UNIVERSITÀ DEGLI STUDI DI PADOVA

DIPARTIMENTO DI MATEMATICA T. LEVI CIVITA

Corso di Laurea Magistrale in MATEMATICA

Syntomic regulators

Relatore:
Prof.
Bruno CHIARELLOTTO

Candidato:
Mattia TISO
Mat. 1207713

16 OTTOBRE 2020

ANNO ACCADEMICO 2019/2020

Dedicata alla mia famiglia e la mia compagna
che mi hanno supportato e aiutato in questo periodo
pieno di imprevisti e di difficoltà.

INTRODUCTION

The purpose of this thesis is to explain how different p-adic cohomology theories are related one whit the other and to show that there are common fundamental aspects that can be shared between different theories. The tool that allows us to investigate such relations is the *regulator map*. It is a map connecting two different type of algebraic/geometric worlds. The source has an intrinsic geometric nature, in which the main objects are *cycles* on a variety. The target is the cohomology theory that we choose to develop.Varieties can be defined over the complexes or over a discrete valuation ring (in equi or mixed characteristic): in all these declinations, cycles have different flavors and they may involve the topology of the variety but also its cohomology. The fact that this map change on the target, but comes by the same source, make it like a bridge between the different theories that we want to study. In the course of this study we have read about different kind of applications in number theory or algebraic geometry and we understood that this kind of applications can be reached after a deep knowledge of these connections. So, in this discussion we want to stress these analogies and where they come from.

In the first chapter we deal with the definition of Higher Chow groups. The most important properties are discussed and the Bloch argument of ([Blo86b]), on the construction of higher cycle maps, is exposed.

In the second, we have described the theory which has motivated our work: that is the Deligne-Beilinson cohomology. For a smooth variety defined over the complex numbers, such a cohomology takes care of the Hodge filtration in the de Rham setting as well as the topological singular cohomology of the associated complex manifold. In the third chapter we deal with an algebraic construction of Abel-Jacobi type maps using the regulator map of the Deligne-Beilinson cohomology. We show how the mixed Hodge structure on the singular cohomology theory is used to define such maps.

As in the classical case we have Deligne-Beilinson cohomology, we would like to find an analogue in the arithmetic setting. Namely, we consider a scheme X defined over a valuation ring in mixed characteristic. For such a situation we would like to have a cohomology theory which mixes the Hodge filtration of the algebraic scheme defined on the generic fiber (in characteristic 0), together with some invariant (under the Frobenius operator) in the cohomology of the special fiber which is a scheme over a characteristic $p > 0$ field. For this reason in the chapters fourth and fifth we deal with the definitions and properties of cohomology theories of varieties in characteristic p which are relevant for our work: the crystalline and the rigid one.

Finally, in the last chapter we consider various definitions of the syntomic cohomology. The original "syntomic cohomology" a' la Fontaine-Messing ([FM87]) involved the crystalline cohomology and its filtration. It admits a link to p-adic étale cohomology. Besser cohomology mixed the rigid cohomology of the special fiber with the Hodge algebraic filtration of the generic one. Gros syntomic cohomology uses the rigid cohomology complex of the special fiber with an analogous of the Hodge filtration (but again on the rigid complex), which has not (a priori) a geometric interpretation. A regulator map has been constructed for the Besser-rigid cohomology ([CCM13]): the aim of our thesis has been to compare all these cohomology theories as well as to study the properties of the Gros syntomic cohomology. Unfortunately, the lack of a geometric interpretation has not allowed us to prove that Gros syntomic cohomology is a Bloch cohomology. However, along the lines of [CCM13] (i.e. using higher Chow groups) a possible construction of a regulator map from the Higher Chow groups has been proposed. Moreover it has been sketched a possible approach to solve the compatibility with the maps to the étale cohomology of all these various syntomic cohomologies.

CONTENTS

CHAPTER 1

HIGHER CHOW GROUPS

In this chapter we introduce a fundamental object in algebraic topology, that is related in a deep way with the different cohomology theories that we will treat. They are the generalization to higher dimension of the classical Chow groups in algebraic geometry. This definition is made and discussed in origin by Bloch, and it is used for proving different properties that are intrinsic and characterize most of the "good" cohomological theories. In fact these objects are considered as the right geometric object to study the so called "motivic cohomology", that is understood as the tool that bring together all different kind of cohomologies and relate them with comparison maps. Actually, Bloch proved that the definition of the higher Chow groups is isomorphic to a portion of K-theory ([Blo86a]) and for this theory there is a way to construct *regulator maps*. This isomorphism suggests to treat the regulator map as an "higher cycle map", but a direct connection between regulator maps and higher cycle maps is not yet so clear. In the course of this thesis, we want to discuss the so called regulator maps, as "higher cycle maps", that is proved to exist in a general setting over "good" cohomological theories, as we will explain soon.
We will see in this chapter some fundamental property of the Higher Chow groups for varieties over the ground field k. During this exposition we will follow essentially [FF84], for the basic definitions in intersection theory and [Blo86a], for the construction of higher Chow groups.

Definition 1.1. *An algebraic cycle on a scheme X is a finite, formal linear combination of the form $\sum n_V[V]$ where $V \subset X$ is a subscheme and n_V are integers. Suppose that X is equidimensional, then $\mathcal{Z}^r(X)$ is the free abelian group generated by $[V]$, with $V \subset X$ a subscheme of codimension r.*

Definition 1.2. *Let W a divisor with complete intersection, so given by equations $f_1 = \cdots = f_n = 0$. for a subscheme $V \subset X$, we say that W and V meet properly, if $codim_V(W \cap V) \geq n$. By classical algebraic geometry we can define a cycle $W \cdot V$.*

Recall that the intersection "$\cdot$"is commutative and associative ([FF84]).

Definition 1.3. *Let Δ^n to be the affine simplex:*

$$\Delta^n := spec(k[t_0, \ldots, t_n]/(t_0 + \ldots .t_n - 1)).$$

Then we can define a map $i : \Delta^m \to \Delta^n$ associate to a map $\rho : [0, m] \to [0, n]$ weakly increasing. More precisely i is defined by the morphism on the algebras, given by

$$i^*(t_k) = \begin{cases} 0 & \text{if } \rho^{-1}(k) = 0 \\ \sum_{\rho(j)=k} t_j & \text{else.} \end{cases}$$

So for an injection ρ, i is the inclusion of Δ^m in Δ^n obtained by intersecting with $t_k = 0$ all the parameters t_k which are not involved in Δ^m, and this map is called *face map*. It is a flat map. While for ρ surjective, all the parameters t_k in which $\rho(k)$ is the same, degenerate in the same parameter in Δ^n, and this map is called *degeneration map*. It is a proper map

Definition 1.4. *Let define $\mathcal{Z}^r(X, n) \subset \mathcal{Z}^r(X \times \Delta^n)$ as the free abelian group generated by irreducible subvarieties $V \subset X \times \Delta^n$ of codimension r such that V meets all faces of $X \times \Delta^n$ properly.*

Lemma 1.5. *We have that $\mathcal{Z}^r(X, \bullet)$ is a simplicial abelian group.*

Proof. We have to find for each $\rho : [0, m] \to [0, n]$ a functorial map

$$\rho^* : \mathcal{Z}^r(X, n) \to \mathcal{Z}^r(X, m).$$

Case 1: ρ is surjective. Then $i_\rho : X \times \Delta^m \to X \times \Delta^n$ is flat. Then there exists the flat pullback

$$\rho^* = i_\rho^* : \mathcal{Z}^r(X \times \Delta^m) \to \mathcal{Z}^r(X \times \Delta^n)$$

. Case 2: ρ is injective. Then for each V that meets properly $i_\rho(X \times \Delta^n)$, define

$$\rho^*(V) = i_\rho(X \times \Delta^m) \cdot V.$$

Case 3: general case. Then for each $\rho : [0,m] \to [0,n]$ there exist ρ_1 injective and ρ_2 surjective such that $\rho_1 \circ \rho_2 = \rho$. Then define $\rho^* = \rho_2^* \circ \rho_1^*$. Then the functoriality is immediate, since when T is closed subvariety of $X \times \Delta^m$ that meets all the faces properly, then the multiplicity of T in $\rho^*(V)$ depends on the length of $\mathcal{O}_{i_\rho^{-1}(V)}$ with respect to the generic point of T and this is a functorial property.
To prove that cycles that meet properly the faces are mapped in the corresponding kind of cycles, let's note that in the injective case, this is satisfied by definition. For the general case, just consider that we have a commutative diagram that, by functoriality, is commutative also at level of higher cycles:

$$\begin{array}{ccc} X \times \Delta^n & \xleftarrow{i_\rho} & X \times \Delta^m \\ {\scriptstyle i_n}\uparrow & \nearrow_{i_m} & \\ X & & \end{array}$$

Since i_m, (i_n) corresponds to an injective $\rho : [0] \to [0,m] \quad ([0,n])$, we have that $i_m^*(V)$ meets X, the face of codimension 0 in $X \times \Delta^m$, properly. But $i_m^*(V) = \rho^*(V) \cdot X$ and so $\rho^*(V)$ meets X properly. By induction on m, fixed a face $\sigma \subset \Delta^m$, we can find an $i_\tau : X \times \Delta^p \to X \times \Delta^m$ such that it is induced by a map $\tau : [0,p] \to [0,n]$ injective with $p < m$ and such that its image is $X \times \sigma$. By induction we have that $\tau^*(V)$ meets properly $X \times \sigma$ and $\rho^*(V) \cdot (X \times \sigma) = \tau^*(V)$, that means that $\rho^*(V)$ meets $X \times \sigma$ properly. □

Definition 1.6. *Let $\partial_i : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,n-1)$ be the pullback map of cycles associate to $i : \Delta^{n-1} \hookrightarrow \Delta^n$ defined by $t_i = 0$*

Now consider $\mathcal{Z}^r(X,\bullet)$ as a complex concentrated in negative degree. Then we can define a map of complex by posing

$$\delta = \sum (-1)^i \partial_i : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,n-1).$$

Then it is a simple computation to verify that $\delta \circ \delta = 0$. Then define

Definition 1.7. *For X an equidimensional k-variety, define*

$$CH^r(X,n) = H^{-n}(\mathcal{Z}^r(X,\bullet))$$

Lemma 1.8. *For $n = 0$, $CH^r(X, 0) \simeq CH^r(X)$*

Proof. The classical Chow groups $CH^r(X)$ modulo rational equivalence are obtained as cokernel of $\mathcal{Z}^r(X,1) \rightrightarrows \mathcal{Z}^r(X,0)$, and so coincide with the definition posed. □

Proposition 1.9. *The complex $\mathcal{Z}^r(X, \bullet)$ is covariant functorial for proper maps and contravariant functorial for flat maps.*

Proof. Let $f : X \to Y$ be a proper map. Then $f \times 1 : X \times 1 \to Y \times 1$ is also a proper map. Now by the properness, we deduce that for a subvariety $Z \subset X \times \Delta^n$ and $m < n$, and $\partial : \Delta^m \to \Delta^n$ that

$$f(Z) \cap \partial(Y \times \Delta^m) = f(Z \cap X \times \Delta^n)$$

and

$$\operatorname{codim}(f(Z \cap X \times \Delta^n)) \geq \operatorname{codim}(Z \cap X \times \Delta^n).$$

Then is well defined the pushforward for proper maps (with the right shift in grading) f_* such that we have the following diagram:

$$\begin{array}{ccc} \mathcal{Z}^*(X,n) & \xrightarrow{\partial^*} & \mathcal{Z}^*(X,m) \\ \downarrow f_* & & \downarrow f_* \\ \mathcal{Z}^*(Y,n) & \xrightarrow{\partial^*} & \mathcal{Z}^*(Y,m) \end{array}$$

The fact that it commutes follows by a theorem of ([FF84], Proposition 6.2), applied to $Z \cap X \times \Delta^m$, $f(Z)$, Z, $f(Z \cap X \times \Delta^m)$, and considered rational equivalence to codimesion 0. The flat case follows also by a theorem in ([FF84] Proposition 1.7). □

Now let Y be a smooth k-variety and $\mathcal{W} = \{W_1, \ldots, W_k\}$ a collection of closed algebraic sets of Y.

Definition 1.10. *Define the subset $\mathcal{Z}^r_{\mathcal{W}}(Y, n) \subset \mathcal{Z}^r(Y, n)$ to be the free abelian group generated by the irreducible cycles of $Y \times \Delta^n$ meeting all $W_i \times \Delta^m$ properly for each $\Delta^m \subset \Delta^n$.*

Proposition 1.11. *Assume Y smooth and $W \subset Y$ a local complete intersection. Then the pullback of cycle induced by the inclusion $W \hookrightarrow Y$, induces a map on the level of complexes*

$$\mathcal{Z}^{r+n}_W(Y, \bullet) \to \mathcal{Z}^r(W, \bullet).$$

Also, if $\mathcal{W} = \{W_1, \dots, W_k\}$ with $W_i \subset W_1 \subset Y$ for $i \geq 2$, then there is a map of complexes:

$$\mathcal{Z}^{r+n}_{\mathcal{W}}(Y, \bullet) \to \mathcal{Z}^r_{\mathcal{W}-W_1}(W_1, \bullet).$$

Proof. The only request to satisfy for being a map of complexes is the commutativity of the suitable diagram, but this is equivalent to request that the intersection of cycles is commutative. □

The next is a proposition that represents a moving lemma for higher cycle classes, and we refer a proof to the article of Bloch ([Blo86a]). We will use it for transporting functoriality on higher Chow groups.

Theorem 1.12. *Let Y be a smooth k-variety, $\mathcal{W}$ a collection as above of closed subsets of Y. Assume Y is either affine or projective. Then the inclusion $\mathcal{Z}^r_{\mathcal{W}}(Y, \bullet) \subset \mathcal{Z}^r(Y, \bullet)$ is a quasi-isomorphism.*

Proof. See ([Blo86a], Lemma 4.2). □

To remove the hypothesis of affine or projective, Bloch at first proved a strong formulation of the moving lemma, with the only quasi-projective hypothesis.

Theorem 1.13. *Let X be a smooth quasi-projective variety and U a Zariski open set of X. Denote $Y = X - U$. Suppose the codimension of Y in X is n. Then we have a quasi-isomorphism:*

$$\mathcal{Z}^{r+n}(X, \bullet)/\mathcal{Z}^r(Y, \bullet) \to \mathcal{Z}^{r+n}(U, \bullet). \tag{1.1}$$

Proof. See ([Blo86a], Theorem 3.3). □

It is important to point up that this quasi isomorphism generates a long exact sequence in Chow cohomology.

Proposition 1.14. *Let $Y \subset X$ a subvariety of codimension d. Then there is a long exact sequence:*

$$\dots \to CH^{*-d}(Y, n) \to CH^*(X, n) \to CH^*(X - Y, n) \to \dots$$

Proof. We have a short exact sequence:

$$0 \to \mathcal{Z}^{*-d}(Y, \bullet) \to \mathcal{Z}^*(X, \bullet) \to \mathcal{Z}^*(X, \bullet)/\mathcal{Z}^{*-d}(Y, \bullet) \to 0.$$

Then by the long exact sequence induced in cohomology, we have the long exact sequence:

$$\cdots \to CH^{*-d}(Y,n) \to CH^{*}(X,n) \to \mathbb{H}^{*}(\mathcal{Z}^{*}(X,\bullet)/\mathcal{Z}^{*-d}(Y,\bullet)) \to \dots$$

Then by the quasi-isomorphism of the theorem (1.13), we have that the proposition holds. □

As corollary we can prove the following:

Corollary 1.15. *Let Y be a smooth quasi-projective variety, and $\mathcal{W}$ as above. Then the inclusion $\mathcal{Z}^{r}_{\mathcal{W}}(Y,\bullet) \subset \mathcal{Z}^{r}(Y,\bullet)$ is a quasi-isomorphism.*

Proof. We proceed with induction on the dimension n of Y. For $n = 0, 1$, Y is affine or projective, so the proposition holds, in this case, by theorem 1.12. For $n > 1$, consider the projective closure $\bar{Y}$ of Y, and call $Z = \bar{Y} - Y$. Then chose an hypersurface $\bar{X}$ such that $Z \subset \bar{X}$ and $X = \bar{X} - Z$ contains a dense Zariski open in $\bar{X}$. Then X has dimension $n-1$ and $Y - X$ is affine (since a complement of an hypersurface in a projective variety is affine). Then by the theorem 1.13 we have a following commutative diagram of distinguished triangles on the rows:

$$\begin{array}{ccccc}
\mathcal{Z}_{\mathcal{W}\cap X}(X,\bullet) & \longrightarrow & \mathcal{Z}_{\mathcal{W}}(Y,\bullet) & \longrightarrow & \mathcal{Z}_{\mathcal{W}}(Y-X,\bullet) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}(X,\bullet) & \longrightarrow & \mathcal{Z}(Y,\bullet) & \longrightarrow & \mathcal{Z}(Y-X,\bullet)
\end{array}$$

Then the external vertical maps are quasi-isomorphism: On the left, by inductive hypothesis. On the right because $Y - X$ is affine. The fact that the rows are distinguished triangles, implies that we can apply the five lemma and it shows that the middle vertical map is a quasi-isomorphism too. □

Proposition 1.16. *Let $f : X \to Y$ be a morphism of k-variety. Suppose that Y is smooth and quasi-projective. Then there is a functorial pullback map:*

$$f^{*} : CH^{*}(Y,n) \to CH^{*}(X,n). \tag{1.2}$$

Proof. Let $1_Y : Y \to Y$ be the identity map. Then we have a map $f \times 1_Y : X \times Y \to Y \times Y$. By the smoothness of Y we have that $Y \times Y$ is a local complete intersection, so also $\Delta_Y \subset Y \times Y$ is local complete intersection. Then the graph $\Gamma_f = (f \times 1_Y)^{-1}(\Delta_Y) \subset X \times Y$ is a local complete intersection. Then define

$$T_i = \overline{\{y \in Y : \dim(f^{-1}(y)) \geq \dim(X) - \dim(Y) + i\}}. \tag{1.3}$$

Then if Z is a closed subvariety of $Y \times \Delta^n$ that meets properly $T_i \times \Delta^n$, then $Z \times X$ meets properly $\Gamma_f \times \Delta^n$. Since $\Gamma_f \sim X$, define

$$f^*(Z) = (Z \times X) \cdot (\Gamma_f \times \Delta^n).$$

Then if $\mathcal{T} = \{T_0, \ldots, T_k\}$ we have by corollary 1.15 that

$$\mathcal{Z}_{\mathcal{T}}(Y, \bullet) \xrightarrow{\sim} \mathcal{Z}(Y, \bullet).$$

Then the definition is well posed since at level of cohomology it is independent on $\mathcal{T}$. The functoriality follows by the fact that $\Gamma_{f \circ g} = \Gamma_f \circ \Gamma_g$. □

Another fundamental property is the homotopy property for higher Chow groups. Its proof constitutes a big work in the article of Bloch ([Blo86a]). We are limited to cite and use the result and try to understand the relationship with cohomological theories.

Theorem 1.17. *Let X a quasi-projective k-variety. Then the projection $X \times \Delta^n \to X$ induces a pullback map*

$$\mathcal{Z}^r(X, \bullet) \to \mathcal{Z}^r(X \times \Delta^n, \bullet),$$

that is a quasi-isomorphism.

Proof. See ([Blo86a], Theorem 2.1) □

The other fact about the complex $\mathcal{Z}^r(X, \bullet)$ is that it is a complex of sheaves over a reasonable topology on X. An interesting fact is the following. Consider X_{zar} a smooth variety with the Zariski topology. Then holds the following:

Theorem 1.18. *The association*

$$U \longmapsto \mathcal{Z}^r(X, \bullet)/\mathcal{Z}^r(X - U, \bullet) =: \Gamma(U, S^r_\bullet) \tag{1.4}$$

constitutes a flabby complex of sheaves.

Proof. The flabby condition for $S^r_\bullet$ is clear by definition. We have to prove that actually they are sheaves. This means that we have to prove the exactness of the following sequence: for open $U, V \subset X$ then we have

$$0 \to \mathcal{Z}^r(X,n)/\mathcal{Z}^r(X-(U\cup V)) \to \mathcal{Z}^r(X,n)/\mathcal{Z}^r(X-U,n) \oplus \mathcal{Z}^r(X,n)/\mathcal{Z}^r(X-V,n)$$
$$\rightrightarrows \mathcal{Z}^r(X,n)/\mathcal{Z}^r(X-(U\cap V),n)$$

Since $X-U$ and $X-V$ are closed sets, we have that $\mathcal{Z}^r(X-(U\cup V)) = \mathcal{Z}^r(X-U)\cap\mathcal{Z}^r(X-V)$, that means, that the first map in the sequence is injective. Then we have to prove that the kernel above is equal to the image of this injective map. Let $(z_U, z_V) \in \mathcal{Z}^r(X,n)/\mathcal{Z}^r(X-U,n) \oplus \mathcal{Z}^r(X,n)/\mathcal{Z}^r(X-V,n)$ such that

$$z_U - z_V \in \mathcal{Z}^r(X-(U\cap V),n).$$

Denote with $\bar{z_U}, \bar{z_V}$ the Zariski closures of respectively z_U, z_V in $X \times \Delta^n$. The condition on z_U and z_V means that they glue together on a section z on $U\cup V$, and the Zariski closure $\bar{z}$ of z in $X\times\Delta^n$ has not any component with support in $X-(U\cup V)$. We must show that $\bar{z} \in \mathcal{Z}^r(X,n)$. To do that, notice that we can write $\bar{z} = \bar{z_U} + w_U = \bar{z_V} + w_V$ where $supp(w_U) \subset X-U$ and $supp(w_V) \subset X-V$ and so $\bar{z_U} - \bar{z_V} = w_V - w_U \in \mathcal{Z}^r(X,n)$. Then we can write $w_U = w'_U + w_{U,V}$ where $w'_U \in \mathcal{Z}^r(X,n)$ and $supp(w'_U) \not\subset X-(U\cup V)$, while $supp(w_{U,V}) \subset X-(U\cup V)$. Then

$$\bar{z} = \bar{z_U} + w'_U + w_{U,V}.$$

But $w_{U,V} = 0$ since there is no support in $X-(U\cup V)$ for $\bar{z}$ and so

$$\bar{z} = \bar{z_U} + w'_U \in \mathcal{Z}^r(X,n).$$

□

A corollary of this fact is the useful spectral sequence that relates the Higher Chow groups:

Corollary 1.19. *There is a spectral sequence of the form:*

$$H^p(X, \mathcal{H}^q(X, \mathcal{Z}^r(-,\bullet))) \implies CH^r(X, -p-q). \tag{1.5}$$

Proof. By the theorem 1.18 we have that the hypercohomology and the cohomology of global sections are isomorphic (it follows by flabby condition). More precisely we have that

$$CH^r(X,n) = H^{-n}(\Gamma(X,S^r_\bullet)) = \mathbb{H}^{-n}(X,S^r_\bullet).$$

Then the Grothendieck spectral sequence for the hyperchomology writes as:

$$H^p(X,\mathcal{H}^q(\mathcal{Z}^r(-,\bullet))) \implies \mathbb{H}^{p+q}(X,S^r_\bullet) = CH^r(X,-p-q).$$

□

With this terms we can deduce a vanishing property for the higher Chow groups:

Proposition 1.20. *Let $\mathcal{CH}^r(n)$ the sheaf associate to the datum*

$$U \longmapsto CH^r(U,n).$$

Then for $i > n$ follows that

$$H^i(X,\mathcal{CH}^r(n)) = 0.$$

Proof. With this notation the spectral sequence above has the form:

$$H^i(X,\mathcal{CH}^r(n)) \implies CH^r(X,n-i).$$

Since $n-i<0$, the proposition follows.

□

This thesis will deal also with a slightly modification of the Higher Chow groups that is referred to smooth schemes defined over a discrete valuation ring $\mathcal{V}$, with residue field k and fraction field K. If $\mathcal{X}$ is a quasi-projective smooth scheme over $\mathcal{V}$ the definition of the group $\mathcal{Z}^r(\mathcal{X}/\mathcal{V},0)$ is given by *flat* and *integral* cycle over $\mathcal{V}$, in a such way there are a non trivial cycle Z_K over the general fiber $X_K \times_{\mathcal{V}} K$ over K, and a cycle Z_k over the special fiber $X_k \times_{\mathcal{V}} k$ over k. In this context an n-simplex is defined as

$$\Delta^n := \mathrm{Spec}(\mathbb{Z}[x_0,\dots,x_n]/(x_0+\cdots+x_n-1)) \tag{1.6}$$

Formally all the previous definitions are the same.

A remark is that the complex $\mathcal{Z}^r(\mathcal{X}/\mathcal{V},\bullet)$ is not only a complex of sheaves for the Zariski topology but also for the étale topology on $\mathcal{X}$ ([Gei04], Lemma 3.1).

Now we want to explain the relationship between higher Chow groups and cohomology. Recall that for "good" cohomology theories, we have a cycle class map, that means a functorial map

$$\mathrm{cl}^r : CH^r(X) \to H^{2r}(X)$$

defined for X smooth. Actually for each $Z \in \mathcal{Z}^r(X)$ is defined a cycle class $\eta_Z \in H^{2r}_Z(X)$ where H_Z denotes the cohomology with support in a closed subset of X. In the higher context, we play with a cohomological theory in "two variables", that means a "good" theory of the form

$$X \longmapsto H^*(X, \bullet)$$

and we expect to construct an "higher cycle map" or "regulator" map [1] that is a functorial map

$$\mathrm{cl}^{r,n} : CH^r(X, n) \to H^{2r-n}(X, r)$$

We will see that this behaviour of double index in cohomology is the common one, also in the singular cohomology for example. In fact often we consider sheaves and their twists that make the cohomological theory bigraded. We will see in some detail further. We want now to explain under which condition a cohomological theory is "good" and when it is possible to construct a regulator map. This is a quite general fact and this generality is the key thanks which it is possible to formulate some connection with the different cohomological theories here exposed. For integers a, b let $H^a(X, b)$ a cohomology theory of schemes X that satisfies the following axioms:

1. Given a subscheme Y of X, there exists a cohomology with support in Y such that satisfies a localization exact sequence of the form:

$$\cdots \to H^a_Y(X, b) \to H^a(X, b) \to H^a(X - Y, b) \to H^{a+1}_Y(X, b) \to \dots$$

   This exact sequence is contravariant functorial for cartesian squares of the form:

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \hookrightarrow & X' \end{array}$$

[1] the terms regulator often is used in the context of K-theory. During this paper our objective is "replace" in some sense the K-theory. The translation of this map in our context is the more appropriate "higher cycle map".

and covariant for $Y' \subset Y \subset X$

2. The homotopy property holds. It means that for any i we have that

$$H^i(X,b) \simeq H^i(X \times \Delta^n, b) \qquad \forall n \geq 0.$$

3. There exists a cycle class map: for a pure codimesion r subscheme $Y \subset X$ there exists a cycle class

$$cl(Y) \in H^{2r}_Y(X,r)$$

such that it is compatible with pullback, i.e for a map $f : X' \to X$ and $Y \subset X$ such that $f^{-1}(Y)$ has codimension r we have that $cl_{X'}(f^*Y) = f^* cl_X(Y)$.

4. The weak purity property holds. It means that for $Y \subset X$ of pure codimension r we have that

$$H^i_Y(X,b) = 0 \qquad \forall i < 2r.$$

When these properties hold, we call a such cohomology a *Bloch cohomology theory*. In fact thess axioms are presented in [Blo86b] were it is introduced the following "method" to construct "higher cycle maps":

Theorem 1.21. *Let $H^a(-,b)$ to be a Bloch cohomology theory in the above sense. Assume that $H^*(X,b)$ is an hypercohomology over a topology in which Godement resolutions there exist, computed by a complex $K^*_X(b)$ contravariant functorial and with $K_X(b)^n$ acyclic for each n (this assumption is endowed in the existence of Godement resolutions). Then there exists a functorial map*

$$cl^{r,n} : CH^r(X,n) \to H^{2r-n}(X,r) \tag{1.7}$$

that in the case $n = 0$ coincides with the classical cycle class map.

Proof. Let's consider the simplicial scheme built below:

$$X \rightrightarrows X \times \Delta^1 \substack{\to\\ \to\\ \to} X \times \Delta^2 \ldots$$

We associate to this diagram the double complex

$$\Gamma(X \times \Delta^{-p}, K_{X \times \Delta^{-p}}(b)^q)$$

with $p \leq 0$. Then for reasons of convergence we choose a truncation of the simplicial complex above, and consider for $N >> 0$ even and $m \geq -N$

$$A^{m,t} := \tau_{m \geq -N} \Gamma(X \times \Delta^{-m}, K_{X \times \Delta^{-m}}(b)^q)$$

for $m \leq 0$. Then there is a spectral sequence

$$E_1^{m,t} := H^t(A^{\bullet,m}) \implies H^{m+t}(\mathrm{tot}(A^{\bullet,\bullet})).$$

Then by the homotopy property we have

$$E_1^{m,t} = H^t(X \times \Delta^{-m}, b) = H^t(X, b).$$

Then the differential map d^1 can be the identity or 0 (as we will see in the proof of c.f. proposition 6.11).

Then we obtain that

$$E_2^{m,t} = \begin{cases} H^t(X, b) & m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that this conclusion on the E_2 terms is the same without truncation, where the only problem is about the convergence of E_1. Then we can define

$$E_{1,c}^{m,t} := \varinjlim_{Z \in \mathcal{Z}^b(X,-m)} H^t_{|Z|}(X \times \Delta^{-m}, b),$$

where $|Z| = supp(Z)$. Now the rest of the construction is the same as we will follow in the proof of proposition 6.11. Since that proof is much detailed and at this stage it depends only by having a Bloch cohomology theory we refer the reader to it.

□

CHAPTER 2

DELIGNE-BEILINSON COHOMOLOGY

In this chapter we introduce the object that permits and motivates the developing of the next theories. It will be the "trace" for the following discussion, and our objective is to stress some peculiar behaviour of this cohomology theory that is expected to share with other suitable theories. We will work over the field of complex numbers $\mathbb{C}$ and the objects of interest are *projective, smooth varieties* over $\mathbb{C}$.
The starting point is the classical de Rham cohomology. The basic idea to formulate a "smaller" cohomology theory, taking the relevant information about the de Rham cohomology, is to "truncate" the de Rham complex. Let X be a smooth, projective variety over $\mathbb{C}$. We endow X by analytical structure, in such way that it is a manifold with differential forms. The main references for this part are [EV88] and [PS08]

Definition 2.1. *Let $\mathbb{Z}(r) = (2\pi i)^r\mathbb{Z} \subset \mathbb{C}$ for $r \geq 0$. Let define the Deligne complex*

$$\mathbb{Z}(r)_{\mathcal{D}} : 0 \to \mathbb{Z}(r) \to \mathcal{O}_X \to \Omega^1_{X/\mathbb{C}} \to \Omega^2_{X/\mathbb{C}} \to \cdots \to \Omega^{r-1}_{X/\mathbb{C}} \to 0. \tag{2.1}$$

The first map is the restriction over $(2\pi i)^r\mathbb{Z}$ of $\mathbb{C} \to \mathcal{O}_X$.

