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Homoclinic chaos and the Poincaré-Melnikov method

Caos omoclino e il metodo di Poincaré-Melnikov

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Introduction

The purpose of the following thesis is to review the use of the Poincaré-Melnikov method in the detection of homoclinic phenomena, and hence chaotic dynamics.

The first chapter gives some basic definitions from the theory of dynamical systems and the mathematical objects necessary for the discussion. The second chapter presents two of the most important theorems in the theory of nonlinear dynamical system and the definitions of homoclinic and heteroclinic points. Chapter 3 deals with chaotic dynamics. This type of dynamics is explained through the horseshoe map, invented by S. Smale in the 1960's, and symbolic dynamics. Furthermore, the Smale-Birkoff homoclinic theorem is presented, that connects the existence of transverse homoclinic points to chaotic dynamics. Chapter 4 concerns the Poincaré-Melnikov method and its extension in the case of heteroclinic orbits. This method introduced by Poincaré in [13], and developed by Melnikov in [9], is still used today to prove the existence of chaotic dynamics in various fields of physics, such as fluid dynamics see [11], mechanics see [8], or astrophysics see [1]. This method is even used in the detection of chaotic orbits in the high energy accelerators like the LHC at CERN, see [10].

An application of the Poincaré-Melnikov method is given in chapter 5, where it is applied to detect the existence of chaotic phenomena in an interesting problem of fluid dynamics, the two-vortex problem. Kozlov in [6] studied the aforementioned problem with a perturbation along the y-axis. In this case the Poincaré-Melnikov method succeeds in predicting chaotic dynamics and this is confirmed by numerical evidence. We also considered a perturbation along the x-axis, where the Poincaré-Melnikov method does not assure that the system is chaotic. However, the presence of chaos is confirmed by numerical evidence.

Acknowledgments I wish to thank prof. Massimiliano Guzzo for some very useful conversations on the point vortex systems.

1. Dynamical systems

We recall some basic definitions, following [4].

1.1 Basic definitions

Definition 1.1. *A* differential dynamical system *is a triplet* $(G, \mathcal{M}, \phi)$*, where* $\mathcal{M}$ *is a differentiable manifold, called the* phase space*,* $G = \mathbb{Z}$ *or* $\mathbb{R}$*, and* $\phi : G \times \mathcal{M} \to \mathcal{M}, (t, x) \mapsto \phi(t, x) =: \phi^t(x)$*, is a differentiable map with the following properties:*

- $\phi^t : \mathcal{M} \to \mathcal{M}$*,* $\forall t \in G$*, is a diffeomorphism;*
- $\phi^0 = \mathrm{id}$*;*
- $\phi^{t+s} = \phi^t \circ \phi^s$ *for each* $t, s \in G$*.*

The *orbit* of $x \in \mathcal{M}$ is the set $\mathcal{O}(x) = \left\{\phi^t(x) | t \in G\right\}$. A point $x^* \in \mathcal{M}$ is a *fixed point* for the dynamical system if $\phi^t(x^*) = x^*$ $\forall t \in G$. A subset $\mathcal{U} \subset \mathcal{M}$ is said to be *invariant* if $\phi^t(\mathcal{U}) \subseteq \mathcal{U}$ $\forall t \in G$. For more information about dynamical system see [14], [4] or [3].

1.2 Iteration of diffeomorphisms

The discrete case $G = \mathbb{Z}$ is the case of *iteration of diffeomorphisms*. If $f : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, let

$$\begin{aligned} &\phi^1 = f; \qquad && \phi^t = \underbrace{f \circ \cdots \circ f}_{t \text{ times}} =: f^t; \\ &\phi^0 = \mathrm{id} \qquad && \phi^{-t} = \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{t \text{ times}} =: f^{-t} \qquad \forall t \in \mathbb{N}; \end{aligned}$$

then $(\mathbb{Z}, \mathcal{M}, \phi)$ is a discrete dynamical system. The fixed points of f are the fixed points of the dynamical system. Fixed points of the k-th composition of f with itself, $f^k(x) = x$, are called *periodic points* of the dynamical system of period k.

Definition 1.2. *A fixed point* x *of a diffeomorphism* f*, is said to be hyperbolic if* df_x*, the linearization of* f *evaluated in* x*, has no eigenvalues of modulus 1.*

Hyperbolic points will play a major role in the following chapters.

Moreover, two dynamical systems, $(\mathbb{Z}, \mathcal{M}, \phi)$ and $(\mathbb{Z}, \mathcal{N}, \psi)$ are *topologically conjugate* if there exists a homeomorphism $h : \mathcal{M} \to \mathcal{N}$ such that

$$h \circ \phi^t = \psi^t \circ h \qquad \forall t \in \mathbb{Z}$$

This implies that h takes the orbits of $(\mathbb{Z}, \mathcal{M}, \phi)$ to orbits of $(\mathbb{Z}, \mathcal{N}, \psi)$.

1.3 Flows

The continuous case $G = \mathbb{R}$ is the case of flows of vector fields. A vector field $X : \mathcal{M} \to T\mathcal{M}$, can be seen as a differential equation

$$\dot{x} = X(x) \tag{1.1}$$

The differential equation (1.1) gives the flow ϕ of X which satisfies

$$\frac{\partial}{\partial t}\phi^t(x) = X(\phi^t(x)), \qquad \phi^0 = \mathrm{id}$$

Therefore, $(\mathbb{R}, \mathcal{M}, \phi)$ is a continuous dynamical system. Fixed points of the dynamical system are the zeros of the vector field, $X(x) = 0$.

Definition 1.3. *A fixed point x of a flow ϕ of X is hyperbolic if dX_x, the linearization evaluated at x, has no eigenvalues with zero real part.*

1.4 Time dependent vector fields

A time dependent vector field, called *non autonomous*, gives a differential equation

$$\dot{x} = X(x, t) \tag{1.2}$$

Let $x^t(x_0; t_0)$ be the solution of (1.2). Due to the time dependence of X, if $t_0 \neq t_0'$, generally $x^{t-t_0}(x_0; t_0) \neq x^{t-t_0'}(x_0; t_0')$. The equation (1.2) does not define a flow, hence a dynamical system.

The dynamics is thus studied through the *extended vector field*

$$\dot{x} = X(x, \tau) \qquad \dot{\tau} = 1 \tag{1.3}$$

In the extended phase space $\mathcal{M} \times \mathbb{R}$, let the flow of equation (1.3) be $\Phi^t(x_0, \tau_0)$, then $\Phi^t(x_0, \tau_0) = (\phi^t(x_0, \tau_0), t + \tau_0)$ where $\phi^t(x_0, \tau_0) = x^t(x_0; \tau_0)$. $(\mathbb{R}, \mathcal{M} \times \mathbb{R}, \Phi)$ is a dynamical system and its study is completely equivalent to the study of (1.2).

1.5 Time periodic vector fields

Let X be a time periodic vector field with period T

$$X(x, t) = X(x, t + T) \qquad \forall x \in \mathcal{M}, \forall t \in \mathbb{R}$$

As we now explain, in this case, the extension can also be done to the *quotient extended phase space* $\mathcal{M} \times S^1_T$, where $S^1_T = \mathbb{R}/T\mathbb{Z}$. Indeed, since the solutions are invariant under time translation of the period T

$$x^t(x_0; t_0 + T) = x^t(x_0; t_0) \qquad \forall x \in \mathcal{M}, \forall t_0 \in \mathbb{R}$$

the extended flow has the property

$$\begin{aligned} \Phi^t(x_0, \tau_0 + T) &= \Big(\phi^t(x_0, \tau_0 + T), \tau_0 + t + T\Big) \\ &= \Big(\phi^t(x_0, \tau_0), \tau_0 + t + T\Big) \\ &= \Phi^t(x_0, \tau_0) + (0, T) \end{aligned}$$

Calling $\tilde{\Phi}$ the flow of the differential equation

$$\dot{x} = X(x, \tau), \dot{\tau} = 1 \qquad \text{with } (x, \tau) \in \mathcal{M} \times S^1_T$$

we have then

$$\tilde{\Phi}^t \circ \pi = \pi \circ \Phi^t$$

where $\pi : \mathcal{M} \times \mathbb{R} \to \mathcal{M} \times S^1_T = (x, \tau) \mapsto (x, \tau \mod T)$. The system can be studied equivalently in $\mathcal{M} \times \mathbb{R}$ or $\mathcal{M} \times S^1_T$.

1.6 Poincaré maps

In [13] Poincaré introduced the idea of analyzing flows through an associated discrete dynamical system, called the *Poincaré map*.

In the case of a time periodic vector field, extended in the phase space $\mathcal{M} \times S^1_T$, the subset

$$\Sigma_{\tau_0} = \Big\{(x, \tau) \in \mathcal{M} \times S^1 : \tau = \tau_0\Big\}$$

is intersected by all orbits of the system. The map at fixed time T of the flow

$$\Phi^T(x_0, \tau_0) = \Big(\phi^T(x_0, \tau_0), \tau_0\Big)$$

is a map from Σ_{τ_0} to itself.

Definition 1.4. *The map*

$$P^{\tau_0}(x_0) = \phi^T(x_0, \tau_0) \qquad P^{\tau_0} : \Sigma_{\tau_0} \to \Sigma_{\tau_0} \tag{1.4}$$

is called Poincaré map.

P^{τ_0} is a diffeomorphism, so the Poincaré map gives a discrete dynamical system on Σ_{τ_0}.

Proposition 1.5. *Poincaré maps relative to different sections are conjugate.*

Proof. Let $\tau_0, \tau_1 \in S^1_T$, with relative Poincaré map $P^{\tau_0} = \Sigma_{\tau_0} \to \Sigma_{\tau_0}$ and $P^{\tau_1} = \Sigma_{\tau_1} \to \Sigma_{\tau_1}$. Since $\psi := \Phi^{(\tau_1 - \tau_0)} : \Sigma_{\tau_0} \to \Sigma_{\tau_1}$ is a diffeomorphism and $\psi \circ P^{\tau_0} = P^{\tau_1} \circ \psi$, hence P^{τ_0} and P^{τ_1} are conjugate. $\square$

This proves that the choice of the section Σ_{τ_0} is not important. The dynamics is conjugate for all sections.

The study of a time periodic system through its Poincaré map P^{τ_0}, gives many insights. For example, fixed point or periodic points for P coincide with periodic orbits in the flow. If the system has an invariant set $\mathcal{U}$, then the Poincaré map has also an invariant set given by $\mathcal{U} \cap \Sigma_0$. Vice Conversely, if the Poincaré map has an invariant set, also the flow has an invariant set.

See [19] for a more in depth analysis of Poincaré maps or [4] for examples of use of Poincaré maps.

2. Stable and unstable manifolds

Following Zehnder in [21], we restrict ourselves to the case of $\mathcal{M} = \mathbb{R}^n$, and consider the dynamical system given by $(G, \mathbb{R}^n, \phi)$, where $G = \mathbb{R}$ or $G = \mathbb{Z}$. If the system has a fixed point p, it is of interest to consider the sets of points that converge or diverge to it.

