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"Portfolio allocation with risk budgeting: evidence of Equal Risk Contribution portfolio in equity markets"

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1. Introduction

In the widespread world of asset management, the asset allocation refers to investment strategies that aim to balance risk and return by apportioning a portfolio's assets according to a given goals, risk tolerance and investment horizon. In simple terms, asset allocation is investing money in different categories of assets, called asset classes, which typically are equity, fixed-income, commodities and cash equivalents, in order to well diversify the investment. An investment strategy should be appropriate for answering the research objective is called for, thus, there is not a unique and optimal approach defined to manage or construct a portfolio because the allocation of assets fluctuates according to the investment objective.

The aim of a good asset allocation plan is to develop an investment portfolio that allows to reach a certain financial objective with a degree of risk suited for the investor. A well-diversified plan will not outperform the top asset class in any given year, but over time, it may be one of the most effective ways to realize long-term goals. The asset allocation is then, one of the most important step in the portfolio construction, which has been always stimulating a strong interest in in both the academic and operative world. The need to support asset managers in the performance of this exercise has generated a vast literature on the topic, focusing on mathematics and statistical interpretation of asset allocation.

Nevertheless, the empirical evidence suggests that the production of formal models of portfolio construction appears huge in comparison with their actual use. In fact, it is more frequent to come across portfolio managers who develop qualitative investment strategies, rather than asset managers that follow rigorous mathematical procedures. This might depend not only from company specific characteristics, such as its investment volume, the market in which it invests, the level of technology and so on, but mainly from the very specific portfolio manager's choices: they usually are reluctant to delegate investment decisions to mathematical models that lead to portfolio compositions often unreasonable, or simply not in line with their *modus operandi*. A mathematical model, even if well structured, it is based on naive assumptions that make the portfolio construction process unable to incorporate all of the operational issues related to the asset allocation choices.

However, even if it looks like a simple and intuitive process, the financial literature includes the asset allocation within a mathematical framework, which requires the estimation of statistical parameters and the development of optimizations aimed at maximizing the

investors' preferences. The financial markets, despite the simple concepts of benefits and rewards, is a complexly volatile industry which requires critical analysis to adequately evaluate risk factors and take decisions about investments. Thus, through a quantitative approach it is possible to develop models that, even if they suffer from being an *ex-post* analysis or back testing, they might still be a solid starting point for investigating this topic. Upon such premises, this research work is an academic insight into different techniques adopted by investors all over the world to build equity portfolios.

In the investment world as of today, there are many different purposes for investment. Certainly, the return on investment is the most striking one, but picking investments based on return alone is not enough. Since, in simple terms, no one can predict the future, investors not only look at the returns but also at the risk associated with it. Thus, financially speaking, performance is an important issue for portfolios and it is often used to rank them, but the ranking procedure takes into consideration different criteria having a significant impact on the allocation decisions. For what concern the last decade, the risk management has become just as important as overall performance and the pressure for more transparency has modified the relationships between investors and portfolio managers: "the time is over when a fund manager could promise the moon. Today, the job of a fund manager is first of all to manage risk" (Bruder, B., & Roncalli, T. 2012, *Managing Risk Exposures using the Risk Budgeting Approach*). The peculiarity of this historical period makes investors reluctant to undertake excessive risks more than ever. In this context, a new risk-based investment style has emerged since the subprime crisis, which is not based anymore on mean-variance optimization as described in the classical portfolio theory and where the minimization of risks is not only simplified in the minimization of volatility. The classic theory leads, in fact, to the construction of portfolios concentrated in terms of weights, very sensible to inputs changes, and with higher transactions costs. Moreover, optimizing the volatility does not necessarily mean optimizing the risk diversification.

It is not the first time that mean-variance portfolio optimization encounters criticisms from investors. What is new today is that some investors do not use optimization methods and may prefer some heuristic solutions based on risk concentration, where diversification is the central parameter to manage. Among these solutions, we consider in this work the Most Diversified and the Equally Risk Contribution portfolios (ERC), with a particular focus on the latter, which belongs to the so-called risk budgeting techniques. Both of these methods benefit from being independent by any forecasts about returns, however it can be shown that the ERC has a better performance overall. In a risk budgeting approach, the investor only chooses the risk repartition between assets of the portfolio, while in the ERC portfolio, the risk contribution

from each portfolio asset is made equal. This portfolio has been extensively studied by Maillard et al. (2010) who had derived several interesting properties such as that this portfolio is located between minimum variance and equally-weighted portfolios. When risk exposures are not managed in a uniform way, risk budgets are made not equal to each other: this could be the case of long-term institutional investor could invest a small part of his strategic portfolio in alternative investments like commodities.

For the purpose of this work, we consider only the Equally Risk Contribution, as it is, by all means, the key model of Risk Budgeting techniques. Only equity markets are taken into account, and, since the sample is composed by 90 large-cap stocks overall, the application of the different models is made a single-stock level.

In the first part of the thesis, a brief theoretical background about Modern Portfolio theories based on Markowitz work is presented. Some of the classic well-known portfolio construction methods, such as the Minimum Variance and the Maximum Sharpe ratio, are investigated. Finally, it will be given a particular attention to market portfolio obtained with the Capital Asset Pricing Method, corresponding in the Empirical Evidence section to the market capitalization weighted portfolio.

In the following part, some alternative and more recent asset allocation techniques are proposed, including the equally weighted, the Most Diversified portfolio developed by Choueifaty and Coignard, and primarily, the Risk Budgeting methods, which find their bases in the work ok Thierry Roncalli.

Finally, The Data and the Empirical Evidence sections represent the heart of this work. Taking large-cap stocks, in well-known equity markets from three different geographical areas, as the sample of this research, we investigate and compare each method on the basis of some pre-defined performance indicators. The goal of this work is to show the benefits and trade-off between the different strategies, demonstrating that, especially in very volatile periods, the risk parity techniques might capture the equity risk premium more effectively than the standard allocation methods.

2. Modern Portfolio Theory

In the 1950s Harry Markowitz (born August 24, 1927), today professor of finance at the Rady School of Management at the University of California, developed a theory of portfolio construction, which allows investors to analyze risk relative to their expected return. For this work, Markowitz shared the 1990 Nobel Memorial Prize in Economic Sciences with William Sharpe and Merton Miller. Markowitz's theory is best known as the Modern Portfolio Theory. The MPT is a theory of investment which attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently minimize risk for a given level of expected return, by carefully choosing the proportions of various assets. Markowitz concluded that there is not only one optimal portfolio, but also a set of optimal portfolios laying on the so-called efficient frontier.

In 1958, James Tobin (March 5, 1918 – March 11, 2002) showed that the efficient frontier becomes a straight line in the presence of a risk-free asset. In this case, the optimal portfolio corresponds to the tangency portfolio (or maximum Sharpe ratio portfolio), a particular combination of risky and riskless assets.

Given the difficulties to accurately define the expected returns of risky assets and their covariance needed to compute the tangency portfolio, in 1964 William Sharpe (born June 16, 1934) developed the CAPM theory, which does not need any assumptions about expected returns, volatilities and correlations. Indeed, by assuming that the market is in equilibrium, he showed that the prices of assets are such that the tangency portfolio is the market portfolio, which is composed of all risky assets in proportion to their market capitalization. This significant contribution of Sharpe brought to the large development of index funds and passive management.

2.1. The Mean–Variance analysis and the Efficient Frontier

Approximately sixty years ago, Markowitz introduced the first mathematical formulation of optimized portfolios. In his opinion *"the investor does (or should) consider expected return a desirable thing and variance of return and undesirable thing"* (Markowitz, 1952). The fundamental goal of Modern Portfolio Theory is to optimally allocate your investments between different assets. Mean variance optimization (MVO) is a quantitative tool that allow this allocation by considering the trade-off between risk and return.

A simple multi-asset investment problem could be as follows. Consider a universe of n assets and a vector of weights in the portfolio $x = (x_1, \dots, x_n)$ and assume that the portfolio is fully invested[1]:

$$\sum_{i=1}^{n} x_i = \mathbf{1}^{\mathrm{T}}x = 1 \quad (1)$$

If the vector of asset returns is $R = (R_1, \dots, R_n)$, then the return of portfolio is equal to:

$$R(x) = \sum_{i=1}^{n} x_i R_i = x^{\mathrm{T}}R$$

Supposing that μ and Σ are respectively the vector of asset returns and the variance-covariance matrix of asset returns:

$$\mu = \mathbb{E}[R]$$

$$\Sigma = \mathbb{E}[(R - \mu)(R - \mu)^{\mathrm{T}}]$$

then it can be proved[2] that the expected return and the variance of the portfolio are as follows:

$$\mu(x) = \mathbb{E}[R(x)] = x^{\mathrm{T}}\mu$$

$$\sigma^2(x) = \mathbb{E}[(R(x) - \mu(x))(R(x) - \mu(x))^{\mathrm{T}}] = x^{\mathrm{T}}\Sigma x$$

In this way, we can define the mean – variance optimization problem as two sides of the same coin:

- Maximizing the expected return of the portfolio under a volatility constraint:

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[1] $\mathbf{1}^{\mathrm{T}}$ is the transpose of a vector of ones (a row vector).
[2] Roncalli, T. (2014). *Introduction to Risk Parity and Budgeting*.

$$\max \mu(x) \quad \text{s.t.} \quad \sigma(x) \leq \sigma^*$$

- Minimizing the volatility of the portfolio under a return constraint:

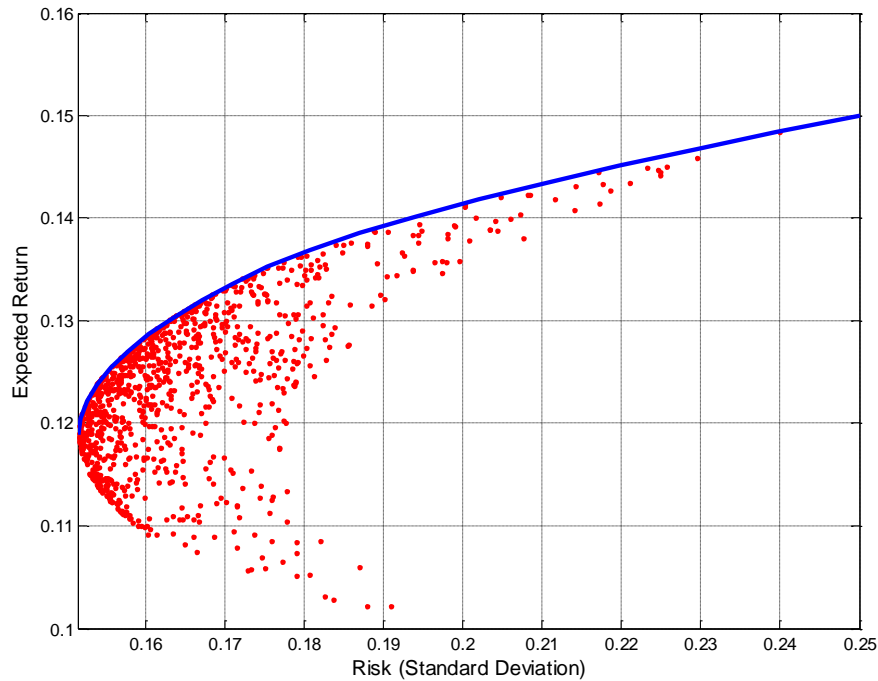$$\min \sigma(x) \quad \text{s.t.} \quad \mu(x) \geq \mu^*$$

Figure 2.1 - Simulated efficient frontier and sub-optimal portfolios.

Approximately sixty years ago, Markowitz introduced the concept of the efficient frontier, the set of all efficient portfolios. Any rational investor using mean variance analysis would choose some portfolio on the efficient frontier that suit their risk preferences.

The Efficient frontier is a visual representation of all possible and optimal sets of portfolios generated by mean-variance analysis. For any preferred risk/return, the corresponding point on efficient frontier denotes the portfolio, which has max return at specified risk or min volatility at given expected return. By repeatedly solving the optimization problems (e.g. with a "*for*" loop using *Matlab* or *Excel VBA*), we can obtain the efficient frontiers as illustrated below.

The area below and to the right of efficient frontier is the feasible area including all portfolios that can be constructed yet not efficient, while portfolios above and to the left are impossible by construction.

An important result to bear in mind is given by the *two-fund separation* theorem, which says that the efficient frontier of risky assets can be formed by any two risky portfolios. All portfolios on the mean-variance efficient frontier can be formed as a weighted average of any two portfolios or funds on the efficient frontier. Thus, having any two points of the portfolio combinations, allows drawing an entire efficient frontier of the risky assets. For example, given the expected return of the minimum variance portfolio and the max Sharpe portfolio, is possible to generate the entire efficient frontier, since both portfolios lie on it.

2.1.1. The Global Minimum Variance portfolio

The global minimum variance portfolio corresponds to the following optimization program:

$$x^* = \arg\min \frac{1}{2} x^{\mathrm{T}} \Sigma x$$

$$\text{u. c. } \mathbf{1}^{\mathrm{T}} x = 1$$

The solution is:

$$x^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mathbf{1}}$$

The GMV portfolio is the only portfolio located on the efficient frontier that is not dependent on any assumptions about expected returns. It is also the tangency portfolio, as defined later on, if and only if expected returns are equal for all stocks.

When the correlation is uniform, the weights may be negative and to define the constrained GMV portfolio (GMVc) we need to impose the long-only constraint:

$$0 \leq x \leq 1$$

The unconstrained GMV (GMVu) portfolio on the other hand, envisages the possibility of going short on assets; therefore, it has a better performance in terms of expected return and volatility compared to the GMVc. Nonetheless, it has higher costs in terms of turnover rate and transaction costs that will be presented aftwerwords.

Graphically the GMV corresponds to the portfolio on the efficient frontier with the lowest volatility. To just have a hint regarding the graphic representations of different GMVu

portfolios and efficient frontiers, the next figures[3] show three portfolios belonging to three different geographical areas (Europe, United States, Japan), each one composed by a sample of 30 large-cap stocks from the main equity indexes of each area[4].
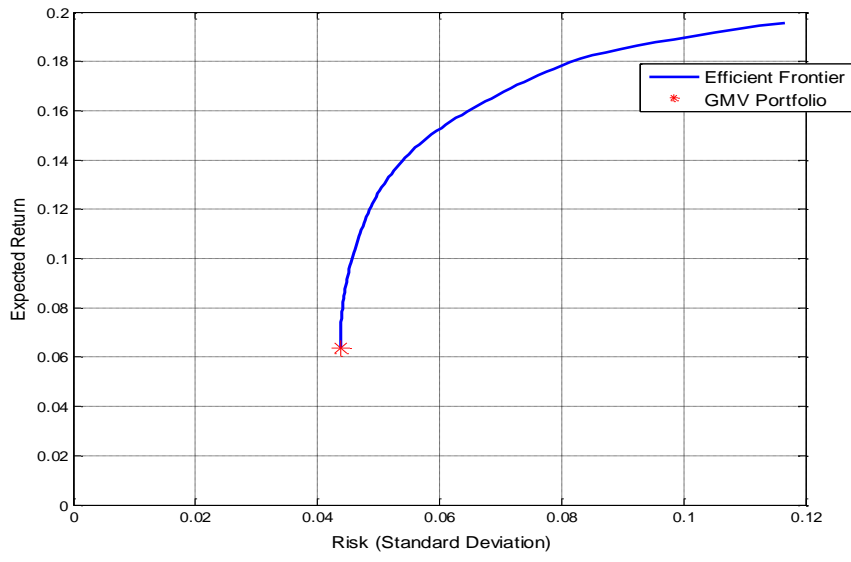
Figure 2.2 – Efficient Frontier (Europe stocks)

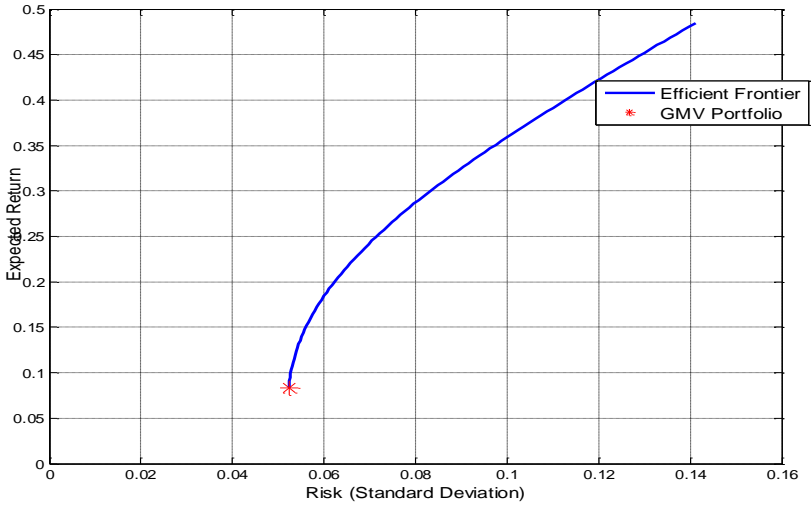*Figure 2.3* - Efficient Frontier (Japan stocks)

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[3] Expected return and standard deviation are annualized.
[4] This sample of 90 stocks is the one used in our research, their historical prices charts are shown in Appendix 6.4 For the purpose of the efficient frontier plots, the stock returns and standard deviations are simply obtained from the stock prices over the past five years.
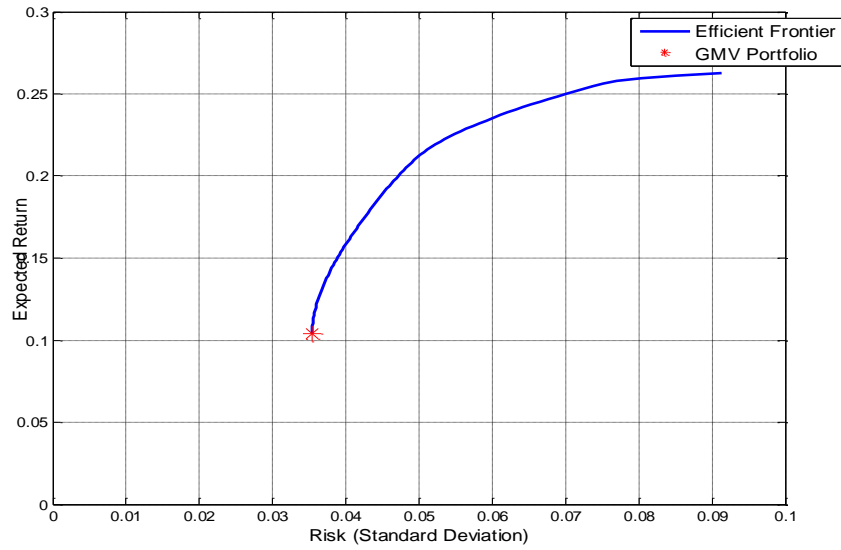
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Figure 2.4 - Efficient Frontier (U.S. stocks)

2.1.2. The Maximum Sharpe Ratio portfolio

Lying on the efficient frontier there is a set of optimized portfolios and one is no better or worse than the other in terms of performance, hence the choice would be based simply on the investor risk aversion. However, in 1958 Tobin showed that, introducing a risk-free asset in this framework, one optimized portfolio dominates all the others. Therefore, investors who wish to optimize the trade off between expected return and risk in their portfolios should combine the portfolio with the risk-free asset such as a Treasury Bonds for U.S. or a German Government Bonds for Europe.

Combining a risk-free asset, with return equal to rf, with a general portfolio x composed by n risky assets we obtain a new portfolio y made of $n + 1$ assets with the following expected return and variance:

$$\mu(y) = \mathbb{E}[R(y)] = (1 - \alpha)rf + \alpha\mu(x) = rf + \alpha(\mu(x) - rf)$$

$$\sigma^2(y) = \alpha^2\sigma^2(x) + (1 - \alpha)^2\sigma^2{}_{rf} + 2\alpha(1 - \alpha)\sigma_{rf}\sigma(x)corr(x,rf)$$

where α is the proportion of money invested in the risky portfolio x and $(1 - \alpha)$ the proportion invested in the risk-free asset.

Since σ_{rf}, the standard deviation of rf, is equal to zero, the variance of portfolio y trivially becomes:

$$\sigma^2(y) = \alpha^2 \sigma^2(x)$$

From which we have:

$$\mu(y) = rf + \frac{\mu(x) - rf}{\sigma(x)} \sigma(y)$$

In other words, the risk-free asset is a pure interest-bearing instrument; its inclusion in a portfolio corresponds to lending or borrowing cash at the risk-free rate: lending (such as the purchase of a bond) corresponds to the risk-free asset having a positive weight, whereas borrowing is equal to have a negative weight for it. From last equation, which is a linear function between the volatility and expected return of the portfolio y, we can derive a commonly used risk-adjusted performance measure for portfolios: the Sharpe ratio (SR) developed by Nobel laureate William Sharpe in 1966.
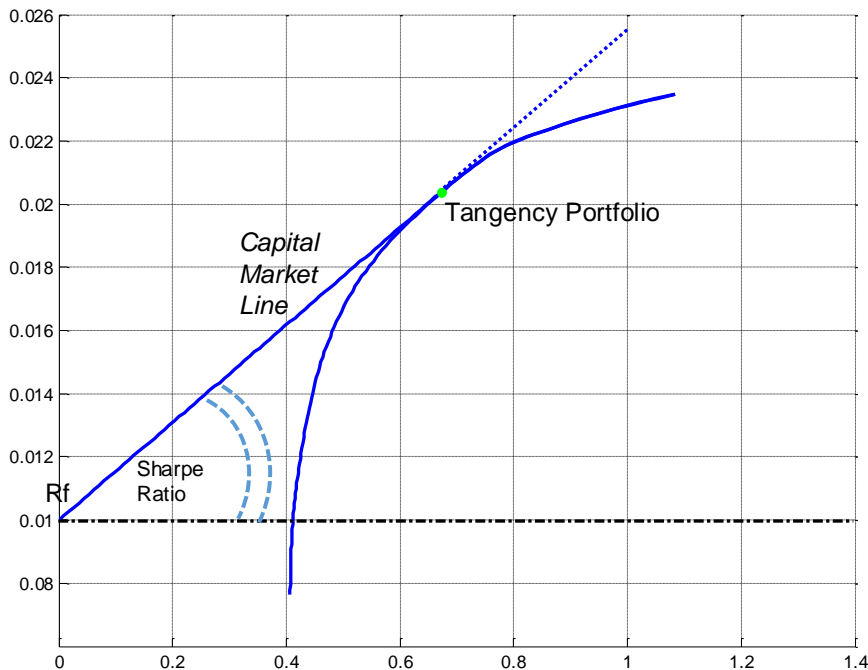

Figure 2.5 - The CML and tangency portfolio

The Sharpe ratio is calculated by subtracting the risk-free asset from the rate of return for a portfolio and dividing the result by the standard deviation of the portfolio returns.

$$SR(x|rf) = \frac{\mu(x) - rf}{\sigma(x)}$$

The Sharpe ratio tells us whether a portfolio's return is due to a smart investment decisions or a result of excess risk. Although one portfolio or fund can reach higher returns than its peers can, it is only a good investment if those higher returns do not come with too much additional risk. The greater a portfolio's Sharpe ratio, the better its risk-adjusted performance has been. A negative SR indicates that a risk-free asset would perform better than the security analyzed.

The Capital Market Line

As shown in figure 5, the SR corresponds to the slope of the so-called Capital Market Line (CML), which dominates all possible choices on the efficient frontier except the tangency portfolio, located at the point where the CML drawn from the point of risk-free asset return is tangent to itself. We deduce that the tangency portfolio is the one that maximizes the slope of the CML, or equivalently, the Sharpe Ratio. Hence, the tangency portfolio is also the risky portfolio on the efficient frontier corresponding to the maximum SR.

Built on Markowitz mean-variance framework, Sharpe Ratio assumes that the mean and standard deviation of the distribution of one-period return are sufficient statistics for evaluating the prospects of an investment portfolio.

The MSR portfolio shares the same drawback of the GMV: the portfolio is concentrated in relatively few stocks that bear all the risk of the portfolio.

To mitigate this problem one can introduce some type of constraints relatively to its diversification. For instance one can set a boundary on volatility, diversification ratio or stocks' risk contributions (measuring how much a single stock or asset affects the total volatility of the portfolio) directly in the maximization function of the MSR portfolio, and this is precisely what has been done in this work: we added a maximum stocks' risk contribution constraint of 6% and 12% to the pure MSR portfolio. The results, which are going to be shown in the Empirical Evidence section, are two additional portfolios with a lower SR, but diversified in a wider range of stocks.

2.2. The CAPM

Assuming that all the investors apply the mean-variance criterion, that all agree on the probabilistic structure and also that there is a single risk-free rate to pay and receive loans, accessible to everyone without transaction costs, then everyone will invest in only one fund of risky assets using at the same time the the riskless one to borrow and lend money.

The proportion between the weights fiven to the two funds depend from the specific risk appetite of the investor: the ones which are more risk-adverse will give a greater weight to the risk-free asset and viceversa. Supposing that everyone purchases this only one fund of risky asset, and the sum of them constitute the whole market, then the fund corresponds to the market portfolio. Thus, the weight of a single security in the market is equal to the proportion between the market capitalization of that secutity and the market capitalization of the whole market.

In this framework, the Capital Market Line becomes the new efficient frontier, which is a linear combination between the market portfolio M and the risk free asset and it is given by:

$$\bar{r} = rf + \frac{\overline{r_M} - rf}{\sigma_M}\sigma$$

Where $\overline{r_M}$ and σ_M are the expected return and standard deviation of the market portfolio, while $\bar{r}$ and σ are the expected return and standard deviation of any efficient portfolio along the Capital Market Line.

The CML relates the expected rate of return of an efficient portfolio to its standard deviation, but it does not show how the expected rate of return of an individual asset relates lo its individual risk. This relation is expressed by the Capital Asset Pricing Model, developed in the early 1960s by William Sharpe (1964)[5], Jack Treynor (1962)[6], John Lintner (1965)[7] and Jan Mossin (1966)[8]. The CAPM is based on the idea that not all risks should affect asset prices.

