Università degli studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Laurea Magistrale in Matematica

Viscosity solutions of the evolutive Hamilton-Jacobi equation by limiting variational methods. With a look to the stationary case.

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20 Aprile 2023

Anno Accademico 2022-2023

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Introduzione

Questa Tesi si concentra sullo studio del problema di Cauchy per l'equazione di Hamilton-Jacobi evolutiva:

$$\begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u(t,x)) = 0 \\ u(0,x) = f(x) \end{cases} \tag{H-J}$$

dove $(t,x) \in [0,T] \times \mathbb{R}^d$ ed H è una Hamiltoniana globalmente a supporto compatto. Questa equazione differenziale alle derivate parziali si trova spesso in Matematica ed in Fisica, ad esempio nei sistemi dinamici, nella teoria dei controlli, nei giochi differenziali, nella meccanica del continuo, nell'ottica geometrica ed in economia.

Per $t > 0$ sufficientemente piccolo, è possibile determinare una soluzione classica di (H-J) utilizzando il metodo delle caratteristiche. Tuttavia, anche per Hamiltoniane e dati iniziali regolari, esiste un tempo "critico" per cui le componenti x di alcune caratteristiche si incrociano. Perciò, in generale, il metodo delle caratteristiche fallisce. Di conseguenza, è naturale chiedersi come definire (e determinare) soluzioni globali deboli di (H-J), ovvero funzioni lipschitziane che soddisfano l'equazione quasi ovunque.

Negli anni Ottanta, M.G. Crandall, L.C. Evans and P.L. Lions introdussero, in un contesto puramente analitico, una prima nozione di soluzione debole, chiamata *soluzione di viscosità*, si vedano [15] e [13]. In seguito, per Hamiltoniane convesse e Liouville-integrabili della forma $H = H(p)$, M. Bardi e L.C. Evans ([2]) fornirono una inf-sup formula per le soluzioni di viscosità.

Alcuni anni dopo, nel 1991, M. Chaperon, J.C. Sikorav e C. Viterbo proposero –in un contesto geometrico– la definizione di *soluzione variazionale (min-max)*. Ci riferiamo, ad esempio, a [11], [31] e [24]. Tale costruzione si basa sulle funzioni generatrici quadratiche all'infinito (GFQI) dei fronti d'onda lagrangiani, dati dall'immagine tramite il flusso hamiltoniano del grafico del dato iniziale df.

In generale, le soluzioni di viscosità e variazionali non coincidono, si vedano, ad esempio, [24][Chapter 7], [6][Section 4.1]. In particolare, per ogni Hamiltoniana integrabile non convessa (non concava), si può costruire un dato iniziale liscio per cui il grafico della soluzione di viscosità non è contenuto nel fronte d'onda associato al problema di Cauchy (ci riferiamo a [25]). Tuttavia, soluzioni min-max e di viscosità coincidono nel caso di Hamiltoniane convesse rispetto alle p (si vedano [22] e [5]). Inoltre, queste soluzioni deboli sono lipschitziane anche per dati iniziali lisci. Di conseguenza, poiché le soluzioni variazionali si basano sul grafico di un dato iniziale regolare (almeno C^2), per poter iterare la procedura min-max occorre definire l'operatore min-max anche per funzioni lipschitziane. Una parte consistente della presente Tesi è dedicata all'introduzione dettagliata di due diverse nozioni di operatore variazionale per dati iniziali non differenziabili.

Il primo approccio, introdotto nel 2014 da Q. Wei (si vedano [33] e [34]), si basa sulle derivate generalizzate di Clarke e produce un operatore

$$R^{s,t} : C^{Lip} \to C^{Lip}, \qquad 0 \leq s < t \leq T,$$

che associa ad un dato iniziale lipschitziano f un selettore min-max generalizzato $R^{s,t}f$ del fronte d'onda lagrangiano che parte da

$$\Gamma(\partial f) := \{(x, y) \,|\, x \in \mathbb{R}^d,\, y \in \partial f(x)\}$$

(qui $\partial f(x)$ denota la derivata generalizzata di Clarke di f in x).

La seconda nozione, dovuta a O. Bernardi e F. Cardin (si veda [6] del 2011), sfrutta la densità delle funzioni lisce nello spazio delle funzioni C^0 rispetto alla norma uniforme e produce un operatore

$$V^{s,t} : C^0 \to C^0, \qquad 0 \leq s < t \leq T.$$

L'idea è semplice: si considerino f, un dato iniziale continuo al tempo s, ed una successione C^2 arbitraria

$$f_n \xrightarrow{\|\cdot\|_\infty} f,$$

allora $V^{s,t}f$ è definito come limite uniforme delle soluzioni min-max "classiche" al tempo t, con dato iniziale f_n al tempo s.

Vale la pena osservare che entrambi gli operatori min-max generalizzati non sono soluzioni del problema di Cauchy (H-J) (si veda anche [34][p.30]). Tuttavia, producono una soluzione di viscosità di (H-J) tramite una procedura "limiting" min-max, come spiegato in seguito.

Sia $\xi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$, con $t \in (0, T]$, una suddivisione dell'intervallo $[0, t]$. Associamo a ξ gli operatori min-max iterati

$$R_\xi^{0,t} =: R^{t_{n-1},t} \circ R^{t_{n-2},t_{n-1}} \circ \ldots \circ R^{0,t_1}$$

e

$$V_\xi^{0,t} =: V^{t_{n-1},t} \circ V^{t_{n-2},t_{n-1}} \circ \ldots \circ V^{0,t_1}.$$

Sia $\{\xi_n\}_{n\in\mathbb{N}}$ una successione di suddivisioni di $[0,t]$ per la quale il massimo delle ampiezze dei passi tende a 0, cioè

$$|\xi_n| := \max |t_{i+1} - t_i| \to 0.$$

Per un dato iniziale lipschitziano f, definiamo gli operatori "limiting" min-max $\bar{R}^{0,t} f$ e $\bar{V}^{0,t} f$ come i limiti uniformi (su sottoinsiemi compatti di $\mathbb{R}^d$) rispettivamente di $R_{\xi_n}^{0,t}$ e $V_{\xi_n}^{0,t}$. Tali limiti non dipendono dalla scelta della suddivisione ξ_n. In questo contesto, dimostriamo il seguente

Teorema. *Siano $H \in C_c^2([0,T] \times T^*\mathbb{R}^d)$ una Hamiltoniana globalmente a supporto compatto ed $f \in C^{Lip}(\mathbb{R}^d)$. Gli operatori "limiting" min-max $\bar{R}^{0,t} f$ e $\bar{V}^{0,t} f$ coincidono con l'unica soluzione di viscosità dell'equazione di Hamilton-Jacobi* (H-J) *con dato iniziale f.*

L'idea originale di costruire soluzioni dividendo un dato intervallo di tempo in piccole porzioni, iterando la procedura min-max passo a passo e prendendo il limite uniforme per l'ampiezza delle porzioni che tende a 0 è dovuta a M. Chaperon. Nel teorema precedente, la parte riguardante l'uguaglianza tra i due operatori è un risultato originale della presente Tesi.

Nella parte conclusiva della Tesi ci concentriamo su (H-J) per Hamiltoniane autonome e globalmente a supporto compatto su $T^*\mathbb{T}^d$. In tale contesto, studiamo dettagliatamente le proprietà delle (ora coincidenti, considerato il teorema precedente) procedure "limiting" min-max sullo spazio dei dati iniziali lipschitziani. In particolare, dimostriamo che l'operatore corrispondente soddisfa tutte le condizioni necessarie a riformulare un risultato á la Weak KAM, originariamente dovuto a A. Fathi (si veda [18][Teorema 4.4.6]). Nello spazio $Lip_L(\mathbb{T}^d)$ delle funzioni lipschitziane con costante di Lipschitz $\leq L$, otteniamo il seguente risultato originale.

Teorema. *Sia $H \in C_c^2(T^*\mathbb{T}^d)$. Esistono una costante $c \in \mathbb{R}$ ed una funzione $\bar{u} \in Lip_L(\mathbb{T}^d)$ tali che*

1. *$u(t,x) := \bar{u}(x) - ct$ è una soluzione su $[0,+\infty) \times \mathbb{T}^d$ dell'equazione di Hamilton-Jacobi evolutiva*

$$\partial_t u + H(x, \partial_x u) = 0$$

2. *$\bar{u}(x)$ è una soluzione globale su $\mathbb{T}^d$ dell'equazione di Hamilton-Jacobi stazionaria*

$$H(x, \partial_x u) = c$$

Concludiamo osservando che, dato il precedente teorema, risulta naturale studiare (i) possibili interpretazioni dinamiche e PDE del livello $c \in \mathbb{R}$ (come nel contesto "Weak KAM") (ii) generalizzazioni ad Hamiltoniane non a supporto compatto.

La Tesi è strutturata come segue.

Nel Capitolo 1, descriviamo le due nozioni di soluzioni deboli per l'equazione di Hamilton-Jacobi: soluzioni di viscosità e variazionali. Forniamo dettagli e diversi esempi.

Il Capitolo 2 è dedicato all'introduzione di due strumenti distinti che utilizzeremo in seguito. Nello specifico, dapprima esponiamo la costruzione delle funzioni generatrici per fronti d'onda lagrangiani tramite il metodo delle "geodetiche spezzate" ("géodésiques brisées") di M. Chaperon. Dopodiché, ne dimostriamo la quadraticità all'infinito. In fine, diamo la definizione tramite omologia del valore critico min-max, già introdotto nel Capitolo 1 tramite coomologia.

Lo scopo del Capitolo 3 è quello di definire un operatore min-max per dati inziali lipschitziani. Seguiamo la costruzione di Q. Wei, che si basa sulle derivate generalizzate di Clarke e su un'estensione al contesto lipschitziano della classica teoria dei punti critici.

Nel Capitolo 4 utilizziamo i precedenti risultati per introdurre le soluzioni min-max iterate per l'equazione di Hamilton-Jacobi evolutiva e per dimostrare alcune stime lipschitziane che risulteranno utili nel seguito. Inoltre, forniamo una dimostrazione dettagliata di un teorema dovuto a Q. Wei: la procedura "limiting" applicata agli operatori min-max iterati converge uniformemente all'unica soluzione di viscosità.

Nel Capitolo 5 introduciamo una definizione delle soluzioni min-max (iterate) adattata a problemi (H-J) con dato iniziale C^0. La costruzione, dovuta a O. Bernardi e F. Cardin, si basa sulla densità delle funzioni lisce nello spazio dell funzioni C^0 munito della norma uniforme. Tale costruzione è svolta per Hamiltoniane globalmente a supporto compatto definite sul fibrato cotangente di una varietà compatta. Inoltre, per dati iniziali lipschitziani, dimostriamo che applicando la procedura min-max "limiting" a questo operatore iterato –in genere distinto da quello definito nel Capitolo 4– otteniamo, come nel caso di Q. Wei, l'unica soluzione di viscosità.

In fine, nel Capitolo 6, studiamo le proprietà dell'operatore min-max "limiting" (che nel capitolo precedente abbiamo dimostrato essere unico) definito sullo spazio dei dati iniziali lipschitziani con costante di Lipschitz fissata. Tali proprietà ci permettono di riformulare la dimostrazione del celebre "Teorema Weak KAM" di A. Fathi. In particolare, dimostriamo che la famiglia degli operatori "limiting" ammette un punto fisso comune (nel tempo). Come nel contesto “Weak KAM”, possiamo definire un valore $c \in \mathbb{R}$ speciale per cui la corrispondente equazione di Hamilton-Jacobi stazionaria ammette una soluzione globale.

Introduction

The Thesis is focused on the study of Cauchy problems for the evolutive Hamilton-Jacobi equation:

$$\begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u(t,x)) = 0 \\ u(0,x) = f(x) \end{cases} \tag{H-J}$$

with $(t,x) \in [0,T] \times \mathbb{R}^d$ and H a globally compactly supported Hamiltonian. This PDE appears in many branches of Mathematics and Physics, as for instance dynamical systems, control theory, differential games, continuum mechanics, geometric optics and economy.

If $t > 0$ is sufficiently small, the unique classical solution of (H-J) can be determined using the method of characteristics. Although, even for smooth Hamiltonians and initial data, there exists a "critical" time for which the x components of some characteristics cross each other. Therefore, the method of characteristics in general fails. As a consequence, it arises the natural question: how to define (and to determine) weak global solutions of (H-J), that is, (Lipschitz-continuous) functions satisfying the equation almost everywhere.

In the Eighties, M.G. Crandall, L.C. Evans and P.L. Lions introduced, in a purely analytical framework, a first notion of weak solution, called *viscosity solution*, see [15] and [13]. Afterwards, in the case of convex Liouville-integrable Hamiltonians of the form $H = H(p)$, M. Bardi and L.C. Evans ([2]) provided an explicit inf-sup Hopf formula for viscosity solutions.

Some years later, in 1991, M. Chaperon, J.C. Sikorav and C. Viterbo proposed –in a genuine geometric framework– the definition of *variational (min-max) solution*. We refer e.g. to [11], [31] and [24]. The construction is based on generating functions quadratic at infinity (GFQI) of the so-called Lagrangian wavefronts given by the image, through the Hamiltonian flow, of the graph of the initial datum df.

Viscosity and variational solutions in general differ, see e.g. [24][Chapter 7] and [6][Section 4.1]. In particular, for any non-convex (non-concave) integrable Hamiltonian, it can be constructed a smooth initial condition such that the graph of the viscosity solution is not contained in the wavefront associated with the Cauchy problem (see [25]). However, min-max and viscous solutions concide for p-convex Hamiltonians ([22] and [5]). Moreover, they are Lipschitz-continuous even for smooth initial data. As a consequence, since variational solutions are based on the graph of a regular (i.e. at least C^2) initial datum, in order to iterate the min-max procedure, it is necessary to define the min-max operator also for Lipschitz-continuous functions. A relevant part of the present Thesis is devoted to introducing and discussing two different notions to variational solutions for weak initial data.

The first approach, introduced in 2014 by Q. Wei (see [33] and [34]), is based on Clarke generalized derivatives and produces an operator

$$R^{s,t} : C^{Lip} \to C^{Lip}, \qquad 0 \leq s < t \leq T,$$

associating to a Lipschitz-continuous initial datum f a generalized min-max selection $R^{s,t}f$ of the Lagrangian wavefront starting from

$$\Gamma(\partial f) := \{(x, y) \,|\, x \in \mathbb{R}^d,\, y \in \partial f(x)\}$$

($\partial f(x)$ here denotes the Clarke generalized derivative of f at x).

The second approach, due to O. Bernardi and F. Cardin (see [6] of 2011), uses the density of smooth functions in the space of C^0 functions with respect to the uniform norm and produces an operator

$$V^{s,t} : C^0 \to C^0, \qquad 0 \leq s < t \leq T.$$

The idea is simple: given a continuous initial datum f at time s, and an arbitrary C^2 sequence

$$f_n \xrightarrow{\|\cdot\|_\infty} f,$$

$V^{s,t}f$ is defined as the uniform limit of the usual min-max solutions at time t starting from f_n at time s.

It is worth noting that both generalized min-max operators are not solutions of the Cauchy problem (H-J) (see also [34][p.30]). However, they produce a viscosity solution of (H-J) by a limiting min-max procedure, as explained in the sequel.

Let $\xi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$, with $t \in (0, T]$, be a subdivision of the interval $[0, t]$. We associate to ξ the iterated min-max operators

$$R^{0,t}_\xi =: R^{t_{n-1},t} \circ R^{t_{n-2},t_{n-1}} \circ \ldots \circ R^{0,t_1}$$

and

$$V^{0,t}_\xi =: V^{t_{n-1},t} \circ V^{t_{n-2},t_{n-1}} \circ \ldots \circ V^{0,t_1}.$$

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,t]$ for which the maximum step tends to 0, that is

$$|\xi_n| := \max |t_{i+1} - t_i| \to 0.$$

For a Lipschitz-continuous initial datum f, we define the limiting min-max operators $\bar{R}^{0,t}f$ and $\bar{V}^{0,t}f$ as the uniform limit (on compact subsets of $\mathbb{R}^d$) of $R^{0,t}_{\xi_n}$ and $V^{0,t}_{\xi_n}$ respectively. These limits do not depend on the choice of the subdivision ξ_n. In this framework, we prove the next

Theorem. *Let $H \in C^2_c([0,T]\times T^*\mathbb{R}^d)$ be a globally compactly supported Hamiltonian and $f \in C^{Lip}(\mathbb{R}^d)$. The limiting min-max operators $\bar{R}^{0,t}f$ and $\bar{V}^{0,t}f$ coincide with the unique viscosity solution of the Hamilton-Jacobi equation* (H-J) *with initial datum f.*

We remark that the original idea to construct solutions by dividing a given time interval into small pieces, performing the min-max step by step, and taking the uniform limit when the amplitude of pieces tends to 0 is due to M. Chaperon. The part of the above theorem concerning the coincidence of the two operators is an original result of the present Thesis.

The last part of the Thesis is focused on (H-J) for autonomous, globally compactly supported Hamiltonians on $T^*\mathbb{T}^d$. In such a setting, we study in detail the properties of limiting min-max procedures (now coinciding, in view of the previous theorem) on the space of Lipschitz-continuous initial data. In particular, we prove that the corresponding operator satisfies all conditions in order to rephrase a result á la Weak KAM, originally due to A. Fathi (see [18][Theorem 4.4.6]). In the space $Lip_L(\mathbb{T}^d)$ of Lipschitz-continuous functions with Lipschitz constant $\leq L$, we obtain the next original

Theorem. *Let $H \in C^2_c(T^*\mathbb{T}^d)$. There exist a constant $c \in \mathbb{R}$ and a function $\bar{u} \in Lip_L(\mathbb{T}^d)$ such that*

1. *$u(t,x) := \bar{u}(x) - ct$ is a solution on $[0,+\infty) \times \mathbb{T}^d$ of the evolutive Hamilton-Jacobi equation*

$$\partial_t u + H(x, \partial_x u) = 0$$

2. *$\bar{u}(x)$ is a global solution on $\mathbb{T}^d$ of the stationary Hamilton-Jacobi equation*

$$H(x, \partial_x u) = c$$

We finally notice that the previous result naturally leads to investigate on (i) possible dynamical and PDE interpretations (as in the Weak KAM framework) of the level $c \in \mathbb{R}$ (ii) generalizations to non-compactly supported Hamiltonians.

The Thesis is organized as follows.

In Chapter 1, we describe the two notions of weak solutions for the Hamilton-Jacobi equation: viscosity and variational solutions. We also provide details and various examples.

Chapter 2 is devoted to the introduction of two distinct tools which will be used in the sequel. More precisely, the first one is the construction of generating functions for Lagrangian wavefronts by means of M. Chaperon "broken geodesics" ("géodésiques brisées") method. We then prove their quadraticity at infinity. Finally, we explain the homology counterpart of the min-max critical value, previously introduced in Chapter 1 by cohomology.

The aim of Chapter 3 is to define a min-max operator for Lipschitz-continuous initial data. We follow Q. Wei construction based on Clarke generalized derivatives and an extension of classical critical point theory to the Lipschitz setting.

In Chapter 4, the previous results are used in order to introduce iterated min-max solutions for the evolutive Hamilton-Jacobi equation and to prove some Lipschitz estimates that are useful in the sequel. Moreover, we give the details of a theorem due to Q. Wei: the limiting procedure applied to the iterated min-max converges uniformly to the unique viscosity solution.

In Chapter 5, we introduce a definition of (iterated) min-max solutions adapted to Hamilton-Jacobi problems with C^0 initial data. The construction, due to O. Bernardi and F. Cardin, is based on the density of smooth functions in the space of C^0 functions equipped with the uniform norm. Such construction is performed for globally compactly supported Hamiltonians on the cotangent bundle of a compact manifold. Moreover, for Lipschitz-continuous initial data, we prove that applying the limiting min-max procedure to this iterated operator –in general distinct from the one defined in Chapter 4– we obtain, as in the Q. Wei case, the unique viscosity solution.

In Chapter 6, we study the properties of the (now unique) limiting min-max operator defined on the space of Lipschitz-continuous initial data with fixed Lipschitz constant. These properties allow us to rephase the proof of A. Fathi celebrated "Weak KAM Theorem". In particular, we prove that the family of the limiting operators admits a common fixed point (in time). As in the "Weak KAM setting", we thus can define a special value $c \in \mathbb{R}$ for which the corresponding stationary Hamilton-Jacobi equation admits a global solution.

Chapter 1

Evolutive Hamilton-Jacobi equations and weak solutions

Abstract. The chapter is devoted to introducing two notions of weak solutions for the Hamilton-Jacobi equation: viscosity and variational (min-max) solutions. We provide details and various examples.

This brief introduction to the evolutionary Hamilton-Jacobi equation and its (generalized) solutions follows rather closely [18][Chapter 7].

Let us consider a Cauchy problem associated to the *evolutive Hamilton-Jacobi equation*:

$$\begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u(t,x)) = 0 \\ u(0,x) = f(x) \end{cases} \tag{H-J}$$

where $H : \mathbb{R} \times T^*\mathbb{R}^n \to \mathbb{R}$ is a continuous Hamiltonian, $f : \mathbb{R}^n \to \mathbb{R}$ is (at least) a C^1 initial datum and $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function.

We first note that (H-J) can be reduced to a *stationary Hamilton-Jacobi equation* by introducing the Hamiltonian

$$\mathscr{H}(t,x,s,p) := s + H(t,x,p) \tag{1.1}$$

on $T^*\mathbb{R}^{n+1}$. Thus, the evolutive Hamilton-Jacobi problem (H-J) becomes

$$\begin{cases} \mathscr{H}(y, d_y u) = 0 \\ u(0,x) = f(x) \end{cases}$$

with $y = (t,x)$. This fact is useful, especially in the next section, since it allows us to use a lighter notation and to give simpler examples.

For small intervals of time, it is possible to obtain a *classical solution* of (H-J) –that is, a C^1 function solving the corresponding Cauchy problem– by applying the well-known *method of characteristics*. We first solve Hamilton equations

$$\begin{cases} \dot{x} = \partial_p H(t,x,p) \\ \dot{p} = -\partial_x H(t,x,p) \end{cases}$$

and obtain the characteristic lines $t \mapsto (x(t), p(t))$. Then the classical solution $u(t,x)$ is such that the graph of $du_t(\cdot) = du(t,\cdot)$ is the section at time t of the union of the characteristic lines passing through $(x, df(x))$ at time $t = 0$, while $\partial_t u(\cdot)$ is given by $\partial_t u = -H(t,x,\partial_x u)$.
In general, e.g. when H is not linear with respect to p, it is not possible to obtain a global classical solution following this procedure –even for small intervals of time– because the union of characteristics is not the graph of a function.

Example 1.0.1. (From [34][Section 2.3, p.24]). For $d = 1$ let consider (H-J) corresponding to the Hamiltonian $H(t,x,y) = \frac{1}{2}y^2$ and initial datum $f(x) = \arctan(x)$. Hamilton equations give

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} = y \\ \dot{y} = \frac{\partial H}{\partial x} = 0 \end{cases}$$

and thus the flow is given by

$$\begin{cases} x(t) = x_0 + ty \\ y(t) = y_0. \end{cases}$$

Since $df(x) = \frac{1}{1+x^2}$, we have that the flow φ_t (at time t) maps the graph $\Gamma(df) = \{(x, df(x)) \,|\, x \in \mathbb{R}\}$ to

$$\varphi_t(\Gamma(df)) = \left\{ \left(x + \frac{t}{1+x^2}, \frac{1}{1+x^2} \right) \Big| \, x \in \mathbb{R} \right\},$$

which is not the graph of a function for $t > 0$ large enough. Let compare two different times, $t = 1, 2$. It is easy to see that the function

$$x \mapsto x + \frac{1}{1+x^2}$$

is bijective, thus the corresponding curve $\varphi_1(\Gamma(df))$ is the graph of a function (see Figure 1.1). However –referring to Figure 1.2– the curve $\varphi_2(\Gamma(df))$ is not the graph of any function.

