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Uniform equilibria in quitting games

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Alla mia famiglia

Contents

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Introduction

The existence of a uniform ε-equilibrium for general stochastic games with three or more players is an open question. The aim of this thesis is to investigate an example of a four-player quitting game that does not have a stationary undiscounted ε-equilibrium and the relation with the fact that in a stochastic game there does not need to exist a stationary uniform ε-equilibrium. In the last part a new class of games is introducted and studied.

In the first two chapters classic definitions and standard results for stochastic games are stated and commented.

In the following one, the definitions are restricted to the class of quitting games with some new results.

In the fourth chapter there is the detailed study of the four-player example that does not have a stationary ε-equilibrium.

In the last chapter, I introduce a new class of games, which may be useful to find counterexamples to the open problem or; at least, to understand it in a deeper way. Then, one can find the proof that every game in a subclass of this new kind of games always has a stationary 0-equilibrium.

Stochastic games

1.1 The model

Stochastic games generalize Markov Decision problems to a several number of decision makers, called players. Each player influences the evolution of the state process and the payoff of the players.

Definition 1.1. A *stochastic game* is the datum of a tuple

$$\Gamma = \left(I, S, \left(A^i(s)\right)_{s\in S}^{i\in I}, q, r\right)$$

where:

- $I = \{1, 2, \dots, N\}$ is a finite set of players. Eventually, players can be denoted by alphabetical letters (i.e. $I = \{A, B, C, \dots\}$).
- S is a finite set of states.
- For every player $i \in I$ and every state $s \in S$, $A^i(s)$ is a finite set of actions avaiable to player i at the state s. Let's introduce the notation for the set of all action profiles at state s:

  $$A(s) = \Pi_{i\in I} A^i(s)$$

  and for the set of all action profiles at all states:

  $$SA = \{(s, a) : s \in S, a \in A(s)\}$$

- $q : SA \to \Delta(S)$ is a transition rule. Here $\Delta(S)$ denotes the set of probabilities on the set S.
- For every player $i \in I$, $r^i : SA \to \mathbb{R}$ is a payoff.

A stochastic game proceeds in the following way.
The game starts at a stage $s_1 \in S$ given. At each stage $t \geq 1$ the following happens:

- The current state s_t is announced to the players.
- Each player $i \in I$ chooses an action $a_t^i \in A^i(s_t)$. These choices are made simultaneously and independently.

- The action profile $a_t = (a_t^i)_{i \in I}$ is announced to all players.
- Each player $i \in I$ receives the corresponding payoff $r^i(s_t, a_t)$
- A new state $s_{t+1} \in S$ is picked according to the transition rule $q(\cdot \mid s_t, a_t)$ and the game proceeds.

It is worth to notice that stochastic games are also a generalization of repeated games where there is only one state an thus the transition rule is the trivial one.

1.2 Histories and strategies

For $t \in \mathbb{N}$ we can define the *set of histories of length* t that is

$$H_t := (SA)^{t-1} \times S$$

and we denote the *set of all histories* by:

$$H := \bigcup_{t \in \mathbb{N}} H_t$$

Finally, we define

$$H_\infty := (SA)^{\mathbb{N}}$$

that is the set of all *infinite histories* or *plays*.

For an history $\tilde{h}_t = (\tilde{s}_1, \tilde{a}_1, \ldots, \tilde{a}_{t-1}, \tilde{s}_t) \in H_t$ we can consider the cylinder $C(\tilde{h}_t) \subset H_\infty$ of length t centered in $\tilde{h}_t$ that is the collection of all plays that are starting by $\tilde{h}_t$, formally:

$$C(\tilde{h}_t) := \{h = (s_1, a_1, \ldots, s_t, a_t, \ldots) \in H_\infty : s_1 = \tilde{s}_1, a_1 = \tilde{a}_1, \ldots, s_t = \tilde{s}_t\}$$

On the set H_∞ we can define the algebra $\mathcal{H}_t$ spanned by the cylinder sets of length t.

Definition 1.2. A *strategy* of player $i \in I$ is a mapping σ^i that assigns to each history $h = (s_1, a_1, \ldots, a_{t-1}, s_t) \in H$ an element of $\Delta(A^i(s_t))$, called *mixed action*.

The set of all strategies of player i will be denoted by Σ^i.

A strategy σ^i of player i is *pure* if it is deterministic:

$$|\operatorname{supp}(\sigma^i(h_t))| = 1 \quad \forall h_t \in H$$

A strategy σ^i of player i is *stationary* if the mixed action assigned ad each history depends only on the current state. Formally, this means that $\sigma^i(h_t)$ is a function of s_t and independent of $(s_1, a_1, \ldots, s_{t-1}, a_{t-1})$.

Remark 1.1. Usually, it is convenient to identify a stationary strategy σ^i of player i with a vector $x^i \in \Pi_{s\in S}\Delta(A^i(s))$. Under this identification $x^i(s)$ is the mixed action that player i implements when the current state is s. Thus the set of stationary strategies of player i is identified with $X^i = \Pi_{s\in S}\Delta(A^i(s))$ that is a compact set.

Definition 1.3. A *strategy profile* is a vector $\sigma = (\sigma^i)_{i\in I}$ of strategies. The set of all strategy profile will be denoted by Σ.

Remark 1.2. The space of stationary strategy profiles can be identified with $X = \Pi_{i\in I}X^i$.

Remark 1.3. Every pair (s, σ), where $s \in S$ is a state and $\sigma \in \Sigma$ is a strategy profile, induces a probability distribution on the space of plays H_∞ equipped with the σ-algebra generated by finite cylinders. This probability measure is obtained thanks Caratheodory extension theorem since this (pre)measure is well defined on the set of cylinders and respects the hypothesis of this theorem thanks the structure of the cylinders. This measure is called $\mathbb{P}_{s,\sigma}$ and the corresponding expectation operator is denoted by $\mathbb{E}_{s,\sigma}$.

1.3 Absorbing games

This is a particular class of stochastic games. Quitting games are a subclass of absorbing games.

Definition 1.4. A *state* $s \in S$ is *absorbing* if for every action profile $a \in A(s)$ we have $q(\cdot \mid s, a) = 1$.

This means that a state is absorbing if once the play reaches that state, it gets stuck there and never leaves that state, no matter which action the players are choosing. Once the game reaches an absorbing state it reduces to a repeated game.

Remark 1.4. It is well known that repeated games admit an equilibrium (for instance the players are playing an equilibrium of the base game). So, if we are interested in the existence of an equilibrium, we can assume without lost of generality that once the game reaches an absorbing state the stream of payoffs is constant (equal to the equilibrium payoff of the base game). In other words, this means that to each absorbing state we can associate an absorbing payoff.

Definition 1.5. An *absorbing game* is a stochastic game Γ in which all states except one are absorbing.

Remark 1.5. Thanks to remark 1.4 in this kind of games it is reasonable to assume that the initial state is the non absorbing one, otherwise we will end up with a repeated game and those kind of games are well known.

Notice that during an absorbing game the state can change at most once along the play and we can think as if the game ends when an absorbing state is reached. This means that the only relevant part of strategy to compute the payoff is how to behave as long as the game remains in the initial state. It is worth to emphasize that if we denote with $s(0)$ the unique non-absorbing state a stationary strategy of a player $i \in I$ reduces to a probability measure in $\Delta(A^i(s(0)))$.

Example 1.1. Alessandra and Bruno during their final exam got some troubles. The supervisior thinks that at least one of those cheated during the exam. They get interrogated by the supervisior in two different rooms and the supervisors belives them both in the case one of them denounces the other. Both of them value most passing the exam rather then be seen as a snitch even though this leads to a penalization. If nobody denounces the other the supervisor will repeat the same question to both of them.

This situation can be described in the following way:

- There are two players: $I = \{A, B\}$
- There are 4 states: $S = \{s(1), s(2), s(3), s(4)\}$. Those are respectively:
  1. Nobody denounces the other.
  2. Alessandra denounces Bruno while he doesn't denounce Alessandra.
  3. Bruno denounces Alessandra while she doesn't denounce Bruno.
  4. Both of them denounce the other.
- In the state $s(1)$ both players have two avaiable action: to denounce the other (D) or not (ND). While in the other states they have only one possible action (just accept the payoff given by the consequence of their choice).
- Payoffs are chosen according their interest to pass the exam but with a penalization if they denounce the other.
- This is an absorbing game so the transition rule is deterministic.

B

A		ND	D
	ND	$0, 0_{(1,0,0,0)}$	$-1, 1_{(0,0,1,0)}$
	D	$1, -1_{(0,1,0,0)}$	$-2, -2_{(0,0,0,1)}$

$s(1)$

B

A	ND
D	$1, -1_{(0,1,0,0)}$

$s(2)$

B

	D
ND	$-1, 1_{(0,0,1,0)}$

$s(3)$

B

	D
D	$-2, -2_{(0,0,0,1)}$

$s(4)$

Figure 1 – We can represent the game played by Alessandra and Bruno in this picture. There are 4 states and we can assume that if a player denounces the other, the payoff of each player is constant in every following stage.

In the Figure 1 the transition probability is represented by a vector (p_1, p_2, p_3, p_4) where p_i represent the probability that the next state will be $s(i)$
To make the notation easier we can just represent the game using a $*$ in the payoff matrix corrisponding to absorbing states and we will represent the only state for which the choice of the actions isn't forced.

	ND	D
ND	$\circlearrowleft$	$-1, 1^*$
D	$1, -1^*$	$-2, -2^*$

Actually this is a quitting games. We will see that the action ND corresponds to the *continuing action*: while both of them are choosing to non-denounce the other (or to continue) the game repeats. Instead, D corresponds to *quitting action*; indeed after one of the two decides to denounce the other (or to quit) the game ends. The standard notation for games in this form will be the following:

		B	
		c^B	q^B
A	c^A	$\circlearrowleft$	$-1, 1$
	q^A	$1, -1$	$-2, -2$

Figure 2 – The standard representation of the quitting game played by Alessandra and Bruno

In the Figure 2 the circle means that if the two players decide to continue the game repeats, while if one of those decides to quit they will get the constant payoff indicated in the matrix in every following stage.

Payoffs and equilibria

Since a stochastic game can last for infinitely many stages there is a problem with the payoff of each player due to its divergent behavior. Assume a player is reciving a constant payoff of 1 at each stage, simply summing those up she would get a total payoff of $+\infty$. Obviously, it would be better to recive; for instance, a constant payoff of 2 at each stage but the sum gives the total payoff of $+\infty$ as if she would recive 1 at each stage. Thus, in this chapter we will see two different ways to solve this problem.

2.1 The discounted payoff

The first way to solve this problem is to weight each stage of the stochastic game. Fix a parameter $\lambda \in (0,1]$ that is called *discount factor*, this is measuring how money grows with time: one dollar today is worth $\frac{1}{1-\lambda}$ dollars tomorrow and $\frac{1}{(1-\lambda)^2}$ dollar the day after tomorrow and so on. So, the discount factor is a way to consider with the same importance $1-\lambda$ dollars today and one dollar tomorrow. This is pretty natural way considering that maybe a player prefers to get $1-\lambda$ dollars today to invest those somewhere else and get a dollar tomorrow.

Lower values of λ represent that players care more about the future rather than the present. Bigger values of λ represent that players care only about the present.

This method is relevant when the play continues indefinitely and usually when λ is small.

Definition 2.1. For every discount factor $\lambda \in (0,1]$, every player $i \in I$, every initial state $s \in S$ and every strategy profile $\sigma \in \Sigma$ the *λ-discounted payoff under strategy profile σ at the initial state s for player i* is

$$\gamma^i_\lambda(s;\sigma) := \mathbb{E}_{s,\sigma}\left[\lambda \sum_{k=1}^{\infty} (1-\lambda)^{k-1} r^i(s_k, a_k)\right]$$

Remark 2.1. The λ in front of the sum is the renormalization factor so that a player that recives a constant payoff of 1 at each stage, has the discounted payoff equals to 1 as well. Since there are finitely many stages and actions, for every player $i \in I$ the payoff function r^i is bounded and therefore γ^i_λ obeys to the same bound that is idependent of λ thanks to the renormalizing factor.

Using dominated convergence theorem (recall that the measure $\mathbb{P}_{s,\sigma}$ is finite and the boundedness of r^i) one gets:

$$\gamma^i_\lambda(s;\sigma) = \lambda \sum_{k=1}^{\infty} (1-\lambda)^{k-1} \mathbb{E}_{s,\sigma}\left[r^i(s_k, a_k)\right]$$

2.2 The T-stage payoff

This is another natural method to deal with the divergent nature of the payoff. This evaluation, conversely to the previous one, is relevant when the process lasts T stages and the main interest is for T big.

Definition 2.2. For every positive integer $T \in \mathbb{N}$, every player $i \in I$, every initial state $s \in S$, every strategy profile $\sigma \in \Sigma$, the T-*stage payoff* is:

$$\gamma^i_T(s;\sigma) := \mathbb{E}_{s,\sigma}\left[\frac{1}{T}\sum_{k=1}^{T} r^i(s_k, a_k)\right]$$

Remark 2.2. The T-stage payoff is the mean of the payoffs the player got in the first T stages. If a player recives a constant payoff of 1 at each stage, she has the T-stage payoff equals to 1 as well. Since there are finitely many stages and actions, for every player $i \in I$ the payoff function r^i is bounded and therefore γ^i_T obeys to the same bound that is idependent of T.