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[5] Sharpe, William F. (1964). "Capital Asset Prices – A Theory of Market Equilibrium Under Conditions of Risk". *Journal of Finance* XIX.
[6] Treynor, Jack L. (1962). "Toward a Theory of Market Value of Risky Assets"
[7] Lintner, Jhon (1965). "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets". *Review of Economics and Statistics*. 47:1, pp. 13–37.
[8] Mossin, Jan (1966) "Equilibrium in a Capital Asset Market ". *Econometrica* Vol.34 pp.768-783.

In particular, a risk that can be diversified away when held along with other investments in a portfolio is, in a very real way, not a risk at all. The specific assumptions on which the CAPM relies, are as follows:

- Investors are rational wealth maximizers who select investments based on expected return and standard deviation (mean - variance criterion).
- Investors can borrow or lend unlimited amounts at a risk-free rate.
- There are no restrictions on short sales of any financial asset.
- All investors have the same expectations related to the market.
- All financial assets are fully divisible (it is possible to buy and sell as much or as little as preferred) and can be sold at any time at the market price.
- There are no transaction costs.
- There are no taxes.
- No investor's activities can influence market prices.
- The quantities of all financial assets are given and fixed.

According to the CAPM, if the market portfolio M is efficient, the expected return r_i of any asset satisfies:

$$r_i - rf = \beta_i(\overline{r_M} - rf)$$

Where

$$\beta_i = \frac{cov(r_i, \overline{r_M})}{\sigma_M{}^2}$$

β_i is the so-called beta coefficient and it measures the systematic risk of a security in comparison to the market as a whole. A beta of 1 indicates that the security's price will move with the market. A beta of less than 1 means that the security will be less volatile than the market. A beta of greater than 1 indicates that the security's price will be more volatile than the market. For example, if a stock's beta is 1.2, it's theoretically 20% more volatile than the market. Many utilities stocks have a beta of less than 1. Conversely, most high-tech, Nasdaq-based stocks have a beta of greater than 1, offering the possibility of a higher rate of return, but also posing more risk[9].

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[9] http://www.investopedia.com/terms/b/beta.asp

The value of $r_i - rf$ is the excess return of the security i with respect to the risk free asset and, similarly, $\overline{r_M} - rf$ is the excess return for the market portfolio. In this way, the CAPM claims that the excess return of any asset is proportionally to the excess return of the market portfolio and the proportionality factor is the beta of the asset.

Moreover, it is very easy to calculate the beta of a portfolio, since it corresponds to the weighted average of the beta's assets that compose the portfolio where the weights (x_i) are the same as the ones which define the portfolio itself:

$$\beta_P = \sum_{i=1}^{n} x_i \beta_i$$

The Security Market Line

As shown in figure 6 the CAPM can be expressed graphically by the Security Market Line. Both graphs show the linear variation of r_i. The first expresses it in covariance terms with the market portfolio: $cov(r_i, \overline{r_M})$ corresponds to the horizontal axis and $\sigma_M{}^2$ defines the market portfolio (M). The second graph

The second graph shows the relation in beta form, with β being the horizontal axis. In this case the market portfolio matches the point where $\beta = 1$. Both of these lines highlight the essence of the CAPM formula: the SML expresses the risk-reward structure of assets according to the CAPM emphasizing that the risk of an asset is a function of its covariance wilt the market or, equivalently, a function of its beta[10]

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[10] Luenberger, D. (1998). *Investment Science*. Oxford Universily Press.
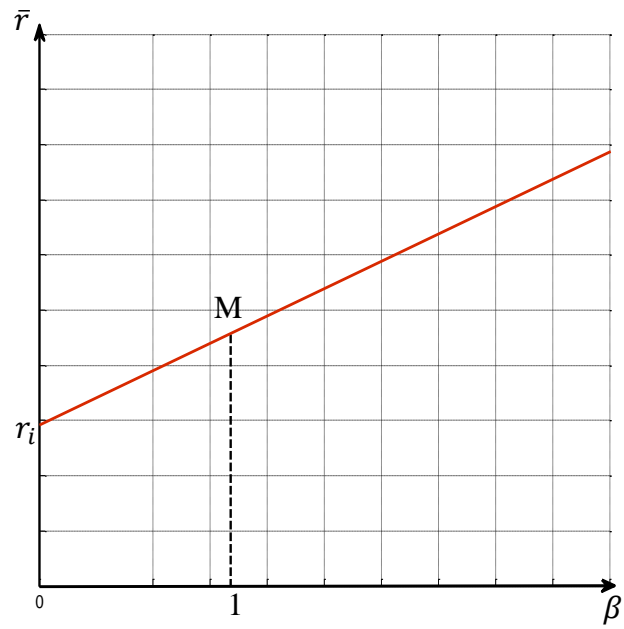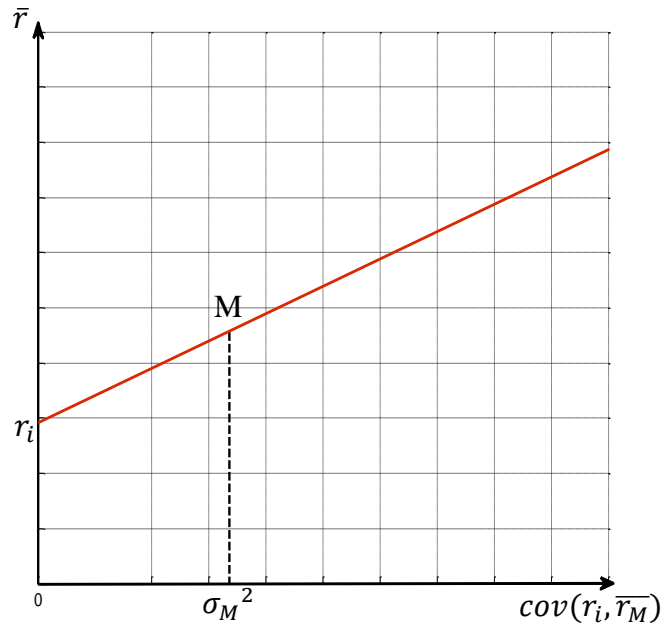
Figure 2.6 - Security Market Line

2.3. Drawbacks

The main drawbacks of the Markowitz mean-variance framework and of the CAPM derive from the assumption which they relies on. In particular, these assumptions are based on the belief that markets are mostly, but not completely, efficient. If markets were completely efficient, investors cannot earn excess returns without bearing extra risk. In an efficient market all information is reflected in current stock price and, for example, any news or relevant information about some company would not allow arbitrage opportunities by anyone in the market. In an efficient market, stock prices move randomly, but when stock prices do move in some predetermined fashion, then investor can make money by trading around the pattern, which means that there is an inefficiency in the market.

In the reality, many investment professionals and academics have observed patterns in historical financial data that contradict the theory of efficient markets (these patterns are well known anomalies such as the calendar effects, size effect, price-to-earning effect and so on). An anomaly suggests that investors habitually fail to consider and correctly interpret all the information relevant to the investment decision, or that institutional barriers prevent them from acting on certain information, or that, even with all the relevant information staring them in the face, they persist in making irrational choices.

This is the essential reason why markets are not perfectly efficient, which is, of course, to the benefit of quantitative equity portfolio managers and other arbitrageurs which otherwise would just prefer passively mirror market indexes.

Moreover, there are many studies and empirical researches that prove the non-consistency of the model: the failure of the markets to satisfy CAPM constraints is explored at great length in many ways in the book *Financial Econometrics* by Campbell, Lo, and McKinley. Other researchers (J.Linter, "Security Prices, Risk, and Maximal Diversification," *Journal of Finance*, 1965; M. H. Miller and M. Scholes, "Rate of Return in Relation to Risk: A Reexamination of Some Recent Findings," *Studies in the Theory of Capital Markets*, 1972) have conducted empirical tests of the CAPM with results that are far from satisfying.

Nevertheless this apparently fatuous model influences the way people think about markets. For example, there are many investment funds (hedge funds mostly) that claim to be "beta neutral". This means that their returns are uncorrelated with some overall market index. In the Gaussian world, this would mean that the fund returns are independent of the market index. If there were many such funds, and if they were independent of each other, this would

be a wondrous investment opportunity. "*Just put a little money in each one and (through diversification) earn a high return at very low risk. It is highly unlikely that there are many brilliant investors with secret and independent high yield beta neutral investment strategies*[11] ".

[11] Risk and Portfolio Management with Econometrics, Courant Institute, Fall 2009
http://www.math.nyu.edu/faculty/goodman/teaching/RPME09/index.html
Jonathan Goodman1, goodman@cims.nyu.edu

3. Heuristic Portfolio Construction Techniques

3.1. Equally Weighted Portfolio

Equal weighting is simply assigning the same weight to every stock. This asset allocation strategy is perhaps the most basic one: the investor simply puts equal capital weights on the different assets and rebalances when appropriate. If there are n stocks, each stock i will have the weight of:

$$x_i = \frac{1}{n}$$

The idea of equally weighted portfolio (EW) is to define a portfolio independently from the estimated statistics and properties of stocks.

On the negative side, this method does not take into account any information regarding the individual assets' characteristics: return, volatility and correlation with other assets. This means that the portfolio probably is not well diversified and therefore the investor takes on risk that he or she is not paid for. Equal weighting makes sense only when the portfolio manager has very poor information about the expected return and the risk of stock selected.

On the positive side, it is so simple that almost anyone can do it, and it ensures very low transaction costs due the minimum turnover rate that it requires by construction. Another appealing property is that the EW (or 1/n portfolio) is the least concentrated portfolio in terms of weights; therefore, the Herfindahl index[12] and Gini coefficient[13] applied to weights reach their minimum for this portfolio. The EW in the technical analysis corresponds to a contrarian strategy with a take-profit scheme, because if one stock has a substantial return between two rebalancing dates, its weight will reset to $\frac{1}{n}$.

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[12] For a probability distribution π the Herfindahl index is defined as $H(\pi) = \sum_{i=1}^{n} \pi_i^2$ where π_i for $i = 1, \dots, n$ are the observations. The Herfindhal index takes value of 1 for a perfectly concentrate probability distribution and $\frac{1}{n}$ for the least concentrated distribution with uniform probabilities.

[13] Similarly to the Herfindahl index, the Gini coefficient measures the concentration among values of a probability distribution. It is based on the famous Lorenz curve of inequality, which is used also in asset management to measure the weight concentration. A Gini coefficient of zero expresses zero concentration where all values are equal, whereas a Gini coefficient of one (or 100%) expresses maximal concentration among values.

From a purely theoretical point of view, the EW portfolio coincides with the Markowitz efficient portfolio if the expected returns and volatilities of stocks are assumed to be equal and correlation is uniform (DeMiguel, Garlappi, & Uppal, 2009).

3.2. Most Diversified Portfolio

The Most Diversied Portfolio (MDP) was introduced by Choueifety and Coignard (2008) and builds upon a measure called the Diversication Ratio (DR). The DR is defined as the ratio between the portfolio's weighted average volatility and the portfolio's overall volatility. Since assets are not perfectly correlated, this ratio will typically be greater than one. Thus, the DR measures the diversification gained by holding non-perfectly correlated assets. By maximizing this measure, one can obtain the Most Diversied Portfolio.

Choueifaty and Coignard (2008) introduce the concept of diversification ratio, which corresponds to the following expression:

$$DR(x) = \frac{\sum_{i=1}^{n} x_i \sigma_i}{\sigma(x)} = \frac{x^{\mathrm{T}} \sigma}{\sqrt{x^{\mathrm{T}} \Sigma x}}$$

The numerator is then the portfolio's volatility ruling out the diversification effect induced by the correlations. By construction a portfolio which is perfectly concentrated in one asset has a diversification ratio equal to one, whereas in general $DR(x) \geq 1$.

The MDP is then defined as the portfolio which maximizes the diversification ratio or equivalently its logarithm:

$$x^* = \arg \max \ln DR(x)$$

$$\text{u. c.} \quad \begin{cases} \mathbf{1}^{\mathrm{T}} x = 1 \\ \mathbf{0} \leq x \leq \mathbf{1} \end{cases}$$

The second constraint $\mathbf{0} \leq x \leq \mathbf{1}$ is added when we consider long-only portfolios.

It can be shown show that if x^* is the long-only MDP, then:

$$\rho(x, x^*) \geq \frac{DR(x)}{DR(x^*)}$$

The last inequality is considered by the authors to be the 'core property' of the MDP:

"The long-only MDP is the long-only portfolio such that the correlation between any other long-only portfolio and itself is greater than or equal to the ratio of their diversification ratios."

Moreover, Choueifaty (2011) concludes that:

"Any stock not held by the long-only MDP is more correlated to the MDP than any of the stocks that belongs to it. Furthermore, all stocks belonging to the MDP have the same correlation to it. [...] This property illustrates that all assets in the universe are effectively represented in the MDP, even if the portfolio does not physically hold them. [...] This is consistent with the notion that the most diversified portfolio is the un-diversifiable portfolio."

Finally yet importantly, Choueifaty and Coignard (2008) find another interesting property concerning the optimality of the MDP. If all assets have the same Sharpe Ratio:

$$\frac{\mu_i - RF}{\sigma_i} = SR$$

the diversification ratio of portfolio x is proportional to its Sharpe ratio:

$$DR(x) = \frac{1}{SR} \frac{\sum_{i=1}^{n}(\mu_i - r)}{\sigma(x)} = \frac{1}{SR} \frac{x^{\mathrm{T}}\mu - r}{\sigma(x)} = \frac{SR(x|r)}{SR}$$

Maximizing the diversification ratio is then equivalent to maximizing the Sharpe ratio. Therefore, the MDP (in the case of uniform SR along all the assets) is the tangency portfolio. This is why the MDP is synonymous with the MSR portfolio presented by Martellini (2008).

The lack of an underlying economic theory, or an underlying utility function of the DR, makes it a bit unclear to why investors should maximize this measure. Also, for the MDP to end up on the efficient frontier, all assets must have identical Sharpe ratios. The assets are typically not required to be identically correlated, but the lack of this assumption at the efficient frontier implies arbitrage opportunities. However, when assets are identically correlated, it can be shown that the MDP and ERC portfolios are the same (Gunnvald & Lagerqvist, 2013).

3.3. Risk Parity Approach

The Risk Parity is a new risk-based investment style that has emerged right after the subprime crisis in a period where investors were lowering their risk tolerance. Although the importance of risk considerations in asset allocation is widely supported even in the classic theories, the idea is often simplified to volatility minimization as described in the GMV case. Even so, it is proved by many authors that mean-variance optimization, generally leads to portfolios concentration in terms of weights. Slight differences in inputs can lead to dramatic changes in allocations and create portfolios heavily invested on very few assets. There is also confusion between optimizing the volatility and optimizing the risk diversification that could be naively described by the general "*don't put all your eggs in one basket*" concept[14]. These ones, together with other problems related to the mean-variance optimization method will be empirically investigated later on and specifically compared with the risk parity strategy.

This new heuristic solution based on risk concentration, is today preferred by many investors that do not want to use optimization methods because it puts diversification as the base of the investment process and, in addition, the corresponding strategies share the property that they do not use any performance forecasts as inputs of the model. In a risk budgeting approach, the investor only chooses the risk repartition between assets of the portfolio, without any consideration of returns. Unfortunately, even if risk budgeting techniques are widely used by market practitioners, there are few results about the behavior of such portfolios in the academic literature.

3.3.1. Risk Budget portfolio

Given a portfolio of n assets, x_i is defined as the weight of the ith asset and $RM(x_1, \dots, x_n)$ as a risk measure for the portfolio $x = (x_1, \dots, x_n)$. If the risk measure is coherent and convex, meaning that it satisfies the properties of monotonicity, sub-additivity, homogeneity, and translational invariance[15], then it verifies the following Euler decomposition:

$$RM(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \frac{\partial RM(x_1, \dots, x_n)}{\partial x_i}$$

[14] Bruder, B., & Roncalli, T. (2012). *Managing Risk Exposures using the Risk Budgeting Approach*.
[15] Philippe Artzner, F. D.-M. (1998, July 22). *Coherent Risk Measure of Risk*.
More information available at http://www.math.ethz.ch/~delbaen/ftp/preprints/CoherentMF.pdf

The risk measure is then the sum of the product of the weight by its marginal risk. Therefore the risk contribution is define for the ith asset as follows:

$$RC_i(x_1, \dots, x_n) = x_i \frac{\partial RM(x_1, \dots, x_n)}{\partial x_i}$$

Considering a set of given risk budgets $(b_1, \dots, b_n)$, where b_i is an amount of risk measured, the risk budgeting portfolio is then defined by the following constraints:

$$\begin{cases} RC_1(x_1, \dots, x_n) = b_1 \\ \vdots \\ RC_i(x_1, \dots, x_n) = b_i \\ \vdots \\ RC_n(x_1, \dots, x_n) = b_n \end{cases}$$

Choosing the volatility as the risk measure we obtained from equation in section 2.1 ($\sqrt{x^{\text{T}} \Sigma x}$):

$$RM(x) = \sigma(x) = \sqrt{x^{\text{T}} \Sigma x}$$

Since the vectorial derivative of $\sigma(x)$ with respect to the weights x_i is equal to:

$$\frac{\partial RM(x)}{\partial x_i} = \frac{\partial \sigma(x)}{\partial x_i} = \frac{(\Sigma x)_i}{\sqrt{x^{\text{T}} \Sigma x}}$$

Thus, it comes that the marginal risk and the risk contribution of the ith asset are respectively:

$$RC_i(x_1, \dots, x_n) = x_i \frac{\partial \sigma(x)}{\partial x_i} = x_i \frac{(\Sigma x)_i}{\sqrt{x^{\text{T}} \Sigma x}}$$

Since the system is too general to define an interesting portfolio from the asset management point of view, the authors (Roncalli *et al*. 2012) add some constraints to specify the right Risk Budget portfolio. In particular they impose that risk budgets are strictly positive (specifying that some assets have a negative risk contribution implies that the risk is highly concentrated in the other assets of the portfolio), that weights are all positive (long-only constraint), and that budgets and weights are defined in relative value. Thus, the new system will result as follow:

$$\begin{cases} x_i \, (\Sigma x)_i = b_i \, \sqrt{x^{\mathrm{T}} \Sigma x} \\ b_i > 0 \\ x_i \geq 0 \\ \sum_{i=1}^{n} b_i = 1 \\ \sum_{i=1}^{n} x_i = 1 \end{cases}$$

It must be noted that if a portfolio manager would like to impose some risk budgets are equal to zero, he first has to reduce the universe of assets by excluding the assets corresponding to these zero risk contributions, because in practice he does not expect to have the corresponding asset in his portfolio. That's why it is important to impose the strict constraint $b_i > 0$. To summarize, the RB portfolio is the solution of the following mathematical problem:

$$x^* = \left\{ x \, \epsilon \, [0,1]^n : \sum_{i=1}^{n} x_i = 1, \ x_i \, (\Sigma x)_i = b_i \, \sqrt{x^{\mathrm{T}} \Sigma x} \right\}$$

Where $b \, \epsilon \,]0,1]^n$ and $\sum_{i=1}^{n} b_i = 1$

3.3.3. Comparison with the mean-variance optimized portfolios

Allocating a certain budget or equal share of risk (e.g. ERC) to each asset class is the idea of risk parity, whose returns have outstripped conventional approaches to strategic asset allocation in the past years16. The key concept is the uncertainty inherent in the return forecasts used in asset allocation, which is additional to standard asset price risks. When this uncertainty is large, ignoring forecasts, as risk parity does, is the optimal response. With less uncertainty, there is useful information in forecasts, and risk parity's advantage recedes.

Apart from that, a crucial drawback of the mean-variance optimized portfolios (MVO) is its sensitivity to the inputs due to the fact that the model are based on optimization techniques. Bruder and Roncalli (2012) empirically prove this result, revealing also that the RB portfolio is more robust and less sensitive. In a dynamic investment strategy, the input parameters (i.e. asset returns, volatilities and correlations) will change from one period to another. In particular, this lack of robustness penalizes so much MVO portfolios because it implies higher transaction costs in rebalancing the portfolio each period, so that they are not used in practice without

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[16] Nottebohm, P. R. (2012, October). *Improving on risk parity - Hedging forecast uncertainty.* Retrieved from http://cameronmckenna.jpmorganassetmanagement.co.uk/Institutional/_documents/JPM5372_Improving_on _Risk_Parity_UK_MINI.pdf

introducing some constraints. As it will be seen in the section 5, the weights of the ERC (a particular type of RB strategy) are much more uniform along the time for every portfolio than any other strategy (excluding, of course the equally weighted).

"The indifference of many investment practitioners to mean-variance optimization technology, despite its theoretical appeal, is understandable in many cases. The major problem with MV optimization is its tendency to maximize the effects of errors in the input assumptions. Unconstrained MV optimization can yield results that are inferior to those of simple equal-weighting schemes" (Michaud, 1989).

For MVO portfolios, the risk approach is marginal and the quantity of interest to study is the marginal volatility. For RB portfolios, the risk approach becomes global by mixing the marginal volatility and the weight.

Comparing the risk budgeting portfolio with the weight budgeting portfolio, which is a naïve interpretation of the RB where the budgets set by the portfolio manage weights directly define the asset weights instead of their risk contribution. Roncalli and Bruder (2012) show that the RB portfolio is located between the global minimum variance portfolio and weight budgeting portfolio. If x_{wb} is the weight budgeting (WB) portfolio, that is $x_i = b_i$ for $i = 1, \dots, n$ it can be prove that:

$$\sigma_{GMV} \leq \sigma_{RB} \leq \sigma_{WB}$$

3.3.4. Equally Risk Contribution Portfolio

As written by Clifford S. Asness, A. F. (2012) in *Leverage Aversion and Risk Parity*, the idea of the ERC can be resumed sum in the concept of *"Diversify, but diversify by risk, not by dollars–that is, take a similar amount of risk in equities and in bonds"*

In a Risk Budgeting portfolio construction method, as we have just seen, an investor determines asset class weights based on their contribution to portfolio risk, which is mostly represented by its volatility. In the special case where the investor wants the assets to contribute equally much to the portfolio risk, an asset allocation approach known as risk parity or equal risk contribution (ERC) is obtained. This portfolio has been extensively studied by Maillard *et al.* (2010) who had derived several interesting properties. In particular, they have shown that this portfolio is located between minimum variance and equally-weighted portfolios.

Equal risk contribution has over the past decade gained attention since it has shown to outperform several classical types of asset allocation strategies. One of the reasons that the ERC approach emerged was that in the standard institutional portfolio consisting of 60 % equity and 40 % bonds/fixed income, the equity part stands for over 90 % of the risk[17]. This, combined with the fact that e.g. a minimum variance approach is based on forecasts of future returns from historical data give rise to a lack of robustness that the ERC approach seems to mitigate. However, the underlying economic theory supporting an ERC approach is somewhat vague[18].

The ERC portfolio corresponds to the portfolio in which the risk contribution from each stock is made equal.

$$RC_i = x_i \cdot \frac{\partial \sigma(x)}{\partial x_i} = x_j \cdot \frac{\partial \sigma(x)}{\partial x_j} = RC_j$$

It is the simplest risk budgeting rule. If we assume volatilities and correlations can reasonably be forecasted but that it is impossible to predict asset returns, then attributing an equal budget of risk to all the portfolio components seems natural. The main advantages of the ERC allocation are the following:

- It defines a portfolio that is well diversified in terms of risk and weights
- Like the previous methods, it does not depend on any expected return hypothesis
- It is less sensitive to small changes in the covariance matrix than the GMV or MDP portfolios

Like the equally weighted portfolio, it is difficult to locate the ERC portfolio on the mean-variance framework, but it corresponds to the optimal portfolio when the correlation is uniform and the assets have the same Sharpe ratio. This is the reason why the ERC portfolio coincides with the MDP when the correlation is uniform.

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[17] Yazann Romahi, K. S. (2012, July). J.P. Morgan Asset Management. *Diversification – still the only free lunch?* Retrieved from jpmorgan.com: https://am.jpmorgan.com/gi/getdoc/1383169204306
[18] Maillard, S., Roncalli, T., & Teiletche, J. (2009). *On the properties of equally-weighted risk contributions portfolios.*

3.3.5. Recent Views on ERC

The risk parity approach and, in particular, the equal risk contribution have become progressively popular because of the properties just seen, getting the attention of both academics and the practitioners. For instance, Romahi and Santiago (2012) from J.P. Morgan in *"Diversification – still the only free lunch?"* have investigated data in various time periods from 1927 to the present day, suggesting that a factor premium risk parity approach outperforms the traditional ERC approach. This risk factor approach is based on the diversification among specific risk factors such as value premia, size and momentum, instead on traditional asset classes (equity, bonds, commodities etc.) defined in a more traditional way. In so doing, risk factors (and their premia) are systematic exposures that are rewarded with a return above the risk free rate uncorrelated to equity returns.

Actually, one concern with the traditional ERC is the increasing correlation between asset classes. In their analysis of U.S. markets from 1927 till 2011, Romahi and Santiago (2012) show that the rolling correlation between risk factors are much lower when compared to traditional asset classes in almost every period.

On the other hand, they claim that some of the risk factors might be difficult to capture and/or illiquid for investors to access, and besides that, much of the benefits may be not achieved if the investor does not have mandate to use leverage or short sell. Nonetheless, according to Romahi and Santiago, the market should be moving in a direction that makes the risk factors more liquid, and that a long-only risk factor approach still would be more beneficial than a traditional equal risk contribution approach.

Furthermore, other authors, Lee (2010)[19] and Rappoport and Nottebohm (2012)[20], see the risk parity, and specifically the ERC, as a starting point for an investor without a clear view of the markets. Rappoport and Nottebohm find out that equal risk contribution approach has performed well during the recent periods of great uncertainty, but allocating equal shares of risk is meaningless as an objective itself. They claim that if an investor has useful information in forecasting returns, it would not be wise to ignore a model built on those estimate but if the investor is a bit uncertain, then he or she might tone down his or her reliance on this forecast,

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[19] Lee, W. (2010). Risk-Based Asset Allocation: A New Answer to An Old Question? *The Journal of Portfolio Management*.