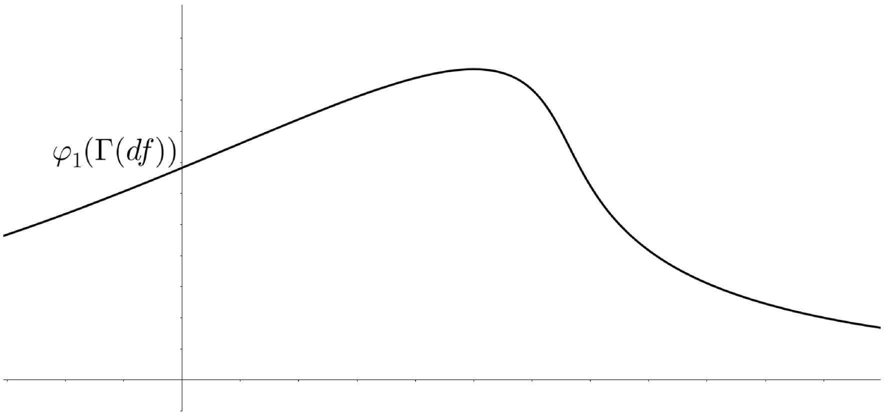


Figure 1.1: $t = 1$

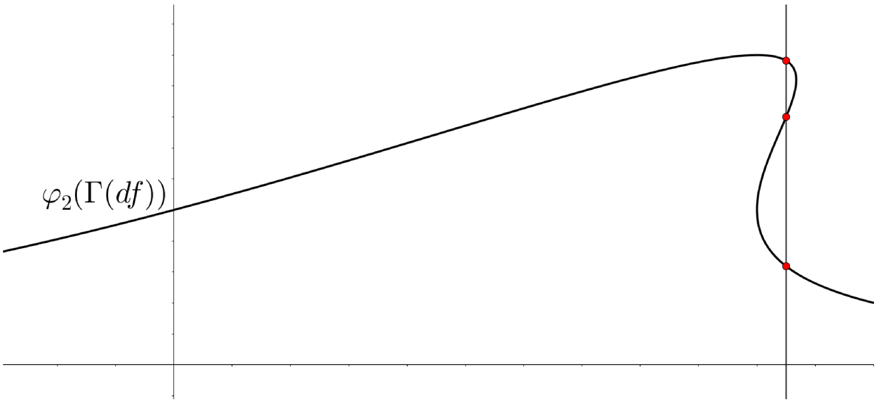


Figure 1.2: $t = 2$

As a consequence, in order to solve (H-J), it becomes natural to be introduce *weak solutions.*

Definition 1.0.1. *A function* $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is a weak solution of* (H-J) *if it is Lipschitz-continuous and satisfies the Cauchy problem almost everywhere.*

Observe that this definition makes sense since the derivative of any Lipschitz function exists almost everywhere by *Rademacher's theorem.* However, as explained in the next example, the notion of weak solution is too general, because it yields too many solutions.

Example 1.0.2. (From [18][Example 7.2.2]). On $T^*\mathbb{R}^1 = \mathbb{R} \times \mathbb{R}$, let us consider the Hamiltonian

$$H(x, p) = p^2 - 1$$

and the corresponding stationary Hamilton-Jacobi equation

$$|d_x u|^2 = 1. \tag{1.2}$$

Of course, any continuous piecewise C^1 function u with derivative taking only the values ± 1 gives a weak solution. This is already a huge amount of solutions, but there are even more. In fact, for any measurable $A \subset \mathbb{R}$, the function

$$f_A(x) = \int_0^x (2\chi_A(t) - 1)dt,$$

where χ_A is the characteristic function of A, is Lipschitz and has derivative ± 1 almost everywhere:

$$f'_A(x) = 2\chi_A(x) - 1 = \begin{cases} 2 - 1 = 1 & \text{if } x \in A \\ 0 - 1 = -1 & \text{if } x \notin A. \end{cases}$$

Thus, for every measurable subset A, also f_A is a weak solution.

A more specific notion of weak solution is the one of *viscosity solution*, introduced in the Eighties by Crandall, Evans and Lions (see [15] and [13]).

1.1 Viscosity solutions

The original idea in order to introduce viscosity solutions was adding to the evolutive Hamilton-Jacobi equation a *viscous term* $-\varepsilon \triangle u$, then applying classic theorems to the new equation and finally making the parameter $\varepsilon \to 0$. In this section we give a definition for viscosity solutions based on C^1 test functions and we state some properties.

Let $H : T^*\mathbb{R}^n \to \mathbb{R}$ be a continuous Hamiltonian function, c a real constant, and consider the stationary Hamilton-Jacobi equation

$$H(x, d_x u) = c. \tag{1.3}$$

Definition 1.1.1. *Let V be an open subset of $\mathbb{R}^n$.*

The function $u : V \to \mathbb{R}$ is a viscosity subsolution of (1.3) *on V if, for every C^1 function $\varphi : V \to \mathbb{R}$ and every point $x_0 \in V$ such that $u - \varphi$ has a (local) maximum in x_0, we have $H(x, d_x\varphi)|_{x=x_0} \leq c$.*

The function u is a viscosity supersolution of (1.3) *on V if, for every C^1 function $\psi : V \to \mathbb{R}$ and every point $y_0 \in V$ such that $u - \psi$ has a (local) minimum at y_0, we have $H(x, d_x\psi)|_{x=y_0} \geq c$.*

Finally, u is a viscosity solution of (1.3) *on V if it is both a subsolution and a supersolution on V.*

Theorem 1.1.1. *A C^1 function $u : V \to \mathbb{R}$ is a viscosity subsolution (respectively supersolution) of* (1.3) *on V if and only if $H(x, d_x u) \leq c$ (respectively $H(x, d_x u) \geq c$) for each $x \in V$. Thus, a C^1 function u is a viscosity solution of the Hamilton-Jacobi equation* (1.3) *if and only if it is a classical solution.*

Proof. Let us prove the statement for viscosity subsolution. The one for viscosity supersolution is analogous.

Suppose that $u \in C^1(V)$ is a viscosity subsolution. Then we can use u itself as a test function. But $u - u \equiv 0$, so every $x \in V$ is a maximum point and, according to the definition,

$$H(x, d_x u) \leq c \quad \forall x \in V.$$

Conversely, suppose that $H(x, d_x u) \leq c$ for each x in V. If $\varphi : V \to \mathbb{R}$ is C^1 and $u - \varphi$ has a maximum at x_0, then the differentiable function $u - \varphi$ must have derivative 0 at the maximum x_0. Therefore $d_x u(x_0) = d_x \varphi(x_0)$, and

$$H(x, d_x \varphi)|_{x=x_0} = H(x, d_x u)|_{x=x_0} \leq 0.$$

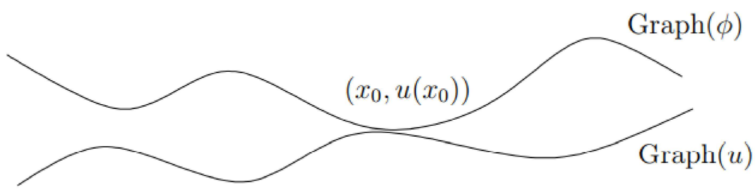
□

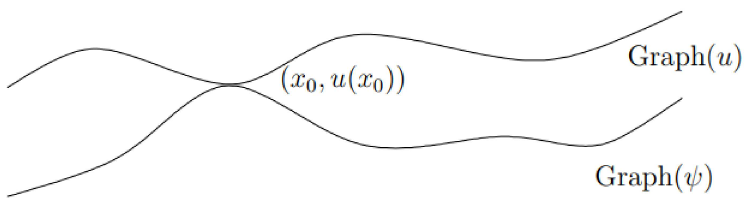
Observe that, since the condition imposed on test functions in the above definition is on the derivative of a C^1 function, its validity does not change if we add a constant to the test function. It is possible to use this fact to formulate definitions in a form that allows a deeper understanding of the concept of viscosity supersolution and subsolution.

Definition 1.1.2. *Let V be an open subset of $\mathbb{R}^n$.*

The function $u : V \to \mathbb{R}$ is a viscosity subsolution of (1.3) *on V if for every $\phi \in C^1(V)$ with $\phi \geq u$, at every point $x_0 \in V$ such that $u(x_0) = \phi(x_0)$, we have $H(x, d_x \phi)|_{x=x_0} \leq c$.*

The function $u : V \to \mathbb{R}$ is a viscosity supersolution of (1.3) *on V if for every $\psi \in C^1(V)$ with $\psi \leq u$, at every point $x_0 \in V$ such that $u(x_0) = \psi(x_0)$, we have $H(x, d_x \psi)|_{x=x_0} \geq c$.*





Let us test the new viscosity conditions on the setting of Example 1.0.2. The following example also proves that not every weak solution is a viscosity solution.

Example 1.1.1. (From [18][Example 7.2.6]. For $n = 1$, let consider the Hamilton-Jacobi equation (1.2). We first remark that the two smooth functions $x \mapsto x$ and $x \mapsto -x$ are the unique classical solutions.

We first prove that any Lipschitz function $u : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant ≤ 1 is a viscosity subsolution of (1.2). Consider φ a C^1 function such that $\varphi \geq u$ everywhere and $\exists x_0 \in \mathbb{R}$ where φ and u assume the same value. As u is Lipschitz, we have that

$$\varphi(x) - \varphi(x_0) \geq u(x) - u(x_0) \geq -|x - x_0|.$$

Thus, for $x > x_0$,

$$\frac{\varphi(x) - \varphi(x_0)}{x - x_0} \geq -1$$

and passing to the limit we find that $\varphi'(x_0) \geq -1$. On the other hand, for $x < x_0$,

$$\frac{\varphi(x) - \varphi(x_0)}{x - x_0} \leq 1,$$

hence $\varphi'(x_0) \leq 1$ and $|\varphi'(x_0)| \leq 1$. Therefore

$$H(x_0, \varphi'(x_0)) = |\varphi'(x_0)|^2 - 1 \leq 0,$$

that is, u is a viscosity subsolution.

In particular, $u(x) = |x|$ is a weak solution and a viscosity subsolution. However, it is not a viscosity solution. In fact, consider the test function $\psi \equiv 0$: $\psi \leq u$ with equality at 0, but $H(0, 0) = -1 < 0$, which violates the supersolution condition. This means that not all weak solutions are viscosity solutions.

We conclude the example by showing that a viscosity solution of (1.2) is $u(x) = -|x|$. As it is a Lipschitz function with Lipschitz constant equal to 1, it is a viscosity subsolution everywhere. Let us consider $\psi \in C^1(\mathbb{R})$ such that $\psi \leq u$ and suppose that there exists $x_0 \in \mathbb{R}$ such that $\psi(x_0) = u(x_0) = -|x_0|$. Note that $x_0 \neq 0$: any function ψ such that $\psi(x) \leq -|x|$ and $\psi(0) = 0$ is not differentiable in 0, thus it cannot be used as a C^1 test function. Then necessarily we have that $x_0 \neq 0$ and, being that ψ is C^1 and $\leq u$ everywhere, we must have that $|\psi'(x_0)| = 1$. Therefore, u is also a viscosity supersolution and thus a viscosity solution of (1.2).

For convex Hamiltonians $H = H(p)$, an example of explicit formulas for viscosity solutions are the *Hopf-Lax formulas* (see [17][Section 3.3.2]). For non-convex Hamiltonians there are formulas that require more hypotheses either on the initial datum or on the Hamiltonian: the initial datum must be convex (see [2]) or either the initial datum or the Hamiltonian must be the difference of convex functions (see [4] and [3]).

In the early Nineties, another type of solutions for Hamilton-Jacobi equations, called *variational solutions*, was introduced by Chaperon, Sikorav and Viterbo (see for example [11], [31] and [24].). Unlike viscosity solutions, variational solutions are constructed using a geometric method that involves Hamiltonian dynamics.

1.2 Variational solutions: preliminaries

Before introducing the concept of variational solution and discussing the corresponding properties, we need some preliminary (standard) notions.

1.2.1 Relative De Rham cohomologies

In this subsection we mostly refer to [7][pp. 78,79] and [9][Section 7.1.1].

Let $N \subset M$ be two manifolds and $\iota : N \hookrightarrow M$ an embedding. Denote with $\Omega^k(M)$ the vector space over $\mathbb{Z}_2$ of the differential k-forms on M. For any positive integer k, define the following complex of forms

$$\Omega^k(M, N) = \Omega^k(M) \oplus \Omega^{k-1}(N) \ni (\omega, \theta)$$

and the following exterior differential

$$d^k : \Omega^k(M, N) \to \Omega^{k+1}(M, N)$$

$$d^k(\omega, \theta) := (d\omega, \iota^*\omega - d\theta)$$

where d is the usual exterior differential for differential k-forms and

$$\iota^* : \Omega^k(M) \to \Omega^k(N)$$

is the pull-back of ι.

We say that the relative form (ω, θ) is *relatively closed* if $d^k(\omega, \theta) = 0$, that is, if ω is a closed form in M, its restriction to N is exact and θ is a primitive. We say that (ω, θ) is *relatively exact* if there is at least one $(\omega', \theta') \in \Omega^{k-1}(M, N)$ such that $d^{k-1}(\omega', \theta') = (\omega, \theta)$, that is, ω is exact on M (with primitive ω') and the restriction of ω' to N is θ up to an exact $k-1$-form.

Similarly to the usual exterior differential defined on differential forms, we have that $d^{k+1} \circ d^k = 0$ and the proof consists of a simple calculation that uses the main properties of the exterior differential.

Definition 1.2.1. *The k-th relative De Rham cohomology with coefficients in $\mathbb{Z}_2$ associated to $\iota : N \hookrightarrow M$ is the quotient*

$$H^k(M, N) := \frac{ker(d^k)}{im(d^{k-1})}.$$

We use the following notation:

$$H^*(M,N) \coloneqq \bigoplus_{k\geq 0} H^k(M,N).$$

The same construction can be performed for arbitrary manifolds M and N and for an arbitrary function $f : N \to M$, as its pull-back is always well defined.

Remark 1.2.1. One can choose as a representative of each class $[(\omega,\theta)]$ in $H^k(M,N)$ an element $(\omega',0)$, where ω' is a closed k-form on M that vanishes on N. As a consequence, $H^*(M,M) = 0$.

In the sequel, we collect some well-known property on the relative De Rham cohomologies.

Proposition 1.2.1 (Invariance under diffeomorphisms)**.** *Let M, M', N, N' be manifolds, $f : N \to M$ an application and $\varphi : M \to M'$, $\psi : N \to N'$ two diffeomorphisms. Define $f' \coloneqq \varphi \circ f \circ \psi^{-1} : N' \to M'$.*

Then

$$H^*(M,N) = H^*(M',N').$$

A direct consequence of this proposition is that, given two diffeomorphic manifolds $N \xrightarrow{\varphi} M$, then $H^*(M,N) = H^*(id_M(M), \varphi(N)) = H^*(M,M) = 0$.

Proposition 1.2.2. *Let $Q \subseteq N \subseteq M$ be manifolds and i, j the inclusions $Q \overset{i}{\hookrightarrow} N \overset{j}{\hookrightarrow} M$. Then the following sequence is exact*

$$H^*(M,N) \xrightarrow{\underline{i}} H^*(M,Q) \xrightarrow{\underline{j}} H^*(N,Q)$$

that is, $im(\underline{i}) = ker(\underline{j})$.

Note that in the previous proposition the function $\underline{i}$ maps $(\omega,\theta) \in H^*(M,N)$ in an element of $H^*(M,Q)$ by restricting the domain of θ from N to Q, while $\underline{j}$ maps (ω',θ'), an element of $H^*(M,Q)$, in an element of $H^*(N,Q)$ by restricting the domain of ω' from M to N.
Other than being invariant under diffeomorphisms, relative cohomologies are invariant under *retraction* and *excision.*

Definition 1.2.2. *Let $N \subset M$ be two manifolds and $\iota : N \hookrightarrow M$ the inclusion. We say that a continuous function $r : M \to N$ is a retraction if it is a left inverse for the inclusion, that is, $r \circ \iota = id_N$.*

Proposition 1.2.3 (Invariance under retractions)**.** *Let M, N be manifolds and let M' and N' be retractions respectively of M and N. Then*

$$H^*(M,N) = H^*(M',N').$$

Proposition 1.2.4 (Invariance under excisions)**.** *Let N be a closed submanifold of M and $U \subset N$ open and disjoint from the boundary of N. Then*

$$H^*(M \setminus U, N \setminus U) = H^*(M,N).$$

1.2.2 Min-max of Palais-Smale functions

In this section all concepts and results are stated in $\mathbb{R}^d$. Otherwise, they can be rephrased for *paracompact* manifolds, that can be endowed with a Riemannian structure, according to a theorem by Whitney. Therefore, on such manifolds it is possible to define for any smooth function f the *gradient vector field* ∇f, which shows up rather frequently in the proofs of the propositions mentioned in this section.

Definition 1.2.3. *A smooth function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies the Palais-Smale condition if every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that*

1. $|f'(x_n)| \to 0$
2. $\{f(x_n)\}_{n\in\mathbb{N}}$ *is bounded*

admits a convergent subsequence.

The Palais-Smale condition grants that, for any $a < b$ in $\mathbb{R}$, the set of critical points of f contained in $f^{-1}([a, b])$ is compact. From now, suppose that f satisfies the Palais-Smale condition and define the sublevel sets of f,

$$f^a := \{x \in \mathbb{R}^d | f(x) \leq a\}.$$

We always suppose that $a < b$ are not critical values of f.

Proposition 1.2.5. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function satisfying the Palais-Smale condition. If f has no critical points in $f^b \setminus f^a$, then f^a is diffeomorphic to f^b and therefore $H^*(f^b, f^a) = 0$.*

This proposition implies that, if $H^*(f^b, f^a) \neq 0$, then there is at least one critical point of f in $f^b \setminus f^a$ with critical value in $[a, b]$.

For $\lambda \in [a, b]$ let i_λ be the inclusion $f^\lambda \hookrightarrow f^b$, that, as seen in the previous section, induces an injective homomorphism

$$\begin{aligned} i_\lambda^* : H^*(f^b, f^a) &\to H^*(f^\lambda, f^a) \\ (\omega, \theta) &\mapsto (\omega|_{f^\lambda}, \theta) \end{aligned}$$

Clearly, for every $(\omega, \theta) = \alpha \in H^*(f^b, f^a)$ we have $i_a^* \alpha = 0$.

Definition 1.2.4. *Let f be as in Proposition 1.2.5. For every $\alpha \in H^*(f^b, f^a)$, $\alpha \neq 0$, we define the min-max*

$$c(\alpha, f) := \inf\{\lambda \in [a, b] | i_\lambda^* \alpha \neq 0\}. \tag{1.4}$$

Theorem 1.2.1. *Let f be as in Proposition 1.2.5. The value $c(\alpha, f)$ is critical for f.*

Proof. Suppose by contradiction that there is $\alpha \in H^*(f^b, f^a)$, $\alpha \neq 0$, such that $c = c(\alpha, f)$ is not critical for f. The Palais-Smale condition on f grants that the set of critical points of f contained in $f^{-1}([a, b])$ is compact, in particular it is closed. As a consequence, $\exists\varepsilon > 0$ such that $[c-\varepsilon, c+\varepsilon]$ does not contain critical values of f. Hence, for Proposition 1.2.5, $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = 0$. Since $f^a \subseteq f^{c-\varepsilon} \subseteq f^{c+\varepsilon}$, according to Proposition 1.2.2, the following sequence is exact

$$0 = H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) \xrightarrow{\underline{i}} H^*(f^{c+\varepsilon}, f^a) \xrightarrow{\underline{j}} H^*(f^{c-\varepsilon}, f^a),$$

that is, $0 = im(\underline{i}) = ker(\underline{j})$. This means that $\underline{j}$ is injective. By the definition of min-max, $i^*_{c+\varepsilon}\alpha \neq 0$, that is, $\alpha \neq 0$ in $H^*(f^{c+\varepsilon}, f^a)$ and, due to the injectivity of $\underline{j}$, we have that $\underline{j}(\alpha) = i^*_{c-\varepsilon}\alpha \neq 0$. This fact contradicts the definition of c. □

1.2.3 A graph selector for Lagrangian submanifolds

On $T^*\mathbb{R}^d$ let us consider the canonical symplectic form ω associated to the Liouville 1-form $\theta = p_i dq^i$,

$$\omega = d\theta = dp_i \wedge dq^i.$$

We say that the submanifold $\Lambda \subseteq T^*\mathbb{R}^d$ is Lagrangian if

$$\omega|_\Lambda = 0 \quad \text{and} \quad dim(\Lambda) = dim(\mathbb{R}^d) = d.$$

Example 1.2.1. Let $s : \mathbb{R}^d \to \mathbb{R}$ be a C^1 function. Then, the graph of ds,

$$\Lambda \coloneqq \Gamma(ds),$$

is a Lagrangian submanifold. In fact, Λ has dimension d and $\omega|_\Lambda = d\theta|_\Lambda = d \circ ds = 0$. Moreover, the set of the critical points of s coincides with the intersection of Λ with the 0-section $0_{\mathbb{R}^d} \subset T^*\mathbb{R}^d$.

Not all Lagrangian submanifolds are the graph of the differential of some function, but, according to a result due to Maslov and refined by Hörmander (see [23], [20]), every Lagrangian submanifold Λ can be locally parametrized by a so-called *generating function*

$$\begin{aligned} S : \mathbb{R}^d \times \mathbb{R}^k &\to \mathbb{R} \\ (x, \eta) &\mapsto S(x, \eta) \end{aligned}$$

as follows. We can describe Λ as

$$\Lambda = \left\{ \left(x, \frac{\partial S}{\partial x}(x, \eta) \right) \in T^*\mathbb{R}^d : \quad \frac{\partial S}{\partial \eta}(x, \eta) = 0 \right\}$$

where 0 is a regular value of the function

$$(x, \eta) \mapsto \frac{\partial S}{\partial \eta}(x, \eta).$$

Moreover, according to Maslov-Hörmander theorem, for every Lagrangian submanifold there is a generating function that is a *Morse function*: a function whose critical points are non-degenerate, that is, the Hessian of the function valued at the critical points has maximal rank.

We shall now define a specific class of generating functions, that are of fundamental importance to the construction of the variational solution and, as a consequence, throughout the following chapters.

Definition 1.2.5. *We say that a differentiable generating function $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is quadratic at infinity (GFQI) if there exists a non-degenerate quadratic form Q such that, for any compact K of $\mathbb{R}^d$, the differential $|\partial_\eta(S(x,\eta) - Q(\eta))|$ is bounded on $K \times \mathbb{R}^k$.*

Remark 1.2.2. Let Λ be a Lagrangian submanifold that admits a GFQI S. Then, according to a result due to Viterbo and Théret (see [30]), the following operations on S give as a result another GFQI for Λ:

1. fiberwise diffeomorphism: if $(x,\eta) \mapsto (x, \varphi(x,\eta))$ is a fiberwise diffeomorphism, then $\tilde{S}(x,\eta) := S(x, \phi(x,\eta))$ are GFQIs for the same Λ;
2. sum of a constant: $\forall C \in \mathbb{R}$, $\tilde{S}(x,\eta) := S(x,\eta) + C$ are GFQIs for the same Λ;
3. stabilization: if p is a non-degenerate quadratic form, $\tilde{S}(x,\eta,\xi) := S(x,\eta) + p(\xi)$ are GFQIs for the same Λ.

As a consequence, any GFQI as in Definition 1.2.5 can be transformed into a generating function *exactly* quadratic at infinity: for any compact $K \subset \mathbb{R}^d$, $(S-Q)|_{K\times\mathbb{R}^k}$ is compactly supported up to a fiberwise diffeomorphism $(x,\eta) \mapsto (x, \varphi(x,\eta))$. Therefore, in this section we use the following definition of GFQI instead of the previous one.

Definition 1.2.6. *A generating function $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is quadratic at infinity if there are a non-degenerate quadratic form Q and a constant $C > 0$ such that*

$$S(x,\eta) = \eta^T Q \eta \quad \text{for} \quad |x| > C.$$

The following lemma states a property of GFQIs that is particularly useful in the construction of variational solutions.

