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Optimal Transport and Sliced Wasserstein Gradient Flow

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Chapter 1

Introduction

In this work we aim to study the gradient flow on the space of probability measures on $\mathbb{R}^d$ generated by the *sliced Wasserstein distance* with respect to a fixed "target" probability measure: this distance was first introduced by M. Bernot and others in [10], and we are going to present it briefly in the following, trying to explain why it is interesting and useful to study its properties. Our goal is to answer the following three questions regarding the curve $(\rho_t)_{t\geq 0}$ which arises as the gradient flow for the functional $F := \frac{SW_2^2(\cdot,\nu)}{2}$, SW_2 being the sliced Wasserstein distance, which will be presented in chapter 3:

- **Conjecture 1.** Does the curve ρ_t converge to any distribution with respect to the sliced Wasserstein or (even better) the Wasserstein distance? In this case, what can we say about the limit measure $\rho_\infty = \lim_{t\to\infty} \rho_t$? Is it true that $\rho_\infty = \nu$?

- **Conjecture 2.**(Lagrangian point of view) Fix an initial particle $x \in \mathbb{R}^d$. We want to study the qualitative behavior of the ODE

$$\begin{cases} \dot{y} = v(t, y), \\ y_0 = x, \end{cases}$$

  where v is the velocity field (4.4). In particular we would like to know existence and uniqueness results, and if the limit map $T = \lim_{t\to\infty} y_t$ is well defined (at least for *a.e.* $x \in \mathbb{R}^d$).

- **Conjecture 3.** Finally, assuming the previous two questions admit a positive answer, what can we say about the map $y_\infty = \lim_{t\to\infty} y_t$? In particular, is it true that this map is optimal in the sense of optimal transport between the initial datum ρ_0 and the target measure ν (which, under these assumptions, coincides with the limit of the curve ρ_t)?

Conjecture 3 was actually the starting point of the present work, since the article [8] by F. Santambrogio and H. Lavenant suggests a strategy to find a counterexample to the optimality in the case of the gradient flow for the Fokker Planck functional. Adapting this article to our case, we managed to prove that the gradient flow generated by the Sliced Wasserstein distance does not provide optimal transport, even if numerical computations in the discrete case suggest that this map is a good approximation of the optimal one. We then focused on the problem of the convergence of the gradient flow, namely on conjecture 1, finding some cases that show different behaviors of the convergence of the gradient flow depending on the initial data (*i.e.* the starting and target measures).

This work is structured as follows: in the two following chapters we will present some basic-knowledge results about optimal transport and Wasserstein spaces, then we will present the sliced Wasserstein distance and its first properties; in the fourth chapter we will discuss the theory of gradient flows in the space of probability measures. Chapters 5 and 6 contain the original results we obtained in this project: we will build the counterexample which proves that conjecture 3 is false and we will present some convergence results, which give a partial answer to conjecture 1. Among the 3 conjectures presented, the second one is the most difficult to treat: some aspects regarding it will be discussed in the final chapter of this work, which will also contain other possible directions for a further research on these topics.

Chapter 2

Optimal transport - basic knowledge

During the last decades, optimal transport has been rediscovered as one of most flourishing branches of mathematics: the theory was born in the Eighteenth century in order to answer some very concrete questions regarding the transport of material from french mines, and nowadays it has become so rich and developed that it has revealed its importance in lots of mathematical disciplines like PDEs, fluid mechanics, geometry, probability theory and also applied mathematics like economics, and image processing (see [7]).

2.1 Monge and Kantorovich problems

Gaspard Monge (1746-1818) is considered to be the father of optimal transport: his "Mémoire sur la théorie des déblais et des remblais" (1781) is the seminal paper for this subject since here, for the first time, the following problem was investigated: we want to transport a fixed mass of material from an initial configuration (or distribution) to another in the optimal way with respect to the linear cost, namely minimizing the average displacement of the moved particles. The problem can be formulated as follows: given two probability densities μ and ν in $\mathbb{R}^d$, we want to find a map $T\colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\nu(A) = \mu(T^{-1}(A)) \tag{2.1}$$

for any Borel set $A \subset \mathbb{R}^d$ (this means that the first distribution is pushed onto the other) and minimizing the quantity

$$\int_{\mathbb{R}^d} |T(x) - x| d\mu(x). \tag{2.2}$$

Condition (2.1) can be written as $T_{\#}\mu = \nu$, where $T_{\#}\colon \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is called the *pushforward* of the map T.

Remark 1 (Properties of the push-forward)**.** *One can check that*

- $(T \circ S)_\# = T_\# \circ S_\#$
- *If T is invertible, then* $(T^{-1})_\# = (T_\#)^{-1}$

Problem (2.2) can also be formulated for a generic cost function $c\colon \mathbb{R}^d \to \mathbb{R}^d$. Solving the Monge problem

$$(\text{MP}) \quad \int_{\mathbb{R}^d} c(T(x), x) d\mu(x)$$

is non trivial. For starters, there may not be feasible (*i.e.* admissible) maps T, as it is shown in the following

Example 1. *Let $a, b \in \mathbb{R}^d$, $a \neq b$. Define $\mu = \delta_a$, $\nu = \frac{\delta_a}{2} + \frac{\delta_b}{2}$. If a feasible map $T\colon \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ exists, then taking the Borel set $A = \{a\}$, we should have*

$$\frac{1}{2} = \mu(T^{-1}(A)) = \begin{cases} 1 & \text{if } a \in T^{-1}(\{a\}), \\ 0 & \text{otherwise}, \end{cases}$$

which is a contradiction.

Moreover, even when the set of admissible maps is not empty, the existence of a minimizer is not guaranteed, since the constraint given by (2.1) is not closed under weak convergence. Here is an example contained in [11] for the quadratic cost $c(x, y) = |x - y|^2$:

Example 2. *Consider $\mu = \mathcal{H}^1 \llcorner A$, $\nu = (\mathcal{H}^1 \llcorner B + \mathcal{H}^1 \llcorner C)/2$ where A, B and C are three vertical parallel segments in $\mathbb{R}^2$ having for abscissas respectively $y = 0, y = 1, y = -1$. Obviously the transport cost can not be less than 1 (that's exactly how much each point needs at least to be displaced horizontally). Consider then the sequence of maps T_n defined in the following way: divide A in $2n$ equal segments $(A_i)_{i=1,\dots,2n}$, and B and C in n equal segments $(B_i)_{i=1,\dots,n}$, $(C_i)_{i=1,\dots,n}$, ordered downward. The map T_n is defined as a piecewise affine map seding A_{2n-1} onto B_i and A_{2i} onto C_i. In this way, the cost of the map T_n is less then $1 + 1/n$, so that the infimum in the Monge problem is 1. On the other hand, a map T realizing this infimum can't exisist, since this would imply that all the points are sent horizontally from A to B and C, but this can't satisfy the condition $T_\# \mu = \nu$.*

Finally, even if the quantity to minimize may seem "simple", the constraint given by (2.1) is highly non linear: this makes the problem really difficult to treat, even from the numerical point of view. These are the main reasons why the natural way to formulate the optimal transport problem is the so called *Kantorovich formulation*. In presenting it, we will also consider two general Polish spaces (*i. e.* complete metric spaces) X and Y in place of $\mathbb{R}^d$ as the ambient space in which the starting and final probabilities μ and ν are defined. Finally, we will denote by $\pi^X : X \times Y \to X$ and $\pi^Y : X \times Y \to Y$ the natural projections. The problem reads as follows: given $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, we aim to find $\gamma \in \Pi(\mu, \nu) := \left\{\rho \in \mathcal{P}(X \times Y) : \pi^X_{\#}\rho = \mu, \pi^Y_{\#}\rho = \nu\right\}$ minimizing

$$\text{(KP)} \quad \int_{X\times Y} c(x, y) d\gamma(x, y).$$

The set $\Pi(\mu, \nu)$ is called the set of *transport plans* between μ and ν. This formulation, the *relaxation* of the Monge problem, is obtained via a duality method:

$$\begin{aligned}
\text{(MP)} \quad & \inf_{T_{\#}\mu=\nu} \left\{ \int c(x, T(x)) d\mu(x) \right\} \\
&= \inf_{T \text{ Borel}} \left\{ \int c(x, T(x)) d\mu(x) + \sup_{\varphi \in C_b(X)} \left\{ \int \varphi(T(x)) d\mu(x) - \int \varphi(y) d\nu(y) \right\} \right\} \\
&\geq \sup_{\varphi \in C_b(X)} \inf_{T \text{ Borel}} \left\{ \int c(x, T(x)) d\mu(x) + \sup_{\varphi \in C_b(X)} \left\{ \int \varphi(T(x)) d\mu(x) - \int \varphi(y) d\nu(y) \right\} \right\} \\
&= \sup_{\varphi \in C_b(X)} \left\{ \int \inf_{y \in Y} (c(x, y) - \varphi(y))\, d\mu(x) + \int \varphi(y) d\nu(y) \right\} \\
&= \sup_{\substack{\varphi \in C_b(X), \psi \in C_b(Y) \\ \varphi(x)+\psi(y) \leq c(x,y)}} \left\{ \int \psi(x) d\mu(x) + \int \varphi(y) d\nu(y) \right\} \text{ (DP)}.
\end{aligned}$$

The condition

$$\varphi \in C_b(X), \psi \in C_b(Y) \text{ and } \varphi(x) + \psi(y) \leq c(x, y) \text{ for any } x \in X, y \in Y$$

is usually denoted by

$$\varphi \oplus \psi \leq c.$$

Finally one can apply the Fenchel-Rockafeller theorem: under some mild assumptions (the cost function should be *l.s.c.* and bounded from below) the dual of the formulation called (DP) - *i.e.* the bidual of the (MP) - is exactly the Kantorovich formulation of the Problem (KP), and so we recover that a minimizer for the Kantorovich problem exists and the duality gap is zero. This motivates the choice of considering (KP) as the natural formulation for the optimal transport problem: minimizers (which are called *optimal plans*) always exist. Furthermore, when they exist, optimal maps (*i.e.* minimizers for (MP)) are optimal plans of the form

$$\gamma(dx, dy) = \delta_0(y - T(x))\mu(dx).$$

Finally, we observe that in the problem (DP) one can choose

$$\psi(y) = \varphi^c(y) := \inf_{x\in X} c(x,y) - \varphi(x).$$

We thus obtain that

$$(\mathrm{DP}) \;=\; \sup_{\varphi\in C_b(X)} \left\{ \int_X \varphi(x)d\mu(x) + \int_Y \varphi^c(y)d\nu(y) \right\}.$$

When they exist, the maps φ realizing the supremum above are called *Kantorovich potentials* of the problem.

2.2 Brenier theorem and Monge - Ampère equations

In the following we will present without giving proofs some classical results on the existence and the properties of (optimal) transport maps and plans. For a complete and detailed discussion on these topics, one can for example consult the books by C. Villani on this subject ([12], [13]).

Theorem 2.2.1. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, and assume μ is atomless. Then there exists at least one transport map T such that $T_\#\mu = \nu$.*

Theorem 2.2.2 (Brenier 1)**.** *Let $K \subset \mathbb{R}^d$ be a compact set such that ∂K is negligible, and let $c(x,y) = h(x-y)$, with h strictly convex. Let $\mu, \nu \in \mathcal{P}(K)$ and assume that μ is absolutely continuous with respect to the Lebesgue measure. Then the Monge Problem (MP) admits a unique solution T which is characterized as the unique Borel map such that $\nu = \mu \circ T^{-1}$ and for which there is a convex function $u\colon K \to \mathbb{R}$ such that $T(x) = \nabla u(x)$ $\mu - a.e.$. Furthermore, there exists a Kantorovich potential φ and we have*

$$T(x) = x - (\nabla h)^{-1}(\nabla\varphi(x)). \tag{2.3}$$

The map $u\colon K \to \mathbb{R}$ introduced above is called *Brenier map* of the problem.

Theorem 2.2.3 (Brenier 2)**.** *Let $c(x,y) = \frac{1}{2}|x-y|^2$. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ have finite second moments and assume that μ does not give mass to $d-1$ surfaces of class C^2. Then the Monge problem (MP) admits a unique solution T which is characterized as the unique Borel map such that $\nu = \mu \circ T^{-1}$ and for which there is a convex function $u\colon \mathbb{R}^d \to \mathbb{R}$ such that $T(x) = \nabla u(x)$ $\mu - a.e.$*

The case in which $c(x,y) = \frac{1}{2}|x-y|^2$ is called the *quadratic-cost case*, and we can deduce by (2.3) that in this situation $\nabla u = \mathrm{id} - \nabla\varphi$.

Remark 2. *Actually, in theorem 2.2.3, one should just check that μ gives no mass to the set $\partial(\{u < \infty\})$ which, actually, is a $(d-1)$-rectifiable set of class C^2, since u is a convex map, and therefore $\overline{\{u < \infty\}}$ is convex too.*

The case of the quadratic cost is the most interesting for our purposes, since (among other reasons) this is exactly the cost involved in the definition of the W_2 and SW_2 distances that we will present in the following chapters. This is also the case in which, by a change of variable prcedure, we can obtain the so called *Monge-Ampère equations* in their most elegant form:

Theorem 2.2.4 (Monge-Ampère equation). *Let* $\Omega \in \mathbb{R}^d$*, and suppose that* $\mu, \nu \in \mathcal{P}(\Omega)$ *are absolutely continuous of densities* f, g *respectively. Let* u *be the Brenier map for the quadratic-cost transport problem from* μ *to* ν*. If we suppose that* $u\colon \Omega \to \mathbb{R}$ *is strictly convex and such that* $\det(D^2u) \neq 0$ *a.e. on* $\{\rho > 0\}$*, then we have*

$$\det(D^2u(x)) = \frac{f(x)}{g(\nabla u(x))} \quad \forall x \in \Omega. \tag{2.4}$$

Equation (2.4) is a fully-non-linear PDE of (possibly degenerate) elliptic type and since the 90's it has been the object of regularity studies: still nowadays lots of questions related to these problems are open, but here we present two remarkable theorems by Caffarelli. For a general survey on the state of art of this subject, see [6].

Theorem 2.2.5 (Caffarelli 1). *Let* $\Omega \subset \mathbb{R}^d$ *be open, and convex, and let* $\mu, \nu \in \mathcal{P}(\Omega)$ *be* $C^{0,\alpha}$ *absolutely continuous densities, bounded from below and above by positive constants on the whole* Ω*. Then the unique solution* u *of* (2.4) *belongs to* $C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

Theorem 2.2.6 (Caffarelli 2). *Let* $K \subset \mathbb{R}^d$ *be a compact set, and let* $c(x, y) = |x - y|^p$*, for* $1 < p < \infty$*. Let* $\mu, \nu \in \mathcal{P}(K)$ *and assume that* $\mu(dx) = \rho_0(x)dx$*,* $\nu(dy) = \rho_1(y)dy$ *with both* $\rho_0, \rho_1 \in C^\infty(K)$*. Assume also that there exists* $\alpha > 0$ *such that* $\rho_0(x) \geq \alpha > 0$*,* $\rho_1(x) \geq \alpha > 0$ *for any* $x \in K$*. Let* $u\colon K \to \mathbb{R}$ *be the Brenier map of the transport problem from* μ *to* ν*. Then*

1. $u \in C^\infty(K)$
2. *u is uniformly strictly convex*
3. $\nabla u\colon K \to \mathbb{R}^d$ *is a diffeomorphism.*

2.3 The one-dimensional case

Theorem 2.2.3 and remark 2 from section 2.2 have an important consequence in dimension one: suppose that $\mu \in \mathcal{P}(\mathbb{R})$ has no atoms. Since every convex $\psi\colon \mathbb{R} \to \mathbb{R}$ is differentiable *a.e.*, it is differentiable μ *– a.e.*. Therefore, an optimal transport map between μ and any $\nu \in \mathcal{P}(\mathbb{R})$ must exist: since it is the derivative of a convex function, it will be a monotone non-increasing map. Furthermore, this transport map can be characterized via the *pseudo-inverse* of the cumulative distribution function (CDF) of the probabilities μ and ν:

Definition 2.3.1 (Cumulative Distribution Function)**.** *Given a probability distribution $\mu \in \mathcal{P}(\mathbb{R})$, we define its cumulative distribution function F_μ as*

$$F_\mu(x) := \mu(-\infty, x].$$

It is well known that this map is monotone non decreasing and right continuous. We are therefore lead to give the following

Definition 2.3.2 (Pseudo-Inverse map)**.** *Given a non decreasing and right continuous map $F \colon \mathbb{R} \to [0,1]$, its pseudo-inverse is the function $F^{[-1]} \colon [0,1] \to \overline{\mathbb{R}}$, defined by*

$$F^{[-1]}(x) := \inf_{t \in \mathbb{R}} \{F(t) \geq x\}.$$

If the above set is empty, the infimum is $+\infty$; if instead the set is not bounded from below, the infimum is $-\infty$.