Since the sum is finite by linearity of $\mathbb{E}_{s,\sigma}$ one has:

$$\gamma^i_T(s;\sigma) = \frac{1}{T}\sum_{k=1}^{T} \mathbb{E}_{s,\sigma}\left[r^i(s_k, a_k)\right]$$

2.3 The θ-payoff

This is a generalization of the previous two method that could be useful in some cases. Given a sequence $\theta = (\theta_k)_{k=1}^{\infty}$ of real non-negative numbers that sums up to 1, every player $i \in I$, every initial state $s \in S$, every strategy profile $\sigma \in \Sigma$, the θ-*payoff* is:

$$\gamma^i_\theta(s;\sigma) := \mathbb{E}_{s,\sigma}\left[\sum_{k=1}^{\infty} \theta_k r^i(s_k, a_k)\right]$$

As done before we can assume that the number θ_k is the weight of interest given by the players to the stage k. If a player recives a constant payoff of 1 at each stage, she has the θ-payoff equals to 1 as well. Since there are finitely many stages and actions, for every player $i \in I$ the payoff function r^i is bounded and therefore γ^i_θ obeys to the same bound that is idependent of θ.

2.4 Uniform equilibrium

2.4.1 ε-equilibria

Roughly speaking, an ε-equilibrium in a stochastic game is a strategy profile such that if a player deviate from that strategy cannot gain more than ε considering any of the previous possible payoffs.

Definition 2.3. Let $\Gamma = (I, S, (A^i(s))^{i\in I}_{s\in S}, q, (r^i)_{i\in I})$ be a stochastic game, let $\varepsilon \geq 0$, let $s \in S$ and let $\lambda \in (0, 1]$. A strategy profile σ_* is a *λ-discounted ε-equilibrium at the initial state s* if for each player $i \in I$ and every strategy $\sigma^i \in \Sigma^i$ one has:

$$\gamma^i_\lambda(s; \sigma^i, \sigma^{-i}_*) \leq \gamma^i_\lambda(s; \sigma_*) + \varepsilon$$

Definition 2.4. Let $\Gamma = (I, S, (A^i(s))^{i\in I}_{s\in S}, q, (r^i)_{i\in I})$ be a stochastic game, let $\varepsilon \geq 0$, let $s \in S$ and let $T \in \mathbb{N}$. A strategy profile σ_* is a *T-stage ε-equilibrium at the initial state s* if for each player $i \in I$ and every strategy $\sigma^i \in \Sigma^i$ one has:

$$\gamma^i_T(s; \sigma^i, \sigma^{-i}_*) \leq \gamma^i_T(s; \sigma_*) + \varepsilon$$

Definition 2.5. Let $\Gamma = (I, S, (A^i(s))^{i\in I}_{s\in S}, q, (r^i)_{i\in I})$ be a stochastic game, let $\varepsilon > 0$ and let $s \in S$. A strategy profile $\sigma_* = (\sigma^i_*)_{i\in I}$ is a *uniform ε-equilibrium at the initial state s* if there exist $\lambda_0 \in (0, 1]$ and $T_0 \in \mathbb{N}$ such that the following conditions hold:

- For every $\lambda \in (0, \lambda_0)$ the strategy profile σ_* is a λ-discounted ε-equilibrium at the initial state s.
- For every $T \geq T_0$ the strategy profile σ_* is a T-stage ε-equilibrium at the initial state s.

A strategy profile σ_* that is uniformly ε-optimal at all initial states is a *uniform ε-equilibrium.*

The main interest in the concept of uniform equilibrium comes from its robustness. If a uniform equilibrium exists, then by playing this strategy profile the players ensure that no player can gain more than ε by deviating, regardless the length of the game (provided it is sufficiently long) and regardless of the value of the discount factor (provided it is sufficiently low).

2.4.2 One condition implies the other

In this section we will see that the second condition in the definition of uniform ε-equilibrium implies the first one, there exist some examples proving the other implication is false.

Now fix $\varepsilon > 0$ and assume that there exists a strategy profile σ_* and $T_0 \in \mathbb{N}$ that for every $T \geq T_0$ is a T-stage ε-equilibrium at the initial state $s \in S$, this means that for each player $i \in I$ and every strategy $\sigma^i \in \Sigma^i$ one has:

$$\gamma_T^i(s; \sigma^i, \sigma_*^{-i}) \leq \gamma_T^i(s; \sigma_*) + \varepsilon$$

for the purpose of clearness for a fixed player i and a fixed strategy $\sigma^i \in \Sigma^i$ define:

$$x_k := \mathbb{E}_{s,\sigma_*}[r^i(s_k, a_k)] \qquad\qquad y_k := \mathbb{E}_{s,\sigma^i,\sigma_*^{-i}}[r^i(s_k, a_k)]$$

Then thanks previous remarks one can write:

$$\|y_k\| \leq \|r\|_\infty \qquad\qquad \|x_k\| \leq \|r\|_\infty$$

$$\gamma_\lambda^i(s; \sigma^i, \sigma_*^{-i}) = \lambda \sum_{k=1}^{\infty} (1-\lambda)^{k-1} y_k \qquad\qquad \gamma_\lambda^i(s; \sigma_*) = \lambda \sum_{k=1}^{\infty} (1-\lambda)^{k-1} x_k$$

$$\gamma_T^i(s; \sigma^i, \sigma_*^{-i}) = \frac{1}{T} \sum_{k=1}^{T} y_k \qquad\qquad \gamma_T^i(s; \sigma_*) = \frac{1}{T} \sum_{k=1}^{T} x_k$$

So the hypothesis becomes that for every $T \geq T_0$ one gets:

$$\frac{1}{T} \sum_{k=1}^{T} y_k \leq \frac{1}{T} \sum_{k=1}^{T} x_k + \varepsilon$$

we should prove the existence of a $\lambda_0 \in (0, 1]$ such that for every $\lambda \in (0, \lambda_0)$ it holds:

$$\lambda \sum_{k=1}^{\infty} (1-\lambda)^{k-1} y_k \leq \lambda \sum_{k=1}^{\infty} (1-\lambda)^{k-1} x_k + \varepsilon$$

Let's start noticing that one can rewrite y_k in the following way:

$$y_k = \sum_{l=1}^{k} y_l - \sum_{l=1}^{k-1} y_l$$

Then, we get:

$$\begin{aligned}
\gamma^i_\lambda(s;\sigma^i,\sigma^{-i}_*) &= \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} y_k = \\
&= \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l - \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k-1} y_l = \\
&= \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l - (1-\lambda) \sum_{k=0}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l = \\
&= \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l - (1-\lambda) \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k} y_= \\
&= (1-(1-\lambda)) \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l = \\
&= \sum_{k=1}^{\infty} \lambda^2(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l = \\
&= \sum_{k=1}^{\infty} \lambda^2(1-\lambda)^{k-1} k \left(\frac{1}{k}\sum_{l=1}^{k} y_l\right) = \\
&= \sum_{k=1}^{\infty} \lambda^2(1-\lambda)^{k-1} k\, \gamma^i_k(s;\sigma^i,\sigma^{-i}_*) = \\
&= \sum_{k=1}^{T_0-1} \lambda^2(1-\lambda)^{k-1} k\, \gamma^i_k(s;\sigma^i,\sigma^{-i}_*) + \sum_{k=T_0}^{\infty} \lambda^2(1-\lambda)^{k-1} k\, \gamma^i_k(s;\sigma^i,\sigma^{-i}_*)
\end{aligned}$$

But, using the hypothesis in the second term of the sum we obtain:

$$\sum_{k=1}^{T_0-1} \lambda^2(1-\lambda)^{k-1} \sum_{l=1}^{k} y_l + \sum_{k=T_0}^{\infty} \lambda^2(1-\lambda)^{k-1} k \underbrace{\gamma^i_k(s;\sigma_*)}_{\left(\frac{1}{k}\sum_{l=1}^{k} x_l\right)} + \sum_{k=T_0}^{\infty} \lambda^2(1-\lambda)^{k-1} k\, \varepsilon$$

Now, let's add and subctract the term

$$\sum_{k=1}^{T_0-1} \lambda^2(1-\lambda)^{k-1} k \left(\frac{1}{k}\sum_{l=1}^{k} x_l\right)$$

To deduce that:

$$\gamma^i_\lambda(s;\sigma^i,\sigma^{-i}_*) \leq \overbrace{\sum_{k=1}^{\infty}\lambda(1-\lambda)^{k-1}x_k}^{\gamma^i_\lambda(s;\sigma_*)} + \varepsilon \overbrace{\sum_{k=T_0}^{\infty}\lambda^2(1-\lambda)^{k-1}k}^{\leq 1} + \\ + \underbrace{\sum_{k=1}^{T_0-1}\lambda^2(1-\lambda)^{k-1}k\left(\frac{1}{k}\sum_{l=1}^{k}y_l - \frac{1}{k}\sum_{l=1}^{k}x_l\right)}_{\leq 2\,\lambda^2(T_0-1)^2\|r\|_\infty}$$

The estimate on the last term is obtained thanks to those on x_l and y_l, thus if λ in small enought such that $\lambda^2(T_0-1)^2\|r\|_\infty < \varepsilon$ then we have that:

$$\gamma^i_\lambda(s;\sigma^i,\sigma^{-i}_*) \leq \gamma^i_\lambda(s;\sigma_*) + 3\varepsilon$$

The reasoning holds for every player $i \in I$ and every strategy $\sigma^i \in \Sigma^i$ at the initial state s. This is exactly what we were looking for.

Quitting games

Quitting games are one of the easiest class of stochastic games, in this chapter we restrict all previous definition to this subclass of stochastic games. They are stochastic games in which players have only two possible actions: to continue or to quit. If all of them decide to continue then the game repeats; otherwise, if at least one player quits the game ends and every player recives a payoff depending on the *quitting set* that is the set of player that decided to quit at that stage.

3.1 The model

Let's start giving a more specific definition of quitting game. Notice that it is a restriction of the one given for stochastic games, as it will be for all the following definitions.

Definition 3.1. A *quitting game* is a pair

$$\big(I, (r_S)_{\emptyset \subset S \subseteq I}\big)$$

where:

- $I = \{1, \dots, N\}$ is a finite set of players.
- for every $\emptyset \subset S \subseteq I$ called the *quitting set* there is a corresponding payoff $r_S \in \mathbb{R}^N$.

The game is a sequential game that proceeds as follows. The set of stages is $\mathbb{N}$ that is the set of all positive integers. At every stage, each player chooses an action: either to *continue* or to *quit*. Let's denote with S the subset of the players who decide to quit. If $S \neq \emptyset$, then the game terminates and each player i recives the payoff r^i_S. If $S = \emptyset$, the games continues to the next stage and repeats. If the game never terminates, each player gets 0.
Let's denote the two actions of player i by $\{c^i, q^i\}$. A *strategy* for player i is a function $\mathbf{x}^i = (x^i_n)_{n \in \mathbb{N}}$ where x^i_n is the probability for player i quits at stage n, provided the game has not terminated before that stage. If $x^i_n = 0$, then at stage n player i plays the pure action c^i, that is to *continue*, while if $x^i_n = 1$, then at stage n player i plays the pure action q^i, that is to *quit*. In particular, we will denote with $\mathbf{c}^i$ (resp. $\mathbf{q}^i$) the strategy of player i by which she always continues (resp. quits).

Remark 3.1. As for general stochastic games, a *strategy profile* $\mathbf{x}$ is a vector of strategy, one for each player. A profile $\mathbf{x} = (x_n)_{n\geq 1}$ induces a probability distribution $\mathbb{P}_{\mathbf{x}}$ and then the expected operator $\mathbb{E}_{\mathbf{x}}$. Notice that this probability distribution does not depend of the initial state since for quitting games, and for absorbing game in gerneral, is natural to start from the non absorbing state.
In quitting games the set of states is given by $Z = \mathcal{P}(I)$ while the space of actions is $A = \{0,1\}^N$ for the non absorbing state. If the game reaches an absorbing state (that means that at least one player quits) then we can assume that every player have only one possible action and that every player is forced to play the last action chosen so that every player will get the same payoff in every following stage. We will usually say that the game ends because when an absorbing state is reached the game becomes a simple repeated game.
As for absorbing games the transition probabilities are even deterministic.
The payoff function is given by $\tilde{r} : A \to \mathbb{R}^N$ that maps $a_S \mapsto \tilde{r}(a_S) := r_S$ where the profile a_S is the profile where players in the quitting set S chose to quit.
Notice that there is no discounting in this model.

Example 3.1. A tipical quitting game form is the one in the example 1.1. The tipical quitting quitting game reprersentation (for a two player quitting game) is the following:

		Player 2	
		c^2	q^2
Player 1	c^1	$\circlearrowleft$	$r^1_{\{2\}}, r^2_{\{2\}}$
	q^1	$r^1_{\{1\}}, r^2_{\{1\}}$	$r^1_{\{1,2\}}, r^2_{\{1,2\}}$

Figure 3 – The standard representation of a two players quitting game

Definition 3.2. Let $\mathbf{x} = (x^i_n)^{i\in I}_{n\geq 1}$ be a strategy profile in a quitting game. The strategy $\mathbf{x}^i = (x^i_n)_{n\geq 1}$ for player i is said to be:

- *pure* if $x^i_n \in \{0,1\}$ for every n.
- *cyclic* if there exists a $k_0 \in \mathbb{N}$ such that $x^i_{n+k_0} = x^i_n$ for every n.
- *stationary* if $x^i_n = x^i_1$ for every n.