Definition 2.2. *The the Deligne cohomology is the bigraded hypercohomology of the Deligne complexes, where X is the variety endowed by the analytical structure:*

$$H^*_{\mathcal{D}}(X, r) := \mathbb{H}^*(X, \mathbb{Z}(r)_{\mathcal{D}}). \tag{2.2}$$

This cohomology theory has a natural cup product. In fact, on the level of complexes it is defined as:

$$\mathbb{Z}(r)_{\mathcal{D}} \otimes \mathbb{Z}(r')_{\mathcal{D}} \to \mathbb{Z}(r+r')_{\mathcal{D}}$$

$$x \cup y = \begin{cases} x \cdot y & \deg(x) = 0 \\ x \wedge dy & \deg(x) > 0, \deg(y) = r' \\ 0 & \text{else} \end{cases} \tag{2.3}$$

Then the differential acts as:

$$d(x \cup y) = dx \cup y + (-1)^{\deg(x)} x \cup dy \tag{2.4}$$

Lemma 2.3. *the $\cup$ product is a morphism of complexes associative and anticommutative. It makes up a ring structure on the group*

$$\bigoplus_{r,n} H^n_{\mathcal{D}}(X, r).$$

Proof. For the associativity, let x, y, z respectively $\in \mathbb{Z}(r)_{\mathcal{D}}, \mathbb{Z}(r')_{\mathcal{D}}, \mathbb{Z}(r'')_{\mathcal{D}}$. Then we have

$$(x \cup y) \cup z = \begin{cases} xy \cup z & deg(x) = 0 \\ (x \wedge dy) \cup z & deg(x) > 0, deg(y) = r' \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} xyz & deg(x) = deg(y) = 0 \\ xy \wedge dz & deg(y) > 0, deg(x) = 0, deg(z) = r'' \\ x \wedge dy \wedge dz & deg(x) > 0, deg(y) = r', deg(z) = r'' \\ 0 & \text{otherwise} \end{cases}$$

while

$$x \cup (y \cup z) = \begin{cases} xyz & deg(x) = deg(y) = 0 \\ x(y \wedge dz) & deg(x) = 0, deg(y) > 0, deg(z) = r'' \\ x \wedge (dy \cup z) & deg(x) > 0, deg(y) = r', deg(z) = r'' \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} xyz & deg(x) = deg(y) = 0 \\ xy \wedge dz & deg(x) = 0, deg(y) > 0, deg(z) = r'' \\ x \wedge dy \wedge dz & deg(x) > 0, deg(y) = r', deg(z) = r'' \\ 0 & \text{otherwise} \end{cases}$$

For the anticommutative property, we have to prove that $x \cup y = (-1)^{deg(x)deg(y)} y \cup x$ where the equality is considered in the homotopy class. An homotopy between $x \cup y$ and $(-1)^{deg(x)deg(y)} y \cup x$ is provided by

$$h : \mathbb{Z}(r)_{\mathcal{D}} \otimes \mathbb{Z}(r')_{\mathcal{D}} \longrightarrow \mathbb{Z}(r+r')_{\mathcal{D}}$$

$$x \otimes y \longmapsto \begin{cases} 0 & \text{if } deg(x) = 0 \text{ or } deg(y) = 0 \\ 0 & \text{if } deg(x) = r, deg(y) = r' \\ (-1)^{deg(x)}(x \wedge y) & \text{otherwise} \end{cases}$$

Then we consider that $d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy$, and that $d(x \wedge y) = dx \wedge y + (-1)^{deg(x)+1} x \wedge dy$. Then by linearity of h on the components of the tensor product, we can compute that

$$\begin{aligned} h(d(x \otimes y)) + d(h(x \otimes y)) &= h(dx \otimes y) + (-1)^{deg(x)} h(x \otimes dy) + dh(x \otimes y) \\ &= x \cup y - (-1)^{deg(x)deg(y)} y \cup x. \end{aligned}$$

In fact it easy to verify that when $deg(x) = 0$ or $deg(y) = 0$ we have 0 on both sides or one between the terms $x \wedge dy$ and $-dx \wedge y$.
When $deg(x) = r$ and $deg(y) = r'$ we have that the LHS $= x \wedge dy + (-1)^{deg(dx)} dx \wedge y$ and the RHS $= x \wedge dy - (-1)^{deg(x)deg(y)} y \wedge dx = x \wedge dy - (-1)^{deg(x)(deg(y)+deg(dy))} dx \wedge y$ and by the fact that $deg(y) + deg(dy)$ is odd, we see that RHS=LHS. In the case that $deg(x) \neq 0$ and $deg(y) \neq 0$ but not in the previous case, we have the simple computation on the LHS:

$$(-1)^{deg(dx)} dx \wedge y + x \wedge dy + (-1)^{deg(x)} dx \wedge y + (-1)^{2deg(x)+1} x \wedge y = 0$$

and by definition the RHS is 0 too. □

Note that in the case $r = 0$, the Deligne cohomology groups $H^*_{\mathcal{D}}(X, 0)$ are just the singular cohomology groups of X.

The relevant aspect of this theory is the following description:

Let

$$F^r\Omega_X^\bullet : 0 \to \Omega_X^r \to \Omega_X^{r+1} \to \dots \tag{2.5}$$

the trivial filtration of the de Rham complex. There is obvious inclusion map

$$i : F^r\Omega_X^\bullet \to \Omega_X^\bullet.$$

Then consider $\mathbb{Z}(r)$ as the trivial complex, in which $\mathbb{Z}(r)$ is in the degree 0. Then there is again an obvious inclusion

$$\epsilon : \mathbb{Z}(r) \to \Omega_X^\bullet.$$

Then define the Deligne complex as the following fibered cone:

$$\mathbb{Z}'(r)_{\mathcal{D}} := \mathrm{Cone}(\epsilon - i : \mathbb{Z}(r) \oplus F^r\Omega_X^\bullet \to \Omega_X^\bullet)[-1].$$

It in not difficult to prove that there is a quasi-isomorphism between the two definitions:

Lemma 2.4. *There is a quasi-ismorphism $\alpha : \mathbb{Z}(r)_{\mathcal{D}} \to \mathbb{Z}'(r)_{\mathcal{D}}$.*

Proof. We have to define $\alpha_i : \Omega_X^{i-1} \to \Omega_X^{i-1}$ such that

$$\begin{array}{ccccccccccc}
\mathbb{Z}(r) & \xrightarrow{d} & \mathcal{O}_X & \xrightarrow{d} & \dots & \longrightarrow & \Omega_X^{r-2} & \xrightarrow{d} & \Omega_X^{r-1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\alpha_1} & & & & \downarrow{\scriptstyle\alpha_{r-1}} & & \downarrow{\scriptstyle\alpha_r} & & \downarrow \\
\mathbb{Z}(r) & \xrightarrow{-\epsilon} & \mathcal{O}_X & \xrightarrow{-\delta_1} & \dots & \longrightarrow & \Omega_X^{r-2} & \xrightarrow{-\delta_{r-1}} & \Omega_X^r \oplus \Omega_X^{r-1} & \xrightarrow{-\delta_r} & \Omega_X^{r+1} \oplus \Omega_X^r
\end{array}$$

is a commutative diagram. In fact let's define $\alpha_i(\omega) = (-1)^i\omega$ for $i = 0, \dots, r-1$, $\alpha_r(\omega) = (-1)^r(d\omega, \omega)$. Then $\delta_{r-1}(\eta) = (0, d\eta)$ and $\delta_r(\eta) = (-d\eta, -\psi + d\eta)$. Then we can see that the diagram above commutes, since

$$-\delta_i(\alpha(\omega)) = -\delta_i((-1)^{i-2}\omega) = (-1)^{i-1}d_i(\omega) = \alpha_{i-1}d_i(\omega).$$

then $\ker(\delta_r)/\mathrm{Im}(\delta_{r-1}) = \{(\psi, \eta) : d\eta = \psi\}$ modulo exact forms η. But this group is the same of $\ker(d_r)/\mathrm{Im}\, d_{r-1}$. □

This description allows to compute the cohomology through long exact sequences induced by the structure of the cone and recalling that in general for complexes

$A^\bullet, B^\bullet, C^\bullet$, we have that

$$\mathrm{Cone}(A^\bullet \oplus B^\bullet \to C^\bullet)[-1] \simeq \tag{2.6}$$

$$\mathrm{Cone}(A^\bullet \to \mathrm{Cone}(B^\bullet \to C^\bullet))[-1] \simeq \tag{2.7}$$

$$\mathrm{Cone}(B^\bullet \to \mathrm{Cone}(A^\bullet \to C^\bullet))[-1]. \tag{2.8}$$

Moreover we notice that $\mathrm{Cone}(F^r\Omega^\bullet_X \to \Omega^\bullet_X) \sim_{q.iso} \Omega^{<r}_X := \Omega^\bullet_X/F^r\Omega^\bullet_X$. Then by the long exact sequences for the cone we have:

Proposition 2.5. *The following sequences are exact:*

1. $\cdots \to H^n_{\mathcal{D}}(X,r) \to H^n(X,\mathbb{Z}(r)) \oplus F^rH^n(X,\mathbb{C}) \to H^n(X,\mathbb{C}) \to \ldots$

2. $\cdots \to H^n_{\mathcal{D}}(X,r) \to H^n(X,\mathbb{Z}(r)) \to H^n(X,\mathbb{C})/F^rH^n(X,\mathbb{C}) \to \ldots$

This kind of description allows to generalize a description of a possible cohomology in the non proper case. Assume then that X is a smooth manifold over $\mathbb{C}$. The idea of Beilinson is to choose a *good compactification* of X in a proper manifold $\bar{X}$, that means that there exists an open inclusion $j : X \hookrightarrow \bar{X}$ such that $D := \bar{X} - X$ is a *normal crossing divisor*, i.e. locally in the analytic topology, D has smooth components that intersect transversally.
When a morphism of manifolds $f : Y \to X$ is given, one can choose compactifications $\bar{X}$ and $\bar{Y}$ of X and Y respectively, such that f lifts to a morhpism $\bar{f} : \bar{Y} \to \bar{X}$. The refining of the Deligne cohomology in this setting is the *Deligne-Beilinson cohomology* that is obtained by replacing the differential $\Omega^\bullet_X$, with $Rj_*\Omega^\bullet_X$, $F^r\Omega^\bullet_X$ with $F^r\Omega^\bullet_{\bar{X}}(\log D)$, and $\mathbb{Z}(r)$ with $Rj_*\mathbb{Z}(r)$. Then we define the following:

Definition 2.6. *The Deligne-Beilinson complex of a smooth variety X is given by the following fibered cone:*

$$\mathbb{Z}(r)_{\mathcal{D}} := Cone(\epsilon - i : Rj_*\mathbb{Z}(r) \oplus F^r\Omega^\bullet_{\bar{X}}(\log D) \to Rj_*\Omega^\bullet_X)[-1]. \tag{2.9}$$

Then the Deligne-Beilinson cohomology of X are the hypercohomology groups :

$$H^*_{\mathcal{D}}(X,r) := \mathbb{H}^*(\bar{X}, \mathbb{Z}(r)_{\mathcal{D}}) \tag{2.10}$$

The first problem is the dependence by the compactifications chosen. Actually this definition does not depend by them. If $(\bar{X}, j)$ and $(\bar{X}', j')$ are different compactifications of X, note that by the definition of the complex of fibered cone, the long exact sequences above still hold. In particular we have the exact sequence

$$\cdots \to \mathbb{H}^n(\bar{X}, \mathbb{Z}_{\mathcal{D}}(r)) \to \mathbb{H}^n(\bar{X}, Rj_*\mathbb{Z}(r)) \oplus \mathbb{H}^n(\bar{X}, F^r\Omega^\bullet_{\bar{X}}(\log D)) \to \mathbb{H}^n(\bar{X}, Rj_*\Omega^\bullet_X) \to \dots \tag{2.11}$$

Then there is the analogous one for $(\bar{X}', j')$. But by the good compactification properties any morphism of manifolds $f : \bar{X}' \to \bar{X}$ induces an isomoprhism f^* on the level of hypercohomology groups

$$\mathbb{H}^n(\bar{X}, Rj_*\mathbb{Z}(r)) \oplus \mathbb{H}^n(\bar{X}, F^r\Omega^\bullet_{\bar{X}}(\log D)) \to \mathbb{H}^n(\bar{X}', Rj'_*\mathbb{Z}(r)) \oplus \mathbb{H}^n(\bar{X}', F^r\Omega^\bullet_{\bar{X}'}(\log D'))$$

and

$$\mathbb{H}^n(\bar{X}, Rj_*\Omega^\bullet_X) \to \mathbb{H}^n(\bar{X}', Rj'_*\Omega^\bullet_X).$$

Then, by the five lemma this implies an isomorphism for the Deligne-Beilinson cohomology. This definition is functorial on X. Given an $f : Y \to X$, and a lift $\bar{f}$, then at the level of complexes we have a pullback map

$$\bar{f}^* : \mathbb{Z}(r)^\bullet_{\mathcal{D},\bar{X}} \to \bar{f}_*\mathbb{Z}(r)^\bullet_{\mathcal{D},\bar{Y}} \tag{2.12}$$

and this induces a map in hypercohomology. It also possible to define a ring structure on this cohomology as done before, but for this construction and details we mention and refer the reader to [EV88].

2.1 Cycle class in Deligne cohomology

The object that we want to introduce, is the key object that permits to construct higher cycle maps or "regulators" maps as discussed in the first chapter. We will interpret this regulators as higher Abel-Jacobi maps in the next chapter. To start, we must recall the main facts about the cycle class in the de Rham cohomology and singular cohomology. Through this section X will be a smooth compact manifold over $\mathbb{C}$. If $Y \subset X$ is a closed irreducible subvariety of codimension r in X then there exists a cycle class $cl(Y) \in H^{2r}(X, \mathbb{Z})$ that is of pure type (r, r): that means that it maps on the level r of the Hodge filtration of $H^n(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}(X)$. Since $\mathbb{H}^{2r}(X, F^r\Omega^\bullet_X) = F^rH^{2r}(X; \mathbb{C})$, $cl(Y)$ induces a fundamental class on the Hodge filtration $cl_{Hdg}(Y) \in \mathbb{H}^{2r}(X, F^r\Omega^\bullet_X)$ and the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ induces a map that sends $cl(Y)$ in $cl_{\mathbb{C}}(Y) \in H^{2r}(X, \mathbb{C})$. Similarly, the inclusion $F^r\Omega^\bullet_X \hookrightarrow \Omega^\bullet_X$ induces a map that sends $cl_{Hdg}(Y)$ in $cl_{\mathbb{C}}(Y)$. To be more precise, upon factors by $(2\pi i)^r$, we can consider $cl(Y) \in H^{2r}(X, \mathbb{Z}(r))$ and the inclusion $\mathbb{Z}(r) \hookrightarrow \mathbb{C}$ is the morphism that make possible all the compatibility above. We also recall that the fundamental class in the singular cohomology comes out by a representative in the local cohomology, that is a Thom class $\tau(Y) \in H^{2r}_Y(X, \mathbb{Z}(r))$. What is less obvious is that all these cycle classes can be refined in the local context with all the compatibility that can be satisfied. More precisely the following theorem holds:

Theorem 2.7. *Let X be a compact algebraic manifold and let $Y \subset X$ an irreducible subvariety of codimension r. Then the following holds:*

1. *It is possible to define a Hodge-Thom class*

$$\tau_{Hdg}(Y) \in \mathbb{H}^{2r}_Y(X, F^r\Omega^\bullet_X)$$

   such that the inclusions $F^r\Omega^\bullet_X \to \Omega^\bullet_X$ and $\mathbb{Z}(r) \to \mathbb{C}$ sends $\tau_{Hdg}(Y)$ and respectively $\tau(Y)$ in the same class $\tau_{\mathbb{C}}(Y) \in H^{2r}_Y(X, \mathbb{C})$

2. *$\tau_{Hdg}(Y)$ maps to $cl_{Hdg(Y)}$ by the forgetful map of support.*

3. *There exists a class $\tau^{r,r} \in H^r_Y(X, \Omega^r_X)$ such that the following relations hold:*

$$\begin{array}{ccccc} \tau^{r,r}(Y) & \longleftarrow & \tau_{Hdg}(Y) & \longmapsto & \tau(Y) \\ \downarrow & & \downarrow & & \downarrow \\ cl^{r,r}(Y) & \longleftarrow & cl_{Hdg}(Y) & \longmapsto & cl_{\mathbb{C}}(Y) \end{array}$$

   where the morphisms are given by the following commutative diagram:

$$\begin{array}{ccccc} H^{2r}_Y(X, \Omega^r_X) & \longleftarrow & \mathbb{H}^{2r}_Y(X, F^r\Omega^\bullet_X) & \longrightarrow & H^{2r}_Y(X, \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2r}(X, \Omega^r_X) & \longleftarrow & F^rH^{2r}(X, \mathbb{C}) & \longrightarrow & H^{2r}(X, \mathbb{C}) \end{array}$$

Proof. See ([PS08], Theorem 2.36)

□

As corollary is not difficult to define a *Deligne-Thom class* that permits to define a cycle class in the Deligne cohomology:

Corollary 2.8. *Let $Y \subset X$ a subvariety of codimension r. Then there exists a Deligne-Thom class*

$$\tau_{\mathcal{D}}(Y) \in H_Y^{2r}(X, \mathbb{Z}(r)_{\mathcal{D}})$$

such that it maps to $\tau(Y)$ and $\tau_{Hdg}(Y)$. By forgetting supports it defines a class $cl_{\mathcal{D}}(Y) \in H_{\mathcal{D}}^{2r}(X, \mathbb{Z}(r))$ such that the natural maps $\mathbb{Z}(r)_{\mathcal{D}} \to F^r\Omega_X^\bullet$ and $\mathbb{Z}(r)_{\mathcal{D}} \to \mathbb{Z}(r)$ respects the fundamental classes.

Proof. Let's observe at first that $H_Y^{2r-1}(X, \mathbb{Z}(r)) = 0$ for dimension reasons. Then the long exact sequence for the Deligne (local) cohomology becomes:

$$0 \to H_Y^{2r}(X, \mathbb{Z}(r)_{\mathcal{D}}) \to \mathbb{H}_Y^{2r}(X, F^r\Omega_X^\bullet) \oplus H_Y^{2r}(X, \mathbb{Z}(r)) \to H_Y^{2r}(X, \mathbb{C}) \to \dots \quad (2.13)$$

Then by the previous theorem the element $(\tau_{Hdg}(Y), \tau(Y))$ maps to 0 by the second map on the sequence above. Then by exactness there exists a unique element $\tau_{\mathcal{D}}(Y) \in H_Y^{2r}(X, \mathbb{Z}(r)_{\mathcal{D}})$ and again, by the previous theorem, it satisfies the claimed compatibility.

□

Remark 1. The definition of the Deligne-Beilinson cohomology depends by the analytical structure of the variety. But this definition can be generalized to define a cohomology theory for schemes endowed with Zariski topology. More precisely in ([EV88],5.5) are described complex of sheaves $\mathbb{Z}(r)_{\mathcal{D},Zar}$ for the Zariski topology, with the property that, when X is smooth (variety over $\mathbb{C}$) and we endow it with the analytic topology, by the GAGA theorem their hypercohomology is isomorphic to the Deligne-Beilinson cohomology. Moreover the definition of these sheaves on the Zariski topology has a product structure in the derived category. This shows a first example where we leave the analytic setting towards a more algebraic setting.

In the next we define an higher cycle map and we see its description in terms of higher Abel-Jacobi maps. We are interested mainly to stress some peculiar behaviour of this cohomology that allows to take out the necessary tools to develop other theories with a more algebraic setting. So it is important to have in mind the different structures on the variety, that we will recall when it is necessary.

CHAPTER 3

HIGHER A-J MAP

The aim of the following discussion is to provide a definition of an Abel-Jacobi map extending one discussed in [EV88]. More precisely we have seen by Bloch's work ([Blo86b]) that there exists a natural generalization of a cycle map for Bloch-Ogus cohomological theories: they arise from the construction of the Higher Chow Groups ([Blo86a]). From this we can derive a simple extension of Abel-Jacobi maps in terms of mixed Hodge structures (see [PS08] for the main definitions and properties of MHS.).

Let X be a projective, smooth variety over $\mathbb{C}$. We have have an exact sequence of sheaves by the definition of Deligne complex as follows:

$$0 \longrightarrow \Omega_X^{<p}[-1] \longrightarrow \mathbb{Z}(p)_{\mathcal{D}} \longrightarrow \mathbb{Z}(p) \longrightarrow 0$$

from which derives the long exact sequence in Hypercohomology:

$$\ldots \to H^{k-1}(X, \mathbb{Z}(p)) \xrightarrow{\alpha} \underbrace{\mathbb{H}^{k-1}(X, \Omega_X^{<p})}_{=\frac{H^{k-1}(X,\mathbb{C})}{F^pH^{k-1}(X,\mathbb{C})}} \to H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to H^k(X, \mathbb{Z}(p)) \xrightarrow{\beta} \underbrace{\mathbb{H}^k(X, \Omega_X^{<p})}_{=\frac{H^k(X,\mathbb{C})}{F^pH^k(X,\mathbb{C})}} \to \ldots \tag{3.1}$$

The equality follows by notice that complex of holomorphic differentials can be viewed as

$$\Omega_X = \mathrm{Cone}(\Omega_X^{<p}[-1] \xrightarrow{0} F^p\Omega_X)$$

from which we have the exact sequence

$$0 \longrightarrow F^p\Omega_X \longrightarrow \Omega_X^{<p} \bigoplus F^p\Omega_X = \Omega_X \longrightarrow \Omega_X^{<p} \longrightarrow 0$$

and by the derived exact sequence

$$... \to \mathbb{H}^{q-1}(X, \Omega_X^{<p}) \xrightarrow{0} \mathbb{H}^q(X, F^p) \to \mathbb{H}^q(X, \Omega_X) \to \mathbb{H}^q(X, \Omega_X^{<p}) \xrightarrow{0} \mathbb{H}^{q+1}(X, F^p) \to ..$$

Recall that $\mathbb{H}^q(X, \Omega_X) \simeq H^q(X, \mathbb{C})$ between the De-Rham and singular cohomology and that $\mathrm{Im}\,(\mathbb{H}^q(X, F^p\Omega_X) \longrightarrow \mathbb{H}^q(X, \Omega_X)) \simeq F^pH^q(X, \mathbb{C})$. We split the long exact sequence to obtain the short exact sequence

$$0 \longrightarrow \operatorname{coker} \alpha \longrightarrow H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow \ker \beta \longrightarrow 0.$$

Now

$$\operatorname{coker} \alpha = \frac{H^{k-1}(X, \mathbb{C})}{F^pH^{k-1}(X, \mathbb{C}) + H^{k-1}(X, \mathbb{Z}(p))}$$

and

$$\ker \beta = H^k(X, \mathbb{Z}(p)) \cap F^pH^k(X, \mathbb{C})$$

and so it rewrites as:

$$0 \to \frac{H^{k-1}(X, \mathbb{C})}{F^pH^{k-1}(X, \mathbb{C}) + H^{k-1}(X, \mathbb{Z}(p))} \to H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to H^k(X, \mathbb{Z}(p)) \cap F^pH^k(X, \mathbb{C}) \to 0. \tag{3.2}$$

Let $k = 2p - n$ and define

$$\mathcal{J}^{p,n}(X) := \frac{H^{2p-n-1}(X, \mathbb{C})}{F^pH^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Z}(p))}.$$

Recall that in (2.1) we have defined a cycle map for the Deligne-Beilinson cohomology:

$$Bl_{p,0} : CH^p(X, 0) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

By the method exposed in (1.21), then it is defined an higher cycle class map

$$Bl_{p,n} : CH^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

Moreover we define

$$CH^p_{hom}(X, n) := \ker \left(CH^p(X, n) \xrightarrow{Bl_{p,n}} H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to H^{2p-n}(X, \mathbb{Z}(p)) \right).$$

We observe that if $[Z] \in CH^p_{hom}(X,n)$ then by the long exact sequence (3.1), it follows that

$$\begin{aligned} Bl_{p,n}([Z]) \in \ker\left(H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p-n}(X,\mathbb{Z}(p))\right) = \\ \operatorname{Im}\left(\mathbb{H}^{2p-n-1}(X,\Omega_X^{<p}) \xrightarrow{\delta} H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p))\right) \\ \simeq \frac{\mathbb{H}^{2p-n-1}(X,\Omega_X^{<p})}{\ker\delta} = \operatorname{coker}\alpha = \mathcal{J}^{p,n}(X). \end{aligned}$$

This means that $Bl_{p,n}$ induces a map

$$\Phi_{p,n} : CH^p_{hom}(X,n) \longrightarrow \mathcal{J}^{p,n}(X) \tag{3.3}$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{J}^{p,n}(X) & \longrightarrow & H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \\ {\scriptstyle\Phi_{p,n}}\uparrow & & {\scriptstyle Bl_{p,n}}\uparrow \\ CH^p_{hom}(X,n) & \longrightarrow & CH^p(X,n) \end{array} \tag{3.4}$$

commutes.

3.1 Review on the case $\Phi_{p,0}$ $(n=0)$

Let's observe that in the case $n=0$ the diagram (3.4) corresponds to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Z}^p_{hom}(X) & \longrightarrow & \mathcal{Z}^p(X) & \longrightarrow & CH^p(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Phi_{p,0}} & & \downarrow{\scriptstyle Bl_{p,0}} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{J}^p(X) & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & H^{2p}(X,\mathbb{Z}(p)) \cap H^{p,p}(X,\mathbb{C}) & \longrightarrow & 0 \end{array}$$

In fact in the case above described we claim that $CH^p_{hom}(X,0) = \mathcal{Z}^p_{hom}(X)/rat.eq.$ and that a cycle rational equivalent to 0 is the 0 class in $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$. The last sentence is proved in [[EV88],Prop.7.6]. In particular we know the cycle map factors through $CH^p(X,0) = \mathcal{Z}^p(X)/rat.eq.$:

$$\begin{array}{ccc} \mathcal{Z}^p(X,0) & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \\ & \searrow \quad \nearrow{\scriptstyle Bl_{p,0}} & \\ & CH^p(X,0) & \end{array}$$

Recall that the composition

$$\mathcal{Z}^p(X,0) \to H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p}(X,\mathbb{Z}(p))$$

is the cycle map for singular cohomology and that

$$\mathcal{Z}^p_{hom}(X) = \mathcal{Z}^p_{hom}(X,0) := \ker\left(\mathcal{Z}^p(X,0) \to H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p}(X,\mathbb{Z}(p))\right)$$

so that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{Z}^p_{hom}(X,0) & \longrightarrow & \mathcal{Z}^p(X,0) & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & H^{2p}(X,\mathbb{Z}(p)) \\ & & \overset{\delta}{\searrow} & \searrow & \nearrow & & & \\ & & & CH^p(X,0) & & & & \end{array}$$

where δ is defined by the obvious composition.
In this way we obtain the exact sequence

$$0 \longrightarrow \operatorname{Im}\delta \longrightarrow CH^p(X,0) \longrightarrow H^{2p}(X,\mathbb{Z}(p))$$

and so $CH^p_{hom}(X,0) = \operatorname{Im}\delta = \mathcal{Z}^p_{hom}(X)/rat.eq.$
This proves that the cycle map defined by Bloch in the case $n=0$ is simply the factorization of the classical cycle map modulo rational equivalence.

3.1.1 Different description of $\Phi_{p,0}$

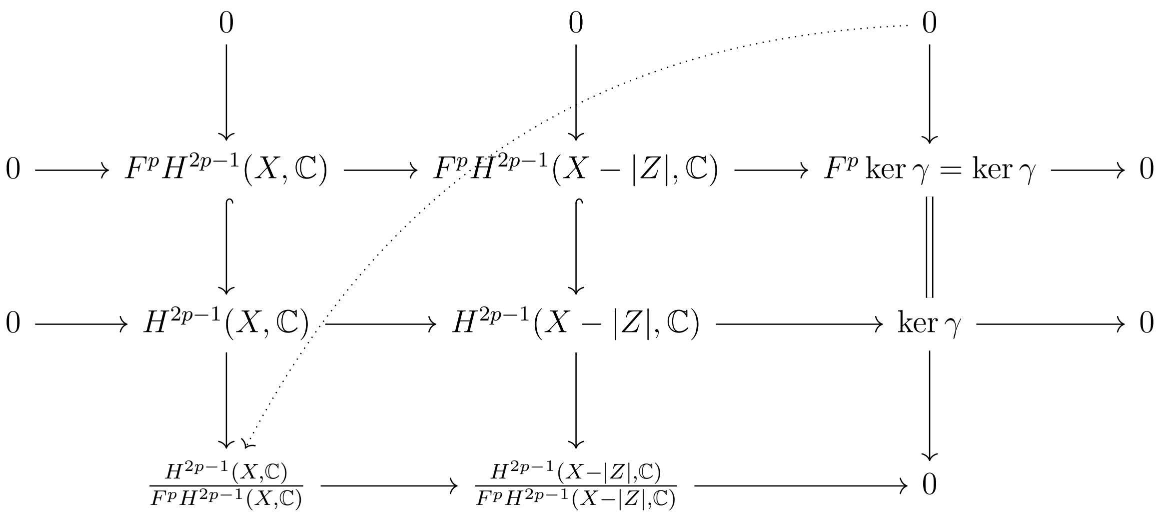
Now we can derive an alternative description of the map $\Phi_{p,0}$ assuming we have a cohomological theory that satisfies:

1. Weak purity: $H^k_{|Z|}(X) = 0$ for $|Z| := supp(Z)$, $Z \in \mathcal{Z}^p(X) \quad \forall k < 2p$
2. Homotopy axiom: $H^k(X \times \mathbb{A}^1) = H^k(X) \quad \forall k$
3. Mixed Hodge structure.