Now, given two probabilities $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that μ has no atoms, we can prove (see [11]) that the map $T_{mon}(x) := F_\nu^{[-1]}(F_\mu(x))$ is the unique monotone non decreasing map such that $(T_{mon})_\# \mu = \nu$. Moreover, if the Kantorivich problem (KP) has a finite value, T_{mon} is the unique optimal transport map. This fact allows us to have an explicit formula for the optimal transport map between two given measures, and this will turn out to be useful when building the counterexample in chapter 4. By the considerations expressed above, it is useful to remark the following result, which summarizes the properties of one dimensional transport maps and whose proof can be found in [2]:

Theorem 2.3.1. *Let $\mathcal{P}_2(\mathbb{R}) := \{\mu \in \mathcal{P}(\mathbb{R}) \colon \int_{\mathbb{R}} |x|^2 d\mu(x) < \infty\}$. Then, $\mathcal{P}_2(\mathbb{R}^1)$ is isometrically isomorphic to a closed convex subset of the Hilbert space $L^2(0,1)$, precisely to the space of square-integrable nondecreasing functions in $(0,1)$.*

Chapter 3

Wasserstein and sliced Wasserstein distances

In this chapter we will introduce the Wasserstein distance on the space of probability measures, and we will define the *sliced Wasserstein distance*, which is the main object involved in our work.

3.1 Wasserstein spaces

Let $\Omega \subset \mathbb{R}^d$. Thanks to the transport value associated with the costs of the form $c(x, y) = |x-y|^p$ for $1 \leq p < \infty$ we can define a distance called *p-Wasserstein distance* over the space $\mathcal{P}_p(\Omega) := \left\{\rho \in \mathcal{P}(\Omega) \colon \int |x|^p d\rho(x) < \infty\right\}$.

Wasserstein distances play a key role in many fields of applications, and seem to be a natural way to describe distances between equal amounts of mass distributed on the same space.

Definition 3.1.1 (Wasserstein distance)**.** *Let* $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^d$. *For* $\mu, \nu \in \mathcal{P}_p(\Omega)$ *define*

$$W_p(\mu, \nu) = \min_{\gamma \in \Pi(\mu,\nu)} \left\{ \int_{\Omega\times\Omega} |x-y|^p d\gamma(x,y) \right\}^{\frac{1}{p}}.$$

It is interesting to compare W_p distances with the common L^p distances between functions: one could observe that the behavior of L^p distances is "vertical" whereas Wasserstein distances are "horizontal". Consider the following

Example 3. *Let f, g be two bounded functions defined on* $[0, 1]$. *Define* $g_h(x) = g(x-h)$*: as soon as* $|h| > 1$, *the L^p distance between f and g_h is* $(\|f\|^p_{L^p} + \|g\|^p_{L^p})^{1/p}$, *not depending on h, the "horizontal" displacement of g_h. On the contrary, the W_p distances keep track of this information, since the distance from f to g_h is of the order of $|h|$, for $|h| \to \infty$.*

Notice that, since the transport plans are probability measures (defined on the product space $\Omega \times \Omega$), for $p \leq q < \infty$ we have

$$W_p(\mu, \nu) \leq W_q(\mu, \nu).$$

Moreover, by Hölder inequality, if Ω has a finite diameter, for any $1 \leq p < \infty$ we have

$$W_p(\mu, \nu) \leq \operatorname{diam}(\Omega)^{\frac{p-1}{p}} W_1(\mu, \nu)^{\frac{1}{p}}.$$

Theorem 3.1.1. *The Wasserstein distance W_p is indeed a distance over $\mathcal{P}_p(\Omega)$.*

Proof. See [11], proposition 5.1. □

We are now interested in the topology generated by this metric. The following theorem links the Wasserstein notion of convergence with the usual weak* convergence. Let us recall that if (X, d) is a Polish space, a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ is said to *weakly* converge* to $\mu \in \mathcal{P}(X)$ if, for any test function $\varphi \in C_b(X)$, we have

$$\lim_{n\to\infty} \int_X \varphi(x) d\mu_n(x) = \int_X \varphi(x) d\mu(x).$$

Remark 3 (Prokhorov compactness criterium)**.** *A subset $K \subset \mathcal{P}(X)$ is (sequentially) relatively compact for the weak* convergence if and only if it is tight: for any $\varepsilon > 0$, there exists a compact subset $Y \subset X$ such that*

$$\sup_{\mu\in K} \mu(Y \setminus K) \leq \varepsilon.$$

Theorem 3.1.2. *The following conditions are equivalent*

1. $\mu_n \to \mu$ *in* W_p,
2. $\{\mu_n\}_{n\in\mathbb{N}} \overset{*}{\rightharpoonup} \mu$ *and*
$$\int_{\mathbb{R}^d} |x|^p d\mu_n(x) \to \int_{\mathbb{R}^d} |x|^p d\mu(x),$$
3. $\{\mu_n\}_{n\in\mathbb{N}} \overset{*}{\rightharpoonup} \mu$ *and*
$$\lim_{R\to\infty} \sup_n \int_{B_R^c(0)} |x|^p d\mu_n(x) = 0.$$

Proof. A sketch of proof for the case $p = 1$ can be found in [1], Proposition 1.1. □

Thanks the previous theorem, we can now characterize the sequentially compact sets in Wasserstein spaces:

Theorem 3.1.3. *Fix $p \geq 1$, $\alpha > p$ and $c > 0$. Then the set*

$$K := \left\{ \mu \in \mathcal{P}_p(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|^\alpha d\mu(x) \leq c \right\}$$

is sequentially compact in the metric space $\left(\mathcal{P}_p(\mathbb{R}^d), W_p\right)$.

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in K. Let us prove that $\{\mu_n\}_{n\in\mathbb{N}}$ is tight: for any $\varepsilon > 0$ there exists a compact set $K_0 \subset \mathbb{R}^d$ such that $\mu_n\left(\mathbb{R}^d \setminus K_0\right) \leq \varepsilon$ for any $n \in \mathbb{N}$: indeed, for any $R > 0$, we have that $\mu_n\left(\mathbb{R}^d \setminus B_R(0)\right) \leq \frac{c}{R^\alpha}$, since

$$c \geq \int_{\mathbb{R}^d} |x|^\alpha d\mu_n(x) \geq \int_{B_R^c(0)} |x|^\alpha d\mu_n(x) \geq R^\alpha \mu_n(B_R^c(0)).$$

Thus $\mu_n(B_R^c(0)) \leq \varepsilon$ for R large enough, and so $\{\mu_n\}_{n\in\mathbb{N}}$ is tight. Therefore, by Prokorov theorem, there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ and a measure $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ such that $\mu_{n_k} \overset{*}{\rightharpoonup}_{k\to\infty} \mu$. Let us check that $\mu \in K$: for any $M > 0$, consider the continuous and bounded map $x \mapsto \min\{|x|^\alpha, M\}$. We have

$$\int_{\mathbb{R}^d} \min\{|x|^\alpha, M\} d\mu(x) = \lim_{k\to\infty} \int_{\mathbb{R}^d} \min\{|x|^\alpha, M\} d\mu_k(x).$$

By monotone convergence, letting $M \to \infty$ we get

$$\int_{\mathbb{R}^d} |x|^\alpha d\mu(x) \leq \lim_{k\to\infty} \int_{\mathbb{R}^d} |x|^\alpha d\mu_k(x) \leq c.$$

By Hölder inequality we have $\int_{\mathbb{R}^d} |x|^p d\mu(x) < \infty$, so that $\mu \in K$. Since we know that $\mu_{n_k} \overset{W_p}{\to} \mu$ if and only if

$$\mu_{n_k} \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad \lim_{k\to\infty} \sup_R \int_{B_R^c(0)} |x|^p d\mu_{n_k}(x) = 0,$$

we now just need to check the second condition. We have

$$c \geq \int_{\mathbb{R}^d} |x|^\alpha d\mu_{n_k}(x) \geq \int_{B_R^c(0)} |x|^p |x|^{\alpha-p} d\mu_{n_k}(x) \geq R^{\alpha-p} \int_{B_R^c(0)} |x|^p d\mu_{n_k}(x).$$

Therefore

$$\sup_k \int_{B_R^c(0)} |x|^p d\mu_{n_k}(x) \leq \frac{c}{R^{\alpha-p}} \to_{R\to\infty} 0.$$

□

3.2 Sliced Wasserstein distance

Computing Wasserstein distances is not an easy task: this is indeed related to the optimal transport problem that, as we already said, is difficult to treat from the numerical point of view. This is why for many applications, in particular for image processing, rather then finding the optimal transport map, people look for "good approximations" of it, that possibly are computationally easy to find and satisfy some monotonicity assumptions. Since, as we saw, in the particular case of dimension $d = 1$, transport maps are nicely characterized and easier to compute, one of the most common way of doing so is to build these approximations via 1D constructions. The *Iterative Distribution Transfer* algorithm (IDT) uses this idea: a source measure μ is given, together with a target ν; in order to approximate the optimal map between μ and ν the following algorithm is developed: a sequence of maps T_n is built and we set $\mu_{n+1} = (T_n)_\#\mu_n$. The idea is that μ_n should converge to ν, so that the map $T_n \circ T_{n-1} \circ \ldots T_1 \circ T_0$ is a transport map from μ to μ_{n+1} (which, for large n, is a measure very close to ν). Each map T_n is constructed as follows: at any step n, an orthonormal basis B_n of $\mathbb{R}^d$ is (randomly) selected. Then we define T_n^j (for $j = 1, \ldots, d$) as the monotone 1D transport map from $(\pi_j)_\#\mu_n$ to $(\pi_j)_\#\nu$, where π_j is the canonical projection in the j-th coordinate of the basis B_n. The map T_n is therefore defined as $T_n(x) = (T_n^1(x_1), \ldots, T_n^d(x))$. The interesting thing is that, if the bases B_n are chosen in a suitable way, then the algorithm stops at a constant measure μ_n only if μ_n and ν have the same projection along all directions. This implies that $\mu = \nu$ thanks to the following

Lemma 3.2.1 (X-Ray transform)**.** *Given two measures μ, ν over $\mathbb{R}^d$, if the two families of measures parametrized by $\vartheta \in \mathbb{S}^{d-1}$*

$$(\pi_\vartheta)_\#\mu \quad \textit{and} \quad (\pi_\vartheta)_\#\nu$$

coincide, then $\mu = \nu$.

Proof. The proof can be found in [11], at Box 2.4. □

From this idea, which was first introduced in [9], M. Bernot defined a similar construction in which instead of choosing T_n as the vector of optimal maps along random directions, one takes $T_n(x) = \fint_{\mathbb{S}^{d-1}} T_n^\vartheta(x\cdot\vartheta)d\mathcal{H}^{d-1}(\vartheta)$, where T_ϑ^n is the optimal map between $(\pi_\vartheta)_\#\mu_n$ and $(\pi_\vartheta)_\#\nu$.

This construction is strictly related to the notion of Sliced Wasserstein distance, which was introduced by M. Bernot himself and now we will present: in order to get a nice approximation of the W_p distance via one-dimensional constructions, the following definition based on the behavior of the measures "direction by direction" is given:

Definition 3.2.1 (Sliced Wasserstein distance)**.** *Let $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, and let π^ϑ be the projection on the direction of the unitary vector $\vartheta \in \mathbb{S}^{d-1}$. We define*

$$SW_p(\mu, \nu) := \left(\fint_{\mathbb{S}^{d-1}} W_p^p((\pi^\vartheta)_\#\mu, (\pi^\vartheta)_\#\nu)d\mathcal{H}^{d-1}(\vartheta)\right)^{\frac{1}{p}}.$$

Theorem 3.2.1. *The Sliced Wasserstein distance is, indeed, a distance.*

Proof. The triangular inequality comes from the triangular inequality property of the usual Wasserstein distance and of the L^p norm. The positivity and the simmetry of SW_p are evident. If $SW_p(\mu,\nu) = 0$, then $W_p^p((\pi^\vartheta)_\#\mu, (\pi^\vartheta)_\#\nu) = 0$ for almost any $\vartheta \in \mathbb{S}^{d-1}$. Since W_p is a distance, $(\pi^\vartheta)_\#\mu = (\pi^\vartheta)_\#\nu$ for almost any $\vartheta \in \mathbb{S}^{d-1}$. Since for any $s \in \mathbb{R}$ we can observe that, denoting by $\mathcal{F}$ the Fourier transform on $\mathbb{R}$, $\mathcal{F}((\pi^\vartheta)_\#\mu)(s) = \mathcal{F}\mu(s\vartheta)$, we have

$$\mathcal{F}\mu(s\vartheta) = \mathcal{F}\nu(s\vartheta),$$

and by the injectivity of the Fourier transform, we get $\mu = \nu$. □

Proposition 1. *If $\mu,\nu \in \mathcal{P}_p(\mathbb{R}^d)$, then $SW_p(\mu,\nu)^p \leq m_{p,d} W_p(\mu,\nu)^p$, where $m_{p,d}$ is the constant defined by*

$$\fint_{\mathbb{S}^{d-1}} |\vartheta \cdot z|^p d\vartheta = m_{d,p}|z|^p$$

for any $z \in \mathbb{R}^d$.

Proof. Let $\gamma \in \Pi(\mu,\nu)$ be an optimal transport plan. Then $(\pi^\vartheta \otimes \pi^\vartheta)_\#\gamma$ is a transport plan between $\pi^\vartheta_\#\mu$ and $\pi^\vartheta_\#\nu$. So

$$W_p(\pi^\vartheta_\#\mu, \pi^\vartheta_\#\nu)^p \leq \int \left|x\cdot\vartheta - y\cdot\vartheta\right|^p d\gamma(x,y).$$

Hence

$$\begin{aligned} SW_p(\mu,\nu)^p &\leq \int \left(\fint \left|x\cdot\vartheta - y\cdot\vartheta\right|^p d\vartheta \right) d\gamma(x,y) \\ &\leq m_{d,p} \int |x-y|^p d\gamma(x,y) \\ &\leq m_{d,p} W_p(\mu,\nu)^p. \end{aligned}$$

□

Proposition 2. *There exists a constant $C_d > 0$ such that, for all μ,ν supported in $B(0,R)$*

$$W_1(\mu,\nu) \leq C_d R^{d/(d+1)} SW_1(\mu,\nu)^{1/(d+1)}.$$

Proof. See [4], Lemma 5.1.4. □

As an immediate consequence of the two previous results, we have the following

Proposition 3 (Equivalence of W_p and SW_p)**.** *There exists a constant $C_{d,p} > 0$ such that, for all μ,ν supported in $B(0,R)$*

$$SW_p(\mu,\nu)^p \leq m_{d,p} W_p^p(\mu,\nu) \leq C_{d,p} R^{p-1/(d+1)} SW_p(\mu,\nu)^{1/(d+1)}.$$

3.3 Curves in Wasserstein spaces

In this section we want to study the properties of Lipschitz and absolutely continuous (AC) curves in Wasserstein spaces. The main difficulty when talking about these objects is that our ambient space is *not* a vector space, hence the classical notion of velocity of a curve has no meaning. Nonetheless we can give the following

Definition 3.3.1 (Metric derivative)**.** *Let (X, d) be a metric space. Let $\omega\colon [0,1] \to X$. We define the metric derivative of ω at time t, denoted by $|\omega'|(t)$, through*

$$|\omega'|(t) := \lim_{h\to 0} \frac{d(\omega(t+h), \omega(t))}{|h|},$$

provided this limit exists.

