phi - 2\theta) = 0 \end{cases} \tag{4.7}$$

We will carry on the discussion using the complex phase $z = e^{i\theta}$. The constraints are so reframed:

$$\begin{cases} P(z) & = 2\gamma e^{-i\Phi} z^4 + z^3 + z + 2\gamma e^{i\Phi} = 0 \\ Q(z) & = 8\gamma e^{-i\Phi} z^3 + 3z^2 + 1 = 0 \end{cases} \tag{4.8}$$

where $Q(z) = P'(z)$. Substituting the $f''(\theta) = 0$ constraint with $Q(z) = 0$ is legitimate since

$$\frac{df'(\theta)}{d\theta} = \frac{dz}{d\theta}\frac{d}{dz}\left(\frac{P(z)}{2z^2}\right) = iz\frac{z^2P'(z) - 2zP(z)}{2z^4} = 0$$

which means that $f''(z) = 0 \iff P'(z) = 0$ when $P(z) = 0$. The reformulation in terms of the complex phase z allows us to employ a resultant argument[15]. The resultant of the two polynomials $P(z)$ and $Q(z)$, defined as the determinant of the Sylvester matrix built from the two polynomials, is equal to zero if and only if the two polynomials have a common root. Applying this, we obtain the expression:

$$\begin{aligned} Res(P,Q) = & \, 2^{15}\gamma^7 e^{-i\Phi} - 3\cdot 2^{11}\gamma^5 e^{-i\Phi} - 3^3 2^3\gamma^3 e^{i\Phi} \\ & - 3\cdot 2^4\gamma^3 e^{-i\Phi} - 3^3 2^3 \gamma^3 e^{-3i\Phi} - 2^3\gamma e^{-i\Phi} \end{aligned}$$

Upon setting this expression equal to zero and rearranging terms, we obtain the following critical curve:

$$\cos^2\Phi = \frac{(16\gamma^2 - 1)^3}{108\gamma^2} \tag{4.9}$$

Imposing that $\cos^2\Phi \in [0,1]$, we obtain the domain $\gamma \in \left[\frac{1}{4}, \frac{1}{2}\right]$. In particular, the sector below the curve is the one in which the coupling function has only two extrema (1$^{\text{st}}$ harmonic regime) while in the region above it the function displays four extrema (2$^{\text{nd}}$ harmonic regime). A sketch of this critical boundary can be found in Fig.(12).

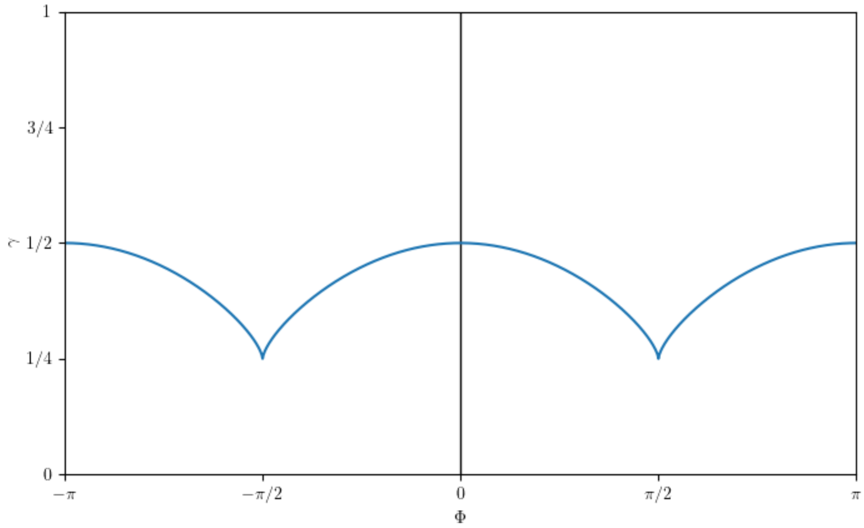


Figure 12: Bifurcation diagram in the (Φ, γ) plane. In the region under the curve, the coupling function has two critical points, while in the region above the curve the function has four.

4.3 The self-consistent equation

We can now investigate the complete Biharmonic model and determine its critical behavior. From our previous considerations, we expect at least three phases to emerge: an incoherent, disordered phase where both order parameters vanish, a 1^{st} harmonic phase, where oscillators form one synchronized cluster, and a 2^{nd} harmonic phase, where oscillators form two antiphase clusters. We know that the information about different synchronized regimes is encoded in the mean-field relative strength γ; conversely, we will investigate the boundary of the incoherent state using Daido's order function. According to Daido's theory[16], an appropriate order-parameter for multi-harmonic models is any L^p norm of the mean-field coupling function. We define the following parameter, which is proportional to the L^2 norm:

$$R = \frac{1}{\pi}\left(\int_0^{2\pi} d\theta (K_1 r_1 \sin\theta + K_2 r_2 \sin(2\theta))^2\right)^{\frac{1}{2}} = \sqrt{(K_1 r_1)^2 + (K_2 r_2)^2} \tag{4.10}$$

Notice that this new order parameter is the correct one to describe the transition from disorder to order since $R \geq 0$ and it vanishes only when both $r_1 = r_2 = 0$. Additionally, the introduction of the R parameter encourages a rewriting of the mean-field couplings in polar form:

$$\begin{cases} K_1 r_1 = R \sin u \\ K_2 r_2 = R \cos u \end{cases} \tag{4.11}$$

for $u \in [0, \frac{\pi}{2}]$, which means we can identify $\cot u = \gamma$ (Komarov-Pikosvky parametrization[4]). We can rewrite the phase velocity relation as:

$$\dot{\theta} = \omega - R(\sin u \sin\theta + \cos u \sin(2\theta))$$

Let us now summarize some results for the coupling function, which are simply the same of the coupling function of the $N = 2$ case provided we exchange c with γ in equations (3.7), (3.8) and (3.9). The coordinates of the global and secondary (which exists only for $\gamma > \frac{1}{2}$) maxima are x_M and $\pi + x_m$ respectively, with:

$$\begin{cases} x_M & = \arccos\left(\frac{-1+\sqrt{1+32\gamma^2}}{8\gamma}\right) \\ x_m & = \arccos\left(\frac{+1+\sqrt{1+32\gamma^2}}{8\gamma}\right) \end{cases} \tag{4.12}$$

We denote by $M(\gamma)$ and $m(\gamma)$ the values of the global and secondary maxima.

$$\begin{cases} M(\gamma) & = +(1 + 2\gamma \cos x_M)\sqrt{1 - \cos^2 x_M} \\ m(\gamma) & = -(1 + 2\gamma \cos(\pi + x_m))\sqrt{1 - \cos^2(\pi + x_m)} \end{cases} \tag{4.13}$$

To be fully general, we also need the coordinate $x^\star$, defined as the value where the function takes the value $m(\gamma)$ and has positive derivative. $x^\star$ can be calculated by imposing that the equation given by the intersection condition between the coupling function and the horizontal line $f(x) = m(\gamma)$ has a double root at $\cos\theta = \cos\pi + x_m$. By matching coefficients of this factorization, we obtain:

$$\begin{cases} \alpha_2(\gamma)\cos^2\theta + \alpha_1(\gamma)\cos\theta + \alpha_0(\gamma) = 0 \\ \alpha_2 = 4\gamma^2 \\ \alpha_1 = 4\gamma(1 + 2\gamma\cos(\pi + x_m)) \\ \alpha_0 = \frac{m^2(\gamma)-1}{\cos^2(\pi+x_m)} \end{cases}$$

which means that the expression for $x^\star$ is:

$$x^\star = \arccos\left(\frac{-\alpha_1(\gamma) + \sqrt{\alpha_1^2(\gamma) - 4\alpha_2(\gamma)\alpha_0(\gamma)}}{2\alpha_2(\gamma)}\right) \tag{4.14}$$

For simplicity, we have left the γ dependence, and possibly the u dependence upon substitution, implicit in all interesting phase-coordinates. We now make a crucial observation: for $\gamma > \frac{1}{2}$ the coupling function has two pairs of opposite extrema; however, unlike in the single-harmonic model, different pairs take on different values. This means that in this γ regime there are "overlapping" regions in the y-axis as well as single-valued regions: we thus deduce that for $\gamma > \frac{1}{2}$ the locked contribution is not made up entirely by indicator-function contribution but there is a "non-arbitrary" contribution as well sitting on the principal stable branch (this is analogous to the third regime we have described in Section (3.2.1)). In Fig.(13) we present a plot of the coupling function where all the interesting points have been marked.

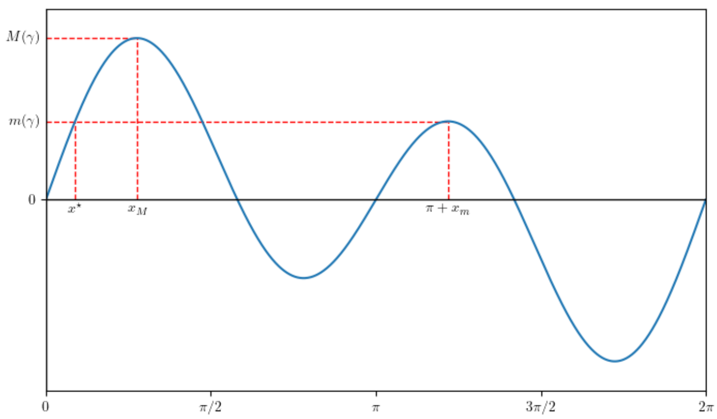


Figure 13: Structure of the Biharmonic coupling function for $\gamma = 2$.