2.1 The stable and unstable sets

Definition 2.1. *The stable and unstable sets for a fixed point p are*

$$\begin{aligned} W^s(p) &= \left\{ x \in \mathbb{R}^n : \lim_{t \to +\infty} \phi^t(x) = p \right\} \\ W^u(p) &= \left\{ x \in \mathbb{R}^n : \lim_{t \to -\infty} \phi^t(x) = p \right\} \end{aligned} \tag{2.1}$$

In general, these sets do not have a well defined structure. They are invariant, non empty since $p \in W^{s,u}(p)$, and, in the case of flows, connected.

If the fixed point is hyperbolic, they have more properties. To state these properties one needs to first consider a local definition of these sets.

Definition 2.2. *In a neighborhood U of p the sets*

$$\begin{aligned} W^s_U(p) &= \left\{ x \in U : \phi^t(x) \in U \ \forall t \geq 0 \text{ and } \lim_{t \to +\infty} \phi^t(x) = p \right\} \\ W^u_U(p) &= \left\{ x \in U : \phi^t(x) \in U \ \forall t \leq 0 \text{ and } \lim_{t \to -\infty} \phi^t(x) = p \right\} \end{aligned} \tag{2.2}$$

are called the local stable *and* local unstable sets *of p in U.*

If p is hyperbolic, these local sets have the following properties

Theorem 2.3 (Hadamard-Perron, 1908)**.** *Assuming that p is a hyperbolic fixed point of $(G, \mathbb{R}^n, \phi)$ fixed point p. Let E^s and E^u be the stable and unstable subspaces of the linearization of the dynamical system in p. Then in a sufficiently small neighborhood U of p,*

1. *$W^s_U(p)$ and $W^u_U(p)$ are embedded submanifolds of $\mathbb{R}^n$ and n_s and n_u their dimension;*

2. *$T_p W^s_U(p) = E^s$ and $T_p W^u_U(p) = E^u$;*

3. $\exists \delta_u, \delta_s, \lambda_u, \lambda_s > 0$, *such that*

$$\begin{aligned} \left\|\phi^t(x) - p\right\| &\leq \delta_s e^{-\lambda_s t}\|x - p\| \qquad \forall t \geq 0 \text{ and } x \in W^s_U(p), \\ \left\|\phi^t(x) - p\right\| &\leq \delta_u e^{-\lambda_u t}\|x - p\| \qquad \forall t \leq 0 \text{ and } x \in W^u_U(p). \end{aligned} \tag{2.3}$$

Proof. See [21]. □

The manifold structure of these sets can be globalized to some extent.

Theorem 2.4 (Smale, 1963)**.** *Let p be a hyperbolic fixed point for a dynamical system $(G, \mathbb{R}^n, \phi)$. Then $W^s(p)$ and $W^u(p)$ are injective immersions of $\mathbb{R}^{n_s}$ and $\mathbb{R}^{n_u}$ in $\mathbb{R}^n$.*

Proof. See [16] or [17]. □

2.2 Parametric dependence

Let $f_\varepsilon : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism, which depends smoothly in the real parameter ε.

Proposition 2.5. *If f_0 has a hyperbolic fixed point p then for ε sufficiently small*

- *f_ε has a hyperbolic fixed point p_ε, $\mathcal{O}(\varepsilon)$ close to p;*
- *$W^s_\varepsilon(p_\varepsilon)$ and $W^u_\varepsilon(p_\varepsilon)$ depend smoothly on ε.*

The case of flows is quite similar,

Proposition 2.6. *Let X_ε be a vector field which smoothly depends in the parameter ε. Then the flow ϕ_ε depends smoothly on ε.*

2.3 Homoclinic and heteroclinic points

Definition 2.7. *Let p a hyperbolic fixed point of a dynamical system $(G, \mathbb{R}^n, \phi)$. A point $q \neq p$ such that*

$$q \in W^s(p) \cap W^u(p)$$

is said to be a homoclinic point *for p.*

If $T_qW^s(p) \oplus T_qW^u(p) = \mathbb{R}^n$ then q is a *transverse homoclinic point.* See figure 2.1(a).

Definition 2.8. *Let $p_1 \neq p_2$ be two hyperbolic fixed points, and $p_1 \neq q \neq p_2$. Then if*

$$q \in W^s(p_1) \cap W^u(p_2)$$

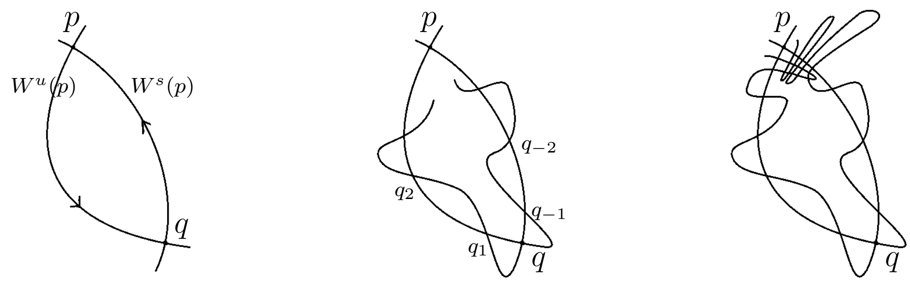
the point q is said to be a heteroclinic point.

If $T_qW^s(p_1) \oplus T_qW^u = \mathbb{R}^n(p_2)$ then q is a *transverse heteroclinic point.*

Since W^s and W^u are invariant sets, each point in the orbit of a homoclinic (resp. heteroclinic) point is homoclinic (resp. heteroclinic), and such an orbit is called homoclinic (resp. heteroclinic) orbit.

2.4 Homoclinic tangle in $\mathbb{R}^2$

Figure 2.1: Examples of homoclinic point (a), the first orbit of homoclinic points (b), and formation of other orbits (c)



a) b) c)

Assuming that the phase space is $\mathbb{R}^2$, we consider the dynamical system given by a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$. Let p be a hyperbolic fixed point for f, we also assume that $W^s(p)$ and $W^u(p)$ are two curves in $\mathbb{R}^2$. If q is a transverse homoclinic point for f, then the whole orbit of q is formed by transverse homoclinic points (transversality follows from the fact that df_x is a isomorphism $\forall x \in \mathbb{R}^2$). The iterations of q, $q^i := f^i(q)$, $i \in \mathbb{Z}$, tend to p for $i \to +\infty$ along $W^s(p)$, converging to p exponentially, by theorem 2.3. The two curves intersect transversally at each point in the orbit of q, so as q^i approaches p along W^s, the curve W^u will cut through W^s creating folds as in figure 2.1(b). As $i \to +\infty$, the folds of $W^u(p)$ arrange parallel to each other other, growing larger and thinner, see figure 2.1(c). Consequently, for i large enough, other homoclinic points, that do not belong to the orbit of q, are formed. In turn, these new points generate other intersections and hence new homoclinic orbits.

A similar situation happens for $i \to -\infty$ with the folds of $W^s(p)$ wrapping around $W^u(p)$ and creating again new homoclinic orbits. The union of all the homoclinic points for p so created form the *homoclinic tangle.* The dynamics of systems presenting homoclinic points is in [21], [18] or [12].

It iss defined, (see [19]),

Definition 2.9. *A set* $\Lambda \subset \mathbb{R}^2$ *which*

- *is closed;*
- *is perfect, (each point in* Λ *is a limit point);*
- *is totally disconnected, (*$\nexists$ *a non-trivial connected subset of* Λ*).*

is a Cantor set.

A cantor set is uncountable and nowhere dense in $\mathbb{R}^2$.

Proposition 2.10. *If p is a hyperbolic fixed point for the dynamical system given by f on $\mathbb{R}^2$, if non-empty, the set of all the transverse homoclinic points for p is a Cantor set.*

Proof. See [12] or [18]. □

3. Homoclinic chaos

According to Wiggins in [19], we explain chaotic dynamics through a model introduced by S. Smale in [15], the horseshoe map.

3.1 The horseshoe map

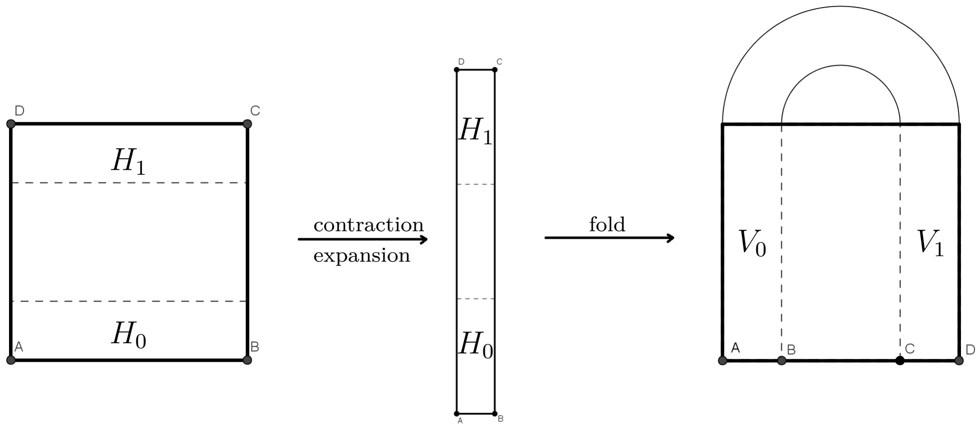


Figure 3.1: On the left is shown U with the subsets H_0, H_1; the contraction and expansion of U, and $\mathcal{F}(U)$ with V_0, V_1 on the right

Let U be the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$. Let $H_0 = [0,1] \times [0,1/4]$ and $H_1 = [0,1] \times [3/4,1]$ be two disjoint horizontal strips; let $V_0 = [0,1/4] \times [0,1]$ and $V_1 = [3/4,1] \times [0,1]$ be two disjoint vertical strips. The map $\mathcal{F} : U \to \mathbb{R}^2$ acts contracting U along x, expanding it along y and folding it, as shown in figure 3.1. The horizontal strips H_0, H_1 are mapped into the vertical strips $V_0 = \mathcal{F}(H_0), V_1 = \mathcal{F}(H_1)$, while the folded part is mapped outside of U. The orientation of the folding is given by the vertexes A, B, C, D. Thus, $U \cap \mathcal{F}(U) = V_0 \cup V_1$; since $\mathcal{F}$ is not onto, $\mathcal{F}^{-1}$ is not globally defined. However $H_0 \cup H_1$ is the preimage of $U \cap \mathcal{F}(U)$, that is $\mathcal{F}^{-1}(U \cap \mathcal{F}(U)) = H_0 \cup H_1$.