All the conditions are satisfied by the singular cohomology, in particular we can assume that the morphisms which involve in the discussion arise from the mixed structure. For the moment consider the case $n=0$. From the long exact sequence of cohomology with support in $|Z|$ and from weak purity we have the following:

$$\underbrace{H^{2p-1}_{|Z|}(X,\mathbb{Z}(p))}_{=0} \to H^{2p-1}(X,\mathbb{Z}(p)) \xrightarrow{\beta} H^{2p-1}(X-|Z|,\mathbb{Z}(p)) \to H^{2p}_{|Z|}(X,\mathbb{Z}(p)) \xrightarrow{i} H^{2p}(X,\mathbb{Z}(p)) \tag{3.5}$$

Observe that $H^{2p}_{|Z|}(X,\mathbb{Z}(p)) \simeq H^0(|Z|,\mathbb{Z}) = \bigoplus_{Y\in|Z|}\mathbb{Z}$ and $F^pH^{2p}_{|Z|} = H^{2p}_{|Z|}$. Moreover the exact sequence is in terms of mixed Hodge structure, so it respect such structure. In particular we have an exact sequences' diagram for which we can apply snake lemma:



In this way we see β induces an isomorphism

$$\frac{H^{2p-1}(X,\mathbb{C})}{F^pH^{2p-1}(X,\mathbb{C})} \simeq \frac{H^{2p-1}(X-|Z|,\mathbb{C})}{F^pH^{2p-1}(X-|Z|,\mathbb{C})}$$

and so in particular

$$\mathcal{J}^{p,0}(X) = \frac{H^{2p-1}(X,\mathbb{C})}{F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z}(p))} \simeq \frac{H^{2p-1}(X-|Z|,\mathbb{C})}{F^pH^{2p-1}(X-|Z|,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z}(p))}. \tag{3.6}$$

If $Z \in \mathcal{Z}^p_{hom}(X)$ there exists a unique (modulo $\operatorname{Im}\beta$) $\tilde{c}_{\mathbb{Z}}(Z) \in H^{2p-1}(X-|Z|,\mathbb{Z}(p))$ that maps in $H^{2p-1}(X-|Z|,\mathbb{C})$ and after take the class in the quotient 3.6, by isomorphism there is a unique $\psi_{p,0}(Z) \in \mathcal{J}^{p,0}(X)$.

Definition 3.1. $\psi_{p,0} : \mathcal{Z}^p_{hom}(X) \longrightarrow \mathcal{J}^{p,0}(X)$ *is called Abel-Jacobi map.*

Theorem 3.2. *We have that* $\psi_{p,0} = \Phi_{p,0}$.

Proof. See [EV88] Prop.7.11. □

3.1.2 Extensions in MHS

Before to go further let's observe the role of the long exact sequence 3.5 in the context of the extensions in the MHS (mixed Hodge structure). An equivalent formulation of the 3.2 is:

$$0 \to \mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(X,\mathbb{Z})(p)) \to H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{2p}(X,\mathbb{Z}(p))) \tag{3.7}$$

from the relation $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(X,\mathbb{Z})(p)) = \mathcal{J}^{p,0}(X)$ ([PS08], Theorem 3.31 (1)). We want to understand how a cycle class defines an extension class. Let's recall that a cycle class is given by a map $\mathbb{Z} \to H^{2p}_{|Z|}(X,\mathbb{Z}(p))$, and when it is nullhomologous it means that when we forget the support it is the zero map (for $Y \in |Z|$ yields $i[Y] = 0$ where i is defined in 3.5). Choosing a pullback of $\mathbb{Z}$ coming from the 3.5, we obtain the following extension :

$$0 \to H^{2p-1}(X,\mathbb{Z}(p)) \to \mathbf{E} \to \mathbb{Z} \to 0 \tag{3.8}$$

where $\mathbf{E} \simeq H^{2p-1}(X,\mathbb{Z}(p)) \oplus \tilde{c}_{\mathbb{Z}}(Y)$ determines a unique extension class. This will be the more general framework on which we will define Abel-Jacobi maps.

3.2 Generic $\Phi_{p,n}$ $(n > 0)$

We want now to generalize this construction for a generic number n. Let

$$\Delta^n := \mathrm{Spec}\,(\mathbb{C}[t_0, ..., t_n]/(t_0 + ... + t_n - 1))$$

the algebraic $n-$*simplex*. The faces are the subsimplexes of Δ^n given by $\{t_i = 0\} =: H_i$. We define

$$\partial\Delta^n := \bigcup_{i=0}^{n} H_i.$$

For the singular cohomology theory we can compute

$$H^i(\partial\Delta^n, \mathbb{Z}(p)) = \begin{cases} \mathbb{Z}(p) & \text{if } i = 0 \text{ or } i = n-1 \\ 0 & \text{else} \end{cases}$$

If $Z \in \mathcal{Z}^p(X, n)$ we define

$$\partial Z := Z \cap (X \times \partial\Delta^n).$$

Let us indicate $U := (X \times \Delta^n) - |Z|$ and $\partial U = U \cap (X \times \partial\Delta^n)$.

We collect some useful remarks:

Lemma 3.3. *With the notation above we have*

1. $H^i(X, \mathbb{Z}(p)) = H^i(U, \mathbb{Z}(p)) \quad i < 2p-1$
2. $H^i(X \times \partial\Delta^n, \mathbb{Z}(p)) = H^i(\partial U, \mathbb{Z}(p)) \quad i < 2p-1$
3. $H^i(X \times \partial\Delta^n, \mathbb{Z}(p)) = H^i(X, \mathbb{Z}(p)) \oplus H^{i-n+1}(X, \mathbb{Z}(p))$.

Proof. 1) We have a long exact sequence in local cohomology with support:

$$0 = H^i_{|Z|}(X\times\Delta^n, \mathbb{Z}(p)) \to H^i(X\times\Delta^n, \mathbb{Z}(p)) \to H^i(U, \mathbb{Z}(p)) \to H^{i+1}_{|Z|}(X\times\Delta^n, \mathbb{Z}(p)) = 0$$

where the 0's come from weak purity and $H^i(X\times\Delta^n) = H^i(X)$ by homotopy axiom.
2) It is analogous to 1) :

$$0 = H^i_{|\partial Z|}(X\times\partial\Delta^n, \mathbb{Z}(p)) \to H^i(X\times\partial\Delta^n, \mathbb{Z}(p)) \to H^i(\partial U, \mathbb{Z}(p)) \to H^{i+1}_{|\partial Z|}(X\times\partial\Delta^n, \mathbb{Z}(p)) = 0$$

where the weak purity is applied to the faces of the simplex that for definition intersect properly the cycle Z.
3) It is just the consequences of the Kunneth formula in cohomology and the computation of cohomology of $\partial\Delta^n$. □

From the long exact sequence for relative cohomology and the lemma 3.3 we obtain:

$$\begin{array}{ccccccccc}
H^{2p-2}(U, \mathbb{Z}(p)) & \longrightarrow & H^{2p-2}(\partial U, \mathbb{Z}(p)) & \longrightarrow & H^{2p-1}(U, \partial U, \mathbb{Z}(p)) & \to & H^{2p-1}(U, \mathbb{Z}(p)) & \to & H^{2p-1}(\partial U, \mathbb{Z}(p)) \\
\|\,{\scriptstyle 1.} & & \|\,{\scriptstyle 2.} & & & & & & \\
H^{2p-2}(X, \mathbb{Z}(p)) & \to & H^{2p-2}(X \times \partial\Delta^n, \mathbb{Z}(p)) & & & & & & \\
 & & \|\,{\scriptstyle 3.} & & & & & & \\
 & & H^{2p-2}(X, \mathbb{Z}(p)) \oplus H^{2p-n-1}(X, \mathbb{Z}(p)) & & & & & &
\end{array}$$

from which derive the following short exact sequence:

$$0 \to H^{2p-n-1}(X, \mathbb{Z}(p)) \to H^{2p-1}(U, \partial U) \to \ker\left(H^{2p-1}(U) \to H^{2p-1}(\partial U)\right) \to 0. \tag{3.9}$$

We want to mimic the construction of an isomorphism for the $\mathcal{J}^{p,n}(X)$ like in the previous section, where we deal with case $n = 0$. For doing that, we need of a certain long exact sequence (of mixed Hodge structure).

Theorem 3.4. *The following is a long exact sequence of mixed Hodge structures:*

$$0 \to H^{2p-n-1}(X, \mathbb{Z}(p)) \to H^{2p-1}(U, \partial U, \mathbb{Z}(p)) \to L(U) \xrightarrow{\gamma} H^{2p-n}(X, \mathbb{Z}(p)) \tag{3.10}$$

where $L(U)$ *is a subgroup of* $H^{2p}_{|Z|}(X \times \Delta^n, \mathbb{Z}(p))$*, so that* $F^pL(U) = L(U)$.

Proof. Let be

$$H^{2p}_{|Z|}(X \times \Delta^n)^\circ := \ker\left(H^{2p}_{|Z|}(X \times \Delta^n) \to H^{2p}(X \times \Delta^n)\right)$$

and

$$H^{2p}_{|\partial Z|}(X \times \partial\Delta^n)^\circ := \ker\left(H^{2p}_{|\partial Z|}(X \times \partial\Delta^n) \to H^{2p}(X \times \partial\Delta^n)\right)$$

Considering the long exact sequence in local cohomology, we obtain the following diagram of exact sequences, for which we apply the snake lemma:

$$\begin{array}{ccccccccc}
 & & 0 & \dashrightarrow & \ker(\alpha) & \dashrightarrow & \ker(\beta) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 = H^{2p-1}_{|Z|}(X \times \Delta^n) & \longrightarrow & H^{2p-1}(X \times \Delta^n) & \longrightarrow & H^{2p-1}(U) & \longrightarrow & H^{2p}_{|Z|}(X \times \Delta^n)^\circ & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \beta & & \\
0 = H^{2p-1}_{|\partial Z|}(X \times \partial\Delta^n) & \to & H^{2p-1}(X \times \partial\Delta^n) & \to & H^{2p-1}(\partial U) & \to & H^{2p}_{|\partial Z|}(X \times \partial\Delta^n)^\circ & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & H^{2p-n}(X) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Put $L(U) := \ker\beta$. In this way we have the short exact sequence

$$0 \to \ker\alpha \to L(U) \to H^{2p-n}(X, \mathbb{Z}(p))$$

that glue together with the (3.9) and it gives the searched one. □

We are now ready just to follow what we have done in the case $n = 0$. In particular if we call

$$\beta' : H^{2p-n-1}(X, \mathbb{Z}(p)) \longrightarrow H^{2p-1}(U, \partial U, \mathbb{Z}(p))$$

then follows, by the fact that $F^p \ker\gamma = \ker\gamma$, that β' induces an isomorphism:

$$\frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C})} \simeq \frac{H^{2p-n-1}(U, \partial U, \mathbb{C})}{F^p H^{2p-n-1}(U, \partial U, \mathbb{C})}$$

and so in particular

$$\mathcal{J}^{p,n}(X) = \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Z}(p))} \simeq \tag{3.11}$$

$$\simeq \frac{H^{2p-n-1}(U, \partial U, \mathbb{C})}{F^p H^{2p-n-1}(U, \partial U, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Z}(p))}. \tag{3.12}$$

If $Z \in CH^p_{hom}(X,n)$ there exists a unique $\tilde{c}_{\mathbb{Z}}(Z) \in H^{2p-1}(U,\partial U,\mathbb{Z}(p))$ (modulo $\mathrm{Im}\,(H^{2p-n-1}(X,\mathbb{Z}(p)) \to H^{2p-1}(U,\partial U,\mathbb{Z}(p)))$ that maps in $H^{2p-1}(U,\partial U,\mathbb{C})$ and after taking the class in the quotient 3.12, by isomorphism there is a unique $\psi_{p,n}(Z) \in \mathcal{J}^{p,n}(X)$.

Definition 3.5. $\psi_{p,n} : CH^p_{hom}(X,n) \longrightarrow \mathcal{J}^{p,n}(X)$ *is called higher Abel-Jacobi map.*

Theorem 3.6. *We have that* $\psi_{p,n} = \Phi_{p,n}$.

The proof deals with the framework of the MHS, so let's observe that as in the case $n = 0$, the nullhomologous cycle class pullbacks to give an extension class in $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-n-1}(X,\mathbb{Z}(p)))$ (3.1.2). This is what we have did explicitly with the construction of the map $\psi_{p,n}$. There is not a direct link to the map $\Phi_{p,n}$ that deals with Deligne cohomology. This connection arises from the fact that the Deligne cohomology is the absolute cohomology for the mixed Hodge category as proved by Beilinson in ([Beĭ86]). There is a different description of the exact sequence 3.10 using a simplicial complex that tracks the simplicial structure of $X \times \Delta^n$. What we need to compare the two descriptions is the following. Any cycle $Z \in CH^p(X,n)$ can be assumed *relative* to $X \times \partial\Delta^n$ in the sense that the intersections of Z with the faces of $X \times \Delta^n$ represent zero cycles. Since $H^{2p}(X \times \Delta^n,\mathbb{Z}(p)) \to H^{2p}(X \times \partial\Delta^n,\mathbb{Z}(p))$ is injective, a fundamental class of Z is 0 in both $H^{2p}(X \times \Delta^n)$ and $H^{2p}_{\partial Z}(X \times \partial\Delta^n)$ and so it arise naturally to an element of $L(U)$. Moreover if Z is actually nullhomologous, then it implies $\gamma([Z]) = 0$. Since we will consider $X \times \Delta^n$ as the geometric realization of a simplicial complex that we will call $\Sigma^n X$, there is a natural map given by the forgetful functor of the realizations:

$$for^* : H^{2p}_{|Z|}(X \times \Delta^n,\mathbb{Z}(p)) \to H^{2p}_{|Z|}(\Sigma^n X,\mathbb{Z}(p))$$

such that the extension classes of Z in both the descriptions are the same. For the definition of the simplicial complex $\Sigma^n X$ we refer to [Sch93]. From this construction we just recall that we have an a long exact sequence of cohomology with support as following:

$$0 \to H^{2p-1}(\Sigma^n X,\mathbb{Z}(p)) \to H^{2p-1}(\Sigma^n X - Z.,\mathbb{Z}(p)) \xrightarrow{\partial} H^{2p}_{Z.}(\Sigma^n X,\mathbb{Z}(p)) \xrightarrow{\gamma^s} H^{2p}(\Sigma^n X,\mathbb{Z}(p)). \tag{3.13}$$

where the s is in place of "simplicial" (to distinguish from the corresponding map in singular cohomology). Moreover $H^j(\Sigma^n X, \mathbb{Z}(p)) = H^{j-n}(X, \mathbb{Z}(p))$. Notice that $\gamma^s \circ for^*([Z]) = \gamma([Z]) = 0$. This compatibility implies a map of the extensions of the couple $(\mathbb{Z}, H^{2n-p-1}(X, \mathbb{Z}(p)))$ given separately by 3.10 and 3.13, so it shows the extension class of Z is the same. Recall now that the cohomology of Deligne is computed as the Yoneda Ext functor for the category of mixed Hodge structures: more specifically if $\mathbb{Z}_X(p)$ is the twisted constant sheaf over X we have that

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) = \mathrm{Ext}^k_{MHS}(\mathbb{Z}, R\Gamma(\mathbb{Z}_X))$$

where $R\Gamma$ is the functor of global sections in the derived category of MHS. ([Beĭ86]). This description yields the existence of a convergent Grothendieck spectral sequence. In fact we have that

$$E_2^{p,q} = R^p\mathrm{Hom}_{MHS}(\mathbb{Z}, -) \circ R^q\Gamma(\mathbb{Z}_X(r)) \implies R^{p+q}(\mathrm{Hom}_{MHS}(\mathbb{Z}, R\Gamma(\mathbb{Z}_X(r)))) = \mathrm{Ext}_{MHS}(\mathbb{Z}, R\Gamma(\mathbb{Z}_X(r))) = H^{p+q}_{\mathcal{D}}(X, \mathbb{Z}(r)). \quad (3.14)$$

Since $R^q\Gamma(\mathbb{Z}_X(p)) = H^q(X, \mathbb{Z}(p))$, by convergence, the natural map $F^0 H^{2p-n}_{\mathcal{D}} \to E_\infty^{0,2p-n} \to E_2^{0,2p-n}$ is the map

$$H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \mathrm{Hom}_{MHS}(\mathbb{Z}, H^{2p-n}(X, \mathbb{Z}(p)))$$

so that the Bloch's map arises naturally by the edge morphism induced by the spectral sequence:

$$\ker\left(H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \mathrm{Hom}_{MHS}(\mathbb{Z}, H^{2p-n}(X, \mathbb{Z}(p)))\right) \xrightarrow{edge} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-n-1}(X, \mathbb{Z}(p))) \quad (3.15)$$

With this notation, we can reformulate the theorem 3.6:

Theorem 3.7. *The composed map*

$$CH^p_{hom}(X, n) \hookrightarrow \ker\left(H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to Hom_{MHS}(\mathbb{Z}, H^{2p-n}(X, \mathbb{Z}(p)))\right) \xrightarrow{edge} Ext^1_{MHS}(\mathbb{Z}, H^{2p-n-1}(X, \mathbb{Z}(p))) \quad (3.16)$$

is equal to $\Phi_{p,n}$.

Proof. Let $Z \in CH^p_{hom}(X, n)$. The fundamental class of Z in the absolute cohomology belongs to $\ker(H^{2p}_{\mathcal{D},|Z|}(\Sigma^n X) \to \mathrm{Hom}(\mathbb{Z}, H^{2p}(\Sigma^n X)))$ and maps to the fundamental class of Z in the singular cohomology, that belongs to $\ker\left(\mathrm{Hom}(\mathbb{Z}, H^{2p}_{|Z|}(\Sigma^n X)) \to \mathrm{Hom}(\mathbb{Z}, H^{2p}(\Sigma^n X))\right)$ and by edge morphism, it maps to the extension class. On the other hand, the map $\Phi_{p,n}$ maps the fundamental class of the absolute cohomology in $\ker\left(H^{2p}_{\mathcal{D}}(\Sigma^n X) \to \mathrm{Hom}(\mathbb{Z}, H^{2p}(\Sigma^n X))\right)$ that by the exact sequence 3.2 is $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(\Sigma^n X))$. Keeping track of this observations, the theorem follows by the theorem [[Jan90], Lemma 9.5] in homological algebra. Actually it implies that we have the following commutative diagram:

$$\begin{array}{ccc}
\ker\left(H^{2p}_{\mathcal{D},|Z|}(\Sigma^n X) \to \mathrm{Hom}(\mathbb{Z}, H^{2p}(\Sigma^n X))\right) & \longrightarrow & \ker\left(H^{2p}_{\mathcal{D}}(\Sigma^n X) \to \mathrm{Hom}(\mathbb{Z}, H^{2p}(\Sigma^n X))\right) \\
\downarrow & & \downarrow \\
\ker\left(H^{2p}_{|Z|}(\Sigma^n X) \to \mathrm{Hom}(\mathbb{Z}, H^{2p}(\Sigma^n X))\right) & \xrightarrow{\ edge\ } & \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(\Sigma^n X))
\end{array} \tag{3.17}$$

and this is precisely the assertion of the theorem. More precisely, the theorem [[Jan90], Lemma 9.5] asserts that for an exact sequence of complexes in a category $\mathcal{A}$ with enough injective: $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and a left exact functor $G : \mathcal{A} \to \mathcal{B}$, for a category $\mathcal{B}$, the following is a commutative diagram:

$$\begin{array}{ccccc}
\ker\left(R^iG(A^\bullet) \to G(H^i(B^\bullet))\right) & \longrightarrow & \ker\left(R^iG(B^\bullet) \to G(H^i(B^\bullet))\right) & & \\
\downarrow & & & \searrow & \\
\ker\left(G(H^i(A^\bullet)) \to G(H^i(B^\bullet))\right) & & & & R^1G(H^{i-1}(B^\bullet)) \\
& \searrow & & \swarrow & \\
& & R^1G(\mathrm{coker}\left(H^{i-1}(A^\bullet) \to H^{i-1}(B^\bullet)\right)) & &
\end{array} \tag{3.18}$$

Now for

$$\begin{aligned}
A^\bullet &= R\Gamma_{|Z|}(\mathbb{Z}_{\Sigma^n X}) \\
B^\bullet &= R\Gamma(\mathbb{Z}_{\Sigma^n X}) \\
C^\bullet &= R\Gamma(\mathbb{Z}_{\Sigma^n X - |Z|}) \\
G &= \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, -),
\end{aligned}$$

we obtain the diagram 3.17.

□

The theorem 3.7 suggests a way to thinking kind of "Abel-Jacobi maps" in a more general setting. In such setting we deal with the formulation of two kind of cohomology theories: one of geometric, and the other with arithmetic flavour. More precisely, for a category of smooth algebraic varieties (or more generally smooth schemes over a field) that include projective varieties on a field K, we can associate a topology on it (analytic, étale, Zariski, etc.) and a functor of global sections Γ. Then the corresponding derived functors are the "geometric" cohomology groups. But as in the example of the singular cohomology, these groups together with the twist with integers , has expected to inherit more structure (as for example the singular cohomology has the MHS), sitting in a abelian tensor category $\mathcal{C}$ with unitary object $\mathbf{1}_{\mathcal{C}}$. If one takes account of the abelian structure of this category, we can form the Yoneda Hom functor:

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, -) \in \mathrm{End}(\mathbf{1}_{\mathcal{C}}) - \mathrm{mod}$$

and the derived functors are the "absolute" (or "arithmetic") cohomologies (as for the Deligne cohomology). Then these cohomology theories are expected to be relate as in 3.14. The edge morphism should provide an Abel-Jacobi type map (as in 3.15): it means that we search for a suitable convergent spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, H^q_{geo}(X)(j)) \implies H^{p+q}_{abs}(X)(j) \qquad j \in \mathbb{Z} \tag{3.19}$$

where j is the twist.

CHAPTER 4

CRYSTALLINE COHOMOLOGY

This chapter is devoted to discuss about a p-adic cohomology theory. Our interest is to obtain similar considerations to the Deligne-Beilinson cohomology, but in the p-adic context. Then, following these similarities we motivate the introduction of syntomic sheaves, and the associated syntomic cohomology, that, for us, represents the analogous relationship between Deligne and singular cohomology. Some bad behaviour on the crystalline theory can represent an obstacle for some purpose, so we choose to discuss a "generalized" form in the next chapter, that has a nice behaviour where the crystalline fails. We follows, for this part of the theory, essentially [BO78] in combination with [Kat85].

4.1 Divided powers

Let R a commutative ring, I and ideal.

Definition 4.1. *A divided power structure (P.D. structure) on I, is a colletion of maps $\{\gamma_i : I \to R\}_i$, for integers $i \geq 0$ such that the following conditions hold:*

1. $\gamma_0(X) = 1 \quad \gamma_1(x) = x \quad \forall x \in I$
2. $\gamma_n(x) \in I \quad n \geq 1$
3. $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y) \quad x, y \in I$

4. $\gamma_n(\lambda x) = \lambda^n \gamma_n(X) \quad \forall \lambda \in R$

5. $\gamma_n(X)\gamma_m(X) = \binom{n+m}{n}\gamma_{n+m}(x) \quad \forall x \in I$

6. $\gamma_m(\gamma_n(X)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x) \quad \forall x \in I.$

Remark 2. The property 1) and 5) implies $n!\gamma_n(x) = x^n$. In fact it follows by a simple induction with the trivial case of $n = 1$, that since $(n+1)\gamma_{n+1}(x) = x\gamma_n(x)$, then $n!(n+1)\gamma_{n+1}(x) = x^{n+1}$. Morover, by 4) $\gamma_n(0) = 0$ for $n \geq 0$.

Definition 4.2. *A morphism of P.D. structure $f : (R, I, \gamma) \to (P, J, \delta)$, is a ring homomorphism $f : R \to P$ such that $f(I) \subset J$ and $\delta_n(f(x)) = f(\gamma_n(x))$ for each $n \geq 0$ and $x \in I$.*

Remark 3. When we looking for a P.D. structure on an ideal, morally we want to verify if the elements of the form $x^n/n!$ lie in the ideal. For example, in the case of a DVR with mixed characteristic $(0, p)$ and ramification e, a classical estimates of $|n!|$ shows that the maximal ideal generated by the uniformizer π, admits P.D. structure iff $e \leq p - 1$.

The ideal of the P.D. structure is called "P.D. ideal".

Definition 4.3. *Let (I, γ) a P.D. ideal, then $J \subset I$ is a sub-P.D. ideal iff $\gamma_n(J) \subset J$ for each $n \geq 1$.*

Under some conditions we can transport P.D. structures.

Lemma 4.4. *Let (R, I, γ) be a P.D. structure, and J an ideal of R. Consider the ideal $\bar{I} = I/I \cap J$ on R/J. Then the P.D. structure γ on I extends (uniquely) on $\bar{\gamma}$ P.D. structure on $\bar{I}$ iff $J \cap I$ is sub-P.D. ideal. Here "extends" means that $(R, I, \gamma) \to (R/J, \bar{I}, \bar{\gamma})$ is a P.D. morphism.*

Proof. If $\bar{\gamma}$ exists, then it is unique. Moreover when $x \in J \cap I$, we have that $\overline{\gamma_n(x)} = \bar{\gamma}_n(\bar{x}) = 0$, where the $^-$symbol denotes the class in $I/I \cap J$. Then $\gamma_n(x) \in J \cap I$.Conversely, define $\bar{\gamma} : \bar{x} \mapsto \overline{\gamma_n(x)}$. It is well defined. In fact if y is another presentation of $\bar{x}$, then $x - y \in I \cap J$. So it holds that

$$\gamma_n(x) = \gamma_n(y + (x - y)) = \sum_{i+j=n} \gamma_i(y)\gamma_j(x - y).$$

Since $I \cap J$ is P.D. sub ideal, we have $\gamma_j(x-y) \in I \cap J$ for $j \geq 1$, but then this implies

$$\overline{\gamma_n(x)} = \overline{\gamma_n(y)}.$$

□

Moreover, to verify that an ideal is a sub ideal (and so verify some extension property) is sufficient to verify that on a set of generators:

Lemma 4.5. *Let (R, I, γ) be a P.D. structure and $J \subset I$ and ideal. Then if $J = \langle S \rangle$ where S is a subset of I, J is sub-P.D. ideal iff $\gamma_n(S) \in S$ for each $n \geq 1$.*

Proof. We have to prove only the sufficiency. Let $J' = \{x \in J : \gamma_n(x) \in J \quad n \geq 1\}$. The proposition claims that J' is an ideal. In fact if $x, y \in J'$ then $\gamma_n(x+y) = \sum_{i+j} \gamma_i(x)\gamma_j(y) \in J$ because at least one of i or j is ≥ 1 and because one of $\gamma_i(x)$ or $\gamma_j(y)$ is in J, that is an ideal. This means that $x + y \in J'$. If $x \in J'$ and $\lambda \in R$, then $\gamma_n(\lambda x) = \lambda^n \gamma_n(x) \in J$, so $\lambda x \in J'$. □

Lemma 4.6. *Let (R, I, γ) and (R, J, δ) P.D. structures . Then IJ is a P.D. sub ideal of both I and J and $\gamma = \delta$ restricted on IJ.*

Proof. Since all the elements xy with $x \in I$ and $y \in J$ are a set of generators of IJ, it suffices to prove that $\gamma_n(xy) = \delta_n(xy) \in IJ$. But we have that

$$\gamma_n(xy) = y^n \gamma_n(x) = n! \delta_n(y) \gamma_n(x) = x^n \delta_n(y) = \delta_n(xy).$$

Moreover is clear that $\gamma_n(xy) \in IJ$, since $n | \gamma_n(x)\delta_n(y) \in IJ$. □

Corollary 4.7. *If (I, γ) is a P.D. ideal, then $I^n \subset I$ a sub-P.D. ideal.*

Now recall the notion of extension of P.D. structures and compatibility in a more general sense.

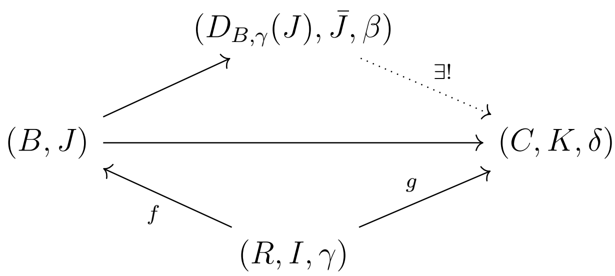
Definition 4.8. *Let (R, I, γ) be a P.D. structure and let B be an R-algebra. Then γ on R extends to $\bar{\gamma}$ on B if and only if $(R, I, \gamma) \to (B, IB, \bar{\gamma})$ is a P.D. morphism.*

Definition 4.9. *Let $(R, I\gamma)$ be a P.D. structure and let B an R-algebra, such that (B, J, δ) is a P.D. structure. Then γ and δ are compatible if and only if γ extends on B and $\bar{\gamma} = \delta$ on $IB \cap J$.*

4.1.1 Divided power envelope

A useful property of the divided power structures is that satisfies a universal property:

Theorem 4.10. *(divided power envelope) Let (R, I, γ) a P.D. structure. Let B be an R-algebra and J an ideal of B. Then for each B-algebra C with P.D. structure (C, K, δ) with $JC \subset K$ and with δ and γ compatible, there exists a unique P.D. morphism of B-algebra $(D_{B,\gamma}(J), \bar{J}, \beta) \to (C, K, \delta)$ such that the following diagram commutes:*



Proof. See ([BO78], Theorem 3.19). □

Remark 4. If $K \subset B$ with $KD_{B,\gamma}(J) = 0$, then by universal property we have $D_{B,\gamma}(J) \simeq D_{B/K,\gamma}(J/J \cap K)$.