[20] Nottebohm, P. R. (2012, October). *Improving on risk parity - Hedging forecast uncertainty.* Retrieved http://cameronmckenna.jpmorganassetmanagement.co.uk/Institutional/_documents/JPM5372_Improving_on _Risk_Parity_UK_MINI.pdf

which is exactly what the risk parity approach does, and hence consider it to be a starting point, just as Lee (2010) does. In their work, the two authors introduce a new asset allocation rule that calibrates the amount of uncertainty in forecasts and weights risk parity and traditional asset allocations accordingly. This new approach called "Forecast Uncertainty Hedge" outperforms both in their tests the traditional ERC in 70 % of the cases. In their view, this forecast hedge rule, and the discipline it places on forecasts, should become integral to strategic asset allocation decisions.

In the case of Lee, the author tries to shed some light on what risk parity approaches attempt to achieve and on the characteristics of the investment universe, as long as these approaches are meant to approximate mean-variance efficiency. Rather than adding to the already large collection of simulation results, Lee uses some simple examples to compare and contrast the portfolio and risk characteristics of these approaches. He also claims that any portfolio which deviates from the market capitalization–weighted one is an active portfolio, concluding that there is no theory to predict, ex ante, that any of these risk-based approaches should outperform the other ones, and this new method should be applied as a starting point for investors which are no well-informed.

3.4. Capitalization Weighted Indexation versus Risk Based Indexation

One of the fastest growing types of mutual fund in the last twenty years are the index funds. One of the largest index fund – the Vanguard 500 Index Fund – has grown in size from about 1 billion dollar in 1988 to more than 75 billion dollars in 2001, reaching now more than 142 billion in terms of total Net Asset Value (NAV). Index Funds are designed to closely track the performance of a stock or bond market index. The most popular index funds are designed to track the S&P500 stock index. The case for index funds can be quite compelling. For one thing, index funds have much lower expenses than actively managed funds. For another, index funds often outperform actively managed mutual funds[21]. The failure of many active managers to generate value has caused investors to shift their focus from active management to passive exposure to the market. This encourages investors to study the alternative ways in which they can gain equity market exposure.

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[21] Hearth Douglas, J. K. (2001). *Contemporary Investiments: Security and Portfolio Analysis*. Harcourt College Publishers.

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Among all the possible allocation method that can be employed by index funds, the most popular is the capitalization weight (CW) or value weight. Value weighted indexes are the most common way to gain access to broad equity market performance because they provide two main advantages: simplicity of management (low turnover and transaction costs) and ease of understanding and replication. The most important ones are probably the MSCI Value Weighted Indexes, present in almost every geographical areas of the world, they usually are the primary benchmarks used by fund managers for performance comparison.

Nevertheless these portfolios are generally concentrated in a few stocks and present some lack of diversification. In order to avoid this drawback or to simply diversify market exposure, alternative indexation methods have recently prompted great interest, both from academic researchers and market practitioners. Fundamental indexation computes weights with regard to economic measures, while risk-based indexation focuses on risk and diversification criteria.

3.4.1. Value Weighting

Value weighting consists in assigning weights proportional to the stock's market capitalizations. For example, if the market capitalizations of the n stocks are $mc_1, mc_2, \ldots, mc_n$, then the weight x_i of stocks i, is

$$x_i = \frac{mc_i}{\sum_{i=1}^{n} mc_i}$$

Just like the equal weighting, value weighting does not reflect the expected returns or the risks of the selected stocks, so it is also not the best method when there is good information about stock returns and risks. In the absence of such information, value weighting may be an improvement on equal weighting. The performance of the value-weighted portfolio is at least guaranteed to match the market average performance because market capitalizations are the weights, so to speak, that the market assigns to stocks. There are also a number of variations on value weighting: for instance, many portfolios and indices can be float-weighted, which has the same sort of characteristics as value weighting, except stocks are weighted by their floating shares, not their shares outstanding.

There are several arguments supporting capitalization-weighted indexation. First, a CW index is comprehensible and data-independent; closing prices and the number of shares are available without any measurement errors. Second, trading costs are low, because it corresponds to a buy-and-hold strategy if the number of shares remains unchanged. The CW

index may be also easily hedged and replicated because liquidity is highly correlated with market capitalizations.

From the theoretical point of view, under the efficient market hypothesis assumed by the CAPM, the theory states that the tangency portfolio is the unique risky portfolio owned by investors, identified by the CW one.

However, value weighting presents some drawbacks. For example, CW indexation is by definition a trend-following strategy where momentum bias leads to bubble risk exposure as weights of best performers increase and the weight of the worst performers decreases: the larger the return of the stock, the greater its weight. In this context, the concept of alternative-weighted indexation emerged after the dot.com bubble. A CW index generally contains a growth bias, because high-valuation multiple stocks weigh more than low valuation multiple stocks with equivalent realized earnings. Moreover, the absence of portfolio construction rules leads to concentration issues (in terms of sectors or stocks) and the index may suffer from a high drawdown risk and a lack of risk diversification.

Finally, instead of equal weighting or value weighting, portfolio managers can also use price weighting. In the price-weighting scheme, the managers buy the same number of shares in each stocks so that the weights are proportional to the prices stocks. If the share price of a stock is high, this stock will have relatively more weight in the portfolio. This is the way the Dow Jones Industrial Avarage and the Nikkei 225 Index are calculated.

3.4.2. Risk Based Indexation

While traditional market cap indexes remain the most efficient tools for capturing market beta as defined in the CAPM, investors are increasingly aware that there are additional sources of return associated with particular investment styles and strategies. Alternatively weighted (AW) indexes, are designed to offer alternative beta exposures, from which the name of "smart beta", such as the Risk Parity ones.

An alternative weighted index is defined as an index in which assets are weighted differently than in the market capitalization approach. Generally they are distinguished in two forms of indexation: fundamental and risk-based. Fundamental indexation defines the weights as a function of economic metrics like dividends or earnings, whereas risk-based indexation defines the weights as a function of individual and common risks and they aim to capture a

broad equity opportunity set with lower risk attributes than comparable market cap weighted indexes.

In some sense, the difference between the two methods comes from the opinion of modifying the risk-adjusted return ratio. In other words, in the case of fundamental indexes, one expects to have superior returns with respect to the CW index. In the case of risk-based indexes, one expects to decrease the risk of the portfolio in either absolute or relative value.

In the particular case of risk budgeting, for a very long time, this technique has been applied to a universe of multi-assets classes to manage and monitor the portfolio risk of large and sophisticated institutional investors like pension funds (Sharpe, 2002). More recently, instead, it has been used to build alternative indexes in order to provide new benchmarks than the traditional market-cap indexes.

In this section, we compare four risk-based indexation methods based on EW, GMV, MDP and ERC portfolios that we have previously seen. Although the four methods are based on different approaches, they present similarities. First, we can compare them in terms of weights and risk contributions and recap what already presented:

- $x_i = x_j$ $\quad\quad for\ i, j = 1, \dots, n$ $\quad(EW)$

- $\dfrac{\partial \sigma(x)}{\partial x_i} = \dfrac{\partial \sigma(x)}{\partial x_j}$ $\quad\quad$ " " $\quad\quad(GMV)$

- $x_i \dfrac{\partial \sigma(x)}{\partial x_i} = x_j \dfrac{\partial \sigma(x)}{\partial x_j}$ $\quad\quad$ " " $\quad\quad(ERC)$

- $\dfrac{1}{\sigma_i} \dfrac{\partial \sigma(x)}{\partial x_i} = \dfrac{1}{\sigma_j} \dfrac{\partial \sigma(x)}{\partial x_j}$ $\quad\quad$ " " $\quad\quad(MDP)$

The weights are equal in the EW portfolio whereas the marginal risk is equal in the MV portfolio. In the case of the ERC portfolio, this is the product of the weight times the marginal which is equal. For the MDP/MSR portfolio, the equality is on the marginal risk divided by the volatility (this measure may be interpreted as relative or scaled marginal risk). We notice that the equalities are verified in the case of the MV or the MDP portfolio only for the assets with a non-zero weight. In fact, these methods can be group into two groups: one where all weights are strictly positive, and the other where some weights may be negative:

- $\forall\, i: x_i > 0$ $\quad\quad(EW/ERC)$
- $\exists\, i: x_i = 0$ $\quad\quad(MDP/GMV)$

Another important result, which will be empirically verified later on, is that the volatility of the MV, ERC and EW portfolios may be ranked in the following order (Maillard et al., 2008):

$$\sigma_{GMV} \leq \sigma_{ERC} \leq \sigma_{EW}$$

The ERC portfolio may then be viewed as a portfolio between the GMV and the $1/n$ portfolios. The volatility of the MDP/MSR portfolio may be either greater or lower than the volatility of the ERC and 1/n portfolios.

It is important to mention that the ERC and MDP/MSR portfolios coincide when the correlation is uniform across assets returns. In this case, the weight of stock x_i is inversely proportional to its volatility σ_i. The MDP portfolio corresponds to the GMV portfolio when the individual volatilities σ_i are equal. Finally, the ERC and GMV portfolios are the same when the correlation is uniform and is equal to the lower bound $\rho = -\frac{1}{n-1}$ meaning that the diversification obtained with the simply correlation is maximum.

4. Data description

4.1. Stocks

The dataset used in this work contains time series, of weekly stock prices for the past 25 years, starting from the 3rdof October 1989 to the 29th of September 2014, for a total of 1305 weeks. Thirty among the most capitalized stocks in well-known equity indices for three different geographic area (United States, Europe and Japan) constitute the investment universe.

In particular, for the United States, stocks belonging to the Standard & Poor's 500 and the Dow Jones Industrial are taken into account, whereas ten stocks from each of the following indices are selected for Europe: CAC40 (France), DAX30 (German), FTSE100 (UK). Finally, stocks are selected from the Nikkei225 constituents for Japan.

The stock prices correspond exactly to the weekly, dividend adjusted, market closing price, and all the data have been obtained from DataStream - Thomson Reuters.

It is widely claimed by practitioners and researchers that daily prices are mostly used to study trends for the past year. For longer range analysis, going back five, ten years or twenty-five as in our case, weekly and monthly charts may be better to employ. Using weekly charts instead of daily or intraday has the advantage of focusing on the predominant longer-term trend, while ignoring the "noise" and volatility of the day-to-day fluctuations. In appendix 1 the price evolution of each single stock in the time considered are presented.

4.2. Methodology

4.2.1. Estimating Returns, Risk and Weights

We first describe the data used to build each portfolio, i.e. how the historical performance indicators have been calculated in order to define the stock weights to each portfolio. The analytical procedures used to calculate the weights instead, are illustrated in the Appendix 2.

The standing point here is to consider the perspective a portfolio manager (PM) who wants to build a mid-long term stock portfolio using different criteria which can be related either to performance optimization (i.e. maximum Sharpe), risk minimization (i.e. minimum variance) or risk parity measures (i.e. equal risk contribution, most diversified portfolio etc.).

In so doing the PM would first focuses on a certain basket of stocks which can be defined by an index, a geographical area, a sector or some common fundamentals. Then he would analyze the historical performances of the assets and find the weights to assign to them accordingly to the rational of the specific portfolio.

For the purpose of this work, the strategy used to build the different portfolios follows the same logic as above. The basket of stocks considered are differentiated on a geographical basis and on their market capitalization. In regards to the analysis of the historical performances, the starting point is the end of September 1989 with a rolling time window of 5 years. In other words, we assess the first portfolio weights in October 1994 using the weekly prices of the stocks in the prior 5 years. From that date onward, the weights are computed on a weekly basis, moving on the 5 years rolling window until the end of September 2014.

Estimating the Expected Returns

Whichever model is used, estimation of the model provides important information about a stock's expected returns. Concerning this purpose, the first step is to consider weekly percentage returns of historical stock prices, where for every stock we have that:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

Where P_t is the price of a stock at time t. The percentage return is then computed for each stock, 30 per geographical area, and for every week along the 25 years.

In the second step a time window of 5 years (corresponding to 260 weeks) is defined, in which computing the mean (μ) of r, with a weekly frequency, as follow:

$$\mu_i = \frac{\sum_{j=1}^{t+j-1} r_j}{t}$$

Where t is equal to 260 weeks, j goes from the week 1 to 1044, and i that goes from the 261th week to the 1304th corresponding to the 29th of September 2014, date of the last observation.

In other words, the expected return (μ) is calculated every week, for the following period, as the average of the past 260 weekly logarithmic returns. This computation has been developed through a Matlab iteration presented in Appendix 2.

In this way, for each single stock, we end up having a vector of 1304 r out of the 1305 weekly (25 years) prices, and 1044 weekly (20 years) expected returns since the first μ is determined after the initial 5 years which, in this analysis, corresponds to the 3rd of October 1994.

The variance - covariance matrix

The relevant inputs for the allocation models presented previously in section 2 (*Modern Portfolio Theory*) and 3 (*Heuristic Portfolio Construction Techniques*), are constituted by expected returns and risk measures. By calculating the variance or the standard deviation of stock returns, it is possible to naively estimate the stock's total risk. The first thing to do is to compute the variance-covariance matrix ($\mathbf{\Sigma}$), analytically:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \dots & \sigma_{1,j} & \dots & \sigma_{1,n} \\ & & \vdots & & \\ \sigma_{i,1} & \dots & \sigma_i^2 & \dots & \sigma_{i,n} \\ & & \vdots & & \\ \sigma_{n,1} & \dots & \sigma_{n,j} & \dots & \sigma_n^2 \end{bmatrix}$$

Where

$i, j = 1, \dots, n$ indicates each different stock (n is equal to 30 in our case);

The diagonal of $\mathbf{\Sigma}$ is composed by the variance of the log returns for each stock (σ_i^2), meanwhile the other elements constitute their covariance ($\sigma_{i,j}$). For each week, these two fundamentals are computed as follow:

$$\sigma_i^2 = \frac{1}{t}\Sigma_k^t (r_{k,i} - \mu_i)^2$$

$$\sigma_{i,j} = \frac{1}{(t-1)}\Sigma_k^t (r_{k,i} - \mu_i)(r_{k,j} - \mu_j)$$

Where t is equal to the time window of 260 weeks. Understandably the time horizon applied in this setting is the same used for expected returns, therefore 1044 variance –

covariance matrixes[22] are generated week by week, from the 261th one to the last 1304th.

Weighting the portfolios

The portfolio manager can use optimization techniques to select the stocks for the portfolio and assign them their respective weights. The weighting of stocks in the portfolio can be set to maximize the portfolio's overall return, minimize its risk, and satisfy other constraints such as diversification requirements and specific investment style requirements. The objective of this work is to compare different strategies in the construction of equity portfolios which, from an operating point of view, means that the first thing to do is to find for each approach the matrix of weights.

Therefore, stock weights are the key unknown variables in this framework because they identify the allocation strategy that we want to asses and, once we obtain them, the rest of the work is rather straightforward.

In the previous sections, we have briefly outlined, for every portfolio construction method, the analytical procedure to find the weights (x_i) for stock $i = 1,..,n$. These procedures have been implemented with Matlab and the functions and codes used in this work, to find the weights of each portfolio are presented in the Annex[23].

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[22] I.e. an Array with 3 dimensions n x n x $time$ (30x30x1044).

[23] For the ERC, MDP and long-only GMV portfolios it has been used a Matlab built-in function "FMINCON" which solve for minimum of constrained nonlinear multivariable function. More information available at http://uk.mathworks.com/help/optim/ug/fmincon.html. For the ERC and MDP functions, the codes used have been developed respectively by Dr Farid Moussaoui (http://mfquant.net/erc_portfolio.html) and Enareta Kurtbegu (http://erasmus-mundus.univ-paris1.fr/fichiers_etudiants/2664_dissertation.pdf) which are both acknowledged and thanked.

The outcome of this process is represented in the figures of the empirical evidence section. Each of them is the plotted area of the weight vectors for each method and geographical area, depicting their evolution on time for all of the 30 stocks. As shown in the next figure for the long-only GMV Japanese portfolio, taken as an example here, the sum of the stock weights at every time is equal to one (or equivalently 100%), meaning that the portfolio is always fully invested, which indeed is an assumption valid for every method that we presented so far.

The x-axis denotes the time from 3-October-1994 to 29-September-2014. Every color
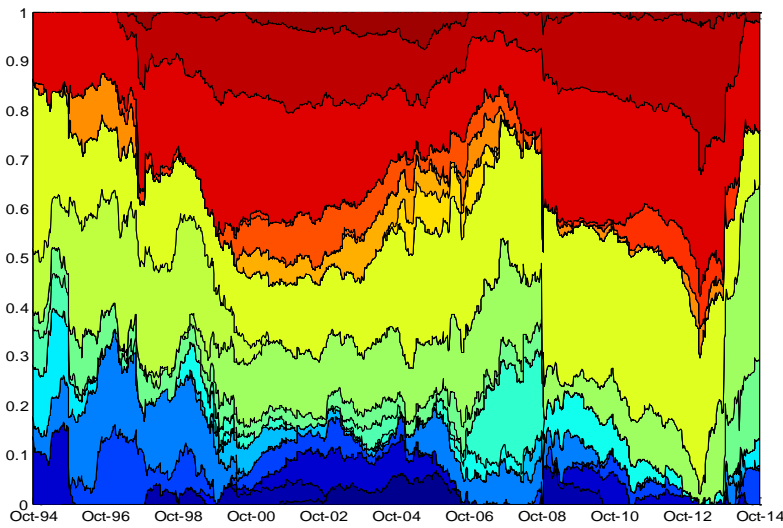
Figure 4.2.1 – Stock weights of GMVc Japanese portfolio

represents each different stocks and clearly, since the GMV portfolio is quite concentrated, not every stock appears in the graph.

Risk Contribution

As already shown in section 3 regarding the Risk Parity approach, the stock risk contribution is calculated likewise section 3.3.1 as follow:

$$RC_{i,t}(x_{1,t}, \dots, x_{n,t}) = x_{i,t} \frac{(\Sigma x)_{i,t}}{\sqrt{x^{\mathrm{T}} \Sigma x}}$$

Differently form $P.Ret_t$ and $P.Vol_t$ which are scalar, $RC_{i,t}$ is vector (30x1).

For every allocation strategy, the three operations are then computed for each of the 1044 weeks of this framework, finally resulting in two vectors of portfolio realized returns and volatilities defined in next paragraph($P.Ret - 1044x1$ and$P.Vol - 1044x1$) and in a matrix of

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stocks risk contribution ($RC - 30x1044$). As an example, the next figure depicts the evolution of the stocks Risk Contribution for the Equally Weighted Japanese portfolio, in the same fashion as the weights graph.
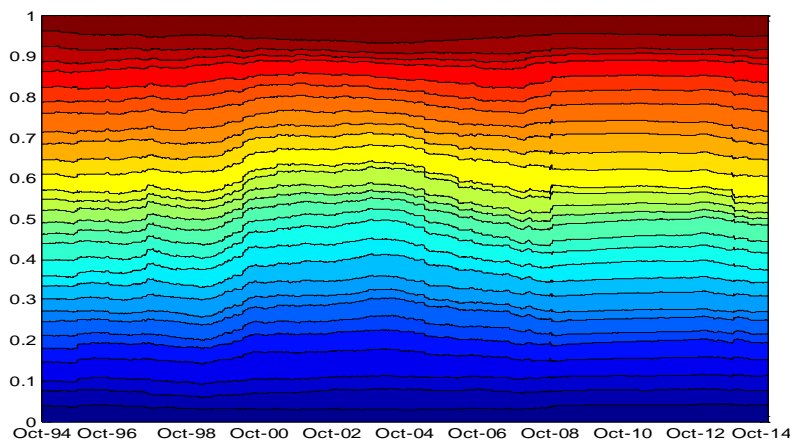

Figure 4.2.2 – Stocks' Risk Contributions for the EW Japanese portfolio

Differently from the previous graph of GMV weights, this time obviously each stock is represented by construction of the equally weighted portfolio itself.

4.2.2. Indicators of portfolio performance

The ratio used for comparing the different portfolios is based on a few performance indicators which are widely used by practitioners. It is important to remind that the portfolios in question correspond to the asset allocation strategy by which they are constructed.

Once we have the weight matrixes for each method, the portfolio realized returns, the historical volatility, the stocks risk contribution, the transaction costs, the historical Value At Risk and Expected Shortfall can be defined for each portfolio every week. The key element here is to apply the weights calculated in "t-1" to the set of information available in "t". For instance, from the point of view of a PM who wants to build the max Sharpe Ratio portfolio with the weights calculated today, he will observe its performance with the stock prices of tomorrow (t+1). Then, in "t+1" he will rebalance the weights and observe the performance given the prices of "t+2" and so on.

Realized Returns

Previously it has been defined $\mu_{i,t}$ as the expected return of stock $i = 1,..n$ at week number $t = 1,...,1044$. The portfolio realized return ($P.Ret$) at any time t, can be simply obtained multiplying each stock's realized return($Real.Ret$) simply defined by the percentage

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difference in stock prices, by their weight(x_i). In a matrix language $P.Ret_t$ is a scalar obtained by the multiplication of the vector of stock return $Real.Ret(30x1)$ and the weight's one $\boldsymbol{x}(30x1)$ calculated one period before:

$$P.Ret_t = Real.Ret_t^T * \boldsymbol{x_{t-1}}$$

Where the realized percentage return for a generic stock at time t is calculated as follows:

$$Real.Ret_t = \left(\frac{P_t}{P_{t-1}} - 1\right)$$

Volatility

Considering the returns standard deviation as a measure for the portfolio volatility, the procedure is then similar to the previous one. Instead of considering the expect return, we take into account the realized return of portfolios $(P.Ret)$ and compute it's standard deviation for the period considered.

Transaction Costs

Transaction costs generally correspond to the commission that brokerages charge to execute orders. These commissions vary among brokers and sometimes depend on the specific trade. The same broker may charge different commission for trading the same stock depending on the number of shares of the trade or the value of the transaction.

Given the complicated and unpredictable nature of transactions costs, it is conventional to model them as a fixed proportion of the total value of the transaction. In our framework a constant of 15 bps has been chosen to approximate the cost per transaction.

The turnover rate has been takes as a measure of transaction, designed as the weekly change in each portfolio weights. Specifically, the turnover rate has been computed as the sum of the absolute value of differences between the stock weights in two close periods:

$$P.Turnover_t = \sum_{i=1}^{n}|x_t - x_{t-1}|$$

In this way, $P.Turnover_t$ has always a value included between 0 (in case of no stock's sale or purchase) and 2 (in the special case of diametrically change the portfolio composition).

Thus, to obtain the transaction cost adjusted return of the portfolio ($P.AdjRet_t$), the constant of 15 bps is multiplied by $P.Turnover_t$ and the result is subtracted from the portfolio expected return ($P.Ret_t$) as follow:

$$P.AdjRet_t = P.Ret_t - 0.0015 * P.Turnover_t$$

Standard arguments imply the computation of $P.AdjRet_t$ for every week starting from the second one (in our settings the 10th of October 1994), resulting in a final vector of returns (**P.AdjRet**)with dimension 1043x1. No matter what conditions or costs arise, the principle of portfolio selection always remains the same. Transaction costs introduce a new variable into the process of determining the optimal portfolio, but they do not alter the selection principle itself.

Finally, it is essential to remind that the EW portfolio is excluded from the transaction costs analysis, since in this setting, it would result in a zero cost strategy which does not make much sense in the reality. Indeed even an equally-weighted portfolio has to be rebalanced: since stocks values change every week, then the weights which are equal at the beginning of the week are not the same anymore at the end of the week.

Diversification Ratio

In recent years, new portfolio construction techniques focused on risk and diversification rather than expected average returns have become quite popular as it has been claimed for the Risk budgeting approach. This success has been due to an increasing acknowledgment that a traditional balanced portfolio, where 60 percent is allocated in equities and the remaining 40 percent is invested in bonds, is not diversified at all. It may look balanced from a capital allocation point of view, but it is not from a risk perspective, as equities are the main risk contributor within such a portfolio. Therefore, it is remarkable to include as a key indicator of portfolio performance the diversification ratio defined in section 3 regarding the Most Diversified Portfolio (MDP) and computed in the same way as section 3.2:

$$P.DR_t = \frac{x_{t-1}{}^T \sigma_t}{\sqrt{x_{t-1}{}^T \Sigma_t x_{t-1}}}$$

where $\boldsymbol{\sigma_t}$ is the vector of stocks standard deviation, corresponding the square root of the diagonal of the variance-covariance matrix $\boldsymbol{\Sigma_t}$, calculated on the basis of the last 5 years weekly stock prices. Thus, following the same logic as before, the index is computed applying the vector of weights obtained in the period before.

5. Empirical Evidence

5.1. Overall results for US portfolios

The next tables present the overall findings of nine different portfolio construction methods[24]. The first indicator "return" is the annual cumulative realized return of the portfolio. Taking the year 1998 as example, it is calculated as follows:

$$Return_{1998} = \prod_{i=1}^{52}(1 + P.Ret_i) - 1$$

Where $P.Ret_i$ as calculated as in paragraph 4.2.2. and i indicates the number of week.

The second indicator "volatility" is the annualized standard deviation of weekly-realized returns:

$$Volatility_{1998} = \sqrt{\frac{\sum_{i=1}^{52}(P.Ret_i - \overline{P.Ret_i})^2}{52}}$$

Where $\overline{P.Ret_i}$ is the simple mean of weekly percentage returns:

$$\overline{P.Ret_i} = \frac{1}{52}\sum_{i=1}^{52}P.Ret_i$$

The following tables[25] highlight the characteristics and performance indicators of each portfolio considering only the United States market, while in the next sections the analysis is extended also to the other two geographical areas.

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[24] Recall the abreviations for the following portfolios: GMVc (long-only global minimum variance), GMVu (unconstrained g.m.v.), MSR (max sharpe ratio), MSR_12 (max sharpe with 12% risk contribution constraint), MCap (market capitalization weighted), EW (equally weighted), MDP (most diversified), ERC (equal risk contribution).

[25] The year 2014 consider the weekly stock prices until the 29th of September.