We want to describe the set of all points in U whose orbits remain in U,

$$\Lambda = \Big\{ p \mid \mathcal{F}^i(p) \in U, i \in \mathbb{Z} \Big\}$$

As shown in figure 3.2, applying $\mathcal{F}$ to $U \cap \mathcal{F}(U)$ we get four vertical strips. Continuing the argument inductively, the set $U \cap \mathcal{F}(U) \cap \cdots \cap \mathcal{F}^n(U)$ is the

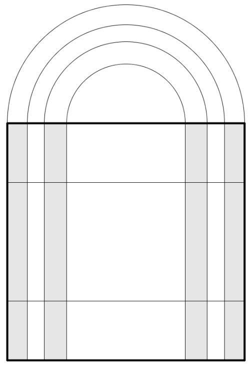


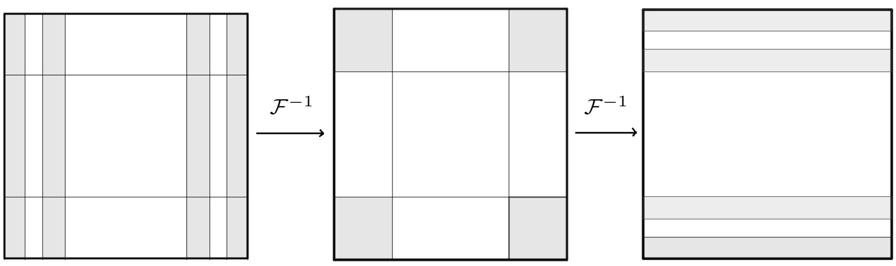
Figure 3.2: The grey area is $U \cap \mathcal{F}(U) \cap \mathcal{F}^2(U)$.

union of 2^n vertical strips, that in the limit $n \to +\infty$ become vertical segments and form the set

$$\Lambda_+ = \left\{ p \mid \mathcal{F}^i(p) \in U, i \geq 0 \right\}$$

which is a Cantor set (see [14]).

If a point $p \in U$, then $\mathcal{F}^{-1}(p) \in H_0 \cup H_1$. If $\mathcal{F}(p) \in U$ then $\mathcal{F}^{-1}(\mathcal{F}(p)) \in H_0 \cup H_1 = \mathcal{F}^{-1}(U \cap \mathcal{F}(U))$. While if $\mathcal{F}^2(p) \in U$, $p \in \mathcal{F}^{-2}(U \cap \mathcal{F}(U) \cap \mathcal{F}^2(U))$. The set $\mathcal{F}^{-2}(U \cap \mathcal{F}(U) \cap \mathcal{F}^2(U))$ is shown in figure below, outlined in grey.



Therefore if $(U \cap \mathcal{F}(U) \cap \cdots \cap \mathcal{F}^n(U))$ is a set of 2^n vertical strips, the n-th preimage $\mathcal{F}^{-n}(U \cap \mathcal{F}(U) \cap \cdots \cap \mathcal{F}^n(U))$ is a set of 2^n horizontal strips. When $n \to +\infty$, the strips form the Cantor set of horizontal segments

$$\Lambda_- = \left\{ p \mid \mathcal{F}^{-i}(p) \in U, i \geq 1 \right\}$$

Since $\Lambda = \Lambda_+ \cap \Lambda_-$, a point $p \in \Lambda$ must belong to both a vertical and horizontal segment. Therefore, Λ is a Cantor set.

So far we have constructed the invariant set Λ for the horseshoe. Now, we want to describe the dynamics given by $\mathcal{F}$ on Λ.

3.2 Symbolic dynamics

Each point $p \in \Lambda$ can be associated with a string of 2 symbols. Recalling [19], we show how it is done.

Let $S = \{0, 1\}$, where element $s \in S$ is called a symbol. A string of symbols is a sequence with index in $\mathbb{Z}$, $\boldsymbol{s} = \{s_i\}_{i \in \mathbb{Z}}$. The set of all the strings of symbols is denoted $\mathcal{S} = \{\{s_i\}_{i \in \mathbb{Z}} | s_i \in S \ \forall i \in \mathbb{Z}\}$.

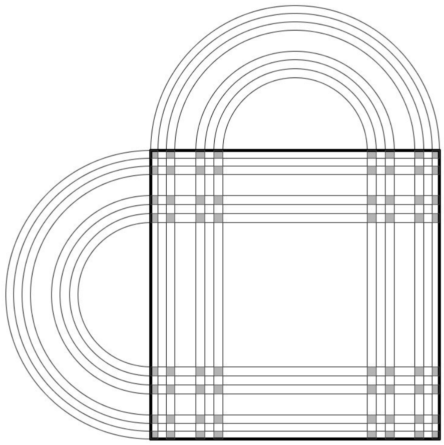


Figure 3.3: Outlined in grey $\bigcap_{n=-2}^{2} \mathcal{F}^n(U)$ to give an idea of the invariant set Λ.

Let $\psi : \Lambda \to \mathcal{S}$ be the map that associates a string of symbols in $\mathcal{S}$ to each point p in Λ as follows

$$\psi(p)_i = a_i \text{ such that } \qquad f^i(p) \in H_{a_i}.$$

The sequence $\psi(p)$ is the *horizontal history* of p. It specifies if, $\forall i \in \mathbb{Z}$, the i-th iteration under f of p is either in H_0 or in H_1.

The set $\mathcal{S}$ can be equipped with the metric

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i \in \mathbb{Z}} 2^{-|i|} \delta(a_i, b_i) \qquad \delta(a_i, b_i) = \begin{cases} 0 & a_i = b_i \\ 1 & a_i \neq b_i \end{cases} \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{S}. \tag{3.1}$$

From now on, the set $\mathcal{S}$ will be considered with the topology induced by the metric (3.1).

Proposition 3.1. *The map $\psi : \Lambda \to \mathcal{S}$ is a homeomorphism.*

Proof. See [4]. □

Definition 3.2. *The* shift map $\sigma : \mathcal{S} \to \mathcal{S}$ *is the map defined* $\forall \boldsymbol{a} \in \mathcal{S}$ *as*

$$\sigma(\boldsymbol{a})_i = a_{i+1} \qquad \forall i \in \mathbb{Z}$$

The map σ is a homeomorphism in $\mathcal{S}$, see [19].

Theorem 3.3. *The dynamical system given by σ on $\mathcal{S}$ is conjugate to the dynamical system given by $\mathcal{F}$ on Λ:*

$$\psi \circ \mathcal{F}|_\Lambda = \sigma \circ \psi.$$

Proof. See [4]. □

The immediate consequence is that

$$\psi \circ \mathcal{F}^n|_\Lambda = \sigma^n \circ \psi$$

Therefore, the unique string associated to $p \in \Lambda$ by ψ contains all the information about the past and future iterations of $\mathcal{F}$.

The dynamics of σ has the following properties (see [2]). A k-periodic point is given by a string $\boldsymbol{a}$, $a_i = a_{i+k}$, $\forall i \in \mathbb{Z}$. If $k = 1$, $\boldsymbol{a}$ is a fixed point and there exist only two fixed points. The most important features of the dynamics of σ are proven in the following theorem.

Theorem 3.4. *In the dynamical system given by σ on $\mathcal{S}$,there exist:*

1. *A countable infinity of periodic orbits, with period arbitrarily large;*
2. *An uncountable infinity of non-periodic orbits.*
3. *An orbit which is dense in $\mathcal{S}$.*

Proof. Periodic points are given by string composed of periodically repeating blocks. The length of the block gives the period. Since for any fixed $k \in \mathbb{N}$, the number of periodic sequences with period k is finite, there exists a countable infinity of periodic orbits with all possible periods.

Since $\mathcal{S}$ and Λ are homeomorphic, $\mathcal{S}$ is a Cantor set and therefore uncountable. Given that there is a countable infinity of periodic orbits, the remaining orbits, which are non-periodic, are uncountable.

To prove the existence of an orbit which is dense in $\mathcal{S}$, it has to be proved that there exists $\boldsymbol{s} \in \mathcal{S}$ for any $\boldsymbol{s}' \in \mathcal{S}$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(\sigma^n(\boldsymbol{s}), \boldsymbol{s}') < \varepsilon$, where d is the metric (3.1). Noting that two string $\boldsymbol{s}', \boldsymbol{s}'' \in \mathcal{S}$ are closer than $\varepsilon > 0$ if $s_i' = s_i''$ for $|i| \leq N$ where $2^{-N} < \varepsilon$.

Let $\boldsymbol{s}$ be the string which contains all possible finite sequence of symbols. Therefore, by construction of $\boldsymbol{s}$, for each $\boldsymbol{s}' \in \mathcal{S}$ there exists $k \in \mathbb{Z}$, where $d(\sigma^k(\boldsymbol{s}), \boldsymbol{s}') < \varepsilon$. Thus, the orbit of $\boldsymbol{s}$ is dense in $\mathcal{S}$. $\square$

3.3 Chaotic dynamics of the horseshoe map

In the previous section it has been shown that the Smale horseshoe map on Λ is conjugate to a shift σ of two symbols. This implies that the horseshoe map shares all the properties listed in theorem 3.4: countable periodic orbits, uncountable non-periodic orbits and an orbit which is dense in the invariant set Λ.

Let $p, q \in \Lambda$ be two points such that $\psi(p)$ and $\psi(q)$ agree on the central block of lenght k such that $\psi(p)_i = \psi(q)_i$ for $|i| < k$ and $\psi(p)_k \neq \psi(q)_k$. Then by the definition of ψ, each one of the first $(k-1)$ forward and backward iterations of p and q belong to the same horizontal strip, H_0 or H_1. The k-th iteration of f separates one point in H_0 and the other in H_1. Thus, no matter how close the two points p and q are in Λ, after some number of iterations they will be separated at least by a certain fixed distance.

Definition 3.5. *A dynamical system $(G, \mathbb{R}^n, \phi)$ is said to have* sensitive dependence on initial condition, *if $\exists \varepsilon > 0$, such that $\forall x \in \mathbb{R}^n$ and for every open neighborhood U of x there is $y \in U$ and $t \in G$ such that $d(\phi^t(x), \phi^t(y)) > \varepsilon$. The dynamic is usually called* chaotic.

Therefore, the horseshoe exhibits sensitive dependence on initial condition and consequently also σ has chaotic dynamic.

3.4 The Smale-Birkoff homoclinic theorem

The symbolic dynamic can be extended easily to the case of N symbols. Let $S = \{1, 2, \ldots, N\}$, then the space of all the strings of N symbol becomes $\mathcal{S}^N$, and it can be equipped with the metric (3.1).

On $\mathcal{S}^N$ it can be defined the shift map $\sigma : \mathcal{S}^N \to \mathcal{S}^N$ in the same manner as above, and the dynamics of σ in $\mathcal{S}^N$ has all the properties discussed above.

The following theorem connects the existence of transverse homoclinic points to chaotic dynamics.

Theorem 3.6 (Smale, 1963)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism, with p a hyperbolic fixed point for f, and let q a transverse homoclinic point for p. Then there exist integers n and $N \geq 2$ so that f^n has an invariant Cantor set Γ, on which f^n is conjugate to a shift of N symbols.*

The theorem proved by Smale in [15] implies that there is an invariant Cantor set Γ in the presence of a homoclinic point, on which the dynamics is conjugate to the shift. Therefore f on Γ displays all the properties discussed in theorem 3.4, in particular exhibits sensitive dependence on initial conditions, and therefore is chaotic.

4. The Poincaré-Melnikov method

In 1963 V. Melnikov (see [9] for the original article) investigated a sufficient condition for the existence of transverse homoclinic points in perturbed Hamiltonian systems. The condition was first studied by Henry Poincaré, see [13], and it is often called "The Poincaré-Melnikov Method".