Lemma 4.11. *Let (R, I, γ) a be a P.D. structure and B a flat algebra over R. Then γ extends uniquely over B.*

Proof. This is a technical lemma and we refer to [stacks-project, Tag 07GZ, Lemma 23.4.2] □

Corollary 4.12. *If $(B, J) \to (C, JC)$ is a flat morphism of (R, I, γ) algebras, then*

$$C \otimes_B D_{B,\gamma}(J) \simeq D_{C,\gamma}(CJ).$$

Proof. By the previous lemma $C \otimes_B D_{B,\gamma}(J)$ has P.D. structure, then by the universal property of P.D. envelopes, the natural map $C \otimes_B D_{B,\gamma}(J) \to D_{C,\gamma}(CJ)$ is an isomorphism. □

Now let (B, J, γ) be a P.D. algebra over R. Then consider the R-module $\Omega^1_{B/R}$ and all the relations

$$d\gamma_n(x) = \gamma_{n-1}(x)dx \quad x \in I, n \geq 1.$$

The quotient by these relations is the *module of divided power Khaler differentials*

$$\Omega^1_{B/S,\gamma} := \Omega^1_{B/S}/\sim \qquad \Omega^r_{B/S,\gamma} = \bigwedge^r \Omega^1_{B/S,\gamma}.$$

Now for a P.D. ideal J, the n-th divided power of J is defined posing

$$J^{[n]} := \{\gamma_{i_1}(x_1)\dots\gamma_{i_r}(x_r)| \quad x_j \in J, \quad i_1+\dots+i_r \geq n\}$$

with $J^{[0]} := B$. Then we can define the de Rham complex by:

$$\Omega^\bullet_{B/R,\gamma} := B \to \Omega^1_{B/S,\gamma} \to \Omega^2_{B/S,\gamma} \to \dots$$

and a filtration, called *divided power Hodge filtration* by the P.D. ideal J of B:

$$\mathrm{Fil}^n\Omega^\bullet_{B/S,\gamma} := J^{[n]} \to J^{[n-1]}\Omega^1_{B/S,\gamma} \to J^{[n-2]}\Omega^2_{B/S,\gamma} \to \dots. \tag{4.1}$$

As an example if (R, I, γ) is a P.D. ring and B is a flat R-algebra, then γ extends to $(B, IB, \bar{\gamma})$. By universal property of divided power envelope, this means that $B = D_{B,\gamma}(IB)$. Then if $b = \sum_{i=1}^r x_i b_i \in IB$ with $x_1,\dots,x_r \in I$ and $b_1,\dots,b_r \in B$ we can verify that $d\gamma_n(b) = \gamma_{n-1}(b)db$. By induction we can suppose $r = 2$. Then the computation follows:

$$d\gamma_n(x_1b_1 + x_2b_2) = d\left(\sum_{i+j=n}\gamma_i(x_1b_1)\gamma_j(x_2b_2)\right) =$$

$$= \sum_{i+j=n} d(\gamma_i(x_1b_1))\gamma_j(x_2b_2) + \gamma_i(x_1b_1)d(\gamma_j(x_2b_2)) =$$

$$= \sum_{i+j=n} d(b_1^i\gamma_i(x_1))\gamma_j(x_2b_2) + \gamma_i(x_1b_1)d(b_2^j\gamma_j(x_2)) =$$

$$\sum_{i+j=n} \gamma_{i-1}(x_1b_1)\gamma_j(x_2b_2)d(x_1b_1) + \gamma_{j-1}(x_2b_2)\gamma_i(x_1b_1)d(x_2b_2) =$$

$$\sum_{i+j=n-1} \gamma_i(x_1b_1)\gamma_j(x_2b_2)d(x_1b_1 + x_2b_2) =$$

$$= \gamma_{n-1}(x_1b_1 + x_2b_2)d(x_1b_1 + x_2b_2),$$

where the equality in the third row follows by the computation

$$d(b_1^i\gamma_i(x_1)) = ib_1^{i-1}\gamma_i(x_1)db_1 = b_1^{i-1}x_1\gamma_{i-1}(x_1)db_1 = \gamma_{i-1}(x_1b_1)d(x_1b_1).$$

This fact proves that in this case $\Omega^1_{B/R,\bar{\gamma}} = \Omega^1_{B/R}$. Then we want to extend the definitions at level of scheme. Let X a topological space within $\mathcal{O}$ a sheaf of rings. Let $\mathcal{J}$ an ideal sheaf of $\mathcal{O}$.

Definition 4.13. *A divided power structure on* $(\mathcal{O}, \mathcal{J})$ *is a sequence of maps* $\gamma_n : \mathcal{O} \to \mathcal{O}$ *such that for each* $U \subset X$ *open subset the induced map* $\gamma_n(U) : \mathcal{O}(U) \to \mathcal{O}(U)$ *is a P.D. ring structure.*

The definition applies to the case of the structural sheaf of regular functions on a scheme X over $spec(R)$. If J is a quasi-coherent ideal sheaf of $\mathcal{O}_X$, then $\Omega^1_{X/R}$ is understood as $\Omega^1_{\mathcal{O}_X/R,\gamma}$ and we have a notion of divided power de Rham complex and divided power Hodge filtration. We have also a divided power envelope:

Definition 4.14. *Let* $(S, \mathcal{O}_S, \mathcal{I}, \gamma)$ *be a P.D. scheme. Let* $(X, \mathcal{O}_X, \mathcal{J})$ *an* S*-scheme. Then the divided power envelope of* X *with respect to* $\mathcal{J}$ *is the following relative spectrum over* $\mathcal{O}_X$*:*

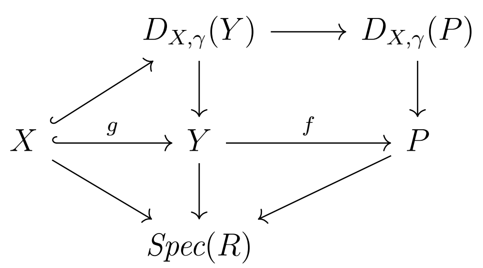
$$Spec_{\mathcal{O}_X}(D_{\mathcal{O}_X,\gamma}(\mathcal{J})).$$

When we have a closed immersion $i : X \to Y$ of S-schemes, there is a corresponding quasi-coherent ideal sheaf $\mathcal{J}$ of $\mathcal{O}_Y$. Then the P.D. envelope of Y with respect to J is simply called P.D. envelope of X in Y and denoted by $D_{X,\gamma}(Y)$.

In the case of interest we are in the situation where $S = \text{spec}(R)$ with (R, I, γ) a P.D. ring, $p > 0$ a prime number and $p \in I$, the ideal (p) locally nilpotent over X. We want to formulate a very useful theorem that permits us to well define the so called crystalline cohomology. To prove this theorem different facts will be involved that we will explain in the course of the proof.

4.2 Crystalline cohomology: definitions

Theorem 4.15. *Let* P *and* Y *be smooth schemes over Spec*(R)*. Let* $f : Y \to P$ *be a map of schemes which is either a closed embedding or smooth and let* $g : X \to Y$ *be a map such that* g *and* fg *are both closed immersion. All is presented in the following diagram:*



Then the following canonical map is a filtered quasi isomorphism:

$$\Omega^{\bullet}_{D_{X,\gamma}(P)} \to \Omega^{\bullet}_{D_{X,\gamma}(Y)}. \tag{4.2}$$

The proof of the theorem comes from different results. We just recall a notation. Define the R-algebra $R\langle X_1, \dots, X_N\rangle$ by

$$R\langle X_1, \dots, X_N\rangle = \bigoplus_{n_1,\dots,n_N \geq 0} RX_1^{[n_1]} \dots X_N^{[n_N]}$$

as R-module. Then the multiplication is defined by extending linearly the relation

$$(X_1^{[n_1]} \dots X_N^{[n_N]}) \cdot (X_1^{[m_1]} \dots X_N^{[m_N]}) = \binom{n_1+m_1}{n_1} \dots \binom{n_N+m_N}{n_N} X_1^{[n_1+m_1]} \dots X_N^{[n_N+m_N]}.$$

Then we denote with

$$R\langle X_1, \dots, X_N\rangle_+ = \bigoplus_{\exists n_j > 0} RX_1^{[n_1]} \dots X_N^{[n_N]}$$

as subalgebra of $R\langle X_1, \dots, X_N\rangle$.

Theorem 4.16. *Let (R, I, γ) be a P.D. ring and $R \to C$ a ring homomorphism. Let $J \subset K$ be an ideal of C and suppose that $J = (x_1, \dots, x_d)$ is generated by a regular sequence. Moreover assume that $R \to C$ and $R \to C/J$ are smooth.*
Then there is an isomorphism of P.D. rings such that the following diagram commutes:

$$\begin{array}{ccc} D_{C,\gamma}(K) & \xrightarrow{\simeq} & D_{C/J,\gamma}(KC/J)\langle X_1, \dots, X_d\rangle \\ \downarrow & \swarrow_{X_i^{[n]} \mapsto 0} & \\ D_{C/J,\gamma}(KC/J) & & \end{array}$$

Proof. Since p is nilpotent in R, this means that $p^m R = 0$ for some $m \geq 0$ integer. Since $x^r = r!\gamma_r(x)$, then $x^r = 0$ for $r \geq p^m$. Since J is finitely generated, there exists n (for example $n = (p^m - 1)d + 1$) such that $J^n D_{C,\gamma}(J) = 0$. This implies that $D_{C,\gamma}(J) \simeq D_{C/J^n,\gamma}(J/J^n)$. Similarly, if $J_0 = (X_1, \dots, X_d)$ is the ideal of $(C/J)[X_1, \dots, X_d]$, then $D_{(C/J)[X_1,\dots,X_d],\gamma}(J_0) \simeq D_{(C/J)[X_1,\dots,X_d]/J_0,\gamma}(J_0/J_0^n)$. Moreover, by the smoothness of C and C/J there is an isomoprhism $(C/J)[X_1, \dots, X_d]/J_0^n \simeq C/J^n$. This lifts to an isomoprhism on the level of P.D. envelopes, and so

$$D_{C,\gamma}(J) \simeq D_{C/J[X_1,\dots,X_d],\gamma}(J_0) \simeq C/J\langle X_1, \dots, X_d\rangle,$$

where the last isomorphism comes from the universal property of P.D. envelope and by flatness of C/J. Then by the universal property of P.D. envelope, we have the following isomorphisms:

$$\begin{aligned} D_{C,\gamma}(K) &\simeq D_{D_{C,\gamma}(J),\bar{\gamma}}(KD_{C,\gamma}(J)+\bar{J}) \\ &\simeq D_{D_{C/J[X_1,\ldots,X_d],\gamma}(J_0),\bar{\gamma}}(K+\bar{J}) \\ &\simeq D_{C/J[X_1,\ldots,X_d],\gamma}(J_0+KC/J[X_1,\ldots,X_d]) \\ &\simeq D_{C/J,\gamma}(KC/J)\langle X_1,\ldots,X_d\rangle \end{aligned}$$

□

Lemma 4.17. *Let (R,I,γ) be a P.D. ring. Let $R\to B\to C$ ring morphisms. $B\to C$ an étale morphism of R-algebras and Let J be an ideal of B and K an ideal of C with $B/J=C/K$ (assume this identification is induced by the étale morphism). Then $B/J^n\simeq C/K^n$ and the ideal JB/J^n corresponds to KC/K^n for each $n\geq 1$. This implies that*

$$D_{B,\gamma}(J)\simeq D_{C,\gamma}(K).$$

Proof. By induction on n and the fact that $B\to C$ is étale, we have a unique morphism from $C\to B/J^n$: it follows by universal property of étale maps:

$$\begin{array}{ccc} C & \longrightarrow & B/J^{n-1} \\ \uparrow & \overset{\exists!}{\searrow} & \uparrow \\ B & \longrightarrow & B/J^n \end{array}$$

It factors through C/K^n: In fact by induction we can identify $B/J^{n-1}\simeq C/K^{n-1}$. Moreover this morphism sends K^n to $0\in B/J^n$, since any $x\in K$ is mapped to 0 by $C\to B/J=C/K$. The map $B\to C/K^n$ induced by the étale morphism $B\to C$ is such that the following diagram commutes:

$$\begin{array}{ccc} B & \longrightarrow & C/K^{n-1} \\ \uparrow & \searrow & \uparrow \\ R & \longrightarrow & C/K^n \end{array}$$

Then it factors through B/J^n. These diagrams yield (identifying $C/K^{n-1}\simeq B/J^{n-1}$ in the bottom square) a factorization $B/J^n\to C/K^n\overset{\exists!}{\to}B/J^n$ of the identity, and by uniqueness it implies $B/J^n\simeq C/K^n$. The sentence about ideal follows immediately and the sentence about the P.D. envelope follows by the nilpotence of p. □

Theorem 4.18. *(Divided power Poincaré lemma) Let (B, J, δ) a P.D. algebra over R.Then the map:*

$$z : \Omega^{\bullet}_{B\langle X_1,\dots,X_d\rangle/R,\delta} \to \Omega^{\bullet}_{B/R,\delta}$$
$$X_i^{[n]} \longmapsto 0$$

is a filtered homotopy equivalence of R-modules.

Proof. By induction we have to prove the assertion for $d = 1$. Denote $X_1 = X$. Note that the inclusion $j : B \to B\langle X\rangle$ satisfies $zj = 1$. Now consider the proposition as B-module differentials. Then $\Omega^{\bullet}_{B\langle X\rangle/B,\delta} = [B\langle X\rangle \to B\langle X\rangle dX]$ and it reduce to prove that $jz \sim id$ (homotopic to identity). This means to find a map $\int : B\langle X\rangle dX \to B\langle X\rangle$, such that $\int d + d\int = id - jz$. In fact the map "$\int$" is the "integral"

$$\int : X^{[i]}dX \to X^{[i+1]}. \tag{4.3}$$

Then

$$\int d(\sum_{i\geq 0} b_i X^{[i]}) + d(\int \sum_{i\geq 0} b_i X^{[i]}dX) = \int (\sum_{i\geq 1} b_i X^{[i-1]}dX) + d(\sum_{i\geq 0} b_i X^{[i+1]}) =$$
$$\sum_{i\geq 1} b_i X^{[i]} + \sum_{i\geq 1} b_i X^{[i]}dX =$$
$$\sum_{i\geq 0} b_i X^{[i]} + \sum_{i\geq 0} b_i X^{[i]}dX - jz\left(\sum_{i\geq 0} b_i X^{[i]})\right) - jz\left(\sum_{i\geq 0} b_i X^{[i]}dX\right).$$

Since

$$\Omega^{\bullet}_{B\langle X\rangle/R,\delta} \simeq \Omega^{\bullet}_{B/R,\delta} \otimes B\langle X\rangle \oplus \Omega^{\bullet}_{B\langle X\rangle/B,\delta}$$

$\int$ extends to an homotopy equivalence for

$$\Omega^{\bullet}_{B/R,\delta} \to \Omega^{\bullet}_{B\langle X\rangle/R,\delta}.$$

By the isomorphism above, $\int$ acts only at level of $B\langle X\rangle$. So it preserves the Hodge filtration. □

Lemma 4.19. *Let R be a ring. Let $R \to C \to B$ be smooth homorphisms of R-algebras. Let J an ideal of B and K an ideal of C with $B/J = C/K = A$. For each $x \in spec(A)$ there exists $f \in B$ and an ideal $J_0 \subset J_f$, (the localization ot f) such that $f(x) \neq 0$ and $C \to B_f/J_0$ is étale.*

Proof. The morphisms $B \to B/J$ and $C \to C/K$ induce the respective exact sequences :

$$J/J^2 \to \Omega^1_{B/R} \otimes_B A \to \Omega^1_{A/R} \to 0$$
$$K/K^2 \to \Omega^1_{C/R} \otimes_C A \to \Omega^1_{A/R} \to 0$$

Moreover, since $C \to B$ is smooth, we have the following exact sequence:

$$\Omega^1_{C/R} \otimes_C A \to \Omega^1_{B/R} \otimes_B A \to \Omega_{B/C} \otimes_B A \to 0.$$

We put all together in the following diagram:

$$\begin{array}{ccccccc}
K/K^2 & \longrightarrow & \Omega^1_{C/R} \otimes A & \longrightarrow & \Omega_{A/R} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
J/J^2 & \longrightarrow & \Omega_{B/R} \otimes A & \longrightarrow & \Omega_{A/R} & \longrightarrow & 0 \\
& \overset{\beta}{\searrow} & \downarrow & & & & \\
& & \Omega_{B/C} \otimes A & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

Then β is surjective: for each $x \in \Omega_{B/C} \otimes A$, by going up we find $x_1 \in \Omega_{B/R} \otimes A$. Then its image x_2 in $\Omega^1_{A/R}$ has a fiber in $\Omega^1_{C/R} \otimes A$. Choose x_3 as an element of this fiber. Then the image x_4 of x_3 in $\Omega^1_{B/R} \otimes A$ is such that the image of $x_1 - x_4$ in $\Omega^1_{A/R}$ is 0. Then, by exactness, there exists an element of J/J^2, say y that maps to $x_1 - x_4$, and the image of $x_1 - x_4$ in $\Omega^1_{B/C} \otimes A$ is equal to the image of x_1 that is x. Now, since β is surjective and $\Omega_{B/C}$ is projective of finite type (by smoothness) of rank n, for some $f \in B$ with $f(x) \neq 0$ can be found a basis of $\Omega_{B_f/C}$ made by $dg_1, \ldots, dg_n$ with $g_1, ..., g_n \in J$. Define $J_0 = (g_1, \ldots, g_n)$. To the morphism $B_f \to B_f/J_0$ is associated the long exact sequence:

$$J_0/{J_0}^2 \to \Omega^1_{B_f/C} \otimes_{B_f} B_f/J_0 \to \Omega^1_{(B_f/J_0)/C} \to 0$$

The first morphism is surjective by construction, and it is a map of module of rank n, since $C \to B_f$ is smooth. Then, it is an isomorphism, and by exactness this yields $\Omega^1_{(B_f/J_0)/C} = 0$, that means $C \to B_f/J_0$ is étale. $\square$

Then the proof of theorem 4.15 is made as following. Since it is a local question, we suppose $P = spec(C)$, $Y = spec(B)$ and $X = spec(C/K)$.

1) f is a closed embedding. Then $B = C/J$, with $J = (x_1, \ldots, x_d)$ a regular sequence. Then we proved $D_{C,\gamma}(K) = D_{C/J,\gamma}(KJ/C)\langle X_1, \ldots, X_d\rangle$, where $d = \dim P - \dim Y$. Then the theorem follows by the divided power Poincaré lemma.
2) f smooth.Let J be the ideal of B such that $B/J = C/K$. Then by a previous lemma we can, after localizing, assume that there is an ideal $J_0 \subset J$ such that $C \to B/J_0$ is étale. Then $\Omega^\bullet_{D_C(K)} \xrightarrow{\sim} \Omega^\bullet_{D_{B/J_0}(JB/J_0)}$, by lemma (4.17). Then $\Omega^\bullet_{D_B(J)} \xrightarrow{\sim} \Omega^\bullet_{D_{B/J_0}(JB/J_0)}$ by the reasoning made previously: since $R \to B/J_0$ is smooth, then $D_{B/J_0}(JB/J_0)\langle X_1, \ldots, X_d\rangle \simeq D_B(J)$. This implies the assertion.
□

Definition 4.20. *Let (R, I, γ) a P.D. ring and X a scheme over R/I that admits lifting Y, P as in the situation of theorem (4.15). Then the crystalline cohomology of X with respect to (R, I, γ) is defined as the hypercohomology groups:*

$$H^*_{crys}(X/R) = \mathbb{H}^*(X, \Omega^\bullet_{D_{X,\gamma}(P)/R})$$

We have to show that this definition is independent of the choices and that is functorial in X. First notice that if Y, Z are smooth schemes over $spec(R)$ and X a closed immersion in both Y and Z, also $Y \times_{spec(R)} Z$ is smooth and Z is closed. The following diagram, by the previous result is made by quasi-isomorphisms:

$$\begin{array}{ccccc} & & \Omega^\bullet_{D_X(Y\times Z)} & & \\ & \overset{\sim}{\nearrow} & & \overset{\sim}{\nwarrow} & \\ \Omega^\bullet_{D_X(Y)} & & & & \Omega^\bullet_{D_X(Z)} \end{array}$$

Moreover they satisfy the cocycle condition, for three scheme X_1, X_2, X_3. This proves the independence of the smooth lifting Y. Let now X, X' two schemes over $spec(R/I)$ and Y, Y' the respective smooth schemes over $spec(R)$ for which X, X' are closed immersion in Y, Y' respectively. Let $f : X \to X'$ be a morphism of schemes and assume to have a lifting $g : Y \to Y'$ of f. Then at level of sheaves there is a pullback map

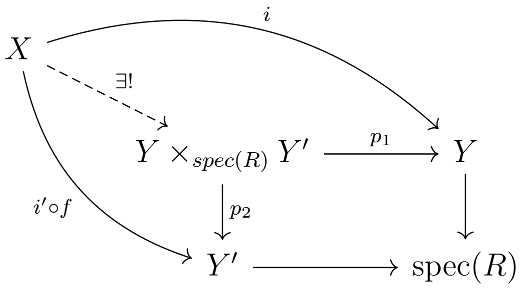
$$f^{-1}\Omega^\bullet_{D_{X'}(Y')} \to \Omega^\bullet_{D_X(Y)}.$$

Then we have to prove that this map does not depend by the choice of g, but only by f. So assume Y_1, Y_1' to be another lift of X, X' respectively and $h : Y_1 \to Y_1'$ is another lifting of f.Then we can form the fiber products $Y \times Y'$ and $Y_1 \times Y_1'$ with p_1

(resp. p_2) the projection on the scheme Y or Y_1 (resp. Y' or Y_1'). Then the following diagram commutes:

$$\begin{array}{ccccc} p_1^{-1}\Omega^\bullet_{D_X(Y)} & \longrightarrow & \Omega^\bullet_{D_X(Y\times Y')} & \longleftarrow & p_2^{-1}\Omega^\bullet_{D_X(Y')} \\ \downarrow & & \downarrow & & \downarrow \\ p_1^{-1}\Omega^\bullet_{D_{X'}(Y_1)} & \longrightarrow & \Omega^\bullet_{D_{X'}(Y_1\times Y_1')} & \longleftarrow & p_2^{-1}\Omega^\bullet_{D_{X'}(Y_1')} \end{array}$$

Moreover there always exists a such lifting. In fact if $i : X \hookrightarrow Y$ and $i' : X' \hookrightarrow Y'$ are the closed immersions, then by the universal property of cartesian squares, we obtain a unique map $X \to Y \times_{spec(R)} Y'$ such that the following diagram commutes:



Then, it yields a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & Y \times_{spec(R)} Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_2} \\ X' & \overset{i'}{\hookrightarrow} & Y'. \end{array}$$

This proves the functoriality with respect to X.

4.2.1 The (crystalline) Frobenius action

For our interest, it is important to study the action of the cohomology on some endomorphism of X. More precisely we are in the following setting. We have a smooth scheme X over R and $X_0 = X \times_{spec(R)} R/I$. We can suppose to embed the scheme X in a smooth and proper scheme P. We have proved that over a flat morphism, the divided power differentials equal the classical differentials. In the case of X smooth we have a flat morphism $R \to \mathcal{O}_X$. Then $D_{X_0,\gamma}(X) = X$ and by flatness we have that

$$\Omega^1_{D_{X_0,\gamma}(X)/R,\delta} = \Omega^1_{D_{X_0,\gamma}(X)/R} = \Omega^1_{X/R}.$$

This means that

$$H^*_{crys}(X_0/R) \simeq H^*_{dR}(X/R) \simeq \mathbb{H}^*(X_0, \Omega^\bullet_{D_{X_0,\gamma}(P)}).$$

Now suppose to have an absolute Frobenius morphism $f_0 : X_0 \to X_0$. It induces a morphism at level of crystalline cohomology over X_0, but by the identification it is an endomorphism over the De Rham cohomology of X, without assumption of lifting of f_0 over X. We are interested in the study of Frobenius morphism in the following case: let k be a perfect field of characteristic $p > 0$, $W = W(k)$ its ring of Witt vectors, X_s a smooth quasi-projective over $W_s = W/p^{s+1}$, $X_0 = X \otimes k$ and $f_0 : X_0 \to X_0$ the absolute Frobenius. For each s choose an embedding of X_s in a projective scheme P_s. Since it is projective, the Frobenius morphism admits a lifting. This lifting induces a morphism

$$f : D_s \to D_s,$$

where $D_s := D_{X_s}(P_s)$. Then f is defined at level of differentials, by pullback

$$f^* : \Omega^\bullet_{D_s} \to \Omega^\bullet_{D_s}.$$

Theorem 4.21. *Let $r < p$ and n with $n + r \leq s$. Then there exists a unique semilinear map*

$$f_r : Fil^r \Omega^\bullet_{D_n} \to \Omega^\bullet_{D_n}$$

such that $p^r f_r = f^*$. Recall that $Fil^r \Omega^\bullet_{D_n}$ is defined in (4.1).

Proof. Let's denote with $\langle m \rangle$ the integer such that $(p^{\langle m \rangle}) = (p)^{[m]}$, whenever $m \geq 0$, otherwise $\langle m \rangle = 0$. By definition of the Frobenius morphism, we have that $f^*(x) \equiv x^p \pmod p$. If $x = \alpha dx_1 \dots dx_i \in \Omega^i_{D_m}$, then

$$f^*(x) = f^*(\alpha) f^*(dx_1) \dots f^*(dx_i) = \alpha^p (p x_1^{p-1} dx_1 + p dw_1) \dots (p x_i^{p-1} dx_i + p dw_i) \in p^i \Omega^i_{D_m}.$$

Then if J is the P.D. ideal of D_m, then follows that if $\gamma_{i_1}(x_1) \dots \gamma_{i_k}(x_k) \in J^{[r-i]}$, we have

$$f^*(\gamma_{i_t}(x_t)) = \gamma_{i_t}(f^*(x_t)) = \gamma_{i_t}(x_t^p + pw) = \gamma_{i_t}(p! \gamma_p(x_t) + pw) = ((p-1)! \gamma_p(x_t))^{i_t} \gamma_{i_t}(p) \in (p)^{[i_t]}.$$

Since $i_1 + \dots + i_k \geq r - i$, by linearity we have that

$$f^*(J^{[r-i]}) \subset (p^{\langle r-i \rangle}).$$

This means that (without the hypothesis $r < p$) holds

$$f^*(\mathrm{Fil}^r \Omega^i_{D_m}) \subset p^{i + \langle r-i \rangle} \Omega^i_{D_m}.$$

When, $r < p$ we have $\langle r-i \rangle = r-i$, that implies

$$f^*(\mathrm{Fil}^r \Omega^i_{D_m}) \subset p^i \Omega^i_{D_m}.$$

Now, we know that

$$p^r : \mathbb{Z}/p^{n+1} \to \mathbb{Z}p^{n+r+1}$$

is injective. We claim that Ω_{D_m} is flat over $\mathbb{Z}/p^{m+1}$. Suppose for the moment that it is true. Then

$$p^r : \Omega^\bullet_{D_n} \to \Omega^\bullet_{D_{r+n}}$$

is injective too. Then, by the following diagram

$$\begin{array}{ccc} \mathrm{Fil}^r\Omega^\bullet_{D_{r+n}} & \xrightarrow{f^*} & \Omega^\bullet_{D_{r+n}} \\ & \searrow^{\exists!\ p^{-r}f^*} & \uparrow p^r \\ & & \Omega^\bullet_{D_n} \end{array}$$

we obtain a factorization map $p^{-r}f^* : \mathrm{Fil}^r\Omega^\bullet_{D_{r+n}} \to \Omega^\bullet_{D_n}$, the reduction modulo p^{n+1} is the map f_r. To see why $\Omega^\bullet_{D_m}$ is flat over $\mathbb{Z}/p^{m+1} = W_m$, we use the flatness of P_m over $spec(W_m[X_1,\dots,X_t])$. In fact the problem is local, so we assume that X_m is generated by a regular sequence $g_1,...,g_t$. Then we have the following cartesian square:

$$\begin{array}{ccc} X_m & \longrightarrow & P_m \\ \downarrow & & \downarrow \\ spec(W_m) & \xrightarrow[0 \leftarrow X_i]{} & spec(W_m[X_1,\dots,X_t]) \end{array}$$

And since $P_m \to spec(W_m[X_1,\dots,X_t])$ is smooth, then flat, we have by a previous lemma that $D_m \simeq P_m \times_{spec(W_m[X_1,\dots,X_t])} D_{spec(W_m)}(spec(W_m[X_1,\dots,X_t]))$ and $\Omega^\bullet_{D_{spec(W_m)}(spec(W_m[X_1,\dots,X_t]))}$ are free over W_m.

□

Remark 5. In the proposition (4.21), actually, the smoothness of X is a strong condition. In fact in the course of the proposition the relevant fact was the existence of a regular local closed immersion of the scheme in a smooth scheme (needed for the independence of the embedding as proved in the theorem (4.15)) and flatness, (needed for extending the P.D. structure of $pW_m \subset W_m$). This led to study the Frobenius endomorphism around *syntomic* schemes over W, that are schemes with local complete intersection and flat. The quasi-projective hypothesis is needed for extending the Frobenius action of the special fiber over the smooth embedding, and so it guarantees the lifting property. We also have seen that the existence of such lifting is not an obstruction to the existence of Frobenius endomorphism at level of the crystalline cohomology, but it helps the study of this action. We will see the really important role of the smoothness in the next chapter.

4.2.2 Syntomic sheaves

We can now define an object that carries the information of the Frobenius action at level of cohomology. Let J_n to be the P.D. ideal of D_n defining X_n, and $J_n^{[r]}$ the r-th divided power. Then the divided power Hodge filtration of $\Omega^\bullet_{D_n}$ becomes

$$\mathrm{Fil}^r\Omega^\bullet_{D_n,X,P} : J_n^{[r]} \to J_n^{[r-1]} \otimes_{\mathcal{O}_{P_n}} \Omega^1_{P_n} \to J_n^{[r-2]} \otimes_{\mathcal{O}_{P_n}} \Omega^2_{P_n} \to \dots . \tag{4.4}$$

Clearly, for $r = 0$ this filtration is the entire complex of differentials, and we have just defined a map f_r between the r-th level of the filtration and the 0 level. Then we consider the map

$$1 - f_r : \mathrm{Fil}^r\Omega^\bullet_{D_n,X,P} \to \mathrm{Fil}^0\Omega^\bullet_{D_n,X,P}$$

where 1 is the inclusion map of $\mathrm{Fil}^r\Omega^\bullet_{D_n} \subset \Omega^\bullet_{D_n}$. Then consider the mapping cone

$$\mathfrak{S}_{n,X,P}(r) := \mathrm{Cone}(\mathrm{Fil}^r\Omega^\bullet_{D_n,X,P} \xrightarrow{1-f_r} \mathrm{Fil}^0\Omega^\bullet_{D_n,X,P})[-1] \tag{4.5}$$

with differential $h = \begin{pmatrix} d & 0 \\ 1-f_r & d \end{pmatrix}$. Then we can see that these sheaves are independent of the choice of f and P. For two lifting P, P' we can consider the fiber product $P \times P'$ and the immersion of $X \hookrightarrow P \times P'$ and we see that $\mathfrak{S}_{n,X,P}(r) \xrightarrow{\sim} \mathfrak{S}_{n,X,P\times P'}(r) \xleftarrow{\sim} \mathfrak{S}_{n,X,P'}(r)$, where the quasi-isomorphism is satisfied by the theorem (4.15). These sheaves have a product structure:

Proposition 4.22. *Let $r, r' \geq 0$ and $r + r' < p$. Then there exists a ring structure on $\mathfrak{S}_{n,X,P}(r)$ defined by a product*

$$\mathfrak{S}_{n,X,P}(r) \times \mathfrak{S}_{n,X,P}(r') \to \mathfrak{S}_{n,X,P}(r + r').$$

Proof. Define the following abelian groups:

$$A = \bigoplus_{0\leq r<p} \mathrm{Fil}^r\Omega^\bullet_{D_n,X,P} \qquad B = \bigoplus_{o\leq r<p} \mathrm{Fil}^0\Omega^\bullet_{D_n,X,P}$$

$$g = \bigoplus_{0\leq r<p} 1 \qquad h = \bigoplus_{0\leq e<p} f_r$$

Then over B, we have a ring structure given by the product

$$(x_1,\dots,x_p)(y_1,\dots,y_p) = (z_1,\dots,z_p)$$

given by

$$z_t = \sum_{i+j=t} x_i y_i.$$

Over A, the product is induced by the inclusion morphism $1 : A \subset B$. Now consider A, and B as complex of abelian groups, with the components given by the filtration $A^r = \mathrm{Fil}^r\Omega^\bullet_{D_n,X,P}$ for A and $B^r = \mathrm{Fil}^0\Omega^\bullet_{D_n,X,P}$, for B. Then the tensor product of the complex $A \otimes A$ defines a ring structure with unit, and such that $A^r A^{r'} \subset A^{r+r'}$ and

$$d(xy) = d(x)y + (-1)^r x d(y),$$

where d is the differential of the complex and $x \in A^r$ and $y \in A^{r'}$. Then if

$$C := \mathrm{Cone}(A \xrightarrow{g-h} B)[-1]$$

define the product structure on $C \otimes C$, by a map $C \otimes C \to C$ such that

$$(x,y)(x',y') = (xx', (-1)^r g(x)y' + yh(x')).$$

Now is a computation to verify that this product define a ring homomorphism with unit $(1,0) \in C^0$.