We can now write the self-consistent equations for both order parameters r_1 and r_2. Their structure is the same as in single-harmonic models:

$$\Psi_m = r_m = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega e^{im\theta} \rho(\theta, \omega) g(\omega)$$

where we have set the mean-field phases to 0 per our previous arguments. As before, we need to distinguish the locked-oscillator statistics from the drifting-oscillator one. By our definition of $M(\gamma)$, we can set identify $\hat{\omega} = K_1 r_1 M(\gamma)$ as the separatrix between the two statistics. Nevertheless, this is a practically inconvenient definition for $\hat{\omega}$ because it is explicitly R-dependent: we can solve this issue by defining the renormalized frequencies $\omega = Rx$, which means that the boundary is at $\hat{x} = M(\gamma(u)) \sin u$. We perform the frequency substitution in the self-consistent equations and obtain:

$$\Psi_m = r_m = R \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dx e^{im\theta} \rho(\theta, Rx) g(Rx) = RF_m(R, u) \tag{4.15}$$

So the main object we need to calculate is:

$$F_m(R, u) = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dx e^{im\theta} \rho(\theta, Rx) g(Rx) \tag{4.16}$$

Let us start from the drifting contribution. We can employ the same ergodicity argument we have used in the regular Kuramoto model to obtain the following stationary phase distribution:

$$\rho_{drift}(\theta, Rx) = \frac{C(x)}{|x - \sin u \sin \theta - \cos u \sin (2\theta)}$$

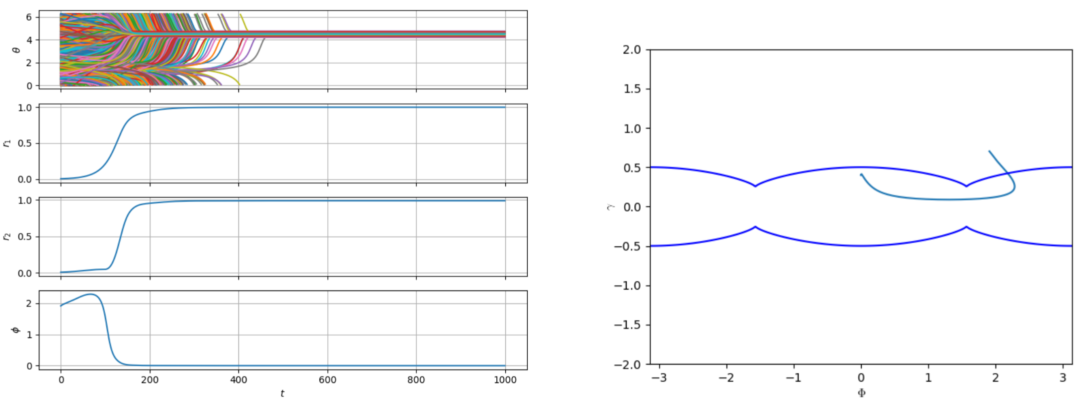
where the normalization constant is simply the reciprocal of the integral along one period of the distribution and it does not admit, unfortunately, a closed form. Notice that, by this definition, the R dependence vanishes from the distribution. This expression is thus valid for $|x| > \hat{x}$

The locked case is more involved. Let us start by defining implicitly the solutions to the fixed-point equation:

$$x = \sin u \sin \bar{\theta} + \cos u \sin (2\bar{\theta})$$

We define $\bar{\theta}_1$ as the solution in the always-existing stable branch (the one between the global extrema of the coupling function, centered in 0) and $\bar{\theta}_2$ as the solution in the other stable branch (the one between the secondary extrema, centered in π, which exists only for $\gamma > \frac{1}{2}$). We can identify two locked ($x < \hat{x}$) regimes:

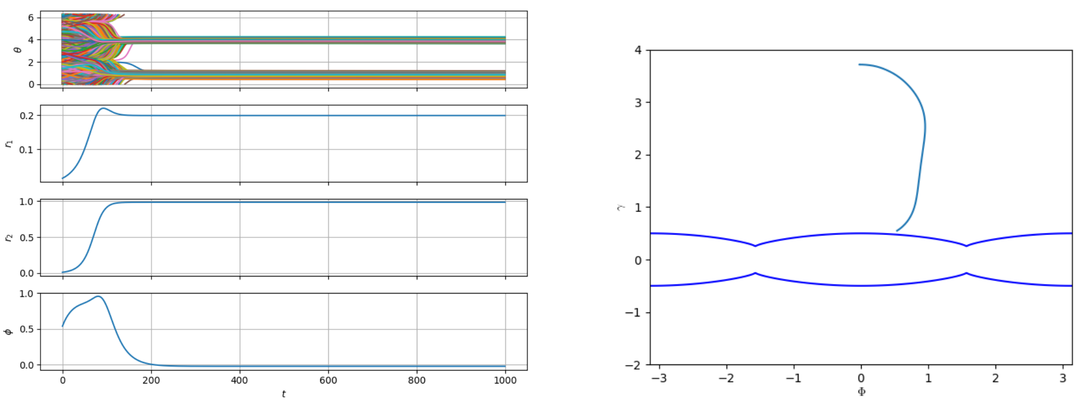
$$\begin{cases} x > m(\gamma(u)) \sin u : \rho_{lock}(\theta, Rx) & = \delta(\theta - \bar{\theta}_1) \\ \omega < m(\gamma(u)) \sin u : \rho_{lock}(\theta, Rx) & = S(x)\delta(\theta - \bar{\theta}_1) + (1 - S(x))\delta(\theta - \bar{\theta}_2) \end{cases}$$



(a) Phase dynamics

(b) Orbit in the (Φ, γ) plane

Figure 14: Example of Biharmonic synchronization for $K_1 = 8$, $K_2 = 3.2$. The synchronous state is a 1st harmonic one and only the $\bar{\theta}_1$ fixed point exists.



(a) Phase dynamics

(b) Orbit in the (Φ, γ) plane

Figure 15: Example of Biharmonic synchronization for $K_1 = 8$, $K_2 = 6$. The synchronous state is a 2nd harmonic one, the phase system displays two equilibria $\bar{\theta}_1$ and $\bar{\theta}_2$.

We can now write both the locked and drifting contributions to the function F. Let us start from noting that by the reflection symmetry $(\theta,\omega)\to(-\theta,-\omega)$ the complex exponential $e^{im\theta}$ can be simplified to its real part. By abuse of notation, we also merge together both γ regimes.

Consider the locked contribution. We eliminate the delta function inside the integrand by integrating over x. The result is:

$$\begin{aligned}F_m^{lock}(R,u) =& 2\int_{x^\star}^{x_M} d\theta \cos{(m\theta)} g(Rz(\theta,u))\left(\frac{\partial z}{\partial\theta}(\theta,u)\right)+\\ & 2\int_0^{x^\star} d\theta \cos{(m\theta)} g(Rz(\theta,u))\left(\frac{\partial z}{\partial\theta}(\theta,u)\right)S(Rz(\theta,u))+\\ & 2\int_0^{x_m} d\theta \cos{(m\theta+m\pi)} g(Rz(\theta+\pi,u))\left(\frac{\partial z}{\partial\theta}(\theta+\pi,u)\right)(1-S(Rz(\theta+\pi,u)))\end{aligned}$$

where the indicator functions $S(x)$ have been assumed to be even functions and we have introduced for convenience the reduced coupling function

$$z(\theta,u)=\sin u\sin\theta+\cos u\sin{(2\theta)}$$

Now, consider the drifting part of F:

$$F_m^{drift}(R,u)=2\int_{\hat{x}}^{+\infty} dx g(Rx)\frac{\int_0^{2\pi} d\theta\frac{\cos{(m\theta)}}{x-z(\theta,u)}}{\int_0^{2\pi} d\theta\frac{1}{x-z(\theta,u)}} \tag{4.17}$$

where the modulus inside the phase distribution has been removed by virtue of the reflection symmetry and by the fact that $\hat{x}$ is, in fact, the global maximum of $z(\theta,u)$. The most important thing to notice in this expression is that there is no symmetry transformation which makes the double integral vanish: neither the shift $\theta\to\theta+\pi$ nor $\theta\to\theta+\frac{\pi}{2}$ make the denominator globally change sign upon $\omega\to-\omega$, which forbids us from performing a Kuramoto-like cancellation. Therefore, the drifting contribution plays a crucial role in this model.

We finally write the auxiliary functions F_m as the sum of both locked and unlocked oscillators contributions:

$$F_m(R,u)=F_m^{lock}(R,u)+F_m^{drift}(R,u) \tag{4.18}$$

The last step in our preliminary analysis is to extract a formula for the couplings $K_{1,2}$ in terms of the parameters R and u. By inverting the definition of the polar form of the mean-field couplings (4.11) and using the expression $r_m=RF_m(R,u)$, we obtain the formulae:

$$\begin{cases}K_1(R,u)=\frac{\sin u}{F_1(R,u)}\\ K_2(R,u)=\frac{\cos u}{F_2(R,u)}\end{cases} \tag{4.19}$$

4.4 Linear stability of the disordered state

In order to correctly understand the model's behavior which emerges when solving the self-constitency conditions, we must first investigate the linear stability of the incoherent state, which is the state characterized by $R = r_1 = r_2 = 0$, whose corresponding phase-density can be easily proved to be $\rho_0(\theta, \omega) = \frac{1}{2\pi}$.

To determine the stability of this configuration, we must solve the eigenvalue problem associated to the perturbed state $\rho(\theta, \omega) = \frac{1}{2\pi} + \delta\rho(\theta, \omega)$. To derive it, we use the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho v) = 0$$

with $v(\theta, \omega, t) = \omega - K_1 r_1 \sin\theta - K_2 r_2 \sin(2\theta)$ the phase velocity. Notice that at the incoherent state we have $v_0(\theta, \omega) = \omega$. We can now expand the perturbation $\delta\rho$ as a Fourier series:

$$\delta\rho(\theta, \omega) = \frac{1}{2\pi} \sum_{m \neq 0} a_m(\omega, t) e^{im\theta}$$

which conversely means that, using Fourier orthogonality, the order parameters can be written as:

$$r_m = \int_{-\infty}^{+\infty} d\omega g(\omega) a_m(\omega, t)$$

We can now work out the linearized continuity equation:

$$\frac{\partial \delta\rho}{\partial t} + \omega \frac{\partial \delta\rho}{\partial \theta} + \frac{1}{2\pi}(-K_1 r_1 \cos\theta - 2K_2 r_2 \cos(2\theta)) = 0$$

Since ρ_0 is a constant, we have that the PDE decouples into simpler PDEs for each harmonic. In particular, we have that:

$$\begin{cases} \frac{\partial a_m}{\partial t} + im\omega a_m = \frac{mK_m}{2} r_m & m = 1, 2 \\ \frac{\partial a_m}{\partial t} + im\omega a_m = 0 & m \geq 3 \end{cases}$$

We now look for solutions with exponential time-behavior, i.e. $a_m(\omega, t) = b_n(\omega) e^{\lambda t}$. The second set of equations ($m \geq 3$) can be solved immediately and yields the imaginary eigenvalues $\lambda_m = -im\omega$. Substituing in the first set of equations, we get the following relation:

$$(\lambda_m + im\omega) b_m = \frac{mK_m}{2} \int_{-\infty}^{+\infty} d\omega g(\omega) b_m(\omega) = \frac{mK_m}{2} R_m$$

If we now multiply both sides by $g(\omega)$ and integrate along the real line, we can simplify R_m on both sides. The final relation we get is:

$$1 = \frac{mK_m}{2} \int_{-\infty}^{+\infty} d\omega \frac{g(\omega)}{\lambda_m + im\omega} \tag{4.20}$$

We are interested in the loss of stability of the incoherent state, which means the transition from a purely imaginary eigenvalue to a complex eigenvalue with positive real part, that means we need to take the limit $\lambda_m \to 0^+$. This is useful because we can use the Dirac identity inside the integral to simplify it:

$$\int_{-\infty}^{+\infty} d\omega \frac{g(\omega)}{im\omega + 0^+} = \int_{-\infty}^{+\infty} d\omega \frac{g(\omega)}{m} \left(\pi\delta(\omega) - i\mathcal{PV}\left(\frac{1}{\omega}\right)\right) = \frac{\pi g(0)}{m}$$

Substituing into the previous equation:

$$1 = \frac{mK_m}{2} \frac{\pi g(0)}{m} = \frac{\pi g(0) K_m}{2}$$

Inverting the expression, we obtain the critical stability thresholds:

$$K_m^{(c)} = \frac{2}{\pi g(0)} = K_c \qquad m = 1, 2 \tag{4.21}$$

which are the same as the usual Kuramoto threshold: this means that when either one of K_1 or K_2 are greater than K_c the incoherent state, corresponding to $R = 0$, loses neutral stability and becomes repulsive.