3.2 The underlying stochastic process

Given a quitting game $\Gamma = \big(I, (r_S)_{\emptyset \subset S \subseteq I}\big)$, $Z = \mathcal{P}(I)$ the corrisponding state space and $A = \{0,1\}^N$ where N is the number of players one obtains a measurable space

$$(\Omega, \mathcal{A})$$

where

$$\Omega := (Z \times A)^{\mathbb{N}}$$

and the sigma algebra is given by

$$\mathcal{A} := \mathcal{P}(Z) \otimes \mathcal{P}(A) \otimes \mathcal{P}(Z) \otimes \mathcal{P}(A) \otimes \dots$$

As noticed before we can assume that the initial state is $S = \emptyset$, then given a strategy profile $\mathbf{x}$ one obtains a unique probability measure $\mathbb{P}_{\mathbf{x}}$ (defined on cylinders and then extended) on $(\Omega, \mathcal{A})$. One also obtains a stochastic process $(X_n, Y_n)_{n \in \mathbb{N}}$ with values in $(Z \times A)$, where:

$$X_n(\omega) = X_n((S_1, a_1, S_2, a_2, \dots)) := S_n$$
$$Y_n(\omega) = X_n((S_1, a_1, S_2, a_2, \dots)) := a_n$$

X_n denotes the random state of the system at time $n \in \mathbb{N}$, $\omega \in \Omega$. Y_n denotes the random action taken at time $n \in \mathbb{N}$, $\omega \in \Omega$.
Starting from the strategy profile one can also define a *stopping time*

$$\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$$

defined by

$$\tau(\omega) := \inf\{n \in \mathbb{N} : Y_n(\omega) \neq (0, \dots, 0)\}$$

concerning the filtration $(\mathcal{U}_n)_{n \in \mathbb{N}}$ with $\mathcal{U}_n := \sigma\{Y_k : 1 \leq k \leq n\}$
The stopping time identifies the state at which the game stops.

3.3 Expected payoff and equilibria

Definition 3.3. Let $\Gamma = \big(I, (r_S)_{\emptyset \subset S \subseteq I}\big)$ be a quitting game and $\mathbf{x}$ a strategy profile. The expected payoff of the game is given by:

$$\gamma(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}\left[\tilde{r}(Y_\tau)\mathbb{1}_{\{\tau < \infty\}}\right]$$

Remark 3.2. Using the definition of $\mathbb{P}_{\mathbf{x}}$ one gets:

$$\begin{aligned}
\gamma(\mathbf{x}) &= \sum_{n\in\mathbb{N}} \mathbb{P}_{\mathbf{x}}(\tau = n)\, \mathbb{E}_{\mathbf{x}}\left[\, \tilde{r}(Y_n) \mid \tau = k \,\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{P}_{\mathbf{x}}(\tau > n-1)\, \mathbb{E}_{\mathbf{x}}\left[\, \tilde{r}(Y_n) \,\right] \\
&= \sum_{n\in\mathbb{N}} \left[\prod_{\substack{k=1 \\ j\in I}}^{n-1} (1 - x_k^j) \cdot \sum_{\emptyset\neq S\subseteq I} \left(\prod_{h\in S} x_k^h \right) \left(\prod_{l\in I\setminus S} (1 - x_k^l) \right) r_S \right]
\end{aligned}$$

This means that under the strategy profile $\mathbf{x}$ player i will get the i-th component of that vector, this means $\gamma^i(\mathbf{x})$.

Remark 3.3. Notice that if the strategies chosen by each player are stationary and we denote the strategy profile as $\mathbf{x} = (x^1, \ldots, x^N)$ then the formula simplifies into:

$$\gamma(\mathbf{x}) = \frac{1}{1 - \prod_{j\in I}(1 - x^j)} \sum_{\emptyset\neq S\subseteq I} \left(\prod_{h\in S} x^h \right) \left(\prod_{l\in I\setminus S} (1 - x^l) \right) r_S$$

This is exactly the formula one would obtain using the implicit technique, indeed if the strategies are stationary one may write:

$$\gamma(\mathbf{x}) = \left(\prod_{j\in I} (1 - x^j) \right) \gamma(\mathbf{x}) + \sum_{\emptyset\neq S\subseteq I} \left(\prod_{h\in S} x_k^h \right) \left(\prod_{l\in I\setminus S} (1 - x_k^l) \right) r_S$$

Definition 3.4. Let $\Gamma = \big(I, (r_S)_{\emptyset\subset S\subseteq I}\big)$ be a quitting game. A strategy profile $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is called *ε-equilibrium* (for $\varepsilon \geq 0$) if for every player $i \in I$ and every strategy $\mathbf{y}^i$ of player i it does hold:

$$\gamma^i(\mathbf{y}^i, \mathbf{x}^{-i}) \leq \gamma^i(\mathbf{x}) + \varepsilon$$

The strategy profile $\mathbf{x}$ is a *Nash equilibrium* or just an *equilibrium* if it is an ε-equilibrium with $\varepsilon = 0$. A game has got an *approximate equilibrium* if for every $\varepsilon > 0$ there exists an ε-equilibrium.

3.4 Relation between equilibria in quitting games and uniform equilibria in stochastic games

There is not a general existence theorem for uniform ε-equilibria in undiscounted stochastic games (for games with three or more players). Since quitting games are the easiest example of stochastic games for which the existence of an ε-equilibrium is still an open question, it is interesting to find some characterization for (eventually stationary) ε-equilibria in quitting games or find some example in which there are no (eventually stationary) ε-equilibria at all.

Proposition 3.1. A uniform ε-equilibrium in a quitting game $\Gamma = \big(I, (r_S)_{\emptyset \subset S \subseteq I}\big)$ is also an ε-equilibrium.

Sketch of Proof. This follows from the fact that in a quitting game for any fixed strategy profile σ and for every player $i \in I$ the T-stage payoff $\gamma^i_T(\sigma)$ converges to some $\gamma^i_\infty(\sigma)$ as $T \to +\infty$; indeed under the strategy profile σ we have only two possible cases:

- For every stage $n \in \mathbb{N}$ we have $S_n = \emptyset$ so the game is never ending and $\gamma^i_T(\sigma) = 0$ for every $T \in \mathbb{N}$ so it converges trivially to $\gamma^i_\infty(\sigma) = 0$ for every player $i \in I$.
- There exists a stage $\hat{T} \in \mathbb{N}$ such that the quitting set is $S_{\hat{T}} \neq \emptyset$ then from that stage on every player $i \in I$ will recive a constant payoff $r^i_{S_{\hat{T}}}$ then we have:

$$\begin{aligned}\gamma^i_T(\sigma) &= \frac{1}{T}\sum_{n=1}^{\hat{T}-1} \mathbb{E}_\sigma\left[r(Y_n)\right] + \frac{1}{T}\sum_{n=\hat{T}}^{T} \mathbb{E}_\sigma\left[r(Y_n)\right] \\ &= \underbrace{\frac{\text{const}}{T}}_{\to 0} + \underbrace{\frac{T-\hat{T}}{T}}_{\to 1} r^i_{\hat{S}} \xrightarrow[T\to+\infty]{} r^i_{\hat{S}}\end{aligned}$$

  Then $\gamma^i_T(\sigma)$ converges to $\gamma^i_\infty(\sigma) = r^i_{\hat{S}}$.

□

This has important applications. For instance, if one proves the non existence of (stationary) ε-equilibria in a quitting game this means that for general stochastic games there does not need to exist (stationary) uniform ε-equilibria. One may also prove a reverse implication that is: any undiscounted ε-equilibrium in a quitting game is also a uniform ε'-equilibrium with $\varepsilon' > \varepsilon$.

3.5 What's known about stochastic games

For discounted games there is a foundamental theorem that can be proved using Kakutani's fixed point theorem (see [3]):

Theorem 3.1. Any stochastic game admits a stationary λ-discounted equilibrium for every $\lambda \in (0, 1]$

The basic difficulty with undiscounted stochastic games is that the undiscounted payoff is not continuous over the strategy space. Indeed; in general, there does not need to exist a stationary undiscounted equilibrium even for quitting games.
Anyway, the existence of a uniform equilibrium for two-player non-zero-sum stochastic games was proved by N. Vieille in [4]. E. Solan in [2] extended it to three-player absorbing games: any 3-player absorbing game has an undiscounted equilibrium payoff.
Moreover, for quitting games E. Solan and N. Vieille [1] proved that:

Theorem 3.2. For every $\varepsilon > 0$ and every quitting game with at most three players, there exists an ε-equilibrium $\mathbf{x} = (x_n^i)_{n\in\mathbb{N}}^{i\in I}$ such that either $\mathbf{x}$ is a stationary profile or $x_n^i \leq \varepsilon$ for every $n \in \mathbb{N}$ and $i \in I$.

The existence of a uniform ε-equilibrium for general stochastic games with three or more players is still an open question.
In the next chapter we will see an example by E. Solan and N. Vieille of a four-players quitting game that does not admit a stationary ε-equilibrium proving; thus, that a general stochastic game does not need to admit a stationary uniform ε-equilibrium.

An example

In this section we will see an important example by E. Solan and N. Vieille [1] of a quitting game with 4 players that does not admit a stationary ε-equilibrium.

4.1 The example

Consider the 4-player quitting game:

<table>
<tr><td></td><td></td><td colspan="2"></td><td></td><td colspan="2">4</td></tr>
<tr><td></td><td></td><td colspan="2">2</td><td></td><td colspan="2">2</td></tr>
<tr><td rowspan="4">3</td><td rowspan="2">1</td><td>$\circlearrowleft$</td><td>4,1,0,0</td><td rowspan="2">1</td><td>0,0,4,1</td><td>1,1,0,1</td></tr>
<tr><td>1,4,0,0</td><td>1,1,1,1</td><td>1,0,1,1</td><td>0,1,0,0</td></tr>
<tr><td rowspan="2">1</td><td>0,0,1,4</td><td>0,1,1,1</td><td rowspan="2">1</td><td>1,1,1,1</td><td>0,0,1,0</td></tr>
<tr><td>1,1,1,0</td><td>1,0,0,0</td><td>0,0,0,1</td><td>-1,-1,-1,-1</td></tr>
</table>

Table 1 – Representation of the four player game

In this game player 1 chooses a row (top row means to continue), player 2 chooses a column (left column means to continue), player 3 chooses either the top two matrices or the two bottom ones (top two matrices means to continue) and player 4 chooses either the two left matrices or the right two ones (two left matrices means to continue).
Notice first the simmetry of the payoff matrix that will avoid lots of computations: for every tuple of actions (a, b, c, d) one has:

- $r^1(a, b, c, d) = r^2(b, a, d, c)$
- $r^1(a, b, c, d) = r^4(c, d, b, a)$
- $r^2(a, b, c, d) = r^3(c, d, b, a)$

where $r^i(a, b, c, d)$ is the payoff of player i if the action combination is (a, b, c, d).

We will see that:

Proposition 4.1. The game does not admit a stationary equilibrium.

Proposition 4.2. For every $\varepsilon > 0$ small enough the game does not admit an ε-equilibrium such that $\|x_n - c\| < \varepsilon$ for every $n \in \mathbb{N}$

With Proposition4.1 and Proposition4.2 it follows that:

Theorem 4.1. The game does not admit a stationary ε-equilibrium provided $\varepsilon > 0$ small enough.

Proof. Assume by contradiction that for every $\varepsilon > 0$ small enough there exists a stationary ε-equilibrium x_ε. By compactness of the set of stationary strategy profiles, let x_* be the stationary strategy profile that is the accumulation point of $(x_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$. Now let's split the discussion into two cases.
Assume first that $x_* \neq \mathbf{c}$. Let's start studying the right hand side. By the property of ε-equilibrium, for every $\varepsilon > 0$ small enough, for every player $i \in I$ and every strategy y^i of player i, one gets that:

$$\gamma^i(y^i, x_\varepsilon^{-i}) \leq \gamma^i(x_\varepsilon) + \varepsilon \tag{1}$$

Thanks Kuhn's theorem we can consider y^i to be c^i or q_t^i for $t \in \mathbb{N}$, where q_t^i is the action of player i where she decides to quit at time t.
Then, by continuity of the payoff (that is continuous up to the action $\mathbf{c}$) one has that

$$\gamma^i(x_\varepsilon) + \varepsilon \xrightarrow{\varepsilon \to 0} \gamma^i(x_*)$$

For the left hand side; if $x_*^{-i} \neq \mathbf{c}^{-i}$ for the same reason one has:

$$\gamma^i(y^i, x_\varepsilon^{-i}) \xrightarrow{\varepsilon \to 0} \gamma^i(y^i, x_*^{-i})$$

So, provided $x_* \neq \mathbf{c}$ taking the limit in 1, one gets that if $x_*^{-i} \neq \mathbf{c}^{-i}$:

$$\gamma^i(y^i, x_*^{-i}) \leq \gamma^i(x_*)$$

Now, let's check what happens if $x_*^{-i} = \mathbf{c}^{-i}$. In this case, since $x_* \neq \mathbf{c}$, it is forced $x_*^i \neq \mathbf{c}^i$. Then $\gamma(x_*) = \gamma(x_*^i, \mathbf{c}^{-i}) = \gamma(q_t^i, \mathbf{c}^{-i}) = r_i^i$ for every $t \in \mathbb{N}$. But, we have that $r_i^i \geq 0$ for every player i. This gives what we were looking for, that is: for every $i \in I$, for every strategy y^i of player i we have:

$$\gamma^i(y^i, x_*^{-i}) \leq \gamma^i(x_*)$$

But, this means that x_* is a stationary equilibrium of the game and this is forbidden by Proposition 4.1.