The following Rademacher-type theorem guarantees the *a.e.* existence of the above limit for Lipschitz continuous curves, *i.e.* maps $\omega\colon [0,1] \to X$ such that there exists a costant $C > 0$ for which $d(\omega(t), \omega(s)) \le C|t-s|$, for any $t, s \in [0,1]$:

Theorem 3.3.1. *Suppose that $\omega\colon [0,1] \to X$ is Lispschitz continuous. Then the metric derivative $|\omega'|(t)$ exists for a.e. $t \in [0,1]$. Moreover, we have, for $t < s$,*

$$d(\omega(t), \omega(s)) \le \int_t^s |\omega'|(\tau) d\tau.$$

Proof. See [3], chapter 12. □

We give now the definition of absolutely continuous curves:

Definition 3.3.2 (AC curves)**.** *A curve $\omega\colon [0,1] \to X$ is defined absolutely continuous if there exists $g \in L^1([0,1])$ such that $d(\omega(t_0), \omega(t_1)) \le \int_{t_0}^{t_1} g(s)ds$ for every $t_0 < t_1$.*

Notice that any Lipschitz curve is also AC. In the following we present a theorem, whose proof can be found in [11], that identifies the absolutely continuous curves in the W_p Wasserstein space as the solutions of a continuity equation for a given L^p velocity field

Theorem 3.3.2. *Let $(\mu_t)_{t\in[0,1]}$ be an absolutely continuous curve in $(\mathcal{P}(\Omega), W_p)$, for $\Omega \subset \mathbb{R}^d$, $p > 1$. Then, for a.e. $t \in [0,1]$, there exists a vector field $v_t \in L^p(\mu_t; \mathbb{R}^d)$ such that*

- *the continuity equation $\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0$ is satisfied in the weak sense: for any test function $\psi \in C^1_c(\overline{\Omega})$, the function $t \mapsto \int \psi d\mu_t$ is absolutely continuous and, for a.e. t we have*
$$\frac{d}{dt}\int_\Omega \psi d\rho_t = \int_\Omega \nabla\psi \cdot v_t d\rho_t.$$

- *for a.e. t, we have $||v_t||_{L^p(\mu_t)} \leq |\mu'|(t)$.*

Conversely, if $(\mu_t)_{t\in[0,1]}$ is a family of measures in $\mathcal{P}_p(\Omega)$ and for each $t \in [0,1]$ we have a vector field $v_t \in L^p(\mu_t;\mathbb{R}^d)$ with $\int_0^1 ||v_t||_{L^p(\mu_t)}dt < +\infty$ solving $\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0$, then $(\mu_t)_{t\in[0,1]}$ is absolutely continuous in $(\mathcal{P}_p(\Omega), W_p)$ and, for a.e. $t \in [0,1]$, we have $|\mu'|(t) \leq ||v_t||_{L^p(\mu_t)}$.

Remark 1. *As a consequence of the second part of the theorem, the vector field v_t must a posteriori satisfy $||v_t||_{L^p(\mu)} = |\mu'|(t)$.*

Definition 3.3.3. *For a curve $\omega\colon [0,1] \to X$, let us define*

$$Length(\omega) := \sup\left\{\sum_{k=0}^{n-1} d(\omega(t_k),\omega(t_{k+1}))\colon n \geq 1, 0 = t_0 < t_1 < \cdots < t_n = 1\right\}.$$

A metric space (X,d) is a length space if, for any $x, y \in X$, it holds

$$d(x,y) = \inf\{Length(\omega)\colon \omega \in AC(X), \omega(0) = x, \omega(1) = y\}.$$

Definition 3.3.4. *Given a length space (X,d), a curve $\omega\colon [0,1] \to X$ is said to be a geodesic between x_0 and $x_1 \in X$ if it minimizes the length among all curves such that $\omega(0) = x_0$, $\omega(1) = x_1$.*
We say that ω is a constant-speed geodesic between x_0 and $x_1 \in X$ if it satisfies

$$d(\omega(t),\omega(s)) = |t-s|d(\omega(0),\omega(1)),$$

for all $t, s \in [0,1]$. A metric space (X,d) is said to be a geodesic space if it holds

$$d(x,y) = \min\left\{\int_0^1 |\omega'|(t)dt\colon \omega \in AC(X), \omega(0) = x, \omega(1) = y\right\}.$$

Remark 2. *We can easily check that any constant-speed geodesic is a geodesic.*

The following proposition holds:

Proposition 4. *Let (X,d) be a geodesic space. Fix an exponent $p > 1$, and consider curves connecting x_0 and x_1. The following facts are equivalent:*

1. *ω is a constant-speed geodesic,*

2. *$\omega \in AC(X)$ and $|\omega'|(t) = d(\omega(0),\omega(1))$ a.e.,*

3. *ω solves*
$$\min\left\{\int_0^1 |\omega'|(t)^p dt\colon \omega(0) = x_0, \omega(1) = x_1\right\}.$$

Theorem 3.3.3. *Let $\Omega \subset \mathbb{R}^d$ be convex, take $\mu, \nu \in \mathcal{P}_p(\Omega)$ and $\gamma \in \Pi(\mu, \nu)$ be an optimal transport plan for the cost $c(x, y) = |x - y|^p$, $p \geq 1$. Define $\pi_t \colon \Omega \times \Omega \to \Omega$ by $\pi_t(x, y) = (1 - t)x + ty$. Then the curve $\mu_t := (\pi_t)_\# \gamma$ is a constant-speed geodesic in the Wasserstein space $(\mathcal{P}_p(\Omega), W_p)$, connecting μ to ν.*

Proof. The proof is contained in [11], Theorem 5.27. □

Remark 3. *If γ comes from an optimal map, then the curve μ_t is obtained as $((1 - t)id + tT)_\# \mu$.*

Theorem 3.3.4. *Consider the geodesic in $\left(\mathcal{P}_2(\mathbb{R}^d), W_2\right)$ from μ to ν given by $\mu_t = ((1 - t)id + tT)_\# \mu$, where T is the optimal map transporting μ into ν. Then the velocity field $v_t(y) := (T - id)(T_t^{-1}(y))$ is well defined on $spt(\mu_t)$ for each $t \in]0, 1[$ and satisfies*

$$\partial_t \mu_t + \nabla \cdot (\mu_t v_t) = 0, \quad \|v_t\|_{L^p(\mu_t)} = |\mu'|(t) = W_p(\mu, \nu).$$

Proof. See [11], Proposition 5.30. □

Remark 4. *The above theorem can be extended even to the case in which the transport plan is not associated with a transport map. See as a reference [11].*

3.4 Geodesic convexity

Thanks to theorem (3.3.3) and (3.3.4), we can see that $(\mathcal{P}_p(\Omega), W_p)$ is a geodesic space for Ω convex, $p \geq 1$. In the following we will give an important notion of convexity, related to the structure of the Wasserstein space $(\mathcal{P}_p(\Omega), W_p)$:

Definition 3.4.1. *In a geodesic metric space X, we define $F \colon X \to \mathbb{R} \cup \{+\infty\}$ to be geodesically convex if for every two points $x_0, x_1 \in X$ there exists a constant-speed geodesic ω connecting $\omega(0) = x_0$ and $\omega(1) = x_1$ such that $[0, 1] \ni t \mapsto F(\omega(t))$ is convex.*

Remark 5. *A functional $[0, 1] \ni t \mapsto G(t)$ is said to be convex if*

$$G((1 - t)x + ty) \geq (1 - t)G(x) + tG(y)$$

for any $x, y \in [0, 1]$.

Remark 6. *Definition (3.4.1) reduces to the usual notion of convexity when the space X is $\mathbb{R}^d$ or any other normed vector space, where segments are the unique geodesics.*

Consider the following functionals defined on $\mathcal{P}_p(\Omega)$:

$$\mathcal{V}(\rho) = \int_\Omega V(x) d\rho(x),$$

$$\mathcal{F}(\rho) = \int_\Omega F(\rho) d\rho.$$

Proposition 5. *Using the notation above, the functional $\mathcal{V}$ is geodesically convex if and only if V is convex.*

Proof. See [11], Proposition 7.24. □

Necessary and sufficient condition for the geodesic convexity of the functional $\mathcal{F}$ are more difficult to get. We can state the following

Theorem 3.4.1. *Using the notation above, suppose that F is convex and superlinear, $F(0) = 0$, and that $s \mapsto s^{-d}F(s^d)$ is convex and decreasing. Suppose that Ω is convex and take $1 < p < \infty$. Then $\mathcal{F}$ is geodesically convex in W_p.*

Proof. See [2], Proposition 9.3.9. □

Example 4. *The following are common convex functionals satisfying the assumptions of theorem* (3.4.1)*:*

- *for any $q > 1$, $F(t) = t^q$,*
- $\mathcal{E}(t) = t \log t$ *(the Entropy),*
- *for any $1 - \frac{1}{d} \leq m < 1$, $F(t) = -t^m$.*

Chapter 4

Gradient flows

Gradient flows are a link between optimal transport theory and the world of evolution PDEs: many evolution equations can be seen as a steepest descent movement in the Wasserstein spaces. Thanks to the preliminaries of the previous section, we are now able to discuss this topic and focus on the main subject of this work.
In the following section we will briefly discuss where gradient flows come from and why it is interesting and worthwhile to study them: without giving rigorous proofs, we will see that they arise as the limit version of the implicit Euler scheme for the minimization of a certain functional F in Wasserstein spaces. In order to get a detailed proof of the construction below, one can consult [2], in which the theory of Minimizing Movements and General Minimizing Movements built by E. De Giorgi is widely presented.

4.1 Gradient flow as the limit of the JKO scheme

In this section we want to give an heuristic derivation of the structure and the equation describing gradient flows on Wasserstein spaces: essentially, gradient flows can be seen as the continuous-in-time version of a discrete minimization scheme. Recall the time-discretization scheme associated with the Cauchy problem

$$\begin{cases} x'(t) & = -\nabla F(x(t)) \quad \text{for } t > 0, \\ x(0) & = x_0 : \end{cases} \tag{4.1}$$

fix a small time step parameter $\tau > 0$ and look for a sequence of points $\{x^\tau_{k\in\mathbb{N}}\}_k$ given by the iterated scheme

$$x^\tau_{k+1} \in \arg\min_x F(x) + \frac{|x - x^\tau_k|^2}{2\tau}.$$

Under mild assumptions, the optimality conditions of the above minimization scheme give

$$\frac{x_{k+1}^\tau - x_k^\tau}{\tau} = -\nabla F(x_{k+1}^\tau),$$

which is the discrete-time *implicit Euler scheme* for (4.1). From this idea, the following discrete minimization scheme can be developed in any metric space (X, d) for a given functional F:

$$\rho_{(k+1)}^\tau \in \arg\min_\rho F(\rho) + \frac{d^2(\rho, \rho_{(k)}^\tau)}{2\tau}. \tag{4.2}$$

This algorithm is called *minimizing movements scheme*, and when (X, d) is the usual euclidean space, we recover the Euler implicit scheme. When instead the space (X, d) is $(\mathcal{P}_2(\Omega), W_2)$ for a given $\Omega \subset \mathbb{R}^d$, the minimizing movements scheme is called *Jordan-Kinderlehrer-Otto scheme* (JKO).

Consider a functional $F\colon \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$. Assume that F and Ω are such that, for any $\tau > 0$ the JKO scheme admits a solution. In order to properly write the optimality conditions for these problems we need the following

Definition 4.1.1 (First Variation). *Given a functional $F\colon \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, we say that $\rho \in \mathcal{P}(\Omega)$ is regular for F if $F((1-\varepsilon)\rho + \varepsilon\tilde{\rho}) < +\infty$ for every $\varepsilon \in [0, 1]$ and every $\tilde{\rho} \in \mathcal{P}(\Omega) \cap L_c^\infty(\Omega)$.*
If ρ is regular for F, we will denote by $\frac{\delta F}{\delta \rho}(\rho)$, if it exists, any measurable function such that

$$\frac{d}{d\varepsilon} F(\rho + \varepsilon\chi)|_{\varepsilon=0} = \int \frac{\delta F}{\delta \rho}(\rho) d\chi$$

for every perturbation $\chi = \tilde{\rho} - \rho$ with $\tilde{\rho} \in L_c^\infty(\Omega) \cap \mathcal{P}(\Omega)$.

We will make use of the following result:

Proposition 6. *Let $\Omega \subset \mathbb{R}^d$ be compact and $c\colon \Omega \times \Omega \to \mathbb{R}$ be continuous. Define*

$$\mathcal{T}_c(\mu, \nu) := \min\left\{ \int c(x, y) d\gamma \colon \gamma \in \Pi(\mu, \nu) \right\}.$$

Then the functional $\mu \mapsto \mathcal{T}_c(\mu, \nu)$ is convex. Moreover, if the Kantorovich potential ψ from μ to ν is unique, then we have

$$\frac{\delta \mathcal{T}_c(\cdot, \nu)}{\delta \rho}(\mu) = \psi.$$

Proof. Consider $\mu_\varepsilon = \mu + \varepsilon\chi$, where $\chi = \tilde{\mu} - \mu$, and estimate the ratio $(\mathcal{T}_c(\mu_\varepsilon, \nu) - \mathcal{T}_c(\mu, \nu))/\varepsilon$:

$$\frac{\mathcal{T}_c(\mu_\varepsilon, \nu) - \mathcal{T}_c(\mu, \nu)}{\varepsilon} \geq \frac{\int \psi d\mu_\varepsilon + \int \psi^c d\nu - \int \psi d\mu - \int \psi^c d\nu}{\varepsilon} = \int \psi d\chi,$$

so that $\liminf_{\varepsilon\to 0}(\mathcal{T}_c(\mu_\varepsilon,\nu)-\mathcal{T}_c(\mu,\nu))/\varepsilon \geq \int \psi d\chi$. Then, consider a sequence of values ε_k realizing the lim sup, *i.e.* $\lim_k(\mathcal{T}_c(\mu_{\varepsilon_k},\nu)-\mathcal{T}_c(\mu,\nu))/\varepsilon_k = \limsup_{\varepsilon\to 0}(\mathcal{T}_c(\mu_\varepsilon,\nu)-\mathcal{T}_c(\mu,\nu))/\varepsilon$. Denoting by ψ_k the Kantorovich potential from μ_{ε_k} to ν, we have

$$\frac{\mathcal{T}_c(\mu_{\varepsilon_k},\nu)-\mathcal{T}_c(\mu,\nu)}{\varepsilon_k} \leq \frac{\int \psi_k d\mu_{\varepsilon_k} + \int \psi_k^c d\nu - \int \psi_k d\mu - \int \psi_k^c d\nu}{\varepsilon} = \int \psi_k d\chi.$$

Recall (see [11], theorem 1.52) that up to further extraction of a subsequence, we have uniform convergence $(\psi_k, \psi_k^c) \to (\tilde{\psi}, \tilde{\psi}^c)$ and that $(\tilde{\psi}, \tilde{\psi}^c)$ must be optimal in the Kantorovich formulation of the problem. By the assumption of uniqueness of the Kantorovich potential, we have $(\tilde{\psi}, \tilde{\psi}^c) = (\psi, \psi^c)$. Passing to the limit for $k \to \infty$, we have also $\limsup_{\varepsilon\to 0}(\mathcal{T}_c(\mu_\varepsilon,\nu)-\mathcal{T}_c(\mu,\nu))/\varepsilon \leq \int \psi d\chi$. □

Consider $\hat{\rho}$ to be the solution to the JKO scheme at time $\tau > 0$, *i.e.* to the minimization problem (4.2) for $(X, d) = (\mathcal{P}_2(\Omega), W_2)$. Taking any perturbation measure of the form $\rho_\varepsilon = \hat{\rho} + \varepsilon(\rho - \hat{\rho}) =: \hat{\rho} + \varepsilon\chi$ and differentiating w.r.t. $\varepsilon > 0$, one gets

$$\begin{aligned} 0 &= \int_\Omega \frac{\delta F}{\delta\rho}(\hat{\rho}) d\chi + \frac{1}{\tau}\int_\Omega \frac{\delta W_2^2(\hat{\rho}, \rho_{(k)}^\tau)}{\delta\rho} d\chi \\ &= \int_\Omega \frac{\delta F}{\delta\rho}(\hat{\rho}) d\chi + \frac{1}{\tau}\int_\Omega \varphi d\chi, \end{aligned}$$

where φ is the Kantorovich potential for the quadratic cost transport problem from $\hat{\rho}$ to $\rho_{(k)}^\tau$. Form this one can prove that

$$\frac{\delta F}{\delta\rho}(\rho) + \frac{\varphi}{\tau} = constant$$

holds $\hat{\rho} - a.e.$. Now, recalling that $T(x) = x - \nabla\varphi(x)$, we get

$$\frac{T(x)-x}{\tau} = \nabla\left(\frac{\delta F}{\delta\rho}(\rho)\right)(x).$$

We will denote by $-v^\tau$ the vector field $\frac{T(x)-x}{\tau}$, since it has the meaning of a velocity, being a ratio between a displacement and a time step. The minus sign can be justified by the fact that it is the displacement associated with the transport from $\hat{\rho} = \rho_{(k+1)}^\tau$ to $\rho_{(k)}^\tau$, so we can see it as a backward velocity. Having seen that any AC curve solves the continuity equation for a given velocity field, we are now led to deduce that the continuity equation ruling the behavior of the curve $(\rho_t)_{t\geq 0}$ which is obtained as a limit curve of the discrete time-step minimization scheme presented above letting $\tau \to 0^+$ must be:

$$\partial_t\rho - \nabla\cdot\left(\rho\nabla\left(\frac{\delta F}{\delta\rho}(\rho)\right)\right) = 0,$$

with no flux boundary conditions on $\partial\Omega$. We can finally give the following

Definition 4.1.2 (Gradient Flow). *Let $\Omega \subset \mathbb{R}^d$. Given a functional $F\colon \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ we say that the curve $(\rho_t)_{t\geq 0}$ is a gradient flow for the functional F if it is the solution of the continuity equation*

$$\partial_t \rho - \nabla \cdot \left(\rho \nabla \left(\frac{\delta F}{\delta \rho}(\rho)\right)\right) = 0.$$

Under mild assumptions, JKO algorithms are known to converge to the minimizer of the considered functional: having this new perspective in mind, the conjectures presented in the introduction of the present work may now appear to be the natural conjectures one can do investigating the gradient flow associated to the sliced Wasserstein distance.