Lemma 1.2.2. *Let $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ be a GFQI. Then, for every fixed x, $S(x,\cdot)$ satisfies the Palais-Smale condition.*

Proof. Let $(\eta_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence, that is,

$$|S(x,\eta_n)| \leq \bar{C} < \infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{\partial S}{\partial \eta}(x,\eta_n) = 0.$$

If the sequence (η_n) is eventually contained in a compact subset of $\mathbb{R}^k$, then necessarily there is a subsequence that converges to a point $\bar{\eta}$, that, due to the smoothness of S, will be a critical point.
Now we shall show that if the sequence is Palais-Smale then nothing other than what we just analyzed could happen. If every compact subset of $\mathbb{R}^k$ contained a finite number of elements of the sequence, then $|\eta_n| \xrightarrow{n\to+\infty} +\infty$, which implies that for n large enough $|\eta_n| > C$. In this case, since S is a GFQI, we have that $S(x, \eta_n) = \eta_n^T Q \eta_n$, thus $\frac{\partial S}{\partial \eta}(x, \eta_n) = 2Q\eta_n$ (for every n large enough). As Q is non-degenerate, that quantity would tend to $+\infty$ as n tends to $+\infty$, which contradicts the hypothesis that the sequence is Palais-Smale. □

As previously stated, every Lagrangian submanifold admits a generating function, but we have not yet specified any condition under which a Lagrangian submanifold admits a GFQI. We start by considering a special Lagrangian submanifold that always admits a GFQI. Let $(\varphi_t)_{t\in[0,1]}$ be the flow associated to a globally *compactly supported*[1] Hamiltonian function H. Thus

$$\Lambda = \varphi_1(0_{\mathbb{R}^d}) \tag{1.5}$$

is a Lagrangian submanifold Hamilton isotopic to the 0-section. The following result can be found in [29], except for the statement about uniqueness, for which we refer to [32][Section 1] and [30].

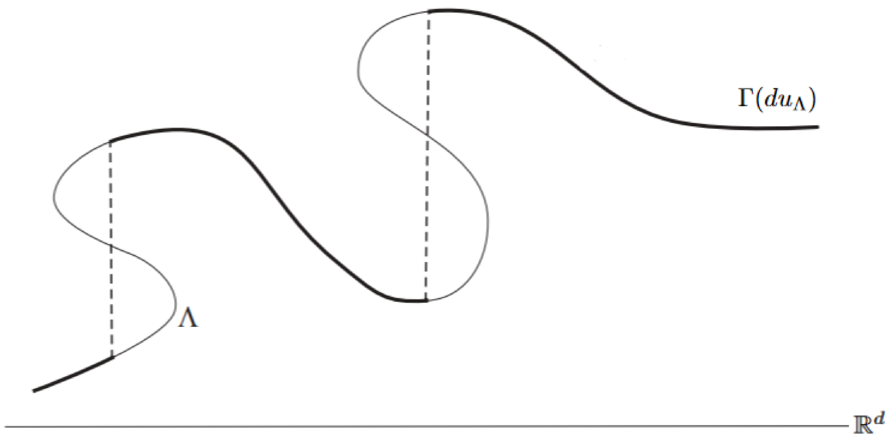
Theorem 1.2.3 (Chaperon-Laudenbach-Sikorav-Viterbo)**.** *Let Λ be a Lagrangian submanifold as in* (1.5)*, then it admits a unique*[2] *GFQI.*

As stated at the beginning of this subsection, in general a Lagrangian submanifold is not the graph of a function. However, the following result (see [28][Theorem 6.1.3]) states that every Lagrangian submanifold Λ as in (1.5) admits a GFQI and thus one can extract a “graph part” from Λ by means of the so-called *graph selector*.

Theorem 1.2.4. *Let Λ be as in* (1.5)*. Then, there exists a Lipschitz function $u_\Lambda : \mathbb{R}^d \to \mathbb{R}$ which is smooth on an open set $M \subset \mathbb{R}^d$ of full measure and such that*

$$(x, du_\Lambda(x)) \in \Lambda \quad \forall x \in M. \tag{1.6}$$

[1]This hypothesis can be omitted if we work on a compact manifold instead of $\mathbb{R}^d$.
[2]Note that the uniqueness is up to the three operations explained in Remark 1.2.2.



Proof. Let $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ be the unique GFQI for Λ given by Theorem 1.2.3. Because of the uniqueness, we can also assume that S is a Morse function. Since S is quadratic at infinity, by Lemma 1.2.2, $S(x,\cdot)$ satisfies the Palais-Smale condition for every fixed $x \in \mathbb{R}^d$. This property allows us to apply Proposition 1.2.5: to verify whether $S(x,\cdot) = S_x$ admits critical points it is sufficient to determine the relative cohomology $H^*(S_x^\infty, S_x^{-\infty})$.

Let Q be the quadratic form associated with S. If $c > 0$ is large enough, then $S_x^c = Q^c$ and $S_x^{-c} = Q^{-c}$, hence $H^*(S_x^c, S_x^{-c}) = H^*(Q^c, Q^{-c})$. Using the properties regarding the invariance of relative de Rham cohomologies stated in subsection 1.2.1 one can prove that, if k_- is the *Morse index* of Q (that is, the number of its negative eigenvalues), then

$$H^i(Q^c, Q^{-c}) = \begin{cases} \mathbb{Z}_2 & \text{if} \quad i = k_- \\ 0 & \text{if} \quad i \neq k_- \end{cases}$$

So far, we have that $H^*(S_x^c, S_x^{-c}) = H^*(Q^c, Q^{-c}) = \mathbb{Z}_2$ for every c large enough, therefore $H^*(S_x^\infty, S_x^{-\infty}) = \mathbb{Z}_2 \neq 0$ and, by Proposition 1.2.5, S_x admits critical points. Thus, we can select one using the min-max. In particular, we can choose the form 1_x, unique generator of $H^{k_-}(Q^c, Q^{-c}) = \mathbb{Z}_2$, and define the following min-max function:

$$u_\Lambda(x) = c(1_x, S(x,\cdot)) \tag{1.7}$$

It follows from Theorem 1.2.1 that each value $u_\Lambda(x)$ is a critical value of $S(x,\cdot)$.

We continue by proving property (1.6). Consider the subset $M \subset \mathbb{R}^d$ consisting of all those x for which S_x is a Morse function whose critical points have pairwise distinct critical values. It is possible to prove that such subset is open and has full measure. In any neighbourhood U of a point in M there exists a smooth function $g : U \to \mathbb{R}^k$ such that $g(x)$ is a critical point of S_x and $u_\Lambda(x) = S(x, g(x))$. Differentiating with respect to x and taking into account that $d_\eta S(x, g(x)) = 0$ we get that $du_\Lambda(x) = d_x S(x, g(x))$. Thus, by definition of generating function, we have $(x, du_\Lambda(x)) \in \Lambda$ for each $x \in M$.

We now prove that M is an open subset of $\mathbb{R}^d$ of full measure. Let $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ be the canonical projection. Then, S_x is a Morse function if and only if $x \in \mathbb{R}^d$ is a regular value of $\pi|_\Lambda$ (see [1][Section 21.2]). Denote the set of these $x \in \mathbb{R}^d$ by M_1: it is an open subset of $\mathbb{R}$ and, by *Sard theorem*[3], it has full measure. Let $U \subset M_1$ be a sufficiently small open subset. The critical points of S_x depend smoothly on $x \in U$. Denote them by $\varphi_1(x), \ldots, \varphi_m(x)$, and define

$$a_{ij}(x) \coloneqq S(x, \varphi_i(x)) - S(x, \varphi_j(x))$$

for $i \neq j$. Note that

$$da_{ij}(x) = d_x S(x, \varphi_i(x)) - d_x S(x, \varphi_j(x)) \neq 0$$

since the map $(x, \eta) \mapsto (x, d_x S(x, \eta))$ is an embedding of $\Gamma(dS) \cap (T^*\mathbb{R}^d \times 0_{\mathbb{R}^k})$ into $T^*\mathbb{R}^d$. Therefore, the sets $\{x \in U \,|\, a_{ij}(x) = 0\}$ are smooth hypersurfaces. It follows from the definition of M that

$$M \cap U = U \setminus \bigcup_{i \neq j} \{x \in U \,|\, a_{ij}(x) = 0\},$$

so $M \cap U$ is an open subset of full measure in $\mathbb{R}^d \cap U$ and indeed M is an open subset of full measure in $\mathbb{R}^d$.

We finish by proving that u_Λ is a Lipschitz function on $\mathbb{R}^d$. Since S is continuously differentiable, $S(\cdot, \eta)$ is locally Lipschitz for every $\eta \in \mathbb{R}^k$. Since the Hamiltonian H is globally compactly supported, the corresponding Hamiltonian flow outside $supp(H)$ coincides with the identity, thus the dynamics "take place" inside $supp(H)$. We can thus restrict the domain of $S(\cdot, \eta)$ to $D \coloneqq \{x \in \mathbb{R}^d \,|\, (x, p) \in supp(H)\, \exists p \in \mathbb{R}^d\}$, which is a compact subset of $\mathbb{R}^d$ because of the compactness of $supp(H)$ in $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, as $S(\cdot, \eta)$ is locally Lipschitz on $\mathbb{R}^d$, it is Lipschitz on D for every fixed $\eta \in \mathbb{R}^k$. Thus, there exists a positive constant C such that for all $x, y \in D$ we have

$$|S(x, \eta) - S(y, \eta)| \leq C|x - y|. \tag{1.8}$$

Fix any $\varepsilon > 0$ and $x \in D$, and set

$$a(y) \coloneqq u_\Lambda(x) + \varepsilon + C|x - y|, \quad y \in D.$$

It follows from inequality (1.8) that $S_x^{a(x)} \subset S_y^{a(y)}$ for all $y \in D$. Note also that $S_x^{u_\Lambda(x)} \subset S_x^{a(y)}$ for all $y \in D$. By definition, $H^*(S_x^{a(x)}, S_x^{-c})$ contains a class represented by 1_x. Therefore, as $S_x^{-c} = S_y^{-c} = Q^{-c}$, the same holds for $H^*(S_y^{a(y)}, S_y^{-c})$. This implies that $u_\Lambda(y) \leq a(y)$, so that

$$u_\Lambda(y) - u_\Lambda(x) \leq C|x - y| + \varepsilon.$$

[3] Sard theorem asserts that the set of critical values of a smooth function between Euclidean spaces or manifolds has Lebesgue measure 0, see [27].

Since $\varepsilon > 0$ is arbitrary we have

$$u_\Lambda(y) - u_\Lambda(x) \leq C|x - y|.$$

Interchanging x and y we prove that u_Λ is Lipschitz-continuous. $\square$

1.3 Variational solutions: definition

We now return to the Cauchy problem (H-J):

$$\begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u(t,x)) = 0 \\ u(0,x) = f(x) \end{cases}$$

In such a setting it is natural to define the so-called *Lagrangian wave front* (at time t)

$$\Lambda_t^f := \varphi_t(\Gamma(df)),$$

where φ_t is the Hamiltonian flow associated with H.
Observe that, supposing that the Hamiltonian is globally compactly supported, Λ_t^f is a Lagrangian submanifold that satisfies the hypotheses of Theorem 1.2.4. It is in fact isotopic to the 0-section through the following composition of Hamiltonian flows:

$$0_{\mathbb{R}^d} \xrightarrow{f} \Gamma(df) \xrightarrow{\varphi_t} \Lambda_t^f.$$

We are now ready to introduce the notion of *variational (or min-max) solution.*

Definition 1.3.1. *The variational solution for the Cauchy problem* (H-J) *is defined as*

$$u(t,x) = c(1_x, S(t,x,\cdot)),$$

where S is the unique GFQI of Λ_t^f.

As proved in Theorem 1.2.4, the function $u(t,x)$ is Lipschitz-continuous with respect to x and, almost everywhere in $\mathbb{R}^d$, it is smooth. Moreover, the graph of $d_x u(t,x)$ is contained in Λ_t^f. The fact that $u(t,x)$ is locally Lipschitz with respect to t and a weak solution for (H-J) has been proved by Chaperon, we refer to [12][Theorem 2].

Chapter 2

Chaperon's generating functions, the min-max critical value in homology

Abstract. In this chapter we first recall Chaperon's construction of generating functions for Lagrangian wavefronts, performed with the so-called "broken geodesics" ("géodésiques brisées") method. Then we prove the quadraticity at infinity of such functions. Moreover, we explain the homology counterpart of the min-max critical value introduced in the previous chapter by cohomology. Both these constructions will be useful in the sequel.

2.1 Generating functions for wavefronts

In the following, we equip $\mathbb{R}^d$ with the usual Euclidean norm, we denote with $Lip(g)$ the Lipschitz constant of a function g and with $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ the canonical projection $\pi(x, y) = x$.
Let $H : [0, T] \times T^*\mathbb{R}^d \to \mathbb{R}$ be an Hamiltonian function that satisfies the following hypothesis:

$$c_H \coloneqq \sup_{\substack{\tau\in[0,T] \\ (x,y)\in T^*\mathbb{R}^d}} |D^2 H_\tau(x, y)| < \infty \tag{2.1}$$

where we use the notation $H_\tau(x, y) \coloneqq H(\tau, x, y)$, $\tau \in [0, T]$, $(x, y) \in T^*\mathbb{R}^d$. Let $X_{H_\tau} = (\partial_2 H_\tau, -\partial_1 H_\tau)$ be the associated Hamiltonian vector field.
From the hypothesis (2.1) follows that $c_H = \max_\tau Lip(DH_\tau) = \max_\tau Lip(X_{H_\tau})$, hence, by the theory of differential equations, the Hamiltonian flow $\varphi_H^{s,t} : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ obtained by integrating X_{H_τ} from s to t is a well defined diffeomorphism for every $s, t \in [0, T]$. For simplicity, we denote $\varphi_H^{s,t} = \varphi_s^t$ omitting H and we denote with (X_s^t, Y_s^t) the components of φ_s^t in $T^*\mathbb{R}^d$.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^1 function and consider the Lagrangian submanifold $\varphi_s^t(\Gamma(df))$. In the sequel we construct a generating function for such submanifold. We start by introducing the concept of generating function for a diffeomorphism.

Definition 2.1.1. *Let $\varphi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ be a diffeomorphism. A C^2 function $\phi : T^*\mathbb{R}^d \to \mathbb{R}$ is a generating function for φ if the graph of φ coincides with the set of the points $((x, y), (X, Y)) \in T^*\mathbb{R}^d \times T^*\mathbb{R}^d$ such that*

$$\begin{cases} x = X + \partial_y \phi(X, y) \\ Y = y + \partial_X \phi(X, y) \end{cases} \tag{2.2}$$

Note that, if it does exist, the generating function of φ is unique up to the addition of a constant.

2.1.1 A generating function for φ_s^t

Using the method described by Chaperon in [11][Chapter 2], we shall now build a generating function for the flow φ_s^t.
The flow φ_s^t is a *resolvent* of the differential equation

$$\frac{dz}{d\tau} + X_{H_\tau}(z) = 0$$

with $z = (x, y)$, $s, t \in [0, T]$, that is, $\varphi_s^t(a)$ is the value at time t of the maximal solution of the equation that, at time s, has value a. Hence, using a theorem on the fixed points of Lipschitz functions (see [11][Theorem 2.1.1]), one can prove that the flow is Lipschitz-continuous and that

$$\lim_{t \to s} Lip(\varphi_s^t - id) = 0$$

uniformly with respect to s. Thus we can fix $\delta > 0$ such that, if $|t - s| < \delta$, then $Lip(\varphi_s^t - id) < \frac{1}{2}$. As a consequence, using a theorem of global inversion, we obtain the following result.

Lemma 2.1.1. *For $|t - s| < \delta$, the map $\alpha_s^t : (x, y) \mapsto (X_s^t(x, y), y)$ is a diffeomorphism.*

Thus, for every $\tau, s, t \in [0, T]$, with $|t - s| < \delta$ e $\tau \in [s, t]$, we can write

$$(X_s^\tau(X, y), Y_s^\tau(X, y)) = \varphi_s^\tau \circ (\alpha_s^t)^{-1}(X, y)$$

and define the following function

$$\phi_s^t(X, y) = \int_s^t \Big[(Y_s^\tau - y) \frac{d}{d\tau} X_s^\tau - H(\tau, X_s^\tau, Y_s^\tau) \Big] d\tau. \tag{2.3}$$

Theorem 2.1.2. *For $|t-s| < \delta$, ϕ_s^t is a generating function for φ_s^t. Moreover, denoting $(X, Y) = \varphi_s^t(x, y)$, we have that*

$$\partial_s \phi_s^t(X, y) = H(s, x, y) \quad \textit{and} \quad \partial_t \phi_s^t(X, y) = H(t, X, Y).$$

Proof. In order to prove that ϕ_s^t is a generating function for φ_s^t, we prove that

$$d\phi_s^t(X, y) = (Y - y)dX + (x - X)dy,$$

where $Y = Y_s^t(X, y)$ and $x = X_s^s(X, y)$. Indeed we have that

$$d\phi_s^t(X, y) = d\int_s^t \Big[(Y_s^\tau - y)\frac{d}{d\tau}X_s^\tau - H(\tau, X_s^\tau, Y_s^\tau)\Big] d\tau.$$

Applying $(\delta X, \delta y) \in T^*\mathbb{R}^d$ to the right side of the equality and denoting

$$(\delta X_s^\tau, \delta Y_s^\tau) = d[\phi_s^\tau \circ (\alpha_s^\tau)^{-1}(X, y)](\delta X, \delta y)$$

we obtain

$$\int_s^t \Big[(\delta Y_s^\tau - \delta y)\frac{d}{d\tau}X_s^\tau + (Y_s^\tau - y)\frac{d}{d\tau}\delta X_s^\tau - \partial_1 H_\tau(X_s^\tau, Y_s^\tau)\delta X_s^\tau - \partial_2 H_\tau(X_s^\tau, Y_s^\tau)\delta Y_s^\tau\Big] d\tau.$$

Then we use the equalities

$$\partial_1 H_\tau(X_s^\tau, Y_s^\tau) = -\frac{d}{d\tau}Y_s^\tau, \qquad \partial_2 H_\tau(X_s^\tau, Y_s^\tau) = \frac{d}{d\tau}X_s^\tau$$

and we find that

$$\begin{aligned} d\phi_s^t(X, y)(\delta X, \delta y) &= -\delta y \int_s^t \frac{d}{d\tau}X_s^\tau d\tau - y\int_s^t \frac{d}{d\tau}\delta X_s^\tau d\tau + \int_s^t \frac{d}{d\tau}(Y_s^\tau \delta X_s^\tau) d\tau \\ &= -\delta y(X_s^t - x) - y(\delta X_s^t - x) + Y_s^t \delta X_s^t - yx \\ &= (Y - y)\delta X + (x - X)\delta y. \end{aligned}$$

Thus, ϕ_s^t is a generating function for φ_s^t.

In order to prove that the equalities regarding the time derivatives hold, we differentiate both sides of (2.3).

$$\begin{aligned} \partial_s \phi_s^t(X, y) &= -[(Y_s^\tau - y)\frac{d}{d\tau}X_s^\tau - H(\tau, X_s^\tau, Y_s^\tau)]|_{\tau=s} + \int_s^t \frac{d}{ds}\Big[(Y_s^\tau - y)\frac{d}{d\tau}X_s^\tau - H(\tau, X_s^\tau, Y_s^\tau)\Big] d\tau \\ &= H(s, x, y) + \int_s^t \Big[\frac{d}{ds}Y_s^\tau \frac{d}{d\tau}X_s^\tau + (Y_s^\tau - y)\frac{d}{ds}\frac{d}{d\tau}X_s^\tau + \frac{d}{d\tau}Y_s^\tau \frac{d}{ds}X_s^\tau - \frac{d}{d\tau}X_s^\tau \frac{d}{ds}Y_s^\tau\Big] d\tau \\ &= H(s, x, y) + \int_s^t (Y_s^\tau - y)\frac{d}{ds}\frac{d}{d\tau}X_s^\tau d\tau + \Big[Y_s^\tau \frac{d}{ds}X_s^\tau\Big]\Big|_s^t - \int_s^t Y_s^\tau \frac{d}{d\tau}\frac{d}{ds}X_s^\tau d\tau \\ &= H(s, x, y) + (Y_s^t - y)\frac{d}{ds}X_s^t \\ &= H(s, x, y), \end{aligned}$$

where we use that $X_s^t \equiv X$ and thus $\frac{d}{ds}X_s^t = 0$. The proof of the second equality is analogous. □

2.1.2 A generating function for $\varphi_s^t(\Gamma(df))$

The following Proposition and its proof can be found in [29][Lemma 1.5].

Proposition 2.1.1 (Composition formula)**.** *Let $\Lambda \subset T^*\mathbb{R}^d$ be a Lagrangian submanifold that admits a generating function $S(x;\eta)$ and let $\varphi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ be a canonical transformation that admits a generating function $\phi(y,z;\mu)$. Then, the Lagrangian submanifold $\varphi(\Lambda)$ admits generating function*

$$\bar{S}(x;\eta,\mu,\xi) \coloneqq S(\xi;\eta) + \phi(\xi,x;\mu).$$

Corollary 2.1.2.1. *For each subdivision $0 \le s = t_0 < t_1 \cdots < t_N = t \le T$ that satisfies $|t_{i+1} - t_i| < \delta \quad \forall i = 0,\dots,N-1$, if $\phi_{t_i}^{t_{i+1}}$ is the generating function of $\varphi_{t_i}^{t_{i+1}}$ as defined in (2.3), the following statements are true for any C^2 function $f : \mathbb{R}^d \to \mathbb{R}$.*

1. *A generating function $S : \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \to \mathbb{R}$ of the Lagrangian submanifold $\varphi_s^t(\Gamma(df))$ is*

$$S(x,\eta) = f(x_0) + \sum_{0\le i<N} \phi_{t_i}^{t_{i+1}}(x_{i+1},y_i) + \sum_{0\le i<N} (x_{i+1}-x_i)y_i, \tag{2.4}$$

   where $x_N \coloneqq x$ and $\eta = \big((x_i,y_i)\big)_{0\le i<N}$.

2. *We can define a C^2 family $S : [s,t] \times \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \to \mathbb{R}$ such that each $S_\tau \coloneqq S(\tau,\cdot)$ is a generating function for $\varphi_s^\tau(\Gamma(df))$ as follows*

$$S(\tau,x,\eta) = f(x_0) + \sum_{0\le i<N} \phi_{\tau_i}^{\tau_{i+1}}(x_{i+1},y_i) + \sum_{0\le i<N} (x_{i+1}-x_i)y_i, \tag{2.5}$$

   where $\tau_j = s + (\tau - s)\frac{t_j - s}{t-s}$.

3. *For each critical point η of $S(\tau,x,\cdot)$, the corresponding critical value is*

$$S_\tau(x,\eta) = f(x_0) + \int_s^\tau \big(Y_s^\sigma \dot{X}_s^\sigma - H(\sigma,X_s^\sigma,Y_s^\sigma)\big)d\sigma,$$

   where $X_s^\sigma \coloneqq X_s^\sigma(x_0,df(x_0))$, $Y_s^\sigma \coloneqq Y_s^\sigma(x_0,df(x_0))$. Hence, if we denote $z \coloneqq (x,y)$ with $y \in \pi^{-1}(x) \cap \varphi_s^\tau(df)$, we have that the critical values of $S(\tau,x,\cdot)$ are the real numbers

$$f(X_\tau^s(z)) + \int_s^\tau \big(Y_\tau^\sigma(z)\dot{X}_\tau^\sigma(z) - H(\sigma,X_\tau^\sigma(z),Y_\tau^\sigma(z))\big)d\sigma. \tag{2.6}$$

Proof.

1. Since φ_s^t is an Hamiltonian flow, the following equality holds:
$$\varphi_s^t = \varphi_{t_{N-1}}^{t_N} \circ \cdots \circ \varphi_{t_0}^{t_1}.$$
Therefore, being that $|t_{i+1} - t_i| < \delta$ for all i, we can apply repeatedly the composition formula stated in Proposition 2.1.1, starting with the Lagrangian submanifold $\Gamma(df)$ and the canonical transformation $\varphi_{t_0}^{t_1}$.