□

Remark 6. It is clear by the discussion above that as consequence of the theorem (4.15) this product is again independent by the choices of P.

Remark 7. These complexes, actually come from the one built in [FM87] where it is called S_n^r. In their theory they deal with the geometric situation in which X is endowed with the syntomic site. What we have defined here in these terms is the direct image of S_n^r by the map of sites $\pi : X_{syn} \to X_{\text{ét}}$, ([Kat85]). We have chosen to not deal with the syntomic topology, but when consider $\mathfrak{S}_n(r)$ as sheaves, we consider X (and X_n) endowed with the étale topology. The smoothness is a strong condition as discussed previously, but the name "syntomic" is motivated in both direction as a definition of sheaves with respect to syntomic topology (i.e. S_n^r) or in the sense that the weaker condition to deal with such definitions is that X can be syntomic (and quasi-projective) as scheme with respect to W.

Why the introduction of these sheaves? They represent an analogy to the Deligne-Beilinson complexes $\mathbb{Z}(r)_{\mathcal{D}}$. Recall that there is an inclusion map $\mathbb{Z} \subset \mathbb{C}$ that is the multiplication by $(2\pi i)^r\mathbb{Z}$ and by this map is constructed the Deligne-Beilinson complex

$$\mathbb{Z}(r) \to \mathcal{O}_X \to \Omega^1_{X/\mathbb{C}} \to \cdots \to \Omega^{r-1}_{X/\mathbb{C}} \to 0$$

for a smooth projective variety over $\mathbb{C}$. The interpretation of this inclusion map in characteristic $p > 0$, is the multiplication by p^r. In the next chapter, we will see that this analogy is supported by the behaviour of the cycle classes. For the moment we want to explain the analogy between the map $\mathbb{Z}(r) \to \Omega_X$ and the map $1 - f_r$.

Lemma 4.23. *Assume that X is a smooth variety over W with a Frobenius morphism f, lifting the one over its reduction modulo p. Then the syntomic sheaves $\mathfrak{S}_{n,X}(r)$ are quasi-isomorphic to the following:*

$$\mathcal{O}_{X_n} \xrightarrow{-d} \Omega^1_{X_n} \xrightarrow{-d} \dots \xrightarrow{-d} \Omega^{r-2}_{X_n} \xrightarrow{(0,-d)} \Omega^r_{X_n} \oplus \Omega^{r-1}_{X_n} \xrightarrow{(1-f_r,-d)} \Omega^r_{X_n} \to 0.$$

Proof. The condition of smoothness and the theorem (4.15) permits to choose $P = X$, and so the P.D. ideal defining X_n, in P_n is the (0). This means that in the explicit writing of the complex $\mathfrak{S}_{n,X}(r)$, the first $r-1$ terms don't contain any term with $J_n^{[k]}$. Since the term $J_n^{[0]} = \mathcal{O}_{D_n}$, it remains to explain, why the complex has a truncation at the level r. In fact it suffices to prove that $\mathcal{H}^q(\mathfrak{S}_{n,X}(r)) = 0$, whenever $q > r$. But this is easy: we may assume $n = 1$, then $f_r = p^{q-r} f_q$ by the property of factorization with f^*. Then $q - r \geq 1$ and so, the reduction modulo p is 0. This means that the component $1 - f_r = 1 : \Omega^q_{X_n} \to \Omega^q_{X_n}$ that is clearly an isomorphism.

□

CHAPTER 5

RIGID COHOMOLOGY

In this chapter we start to recall the principal definitions and theorems of the rigid cohomology of Berthelot. The main references are [Ber96],[Ber97b], [Pet03]. This cohomology theory has some similarity with the de Rham cohomology, but the essential difference is to associate a different topology to a space, for which it is called *rigid*. The purpose for introducing such theory is to allow the study of a p-adic cohomology theory that has a nice behaviour for non proper e non smooth schemes, for which the crystalline cohomology in general is not finite dimensional.

Definition 5.1. *Let X be a set. A Grothendieck topology on X, denoted with τ, is a subset $\mathcal{S}$ of parts of X, such that it contains $\emptyset, X$, it is stable for finite intersections and for each $U \in \mathcal{S}$ there exist a set $Cov(U)$ of covering of U, such that the following holds:*

1. $(U) \in Cov(U)$
2. *if* $U, V \in \mathcal{S}$, $V \subset U$, *and if* $(U_i)_i \in Cov(U)$ *then* $U_i \cap V \in Cov(V)$
3. *if* $(U_i)_i \in Cov(U)$ *and for each* i, $(U_{i,j})_j \in Cov(U_i)$, *then* $(U_{i,j})_{i,j} \in Cov(U)$.

Definition 5.2. *The elements of S are called open; the covering $(U_i)_i \in Cov(U)$ is called admissible covering.*

Then we work in the following context. We assume to have a discrete valuation ring W with residue field k of characteristic $p > 0$, and $Frac(W) =: K$. We assume π is a uniformizer of W. To start, fix A a Tate algebra and $X = \mathrm{spm}(A)$. We define the following *strong Grothendieck topology* on X.

Definition 5.3. *Let $X = Spm(A)$ as above.*
An open $U \in \mathcal{S}(X)$ is a subset of X such that:
1) There exists a covering $(V_i)_i$, for which the elements are called special domains, such that

$$V_i = \{x \in X : s.t.\ there\ exists$$
$$f_0, \dots, f_n \quad generating \quad A \quad and \quad |f_j(x)| \leq |f_0(x)| \quad \forall j = 1, \dots, n\}$$

and for each morphism of Tate algebras $A \to B$, such that the induced $f : spm(B) \to spm(A)$, satisfies $f(spm(B)) \subset U$, then there exists a finite number of $(V_i)_i$ that covers $f(spm(B))$
2) $(U_i)_i \in Cov(U)$ if and only if for all morphisms as in 1) there exists a finite covering of $spm(B)$ of special domains, much finer then $f^{-1}(U_i)$

Lemma 5.4. *The strong topology defined above is satured, that means: 1) If U is open and $V \subset U$ such that there exists $(U_i)_i \in Cov(U)$ with $U_i \cap V$ open, then V is open.*
2) if an open $U = \bigcup_i U_i$ is given with each U_i open, and $(U_i)_i$ admits a refining by an admissible covering, then $(U_i)_i$ is admissible

Proof. The assertion in 2) is obvious, so we have to prove 1). Since $U_i \cap V \subset V \subset U$, let's choose a morphism of Tate algebras $\phi : \mathrm{Spm}(C) \to \mathrm{Spm}(A)$ such that $\phi(\mathrm{Spm}(C)) \subset V \subset U$. By the fact that $(U_i)_i$ is an admissible covering of U, this means that $(\phi^{-1}(U_i) \cap \mathrm{Spm}(C))_i$ admits a finite refining by special domains, $(Z_{i,j})_j$ for each i. Since $U_i \cap V$ is open, by ([FP04], section 9.1.2, lemma 4 (ii)) follows that the collection $(Z_{i,j})_{i,j})$ is a covering of V that admits a refining similar to $(U_i)_i$, i.e. for each $f : \mathrm{Spm}(D) \to \mathrm{Spm}(A)$ morphism of Tate algebras, there exists a finite refining of $(f^{-1}(Z_{i,j}) \cap \mathrm{Spm}(D))_{i,j}$ by finite number of special domains and so V is open. □

Next we can define the ring of functions for X. For each special domain V, we simply define

$$\Gamma(V, \mathcal{O}_X) := A\{t_1, \dots, t_n\}/(f_0 - f_i t_i)_{i=1,\dots,n}.$$

By the acyclicity theorem of Tate ([Ber96], loc.ref. [Ta1,8.2]) we can define for an open U, the sheaf of analytic functions

$$\Gamma(U, \mathcal{O}_X) := \ker \left(\prod_i \Gamma(V_i, \mathcal{O}_X) \rightrightarrows \prod_{i,j} \Gamma(V_i \cap V_j, \mathcal{O}_X) \right)$$

where $(V_i)_i \in \mathrm{Cov}(U)$.

Definition 5.5. *The ringed space $(Spm(A), \mathcal{O}_{Spm(A)})$ is called affinoid analytic space.*

Definition 5.6. *Let K a non-archimedean field. A (rigid) analytic space X over K is a set X endowed with a Grothendieck satured topology, and a sheaf of K-algebras $\mathcal{O}_X$ such that there exists an admissible covering X_i of open sets, such that $(X_i, \mathcal{O}_{X|X_i})$ is isomorphic as analytic space to an affinoid analytic space.*

Definition 5.7. *A morphism $f : X \to Y$ between K-analytic spaces is the datum of a continuous map from X to Y and an homomorphism of sheaves of local rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.*
The morphism f is finite if there exists an admissible covering $(X_i)_i$ such that $f^{-1}(X_i)$ is affinoid for each i and $\Gamma(f^{-1}(X_i), \mathcal{O}_Y)$ is a finite algebra over $\Gamma(X_i, \mathcal{O}_X)$.
The morphism f is smooth (resp. étale) if there exists an admissible covering $(X_i)_i$ of X and an admissible covering $(Y_i)_i$ of Y such that:
1) $f(Y_i) \subset X_i$ are affinoid spaces,
2)

$$\Gamma(Y_i, \mathcal{O}_Y) \simeq \Gamma(X_i, \mathcal{O}_X)\{t_1, \dots, t_n\}/(f_1, \dots, f_r) \quad \textit{(resp. } r = n\textit{)},$$

with $\det(\partial_{t_k}(f_j)))_{1 \le k,j \le r}$ invertible.

Similar to classical algebraic geometry, we can consider the immersion of $Y \hookrightarrow Y \times_X Y$ and the $\mathcal{O}_Y$-module J defining Y. Then we define

$$\Omega^1_{X/K} := J/J^2, \qquad \Omega^r_{X/K} = \bigwedge^r \Omega^1_{X/K}.$$

5.1 Formal schemes and rigid spaces

Now let A to be a Noetherian topological ring. An ideal of definition I of A is a system of neighborhood of 0 of A of the form $\{I^n\}_{n \in \mathbb{N}}$. If A is separated and complete, it is called adic ring. Assume A is adic and I its ideal of definition. Let

$$X_n := \operatorname{spec}(A/I^{n+1}).$$

Clearly, the underlying space for each n is the same, since A is separated. Then consider the structural sheaves $\mathcal{O}_{X_n}$. The projection maps

$$A/I^{n+1} \xrightarrow{\pi_n} A/I^n$$

together give rise to a projective system, so the induced maps on the structural sheaves give rise again to a projective system. So we can define

$$\varprojlim_n \mathcal{O}_{X_n} =: \mathcal{O}_{\operatorname{Spf}(A)},$$

and the ringed space $(\operatorname{Spf}(A), \mathcal{O}_{\operatorname{Spf}(A)})$ is the *formal spectrum* of A. Let's immediately observe that

$$\mathcal{O}_{\operatorname{Spf}(A)}(\operatorname{Spf}(A)) = \varprojlim_n \Gamma(\operatorname{Spf}(A), \mathcal{O}_{\operatorname{Spf}(A)}) = \varprojlim_n A/I^{n+1} = \hat{A} = A.$$

Definition 5.8. *If a ringed space $(X, \mathcal{O}_X)$ is isomorphic to a formal spectrum Spf(A), then it is called affine formal scheme.*
A formal scheme $(X, \mathcal{O}_X)$ is a ringed space such that there exists a covering of affine formal schemes.

A construction with formal schemes arise directly by the schemes.
Let X a Noetherian scheme and $X_0 \hookrightarrow X$ a closed subscheme of X. Then let J be the ideal of $\mathcal{O}_X$ defining X_0. Then the *completion of X along* X_0, denoted with $\hat{X}$ is defined as following.
Let $X_n := \operatorname{spec}(\mathcal{O}_X/J^{n+1})$, then we form the projective limit over $\mathcal{O}_{X_n}$ and define $\mathcal{O}_{\hat{X}} := \varprojlim_n \mathcal{O}_{X_n}$. Observe that the underlying space of $\hat{X}$ is the same as X_0.
For example if A is a discrete valuation ring with uniformizer p, in the case of $X = \operatorname{spec}(A)$ we can complete along the *special fiber*

$$X_0 = \operatorname{Spec}(A/(p))$$

and we obtain that

$$\mathcal{O}_{\hat{X}} = \operatorname{Spf}(\hat{A}).$$

5.2 Generic fiber in rigid geometry

Now we establish a connection with the analytic rigid spaces defined above. Let k be a perfect field of characteristic $p > 0$. Let W the ring of Witt vectors of k and π a uniformizer of the maximal ideal of W. Let $K := Frac(W)$. We consider the formal schemes over W. Let A be a W-algebra of finite type over W. Then we pose $X = \mathrm{Spf}(A)$ a W-formal scheme of finite type. Since A is an adic ring, $A \otimes_W K$ is a Tate algebra over K. Then we define the *generic fiber* X_K, as the affinoid analytic space associated to $\mathrm{Spm}(A \otimes_W K)$.

Lemma 5.9. *The points of the space X_K are in bijection with the quotients of the form A/I that are integral, finite and flat over W*

Proof. For each point of X_K corresponds a maximal ideal of $A \otimes_W K$, then the corresponding quotient is an extension of K, let's say K'. The image of A in K' is integral and flat, because it is torsion free and it is a module over a discrete valuation ring. Then, keeping a finite number of generators of A (it is of finite type), their images in K' lie in the ring of valuation of W' of K', so they are integral also over W.
Viceversa, an integral, flat, finite quotient of A of the form A/I, means that $A/I \otimes K$ is a finite extension of K. □

Corollary 5.10. *If the quotient A/I is given as above, then the support of Spf$(A/I) \subset$ Spf$(A) = X$ is a closed point x_o.*

Proof. Since K is a complete non archimedean field, W is henselian. Since A/I is a finite, integral W-algebra, it is a finite product of integral local ring, so it is local. □

The described point x_o, is called *specialization* of the point $x_K \in X_K$ corresponding to A/I. The specialization induces a map at the level of sets,

$$\mathrm{sp} : X_K \to X$$

that we can define for a general W-formal schemes X. So we define X_K as the set of closed subschemes of X finite, flat, integral over W, with support in a closed point x_o. Then the specialization map sends a such subscheme to the corresponding closed point. In particular, at level of affine open cover of X, that is a composed by affine formal schemes of the form $U_i = \mathrm{Spf}(A_i)$ for A_i of finite type, then $sp^{-1}(U_i)$ is in bijection with $\mathrm{Spm}(A_i \otimes K)$. Then, since the last scheme has an analytic structure, by transporting the structure, we can consider each $sp^{-1}(U_i)$ like analytic space. Then a theorem of Berthelot follows.

Theorem 5.11. *Let X, X' be given W-formal scheme of finite type. Then the following holds: 1) There exist on X_K a unique structure of analytic rigid space, such that sp^{-1} is an open map and for each affine open covering $(U_i)_i$ of X, $sp^{-1}(U_i)$ has the analytic structure induced by the respective affinoid analytic space. 2) The analytic space X_K is functorial in X, and for each morphism $f : X \to X'$ of formal W-scheme, there is a commutative diagram of sites:*

$$\begin{array}{ccc} X_K & \xrightarrow{sp_X} & X \\ \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle f} \\ X'_K & \xrightarrow{sp_{X'}} & X' \end{array}$$

Proof. See ([Ber96],Prop 0.2.3). □

Theorem 5.12. *Let X be a W-formal scheme and let J be an ideal of definition. Denote with X_0 the scheme defined by J. Let Z a closed subscheme of X_0 and denote by $\hat{X}$ the completion of X by the subscheme Z. If $sp : X_K \to X$ is the specialization morphism, then $sp^{-1}(Z)$ is open in X_K and the cononical morphism $\hat{X}_K \to X_K$ induces an isomorphism of analytic spaces*

$$\hat{X}_K \xrightarrow{\simeq} sp^{-1}(Z)$$

Proof. See ([Ber96],0.2.7). □

5.2.1 The tube of a k-scheme

The next tools needed to define the rigid cohomology is the notion of *tube*. Let X a k-scheme separated of finite type, $\mathcal{P}$ a W-formal scheme of finite type and a locally closed immersion $X \xrightarrow{i} \mathcal{P}$. We call this datum, *formal embedding*. Denote with P_K the generic fiber, and P_k the special fiber. The notion of the tube deals with the relation between k-scheme and its "lifting" over the generic fiber, endowed with the topology structure defined previously.

Definition 5.13. *Let $X \xrightarrow{i} \mathcal{P}$ as above. The tube of X in $\mathcal{P}$, denoted by $]X[_{\mathcal{P}}$ is the set defined by $sp^{-1}(i(X))$. By the theorem (5.11), $]X[_{\mathcal{P}}$ has analytical structure.*

We will see some properties of the tube.

Lemma 5.14. *Assume $X = \bigcup_i X_i$ a k-scheme of finite type union of k-subschemes of P_k. Then $]X[_{\mathcal{P}}= \bigcup_i]X_i[_{\mathcal{P}}$. The analogous result holds for intersections.*

Proof. We just notice that sp^{-1} respect intersections and unions. □

Then we can see that the definition of the tube is independent by open subsets of the formal scheme.

Lemma 5.15. *Let $i : X \hookrightarrow \mathcal{P}$. Then there exists $i' : X \hookrightarrow \mathcal{P}'$ formal embedding, with $\mathcal{P}' \xrightarrow{u} \mathcal{P}$ open immersion that factors through u. Moreover $]X[_{\mathcal{P}}=]X[_{\mathcal{P}'}$.*

Proof. Since $\mathcal{P}$ and P_k have the same underlying spaces, if X' is an open subset of P_k, we can write $X' = P'_k$ with $\mathcal{P}'$ open in $\mathcal{P}$ Since holds that $P'_K = sp^{-1}(\mathcal{P}')$, then follows

$$i : X \hookrightarrow X' = P'_k \hookrightarrow \mathcal{P}' \hookrightarrow \mathcal{P}$$

Moreover $]X[_{\mathcal{P}}= sp^{-1}(i(X)) = sp^{-1}(u \circ i'(X)) =]X[_{\mathcal{P}'}$.

□

Proposition 5.16. *Let be a following commutative diagram of morphisms of scheme:*

$$\begin{array}{ccc} X & \xhookrightarrow{i_X} & \mathcal{P} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} \\ X' & \xhookrightarrow{i_{X'}} & \mathcal{P}' \end{array}$$

Then holds that $]X[_{\mathcal{P}}\subset u_K^{-1}(]X'[_{\mathcal{P}'})$ and the equality holds when the diagram is cartesian.

Proof. Let's consider the following commutative diagram,

$$\begin{array}{ccccc} X & \xhookrightarrow{i_X} & \mathcal{P} & \xleftarrow{\mathrm{sp}} & P_K \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_K} \\ X' & \xhookrightarrow{i_{X'}} & \mathcal{P}' & \xleftarrow{\mathrm{sp}} & P'_K \end{array}$$

If $x \in]X[_{\mathcal{P}}$ we have to prove by definition of tube, that $(sp \circ u_K)(x) \in i_{X'}(X')$. By the commutative diagram above follows that

$$(sp \circ u_k)(x) = (u \circ sp)(x) \in u(sp(]X[_{\mathcal{P}})) = u(i_X(X)) = i_{X'}(f(X)) \in i_{X'}(X').$$

If the initial diagram is cartesian, then $X = X' \times_{\mathcal{P}'} \mathcal{P}$ and so

$$u^{-1}(i_{X'}(X')) = i_X(X).$$

We need to prove that

$$(sp \circ u_K)^{-1}(i_{X'}(X')) \subset sp^{-1}(i_X(X)).$$

So let $x \in P_K$ such that $(sp \circ u_K)(x) \in i_{X'}(X')$, then

$$(sp \circ u_K)(x) = (u \circ sp)(x) \in i_{X'}(X'),$$

that implies $sp(x) = u^{-1}(u(sp(x))) \in i_X(X)$.

□

We assume now that the W-formal scheme $\mathcal{P}$ is affine of the form $\mathcal{P} = \mathrm{Spf}(A)$, with A a W-algebra of finite type. If we consider an immersion $i : X \hookrightarrow \mathcal{P}$, for a k-scheme X, we call a *presentation* of X in $\mathcal{P}$ an identity of the form

$$i(X) = V(f_1, ..., f_r) \cap D(g_1, ..., g_m) \cap P_k$$

where, $f_1, \ldots, f_r$ (resp. $g_1, \ldots, g_m$) are elements of A and $V(f_1, \ldots, f_r)$ is the generated closed formal subscheme, and $D(g_1, \ldots, g_m)$ is the complement of $V(g_1, \ldots, g_m)$.

Proposition 5.17. *If X has a presentation in an affine formal scheme $\mathcal{P}$ then the tube $]X[_{\mathcal{P}}$ can be written as*

$$]X[_{\mathcal{P}} = \{x \in P_K : |f_i(x)| < 1 \quad \forall i = 1, \ldots, r \quad \textit{and} \quad |g_j(x)| = 1 \quad \textit{for some} \quad j = 1, \ldots, m\}.$$

Proof. Let's assume $X = \mathrm{Spf}(A)$. By induction over the number of components of the variety, we have to verify the assertion for the case $j = 1$, $i = 0$, and $j = 0$, $i = 1$.

In the first case we have

$$i(X) = V(f) \cap P_k \quad f \in A,$$

then follows that

$$x \in]X[_{\mathcal{P}} \iff sp(X) \in i(X) \iff sp(f(sp(x))) = 0 \iff |f(x)| < 1.$$

For the second case we have, for $g \in A$

$$i(X) = D(g) \cap P_k$$

that, by the previous computation, it implies

$$]X[_{\mathcal{P}} = \{x \in P_K : |g(x)| \geq 1\} = \{x \in P_K : |g(x)| = 1\}. \tag{5.1}$$

The last equality holds by normalizing. □

Proposition 5.18. *Let $i : X \hookrightarrow \mathcal{P}$ a formal embedding. Then $]X[_{\mathcal{P}}$ is an admissible open subset of P_K.*

Proof. Let's remark that this is a local property. In fact it is a consequence of the saturated condition on the strong Grothendieck topology on $\mathcal{P}_K$. To verify the admissible condition, it is sufficient to choose an admissible covering of the big space and prove it on the restriction of each set of the covering. This means that we can assume that $\mathcal{P}$ is an affine formal scheme. Then by the previous result, we choose a presentation of X. The assertion is equivalent to verify that

$$]X[_{\mathcal{P}} = \{x \in P_K : |f_i(x)| < 1 \quad \forall i = 1, \ldots, r$$
$$\text{and} \quad |g_j(x)| = 1 \quad \text{for some} \quad j = 1, \ldots, m\}.$$

is admissible. Note that $\{|g_j| = 1\}$ is intersection of special domains, so it is admissible. By intersection property, it is sufficient to verify that

$$\{x \in P_K : |f_i(x)| < 1 \quad \forall i = 1, \ldots, r\}$$

is admissible. Again for the strong property of the topology, this set is a countable union of special domains, that are admissible, so it is so. □

We can extend this result to *covering of formal embedding.* More precisely, if it is given a commutative diagram

$$\begin{array}{ccc} X_i & \xrightarrow{i_{X_i}} & \mathcal{P}_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{i_X} & \mathcal{P} \end{array} \tag{5.2}$$

with $(X_i)_i$ an open covering (resp. $(\mathcal{P}_i)_i$), with the open inclusion $X_i \hookrightarrow X$ (resp. $\mathcal{P}$), we call it a covering of the the formal embedding $i : X \hookrightarrow \mathcal{P}$.

Proposition 5.19. *Let be given a diagram as in (5.2). Then* $]X[_{\mathcal{P}}=]X_i[_{\mathcal{P}_i}$ *and it is an admissible covering.*

Proof. As discussed in the previous proof, the statement is local. Moreover the notion of tube is independent by open embedding and the equality follows. So we can assume $\mathcal{P}_i = \mathcal{P}$ and affine, for each i. Since $]X_i[_{\mathcal{P}}$ is admissible open and X Noetherian, we can assume on induction on irreducible components of X, that $X = X_1 \cup X_2$. In particular the affine description of the $]X_1[_{\mathcal{P}}$ and $]X_2[_{\mathcal{P}}$ implies that X is covered by $X_1^n \cup X_2^n$, that is a union of special domains so an admissible open; by strong property of the topology, $(]X_i[_{\mathcal{P}_i})_i$ it is an admissible covering.

□

Proposition 5.20. *Fix* $\eta < 1$*. Let* $i : X \hookrightarrow \mathcal{P}$ *be a formal embedding with* $\mathcal{P}$ *affine and fix a presentation of* X*. Let's define*

$$]X[_{\mathcal{P}_\eta} := \{x \in P_K : |f_i(x)| < \eta \quad \forall i = 1, \ldots, r \quad \textit{and} \quad |g_j(x)| = 1 \quad \textit{for some} \quad j = 1, \ldots, m\}.$$

Call it open tube of radius $\eta < 1$*. Then the open tubes of radius* $\eta < 1$ *form an admissible covering of* $]X[_{\mathcal{P}}$*.*

Proof. By the satured property of Grothendieck topology, the assertion is proved after showing a admissible refinement, since the open tubes defined form a covering. But this refining can be obtained by special domains $V_{\eta,n}$ for each fixed η and $n > 0$, and this collection for the variables n and η form an admissible covering. □

Another important concept for defining the rigid cohomology is the notion of *strict neighborhoods* that is defined for make possible a discussion on non proper varieties. We will work in the following situation.

Definition 5.21. *A frame is a diagram* $X \hookrightarrow Y \hookrightarrow \mathcal{P}$ *given an open immersion of an algebraic* k*-variety (*k*-scheme separated and of finite type)* X *into another algebraic* k*-variety* Y *and a closed immersion of* Y *into a formal* W*-scheme* $\mathcal{P}$

In the situation of varieties in characteristc $p > 0$, a theorem of Nagata ([Nag62]) holds on the compactifications, and it always permits to have an open embedding of X in a proper variety $\bar{X}$. Consider $\mathcal{P} = \hat{\bar{X}}$ as the completion along some closed subvariety. Since we will concerned with (flat) W-scheme, often for our interest the completion is made along the special fiber. In the next, we will interested to the smooth case for X, but now we discuss more generally.

Definition 5.22. *Let $(X, Y, \mathcal{P})$ a frame. $]Y[_\mathcal{P}$ the rigid analytic variety associate to the tube. Let U be an admissible open of $]Y[_\mathcal{P}$. Then U is called a strict neighborhood of $]X[_\mathcal{P}$ in $]Y[_\mathcal{P}$ if $]X[_\mathcal{P} \subset U$ and the covering $(U,]Y - X[_\mathcal{P})$ is an admissible covering of $]Y[_\mathcal{P}$.*

Lemma 5.23. *Let U be a strict neighborhood as above.*
1) Suppose that U' is an admissible open subset of $]Y[_\mathcal{P}$ such that $U \subset U'$. Then U' is a strict neighborhood.
2) Any finite intersection of strict neighborhoods is a strict neighborhood.

Proof. The first assertion results from the fact that a covering by admissible open subsets that has an admissible refinement is already admissible. To prove the second assertion, by induction we suppose to have two strict neighborhood V_1, V_2. We have to prove that

$$(V_1 \cap V_2,]Y - X[)$$

is admissible. since $(V_2,]Y - X[)$ is admissible, by intersection on V_1 follows that

$$(V_2 \cap V_1,]Y - X[\cap V_1)$$

is admissible. Then

$$(V_1 \cap V_2 \cap]Y - X[,]Y - X[)$$

is also admissible, since it admits the trivial refining by $(]Y - X[)$. Since $(V_1,]Y - X[)$ is admissible by hypothesis, then the following covering obtained by gluing that admissible above, is admissible:

$$(V_1 \cap V_2,]Y - X[\cap V_1, V_1 \cap V_2 \cap]Y - X[,]Y - X[).$$

But it is also an admissible refining of $(V_1 \cap V_2,]Y - X[)$, so it is admissible too. □

With similar kind of reasoning one can show different properties of the the tube. Since it is not the principal argument of this thesis, we limit ourselves to cite the next fact, that can be useful on the next.

Proposition 5.24. *Let $(X, Y, \mathcal{P})$ be a frame with $\mathcal{P}$ affine. Let $\lambda < 1$ and $U =]Y_X[_{\mathcal{P}_\lambda}$. Then fix a presentation of $Y - X = V(g_1, ..., g_r) \cap P_k$. Then*

1) $U_\lambda = \{x \in]Y[_{\mathcal{P}}: \text{and} \quad |g_j(x)| \geq \lambda \quad \text{for some} \quad j = 1, \ldots, r\}$

2) Let V an admissible open of $]Y[_{\mathcal{P}}$ containing $]X[_{\mathcal{P}}$. V is a strict neighborhood if and only if there exists $\lambda_0 < 1$ such that for all affinoid $W \subset]Y[_{\mathcal{P}}$ and $\lambda_0 \leq \lambda < 1$ then $W \cap U_\lambda \subset V$.

3) for X quasi-compact each U_λ is a strict neighborhood.

Proof. See [Ber96]. □

We can extend this proposition for general formal schemes, but for details we refer to [Ber96]. Let's conclude citing a result of Berthelot used to verify some property on the rigid cohomology the we will define, but we link the proof to [Ber96].

Theorem 5.25. *(strong fibration theorem) Given a commutative diagram of frames as follows,*

$$\begin{array}{ccccc} & & & & \mathcal{P}' \\ & & & \nearrow & \downarrow u \\ X & \hookrightarrow & Y & \hookrightarrow & \mathcal{P} \end{array}$$

with u smooth, there is an isomorphism between a strict neighborhood V' of $]X[_{\mathcal{P}'}$ in $]Y[_{\mathcal{P}'}$ and a strict neighborhood V'' of $]X[_{\mathcal{P}} \times \mathbb{B}^d(0, 1)$ in $]Y[_{\mathcal{P}} \times \mathbb{B}^d(0, 1)$.