4.2 Sliced Wasserstein gradient flow

In this section we will present the model we studied and to which the rest of this work is devoted. Fix a target measure $\nu \in \mathcal{P}_2(\mathbb{R}^d)$: the Sliced Wasserstein gradient flow (SWGF) is the gradient flow associated with the functional

$$F(\rho) := \frac{SW_2^2(\rho, \nu)}{2}.$$

From now on we will use the following notation: for any $\vartheta \in \mathbb{S}^{d-1}$,

$$\rho_\vartheta := (\pi_\vartheta)_\# \rho,$$

where π_ϑ is the canonical projection on the direction $\vartheta\mathbb{R}$. Recall that ρ_ϑ is a one-dimensional probability measure, and one has

$$\begin{aligned} \rho_\vartheta(t) &= \int_{\vartheta^\perp} \rho(t\vartheta + y)dy, \\ \nu_\vartheta(t) &= \int_{\vartheta^\perp} \nu(t\vartheta + y)dy. \end{aligned} \tag{4.3}$$

Using (6), and calling $\varphi_{\vartheta,t}$ the Kantorovich potential for the quadratic-cost transport problem from $(\mu_t)_\vartheta$ to ν_ϑ, we can derive the associated continuity equation: the velocity field v_t is given by

$$\begin{aligned} v_t(x) &= -\nabla_x \left(\frac{\delta SW_2^2(\rho_t, \nu)}{2\delta\rho}(\rho)\right) \\ &= -\fint_{\mathbb{S}^{d-1}} \varphi'_{\vartheta,t}(x\cdot\vartheta)\vartheta d\mathcal{H}^{d-1}(\vartheta). \end{aligned} \tag{4.4}$$

The ODE describing the Lagrangian interpretation of this model, for a starting point $x \in \mathbb{R}^d$ is

$$\begin{cases} y'(t) = & v_t(y(t)) = -\displaystyle\fint_{\mathbb{S}^{d-1}} \varphi'(y(t)\cdot\vartheta)\vartheta d\mathcal{H}^{d-1}(\vartheta) \\ & = \displaystyle\fint_{\mathbb{S}^{d-1}} \vartheta\,(T_\vartheta - \mathrm{id})\,(y(t)\cdot\vartheta) d\mathcal{H}^{d-1}(\vartheta), \\ y(0) = & x. \end{cases} \tag{4.5}$$

Chapter 5

SWGF does not provide optimal transport

This chapter contains a counterexample to the conjecture 3 presented in the introduction, namely we show the following:

It is not true that, for any initial distribution ρ_0 and any target distribution ν, the sliced Wasserstein gradient flow converges to the target measure ν itself and the limit of the flow map arising from the Lagrangian description of the model exists and is the optimal transport map from ρ_0 to ν.

Using the idea developed in [8], we can write a necessary condition which must hold if we want conjecture 3 to be true: take ρ_0 sufficiently smooth and quickly decaying at infinity, and assume that conjecture 3 holds not only for ρ_0 but for all $(\rho_t)_{t\geq 0}$. This means that the flow y_t is well defined for any $t \geq 0$, that the limit $\lim_{t\to\infty} y_t =: T$ exists, and that it is the optimal transport map between ρ_0 and ν. Under these assumptions, we have that for any $t \geq 0$ the SWGF provides optimal transport from ρ_t to ν. That is, the map $T \circ y_t^{-1}$ is an optimal transport map between ρ_t and ν. Using remark 1 from section 2.1, $y_t \circ T^{-1}$ is also an optimal transport map between ν and ρ_t. Let us denote by S the map T^{-1}. Making use of theorem (2.2.3), we have that the following condition must hold

$$\forall t \geq 0, \forall x \in \mathbb{R}^d, \quad \text{the Jacobian of } y_t \circ S(x) \text{ is a symmetric matrix .} \tag{5.1}$$

The Jacobian matrix reads $Dy_t(S)DS$. Moreover, differentiating (4.5) with respect to x, we see that

$$\frac{\partial Dy_t}{\partial t} = -\left(\fint_{\mathbb{S}^{d-1}} \varphi''_{t,\vartheta}(x \cdot \vartheta)\vartheta \otimes \vartheta \quad d\vartheta \right),$$

together with $Dy_0 = \mathrm{Id}$. Thus, differentiating the Jacobian of $y_t \circ S$ with respect to time, we have that condition (5.1) implies that

$$\forall t \geq 0, \forall x \in \mathbb{R}^d, \quad \left[\fint_{\mathbb{S}^{d-1}} \varphi''_{t,\vartheta} \left(S(x) \cdot \vartheta \right) \vartheta \otimes \vartheta \right] DS(x) \quad \text{is a symmetric matrix.}$$

Valuating this expression at $t = 0$ and remembering that $DS(x)$ is symmetric for any x, since our assumption is that S is an optimal transport map between ν and ρ_0, we conclude that the matrices

$$\fint_{\mathbb{S}^{d-1}} \varphi''_{\vartheta} \left(S(x) \cdot \vartheta \right) \vartheta \otimes \vartheta \quad \text{and} \quad DS(x) \quad \text{should commute for any } x$$

Composing both the matrices with $S^{-1} = T$ on the right hand side and using the identity $DS(S^{-1}) = [DT]^{-1}$, and the fact that a symmetric matrix A commutes with an invertible matrix B if and only if it commutes with B^{-1}, we conclude that:

$$\forall x \in \mathbb{R}^d, \quad \fint_{\mathbb{S}^{d-1}} \varphi''_{\vartheta} \left(x \cdot \vartheta \right) \vartheta \otimes \vartheta \quad \text{and} \quad DT(x) \quad \text{commute.}$$

Using again the fact that $T = Du$ for a convex map $u \colon \mathbb{R}^d \to \mathbb{R}$ (the Brenier map of the problem), we can write the necessary condition for which we will build a counterexample in the following section:

$$\forall x \in \mathbb{R}^d, \quad \fint_{\mathbb{S}^{d-1}} \varphi''_{\vartheta} \left(x \cdot \vartheta \right) \vartheta \otimes \vartheta \quad \text{and} \quad D^2 u(x) \quad \text{commute.}$$

5.1 Counterexample

For the seek of simplicity we will call $\rho_0 = \rho$ in the following. We will also denote by φ_ϑ the Kantorovich potential between ρ_ϑ and ν_ϑ with respect to the 2-Wasserstein distance and by T_ϑ the quadratic-cost optimal transport map between the two measures. By the previous discussion we aim to show that the matrix

$$M_{i,j}(x) := \left(\fint_{\mathbb{S}^{d-1}} \varphi''_{\vartheta}(x \cdot \vartheta) \vartheta_i \vartheta_j \quad d\mathcal{H}^{d-1}(\vartheta) \right)_{i,j}$$

does not commute with $D^2 u$, where $u : \mathbb{R}^d \to \mathbb{R}$ is a convex function whose gradient provides the transport map $T : \mathbb{R}^d \to \mathbb{R}^d$ between ρ and ν.

Call u_ϑ the Brenier map for the transport between ρ_ϑ and ν_ϑ. Writing the Monge-Ampere equation for ρ_ϑ and ν_ϑ we have

$$u''_{\vartheta}(t) = \frac{\rho_\vartheta(t)}{\nu_\vartheta(u'_\vartheta(t))}.$$

Since we have that $\varphi''_\vartheta(t) = 1 - u''_\vartheta(t)$, we get that

$$M_{i,j}(x) = \fint \left(1 - \frac{\rho_\vartheta(x\cdot\vartheta)}{\nu_\vartheta(T_\vartheta(x\cdot\vartheta))}\right)\vartheta_i\vartheta_j d\vartheta = \frac{\mathrm{Id}}{d} - \fint \frac{\rho_\vartheta(x\cdot\vartheta)}{\nu_\vartheta(T_\vartheta(x\cdot\vartheta))}\vartheta_i\vartheta_j d\vartheta.$$

If we choose ρ and ν to be symmetric measures ($\rho(x) = \rho(-x)$, $\nu(x) = \nu(-x)$ for all $x \in \mathbb{R}^d$), we have that $T_\vartheta(0) = 0$ for all $\vartheta \in \mathbb{S}^{d-1}$. So our goal can be rewritten as follows: we want that the matrix

$$\mathrm{Id} - M(0) = \fint \rho_\vartheta(0)(\vartheta\otimes\vartheta)d\vartheta \quad \text{does not commute with} \quad D^2u(0). \tag{5.2}$$

Since we are free to choose any (symmetric) initial measure ρ, let us be free to impose $\rho = (\nabla u)^{-1}_{\#}\nu$, for $u(x) = |x|^2/2 + \varepsilon\varphi(x)$, with φ smooth, compactly supported and symmetric. It is well known that, for ε small enough, u is convex and ∇u is a C^∞ diffeomorphism that coincides with the identity outside of a compact set. Finally, asking φ to be symmetric guarantees that ρ is symmetric, since indeed in this case the transport map $T = \nabla u$ is symmetric itself, and the pushforward of a symmetric distribution via (the invrese of) a symmetric map is symmetric. We seek for the expression of $\rho_\vartheta(0)$ in terms of φ. We start by writing the Monge-Ampere equation for u:

$$\det D^2u(x) = \frac{\rho(x)}{\nu(\nabla(u(x)))},$$

for every $x \in \mathbb{R}^d$. Thus

$$\rho(x) = \det D^2u(x)\nu(\nabla u(x)).$$

In our case $\nabla u(x) = x + \varepsilon\nabla\varphi(x)$ and $\det D^2u(x) = 1 + \varepsilon\Delta\varphi(x) + O(\varepsilon^2)$. By Taylor expansion, for any $x, z \in \mathbb{R}^d$:

$$\nu(x + \varepsilon z) = \nu(x) + \varepsilon D\nu(x)\cdot z + O(\varepsilon^2).$$

Thus

$$\begin{aligned}\rho(x) =& \left(1 + \varepsilon\Delta\varphi + O(\varepsilon^2)\right)(\nu(x) + \varepsilon D\nu(x)\cdot\nabla\varphi(x)) + O(\varepsilon^2)\\ =&\nu(x) + \varepsilon\left(D\nu(x)\cdot\nabla\varphi(x) + \Delta\varphi(x)\nu(x)\right) + O(\varepsilon^2).\end{aligned}$$

So, remembering the expressions (4.3), we have

$$\rho_\vartheta(0) = \int_{\vartheta^\perp}\rho(y)dy = \int_{\vartheta^\perp}\left(\nu(y) + \varepsilon\left(D\nu(y)\cdot\nabla\varphi(y) + \Delta\varphi(y)\nu(y)\right)\right)dy + O(\varepsilon^2).$$

We now impose $\varphi(x) = \varepsilon\psi(x) + \eta(x)$, with ψ, η symmetric functions (so that φ is symmetric) and such that $\mathrm{supp}(\psi) \subseteq B(0,1)$ and $\mathrm{supp}(\eta) \subseteq B(Re_d, 1) \cup B(-Re_d, 1)$ for a fixed $R >> 1$, e_d being the last vector of the canonical basis in $\mathbb{R}^d$. In this way $D^2u(0) = \varepsilon^2 D^2\psi(0)$ and so, by (5.2), our aim becomes to prove that the matrix $⨍ \rho_\vartheta(0)\vartheta \otimes \vartheta$ does not commute with $D^2\psi(0)$. We have

$$\begin{aligned}\rho_\vartheta(0) &= \int_{\vartheta^\perp} \nu(y)dy + \varepsilon \int_{\vartheta^\perp} (D\nu(y)\cdot\nabla\eta(y) + \Delta\eta(y)\nu(y))\,dy + O(\varepsilon^2)\\ &= \int_{\vartheta^\perp} \nu(y)dy + \varepsilon \int_{\vartheta^\perp} \nu(y)\eta_{\vartheta\vartheta}(y)dy + O(\varepsilon^2),\end{aligned}$$

after integration by parts. We deduce that our goal becomes now to prove the following fact: "$D^2\psi(0)$ must commute with A", where

$$\begin{aligned}A &= ⨍\int_{\vartheta^\perp} \nu(y)\eta_{\vartheta\vartheta}(y)(\vartheta\otimes\vartheta)dyd\vartheta\\ &= ⨍\int_{\vartheta^\perp} \nu(y)(\vartheta\otimes\vartheta)D^2\eta(y)(\vartheta\otimes\vartheta)dyd\vartheta\\ &=2⨍\int_{\vartheta^\perp} \nu(y)(\vartheta\otimes\vartheta)D^2\tilde{\eta}(y)(\vartheta\otimes\vartheta)dyd\vartheta,\end{aligned} \tag{5.3}$$

and $\tilde{\eta}$ is the restriction of η to $B(Re_d, 1)$. For convenience, with a slight abuse of notation, in the following we will call η this very function. Notice therefore that $\eta(x) = 0$ if $x_d \leq 0$. Since the choice of $D^2\psi(0)$ is now completely free, we just have to ensure that the matrix A is not a multiple of the identity matrix, in order to get our claim. To do so, we can for example check that the diagonal elements of A are not equal.

Counterexample in $\mathbb{R}^2$ If $d = 2$ we can just check that $A_{11} - A_{22} \neq 0$. Take $\eta(y_1, y_2) = a^r(y_1)b(y_2)$ for a^r and b positive functions, even on their support and of unitary integrals, supported respectively in $[-r, r]$, $R + [0, 1/2]$ (so that, if r is small, $\mathrm{supp}(\eta) \subseteq B(Re_2, 1)$). Observe that $a^r \rightharpoonup \delta_0$ when $r \to 0$.