Let's now investigate the case $x_* = \mathbf{c}$. This means that for $\varepsilon > 0$ small enough there is an ε-equilibrium $\mathbf{x}$ such that $\|x_n - c\| < \varepsilon$ for every $n \in \mathbb{N}$ that is ruled out by Proposition 4.2.

□

4.2 Non-existence of stationary equilibria

Let's check that the game does not admit a stationary equilibrium. We organize the discussion according to the number of players who play both actions (continue and quit) with positive probability.

4.2.1 No non-fully mixed stationary equilibrium

It's easy to check (watch at the matrix) that there does not exists a stationary equilibrium where all 4 players are playing pure strategies. With the same reasoning it is immediate that there is no stationary equilibrium in which 3 players are playing pure strategies. Let's verify that there is no stationary equilibrium where two players play pure stationary strategies. Using the symmetries of the payoff function it is enough to consider the cases where either player 3 and player 4 play pure strategies, or player 2 and player 4 play pure strategies.
Assume first that there is an equilibrium in which players 3 and 4 play pure syrategies. The strategies of player 1 and 2 will then form an equilibrium of a 2×2 game.

1. Players 3 and 4 play (q^3, q^4): the induced game is:

	c^2	q^2
c^1	$1, 1$	$0, 0$
q^1	$0, 0$	$-1, -1$

Figure 4 – The unique equilibrium in the induced game is (c^1, c^2).

2. Players 3 and 4 play (c^3, q^4): the induced game is:

	c^2	q^2
c^1	$0, 0$	$1, 1$
q^1	$1, 0$	$0, 1$

Figure 5 – The unique equilibrium in the induced game is (c^1, q^2).

3. Players 3 and 4 play (q^3, c^4): the induced game is:

	c^2	q^2
c^1	$0, 0$	$0, 1$
q^1	$1, 1$	$1, 0$

Figure 6 – The unique equilibrium in the induced game is (q^1, c^2).

4. Players 3 and 4 play (c^3, c^4): the induced game is:

	c^2	q^2
c^1	$\circlearrowleft$	$4, 1$
q^1	$1, 4$	$1, 1$

Figure 7 – The equilibria in the induced game are only (q^1, c^2) and (c^1, q^2)

But in each case the equilibria in the induced game are pure, that would give an equilibrium for the four-player game in pure stationary strategies that is a contradiction.

In a similar way we will investigate that there is no stationary equilibrium where players 2 and 4 play pure actions. By analyzing the induced game between players 1 and 3 one gets:

1. Players 2 and 4 play (c^2, c^4): the induced game is:

	c^3	q^3
c^1	$\circlearrowleft$	$0, 1$
q^1	$1, 0$	$1, 1$

Figure 8 – The unique equilibrium in the induced game is (q^1, q^3).

So, as in the previous cases the four-player game will have a stationary equilibrium in pure strategies that is a contradiction.

2. Players 2 and 4 play (q^2, c^4): the induced game is:

	c^3	q^3
c^1	$4, 0$	$0, 1$
q^1	$1, 1$	$1, 0$

Figure 9 – The unique equilibrium in the induced game is $\left(\frac{1}{2}\,c^1 + \frac{1}{2}\,q^1, \frac{1}{4}\,c^3 + \frac{3}{4}\,q^3\right)$.

In this case Player 2 would recive $\frac{5}{8}$ while he would get 1 by playing c^2.

3. Players 2 and 4 play (c^2, q^4): the induced game is:

	c^3	q^3
c^1	$0, 4$	$1, 1$
q^1	$1, 1$	$0, 0$

Figure 10 – The unique equilibrium in the induced game is (q^1, c^3).

So, as in the previous cases the four-player game will have a stationary equilibrium in pure strategies that is a contradiction.

4. Players 2 and 4 play (q^2, q^4): the induced game is:

	c^3	q^3
c^1	$1, 0$	$0, 1$
q^1	$0, 0$	$-1, -1$

Figure 11 – The unique equilibrium in the induced game is (c^1, q^3).

So, as in the previous cases the four-player game will have a stationary equilibrium in pure strategies that is a contradiction.

Now, we need to check that there in no stationary equilibrium where one player (by symmetry say player 4) plays a pure strategy and all other players play a fully mixed strategy. Let's denote with (x, y, z) the fully mixed

stationary equilibrium in the three-player game when player 4 plays some pure stationary strategy. With this notation x (resp. y,z) represents the probability that player 1 (resp. 2,3) quits.
Assume first that player 4 plays q^4. Then, in order to have player 2 indifferent, we should have:

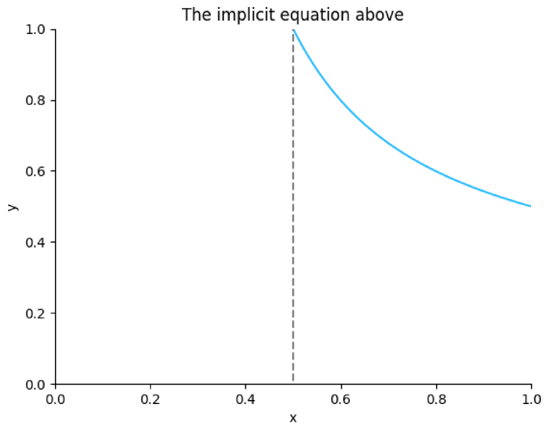
$$z(1-x) = (1-z) - xz$$

This gives the condition $z = \frac{1}{2}$. In order to have player 1 indifferent, one gets:

$$y(1-z) + (1-y)z = (1-y)(1-z) - yz$$

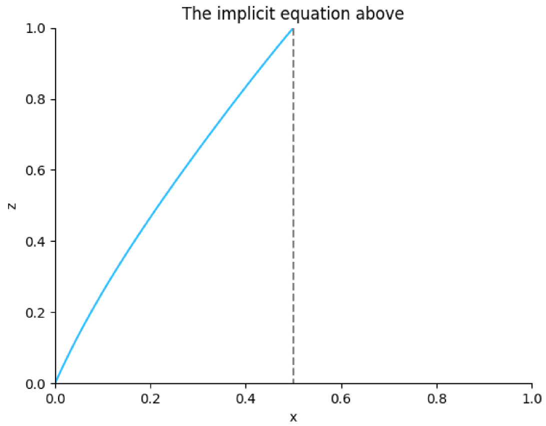
But, using that $z = \frac{1}{2}$ we obtain that $y = 0$, which is pure.
Assume now that player 4 plays c^4. In order to have player 3 indifference:

$$\gamma^3(x,y,0,0) = \frac{xy}{1-(1-x)(1-y)} = 1 - xy = \gamma^3(x,y,1,0)$$



This gives that any solution to this equation for $y \in (0,1)$ is such that $x > \frac{1}{2}$.
But then, using indifference for player 2:

$$\gamma^2(x,0,z,0) = \frac{4x(1-z) + xz}{1-(1-x)(1-z)} = 1 - xz = \gamma^2(x,1,z,0)$$



This have no solution such that $z \in (0,1)$ if $x > \frac{1}{2}$.
This proves that there is no stationary equilibrium where one player plays a pure strategy and the other players play a fully mixed strategy.

4.2.2 No fully mixed stationary equilibrium

To prove that there is no fully mixed stationary equilibrium, we will write best-reply conditions and then check that these cannot be satisfied simultaneously. To start let's focus just on player 1.
Let $(y,z,t) \in (0,1)^3$ be a given fully mixed profile of players 2,3 and 4. In this subsection and this subsection only we will consider y, (resp. z,t) as the probability according with player 2 (resp. 3,4) continues.

By playing c^1 at the first stage and then the mixed action x in all subsequent stages player 1's payoff is:

$$\alpha(y,z,t) := yzt(\gamma^1(x,y,z,t) - 2) - 2yz + 3zt - yt + y + z$$

On the other hand by playing q^1 at the first stage (and in all the subsequent stages), player 1's expected payoff is:

$$\beta(y,z,t) := t + (1-t)(y+z-1)$$

If $x \in (0,1)$ is a stationary best reply to (y,z,t) the two payoffs are equal, and equal to the player 1's payoff given under the strategy profile (x,y,z,t):

$$\alpha_x(y,z,t) = \beta(y,z,t) = \gamma^1(x,y,z,t)$$

In particular, we can define the polynomial Δ_1 defined by:

$$\Delta_1(\tilde{y},\tilde{z},\tilde{t}) := \alpha(\beta(\tilde{y},\tilde{z},\tilde{t}); \tilde{y},\tilde{z},\tilde{t}) - \beta(\tilde{y},\tilde{z},\tilde{t})$$

it has a zero in the point (y,z,t).
Notice that $-1 \leq \beta(y,z,t) = \gamma^1(x,y,z,t) \leq 1$ and that $\gamma^1(c^1,y,z,t) \geq 0$ then $\gamma^1(x,y,z,t) \in [0,1]$.
Reasoning as above (or using the game's symmetries) construct the other polynomials Δ_2,Δ_3,Δ_4.
Notice that we have just proved the following:

Lemma 4.1. If $(x,y,z,t) \in (0,1)^4$ is a fully mixed stationary equilibrium, then:

- $\Delta_1(y,z,t) = \Delta_2(x,z,t) = \Delta_3(x,y,t) = \Delta_4(x,y,z) = 0$
- $\gamma^i(x,y,z,t) \in [0,1]$ for every $i = 1,2,3,4$

Thank to the symmetries of the game we just need to prove that there is no $(x,y,z,t) \in (0,1)^4$ such that:

1. $y = \min\{x,y,z,t\}$
2. $\Delta_1(y,z,t) = \Delta_4(x,y,z) = 0$
3. $\gamma^1(x,y,z,t), \gamma^4(x,y,z,t) \in [0,1]$

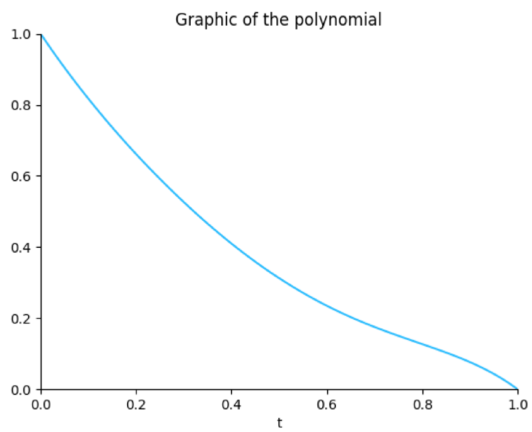
The first condition can be assumed w.l.o.g. by symmetry of the game. But, using lemma 4.1, this would imply that the game has no fully mixed stationary equilibrium.

Lemma 4.2. $\Delta_1(t,t,t) > 0$ for every $t \in [0,1)$.

Proof.

$$\Delta_1(t,t,t) = -2t^5 + 4t^4 - 3t^3 + 2t^2 - 2t + 1$$

Using any method (for instance Strum's method) that counts zeros of a polynomial in an interval.



$\square$

Let's state two useful fact:

Lemma 4.3. β is separately increasing in each variable in $(0,1)^3$

Proof.

$$\beta(y,z,t) = t + (1-t)(y+z-1)$$

Obviously this affine function in y (resp z) is increasing since the slope is $(1-t) > 0$. A similar observation holds for the variable t rewriting it as:

$$\beta(y,z,t) = (\underbrace{2-y-z}_{>0})t + (y+z-1)$$

$\square$

Lemma 4.4. Δ_1 is decreasing in y and increasing in z on the set $\{y \leq \min\{z,t\}\}$.

Proof.

$$\Delta_1(y,z,t) = yzt(\beta(y,z,t)-2) - 2yz + 3zt - yt + y + z - \beta(y,z,t)$$

Computing the derivatives and using the fact that we are working on $\{y \leq \min\{z,t\}\}$ one gets:

$$\frac{\partial \Delta_1}{\partial y}(y,z,t) = z\left(t(t-2) + t(1-t)(2y+z-1) - 2\right) \leq z(0+2-2) \leq 0$$

This proves that, in the set we are working on, Δ_1 is decreasing in y.

$$\frac{\partial \Delta_1}{\partial z}(y,z,t) \geq 3y^3 - 3y^2 + 2y \geq 0$$

This proves that, in the set we are working on, Δ_1 is increasing in z. $\square$

Let's proceed proving that there is no (x,y,z,t) satisfying lemma 4.1.