Proof. See ([Ber96], Theorem 1.3.7). □

5.3 Rigid cohomology is well defined

Now suppose to have a frame $(X, Y, \mathcal{P})$ and after Nagata assume $Y = \bar{X}$. We want to define a functor from abelian sheaves on $]\bar{X}[_{\mathcal{P}}$ to itself. Let $\mathcal{F}$ be a such sheaf. Let consider V a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\bar{X}[_{\mathcal{P}}$. Then, denote with

$$\alpha_V : V \hookrightarrow]\bar{X}[_{\mathcal{P}}$$

$$j_X : X \hookrightarrow \bar{X}$$

the respective open inclusions. Then consider an inductive system of strict neighborhoods as above. Define

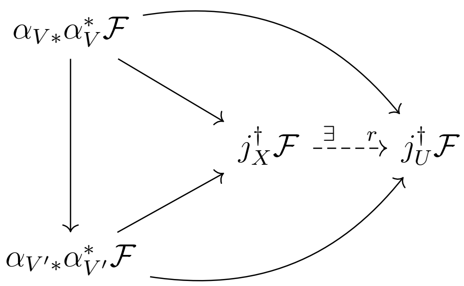
$$j_X^\dagger \mathcal{F} := \varinjlim_V \alpha_{V*} \alpha_V^* \mathcal{F}$$

where obviously the maps α_{V*} and α_V^* denotes respectively, pushforward and pullback of sheaves.

Now let Z a closed subscheme of X. Let $U = X - Z$ be the open complement, with an open immersion $j_U : U \hookrightarrow \bar{X}$ (again by Nagata). In a similar way, we can define $j_U^\dagger$. More precisely, let V a strict neighborhood of $]U[_{\mathcal{P}}$ in $]\bar{X}[_{\mathcal{P}}$, then one chooses an inductive fundamental system of strict neighborhoods V and define

$$j_U^\dagger \mathcal{F} := \varinjlim_{V'} \alpha_{V*}\alpha_V^* \mathcal{F}$$

Now let V a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\bar{X}[_{\mathcal{P}}$ and V' a strict neighborhood with of $]U[_{\mathcal{P}}$ in $]\bar{X}[_{\mathcal{P}}$, then $(V',]\bar{X} - X[_{\mathcal{P}})$ is again an admissible covering since admits a refining admissible. The by the universal property of inductive limits and by a choice of an inductive system of strict neighboorhoods like V and V', there is a map given by the following diagram



where $r : j_X^\dagger \mathcal{F} \to j_U^\dagger \mathcal{F}$ is the *restriction map.*

We consider the sheaves of differential forms on $]\bar{X}[_{\mathcal{P}}$ and consider the complex:

$$\Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} := \mathcal{O}_{]\bar{X}[_{\mathcal{P}}} \to \Omega^1_{]\bar{X}[_{\mathcal{P}}} \to \dots$$

The restriction induces a map

$$r : j_X^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \to j_U^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}}.$$

Then we can form the fibered cone, such that there is a distinguished triangle of the form:

$$\text{Cone}\left(r : j_X^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \to j_U^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}}\right)[-1] \to j_X^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \xrightarrow{r} j_U^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \xrightarrow{+1} \dots . \qquad (5.3)$$

The functor $j^\dagger$ is exact ([Ber96]) and the fundamental system of strict neighborhoods can made by pseudo-compact spaces (i.e. locally on affinoid subset is quasi-compact). This means that $j^\dagger$ commutes with filtrant inductive limits.

Definition 5.26. *Let $(X, \bar{X}, \mathcal{P})$ be a frame with $\bar{X}$ a compactification of X and assume that $\mathcal{P}$ is smooth on a neighborhood of X. Define the rigid cohomology of X on K as the hypercohomology groups given by*

$$H^*_{rig}(X/K) := \mathbb{H}^*(]\bar{X}[_{\mathcal{P}}, j^\dagger_X \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}}).$$

The cohomology with support in the closed subscheme Z is given by the hypercohomology groups defined on the fibered cone:

$$H^*_{rig,|Z|}(X/K) := \mathbb{H}^*(]\bar{X}[_{\mathcal{P}}, Cone\left(r : j^\dagger_X \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \to j^\dagger_U \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}}\right)[-1]).$$

A simple remark is that when $X = Z$ the definitions coincide.
Moreover by the long exact sequence in hypercohomology we deduce a long exact sequence of the form:

$$\cdots \to H^i_{rig,|Z|}(X/K) \to H^i_{rig}(X/K) \to H^i_{rig}(X - Z/K) \to \ldots \qquad (5.4)$$

The major work of Berthelot has been to prove that this definition is independent by the choices of the compactification and the formal scheme $\mathcal{P}$. Here we will see what tools are needed to prove such facts, but the details go out from our discussion and the purpose of the thesis, so what we do is to collect and discuss some useful and remarkable facts on the rigid cohomology, that permits us to consider it as a good cohomological theory. We will use such results to follow the reasoning in the last chapter. We will refer on this part to [Ber97b]. To prove the independence of the formal scheme, Berthelot used the following result:

Theorem 5.27. *Let $X \to \bar{X}$ an open immersion of k-scheme of finite type. Then consider the following commutative diagram*

$$\begin{array}{ccccc} & & & & \mathcal{P}' \\ & & & \overset{i'}{\nearrow} & \downarrow u \\ X & \overset{j}{\hookrightarrow} & \bar{X} & \overset{i}{\hookrightarrow} & \mathcal{P} \end{array}$$

where j is an open immersion and i, i' closed immersions, smooth on X. Let $j^\dagger$ and $j'^\dagger$ the respective functor on $]\bar{X}[_{\mathcal{P}}$ and $]\bar{X}[_{\mathcal{P}'}$. Then assume u smooth over X and $u_K : P'_K \to P_K$ is the morphism of the analytic varieties corresponding to $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then the following canonical morphism is an isomoprhism of complexes:

$$j^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \xrightarrow{\simeq} \mathbb{R}u_{K*} j'^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}'}} \qquad (5.5)$$

Then after applying $\mathbb{R}sp_*$ *we obtain an isomorphism:*

$$\mathbb{R}sp_* j^\dagger \Omega^\bullet_{]\bar{X}[_\mathcal{P}} \xrightarrow{\simeq} \mathbb{R}sp_* j'^\dagger \Omega^\bullet_{]\bar{X}[_{\mathcal{P}'}}. \tag{5.6}$$

An analogous result holds for cohomology with support.

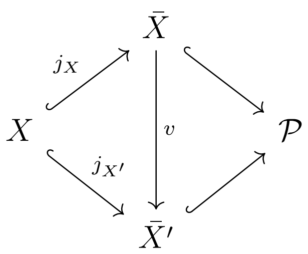
Proof. See ([Ber97b], Theorem 1.4). □

Moreover the complex $j^\dagger_X \Omega^\bullet_{]\bar{X}[_\mathcal{P}}$ is functorial on the couple $(X, \bar{X})$ in the following sense: given an $(f, \bar{f}) : (X, \bar{X}) \to (X', \bar{X}')$ then after a choice of $\bar{X} \hookrightarrow \mathcal{P}$ and $\bar{X}' \hookrightarrow \mathcal{P}'$, we can form the product $\mathcal{P}'' = \mathcal{P} \times \mathcal{P}'$ and consider the projection morphism $q_1 : \mathcal{P}'' \to \mathcal{P}$ and $q_2 : \mathcal{P}'' \to \mathcal{P}'$. Then we can compose and find the morphism of associated complexes :

$$\mathbb{R}sp_* j^\dagger_{X'} \Omega^\bullet_{]\bar{X}'[_{\mathcal{P}'}} \xrightarrow{q_2^*} \mathbb{R}\bar{f}_* \mathbb{R}sp_* j^\dagger_X \Omega^\bullet_{]\bar{X}[_{\mathcal{P}''}} \underset{\simeq}{\xrightarrow{(q_1^*)^{-1}}} \mathbb{R}\bar{f}_* \mathbb{R}sp_* j^\dagger_X \Omega^\bullet_{]\bar{X}[_\mathcal{P}}. \tag{5.7}$$

This morphism does not depend by the choice of the formal scheme and one can verify the functoriality. Then the independence of the compactification follows by the theorem

Theorem 5.28. *Let have the following commutative diagram:*



where $j_X, j_{X'}$ *are open immersion in the two compactifications of* X*. Suppose* v *is a proper morphism and* $\mathcal{P}$ *smooth in a neighborhood of* X*. Then we have the following isomoprhism of complexes:*

$$\mathbb{R}sp_* j^\dagger_X \Omega^\bullet_{]\bar{X}[_\mathcal{P}} \xrightarrow{\simeq} \mathbb{R}v_* \mathbb{R}sp_* j^\dagger_{X'} \Omega^\bullet_{]\bar{X}'[_\mathcal{P}}. \tag{5.8}$$

Proof. See ([Ber97b],Theorem 1.6). □

The theorem implies that when one keeps the global sections with respect to $\bar{X}$ and $\bar{X}'$ we have an isomoprhism at the level of cohomology:

$$H^*_{rig}(X/K) \xrightarrow{\simeq} \mathbb{H}^*(]\bar{X}'[_\mathcal{P}, j^\dagger_{X'} \Omega^\bullet_{]\bar{X}[_\mathcal{P}}). \tag{5.9}$$

Now it's easy to define a pullback for the cohomology and verify its functoriality. So to a morphism of k-scheme $f : X \to X'$ we can choose compactifications $\bar{X}$ and $\bar{X}'$, consider the product $\bar{X} \times \bar{X}'$ and denote with $\bar{X}''$ the schematic closure of X in $\bar{X} \times \bar{X}'$. Then $p_1 : \bar{X}'' \to \bar{X}$ and $p_2 : \bar{X}'' \times \bar{X}' \to \bar{X}'$ denote the projections. Then we define f^* as

$$f^* : H^\bullet_{rig}(X'/K) \xrightarrow{p_2^*} \mathbb{H}^\bullet(\bar{X}'', j^\dagger_X \Omega^\bullet_{]\bar{X}''[_{\mathcal{P}}}) \underset{\simeq}{\xrightarrow{(q_1^*)^{-1}}} H^\bullet_{rig}(X/K). \tag{5.10}$$

It is possible to extend this functoriality property to cohomology with support, under the condition that for a given morphism $f : X \to X'$, Z a closed subscheme of X, and T closed subscheme of X', then $f^{-1}(T) \subset Z$. Then there is a functorial pullback

$$f^* : H^\bullet_{rig,|T|}(X'/K) \to H^\bullet_{rig,|Z|}(X/K). \tag{5.11}$$

5.3.1 Rigid cohomology: fundamental properties

Since the technical points are almost discussed, we can prove some elementary property following by the "well" (at posteriori) posed definitions. In particular the rigid cohomology is a Bloch-Ogus cohomology theory.
We can say that an excision property holds:

Proposition 5.29. *Let X a k-variety. Let $T \subset Z \subset X$, closed subschemes. Then there is a long exact sequence:*

$$\cdots \to H^i_{rig,|T|}(X/K) \to H^i_{rig,|Z|}(X/K) \to H^i_{rig,|Z-T|}((X-T)/K) \to \dots \tag{5.12}$$

Proof. We choose a compactification $\bar{X}$ of X and $\mathcal{P}$ a formal scheme, smooth on a neighborhood of X. Then, for any sheaf $\mathcal{F}$ on $]\bar{X}[_{\mathcal{P}}$ we have the following short exact sequence of the cone:

$$0 \to j_{\bar{X}-\bar{T}}\mathcal{F}[-1] \to \mathrm{Cone}(\mathcal{F} \to j_{\bar{X}-\bar{T}}\mathcal{F})[-1] \to \mathcal{F} \to 0.$$

The terms $\bar{T}$ and $\bar{Z}$ are the schematic closure of T and Z respectively, in $\bar{X}$. The we can apply this exact sequence to the complex of sheaves

$$\mathcal{F} = \mathrm{Cone}(j^\dagger_X \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}} \to j^\dagger_{\bar{X}-\bar{Z}} j^\dagger_X \Omega^\bullet_{]\bar{X}[_{\mathcal{P}}})[-1].$$

Since $j^\dagger_{\bar{X}-\bar{Z}} j^\dagger_X = j^\dagger_{X-Z}$, we notice that $\mathcal{F}$ is the complex the computes the rigid cohomology of X relative to the closed subset Z. Moreover, since the $j^\dagger$ commutes with fibered cones, and by the fact that $j^\dagger_{\bar{X}-\bar{T}} j^\dagger_X = j^\dagger_{X-T}$ and $j^\dagger_{\bar{X}-\bar{T}} j^\dagger_{\bar{X}-\bar{Z}} j^\dagger_X = j^\dagger_{(X-T)-(Z-T)}$, we see that

$$j^\dagger_{\bar{X}-\bar{T}}\mathcal{F} = \mathrm{Cone}(j^\dagger_{X-T}\Omega^\bullet_{]\bar{X}[_\mathcal{P}} \to j^\dagger_{(X-T)-(Z-T)}\Omega^\bullet_{]\bar{X}[_\mathcal{P}})[-1]$$

and this is the complex that computes the rigid cohomology of $X-T$ relative to $Z-T$. Now observe that holds

$$\mathrm{Cone}(\mathcal{F} \to j^\dagger_{\bar{X}-\bar{T}}\mathcal{F})[-1] \simeq$$
$$\mathrm{Cone}(\mathrm{Cone}(j^\dagger_X\Omega^\bullet_{]\bar{X}[_\mathcal{P}} \to j^\dagger_{X-T}\Omega^\bullet_{]\bar{X}[_\mathcal{P}})[-1] \to \mathrm{Cone}(j^\dagger_{X-Z}\Omega^\bullet_{]\bar{X}[_\mathcal{P}} \to j^\dagger_{(X-T)-(Z-T)}\Omega^\bullet_{]\bar{X}[_\mathcal{P}})[-1])[-1].$$

This follows by the fact that $j^\dagger_{\bar{X}-\bar{T}}$ commutes with the cone and by interchanging the fibered cones. Then by the fact that $X-Z=(X-T)-(Z-T)$, follows that

$$\mathrm{Cone}(\mathcal{F} \to j^\dagger_{\bar{X}-\bar{T}}\mathcal{F})[-1] = \mathrm{Cone}(j^\dagger_X\Omega^\bullet_{]\bar{X}[_\mathcal{P}} \to j^\dagger_{X-T}\Omega^\bullet_{]\bar{X}[_\mathcal{P}})[-1].$$

This complex computes the rigid cohomology of X relative to the closed T. Then the long exact sequence (5.12) follows by the long exact sequence in hypercohomology induced by the short exact sequence of the cone.

□

Corollary 5.30. *Let Z a closed subscheme of X.*
1) Suppose to have an open X' of X, such that it contains Z. Then the open immersion induces an isomorphism:

$$H^i_{|Z|,rig}(X/K) \xrightarrow{\simeq} H^i_{|Z|,rig}(X'/K) \tag{5.13}$$

2) If $Z = Z_1 \cup Z_2$ and $Z_1 \cap Z_2 = \emptyset$ then

$$H^i_{rig,|Z|}(X/K) \simeq H^i_{rig,|Z_1|}(X/K) \oplus H^i_{rig,|Z_2|}(X/K).$$

Proof. 1) follows by the long exact sequence (5.12) for $T=\emptyset$. 2) since $Z_1 \cap Z_2 = \emptyset$, and $(X-Z_1, X-Z_2)$ is an open cover of X, by the property of the functor $j^\dagger$, it follows that $j^\dagger_X\mathcal{F} \xrightarrow{\sim} j^\dagger_{X-Z_1}\mathcal{F} \oplus j^\dagger_{X-Z_2}\mathcal{F}$ for each abelian sheaf. Then

$$\mathrm{Cone}\left(j^\dagger_X\mathcal{F} \to j^\dagger_{X-Z_1}\mathcal{F} \oplus j^\dagger_{X-Z_2}\mathcal{F}\right)[-1] = 0.$$

Then the isomorphism follows by the long exact sequence for the cone. □

Another important fact on the rigid cohomology is the fact that is finite dimensional for each k-variety X. The crystalline cohomology does not preserves this property in the non proper or not smooth case. We will concern about the smooth case, so for the non proper case, the rigid cohomology is more suitable then crystalline. Moreover is a fact that the two cohomology theories are canonical isomorphic when X is smooth and proper ([Ber97b]). Before to go further we want just to see some terminology and example of the rigid cohomology in the affine case, which will be used in the last chapter, for the construction of a map between cohomological theories.
Let A be an R-algebra of finite type for R a commutative ring.

Definition 5.31. *Let I be an ideal of R. The weak completion of A with respect to I is the subset of the completion $\hat{A} = \varprojlim_n A/I^n A$ defined by:*

$$A^\dagger := \{z \in \hat{A} \quad | \quad \exists x_1, \dots, x_n \in A \quad \textit{and} \quad \exists p_j \in I^j R[X_1, \dots, X_n] \quad \textit{s.t.}$$
$$\exists c \quad \textit{s.t.} \quad \textit{deg}(p_j) \leq c(j+1) \quad \forall j$$
$$\textit{and} \quad z = \sum_{j \geq 0} p_j(x_1, \dots, x_n)\}.$$

In our case we let $R = W(k) = W$ and consider the weak completion respect to (π). If $X = \text{spec}(A)$ where a representation of A is

$$A = W[x_1, \dots, x_n]/(f_1, \dots, f_m),$$

then we find that

$$A^\dagger = W[x_1, \dots, x_n]^\dagger/(f_1, \dots, f_m)$$

where $W[x, \dots, x_n]^\dagger$ is equal to

$$W[x_1, \dots, x_n]^\dagger = \{\sum_{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}} \quad \mathbf{i} = (i_1, \dots, i_n) | a_{\mathbf{i}} \in W$$
$$\text{s.t.} \quad \exists c \quad \text{and} \quad \rho \in (0,1)$$
$$\text{s.t.} \quad |a_{\mathbf{i}}| \leq c\rho^{|\mathbf{i}|} \quad |\mathbf{i}| = i_1 + \dots + i_n\}.$$

Then suppose that X is smooth. We want to compute $H^*_{rig}(\text{spec}(A_k))$.
Let $X_k = \text{spec}(A_k)$, then consider $\bar{X}_k$ as the reduction $\pmod{\pi}$ of the projective closure of $X \hookrightarrow \mathbb{P}^n_W$. Then we complete $\bar{X}$ along the ideal defining X_k. Then denote it with $\mathcal{P} = \hat{\bar{X}}$. Then $(X_k, \bar{X}_k, \mathcal{P})$ is a frame.
By definition

$$]\bar{X}_k[_{\mathcal{P}} = P_K$$

and by representation of X_k, it follows that

$$]X_k[_{\mathcal{P}}= P_K \cap \mathbb{B}^n(0, 1^+).$$

For $\lambda > 1$, the open admissible sets

$$U_\lambda := \{x \in P_K : \text{and} \quad |g_j(x)| \leq \lambda \quad \text{for some} \quad j\}$$

form a quasi-compact admissible covering of P_K. Then by definition of the sheaf of regular functions $\mathcal{O}_{P_K}$, we have

$$\Gamma(]X_k[_{\mathcal{P}}, \mathcal{O}_P) = \hat{A} \otimes K.$$

Then the inclusion $]X_k[_{\mathcal{P}} \subset U_\lambda$ induces an homomorphism

$$\Gamma(U_\lambda, \mathcal{O}_{P_K}) \to \hat{A} \otimes K.$$

If I is the ideal in $\mathcal{O}_{\mathbb{P}^n_K}$ defining P_K, then

$$\Gamma(U_\lambda, \mathcal{O}_{P_K}) \simeq \Gamma(U_\lambda, \mathcal{O}_{\mathbb{P}^n_K})/I\Gamma(U_\lambda, \mathcal{O}_{\mathbb{P}^n_K}).$$

Since on the right side there are regular functions defined on U_λ, that is the ball centered in 0 and with radius $\lambda > 1$, its image in $\hat{A} \otimes K$ is contained in $A^\dagger \otimes K$, since its power series's develop has to be convergent on this ball. On the other hand, any element of $A^\dagger \otimes K$ converges on a such ball, for some $\lambda > 1$. For these reasons, we have an isomorphism

$$\varinjlim_{\lambda \to 1^+} \Gamma(U_\lambda, \mathcal{O}_{P_K}) \simeq A^\dagger \otimes K.$$

Now since P_K admits an admissible covering of quasi-compact spaces, then each finite intersection of the elements of the covering is a quasi-compact space, and this implies that the left exact functor Γ commutes with inductive limits. For any $\mathcal{F}$ $\mathcal{O}_{P_K}$-module, then follows that

$$\Gamma(P_K, j^\dagger \mathcal{F}) = \Gamma(P_K, \varinjlim_{U_\lambda} \alpha_{U_\lambda,*}\alpha^*_{U_\lambda}\mathcal{F}) = \varinjlim_{\lambda} \Gamma(P_K \cap U_\lambda, \mathcal{F}).$$

Then the following proposition holds:

Proposition 5.32. *Let $X = spec(A)$ an affine smooth W-scheme, with reduction X_k. Then*

$$H^*_{rig}(X/K) = \mathbb{H}^*(A^\dagger \to \mathcal{O}_{A^\dagger} \otimes K \to \Omega^1_{A^\dagger/W} \otimes K \to \dots). \tag{5.14}$$

Proof. It follows by the previous discussion and by noting that

$$\varinjlim_{\lambda \to 1^+} \Gamma(U_\lambda, \Omega^\bullet_{P_K} \otimes K) = \Omega^\bullet_{A^\dagger/W} \otimes K.$$

□

There is also a notion of rigid cohomology with compact support.
Let $(X, \bar{X}, \mathcal{P})$ be a frame, Z a closed subscheme of X. Then $]Z[_\mathcal{P} \subset]\bar{X}[_\mathcal{P}$ is admissible open. Denote by i the inclusion map. For an abelian sheaf $\mathcal{F}$ of $]\bar{X}[_\mathcal{P}$ Then define

$$\Gamma_X \mathcal{F} := \ker(\mathcal{F} \to i_* i^* \mathcal{F}).$$

Γ_X is a left exact functor and admits right derived functors. Then the cohomology with compact support is given by

$$H^*_{c,rig}(X/K) := \mathbb{H}^*(]\bar{X}[_\mathcal{P}, \mathbb{R}\Gamma_X \Omega^\bullet_{]\bar{X}[_\mathcal{P}}). \tag{5.15}$$

A first look to the latter definition seems to be a strong dependence by the choices of the closed scheme Z. But a particular case of ([Ber86],section 3, Theorem 4) proves that is not actually true.

Theorem 5.33. *Let $v : \bar{X} \to \bar{X}$ a proper morphism. Let Z, Z' two closed subscheme of $\bar{X}$ such that $v(Z') \subset Z$. Then the rigid cohomology with compact support relative to the inclusion $i :]Z[_\mathcal{P} \to]\bar{X}[_\mathcal{P}$ and $i' :]Z'[_\mathcal{P} \to]\bar{X}[_\mathcal{P}$ are isomorphic.*

Proof. See ([Ber86],section 3, Theorem 4). □

The consequence is that "bring together" all the closed subschemes of X, that means to choose an inductive system of such set, does not change the cohomology.

Theorem 5.34. *The rigid cohomology with compact support satisfies the following properties:*

1) There exists a natural map $H_{c,rig}(X/K) \to H_{rig}(X/K)$ that is an isomoprhism when X is proper.

2) It is contravariant for proper morphisms and covariant for open immersions

3) There is an excision long exact sequence: For Z a closed subscheme of X and $U = X_Z$, then

$$\cdots \to H^i_{c,rig}(U/K) \to H^i_{c,rig}(X/K) \to H^i_{c,rig}(Z/K) \to \dots \tag{5.16}$$

holds.

Proof. See ([Ber86], 3.1). □

Now suppose we are in the case with X a smooth scheme and to have a closed subscheme Z of X of codimension r. Then one can define a Gysin morphism ([Pet03]) that induce the following isomorphism:

$$H^i_{|Z|,rig}(X/K) \simeq H^{i-2r}_{rig}(Z/K) \tag{5.17}$$

This implies the weak purity condition on the rigid cohomology.

Proposition 5.35. *(weak purity) If Z is a closed subscheme of codimension r of a smooth k-variety X, then holds that*

$$H^i_{|Z|,rig}(X/K) = 0 \quad \forall i < 2r. \tag{5.18}$$

Moreover, there exists a *trace* map

$$Tr_X : H^{2n}_{c,rig}(X/K) \to K \tag{5.19}$$

defined for a k-variety X of dimension n.
For a subscheme Z of codimension r of a smooth k-variety X we then define a fundamental rigid class $\eta_{Z,rig}$ as

$$\eta_{Z,rig} := G_{Z/X}(1), \tag{5.20}$$

where $G_{Z/X} : H^0_{rig}(Z/K) \to H^{2r}_{|Z|,rig}(X/K)$ is defined by the isomorphism (5.17). Following Petrequin we collect all the basic facts on rigid cohomology that make it a Bloch-Ogus cohomology theory.

Theorem 5.36. *Let k be a perfect field of characteristic $p > 0$, let $W(k) = W$ its ring of Witt vectors and $K = Frac(W)$ the fraction field of W with charcateristic 0. Let X a k-scheme reduced of finite type. The following properties holds:*

1. *$H^i_{rig}(X/K)$ and $H^i_{rig,c}(X/K)$ are K-vector spaces of finite dimension.*

2. *If X is equidimensional of dimension n, then the rigid cohomology and the rigid cohomology with compact support is 0 for each $i \notin \{0, \dots, 2n\}$.*

3. *The weak purity property holds for X smooth.*

4. *When X is smooth there is a perfect pairing, given by the trace morphism :*

$$H^i_{|Z|,rig}(X/K) \times H^{2n-i}_{c,rig}(Z/K) \to K$$

5. *If X, Y are smooth varieties then there is an isomorphism:*

$$H^*_{rig}(X/K) \otimes H^*_{rig}(Y/K) \xrightarrow{\simeq} H^*_{rig}(X \times_k Y/K).$$

It is compatible with fundamental classes:

$$\eta_{X,rig} \otimes \eta_{Y,rig} = \eta_{X\times_k Y,rig}$$

Proof. See ([Pet03],Theorem 2.10). □

Note that in the smooth case we can define a functorial pushforward for rigid cohomology. If $f : X \to Y$ a morphism of smooth varieties then we have a pullback on the support:

$$f^* : H^i_{c,rig}(Y) \to H^i_{c,rig}(X),$$

then by the perfect pairing (4), if n, m are the dimension respectively of X, Y, then they induce a morphism

$$f_* : H^{2n-i}_{rig}(X) \to H^{2m-i}_{rig}(Y). \tag{5.21}$$

By these results we can deduce the homotopy property for the rigid cohomology:

Proposition 5.37. *(Homotopy property) Let X a smooth k-variety, then for each natural number i holds that*

$$H^i_{rig}(X/K) \simeq H^i_{rig}(X \times_k \mathbb{A}^1_k/K). \tag{5.22}$$

Proof. By the property 5) of the Theorem (5.36), it suffices to prove that the rigid cohomology of $\mathbb{A}^1_k$ is 0 for $i > 0$. By the isomorphism (5.14) we have in this case $A^\dagger = W^\dagger[t]$. Now we claim that the sequence (like de De Rham)

$$0 \to A^\dagger \otimes K \to \mathcal{O}_{A^\dagger} \otimes K \to \Omega^1_{A^\dagger} \otimes K \to 0 \tag{5.23}$$

is exact. But by a remark on the Poincaré lemma on the characteristic 0 case, it is sufficient to verify that there exist the "integrals of $\sum a_n t^n dt$", i.e. that the elements

$$\sum \frac{a_n t^{n+1}}{n+1} \in \mathcal{O}_{A^\dagger} \otimes K.$$

By the definition of the weak completion it is sufficient to verify that $|\frac{a_n}{n+1}| \leq c\rho^n$ for some c and $\rho < 1$ independently by n. For simplicity suppose to have a p-adic valuation $v_p(.)$. Since there exist a $\rho_0 < 1$ such that $|a_n| \leq c\rho_0^n$, then $|c\rho_0^n/(n+1)| \leq c\rho_0^n p^{v_p(n+1)}$. Then choose an a such that $1/{\rho_0}^a > p$ and let $n = am$. For $m \to \infty$ then we have

$$c\rho_0^n p^{v_p(n+1)} = c{\rho_0}^{am} p^{v_p(am+1)} \leq cp^{v_p(am+1)-m} \xrightarrow[m\to\infty]{} 0.$$

Then there exists some suitable $\tilde{c}$ and $\rho = \rho_0$ such that the claim holds. □

5.3.2 The (rigid) Frobenius action

Let X be a smooth k-variety. For a closed subscheme Z of codimensior r of X we have defined the fundamental class. Now suppose to have a cycle $\sum n_i[Z_i] \in \mathcal{Z}^r(X)$. Then if $\alpha_i : Z_i \hookrightarrow X$ denotes the inclusion morphism, extending by linearity we obtain a class in cohomology, given by

$$\gamma_X : \mathcal{Z}^r(X) \to H^{2r}_{rig}(X/K)$$
$$\sum n_i[Z_i] \mapsto \sum n_i \alpha_{i*}\eta_{Z_i,rig}$$

The following propositions hold:

Proposition 5.38. *1. If X, X' are smooth k-varieties and $f : X \to X'$ is a proper morphism. For each $x \in Z^r(X)$, then pushforward f_* commutes with cycle class:*

$$\gamma_{X'}(f_*(x)) = \gamma_X(f_*(x))$$

2. if $x \in Z^r(X)$ is a cycle rationally equivalent to 0, then $\gamma_X(x) = 0$

3. γ_X factorizes with respect to Chow groups $CH^r(X)$ and it is a ring homomorphism.

4. if f is morphism of smooth varieties then the pullback commutes with cycle classes: if $x \in CH^r(X')$

$$\gamma_X(f^*(x)) = \gamma_{X'}(f^*(x)).$$

Proof. See Petrequin [Pet03].