2. It is analogous to 1.

3. Being S_τ a generating function for $\varphi_s^\tau(\Gamma(df))$, $\frac{\partial}{\partial\eta}S_\tau(x,\eta) = 0$ in every point belonging to $\varphi_s^\tau(\Gamma(df))$. Thus, in order to obtain the critical values of S_τ, it is enough to evaluate S_τ on points of $\varphi_s^\tau(\Gamma(df))$.

□

2.2 Quadraticity at infinity

As seen in Section 1.2, a property of generating functions that is crucial for the construction of variational solutions is quadraticity at infinity. We now state a result regarding the quadraticity at infinity of the generating functions defined in Corollary 2.1.2.1.

Lemma 2.2.1. *If H is globally compactly supported and f is Lipschitz-continuous, then the generating functions* (2.4) *and* (2.5) *constructed in Corollary 2.1.2.1 are quadratic at infinity.*

Proof. We prove the statement for S as in (2.4), the proof for (2.5) is analogous. Let us write $S(x,\eta) = \psi(x,\eta) + Q(\eta)$, where

$$\psi(x,\eta) = f(x_0) + \sum_{0\le i<N} \phi_{t_i}^{t_{i+1}}(x_{i+1}, y_i) + x y_{N-1},$$

$$Q(\eta) = \sum_{0\le i<N} (x_{i+1} - x_i) y_i - x_{N-1} y_{N-1}.$$

Clearly, Q is a non-degenerate quadratic form.
Since $supp(H)$ is compact, each $\phi_{t_i}^{t_{i+1}}$ is compactly supported and therefore has bounded derivatives. Thus, being also that f is Lipschitz-continuous, for x belonging to any compact $K \subset \mathbb{R}^d$

$$|\partial_\eta(S(x,\eta) - Q(\eta))| = |\partial_\eta \psi(x,\eta)|$$

is bounded on $K \times (T^*\mathbb{R}^d)^N$. Thus, we conclude that S is quadratic at infinity. □

For the sake of brevity, let us denote $L_s^t := \varphi_s^t(\Gamma(df))$. In general, a function on $\mathbb{R}^k$, whose differential on a compact set equals that of a non-degenerate quadratic form up to a bounded map, must have critical points. Thus, a necessary condition for L_s^t to admit a GFQI is that, for every compact subset K of $\mathbb{R}^d$, $L_s^t \cap \pi^{-1}(K)$ is compact and non-empty.
We shall now give a couple of examples where L does not admit a GFQI: in one case H is not compactly supported, in the other f is not Lipschitz.

Example 2.2.1 (see [34][Example 2.10])**.** Consider the Hamiltonian $H(t,x,y) = x^2 + y^2$, with initial data $f \equiv 0$. We compute the flow:

$$\varphi_0^t(x,y) = (x\cos 2t - y\sin 2t, y\cos 2t + x\sin 2t).$$

Thus, $L_0^{\frac{\pi}{4}} = \{0\} \times \mathbb{R}$, which has empty intersection with $\pi^{-1}(x) = \{x\} \times \mathbb{R}$ for every $x \neq 0$ and its intersection with $\pi^{-1}(0)$ is not compact. Thus, $L_0^{\frac{\pi}{4}}$ does not admit a GFQI.

Example 2.2.2 (see [34][Example 2.11])**.** Consider the compactly supported Hamiltonian

$$H(t,x,y) = h(x,y) = \frac{1}{2}y^2 \quad \text{for} \quad |y| \leq 1.$$

Note that, for $|y| \leq 1$, $|\partial_x h| \leq \frac{1}{2}$ and $|\partial_y h| \leq \frac{1}{2}$. If $f = \frac{1}{3}x^3$,

$$\pi \circ L_0^1 \subset \{x + x^2 \big| |x| \leq 1\} \cup \left[\frac{1}{2}, +\infty\right] \cup \left[-\infty, -\frac{1}{2}\right] \subset \mathbb{R}.$$

Thus, $\exists z \in \mathbb{R} \setminus \pi(L_0^1)$ and, being that $\{z\}$ is a compact subset of $\mathbb{R}$ and $L_0^1 \cap \pi^{-1}(\{z\})$ is empty, L_0^1 does not admit a GFQI.

We observe that the main ingredient in the construction of the generating functions in Corollary 2.1.2.1 is the Hamiltonian flow. Therefore, when studying the properties of the generating functions of $L_s^t = \varphi_s^t(\Gamma(df))$, what matters over a given compact subset K of $\mathbb{R}^d$ is the region swept by the Hamiltonian flow: this is the idea of what is called the *property of finite propagation speed* in [10][Appendix B].

Proposition 2.2.1 (Property of finite propagation speed)**.** *Consider* $[s,t] \subset [0,T]$ *and* $L_s^t = \varphi_s^t(\Gamma(df))$*. If for any compact* $K \subset \mathbb{R}^d$ *the set*

$$\mathscr{U}_K := \bigcup_{\tau \in [s,t]} \{\tau\} \times \left\{\varphi_s^\tau\big(\varphi_t^s\big(\pi^{-1}(K)\big) \cap df\big)\right\}$$

is non-empty and compact, then L_s^t *admits a GFQI in the sense that each* $L_s^t|_K := L_s^t \cap \pi^{-1}(K)$ *admits a GFQI.*

Proof. For any fixed K compact subset of $\mathbb{R}^d$, consider the Hamiltonian $\tilde{H} = \chi H$, where χ is a compactly supported smooth function on $[0,T] \times T^*\mathbb{R}^d$ that is equal to 1 on a neighbourhood of $\mathscr{U}_K$. Then we can apply the formula (2.4) with Hamiltonian $\tilde{H}$ to obtain a GFQI $S_{\tilde{H}}$ for $L|_K$. □

As seen in the proof of Theorem 1.2.4, for every GFQI $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$, one can define the associated min-max function $R_S : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$R_S(x) \coloneqq c(1_x, S(x,\cdot)) \quad \forall x \in \mathbb{R}^d. \tag{2.7}$$

Lemma 2.2.2. *If two generating functions S and S' are quadratic at infinity with $\|S - S'\|_{C^0} < \infty$, then for every $x \in \mathbb{R}^d$ the associated min-max functions R_S and $R_{S'}$ satisfy*

$$|R_S(x) - R_{S'}(x)| \leq \|S - S'\|_{C^0}.$$

Proof. If $S \leq S'$, then by definition $R_S(x) \leq R_{S'}(x)$. In general we have that $S \leq S' + \|S - S'\|_{C^0}$, hence $R_S(x) \leq R_{S'}(x) + \|S - S'\|_{C^0}$. Similarly, $R_{S'}(x) \leq R_S(x) + \|S - S'\|_{C^0}$ and therefore we proved the inequality stated above. □

Lemma 2.2.3. *The min-max R_S associated to S given by* (2.4) *or* (2.5) *does not depend on the subdivision of the time interval $[s,t]$ chosen in the construction of S.*

Proof. We start by assuming that $t - s < \delta$. For $\tau \in (s,t)$, consider the family of subdivisions $\xi_\mu \coloneqq \{s \leq s + \mu(\tau - s) < t\}$ with $\mu \in [0,1]$. Then,

$$S_\mu(x, x_0, y_0, x_1, y_1) = f(x_0) + \phi_s^{s+\mu(\tau-s)}(x_1, y_0) + (x_1 - x_0)y_0 + \phi^t_{s+\mu(\tau-s)}(x, y_1) + (x - x_1)y_1$$

is the generating function defined by (2.4) associated to the subdivision ξ_μ. The function S_μ is continuous in μ and the min-max $R_{S_\mu}(x)$ is a critical value of the map $\eta \mapsto S_\mu(x,\eta)$, where $\eta = (x_0, y_0, x_1, y_1)$. By point 3 of Corollary 2.1.2.1, the set of such critical values does not depend on μ and, by Sard theorem, it has measure zero, thus R_{S_μ} is constant for $\mu \in [0,1]$ and

$$R_S(x) = R_{S_0}(x) = R_{S_1}(x).$$

Let us now consider the general case. For a subdivision $\xi = \{t_0 < \cdots < t_n\}$ we define $|\xi| \coloneqq \max_i |t_{i+1} - t_i|$. Given any two subdivisions ξ', ξ'' of $[s,t]$ with $|\xi'|,|\xi''| < \delta$, denote by $\xi = \xi' \cup \xi'' = \{s = t_0 < \cdots < t_n = t\}$ the subdivisions whose points are contained in ξ' or ξ''. Suppose t_j is not contained in ξ' and consider the family of subdivisions

$$\xi_\mu(j) \coloneqq \{t_0 < \cdots < t_{j-1} \leq t_{j-1} + \mu(t_j - t_{j-1}) < t_{j+1} < \cdots < t_n\},$$

with $\mu \in [0,1]$. The same arguement as before proves that the min-max relative to $\xi_0(j)$ is equal to the min-max relative to $\xi_1(j)$. Repeating this procedure for each t_i that belongs to only one of the two subdivisions, we get that $R_{S_{\xi'}}(x) = R_{S_\xi}(x)$ and $R_{S_{\xi''}}(x) = R_{S_\xi}(x)$, therefore $R_{S_{\xi'}}(x) = R_{S_{\xi''}}(x)$. □

Proposition 2.2.2. *Under the hypotheses of Proposition 2.2.1 and with the notation used in its proof, the Lagrangian submanifold L determines a min-max function defined by*

$$R(x) = R_{S_{\tilde{H}}}(x) \quad \textit{for } x \in K,$$

which does not depend on the truncation $\tilde{H}$ of the Hamiltonian H or on the subdivision of $[s,t]$ used to define $S_{\tilde{H}}$.

Proof. Let $\tilde{H}$ and $\tilde{H}'$ be two truncations for H on $\mathscr{U}_K$ and consider the following family of Hamiltonian functions:

$$H^\mu = \mu\tilde{H} + (1-\mu)\tilde{H}', \quad \text{for } \mu \in [0,1].$$

Each one of them satisfies the condition (2.1) with the constant c_{H^μ} and $\{c_{H^\mu}\}_{\mu\in[0,1]}$ is uniformly bounded, thus we can find a subdivision $s = t_0 < t_1 < \cdots < t_N = t$ that satisfies $|t_{i+1} - t_i| < \delta_{H_\mu}$ for all μ. Let S_μ be the GFQI of L_K obtained for the truncation H_μ. By Lemma 2.2.2, as S_μ depends continuously on μ, so does the min-max function $R_{S_\mu}(x)$. On the other hand, $R_{S_\mu}(x)$ is a critical value of the map $\eta \mapsto S_\mu(x,\eta)$ and, due to statement 3 of Corollary 2.1.2.1, the set of all such critical values is independent of μ and the subdivision, it only depends on $\mathscr{U}_K$. Moreover, by Sard theorem, it has Lebesgue measure equal to zero. Thus, $R_{S_\mu}(x)$ is constant for $\mu \in [0,1]$.

By Lemma 2.2.3, $R_{S_{\tilde{H}}}(x)$ does not depend on the subdivision of $[s,t]$ used in the construction of $S_{\tilde{H}}$. □

2.3 The min-max critical value in homology

In this section, we introduce a different yet equivalent definition to the min-max critical value, in order to prove some of the results presented in the next chapter. We start by defining some preliminary notions.

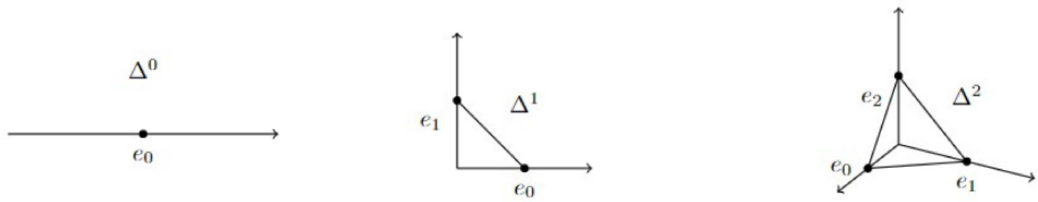
Definition 2.3.1. *Consider in $\mathbb{R}^\infty$ a basis $\{e_0,\ldots,e_n\}$ of $\mathbb{R}^{n+1} \subset \mathbb{R}^\infty$. The n-standard simplex is*

$$\Delta^n := \left\{ t_0e_0 + \ldots t_ne_n \,\middle|\, \sum_{i=0}^{n} t_i = 1 \right\}.$$

The i-th face of Δ^n is

$$F_i := [e_0,\ldots,e_{i-1},e_{i+1},\ldots,e_n] \cong \Delta^{n-1}.$$

Example 2.3.1. The following picture shows some examples of simplices for small n.



Let X be a topological space.

Definition 2.3.2. *A n-singular simplex is a continuous map $\sigma : \Delta^n \to X$.*

Definition 2.3.3. *The n-chain group $C_n(X)$ is the free abelian group generated by the set of n-singular simplices of X, that is*

$$C_n(X) \coloneqq \{\lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k \,|\, k \geq 0,\ \lambda_i \in \mathbb{Z}_2,\ \sigma_i \text{ n-singular simplex}\}.$$

Definition 2.3.4. *The boundary map is defined as*

$$\begin{aligned} \partial_n :\ C_n(X) &\to C_{n-1}(X) \\ \sigma &\mapsto \sigma|_{F_0} + \cdots + \sigma|_{F_n} \end{aligned}$$

for every $n \geq 1$, while $\partial_k = 0$ for every $k \leq 0$.

Proposition 2.3.1. *For every integer n,*

$$\partial_{n-1} \circ \partial_n = 0.$$

Proof. Consider $\sigma \in C_n(X)$,

$$\begin{aligned} \partial_{n-1} \circ \partial_n(\sigma) &= \sum_{0 \leq j < i}^{n} \sigma|_{[e_0,\ldots,e_{j-1},e_{j+1},\ldots,e_{i-1},e_{i+1},\ldots,e_n]} + \sum_{0 \leq i < j}^{n} \sigma|_{[e_0,\ldots,e_{i-1},e_{i+1},\ldots,e_{j-1},e_{j+1},\ldots,e_n]} = \\ &= 0, \end{aligned}$$

since all the coefficients belong to $\mathbb{Z}_2$. □

This fact implies that we can build the following *chain complex*:

$$\ldots \xrightarrow{\partial_n+1} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Moreover, we call

- *n-singular cycle* any $\sigma \in C_n(x)$ such that $\partial_n(\sigma) = 0$;
- *n-singular boundary* any $\sigma \in C_n(x)$ that is the image through ∂_{n+1} of some $\sigma' \in C_{n+1}(x)$.

Definition 2.3.5. *The n-th singular homology of the topological space X with $\mathbb{Z}_2$ coefficients is the following quotient*

$$H_n(X) \coloneqq \frac{ker(\partial_n)}{im(\partial_{n+1})},$$

in other words, it is generated by the n-singular cycles that are not boundaries.
The singular homology of X is

$$H_*(X) \coloneqq \bigoplus_{n \geq 0} H_n(X).$$

Let us consider $Y \subseteq X$ and construct the *relative homology.*

Definition 2.3.6. *Given two topological spaces $Y \subseteq X$, the n-th relative chain group is the following quotient*

$$C_n(X,Y) \coloneqq \frac{C_n(X)}{C_n(Y)}.$$

We can construct the boundary maps

$$\partial_n : C_n(X,Y) \to C_{n-1}(X,Y)$$

for which the equality $\partial_n \circ \partial_{n-1}$ still holds, therefore we can build a chain complex.

Definition 2.3.7. *The n-th relative homology of X, Y with coefficients in $\mathbb{Z}_2$ is the quotient*

$$H_n(X,Y) \coloneqq \frac{ker(\partial_n)}{im(\partial_{n+1})}$$

and the relative homology of X, Y is

$$H_*(X,Y) \coloneqq \bigoplus_{n \geq 0} H_n(X,Y).$$

Of course, being that a manifold is a topological space, we can talk about relative homologies of couples of manifolds, similarly to the De Rham relative cohomologies introduced in Section 1.2.1. Moreover, one can prove for relative homologies properties analogous to those of relative cohomologies.

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a continuous function quadratic at infinity and let Q be the non-degenerate quadratic form that coincides with f outside some compact set of $\mathbb{R}^k$.
Similarly to the relative cohomology case (see the first part of the proof of Theorem 1.2.4), one can prove that

$$H_*(f^\infty, f^{-\infty}) = H_*(Q^\infty, Q^{-\infty}) \cong \mathbb{Z}_2,$$

thus it admits a unique generator 1_f.

Definition 2.3.8. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a continuous function that is quadratic at infinity. We let*

$$c'(1_f, f) \coloneqq \inf\{\lambda \in \mathbb{R} \,|\, 1_f \in im(i_{\lambda *})\},$$

where $i_{\lambda *} : H_*(f^\lambda, f^{-\infty}) \to H_*(f^\infty, f^{-\infty})$ is the homomorphism induced by the inclusion $i_\lambda : (f^\lambda, f^{-\infty}) \hookrightarrow (f^\infty, f^{-\infty})$, that is

$$\begin{aligned} i_{\lambda *} : H_*(f^\lambda, f^{-\infty}) &\to H_*(f^\infty, f^{-\infty}) \\ [\sigma] &\mapsto [\sigma|_{f^\lambda}]. \end{aligned}$$

Lemma 2.3.1. *For every function $f : \mathbb{R}^k \to \mathbb{R}$ continuous and quadratic at infinity,*

$$c'(1_f, f) = \inf_{[\sigma]=1_f} \max_{x \in |\sigma|} f(x),$$

where $|\sigma|$ denotes the imagine of the relative singular cycle σ.
We call the relative singular cycles σ such that $[\sigma] = 1_f$ descending cycles.

Proof. A descending cycle σ defines a homology class in $H_*(f^\lambda, f^{-\infty})$ if and only if $|\sigma| \subset f^\lambda$, in which case one has $\max_{x\in|\sigma|} f(x) \leq \lambda$, hence $c'(f, 1_f) \geq \inf_{[\sigma]=1_f} \max_{x\in|\sigma|} f(x)$; choosing $\lambda = \max_{x\in|\sigma|} f(x)$, we obtain the equality. □

The next result can be found in [33][Section 3.2].

Theorem 2.3.2. *For every function $f : \mathbb{R}^k \to \mathbb{R}$ continuous and quadratic at infinity*

$$c(1_f, f) = c'(1_f, f).$$

Chapter 3

The Lipschitz setting

Abstract. In this chapter we start by introducing Clarke generalized derivatives. Moreover we discuss critical points, quadraticity at infinity and min-max critical values in this Lipschitz setting. These preliminaries allow us to give a notion of min-max solution for the evolutive Hamilton-Jacobi equation for Lipschitz-continuous initial data. This construction has been introduced by Q. Wei in [33] (see also [34], [26]).

3.1 Clarke generalized derivatives

Let us return to Example 1.0.1, where we considered for $d = 1$ the Hamiltonian function $H(t, x, y) = \frac{1}{2}y^2$ and the initial datum $f(x) = \arctan(x)$. We observed that the Lagrangian submanifold

$$\varphi_t(\Gamma(df)) = \left\{ \left(x + \frac{t}{1+x^2}, \frac{1}{1+x^2} \right) \Big| \, x \in \mathbb{R} \right\}$$

is not the graph of a function for $t > 0$ large enough. As a consequence, the generating function for $\varphi_t(\Gamma(df))$,

$$S_t(x; x_0, y_0) = \arctan(x_0) + \frac{t}{2}y_0^2 + (x - x_0)y_0,$$

has an associated min-max function R_{S_t} that is not C^1, though it is locally Lipschitz (as stated and proved for the min-max function in Theorem 1.2.4).
Hence, in order to iterate the min-max procedure, we are interested in defining the min-max function when the initial datum f of the Cauchy problem (H-J) is only a Lipschitz-continuous function. This notion, that was first introduced in [33][Section 1.3], is based on the concept of *Clarke generalized derivative*.

From now on, we refer to [33][Appendix A]. Let $g : \mathbb{R}^k \to \mathbb{R}$ be a locally Lipschitzian function. In order to give the definition of the Clarke generalized derivative of g in $a \in \mathbb{R}^k$, we remind that:

- given a Euclidean space X and $A \subset X$, the *convex hull* of A is the smallest convex set that contains A and we denote it with $co(A)$;
- by Rademacher's theorem, being that g is locally Lipschitz, the set of the differentiable points of g, $dom(dg)$, is dense in $\mathbb{R}^k$.

Definition 3.1.1. *The Clarke generalized derivative $\partial g(a)$ of g in $a \in \mathbb{R}^k$ is the following subset of $T_a^*\mathbb{R}^k$:*

$$\partial g(a) \coloneqq co\{y \in T_a^*\mathbb{R}^k \,|\, (a, y) \in \overline{\Gamma(dg)}\},$$

where $\Gamma(dg) = \{(x, dg(x)) \,|\, x \in dom(dg)\}$.

In other words, $\partial g(a)$ is the smallest convex set that contains the set of limits of the convergent sequences $dg(x_n)$ such that $x_n \xrightarrow{n \to +\infty} a$. It is thus clear that, if g is C^1 in a then $\partial g(a) = \{dg(a)\}$ and viceversa.
Note that, for x close to a, $|dg(x)|$ is bounded by the local Lipschitz constant of g, therefore every sequence $dg(x_n)$ with $x_n \to a$ is bounded and thus has a convergent subsequence. This implies that, for every locally Lipschitzian function g and for every $a \in \mathbb{R}^k$, $\partial g(a) \neq \emptyset$.
Moreover, $\partial g(a)$ is the convex hull of a compact subset of a finite dimensional space, thus it is compact.

We may also define a generalization of a graph for the set-valued function ∂g, the set

$$\Gamma(\partial g) \coloneqq \{(x, y) \,|\, x \in \mathbb{R}^k,\ y \in \partial g(x)\}.$$

In simple one-dimensional cases, it can be obtained from the graph of dg by adding a vertical segment where g is not differentiable. For example, if we consider $g(x) = |x|$, $\Gamma(\partial g)$ would be as pictured in Figure 3.1.

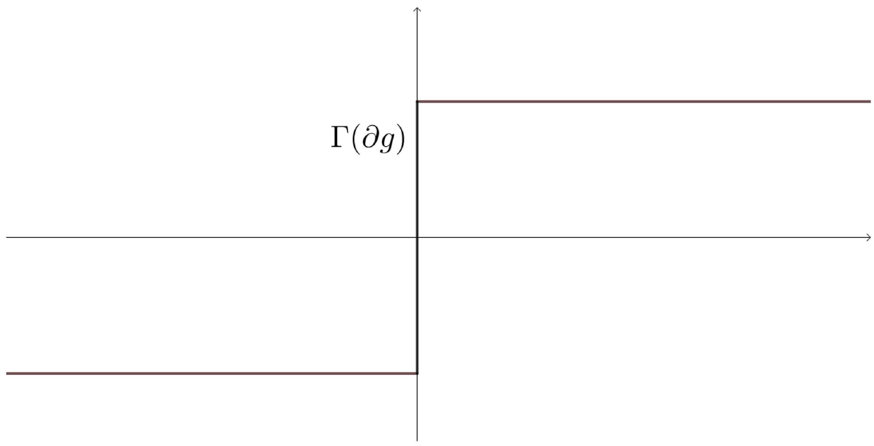


Figure 3.1

For the relation with partial derivatives, if $g : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$ is Lipschitz, in general it is not true that

$$\partial g(x, y) = (\partial_x g(x, y), \partial_y g(x, y)) \coloneqq \partial_x g(x, y) \times \partial_y g(x, y). \tag{3.1}$$

Let us see an example.

Example 3.1.1 (see [33][Example A.5])**.** Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g(x, y) = |x - y|$. We have that $\partial_1 g(0, 0) \times \partial_2 g(0, 0) = [-1, 1] \times [-1, 1]$, but

$$\begin{aligned}\partial g(0, 0) &= co\{\lim_n dg(x_n, y_n) \,|\, (x_n, y_n) \to (0, 0)\}\\ &= co\{\lim_n d_1 g(x_n, y_n) \times d_2 g(x_n, y_n) \,|\, (x_n, y_n) \to (0, 0)\}\\ &= co\{(1, -1), (-1, 1)\},\end{aligned}$$

which is the straight line segment in $\mathbb{R}^2$ that joins the points $(1, -1)$ and $(-1, 1)$.