4.1 Hamiltonian systems

The Hamiltonian systems are the class of dynamical systems, whose differential equation is of the form:

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial y_i}(x_1, \ldots, x_n, y_1, \ldots, y_n) \\ \dot{y}_i &= -\frac{\partial H}{\partial y_i}(x_1, \ldots, x_n, y_1, \ldots, y_n) \end{aligned} \tag{4.1}$$

where the phase space is $\mathbb{R}^{2n}$, and the function $H : \mathbb{R}^{2n} \to \mathbb{R}$ is called the *Hamiltonian*. Or in a more compact form, where $q = (x_1, \ldots, x_n, y_1, \ldots, y_n)$:

$$\dot{q} = \mathbb{J}\nabla H(q) \qquad \text{where} \quad \mathbb{J} = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_n \\ \mathbb{O}_n & -\mathbb{I}_n \end{pmatrix}$$

The function H is a first integral:

$$\dot{H} = \nabla H \cdot \dot{q} = \nabla H \cdot \mathbb{J}\nabla H = 0$$

In the simplest form the method is applied to bi-dimensional Hamiltonian systems, perturbed with a small perturbation that is possibly non Hamiltonian, which depends periodically on time.

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x, y) + \varepsilon g_x(x, y, t) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x, y) + \varepsilon g_y(x, y, t) \end{aligned} \tag{4.2}$$

For $\varepsilon = 0$, the system is called *unperturbed system* and it is assumed that it has a hyperbolic fixed point p with a homoclinic orbit γ_0. See figure 4.1. The

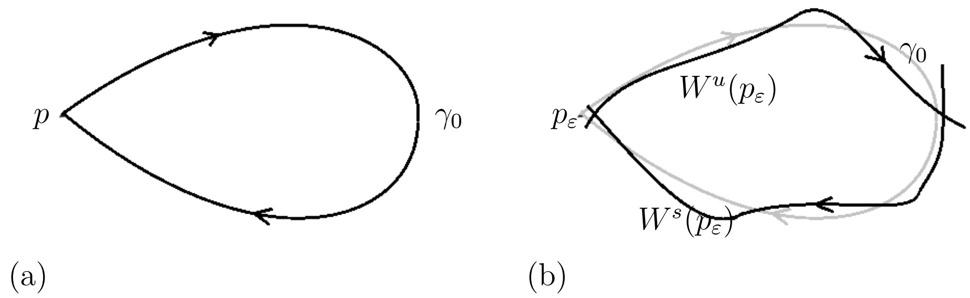


Figure 4.1: (a) The fixed point p and the homoclinic orbit γ_0. (b) For ε small, the stable and unstable manifolds of p_ε, in the Poincaré map.

study of this system can be reduced to that of one of its Poincaré maps. If T is the period of g the Poincaré map $P_\varepsilon(x,y) : \Sigma_0 \to \Sigma_0$ is

$$P_\varepsilon(x,y) = \phi_\varepsilon^T(x,y,0)$$

When $\varepsilon = 0$, P_0 can be considered as the Poincaré map of the unperturbed system. The point p is a hyperbolic fixed point for P_0, and γ_0 the homoclinic orbit for p.

By propositions 2.5 and 2.6, the Poincaré map depends smoothly on ε. For ε sufficiently small, P_ε has a hyperbolic fixed point p_ε, close to p. The invariant manifolds, $W_\varepsilon^s(p_\varepsilon)$ and $W_\varepsilon^u(p_\varepsilon)$, will not necessarily coincide anymore, see figure 4.1(b).

The Poincaré-Melnikov method gives a sufficient condition to prove the existence of a transverse intersection between these two manifolds. From now we write $W_\varepsilon^s(p_\varepsilon) =: W^s$ and $W_\varepsilon^u(p_\varepsilon) =: W^u$.

4.2 The Melnikov function

In the unperturbed system, the curve γ_0, being the image of a solution, can be parametrized with the time along that solution. Let us choose $q^* \in \gamma_0$ and a parametrization

$$\lambda \mapsto \gamma_0(\lambda) := \phi_0^\lambda(q^*) \tag{4.3}$$

At each point of γ_0 the vector $\upsilon(\lambda) = \nabla H_0(\gamma_0(\lambda))$ is perpendicular to γ_0. This vector, gives a segment r:

$$r(\lambda) = \{\gamma_0(\lambda) + t\,\upsilon(\lambda) \,|\, t \in]-1, 1[\}$$

For $\varepsilon = 0$, W^s and W^u coincide with γ_0 and intersect r perpendicularly. For ε sufficiently small, W^s and W^u still intersect r transversely. However, W^s and W^u may intersect r many times, not just once. The choice of the intersection point with which continue the study is given by the following lemma.

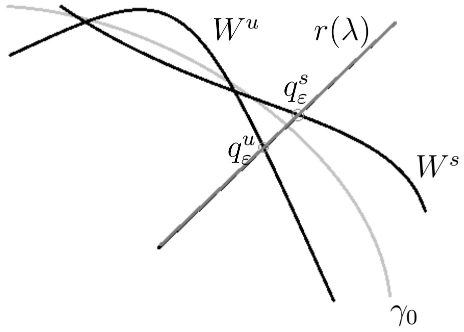


Figure 4.2: The segment r_0 with the intersections q_ε^s and q_ε^u.

Lemma 4.1. *For ε sufficiently small, $\exists!\, q_\varepsilon^s \in W^s \cap r(\lambda)$ where*

$$\phi_\varepsilon^t(q_\varepsilon^s(\lambda)) \cap r(\lambda) = \emptyset \qquad \forall t \geq 0$$

$$\left\| \phi_\varepsilon^t(q_\varepsilon^s(\lambda)) - \gamma_0(t+\lambda) \right\| = O(\varepsilon) \qquad \forall t \geq 0$$

$$\left\| \dot{\phi}_\varepsilon^t(q_\varepsilon^s(\lambda)) - \dot{\gamma_0}(t+\lambda) \right\| = O(\varepsilon) \qquad \forall t \geq 0$$

The same holds true for q_ε^u for $t \leq 0$.

Proof. See [20]. □

It is possible to compute the signed distance between the two points $q_\varepsilon^u(\lambda)$ and $q_\varepsilon^s(\lambda)$ as follows

$$d(\lambda, \varepsilon) = \frac{(q_\varepsilon^u(\lambda) - q_\varepsilon^s(\lambda)) \cdot \upsilon(\lambda)}{\|\upsilon(\lambda)\|} \tag{4.4}$$

In the unperturbed system the invariant manifolds coincide, $d(\lambda, 0) = 0\ \forall \lambda \in \mathbb{R}$. Moreover $q_\varepsilon^{u,s}$ are differentiable in ε, for ε small, since $W^{u,s}$ are smooth in ε and for $\varepsilon = 0$ $W^{u,s}$ intersect transversally r.

The expansion in a Taylor series of (4.4) about $\varepsilon = 0$ gives

$$d(\lambda, \varepsilon) = d(\lambda, 0) + \varepsilon \frac{\partial d}{\partial \varepsilon}(\lambda, 0) + O(\varepsilon^2)$$

where

$$\frac{\partial d}{\partial \varepsilon}(\lambda, 0) = \left(\left. \frac{\partial q_\varepsilon^u(\lambda)}{\partial \varepsilon} \right|_{\varepsilon=0} - \left. \frac{\partial q_\varepsilon^s(\lambda)}{\partial \varepsilon} \right|_{\varepsilon=0} \right) \cdot \frac{\nabla H(\lambda)}{\|\nabla H(\lambda)\|}$$

Definition 4.2. *The* Melnikov function $M : \mathbb{R} \to \mathbb{R}$ *is*

$$M(\lambda) = \left(\left. \frac{\partial q_\varepsilon^u(\lambda)}{\partial \varepsilon} \right|_{\varepsilon=0} - \left. \frac{\partial q_\varepsilon^s(\lambda)}{\partial \varepsilon} \right|_{\varepsilon=0} \right) \cdot \nabla H(\gamma_0(\lambda)) \tag{4.5}$$

The Melnikov function is the first order term of the Taylor expansion about $\varepsilon = 0$ of the distance between W^s and W^u, except for the norm of the gradient of H, which is always nonzero along γ_0.

4.3 The Melnikov method

In [9], Melnikov found a commutable expression for the function above.

Proposition 4.3. *The* Melnikov function (4.5) *can be written in the form*

$$M(\lambda) = \int_{-\infty}^{\infty} \nabla H(\gamma_0(t)) \cdot g(\gamma_0(t), t + \lambda) dt \tag{4.6}$$

Proof. The proof is given in 4.6. □

Theorem 4.4. *Let $M(\lambda)$ be the Melnikov function for a system of the form* (4.2)*, with the hypotheses stated above. If $\exists \lambda^*$ such that $M(\lambda^*) = 0$, $\frac{dM}{d\lambda}(\lambda^*) \neq 0$, then for ε sufficiently small in the dynamical system given by the Poincaré map of* (4.2)*, $W^u(p_\varepsilon)$ and $W^s(p_\varepsilon)$ intersect transversally at $\gamma_0(\lambda^*) + O(\varepsilon)$.*

Proof. Using the distance between the two manifolds (4.4)

$$d(\lambda, \varepsilon) = \varepsilon \frac{M(\lambda)}{\|\nabla H(\gamma_0(\lambda))\|} + O(\varepsilon^2)$$

If $M(\lambda^*) = 0$ then $d(\lambda^*, 0) = 0$. Since $\frac{\partial M}{\partial \lambda}(\lambda^*) \neq 0$ by the implicit function theorem, for ε sufficiently small, there exists $\lambda(\varepsilon)$ so that $d(\lambda(\varepsilon), \varepsilon) = 0$. This means that $W^s(p_\varepsilon)$ and $W^u(p_\varepsilon)$ intersect at $\gamma_0(\lambda^*) + O(\varepsilon)$.

In a small neighborhood of the homoclinic point $\gamma_0(\lambda) + O(\varepsilon)$, q_ε^s and q_ε^u can be parametrized by λ. A sufficient condition for transversality is

$$\frac{\partial q_\varepsilon^s}{\partial \lambda} \neq \frac{\partial q_\varepsilon^u}{\partial \lambda}$$

Since

$$\frac{\partial M}{\partial \lambda}(\lambda) = \nabla H(\gamma_0(\lambda)) \cdot \left(\frac{\partial q_\varepsilon^s}{\partial \lambda} - \frac{\partial q_\varepsilon^u}{\partial \lambda} \right),$$

a sufficient condition for transversality is $\frac{\partial M}{\partial \lambda}(\lambda^*) \neq 0$. □

4.4 The heteroclinic case

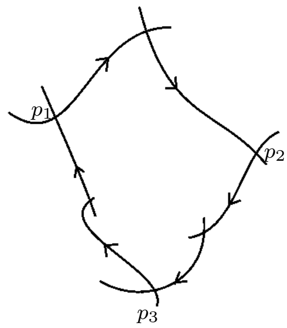
The Melnikov method can be extended to the case of heteroclinic orbits. In the proof, it is assumed that γ_0 is a homoclinic orbit of the unperturbed system, but the fact that $\lim_{t\to+\infty} \gamma_0(t) = \lim_{t\to-\infty} \gamma_0(t)$ is never used. As seen in [20], the Poincaré-Melnikov method can be used to detect transverse heteroclinic points.