Table 1 - Returns

	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	32.96	31.66	38.77	36.58	36.53	38.57	41.52	36.36	38.87
1996	32.20	31.95	37.49	36.02	28.97	33.31	33.84	33.98	33.48
1997	19.11	15.48	17.04	17.95	17.64	31.50	30.47	23.43	28.82
1998	20.66	41.80	48.84	37.38	37.04	37.54	34.13	31.00	31.57
1999	9.70	-1.44	16.56	15.89	15.91	27.56	24.91	15.05	19.33
2000	5.12	-3.93	-23.99	-17.52	-8.64	-10.33	11.50	13.06	11.55
2001	5.58	7.08	-11.92	-9.23	-8.80	-3.12	-3.45	-1.38	-2.07
2002	-14.04	-11.44	-25.62	-25.84	-20.53	-19.90	-15.03	-11.84	-14.28
2003	19.56	13.35	24.74	21.88	23.31	18.57	27.54	22.33	25.72
2004	12.12	10.48	6.66	8.69	9.77	7.47	10.94	9.66	11.08
2005	13.36	10.95	21.66	14.23	14.51	1.87	5.87	14.69	7.13
2006	14.56	16.23	6.66	11.27	16.56	13.43	16.53	21.34	16.18
2007	13.91	15.82	21.92	22.07	13.92	4.77	6.02	12.69	7.03
2008	-22.50	-15.38	-26.81	-31.45	-29.03	-31.08	-27.81	-25.30	-25.26
2009	11.76	-4.49	0.90	17.74	11.59	14.55	27.63	19.16	22.97
2010	7.62	3.61	21.80	17.44	14.10	8.71	13.67	12.47	11.80
2011	11.91	17.98	22.82	12.36	9.21	1.72	2.84	11.20	8.08
2012	4.84	-3.23	-7.63	8.63	11.78	12.55	17.56	11.94	12.81
2013	21.63	15.07	21.46	21.67	24.92	25.28	29.45	24.65	26.66
2014	6.13	5.85	12.76	10.54	7.64	5.20	6.84	7.49	6.78

The portfolio returns is clearly the first and main performance indicator that anyone would look at. Reasonably, it is possible to observe how all the portfolios show a common trend during both periods of market crisis or economic growth. For example, in 2002 with the dot-com bubble, and in 2008 with the real estate bubble and consequently with the financial crisis, every portfolios show a negative cumulative annual returns. Similarly, in flourishing periods of U.S. economy, such as in the late 90s, or in a year of economic upturn like 2013 (at least from the financial market point of view), the performance of portfolios are all extremely positive. Nonetheless, it comes immediately evident how the methods based on measures of performance optimization are completely outperformed, in several years, by the ones based on risk diversification. The most significant example are the years 2000, 2001 and 2012. In 2000, for instance, the MSR portfolio is outperformed by the EW, MDP and ERC by more than 30% and this result gets only slightly better if considering the MSR with risk contribution constraints. However, the three MSR portfolios along with the MCap and the GMVu reveal negative returns in a period in which all the risk based portfolios gain around 10%. The straightforward explanation to this behavior is that, in these periods, the MSR and the GMVu simply fail to pick up the few stocks on which they put all the risk. This occurs because, at least in our model, the weights allocation is based on the historical performance of the assets in the previous 5 years.

An emblematic example is given by the composition of the MSR portfolio in 2012, which concentrates always more than 60% in the McDonalds stock, simply because the performance of that stock was outstanding in the preceding years, as shown in the next figure.
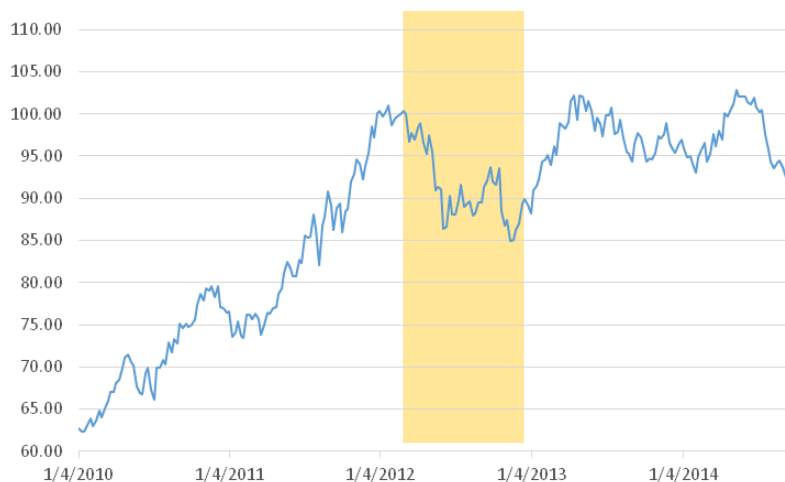

Figure 5.1 – McDonald's stock prices (USD)

Thus, sometimes it might happen that stocks, which performed the best in the recent past, in terms of returns and volatility, will fall down right after. Of course, this mechanism is amplified in periods of financial bubbles, with dramatic consequences in terms of return of some portfolios as presented in the table.

On the other hand, there are some periods in which the optimized methods have much higher, such as in 1998 where the MSR outperformed the diversified portfolios (EW, ERC, MDP) by almost 18%. Nevertheless, it seems quite indisputable that, overall, this last category of portfolios beats the other ones.

Moreover, this result is confirmed by considering the volatility and the Sharpe Ratio shown in Table 2 and Table 3. The Sharpe Ratio indicated is the simple ratio between return and volatility (e.g. zero *risk-free* rate), which gives an immediate hint of the risk/reward value of the investment. In Table 2 the value of the volatility is represented with different colors and intensity: red if it is higher than the average of all portfolios in the year considered, blue if it is lower; the more is the color intensity, the more is the value far from the average.

Unsurprisingly, the volatility of the GMVc and GMVu are the lowest in most of the years (1996, 1997, 1998, 1999, 2001, 2004, 2006, 2007, 2008, 2009, 2010, and 2012).

These two portfolios show also a similar trend in their Sharpe Ratio (Table 3) with the remarkable differences that in 2000, 2009 and 2013 the GMVu delivers negative returns, while in 1998 it has an outstanding performance with more than 20% return higher than the GMVc. The first hint that is possible to gather from these figures is that the build a minimum variance long/short portfolio does not seem the optimal choice, since the long-only GMV delivers the same roughly the same results.

Table 2 - Volatility

	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	8.93	8.94	12.25	10.11	8.55	8.08	8.17	9.08	7.80
1996	13.27	13.47	17.57	16.66	15.19	14.62	14.17	13.50	13.69
1997	16.59	15.50	21.57	21.18	19.08	20.31	18.66	17.69	18.11
1998	17.43	17.32	24.97	22.92	22.94	22.24	22.84	20.95	21.34
1999	15.55	16.59	24.07	20.99	18.26	19.95	17.97	15.70	16.35
2000	16.33	17.83	18.15	17.88	16.94	18.08	16.36	15.52	15.52
2001	12.78	12.92	15.54	15.16	14.07	15.63	14.33	13.11	13.57
2002	23.91	23.65	27.92	27.49	27.14	28.22	26.70	23.01	25.74
2003	14.94	14.44	15.45	13.68	13.55	15.43	15.41	15.82	14.74
2004	9.52	9.69	13.29	11.19	9.97	10.46	10.38	11.11	10.14
2005	10.02	11.38	11.92	10.40	9.36	7.90	8.48	10.12	8.61
2006	8.15	9.05	11.00	10.23	9.01	9.41	9.29	9.27	9.04
2007	11.33	10.43	13.86	13.32	13.37	14.03	13.75	12.68	13.49
2008	20.03	21.23	22.87	22.94	24.54	28.89	30.35	24.84	27.01
2009	17.43	15.10	18.66	24.60	23.22	26.76	33.27	25.27	26.21
2010	10.19	9.99	13.53	14.61	14.71	17.01	17.16	12.56	14.63
2011	12.27	9.65	15.28	18.62	19.87	23.46	24.68	16.08	19.96
2012	8.24	9.92	10.38	9.36	9.30	10.84	11.09	9.21	9.15
2013	9.51	9.84	12.10	9.83	9.66	9.76	10.03	10.23	9.34
2014	8.41	8.69	10.38	8.73	8.20	9.21	8.95	7.18	8.45

In general, the portfolios based on asset diversification show low values of volatility except the EW. In particular, the MDP and the ERC both deliver significantly higher returns and lower volatility. These two portfolios can be considered the "winners" from any point of vire, as it will be also proved later. The ERC, in turn, functions as a risk-weighted version of the EW, with significantly lower risk. On the other hand, the EW outperforms the market cap-weighted index with comparable volatility.

	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	3.69	3.54	3.16	3.62	4.27	4.78	5.08	4.00	4.98
1996	2.43	2.37	2.13	2.16	1.91	2.28	2.39	2.52	2.45
1997	1.15	1.00	0.79	0.85	0.92	1.55	1.63	1.32	1.59
1998	1.19	2.41	1.96	1.63	1.61	1.69	1.49	1.48	1.48
1999	0.62	-0.09	0.69	0.76	0.87	1.38	1.39	0.96	1.18
2000	0.31	-0.22	-1.32	-0.98	-0.51	-0.57	0.70	0.84	0.74
2001	0.44	0.55	-0.77	-0.61	-0.63	-0.20	-0.24	-0.11	-0.15
2002	-0.59	-0.48	-0.92	-0.94	-0.76	-0.71	-0.56	-0.51	-0.55
2003	1.31	0.92	1.60	1.60	1.72	1.20	1.79	1.41	1.74
2004	1.27	1.08	0.50	0.78	0.98	0.71	1.05	0.87	1.09
2005	1.33	0.96	1.82	1.37	1.55	0.24	0.69	1.45	0.83
2006	1.79	1.79	0.61	1.10	1.84	1.43	1.78	2.30	1.79
2007	1.23	1.52	1.58	1.66	1.04	0.34	0.44	1.00	0.52
2008	-1.12	-0.72	-1.17	-1.37	-1.18	-1.08	-0.92	-1.02	-0.94
2009	0.67	-0.30	0.05	0.72	0.50	0.54	0.83	0.76	0.88
2010	0.75	0.36	1.61	1.19	0.96	0.51	0.80	0.99	0.81
2011	0.97	1.86	1.49	0.66	0.46	0.07	0.12	0.70	0.40
2012	0.59	-0.33	-0.74	0.92	1.27	1.16	1.58	1.30	1.40
2013	2.27	1.53	1.77	2.21	2.58	2.59	2.94	2.41	2.85
2014	0.73	0.67	1.23	1.21	0.93	0.56	0.76	1.04	0.80

Observing the figures in Table 3, one can see how the actual maximum Sharpe Ratio is obtained, ex-post, only in three years out of twenty: 2005, 2010, 2014, plus 2007 if considering also the MSR_12. The last three portfolios on the right of the table seem, overall, to capture the best risk-adjusted return. Nonetheless, it is worth to note that the GMVu presents the best ratios in seven years (1998, 2001, 2002, 2007, 2008 and 2011) more times than any other portfolio. In particular, it tends to reach the best results in periods of market depression and poor stock performance such as the early 2000 and during the crisis: indeed, in 2002 and 2008 the GMVu has the highest Sharpe Ratio with negative sign, while in 2001, along with the GMVc, they are the only portfolios delivering positive returns.

Table 4 - Diversification Ratio

	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	1.90	1.80	1.70	1.80	1.88	1.86	1.95	2.19	2.02
1996	2.18	2.13	1.83	1.91	2.01	2.07	2.19	2.40	2.23
1997	2.14	2.17	1.82	1.86	1.96	1.94	2.09	2.36	2.14
1998	1.93	1.95	1.57	1.69	1.80	1.73	1.88	2.11	1.92
1999	1.96	1.93	1.60	1.72	1.78	1.64	1.81	2.05	1.87
2000	1.61	1.60	1.50	1.60	1.65	1.55	1.65	1.86	1.68
2001	2.14	2.18	1.83	1.85	1.92	1.79	1.96	2.28	2.05
2002	2.13	2.17	1.54	1.74	1.87	1.80	1.95	2.27	2.03
2003	2.06	2.12	1.46	1.71	1.84	1.74	1.88	2.19	1.95
2004	2.03	2.13	1.47	1.75	1.87	1.76	1.90	2.20	1.95
2005	1.86	1.93	1.60	1.67	1.74	1.68	1.80	2.06	1.84
2006	1.61	1.60	1.50	1.60	1.65	1.55	1.65	1.86	1.68
2007	1.61	1.59	1.62	1.68	1.68	1.59	1.66	1.86	1.69
2008	1.71	1.66	1.62	1.73	1.77	1.70	1.74	1.95	1.80
2009	1.51	1.29	1.24	1.47	1.51	1.48	1.49	1.69	1.56
2010	1.49	1.28	1.19	1.43	1.46	1.42	1.45	1.65	1.52
2011	1.44	1.23	1.19	1.39	1.41	1.38	1.40	1.60	1.46
2012	1.40	1.16	1.28	1.41	1.41	1.34	1.35	1.54	1.41
2013	1.41	1.14	1.35	1.43	1.41	1.33	1.34	1.53	1.40
2014	1.45	1.29	1.36	1.45	1.48	1.37	1.38	1.60	1.45

Table 5 - Turnover

	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	4.05	8.19	14.57	13.31	12.22	2.04	n. r.	4.97	0.74
1996	5.44	8.95	11.84	10.39	13.29	2.08	n. r.	5.05	0.89
1997	5.31	8.49	14.74	11.07	9.01	2.31	n. r.	4.08	0.78
1998	5.06	10.43	15.07	14.31	10.82	2.95	n. r.	4.90	0.81
1999	4.08	7.84	15.05	12.41	9.49	3.28	n. r.	3.95	0.58
2000	3.75	7.68	14.96	13.01	5.21	1.55	n. r.	3.45	0.40
2001	2.55	5.12	15.73	11.79	10.15	2.93	n. r.	2.09	0.39
2002	4.13	7.77	18.01	17.37	11.28	2.76	n. r.	2.95	0.66
2003	3.51	6.96	19.98	14.46	8.05	1.90	n. r.	2.79	0.47
2004	3.51	6.77	21.38	13.57	7.87	1.59	n. r.	3.10	0.47
2005	6.36	11.68	15.97	13.03	7.07	1.57	n. r.	4.95	0.55
2006	4.46	9.61	21.41	9.99	5.21	1.55	n. r.	3.45	0.40
2007	7.28	15.88	20.72	14.40	10.94	1.64	n. r.	5.89	1.08
2008	7.49	17.66	18.25	13.99	7.91	3.28	n. r.	8.66	1.35
2009	3.46	10.30	11.09	9.69	10.59	2.95	n. r.	5.84	0.58
2010	1.37	5.81	8.90	13.82	9.15	1.67	n. r.	2.25	0.20
2011	2.12	6.04	8.78	13.09	9.83	1.83	n. r.	2.26	0.24
2012	1.99	6.96	14.19	5.51	9.79	1.42	n. r.	1.89	0.19
2013	3.38	10.89	18.22	13.72	8.19	1.42	n. r.	4.33	0.51
2014	3.92	11.60	18.37	15.88	10.95	1.31	n. r.	5.53	0.81

Table 4 and Table 5 are strictly connected to each other as they both illustrate where the MDP and ERC portfolios find most of their benefits. Indeed, the Diversification ratio (DR) and the Turnover rate are key indicators, which every investor would look at when managing a portfolio of stocks: the DR because, being the ratio between portfolios' weighted average asset volatility and its actual volatility, it measures the essence of diversification. The Turnover rate, on the other hand, gives an important information regarding the need and the frequency of rebalancing the portfolio with all its costs associated.

The DR in the table is calculated as the annual average of the 52 weekly values along every year. As expected, the MDP, built on the maximization of this ratio, delivers the highest values even ex-post, followed closely by the ERC, and in some years in the early 2000 by the minimum variance portfolios. Furthermore, it is interesting to note how the other two least concentrated portfolios, the MCap and the EW, cannot compete with the other two in terms of Diversification Ratio, with the MCap which has sometimes even lower values than the MSR with risk contribution constraints.

On the other hand, the MCap presents, obviously, optimal values of Turnover rates, even slightly better than the MDP. Only the ERC does better, with values closer to zero, which, in this framework is the hypothetical, but not real (as explained above) value of the EW portfolio. Finally yet importantly, Table 4 and especially Table 5 show the greatest discrepancy between the two minimum variance portfolios: the long-only portfolios is more diversified and needs less rebalancing, than the long/short one. Hence, as far as can be depicted from these figures, the slightly better returns adjusted for volatility results delivered by the GMVu are not good enough to compensate the drawbacks in terms of DR and Turnover; therefore anyone should prefer, in our case, to build a long-only strategy for the minimum variance portfolio.

5.1.1. Risk Contribution and Weight Distribution

Comparing the evolution of the stock's risk contribution (RC) among the different methods is highly important, not only to get a measure about the transaction and rebalancing cost associated to that specific portfolio, but mainly an idea of its diversification. The RC goes hand in hand with the distribution of weights: *ceteris paribus,* the higher the weight of the stock in the portfolio, the higher its risk contribution.

The following graphs represent stacked area chart of the paths of the thirty stock's risk contribution and weights distribution along time for each portfolio in the United States[26]. The stocks picked are, as already said, among the largest capitalized ones in S&P500 and DJ30 of the last 25 years, and are the following:

EXXON MOBIL
MICROSOFT
JOHNSON & JOHNSON
GENERAL ELECTRIC
WELLS FARGO & CO
JP MORGAN CHASE & CO.
PROCTER & GAMBLE
CHEVRON
VERIZON COMMUNICATIONS
PFIZER
BANK OF AMERICA
COCA COLA
IBM
CITIGROUP
PEPSICO
COMCAST 'A'
WALT DISNEY
SCHLUMBERGER
ORACLE
HOME DEPOT
WAL MART STORES
AMGEN
UNION PACIFIC
CVS HEALTH
ALTRIA GROUP
CONOCOPHILLIPS
MCDONALDS
3M
UNITED TECHNOLOGIES
BOEING

*Figure 5.1.1 – Stock's **RISK CONTRIBUTION** in the **GMVc** portfolio (U.S.)*

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[26] For the purpose of this section, only the United States equity portfolios are taken as the key figures. The European and Japanese RC and weight distribution figures (which are shown in annex #) follow essentially the same trends.
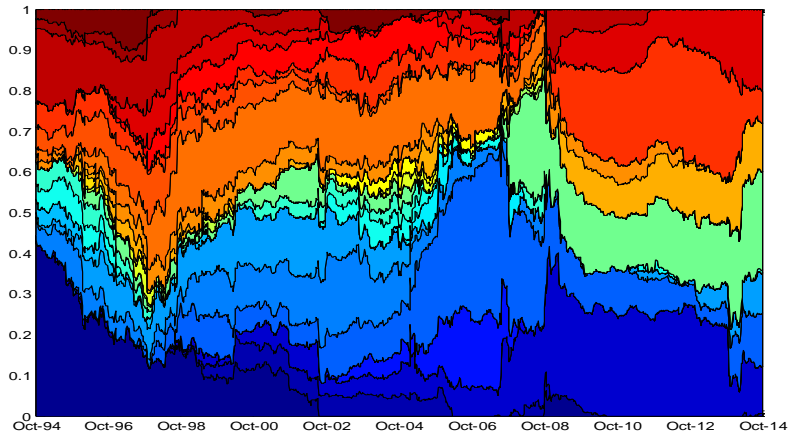
Figure 5.1.2 – Stock's **WEIGHT DISTRIBUTION** in the **GMVc** portfolio (U.S.)
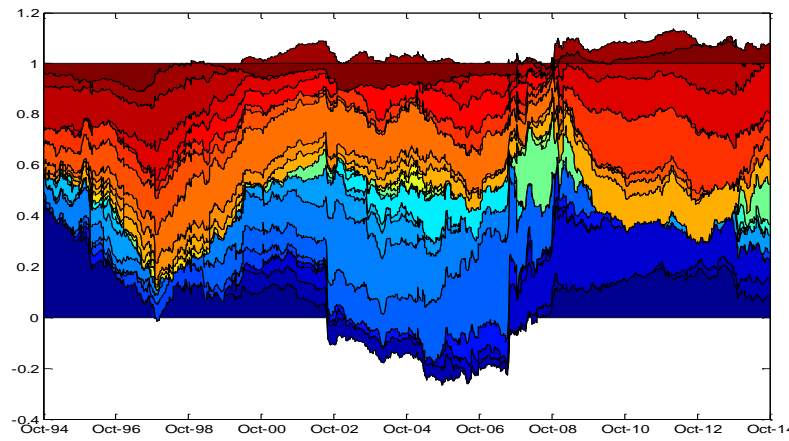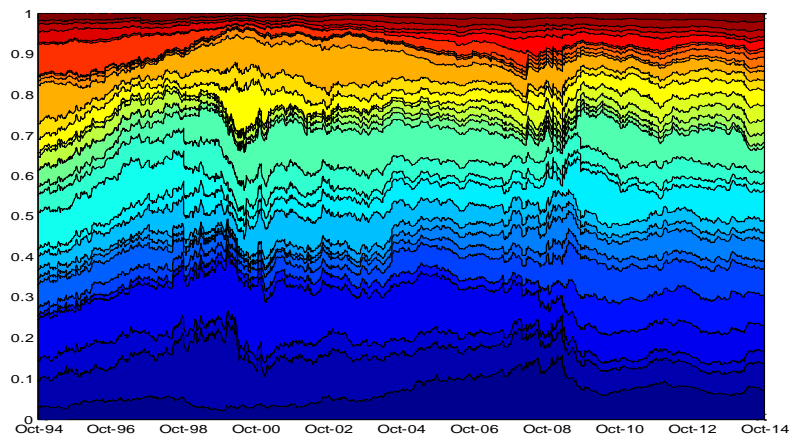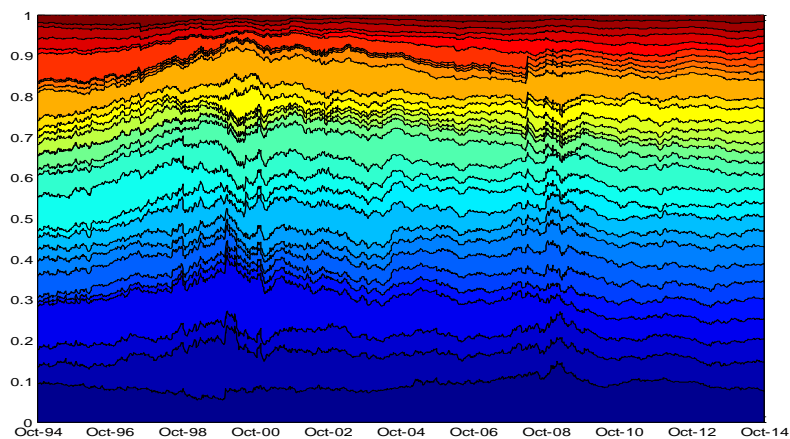