Lemma 4.5. $\Delta_1 > 0$ on $\{y \leq t \leq z\}$

Proof. Thank to the monotonicity of Δ_1 one has:

$$\Delta_1(y,z,t) \geq \Delta_1(t,z,t) \geq \Delta_1(t,t,t) > 0$$

$\square$

Lemma 4.6. $\Delta_4 > 0$ on $\left\{y \leq z \leq \frac{1}{2}\right\} \cap \{\gamma^4 \geq 0\}$

Proof. Computing Δ_4 one gets:

$$\Delta_4(x,y,z) = \alpha_4(\beta_4; x,y,z) - \beta_4(x,y,z)$$

where

$$\alpha_4(x,y,z) = xyz(\gamma^4(x,y,z,t) - 2) + 3xy - 2xz - yz + x + y$$
$$\beta_4(x,y,z) = y + (1-y)(x+z-1)$$

If $\gamma^4 \geq 0$ then:

$$\Delta_4(x,y,z) \geq (4x - 2xz - 2)y + 1 - 2xz$$

that is an affine function in y and in $y = 0$ is equal to $1 - 2xz > 0$ while in $y = z$ is equal to $(1-2z) + 2xz(1-z) > 0$ thus is positive on $\left\{y \leq z \leq \frac{1}{2}\right\}$. $\square$

Lemma 4.7. $\Delta_1 > 0$ on the set $\left\{\max\left\{y, \frac{1}{2}\right\} \leq z \leq t\right\}$

Proof. It is convenient to split the proof in several steps.

Step 1: $\Delta_1 > 0$ on $\left\{y < \frac{1}{2} \leq z \leq t\right\}$.

Indeed, by monotonicity of Δ_1 we have

$$\Delta_1(y,z,t) \geq \Delta_1\left(\frac{1}{2}, \frac{1}{2}, t\right) = \frac{1}{2} - \frac{t}{2} + \beta\frac{t}{4} > 0$$

Step 2: $\Delta_1 > 0$ on $\left\{\frac{1}{2} \leq y \leq z \leq t \leq \frac{2}{3}\right\}$.

Indeed, by monotonicity of Δ_1 we have $\Delta_1(y,z,t) \geq \Delta_1(z,z,t)$. Notice that $\Delta_1(z,z,t)$ is decreasing in z since:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\Delta_1(z,z,t) = 2z(\beta - 2) + 2zt(2 - y - z) + 2z^2(1-t) - 2z^2t + 4$$

This means that $\frac{\partial}{\partial z}\Delta_1(z,z,t)$ is increasing in t and so:

$$\frac{\partial}{\partial z}\Delta_1(z,z,t) \leq \frac{\partial}{\partial z}\Delta_1\left(z, z, \frac{2}{3}\right) = \frac{4}{3}z(\beta - 2) + \frac{4}{9}z^2 - 4z + \frac{8}{3}$$

The right-hand side is decreasing in z. It is therefore maximal for $z = \frac{1}{2}$, it is then equal to $\frac{2}{3}(\beta - 1) + \frac{1}{9} < 0$ that is negative thanks monotonicity of β: $\beta(y,z,t) \leq \beta(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) = \frac{7}{9}$.
Since $\Delta_1(z,z,t)$ is decreasing in z we have:

$$\Delta_1(y,z,t) \geq \Delta_1(z,z,t) \geq \Delta_1(t,t,t) > 0$$

Step 3*:* $\Delta_1 > 0$ on $\left\{\frac{1}{2} \le y < \frac{2}{3} \le z \le t\right\}$.

By monotonicity of β one has $\beta(\frac{2}{3}, \frac{2}{3}, t) \ge \beta(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \ge \frac{3}{4}$. Hence, by monotonicity of Δ_1:

$$\Delta_1(y,z,t) \ge \Delta_1\left(\frac{2}{3}, \frac{2}{3}, t\right) = \frac{4}{9}t\left(\beta\left(\frac{2}{3}, \frac{2}{3}, t\right) - 2\right) + \frac{1}{9} + \frac{2}{3}t \ge \frac{t+1}{9}$$

Step 4*:* $\Delta_1 > 0$ on $\left\{\frac{2}{3} \le y \le z \le t\right\}$.

By monotonicity of β we have $\beta \ge \frac{3}{4}$. Then:

$$\frac{\partial \Delta_1}{\partial t} = (\beta - 2)yz + yzt(2 - y - z) + 4z - 2 \ge (\beta - 2)yz + 4z - 2$$

The right-hand side is decreasing in z. Therefore, it is minimal when $z = y$ then it is at least $-\frac{5}{4}y^2 + 4y - 2$. This last expression is minimized when $y = \frac{2}{3}$ and in this case it equals $\frac{1}{9}$. This proves that Δ_1 is increasing in t. By monotonicity of Δ_1 one obtains $\Delta_1(y,z,t) \ge \Delta_1(z,z,z) > 0$.

Step 5*:* $\Delta_1 > 0$ on $\left\{\frac{1}{2} \le y \le z \le \frac{2}{3} \le t\right\}$.

By monotonicity of β and Δ_1 we have $\beta(y,z,t) \ge \frac{2}{3}$ and $\Delta_1(y,z,t) \ge \Delta_1(z,z,t)$. Therefore,

$$\Delta_1(y,z,t) \ge -\frac{4}{3}z^2 t - 2z^2 + 4zt + 1 - 2t$$

For $t \ge \frac{2}{3}$ the right-hand side is a quadratic concave function in z that is positive on the boundary of the interval $\left[\frac{1}{2}, \frac{2}{3}\right]$ and thus on the whole interval. $\square$

4.2.3 No perturbated ε-equilibrium - Preliminaries

We need to check that there is no ε-equilibrium that is near to the continuing action of every player.
Let $\rho = 2\,\|r\|_\infty = 8$, twice the maximum value attained by the payoff and $N = 4$, the number of players. We will denote the quitting set at the stage $k \in \mathbb{N}$ with S_k. Denote with t the stage at which the game terminates (the stopping time).
It is convenient to consider strategy profiles where at most one player has a positive probability to quit at each stage. In order to prove that we can restrict the discussion to this case consider the following construction:
Given a strategy profile $\mathbf{x} = (x_n^i)_{n\in\mathbb{N}}^{i\in I}$ consider the associated strategy profile $\mathbf{y} = (y_n^i)_{n\in\mathbb{N}}^{i\in I}$ defined by:

$$y^i_{(n-1)N+j} = \begin{cases} x^i_n & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for every $n \in \mathbb{N}$ and for every $i, j \in I$
To represent what kind of strategy it is, assume

$$\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4) = ((x^1_1, x^1_2, \dots), (x^2_1, x^2_2, \dots), (x^3_1, x^3_2, \dots), (x^4_1, x^4_2, \dots))$$

Then the strategy profile $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4)$ splits the strategy profile $\mathbf{x}$ into 4 blocks in which (at most) one player has positive probability to quit at each stage:

$$\begin{aligned}
\mathbf{y}^1 =&(x^1_1, 0, 0, 0 \mid x^1_2, 0, 0, 0 \mid x^1_3, 0, 0, 0, \dots) \\
\mathbf{y}^2 =&(0, x^2_1, 0, 0 \mid 0, x^2_2, 0, 0 \mid 0, x^2_3, 0, 0, \dots) \\
\mathbf{y}^3 =&(0, 0, x^3_1, 0 \mid 0, 0, x^3_2, 0 \mid 0, 0, x^3_3, 0, \dots) \\
\mathbf{y}^4 =&(0, 0, 0, x^4_1 \mid 0, 0, 0, x^4_2 \mid 0, 0, 0, x^4_3, \dots)
\end{aligned}$$

The following lemma will allow to consider only the case in which at most one player quits with positive probability at each stage.

Lemma 4.8. Let $\varepsilon \leq \frac{1}{8}$ and $\mathbf{x}$ be an ε-equilibrium such that $\|x_n - \mathbf{c}\| < \varepsilon$ for every $n \in \mathbb{N}$.
Then there exists a $12N\rho\varepsilon$-equilibrium $\mathbf{y}$ such that for every $n \in \mathbb{N}$ we have:

- $\|y_n - \mathbf{c}\| < \varepsilon$
- $|\{i \in I : y^i_n > 0\}| \leq 1$

Proof. Define the strategy profile $\mathbf{y} = (y^i_n)^{i \in I}_{n \in \mathbb{N}}$ as above:

$$y^i_{(n-1)N+j} = \begin{cases} x^i_n & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for every $n \in \mathbb{N}$ and for every $i, j \in I$.
The aim is to prove that there are no profitable deviation; but, we will first compare the payoffs given by this two strategy profiles: $\gamma^i(\mathbf{x})$ and $\gamma^i(\mathbf{y})$.
Notice that :

$$\begin{aligned}
\mathbb{P}_{\mathbf{y}}(S_t = \{i\} \mid (n-1) < t \leq nN) =& \frac{x^i_n \prod_{j<i}(1 - x^j_n)}{1 - \prod_{j \in I}(1 - x^j_n)} \\
\mathbb{P}_{\mathbf{x}}(S_t = \{i\} \mid t = n) =& \frac{x^i_n \prod_{j \neq i}(1 - x^j_n)}{1 - \prod_{j \in I}(1 - x^j_n)}
\end{aligned}$$

Their difference is bounded by:

$$\left|\frac{x_n^i \prod_{j<i}(1-x_n^j) - x_n^i \prod_{j\neq i}(1-x_n^j)}{1-\prod_{j\in I}(1-x_n^j)}\right| \leq \frac{x_n^i \left(\prod_{j<i}(1-x_n^j) - \prod_{j\neq i}(1-x_n^j)\right)}{1-x_n^i-(1-x_n^i)\prod_{j\neq I}(1-x_n^j)}$$
$$\leq \frac{x_n^i}{1-x_n^i} \leq 2\varepsilon$$

Now, let's compute the following:

$$\mathbb{P}_{\mathbf{x}}(S_t = \{i\} \mid t<\infty) = \frac{\mathbb{P}_{\mathbf{x}}(S_t=\{i\}, t<\infty)}{\mathbb{P}_{\mathbf{x}}(t<\infty)} = \sum_{n=1}^{\infty} \frac{\mathbb{P}_{\mathbf{x}}(t=n)}{\mathbb{P}_{\mathbf{x}}(t<\infty)} \mathbb{P}_{\mathbf{x}}(S_t=\{i\} \mid t=n)$$

A similar computation for $\mathbf{y}$ brings to:

$$\mathbb{P}_{\mathbf{y}}(S_t = \{i\} \mid t<\infty) = \frac{\mathbb{P}_{\mathbf{y}}(S_t=\{i\}, t<\infty)}{\mathbb{P}_{\mathbf{y}}(t<\infty)}$$
$$= \sum_{n=1}^{\infty} \frac{\mathbb{P}_{\mathbf{y}}((n-1)N<t\leq nN)}{\mathbb{P}_{\mathbf{y}}(t<\infty)} \mathbb{P}_{\mathbf{y}}(S_t=\{i\} \mid (n-1)N<t\leq nN)$$

Then, notice that:

$$\mathbb{P}_{\mathbf{x}}(t=n) = \prod_{\substack{k=1\\ j\in I}}^{n-1}(1-x_k^j)\left[1-\prod_{j\in I}(1-x_n^j)\right] = \mathbb{P}_{\mathbf{y}}((n-1)N<t\leq nN)$$
$$\mathbb{P}_{\mathbf{x}}(t<\infty) = \sum_{n=1}^{\infty}\mathbb{P}_{\mathbf{x}}(t=n) = \sum_{n=1}^{\infty}\mathbb{P}_{\mathbf{y}}((n-1)N<t\leq nN) = \mathbb{P}_{\mathbf{y}}(t<\infty)$$

This implies that

$$\frac{\mathbb{P}_{\mathbf{x}}(t=n)}{\mathbb{P}_{\mathbf{x}}(t<\infty)} = \frac{\mathbb{P}_{\mathbf{y}}((n-1)N<t\leq nN)}{\mathbb{P}_{\mathbf{y}}(t<\infty)} =: a_n \geq 0 \quad \text{with} \quad \sum_n a_n = 1$$

Now, estimate the following:

$$\left|\mathbb{P}_{\mathbf{y}}(S_t=\{i\}, t<\infty) - \mathbb{P}_{\mathbf{x}}(S_t=\{i\}, t<\infty)\right| \leq$$
$$\leq \sum_{n=1}^{\infty} a_n \left|\mathbb{P}_{\mathbf{y}}(S_t=\{i\} \mid (n-1)<t\leq nN) - \mathbb{P}_{\mathbf{x}}(S_t=\{i\} \mid t=n)\right|$$
$$\leq \sum_{n=1}^{\infty} a_n\, 2\varepsilon = 2\varepsilon$$

Since under strategy profile $\mathbf{y}$ two players cannot quit simultaneously, summing the previous equation over $i\in I$ one gets:

$$1-\mathbb{P}_{\mathbf{x}}(|S_t|=1 \mid t<\infty) \leq 2n\varepsilon \implies \mathbb{P}_{\mathbf{x}}(|S_t|>1 \mid t<\infty)$$