Theorem 4.5. *Let $M(\lambda)$ be the Melnikov function for a system of the form* (4.2)*, where γ_0 is a heteroclinic orbit that connects two hyperbolic fixed points p_1 and p_2. If $\exists\lambda^*$ such that $M(\lambda^*) = 0$, $\frac{dM}{d\lambda}(\lambda^*) \neq 0$, then for ε sufficiently small, in the dynamical system given by the Poincaré map of* (4.2)*, $W^u(p_{1_\varepsilon})$ and $W^s(p_{2_\varepsilon})$ intersect transversally at $\gamma_0(\lambda^*) + O(\varepsilon)$.*

The existence of the heteroclinic point found with the theorem above, is not sufficient for the system to show chaotic phenomena.

Definition 4.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. Let $\{p_i\}_{i=1,\dots,n}$ be a collection of hyperbolic fixed points for f, with $p_1 \equiv p_n$. If, for all i , $W^u(p_i)$ intersects transversally $W^s(p_{i+1})$ then the fixed points are said to form a* heteroclinic cycle*. See figure 4.3.*

Figure 4.3: Three hyperbolic fixed points p_1, p_2, p_3, with their stable and unstable manifolds, forming a heteroclinic cicle.



Theorem 4.7. *Let the diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ have a heteroclinic cycle. Then $W^u(p_i)$ intersects transversally $W^s(p_i)$, for all $i = 1, \dots, n$*

Proof. See [19] or [12]. □

The above theorem states that in the presence of n fixed points for a diffeomorphism, whose stable and unstable manifolds intersect transversally, and form a heteroclinic cycle, then the stable and unstable manifolds of the same fixed point intersect; hence homoclinic phenomena are present.

These facts reconnect to the Melnikov method as follows:

Theorem 4.8. *Let a system of the form* (4.2) *have n hyperbolic fixed points, $p_1, p_2, \dots, p_n$ connected by heteroclinic orbits $\gamma_{12}, \gamma_{23}, \dots, \gamma_{n1}$. If for each $\gamma_{i,i+1}$, the Melnikov Function $M(\lambda)$ has a transverse zero, then, for ε sufficiently small, in the dynamical system given by the Poincaré map, $W^u(p_i)$ and $W^s(p_i)$ intersect transversally for each $i = 1, \dots, n$.*

Proof. From the statements above, the Melnikov method is a sufficient condition for a heteroclinic orbit to have a transverse intersection, if the existence of a transverse heteroclinic point is proved for each heteroclinic orbit $q_{12}, \dots q_{n1}$ then the Poincaré Map shows a heteroclinic cycle. The presence of a heteroclinic cycle is a sufficient condition for $W^u(p_i)$ to intersect transversally $W^s(p_i$, by Theorem 4.7. □

4.5 Consequences

- In case of a Hamiltonian perturbation, that is $g = \mathbb{J}\nabla H_1$, $M(\lambda)$ can be written
$$M(\lambda) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\gamma_0(t), t+\lambda)dt$$
where $\{\cdot, \cdot\}$ are the Poisson brackets.

- In the heteroclinic case, it is sufficient to prove the existence of a heteroclinic point for each orbit in the cycle, for P to display chaotic dynamics.

- Transverse zeros of M correspond to a given homoclinic point, but not all the homoclinic points correspond to a transverse zero of M, since M is valid only ε-near to the unperturbed homoclinic orbit.

- For ε small, M is the signed distance between W^s and W^u. If $M(\lambda) \neq 0$ for all λ then W^s and W^u do not intersect.

- $M(\lambda)$ is periodic in λ with period T. That follows from the periodicity of the perturbation g.

- If the perturbation g is autonomous, then M is a number, not a function of λ. This makes sense since for autonomous vector fields W^s and W^u either coincide or do not intersect at all.

- The choice of the section of the Poincaré map is not important, since they are all conjugate. If a homoclinic point is found for a certain section, all other sections have a homoclinic point.

4.6 Sketch of the proof

The Melnikov function can be written as

$$M(\lambda) = \Delta^u(0, \lambda) - \Delta^s(0, \lambda),$$

where

$$\Delta^u(t, \lambda) := \nabla H(\gamma_0(t+\lambda)) \cdot u(t, \lambda) \qquad \Delta^s(t, \lambda) := \nabla H(\gamma_0(t+\lambda)) \cdot s(t, \lambda)$$

and

$$u(t, \lambda) := \left.\frac{\partial q_\varepsilon^u(t, \lambda)}{\partial \varepsilon}\right|_{\varepsilon=0} \qquad s(t, \lambda) := \left.\frac{\partial q_\varepsilon^s(t, \lambda)}{\partial \varepsilon}\right|_{\varepsilon=0}$$

We compute only Δ^u, since the argument presented is equally valid for Δ^s. For a fixed $\lambda \in \mathbb{R}$, note that

$$\frac{d}{dt}\Delta^u(t) = \frac{d}{dt}\left(\nabla H(\gamma_0(t))\right) \cdot u + \nabla H(\gamma_0(t)) \cdot \frac{d}{dt}u(t) \tag{4.7}$$

then

$$\frac{d}{dt}u(t) = \frac{d}{dt}\left.\frac{\partial q^u_\varepsilon(t)}{\partial\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}\frac{dq^u_\varepsilon(t)}{dt}.$$

The quantity $\frac{dq^u_\varepsilon(t)}{dt}$ is simply the vector field at that point:

$$\frac{d}{dt}q^u_\varepsilon(t) = \mathbb{J}\nabla H(q^u_\varepsilon(t))) + \varepsilon g(q^u_\varepsilon(t), t),$$

hence

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}\frac{d}{dt}q^u_\varepsilon(t) = \mathbb{J}H''(\gamma_0(t))\left.\frac{\partial q^u_\varepsilon(t)}{\partial\varepsilon}\right|_{\varepsilon=0} + g(\gamma_0(t), t),$$

where H'' is the hessian of H. By the definition of $u(t)$

$$\frac{d}{dt}u(t) = \mathbb{J}H''(\gamma_0(t)) \cdot u(t) + g(\gamma_0(t), t)$$

substituting into (4.7),

$$\begin{aligned}\frac{d}{dt}\Delta^u(t) =& \frac{d}{dt}\left(\nabla H(\gamma_0(t))\right) \cdot u(t) + \nabla H(\gamma_0(t)) \cdot \mathbb{J}H''(\gamma_0(t))\, u(t)\\ &+ \nabla H(\gamma_0(t)) \cdot g(\gamma_0(t), t)\end{aligned}$$

By the properties of $\mathbb{J}$

$$\frac{d}{dt}(\nabla H) \cdot u = H''\mathbb{J}\nabla H \cdot u = -\nabla H \cdot \mathbb{J}H''u$$

Therefore the first two addenda sum up to zero

$$\frac{d}{dt}\Delta^u(t) = \nabla H(\gamma_0(t)) \cdot g(\gamma_0(t), t) \tag{4.8}$$

In conclusion, for any $\xi > 0$

$$\Delta^u(0) - \Delta^u(-\xi) = \int_{-\xi}^{0} \nabla H(\gamma_0(t+\lambda) \cdot g(\gamma_0(t+\lambda), t)dt$$

and similarly

$$\Delta^s(\xi) - \Delta^s(0) = \int_{0}^{\xi} \nabla H(\gamma_0(t+\lambda)) \cdot g(\gamma_0(t+\lambda), t)dt$$

These results can be added to get

$$M(\lambda) = \Delta^u(0) - \Delta^s(0) = \int_{-\xi}^{\xi} \nabla H(\gamma_0(t+\lambda)) \cdot g(\gamma_0(t+\lambda), t)dt + \Delta^u(-\xi) - \Delta^s(\xi) \tag{4.9}$$

Lemma 4.9.

$$\lim_{\xi\to+\infty} \Delta^u(-\xi) = \lim_{\xi\to+\infty} \Delta^s(\xi) = 0$$

Proof. By the definition of Δ^s

$$\Delta^s(t) = \nabla H(\gamma_0(t+\lambda)) \cdot s(t)$$

As $t \to +\infty$, γ_0 converges to p. The local stable manifold theorem 2.3 states that $\exists \lambda, \delta > 0$ such that $\dot{\gamma_0}(t-\lambda) \leq \delta e^{-\lambda t}$ for $t \geq 0$. The gradient of H, calculated along the orbit, is in norm equal to $\dot{\gamma_0}(t-\lambda)$.

Taking the limit of the second factor $s(t)$, it is possible to swap the derivation and the limit

$$\lim_{t\to\infty} s(t) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \lim_{t\to\infty} q^s_\varepsilon(t)$$

The point q^s_ε approaches p_ε under iterations of the Poincaré map. The fixed point p_ε is necessarily a periodic point of ϕ^t_ε, thus $\phi^t_\varepsilon(p_\varepsilon)$ is bounded for all t. Expanding around $\varepsilon = 0$,

$$\phi^t_\varepsilon(p_\varepsilon) = p + \varepsilon \left.\frac{\partial \phi^t_\varepsilon(p_\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} + O(\varepsilon^2)$$

This means that $\left.\frac{\partial \phi^t_\varepsilon(p_\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}$ is bounded. As a consequence $\lim_{\lambda\to+\infty} \Delta^s(\xi) = 0$. A similar argument gives $\lim_{\xi\to\infty} \Delta^u(-\xi) = 0$. □

Therefore, taking the limit for $\xi \to +\infty$ of expression (4.9), we have

$$M(\lambda) = \int_{-\infty}^{\infty} \nabla H(\gamma_0(t+\lambda)) \cdot g(\gamma_0(t+\lambda), t)dt$$

or

$$M(\lambda) = \int_{-\infty}^{\infty} \nabla H(\gamma_0(t)) \cdot g(\gamma_0(t), t-\lambda)dt$$

We note that the improper integral in the Melnikov function converges absolutely as follows from the local stable manifold theorem and the boundedness of g along γ_0.

5. Chaos in fluid dynamics

In [6] Kozlov studied the motion of fluid particles in the velocity vector field given by two point vortexes, perturbed with a small time periodic perturbation along the y-axis. In this case the Poincaré-Melnikov method succeed in proving the existence of chaos.

Following [7], we retrace the theory behind the point vortexes.

5.1 Theory of point vortexes

The motion of inviscid fluids is governed by the Euler equation and by the incompressibility condition:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{5.1}$$

here $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity vector field, $p(\boldsymbol{x}, t)$ is the pressure and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla)$ the material derivative.

The curl of $\boldsymbol{u}$, $\nabla \times \boldsymbol{u} =: \boldsymbol{\omega}$, is called the *vorticity*, and gives a measure of the amount of local rotation. Taking the curl of equations (5.1) we get:

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{\omega} = 0. \tag{5.2}$$

The vorticity equations (5.2) contain as much information as the Euler equation above, that solving (5.2) is completely equivalent to solving (5.1) . See [7].