Figure 5.1.3 – Stock's **RISK CONTRIBUTION** in the **GMVu** portfolio (U.S.)
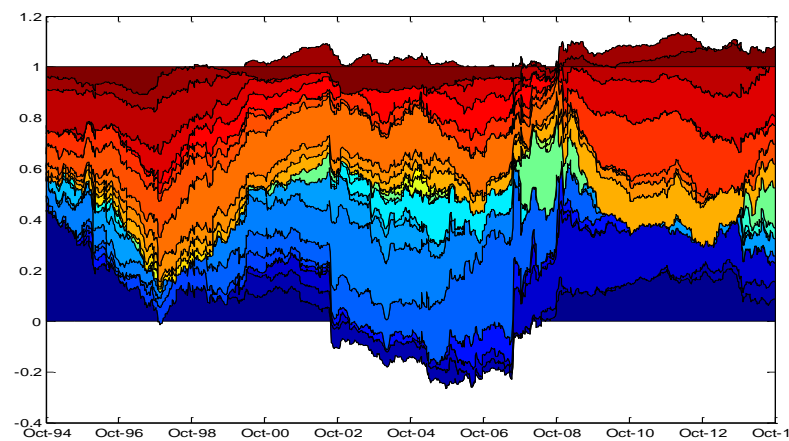
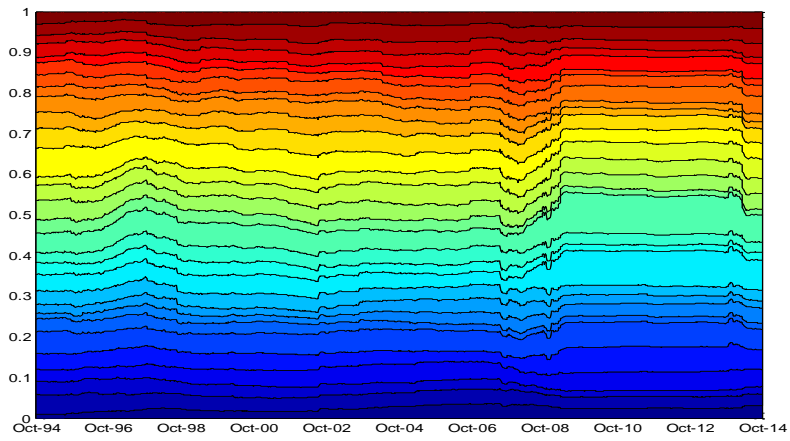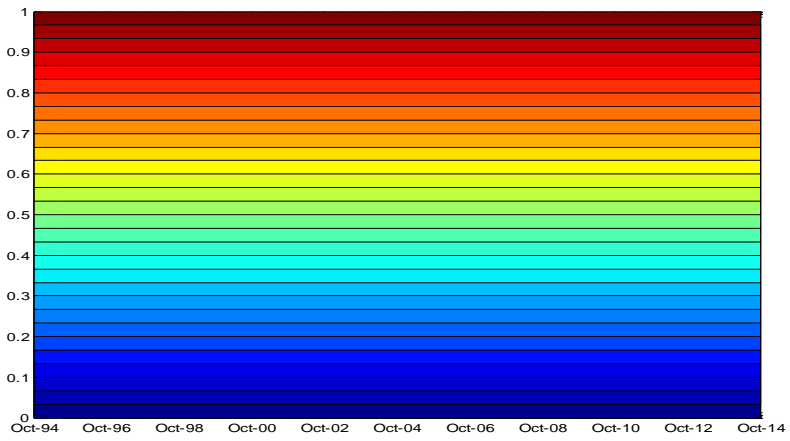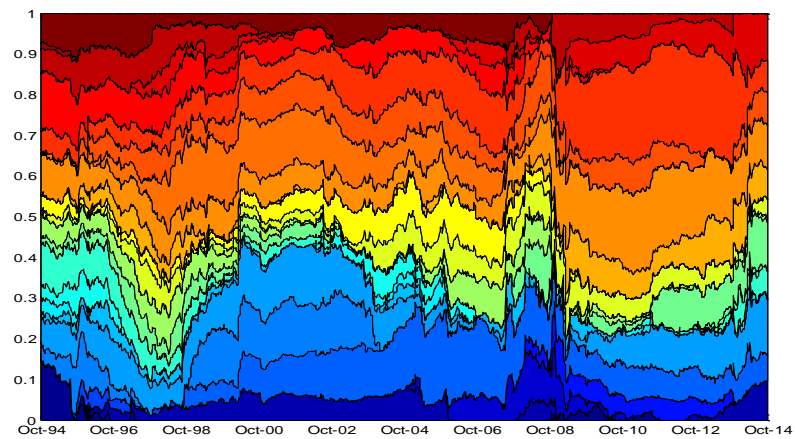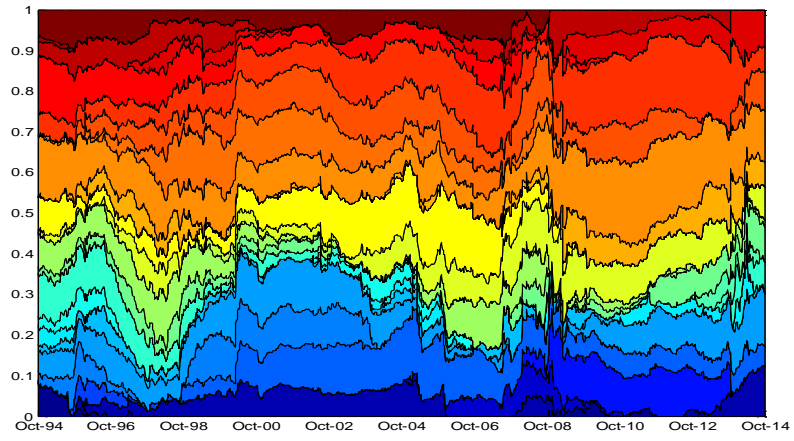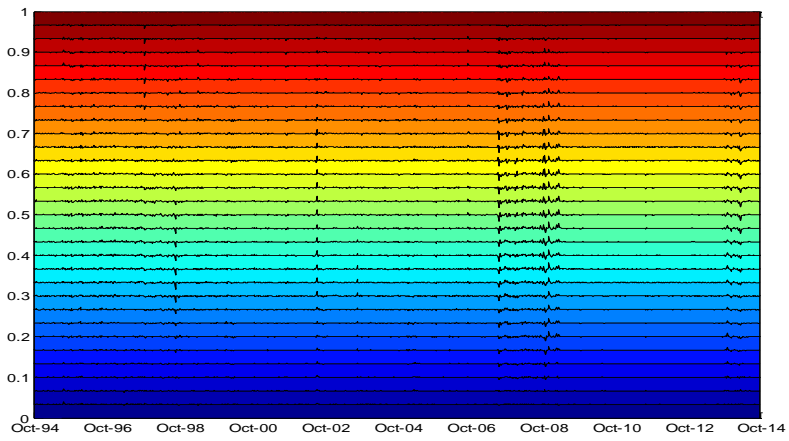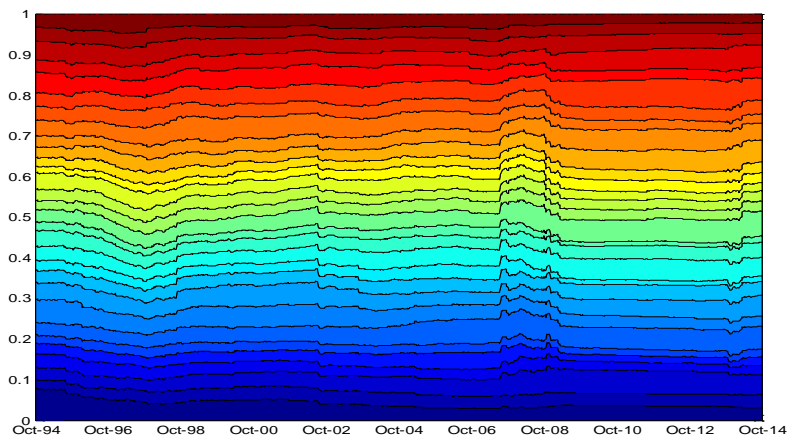*Figure 5.1.4 –* Stock's **WEIGHT DISTRIBUTION** in the **GMVu** portfolio (U.S.)

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As shown in the first four graphs, the distribution over time of the weights and risk contribution of the two minimum variance portfolio are very similar. Each components seem to follow approximately the same trend, with the only obvious difference that the weights – and consequently, the risk contributions – of the GMVu can be negative. Nevertheless, the absolute sum of weight differences in the two portfolios reaches quite high values, especially in 2007, 2011 and 2010 - as revealed in the next figure - justifying the large differences in Turnover rates presented in Table 5.

Figure 5.1.5 shows this evolution, where the weight differences between the GMVc and the GMVu portfolio are calculated at every week as follows:

$$w.diff = \sum_{i=1}^{n} \left| x_i^{GMVu} - x_i^{GMVc} \right|$$

where $n = 30$ and x_i is the weight of the stock i.

Next figures show the distributions of the MSR portfolio that, predictably, present the most chaotic pattern due to the need of persistent rebalancing. Moreover it can be seen how the portfolio concentrates more than 60% of the risk in only one stock for long periods, like *McDonalds* (the red one) from October 2008 to July 2012.

Figure 5.1.5 –Sum of GMVu and GMVc weights differences on time

The remaining graphs highlight the great dissimilarity between the more concentrated portfolios – GMV, MSR - and the ones diversified in every assets - MCap, ERC, EW – or almost every assets – MDP. Indeed, recalling what already illustrated in Section 3, the core property of the MDP shows that any stock not held by the MDP is more correlated to the MDP than any of the stocks that belong to it. Furthermore, all stocks belonging to the MDP have the same correlation to it. Thus, this property illustrates that all assets in the universe considered are effectively represented in the MDP, even if the portfolio does not physically hold them. Nonetheless these portfolios show a steady behavior in the distribution of weights and risk contribution, explaining the low turnover rates in Table 5.
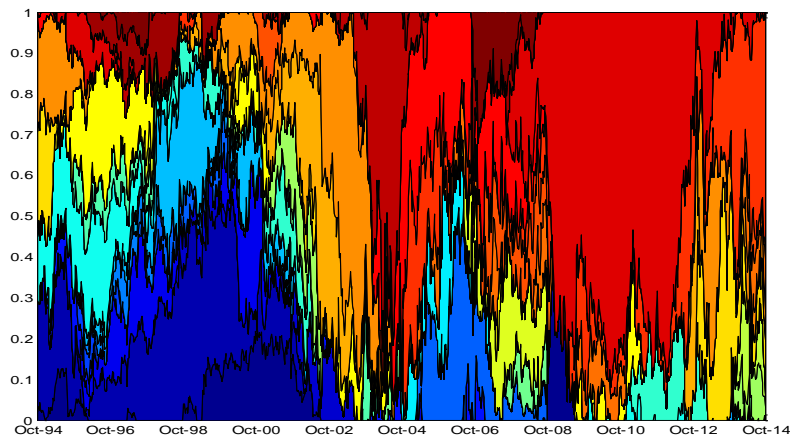
Figure 5.1.6 – Stock's **RISK CONTRIBUTION** in the **MSR** portfolio (U.S.)
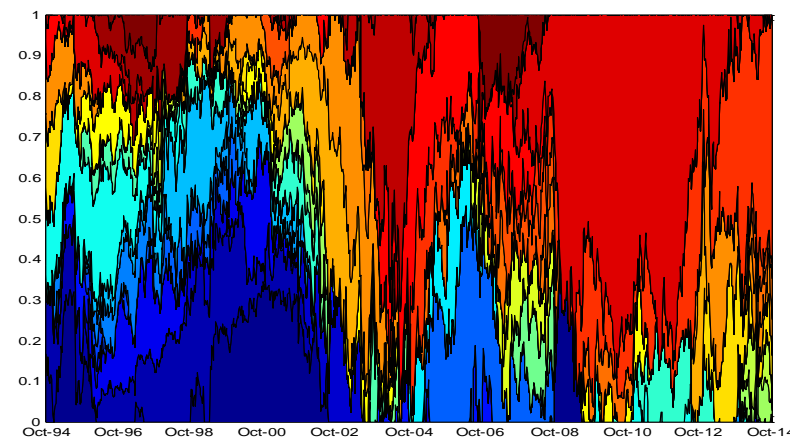
Figure 5.1.7 – Stock's **WEIGHT DISTRIBUTION** in the **MSR** portfolio (U.S.)

Undoubtedly, the ERC (Figure 5.1.14 - 15) and the EW (Figure 5.1.10 - 11) portfolios show the most similar trends, with the former that adjust the weights on basis of constant stock's risk contributions and the latter that takes constant weights as input. The ERC risk contributions bars are not perfectly constant () like the weights of the EW portfolio, simply because we the weights obtained in "t", are used to calculate the risk contributions in "$t + 1$", just as like it has been done for the calculation of portfolio returns and the other indicators.

Figure 5.1.8 – Stock's **RISK CONTRIBUTION** in the **MCap** portfolio (U.S.)

Figure 5.1.9 – Stock's **WEIGHT DISTRIBUTION** in the **MCap** portfolio (U.S.)

Figure 5.1.10 – Stock's **RISK CONTRIBUTION** in the **EW** portfolio (U.S.)

Figure 5.1.11 – Stock's **WEIGHT DISTRIBUTION** in the **EW** portfolio (U.S.)

Figure 5.1.12 – Stock's **RISK CONTRIBUTION** in the **MDP** portfolio (U.S.)

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Figure 5.1.13 – Stock's **WEIGHT DISTRIBUTION** in the **MDP** portfolio (U.S.)

Figure 5.1.14 – Stock's **RISK CONTRIBUTION** in the **ERC** portfolio (U.S.)

Figure 5.1.15 – Stock's **WEIGHT CONTRIBUTION** in the **ERC** portfolio (U.S.)

5.1.1. Value at Risk and Expected Shortfall

Finally, we take into account a measurement for the ex-post, Value at Risk (VaR) and Expected Shortfall (ES). Starting from the weekly portfolio realized returns ($P.Ret$) for each year, we consider their differences from one week to the next one, then sorting them year by year, we end up having a distribution of 52 return's differences for every year. In this way, it is possible to capture the highest weekly portfolio drawdowns for every year, considering the percentile of the 99th and 95th of the distribution; meanwhile for the Expected Shortfall we consider the simple mean of the return's differences tail distribution.

Since the distribution includes only 52 returns at each periods, Table VaR-99 and Table ES-99 show very similar results. The four tables generally resemble the volatility figures in Table 2, in the way that the minimum variance portfolio show most of the times the lower values, followed by ERC and MDP. There are however some exceptions for example in the years of 1996 and 2001 where the MSR portfolio and the MSR 12, the most concentrated portfolios, show lower values of VaR and shortfall. As expected, during the financial turmoil of 2007-2009, the capitalization weighted portfolio, along with the other diversified portfolios, tend to be more risky with high level of drawdowns. Remarkably, in the same period, the minimum variance portfolios in particular, but also the MSR unconstrained, present good results, meaning that, in this case, the stock selection process based on historical values worked well.

Table 6 - Value at Risk (95%)

VaR (95%)	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	3.15	3.37	5.47	3.46	2.96	2.68	3.18	3.17	2.81
1996	4.13	4.21	4.44	3.94	4.22	4.42	3.83	4.49	3.95
1997	6.04	5.32	8.17	7.82	6.97	6.62	6.30	5.70	6.26
1998	5.87	4.96	7.91	8.23	6.98	7.14	6.80	6.39	6.26
1999	5.87	4.96	7.91	8.23	6.98	7.14	6.80	6.39	6.26
2000	4.54	4.73	6.53	5.78	5.31	5.65	5.18	5.23	4.83
2001	3.16	3.78	4.07	3.71	3.34	4.05	3.28	3.14	3.55
2002	6.14	6.62	6.45	6.74	8.86	8.05	7.48	6.30	6.94
2003	4.69	4.11	4.89	4.48	4.19	4.93	4.95	4.55	4.74
2004	3.24	2.98	3.82	3.24	2.88	3.01	3.01	3.59	3.02
2005	3.75	4.12	3.71	3.66	3.69	2.82	3.25	4.09	3.43
2006	2.47	2.95	3.18	2.95	2.53	2.57	2.67	2.83	2.63
2007	3.52	3.47	4.20	4.21	4.31	4.78	4.45	3.74	4.43
2008	6.25	8.50	7.11	6.95	8.66	10.38	9.27	7.04	8.48
2009	5.08	4.45	7.19	11.17	8.11	9.29	10.69	10.37	8.91
2010	3.42	3.48	4.96	4.66	4.20	4.52	4.63	3.95	4.18
2011	4.30	3.14	4.89	6.39	6.77	8.34	8.91	6.15	7.25
2012	2.46	3.16	3.18	3.46	3.39	3.18	3.26	3.08	2.85
2013	2.14	2.75	3.36	2.70	2.67	2.28	2.46	2.56	2.20
2014	2.17	2.65	3.18	2.30	2.57	2.72	2.68	2.21	2.38

Table 7 – Expected Shortfall (95%)

ES (95%)	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	4.63	4.40	5.83	4.57	3.92	3.61	3.62	4.23	3.53
1996	5.94	6.58	4.71	4.67	4.68	4.81	4.55	5.09	4.67
1997	6.78	6.20	8.84	8.72	7.89	8.21	7.74	7.49	7.47
1998	8.20	8.11	11.22	10.58	9.71	9.60	9.43	9.13	8.84
1999	6.09	7.06	11.69	9.73	8.22	8.31	6.92	6.21	6.26
2000	5.22	6.28	6.91	6.48	5.69	6.10	5.47	5.48	5.40
2001	4.30	4.51	4.47	4.46	4.91	4.85	4.91	4.56	4.80
2002	10.45	9.85	11.75	12.40	12.69	12.56	11.74	9.41	11.63
2003	5.06	5.00	8.14	6.30	5.63	6.01	6.21	5.78	5.70
2004	3.92	4.03	5.71	4.40	3.89	3.99	4.08	4.63	3.99
2005	4.81	5.50	5.44	4.82	4.20	3.14	3.55	4.99	3.72
2006	2.75	3.19	3.70	3.52	2.61	3.15	2.85	3.11	2.82
2007	3.78	3.83	4.69	4.56	4.96	5.71	5.32	4.03	5.18
2008	8.73	9.55	9.65	10.25	12.44	15.04	15.73	12.39	13.88
2009	6.46	5.83	8.91	12.83	10.29	11.57	13.59	11.01	11.00
2010	4.13	4.37	5.73	5.25	5.04	5.81	5.66	4.09	4.93
2011	4.64	3.69	6.12	7.90	8.15	10.11	10.64	6.31	8.41
2012	3.53	4.13	4.34	4.04	3.91	4.45	4.60	4.02	4.07
2013	2.42	3.03	3.97	2.93	3.14	2.90	3.06	2.72	2.83
2014	2.36	2.87	3.57	2.82	2.65	3.37	3.11	2.44	2.78

Table 8 - Value at Risk (99%)

VaR (99%)	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	6.27	5.77	6.14	5.79	4.88	4.31	3.91	4.71	3.88
1996	6.79	7.59	4.94	5.08	5.28	5.17	4.88	5.88	5.30
1997	7.16	6.88	9.43	9.15	8.66	9.98	9.39	8.97	9.15
1998	10.61	11.00	13.62	13.01	13.39	12.87	12.72	12.75	12.41
1999	6.70	8.92	15.92	12.07	10.48	10.25	7.44	6.68	6.52
2000	5.50	7.98	7.13	6.89	6.06	6.43	5.65	5.63	6.11
2001	5.45	5.06	4.86	4.87	6.30	5.62	6.61	6.41	6.36
2002	14.23	12.71	18.75	18.80	17.54	18.06	16.93	12.67	16.76
2003	5.27	5.91	9.91	7.21	6.23	6.57	6.88	6.91	6.12
2004	4.82	5.32	6.84	5.31	5.20	4.68	5.44	5.69	5.28
2005	5.45	6.06	6.62	5.29	4.40	3.32	3.72	6.10	3.87
2006	2.88	3.52	4.17	3.83	2.70	3.37	3.00	3.41	2.90
2007	4.01	4.06	5.31	4.98	5.66	6.51	6.38	4.20	6.13
2008	11.46	10.14	12.30	13.58	17.06	17.01	19.10	16.90	17.43
2009	7.97	6.93	10.65	14.28	11.62	12.69	15.46	11.41	11.92
2010	4.53	5.39	6.46	6.00	6.08	6.75	6.41	4.26	5.40
2011	4.94	4.29	6.61	9.74	9.13	11.47	11.78	6.38	9.26
2012	4.34	4.70	5.30	4.43	4.19	5.13	5.19	4.83	4.70
2013	2.69	3.34	4.47	3.22	3.44	3.40	3.50	2.83	3.24
2014	2.52	3.05	3.83	3.23	2.73	3.99	3.52	2.58	3.10

Table 9 – Expected Shortfall (99%)

ES (99%)	GMVc	GMVu	MSR	MSR_12	MSR_6	Mcap	EW	MDP	ERC
1995	6.29	5.80	6.15	5.81	4.89	4.32	3.92	4.71	3.88
1996	6.79	7.59	4.95	5.08	5.30	5.18	4.89	5.90	5.31
1997	7.16	6.89	9.43	9.15	8.67	10.01	9.42	8.99	9.18
1998	10.65	11.04	13.65	13.05	13.45	12.93	12.77	12.81	12.47
1999	6.70	8.95	15.99	12.10	10.52	10.28	7.45	6.69	6.53
2000	5.51	8.00	7.13	6.89	6.07	6.43	5.65	5.64	6.13
2001	5.48	5.07	4.87	4.88	6.33	5.64	6.65	6.46	6.40
2002	14.28	12.75	18.87	18.90	17.62	18.15	17.02	12.72	16.84
2003	5.27	5.92	9.92	7.22	6.23	6.58	6.89	6.93	6.12
2004	4.84	5.34	6.85	5.32	5.22	4.69	5.47	5.71	5.30
2005	5.46	6.06	6.64	5.29	4.40	3.32	3.72	6.12	3.87
2006	2.88	3.53	4.18	3.83	2.70	3.37	3.00	3.41	2.90
2007	4.01	4.06	5.33	4.99	5.68	6.53	6.41	4.20	6.16
2008	11.50	10.15	12.34	13.63	17.13	17.01	19.13	16.96	17.46
2009	7.99	6.95	10.67	14.30	11.64	12.70	15.48	11.41	11.92
2010	4.54	5.40	6.48	6.01	6.10	6.76	6.42	4.26	5.41
2011	4.95	4.30	6.62	9.77	9.14	11.49	11.80	6.38	9.27
2012	4.35	4.71	5.32	4.44	4.19	5.13	5.19	4.85	4.71
2013	2.69	3.34	4.47	3.22	3.45	3.41	3.50	2.83	3.25
2014	2.52	3.05	3.83	3.23	2.73	3.99	3.52	2.58	3.10

5.2. Cumulative Return Analysis

The cumulative return represents the aggregate amount that an investment has gained or lost over time, independent of the period involved. Investors are more likely to see a compound return than a cumulative return, as the compound return figure will be annualized. This helps investors to compare different investment choices. However, it could be also significant to observe the total percentage increase in the value of an investment from the time it was purchased, and so, the evolution of the total return of different the portfolios as shown in the following graphs.

The total cumulative return (TCR) for a generic portfolio is defined as in paragraph 5.1. with the difference that now the cumulated return is not reset every year:

$$TCR = \prod_{i=1}^{1044} (1 + P.Ret_i) - 1$$

TCR gives a measure of realized return, meaning what an investor would have actual obtained if he applied the weights calculated using, as inputs, the historycal data in the preceding 5 years.

It is worth to compare the portfolios cumulative returns arising from the TCR calculated as above with a purely theoretical situation in which the investor applies the weights calculated in t having as inputs also the stock 's return in t (e.g. the stock prices of $t+1$). Analytically this hypothetical TCR represented in the next three figures has been calculated using the portfolio returns as follows:

$$Hyp.Ret_t = Real.Ret_t^T * x_t$$

While the realized portfolio returns, as already said, are computed as

$$P.Ret_t = Real.Ret_t^T * x_{t-1}$$

In the scenario of hypothetical, anyone would use a max Sharpe criterion to build its portfolio, which dominates all the others in every geographical area.
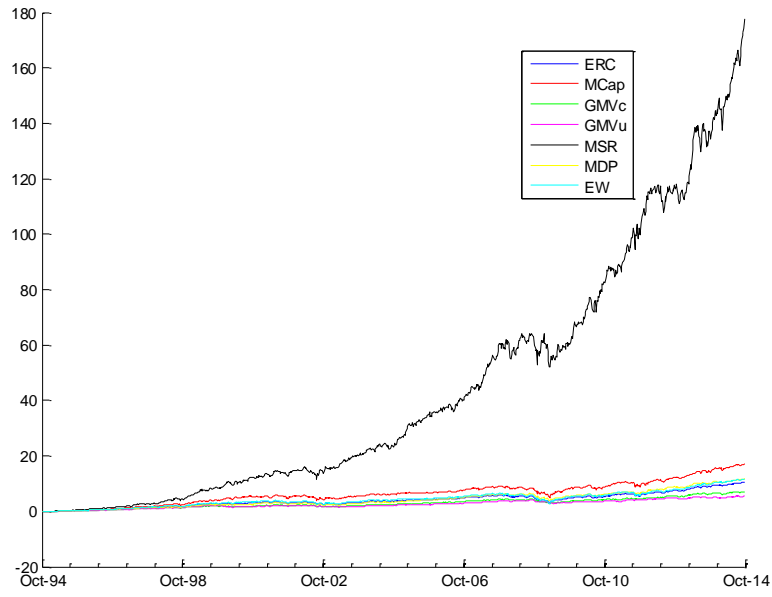
Figure 5.2.1 – Hypothetical portfolios **Cumulative Return**s in U.S.
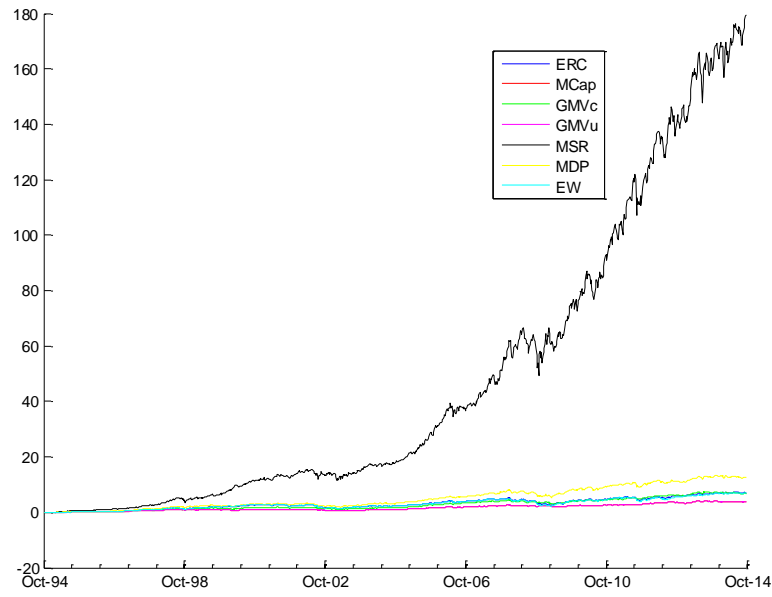
Figure 5.2.2 – Hypothetical portfolio **Cumulative Return**s in Europe
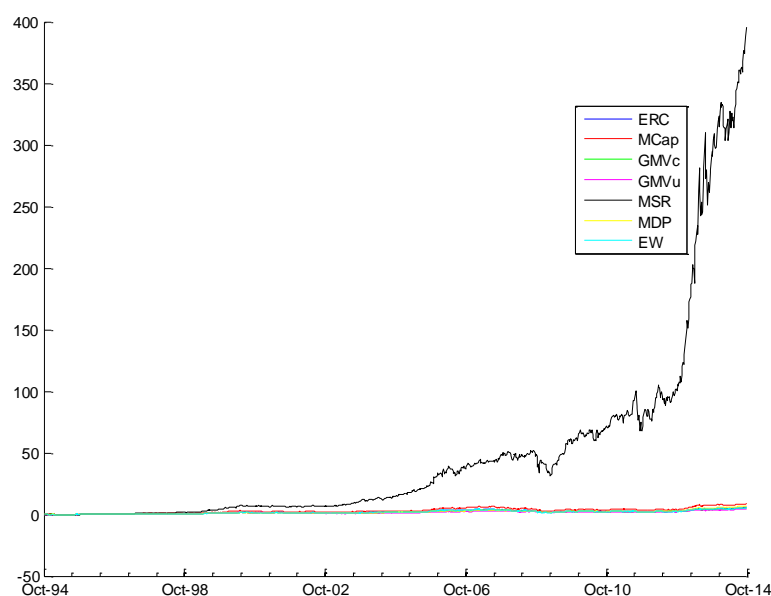
Figure 5.2.3 – Hypothetical portfolio **Cumulative Return**s in Japan

The unit of measure in the vertical axis is not percentage, thus according to Figure 5.2.3, 1 euro invested the MSR portfolio in 1994 would return 400 euros in 2014. Moreover, the spike of the black line on the right side of the graph is due to the stock price of Fuji Heavy Industries, which increased from 423 Yen at the end of September 2011 to 3578 in the same period of 2014. Even if not so striking, this positive result is achieved by the MSR also in terms of actual realized cumulative returns (Figure 5.2.6). This is one of the few cases in which the concentration of the MSR portfolio in one stock, with a Diversification Ratio equal to 1 (its minimum), delivers positive results.