We now know that in the region defined by $K_1 > K_c \vee K_2 > K_c$ the system always reaches synchronicity, with the distinction between different harmonic phases to be determined by the constraint $\gamma = \frac{1}{2}$.

4.5 Phase diagram and critical properties

What is left to be determined is the behavior inside the square region determined by $K_1, K_2 < K_c$. To investigate this, we need to solve the self-consistent equations, which, as we will see, give rise to a non-trivial behavior.

Before proceeding, there are two caveats to be considered. The first one is that the indicator function $S(\omega)$ in principle does not need to be a continuous one (in fact it could also be nowhere continuous); it is, however, convenient to assume it is well-behaved. The second caveat is that $S(\omega)$ should not be, in principle, arbitrary, but rather dependent on the initial distribution of phases (as it has already been pointed out in [17], though in the context of the classical Kuramoto model); however, computing the exact indicator function from first principles is difficult due to the fact that the basins of attraction depend on Φ, whose dynamics is unknown. Moreover, the fixed points of the phase velocity, as we have discusses, are given by a quartic equation: this fact further complicates the calculation. For all these reasons, we will resort to studying the simpler problem in which $S(\omega)$ is a constant. In particular, we are going to assume that $S(\omega) = 1$

As in the Kuramoto model, we look for solutions to the self consistent equation which bifurcate from the incoherent state $R = 0$. In our case, we have a parametrization for the couplings where the R dependence is only inside the auxiliary functions F_m, which are well-defined in the limit $R \to 0^+$. Therefore, the non-trivial region where both order parameters vanish is simply given by the following curve in parameter space parametrized by u:

$$\begin{cases} K_1^{(2)} = \frac{\sin u}{F_1(0,u)} \\ K_2^{(2)} = \frac{\cos u}{F_2(0,u)} \end{cases} \tag{4.22}$$

Let us now comment on the content of the function $F_m(0, u)$. We focus on the locked part first, which can be easily computed. For example, the $m = 1$ part is equal to:

$$\begin{aligned} F_1^{lock}(0,u) =& g(0)\int_{-x_M}^{+x_M} d\theta \cos\theta(\sin u\cos\theta + 2\cos u\cos(2\theta)) = \\ & g(0)\left(\sin u\left(x_M + \frac{\sin(2x_M)}{2}\right) + 2\cos u\left(\sin x_M + \frac{\sin(3x_M)}{3}\right)\right) \end{aligned}$$

Now for the drifting part:

$$F_m^{drift}(0,u) = 2\lim_{R\to 0^+}\int_{\hat{x}}^{+\infty} dx g(Rx)\Phi_m(x) = 2g(0)\int_{\hat{x}}^{+\infty} dx\Phi_m(x)$$

where in the last step we have used the dominated convergence theorem to exchange the integration with the limit. $\Phi_m(x)$ is simply the ratio:

$$\Phi_m(x) = \frac{\int_0^{2\pi} d\theta \frac{\cos(m\theta)}{x - \sin u\sin\theta - \cos u\sin(2\theta)}}{\int_0^{2\pi} d\theta \frac{1}{x - \sin u\sin\theta - \cos u\sin(2\theta)}} \tag{4.23}$$

Unfortunately, Φ_m does not admit a simple closed form. One can rewrite the phase-integrals as a sum of hypergeometric functions using the geometric-series expansion of the integrand, for example:

$$\begin{cases} N(x,u) & = 2\sum_{k=1}^{\infty}\left(\frac{1}{x}\right)^{2k}(\cos u)B(\frac{1}{2}+k,\frac{3}{2})_2F_1(1-k,\frac{3}{2}-k,\frac{5}{2};4\cot^2 u) \\ D(x,u) & = \sum_{k=0}^{\infty}\left(\frac{1}{x}\right)^{2k}(\sin u)^{2k}B(\frac{1}{2},k+\frac{1}{2})_2F_1(-k,\frac{1}{2}-k,k+1;4\cot^2 u) \\ \Phi_1(x) & = \frac{N(x,u)}{D(x,u)} \end{cases}$$

with $B(\alpha, \beta)$ being Euler's Beta function.
Since this formulation is rather convoluted, for our numerical results we have employed a simple numerical evaluation of the integrals instead of the hypergeometric-function-heavy formulae.

We now make a crucial observation. Let us start by observing that $F_m(R, u)$ is a positive-valued function: this is obvious by the relation $r_m = RF_m(R, u)$. Let us evaluate the derivative of the auxiliary function with respect to R while keeping u fixed:

$$\frac{\partial F_m}{\partial R} = 2\int_0^{x_M} d\theta\cos(m\theta)\left(\frac{\partial z}{\partial\theta}(\theta,u)\right)(z(\theta,u))g'(Rz(\theta,u)) + 2\int_{\hat{x}}^{+\infty} dx\left(xg'(Rx)\right)\Phi_m(x)$$

In order to evaluate this at $R = 0^+$ we need to discuss the convergence of the drift-integral since at the upper boundary the argument of the distribution is ambiguous. To do so, let us reintroduce the frequency $\omega = Rx$:

$$\int_{R\hat{x}}^{+\infty} \frac{d\omega}{R}\frac{\omega}{R} g'(\omega)\Phi_m\left(\frac{\omega}{R}\right)$$

we see that at the lower boundary, the integrand is $\sim \frac{\hat{x}}{R} g'(R\hat{x})\Phi_m(\hat{x}) \sim \hat{x}^2 g''(0)\Phi_m(\hat{x})$, where:

$$\Phi_m(\hat{x}) = \lim_{x\to\hat{x}^+} \Phi_m(x) = \cos(m x_M)$$

by the fact that, as x approaches $\hat{x}$, the measure gets denser about the global maximum of $z(\theta, u)$. In order to check convergence at the upper boundary, we need an asymptotic expansion of $\Phi_m(x)$, which can be easily constructed using a geometric series truncated at second order in its numerator and a geometric series truncated at zeroth order in its denominator:

$$\Phi_m(x) \sim \frac{\int_0^{2\pi} d\theta \cos(m\theta)\left(1 + \frac{z(\theta,u)}{x} + \left(\frac{z(\theta,u)}{x}\right)^2\right)}{\int_0^{2\pi} d\theta} = \frac{A_m}{x^2}$$

with:

$$\Phi_1 = \frac{\sin u \cos u}{2} \qquad\qquad \Phi_2 = -\frac{\sin^2 u}{4}$$

We notice that the appropriate expressions correctly vanish in the trivial cases $u = 0, \frac{\pi}{2}$, i.e. the single-harmonic cases. Evaluating the integrand at the upper boundary, we have $\sim \frac{\omega}{R^2} g'(\omega)\frac{A_m R^2}{\omega^2} = \frac{g'(\omega)}{\omega} A_m$ which is generically well behaved since $g(\omega)$ decays fast enough even in the case of heavy-tailed distributions.

Going back to the dx integral, we can naively see that all the relevant contribution to its value comes from the $x \to \infty$ sector, which means that its sign is dictated by the asymptotic behavior of $\Phi_m(x)$, whose sign is the same as A_m, namely $(-1)^{1+m}$ (as a side note, $\Phi_m(x)$ is a monotone function which has a fixed sign along of all its domain). We finally write the derivative evaluated at $R = 0^+$ symbolically as:

$$\frac{\partial F_m}{\partial R}(0, u) = 2Q_m(-1)^m \tag{4.24}$$

where the extra minus sign comes from the fact that $g'(\omega)$ is negative for $\omega = Rx > 0$ and Q_m is a positive quantity.

We now examine briefly the behavior of the order parameters and of the couplings along constant $u^\star$ curves through their derivatives with respect to R. Starting from the order parameters:

$$r'_m(0, u^\star) = F_m(0, u^\star) > 0$$

Which shows that the order parameters grow linearly from the $R = 0$ branch. One could further show that the derivative stays positive for every R, which means that r_m are monotonic functions for fixed $u^\star$ values.

Denoting by t_m the appropriate trigonometric function, for the couplings we have:

$$K'_m(0, u^\star) = -t_m(u)\left(\frac{F'_m}{F_m^2}\right)(0, u^\star) \propto (-1)^{1+m}$$

Which means that moving along $u = u^\star$ curves the coupling K_1 increases while the coupling K_2 initially decreases[1]; though, by compatibility with the large coupling behavior, K_2 must start increasing at some point. This is crucial because it indicates the presence of a global minimum for $K_2(R, u^\star)$, which we denote by $\chi = K_2(R^\star, u^\star)$: if we now take a constant value $\varepsilon = K_1(R^\star, u^\star)$, we have two interesting K_2 values, namely the one which corresponds to $R = 0^+$ and a certain $\bar{u}$, i.e. $K_2^{(2)} = K_2(0, \bar{u})$, and χ. By the fact that χ is a minimum we deduce that $\chi < K_2^{(2)}$ and, as such, our previous critical line ($R = 0^+$) is not the boundary of the incoherent state. We thus define the proper border of synchronicity as the pairs $(K_1^{(1)}, K_2^{(1)})$:

$$\begin{cases} R(u) : K_1^{(1)} = K_1(R(u), u) & K_1^{(1)} \in [0, K_c] \\ K_2^{(1)} = \min\limits_u \left[K_2(R(u), u)\right] \end{cases} \tag{4.25}$$

This serves as a pragmatic definition; however, let us formulate a more meaningful equivalent. Along the $R(u)$ curve, we have the following relation among the order parameters and the couplings:

$$\frac{dr_m}{dK_2} = \frac{dr_m}{du}\frac{du}{dK_2}$$

But we know that the derivative of r_m in general does not vanish, while the minimum condition is equivalent to $\frac{dK_2}{du} = 0$; therefore we can guess that the border of synchronicity is given by:

$$\frac{dr_m}{dK_2} = \infty$$

More rigorously, by the optimization requirement, we can define a Lagrange multiplier λ such that:

$$\nabla K_2 = \lambda \nabla K_1$$

where the gradient is taken with respect to the variables R and u. The Jacobian of the map must then be singular and thus the inverse derivatives $\frac{\partial R_m}{\partial K_n}$ blow up at that point.