Finally, we can compare the two payoffs:

$$\left|\gamma^i(\mathbf{y}) - \gamma^i(\mathbf{x})\right| \leq \rho \sum_{S\subseteq I} \left|\mathbb{P}_{\mathbf{y}}(S_t = S) - \mathbb{P}_{\mathbf{x}}(S_t = S)\right| \leq$$

$$\leq \rho \left[\sum_{j\in I} \left[\mathbb{P}_{\mathbf{y}}(S_t = \{i\}) - \mathbb{P}_{\mathbf{y}}(S_t = \{i\})\right] + \sum_{|S|>1} \mathbb{P}_{\mathbf{x}}(S_t = S)\right]$$

One can estimate this with the following:

$$\rho \left[\mathbb{P}_{\mathbf{x}}(|S_t| > 1 \mid t < \infty) + \sum_{j\in I} \left[\mathbb{P}_{\mathbf{y}}(S_t = \{i\} \mid t < \infty) - \mathbb{P}_{\mathbf{y}}(S_t = \{i\} \mid t < \infty)\right]\right] \underbrace{\mathbb{P}_{\mathbf{x}}(t < \infty)}_{\leq 1}$$

This brings to the final:

$$\left|\gamma^i(\mathbf{y}) - \gamma^i(\mathbf{x})\right| \leq \rho\left[2N\varepsilon + 2N\varepsilon\right] \leq 4N\rho\varepsilon$$

This will help us to prove that there is no unilateral profitable deviation from $\mathbf{y}$. Thanks to Kuhn's theorem we just need to check the ε-equilibrium property only for pure deviations. Notice first that the previous reasoning does not involve the property of $\mathbf{x}$ to be an ε-equilibrium but just his property to be ε-near the continuing strategy profile. Then, if player i deviates from $\mathbf{y}$ by picking c^i, the previous computation brings to:

$$\gamma^i(c^i, \mathbf{y}^{-i}) - \gamma^i(c^i, \mathbf{x}^{-i}) \leq 4N\rho\varepsilon$$

By using the ε-equilibrium property for $\mathbf{x}$ one obtains:

$$\gamma^i(c^i, \mathbf{y}^{-i}) \leq \gamma^i(c^i, \mathbf{x}^{-i}) + 4N\rho\varepsilon \leq \gamma^i(\mathbf{x}) + \varepsilon + 4N\rho\varepsilon \leq \gamma^i(\mathbf{y}) + 4N\rho\varepsilon + \varepsilon + 4N\rho\varepsilon$$

Now consider the pure deviation $q^i_{(n-1)N+k}$, this means that player i decides to continue until stage $(n-1)N + k$ in which she quits with probability 1. Fix $n \in \mathbb{N}$ and $k = 1, 2, 3, 4$. We will use a similar reasoning as before, let's compare the payoffs $\gamma^i(q^i_n, \mathbf{x}^{-i})$ and $\gamma^i(q^i_{(n-1)N+k}, \mathbf{y}^{-i})$. Define $\tilde{q}^i := q^i_{(n-1)N+k}$

$$\begin{aligned}\gamma^i(\tilde{q}^i, \mathbf{y}^{-i}) =& \mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}(t \leq (n-1)N)\, \mathbb{E}_{\tilde{q}^i,\mathbf{y}^{-i}}\left[r^i_{S_t} \mid t \leq (n-1)N\right] + \\ &+ \mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}((n-1)N < t \leq nN)\, \mathbb{E}_{\tilde{q}^i,\mathbf{y}^{-i}}\left[r^i_{S_t} \mid (n-1)N < t \leq nN\right]\end{aligned}$$

$$\begin{aligned}\gamma^i(q^i_n, \mathbf{x}^{-i}) =& \mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(t \leq (n-1)N)\, \mathbb{E}_{q^i_n,\mathbf{x}^{-i}}\left[r^i_{S_t} \mid t \leq (n-1)N\right] + \\ &+ \mathbb{P}_{q^i_n,\mathbf{x}^{-i}}((n-1)N < t \leq nN)\, \mathbb{E}_{q^i_n,\mathbf{x}^{-i}}\left[r^i_{S_t} \mid (n-1)N < t \leq nN\right]\end{aligned}$$

So, following the previous reasoning we are interested in computing:

$$\mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}(S_t = \{i\} \mid t \leq (n-1)N) =$$
$$\sum_{k=1}^{n-1} \mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}((k-1)N < t \leq kN \mid t \leq (n-1)N)\, \mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}(S_t = \{i\} \mid (k-1)N < t \leq kN)$$

$$\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(S_t=\{i\}\mid t\le(n-1)N)=\sum_{k=1}^{n-1}\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(t=k\mid t\le n-1)\,\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(S_t=\{i\}\mid t=k)$$

But, notice that:

$$\mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}((k-1)N<t\le kN\mid t\le(n-1)N)=\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(t=k\mid t\le n-1)=:b_k\ge 0$$

with $\sum_k b_k=1$.
Then, as before, we can estimate the difference between the two following probabilities:

$$|\mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}(S_t=\{i\}\mid(k-1)N<t\le kN)-\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(S_t=\{i\}\mid t=k)|\le 2\varepsilon$$

In a similar way, if $t\le(n-1)N$ two player cannot quit simultaneously then by summing over $i\in I$ the previous equation we get:

$$\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(|S_t|>1\mid t\le n-1)\le 2N\varepsilon$$

Now, consider the contribution given by the fact that someone may quit in the first $k-1$ substages of stage n and that some other player other that i may quit in the substage k:

$$\text{If } i\ne j\implies \mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}(S_t=\{j\}\mid(n-1)N<t\le nN)\le(N-1)\varepsilon$$

$$\mathbb{P}_{\tilde{q}^i,\mathbf{y}^{-i}}(S_t=\{i\}\mid(n-1)N<t\le nN)-\mathbb{P}_{q^i_n,\mathbf{x}^{-i}}(S_t=\{i\}\mid t=n)\le\varepsilon$$

Collecting all those information we finally deduce that:

$$\gamma^i((q^i_{(n-1)N+k},\mathbf{y}^{-i}))\le\gamma^i(q^i_n,\mathbf{x}^{-i})+6N\rho\varepsilon\le\gamma^i(\mathbf{x})+7N\rho\varepsilon\le\gamma^i(\mathbf{y})+\varepsilon+11N\rho\varepsilon$$

This proves that $\mathbf{y}$ is a $12N\rho\varepsilon$-equilibrium. □

Therefore, from now on we may assume that $\mathbf{x}$ is an ε-equilibrium such that $|\{i\in I: y^i_n>0\}|\le 1$ and $\|y_n-\mathbf{c}\|<\varepsilon$ for every $n\in\mathbb{N}$. We will refer to such a profile as a *perturbated ε-equilibrium.*

Lemma 4.9. For every perturbated ε-equilibrium $\mathbf{x}$ one has:

1. $\mathbb{P}_{\mathbf{x}}(t<\infty)\ge 1-\varepsilon$.

2. $\gamma^i(\mathbf{x})\ge 1-\rho\varepsilon-\varepsilon$ for every $i\in I$ and $\gamma^i(\mathbf{x})\ge\frac{5}{4}-2\varepsilon$ for some $i\in I$.

3. $\mathbb{P}_{\mathbf{x}}(S_t=\{i\})\ge\frac{2}{15}-\rho\varepsilon$ for every $i\in I$.

Proof. Given $n \in \mathbb{N}$ consider the strategy $y^{i,n}$ for player i that coincides with x^i the first n stages and quits with probability 1 at stage $n+1$. Then,

$$\gamma^i(y^{i,n}, \mathbf{x}^{-i}) = \sum_{k=1}^{n} \sum_{S \subseteq I} r_S^i \, \mathbb{P}_{\mathbf{x}}(t = k, S_t = S) + \sum_{\substack{S \subseteq I \\ S \ni i}} r_S^i \mathbb{P}_{y^{i,n}, \mathbf{x}^{-i}}(S_{n+1} = S) \, \mathbb{P}_{\mathbf{x}}(t > n)$$

Considering the subsequence y^{i,n_k} for n_k such that player i is not quitting with other players, in the limit along these subsequences we have that the fist term of the sum converges to $\gamma^i(\mathbf{x})$ while the second one to $\mathbb{P}_{\mathbf{x}}(t = \infty)$, using ε-equilibrium property we get:

$$\gamma^i(\mathbf{x}) + \mathbb{P}_{\mathbf{x}}(t = \infty) \leftarrow \gamma^i(y^{i,n_k}, \mathbf{x}^{-i}) \leq \gamma^i(\mathbf{x}) + \varepsilon$$

That implies

$$\mathbb{P}_{\mathbf{x}}(t < \infty) \geq 1 - \varepsilon$$

This proves **1.**
W.l.o.g. assume that there exists $j \in I$ such that $x_1^j > 0$. By quitting at the first stage player i will recive 1 if $i = j$, otherwise if $i \neq j$:

$$\begin{aligned}
\gamma^i(q_1^i, \mathbf{x}^{-i}) &= (1 - x_n^j) r_{\{i\}}^i + x_1^j r_{\{j,i\}}^i \\
&= \underbrace{r_{\{i\}}^i}_{=1} + x_1^j (r_{\{j,i\}}^i - r_{\{i\}}^i) \\
&\geq 1 - \rho\varepsilon
\end{aligned}$$

Then, by ε-equilibrium property we get:

$$1 - \rho\varepsilon \leq \gamma^i(q_1^i, \mathbf{x}^{-i}) \leq \gamma^i(\mathbf{x}) + \varepsilon$$

Since the quitting set can only be a singleton under a perturbated ε-equilibrium the total payoff sums up to $5 = 0 + 0 + 1 + 4$ in any case, thus:

$$\sum_{i \in I} \gamma^i(\mathbf{x}) = 5 \, \mathbb{P}_{\mathbf{x}}(t < \infty) \geq 5 - 5\varepsilon$$

So, at least one term of the sum must be greater that $\frac{5}{4} - \frac{5}{4}\varepsilon$. This proves **2.**
Set

$$p^i := \mathbb{P}_{\mathbf{x}}(S_t = \{i\})$$

we can write:

$$\begin{aligned}
\gamma^1(\mathbf{x}) &= p^1 + 4p^2 & \qquad \gamma^3(\mathbf{x}) &= p^3 + 4p^4 \\
\gamma^2(\mathbf{x}) &= p^2 + 4p^1 & \qquad \gamma^4(\mathbf{x}) &= p^4 + 4p^3
\end{aligned}$$

By **2.** we have that summing up the first column:

$$
\begin{aligned}
p^1 + p^2 &\geq \frac{2}{5} - \frac{4}{5}\rho\varepsilon \geq \frac{2}{5} - \rho\varepsilon \\
p^3 + p^4 &\geq \frac{2}{5} - \rho\varepsilon
\end{aligned}
$$

But we have that $p^1 + p^2 + p^3 + p^4 + \mathbb{P}_{\mathbf{x}}(t = \infty) = 1$.
Then $p^1 + p^2 \leq 1 - p^3 - p^4 \leq \frac{3}{5} + \rho\varepsilon$. But, any solution to the system

$$
\begin{cases}
p^1 + 4p^2 \geq 1 - 2\rho\varepsilon \\
4p^1 + p^2 \geq 1 - 2\rho\varepsilon \\
p^1 + p^2 \leq \frac{3}{5} + \rho\varepsilon
\end{cases}
$$

Will satisfy $p^1, p^2 \geq \frac{2}{15} - \rho\varepsilon$, a symmetric reasoning will work for other players proving **3.** $\square$

Let's introduce a notation. Given a strategy profile $\mathbf{x}$, $i \in I$, $n \in \mathbb{N}$ we denote by

$$
x^i(n) := (c^i_1, \ldots, c^i_{n-1}, x^i_n, x^i_{n+1}, \ldots)
$$

the strategy for player i that coincides with the strategy x^i after stage n and with

$$
\mathbf{x}_n := \left(\left(x^i_n\right)^{i \in I}, \left(x^i_{n+1}\right)^{i \in I}, \left(x^i_{n+2}\right)^{i \in I}, \ldots \right)
$$

the stretegy profile induced by $\mathbf{x}$ after stage n, this is like to let the game start from stage n.
Then, define:

$$
p^i_n := \mathbb{P}_{\mathbf{x}}(t < n, S_t = \{i\}) \nearrow p^i := \mathbb{P}_{\mathbf{x}}(S_t = \{i\})
$$

Notice that with this notation we lose the dependence of the strategy profile $\mathbf{x}$. If there will be any possible misunderstanding the strategy profiles will be specified by denoting it with $p^i_n(\mathbf{x})$ or $p^i(\mathbf{x})$.

Definition 4.1. We say that players 1 and 2 (resp. 3 and 4) are *partners*. The partner of player i is denoted by $\tilde{\imath}$.

Remark 4.1. Notice that this structure of partners helps the analysis of the game, the definition is pretty natural since the best outcome for player i is obtained when her partner $\tilde{\imath}$ quits alone.

Now, we will prove that under a perturbated ε-equilibrium while the expected payoff of a player $i \in I$ exceeds 1 its contribution in the probability of termination is small.