For a planar fluid, the vector fields are $\boldsymbol{u} = (u_x(x, y, t), u_y(x, y, t), 0)$ and $\boldsymbol{\omega} = (0, 0, \omega(x, y, t))$. Equations (5.2) can be written in the simple form:

$$\frac{D\omega}{Dt} = 0. \tag{5.3}$$

This means that the vorticity is carried by the fluid.

In this case, exists a function $H(x, y, t)$, see [7], such that

$$u_x(x, y, t) = \frac{\partial H}{\partial y}(x, y, t) \qquad u_y(x, y, t) = -\frac{\partial H}{\partial x}(x, y, t)$$

The motion of fluid particles then becomes

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x, y, t) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x, y, t) \end{aligned} \tag{5.4}$$

Therefore, the dynamics has a Hamiltonian structure.

In the limit case (see [11] for details and references), where the vorticity is highly localized about (x_1, y_1):

$$\omega(x, y, t) = \frac{\Gamma}{2\pi}\delta(x - x_1, y - y_1)$$

is called a *point vortex*. Here the δ is the Dirac function, and Γ is referred as the *strength of the vortex*. The velocity field associated with an isolated point vortex fixed in time at (x_1, y_1) exists and it is:

$$\begin{aligned} u_x(x,y) &= \frac{\Gamma}{2\pi}\frac{y - y_1}{((x - x_1)^2 + (y - y_1)^2)} \\ u_y(x,y) &= -\frac{\Gamma}{2\pi}\frac{x - x_1}{((x - x_1)^2 + (y - y_1)^2)} \end{aligned} \tag{5.5}$$

.

We want to study the dynamics of two point vortex with equal and opposite strength. Since vorticity is carried by the fluid, their position will change in time according to (5.4). Let the position of the two point vortexes be $\boldsymbol{x}_i(t) = (x_i(t), y_i(t))$ where $i = 1, 2$. We want to prove the existence of a $\boldsymbol{u}$ vector field which has a vorticity:

$$\omega(x, y, t) = \frac{\Gamma}{2\pi}\left(\delta(\boldsymbol{x} - \boldsymbol{x}_1(t)) - \delta(\boldsymbol{x} - \boldsymbol{x}_2(t))\right). \tag{5.6}$$

where $\boldsymbol{x}_{1,2}(t)$ are given by:

$$\begin{aligned} \dot{x}_i &= u_x(x_i, y_i, t) \\ \dot{y}_i &= u_y(x_i, y_i, t). \end{aligned} \tag{5.7}$$

The interactions are governed by the velocity that one of the vortexes induces in the position of the other. Since $\boldsymbol{u} = \mathbb{J}\nabla H$, and the function H satisfies a Poisson equation with vorticity (see [11]),

$$\nabla^2 H(x, y, t) = -\omega(x, y, t). \tag{5.8}$$

An isolated point vortex $\boldsymbol{x}_i(t)$, $\omega(x, y, t) = \frac{\Gamma}{2\pi}\delta(\boldsymbol{x} - \boldsymbol{x}_i(t))$, induces

$$\boldsymbol{u}_i = \mathbb{J}\nabla H_i$$

where H_i is found using the Green's functions

$$H_i(x, y, t) = \frac{\Gamma}{2\pi}\log\Big(\|\boldsymbol{x} - \boldsymbol{x}_i(t)\|\Big). \tag{5.9}$$

Since each vortex moves in the velocity vector field induced by the other

$$\dot{\boldsymbol{x}}_1 = \mathbb{J}\nabla H_2, \qquad \dot{\boldsymbol{x}}_2 = \mathbb{J}\nabla H_1;$$

then

$$\begin{aligned} \dot{x}_1 &= \frac{\Gamma}{2\pi}\frac{y_2 - y_1}{d^2} \qquad & \dot{x}_2 &= \frac{\Gamma}{2\pi}\frac{y_2 - y_1}{d^2} \\ \dot{y}_1 &= \frac{\Gamma}{2\pi}\frac{x_2 - x_1}{d^2} \qquad & \dot{y}_2 &= \frac{\Gamma}{2\pi}\frac{x_2 - x_1}{d^2} \end{aligned}$$

where d is the distance between the two vortexes, $d = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$, and clearly it remains fixed in time $\dot{d} = 0$.

Moreover, we assume that the initial conditions of the two vortexes are

$$\boldsymbol{x}_1(0) = (0, a) \qquad \boldsymbol{x}_2(0) = (0, -a),$$

then the equations of motion are

$$\begin{aligned} x_1(t) &= -\frac{\Gamma}{4\pi a}t \qquad & x_2(t) &= -\frac{\Gamma}{4\pi a}t \\ y_1(t) &= a & y_2(t) &= -a. \end{aligned}$$

The resulting Hamiltonian is obtained by superposition $H = H_1 + H_2$, (since vorticity and velocity are related linearly),

$$H(x, y, t) = \frac{\Gamma}{4\pi}\left(\log\frac{(x + \frac{\Gamma}{4\pi a}t)^2 + (y-a)^2}{(x + \frac{\Gamma}{4\pi a}t)^2 + (y+a)^2}\right),$$

which gives the vector field

$$\boldsymbol{u} = \mathbb{J}\nabla H$$

with vorticity in the form (5.6).

Transforming the coordinate system with $x' = x + \frac{\Gamma}{4\pi a}t, y' = y$, we get the autonomous Hamiltonian

$$H(x, y) = \frac{\Gamma}{4\pi}\left(\frac{y}{a} + \log\frac{x^2 + (y-a)^2}{x^2 + (y+a)^2}\right) \tag{5.10}$$

that is the unperturbed system studied in [6].

The phase portrait of the Hamiltonian system given by (5.10) is shown in figure 5.1. The system has four fixed points: two elliptic fixed points e_1, e_2 at $(0, \pm a)$, in the position of the two point vortexes, encircled by period orbits; and two hyperbolic fixed points p^-,p^+ at $(\pm\sqrt{3}a, 0)$, connected by three heteroclinic orbits, γ^+, γ^0, γ^- (thicker in figure 5.1) that divide the phase space into four invariant subsets. Fluid particles with initial condition on one of these subsets will remain in it. The two central invariant sets that includes the periodic orbits will be referred to as the *vortex area.*

5.2 Kozlov perturbation

In [6], Kozlov perturbed the system given by (5.10) with a Hamiltonian perturbation $H_1 = -x\sin(\omega t)$ or $g = (0, \sin(\omega t))$.

There are two possible cycles that can be studied with the Poincaré-Melnikov method, $\gamma^+ \to \gamma^0$ and $\gamma^- \to \gamma^0$. However, since the system has a y reflectional symmetry, the results found for the cycle $\gamma^+ \to \gamma^0$ will be the same for the other.

In this case, the Poincaré-Melnikov method is sufficient to prove the existence of homoclinic points and thus chaotic dynamics, and the Melnikov

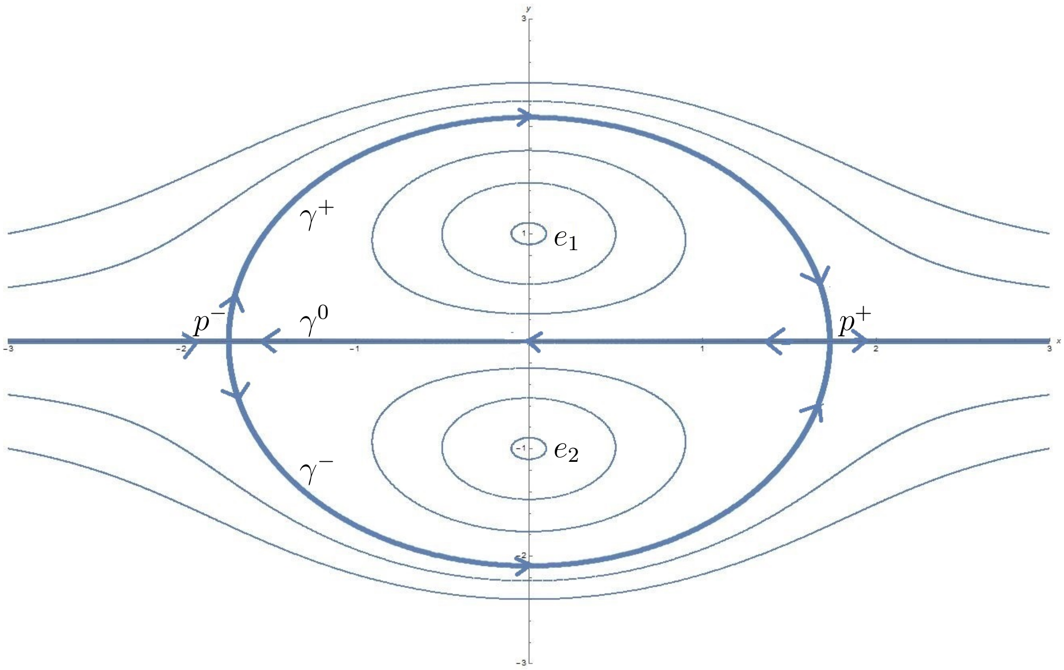


Figure 5.1: Phase portrait of the hamiltonian system given by (5.10), with $\Gamma = 2\pi, a = 1$.

function (4.6) takes the form

$$M(\lambda) = \int_{-\infty}^{\infty} -\dot{x}(t) \cdot \sin(\omega(t-\lambda))dt$$

On γ^+, fixing q^* in the intersection of γ^+ with the y-axis, $\lambda = 0$ is a transverse zero for M, since $\sin(\omega t)$ is an odd function and $\dot{x}(t)$ is even. In γ^0, fixing $q^* = (0,0)$, also M has a transverse zero for $\lambda = 0$. Consequently, by theorem 4.7, the Poincaré map will exhibit chaotic dynamics.

Figure 5.2 shows the phase portraits for various values of ε. As expected for this type of problems, the black region which is densely filled with chaotic orbits, grows larger with ε. Moreover, are also plotted periodic orbits in the neighborhood of both e^+ and e^- which persisted the perturbation, and orbits with initial conditions outside of the vortex area. Note: It is not possible to follow the path of just one point since the particles tends to escape the vortex area towards the positive x-axis. Therefore, in figure 5.2, plots one thousand orbits in the chaotic region.

It is important to notice how the stable and unstable manifolds of the two fixed points, no longer divide the phase space into the four invariant sets as before. Thus, particles with initial condition in the vortex area are able to escape, as well for particles outside to enter in.

In figure 5.3 are instead plotted 50 forward and backwards iterations of one thousand initial conditions. It can be seen how points on the outside of the region, from the negative x-direction, can enter the vortex area and leave towards positive x direction.