Using the notation $\eta_{ij} = (D^2\eta)_{ij}$, we have that

$$A_{11} = 2⨍\int_{\vartheta^\perp} \vartheta_1^2(\vartheta_1^2\eta_{11} + 2\vartheta_1\vartheta_2\eta_{12} + \vartheta_2^2\eta_{22})\nu(y)dyd\vartheta,$$

$$A_{22} = 2⨍\int_{\vartheta^\perp} \vartheta_2^2(\vartheta_1^2\eta_{11} + 2\vartheta_1\vartheta_2\eta_{12} + \vartheta_2^2\eta_{22})\nu(y)dyd\vartheta.$$

Since in dimension $d = 2$, for any $y \in \vartheta^\perp$ one has

$$(\vartheta_1, \vartheta_2) = \left(-\frac{y_2}{|y|}, \frac{y_1}{|y|}\right) \tag{5.4}$$

(notice we could have chosen the opposite vector for ϑ, but by symmetry of the problem, this choice is not relevant), we have

$$
\begin{aligned}
A_{11} =& 2 \fint \int_{\vartheta^\perp} \frac{y_2^2}{|y|^4}(y_2^2\eta_{11} - 2y_1y_2\eta_{12} + y_1^2\eta_{22})\nu(y)dyd\vartheta, \\
A_{22} =& 2 \fint \int_{\vartheta^\perp} \frac{y_1^2}{|y|^4}(y_2^2\eta_{11} - 2y_1y_2\eta_{12} + y_1^2\eta_{22})\nu(y)dyd\vartheta.
\end{aligned} \tag{5.5}
$$

We now need the following

Lemma 5.1.1. *Given* $f\colon \mathbb{R}^d \to \mathbb{R}$, $g\colon \mathbb{S}^{d-1}\colon \mathbb{R}$ *such that the following integrals are well defined, we have*

$$
\int_{\mathbb{S}^{d-1}} \int_{\vartheta^\perp} f(y)g(\vartheta)dyd\vartheta = \int_{\mathbb{R}^d} \frac{f(x)H(x;g)}{|x|}dx,
$$

where

$$
H(x;g) := \int_{\mathbb{S}^{d-2}_{x^\perp}} g(\vartheta)d\mathcal{H}^{d-2}(\vartheta),
$$

$\mathbb{S}^{d-2}_{x^\perp}$ *being the* $d-2$ *dimensional unitary circle having center in* x *and orthogonal to the vector* x *itself.*

Proof. Split the sphere in the usual two charts covering it: $\mathbb{S}^{d-1}_+$ and $\mathbb{S}^{d-1}_-$. In each of these domains we can define an orthonormal tangent smooth vector field to the sphere itself $\Phi(\vartheta) = (\phi_1(\vartheta), \ldots, \phi_{d-1}(\vartheta)) \in M_{d\times(d-1)}$. We can ask each map ϕ_i to be a diffeomorphism with $|\det D\phi_i|= 1$ on the domain. Furthermore, notice that for fixed $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d-1}$, there exists only one $\vartheta \in \mathbb{S}^{d-1}_+$ (or $\mathbb{S}^{d-1}_-$) such that $\Phi(\vartheta)z = y$, so that the map $A_z\colon \mathbb{R}^d \to \mathbb{S}^{d-1}_+$ such that $A_z(y) = \vartheta$ is well defined. Moreover, if $|z|= 1$, $|\det D_z A_z(y)|= 1$ for any y. Using the formula

$$
\int_{\mathbb{R}^d} h(x)dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} h(\xi\rho)\rho^{d-1}d\xi d\rho,
$$

we have

$$
\begin{aligned}
\int_{\mathbb{S}^{d-1}} \int_{\vartheta^\perp} f(y)g(\vartheta)dyd\vartheta &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^{d-1}} f(\Phi(\vartheta)z)g(\vartheta)dzd\vartheta \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-2}} f(\rho\Phi(\vartheta)\xi)g(\vartheta)d\xi d\vartheta \rho^{d-2}d\rho.
\end{aligned} \tag{5.6}
$$

Defining also $(h/r)(x) := h(x/r)$, by (5.6), we deduce

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \int_{\vartheta^\perp} f(y)g(\vartheta)dyd\vartheta \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-2}} f(\rho\Phi(\vartheta)\xi)\left(\frac{g}{\rho} \circ A_\xi\right)(\rho\Phi(\vartheta)\xi)d\xi d\vartheta \rho^{d-2}d\rho.
\end{aligned} \tag{5.7}
$$

Define the change of variables

$$y = \Psi_\xi(\vartheta) := \Phi(\vartheta)\xi.$$

We have that $\Psi_\xi \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ for every $\xi \in \mathbb{S}^{d-2}$. Ψ_ξ is a diffeomorphism for every $\xi \in \mathbb{S}^{d-2}$ and

$$|\det D\Psi_\xi(\vartheta)| = 1$$

for every $\xi \in \mathbb{S}^{d-2}$, $\vartheta \in \mathbb{S}^{d-1}$. By (5.7) and since $|\det D_\xi A_\xi(y)| = |\det \Phi(y)| = 1$ for every $y \in \mathbb{S}^{d-1}$, we get

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} \int_{\vartheta^\perp} f(y) g(\vartheta) dy d\vartheta &= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\rho y) \left(\int_{\mathbb{S}^{d-2}} \left(\frac{g}{\rho} \circ A_\xi \right) (\rho y) d\xi \right) dy \rho^{d-2} d\rho \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\rho y) \left(\int_{\mathbb{S}^{d-2}} g(A_\xi(y)) d\xi \right) dy \rho^{d-2} d\rho \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\rho y) \left(\int_{\mathbb{S}^{d-2}_{y^\perp}} g(\eta) d\eta \right) dy \rho^{d-2} d\rho \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\rho y) G(y) dy \rho^{d-2} d\rho \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\rho y) H(\rho y) dy \rho^{d-2} d\rho \\
&= \int_{\mathbb{R}^d} \frac{f(x) H(x)}{|x|} dx,
\end{aligned}$$

where $\mathbb{S}^{d-2}_{y^\perp}$ is the $d-2$ dimensional unit sphere orthogonal to the direction $y \in \mathbb{R}^d$ and $\int_{\mathbb{S}^{d-2}_{y^\perp}} g(\eta) d\eta =: G(y)$ and therefore $G(y/|y|) =: H(y)$ is the integral of the function g in the very same set $\mathbb{S}^{d-2}_{y^\perp}$. □

Since in our 2-dimensional case, in equations (5.5) we have $g = 1$, we deduce

$$⨍_{\mathbb{S}^1} \int_{\vartheta^\perp} g(y) dy d\vartheta = \frac{\mathscr{H}^{d-2}(\mathbb{S}^{d-2})}{|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} \frac{g(x)}{|x|} dx = \frac{1}{\pi} \int_{\mathbb{R}^d} \frac{g(x)}{|x|} dx.$$

Using these formulas, the difference between diagonal elements of the 2×2- matrix A, up to multiplication by a constant, reads

$$A_{11} - A_{22} = \int_{\mathbb{R}^2} \frac{x_2^2 - x_1^2}{|x|^5} (x_2^2 \eta_{11} - 2x_1 x_2 \eta_{12} + x_1^2 \eta_{22}) \nu(x) dx_1 dx_2.$$

Take $\eta_1(y_1, y_2) = a^r(y_1) b(y_2)$ for a^r and b positive functions, even on their support and of unitary integrals, supported respectively in $[-r, r]$, $R + [0, 1/2]$ for $R > 1/2$, so that, if r is small enough, $\mathrm{supp}(\eta) \subseteq B(Re_2, 1)$. Observe that $a^r \rightharpoonup \delta_0$.
Since

$$g_i(x) := \frac{x_i^2}{|x|^5} \nu(x)$$

can be assumed smooth enough in $\mathbb{R} \setminus \{0\}$, when we send $r \to 0$ the integrals defining A_{11} and A_{22} stay bounded (remembering that η is compactly supported). Defining

$$f(x_1, x_2) := g_2(x) - g_1(x) = \frac{x_2^2 - x_1^2}{|x|^5} v(x_1, x_2),$$

we have that

$$A_{11} - A_{22} = \int_{\mathbb{R}^2} f(x)(x_2^2 a_{11}^r b - 2x_1 x_2 a_1^r b_2 + x_1^2 a^r b_{22}) dx_1 dx_2.$$

We need to be sure that this quantity is not zero as soon as $r \to 0$. Integrating by parts with respect to the variable x_1,

$$A_{11} - A_{22} = \int \left(x_2^2 b \int f_{11} a^r + 2x_2 b_2 \int (f_1 x_1 + f) a^r + b_{22} \int f(x) x_1^2 a^r \right) dx_2$$

Sending $r \to 0$, and (with a slight abuse of notation) still denoting $f = f(0, x_2)$,

$$A_{11} - A_{22} = \int x_2^2 b f_{11} + 2x_2 b_2 f dx_2.$$

Integrating by parts with respect to the remaining variable,

$$A_{11} - A_{22} = \int b \left(f_{11} x_2^2 - 2f - 2x_2 f_2 \right) dx_2.$$

Remembering that x_2 can be assumed strictly positive (due to the fact that η is symmetric and supported away from zero), we have that

$$f_1 = v x_1 \frac{-2|x|^2 - 5x_2^2 + 5x_1^2}{|x|^7} + \frac{x_2^2 - x_1^2}{|x|^5} v_1,$$

then

$$f_{11}|_{(0,x_2)} = -7\frac{v}{|x|^5} + \frac{v_{11}}{|x|^3}$$

and, by similar computations,

$$f_2|_{(0,x_2)} = -3\frac{v}{|x|^4} + \frac{v_2}{|x|^3}.$$

Therefore

$$A_{11} - A_{22} = \int b \left(-3\frac{v}{|x_2|^3} + \frac{v_{11}}{|x_2|} - 2\frac{v_2}{|x_2|^2} \right) dx_2.$$

Under the further assumption of v to be radial, and thanks to the relations

$$v_i = v' \frac{x_i}{|x|} \tag{5.8}$$

and

$$v_{ij} = \frac{v'}{|x|}\delta_{ij} + \left(\frac{v''}{|x|^2} - \frac{v'}{|x|^3} \right) x_i x_j, \tag{5.9}$$

we get that $A_{11} - A_{22} = 0$ for any choice of b if and only if

$$-3\frac{\nu}{|x|^3} - \frac{\nu'}{|x|^2} = 0$$

holds for any $x \in \{0\} \times \mathbb{R}$: indeed, since b is free to be chosen, so is its support (which we defined to be of the form $R + [0, 1/2]$). Still remembering that x_2 can be assumed strictly positive, we get that this is equivalent to

$$3\nu + |x|\nu' = 0,$$

but this can occur only if $\nu \propto |x|^{-3}$, which implies that ν is not a probability distribution, being not even integrable. We deduce that, for any radial target measure ν, in dimension $d = 2$ the sliced Wasserstein gradient flow does not provide an optimal transport map from the initial distribution to the target measure ν itself.

Counterexample in $\mathbb{R}^d$ Obtaining a similar result for any dimension $d \geq 2$ is more difficult: the turning point in $\mathbb{R}^2$ was (5.5), namely the fact that we can uniquely express the orthogonal direction to a given vector thanks to the coordinates of the vector itself, and this was done in (5.4). In higher dimension this "trick" can not longer be applied. The computation machinery will thus be more sophisticated. For starters, still assuming ν to be radial, from relations (5.3) we can now deduce that, for any $1 \leq n \leq d$,

$$\begin{aligned}
A_{nn} &= 2 \fint \int \nu \vartheta_n^2 \left(\sum_{i,j} \vartheta_i \vartheta_j \eta_{ij} \right) dy d\vartheta \\
&= \frac{2}{|\mathbb{S}^{d-1}|} \sum_{i,j} \int_{\mathbb{R}^d} H(x; \vartheta_d^2 \vartheta_j \vartheta_i) \frac{\eta_{ij}}{|x|} \nu(x) dx \\
&= \frac{2}{|\mathbb{S}^{d-1}|} \left(\sum_{i<d} \int H(x; \vartheta_n^2 \vartheta_i^2) \frac{\eta_{ii}\nu(x)}{|x|} + \int H(x; \vartheta_n^2 \vartheta_d^2) \frac{\eta_{dd}\nu(x)}{|x|} \right. \\
&\quad \left. + 2 \sum_{i<j<d} \int H(x; \vartheta_n^2 \vartheta_i \vartheta_j) \frac{\eta_{ij}\nu}{|x|} + 2 \sum_{i<d} \int H(x; \vartheta_n \vartheta_i \vartheta_d) \frac{\eta_{id}\nu}{|x|} \right).
\end{aligned}$$

In particular, for $n = 1$, after integration by parts and letting $r \to 0$ we have

$$\begin{aligned}
\frac{|\mathbb{S}^{d-1}|}{2} A_{11} &= \sum_{i \neq 1,d} \int_{\mathbb{R}} b \partial_{ii} \left(H(x; \vartheta_1^2 \vartheta_i^2) \frac{\nu}{|x|} \right) \Big|_{(0,x_d)} + \int_{\mathbb{R}} b \partial_{11} \left(H(x; \vartheta_1^4) \frac{\nu}{|x|} \right) \Big|_{(0,x_d)} \\
&+ \int_{\mathbb{R}} b \partial_{dd} \left(H(x; \vartheta_1^2 \vartheta_d^2) \frac{\nu}{|x|} \Big|_{(0,x_d)} \right) + \sum_{i \neq j; i,j \neq 1,d} \int_{\mathbb{R}} b \partial_{ij} \left(H(x; \vartheta_1^2 \vartheta_i \vartheta_j) \frac{\nu}{|x|} \right) \Big|_{(0,x_d)} \\
&+ 2 \sum_{i \neq 1,d} \int_{\mathbb{R}} b \partial_{i1} \left(H(x; \vartheta_1^3 \vartheta_i) \frac{\nu}{|x|} \right) \Big|_{(0,x_d)} + 2 \sum_{i \neq 1,d} \int_{\mathbb{R}} b \partial_d \left(\left(\partial_i H(x; \vartheta_1^2 \vartheta_i \vartheta_d) \frac{\nu}{|x|} \right) \Big|_{(0,x_d)} \right) \\
&+ 2 \int_{\mathbb{R}} b \partial_d \left(\left(\partial_1 H(x; \vartheta_1^3 \vartheta_d) \frac{\nu}{|x|} \right) \Big|_{(0,x_d)} \right),
\end{aligned} \tag{5.10}$$

and, for $n = d$

$$\frac{|\mathbb{S}^{d-1}|}{2}A_{dd} = \sum_{i\neq d}\int_{\mathbb{R}} b\partial_{ii}\left(H(x;\vartheta_d^2\vartheta_i^2)\frac{\nu}{|x|}\right)\Big|_{(0,x_d)} + \int_{\mathbb{R}} b\partial_{dd}\left(H(x;\vartheta_d^4)\frac{\nu}{|x|}\Big|_{(0,x_d)}\right) \tag{5.11}$$
$$+ \sum_{i\neq j;i,j\neq d}\int_{\mathbb{R}} b\partial_{ij}\left(H(x;\vartheta_d^2\vartheta_i\vartheta_j)\frac{\nu}{|x|}\right)\Big|_{(0,x_d)} + 2\sum_{i\neq d}\int_{\mathbb{R}} b\partial_d\left(\partial_i\left(H(x;\vartheta_d^3\vartheta_i)\frac{\nu}{|x|}\right)\Big|_{(0,x_d)}\right).$$

We can then continue thanks to the following

Lemma 5.1.2. *It holds*

$$\frac{|\mathbb{S}^{d-1}|}{2}A_{11} = \int_{\mathbb{R}} bC_{(2,2),d}\left((1-d^2)\frac{\nu}{|x|^3} - (1+d)\frac{\nu'}{|x|^2}\right),$$

$$\frac{|\mathbb{S}^{d-1}|}{2}A_{dd} = \int_{\mathbb{R}} b6C_{(2,2),d}(d-1)^2\frac{\nu}{|x|^3},$$

where

$$C_{(2,2),d} = \int_{\mathbb{S}^{d-2}_{e_d^\perp}} \vartheta_i^2\vartheta_j^2 d\mathcal{H}^{d-2}(\vartheta)$$

for any $i \neq j; i, j \neq d$.