Lemma 4.10. Let $\mathbf{x}$ be a strategy profile such that $|\{i \in I : y^i_n > 0\}| \leq 1$ for every $n \in \mathbf{N}$. If $\gamma^i(\mathbf{x}_n) \geq 1 + \sqrt{\varepsilon}$ for some $i \in I$, for every $n \leq n_0$, then

$$\gamma^i(x^i(n), \mathbf{x}^{-i}) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}\, p^i_n \quad \forall n \leq n_0$$

If $\mathbf{x}$ is an ε-equilibrium, then

$$p^i_n \leq \sqrt{\varepsilon} \quad \forall n \leq n_0$$

Proof. By induction over n.
Base step: if $x^i_1 = 0$ the result obviously holds. If $x^i_1 > 0$, since nobody else can quit in the first stage we have:

$$\gamma^i(\mathbf{x}) = x^i_1 \cdot 1 + (1 - x^i_1)\gamma^i(x^i(1), \mathbf{x}^{-i})$$

Rewriting this we get:

$$\gamma^i(x^i(1), \mathbf{x}^{-i}) = \gamma^i(\mathbf{x}) + \frac{x^i_1}{1 - x^i_1}(\gamma^i(\mathbf{x}) - 1) \geq \gamma^i(\mathbf{x}) + \frac{x^i_1}{1 - x^i_1}\sqrt{\varepsilon} \geq \gamma^i(\mathbf{x}) + x^i_1 \sqrt{\varepsilon} \quad (2)$$

Then obviously $\gamma^i(x^i(1), \mathbf{x}^{-i}) \geq \gamma^i(\mathbf{x}) + 0 \cdot \sqrt{\varepsilon}$, since $p^i_1 = 0$ the thesis holds.
Induction step: assume the tesis holds for $n - 1$, we will prove it for $1 < n \leq n_0$. Let's apply equation 2 to the strategy profile $\mathbf{x}_{n-1}$ to get:

$$\begin{aligned}\gamma^i(x^i_{n-1}(1), \mathbf{x}^{-i}_{n-1}) = \gamma^i(x^i(n)_{n-1}, \mathbf{x}^{-i}_{n-1}) &\geq \gamma^i(\mathbf{x}_{n-1}) + (\mathbf{x}_{n-1})^i_1 \cdot \sqrt{\varepsilon} \\ &\geq \gamma^i(\mathbf{x}_{n-1}) + x^i_{n-1} \cdot \sqrt{\varepsilon} \qquad (3)\end{aligned}$$

since $(\mathbf{x}_{n-1})^i_1 = x^i_{n-1}$.
Now, notice that using the relation 3

$$\begin{aligned}\gamma^i(x^i(n), \mathbf{x}^{-i}) &= 4p^{\tilde{i}}_{n-1} + \mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1)\, \gamma^i(x^i(n)_{n-1}, \mathbf{x}^{-i}_{n-1}) \\ &\geq 4p^{\tilde{i}}_{n-1} + \mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1)(\gamma^i(\mathbf{x}_{n-1}) + x^i_{n-1}\sqrt{\varepsilon}) \\ &= \underbrace{\gamma^i(x^i(n-1), \mathbf{x}^{-i})}_{\text{use induction hypothesis}} + x^i_{n-1}\sqrt{\varepsilon}\, \mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1) \\ &\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}(p^i_{n-1} + x^i_{n-1}\, \mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1))\end{aligned}$$

Notice that:

$$\mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1) = \prod_{j \neq i} \prod_{k=1}^{n-2} (1 - x^j_k)$$

$$\mathbb{P}_{x^i,\mathbf{x}^{-i}}(t \geq n-1) = \prod_{j \in I} \prod_{k=1}^{n-2} (1 - x^j_k)$$

that gives:

$$\mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1) \geq \mathbb{P}_{\mathbf{x}}(t \geq n-1)$$

then, resuming the previous computations:

$$\begin{aligned}
\gamma^i(x^i(n), \mathbf{x}^{-i}) &\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}(p^i_{n-1} + x^i_{n-1}\,\mathbb{P}_{c^i,\mathbf{x}^{-i}}(t \geq n-1)) \\
&\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}(p^i_{n-1} + x^i_{n-1}\,\mathbb{P}_{\mathbf{x}}(t \geq n-1)) \\
&\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}(\mathbb{P}_{\mathbf{x}}(t < n-1, S_t = \{i\}) + \mathbb{P}_{\mathbf{x}}(t = n-1, S_t = \{i\})) \\
&= \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}\,\mathbb{P}_{\mathbf{x}}(t < n, S_t = \{i\}) \\
&= \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}\,p^i_n
\end{aligned}$$

If we add the hypothesis that $\mathbf{x}$ is an ε-equilibrium we get:

$$\gamma^i(\mathbf{x}) + \sqrt{\varepsilon}\,p^i_n \leq \gamma^i(x^i(n), \mathbf{x}^{-i}) \leq \gamma^i(\mathbf{x}) + \varepsilon$$

for every $n \leq n_0$, that gives:

$$p^i_n \leq \sqrt{\varepsilon} \quad \forall n \leq n_0$$

□

We will now prove that under a perturbated ε-equilibrium whenever a player $i \in I$ gets a payoff higher than one, player i will not contribute to the probability of termination, while the partner $\tilde{\imath}$ will contribute, until a stage in which the continuation payoff of player i is close to one is reached.

Lemma 4.11. Let $a > 0$, $\varepsilon \in (0, \frac{1}{900})$ and $i \in I$. Let $\mathbf{x}$ be a perturbated ε-equilibrium such that $\gamma^i(\mathbf{x}) \geq 1 + a$. Then there exists a stage $n_1 > 1$ such that:

(A) $\gamma^i(\mathbf{x}_{n_1}) < 1 + \sqrt{\varepsilon}$

(B) $p^i_{n_1} \leq 2\sqrt{\varepsilon}$

(C) $3p^{\tilde{\imath}}_{n_1} \geq a - \sqrt{\varepsilon}$

Proof. Assume w.l.o.g. $i = 1$. By lemma 4.9 $p^i = \mathbb{P}_{\mathbf{x}}(S_t = \{i\}) \geq \frac{2}{15} - \rho\varepsilon$ for every $i \in I$, this means that it is bounded away from 0 for every $i \in I$. To avoid contradiction with lemma 4.10 there exists a stage $\tilde{n} \in \mathbb{N}$ such that $\gamma^1(\mathbf{x}_{\tilde{n}}) < 1+\sqrt{\varepsilon}$. Define

$$n_1 := \inf\{n \in \mathbb{N} : \gamma^1(\mathbf{x}_n) < 1 + \sqrt{\varepsilon}\}$$

that is well posed. Obviously, $n_1 > 1$ since $\gamma^1(\mathbf{x}_1) = \gamma^1(\mathbf{x}) \geq 1 + a$. By definition condition (A) holds.

Moreover, $\gamma^1(\mathbf{x}_n) \geq 1+\sqrt{\varepsilon}$ for every $n \leq n_1 - 1$, using lemma 4.10 we obtain that $p^1_{n_1-1} \leq \sqrt{\varepsilon}$. But, then:

$$p^1_{n_1} \leq p^1_{n_1-1} + x^1_{n_1} \leq \sqrt{\varepsilon} + \varepsilon \leq 2\sqrt{\varepsilon}$$

Then, also condition (B) is satisfied.
Since $\gamma^1(\mathbf{x}_{n_1}) < 1 + \sqrt{\varepsilon}$ one has that:

$$\begin{aligned}
1 + a \leq \gamma^1(\mathbf{x}) &= p^1_{n_1} + 4p^2_{n_1} + (1 - p^1_{n_1} - p^2_{n_1} - p^3_{n_1} - p^4_{n_1})\gamma^1(\mathbf{x}_{n_1}) \\
&\leq p^1_{n_1} + 4p^2_{n_1} + (1 - p^1_{n_1} - p^2_{n_1})\gamma^1(\mathbf{x}_{n_1}) \\
&\leq 3p^2_{n_1} + 1 + \sqrt{\varepsilon}
\end{aligned}$$

Then, with the above definition of n_1 also condition (C) is satisfied. □

Now, it's time to prove that there does not exist an ε-equilibrium in which two partners get both a payoff substantially higher than 1.

Proposition 4.3. Let $\varepsilon \in (0, \frac{1}{900})$ and $a > 7\sqrt{\varepsilon}$. There is no perturbated ε-equilibrium $\mathbf{x}$ such that

$$\gamma^i(\mathbf{x}), \gamma^{\tilde{i}}(\mathbf{x}) \geq 1 + a \quad \text{for some } i \in I$$

Proof. Assume, by contradiction, that there exists $i \in I$ such that

$$\gamma^i(\mathbf{x}), \gamma^{\tilde{i}}(\mathbf{x}) \geq 1 + a$$

W.l.o.g. we can assume $i = 1$. Applying lemma 4.11 to players 1 and 2 we obtain two stages n_1 and n_2 respectively. Assume w.l.o.g. that $n_1 \leq n_2$, this implies that $p^2_{n_1} \leq p^2_{n_2}$. By condition (C) of lemma 4.11 applied to player 1 we get:

$$p^{\tilde{1}}_{n_1} = p^2_{n_1} \geq \frac{a}{3} - \frac{\sqrt{\varepsilon}}{3}$$

On the other hand, by condition (B) of lemma 4.11 applied to player 2 we obtain:

$$p^2_{n_2} \leq 2\sqrt{\varepsilon}$$

But, since $p^2_{n_1} \leq p^2_{n_2}$ we have $a \leq 7\sqrt{\varepsilon}$ that is a contradiction. □

4.2.4 No perturbated ε-equilibrium - Proof

Finally, we are ready to prove the non existence of a perturbated ε-equilibrium.

Theorem 4.2. For every $\varepsilon > 0$ small enough the game in Table 1 does not admit an ε-equilibrium such that $\|x_n - c\| < \varepsilon$ for every $n \in \mathbb{N}$

Proof. Let $\varepsilon > 0$ small enough, assume w.l.o.g. $\gamma^1(\mathbf{x}) \geq \frac{5}{4} - 2\varepsilon$.
The aim is to find a stage n_2 such that:

- $\mathbf{x}_{n_2}$ is an 8ε-equilibrium
- $\gamma^3(\mathbf{x}_{n_2}), \gamma^4(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{12}$

In such a way this contraddicts proposition 4.3 proving the non existence of a ε-equilibrium.
Let's apply lemma 4.11 to $\mathbf{x}$ with $i = 1$ to get a stage n_1 such that:

$$p^1_{n_1} \leq 2\sqrt{\varepsilon}$$
$$p^2_{n_1} = p^{\tilde{1}}_{n_1} \geq \frac{1}{3}\left(\frac{1}{4} - 3\sqrt{\varepsilon}\right) \geq \frac{1}{12} - \sqrt{\varepsilon}$$

Here, to avoid contradictions with lemma 4.10, since $p^2_{n_1}$ is bounded away from 0, we have the existence of a stage $\tilde{n} \leq n_1$ such that $\gamma^2(\mathbf{x}_{\tilde{n}}) < 1 + \sqrt{\varepsilon}$.
Set

$$n_2 := \max\{n \leq n_1 : \gamma^2(\mathbf{x}_n) \leq 1 + \sqrt{\varepsilon}\}$$

By definition $n_2 \leq n_1$, this implies

$$p^1_{n_2}(\mathbf{x}) \leq p^1_{n_1}(\mathbf{x}) \leq 2\sqrt{\varepsilon}$$

Thanks lemma 4.9 $p^1 \geq \frac{2}{15} - \rho\varepsilon$. So, we obtain:

$$\begin{aligned}
\mathbb{P}_{\mathbf{x}}(t < n_2) &= 1 - \mathbb{P}_{\mathbf{x}}(t \geq n_2) \\
&= 1 - \underbrace{\mathbb{P}_{\mathbf{x}}(t \geq n_2, S_t = \{1\})}_{=p^1(\mathbf{x}) - p^1_{n_2}(\mathbf{x})} \\
&= 1 + p^1_{n_2}(\mathbf{x}) - p^1(\mathbf{x}) \\
&\leq 1 - \frac{2}{15} + \rho\varepsilon + 2\sqrt{\varepsilon} \\
&\leq \frac{7}{8} \quad \text{for } \varepsilon > 0 \text{ small enough}
\end{aligned}$$

Then, since $\mathbf{x}$ is an ε-equilibrium, $\mathbf{x}_{n_2}$ is an 8ε-equilibrium; indeed: consider any possible deviation y^i for player $i \in I$ and denote with $\tilde{y}^i$ the extended strategy that coincides with x^i in the first $n_2 - 1$ stages and $\tilde{y}^i_{n_2} = y^i$, then

$$\begin{aligned}
\varepsilon \geq \gamma^i(\tilde{y}^i, \mathbf{x}^{-i}) - \gamma^i(\mathbf{x}) &= \mathbb{P}_{\mathbf{x}}(t \geq n_2)\left(\gamma^i(y^i, \mathbf{x}^{-i}_{n_2}) - \gamma^i(\mathbf{x}_{n_2})\right) \\
&\geq \frac{1}{8}\left(\gamma^i(y^i, \mathbf{x}^{-i}_{n_2}) - \gamma^i(\mathbf{x}_{n_2})\right)
\end{aligned}$$

Now, the aim is to prove that $p^2_{n_2}(\mathbf{x}) \geq \frac{1}{12} - 17\sqrt{\varepsilon}$.
If $n_2 = n_1$ it is trivial since $p^2_{n_2}(\mathbf{x}) = p^2_{n_1}(\mathbf{x}) \geq \frac{1}{12} - \sqrt{\varepsilon}$.
If $n_2 < n_1$, by definition of n_2 this implies that $\gamma^2(\mathbf{x}_k) > 1 + \sqrt{\varepsilon}$ for every $n_2 < k \leq n_1$.