Finally, consider the following: along the constraint $K_1(R(u), u) = \varepsilon$ and near the minimum we have discussed, which we say is realized at $u^\star$, the order parameters and the coupling K_2 can be generically written as:

[1] Numerical simulations for the couplings derivatives show that K_2 decreases more steeply along constant u curves compared to constant R curves at $R = 0^+$.

$$\begin{cases} K_2(R(u),u) & \sim K_2^{(1)} + a_1 L^2 \\ r_m(R(u),u) & \sim r_m(R(u^\star),u^\star) + a_2 L \end{cases}$$

for some coefficients a_1, a_2 and where L is the arclength along the constraint. By eliminating L in either equation, we obtain the following scaling relation:

$$R_m - R_m^\star \sim \pm a_2 \sqrt{\frac{K_2 - K_2^{(1)}}{a_1}} \tag{4.26}$$

This asymptotic behavior is most interesting for two reasons. First, it resembles Kuramoto's scaling law, which is good since it must be recovered in the degenerate cases $K_1 = 0 \vee K_2 = 0$. Lastly and most importantly, it shows that, when deviating from $K_2^{(1)}$ along the constraint, the order parameters become multivalued, with one branch being decreasing and the other being increasing. This is different from the classical Kuramoto model since, in that case, there is only one physical branch, which is the increasing one. We can interpret this phenomenon as follows: the increasing branch is the analog of the $r_m^{(\infty)}$ branch in the single-harmonic model, while the other branch is the one which reconnects with the $R = 0^+$ solution. The obvious question now is whether these solutions represent stable configurations of the system or not. We have addressed this problem numerically and simulations have shown that the decreasing branch can never be observed, and, as such, we regard it as an unstable solution, while the increasing branch can be reached and is stable.

What we have discussed allows us to finally trace out the phase diagram of the model, which consists of five critical lines. The line L_1 corresponds to the pairs $(K_1^{(1)}, K_2^{(1)})$ and separates the incoherent phase from the ordered ones, while the line L_2 corresponds to the pairs $(K_1^{(2)}, K_2^{(2)})$, i.e. $R = 0^+$. We now define the line L_3 as the pairs $(K_1^{(3)}, K_2^{(3)})$ which satisfy $K_m^{(3)} = K_m(R, u = \arctan 2)$ for $R \in [0, +\infty)$: this line is simply the separatrix between the 1st and 2nd harmonic phases since $\tan u = 2 \iff \gamma = \frac{1}{2}$. The last two lines correspond to the stability thresholds of the incoherent solution, i.e. the lines defined by $(t, 0)$ and $(0, t)$, where $t \in [0, K_c]$.

From a thermodynamic point of view, the Biharmonic model undergoes a different phase transition compared to the single-harmonic model: as we have discussed, on the line L_1 the order parameters are non-vanishing, therefore we must conclude that this is a discontinuous phase transition. Between the lines L_1 and L_2 there is a coexistence region: there are two stable solutions to the self-consistent equation, namely the $R = 0$ solution (neutrally stable) and the increasing r_m branch (attracting). This is a limiting example of hysteresis in the (r_1, r_2) plane, which usually requires the two stable fixed points to also be attracting. Beyond the line L_2 the hysteresis analogy does not hold anymore since there is no unstable solution in between the stable branches, which maintain their stability properties nonetheless: the interesting part about this region is the fact that finite-size corrections (which we can model as stochastic white noise in the continuum realization) destroy the neutral stability of the $R = 0^+$ solution if the system is given enough time to evolve. In particular, the average time it takes a system initialized near $R = 0$ to escape toward the attracting solution scales as a power law of the number of oscillators (see [4] for a numerical validation of this claim).

In Fig.(16) we present the phase diagram for the Biharmonic model with all the critical lines we have discusses. This is to be compared with Fig.(17), which shows a numerically-derived phase diagram starting from two different initial phases distributions (a uniform distribution and a delta one): this dual approach is necessary to verify the actual presence of the hysteresis region. Finally, in Fig.(18) we present the numerically-solved self-consistent equations for both order parameters with fixed $K_1 = 0.9K_c$ and moving K_2, along with the numerical data obtained from independent runs initialized at the same phase.

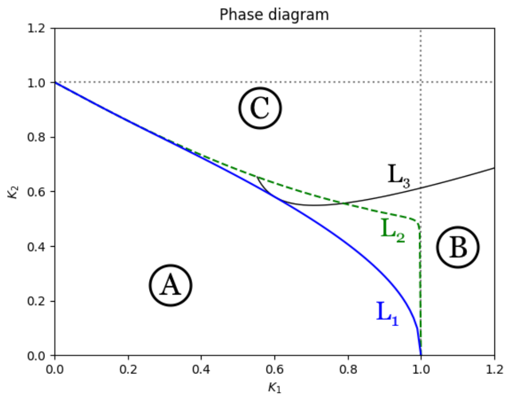
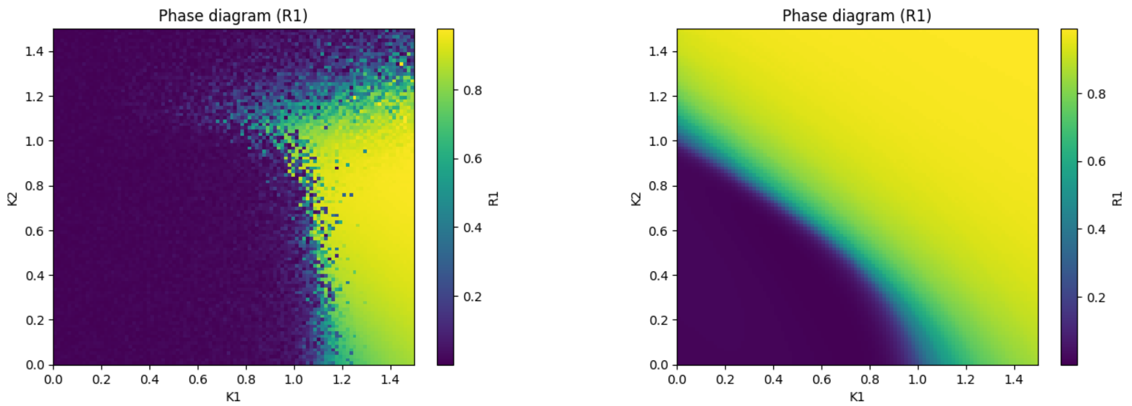


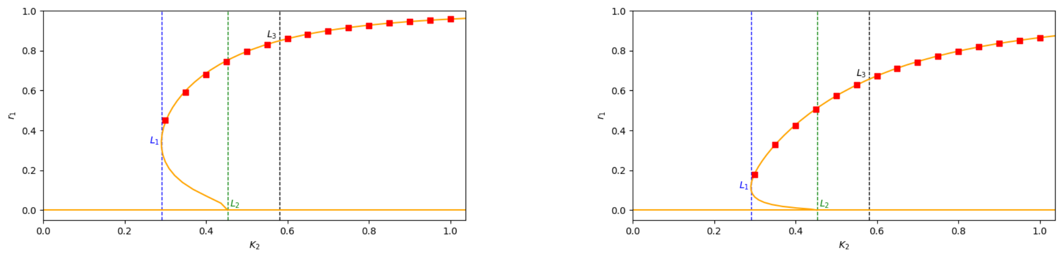
Figure 16: Phase diagram for the Biharmonic model in the case $S(\omega) = 1$ with the axes normalized by K_c. The line L_1 marks the onset of synchronicity, the line L_2 marks the non-trivial curve where $R = 0^+$, the line L_3 separates different harmonic phases. The gray dashed lines are the linear stability thresholds of the incoherent solution. A: disordered phase. B: 1^{st} harmonic phase. C: 2^{nd} harmonic phase. Between L_1 and L_2 there is a hysteresis-like region while between L_2 and the gray dashed lines there is an effectively metastable region.



(a) System initialized at random phases

(b) System initialized at synchronicity

Figure 17: Numerical phase diagram for the Biharmonic model with different initial conditions: the left diagram is initialized with uniformly random phases while the right one is initialized with equal phases. Axes are normalized by K_c. The left diagram matches approximately with the $R = 0$ neutral stability region. Notice how outside the $[0, 1]^2$ region the r_1 order parameter takes values which are generically different from $r_1 \approx 1$: this is due to the dynamics which evolved toward synchronized states with $S(\omega) \neq 1$. The right diagram matches the boundary of incoherent solutions, showing that the $R > 0$ solutions are attracting.



(a) r_1 plot

(b) r_2 plot

Figure 18: Solutions of the self-consistent equations for $K_1 = 0.9K_c$ and varying K_2 (normalized by K_c). The data is obtained by mediating independent runs of $N = 2 \cdot 10^5$ oscillators initialized at the same phase in order to ensure the $S(\omega) = 1$ condition. The dashed lines mark the corresponding interesting order parameters. The irregularity of the curve near L_2 is due to the difficul numerical integration of $\Phi_m(x)$.

4.6 Pseudocritical exponents

We conclude our study of the Biharmonic model with the calculations of some pseudocritical exponents[2].

We start from the β exponent, which we define as the asymptotic behavior when approaching L_1 of the order parameters with respect to the couplings:

$$\delta r_m \sim \delta K_n^{\beta}$$

where δr_m denotes the distance of the order parameter from a reference value on the line L_1 and similarly δK_n denotes the distance $K_n - K_n^{(1)}$ where $K_n^{(1)}$ realizes said r_m value. We have already discussed that along constant K_1 slices, we recover Kuramoto's scaling, which means that $\beta = \frac{1}{2}$ (one can explicitly observe it in Fig.(18)). This can be shown to hold along analogous K_2 slices as well.

We now turn to the pseudoscaling properties of the L_2 curve: we are going to calculate the $\bar{\beta}$ and $\bar{\nu}$ pseudocritical exponents. We must first clarify that this is actually just a theoretical question, since the r_m branch bifurcating from L_2 (i.e. from incoherence) is unstable and thus can never be observed.

We can straightforwardly calculate the $\bar{\beta}$ exponent by considering the critical behavior of the derivatives. In fact, we expect that near L_2 the following scaling law holds:

$$r_m \sim \delta K_n^{\bar{\beta}}$$

For simplicity, let us consider constant $u = u^{\star}$ curves. In that case, we can approximate:

$$\begin{cases} r_m \sim RF_m(0, u^{\star}) \\ K_n \sim K_n^{(2)} + (-1)^{1+m} G_n R \end{cases}$$

[2]The "pseudo" prefix is meaningful since the phase transition is a first order one, while the notions of critical exponents and universality are rigorously exclusive to continuous phase transitions.

where G_n encodes all prefactors coming from the derivative. From the first relation, we obtain $R \sim \frac{r_m}{F_m^\star}$, which we can susbtitute in the second equation:

$$r_m \sim \delta K_n \tag{4.27}$$

This relation requires the scaling exponent to be $\bar{\beta} = 1$. We can generalize this result to the orthogonal directions of constant K_1 and K_2 by simply crossing criticality through different u curves, i.e. writing $u = u^\star + tR$ and expanding the couplings appropriately.

This scaling of the order parameters from the incoherent branch has already been discovered by Daido[18], however we have now an interesting insight on it: it is associated to an experimentally inaccessible branch. Conversely, we have recovered the typical mean-field exponent from the saddle-node bifurcation on the line L_1 which corresponds to the true onset of synchronicity and can be effectively observed.