□

Then we can focus on the absolute Frobenius morphism $f : X \to X$. Then the pullback $f^* = \phi$, induces a morphism

$$\phi : H^i_{rig}(X/K) \to H^i_{rig}(X/K).$$

Then the following proposition holds:

Proposition 5.39. *([Pet03],Proposition 7.13) If Z is a subvariety of X smooth of codimension r, then*

$$\phi(\gamma_X(Z)) = p^r \gamma_X(Z). \tag{5.24}$$

Proof. Assume Z is irreducible, the general case follows by linearity. The first claim is that when X is smooth then $\phi = f^*$ is flat. Assume that. Then by the definition of flat pullback for the cycle on Chow groups, it follows that

$$f^*[Z^{(p)}] = [f^{-1}(Z^{(p)})] = s[Z] \qquad s = \text{leght}_{\mathcal{O}_{X,\zeta}}(\mathcal{O}_{f^{-1}(Z^{(p)}),\zeta}),$$

where $Z^{(p)}$ is the image of Z by f, that like spaces are the same, but the local function are obtained by those of Z with power of p.Here ζ is the generic point corresponding to $f^{-1}(Z^{(p)})$. Then $\mathcal{O}_{X,\zeta}$ is a regular (since X smooth) local ring of dimension r and so the length of $\mathcal{O}_{f^{-1}(Z^{(p)}),\zeta}$ is equal to p^r. Then by compatibility of the cycle class with the pullback, we obtain:

$$\phi(\gamma(Z)) = f^*(\gamma(Z)) = \gamma(f^*(Z)) = p^r\gamma(Z).$$

Now we prove the claim. Actually, since it is a local question, we suppose X to be a smooth k-scheme of relative dimension r. This means that locally there is a factorization

$$\begin{array}{ccc} U & \xrightarrow{\quad\pi\quad} & \mathbb{A}^r_k \\ & \searrow \quad \swarrow & \\ & spec(k) & \end{array}$$

with π un étale map and U a Zariski open of X. We claim that actually the relative Frobenius $f_{X/k} : X \to X^{(p)}$ is locally free of rank p^r. With this in mind, the absolute Frobenius is the composition of relative and arithmetical Frobenius. Then by base change, since $f : spec(k) \to spec(k)$ is obviously locally free, also the arithmetic Frobenius is so. Then the absolute Frobenius is locally free, then flat. The assertion is local so we can assume to have $\pi : X \to \mathbb{A}^r_k$ étale. A fact in algebraic geometry is that when X is k-scheme étale then the relative Frobenius $f_{X/k}$ is an isomorphism. This means we have a cartesian square

$$\begin{CD} X @>{f}>> X \\ @VVV @VVV \\ spec(k) @>{a\mapsto a^p}>> spec(k) \end{CD}$$

where f is the absolute Frobenius. Then we have other two cartesian squares (by construction)

$$\begin{CD} X^{(p)} @>>> X \\ @VVV @VVV \\ spec(k) @>{a\mapsto a^p}>> spec(k) \end{CD} \qquad \begin{CD} \mathbb{A}_k^{r(p)} @>>> \mathbb{A}_k^r \\ @VVV @VVV \\ spec(k) @>>> spec(k) \end{CD}$$

The universal property applied to the second cartesian square provides a map $\pi^{(p)} : X^{(p)} \to \mathbb{A}_k^{r(p)}$ such that

$$\begin{array}{ccccc} X^{(p)} & \longrightarrow & X & \xrightarrow{\pi} & \\ & \searrow^{\exists!\pi^{(p)}} & & & \downarrow \\ & & \mathbb{A}_k^{r(p)} & \longrightarrow & \mathbb{A}_k^r \\ & \searrow & \downarrow & & \downarrow \\ & & spec(k) & \longrightarrow & spec(k) \end{array}$$

Moreover

$$\begin{CD} X @>{f_{X/k}}>> X^{(p)} \\ @V{\pi}VV @VV{\pi^{(p)}}V \\ \mathbb{A}_k^r @>{f_{\mathbb{A}_k^r/k}}>> \mathbb{A}_k^{r(p)} \end{CD}$$

commutes. In fact, by universal property, there exists a unique map $f_{X/k}$ and $\pi^{(p)}$ such that

$$\begin{array}{ccccc} X & \xrightarrow{f_{X/k}} & X^{(p)} & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \pi^{(p)}} & & \downarrow{\scriptstyle \pi} \\ \mathbb{A}_k^r & & \mathbb{A}_k^{r(p)} & \longrightarrow & \mathbb{A}_k^r \\ & \searrow & \downarrow & & \downarrow \\ & & spec(k) & \longrightarrow & spec(k) \end{array}$$

but again, by universal property also the map

$$X \to \mathbb{A}_k^r \xrightarrow{f_{\mathbb{A}_k^r/k}} \mathbb{A}_k^{r(p)} \to spec(k)$$

make the same diagram commutes. Then, by uniqueness, the claimed diagram is commutative. Moreover it is a cartesian square because all squares written above are cartesian, so it follows at fortiori. This means that it suffices to verify the claim for the case $X = \mathbb{A}^r_k$ and it is trivial. □

CHAPTER 6

SYNTOMIC REGULATOR IN ÉTALE AND RIGID COHOMOLOGY

Until now, we have seen two cohomology theories for the special fiber. Through them, we can construct some syntomic theories: one is the original ”Fontaine-Messing” ([FM87]) which deals with crystalline cohomology, the others are its development using rigid cohomology: rigid-Gros and rigid-Besser cohomology. The Fontaine-Messing and rigid-Besser cohomology have their own way of constructing a regulator map. In particular, the rigid-Besser cohomology admits a regulator map arising from the classical cycle map, hence having as a source the Higher Chow groups. This has been done in the article [CCM13]: the proof involved some properties of this cohomology theory as, for example, the homotopy property. We plan in this thesis to follow the same path in the possible construction of a regulator map for the rigid-Gros cohomology.

On the other hand, one of the important features of the Fontaine-Messing cohomology was its link to the étale p-adic cohomology. Gros was able to conctruct a similar map to (the syntomic- étale cohomology hence to) the étale p-adic cohomology for his cohomology, as well as to the crystalline cohomology. We address a similar problem for the Besser-Rigid cohomology and we ask about a possible compatibility of the regulator maps between rigid-Besser and rigid-Gros cohomology.

This was initially motivated by the following compatibility for the K-theory. In fact in [[GK90], Theorem 0.1] it is proved the following (let p a prime number):

Theorem 6.1. *If k is a perfect field of characteristic $p > 0$ and $W(k)$ denotes its ring of Witt vectors, then for all W-schemes quasi-projective and flat X there exist for all $i < p$ Chern maps*

$$c_{i,j} : K_j(X) \to H^{2i-j}_{ét}(X_n, s_n(i)_X)$$

such that they factor the higher Chern class in crystalline cohomology, i.e. $c_{i,j}$: $K_j(X) \to H^{2i-j}_{crys}(X_0/W_n)$.
Here $X_n = X \times_W W_n$ and $W_n = W/p^{n+1}$.

Here $s_n(r)_X$ is the same of the syntomic sheaves (4.5). For convenience, we also use the same notation of Gros. Actually can be already seen that a kind of generalisation appears in ([Gro94]): we will use its definition of the "rigid-syntomic" sheaf.
In the course of the exposition we notice that there are some problems with the analogue in the rigid setting of the Hodge filtration in de Rham cohomology. So, we try to understand which kind of behaviour such filtration has to satisfy to build a nice cohomology theory.
Notation: We have k, perfect field of characteristic $p > 0$, W its ring of Witt vectors, we suppose there is not ramification, and $K := Frac(W)$. For a W-scheme Z, $Z_n = Z \otimes W_n$, where $W_n = W/p^{n+1}$.

In this discussion we will assume X smooth. Our first task is to replace the K-theory in 6.1 with the "motivic" interpretation of the higher Chow groups. This led to approach the question not anymore on the crystalline theory, but more generally on a rigid point of view. The motivation of such choice, a part of his natural generalization of the crystalline cohomology, lies on the fact that the rigid cohomology has a more classical treating of the existence of cycle class map, like in the étale, de Rham, singular and other classical Weil cohomology theories. Moreover it is a better cohomology theory for non-proper schemes (since the crystalline cohomology is not finite for the non proper case). However, the crystalline setting is easier to connect to étale cohomology, while in the rigid case, this relationship seems less clear. So we notice that a satisfactory theory bringing such connection is the "rigid" cohomology of Gros ([Gro94]), defined by using a slightly modification of the syntomic sheaf of Fontaine and Messing in the rigid case. Then, our task is to prove that in the case of smooth schemes (and possibly with other known hypothesis), there exists a cycle map and that this map can be extended to higher cycle maps, as in the language of Bloch ([Blo86b]). As consequence we just looking for deriving the analogous of theorem 3.4 in ([Gro94]) but with the higher Chow groups instead of K-theory. Finally, when this is true, it shows that the map between rigid-Besser and rigid-Gros cohomology defined in the proof of Proposition 9.5 of ([Bes00]) is compatible with higher Chow groups (for the higher cycle maps in the Besser-rigid cohomology we refer to [CCM13]).

6.1 A remark on a method of Bloch

In this section we want to remark a simple general fact on Bloch cohomology theories (that means cohomology theories that satisfy the axioms described in [Blo86b]).
Let $H_1^a(-,b), H_2^a(-,b)$ be two Bloch cohomology theories.

Lemma 6.2. *Suppose that there are cycle maps*

$$cl_1 : CH^r(X) \to H_1^{2r}(X,r)$$
$$cl_2 : CH^r(X) \to H_2^{2r}(X,r)$$

and a functorial map $f_X : H_1^{2r}(X,r) \to H_2^{2r}(X,r)$ compatible with the cycle maps. Then the map $f_X^{m,r} : H_1^{2r-m}(X,r) \to H_2^{2r-m}(X,r)$ is compatible with the higher cycle maps of Bloch

$$cl_1^{m,r} : CH^r(X,m) \to H_1^{2r-m}(X,r)$$
$$cl_2^{m,r} : CH^r(X,m) \to H_2^{2r-m}(X,r)$$

Proof. It is a consequence of the fact that the higher cycle maps depend only from cycle maps. In fact in the argument of Bloch there is a map of complexes

$$\tau_N \mathcal{Z}^r(X,*) \to \varinjlim_{Z \in \mathcal{Z}^r(X,*)} H_{2,|Z|}^{2r}(X \times \Delta^*, r)$$

induced by cycle class map and by hypothesis, that map factors throw f_X. Then the rest of the argument depends only by the weak purity property, so the defined higher cycle maps are compatible by construction. □

This is a useful remark that permits to reduce compatibility at higher dimension essentially to dimension 0. What we have in mind is to compare the cohomology of Gros that we will recall in the next section, with the étale-syntomic cohomology $H^{2r}_{\text{ét}}(X, s_\infty(r)_X)$ where $s_\infty(r)_X = R\varprojlim_n s_n(r)_X$. Since we can define a map connecting these cohomology theories ([Gro94], section 3), we want to verify that there exist compatible cycle maps.

6.2 Rigid-Gros cohomology

Let us consider the following situation: Let be X a smooth W-scheme, and X_0 its special fiber. Suppose there exists an open immersion $j_{X_0} : X_0 \to Y$ of finite type k-schemes and a closed immersion of the k-scheme Y in a formal Spf(W)-scheme $\mathcal{P}$ with (rigid) generic fiber P_K, namely $i : Y \to \mathcal{P}$. Let $\tilde{Y}$ be a W-scheme such that its special fiber is Y. Then we can complete $\tilde{Y}$ along Y and we call it the formal scheme associate to $\tilde{Y}$. Denote it by $\mathcal{Y} := \hat{\tilde{Y}}$. We denote with $\tilde{Y}_K$ its (rigid) generic fiber. The tube of Y in $\mathcal{P}$ defines an open subset of P_K. The ring $\mathcal{O}_{]Y[_\mathcal{P}}$ admits an ideal I defining the generic fiber $\tilde{Y}_K$. For any strict neighborhood V of $]X_0[$ in $]Y[$, we denote the inclusion with $\alpha_V : V \hookrightarrow]Y[_\mathcal{P}$. Berthelot defined for any sheaf $\mathcal{F}$ on $]Y[$ the functor

$$j^\dagger_{X_0}\mathcal{F} := \varinjlim_V \alpha_V * \alpha_V^* \mathcal{F}$$

where the inductive limit is over the strict neighborhoods above. Let's consider a decreasing filtration of the differential forms on the tube $]Y[$. We define for an integer r

$$\mathrm{Fil}^r \Omega^\bullet_{]Y[_\mathcal{P}} := I^r \to I^{r-1}\Omega^1_{]Y[_\mathcal{P}} \to I^{r-2}\Omega^2_{]Y[_\mathcal{P}} \to \dots$$

with the convention that $I^r = \mathcal{O}_{]Y[}$ for $r \leq 0$. Then we can define a complex of sheaves as following:

$$\mathcal{J}^r_{X_0,Y,\mathcal{P}} := \mathbb{R}\mathrm{sp}_*(j^\dagger_{X_0}\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}}),$$

where sp : $\tilde{Y}_K \to \mathcal{Y}$. It represents a complex of sheaves on an open subset of P_K. For our purpose, we want to define a relative version of this complex, with analogy of rigid cohomology with support in a closed subscheme.

Let Z be a closed subscheme of X_0 and let $U = X_0 - Z$. Denote with $j_U : U \hookrightarrow Y$ the respective open immersion. As done previously, we can form the Berthelot $j^\dagger_U$

functor. Since a strict neighborhood of $]U[$ in $]Y[$ is a strict neighborhood of $]X_0[$ in $]Y[$, by the universal property of the inductive limits, we can form a restriction map $r : j^\dagger_{X_0}\mathcal{F} \to j^\dagger_U\mathcal{F}$. Then we can define a map between complexes

$$j^\dagger_{X_0}\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}} \xrightarrow{r} j^\dagger_U\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}}.$$

For convention with the notation in [Pet03], we define the fibered cone of r and denote it as follows. Define

$$\left(j^\dagger_{X_0}\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}} \xrightarrow{r} j^\dagger_U\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}}\right)_s := \mathrm{Cone}\left(j^\dagger_{X_0}\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}} \xrightarrow{r} j^\dagger_U\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}}\right)[-1]$$

and note that by definition of the cone

$$\mathrm{Cone}(r)[-1] = \begin{pmatrix} d_{X_0} & 0 \\ r & -d_U \end{pmatrix}.$$

Now we are ready to define the relative complex of sheaves as following:

$$\mathcal{J}^r_{X_0,Y,\mathcal{P},|Z|} := \mathbb{R}\mathrm{sp}_*\left(j^\dagger_{X_0}\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}} \xrightarrow{r} j^\dagger_U\mathrm{Fil}^r\Omega^\bullet_{]Y[_\mathcal{P}}\right)_s.$$

Now we note that for $r = 0$ the global sections of the complex above represent the rigid cohomology with support in Z defined in Berthelot ([Ber97b]) and Petrequin ([Pet03]), and for $X = Z$ it is simply the definition of rigid cohomology. In particular under some assumptions that can be always satisfied, we can kill some dependency's condition. For the rest of the discussion, we then suppose to be in the following situation:

Situation 1. We can assume that X_0 can be embedded in proper scheme $\bar{X}_0$ by a theorem of Nagata ([Nag62]) and suppose that there exists a formal scheme $\mathcal{P}$ smooth on a neighborhood of X_0: in particular we assume that X is endowed with étale topology and quasi-projective, so that we can lift the Frobenius morphism f of X_0 on $\mathcal{P}$.

Remark 8. In this setting Berthelot proved the independence of the definition of $\mathcal{J}^0_{X_0,Y,\mathcal{P}}$, and $\mathcal{J}^r_{X_0,Y,\mathcal{P}}$ for each r, with respect to Y and $\mathcal{P}$. This is claimed in ([Gro94], Propositions 3.3 and 3.5). With this remark we can denote it simply by $\mathcal{J}^r_{X_0}$. However we want to advert the reader that what is claimed in Gros is not supported by a verifiable reference. For us, it has been impossible to find it as well as the concerned proof.

As done in [Bes00] we can take together all such choices.

Definition 6.3. *If $(X, \bar{X}, j, \mathcal{P})$ with $j : X \hookrightarrow \bar{X}$ are W-scheme with the hypothesis as in the situation (1), we call a such collection rigid datum. A morphisms between the rigid data $(X, \bar{X}, j, \mathcal{P})$ and $(X, \bar{X}'.j', \mathcal{P}')$ is the data of a proper morphism $\alpha : \bar{X} \to \bar{X}'$ and a smooth morphism $u : \mathcal{P} \to \mathcal{P}'$ compatible with the obvious morphisms.*

These objects form a category $\mathcal{RD}(X, W)$ which is filtered as proved in [Bes00]. Moreover we can refine this definition, requiring a compatibility with strict neighborhoods. If to a rigid datum we choose a strict neighborhood V of $]X_0[_{\mathcal{P}}$ in $]\bar{X}_0[_{\mathcal{P}}$, we call this datum, *extended rigid datum.* Then a morphism of extended rigid data, with V and V' their corresponding strict neighborhoods, is a morphism of rigid data such that the induced map on the tubes sends V in V'. Again this is a filtered category that we note $\mathcal{ED}(X, W)$ so we have direct limits taking all the rigid data or extended rigid data. We remark that all this constructions induce quasi isomorphism on each particular chosen datum. Under these assumptions we can form the following so called rigid syntomic sheaves. First define

$$\mathcal{J}'^r_{X_0} := \varinjlim_{(\bar{X}, j, \mathcal{P}, V) \in \mathcal{ED}(X,W)} \mathcal{J}^r_{X_0, \bar{X}, \mathcal{P}, V}.$$

$$\mathcal{J}'^r_{X_0,|Z|} := \varinjlim_{(\bar{X}, j, \mathcal{P}, V, |Z|) \in \mathcal{ED}(X,W)} \mathcal{J}^r_{X_0, \bar{X}, \mathcal{P}, V, |Z|}$$

Here V means that the sheaves $\mathcal{J}^r_{X_0, \bar{X}, \mathcal{P}, V}$, $\mathcal{J}^r_{X_0, \bar{X}, \mathcal{P}, V, |Z|}$ are defined more generally with $\Omega^\bullet_V$. Since the Frobenius on X_0 admits a lifting on the formal scheme $\mathcal{P}$ that we denote still by f, we define the following sheaves for each integer r:

$$s(r)_{X/K,f,rig} := \mathrm{Cone}(1 - p^{-r} f : \mathcal{J}'^r_{X_0} \to \mathcal{J}'^0_{X_0})[-1]$$

$$s(r)_{X/K,|Z|,f,rig} := \mathrm{Cone}(1 - p^{-r} f : \mathcal{J}'^r_{X_0,|Z|} \to \mathcal{J}'^0_{X_0,|Z|})[-1].$$

It is important to remark that these sheaves depend by f but by the theorem 1.5 in ([Gro94]), they are independent by homotopy class. Since we are interested in the study of their cohomology, we can just forget the map f by the notation, and we consider them as in the homotopy category. We are now ready to define the Gros cohomology theory.

Definition 6.4. *Let's $X, X_0, \bar{X}_0, \mathcal{P}$ as above. We define $H^a_{Gr}(X, r)$ as the hypercohomology groups*

$$H^a_{Gr}(X, r) := \mathbb{H}^a(\bar{X}, s(r)_{X/K,rig})$$

and in the relative case, we define

$$H^a_{Gr,|Z|}(X,r) := \mathbb{H}^a(\bar{X}, s(r)_{X/K,|Z|,rig}).$$

Remark 9. Note that the filtration

$$\mathcal{J}'^{r}_{X_0} \subset \mathcal{J}'^{r-1}_{X_0} \subset \cdots \subset \mathcal{J}'^{0}_{X_0} = \varinjlim_{\mathcal{ED}(X,W)} \mathbb{R}\mathrm{sp}_* j^\dagger_{X_0}\Omega^\bullet_{]\bar{X}_0[}$$

induces K-vector spaces of the rigid cohomology that we denote with $\mathrm{Fil}^r H^*_{rig}(X_0/K) := \mathbb{H}^*(\bar{X}, \mathcal{J}'^{r}_{X_0})$. In the definition of rigid-Besser cohomology in [Bes00], it is defined a filtration on the De Rham cohomology, that play the role of the filtration defined above. Actually Besser provides a map between them in a more general situation. The relevant fact is that this map is functorial with respect to X. This allows us to define a cycle class on the Gros filtration as image of the cycle class of the de Rham filtration.

Remark 10. We have to underline that in the definition of these K-vector spaces we don't know if they actually are sub-objects. In this discussion we assume (or better, we have conjectured) that: *under suitable conditions on X, we have that*

$$\mathrm{Fil}^r H^*_{rig}(X_0/K) \hookrightarrow H^*_{rig}(X_0/K)$$

is an injection.

We stress that in the smooth and proper case, the identification of rigid and de Rham cohomology follows by the GAGA theorem (where the terms analytic is intended rigid-analytic) and so we apply this principle on the filtration both on rigid that and on the de Rham cohomology, to obtain an isomorphism. However when we don't have properness, than we don't know the behaviour of this filtration. For example, in the case $X_0 = \mathrm{Spec}(k[t])$, the injection cannot be hold, since the filtration on the left makes the cohomology an infinite dimensional K-space, meanwhile on the right we have one that is finite dimensional. In fact, we can choose a frame $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k \hookrightarrow \hat{\mathbb{P}}_W$. Also $\tilde{Y} = \mathbb{P}^1_W$ and so $\hat{\tilde{Y}}_K = \hat{\mathbb{P}}^1_K$. So we have the stupid filtration relative to the rigid complex

$$0 \to W^\dagger[t] \otimes K \to \mathcal{O}_{W^\dagger[t]\otimes K} \to \Omega^1_{W^\dagger[t]\otimes K} \to 0. \tag{6.1}$$

Then we see that $\mathrm{Fil}^1 H^1_{rig}(X/K)$ is infinite dimensional, while $H^1_{rig}(X/K)$ is finite dimensional.

Lemma 6.5. *For X smooth (and suitable conditions as described above), and W a flat integral subscheme of codimension r, there exists a functorial map*

$$Fil^r H^{2r}_{dR,W_K}(X_K/K) \xrightarrow{c_X} Fil^r H^{2r}_{rig,W_0}(X_0/K)$$

such that,

$$Fil^r H^{2r}_{dR,W_K}(X_K/K) \xrightarrow{c_X} Fil^r H^{2r}_{rig,W_0}(X_0/K) \to H^{2r}_{rig,W_0}(X_0/K)$$

is compatible with the cycle classes (betweeen the de Rham and rigid one). That means: if η_{dR,W_K} denote a cycle class of a closed subscheme W_K of codimension r of X_K, then $c_X(\eta_{dR,W_K})$ is compatible with $\eta_{rig,W_0} \in H^{2r}_{rig,W_0}(X_0/K)$ (defined in chapter 5*).*

Proof. The existence of the map is proved in [Bes00] proposition 9.5. But after [[CCM13], section 1.5] we have a *specialization map*

$$H^{2r}_{dR,W_K}(X_K/K) \xrightarrow{sp} H^{2r}_{rig,W_0}(X_0/K)$$

$$\eta_{dR,W_K} \longmapsto \eta_{rig,W_0}$$

The Hodge theory for the de Rham cohomology provides an equality $\mathrm{Fil}^r H^{2r}_{dR,W_K}(X_K/K) = H^{2r}_{dR,W_K}(X_K/K)$. Then actually the de Rham fundamental class lies in the r-th part of the filtration and so, c_X is the restriction of the specialization map to this filtration. This means that we have the following commutative diagram (where we stress the injectivity hypothesis):

$$\begin{array}{ccc} H^{2r}_{dR,W_K}(X_K/K) & \xrightarrow{sp} & H^{2r}_{rig,W_0}(X_0/K) \\ \uparrow & & \uparrow \text{remark 10} \\ \mathrm{Fil}^r H^{2r}_{dR,W_K}(X_K/K) & \xrightarrow{c_X} & \mathrm{Fil}^r H^{2r}_{rig,W_0}(X_0/K) \end{array}$$

□

This lemma guarantees the existence of a cycle class in the rigid filtration of Gros.

Definition 6.6. *Let W be a flat closed subscheme of X of pure codimension r. Then we define a fundamental class in the rigid filtration as*

$$\tilde{\eta}_{rig,W_0} := c_X(\eta_{dR,W_K}). \tag{6.2}$$

Proposition 6.7. *The Rigid-Gros cohomology satisfies the following long exact sequences:*

1. $\cdots \to H^{a-1}_{rig}(X_0/K) \oplus H^{a-1}_{rig}(X_0/K) \to H^a_{Gr}(X,r) \to H^a_{rig}(X_0/K) \oplus Fil^r H^a_{rig}(X_0/K) \to H^a_{rig}(X_0/K) \oplus H^a_{rig}(X_0/K) \to \dots$
2. $\cdots \to H^{a-1}_{|Z|,rig}(X_0/K) \oplus H^{a-1}_{|Z|,rig}(X_0/K) \to H^a_{Gr,|Z|}(X,r) \to H^a_{|Z|,rig}(X_0/K) \oplus Fil^r H^a_{|Z|,rig}(X_0/K) \to H^a_{|Z|,rig}(X_0/K) \oplus H^a_{|Z|,rig}(X_0/K) \to \dots$
3. $\cdots \to H^a_{Gr,|Z|}(X,r) \to H^a_{Gr}(X,r) \to H^a_{Gr}(X-Z,r) \to \dots.$

Moreover the last exact sequence is contravariant for cartesian squares.

Proof. Let's observe that we have a quasi isomorphism

$$\mathrm{Cone}(1-p^{-r}f : \mathcal{J}'^r_{X_0} \to \mathcal{J}'^0_{X_0})[-1] \sim \mathrm{Cone}(\mathcal{J}'^r_{X_0} \oplus \mathcal{J}'^0_{X_0} \xrightarrow{h} \mathcal{J}'^0_{X_0} \oplus \mathcal{J}'^0_{X_0})[-1],$$

where $h = \begin{pmatrix} 1-p^{-r}f & 0 \\ 0 & 1 \end{pmatrix}$. The same holds in the relative case. Then 1) and 2) follows by the long exact sequence associates to a mapping cone. For the property 3), recall that in general holds that

$$\mathrm{Cone}(\mathrm{Cone}(A^\bullet \to B^\bullet)[-1] \to \mathrm{Cone}(C^\bullet \to D^\bullet)[-1])[-1]$$
$$\simeq \mathrm{Cone}(\mathrm{Cone}(A^\bullet \to C^\bullet)[-1] \to \mathrm{Cone}(B^\bullet \to D^\bullet)[-1])[-1],$$

and in particular it applies to the sheaves $s(r)_{X/K,|Z|,f,rig}$. Then 3) follows again by the long exact sequence of the mapping cone. The last sentence follows by the fact that $s(r)_{X/K,|Z|,f,rig}$ is contravariant for cartesian squares. □

Conjecture 6.1. *The rigid Gros cohomology satisfies the homotopy property, i.e. there exist a canonical isomorphism such that*

$$H^a_{Gr}(X,r) \simeq H^a_{Gr}(X \times \mathbb{A}^1_{W(k)}, r)$$

A possible proof has been attempted as following.
We consider the long exact sequence on Gros cohomology

$$H^{a-1}_{rig}(X_0/K) \oplus H^{a-1}_{rig}(X_0/K) \to H^a_{Gr}(X,r) \to H^a_{rig}(X_0/K) \oplus \mathrm{Fil}^r H^a_{rig}(X_0/K)$$
$$\to H^a_{rig}(X_0/K) \oplus H^a_{rig}(X_0/K)$$

and the similar one corresponding to the scheme $X \times \mathbb{A}^1_{W(k)}$. The projection map $X \times \mathbb{A}^1_{W(k)} \to X$ induces an isomorphism on the rigid cohomology groups appearing in the exact sequences since the rigid cohomology satisfies the homotopy property. To prove that the homotopy property holds for the filtration, we consider that in the affine case, with $X = \mathrm{Spec}(A)$ we have the short exact sequence

$$0 \to \mathrm{Fil}^r \Omega^\bullet_{A^\dagger} \to \Omega^\bullet_{A^\dagger} \to \Omega^{<r,\bullet}_{A^\dagger} \to 0$$

that induces a long exact sequence in cohomology. By the five lemma it suffices to prove the isomorphism $H^j(\Omega^{<r,\bullet}_{A^\dagger}) \simeq H^j(\Omega^{<r,\bullet}_{(A \times W[t])^\dagger})$. If we consider the Kunneth formula in ([Ber97a], section 3.1, ref. (3.1.1)), we observe that there is an isomorphism at level of complexes, so it is an isomorphism also only considering the first $r-1$ terms. Since the computation of cohomology of $\Omega^\bullet_{W^\dagger[t]}$ is trivial ($H^0 = K$, otherwise is 0), by Kunneth formula, the only term of the direct sum is the cohomology relative to X. This reasoning proves the searched isomorphism for $j \leq r-2$ and for $j \geq r$. What about $j = r-1$? We note that this corresponds to the case where $\mathrm{Fil}^1 H^1(\mathrm{Spec}(k[t])/K)$ is infinite dimensional discussed in remark (10). When this is verified, then by four lemma, also the map between the rigid Gros cohomology groups is an isomorphism.

Proposition 6.8. *Let Z be a closed W-subscheme of X of pure codimension q. Then the rigid Gros cohomology satisfies the weak purity condition, i.e.*

$$H^a_{Gr,|Z|}(X, r) = 0 \quad a < 2q.$$

Proof. By the long exact sequence

$$H^{a-1}_{rig,|Z|}(X_0/K) \oplus H^{a-1}_{rig,|Z|}(X_0/K) \to H^a_{Gr,|Z|}(X, r) \to H^a_{rig,|Z|}(X_0/K) \oplus \mathrm{Fil}^r H^a_{rig,|Z|}(X_0/K)$$

and by weak purity of rigid cohomology (the weak purity holds on the filtration, by the injectivity conjectured hypothesis), it follows that the first and last term of the sequence are 0. The proposition follows. □

Now assume that X_0 is smooth. As described in [Pet03], if Z_0 is a closed k-subscheme of X_0 of codimension q we can define a fundamental class $\eta_{Z_0,rig} \in H^{2q}_{|Z_0|,rig}(X_0/K)$. Starting from such class we can define the analogous one for the rigid Gros cohomology.

Proposition 6.9. *Let Z a closed W-subscheme of codimension r. There exist a unique syntomic fundamental class in the rigid Gros cohomology, $\eta_{Z,Gr} \in H^{2r}_{|Z|,Gr}(X,r)$ compatible with the rigid fundamental class $\eta_{Z_0,rig}$.*

Proof. By weak purity we have the following exact sequence:

$$0 \to H^{2r}_{|Z|,Gr}(X,r) \to H^{2r}_{rig,|Z_0|}(X_0/K) \oplus \mathrm{Fil}^r H^{2r}_{rig,|Z_0|}(X_0/K) \xrightarrow{h} H^{2r}_{|Z_0|,rig}(X_0/K) \oplus H^{2r}_{|Z_0|,rig}(X_0/K).$$

In particular by exactness there is a unique class in the Gros cohomology corresponding to an element of the kernel of h. We have defined $\tilde{\eta}_{Z_0,rig} = c_X(\eta_{Z_K,dR}) \in \mathrm{Fil}^r H^{2r}_{rig,|Z_0|}(X_0/K)$. Moreover the Frobenius action on $\eta_{Z_0,rig}$ is the multiplication by p^r and it is compatible with the r-th level of the de Rham filtration, so that the couple $(\eta_{Z_0,rig}, \tilde{\eta}_{Z_0,rig}) \in \ker(h)$. This proves that there exists a unique corresponding $\eta_{Z,Gr} \in H^{2r}_{|Z|,Gr}(X,r)$. □

Now by linearity we can extend the definition of Gros fundamental class, to Gros cycle class.