Figure 5.4 shows the stable and unstable manifolds of p_ε^+ and p_ε^-, for $\varepsilon = .5$. The Melnikov method predicts transverse heteroclinic points for

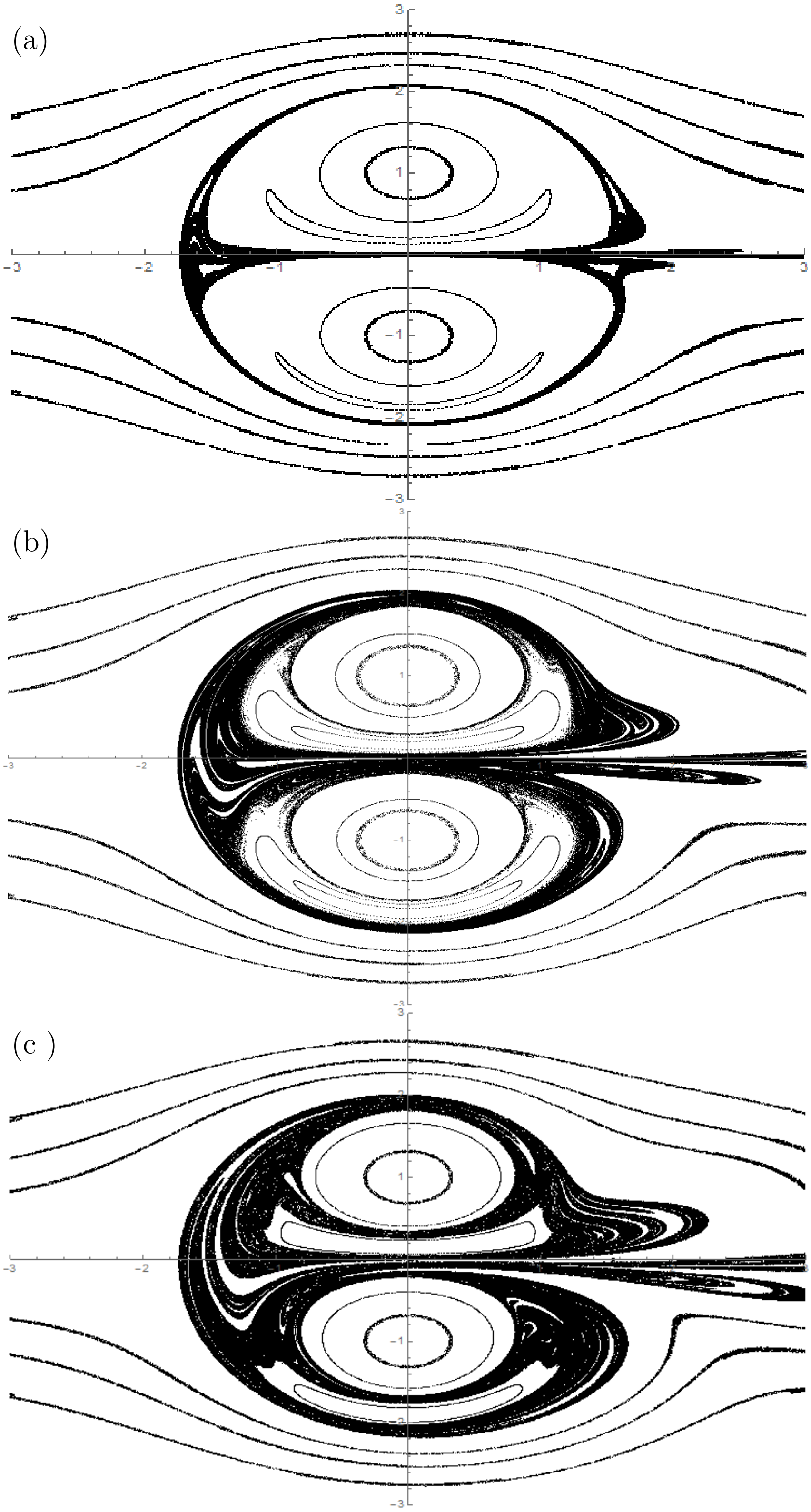


Figure 5.2: Phase portrait of the perturbed system, with $\Gamma = 2\pi$, $a = 1$ where $\varepsilon = .01$ in (a), $\varepsilon = .05$ in (b), $\varepsilon = .1$ in (c).

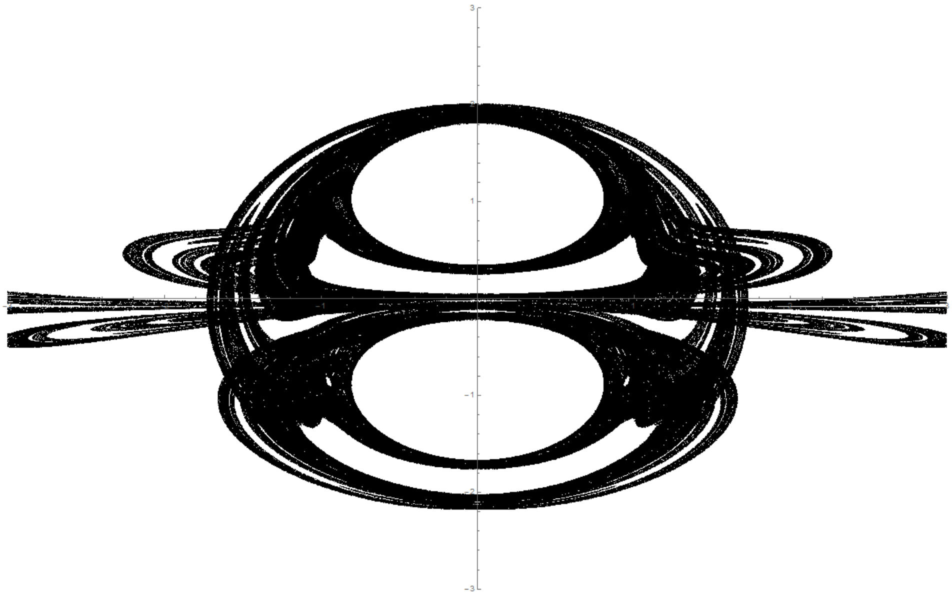


Figure 5.3: Forward and backward iterations of few close initial conditions. ($\varepsilon = .5$, $a = 1$)

$W^u(p^+), W^s(p^-)$ and $W^s(p^+), W^u(p^-)$, the two heteroclinic tangles are showed in figure 5.5. A heteroclinic cycle is formed. This this is a sufficient condition for the existence of homoclinic points, in fact figure 5.6 shows various intersections between the stable and unstable manifolds of both fixed points.

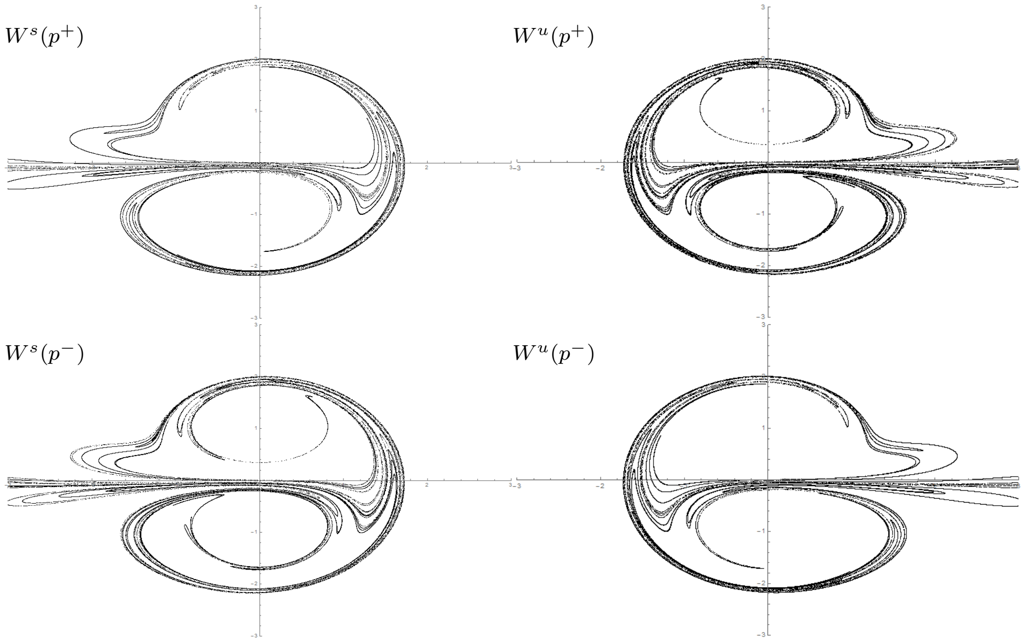


Figure 5.4: The stable and unstable manifolds of the two fixed points. With $\varepsilon = .5$.

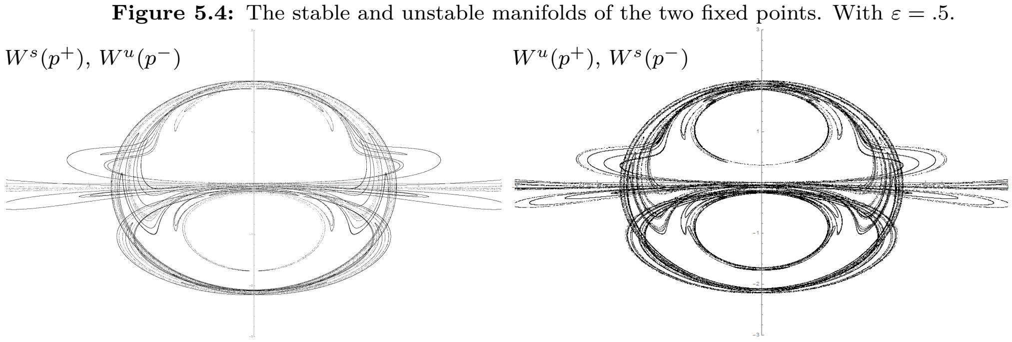


Figure 5.5: The two heteroclinic tangles. All the intersection represent heteroclinic points. ($\varepsilon = .5$)

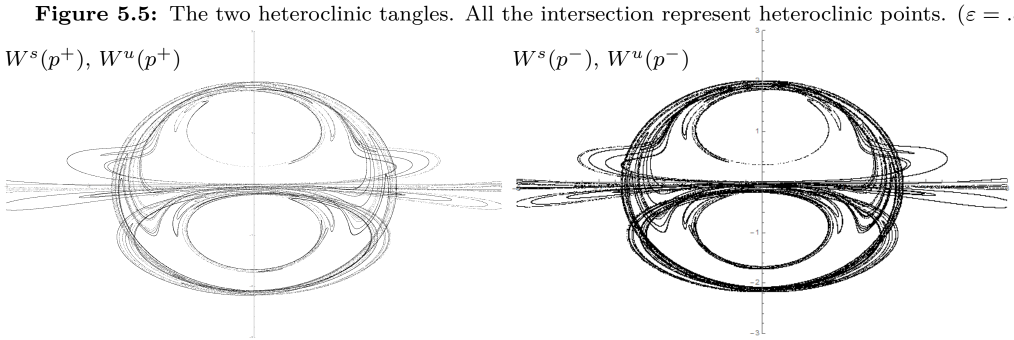


Figure 5.6: The homoclinic tangles for both fixed points.($\varepsilon = .5$, $a = 1$)

5.3 Perturbation on the x-axis

If the perturbation is $g = (\sin(\omega t), 0)$ or in Hamiltonian form $H_1 = y \sin(\omega t)$, then the Melnikov function has the form:

$$M(\lambda) = \int_{-\infty}^{\infty} -\dot{y}(t) \cdot \sin(\omega(t - \lambda))dt$$

The Melnikov function for γ^+ has a transverse zero, for the same arguments of the previous case. So $W^u(p^-)$ and $W^s(p^+)$ intersect transversally forming heteroclinic points, by theorem 4.5.