However, in the special case of the generating function $S(x, \eta)$ defined in Corollary 2.1.2.1, equality (3.1) holds.

Example 3.1.2 (see [33][Example A.6])**.** Using the notation $\eta = ((x_i, y_i))_{0 \le i < N}$, consider the generating function $S : \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \to \mathbb{R}$ defined in Corollary 2.1.2.1 (either in (2.4) or (2.5)) with initial function f Lipschitz: it can be written as

$$S(x, \eta) = f(x_0) + g(x, \eta).$$

Note that f only depends on x_0, thus $\partial_x f(x_0) = \{0_{\mathbb{R}^d}\}$, and that, since g is C^2,

$$\partial g(x, \eta) = (\partial_x g(x, \eta), \partial_\eta g(x, \eta)) = dg(x, \eta).$$

Then we have that

$$\begin{aligned}\partial S(x, \eta) = \partial f(x_0) + \partial g(x, \eta) &= (0_{\mathbb{R}^d}, \partial_\eta f(x_0)) + \partial g(x, \eta) =\\ &= (\partial_x g(x, \eta), \partial_\eta S(x, \eta)) = (\partial_x S(x, \eta), \partial_\eta S(x, \eta)).\end{aligned}$$

Proposition 3.1.1. *The set-valued function $x \mapsto \partial g(x)$ is upper semi-continuous, that is, for every convergent sequence $(x_n, y_n) \to (x, y)$ with $y_n \in \partial g(x_n)$, one has $y \in \partial g(x)$.*

Proof. For every $n \in \mathbb{N}$, by convexity of $\partial g(x_n)$, we can write

$$y_n = t_{n,1} v_{n,1} + \cdots + t_{n,k+1}, v_{n,k+1},$$

where, for $i = 1, \ldots, k+1$, $t_{n,i} \in [0, 1]$, $\sum_i t_{n,i} = 1$ and $v_{n,i} \in \partial f(x_n)$.
Since g locally Lipschitz, there exists a compact subset $K \subset T^*\mathbb{R}^k$ that contains every $v_{n,i}$ for n large enough. Therefore, extracting subsequences, we can assume that the sequences $(v_{n,i})_n$ and $(t_{n,i})_n$ converge respectively to $v_i \in K$ and $t_i \in [0, 1]$.

Being that $(x_n, v_{n,i}) \in \overline{\Gamma(dg)}$ for every n, there are points of $\Gamma(dg)$ arbitrarily close to $(x_n, v_{n,i})$ and thus we can assume $(x_n, v_{n,i}) \in \Gamma(dg)$, that is, $x_n \in dom(dg)$ and $v_{n,i} = dg(x_n)$. Hence, for every $i = 1, \ldots, k+1$, $v_i = \lim_n v_{n,i} \in \partial g(x)$ and

$$y = \lim_n y_n = t_1 v_1 + \ldots t_{k+1} v_{k+1} \in \partial g(x).$$

□

As a consequence, the set $\Gamma(\partial g)$ is a closed subset of $T^*\mathbb{R}^k$.

We conclude this brief introduction to Clarke generalized derivatives by stating a couple of properties.

Lemma 3.1.1 (Chain rule)**.** *If $g : \mathbb{R}^k \to \mathbb{R}$ is a Lipschitz function, $G : \mathbb{R}^k \to \mathbb{R}^k$ a C^1 diffeomorphism, then*

$$\partial(g \circ G)(x) = \partial g(G(x)) \circ dG(x) := \{dG(x)(\xi) \mid \xi \in \partial g(G(x))\}.$$

Proof. Let $h(x) = g \circ G(x)$, then $h : \mathbb{R}^k \to \mathbb{R}$ is Lipschitz, hence differentiable almost everywhere. We start by showing that, if h is differentiable at x, then $dh(x) = dG(x) dg(G(x))$. For any $v \in T^*\mathbb{R}^k$ we have

$$\begin{aligned} dh(x)(v) &= \lim_{t \to 0} \frac{g(G(x+tv)) - g(G(x))}{t} \\ &= \lim_{t \to 0} \frac{g(G(x) + t dG(x)(v)) - g(G(x))}{t} \end{aligned}$$

and, being that $dG(x)$ is a bijective linear map, we have that g is differentiable at $G(x)$ by definition and thus $dh(x) = dG(x) dg(G(x))$.

$$\begin{aligned} \partial(g \circ G)(x) &= co\{\lim_n d(g \circ G)(x_n) \mid x_n \to x\} \\ &= co\{\lim_n dG(x_n)(dg(G(x_n))) \mid x_n \to x\} \\ &= co\{\lim_n dG(x)(dg(G(x_n))) \mid x_n \to x\} \\ &\subset co\{\lim_n dG(x)(dg(y_n)) \mid y_n \to G(x)\} \\ &= \partial g(G(x)) \circ dG(x), \end{aligned}$$

where the inclusion becomes an equality since G is surjective. □

Lemma 3.1.2. *If $g, h : \mathbb{R}^k \to \mathbb{R}$ are Lipschitz functions, then*

$$\partial(gh)(x) \subset g(x)\partial h(x) + h(x)\partial g(x).$$

Proof. By definition,

$$\begin{aligned}\partial(gh)(x) &= co\{\lim_n d(gh)(x_n) \mid x_n \to x\} \\ &= co\{\lim_n (g(x_n)dh(x_n) + h(x_n)dg(x_n)) \mid x_n \to x\} \\ &= co\{g(x)\lim_n dh(x_n) + h(x)\lim_n dg(x_n) \mid x_n \to x\} \\ &\subset g(x)\partial h(x) + h(x)\partial g(x).\end{aligned}$$

□

3.2 Lipschitz critical point theory

Definition 3.2.1. *A point $x \in \mathbb{R}^k$ is a critical point of g if $0 \in \partial g(x)$. In such a case, $g(x)$ is called critical value of g. We denote with $Crit(g)$ the critical set of g, that is, the set of critical points of g.*

We observe that, by Proposition 3.1.1, $Crit(g) = \{x \in \mathbb{R}^k \mid 0 \in \partial g(x)\}$ is closed in $\mathbb{R}^k$.

We now give an analogous definition of the Palais-Smale condition for locally Lipschitz functions.
Let us consider the following function for $x \in \mathbb{R}^k$

$$\lambda(x) := \min_{w \in \partial g(x)} \|w\|_{T_x^*\mathbb{R}^k}.$$

Definition 3.2.2. *A locally Lipschitz function g satisfies the Palais-Smale condition if every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ such that*

1. $\lambda(x_n) \to 0$
2. $\{g(x_n)\}_{n\in\mathbb{N}}$ *is bounded*

admits a convergent subsequence.

Proposition 3.2.1. *Let g be a locally Lipschitz function,*

$$Crit(g) = \{x \in \mathbb{R}^k \mid \exists\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k \text{ s.t. } x_n \to x \text{ and } \lambda(x_n) \to 0\}.$$

Proof.

"$\subseteq$": Let $x \in Crit(g)$, then $\lambda(x) = 0$ and thus we can simply consider the sequence $x_n \equiv x$ to conclude that x is also contained in the set on the right hand side.

"$\supseteq$": Let $x \in \mathbb{R}^k$ be an element of the set on the right hand side and consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that, for every n, $y_n \in \partial g(x_n)$ and $\|y_n\|_{T_x^*\mathbb{R}^k} = \lambda(x_n)$. Then, $(x_n, y_n) \to (x, 0)$ and, by Proposition 3.1.1, $0 \in \partial g(x)$. Thus, $x = \lim_n x_n \in Crit(g)$.

□

In particular, if g satisfies the Palais-Smale condition, $\{x_n\}_{n\in\mathbb{N}}$ is a sequence such that $\lambda(x_n) \to 0$ and $\{g(x_n)\}_{n\in\mathbb{N}}$ is bounded and $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a convergent subsequence, then $\lim_k x_{n_k} \in Crit(g)$.

Proposition 3.2.2. *Let g be a locally Lipschitz function. If there exists a non-degenerate quadratic form Q on $\mathbb{R}^k$ such that $Lip(g - Q) < \infty$, then g satisfies the Palais-Smale condition. Moreover, in that case, $Crit(g)$ is compact.*

Proof. Let $\psi \coloneqq g - Q$, which is a Lipschitz function. For every $x \in \mathbb{R}^k$, $\partial g(x) = \partial\psi(x) + dQ(x)$ contains vectors whose norm is at least $|dQ(x)| - Lip(\psi)$. Therefore,

$$\lambda(x) = \min_{w\in\partial g(x)} \|w\|_{T^*\mathbb{R}^k} \geq |dQ(x)| - Lip(\psi)$$

and, if $|x_n| \to \infty$, $\lambda(x_n) \to \infty$.
As a consequence, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ with $\lambda(x_n) \to 0$, there necessarily exists $R > 0$ such that $|x_n| \leq R$. This means that $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence and thus g satisfies the Palais-Smale condition.

Consequently, the limits of each sequence $\{x_n\}_{n\in\mathbb{N}}$ with $\lambda(x_n) \to 0$ are contained in the closed ball $B(0_{\mathbb{R}^k}, R]$. As proved in Proposition 3.2.1, the set $\{x \in \mathbb{R}^k \mid \exists\{x_n\}_{n\in\mathbb{N}} : x_n \to x, \lambda(x_n) \to 0\}$ coincides with $Crit(g)$, thus $Crit(g)$ is bounded. Since $Crit(g)$ is also closed, this implies that $Crit(g)$ is compact. □

We now state two versions of the *Deformation Lemma* in the locally Lipschitz setting.

Theorem 3.2.1 (Deformation Lemma I)**.** *Let g be a locally Lipschitz function that satisfies the Palais-Smale condition. If $c \in \mathbb{R}$ is not a critical value of g, then there exists $\varepsilon > 0$ and a bounded smooth vector field V on $\mathbb{R}^k$ equal to 0 off $g^{c+2\varepsilon} \setminus g^{c-2\varepsilon}$, whose flow φ_t^V satisfies $\varphi_1^V(g^{c+\varepsilon}) \subset g^{c-\varepsilon}$.*

Theorem 3.2.2 (Deformation Lemma II)**.** *Let g be a locally Lipschitz function that satisfies the Palais-Smale condition. If $c \in \mathbb{R}$ is a critical value of g and N is a neighbourhood of $K_c \coloneqq Crit(g) \cap g^{-1}(c)$, then there exists $\varepsilon > 0$ and a bounded smooth vector field V on $\mathbb{R}^k$ equal to 0 off $g^{c+2\varepsilon} \setminus g^{c-2\varepsilon}$, whose flow φ_t^V satisfies $\varphi_1^V(g^{c+\varepsilon} \setminus N) \subset g^{c-\varepsilon}$.*

3.3 GFQI and min-max in the Lipschitz setting

We shall now work under the hypothesis (2.1):

$$c_H \coloneqq \sup_{\substack{\tau\in[0,T] \\ (x,y)\in T^*\mathbb{R}^d}} |D^2 H_\tau(x, y)| < \infty$$

and using the notation of Corollary 2.1.2.1:

$$\eta = ((x_i, y_i))_{0\le i<N}, \quad x_N = x \quad \text{and} \quad 0 \le s = t_0 < t_1 < \cdots < t_N = t \le T.$$

Proposition 3.3.1. *Let $H : \mathbb{R} \times T^*\mathbb{R}^d$ be a C^2 Hamiltonian satisfying* (2.1) *and $f : \mathbb{R}^d \to \mathbb{R}$ a Lipschitz function. The function S given by*

$$S(x,\eta) = f(x_0) + \sum_{0\le i<N} \phi_{t_i}^{t_{i+1}}(x_{i+1}, y_i) + \sum_{0\le i<N} (x_{i+1} - x_i) y_i,$$

generates $L = \varphi_s^t(\Gamma(\partial f))$ in the sense that

$$L = \{(x, \partial_x S(x,\eta)) \,|\, 0 \in \partial_\eta S(x,\eta)\}. \tag{3.2}$$

Proof. Note that, $S(x,\eta)$ only differs from the function constructed in Corollary 2.1.2.1 because f is now only locally Lipschitz. As f does not depend on x and ϕ is a differentiable function, $\partial_x S(x,\eta)$ is a derivative in the strong sense and, to conclude the proof, it is sufficient to investigate the meaning of $0 \in \partial_\eta S(x,\eta)$.
In particular, f only depends on x_0, thus $\partial_{y_0} S(x,\eta)$ and $\partial_{(x_i,y_i)} S(x,\eta)$ for $i = 1,\dots,N-1$ as Clarke derivatives are singletons that contain the respective strong partial derivative and the conditions $0 \in \partial_{y_0} S(x,\eta)$, $0 \in \partial_{(x_i,y_i)} S(x,\eta)$ imply that

$$\begin{cases} y_{i+1} = y_i + \partial_{x_{i+1}} \phi_{t_i}^{t_{i+1}}(x_{i+1}, y_i) \\ x_i = x_{i+1} + \partial_{y_i} \phi_{t_i}^{t_{i+1}}(x_{i+1}, y_i) \end{cases}$$

for $0 \le i < N$, that is, as $\phi_{t_i}^{t_{i+1}}$ is a generating function for $\varphi_{t_i}^{t_{i+1}}$, $((x_i, y_i), (x_{i+1}, y_{i+1})) \in \Gamma(\varphi_{t_i}^{t_{i+1}})$.
Finally, $0 \in \partial_{x_0} S(x,\eta)$ implies that $0 \in \partial f(x_0) - y_0$, thus $y_0 \in \partial f(x_0)$ and $(x_0, y_0) \in \Gamma(\partial f)$. □

However, this definition of a generating family is not invariant by fiberwise diffeomorphisms, as can be seen in the following example.

Example 3.3.1 (see [34][p.24]). Consider the following diffeomorphism:

$$(x; (x_i, y_i)_{0\le i<N}) \mapsto (x; (x_{i+1} - x_i, y_i)_{0\le i<N}) =: (x; (\xi_i, y_i)_{0\le i<N}).$$

It transforms the family S given in (2.4) into

$$S'(x; (\xi_i, y_i)_{0\le i<N}) := f\Big(x - \sum_{0\le i<N} \xi_i\Big) + \sum_{0\le i<N} \phi_{t_i}^{t_{i+1}}\Big(x - \sum_{i<j<N} \xi_j, y_i\Big) + \sum_{0\le i<N} \xi_i y_i,$$

for which $\partial_x S'(x; (\xi_i, y_i)_{0\le i<N})$ is not a point, but the subset

$$\partial f\Big(x - \sum_{0\le i<N} \xi_i\Big) + \sum_{0\le i<N} \partial_1 \phi_{t_i}^{t_{i+1}}\Big(x - \sum_{i<j<N} \xi_j, y_i\Big).$$

We can overcome this inconvenience by giving an adapted definition.

Definition 3.3.1. *A Lipschitz function $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is a generating function for a Lagrangian submanifold $L \subset T^*\mathbb{R}^d$ if*

$$L = \{(x, y) \in T^*\mathbb{R}^d \,|\, \exists \eta \in \mathbb{R}^k \ s.t.\ (y, 0) \in \partial S(x, \eta)\}.$$

Lemma 3.3.1. *The Definition 3.3.1 of a Lipschitz-continuous generating function for a Lagrangian submanifold is invariant by the three operations:*

1. *fiberwise C^1 diffeomorphisms;*
2. *sum of a constant;*
3. *stabilization.*

Proof. Invariance by sum of a constant and stabilization is clear, let us prove the invariance by fiberwise diffeomorphisms.

Let S be a Lipschitz-continuous generating function for a Lagrangian submanifold. If $\Psi(x, \eta') = (x, \psi(x, \eta'))$ is a fiberwise diffeomorphism of $\mathbb{R}^d \times \mathbb{R}^k$ and $S' := S \circ \Psi$, then the chain rule stated in Lemma 3.1.1 implies that

$$\begin{aligned}\partial S'(x, \eta') &= \{d\Psi(x, \eta')(\xi) \,|\, \xi \in \partial S(\Psi(x, \eta'))\} \\ &= \left\{ \left(y + \xi' \frac{\partial}{\partial x}\psi(x, \eta'), \xi' \frac{\partial}{\partial \eta'}\psi(x, \eta') \right) \,\middle|\, (y, \xi') \in \partial S(x, \psi(x, \eta')) \right\}.\end{aligned}$$

Being that $\eta' \mapsto \psi(x, \eta')$ is a diffeomorphism, the conditions

$$\exists \eta \in \mathbb{R}^k \, s.t.\, (y, 0) \in \partial S(x, \eta) \quad \text{and} \quad \exists \eta' \in \mathbb{R}^k \, s.t.\, (y, 0) \in \partial S'(x, \eta')$$

are equivalent. □

Note that S as in Proposition 3.3.1 satisfies Definition 3.3.1. In fact, by Example 3.1.2,

$$\partial S(x, \eta) = (\partial_x S(x, \eta), \partial_\eta S(x, \eta)) = (\partial_x g(x, \eta), \partial_\eta S(x, \eta))$$

where $g(x, \eta) = S(x, \eta) - f(x_0)$ is differentiable, therefore $\partial_x g(x, \eta)$ is a partial derivative in the strong sense. Hence, by Proposition 3.3.1,

$$\{(x, y) \in T^*\mathbb{R}^d \,|\, \exists \eta \in T^*\mathbb{R}^d \ s.t.\ (y, 0) \in \partial S(x, \eta)\} = \{(x, \partial_x S(x, \eta)) \,|\, 0 \in \partial_\eta S(x, \eta)\} = L.$$

We now prove that -when the Hamiltonian is globally compactly supported- the Lagrangian submanifold $L = \varphi_s^t(\Gamma(\partial f))$ considered above admits a generating function which results quadratic at infinity.

Proposition 3.3.2. *Let $H : \mathbb{R} \times T^*\mathbb{R}^d \to \mathbb{R}$ be a C^2, globally compactly supported Hamiltonian and $f : \mathbb{R}^d \to \mathbb{R}$ a Lipschitz function. Consider the generating function of $L = \varphi_s^t(\Gamma(\partial f))$ given by*

$$S(x, \eta) = f(x_0) + \sum_{0 \le i < N} \phi_{t_i}^{t_{i+1}}(x_{i+1}, y_i) + \sum_{0 \le i < N} (x_{i+1} - x_i) y_i$$

and the quadratic form

$$Q(\eta) = -x_{N-1} y_{N-1} + \sum_{0 \le i < N-1} (x_{i+1} - x_i) y_i = \frac{1}{2} \eta^T B \eta.$$

It holds that:

1. *$S(x, \eta)$ is quadratic at infinity in the Lipschitz sense, that is, the Lipschitz constant of $S(x, \cdot) - Q(\cdot)$ is uniformly bounded with respect to x on each compact subset of $\mathbb{R}^d$.*

2. *For every compact $K \subset \mathbb{R}^d$, let $\theta \in C_c^\infty(\mathbb{R}^d, [0, 1])$ be a function equal to 1 in a neighbourhood of 0. Then, there exists a constant $a_K > 0$ such that*

$$S_K(x, \eta) \coloneqq \theta(\eta / a_K)(S(x, \eta) - Q(\eta)) + Q(\eta) = \psi_K(x, \eta) + Q(\eta) \qquad (3.3)$$

is a GFQI of $L_K = L \cap \pi^{-1}(K)$.

Proof. Denote $\psi(x, \eta) = S(x, \eta) - Q(\eta)$. Fix a compact $K \subset \mathbb{R}^d$, let $c = \max_{x \in K} Lip(\psi(x, \cdot))$ and assume that $|D\theta| \le 1$.
Then $S_K(x, \eta) = \theta(\eta / a_K)\psi(x, \eta) + Q(\eta)$ and by Lemma 3.1.2,

$$\begin{aligned} \partial_\eta S_K(x, \eta) &= \partial_\eta \left(\theta \left(\frac{\eta}{a_K} \right) \psi(x, \eta) \right) + DQ(\eta) \\ &\subset \frac{1}{a_K} D\theta \left(\frac{\eta}{a_K} \right) \psi(x, \eta) + \theta \left(\frac{\eta}{a_K} \right) \partial_\eta \psi(x, \eta) + DQ(\eta). \end{aligned}$$

Since ψ is continuous and K is compact, there exists $b \coloneqq \max_{x \in K} |\psi(x, 0)|$ and we have that

$$|\psi(x, \eta)| \le |\psi(x, 0)| + |\psi(x, \eta) - \psi(x, 0)| \le b + c|\eta|.$$

Therefore,

$$\begin{aligned} \max_{x \in K} \left| \frac{1}{a_K} D\theta \left(\frac{\eta}{a_K} \right) \psi(x, \eta) + \theta \left(\frac{\eta}{a_K} \right) \partial_\eta \psi(x, \eta) \right| &\le \frac{1}{a_K}(b + c|\eta|) + c \\ &\le \frac{1}{2} |B^{-1}|^{-1} |\eta| < |DQ(\eta)| \end{aligned} \qquad (3.4)$$

for $|\eta| \ge d_K$ for some d_K, with a_K and d_K large enough.

The inequality (3.4) implies that the zero vector does not belong to $\partial_\eta S_K(x,\eta)$ if $|\eta| \geq d_K$, thus for every $x \in K$ all the critical points of $S_K(x,\cdot)$ are contained in $\{\eta \in \mathbb{R}^k \,|\, |\eta| \leq d_K\}$.
We can also choose a_K and d_K such that, for $|\eta| \leq d_K$, $\theta(\eta/a_K) = 1$. Therefore, $S_K = S|_{x\in K}$ for $|\eta| \leq d_K$. Since S generates L by Proposition 3.3.1, this implies that $L_K = L \cap \pi^{-1}(K) = \{(x, \partial_x S_K(x,\eta)) \,|\, 0 \in \partial_\eta S_K(x,\eta)\}$. □

In the following, unless otherwise specified, we consider generating functions S of the form (2.4) and S_K of the form (3.3).

Note that if H is compactly supported, by Lemma 2.2.1, a generating function of the form (2.4) with f C^2 and Lipschitz-continuous is quadratic at infinity. Hence, up to a fiberwise diffeomorphism ζ, it is exactly quadratic at infinity, that is, as in Definition 1.2.6, there exists a constant $C > 0$ such that for $|x| > C$ the generating function coincides with the quadratic form Q.
Let us now consider the generating function S as in the previous proposition, with f only Lipschitz-continuous. Since S is given by (2.4), $S' := S\circ\zeta$ is exactly quadratic at infinity and, by Lemma 3.3.1, S and S' generate the same Lagrangian submanifold. Therefore, S is exactly quadratic at infinity up to a fiberwise diffeomorphism and, as seen in the proof of Theorem 1.2.4, we can define the associated min-max function.