Nevertheless, in the real world, the performance of the MSR portfolio dramatically falls and the risk parity based portfolios outperform it by a long way in every geographical area considered in the analysis.

In particular, the EW, MDP and the ERC show very similar trend in United States, Figure 5.2.4 outperforming all other portfolios. In Europe (Figure 5.2.5) , the situation is a bit different, since the "winner" in terms of cumulative returns is the MDP, while the ERC and the EW are on the same level as the MSR and the GMVc. In Japan (Figure 5.2.6), where the trends are much more volatile than in the other two regions, the best cumulative returns are obtained not only by the MDP, ERC and EW but also by the GMVu.
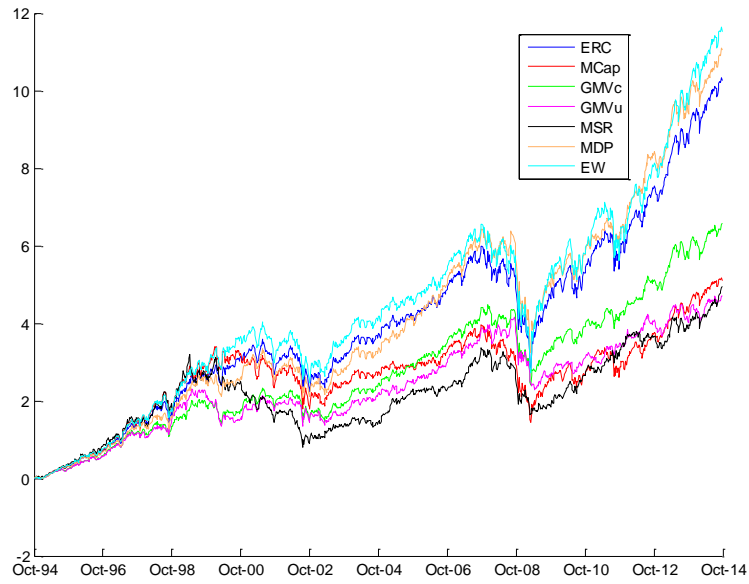
Figure 5.2.4 – Realized portfolio **Cumulative Return**s in U.S.
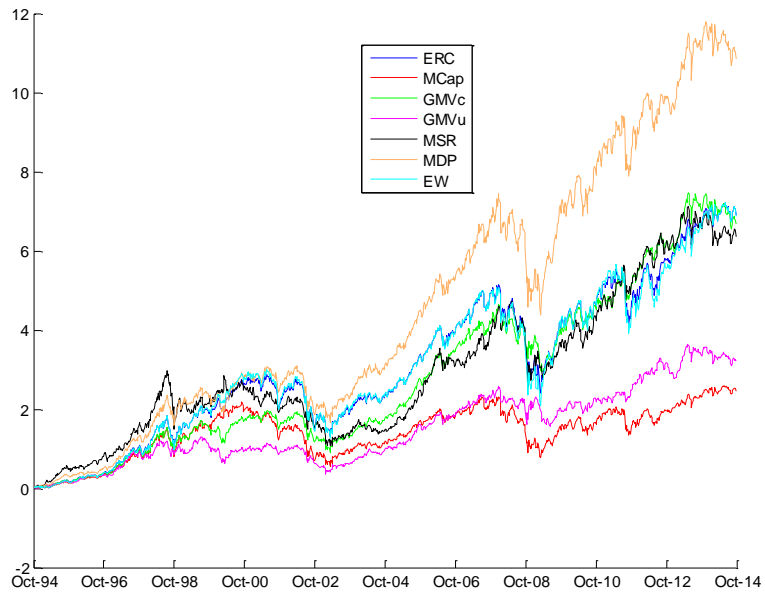
Figure 5.2.5 – Realized portfolio **Cumulative Return**s in Europe

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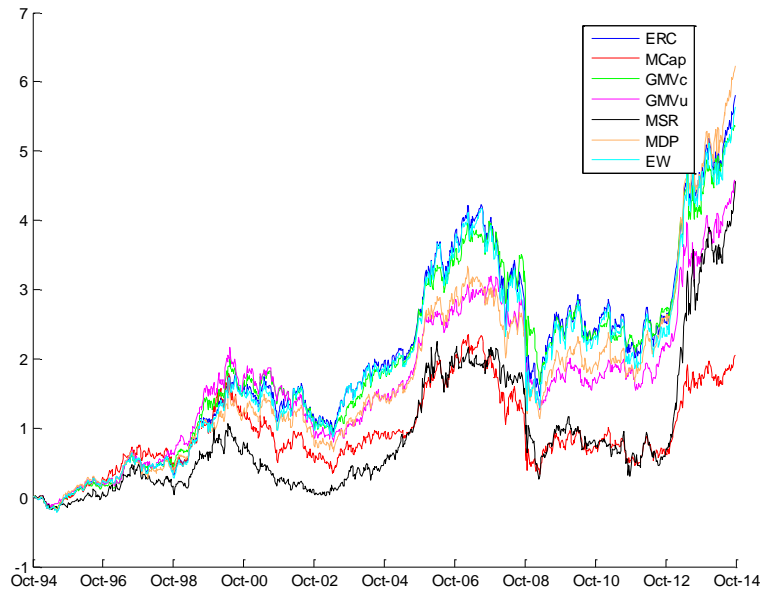
Figure 5.2.6 – Realized portfolio **Cumulative Return**s in Japan

5.2.1. **Transaction Costs and Diversification Ratio**

Considering the transaction costs, as estimated in paragraph 4.2.2, the situation little changes, but the *leitmotif* remains the same. The 15 bps "fee" applied to the weekly change in the portfolio weights (rebalancing) does not affect the allocation methods with low turnover rate such as the MCap and the ERC, it slightly affects the MDP, while it worsen the situation of the MSR and the GMVu because of their high level of turnover.
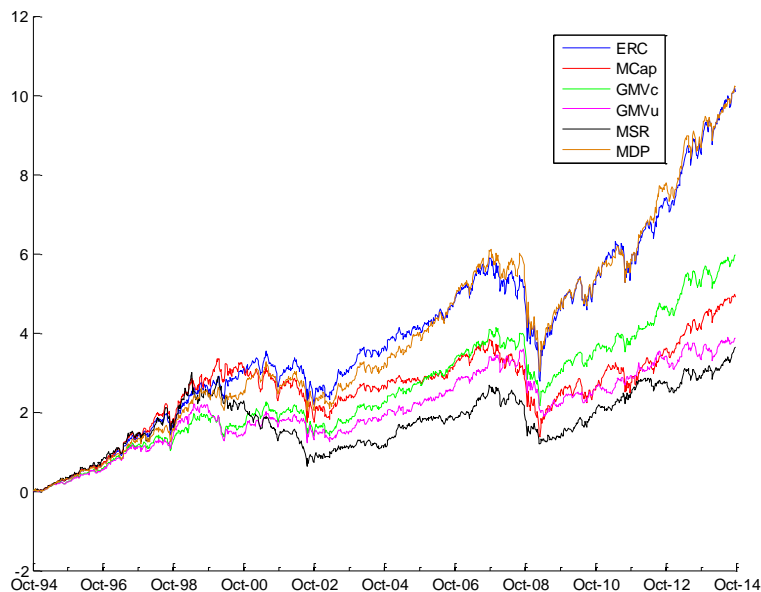
Figure 5.2.8 – Cumulative Returns net of transaction costs (U.S.)
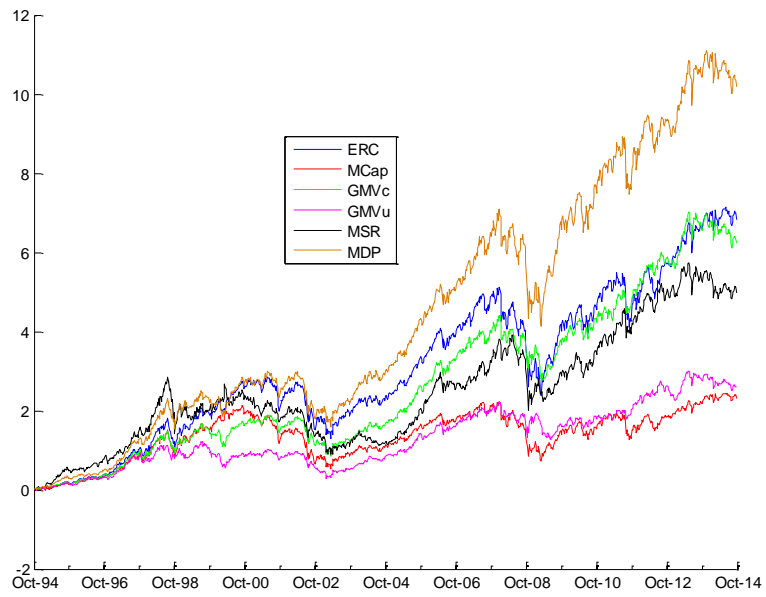
Figure 5.2.7 – Cumulative Returns net of transaction costs (Europe)
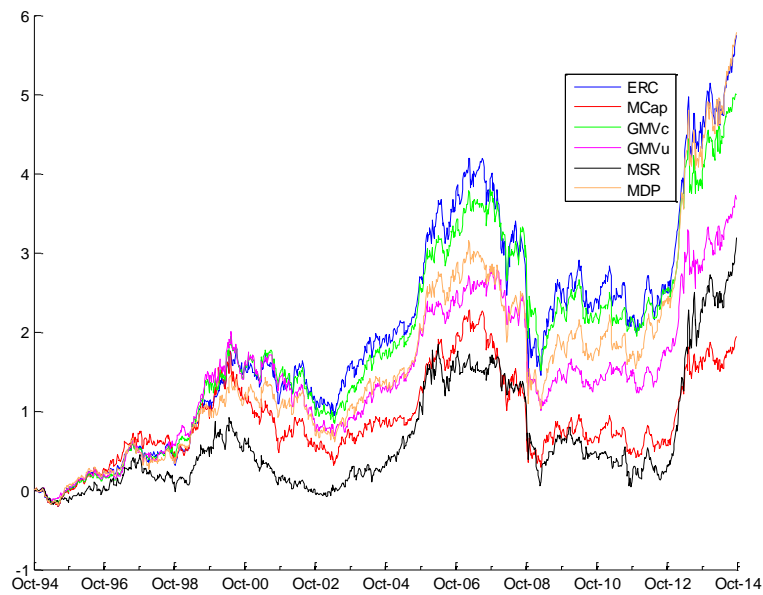
Figure 5.2.9 – Cumulative Returns net of transaction costs (Japan)

67
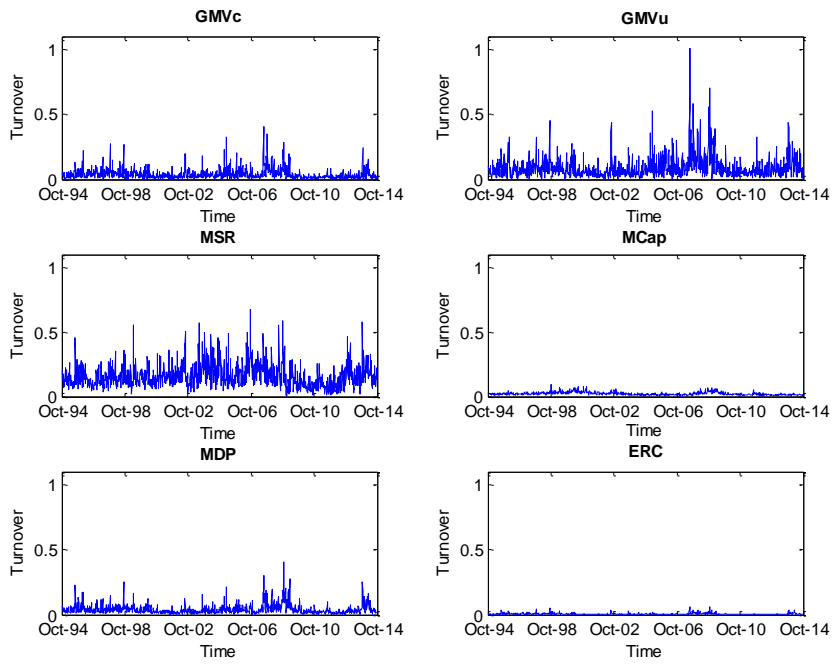
Figure 5.2.10 – Portfolio **turnover rates** in US
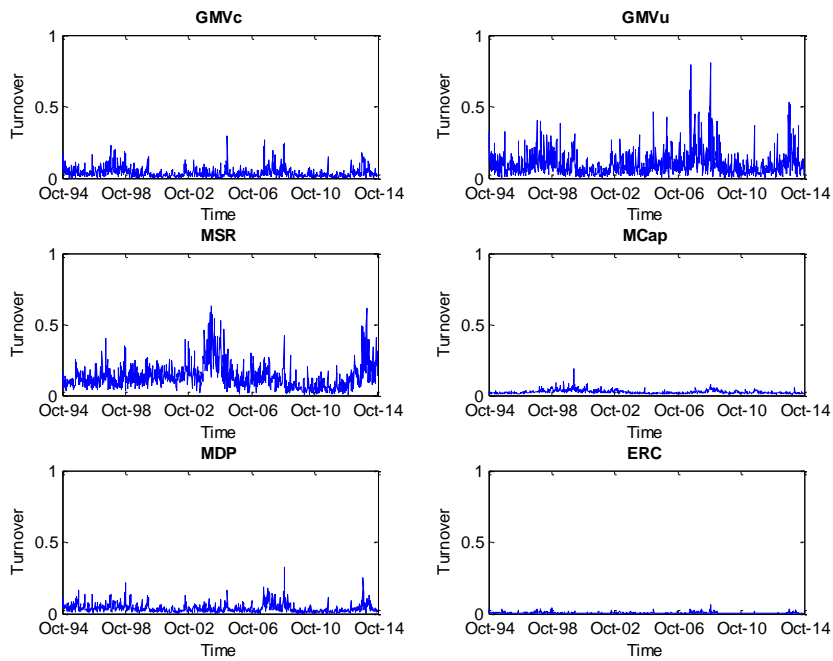
Figure 5.2.11 – Portfolio **turnover rates** in Europe

68
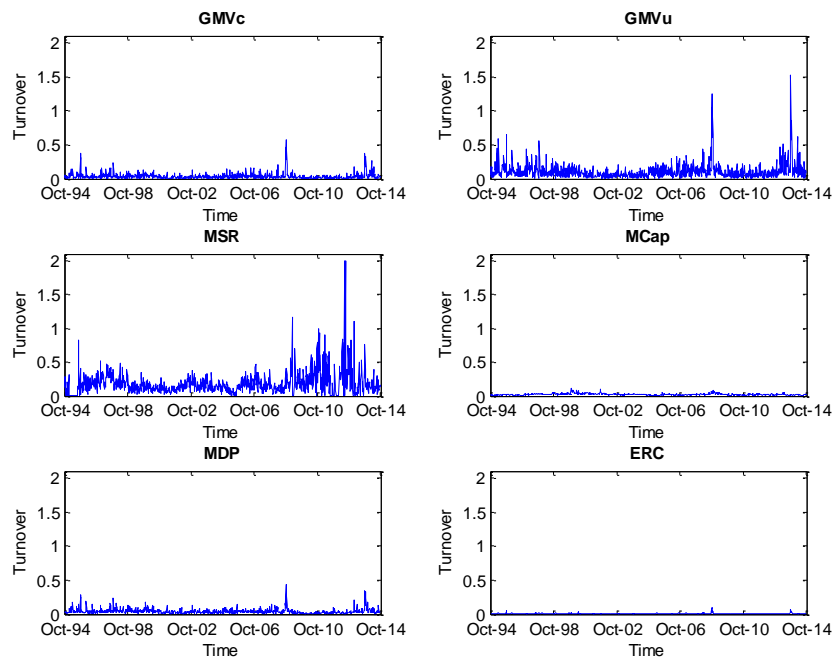
*Figure 5.2.12 – Portfolio **turnover rates** in Japan*

It is remarkable to note that the maximum level of 2 in the turnover rate is reached only by the Japanese MSR portfolio. This happens because for three times, in the middle of 2012 the portfolio is fully rebalanced from one week to another. Specifically, the weights are concentrated at 100% in one stock (Fanuc Corp) on the 18 of June, then in another stock (Fujifilm Holdings) for three weeks, then in another one (JGC Corp) and then back in the Fujifilm Holdings on the 23 of July. This kind of behavior, which is not presented in any other portfolios or regions, it is due to the great volatility of the Japanese Nikkei constituents that affects, in this way, the inputs (historical stock returns and their variance/covariance) for the calculation of the MSR weights.

Furthermore, only in the MSR Japanese portfolio happens that, sometimes, all the weight is concentrated in one stock. Because of this, the Diversification Ratio reaches its minimum value of 1 as shown in the figure 5.2.13.

Obviously, it is not the optimal choice to concentrate the portfolio in only one asset, especially in terms of risk management, but also in terms of total return. Indeed, it was observed that the MSR portfolios in all the three regions deliver poor results, and the stock picking method based on this mechanism does not work properly.
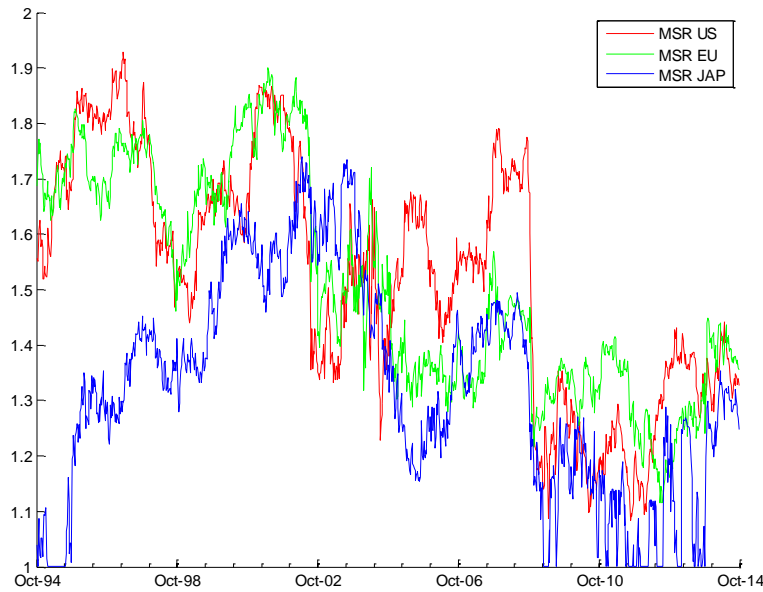
*Figure 5.2.13 – **Diversification ratio** of MSR portfolios*

5.2.2. Max Sharpe Ratio with risk contribution constraints

To limit the shortcomings of the MSR seen above, it might be helpful to introduce some boundaries on the maximum level of single stock risk contribution allowed at any time in the portfolio. Specifically two constraints are introduced in the analysis, providing that any stock held in the portfolio cannot exceed the 12% or 6% of its total risk. In this way, the resulting portfolio will always hold at least 9 and 17 stocks, respectively for the 12% and 6% boundaries. Therefore, some of the problems related to the poor diversification of the MSR could be limited.

For a stock i , at every week we have that

$$x = \arg\max \frac{x^T}{\sqrt{x^T \Sigma x}}$$

Subject to

$$RC_i(x_1, \dots, x_n) = x_i \frac{\partial \sigma(x)}{\partial x_i} = x_i \frac{(\Sigma x)_i}{\sqrt{x^T \Sigma x}} \le 12\% \ (or\ 6\%$$

These restrictions are clearly visible in the next two figures showing the distribution of stocks risk contribution for the two constrained portfolios in United States.

70
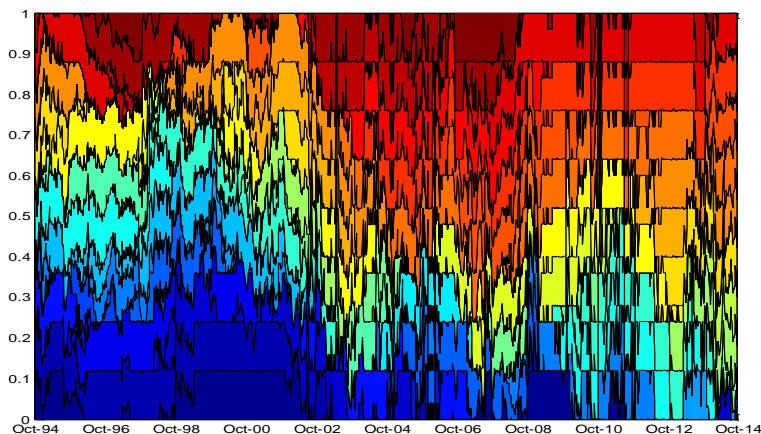
Figure 5.2.14 – Stocks Risk Contribution MSR 12 (US)
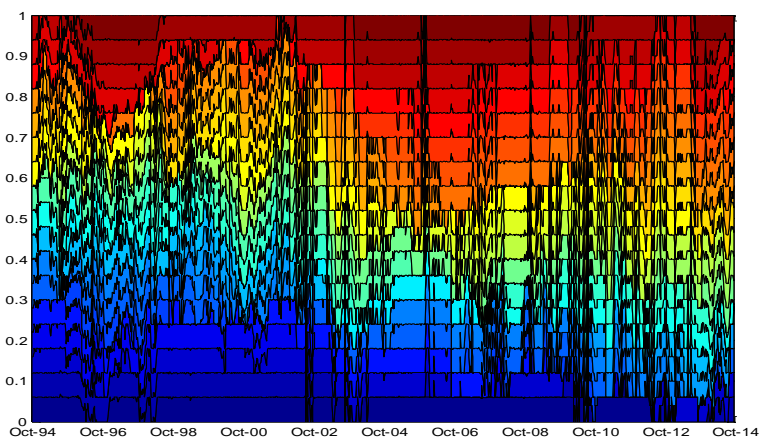
Figure 5.2.15 – Stocks Risk Contribution MSR 6 (US)

It is possible to observe and compare, in the following three figures, the hypothetical cumulative return obtained if the stock prices of *t+1* are known in *t,* with the actual realized cumulative returns (not considering the transaction costs). Clearly, from the theoretical point of view represented in the left-hand side of the figures, the unconstrained portfolio always beats the other two, and the MSR with 12% risk contribution boundary (MSR 12) always outperforms the one with the tighter boundary at 6% (MSR 6). This happens because the maximization of returns over the volatility imposed by the Sharpe Ratio criterion delivers inferior results whenever it has some restrictions preventing it to pick only the stocks with the higher Sharpe Ratio.
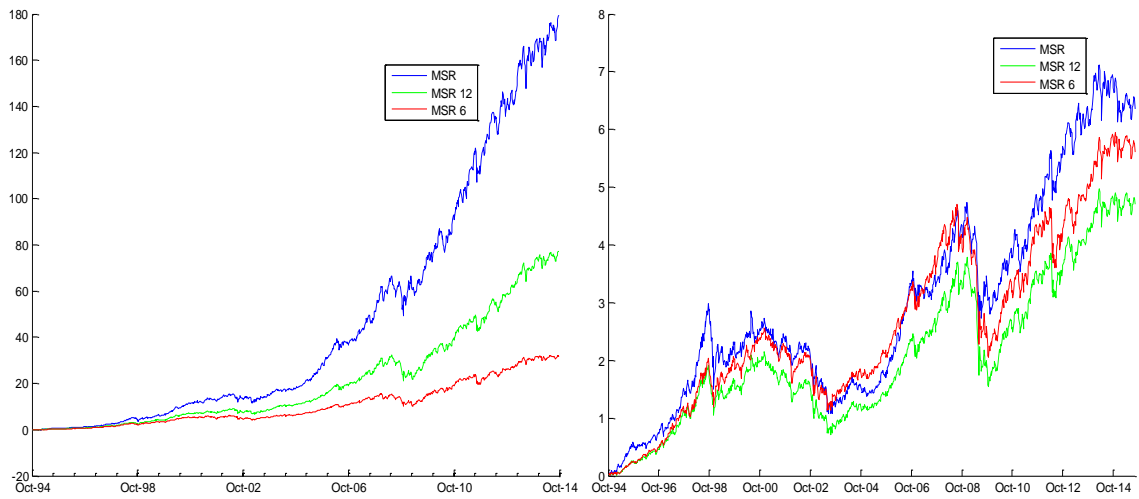
Figure 5.2.16 – Hypothetical and Realized Cumulative Returns in Europe
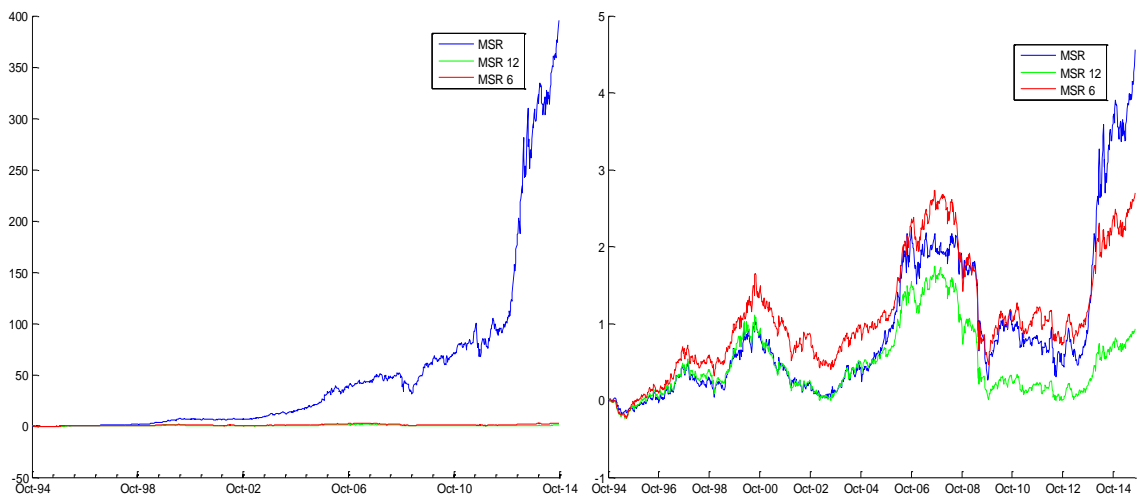
Figure 5.2.17 – Hypothetical and Realized Cumulative Returns in Japan
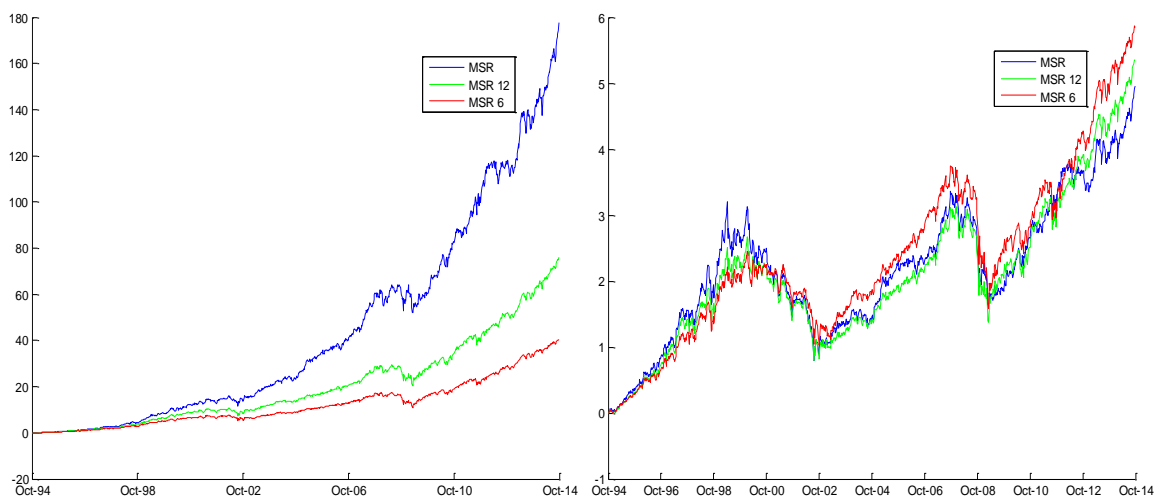
Figure 5.2.18 – Hypothetical and Realized Cumulative Returns in U.S.