Let's apply lemma 4.10 to the strategy profile $\mathbf{y} := \mathbf{x}_{n_2}$ at the stage $n = n_1 - n_2$. In this way, mimicking the last step in the proof of lemma 4.10 we obtain:

$$\gamma^2(\mathbf{x}_{n_2}) + \sqrt{\varepsilon}\, p_n^2(\mathbf{y}) \leq \gamma^2(\mathbf{x}_{n_2}) + 8\varepsilon \implies p_n^2(\mathbf{y}) \leq 8\sqrt{\varepsilon}$$

Let t be the stopping time associated to the strategy profile $\mathbf{x}$ and $\tilde{t} = t - n_2$ be the translated stopping time for $\mathbf{y}$. Rewriting in another form $p_n^2(\mathbf{y})$ we notice that:

$$\begin{aligned} p_n^2(\mathbf{y}) &= \mathbb{P}_{\mathbf{y}}(\tilde{t} < n_1 - n_2, S_{\tilde{t}} = \{2\}) \\ &\mathbb{P}_{\mathbf{x}_{n_2}}(t < n_1, S_t = \{2\} \mid t \geq n_2) \leq 8\sqrt{\varepsilon} \end{aligned}$$

Therefore, one may deduce that:

$$p_{n_1}^2(\mathbf{x}) - p_{n_2}^2(\mathbf{x}) = \mathbb{P}_{\mathbf{x}}(n_2 \leq t < n_1, S_t = \{2\}) \leq 8\sqrt{\varepsilon}$$

This gives:

$$p_{n_2}^2(\mathbf{x}) \geq p_{n_1}^2(\mathbf{x}) - 8\sqrt{\varepsilon} \geq \frac{1}{12} - 9\sqrt{\varepsilon}$$

Now, we will use this result to prove that $\gamma^3(\mathbf{x}_{n_2}), \gamma^4(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{12}$. As before, since $\gamma^i(\mathbf{x}) \geq 1 - 2\rho\varepsilon$ for every $i \in I$, we have:

$$1 - 2\rho\varepsilon \leq \gamma^2(\mathbf{x}) = 4p_{n_2}^1(\mathbf{x}) + p_{n_2}^2(\mathbf{x}) + (1 - p_{n_2}^1(\mathbf{x}) - p_{n_2}^2(\mathbf{x}) - p_{n_2}^3(\mathbf{x}) - p_{n_2}^4(\mathbf{x}))\gamma^2(\mathbf{x}_{n_2})$$

Since $\gamma^2(\mathbf{x}_{n_2}) \leq 1 + \sqrt{\varepsilon}$ and $p_{n_2}^1 \leq p_{n_1}^1 \leq 2\sqrt{\varepsilon}$ we can deduce that:

$$\begin{aligned} 1 - 2\rho\varepsilon &\leq 3p_{n_2}^1(\mathbf{x}) + 1 + \sqrt{\varepsilon} - p_{n_2}^3(\mathbf{x}) - p_{n_2}^4(\mathbf{x}) \\ p_{n_2}^3(\mathbf{x}) + p_{n_2}^4(\mathbf{x}) &\leq 2\rho\varepsilon + 6\sqrt{\varepsilon} \leq 8\sqrt{\varepsilon} \end{aligned}$$

But, on the other hand we get:

$$1 - 2\rho\varepsilon \leq \gamma^3(\mathbf{x}) = 4p_{n_2}^4(\mathbf{x}) + p_{n_2}^3(\mathbf{x}) + (1 - p_{n_2}^1(\mathbf{x}) - p_{n_2}^2(\mathbf{x}) - p_{n_2}^3(\mathbf{x}) - p_{n_2}^4(\mathbf{x}))\gamma^3(\mathbf{x}_{n_2})$$

using the fact that $p_{n_2}^2 \geq \frac{1}{12} - 17\sqrt{\varepsilon}$ we obtain:

$$1 - 2\rho\varepsilon \leq 4(p_{n_2}^3(\mathbf{x}) + p_{n_2}^4(\mathbf{x})) + \left(1 - \frac{1}{12} + \frac{25}{3}\sqrt{\varepsilon}\right)\gamma^3(\mathbf{x}_{n_2})$$

Finally, rewriting it we notice that:

$$\gamma^3(\mathbf{x}_{n_2}) \geq \frac{1 - 2\rho\varepsilon - 32\sqrt{\varepsilon}}{1 - \frac{1}{12} + \frac{25}{3}\sqrt{\varepsilon}} \xrightarrow{\varepsilon \to 0} \frac{1}{1 - \frac{1}{12}} = \frac{1 + \frac{1}{12}}{1 - \frac{1}{12^2}} > 1 + \frac{1}{12}$$

Then for $\varepsilon > 0$ small enough, it holds $\gamma^3(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{12}$. A symmetric reasoning proves the same for $\gamma^4(\mathbf{x}_{n_2})$. But, $\mathbf{x}_{n_2}$ is a 8ε-equilibrium, this contraddicts proposition 4.3. $\square$

An example of a two player generalized quitting game

The aim of this section is to introduce a new class of games that generalizes quitting games and to prove the existence of a stationary 0-equilibrium for a subclass of this new type of games.
The initial aim of this section was to find a 2-player game that does not admit a stationary ε-equilibrium, because moving out from the class of quitting games (for example adding other possible quitting actions for the players we still obtain a 2-player absorbing game for which it is well known the existence of a ε-equilibrium but nothing about the existence of a stationary one). Then, it may be helpful to add even other kind of continuing actions in order to move out from the class of absorbing games (or add at least two more players keeping the structure of an absorbing game but; in this case, at the moment, there are no theorem about the existence of ε-equilibria) but still keeping the game simpler than a general stochastic game. It this direction one may find some example that does not admit an uniform ε-equilibrium.

Generalized quitting games

A *generalized quitting game* is a quitting game in which each player have two different **set** of actions avaiable. One of continuing actions and one of quitting actions. That means that the set of actions at the starting state (the non absorbing one) of each player is:

$$A^i = C^i \cup Q^i$$

where C^i is the continuing actions set while Q^i is the quitting one.
The game proceeds exactly as a quitting game but in this case the game ends as soon as at least one player i chooses an action in Q^i, while if every player $i \in I$ choose an action in their continuing set C^i the game proceeds.

The example

Consider the following two player generaized quitting game:

		B	
		c^B	q^B
	c^A	$\circlearrowleft$	$2, 1$
A	q^A	$1, 2$	$3, -1$
	$\tilde{q}^A$	$4, 0$	$-2, 1$

Figure 12 – Generalized quitting game

In this example player A have only one continuing action denoted by c^A and two quitting actions denoted by q^B and $\tilde{q}^B$ while player B have just one continuing action and one quitting one that are denoted as usual with c^B and q^B.

Equilibria of the game

4.2.5 Equilibria near the continuing action

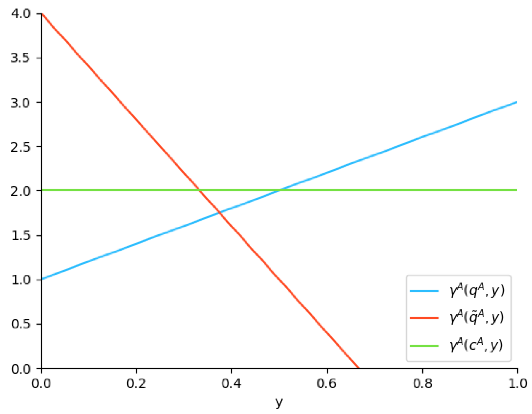
The theorem 4.1 can be generalized even to this game provided proposition 4.2 ask in addition the non existence of an equilibrium in which player B plays ε-near to the continuing action.

This is the case of this game, indeed if player B plays near the continuing action then player A can ensure almost 4 by picking $\tilde{q}^A$, in that way player B would recive slightly more than 0 but by choosing q^B he would obtain 1.

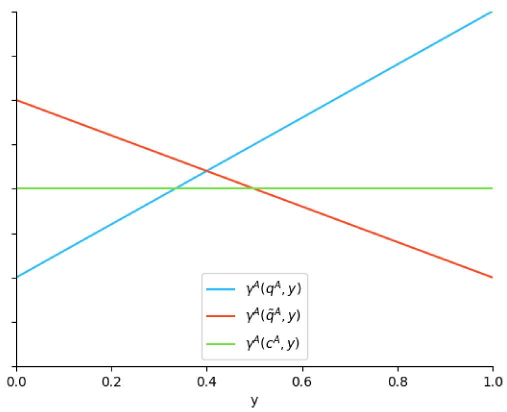
4.2.6 Stationary equilibria

One can check that the only stationary equilibria in this game are (c^A, y) where $y \in \left[\frac{1}{3}, \frac{1}{2}\right]$. Let's reason in general, with this geometric approach, to check what happens if one tries to avoid a stationary equilibrium of this kind.

To avoid an equilibrium of the form (c^A, y) for some $y > 0$ we need that for every $y \in (0, 1]$ there exists an action between q^A and $\tilde{q}^A$ for player A that guarantees $r^A_{c^A, q^B} = 2$. Let's draw the payoffs $\gamma^A(q^A, y), \gamma^B(\tilde{q}^A, y)$ as functions of y:



Since the intersection between the payoffs of player A under the strategies q^A and $\tilde{q}^A$ lies under the line $\gamma^A(c^A, y)$ then there does exist some y for which is not possible to obtain a profitable deviation from (c^A, y).
In the case the intersection between the payoffs of player A under the strategies q^A and $\tilde{q}^A$ lies above the line $\gamma^A(c^A, y)$ the situation is as follows:



Then, the game has an equilibrium where player A plays a mixed action supported in $\{q^A, \tilde{q}^A\}$ and this Nash equilibrium of the reduced game is a stationary 0-equilibrium for the initial game since player A would not profit by moving to the action c^A.

One can reason in a similar way with all other similar configuration. Notice that if $r^B_{c^A,q^B} \geq 0$ in order to do not have a pure stationary equilibrium we need to ask that $r^A_{q^A,q^B} \geq r^A_{c^A,q^B}$ or $r^A_{\tilde{q}^A,q^B} \geq r^A_{c^A,q^B}$. This translates into the fact that the ending point of the blue line or the red one is above the green one. W.l.o.g. we can assume that $r^A_{q^A,q^B} = \max\{r^A_{q^A,q^B}, r^A_{\tilde{q}^A,q^B}\}$ and so we have the following possibilities:

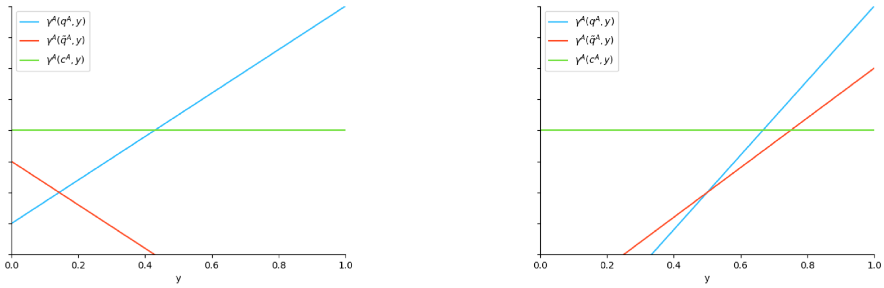


Figure 13 – First case

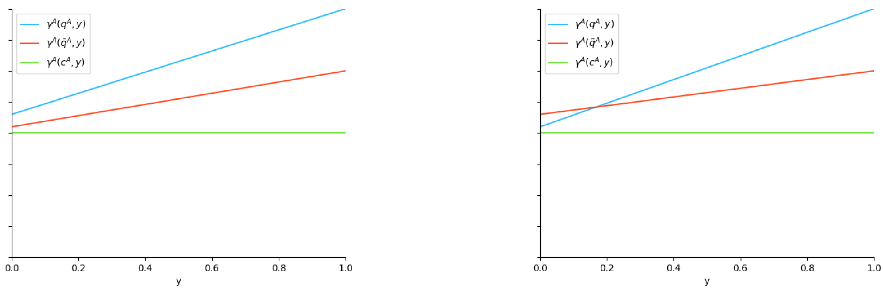


Figure 14 – Second case

In the case of Figure 13 the stationary equilibrium is given by (c^A, y) where y is less or equal to the point of intersection between the blue and the green line. In the case of Figure 14 one may have $r^A_{q^A,c^B} \geq r^A_{\tilde{q}^A,c^B}$ in such a case the action $\tilde{q}^A$ is dominated by q^A for player A. In this case, the game reduces to a usual two-player quitting game; so, we may avoid to consider this case since we fall into an easier class of games that are already well known. Anyway, it is known that any two player quitting game has a stationary ε-equilibrium.

In the other case, $r^A_{\tilde{q}^A,c^B} \geq r^A_{q^A,c^B}$ that means the blu line and the red one intersects; so one can use the above reasoning.
This proves that any kind of those game has a stationary 0-equilibrium.

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