Definition 3.3.2. *Let $S(x,\eta)$ be a Lipschitz generating function of the form* (2.4). *The associated min-max function is*

$$R_S(x) := c(1_x, S_x) = \inf_{[\sigma]=1_x} \max_{\eta\in|\sigma|} S(x,\eta)$$

Proposition 3.3.3. *The min-max $R_S(x)$ is a critical value (in the sense of Definition 3.2.1) of $S(x,\cdot)$. For each compact $K \subset \mathbb{R}^d$ and each S_K of the form* (3.3) *generating L_K, we have that $R_S(x) = R_{S_K}(x)$ for $x \in K$.*

Proof. If $c = R_S(x)$ was not a critical value of S_x, by Theorem 3.2.1 there would exist a flow φ_t^V such that $\varphi_t^V(S_x^{c+\varepsilon}) \subset S_x^{c-\varepsilon}$, therefore φ_t^V would deform the descending cycles in $S_x^{c+\varepsilon}$ into descending cycles in $S_x^{c-\varepsilon}$. We would thus have a contradiction:

$$c = \inf_{[\sigma]=1_x} \max_{\eta\in|\sigma|} S(x,\eta) \leq c - \varepsilon.$$

To prove that $R_S|_K = R_{S_K}$, we observe that every descending cycle σ of $S(x,\cdot)$ or $S_K(x,\cdot)$ with $x \in K$ can be deformed into a descending cycle σ' such that

$$\max_{\eta\in|\sigma'|} S(x,\eta) = \max_{\eta\in|\sigma'|} S_K(x,\eta)$$

by using the gradient flow of Q suitably truncated. □

Proposition 3.3.4. *The min-max $R_S(x)$ is a locally Lipschitz function.*

Proof. Let $K \subset \mathbb{R}^d$ be compact. Being S locally Lipschitz, it is Lipschitz on K. The proof of the Lipschitz-continuity of $R_S(x)$ on K is analogous to the proof of the Lipschitz-continuity of u_Λ in Theorem 1.2.4. □

Proposition 3.3.5. *The set $C(x) := \{\eta \in \mathbb{R}^k \,|\, 0 \in \partial_\eta S(x,\eta),\ S(x,\eta) = R_S(x)\}$ is compact and the set-valued map $x \mapsto C(x)$ is upper semi-continuous. In other words, $\Gamma(C) = \{(x,\eta) \,|\, \eta \in C(x)\}$ is closed.*

Proof. By Proposition 3.2.2, $Crit(S_x)$ is compact for every $x \in \mathbb{R}^d$. Therefore, $C(x) = Crit(S_x) \cap \{\eta \in \mathbb{R}^k \,|\, S_x(\eta) = R_S(x)\}$ is compact. In order to prove the upper semi-continuity, we must show that, given a convergent sequence $(x_k,\eta_k) \to (x,\eta)$ with $\eta_k \in C(x_k)$, we have $\eta \in C(x)$.
As seen in Example 3.1.2, $\partial S = (\partial_x S, \partial_\eta S)$ and, by Proposition 3.1.1, the set-valued function $\partial S : (x,\eta) \mapsto (\partial_x S(x,\eta), \partial_\eta S(x,\eta))$ is upper semi-continuous. Hence, the sequence $(\partial_x S(x_k,\eta_k), 0) \in \partial S(x_k,\eta_k)$ is convergent since S is C^1 with respect to x and its limit $(\partial_x S(x,\eta), 0)$ belongs to $\partial S(x,\eta)$. Thus, $0 \in \partial_\eta S(x,\eta)$. By the continuity of S and R_S, $S(x_k,\eta_k) \to S(x,\eta)$ and $R_S(x_k) \to R_S(x)$ and therefore $\eta \in C(x)$. □

Lemma 3.3.2. *For any $\delta > 0$ there exists $\varepsilon > 0$ such that*

$$R_S(x) = \inf_{\sigma \in \Sigma_\varepsilon} \max\{S(x,\eta) \,|\, \eta \in |\sigma| \cap C_\delta(x)\},$$

where $\Sigma_\varepsilon = \{\sigma \in H_*(S_x^\infty, S_x^{-\infty}) \,|\, \max_{\eta \in |\sigma|} S(x,\eta) \leq R_S(x) + \varepsilon\}$ and $C_\delta(x) = B(C(x),\delta)$ denotes the open ball of radius δ around $C(x)$.

Proof. This is a direct consequence of the Deformation Lemma (Theorem 3.2.2) for $g = S_x$ and $c = R_S(x)$: for any $\delta > 0$ there exists $\varepsilon > 0$ and a bounded smooth vector field V whose flow φ_t^V is such that $\varphi_1^V(S_x^{c+\varepsilon} \setminus C_\delta(x)) \subset S_x^{c-\varepsilon}$.
In particular, for $\sigma \in \Sigma_\varepsilon$, the intersection $|\sigma| \cap C_\delta(x)$ is not empty, otherwise φ_1^V may map σ to a descending cycle σ' such that $\max_{\eta \in |\sigma'|} S(x,\eta) \leq R_S(x) - \varepsilon$, which contradicts the definition of min-max. □

Proposition 3.3.6. *The Clarke generalized derivative of R_S satisfies*

$$\partial R_S(x) \subset co\{\partial_x S(x,\eta) \,|\, \eta \in C(x)\}. \tag{3.5}$$

Proof. Since R_S is locally Lipschitz, it is differentiable almost everywhere by Rademacher theorem. Let $\bar{x} \in \mathbb{R}^d$ be a point where R_S is differentiable. Let us prove that

$$dR_S(\bar{x}) \subset co\{\partial_x S(\bar{x},\eta) \,|\, \eta \in C(\bar{x})\}. \tag{3.6}$$

For $\bar{x}$, given $\delta > 0$, let $\varepsilon > 0$ be as in Lemma 3.3.2. Consider the compact set $K = B(\bar{x},1]$: $R_S|_K = R_{S_K}$ is Lipschitz-continuous. Since S is Lipschitz-continuous, one can chose $\rho \in (0,1)$ such that for $x \in B(\bar{x},\rho)$

$$|S(x,\cdot) - S(\bar{x},\cdot)|_{C^0} \leq \frac{\varepsilon}{4}.$$

Let $y \in \mathbb{R}^d$ and $\lambda > 0$ such that $\lambda^2 < \varepsilon/4$ and $x_\lambda := \bar{x} + \lambda y \in B(\bar{x}, \rho)$. By Lemma 3.3.2, there exists a descending cycle σ_λ such that

$$\max_{\eta \in |\sigma_\lambda|} S(x_\lambda, \eta) \leq R_S(x_\lambda) + \lambda^2.$$

Therefore,

$$\max_{\eta \in |\sigma_\lambda|} S(\bar{x}, \eta) \leq \max_{\eta \in |\sigma_\lambda|} S(x_\lambda, \eta) + \frac{\varepsilon}{4} \leq R_S(x_\lambda) + \frac{\varepsilon}{2} \leq R_S(\bar{x}) + \frac{3\varepsilon}{4}.$$

By Lemma 3.3.2,

$$R_S(\bar{x}) \leq \max\{S(\bar{x}, \eta) \,|\, \eta \in |\sigma_\lambda| \cap C_\delta(\bar{x})\} = S(\bar{x}, \eta_\lambda) \quad \exists \eta_\lambda \in |\sigma_\lambda| \cap C_\delta(\bar{x}).$$

Hence, since we also have that $S(x_\lambda, \eta_\lambda) \leq R_S(x_\lambda) + \lambda^2$,

$$\begin{aligned} \frac{R_S(\bar{x}) - R_S(x_\lambda)}{\lambda} &\leq \frac{S(\bar{x}, \eta_\lambda) - S(x_\lambda, \eta_\lambda)}{\lambda} + \lambda \\ &= \langle \partial_x S(x'_\lambda, \eta_\lambda), y \rangle + \lambda, \end{aligned}$$

where the equality is given by the mean value theorem for some x'_λ belonging to the line segment between $\bar{x}$ and x_λ. Taking the limsup for $\lambda \to 0$ and letting $\delta \to 0$, we get

$$\langle dR_S(\bar{x}), y \rangle \leq \max_{\eta \in C(\bar{x})} \langle \partial_x S(\bar{x}, \eta), y \rangle \quad \forall y \in \mathbb{R}^d.$$

This implies that $dR_S(\bar{x})$ belongs to the subderivative of the convex function $f(y) := \max_{\eta \in C(\bar{x})} \langle \partial_x S(\bar{x}, \eta), y \rangle$ at $y = 0$, that is,

$$\begin{aligned} dR_S(\bar{x}) \in \partial f(0) &= \{\xi \in \mathbb{R}^d \,|\, f(y) - f(0) \geq \langle \xi, y - 0 \rangle \ \forall y \in \mathbb{R}^d\} \\ &= \{\xi \in \mathbb{R}^d \,|\, \max_{\eta \in C(\bar{x})} \langle \partial_x S(\bar{x}, \eta), y \rangle \geq \langle \xi, y \rangle \ \forall y \in \mathbb{R}^d\} \end{aligned}$$

Through simple computations, one can find that

$$\partial f(0) = co\{\partial_x S(\bar{x}, \eta) \,|\, \eta \in C(\bar{x})\},$$

thus we get (3.6).

In general, for any $x \in \mathbb{R}^d$, by the definition of Clarke generalized derivative we have that

$$\begin{aligned} \partial R_S(x) &= co\{\lim_{n \to \infty} dR_S(x_n) \,|\, x_n \to x, \ R_S \text{ is differentiable in } x_n\} \\ &\subset co\{\lim_{n \to \infty} \{\partial_x S(x_n, \eta_n) \,|\, \eta_n \in C(x_n), \ x_n \to x, \ R_S \text{ is differentiable in } x_n\}\} \\ &\subset co\{\partial_x S(x, \eta) \,|\, \eta \in C(x)\}, \end{aligned}$$

where we use the upper semi-continuity of $x \mapsto C(x)$ and the continuity of $\partial_x S$. □

Note that the formula (3.5) can be interpreted as a generalized graph selector, whereas we remind that the classical notion requires that for almost every $x \in \mathbb{R}^d$

$$dR_S(x) = \partial_x S(x, \eta) \quad \exists \eta \in C(x).$$

3.4 Variational solutions in the Lipschitz setting

Definition 3.4.1. *Let $0 \le s < t \le T$ and let $S : [s,t] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ be given by*

$$S(\tau, x, \eta) = f(x_0) + \sum_{0 \le i < N} \phi_{\tau_i}^{\tau_{i+1}}(x_{i+1}, y_i) + \sum_{0 \le i < N} (x_{i+1} - x_i) y_i$$

where $\tau_j = s + (\tau - s)\frac{t_j - s}{t - s}$. For any $\tau \in [s,t]$ we define the min-max operator $R_H^{s,\tau}$ for the Cauchy problem (H-J) *as*

$$R_H^{s,\tau} f(x) = c(1_x, S(\tau, x, \cdot)).$$

Remark 3.4.1. Let $s = 0$. If $f \in C^2(\mathbb{R}^d) \cap C^{Lip}(\mathbb{R}^d)$, since for every $\tau \in [0,t]$ S_τ is a GFQI for the Lagrangian wave front $\varphi_0^\tau(\Gamma(df))$ in the classical sense, then, by Theorem 1.2.4, the min-max $R_H^{0,t} f$ satisfies the Cauchy problem (H-J) almost everywhere in $\mathbb{R}^d$.

Lemma 3.4.1. *If f is C^2 with bounded second derivative, then there exists an $\varepsilon > 0$ such that for $t \in [0, \varepsilon)$ the min-max $R_H^{0,t} f$ is C^2.*

Proof. The map $g_t : x_0 \mapsto X_0^t(x_0, df(x_0))$ is a diffeomorphism and, for t small enough,

$$Lip(g_t - id) \le Lip(\alpha_0^t - id)(1 + Lip(df)) < 1,$$

where $\alpha_0^t : (x,y) \mapsto (X_0^t(x,y), y)$ is the diffeomorphism defined in Lemma 2.1.1. This implies that the projection map

$$\begin{aligned} \varphi_0^t(\Gamma(df)) &\to \mathbb{R}^d \\ (x,p) &\mapsto x \end{aligned}$$

is a diffeomorphism. Hence for t small enough the characteristics beginning from the graph of df do not intersect and $\varphi_0^t(\Gamma(df)) = \{(x, dR_H^{0,t} f(x)) \,|\, x \in \mathbb{R}^d\}$, which implies that $R_H^{0,t} f$ is C^2. □

Chapter 4

Iterated min-max and viscosity solutions

Abstract. By using the Lipschitz setting of the previous chapter, essentially based on Clarke generalized derivatives, here we introduce iterated min-max solutions for the evolutive Hamilton-Jacobi equation. Moreover, we prove in detail a theorem by Q. Wei: the limiting procedure of iterated min-max, that is for time intervals tending to zero, uniformly converges to the unique viscosity solution.

4.1 Iterated min-max as introduced by Q. Wei

In the following, $|\cdot|_K$ denotes the sup norm on a compact K and, if f is a globally Lipschitz function, $\|\partial f\|$ denotes its Lipschitz constant.

Proposition 4.1.1. *Assume $H \in C^2_c([0,T] \times T^*\mathbb{R}^d)$ and $f \in C^{Lip}(\mathbb{R}^d)$, then the following estimates are satisfied.*

1. *The operator $R^{s,t}_H$ maps globally Lipschitz functions in other globally Lipschitz functions and, in particular,*

$$\|\partial(R^{s,t}_H f)\| \leq \|\partial f\| + \|\partial_x H\| |t-s|.$$

2. *For any $0 \leq s < t_i \leq T$, with $i = 1, 2$,*

$$|R^{s,t_1}_H f(x) - R^{s,t_2}_H f(x)| \leq |t_1 - t_2| \max_{t \in [t_1,t_2]} |H(t,x,\cdot)|_Y,$$

where $Y = \{y \in \mathbb{R}^d \,|\, |y| \leq \|\partial f\| + \|\partial_x H\| \max_i |t_i - s|\}$.

3. *Let $H_0, H_1 \in C^2_c([0,T] \times T^*\mathbb{R}^d)$ be two Hamiltonians, then*

$$|R^{s,t}_{H_0} f - R^{s,t}_{H_1} f|_{C^0} \leq |t-s| \max_{\tau \in [s,t], y \in Y'} |(H_0 - H_1)(\tau, \cdot, y)|_{C^0},$$

where $Y' = \{y \in \mathbb{R}^d \,|\, |y| \leq \|\partial f\| + \max_i \|\partial_x H_i\| |t-s|\}$.

4. *If $f_0, f_1 \in C^{Lip}(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ is compact, then there exists $\tilde{K}$, a bounded subset of $\mathbb{R}^d$, which depends on $K \times [0,T]$ and $\|\partial f_i\|$, such that, for $0 \le s < t \le T$,*
$$|R_H^{s,t} f_0 - R_H^{s,t} f_1|_K \le |f_0 - f_1|_{\tilde{K}}.$$

Proof. For simplicity, let us assume that $|t-s| < \delta$, so that

$$S(x, x_0, y_0) = f(x_0) + \phi_s^t(x, y_0) + (x - x_0)y_0$$

is a generating function of $\varphi_s^t(\Gamma(df))$. Let $(x(\tau), y(\tau))$ denote the Hamiltonian flow φ and
$$C(x) = \{(x_0, y_0) \in Crit(S_x) \,|\, S(x, x_0, y_0) = R_H^{s,t} f(x)\}.$$

1. For $(x_0, y_0) \in C(x)$, being that ϕ is a generating function for the Hamiltonian flow φ, we have that

   $$\partial_x S(x, x_0, y_0) = \partial_x \phi_s^t(x, y_0) + y_0 = y(t),$$

   where
   $$y(t) = y_0 - \int_s^t \partial_x H(\tau, x(\tau), y(\tau)) d\tau,$$

   with $y_0 \in \partial f(x_0)$. Therefore, by Proposition 3.3.6,

   $$\partial R_H^{s,t} f(x) \subset co\{y(t) \,|\, y_0 \in \partial f(x_0)\},$$

   thus
   $$\|\partial R_H^{s,t} f\| \le \sup_{y_0 \in \partial f(x_0)} |y(t)| \le \|\partial f\| + \|\partial_x H\| |t - s|.$$

2. For $(x_0, y_0) \in C(x)$, by Theorem 2.1.2,

   $$\partial_t S(x, x_0, y_0) = \partial_t \phi_s^t(x, y_0) = -H(t, x, y(t)).$$

   Adapting the proof of Proposition 3.3.6 one can obtain

   $$\begin{aligned} \partial_t R_H^{s,t} f(x) \subset co\{\partial_t S(x, x_0, y_0) \,|\, (x_0, y_0) \in C(x)\} = \\ = co\{-H(t, x, y(t)) \,|\, y_0 \in \partial f(x_0)\}. \end{aligned}$$

   Therefore, for any $t_1, t_2 \in (s, T]$ we have that

   $$|R_H^{s,t_1} f(x) - R_H^{s,t_2} f(x)| \le |t_1 - t_2| \max_{t \in [t_1, t_2], y \in Y} |H(t, x, y)|,$$

   where $Y = \{y \in \mathbb{R}^d \,|\, |y| \le \|\partial f\| + \|\partial_x H\| \max_i |t_i - s|\}$.

3. Consider, for $\lambda \in [0,1]$, $H_\lambda := (1-\lambda)H_0 + \lambda H_1$, let S_λ be the corresponding generating function and $C_\lambda(x) = \{(x_0, y_0) \in Crit(S_{\lambda,x}) \,|\, S_\lambda(x, x_0, y_0) = R^{s,t}_{H_\lambda} f(x)\}$. For a fixed λ and $(x_0, y_0) \in C_\lambda(x)$, through computations similar to those presented in the proof of Theorem 2.1.2 we obtain that

$$\partial_\lambda S_\lambda(x, x_0, y_0) = \partial_\lambda (\phi_{H_\lambda})^t_s(x, y_0) = \int_s^t (H_0 - H_1)(\tau, x_\lambda(\tau), y_\lambda(\tau))d\tau.$$

Similarly to the previous point, we may adapt the proof of Proposition 3.3.6 in order to obtain that

$$\begin{aligned}\partial_\lambda R^{s,t}_{H_\lambda} f(x) &\subset co\{\partial_\lambda S_\lambda(x, x_0, y_0) \,|\, (x_0, y_0) \in C_\lambda(x)\} = \\ &= co\left\{\int_s^t (H_0 - H_1)(\tau, x_\lambda(\tau), y_\lambda(\tau))d\tau \,\middle|\, y_0 \in \partial f(x_0)\right\}.\end{aligned}$$

Therefore, by the mean value inequality,

$$\begin{aligned}|R^{s,t}_{H_0} f(x) - R^{s,t}_{H_1} f(x)| &\leq \int_0^1 \int_s^t (H_0 - H_1)(\tau, x_\lambda(\tau), y_\lambda(\tau))d\tau d\lambda \\ &\leq |t-s| \max_{\tau \in [s,t], y \in Y'} |(H_0 - H_1)(\tau, \cdot, y)|_{C^0},\end{aligned}$$

where $Y' = \{y \in \mathbb{R}^d \,|\, |y| \leq \|\partial f\| + \max_i \|\partial_x H_i\| \max_i |t-s|\}$.

4. Consider, for $\lambda \in [0,1]$, $f_\lambda := (1-\lambda)f_0 + \lambda f_1$ and denote with S_λ the corresponding generating function. Then,

$$\partial S_\lambda(x, x_0, y_0) = f_1(x_0) - f_0(x_0).$$

As in the previous points,

$$\partial_\lambda R^{s,t}_H f_\lambda(x) \subset co\{f_1(x_0) - f_0(x_0) \,|\, (x_0, y_0) \in C_\lambda(x)\},$$

where $C_\lambda(x) \subset \{(x_0, y_0) \,|\, |x_0| \leq |x|_K + T\|\partial_y(H|_{y \in Y''})\|\}$, $Y'' := \{y \in \mathbb{R}^d \,|\, |y| \leq \|\partial f_0\| + \|\partial f_1\| + T\|\partial_x H\|\}$. If we take $\tilde{K} = \{x_0 \in \mathbb{R}^d \,|\, |x_0| \leq |x|_K + T\|\partial_y(H|_{y \in Y''})\|\}$, we obtain

$$|R^{s,t}_H f_0 - R^{s,t}_H f_1|_K \leq |f_0 - f_1|_{\tilde{K}}.$$

□

Now, given any compact set $K \subset \mathbb{R}^d$, let us consider $(t, x) \in [0,T] \times K$. Let $\xi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a subdivision of $[0,T]$, then to each $s \in [0,T]$ we associate the number

$$[s]_\xi := t_i \quad \text{if } t_i \leq s < t_{i+1}.$$

Definition 4.1.1. *The iterated min-max solution operator for the Hamilton-Jacobi equation with respect to the subdivision ξ for $0 \leq s' < s \leq T$ is defined as*

$$R^{s',s}_{H,\xi} \coloneqq R^{t_j,s}_H \circ R^{t_{j-1},t_j}_H \circ \cdots \circ R^{t_{i+1},t_{i+2}}_H \circ R^{s',t_{i+1}}_H, \quad \textit{where } t_j = [s]_\xi,\ t_i = [s']_\xi,\ i \leq j.$$

In the sequel, we fix the Hamiltonian H and thus we can denote:

$$R^{s,t}_H = R^{s,t}, \quad R^{s',s}_{H,\xi} = R^{s',s}_\xi.$$

For each subdivision ξ of $[0,T]$ we define its *length* as

$$|\xi| \coloneqq \max_{i=0,\dots,n-1} |t_{i+1} - t_i|.$$

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$ such that $|\xi_n| \xrightarrow{n\to\infty} 0$ and let $\{R^{0,s}_{\xi_n} f\}_{n\in\mathbb{N}}$ be the corresponding sequence of iterated min-max solutions for the initial datum $f \in C^{Lip}(\mathbb{R}^d)$.

Lemma 4.1.1. *The sequence of functions $u_n(s,x) = R^{0,s}_{\xi_n} f(x)$ is equi-Lipschitz and uniformly bounded for $(s,x) \in [0,T] \times K$.*

Proof. By Proposition 4.1.1, point 1, we have that

$$\|\partial(R^{0,s}_{\xi_n} f)\| \leq \|\partial f\| + T\|\partial_x H\|,$$

that is, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is equi-Lipschitz with respect to x.
By Proposition 4.1.1, point 2, we have that

$$|R^{0,s}_{\xi_n} f - R^{0,t}_{\xi_n} f|_K \leq |H|_{\mathscr{K}} |s-t|,$$

where $s,t \in [0,T]$ and $\mathscr{K} = \{(t',x,y) \,|\, t' \in [0,T],\ x \in K,\ |y| \leq \|\partial f\| + T\|\partial_x H\|\}$.
Thus, $\{u_n\}_{n\in\mathbb{N}}$ is equi-Lipschitz with respect to the time s.
Moreover, for $t = 0$, using the converse of the triangle inequality we get

$$|R^{0,s}_{\xi_n} f|_K \leq |f|_K + T|H|_{\mathscr{K}}$$

for each $s \in [0,T]$, which means that the sequence $\{R^{0,s}_{\xi_n} f(x)\}_{n\in\mathbb{N}}$ is uniformly bounded for $(s,x) \in [0,T] \times K$. □

4.2 Limiting procedure of iterated min-max

In this section we prove in detail that the limiting procedure of iterated min-max, that is for time intervals tending to zero, uniformly converges to the viscosity solution.

Proposition 4.2.1. *For any sequence* $\{\xi_n\}_{n\in\mathbb{N}}$ *of subdivisions of* $[0,T]$ *such that* $|\xi_n| \to 0$ *and any compact* $K \subset \mathbb{R}^d$*, the sequence* $\{u_n(s,x)\}_{n\in\mathbb{N}} = \{R^{0,s}_{\xi_n} f(x)\}_{n\in\mathbb{N}}$ *admits a subsequence that converges uniformly on* $[0,T]\times K$ *to the viscosity solution of the Cauchy problem* (H-J).

Proof. By Lemma 4.1.1, the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the hypotheses of the Arzelà-Ascoli theorem, therefore it admits a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly in $C^0([0,T]\times K)$. Denote by $\bar{R}^{0,s}f(x)$ its limit. We start by proving that

$$\bar{R}^{0,s}f(x) = \lim_{k\to\infty} R^{s',s}_{\xi_{n_k}} \circ \bar{R}^{0,s'} f(x) \quad \forall\, 0 \le s' < s \le T. \tag{4.1}$$

Let $\tilde{K} \supset K$ be as in point 4 of Proposition 4.1.1, we may suppose that $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges uniformly in $C^0([0,T],\tilde{K})$.

For simplicity, in the following we omit the subindex k of n_k.