However, as it could be expected considering the previous results, the situation change completely considering the realized returns indicated in the right-hand side of the figures. Specifically, the unconstrained MSR delivers higher total return both in Europe and Japan, while in the US it is the least profitable, even though all the three have quite similar trends. It can be observe how the green line of MSR 12 tends to stay below the other two (at least in Europe and Japan), while in the theoretical framework it is always in the middle. It is also quite interesting to see how the MSR 6 outperforms the MSR 12 in every region, suggesting that a more diversification may lead to higher returns, at least in the U.S. case. Considering also the Tables in Section 5.1, regarding U.S. we can observe, as expected, the improvement that the MSR portfolio has when adding the constraints, in terms of volatility, diversification ratio, turnover ratio, VaR and ES.

5.3. Conclusions

In the first part of this Section we analyzed the performance of the different portfolio allocation methods with regards to the United States investment universe. From those results it was possible to observe how all the portfolios show a common trend in terms of returns and volatility, during both periods of market crisis or economic growth, nonetheless risk based portfolios present overall slightly better performance, translated in higher Sharpe ratio values. Considering the Diversification Ratio (Choueifaty & Coignard) and the turnover rate, we saw how the MDP maximizes the first and the ERC minimizes the second, delivering the best results in terms of diversification and transaction costs. On the other hand, the GMVc and GMVu seem to be the best choices when taking into account measures of drawdown and shortfall.

In the second part of the Empirical Evidence we focused on the cumulative return analysis considering also the results from the other geographical area. On one hand, we considered the return of a hypothetical portfolio built as if the future stock prices at "t+1" were known to the portfolio managers. In a case like that, the MSR portfolio would largely outperform all the other strategies. Nevertheless, in the real world, the return of the MSR portfolio dramatically falls and the risk parity based portfolios greatly outperform, in every geographical area considered in the analysis. This difference is striking especially considering the fact that the weights of the portfolios are calculated on the basis of 5 years of weekly historical stock prices. Therefore, only by moving one week forward the range considered, the portfolios' performance radically change.

In the last part of this section we showed the results of cumulative returns adjusted for the transaction costs, confirming the benefits of the risk parity strategies and their superiority also in Europe and Japan. Finally, we introduced a new technique of portfolio construction consisting in putting some limits of single stock risk contribution to the MSR portfolio. The aim was to try to reduce the shortcomings of this optimized method, both in terms of high concentration and poor performance. By construction the portfolios obtained become more diversified in terms of weights and risk contribution, while their cumulative returns show different trends across the regions, but with an overall improvement. In the United States case, the benefits grasped by the constrained MSR are the highest, as demonstrated by the level of the Sharpe ratio which is consistently superior to the original portfolio in most of the periods (in particular in very volatile market phases).

From what seen in the Empirical Evidence, the risk parity approach seems a good way to obtain well diversified portfolio delivering competitive returns. Indeed, this is the main reason of the success of risk parity funds. It is not a coincidence that many operators from the buy-side financial industry are changing their investment views, becoming more sensitive to the risk management, in the recent crisis of credit, equity, hedge funds and sovereign bond markets.

In particular, from the evidence obtained in our analysis two portfolios could be pick as potentially "winners": the MDP and the ERC. These two methods share the logic of aiming to manage the portfolio risks, but with a diverse approach; in fact, the weights obtained by optimizing the functions on which they are built on, are completely different (i.e. they pick different assets: the ERC is by construction always diversified in all the stocks, while the MDP is , from this point of view, more "concentrated"). Nonetheless, both of them deliver extraordinary results not only in terms of low volatility, high diversification, low transaction costs and fairly low drawdown risks, but they also deliver a satisfying rate of return. In particular, they systematically outperform across all the three regions considered, not only the MSR, but also the capitalization-weighted and the minimum variance portfolios.

6. Appendix

6.1. Historical stock prices

Next figures show the price evolution of the stocks included in the investment universe of the analysis. The stocks picked are thirty large-cap from each geographical area. The prices are expressed in local currency. In the case of Europe, the prices of the constituents of CAC40 and DAX30 are expressed in Euro, even before its introduction in 1999 (the exchange rate used, is the official one adopted in the moment of the introduction in France and Germany). Meanwhile the prices in the British FTSE100 are represented in pound sterling (GBP), although the stocks are usually traded in pence sterling (GBX or GBp).

United States (USD $) - DJ30 and S& P500
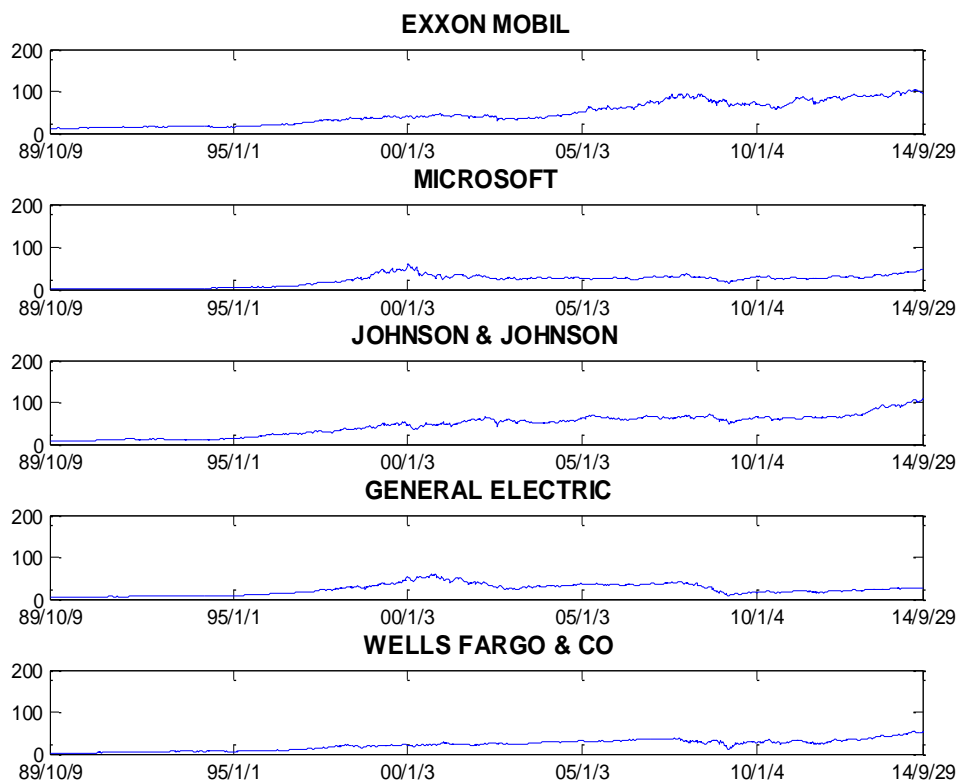
JP MORGAN CHASE & CO.
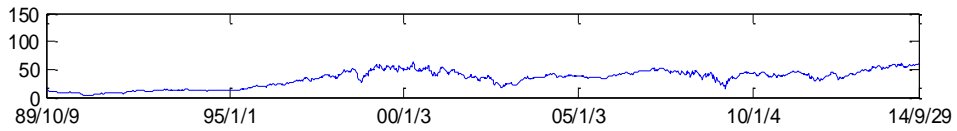
PROCTER & GAMBLE
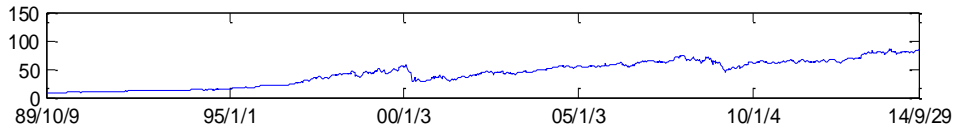
CHEVRON
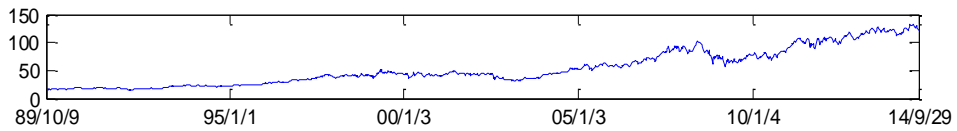
VERIZON COMMUNICATIONS
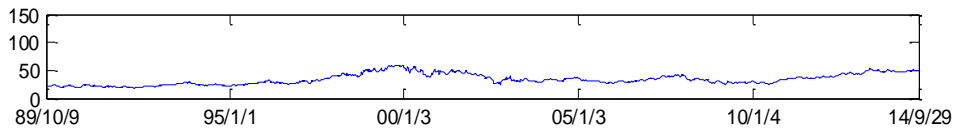
PFIZER

WAL MART STORES

AMGEN

UNION PACIFIC

CVS HEALTH

ALTRIA GROUP

BANK OF AMERICA

COCA COLA

INTERNATIONAL BUS.MCHS.

CITIGROUP

PEPSICO

COMCAST 'A'

WALT DISNEY

SCHLUMBERGER

ORACLE

HOME DEPOT

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CONOCOPHILLIPS

MCDONALDS

3M

UNITED TECHNOLOGIES

BOEING
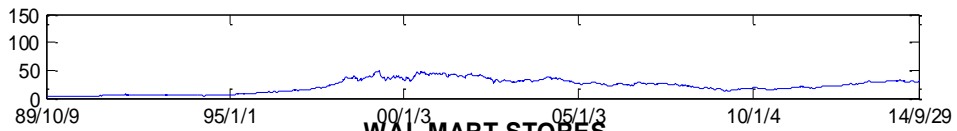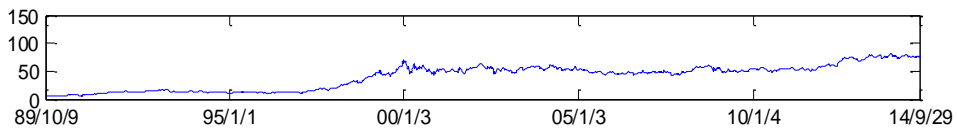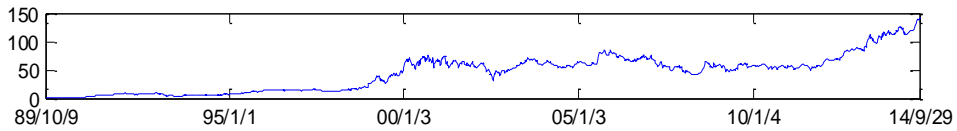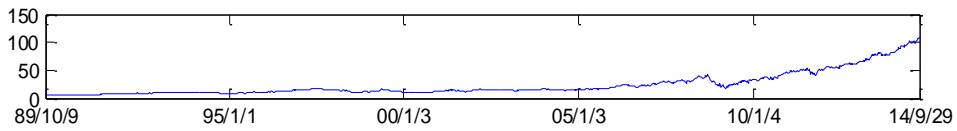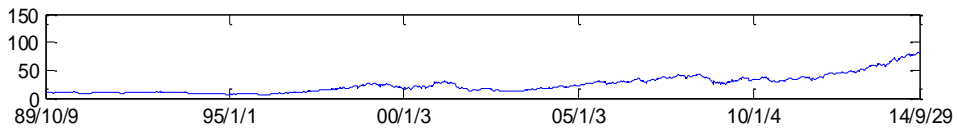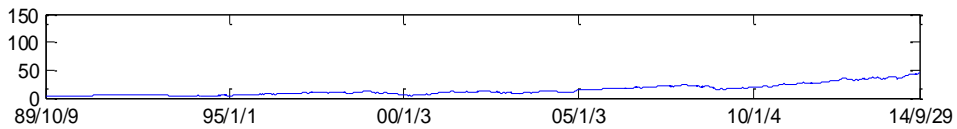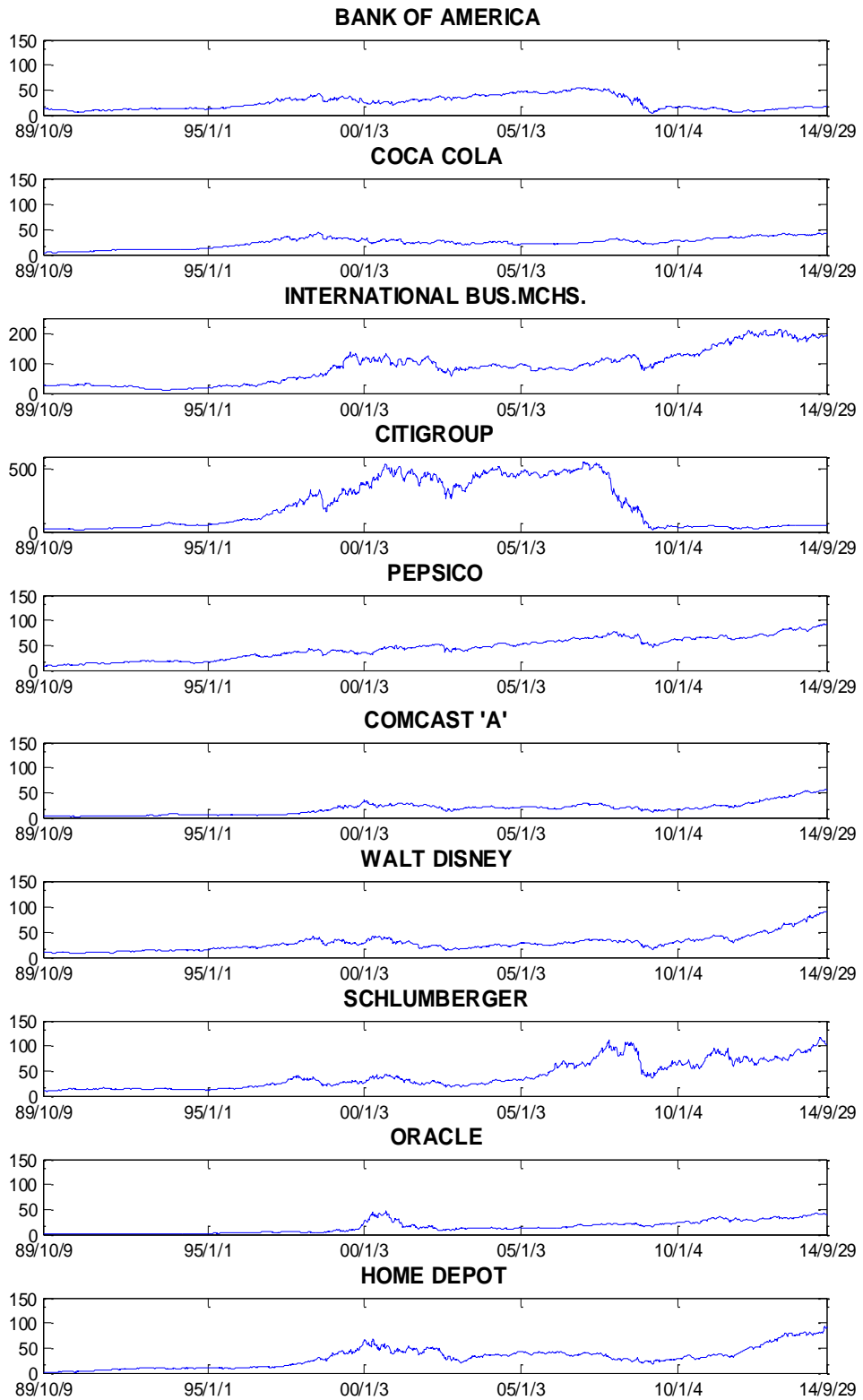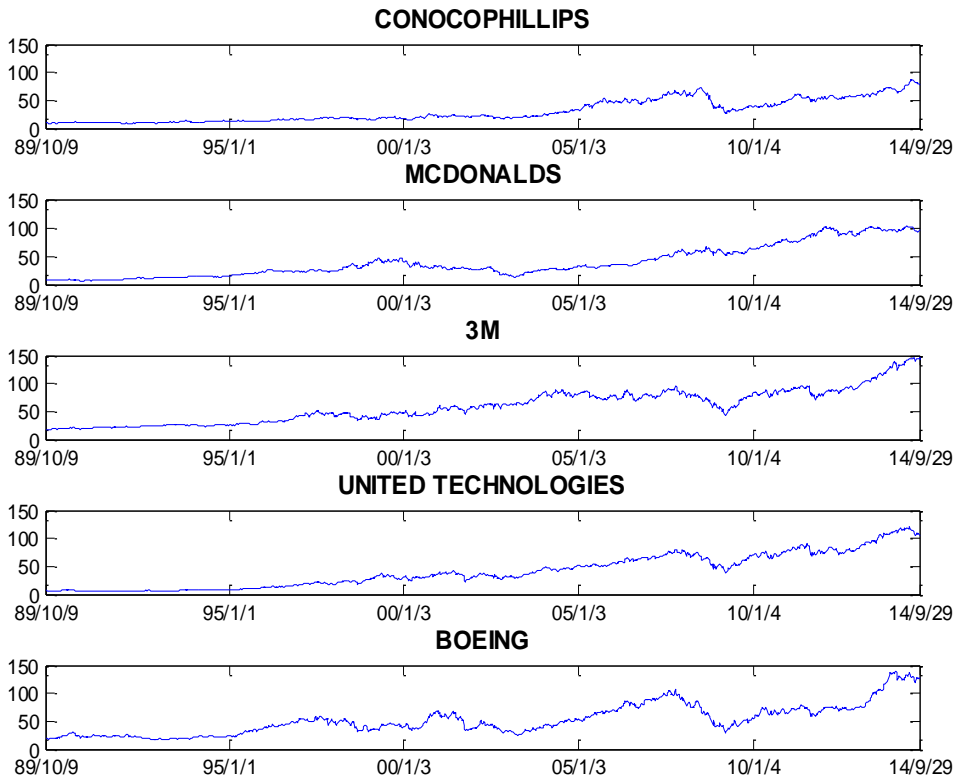
Europe (EUR €) – DAX30 Germany

BAYER

SIEMENS

SAP

BASF

ALLIANZ

78

BMW

DEUTSCHE BANK

E ON

LINDE

VOLKSWAGEN PREF.

Europe (EUR €) – CAC40 France

TOTAL

SANOFI

AXA

LVMH

SCHNEIDER ELECTRIC SE

AIR LIQUIDE

L'OREAL

DANONE

SOCIETE GENERALE

VIVENDI

Europe (GBP €) – FTSE100 UK

ROYAL DUTCH SHELL B

BP

GLAXOSMITHKLINE

BRITISH AMERICAN TOBACCO

VODAFONE GROUP

80

DIAGEO

BG GROUP

RIO TINTO

BARCLAYS

PRUDENTIAL

Japan (YEN - ¥) – NIKKEI225

SECOM

SHIN-ETSU CHEMICAL

ASTELLAS PHARMA

HONDA MOTOR

KYOCERA

81

DAIKIN INDUSTRIES

FANUC

TOYOTA MOTOR

TOKYO ELECTRON

TDK

CANON

NITTO DENKO

TERUMO

DENSO

EISAI

TOYOTA TSUSHO

x 10⁴

BRIDGESTONE

x 10⁴

FUJI HEAVY INDS.

x 10⁴

FUJIFILM HOLDINGS

x 10⁴

SUZUKI MOTOR

x 10⁴

TAKEDA PHARMACEUTICAL

x 10⁴

SEVEN & I HDG.
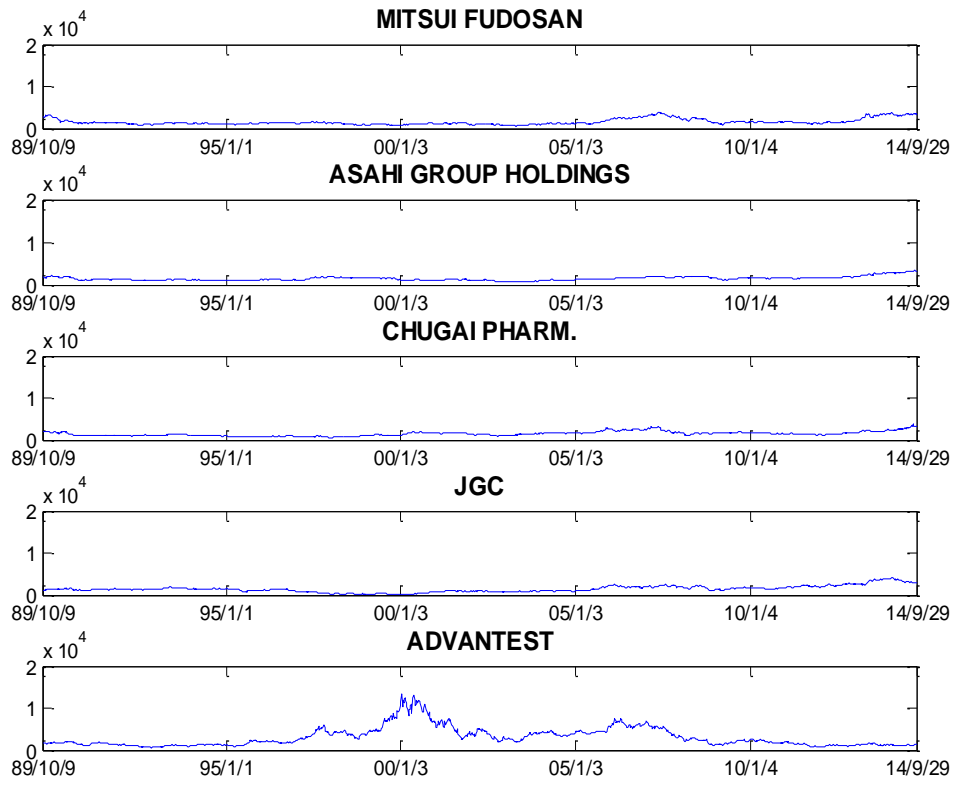
x 10⁴

KAO

x 10⁴

SUMITOMO REAL.&DEV.

x 10⁴

OLYMPUS

x 10⁴

6.2. Diversification ratio charts

Next figures[27] show the evolution, in the three geographical regions, of portfolio Diversification Ratio, the measure introduced by Choueifaty and Coignard in 2008. Considering what already stated in the previous sections, it is not surprising to find the black line of the MSR portfolio at the bottom in all the graphs. In Japan it even touches, more than once, the lower bound value of one. On the other hand, it is quite interesting to observe how the GMV achieves decent level of diversification in this sense, even though this portfolio is not diversified at all: indeed, in our analysis the GMVc usually picks about 25% - 40% of the available stocks in the universe (from 7 to 12 out of 30). Nonetheless, it does much better than the EW and the MCap, which by constructions are always diversified in all the assets. This happens, of course, because the DR is the ratio between the portfolio's weighted average asset volatility and its actual volatility.
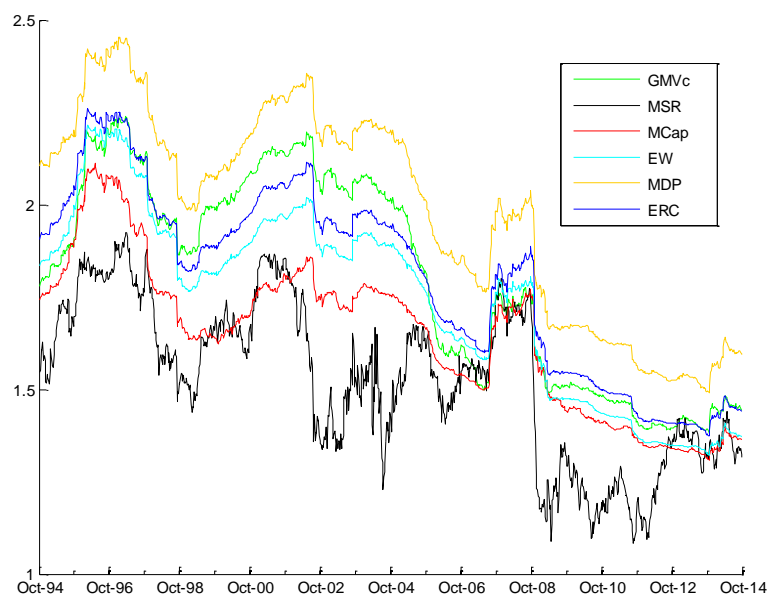

Figure 6.2.1 – Portfolio Diversification Ratio in United States.