Remark 4. *Observe that, using the standard parametrization of the sphere,*

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \vartheta_i^2\vartheta_j^2 d\vartheta \\
&= \int_0^\pi \cdots \int_0^{2\pi} \sin(\varphi_1)^2\cdots\sin(\varphi_{d-1})^2\cdot\sin(\varphi_1)^2\cdots\cos(\varphi_{d-1})^2 d\varphi_1\cdots d\varphi_{d-1} \\
&= \int_0^\pi \cdots \int_0^\pi \sin(\varphi_1)^4\cdots\sin(\varphi_{d-2})^4 d\varphi_1\cdots d\varphi_{d-2}\int_0^{2\pi}\sin(\varphi_{d-1})^2\cos(\varphi_{d-1})^2 d\varphi_{d-1} \\
&= \frac{1}{3}\int_0^\pi \cdots \int_0^{2\pi} \sin(\varphi_1)^4\cdots\sin(\varphi_{d-1})^4 d\varphi_1\cdots d\varphi_{d-1} \\
&= \int_{\mathbb{S}^{d-1}} \vartheta_i^4 d\vartheta =: C_{(4),d},
\end{aligned}$$

so that $C_{(4),d} = 3C_{(2,2),d}$.

Proof. Recalling the notation used in lemma 5.1.1, with a little abuse of notation, denoting $H(x, g) = H$, observe we have

$$\begin{aligned}
\partial_{jk}\left(\frac{H}{|x|}\nu\right)\Big|_{(0,x_d)} &= \left(\partial_k\left(\frac{\partial_j H}{|x|}\nu + \frac{H}{|x|}\partial_j\nu - \frac{H\nu}{|x|^3}x_j\right)\right)\Big|_{(0,x_d)} \\
&= \partial_{jk}H\frac{\nu}{|x|} + H\frac{\partial_{jk}\nu}{|x|} - \delta_{jk}H\frac{\nu}{|x|^3}.
\end{aligned}$$

Since ν is radial, we make use of (5.8) and (5.9) obtaining

$$\partial_{jk}\left(\frac{H}{|x|}\nu\right)\Big|_{(0,x_d)} = \partial_{jk}H(0,x_d)\frac{\nu}{|x_d|} + \frac{H(0,x_d)}{|x_d|^2}\left(\nu' - \frac{\nu}{|x_d|}\right)\delta_{jk}, \tag{5.12}$$

and

$$\partial_{j}\left(\frac{H}{|x|}\nu\right)\Big|_{(0,x_d)} = \partial_{j}H(0,x_d)\frac{\nu}{|x_d|}. \tag{5.13}$$

Now we proceed with an explicit computation of the terms $\partial_{jk}H(x,g)\Big|_{(0,x)}$ and $\partial_{j}H(x,g)\Big|_{(0,x)}$. The result will thus follow by computing the expressions (5.10) and (5.11) making use of relations (5.12) and (5.13). We start by computing $\partial_{j}H(x,g)\Big|_{(0,x)}$: since the matrix

$$R_1^j = \begin{pmatrix} \mathbb{I}_{(j-1)\times(j-1)} & & & \\ & \frac{x}{\sqrt{x^2+h^2}} & & -\frac{h}{\sqrt{x^2+h^2}} \\ & & \mathbb{I}_{(d-j-1)\times(d-j-1)} & \\ & \frac{h}{\sqrt{x^2+h^2}} & & \frac{x}{\sqrt{x^2+h^2}} \end{pmatrix}$$

provides a rotation that alignes the direction $e_j\frac{h}{\sqrt{x^2+h^2}} + e_d\frac{x}{\sqrt{x^2+h^2}}$ into the direction e_d, we have

$$\begin{aligned}
\partial_j H(xe_d, g) &= \lim_{h\to 0}\frac{H(xe_d + he_j, g) - H(xe_d, g)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_{\mathbb{S}^{\perp}_{x+he_j}} g(\vartheta)d\vartheta - \int_{\mathbb{S}^{\perp}_{\hat{x}}} g(u)du\right) \\
&== \lim_{h\to 0}\frac{1}{h}\left(\int_{\mathbb{S}^{\perp}_{e_d}} g((R_1^j)^{-1}u)du - \int_{\mathbb{S}^{\perp}_{e_d}} g(u)du\right) \\
&= \int_{\mathbb{S}^{\perp}_{e_d}} \lim_{h\to 0}\frac{1}{h}\left(g\begin{pmatrix} u_1 \\ \vdots \\ \frac{x}{\sqrt{x^2+h^2}}u_j \\ \vdots \\ -\frac{h}{\sqrt{x^2+h^2}}u_j \end{pmatrix} - g\begin{pmatrix} u_1 \\ \vdots \\ 0 \end{pmatrix}\right)
\end{aligned}$$

Thanks to a direct computation, we can then check that, for $i \neq j$ and $i, j \neq 1, d$, the following expressions hold true:

$$
\begin{aligned}
&\partial_j H(xe_d, \vartheta_d^3 \vartheta_j) = 0,\\
&\partial_j H(xe_d, \vartheta_1^2 \vartheta_i \vartheta_j) = -\frac{C_{(2,2),d}}{|x|},\\
&\partial_j H(xe_d, \vartheta_1^2 \vartheta_j^2) = 0,\\
&\partial_j H(xe_d, \vartheta_d^2 \vartheta_j^2) = 0,\\
&\partial_j H(xe_d, \vartheta_1^2 \vartheta_i \vartheta_j) = 0,\\
&\partial_1 H(xe_d, \vartheta_1^3 \vartheta_d) = -\frac{C_{(4),d}}{|x|},\\
&\partial_1 H(xe_d, \vartheta_1^3 \vartheta_i) = 0,\\
&\partial_i H(xe_d, \vartheta_1^2 \vartheta_i \vartheta_d) = -\frac{C_{(2,2),d}}{|x|},\\
&\partial_d H(xe_d, g) = 0 \ \text{ for any } g.
\end{aligned}
$$

We now continue with the explicit computation of the terms $\partial_{ii} H(x,g)\Big|_{(0,x)}$: to do so, we start by computing $\partial_i H(xe_d + me_i, g)$. Since the matrix

$$
R_2^i = \begin{pmatrix}
\mathbb{I}_{(i-1)\times(i-1)} & & & \\
 & \cos\alpha & & -\sin\alpha \\
 & & \mathbb{I}_{(d-i-1)\times(d-i-1)} & \\
 & \sin\alpha & & \cos\alpha
\end{pmatrix}
$$

with

$$
\cos\alpha = \frac{m(m+h)+x^2}{\sqrt{x^2+m^2}\sqrt{x^2+(m+h)^2}} \quad \sin\alpha = \frac{hx}{\sqrt{x^2+m^2}\sqrt{x^2+(m+h)^2}}
$$

provides a rotation that alignes the direction $\left(e_i \frac{m+h}{\sqrt{x^2+(m+h)^2}} + e_d \frac{x}{\sqrt{x^2+(m+h)^2}}\right)$ into the direction $\left(e_i \frac{m}{\sqrt{x^2+m^2}} + e_d \frac{x}{\sqrt{x^2+m^2}}\right)$, we have

$$
\begin{aligned}
\partial_i H(xe_d + me_i, g) &= \lim_{h\to 0} \frac{H(x+(m+h)e_i, g) - H(xe_d + me_i, g)}{h}\\
&= \int_{\mathbb{S}^{\perp}_{xe_d+me_i}} \lim_{h\to 0} \frac{1}{h} \left(g \begin{pmatrix} u_1 \\ \vdots \\ \cos\alpha u_j + \sin\alpha u_d \\ \vdots \\ \sin\alpha u_j - \cos\alpha u_d \end{pmatrix} - g \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \right) du.
\end{aligned}
$$

Choosing $g = \vartheta_i^2\vartheta_1^2$, we obtain

$$\partial_i H(xe_d + me_i, g) = \frac{2x}{m^2 + x^2}\int_{\mathbb{S}^{\perp}_{xe_d+me_i}} u_1^2 u_i u_d du =: \frac{2x}{m^2 + x^2}\int_{\mathbb{S}^{\perp}_{xe_d+me_i}} f(u)du.$$

Since for $g = \vartheta_i^2\vartheta_1^2$, it holds $\partial_i H = 0$, we have that

$$\begin{aligned}
\partial_{ii} H(xe_d, \vartheta_i^2\vartheta_1^2) &= \lim_{m\to 0}\frac{\partial_i H(xe_d + me_i)}{m} \\
&= \int_{\mathbb{S}^{\perp}_{e_d}} f((R_1^i)^{-1}v)dv \\
&= \int_{\mathbb{S}^{\perp}_{e_d}} \lim_{m\to 0}\frac{2x}{m(m^2+x^2)}\left(-v_1^2\frac{xv_i}{\sqrt{x^2+m^2}}\frac{mv_i}{\sqrt{x^2+m^2}}\right)dv \\
&= -\frac{2C_{(2,2),d}}{|x|^2}
\end{aligned} \tag{5.14}$$

Choosing $g = \vartheta_i^2\vartheta_d^2$, we obtain

$$\partial_i H(xe_d + me_i, g) = -\frac{2x}{m^2 + x^2}\int_{\mathbb{S}^{\perp}_{xe_d+me_i}} (u_i^3 u_d - u_d^3 u_i)du.$$

Since, for $g = \vartheta_i^2\vartheta_d^2$, it holds $\partial_i H = 0$, with a computation similar to the one done in (5.14), we have that

$$\partial_{ii} H(xe_d, \vartheta_i^2\vartheta_d^2) = \lim_{m\to 0}\frac{\partial_i H(xe_d + me_i)}{m} = \frac{2C_{(4),d}}{|x|^2}.$$

Finally, choosing $g = \vartheta_1^4$, we obtain

$$\partial_1 H(xe_d + me_1, g) = \frac{4x}{x^2 + m^2}\int_{\mathbb{S}^{\perp}_{xe_d+me_1}} u_1^3 u_d.$$

Since, for this $g = \vartheta_1^4$, it holds $\partial_1 H = 0$, we have that

$$\partial_{11} H(xe_d, \vartheta_1^4) = \lim_{m\to 0}\frac{\partial_1 H(xe_d + me_1)}{m} = -\frac{4C_{(4),d}}{|x|^2}.$$

Finally, notice that

$$\partial_{dd} H(xe_d, g) = 0$$

for any g. To conclude, we compute the second derivative of the form ∂_{ij} for $i \neq j$.

We start by computing the expression $\partial_j H(xe_d + me_i, g)$. Since the matrix

$$R_3^{j,i} = \begin{pmatrix} \mathbb{I}_{(j-1)\times(j-1)} & & & & & \\ & 1-\frac{h^2}{S} & & -\frac{hm}{S} & & -\frac{h}{R} \\ & & \mathbb{I}_{(i-1-j)\times(i-1-j)} & & & \\ & -\frac{hm}{S} & & 1-\frac{m^2}{S} & & -\frac{m}{R} \\ & & & & \mathbb{I}_{(d-1-j)\times(d-1-j)} & \\ & \frac{h}{R} & & \frac{m}{R} & & 1-\frac{h^2+m^2}{S} \end{pmatrix}$$

with

$$R := \sqrt{x^2+m^2+h^2}, \quad S := R(R+x)$$

provides a rotation that alignes the direction $\left(e_j\frac{h}{R} + e_i\frac{m}{R} + e_d\frac{x}{R}\right)$ into the direction e_d, we have

$$\begin{aligned} \partial_j H(xe_d + me_i, g) &= \lim_{h\to 0} \frac{H(xe_d + me_i + he_j, g) - H(xe_d + me_i, g)}{h} \\ &= \int_{\mathbb{S}^{\perp}_{e_d}} \lim_{h\to 0} \frac{1}{h}\left(g\left(\left(R_3^{ji}\right)^{-1} v\right) - g\left(\left(R_1^{i}\right)^{-1} v\right)\right) \\ &= \int_{\mathbb{S}^{\perp}_{e_d}} \lim_{h\to 0} \frac{1}{h}\left(g\begin{pmatrix} v_1 \\ \vdots \\ \left(1-\frac{h^2}{S}\right)v_j - \frac{hm}{S}v_i \\ \vdots \\ -\frac{hm}{S}v_j + \left(1-\frac{m^2}{S}\right)v_i \\ \vdots \\ -\frac{h}{R}v_j - \frac{m}{R}v_i \end{pmatrix} - g\begin{pmatrix} v_1 \\ \vdots \\ \frac{x}{r}v_i \\ \vdots \\ v_j \\ \vdots \\ \frac{-m}{r}v_i \end{pmatrix}\right), \end{aligned}$$

with

$$r := \sqrt{x^2+m^2}.$$

Choose $g = \vartheta_1^2\vartheta_i\vartheta_j$ and we obtain

$$\partial_j H(xe_d + me_i, g) = -\frac{mC_{(2,2),d}}{(r+x)r}\left(2 - \frac{m^2}{(r+x)r}\right).$$

Since, for $g = \vartheta_1^2\vartheta_i\vartheta_j$, it holds $\partial_i H = 0$, we have that

$$\partial_{ij} H(xe_d, \vartheta_1^2\vartheta_i\vartheta_j) = \lim_{m\to 0} \frac{\partial_j H(xe_d + me_i)}{m} = -\frac{C_{(2,2),d}}{|x|^2}$$

.

Similarly, choose $g = \vartheta_1^3\vartheta_i$. We obtain

$$\partial_1 H(xe_d + me_i, g) = -\frac{m}{(r+x)r}\left(C_{(4),d} + 3\left(1 - \frac{m^2}{(r+x)r}C_{(2,2),d}\right)\right).$$

Notice that also for this $g = \vartheta_1^3\vartheta_i$, it holds $\partial_1 H = 0$. Therefore, recalling that $C_{(4),d} = 3C_{(2,2),d}$, we have

$$\partial_{1i} H(xe_d, \vartheta_1^3\vartheta_i) = \lim_{m\to 0}\frac{\partial_1 H(xe_d + me_i)}{m} = -\frac{C_{(4),d}}{|x|^2}.$$

Finally choose $g = \vartheta_d^2\vartheta_i\vartheta_j$ and obtain

$$\partial_j H(xe_d + me_i, g) = \frac{1}{r^2}\left(2mC_{(2,2),d} + o(m)\right).$$

For this $g = \vartheta_d^2\vartheta_i\vartheta_j$ it holds $\partial_j H = 0$, so that we have

$$\partial_{ij} H(xe_d, \vartheta_d^2\vartheta_i\vartheta_j) = \lim_{m\to 0}\frac{\partial_j H(xe_d + me_i)}{m} = \frac{2C_{(4),d}}{|x|^2}.$$

□

Thanks to the above result, we deduce that

$$\frac{|\mathbb{S}^{d-1}|}{2}(A_{11} - A_{dd}) = \int_{\mathbb{R}} bC_{(2,2),d}\left[\left((1-d^2) - 6(d-1)^2\right)\frac{\nu}{|x|^3} - (d+1)\frac{\nu'}{|x|^2}\right],$$

and therefore, by the arbitrariness of b, $A_{11} - A_{dd} = 0$ if and only if

$$M_{1,d}\frac{\nu}{|x|^3} + M_{2,d}\frac{\nu'}{|x|^2} = 0,$$

for any $x \in \{0\} \times \mathbb{R}^{d-1}$ for $M_{i,d} > 0$, but this can occur only if

$$\nu = \frac{c}{|x|^{\frac{M_{1,d}}{M_{2,d}}}}$$

for $c \in \mathbb{R}$. This implies that ν is not integrable on the whole $\mathbb{R}^d$ (since it is a negative power) and therefore it is not a probability measure.

Chapter 6

Global and asymptotic properties of the flow

This chapter is devoted to the study of the convergence of the SWGF. We are interested in showing that (at least) two different situations can happen. The flow

- could converge, but not to the target measure,

- could converge to the target measure ν.

In the first section of this chapter we will give some estimates on the p-moments of the gradient flow, which will turn out to be useful in the following. Then we will present the example in which the flow doesn't converge to the target measure. After that, we will discuss the main example of SWGF we studied, namely the Gaussian-target case, in which the flow converges to the target measure itself. Finally we will treat the more general case in which the initial and target measures are radial functions: our claim is that in that case the only radial stationary point must be the target measure itself.