We now turn to the $\bar{\nu}$ exponent and calculate it using Finite Size Scaling methods associated to the L_2 continuous branching. We will carry out the analysis using a schematic approach and the original parameters of the model. First of all, we describe the discrete realization of order parameters as:

$$\begin{cases} r_1 &= \frac{1}{N}\sum_{|\omega_j|<K_1 r_1 M(\gamma)} \sqrt{1 - w^2\left(\frac{\omega_j}{K_1 r_1}\right)} + \frac{1}{N}\sum_{|\omega_j|>K_1 r_1 M(\gamma)} \Phi_1(\omega) \\ r_2 &= \frac{1}{N}\sum_{|\omega_j|<K_1 r_1 M(\gamma)} \sqrt{1 - v^2\left(\frac{\omega_j}{K_1 r_1}\right)} + \frac{1}{N}\sum_{|\omega_j|>K_1 r_1 M(\gamma)} \Phi_2(\omega) \end{cases}$$

where $w(x)$ and $v(x)$ are the appropriate functions which express $\sin(m\theta)$ in terms of $\left(\frac{\omega}{K_1 r_1}\right)$ in the fixed point equation. We can express the variance of the finite-size fluctuation $\delta\tilde{\Psi}(R)$ symbolically as:

$$\mathrm{var}(\delta\tilde{\Psi}(R)) \sim \frac{K_1^2 r_1}{N} + \frac{K_2^2 r_2}{N} + \frac{\sqrt{(K_1^2 r_1)(K_2^2 r_2)}}{N} + \tag{4.28}$$

$$+\left(\frac{K_1}{N}\right)\left(\frac{(K_1 r_1)(K_2 r_2)}{R}\right) + \left(\frac{K_2}{N}\right)\left(\frac{(K_1 r_1)^2}{R}\right) \sim \frac{R}{N} \tag{4.29}$$

which means that, as in the Kuramoto model, we have $\delta\tilde{\Psi}(R) \sim \sqrt{\frac{R}{N}}$. Writing the self-consistency of the R parameter near the line L_2 we have:

$$0 \approx A\mu R + BR^2 + \delta\tilde{\Psi}(R)$$

where the parameters A and B encode the factors coming from the expansions of the individual order parameters and of the couplings, μ is a distance from the line L_2 and the scaling R^2 is justified by virtue of the $\bar{\beta}$ exponent calculation. As in the ordinary Kuramoto model, we proceed by making an ansatz for the order parameter $R \sim N^{-\alpha}$ and consider the competition between the quadratic term and the finite-size fluctuations at criticality ($\mu = 0$):

$$R^2 \sim N^{-2\alpha} \sim N^{-\frac{1}{2}(\alpha+1)} \sim \delta\tilde{\Psi}(R)$$

This asymptotic calculation yields $\alpha = \frac{1}{3}$. Next, we employ the FSS ansatz to write the scaling relation:

$$R(N,\mu) = b^{-\frac{\bar{\beta}}{\bar{\nu}}} R_b \left(b^{-1}N, b^{\frac{1}{\bar{\nu}}}\mu\right)$$

Setting $b = N$, we obtain:

$$R = N^{-\frac{\bar{\beta}}{\bar{\nu}}} f(\mu N^{\frac{1}{\bar{\nu}}}) \tag{4.30}$$

where f is a scaling function. If we compare terms with our previous result for α, we obtain the pseudocritical correlation exponent $\bar{\nu} = 3$, which is consistent with the fact that the combination $\mu N^{\frac{1}{3}}$ should be scale invariant. In fact, if we consider competition between the linear and quadratic term:

$$\mu R \sim \mu N^{-\frac{1}{3}} \sim N^{-\frac{2}{3}} \sim R^2$$

we find perfect agreement with our previous result. We observe that $n\bar{u}$ is higher in the Biharmonic case compared to the classical Kuramoto model, meaning that finite-size fluctuations are even more relevant

4.7 A glimpse into three-body interactions

We conclude our study of the Biharmonic model with an application. We consider an extension of the Kuramoto model which adds three-body interactions (which, as we have anticipated, might create higher-harmonic effective couplings) between oscillators in a specific way[19][20]:

$$\dot{\theta}_i = \omega_i + \frac{K_1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \frac{K_2}{2N^2}\sum_{j=1}^{N}\sum_{k=1}^{N} \sin(\theta_j + \theta_k - 2\theta_i) \tag{4.31}$$

We can rewrite this system in explicit mean-field form by using only the Ψ_1 order parameter:

$$\dot{\theta} = \omega + K_1 r_1 \sin(\varphi_1 - \theta) + \frac{K_2 r_1^2}{2} \sin(2\varphi_1 - 2\theta) \tag{4.32}$$

We can interpret it this as a Kuramoto model with a higher-order frustrated interaction between oscillators. To perform the analysis at stationarity, it is sufficient to perform the usual formal phase shift by φ_1, thus we recover the same expression we had in the Biharmonic case provided we send $K_2 r_2 \to \frac{K_2 r_1^2}{2}$.

Let us start from the instability threshold of the incoherent state: by the mere fact that the second-harmonic interaction is higher order in r_1, we can deduce that the disordered state becomes unstable at $K_1 = K_c$, provided we make the usual assumption of even $g(\omega)$. To investigate the region $K_1 < K_c$ we can exploit the KP parametrization we have previously introduced:

$$\begin{cases} R = \sqrt{(K_1 r_1)^2 + \left(\frac{K_2 r_1^2}{2}\right)^2} \\ K_1(R,u) = \frac{\sin u}{F_1(R,u)} \\ K_2(R,u) = \frac{2\cos u}{R F_1^2(R,u)} \end{cases} \tag{4.33}$$

where $F_1(R,u)$ is defined in the same way as before. If we send $R \to 0^+$ varying $u \in [0, \frac{\pi}{2}]$, we obtain finite values for K_1 while K_2 is identically sent to $+\infty$, except for $u \to \frac{\pi}{2}$, where the limit does not exist. We can heuristically interpret this fact as follows: we take the double limit along the curve $\cos u = \eta R$ for $\eta > 0$, which yields a multiplicity of values for $K_2 \in [0, +\infty)$ corresponding to $K_1 = K_c$. Therefore, the r_m branch bifurcating from $R = 0$ emerges exactly at the incoherent stability threshold. Since we know the second-harmonic introduces a hysteresis-like behavior, the line $K_1 = K_c$ is also the analog of L_2 in the Biharmonic model. By the same methods, we can also identify the lines L_1, which marks the onset of synchronicity, and L_3, the separatrix between different harmonic phases. The order parameters maintain the same properties we have deduced for the Biharmonic model, like the divergence of the derivative at L_1 and the scaling behavior at L_1 and L_2.

In Fig.(19) we present the analytical phase diagram for this model in the case $S(\omega) = 1$, while in Fig.(20) we present the same phase diagram obtained numerically in the case of both uniform and delta distribution of initial phases. For other numerical confirmations of our solution, one can refer to [19].

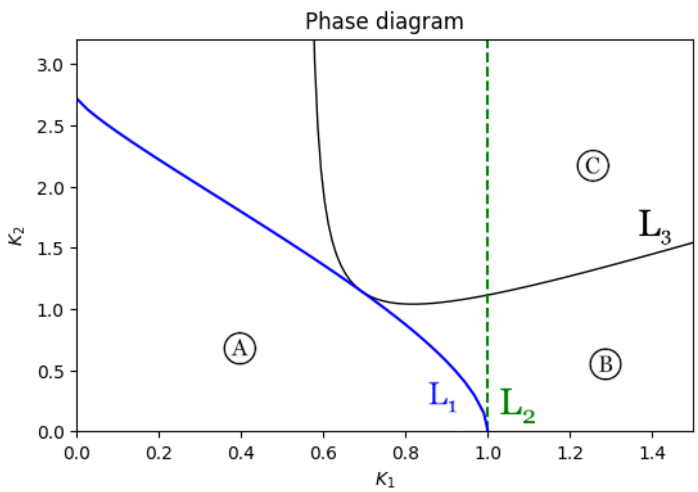
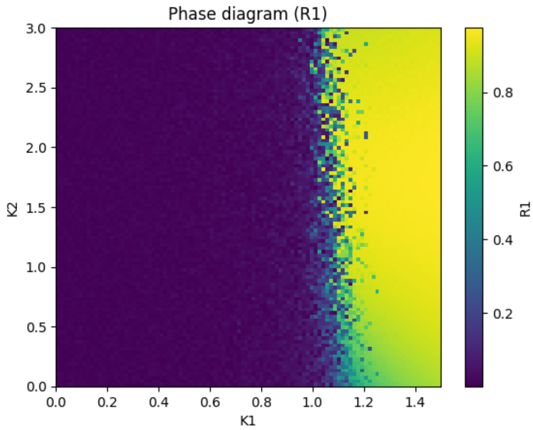
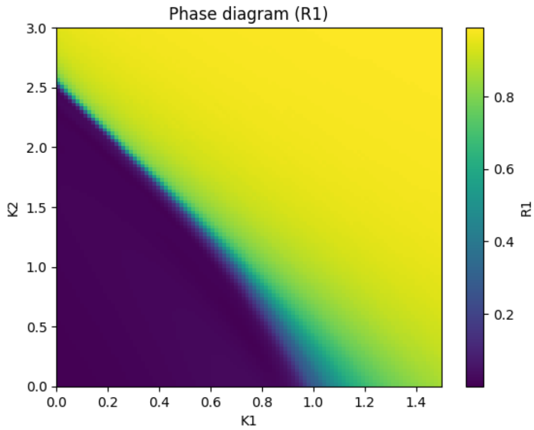


Figure 19: Phase diagram for the three-body Kuramoto model in the case $S(\omega) = 1$, each axis is normalized by K_c. The line L_1 is the border of incoherence, the line L_2 marks the curve where $R = 0$ non-trivially and is also the linear instability threshold for $R = 0$, the line L_3 separates different harmonic phases. A: Disordered phase. B: 1^{st} harmonic phase. C: 2^{nd} harmonic phase. In the region between L_1 and L_2 the system exhibits a hysteresis-like behavior.



(a) System initialized at random phases



(b) System initialized at synchronicity

Figure 20: Numerical phase diagram for the three-body Kuramoto model with different initial conditions: the left diagram is initialized with uniformly random phases while the right one is initialized with equal phases. Axes are normalized by K_c.