Denote $[s]_n \coloneqq [s]_{\xi_n}$. By point 2 of Proposition 4.1.1, we have that

$$\begin{aligned}|R^{0,s}_{\xi_n} f(x) - R^{0,[s]_n}_{\xi_n} f(x)| &= |R^{[s]_n,s} \circ R^{0,[s]_n}_{\xi_n} f(x) - R^{0,[s]_n}_{\xi_n} f(x)|\\ &\le |H|_{\mathscr{K}}(s-[s]_n) \le |H|_{\mathscr{K}}|\xi_n| \xrightarrow{n\to\infty} 0\end{aligned}$$

therefore,

$$\bar{R}^{0,s}f(x) = \lim_{n\to\infty} R^{0,[s]_n}_{\xi_n} f(x). \tag{4.2}$$

As a consequence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $i,j > N$,

$$|R^{0,[s]_i}_{\xi_i} f - R^{0,[s]_j}_{\xi_j} f|_{\tilde{K}} < \varepsilon \quad \forall\, s \in [0,T].$$

Hence, by Definition 4.1.1 and point 4 of Proposition 4.1.1,

$$\begin{aligned}|R^{[s']_i,[s]_i}_{\xi_i} \circ R^{0,[s']_j}_{\xi_j} f - R^{0,[s]_i}_{\xi_i} f|_{\tilde{K}} &= |R^{[s']_i,[s]_i}_{\xi_i} \circ R^{0,[s']_j}_{\xi_j} f - R^{[s']_i,[s]_i}_{\xi_i} \circ R^{0,[s']_i}_{\xi_i} f|_{\tilde{K}}\\ &\le |R^{0,[s']_j}_{\xi_j} f - R^{0,[s']_i}_{\xi_i} f|_{\tilde{K}} < \varepsilon\end{aligned}$$

for every $0 \le s' < s \le T$.

For j tending to infinity, we have

$$|R^{[s']_i,[s]_i}_{\xi_i} \circ \bar{R}^{0,s'} f - R^{0,[s]_i}_{\xi_i} f|_{\tilde{K}} < \varepsilon \quad \forall\, i > N.$$

Thus,

$$\lim_{i\to\infty} R^{[s']_i,[s]_i}_{\xi_i} \circ \bar{R}^{0,s'} f(x) = \lim_{i\to\infty} R^{0,[s]_i}_{\xi_i} f(x) = \bar{R}^{0,s} f(x) \quad \forall\, x \in K.$$

Similarly to (4.2), one can show that

$$\lim_{i\to\infty} R^{s',s}_{\xi_i} \circ \bar{R}^{0,s'} f(x) = \lim_{i\to\infty} R^{[s']_i,[s]_i}_{\xi_i} \circ \bar{R}^{0,s'} f(x)$$

and therefore we get (4.1).

Now we prove that $\bar{R}^{0,s}f$ is a viscosity solution of the problem (H-J). First we show that it is a viscosity subsolution.
For any $(t,x) \in [0,T] \times K$, let ψ be a C^2 function defined in an open neighbourhood of (t,x) such that it has bounded second derivative and $\psi(s,y) =: \psi_s(y) \geq \bar{R}^{0,s}f(y)$ with equality at (t,x). Then,

$$\psi_t(x) = \bar{R}^{0,t}f(x) = \lim_{n\to\infty} R^{\tau,t}_{\xi_n} \circ \bar{R}^{0,\tau}f(x) \leq \lim_{n\to\infty} R^{\tau,t}_{\xi_n}\psi_\tau(x) = R^{\tau,t}\psi_\tau(x), \tag{4.3}$$

where we use the monotonicity of the min-max operator and the last equality holds for $t-\tau$ small enough that the characteristics originating from $d\psi_\tau$ do not intersect, thus the iterated min-max coincides with the 1-step min-max, which is the classical C^2 solution of the Hamilton-Jacobi equation. Hence,

$$R^{\tau,t}\psi_\tau(x) = \psi_\tau(x) - \int_\tau^t H(s,x,\partial_x R^{\tau,s}\psi_\tau(x))ds.$$

By (4.3), $R^{\tau,t}\psi_\tau(x) \geq \psi_t(x)$, therefore

$$\psi_t(x) \leq \psi_\tau(x) - \int_\tau^t H(s,x,\partial_x R^{\tau,s}\psi_\tau(x))ds.$$

Subtracting $\psi_t(x)$ to each side, dividing by $t-\tau$ and letting τ tend to t, we get

$$0 \leq -\partial_t\psi_t(x) - H(t,x,\partial_x\psi_t(x)).$$

Thus, by definition, $\bar{R}^{0,s}f$ is a viscosity subsolution of (H-J). Similarly, we can prove that it is a viscosity supersolution, hence it is a viscosity solution. □

Given a Hamiltonian function H and an initial datum f, we say that the limit of the iterated min-max solutions exists in $[0,T]$ if, for any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of subdivisions of $[0,T]$ such that $|\xi_n| \to 0$, the sequence $\{R^{0,t}_{H,\xi_n}f\}_{n\in\mathbb{N}}$ with $t \in [0,T]$ converges uniformly on compact subsets of $\mathbb{R}^d$ to a limit which is independent on the choice of subdivisions. In such a case, we also denote

$$\lim_{n\to\infty} R^{0,t}_{H,\xi_n}f(x) = \bar{R}^{0,t}_H f(x).$$

Theorem 4.2.1 ([14])**.** *If $f \in C^{Lip}(\mathbb{R}^d)$ and $H \in C^2_c([0,T] \times T^*\mathbb{R}^d)$, then there exists a unique viscosity solution of the Cauchy problem* (H-J)*.*

Theorem 4.2.2. *Suppose $H \in C^2_c([0,T] \times T^*\mathbb{R}^d)$ and $f \in C^{Lip}(\mathbb{R}^d)$. Then, for the Cauchy problem of the evolutive Hamilton-Jacobi equation*

$$\begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u(t,x)) = 0 \\ u(0,x) = f(x) \end{cases} \tag{H-J}$$

the limit of iterated min-max solutions exists in $[0,T]$ and coincides with the viscosity solution.

Proof. Let K be a compact subset of $\mathbb{R}^d$ and $\{\xi_n\}_{n\in\mathbb{N}}$ a sequence of subdivisions of $[0,T]$ such that $|\xi_n| \to 0$. By Theorem 4.2.1, there exists a unique viscosity solution of the Cauchy problem (H-J), let us denote it by $u(t,x)$. Denote $u_n(t,x) := R^{0,t}_{\xi_n} f(x)$. Suppose that $\{u_n\}_{n\in\mathbb{N}}$ does not converge uniformly to u on $[0,T]\times K$, then there exists $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$|u_{n_k} - u|_{[0,T]\times K} > \varepsilon \quad \forall\, k \in \mathbb{N}. \tag{4.4}$$

However, $\{u_{n_k}\}_{k\in\mathbb{N}}$ is associated to the sequence of subdivisions $\{\xi_{n_k}\}_{k\in\mathbb{N}}$, which is such that $|\xi_{n_k}| \to 0$. Therefore, by Proposition 4.2.1, there exists a subsequence $\{u_{n_{k_j}}\}_{j\in\mathbb{N}}$ that converges uniformly to u on $[0,T]\times K$, which contradicts (4.4). $\square$

Chapter 5

Iterated min-max in the continuous setting

Abstract. For evolutive Hamilton-Jacobi equations, in this chapter we follow O. Bernardi and F. Cardin (see [6]) and propose an alternative definition of min-max solution, adapted to Cauchy problems for only continuous initial data. This definition is performed for globally compactly supported Hamiltonians, defined on the cotangent bundle of a compact manifold (tipically $T^*\mathbb{T}^d$). Moreover we prove that –for Lipschitz-continuous initial data– the limiting min-max operator obtained in this way coincides with the one constructed by Q. Wei and (as a consequence) to the unique viscosity solution. This is an original result of the Thesis.

5.1 Min-max solutions in the continuous setting

Let us consider the Cauchy problem (H-J) for a Hamiltonian function $H \in C^2_c(T^*\mathbb{T}^d)$ and initial datum $f \in C^2(\mathbb{T}^d)$.

$$\begin{cases} \partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0 \quad \text{with } t \in [0,T] \\ u(0,x) = f(x) \end{cases}$$

Note that in such setting, f and H are Lipschitz. In the following we denote with k and k' the Lipschitz constants of H relative to x and p respectively.
By Lemma 2.2.1, for any $u \in C^2(\mathbb{T}^d)$, $\varphi^t_s(\Gamma(du))$ admits GFQI of the forms (2.4) and (2.5), that is,

$$S(x,\eta) = u(x_0) + \sum_{0\le i<N} \phi^{t_{i+1}}_{t_i}(x_{i+1}, y_i) + \sum_{0\le i<N} (x_{i+1} - x_i) y_i$$

and

$$S(\tau, x, \eta) = u(x_0) + \sum_{0\le i<N} \phi^{\tau_{i+1}}_{\tau_i}(x_{i+1}, y_i) + \sum_{0\le i<N} (x_{i+1} - x_i) y_i$$

where we use the notation of Corollary 2.1.2.1:

$$\eta = ((x_i, y_i))_{0\le i<N}, \quad x_N = x \quad \text{and} \quad 0 \le s = t_0 < t_1 < \cdots < t_N = t \le T.$$

Hence, for any $x \in \mathbb{T}^d$ we can define the min-max function

$$R^{s,t}u(x) = c(1_x, S_x) = \inf_{[\sigma]=1_x} \max_{\eta\in|\sigma|} S(x,\eta)$$

and the min-max operator

$$\begin{aligned} R^{s,t} : C^2(\mathbb{T}^d) &\to C^{Lip}(\mathbb{T}^d) \\ u &\mapsto R^{s,t}u. \end{aligned}$$

We shall now introduce concepts and results that are useful in the sequel. More details can be found in [21][Section 1.1.2].
Let $\mathscr{L}$ be the set of Lagrangian submanifolds of $T^*\mathbb{T}^d$ which are Hamiltonian isotopic to the zero section and $L_1, L_2 \in \mathscr{L}$ be generated respectively by the GFQIs $S_1(x,\eta)$ and $S_2(x,\xi)$. Denote by $S_1 \sharp S_2$ the GFQI

$$(S_1\sharp S_2)(x,\eta,\xi) \coloneqq S_1(x,\eta) + S_2(x,\xi).$$

Definition 5.1.1. *Viterbo's distance between two Lagrangian submanifolds $L_1, L_2 \in \mathscr{L}$ is given by*

$$\gamma(L_1,L_2) \coloneqq c(\mu, S_1\sharp(-S_2)) - c(1, S_1\sharp(-S_2)),$$

where $1 \in H^0(\mathbb{T}^d)$ and $\mu \in H^d(\mathbb{T}^d)$ are generators.

Moreover, for any Hamiltonian isotopies ϕ_1, ϕ_2 we set

$$\tilde\gamma(\phi_1,\phi_2) \coloneqq \sup\{\gamma(\phi_1(L), \phi_2(L)) \mid L \in \mathscr{L}\}.$$

One can prove that $\tilde\gamma$ defines a metric on the group of Hamiltonian diffeomorphisms of $T^*\mathbb{T}^d$.

Proposition 5.1.1.

1. *For any $L_1, L_2 \in \mathscr{L}$, let u_{L_1} and u_{L_2} be the respective graph selectors. Then,*

$$\|u_{L_1} - u_{L_2}\|_\infty \le \gamma(L_1,L_2).$$

2. *For any $L_1, L_2 \in \mathscr{L}$ and for any Hamiltonian diffeomorphism ψ*

$$\gamma(\psi(L_1),\psi(L_2)) = \gamma(L_1,L_2).$$

3. *Let H, K be Hamiltonians and φ_H, φ_K be respectively their Hamiltonian flows, then*

$$\tilde\gamma(\varphi_H,\varphi_K) \le \|H-K\|_\infty.$$

In the following, we denote with $\mathbb{1}$ the function that is constantly equal to 1 on $\mathbb{T}^d$.

Proposition 5.1.2. *The min-max operator $R^{s,t}$ satisfies the following properties for any $0 \leq s < t \leq T$.*

1. *For every $u, v \in C^2(\mathbb{T}^d)$ such that $u \leq v$, $R^{s,t}u \leq R^{s,t}v$ (Monotonicity).*

2. *For every constant $c \in \mathbb{R}$ and for every $u \in C^2(\mathbb{T}^d)$, $R^{s,t}(u+\mathbb{1}\cdot c) = R^{s,t}u+\mathbb{1}\cdot c$.*

3. *For every $u, v \in C^2(\mathbb{T}^d)$, $\|R^{s,t}u - R^{s,t}v\|_\infty \leq \|u - v\|_\infty$ (Non-expansivity).*

4. *For every $u \in C^2(\mathbb{T}^d)$ and for every $x, y \in \mathbb{T}^d$*
$$|R^{s,t}u(x) - R^{s,t}u(y)| \leq (Lip(u) + (t-s)k)d_{\mathbb{T}^d}(x, y).$$

5. *For every $u \in C^2(\mathbb{T}^d)$, the mapping $t \mapsto R^{0,t}u$ is Lipschitz-continuous.*

Proof.

1. Let $u, v \in C^2(\mathbb{T}^d)$ with $u \leq v$ and denote with S_u and S_v respectively GFQIs of the form (2.4) of the Lagrangian submanifolds $\varphi_s^t(\Gamma(du))$ and $\varphi_s^t(\Gamma(dv))$. We can write them as
$$S_u(x, \eta) = u(x_0) + g(x, \eta), \quad S_v(x, \eta) = v(x_0) + g(x, \eta)$$
where $g(x, \eta)$ does not depend on the initial datum. Therefore, since $u \leq v$, $S_u \leq S_v$.
Thus, by definition, $R^{s,t}u \leq R^{s,t}v$.

2. Let $u \in C^2(\mathbb{T}^d)$, $c \in \mathbb{R}$. Let $S_{u+\mathbb{1}\cdot c}$ be the GFQI of $\varphi_s^t(\Gamma(d(u + \mathbb{1} \cdot c)))$ of the form (2.4), thus
$$S_{u+\mathbb{1}\cdot c}(x, \eta) = u(x_0) + c + g(x, \eta) = S_u(x, \eta) + c.$$
Then,
$$R^{s,t}(u + \mathbb{1} \cdot c)(x) = \inf_{[\sigma]=1_x} \max_{\eta\in|\sigma|} S_u(x, \eta) + c = R^{s,t}u(x) + c.$$

3. Let $u, v \in C^2(\mathbb{T}^d)$. Then, by points 1 and 2 of Proposition 5.1.1,
$$\|R^{s,t}u - R^{s,t}v\|_\infty \leq \gamma(\varphi_s^t(\Gamma(du)), \varphi_s^t(\Gamma(dv))) = \gamma(\Gamma(du), \Gamma(dv)).$$
Consider the Hamiltonian $K(z, p) := u(z)-v(z)$. Solving the associated Hamilton equations
$$\begin{cases} \dot{z} = 0 \\ \dot{p} = -du(z) + dv(z) \end{cases}$$

we obtain the Hamiltonian flow

$$\begin{aligned} \psi_0^1 : \quad & \Gamma(du) \to \Gamma(dv) \\ & (z, du(z)) \mapsto (z, dv(z)). \end{aligned}$$

Hence, using the definition of $\tilde{\gamma}$ and point 3 of Proposition 5.1.1, we can write

$$\gamma(\Gamma(du), \Gamma(dv)) = \gamma(id(\Gamma(du)), \psi_0^1(\Gamma(du))) \leq \tilde{\gamma}(id, \psi_0^1) \leq \|K\|_\infty = \|u - v\|_\infty.$$

4. Let $u \in C^2(\mathbb{T}^d)$. Since the Hamiltonian H is globally Lipschitz, for every $(x, p) \in T^*\mathbb{T}^d$

$$|\partial_x H(x,p)| \leq k < \infty \quad \text{and} \quad |\partial_p H(x,p)| \leq k' < \infty,$$

which implies that $\varphi_s^t(\Gamma(du))$ exists for every $0 \leq s < t \leq T$. Since u is Lipschitz, denoting with L its Lipschitz constant, for every $x \in \mathbb{T}^d$

$$|du(x)| \leq L < \infty.$$

By Theorem 1.2.4, $(x, d_x R^{s,t} u(x)) \in \varphi_s^t(\Gamma(du))$ for every $x \in \mathbb{T}^d$, therefore

$$|d_x R^{s,t} u(x)| = |p| = \left| p_0 + \int_s^t \dot{p} d\tau \right| = \left| du(x) + \int_s^t -\partial_x H(x,p) d\tau \right| \leq L + |t-s| k.$$

5. As stated in Chapter 1, according to a theorem that can be found in [12][Theorem 2], $R^{0,t}u$ is locally Lipschitz with respect to t. Therefore, the mapping

$$[0, T] \ni t \mapsto R^{0,t} u$$

is Lipschitz-continuous.

□

Now we consider a C^0 initial datum f. Since $C^2(\mathbb{T}^d)$ is dense in $C^0(\mathbb{T}^d)$ with the uniform norm, then there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C^2(\mathbb{T}^d)$ that is uniformly convergent to f.

Proposition 5.1.3. *For $0 \leq s < t \leq T$, consider the operator*

$$\begin{aligned} V^{s,t} : C^0(\mathbb{T}^d) &\to C^0(\mathbb{T}^d) \\ f &\mapsto V^{s,t} f \end{aligned}$$

where $V^{s,t} f$ is the uniform limit of any sequence $\{R^{s,t} f_n\}_{n \in \mathbb{N}}$ such that $\{f_n\}_{n \in \mathbb{N}} \subset C^2(\mathbb{T}^d)$ converges uniformly to f. Such operator is well-defined.

Proof. We start by proving that, if $f_n \xrightarrow{\|\cdot\|_\infty} f$, then the sequence $\{R^{s,t}f_n\}_{n\in\mathbb{N}}$ converges uniformly in $C^0(\mathbb{T}^d)$. Since $(C^0(\mathbb{T}^d), \|\cdot\|_\infty)$ is a Banach space, it is sufficient to show that $\{R^{s,t}f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the uniform norm.
Fix $\varepsilon > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ is convergent, it is a Cauchy sequence in $(C^0(\mathbb{T}^d), \|\cdot\|_\infty)$: there exists $n_\varepsilon \in \mathbb{N}$ such that, for every $n, m \geq n_\varepsilon$, $\|f_n - f_m\|_\infty \leq \varepsilon$. By point 3 of Proposition 5.1.2,

$$\|R^{s,t}f_n - R^{s,t}f_m\|_\infty \leq \|f_n - f_m\|_\infty \leq \varepsilon$$

for any $n, m \geq n_\varepsilon$, therefore $\{R^{s,t}f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let $V^{s,t}f \in C^0(\mathbb{T}^d)$ be the uniform limit of $\{R^{s,t}f_n\}_{n\in\mathbb{N}}$, let us prove that if $\{g_n\}_{n\in\mathbb{N}}$ is another sequence in C^2 that converges uniformly to f, then $\{R^{s,t}g_n\}_{n\in\mathbb{N}}$ is also uniformly convergent to $V^{s,t}f$.
Let us denote by G the uniform limit of $\{R^{s,t}g_n\}_{n\in\mathbb{N}}$. Then,

$$\|V^{s,t}f - G\|_\infty \leq \|V^{s,t}f - R^{s,t}f_n\|_\infty + \|G - R^{s,t}g_n\|_\infty + \|R^{s,t}f_n - R^{s,t}g_n\|_\infty$$

where the first two terms on the right hand side tend to 0 as $n \to \infty$. By point 3 of Proposition 5.1.2,

$$\|R^{s,t}f_n - R^{s,t}g_n\|_\infty \leq \|f_n - g_n\|_\infty \leq \|f_n - f\|_\infty + \|g_n - g\|_\infty$$

therefore this also tends to 0 as $n \to \infty$. This concludes that $\|V^{s,t}f - G\|_\infty = 0$, thus $\{R^{s,t}g_n\}_{n\in\mathbb{N}}$ also converges to $V^{s,t}f$. □

5.2 Limiting procedure of iterated C^0 min-max

We start by proving that the properties of $R^{s,t}$ stated in Proposition 5.1.2 are also satisfied by the operator $V^{s,t}$.

Proposition 5.2.1. *For $0 \leq s < t \leq T$, the operator $V^{s,t}$ satisfies the following properties.*

1. *For every $u, v \in C^0(\mathbb{T}^d)$ be such that $u \leq v$, $V^{s,t}u \leq V^{s,t}v$ (Monotonicity).*
2. *For every constant $c \in \mathbb{R}$ and for every $u \in C^0(\mathbb{T}^d)$, $V^{s,t}(u+\mathbb{1}\cdot c) = V^{s,t}u+\mathbb{1}\cdot c$.*
3. *For every $u, v \in C^0(\mathbb{T}^d)$, $\|V^{s,t}u - V^{s,t}v\|_\infty \leq \|u - v\|_\infty$ (Non-expansivity).*
4. *For every $u \in C^{Lip}(\mathbb{T}^d)$ with $Lip(u) = L$ and for every $x, y \in \mathbb{T}^d$*
$$|V^{s,t}u(x) - V^{s,t}u(y)| \leq (L + (t - s)k)d_{\mathbb{T}^d}(x, y).$$
5. *For any $u \in C^0(\mathbb{T}^d)$, the mapping $t \mapsto V^{0,t}u$ is Lipschitz-continuous. Moreover, there exists $C > 0$ such that for every $u \in C^0(\mathbb{T}^d)$ the Lipschitz constant of $t \mapsto V^{0,t}u$ is bounded by C.*

Proof.

1. Let $u \leq v$. We can take $\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}}$ sequences in $C^2(\mathbb{T}^d)$ such that $\{u_n\}_{n\in\mathbb{N}}$ converges to u from below and $\{v_n\}_{n\in\mathbb{N}}$ converges to v from above. As a consequence, for every $n \in \mathbb{N}$, $u_n \leq v_n$. By point 1 of Proposition 5.1.2, $R^{s,t}u_n \leq R^{s,t}v_n$ for every n, thus

$$V^{s,t}u(x) = \lim_{n\to\infty} R^{s,t}u_n(x) \leq \lim_{n\to\infty} R^{s,t}v_n(x) = V^{s,t}v(x)$$

for every $x \in \mathbb{T}^d$.

2. Let $u \in C^0(\mathbb{T}^d)$, $c \in \mathbb{R}$ and $\{u_n\}_{n\in\mathbb{N}} \subset C^2(\mathbb{T}^d)$ that converges uniformly to u. Then, the sequence $\{f_n + \mathbb{1} \cdot c\}_{n\in\mathbb{N}}$ is contained in $C^2(\mathbb{T}^d)$ and it converges uniformly to $f + \mathbb{1} \cdot c \in C^0(\mathbb{T}^d)$. Thus, by point 2 of Proposition 5.1.2,

$$V^{s,t}(f + \mathbb{1} \cdot c)(x) = \lim_{n\to\infty} R^{s,t}(f_n + \mathbb{1} \cdot c)(x) = \lim_{n\to\infty} R^{s,t}f_n(x) + c = V^{s,t}f(x) + c$$

for every $x \in \mathbb{T}^d$.

3. Let $u, v \in C^0(\mathbb{T}^d)$ and $\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}} \subset C^2(\mathbb{T}^d)$ such that $u_n \xrightarrow{\|\cdot\|_\infty} u$ and $v_n \xrightarrow{\|\cdot\|_\infty} v$. By point 3 of Proposition 5.1.2, for every $n \in \mathbb{N}$,

$$\|R^{s,t}u_n - R^{s,t}v_n\|_\infty \leq \|u_n - v_n\|_\infty.$$

Thus, by continuity of the uniform norm,

$$\|V^{s,t}u - V^{s,t}v\|_\infty = \lim_{n\to\infty} \|R^{s,t}u_n - R^{s,t}v_n\|_\infty \leq \lim_{n\to\infty} \|u_n - v_n\|_\infty = \|u - v\|_\infty.$$

4. Let u be a Lipschitz function on $\mathbb{T}^d$ with Lipschitz constant L. We can take $\{u_n\}_{n\in\mathbb{N}} \subset C^2(\mathbb{T}^d)$ such that $u_n \xrightarrow{\|\cdot\|_\infty} u$ and $Lip(u_n) = L_n \xrightarrow{n\to\infty} L$. Then, by point 4 of Proposition 5.1.2,

$$\begin{aligned} |V^{s,t}u(x) - V^{s,t}u(y)| &= \lim_{n\to\infty} |R^{s,t}u_n(x) - R^{s,t}u_n(y)| \leq \lim_{n\to\infty} (L_n + (t-s)k) d_{\mathbb{T}^d}(x,y) \\ &= (L + (t-s)k) d_{\mathbb{T}^d}(x,y). \end{aligned}$$

5. Let $u \in C^0(\mathbb{T}^d)$ and $\{u_n\}_{n\in\mathbb{N}} \subset C^2(\mathbb{T}^d)$ such that $u_n \xrightarrow{\|\cdot\|_\infty} u$. By point 5 of Proposition 5.1.2, the mapping $t \mapsto R^{0,t}u_n$ with $t \in [0,T]$ is Lipschitz for every n.