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[27] The GMVu portfolio is not represented in the graphs because its value in terms of Diversification Ratio is not much significant since we are comparing long-only portfolios: its values would be, anyways, slightly lower than the GMVc.
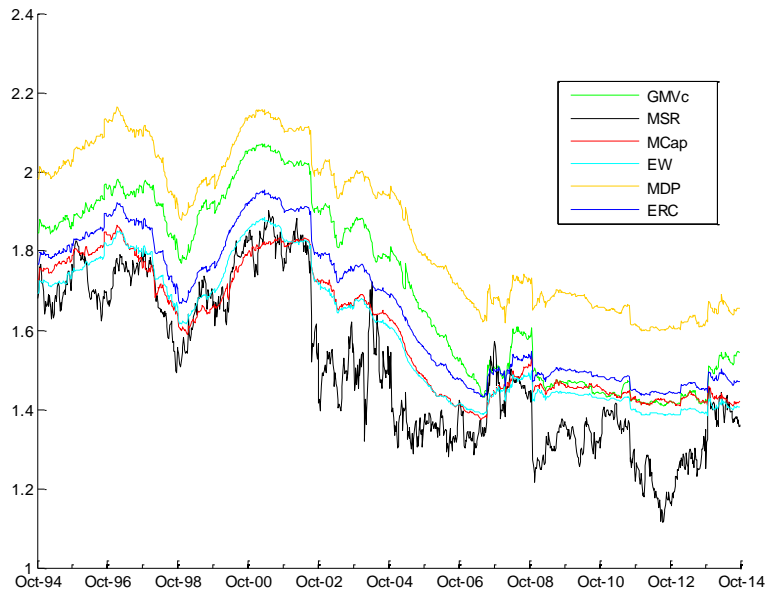
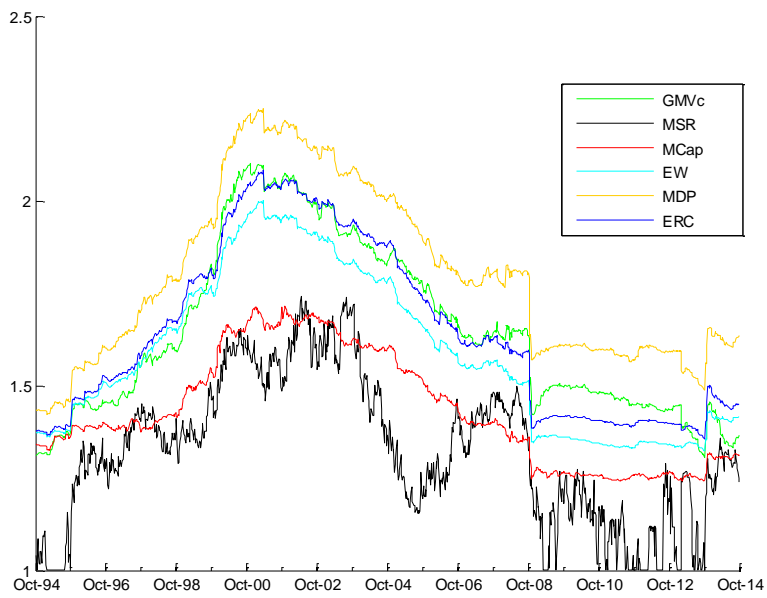*Figure 6.2.2* – Portfolio Diversification Ratio in Europe.

Figure 6.2.3 – Portfolio Diversification Ratio in Japan

6.3. Rolling volatility charts

The portfolio volatility has already been presented in Table 2 (5.1) for the United States. In this section, we show the graphs for each allocation method in every region. Starting from the weekly portfolio returns, the volatility is calculated as the annualized standard deviation on a one-year rolling base. The values are slightly different from the Table 2, because now we start the analysis from 3 October 1994, instead of starting from 2 January 1995.
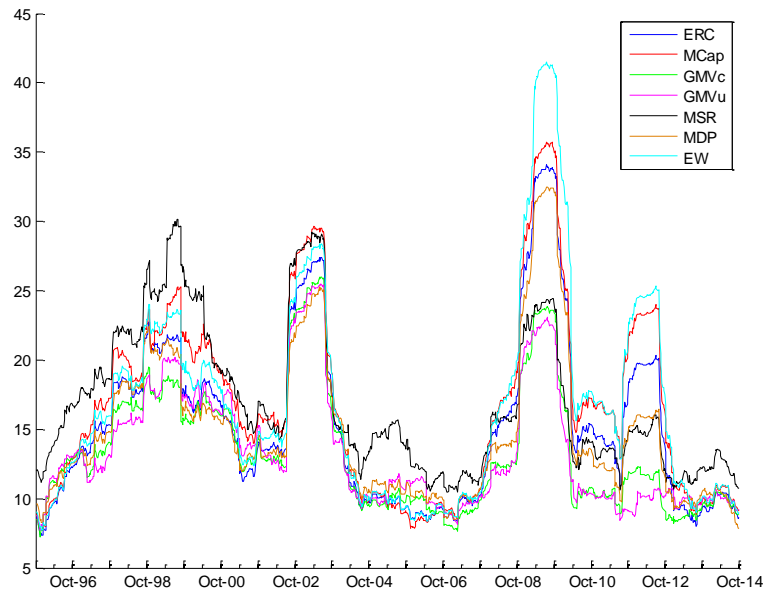

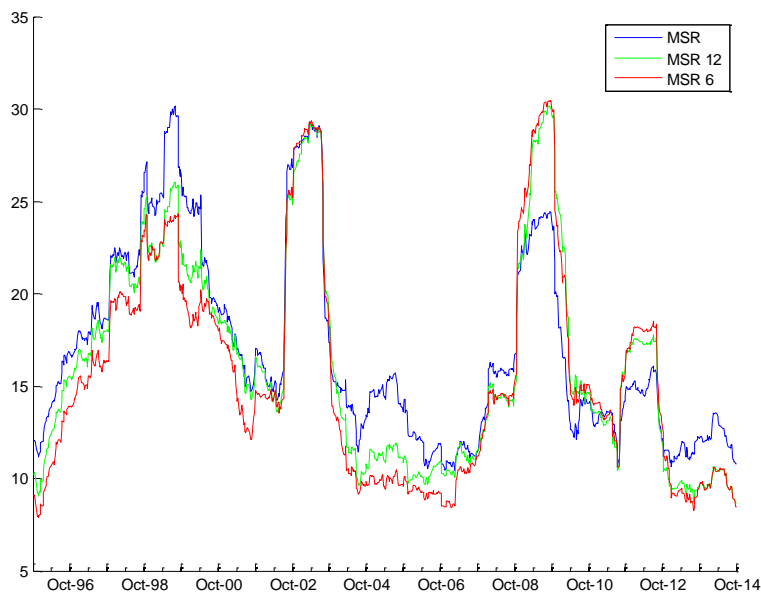*Figure 6.3.1* – Portfolio volatility in United States

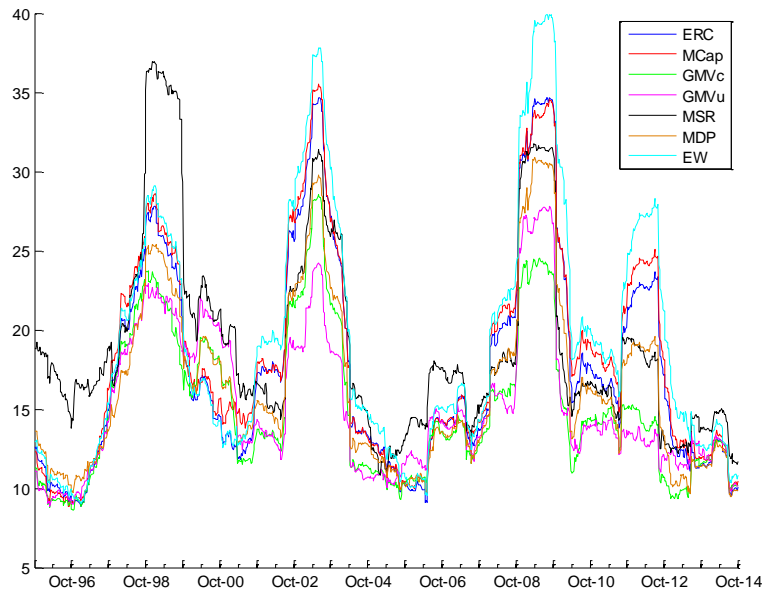*Figure 6.3.2* – MSR portfolios volatility in United States
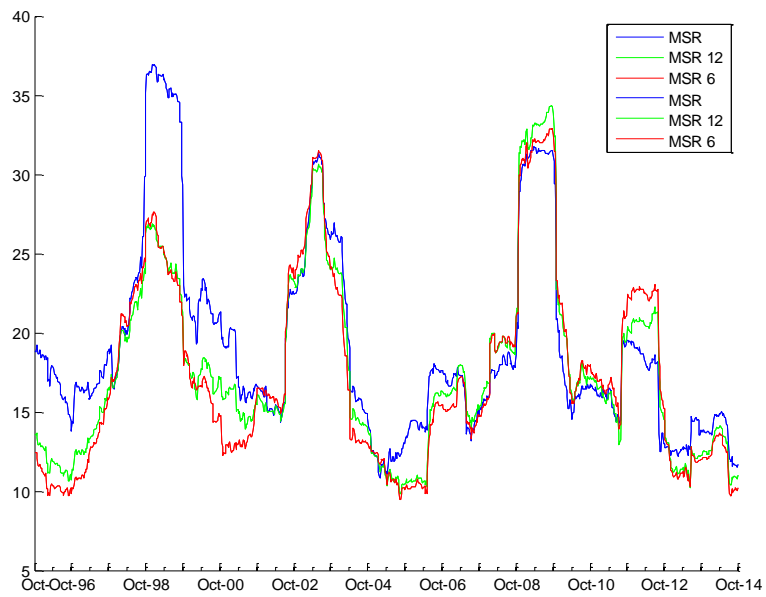
Figure 6.3.3 – Portfolio volatility in Europe

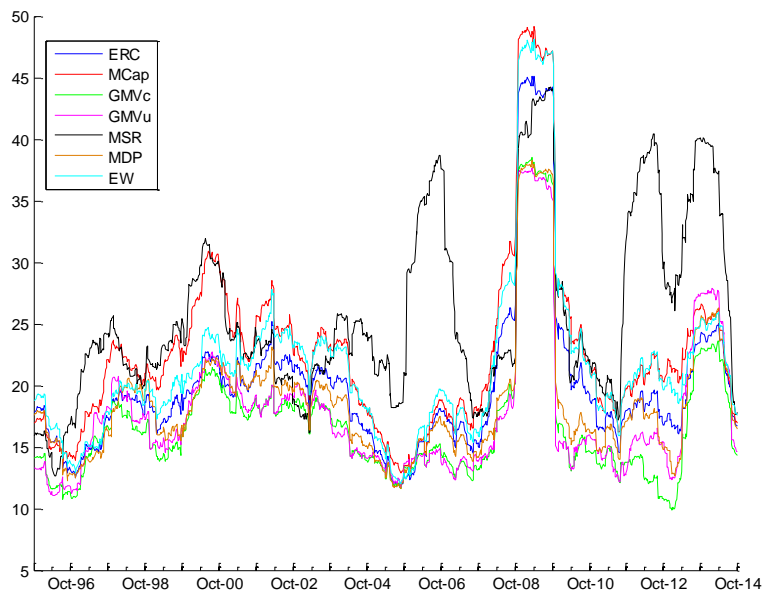*Figure 6.3.4 – MSR* Portfolio volatility in Europe

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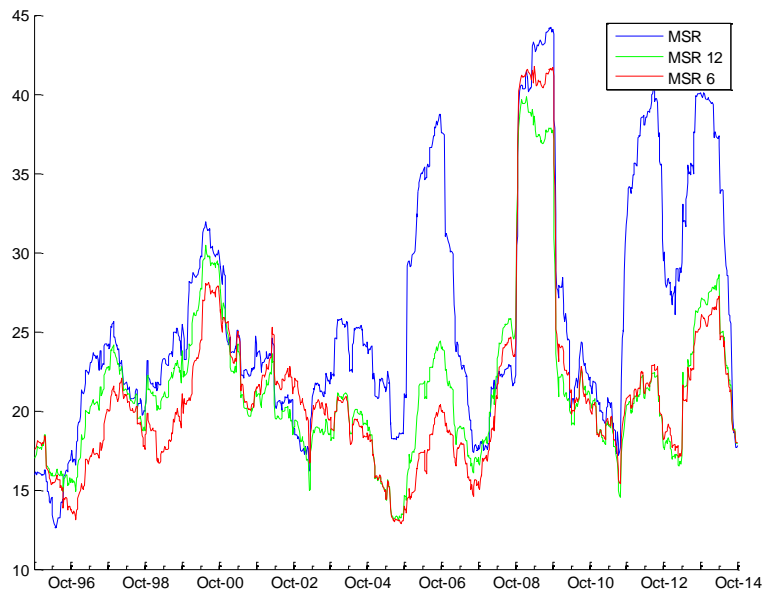*Figure 6.3.5* – Portfolio volatility in Japan

Figure 6.3.6 – MSR Portfolio volatility in Japan

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6.4. *Matlab* Codes

In this section, we will include the main part of the Matlab codes implemented to achieve the empirical evidence outcomes, divided in *scripts* and *functions*. It must be noted that for some computations, only *Excel* have been used (i.e. for the calculation of portfolio cumulative returns and single stock percentage returns).

❖ **Portfolio weights**

```
clc
clear all
tic,

[A,B]     =   xlsread('US_stocks') ; % Weekly Prices
prices    =   A;
names     =   B(6,2:end); % Stock Names and Ticker
ret       =   price2ret(prices,'','periodic');
[T,N]     =   size(ret); %1304 Weekly returns and 30 Stocks
wind      =   fix(T/5); % 260 = 5 years

sigma_all      =        zeros(N,N,T-wind);
std_all        =        zeros(N,T-wind);
mu_all         =        zeros(T-wind,N);
wGMVc          =        zeros(N,T-wind);
wGMVu          =        zeros(N,T-wind);
wEW            =        zeros(N,T-wind);
wERC           =        zeros(N,T-wind);
wMDP           =        zeros(N,T-wind);
wMSR           =        zeros(N,T-wind);
wMSR_12        =        zeros(N,T-wind);
wMSR_6         =        zeros(N,T-wind);

LB    = zeros(1,N);
UB    = ones(1,N);
beq   = 1;
b     = -eye(N);
opts  = optimset('Display','off','algorithm','active-set');
opts2 = optimset('Display','off','algorithm','sqp');

for i                  = 1:(T-wind)
    sigma_all(:,:,i)   = cov(ret(i:i+wind,:));
    mu_all(i,:)        = mean(ret(i:i+wind,:));
    std_all(:,i)       = sqrt(diag(sigma_all(:,:,i))); %Var/Cov
Matrix

%% Long-Only GMV (GMVc)
wGMVc(:,i) =  quadprog(sigma_all(:,:,i),LB,b,LB', ...
              UB,beq,LB',UB',UB'/N,opts);
```

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```matlab
        for j = 1:size(wGMVc,1)
                for k = 1:size(wGMVc,2)
                        if      wGMVc(j,k)<0
                                wGMVc(j,k)=0;
                        end
                end
        end
```

%% Unconstrained GMV (GMVu)
```matlab
wGMVu(:,i)    =    (inv(sigma_all(:,:,i))*ones(N,1))/ ...
                  (ones(1,N)*inv(sigma_all(:,:,i))*ones(N,1));
```

%% Equally Weighted (EW)
```matlab
wEW          =    (1/N)*ones(N,T-wind);
```

%% Equally Risk Contribution (ERC)
```matlab
f1           =    @(w1) ERC_FUNCTION(w,sigma_all(:,:,i));
w0           =    1/N*ones(N,1);
wERC(:,i)    =    fmincon(f1,w0,[],[],UB,beq,LB,UB',[],opts2);
```

%% Most Diversified Portfolio (MDP)
```matlab
f2        = @(w2)
MDP_FUNCTION(w2,std_all(:,i),sigma_all(:,:,i));
w0        = 1/N*ones(N,1);
wMDP(:,i) = fmincon(f2,w0,[],[],UB,beq,LB',UB',[],opts2);
```

%% Max Sharpe Ratio (MSR)
```matlab
f3  = @(w3) MSR_FUNCTION(w3,mu_all(:,i),sigma_all(:,:,i));
w0        = 1/N*ones(N,1);
wMSR(:,i  = fmincon(f3,w0,[],[],UB,beq,LB',UB',[],opts);
```

%% MSR with RC constraints(MSR_12 and MSR_6)
```matlab
F4  = @(w4) MSR_con_fun(w4,sigma_all(:,:,i));
w0           = 1/N*ones(N,1);
wMSR_12(:,i) = fmincon(f,w0,[],[],UB,beq,LB',UB',f4,opts);

end
```

❖ **ERC Function**

```matlab
function [x]=ERC_FUNCTION(w1,Sigma)
x = 0;
R = Sigma*w1;
for i=1:size(w1)
    for j=1:size(w1)
        x = x + (w1(i)*R(i)-w1(j)*R(j))^2;
    end
end
x = x/(w1'*R);
```

```matlab
end
```

❖ MDP Function

```matlab
function f = MDP_FUNCTION(w2,std,sigma)
f = -(w2'*std)*(w2'*sigma*w2)^(-.5);
end
```

❖ MSR Function

```matlab
function f = MSR_FUNCTION(w3,mu,sigma)
f = -((mu*w3)/sqrt(w3'* sigma*w3));
end
```

❖ MSR constrained Function

```matlab
function [c,ceq] = MSR_con_fun(w4,z)
ceq=0;
RC=(w4.*(z*w4))/(w4'*z*w4);
URC = 0.12*ones(size(RC)); % 12% RC constraint
c = RC-URC;
end
```

❖ Portfolio percentage returns

```matlab
ren         = ren(2:end,:);

wERC        = wERC(:,1:end-1);
wMSR        = wMSR(:,1:end-1);
wGMVc       = wGMVc(:,1:end-1);
wGMVu       = wGMVu(:,1:end-1);
wEW         = wEW(:,1:end-1);
wMDP        = wMDP(:,1:end-1);
wMCap       = wMCap(:,1:end-1);
wMSR_12     = wMSR_12(:,1:end-1);
wMSR_6      = wMSR_6(:,1:end-1);

for i=1:length(ren)
    muGMVu(i)           =           ren(i,:)*wGMVu(:,i);
    muERC(i)            =           ren(i,:)*wERC(:,i);
    muGMVc(i)           =           ren(i,:)*wGMVc(:,i);
    muEW(i)             =           ren(i,:)*wEW(:,i);
    muMSR(i)            =           ren(i,:)*wMSR(:,i);
    muMDP(i)            =           ren(i,:)*wMDP(:,i);
    muMCap(i)           =           ren(i,:)*wMCap(:,i);
    muMSR_12(i)         =           ren(i,:)*wMSR_12(:,i);
    muMSR_6(i)          =           ren(i,:)*wMSR_6(:,i);
```

```
    end


    ❖ Portfolio cumulative returns

CumRet  = xlsread('PTF_CumRet');

% PTF_CumRet are the portfolio cumulative returns computed
throush Excel, starting from the percentage returns as above.

ERC      = CumRet(:,1);
EW       = CumRet(:,2);
GMVc     = CumRet(:,3);
GMVu     = CumRet(:,4);
MCap     = CumRet(:,5);
MDP      = CumRet(:,6);
MSR      = CumRet(:,7);
MSR_12   = CumRet(:,8);
MSR_6    = CumRet(:,9);

%% Plots
figure (1)
hold on
plot(ERC,'b','DisplayName','ERC');
plot(MCap,'r','DisplayName','MCap');
plot(GMVc,'g','DisplayName','GMVc');
plot(GMVu,'m','DisplayName','GMVu');
plot(MSR,'k','DisplayName','MSR');
plot(MDP,'y','DisplayName','MDP');
plot(EW,'c','DisplayName','EW');
hold off

figure (2)
hold on
plot(MSR,'b','DisplayName','MSR');
plot(MSR_12,'g','DisplayName','MSR 12');
plot(MSR_6,'r','DisplayName','MSR 6');
hold off


    ❖ Portfolio volatility

wind=52; % 1-year rolling base

for i = 1:1043-wind
SigERC(i)       = std(muERC(i:i+wind))*sqrt(52)*100;
SigMCap(i)      = std(muMCap(i:i+wind))*sqrt(52)*100;
SigGMVc(i)      = std(muGMVc(i:i+wind))*sqrt(52)*100;
SigGMVu(i)      = std(muGMVu(i:i+wind))*sqrt(52)*100;
SigMSR(i)       = std(muMSR(i:i+wind))*sqrt(52)*100;
SigMDP(i)       = std(muMDP(i:i+wind))*sqrt(52)*100;
SigEW(i)        = std(muEW(i:i+wind))*sqrt(52)*100;
```

```matlab
SigMSR_6(i)    = std(muMSR_6(i:i+wind))*sqrt(52)*100;
SigMSR_12(i)   = std(muMSR_12(i:i+wind))*sqrt(52)*100;
end

%% Plots
figure (1)
hold on
plot(SigERC,'b','DisplayName','ERC');
plot(SigMCap,'r','DisplayName','MCap');
plot(SigGMVc,'g','DisplayName','GMVc');
plot(SigGMVu,'m','DisplayName','GMVu');
plot(SigMSR,'k','DisplayName','MSR');
plot(SigMDP,'y','DisplayName','MDP');
plot(SigEW,'c','DisplayName','EW');
hold off
```

❖ **Transaction Costs**

```matlab
%% Turnover
TO_ERC_US   = sum(abs(diff(wERC_US')'));
TO_GMVc_US  = sum(abs(diff(wGMVc_US')'));
TO_GMVu_US  = sum(abs(diff(wGMVu_US')'));
TO_MSR_US   = sum(abs(diff(wMSR_US')'));
TO_MSR_12   = sum(abs(diff(wMSR_12')'));
TO_MSR_6    = sum(abs(diff(wMSR_6')'));
TO_MDP_US   = sum(abs(diff(wMDP_US')'));
TO_MCap_US  = sum(abs(diff(wMCap_US')'));

%% Treansaction costs
Adj_muERC      =   muERC(1:end)-(TO_ERC.*0.15e-2)';
Adj_muGMVc     =   muGMVc(1:end)-(TO_GMVc.*0.15e-2)';
Adj_muGMVu     =   muGMVu(1:end)-(TO_GMVu.*0.15e-2)';
Adj_muMSR      =   muMSR(1:end)-(TO_MSR.*0.15e-2)';
Adj_muMSR_12   =   muMSR_12(1:end)-(TO_MSR_12.*0.15e-2);
Adj_muMSR_6    =   muMSR_6(1:end)-(TO_MSR_6.*0.15e-2);
Adj_muMDP      =   muMDP(1:end)-(TO_MDP.*0.15e-2)';
Adj_muMCap     =   muMCap(1:end)-(TO_MCap.*0.15e-2)';

%% Plots
figure(1);
ax_size=[0 1045 0 1];

subplot 326;
plot(TO_ERC);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('ERC');

subplot 324;
```

```matlab
plot(TO_MCap);
axis(ax_size);
ylabel('Turnover')
xlabel('Time');
title('MCap');

subplot 321;
plot(TO_GMVc);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('GMVc');

subplot 322;
plot(TO_GMVu);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('GMVu');

subplot 325;
plot(TO_MDP);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('MDP');

subplot 323;
plot(TO_MSR);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('MSR');

figure(2);
ax_size=[0 1045 0 1.1];

subplot 311;
plot(TO_MSR);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('MSR');

subplot 312;
plot(TO_MSR);
axis(ax_size);
ylabel('Turnover')
xlabel('Time');
```

```matlab
title('MSR 12%');

subplot 313;
plot(TO_MSR);
axis(ax_size);
ylabel('Turnover');
xlabel('Time');
title('MSR 6%');
```

❖ **VaR and ES**

```matlab
a       =       (muERC(13:end)); %starting from 2 Jan 1995
b       =       zeros(1030,1);

for i   =       1:length(b)
    b(i) =      a(i+1)-a(i);
end

% y=99 x=95
b1          =   sort(b(1:52)); % 1995
VaRx1       =   abs(prctile(b1,5));
VaRy1       =   abs(prctile(b1,1));
ESx1        =   abs(prctile(b1,0)+prctile(b1,1)+prctile(b1,2)+ ...
                prctile(b1,3)+prctile(b1,4)+prctile(b1,5))/6;
ESy1        =   abs(prctile(b1,0)+prctile(b1,1))/2;

b2          =   sort(b(53:105)); % 1996
VaRx2       =   abs(prctile(b2,5));
VaRy2       =   abs(prctile(b2,1));
ESx2        =   abs(prctile(b2,0)+prctile(b2,1)+prctile(b2,2)+ ...
                prctile(b2,3)+prctile(b2,4)+prctile(b2,5))/6;
ESy2        =   abs(prctile(b2,0)+prctile(b2,1))/2;

                            …

b20         =   sort(b(992:1030)); % 2014
VaRx20      =   abs(prctile(b20,5));
VaRy20      =   abs(prctile(b20,1));
ESx20       =
abs(prctile(b20,0)+prctile(b20,1)+prctile(b20,2)+...
                prctile(b20,3)+prctile(b20,4)+prctile(b20,5))/6;
ESy20       =   abs(prctile(b20,0)+prctile(b20,1))/2;
```

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