5 More general coupling functions

To finish our work, we will derive some results for more general multi-harmonic models. The idea is to consider the following class of systems of N oscillators with intrinsic frequencies sampled from an even probability distribution $g(\omega)$:

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} h(\theta_j - \theta_i) \tag{5.1}$$

where $h(x)$ is an odd periodic function with period 2π. The oddness of $h(x)$ is a useful accessory condition since it forces the "center of mass" of the systems to run at a constant speed (this can be checked by considering the time evolution of the sum of all phases), but most importantly because it allows us to expand the function in a Fourier series made up exclusively of sine functions:

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N}\sum_{m=1}^{M} K_m \sin\left(m(\theta_j - \theta_i)\right) \tag{5.2}$$

Notice that we have truncated the series at the M-th harmonic. We will also make the extra assumption of positive series coefficients, i.e. positive physical couplings $K_m > 0$. Using Daido's order parameters, we can straightforwardly rewrite the equation in explicitly mean-field form:

$$\dot{\theta} = \omega + \sum_{m=1}^{M} K_m r_m \sin\left(\varphi_m - m\theta\right)$$

The final strong assumption we make is that all the mean-field phases φ_m can be formally set to zero at synchronicity:

$$\dot{\theta} = \omega - \sum_{m=1}^{M} K_m r_m \sin\left(m\theta\right) \tag{5.3}$$

5.1 Linear stability of the disordered state

The first result we will prove is that the incoherent state, which is given by the phase distribution $\rho_0(\theta,\omega) = \frac{1}{2\pi}$, has a universal instability threshold. This follows from the same kind of argument we have carried out for the Biharmonic model in Section (4.4).

Let us start from observing that every order parameter is trivially vanishing for this distribution:

$$r_m = \langle e^{im\theta}\rangle = \int_{-\infty}^{+\infty} d\omega g(\omega) \int_0^{2\pi} d\theta \frac{1}{2\pi} e^{im\theta} = 0 \tag{5.4}$$

For the next step, we need to study the eigenvalue problem associated to the linearized system about the incoherent state. Using the same setup we have employed in the Biharmonic model, we define a perturbation $\delta\rho(\theta,\omega)$, we expand it in Fourier modes and we look for solutions with exponential time-behavior. The relation we obtain is:

$$1 = \frac{K_m}{2}\int_{-\infty}^{+\infty} d\omega \frac{g(\omega)}{\lambda_m + i\omega} \qquad m = 1, \ldots, M \tag{5.5}$$

Taking the limit $\lambda_m \to 0^+$ and expanding the integrand using the Dirac identity, we recover the relation:

$$K_m^{(c)} = \frac{2}{\pi g(0)} = K_c \qquad m = 1, \ldots, M \tag{5.6}$$

As we have seen with the Biharmonic model, this result about the lineary instability threshold is not enough to describe accurately the overall behavior of the whole model.

5.2 Odd-multiple model

The second result we want to prove regards a specific model which we will refer to as "Odd-multiple model": it is defined as the polyharmonic model in which only odd-multiples of the lowest harmonic appear. The governing mean field equation is:

$$\dot{\theta} = \omega - \sum_{m=0}^{M} K_{(2m+1)n} r_{(2m+1)n} \sin\left((2m+1)n\theta\right) \tag{5.7}$$

Notice that if we perform the rescalings $\psi = n\theta$ and $\tau = nt$, the model can be formally simplified to the case $n = 1$ (this reduction is analogous to the equivalence relation between the single-harmonic model and the Kuramoto model). So it is sufficient to study the model:

$$\dot{\theta} = \omega - \sum_{m=0}^{K} K_{2m+1} r_{2m+1} \sin\left((2m+1)\theta\right) \tag{5.8}$$

We want to prove that in the 1^{st} harmonic regime the self-consistent equations for all the order parameters are decoupled and can be solved independently. Before venturing into this, we need to study the following reduced coupling function:

$$f(x) = \sin x + \sum_{k=1}^{K} a_{2k+1} \sin\left((2k+1)x\right) \tag{5.9}$$

The regime we are interested in is the one in which this function displays only one maximum and one minimum. Since the only parameters are $\{a_i\}$, we deduce that, in this regime, $f(x)$ "follows" $\sin x$ and this is realized for sufficiently small parameters. Let us now observe that the points $x = \pm\frac{\pi}{2}$ are always stationary points. In fact:

$$f'(x) = \cos x + \sum_{k=1}^{K} (2k+1) a_{2k+1} \cos((2k+1)x)$$

which always vanishes at $x = \pm\frac{\pi}{2}$. We factor this expression as:

$$f'(x) = H(\cos x) \cdot \cos x$$

where the function $H(t)$ is defined using Chebyshev's polynomials of the first kind:

$$H(t) = 1 + \sum_{k=1}^{K}(2k+1)a_{2k+1}\sum_{p=0}^{m}\binom{2k+1}{2p}(t^2-1)^p t^{2(k-p)}$$

The condition we have mentioned is equivalent to the following requirement:

$$\mathrm{sgn}(f'(x)) = \mathrm{sgn}(\cos x) \quad \forall x \in [0, 2\pi) \iff \mathrm{sgn}\left(H(t)\right) > 0 \quad \forall t \in [0,1]$$

which is ultimately a matter of choosing the a_{2k+1} parameters, which can be done in several ways. We can directly derive a sufficient condition for the sequence $\{a_{2k+1}\}$ by imposing that the second derivative of the coupling function is strictly negative at $x = \frac{\pi}{2}$:

$$f''\left(\frac{\pi}{2}\right) = -1 + \sum_{k=1}^{K}(2k+1)^2 a_{2k+1}(-1)^{k+1} \tag{5.10}$$

If we choose the coefficients in such a way that $b_{2k+1} = (2k+1)^2 a_{2k+1}$ defines a montonically decreasing sequence, we can apply the alternating series criterion (though this is not actually a full series) and we are thus sure that, for a suitable a_3, the second derivative has negative sign. Another way of constructing an appropriate coupling function is to use a Fejér-Riesz approach, which means building a complex polynomial on the unit circle and taking its square modulus (this is valid due to the Fejér-Riesz theorem[21]).

As an example, let us consider the simplest non-trivial model:

$$H(\cos x) = 1 + \lambda \cos(2x)$$

with $\lambda \in (-1, +1)$, which assures that the sign of this function is positive for all x. Constructing the derivative of the coupling function, we obtain:

$$f'(x) = \cos x(1 + \lambda\cos(2x)) = \cos x + \frac{\lambda}{2}(\cos x + \cos(3x))$$

We renormalize all coefficients by a factor of $\left(1 + \frac{\lambda}{2}\right)$ in order to have a parameter-free first harmonic, we obtain:

$$f'(x) = \cos x + \frac{\lambda}{2+\lambda}\cos(3x)$$

Matching with our general expression for the derivative, we obtain:

$$a_3 = \frac{\lambda}{6+3\lambda} \in \left(-\frac{1}{3}, +\frac{1}{9}\right) \subset \left[0, \frac{1}{9}\right)$$

Let us now consider the most relevant feature of the Odd-multiple model, which is the absence of a drift contribution to the self-consistent equations. We start from its bare expression for the n-th order parameter:

$$\langle e^{in\theta}\rangle_{drift} = \int_{|\omega|>\omega_0} d\omega g(\omega)C(\omega)\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{\left|\omega - \sum_{m=0}^{M} K_{2m+1}r_{2m+1}\sin(m\theta)\right|}$$

where ω_0 is the maximum value of the mean-field coupling function, $g(\omega)$ is the probability distribution associated to the frequency of each oscillator, $C(\omega)$ is the normalization for the phase probability distribution, which follows from the usual ergodic argument. In this case, Kuramoto's cancellation holds since the integrand changes sign under the transformation $\theta \to \theta + \pi$ and the integration domain is symmetric by the fact that ω_0 is the absolute value of both extrema of the coupling function due to its parity (these properties can also be used as the definition of the Odd-multiple model starting from the general expression of the multi-harmonic Kuramoto model). Thus, the drifting contribution vanishes entirely by symmetry.

We can finally prove our result. In the 1$^{\text{st}}$ harmonic regime, the self consistent equation only has one solution, which satisfies:

$$\omega = \sum_{m=1}^{M} K_{2m+1}r_{2m+1}\sin((2m+1)\bar{\theta})$$

therefore, we can write the self-consistent equation for the n-th order parameter simply as:

$$r_n = \langle e^{in\theta}\rangle_{lock} = \int_0^{2\pi} d\theta \int_{-\omega_0}^{+\omega_0} d\omega g(\omega)\delta\left(\theta - \bar{\theta}(\omega)\right)\cos(n\theta)$$

integrating over the frequencies, we obtain:

$$r_n = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta \cos(n\theta) g\left(\sum_{m=0}^{M} K_{2m+1}r_{2m+1}\sin((2m+1)\theta)\right)\left(\sum_{m=0}^{M}(2m+1)K_{2m+1}r_{2m+1}\cos((2m+1)\theta)\right)$$

Since we are investigating the onset of synchronicity, we take the limit $r_{2m+1} \to 0^+$ as per the usual procedure: all the relevant contribution comes from $g(\omega) \approx g(0)$. The next thing we need to do is to evaluate a sum of integrals of the type:

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} dx \cos(ax)\cos(bx) = \int_0^{+\frac{\pi}{2}} dx\left(\cos((a-b)x) + \cos((a+b)x)\right) = \frac{\sin\left((a-b)\frac{\pi}{2}\right)}{a-b} + \frac{\sin\left((a+b)\frac{\pi}{2}\right)}{a+b}$$

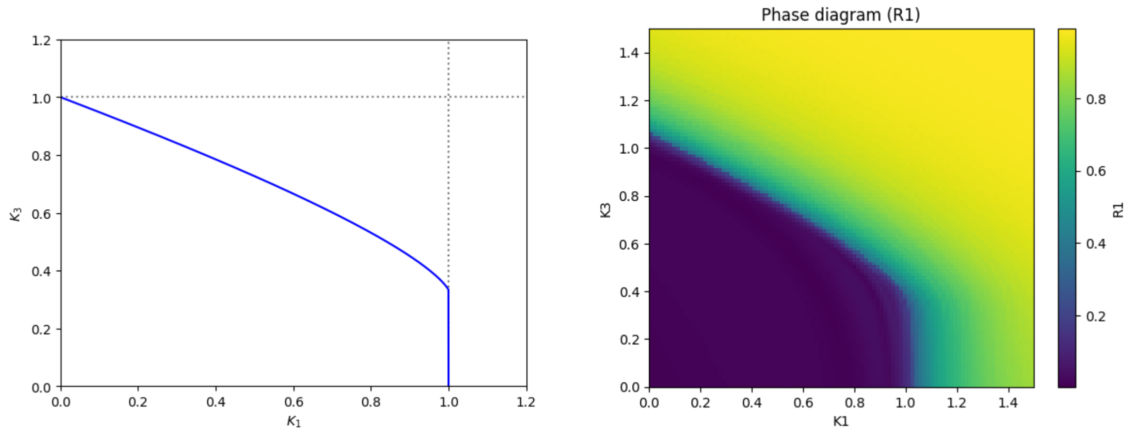
in our case, we have $a = n$ and $b = 2m + 1$. Upon simplification (carried out using the fact that $\lim_{x\to 0}\frac{\sin x}{x} = 1$) we obtain:

$$r_q = \left(\frac{K_q}{K_c}\right) q \cdot r_q \tag{5.11}$$

where q stands for an odd natural number. As in the classical Kuramoto model (which is, in fact, a special case of this model), we have the usual incoherent solution $r_q = 0 \quad \forall q$ and a non-trivial solution associated to a critical coupling, namely:

$$K_q^{(c)} = \frac{K_c}{q} \tag{5.12}$$

The correct way to interpret this result is to treat it like in the Kuramoto model, which means that the region $[0, K_1^{(c)}] \times [0, K_3^{(c)}] \times \dots$ in the $(K_1, K_3, \dots)$ parameter space is part of the incoherent phase and if we cross this critical hypersurface varying K_1 we end up in the 1^{st} harmonic phase[16]. In Fig.(21) we present both an analytical (obtained by using a KP-parametrization-like approach with all oscillators on the principal stable branch) and a numerical phase diagram for the simple case of the $1^{\text{st}} + 3^{\text{rd}}$ harmonics model.