6.1 Bounds on the moments along the flow

In this section we provide some bounds to the p-moments and the ∞-moment of the sliced Wasserstein gradient flow. We start by computing the derivative with respect to time of the p-moment of the sliced Wasserstein gradient flow ρ_t

associated with a given target measure ν: defining $\mathcal{M}_p(\rho) := \int_{\mathbb{R}^d} |x|^p d\rho(x)$, we have

$$
\begin{aligned}
\partial_t \mathcal{M}_p(\rho) &= \partial_t \int \frac{|x|^p}{p} \rho \\
&= -\int \frac{|x|^p}{p} \nabla(v\rho) \\
&= \int |x|^{p-2} x \cdot v\rho \\
&= \int |x|^{p-2} x \cdot \left(\fint (T_\vartheta(x\cdot\vartheta) - x\cdot\vartheta)\vartheta \right) \rho \\
&= \fint \int |x|^{p-2} x\cdot\vartheta (T_\vartheta(x\cdot\vartheta) - x\cdot\vartheta) \rho d\vartheta \\
&\leq -\fint \int |x|^{p-2} \frac{|x\cdot\vartheta|^2}{2} \rho + \fint \int |x|^{p-2} \frac{|T_\vartheta|^2}{2} \rho d\vartheta \\
&= -\frac{1}{2} \int |x|^{p-2} \left(\fint |x\cdot\vartheta|^2 \right) \rho d\vartheta + \frac{1}{2} \fint \int |x|^{p-2} |T_\vartheta(x\cdot\vartheta)|^2 \rho d\vartheta \\
&\leq -\int |x|^{p-2} \frac{|x|^2}{2d} \rho + \frac{1}{2} \fint \left[\left(\int |x|^p \rho \right)^{\frac{p-2}{p}} \left(\int |T_\vartheta(x\cdot\vartheta)|^p \rho \right)^{\frac{2}{p}} \right] d\vartheta \\
&\leq -\frac{1}{2d} \mathcal{M}_p(\rho) + \frac{1}{2} \mathcal{M}_p(\rho)^{1-\frac{2}{p}} \left(\int \fint |y\cdot\vartheta|^p \nu \right)^{\frac{2}{p}} \\
&\leq -\frac{1}{2d} \mathcal{M}_p(\rho) + \frac{1}{2} \mathcal{M}_p(\rho)^{1-\frac{2}{p}} c_{p,d}^{\frac{2}{p}} \mathcal{M}_p(\nu)^{\frac{2}{p}},
\end{aligned}
\tag{6.1}
$$

where we used the inequalities

$$x\cdot\vartheta(T_\vartheta(x\cdot\vartheta) - x\cdot\vartheta) \leq \frac{|T_\vartheta(x\cdot\vartheta)|^2}{2} - \frac{|x\cdot\vartheta|^2}{2},$$

along with Jensen and Hölder's inequalities, the equality

$$\fint |x\cdot\vartheta|^2 d\vartheta = \frac{|x|^2}{d},$$

and the inequality

$$\fint |x\cdot\vartheta|^p d\vartheta \leq |x|^p \fint |\vartheta|^p d\vartheta =: c_{p,d}|x|^p.$$

From (6.1) we deduce that

$$\mathcal{M}_p(\rho_t) \leq \max\left\{ \mathcal{M}_p(\rho_0), d^{p/2} c_{p,d} \mathcal{M}_p(\nu) \right\}. \tag{6.2}$$

Indeed, (6.1) gives a differential inequality of the form

$$x' \leq -\frac{1}{2d}x + x^{1-2/p} \frac{C^{2/p}}{2},$$

and the left hand side is negative for

$$x \geq Cd^{p/2}.$$

Moreover, denoting by

$$\mathrm{diam}(\mu) := \inf\{r > 0 \colon \mathrm{supp}(\mu) \subset C, \text{ where } C \text{ is a ball of radius } r\},$$

and recalling that $\mathrm{diam}(\mu) = \lim_{p\to\infty} \mathcal{M}_p^{1/p}(\mu)$, we can see that

$$\begin{aligned}\mathrm{diam}(\rho_t) = \lim_{p\to\infty} \mathcal{M}_p(\rho)^{1/p} &\leq \max\left\{\lim_{p\to\infty} \mathcal{M}_p(\rho_0)^{1/p}, d^{1/2} \lim_{p\to\infty} c_{p,d}^{1/p} \mathcal{M}_p(\nu)^{1/p}\right\} \\ &= \max\left\{\mathrm{diam}(\rho_0), \sqrt{d}\mathrm{diam}(\nu)\right\},\end{aligned}$$

since

$$\lim_{p\to\infty} c_{d,p} = \lim_{p\to\infty} \|\vartheta\|_{L^p(\mathbb{S}^{d-1})} = \|\vartheta\|_{L^\infty(\mathbb{S}^{d-1})} = 1$$

Obviously this estimate is vacuous when the initial and target measures are not compactly supported. Notice however that in this case we can prove a slightly better estimate: consider $B(0, R)$ to be the smallest ball centered at zero in which ν, the target measure, is supported. Observe that in this model, $|T_\vartheta(x \cdot \vartheta)| \leq R$ for every $x \in \mathbb{R}^d$. Let $x \in \mathbb{R}^d$, with $|x| = c$. By the symmetry in this construction, we can consider $x = ce_1$ without loss of generality. We want to compute what is the maximum value of c for which the velocity field v_t is pointing outwards with respect to the ball $B(0, c)$, *i.e.* the maximum radius of the trajectories. To do so, we can just look at the behavior of the component e_1 of the velocity field:

$$\begin{aligned} v_t^1(x) &= \fint (T_\vartheta(x \cdot \vartheta) - x \cdot \vartheta)\, \vartheta_1 \\ &\leq \fint R|\vartheta_1| - c\vartheta_1^2 \\ &= R \fint |\vartheta_1| - c \fint \vartheta_1^2 \\ &= R \fint |\vartheta_1| - c \fint |\vartheta_1|^2. \end{aligned}$$

Therefore, if $c \geq \frac{R \fint |\vartheta_1|}{\fint |\vartheta_1|^2}$, $v_t^1(x)$ is negative and hence we deduce that in this setting, if the initial datum ρ_0 is a compactly supported measure,

$$\mathrm{diam}(\rho_t) \leq \max\left\{\mathrm{diam}(\rho_0), \frac{b_{1,d}}{b_{2,d}} \mathrm{diam}(\mu)\right\},$$

where $b_{p,d}$ is the p-barycenter of the d-dimensional hemisphere:

$$b_{p,d} := \fint |\vartheta_1|^p d\vartheta.$$

This value can be computed explicitly: denoting by $w_d(r)$ the surface area of the d-dimensional sphere of radius r,

$$\begin{aligned}
b_{p,d} &= 2\frac{\int_0^1 y^p w_{d-1}(\sqrt{1-y^2})d\sigma}{w_d(1)} \\
&= 2\frac{\int_0^1 y^p w_{d-1}(\sqrt{1-y^2})\frac{dy}{\sqrt{1-y^2}}}{w_d(1)} \\
&= 2\Gamma\left(\frac{d}{2}\right)\frac{\int_0^1 y^p \frac{2\pi^{\frac{d-1}{2}}(1-y^2)^{\frac{d-2}{2}}}{\Gamma(\frac{d-1}{2})}\frac{dy}{\sqrt{1-y^2}}}{2\pi^{d/2}} \\
&= 2\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})}\frac{\int_0^1 y^p(1-y^2)^{\frac{d-3}{2}}dy}{\pi^{1/2}} \\
&= \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})}\frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{p+1}{2})}{2\Gamma(\frac{d+p}{2})} \\
&= \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{d}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{d+p}{2})}.
\end{aligned}$$

So in our case:

$$b_{1,d} = \frac{\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d+1}{2})},$$

and

$$b_{2,d} = \frac{\Gamma(\frac{d}{2})}{2\Gamma(\frac{d}{2}+1)} = \frac{1}{2d}.$$

One can observe that

$$\frac{2d\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d+1}{2})} < \sqrt{d}$$

for all values of $d > 2$, proving that this estimate is more accurate than the previous one.

6.2 The flow may not converge to the target

We consider the following construction in $\mathbb{R}^2$: let the starting measure be

$$\rho_0 = \frac{\delta_{(-1,0)}}{2} + \frac{\delta_{(1,0)}}{2}$$

and the target measure

$$\nu = \frac{\delta_{(0,a)}}{2} + \frac{\delta_{(0,-a)}}{2},$$

for a certain $a > 0$ to be defined. We can show that there exists $a > 0$ such that the configuration ρ_0 is stable, *i.e.* the velocity field associated to the SWGF at time zero is $v_0 = 0$ and therefore the gradient flow is $\rho_t = \rho_0$ for any $t \geq 0$, so that the flow does not converge to the target. Calling T_ϑ the optimal transport map from ρ_ϑ and ν_ϑ, we have

$$\begin{aligned}
v_0 &= \fint (T_\vartheta(x \cdot \vartheta) - x \cdot \vartheta)\, \vartheta \\
&= 2 \int_0^{\frac{\pi}{2}} (\cos\vartheta - a \sin\vartheta)(\cos\vartheta, \sin\vartheta) - 2 \int_{-\frac{\pi}{2}}^{0} (\cos\vartheta + a\sin\vartheta)(\cos\vartheta, \sin\vartheta) \\
&= 2 \int_0^{\frac{\pi}{2}} \begin{pmatrix} \cos^2\vartheta - a \sin\vartheta\cos\vartheta \\ \cos\vartheta\sin\vartheta - a\sin^2\vartheta \end{pmatrix} - 2 \int_{\frac{-\pi}{2}}^{0} \begin{pmatrix} \cos^2\vartheta + a\cos\vartheta\sin\vartheta \\ \cos\vartheta\sin\vartheta + a\sin^2\vartheta \end{pmatrix} \\
&= 2 \begin{pmatrix} 0 \\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2\vartheta)}{2}\operatorname{sgn}(\vartheta) - a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\vartheta \end{pmatrix}.
\end{aligned}$$

All we have to do now is imposing the second row of the above matrix equal to zero:

$$a = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2\vartheta)}{2}\operatorname{sgn}(\vartheta) = \frac{2}{\pi}.$$

6.3 The Gaussian-target case

We prove that if the target measure ν used in the definition of

$$v(t,x) = -\nabla_x \frac{\delta SW(\rho_t,\nu)}{\delta\rho} = \fint_{\mathbb{S}^{d-1}} (T_{t,\vartheta}(x\cdot\vartheta) - x\cdot\vartheta)d\vartheta$$

is the gaussian distribution, then $\lim_{t\to\infty} S_t\rho := \rho_\infty$ exists and coincides with ν, the guassian distribution itself.
We know that $\partial_t\rho + \nabla(v\rho) = 0$:

$$\begin{aligned}
\partial_s \int \rho_s \log\rho_s dx &= \int [\partial_s\rho_s \log\rho_s + \partial_t\rho_s]dx \\
&= -\int [\nabla(v_s\rho_s)\log\rho_s - \nabla(v_s\rho_s)]dx \\
&= -\int \nabla(\rho_s v_s)\log\rho_s dx \\
&= \int \rho_s v_s \frac{1}{\rho_s}\nabla\rho_s dx \\
&= \int v_s\nabla\rho_s dx \\
&= \fint \left[\int \nabla\rho_s\vartheta(T_{s,\vartheta}(x\cdot\vartheta) - x\cdot\vartheta)dx\right] d\vartheta \\
&= \fint\int \rho'_{s,\vartheta}(r)(T_{s,\vartheta}(r) - r)drd\vartheta \\
&\le \fint [\mathcal{E}(\nu_\vartheta) - \mathcal{E}(\rho_\vartheta)]\, d\vartheta,
\end{aligned} \tag{6.3}$$

where $\mathcal{E}(\mu) = \int \mu\log\mu$ is the entropy of the (probability) measure μ. The last inequality comes from the geodesic convexity of the entropy in dimension $d = 1$: consider the two one-dimensional measures $\rho_{s,\vartheta}$ and ν_ϑ and define the geodesic $\omega : [0,1] \mapsto \mathcal{P}(\mathbb{R})$ with respect to the W_2 distance connecting them. Since ω is a geodesic, we know by (3.3.4) that it solves the the continuity equation

$$\partial_t\omega + \partial_r(\mathbf{v}_t\omega) = 0$$

where the velocity field is $\mathbf{v}(t,r) = T_{s,\vartheta}(S_t^{-1}(r)) - S_t^{-1}(r)$, with $S_t(y) := (1-t)y + tT_{s,\vartheta}(y)$ and $T_{s,\vartheta}$ still being the transport map between $\rho_{\vartheta,s}$ and ν^ϑ. We thus have

$$\begin{aligned}
\partial_t \int_{\mathbb{R}} \omega_t\log\omega_t dr &= \int_{\mathbb{R}} \partial_t\omega_t\log\omega_t dr \\
= -\int_{\mathbb{R}} \partial_r(\mathbf{v}_t\omega_t)\log\omega_t dr &= \int_{\mathbb{R}} \mathbf{v}_t\partial_r\omega_t \\
= \int_{\mathbb{R}} [T_{s,\vartheta}(S_t^{-1}(r)) - S_t^{-1}(r)]\partial_r\omega_t dr,
\end{aligned}$$

and valuating at $t = 0$ we obtain:

$$\mathcal{E}'(\rho_{s,\vartheta}) = \int_{\mathbb{R}} \rho'_{s,\vartheta}(r)(T_{s,\vartheta}(r) - r)dr.$$

Since the entropy is geodetically convex, and following the property that for a convex function f one has $f'(0) \leq f(1) - f(0)$, we have that

$$\mathcal{E}'(\omega(0)) = \mathcal{E}'(\rho_{\vartheta,s}) = \int_{\mathbb{R}} \rho'_{\vartheta,s}(r)(T_{s,\vartheta}(r) - r)dr \leq \mathcal{E}(\omega(1)) - \mathcal{E}(\omega(0)) = \mathcal{E}(\nu_\vartheta) - \mathcal{E}(\rho_{s,\vartheta}),$$

thus the last inequality in (6.3) is proved. Consider then the following:

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} \frac{|x|^2}{2}\rho_s &= \int_{\mathbb{R}^d} \frac{|x|^2}{2}\partial\rho_s \\
&= \fint \int_{\mathbb{R}^d} (x \cdot \vartheta)(T_{s,\vartheta}(x \cdot \vartheta) - x \cdot \vartheta)\rho_s(x) = \fint \int_{\mathbb{R}} r(T_{s,\vartheta}(r) - r)\rho_{s,\vartheta}(r)
\end{aligned}$$

and, using $r(R - r) = R^2/2 - r^2/2 - |R - r|^2/2$, we obtain

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} \frac{|x|^2}{2}\rho_s &= \fint \int_{\mathbb{R}} \frac{|T_{s,\vartheta}(r)|^2}{2}\rho_{s,\vartheta}(r) - \fint \int_{\mathbb{R}} \frac{r^2}{2}\rho_{s,\vartheta}(r) - \frac{1}{2}\fint \int_{\mathbb{R}} |T_{s,\vartheta}(r) - r|^2\rho_{s,\vartheta}(r) \\
&= \fint \int \frac{|T_{s,\vartheta}(r)|^2}{2}\rho_{s,\vartheta}(r) - \fint \int \frac{r^2}{2}\rho_{s,\vartheta}(r) - \frac{1}{2}SW_2^2(\rho_s, \nu).
\end{aligned} \tag{6.4}$$

Now notice that the gaussian measure minimizes the functional

$$\rho \mapsto \int \left[\rho \log \rho + \rho\frac{|x|^2}{2}\right].$$

Indeed $f(s) = s \log s$ is a convex function, and thus $f(t) \geq f(s) + f'(s)(t - s) = f(s) + (\log s + 1)(t - s)$. Thus, for every ρ, and if ν is the gaussian distribution,

$$\begin{aligned}
\int \rho \log \rho &\geq \int \nu \log \nu + \int \log \nu(\rho - \nu) + \int (\rho - \nu) \\
&= \int \nu \log \nu - \int \frac{|x|^2}{2}\rho + \int \frac{|x|^2}{2}\nu.
\end{aligned}$$

Therefore

$$\int \left[\rho \log \rho + \frac{|x|^2}{2}\rho\right] \geq \int \left[\nu \log \nu + \frac{|x|^2}{2}\nu\right] = 0.$$

Summing (6.3) and (6.4), and using the fact that ν_ϑ is still a (one dimensional) gaussian, we obtain

$$\begin{aligned}
&\partial_t\left(\mathcal{E}(\rho_s)+\int\frac{|x|^2}{2}\rho_s\right)\\
&\leq \fint(\mathcal{E}(\nu_\vartheta)-\mathcal{E}(\rho_{s,\vartheta}))+\fint\int\frac{|r|^2}{2}\nu_\vartheta-\fint\int\frac{|r|^2}{2}\rho_{s,\vartheta}-\frac{1}{2}SW_2^2(\rho_s,\nu)\\
&=\fint\left[\left(\mathcal{E}(\nu_\vartheta)+\int\frac{|r|^2}{2}\nu_\vartheta\right)-\left(\mathcal{E}(\rho_{s,\vartheta})+\int\frac{|r|^2}{2}\rho_{s,\vartheta}\right)\right]-\frac{1}{2}SW_2^2(\rho_s,\nu)\\
&\leq -\frac{1}{2}SW_2^2(\rho_s,\nu).
\end{aligned}$$

So, finally,

$$\begin{aligned}
\int_0^T SW_2^2(\rho_t,\nu)dt &\leq -\left[\mathcal{E}(\rho_T)-\mathcal{E}(\rho_0)\right]-\int\left[\frac{|x|^2}{2}\rho_T-\frac{|x|^2}{2}\rho_0\right]dt\\
&\leq \mathcal{E}(\rho_0)+\int\frac{|x|^2}{2}\rho_0 := C
\end{aligned}$$

for every $T \geq 0$. Thus

$$\int_0^\infty SW_2^2(\rho_t,\nu)dt \leq C,$$

and therefore, since SW_2 is decreasing along the flow,

$$\lim_{t\to\infty} SW_2(\rho_t,\nu)=0.$$

Since, by (6.1), we have an information about the compactness of the measures set $\{\rho_t\}_{t\geq 0}$ in the space $\mathcal{P}(\mathbb{R}^d)$, W_p for every $p \geq 2$ (recall theorem 3.1.3),we can deduce that

$$\rho_t \xrightarrow[t\to\infty]{W_p} \nu$$

for every $p \geq 2$.