We can verify the compatibility on the pullback of cycle.

Lemma 6.10. *Let $f : X' \to X$ be a closed immersion of smooth W-schemes, and let $Z \in \mathcal{Z}^r(X)$ a relative cycle of codimension r. We Assume that the $f^{-1}(Z)$ lies in $\mathcal{Z}^r(X')$. Then holds that*

$$f^*\eta_{Gr,Z} = \eta_{gr,f^*Z}$$

Proof. Since $\eta_{Gr,Z}$ is defined as the unique corresponding element of $(\eta_{Z_0,rig}, c_X(\eta_{Z_K,dR}))$, and c_X is a functorial map, the assertion follows by the compatibility of the de Rham cycle class with respect to pullback, as proved in ([CCM13], lemma 1.6.3).

□

The previous assertions prove that the Gros cohomology is a Bloch (in our sense) cohomology theory. As seen in ([Blo86b]), on such kind of cohomology it is possible to construct an higher class cycle map. We will just apply the Bloch's argument to the Gros cohomology here defined.

Proposition 6.11. *Let X be a smooth W-scheme. Then the Gros cycle class induces an higher cycle class map (regulator)*

$$cl_{syn}^{r,m} : CH^r(X/W,m) \to H_{Gr}^{2r-m}(X,r) \tag{6.3}$$

compatible with the regulator map for the rigid-Besser cohomology defined in ([CCM13],Proposition 1.6.6).

Proof. Let's choose a resolution of the complex that computes the Gros cohomology. In particular we denote

$$\mathbb{R}_{Gr}(X \times \Delta^{-m}, r) := \mathbb{R}\Gamma(X \times \Delta^{-m}, s(r)_{X\times\Delta^{-m}/K,rig})$$
$$\mathbb{R}_{Gr,|Z|}(X \times \Delta^{-m}, r) := \mathbb{R}\Gamma(X \times \Delta^{-m}, s(r)_{X\times\Delta^{-m}/K,|Z|,rig})$$
$$\mathbb{R}_{Gr,c}(X \times \Delta^{-m}, r) := \varinjlim_{Z\in\mathcal{Z}^r(X,-m)} \mathbb{R}_{Gr,|Z|}(X \times \Delta^{-m}, r)$$

where $m \leq 0$ and $Z \in \mathcal{Z}^r(X,-m)$. Since it can be some problem of convergence about unbounded spectral sequences, we choose an $N >> 0$ even and suppose $m \geq -N$, then we pose

$$K^{m,t} := \tau_{m\geq -N}\mathbb{R}_{Gr}(X \times \Delta^{-m}, r)^t$$
$$K_c^{m,t} := \tau_{m\geq -N}\mathbb{R}_{Gr,c}(X \times \Delta^{-m}, r)^t.$$

with $t \geq 0$. In this way we can consider the "filtration bete" spectral sequences associate to $K^{\bullet,\bullet}$ and $K_c^{\bullet,\bullet}$, namely:

$$E_1^{m,t} := H^t(K^{\bullet,m}) \implies H^{m+t}(\mathrm{tot}(K^{\bullet,\bullet}))$$
$$E_{1,c}^{m,t} := H^t(K_c^{\bullet,m}) \implies H^{m+t}(\mathrm{tot}(K_c^{\bullet,\bullet}))$$

By a closer inspection on the first spectral sequence we can notice that:

$$E_1^{m,t} = H_{Gr}^t(X \times \Delta^{-m}, r) = H_{Gr}^t(X,r)$$

by homotopy property. Moreover, if $D_2^{\bullet,m} : K^{\bullet,m} \to K^{\bullet,m+1}$ denotes the differential, (that is induced by the face inclusion maps $X \times \Delta^m \overset{\hookrightarrow}{\underset{\hookrightarrow}{\vdots}} X \times \Delta^{m+1}$) then we have that $d_1^{m,t} = (-1)^t H^t(D_2^{\bullet,m})$ and so we deduce

$$d_1^{m,t} = \begin{cases} \text{id} & \text{if } -N \leq m < 0 \text{ , } m \text{ even and } t \text{ even} \\ \text{-id} & \text{if } -N \leq m < 0, m \text{ even, } t \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

This means that

$$E_2^{m,t} = \begin{cases} H^t_{Gr}(X,r) & m=0 \\ 0 & \text{otherwise} \end{cases}$$

and so $H^i(\mathrm{tot}(K^{\bullet,\bullet})) \simeq H^i_{Gr}(X,r)$. Now, the Gros cycle class map induces the following map of complexes

$$\tau_{m\geq -N}\mathcal{Z}^r(X,\bullet) \to E_{1,c}^{-\bullet,2r}. \tag{6.4}$$

From the weak purity property we deduce that $E_{q,c}^{m,t} = 0$ for $t < 2r$ and for $q > 1$. This means that we have a map

$$\begin{aligned} E_{2,c}^{-m,2r} \to E_{\infty,c}^{-m,2r} \to H^{2r-m}(\mathrm{tot}(K_c^{\bullet,\bullet})) \\ \to H^{2r-m}(\mathrm{tot}(K^{\bullet,\bullet})) \simeq H^{2r-m}_{Gr}(X,r). \end{aligned}$$

Then composing with the cycle map 6.4 after taking the cohomology, we obtain the desired map. The compatibility with Besser syntomic regulator follows by the compatibility of cycle maps (6.5) and lemma (6.2).

□

6.3 A map from Gros cohomology to (étale) syntomic cohomology

The previous sections show that the Gros cohomology has good properties, like to the classical étale cohomology. Here we will see that a connection between the Gros cohomology and étale-syntomic cohomology arise naturally. Unfortunately it is important to remark that the Fontaine-Messing étale sheaves $s_n(r)_X$ have not the same kind of behaviour. We don't know in general if the homotopy property holds. Actually, a modified version of these sheaves appears in [Sat10], where it is claimed a lack of the homotopy invariant, so the construction of an higher cycle map needs some care and a little different approach. We will see it in the next section.

In this section we are going to use the differential description of the syntomic sheaves of Fontaine and Messing, as described in ([Kat85]) or in section 4.2.2 .

Let X be a smooth, quasi-projective W-scheme. Let $X \hookrightarrow Z$ a closed immersion in a smooth (in a neighborhood of X) W-scheme Z, such that the Frobenius morphism of X lifts to Z. Let $D_n := D_{X_n}(Z_n)$ the divided power envelope with respect to the divided power ideal $pW_n \subset W_n$, and let $J_n := J_{D_n}$ the ideal of $\mathcal{O}_{D_n}$ defining X_n. Let $J_n^{[r]}$ denote the r-th divided power of J_n, where for $r \leq 0$ we denote $J_n^{[r]} := \mathcal{O}_{D_n}$. We can define a complex of sheaves over $(X_0)_{\text{ét}}$:

$$\mathbf{J}^{[r]}_{n,X,Z} : J_n^{[r]} \to J_n^{[r-1]} \otimes_{\mathcal{O}_{Z_n}} \Omega^1_{Z_n} \to J_n^{[r-2]} \otimes_{\mathcal{O}_{Z_n}} \Omega^2_{Z_n} \to \dots .$$

Moreover there exists a unique f_r that represents the "division by p^r" of the Frobenius f. Then we can define the fibered cone

$$s_n(r)_{X,Z} := \text{Cone}(\mathbf{J}^{[r]}_{n,X,Z} \xrightarrow{1-f_r} \mathbf{J}^{[0]}_{n,X,Z})[-1].$$

Let's denote the image of these sheaves (étale) on the derived category with $s_n(r)_X$. All the sentences are verified in ([Kat85], and in loc. section on Crystalline cohomology). Now let's assume $X = \text{spec}(A)$ is an affine smooth scheme, where A is a finite type W-algebra. If $A^\dagger$ denotes the weak completion respect to (p), Berthelot ([Ber97b]) proved that there is an isomorphism

$$\mathbb{R}^i\Gamma(\Omega^\bullet_{]\bar{X}_0[_{\mathcal{P}}}) \simeq \mathbb{R}^i\Gamma(\Omega^\bullet_{A^\dagger/W} \otimes K).$$

Moreover there is a natural inclusion (by definition) of $A^\dagger \hookrightarrow \hat{A} := \varprojlim_n A/p^n$. By covariant functoriality of the differential Ω^i_* for each i, the inclusion induces a morphism

$$\Omega^\bullet_{A^\dagger/W} \otimes K \to (\varprojlim_n \Omega^\bullet_{A_n/W_n}) \otimes K. \tag{6.5}$$

Moreover notice that there is an inclusion $\text{spec}(A_n) \hookrightarrow Z_n$ and the divided power envelope of $\text{spec}(A_n)$ in Z_n with respect to pW_n yields a map

$$\Omega^\bullet_{A_n/W_n} \to \Omega^\bullet_{D_{\text{spec}(A_n)}(Z_n)}.$$

Keeping the projective limits on each terms, then extending to the scalars K and composing the obtained map with (6.5), we obtain a map

$$\Omega^\bullet_{A^\dagger/W} \otimes K \to (\varprojlim_n \Omega^\bullet_{D_{\text{spec}(A_n)}(Z_n)}) \otimes K.$$

So in the case of affine smooth schemes we have a map

$$\mathbb{R}\Gamma(X, s(0)_{X/K,rig}) \to \mathbb{R}\Gamma(X, s_\infty(0)_X) \otimes K.$$

where in this context $\mathbb{R}\Gamma(X, s(r)_{X/K,rig})$ is the complex such that $\mathbb{R}^i\Gamma(X, s(r)_{X/K,rig}) = H^i_{Gr}(X, r)$ With an analogous reasoning, we deduce the existence of a map [1]

$$\mathbb{R}\Gamma(X, s(r)_{X/K,rig}) \to \mathbb{R}\Gamma(X, s_\infty(r)_X) \otimes K \tag{6.6}$$

for each integer r. We underline that in the map 6.6, X is endowed with Zariski topology on the LHS, while X is endowed with étale topology in RHS.
Now for a generic X smooth, we choose a covering of affine schemes. The universal property of divided power envelopes guarantees the compatibility of these maps on finite intersections of the covering. So by the definition of sheaf, there exist a unique map extending (6.6) for a generic X smooth.

6.4 Cycle map in étale cohomology

In the previous section, we observed that the Fontaine-Messing sheaves $s_n(r)_X$ loss the homotopy property. This means there is a gap in the construction of an higher cycle map in the sense of Bloch. However, the classical étale cohomology is a good Bloch cohomology theory. We ask then if it possible to define, in a functorial way, an higher syntomic cycle map, starting from the classical étale point of view. In ([Sat10]) is described a modified version of the Fontaine-Messing sheaves to make up for the lack of the homotopy property. His discussion is more general, but it has some difficulty that make less clear the connection with the maps built here. So, for simplicity we choose to deal with the classical étale cohomology and then, through a result of Kurihara ([Kur87]) and an argument of Geisser ([Gei04]), we make possible the desired connection. Under the assumption of smoothness of X, for the author this seems the shortest way to approach the argument.
In this section we study the cohomology of the generic fiber X_K of a smooth W-scheme X. Let $m = p^r$ and denote with $\mathbb{Z}/m(q)$ the p-adic sheaves

$$\mathbb{Z}/m(q) = \begin{cases} \mu_m^{\otimes q} & \text{if } q \geq 1 \\ \mathbb{Z}/m & \text{if } q = 0. \end{cases}$$

[1] We notice that in the affine case there is no information on the ideal I, i.e. $I = (0)$. This beacause we can choose the frame $spec(A_k) \hookrightarrow \mathbb{P}^n_k \hookrightarrow \hat{\mathbb{P}}^n_W$ and if $Y = \mathbb{P}^n_k$ and $\mathcal{P} = \hat{\mathbb{P}}^n_W$, then $\tilde{Y}_K = P_K$.

These sheaves satisfy the purity condition from classical étale theory: let Z_K be a closed subscheme of pure codimension q, then there is a canonical isomorphism

$$H^i_{Z_K}(X_{K,\text{ét}}, \mathbb{Z}/m(q)) \simeq H^{i-2q}((Z_K)_{\text{ét}}, \mathbb{Z}/m). \tag{6.7}$$

In particular $H^{2q}_{Z_K}(X_{\text{ét}}, \mathbb{Z}/m(q))$ is generated by the irreducible components of Z_K. A cycle map is then defined extending by linearity the fundamental classes of such Z_K, i.e. the preimage of $[Z_K] \in H^0(Z_K, \mathbb{Z}/m)$ by the isomorphism (6.7). Moreover the étale cohomology has the homotopy property. To apply the Bloch's argument, we need of a resolution that computes the étale cohomology and the functoriality of the class cycle map. For the resolution, choose a Godement resolution of $\mathbb{Z}/m(q)$ on X_K, namely $G^\bullet(X_K, \mathbb{Z}/m(q))$. We note that it is quasi isomorphic with $R\Gamma(X_K, \mathbb{Z}/m(q))$, so the cohomology of both complexes is the same. The functoriality of cycle map follows by the following lemma in étale cohomology:

Lemma 6.12. *Let $cl^q_{ét} : \mathcal{Z}^q(X_K, 0) \to H^{2q}(X_{K,ét}, \mathbb{Z}/m(q))$ the cycle map for the étale cohomology. If $f : Y_K \to X_K$ is a morphism of smooth K-scheme, $W_K \subset X_K$ a closed subscheme of codimension $\geq q$, $T_K \subset Y_K$ a closed subscheme such that $f^{-1}(W_K) \subset T_K$ and T_K has codimension $\geq q$, then*

$$f^*(cl^q_{ét}([W_K])) = cl^q_{ét}(f^*[W_K])$$

Proof. See ([GL01], proposition 3.5 (1)). □

Now, by the argument of Bloch, we can construct an higher cycle map (regulator) from higher Chow groups to étale cohomology. More precisely it is the map in cohomology induced by a map

$$\text{cl}^q_{\text{ét}} : \mathcal{Z}^q(X_K, \bullet) \otimes \mathbb{Z}/m \to G^{2q-\bullet}(X_K, \mathbb{Z}/m(q)). \tag{6.8}$$

To see it, we note that the argument of Bloch, at first step provide a map of complexes induced by cycle map as following:

$$\mathcal{Z}^q(-, 0) \otimes \mathbb{Z}/m \to H^{2q}(G^\bullet_c(-_{\text{ét}}, \mathbb{Z}/m(q))) \tag{6.9}$$

where $G^\bullet_c(-_{\text{ét}}, \mathbb{Z}/m(q))$ is obtained by $G^\bullet$ taking the inductive limit over the subgroups of $\mathcal{Z}^q(-, 0)$ of closed subschemes that meet properly. Moreover by the weak purity in étale cohomology we have a natural quasi isomorphism

$$\tau_{\leq 2q} G^\bullet_c(-, \mathbb{Z}/m(q)) \xrightarrow{\alpha} H^{2q}(G^\bullet_c(-_{\text{ét}}, \mathbb{Z}/m(q)))[-2q]. \tag{6.10}$$

We have also a map

$$\tau_{\leq 2q} G_c^\bullet(-, \mathbb{Z}/m(q)) \to G_c^\bullet(-, \mathbb{Z}/m(q)) \to G^\bullet(-, \mathbb{Z}/m(q)) \qquad (6.11)$$

obtained by composing with forgetful support functor. Composing all these maps, then applying to $X \times \Delta^*$, and by homotopy property, we get (6.8).

6.4.1 Looking for compatibility and questions

What we have done until now is provide a strategy to formulate an analogous of the theorem (3.4) ([Gro94]) in terms of higher cycle maps. More precisely we hope that the following holds:

Conjecture 6.2. *Let X a smooth W-scheme, with X_0 smooth special fiber and X_K the (smooth) generic fiber. Then there exists a functorial (in X) regulator map*

$$cl^{r,m}_{\acute{e}t\text{-}syn} : CH^r(X, m) \otimes \mathbb{Q}_p \to H^{2r-m}(X_{\acute{e}t}, s_\infty(r)_X) \otimes \mathbb{Q}_p \qquad (6.12)$$

making the following diagram commutative:

$$\begin{array}{ccc} CH^r(X,m) \otimes \mathbb{Q}_p & \xrightarrow{cl^{r,m}_{syn}} & H^{2r-m}_{Gr}(X,r) \\ & \searrow^{cl^{r,m}_{\acute{e}t\text{-}syn}} & \downarrow \\ & & H^{2r-m}(X_{\acute{e}t}, s_\infty(r)_X) \otimes \mathbb{Q}_p \end{array} \qquad (6.13)$$

where the vertical map is induced by (6.6).

As a consequence, we can express the relation between the (rigid) syntomic cohomology in the sense of Besser, and the étale cohomology, in a similar fashion of the Deligne-Beilinson cohomology.

Corollary 6.13. *Let X as in the theorem (6.2). Then there is a commutative diagram*

$$\begin{array}{ccc} CH^r(X,m) \otimes \mathbb{Q}_p & \longrightarrow & H^{2r-m}_{syn\text{-}Bess}(X,r) \\ \downarrow & & \downarrow \\ CH^r(X_K,m) \otimes \mathbb{Q}_p & \longrightarrow & H^{2r-m}_{\acute{e}t}(X_K, \mathbb{Q}_p(r)) \end{array} \qquad (6.14)$$

where the right vertical map is given by the following composition:

$$H^{2r-m}_{syn\text{-}Bess}(X,r) \to H^{2r-m}_{Gr}(X,r) \to H^{2r-m}(X_{\acute{e}t}, s_\infty(r)_X) \otimes \mathbb{Q}_p \to H^{2r-m}_{\acute{e}t}(X_K, \mathbb{Q}_p(r)). \qquad (6.15)$$

Here, the first map on the left is given by one defined in ([Bes00], proof lemma 9.5). The last map on the right is the one given in ([FM87],3, 5.1).

Proof. By the theorem 6.2 and by construction of the syntomic-Besser regulator map given in [CCM13], we have just said that $CH^r(X,m) \to H^{2r-m}_{\text{ét}}(X_K, \mathbb{Q}_p(r))$ factorizes through $H^{2r-m}_{syn-Bess}(X,r)$. In the case $m=0$ the assertion of the corollary is true: In fact by the weak purity property $H^{2r}_{\text{ét},Z_K}(X_K, \mathbb{Q}_p(r))$ is generated by the irreducible components of Z_K. The elements of $CH^r(X)$ are flat subschemes of pure codimension r, so in the generic fiber the irreducible components relative to W are irreducible relative to K, then also the respective classes in cohomology are the same. Since the syntomic-Besser cohomolgy and the étale cohomology are Bloch type, by the lemma 6.2 the commutativity follows for $m>0$.

□

The rest of the section is devoted to see how the "conjecture" 6.2 has been approached.

Let start with some notation. Let

$$i : X_0 \hookrightarrow X$$

the closed immersion of the special fiber, denote with

$$j : X_K \hookrightarrow X$$

the open immersion of the generic fiber. Denote with $\Gamma_{X,\text{ét}}(r)$ the complex of étale sheaves associate to the presheaves $U \to \mathcal{Z}^r(U, \bullet)$. Analogously we denote $\Gamma_{X,Zar}(r)$ the corresponding Zariski complex of sheaves. Denote by $\epsilon : X_{\text{ét}} \to X_{Zar}$, the map of sites. Now we sketch an argument of Geisser . By a result of Kurihara ([Kur87]) there is a map

$$s_n(r)_X \to \tau_{\leq r} i^* Rj_* \mathbb{Z}/p^n(r). \tag{6.16}$$

Moreover there is the following map of complexes:

$$\tau_{\leq r} i^* \epsilon^* \Gamma_{X,Zar}(r) \otimes \mathbb{Z}/p^n \xrightarrow{c} \tau_{\leq r} i^* \epsilon^* Rj_* \Gamma_{X_K}(r) \otimes \mathbb{Z}/p^n. \tag{6.17}$$

Then the adjoint map $\epsilon^* Rj_* \to Rj_* \epsilon^*$ composed with (6.8) yields a map

$$\tau_{\leq r} i^* \epsilon^* Rj_* \Gamma_{X_K}(r)[-2r] \otimes \mathbb{Z}/p^n \to \tau_{\leq r} i^* Rj_* \mathbb{Z}/p^n(r). \tag{6.18}$$

Then the main result of Geisser ([Gei04] proof of theorem 1.3) is to prove that there exists a map

$$\tau_{\leq r} i^* \epsilon^* \Gamma_{X,Zar}(r)[-2r] \otimes \mathbb{Z}/p^n \to s_n(r)_X \tag{6.19}$$

such that the following diagram

$$\begin{array}{ccc} \tau_{\leq r} i^* \epsilon^* \Gamma_{X,Zar}(r)[-2r] \otimes \mathbb{Z}/p^n & \longrightarrow & \tau_{\leq r} i^* \epsilon^* R j_* \Gamma_{X_K}(r)[-2r] \otimes \mathbb{Z}/p^n \\ \downarrow & & \downarrow \\ s_n(r)_X & \longrightarrow & \tau_{\leq r} i^* R j_* \mathbb{Z}/p^n(r) \end{array} \tag{6.20}$$

commutes. Now we consider the following extended diagram:

$$\begin{array}{ccccc} \tau_{\leq r} i^* \epsilon^* \Gamma_{X,Zar}(r)[-2r] \otimes \mathbb{Q}_p & \longrightarrow & \tau_{\leq r} i^* \epsilon^* \Gamma_{X,Zar}(r)[-2r] \otimes \mathbb{Q}_p & \longrightarrow & \tau_{\leq r} i^* \epsilon^* R j_* \Gamma_{X_K}(r) \otimes \mathbb{Q}_p \\ \downarrow & & \downarrow & & \downarrow \\ s(r)_{X/K,rig} & \longrightarrow & s_\infty(r)_X \otimes \mathbb{Q}_p & \longrightarrow & \tau_{\leq r} i^* R j_* \mathbb{Q}_p(r) \end{array} \tag{6.21}$$

The claim is that, at the level of cohomology, it is commutative. In fact we note that all the vertical maps represent the suitable higher cycle maps, since by vanishing of the cohomology of the higher Chow groups, the truncation at level r is not restrictive. Moreover the composition of the maps on the bottom is a non trivial functorial map between two Bloch cohomology theories, so we can ask if they are compatible with respect to the cycle maps. Is this true? Since the weak purity property forces a non trivial map to send fundamental class in fundamental class, this sentence is verified only for the relative cohomology. We know by ([Gro94]) that the map 6.6 is compatible with Chern classes, but we don't know if we can refine these classes to "local Chern classes" with target in the relative cohomology. On the other hand, without this assumption we can ask if the map on the relative cohomology induced by 6.6 is an isomorphism. In this case, by weak purity the compatibility follows at fortiori. If assume such compatibility for the cycle classes, then by the lemma 6.2, this means that all the bigger rectangle is commutative. Together with the commutative diagram in (6.20), this not implies yet that the square on the left is commutative on the cohomology. But for this purpose it suffices that the map

$$s_\infty(r)_X \otimes \mathbb{Q}_p \to \tau_{\leq r} i^* R j_* \mathbb{Q}_p(r) \tag{6.22}$$

is injective. This follows by a simple fact on commutative diagrams: we are in the situation in which the following diagram in a category $\mathcal{C}$

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k} \\ D & \xrightarrow{d} & E & \xrightarrow{e} & F \end{array}$$

is such that all the bigger rectangle and the right square commute, i.e. the following diagrams are commutative:

$$\begin{array}{ccc} A & \xrightarrow{b\circ a} & C \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ D & \xrightarrow{e\circ d} & F \end{array} \qquad\qquad \begin{array}{ccc} B & \xrightarrow{b} & C \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k} \\ E & \xrightarrow{e} & F \end{array}$$

If e is a monomorphism then, also the square on the left is commutative: it follows simply by the relations

$$e \circ d \circ h = k \circ b \circ a = e \circ l \circ a$$

and since e is a monomorphism, $d \circ h = l \circ a$. So the question is:
Under which hypothesis, the map (6.22) is injective? Is it possible for example in the case $r = 1$?
Does exist a refining of Chern classes as ”local Chern classes” compatible with the classical one?
Is it possible, anyway, to prove that 6.6 induces an isomorphism at level of relative cohomology?

REFERENCES

[Beĭ86] A. Beĭlinson. "Notes on absolute Hodge cohomology". In: *Applications of algebraic K-theory to algebraic geometry and number theory, Part I (Boulder, CO, 1983)*. Ed. by S. J. Bloch et al. Contemp. Math. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 35–68.

[Ber86] Pierre Berthelot. "Géométrie rigide et cohomologie des variétés algébriques de caractéristique p". mul. In: *Introductions aux cohomologies p-adiques*. Ed. by Daniel Barsky and Philippe Robba. Mémoires de la Société Mathématique de France 23. Société mathématique de France, 1986, pp. 7–32. URL: http://www.numdam.org/item/MSMF_1986_2_23__R3_0.

[Ber96] P. Berthelot. "Cohomologie rigide et cohomologie rigide à supports propres. Première partie". In: *Prépublication IRMAR 96-03, 89 pages* (1996).

[Ber97a] Pierre Berthelot. "Dualité de Poincaré et formule de Künneth en cohomologie rigide". In: *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics* 325.5 (1997), pp. 493–498. ISSN: 0764-4442. URL: http://www.sciencedirect.com/science/article/pii/S0764444297888957.

[Ber97b] Pierre Berthelot. "Finitude et pureté cohomologique en cohomologie rigide avec un appendice par Aise Johan de Jong". In: *Inventiones mathematicae* 128.2 (1997), pp. 329–377.

[Bes00] Amnon Besser. “Syntomic Regulators and p-adic Integration I: Rigid Syntomic Regulators”. In: *Israel Journal of Mathematics* 120 (2000), pp. 291–334.

[Blo86a] Spencer Bloch. “Algebraic cycles and higher K-theory”. In: *Advances in Mathematics* 61.3 (1986), pp. 267–304.

[Blo86b] Spencer Bloch. “Algebraic cycles and the Beilinson conjectures”. In: *Contemporary Mathematics* 58.1 (1986), pp. 65–79.

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on Crystalline Cohomology.(MN-21)*. Princeton University Press, 1978.

[CCM13] Bruno Chiarellotto, Alice Ciccioni, and Nicola Mazzari. “Cycle classes and the syntomic regulator”. In: *Algebra & Number Theory* 7.3 (Aug. 2013), pp. 533–566. URL: https://doi.org/10.2140%2Fant.2013.7.533.

[EV88] Hélène Esnault and Eckart Viehweg. “DELIGNE-BEILINSON COHOMOLOGY”. In: *Beilinson's Conjectures on Special Values of L-Functions*. Ed. by M. Rapoport, N. Schappacher, and P. Schneider. Academic Press, 1988, pp. 43–91. ISBN: 978-0-12-581120-0. URL: http://www.sciencedirect.com/science/article/pii/B9780125811200500094.

[FF84] W. Fulton and B. Fulton. *Intersection Theory:* Ergebnisse der Mathematik und ihrer Grenzgebiete : a series of modern surveys in mathematics. Folge 3. Springer-Verlag, 1984. ISBN: 9783540121763. URL: https://books.google.it/books?id=cmoPAQAAMAAJ.

[FM87] Jean-Marc Fontaine and William Messing. “p-adic periods and p-adic étale cohomology”. In: *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*. Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987, pp. 179–207. URL: https://doi.org/10.1090%2Fconm%2F067%2F902593.

[FP04] Jean Fresnel and Marius van der Put. *Rigid Analytic Geometry and Its Applications*. Birkhäuser Boston, 2004. URL: https://doi.org/10.1007%2F978-1-4612-0041-3.

[Gei04] Thomas Geisser. "Motivic cohomology over Dedekind rings". In: *Mathematische Zeitschrift* 248.4 (Aug. 2004), pp. 773–794. URL: https://doi.org/10.1007%2Fs00209-004-0680-x.

[GK90] Michel Gros and Masato Kurihara. "Régulateurs syntomiques et valeurs de fonctions L p-adiques I". In: *Inventiones Mathematicae* 99 (Dec. 1990), pp. 293–320.

[GL01] Thomas Geisser and M. Levine. "The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky". In: *Journal fur die Reine und Angewandte Mathematik* 530 (Jan. 2001), pp. 55–103.

[Gro94] Michel Gros. "Régulateurs syntomiques et valeurs de fonctions L p-adiques II". In: *Inventiones Mathematicae* 115 (Jan. 1994), pp. 61–79.

[Jan90] Uwe Jannsen. "Algebraic cycles, K-theory, and extension classes". In: *Mixed Motives and Algebraic K-Theory.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1990, pp. 57–188. ISBN: 978-3-540-46941-4. URL: https://doi.org/10.1007/BFb0085082.

[Kat85] Kazuya Kato. "On p-adic Vanishing Cycles (Application of ideas of Fontaine-Messing)". In: *Algebraic Geometry, Sendai, 1985.* Mathematical Society of Japan, 1985, pp. 207–251. URL: https://doi.org/10.2969%2Faspm%2F01010207.

[Kur87] Masato Kurihara. "A note on p-adic etale cohomology". In: *Proc. Japan Acad. Ser. A Math. Sci.* 63.7 (1987), pp. 275–278. URL: https://doi.org/10.3792/pjaa.63.275.

[Nag62] Masayoshi Nagata. "Imbedding of an abstract variety in a complete variety". In: *Journal of Mathematics of Kyoto University* 2.1 (1962), pp. 1–10. URL: https://doi.org/10.1215%2Fkjm%2F1250524969.

[Pet03] Denis Petrequin. "Classes de Chern et classes de cycles en cohomologie rigide". In: *Bulletin de la Société mathématique de France* 131 (Jan. 2003), pp. 59–121.

[PS08] Chris A.M. Peters and Joseph H.M. Steenbrink. "Mixed Hodge Structures". In: *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Vol. 52. 3. Springer Berlin Heidelberg, 2008. URL: https://doi.org/10.1007%2F978-3-540-77017-6.

[Sat10] Kanetomo Sato. "Cycle Classes for p-Adic Etale Tate twists and The Image of p-Adic Regulators". In: *Documenta Mathematica* 18 (Apr. 2010), pp. 177–247.

[Sch93] A. J. Scholl. "Extensions of motives, higher Chow groups and special values of L-functions". In: *Séminaire de Théorie des Nombres, Paris 1991–92*. Ed. by Sinnou David. Boston, MA: Birkhäuser Boston, 1993, pp. 279–292.