(a) Analytical diagram

(b) Numerical diagram

Figure 21: Phase diagram of the $1^{\text{st}} + 3^{\text{rd}}$ harmonics model with all oscillators on the principal stable branch. All axes are normalized by the usual critical coupling K_c. On the analytical diagram, the blue line separates the disordered phase from the ordered one while the gray dotted lines are the linear instability thresholds of the incoherent state. We can find good accordance with the numerical diagram, which has been obtained by initializing all phases to a common value (both to assure $S(\omega) \sim 1$ synchronization and to escape a priori the $R = 0$ stable solution).

This phase transition is in fact a continuous one, so we are motivated to look for the critical exponent β. This can be calculated simply by expanding $g(\omega)$ up to second order in the self-consistent equation. If we now rewrite the mean-field couplings using hyperspherical coordinate with radius R such that:

$$R = \sqrt{\sum_{m=0}^{M} (K_{2m+1} r_{2m+1})^2}$$

we obtain a structure which is schematically:

$$r_n = \left(\frac{K_n}{K_c}\right) n r_n + R^3 g''(0) \left(\frac{\pi}{16}\right) \sum_{m,r,s} \alpha_{m,r,s} \Im_{n,m,r,s} \tag{5.13}$$

where $\alpha_{m,r,s}$ are constants involving the couplings $\{K_{2j+1}\}$ and the angular coordinates in parameter space, $\mathcal{J}_{n,m,r,s}$ is a sum of Kronecker's deltas involving the n, m, r, s indices and said indices are to be understood as positive odd numbers. From this relation, the scaling law can be stated straightforwardly as $R \sim \mu^{\frac{1}{2}}$ where μ is a distance from the critical hypersurface in coupling space. Thus, we can set the critical exponent to be $\beta = \frac{1}{2}$. Notice that the spirit of this calculation holds for other harmonic phases of the model as well since it only relies on the absence of the drifting term, though a proper calculation for the other phases should take into account the partition parameters $\{p_j\} \in \mathbf{\Delta}^\sigma$ where Δ^n is the n-th dimensional simplex and $\sigma + 1$ is the harmonic which rules the considered phenomenology.

5.3 Generic scaling of the order parameter

The final result we want to prove is that the scaling of the order parameters near $R = 0^+$ is generically ruled by a power law with exponent $\beta = 1$. The order parameter R is defined as:

$$R = \sqrt{\sum_{m=1}^{M} (K_m r_m)^2} \tag{5.14}$$

and, as we have observed for the Biharmonic model: $R = 0 \iff r_m = 0 \quad \forall m$. We also parametrize the mean-field couplings using hyperspherical coordinates, which we write symbolically as:

$$K_m r_m = R \cdot f_m(\mathbf{u}) \tag{5.15}$$

where $\mathbf{u}$ is the vector containing all angular parameters.

We now need to evaluate the self-consistent equations for $R \to 0^+$. To do so, let us assume to be in an "effective" 1$^{\text{st}}$ harmonic phase, that is all oscillators are locked on the stable branch defined by $\theta \in [-x_M, +x_M]$ where $\theta = x_M$ is the $\mathbf{u}$-dependent coordinate of the maximum which is nearest to the origin of the mean-field coupling function. We will also refer to the value of said maximum as $K_1 r_1 \mathcal{M}(\mathbf{u})$.

By the usual δ-integration, we can write the locked contribution as:

$$\langle e^{in\theta} \rangle_{lock} = \int_{-x_M}^{+x_M} d\theta \cos{(n\theta)} g\left(R \sum_{m=1}^{M} f_m(\mathbf{u}) \sin{(m\theta)}\right) \left(R \sum_{m=1}^{M} m f_m(\mathbf{u}) \cos{(m\theta)}\right)$$

when taking the $R \to 0^+$ limit, the integrand become simply a sum of products of trigonometric functions, which can be analytically integrated. We write the result schematically as:

$$\langle e^{in\theta} \rangle_{lock} = \mathcal{A}_n(\mathbf{u}) R + \mathcal{B}_n(\mathbf{u}) R^3 \tag{5.16}$$

This functional form is obviously due to the expansion of $g(\omega)$, which is even, up to second order in ω.
We now turn to the unlocked contribution. We define the ratio $\Phi_m(\omega)$ as:

$$\Phi_n(\omega) = \frac{\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{\omega - \sum_{m=1}^{M} K_m r_m \sin(m\theta)}}{\int_0^{2\pi} d\theta \frac{1}{\omega - \sum_{m=1}^{M} K_m r_m \sin(m\theta)}}$$

such that we can write the drifting part as:

$$\langle e^{in\theta} \rangle_{drift} = 2 \int_{K_1 r_1 \mathcal{M}(\mathbf{u})}^{+\infty} d\omega g(\omega) \Phi_n(\omega)$$

This is analogous to the discussion of the drifting contribution in the Biharmonic model, and the same conclusions hold provided that $\Phi_n(\omega)$ has the same asymptotic structure of the Biharmonic analog. This is indeed the case, in fact $\Phi_n(\omega \to K_1 r_1 \mathcal{M}(\mathbf{u}))$ is well defined and by the same measure-accumulation argument it is equal to $\cos(n x_M)$; on the other hand, if we expand for large frequencies the numerator up to second order and the denominator up to zeroeth order in ω, we obtain:

$$\begin{aligned}
\Xi_n(\omega) &= \int_0^{2\pi} dx \cos(nx) \left(\sum_{m=1}^{M} K_m r_m \sin(mx) \right)^2 \\
&= \sum_{m=1}^{M} \sum_{s=1}^{M} (K_m r_m)(K_s r_s) \int_0^{2\pi} dx \cos(nx) \sin(mx) \sin(sx) = \\
&= \sum_{m=1}^{M} \sum_{s=1}^{M} (K_m r_m)(K_s r_s)(\delta_{n,m+s} + \delta_{n,|m-s|}) \frac{\pi}{2} = \\
&= \sum_{m=1}^{M} \sum_{s=1}^{M} (K_m r_m)(K_s r_s)(\delta_{n,m+s} + \delta_{s,n+m} + \delta_{m,n+s}) \frac{\pi}{2} = \\
&= \pi \left(\sum_{m=1}^{M-n} (K_m r_m)(K_{m+n} r_{m+n}) + \frac{1}{2} \sum_{m=1}^{n-1} (K_m r_m)(K_{n-m} r_{n-m}) \right)
\end{aligned}$$

which means that the asymptotic expansion for $\Phi_n(\omega)$ is:

$$\Phi_n(\omega) \sim \frac{\Xi_n}{2\pi\omega^2} = \frac{\sum_{m=1}^{M-n} (K_m r_m)(K_{m+n} r_{m+n}) + \frac{1}{2} \sum_{m=1}^{n-1} (K_m r_m)(K_{n-m} r_{n-m})}{2\omega^2}$$

Therefore, the same argument of the Biharmonic model holds, which means that we can overall write the unlocked contribution as:

$$\langle e^{in\theta} \rangle_{drift} = \mathcal{C}_n(\mathbf{u}) R + \mathcal{D}_n(\mathbf{u}) R^2 \tag{5.17}$$

We can finally write down the final expansion as:

$$r_n = \frac{\mathcal{K}_n(\mathbf{u})}{\mathcal{K}_n^{(c)}(\mathbf{u})} R + \mathcal{D}_n(\mathbf{u}) R^2 \tag{5.18}$$

Where $\mathcal{K}_n(\mathbf{u})$ is a function of the couplings $\{K_m\}$ and $\mathcal{K}_n(\mathbf{u}) = \mathcal{K}_n^{(c)}(\mathbf{u})$ defines the critical hypersurface in coupling space. Defining a distance μ from this hypersurface, rewriting formally the order parameter as $r_n = R\psi_n(\mathbf{u})$ and simplifying R from both sides of the equation (since $R = 0$ is simply the incoherent state), we easily obtain the scaling relation:

$$R \sim \mu \tag{5.19}$$

which, when inverting the relation expliciting the order parameters, means that the scaling exponent is simply $\beta = 1$[18]. However, this is not sufficient to conclude that the global phase transition of the model is a continuous one: this branch might simply be an unstable one for a certain R interval, like we have verified in the Biharmonic model.

6 Conclusions

In this dissertation, we have investigated the effects of higher harmonic interactions on the collective dynamics of populations of coupled phase oscillators. Starting from the classical Kuramoto paradigm, we have considered progressively more general coupling functions and studied how the introduction of additional harmonic contributions modifies the synchronization properties of the system.

As a preliminary step, we reviewed the main analytical results of the Kuramoto model and discussed the role played by harmonic couplings. In particular, we showed how single-harmonic interactions of arbitrary order preserve many of the properties of the original model through a suitable phase transformation, while simultaneously introducing clustered synchronized states.

The core of this work was devoted to the Biharmonic model, representing the simplest genuinely nontrivial extension of the standard Kuramoto interaction. Through the analysis of finite-dimensional systems, we characterized the equilibrium structure of the model and identified the mechanisms responsible for the onset of different harmonic synchronizations, providing an insight into the bifurcation structure generated by the competition between first- and second-harmonic couplings.

The investigation was then extended to the thermodynamic limit, where a mean field description becomes rigorous. Within this framework, we derived the corresponding self-consistency equations and analyzed the phase diagram of the model. A central result of the thesis is the emergence of a discontinuous phase transition between the disordered phase and the synchronized one, unlike in the classical Kuramoto model. In addition, the presence of multiple harmonic contributions has been shown to enrich the phenomenology substantially, leading to multistability, clustered states and a more intricate organization of the phase space compared to the standard Kuramoto model.

Finally, we generalized the discussion to polyharmonic coupling functions. This broader perspective allowed us to identify several features that transcend the specific Biharmonic case and appear to be general properties of higher harmonic interactions. In particular, the stability of the incoherent state and the scaling properties of the branching from incoherence were shown to depend sensitively on the harmonic content of the coupling function, which makes the drifting contribution in the self-consistent equation nontrivial. These results highlight the fact that the critical behavior of coupled oscillators systems cannot always be understood solely within the framework of the original Kuramoto model; though, using the Odd-multiple model as a case study, we have proved that this reduction might occur for specific structures of the coupling.