   Moreover, the Lipschitz constant does not depend on the initial datum, in fact

$$\partial_t R^{0,t}u(x) = \partial_t S(t,x,\eta) = \partial_t \phi^{t_{N-1},t} = -H(t,x,y(t)),$$

thus, the Lipschitz constant is always bounded by $C = \|H\|_\infty$, which is finite because H is C^2 and globally compactly supported.

Therefore,

$$\|V^{0,t}u - V^{0,t'}u\|_\infty = \lim_{n\to\infty} \|R^{0,t}u_n - R^{0,t'}u_n\|_\infty \le C|t-t'|$$

that is, $[0,T] \ni t \mapsto V^{0,t}u$ is Lipschitz.

□

Note that, under the hypotheses we assume at the beginning of the chapter, it is possible to repeat the same procedure shown in Chapter 4 and obtain for each Lipschitz initial datum f the unique viscosity solution in $[0,T] \times \mathbb{T}^d$,

$$u(t,x) = \bar{R}^{0,t} f(x).$$

We now apply the same procedure to the operator $V^{s,t}$.
Let $\xi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a subdivision of $[0,T]$. To each $s \in [0,T]$ we associate the number

$$[s]_\xi := t_i \quad \text{if } t_i \le s < t_{i+1}.$$

The iterated operator with respect to the subdivision ξ for $0 \le s < t \le T$ is defined as

$$V_\xi^{s,t} := V^{t_j,t} \circ V^{t_{j-1},t_j} \circ \cdots \circ V^{t_{i+1},t_{i+2}} \circ V^{s,t_{i+1}}, \quad \text{where } t_j = [t]_\xi,\ t_i = [s]_\xi,\ i \le j.$$

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$ such that $|\xi_n| \xrightarrow{n\to\infty} 0$ and let $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$ be the corresponding sequence of iterated min-max solutions for the initial datum $f \in C^{Lip}(\mathbb{T}^d)$, with $(t,x) \in [0,T] \times \mathbb{T}^d$.

Lemma 5.2.1. *The sequence of functions $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$ is equi-Lipschitz and uniformly bounded for $(t,x) \in [0,T] \times \mathbb{T}^d$.*

Proof. By point 4 of Proposition 5.2.1, for every $x, y \in \mathbb{T}^d$,

$$|V_{\xi_n}^{0,t} f(x) - V_{\xi_n}^{0,t} f(y)| \le (L + tk) d_{\mathbb{T}^d}(x,y)$$

that is, $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$ is equi-Lipschitz with respect to x.
By point 5 of Proposition 5.2.1, for every $t, t' \in [0,T]$,

$$\|V_{\xi_n}^{0,t} f - V_{\xi_n}^{0,t'} f\|_\infty \le C|t-t'|$$

that is, $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$ is equi-Lipschitz with respect to t.
Moreover, taking $t' = 0$, by the inverse of the triangular inequality we get

$$\|V_{\xi_n}^{0,t} f\|_\infty \le \|f\|_\infty + C|t-t'|$$

that is, $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$ is uniformly bounded. □

Thus, the hypotheses of Ascoli-Arzelà theorem are satisfied and the following statement can be simply proved by adapting the proof of Proposition 4.2.1.

Proposition 5.2.2. *For any sequence* $\{\xi_n\}_{n\in\mathbb{N}}$ *of subdivisions of* $[0,T]$ *such that* $|\xi_n| \to 0$*, the sequence* $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$ *admits a subsequence that converges uniformly on* $[0,T] \times \mathbb{T}^d$ *to the viscosity solution of the Cauchy problem* (H-J) *with initial datum* f*.*

Remark 5.2.1. By Proposition 4.2.1, the viscosity solution in $[0,T] \times \mathbb{T}^d$ for the initial datum f coincides with the limit operator $\bar{R}^{0,t} f(x)$. If we denote with $\bar{V}^{0,t} f(x)$ the limit operator of the converging subsequence of $\{V_{\xi_n}^{0,t} f(x)\}_{n\in\mathbb{N}}$, the uniqueness of the viscosity solution in our setting implies that $\bar{V}^{0,t} f(x) = \bar{R}^{0,t} f(x)$ for every $(t,x) \in [0,T] \times \mathbb{T}^d$. In the following we shall denote both limit operators by $\bar{R}^{0,t} f(x)$.

Chapter 6

A look to the stationary case

Abstract. For globally compactly supported Hamiltonians on $T^*\mathbb{T}^d$, we study the properties of the limiting min-max operator defined on the space of Lipschitz-continuous initial data. Moreover, following the proof of A. Fathi celebrated "Weak KAM Theorem", we prove that this operator admits a common (in $t \in \mathbb{R}$) fixed point. As in the "Weak KAM" setting, this result allows us to define a special value $c \in \mathbb{R}$ for which the corresponding stationary Hamilton-Jacobi equation admits a global solution. This chapter represents an original part of the Thesis.

6.1 Properties of the limiting min-max operator

In the following, we denote with $Lip_M(\mathbb{T}^d)$ the set of the Lipschitz-continuous functions $f : \mathbb{T}^d \to \mathbb{R}$ with $Lip(f) \leq M$. Moreover, let $H \in C^2_c(T^*\mathbb{T}^d)$ and $k > 0$ the Lipschitz constant of H with respect to x.
We start by proving some properties of the operator

$$\bar{R}^{s,t} : f \mapsto \bar{R}^{s,t} f$$

where $\bar{R}^{s,t} f$ is the (unique) viscosity solution at time t for the evolutive Hamilton-Jacobi equation

$$\partial_t u + H(x, \partial_x u) = 0$$

with initial condition f at time s. $\bar{R}^{s,t} f$ is equivalently obtained by one of the limiting procedures explained in the previous sections.
We remind that we denote by $\mathbb{1}$ the function that is constantly equal to 1 on $\mathbb{T}^d$.

Proposition 6.1.1. *For any $0 \leq s < t \leq T$, the operator $\bar{R}^{s,t}$ satisfies the following properties:*

1. $\bar{R}^{0,t} : Lip_L(\mathbb{T}^d) \to Lip_{L+tk}(\mathbb{T}^d)$.

2. $\bar{R}^{s,t} \circ \bar{R}^{0,s} = \bar{R}^{0,t}$ *(Semigroup).*

3. *For every* $u, v \in Lip_L(\mathbb{T}^d)$ *such that* $u \le v$, $\bar{R}^{0,t}u \le \bar{R}^{0,t}v$ *(Monotonicity).*

4. *For every constant* $c \in \mathbb{R}$ *and* $u \in Lip_L(\mathbb{T}^d)$, $\bar{R}^{0,t}(u + \mathbb{1} \cdot c) = \bar{R}^{0,t}u + \mathbb{1} \cdot c$.

5. *For every* $u, v \in Lip_L(\mathbb{T}^d)$, $\|\bar{R}^{0,t}u - \bar{R}^{0,t}v\|_\infty \le \|u - v\|_\infty$ *(Non-expansivity).*

6. *For every* $u \in Lip_L(\mathbb{T}^d)$*, the mapping* $t \mapsto \bar{R}^{0,t}u$ *is Lipschitz-continuous.*

Proof. We start by reminding that, for any $u \in C^{Lip}(\mathbb{T}^d)$ and $t \in [0, T]$, $\bar{R}^{0,t}u(x)$ is the uniform limit in $[0, T] \times \mathbb{T}^d$ of a subsequence of $\{R^{0,t}_{\xi_n}u(x)\}_{n\in\mathbb{N}}$, where $\{\xi_n\}_{n\in\mathbb{N}}$ is any sequence of subdivisions of the time interval $[0, T]$ such that the lengths $|\xi_n| \to 0$ as $n \to \infty$. Equivalently, it is the uniform limit of a subsequence of $\{V^{0,t}_{\xi_n}u(x)\}_{n\in\mathbb{N}}$.

Note that, since $\bar{R}^{0,t}u(x)$ is the unique viscosity solution on $[0, T] \times \mathbb{T}^d$ of the Cauchy problem associated with the Hamilton-Jacobi equation, it does not depend on the choice of the sequence of subdivisions of $[0, T]$, as stated in Theorem 4.2.2. Therefore, we may choose the same sequence $\{\xi_n\}_{n\in\mathbb{N}}$ for every $u \in C^{Lip}(\mathbb{T}^d)$.

For simplicity, for each $u \in C^{Lip}(\mathbb{T}^d)$, we denote by $\{R^{0,t}_{\xi_n}u(x)\}_{n\in\mathbb{N}}$ a subsequence that converges to $\bar{R}^{0,t}u(x)$.

1. Let $u \in Lip_L(\mathbb{T}^d)$. By point 1 of Proposition 4.1.1, we get that

$$|R^{0,t}_{\xi_n}u(x) - R^{0,t}_{\xi_n}u(y)| \le (L + tk)d_{\mathbb{T}^d}(x, y)$$

for every $n \in \mathbb{N}$. Therefore, for any $x, y \in \mathbb{T}^d$

$$|\bar{R}^{0,t}u(x) - \bar{R}^{0,t}u(y)| = \lim_{n\to\infty} |R^{0,t}_{\xi_n}u(x) - R^{0,t}_{\xi_n}u(y)| \le (L + tk)d_{\mathbb{T}^d}(x, y).$$

Thus, we conclude that $\bar{R}^{0,t}u \in Lip_{L+tk}(\mathbb{T}^d)$.

2. The proof of the semigroup property is contained in the proof of Proposition 4.2.1.

3. Let $u, v \in Lip_L(\mathbb{T}^d)$ be such that $u \le v$. Using the same arguments used in the proof of point 1 of Proposition 5.1.2, we get that, for every $0 \le s < t \le T$, $R^{s,t}u \le R^{s,t}v$. Hence, for every $n \in \mathbb{N}$, $R^{0,t}_{\xi_n}u \le R^{0,t}_{\xi_n}v$ and, for $n \to \infty$, $\bar{R}^{0,t}u \le \bar{R}^{0,t}v$.

4. Let $c \in \mathbb{R}$, $u \in Lip_L(\mathbb{T}^d)$. Using the same arguments used in the proof of point 2 of Proposition 5.1.2, we get that, for every $0 \le s < t \le T$, $R^{s,t}(u + \mathbb{1} \cdot c) = R^{s,t}u + \mathbb{1} \cdot c$. Hence, for every $n \in \mathbb{N}$, $R^{0,t}_{\xi_n}(u + \mathbb{1} \cdot c) = R^{0,t}_{\xi_n}u + \mathbb{1} \cdot c$. Therefore,

$$\bar{R}^{0,t}(u + \mathbb{1} \cdot c) = \lim_{n\to\infty} R^{0,t}_{\xi_n}(u + \mathbb{1} \cdot c) = \lim_{n\to\infty} R^{0,t}_{\xi_n}u + \mathbb{1} \cdot c = \bar{R}^{0,t}u + \mathbb{1} \cdot c.$$

5. Let $u, v \in Lip_L(\mathbb{T}^d)$. By point 4 of Proposition 4.1.1, we get that

$$\|R^{s,t}u - R^{s,t}v\|_\infty \leq \|u - v\|_\infty \quad \forall\, 0 \leq s < t \leq T.$$

Applying this result repeatedly, we obtain that for every $n \in \mathbb{N}$

$$\|R^{0,t}_{\xi_n}u - R^{0,t}_{\xi_n}v\|_\infty \leq \|u - v\|_\infty.$$

Therefore,

$$\|\bar{R}^{0,t}u - \bar{R}^{0,t}v\|_\infty = \lim_{n\to\infty} \|R^{0,t}_{\xi_n}u - R^{0,t}_{\xi_n}v\|_\infty \leq \|u - v\|_\infty.$$

6. Let $u \in Lip_L(\mathbb{T}^d)$. By point 2 of Proposition 4.1.1, we have that there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and for any $t, t' \in [0, T]$

$$\|R^{0,t}_{\xi_n}u - R^{0,t'}_{\xi_n}u\|_\infty \leq C|t - t'|.$$

Hence,

$$\|\bar{R}^{0,t}u - \bar{R}^{0,t'}u\|_\infty = \lim_{n\to\infty} \|R^{0,t}_{\xi_n}u - R^{0,t'}_{\xi_n}u\|_\infty \leq C|t - t'|.$$

□

Let us now consider the quotient space

$$E_L := \frac{Lip_L(\mathbb{T}^d)}{\mathbb{R}\cdot\mathbb{1}}$$

that is, the space of the Lipschitz functions with Lipschitz constant bounded by L, where we identify two functions $f, g \in Lip_L(\mathbb{T}^d)$ if there exists a constant C such that $f = g + C\cdot\mathbb{1}$.
We can equip E_L with its quotient norm $\|\cdot\|_Q$:

$$\|[u]\|_Q = \inf_{a\in\mathbb{R}} \|u + a\cdot\mathbb{1}\|_\infty \quad \forall\, u \in Lip_L(\mathbb{T}^d).$$

Proposition 6.1.2. *$(E_L, \|\cdot\|_Q)$ is a Banach space.*

Proof. We use a known result of functional analysis: let $(E, \|\cdot\|)$ be a Banach space and M be a closed subspace of E, then the quotient space E/M equipped with the quotient norm $\|\cdot\|_{E/M}$ is a Banach space (a proof of this result can be found, for instance, in [8][Proposition 11.8]).

We start by proving that $(Lip_L(\mathbb{T}^d), \|\cdot\|_\infty)$ is a Banach space.
Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $Lip_L(\mathbb{T}^d)$ with respect to the uniform norm. Since $(C^0(\mathbb{T}^d), \|\cdot\|_\infty)$ is a Banach space, there exists $f \in C^0(\mathbb{T}^d)$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f. In particular, the convergence is also punctual. Therefore, since $f_n \in Lip_L(\mathbb{T}^d)$ for every $n \in \mathbb{N}$, we have that for every $x, y \in \mathbb{T}^d$

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq L d_{\mathbb{T}^d}(x, y)$$

thus, $f \in Lip_L(\mathbb{T}^d)$ and $Lip_L(\mathbb{T}^d)$ is a Banach space with the uniform norm.

Clearly, since every convergent sequence of constant functions necessarily converges to a constant function with respect to the uniform norm, $\mathbb{R} \cdot \mathbb{1}$ is a closed subspace of $Lip_L(\mathbb{T}^d)$.

Therefore we conclude that E_L equipped with its quotient norm is a Banach space. □

By point 4 of Proposition 6.1.1, for every constant $a \in \mathbb{R}$ and every $u \in Lip_L(\mathbb{T}^d)$,

$$\bar{R}^{0,t}(u + a \cdot \mathbb{1}) = \bar{R}^{0,t}u + a \cdot \mathbb{1}.$$

Thus, $\bar{R}^{0,t}$ passes to the quotient to the operator

$$\widetilde{R}^{0,t} : E_L \to E_{L+tk}$$

for which the properties of $\bar{R}^{0,t}$ states in Proposition 6.1.1 still hold.

In order to gain a common (in $t \in \mathbb{R}$) fixed point for the family of operators $\{\widetilde{R}^{0,t}\}_{t\in[0,T]}$, we will use the next more general version of Ascoli-Arzelà Theorem.

Theorem 6.1.1 (see [16][Theorem 6.4]). *Let (Z, d) be a metric space, Y an arbitrary space and $\mathscr{F}$ a family of continuous functions from Y to Z. Assume that*

1. *$\mathscr{F}$ is equicontinuous on Y;*
2. *$\overline{\{f(y) \,|\, f \in \mathscr{F}\}}$ is compact for every $y \in Y$.*

Then $\overline{\mathscr{F}}$ is compact on Y.

Proposition 6.1.3. *The family $\{\widetilde{R}^{0,t}\}_{t\in[0,T]}$ is relatively compact on E_L.*

Proof. $\mathscr{F} = \{\widetilde{R}^{0,t}\}_{t\in[0,T]}$ is relatively compact if it satisfies conditions 1 and 2 of Theorem 6.1.1.

1. $\mathscr{F}$ is equicontinuous in $[u] \in E_L$ if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $t \in [0, T]$, if $\|[u] - [v]\|_Q \leq \delta$ then $\|\widetilde{R}^{0,t}[u] - \widetilde{R}^{0,t}[v]\|_Q \leq \varepsilon$. By the non-expansivity of $\widetilde{R}^{0,t}$, it is enough to take $\delta = \varepsilon$.

2. Let $[u] \in E_L$ and $\{f_n\}_{n\in\mathbb{N}} \subset \{\widetilde{R}^{0,t}[u] \,|\, t \in [0, T]\}$. For each $n \in \mathbb{N}$ there exists $t_n \in [0, T]$ such that $f_n = \widetilde{R}^{0,t_n}[u]$. $\{t_n\}_{n\in\mathbb{N}}$ is a sequence in $[0, T]$, thus there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ that converges to $t' \in [0, T]$. Since the mapping $[0, T] \ni t \mapsto \widetilde{R}^{0,t}[u]$ is continuous, the subsequence $\{f_{n_k}\}_{k\in\mathbb{N}} = \{\widetilde{R}^{0,t_{n_k}}[u]\}_{k\in\mathbb{N}}$ converges to $\widetilde{R}^{0,t'}[u]$ with respect to the norm $\|\cdot\|_Q$. Since $\widetilde{R}^{0,t'}[u]$ is an element of $\{\widetilde{R}^{0,t}[u] \,|\, t \in [0, T]\}$, we may conclude that such set is compact and clearly so is its closure.

□

Following A. Fathi proof of the well-known "Weak KAM Theorem", we use the previous version of Ascoli-Arzelà Theorem and Schauder Fixed Point Theorem in order to prove the next result.

Theorem 6.1.2. *The family* $\{\widetilde{R}^{0,t}\}_{t\in[0,T]}$ *admits a common fixed point* $\bar{u} \in E_L$.

Proof. By Proposition 6.1.3, for every $t \in [0,T]$ the image of $\widetilde{R}^{0,t}$ is relatively compact in E_L. Thus, by Schauder's fixed point theorem (see [19][Theorem 3.1]), for every $t \in [0,T]$ there exists $u_t \in E_L$ such that

$$\widetilde{R}^{0,t} u_t = u_t.$$

Since $\widetilde{R}^{0,t}$ satisfies the semigroup property, it is clear that for every $t \in [0,T]$ and $k \in \mathbb{N}$ such that $kt \in [0,T]$,

$$\widetilde{R}^{0,kt} u_t = u_t.$$

Consider $t_n = 2^{-n}$ (clearly $t_n \in [0,T]$ for n large enough) and let $u_{t_n} \in E_L$ be a fixed point of $\widetilde{R}^{0,2^{-n}}$, then for every $k \in \mathbb{N}$

$$\widetilde{R}^{0,k2^{-n}} u_{t_n} = u_{t_n}.$$

By choosing different values of k one can prove that for every $t \in [0,T]$

$$\widetilde{R}^{0,t} u_{t_n} = u_{t_n}.$$

Therefore, as a consequence of the continuity of the operator with respect to the time, the family $\{\widetilde{R}^{0,t}\}_{t\in[0,T]}$ admits a common fixed point $\bar{u} = u_{t_n} \in E_L$. □

6.2 A result á la Weak KAM

In this last section we discuss two main consequences of Theorem 6.1.2 for the Hamilton-Jacobi equation, both in the evolutive and in the stationary case. These results are resumed in the next theorem. We refer to [18][Theorem 4.4.6] for the same result in the framework of Tonelli Hamiltonians, essentially based on the Least Action Principle and the so-called Lax-Oleinik semigroup.

Theorem 6.2.1. *Let* $H \in C^2_c(T^*\mathbb{T}^d)$. *There exist a constant* $c \in \mathbb{R}$ *and a function* $\bar{u} \in Lip_L(\mathbb{T}^d)$ *such that*

1. $u(t,x) := \bar{u}(x) - ct$ *is a solution on* $[0,+\infty) \times \mathbb{T}^d$ *of the evolutive Hamilton-Jacobi equation*

$$\partial_t u + H(x, \partial_x u) = 0$$

2. $\bar{u}(x)$ is a global solution on $\mathbb{T}^d$ of the stationary Hamilton-Jacobi equation

$$H(x, \partial_x u) = c$$

Proof. By Theorem 6.1.2, the family $\{\widetilde{R}^{0,t}\}_{t\in[0,T]}$ admits a common fixed point $\bar{u}$, that is

$$\widetilde{R}^{0,t}\bar{u} = \bar{u} \quad \forall\, t \in [0, T].$$

Let $t \in [0, T]$ and $t < s < 2t$, so that $s - t \leq T$. By the semigroup property of $\widetilde{R}$, we get that

$$\widetilde{R}^{0,s}\bar{u} = \widetilde{R}^{t,s} \circ \widetilde{R}^{0,t}\bar{u} = \widetilde{R}^{t,s}\bar{u} = \widetilde{R}^{0,s-t}\bar{u} = \bar{u}$$

where the equality $\widetilde{R}^{t,s} = \widetilde{R}^{0,s-t}$ follows from the fact that the Hamiltonian H does not depend on time. Thus, $\bar{u}$ is a common fixed point for the family $\{\widetilde{R}^{0,t}\}_{t\in[0,2T]}$. Iterating this argument we get that $\bar{u}$ is a fixed point of $\widetilde{R}^{0,t}$ for every $t \geq 0$.

Since the operator $\widetilde{R}$ is obtained by passing to the quotient the operator $\bar{R}$, the fact that $\widetilde{R}^{0,t}\bar{u} = \bar{u}$ for every $t \geq 0$ implies that

$$\bar{R}^{0,t}\bar{u} = \bar{u} + c(t) \quad \forall\, t \geq 0$$

for some function $c : [0, +\infty) \to \mathbb{R}$ that is continuous since $\bar{R}^{0,t}$ is continuous with respect to t.
Let $t, t' \in [0, +\infty)$. By the semigroup property of $\bar{R}$,

$$\bar{u} + c(t + t') = \bar{R}^{0,t+t'}\bar{u} = \bar{R}^{t,t+t'} \circ \bar{R}^{0,t}\bar{u} = \bar{u} + c(t) + c(t')$$

thus, $c(t+t') = c(t)+c(t')$. Therefore there necessarily exists a constant c such that $c(t) = -ct$ for every $t \geq 0$.
Thus $\bar{R}^{0,t}\bar{u}(x) = \bar{u}(x) - ct$ is the viscosity solution on $[0, +\infty) \times \mathbb{T}^d$ of the Cauchy problem associated with the evolutive Hamilton-Jacobi equation

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 \\ u(0, x) = \bar{u}(x) \end{cases}$$

therefore, point 1. of the statement is proved.

In particular, this also proves point 2. In fact, substituting $u(t, x) = \bar{u}(x) - ct$ into the evolutive Hamilton-Jacobi equation we get

$$-c + H(x, \partial_x \bar{u}) = 0.$$

Thus, $\bar{u}(x)$ is a global solution of $\mathbb{T}^d$ of the stationary Hamilton-Jacobi equation

$$H(x, \partial_x u) = c.$$

□

Ringraziamenti

Ritengo doveroso concludere la Tesi con dei sentiti ringraziamenti alle persone che mi hanno guidata in questi ultimi passi del mio percorso universitario.

Ringrazio la professoressa Olga Bernardi, non solo per le idee su cui si basa questo elaborato, ma anche e soprattutto per l'instancabile impegno e disponibilità. Farò tesoro dell'entusiasmo e della determinazione che ha riversato durante i nostri ricevimenti, qualità che spero possano accompagnermi anche nei miei progetti futuri.

Ringrazio la professoressa Anna Florio per i suoi preziosi contributi, sempre forniti con gentilezza e precisione.

A entrambe va infine la mia gratitudine per avermi dato un assaggio della collaborazione che è alla base del lavoro di ricerca, per cui non posso che rinnovare la mia profonda stima.

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