6.4 The radial-target case

In the present section we will use the notation and the results contained in [5] for what concerns the theory of dynamical systems in Wasserstein spaces. Recall that any gradient flow is a dynamical system and that the notion of *curve of maximal slope* is precisely the one needed in order to discuss these topics properly: we can cite [2] for a detailed presentation.

Beside the lucky Gaussian target case, we have another interesting situation in which we claim that the only stationary point should be the target measure itself, precisely the one in which the gradient flow is well posed (it admits existence and uniqueness for any $t \geq 0$) and both the starting and target measure are radial. In our setting, the function

$$E(\rho) = \frac{SW_2^2}{2}(\rho, \nu)$$

is a free energy of the dynamical system and the function

$$G(\rho) = \left(\int \left| \nabla \frac{\delta SW_2^2(\rho, \nu)}{2\delta\rho} \right|^2 d\rho \right)^{1/2}$$

is a weak upper gradient both satisfying assumptions of theorem 2.12 of [5]. The sliced Wasserstein gradient flow $(\rho_t)_{t\geq 0}$ is therefore a curve of maximal slope. As such, it must converge to the set of ω-limits of the system, which is a subset of the set of stationary points.

Remark 7. *Notice that one can prove the following two conditions to be equivalent:*

1. $\rho > 0$ *a.e. and* $G(\rho) = 0$,
2. $\rho = \nu$.

We could therefore be tempted to say that ν *is the only stationary point, but we are not considering all the stationary points that are not strictly positive.*

By the theory of dynamical systems developed in [5], we know that the set of stationary points coincides with the zeros of the function G, namely the set

$$\mathcal{S} := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \text{ s. t. } \frac{\bar{T}^{\mu\to\nu}(x)}{d} := \fint T^{\mu_\vartheta \to \nu_\vartheta}(x \cdot \vartheta)\vartheta d\vartheta = \frac{x}{d} \right\}.$$

If ν is radial, we claim that the only radial measure in $\mathcal{S}$ is ν itself. Indeed, suppose that $\mu \in \mathcal{S}$ is radial. For starters, recall the *coarea formula*

$$\int_E h(\vartheta)|\nabla f(\vartheta)| d\vartheta = \int_{\mathbb{R}} \left(\int_{E \cap f^{-1}(s)} h(y) d\mathcal{H}^{d-1} \right) ds,$$

where $E = \mathbb{S}^{d-1}$, $f(\vartheta) = x \cdot \vartheta$ and $h(\vartheta) = \frac{T(x\cdot\vartheta)\vartheta}{|\nabla f(\vartheta)|}$. We have therefore (notice that for any ϑ, T_ϑ is always the same map, that we will denote by T)

$$\begin{aligned} \frac{x}{d} &= \fint T_\vartheta(x \cdot \vartheta)\vartheta d\vartheta \\ &= \frac{1}{\omega_d} \int_{\mathbb{R}} \left(\int_{\{y \in \mathbb{S}^{d-1} : y \cdot x = s\}} \frac{T(x \cdot y) y}{|\nabla f(y)|} d\mathcal{H}^{d-2}(y) \right) ds, \end{aligned}$$

and by a direct computation of the Hausdorff $(d-2)$-dimensional measure of the set $E \cap f^{-1}(s)$, we have

$$\begin{aligned}\frac{x}{d} =& \frac{\omega_{d-1}}{\omega_d} \int_{\mathbb{R}} \left(\int_{\{y \in \mathbb{S}^{d-1} : x \cdot y = s\}} \frac{T(x \cdot y) y}{|\nabla f(y)|} \left(1 - \frac{s^2}{|x|^2}\right)^{\frac{d-2}{2}} dy \right) ds \\ =& \frac{\omega_{d-1}}{\omega_d} \int_{\mathbb{R}} \left(1 - \frac{s^2}{|x|^2}\right)^{\frac{d-2}{2}} T(s) \left(\int_{\{y \in \mathbb{S}^{d-1} : x \cdot y = s\}} \frac{y}{|\nabla f(y)|} dy \right) ds.\end{aligned}$$

Since on the set $E \cap f^{-1}(s)$ one has $|\nabla f(y)| = \sqrt{|x|^2 - s^2}$, we get

$$\begin{aligned}\frac{x}{d} =& \frac{\omega_{d-1}}{\omega_d} \int_{\mathbb{R}} \left(1 - \frac{s^2}{|x|^2}\right)^{\frac{d-3}{2}} \frac{T(s)}{|x|} \left(\int_{\{y \in \mathbb{S}^{d-1} : x \cdot y = s\}} y dy \right) ds \\ =& 2 \frac{\omega_{d-1}}{\omega_d} \int_{-|x|}^{|x|} \left(1 - \frac{s^2}{|x|^2}\right)^{\frac{d-3}{2}} \frac{T(s)}{|x|^2} s \hat{x} ds.\end{aligned}$$

Then, using the change of variables $t = s|x|$, we have

$$\frac{1}{d} = 2 \frac{\omega_{d-1}}{\omega_d} \int_{-1}^{1} (1 - t^2)^{\frac{d-3}{2}} \frac{T(t|x|)}{|x|} t dt = 4 \frac{\omega_{d-1}}{\omega_d} \int_0^1 (1 - t^2)^{\frac{d-3}{2}} \frac{T(t|x|)}{|x|} t dt.$$

Remark 8. *From now on, our computations are formal, and there are some problems in making them rigorous which we will explain in the following.*

Take the derivative with respect to $\lambda := |x|$ of the above expression, and deduce that

$$\int_0^1 (t\lambda T'(t\lambda) - T(t\lambda))\, t(1 - t^2)^{\frac{d-3}{2}} dt = 0.$$

This expression must hold for all the $x \in \mathrm{supp}(\mu)$ such that there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset \mathrm{supp}(\mu)$ such that $|y_n| \to |x|$. We call $\Lambda_\mu := \{\lambda \in [0, \infty) : \exists x \in \mathrm{supp}(\mu), \exists \{y_n\}_{n \in \mathbb{N}} \subset \mathrm{supp}(\mu) \text{ s. t. } |y_n| \to |x| = \lambda\}$. Using the change of variables $t\lambda = s$, and calling $g(t) := t\lambda T'(t\lambda) - T(t\lambda)$, we get that

$$\begin{aligned}0 =& \int_0^\lambda g(s) s (\lambda^2 - s^2)^{\frac{d-3}{2}} ds \\ =& \int_{\mathbb{R}} g(s) s (\lambda^2 - s^2)_+^{\frac{d-3}{2}}\end{aligned}$$

If the dimension d is even, by taking further derivatives with respect to λ and using the Lebesgue derivation lemma under the integral sign, we obtain

$$0 = \int_{\mathbb{R}} g(s) s (\lambda^2 - s^2)_+^{\frac{d-1}{2} - k} ds$$

and therefore

$$\int_0^\lambda g(s) s (\lambda^2 - s^2)^{\frac{d-1}{2} - k} = 0$$

for any $\lambda \in \Lambda_\mu$, $k \in \mathbb{N}$.

By the change of variables $u = \lambda^2 - s^2$, we get

$$\int_0^\lambda g(\sqrt{\lambda^2 - u})u^{\frac{d-1}{2}-k}du.$$

Use then the change of variables $u = \frac{1}{x}$, to deduce

$$\int_{1/\lambda^2}^\infty g\left(\sqrt{\lambda^2 - \frac{1}{x}}\right)\frac{x^k}{x^{\frac{d-1}{2}+2}}dx =: \int_{1/\lambda^2}^\infty f_\lambda(x)x^k dx = 0.$$

Now, what we would like to do is to invoke the Stone-Weierstrass theorem, in order to deduce that the function f_λ should be the zero function for any $\lambda \in \Lambda_\mu$. This would give us the claim, since it would imply $g(x) = 0$ for any $x \in \mathbb{R}$ and therefore that the transport map T was linear, which in turn would imply $T = \mathrm{id}$, by a direct computation. As already pointed out, the above argument is just formal, since some difficulties arise:

1. We don't know if the function g is integrable and if the above integrals in which it is involved make sense

2. For the same reason, the derivation under the integral sign could be a problem

3. We are trying to apply the Stone-Weierstrass theorem in a non bounded interval: this could be done employing some more abstract tools, but nonetheless doing this is a problem, since we are trying to use the theorem using with not bounded functions.

Chapter 7

Further perspectives

In this chapter we will present some issues that weren't treated or fully understood during this work, but for which we tried to develop some ideas.

Conjecture 2 We still haven't discuss conjecture 2: the statement of this conjecture is indeed very strong, as a positive answer to it would imply conjecture 1 is true. Still, saying something about the convergence of the particles moved along the sliced Wasserstein gradient flow is difficult. One of the approaches we tried to develop was the following: subdivide the interval $(0, \infty)$ in intervals (t_k, t_{k+1}), $k \in \mathbb{N}$ of a length $|t_{k+1} - t_k|$ to be determined. A standard result in gradient flows theory tells us that

$$\int_{t_k}^{t_{k+1}} |\rho'|^2(t)dt \leq \frac{SW^2}{2}(\rho_{t_k}) - \frac{SW^2}{2}(\rho_{t_{k+1}}).$$

This property, indeed, essentially follows from the fact that, for the ODE

$$x'(t) = -\nabla F(x(t)),$$

which represents the $\mathbb{R}^d$ model for gradient flows, one has

$$\int_{t_k}^{t_{k+1}} |x'(t)|^2 dt = -\int_{t_k}^{t_{k+1}} x'(t) \cdot \nabla F(x(t))dt = -\int_{t_k}^{t_{k+1}} \frac{d}{dt} F(x(t))dt = F(x(t_k)) - F(x(t_{k+1}))$$

By Hölder inequality we have therefore the following estimate:

$$\int_0^{\infty} |\rho'|(t)dt = \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} |\rho'|(t) \leq \sum_{k=0}^{\infty} \sqrt{t_{k+1} - t_k} \sqrt{\int_{t_k}^{t_{k+1}} |\rho'|^2(t)dt}$$
$$\leq \sum_{k=0}^{\infty} \sqrt{t_{k+1} - t_k} \sqrt{\frac{SW^2}{2}(\rho_{t_k}) - \frac{SW^2}{2}(\rho_{t_{k+1}})}.$$

If we assume the decay of the term $SW^2(\rho_{t_k}) - SW^2(\rho_{t_{k+1}})$ to be fast enough to have $\int_0^\infty |\rho'|(t)dt < \infty$ (and to obtain this we are free to model the choice of the intervals

(t_k, t_{k+1}) in the most convenient way), thanks to theorem (3.3.2), we deduce

$$\int_0^\infty \|v_t\|_{L^2(\rho_t)} dt = \int_0^\infty |\rho'|(t) dt < \infty, \tag{7.1}$$

whereas we recall that the result we would like to obtain in order give a positive answer to conjecture 2 is

$$\int_0^\infty \|v_t\|_{L^\infty} dt < \infty.$$

Even if it is still not clear how to deduce bounds on the L^∞ norm from estimates on the $L^2(\rho_t)$ norm, we think that obtaining (7.1) could be a starting point for a research which aims to furnish a positive answer to conjecture 2. The biggest problem in this plan is that providing an estimate for the decrease of the term SW^2 along the flow is very hard. One could for example use the following argument: by equation (6.4) and using the notation $x(t) := \mathcal{M}_2^2(\rho_t)/2 - \mathcal{M}_2^2(\nu)/2$, $g(t) := SW^2(\rho_t)/2$, we have that

$$x' = -\frac{x}{d} - g.$$

Thanks to the monotonicity of g (recall that a functional F decreases along its gradient flow) one can prove that x is either decreasing or (eventually) monotone increasing, hence admits a limit x_∞, that must be finite thanks to estimate (6.2). Using the shortcuts $y := x - x_\infty$ and $w := g - g_\infty$, we have therefore the relation

$$w' = -\frac{w}{d} - h.$$

Now, if we could prove that $w > 0$, we would deduce the integrability of the term $SW^2(t) - \lim_{t\to\infty} SW^2(t)$, and this would be sufficient to obtain (7.1). This is equivalent to ask that, along the flow, the second moment of the measure at time t is larger than the second moment of the target measure. Under which assumptions this happens?

The SWGF could not converge(?) Another interesting aspect we didn't fully treat is related to conjecture 1: we would like to provide an example in which the flow does not converge to any measure. To fix the ideas, consider the following

Example 5. *In $\mathbb{R}^2$ consider the following autonomous ODE expressed in polar coordinates:*

$$\frac{d}{dt}(r\cos\vartheta, r\sin\vartheta) = -\nabla f(r,\vartheta),$$

where

$$f(r,\vartheta) = \begin{cases} 1, & r \le 1 \\ 1 + e^{-\frac{1}{r-1}}\left[\sin\left(\frac{1}{r-1} + \vartheta\right) + 2\right], & r > 1. \end{cases}$$

Solutions starting in $\mathbb{R}^2 \setminus \bar{B}_1$ spiral toward the unit circle, and we can find subsequences along this trajectory that converge to different limits. From this example, we can construct a gradient flow in $(\mathcal{P}_2(\mathbb{R}^2), W_2)$ which exhibits similar spiralling behavior, namely the solution to the continuity equation

$$\partial_t \rho - \nabla \cdot (\rho \nabla f) = 0,$$

with initial datum δ_{x_0} for $x_0 \in \mathbb{R}^2 \setminus \bar{B}_1$, which is given by $\rho(t) = \delta_{x(t)}$.

Is it possible to choose an initial and target measure in such a way that the velocity field associated with the SWGF, namely

$$v_t(x) = \fint_{\mathbb{S}^{d-1}} (T_{t,\vartheta}(x \cdot \vartheta) - x \cdot \vartheta) d\vartheta$$

behaves similarly to the function f in the above example? Of course, here we have a different situation from the one presented before, since the velocity field v is depending on time and position and even on the curve ρ_t itself, namely the ODE providing the Lagrangian description of the SWGF is non autonomous.

The positive measure case Beside the problems we already pointed out in the last part of the work, let us mention a possible way (still unexplored by us) to deduce a convergence result for positive measures: recalling remark 7 from section 6.4, we would like to know under which hypothesis the following fact is true: given an initial and a target measure both positive on the entire space, the associated gradient flow is (uniformly in t) positive on the entire space. A positive answer to this result (or to any similar result ensuring the uniform positivity of the gradient flow) would give a straight-forward answer to the question of the convergence of the SWGF associated with positive measures.

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