This work has brought to light three modest original contributions, which are specific to the Biharmonic model. The first one is the theorem about the synchronized microstates in the identical-oscillators case, which allowed us to rigorously characterize the stability of the solutions to the self-consistent equations. The second contribution is the characterization of the critical curve separating the disordered phase from the synchronized one in the full model: we derived an analytical justification for the presence of a different critical line from the $R = 0^+$ one, we characterized said line via a strict condition on the derivative of the order parameters with respect to the couplings and finally we have shown that the Kuramoto scaling holds on this critical line. The last contribution regards the calculation of the correlation pseudocritical exponent on the $R = 0^+$ line, which rules the finite-size corrections to the critical curve delimiting the hysteresis region.

Possible future research directions concern the rigorous characterization of the indicator function $S(\omega)$, which is deeply connected to the dynamics of the system, in both single-harmonic models and in the Biharmonic model, and the investigation of the nature of the transition from disorder to synchronization for multi-harmonic-sine models, which is suggested to be generically a discontinuous one, with the $\beta = 1$ scaling we have derived related to an inaccessible branch of a hysteretic region in the phase diagram. Understanding these questions would contribute to a more complete theory of synchronization in systems with complex interaction structures and further clarify the role played by higher harmonics in shaping collective behavior.

A Hypergeometric functions

In this section we will sketch the derivation of the formulation of $\Phi_m(x)$, as defined in equation (4.23), in terms of hypergeometric functions. We will consider the denominator as an example.

$$D(x,u) = \int_0^{2\pi} \frac{d\theta}{x - z(\theta,u)} = \frac{1}{x}\int_0^{2\pi} \frac{d\theta}{1 - \frac{z(\theta,u)}{x}} = \frac{1}{x}\int_0^{2\pi} d\theta \sum_{n=0}^{\infty} \left(\frac{z(\theta,u)}{x}\right)^n = \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \int_0^{2\pi} d\theta (z(\theta,u))^n$$

Therefore the main object we need to calculate is:

$$I_n = \int_0^{2\pi} d\theta (\sin u \sin\theta + \cos u \sin(2\theta))^n$$

Notice that by symmetry the integral vanishes for n odd, therefore it is sufficient to consider the case $n = 2m$. We perform the following change of variable:

$$\begin{cases} x = \cos\theta \\ dx = \sqrt{1-x^2}d\theta \end{cases}$$

so that:

$$I_{2m} = 2\int_{-1}^{+1} dx (1-x^2)^{m-\frac{1}{2}} (\sin u + 2\cos u x)^{2m}$$

Applying the binomial theorem, we can rewrite it as:

$$\begin{aligned}
I_{2m} =& 2\sum_{j=1}^{m} \binom{2m}{2j} (\sin u)^{2(m-j)} (2\cos u)^{2j} \int_0^1 dx (1-x^2)^{m-\frac{1}{2}} x^{2j} = \\
=& \sum_{j=1}^{m} \binom{2m}{2j} (\sin u)^{2(m-j)} (2\cos u)^{2j} \int_0^1 dt (1-t)^{m-\frac{1}{2}} t^{j-\frac{1}{2}} = \\
=& \Gamma\left(m+\frac{1}{2}\right) \sum_{j=0}^{m} \binom{2m}{2j} \frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(j+m+1)} (\sin u)^{2(m-j)} (2\cos u)^{2j} = \\
=& \frac{\sqrt{\pi}\Gamma\left(m+\frac{1}{2}\right)(\sin u)^{2m}}{m!} \sum_{j=0}^{m} \binom{2m}{2j} \frac{\left(\frac{1}{2}\right)_j}{(m+1)_j} (4\cot^2 u)^j = \\
=& \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} (\sin u)^{2m} \sum_{j=0}^{m} \frac{(-m)_j \left(\frac{1}{2}-m\right)_j}{j!(m+1)_j} (4\cot^2 u)^j = \\
=& (\sin u)^{2m} B\left(\frac{1}{2}, m+\frac{1}{2}\right) {}_2F_1\left(-m, \frac{1}{2}-m, m+1; 4\cot^2 u\right)
\end{aligned}$$

where $\Gamma(x)$ denotes Euler's Gamma function and $(k)_j$ denotes the Pochhammer's symbol. In the derivation we have also employed the Legendre-Pochhammer reflection property of Pochhammer's symbols.

B Different indicator-function values

In this section we will briefly present some basic results regarding how the L_2 line in the phase diagram changes for different choices of constant indicator function $S(\omega) = p$. In particular, in the following figure we show the plots of different L_2 lines, as defined in Section (4.5), corresponding to p chosen from the set $\left\{1, \frac{4}{5}, \frac{3}{5}, \frac{1}{2}, \frac{2}{5}, \frac{1}{5}, 0\right\}$:

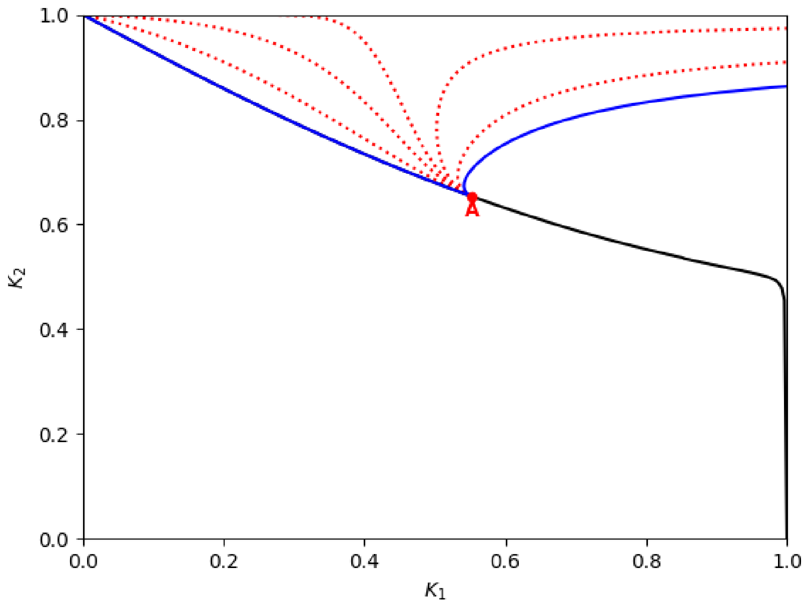


Figure 22: Different L_2 lines corresponding to the p values mentioned in the text. The axes are normalized by K_c. In particular, the black line is the p-independent part of each L_2 line ($\gamma < \frac{1}{2}$), the point $A \approx (0.55, 0.65)$ separates the p-independent section from the p-dependent ones, the blue line on the left corresponds to $p = 1$ and the one on the right to $p = 0$, between them are all the intermediary p values as mentioned in the text.

We can make two observations:

- The line L_1, which has not been plotted for the sake of clarity, is, in principle, p-dependent; however, it can be shown that the one corresponding to $p = 1$ (i.e. the one we investigated in Section (4.5)) is the one which is nearest to the origin, meaning that it is the true line separating the disordered phase from the synchronized ones.
- Lines with $p < p^*$ reach the point $(0, 1)$ within the $[0, 1]^2$ region, while lines with $p > p^*$ diverge to $K_1 = +\infty$ and reach the same point from the left, looping back from $K_1 = -\infty$, where $p^* \in [0.5, 0.6]$ is a special value separating the divergent lines from the regular ones, presumably making $r_1 = 0$ and thus simplifying the mean-field coupling function to a purely 2^{nd} harmonic one. The reason why this special value exceeds 0.5 is due to the fact that the region between the maxima $M(\gamma)$ and $m(\gamma)$ is not p-dependent, therefore its contribution is constant and adds to the locked contribution on the principal stable branch.

C Qualitative Biharmonic behavior for negative couplings

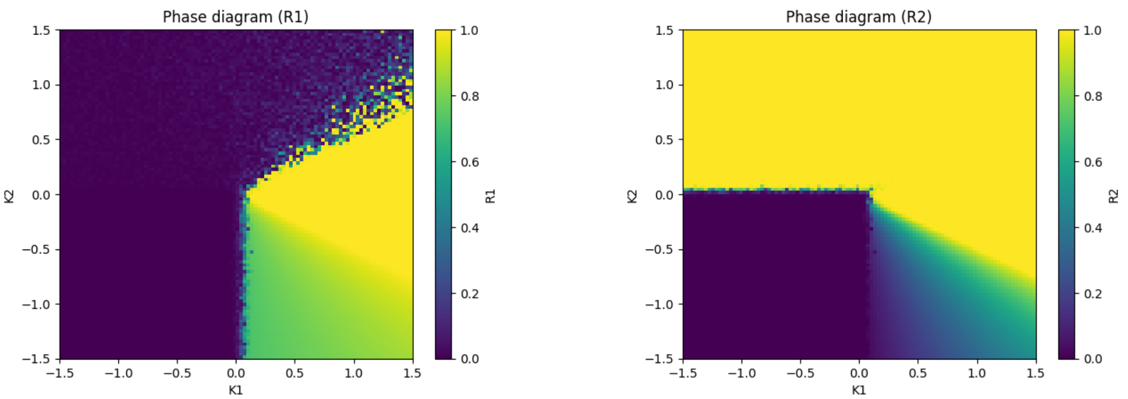
In this section, we will discuss qualitatively the behavior of the Biharmonic model when the couplings K_1 and K_2 are negative. To carry out this discussion, we will refer to the identical-oscillators case.

Let us start from the fact that $K_1 = 0$ and $K_2 = 0$ are the linear instability thresholds for the disordered state: if we take both $K_1 < 0$ and $K_2 < 0$ then the incoherent state is attracting by the fact that this corresponds to the time-inverted dynamics of the positive-couplings case.

Now we take $K_1 < 0$ and $K_2 > 0$. In this regime, the first-harmonic part of the coupling favors disorder while the second-harmonic part favors antiphase alignment. This interplay results in the display of 2^{nd} harmonic order with extremely low values of r_1 (which is symptomatic of the repulsive first harmonic).

The last regime is $K_1 > 0$ and $K_2 < 0$. Here the first harmonic favor in-phase alignment while the second harmonic favors disorder, which complicates the investigation by introducing non trivial competition. The resulting phenomenology is dictated by the presence of a critical line along $K_2 = -\frac{K_1}{2}$: this follows from the stability analysis of the in-phase configuration, which becomes unstable at $c = -\frac{1}{2}$ (see equation (3.13)). Therefore, we can identify two phases in this regime: the 1^{st} harmonic phase, which is connected with the one discussed for positive couplings, and a peculiar phase where both order parameters are non vanishing and less than 1 and where the stable equilibria are the non trivial zeros of the fixed-point equation.

We summarize what we have found with the following picture, showing a numerical investigation of the system we have studied.



(a) r_1 phase diagram

(b) r_2 phase diagram

Figure 23: Numerical phase diagrams for the identical-oscillator Biharmonic model with negative couplings.

Finally, let us remark that we expect this description to match the general case only in the large coupling limit. More thoroughly, we expect the region where both couplings are negative to have the same properties, while we expect the two region with only one negative coupling to also contain a portion of the disordered phase, which connects with the disordered phase of the first and third quadrants of the coupling space. We can see a partial confirmation of this in [4] for the $K_1 < 0$ region.

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