However, since γ_0 is horizontal $\dot{y}(t) \equiv 0$, the Melnikov function for γ^0 is identically zero. Thus the Melnikov method fails. However, the Melnikov function is the first order approximation of the distance between the two manifolds, so its vanishing does not necessarily means that the stable and unstable manifolds will not split.

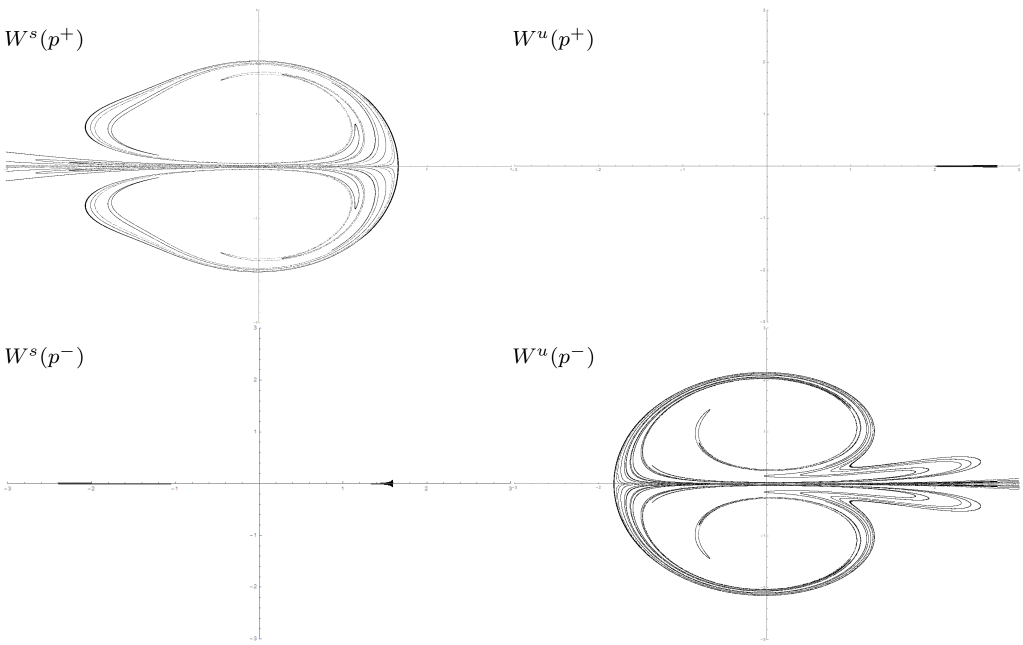


Figure 5.7: Stable and unstable manifolds for the two fixed points.($\varepsilon = .5$, $a = 1$)

Figure 5.7 plots the stable and unstable manifolds of the fixed points. The heteroclinic orbit γ^+ splits, while the horizontal heteroclinic orbit γ^0 does not split and remains horizontal. Showing heteroclinics points, in figure 5.8

Figure 5.8 shows $W^s(p^+)$ and $W^u(p^-)$ together and it is noticed the presence of transverse heteroclinic points; which instead are not present in the intersection between $W^u(p^+)$ and $W^s(p^-)$.

Figure 5.9 shows that no transverse homoclinic points are formed, for both p^+ and p^-.

Since the horizontal manifolds does not split, the phase space is still divided into two parts. Points outside the vortex area can enter and then exit again, but initial conditions with positive y will remain in the positive y sector, and similarly for the negative y.

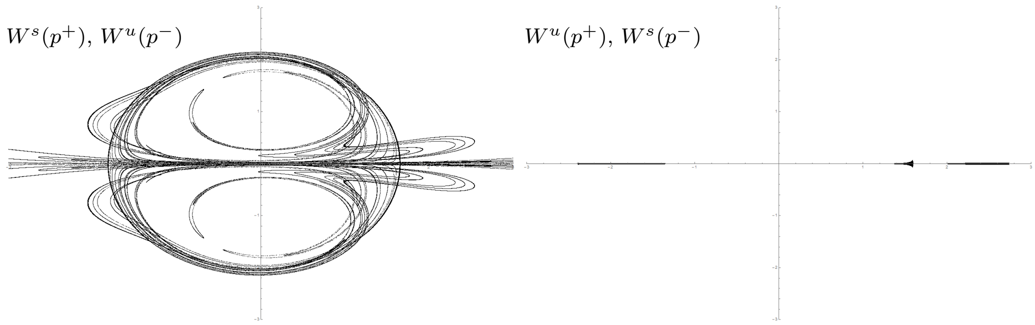


Figure 5.8: The formation of the heteroclinic tangle for γ^+, on the left. For γ_0 instead, no transverse heteroclinic point is formed.($\varepsilon = .5$, $a = 1$)

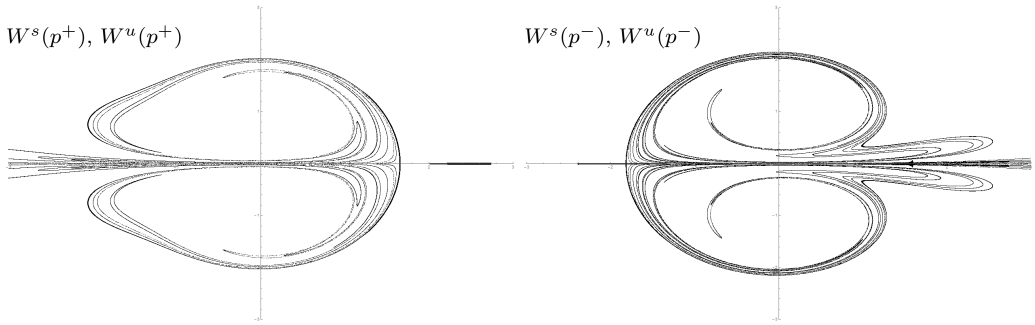


Figure 5.9: No intersections are showed, thus no homoclinic points are present.($\varepsilon = .5$, $a = 1$)

As shown in figure 5.10, the regions in $y > 0$ and $y < 0$, display chaotic dynamics. The chaotic orbits fill densely the region in the proximity of the invariant manifolds, but never cross the horizontal line $y = 0$.

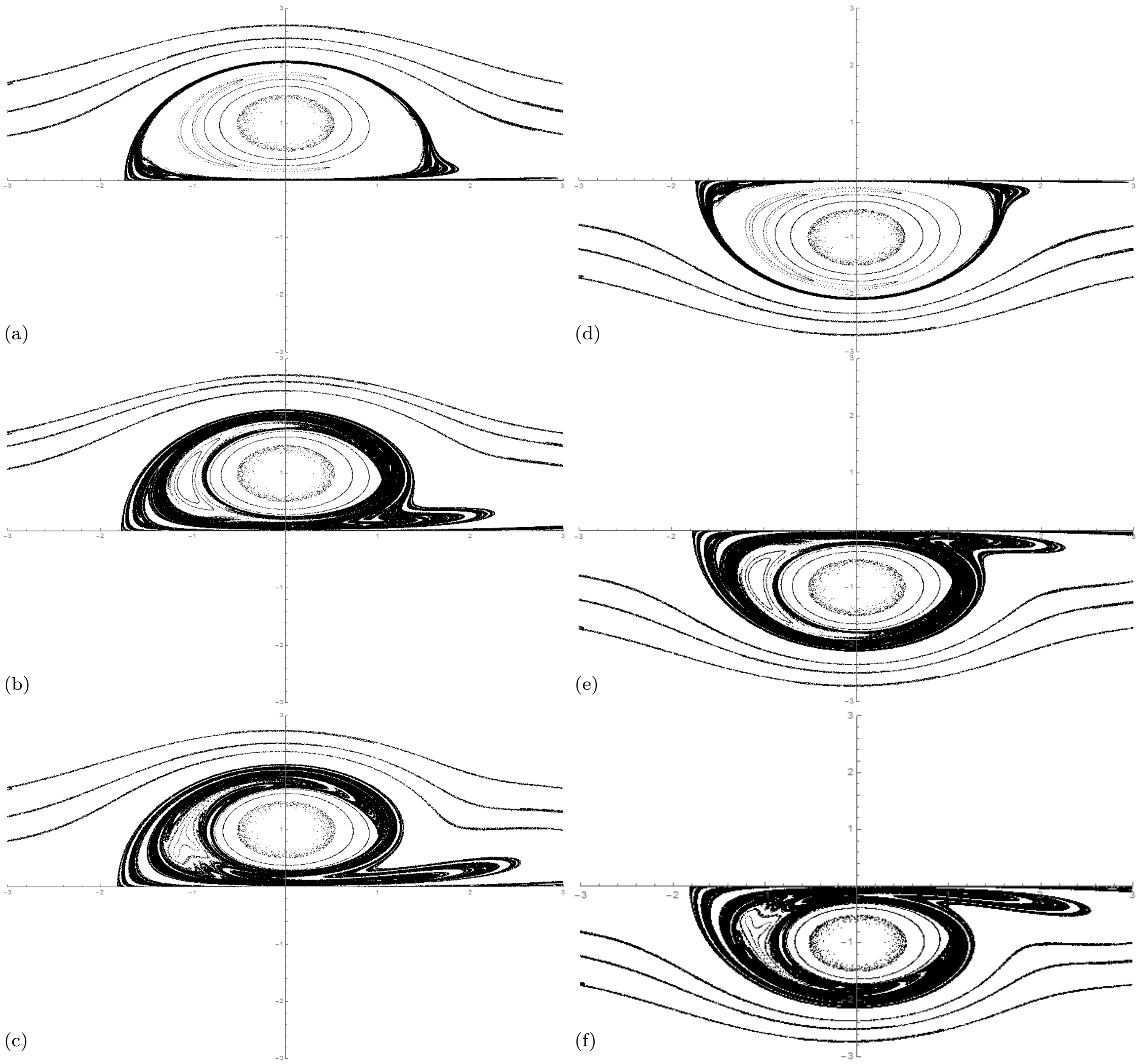


Figure 5.10: Phase portrait of the perturbed system with $\Gamma = 2\pi$, $a = 1$. For positive y initial condition where $\varepsilon = .01$ in (a), $\varepsilon = .05$ in (b), $\varepsilon = .1$ in (c).
For negative y initial conditions where $\varepsilon = .01$ in (d), $\varepsilon = .05$ in (e), $\varepsilon = .1$ in (f).

Conclusion

We showed how the Poincaré-Melnikov method is a sufficient condition for the existence of transverse homoclinic points, which in turn are a sufficient condition for a system to display chaotic dynamics. However, the presence of heteroclinic orbits for the unperturbed system, can be sufficient to apply the Poincaré-Melnikov method and prove the existence of chaotic orbits. We showed two example of the same system perturbed in two different ways. The first, examined in [6], the Poincaré-Melnikov method succeed in detecting chaotic dynamics; in the second, the method fails. In this latter case, the presence of chaotic dynamics is confirmed by numerical evidence.

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