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The infinite-spin representations of the Poincaré group

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# List of symbols

$\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes SO^\uparrow(1,3)$ Restricted (proper orthocronous) Poincaré group

$\partial V^+$ Forward light cone

$D$ Unitary (irreducible) representation of the Poincaré group

$d$ Unitary (irreducible) representation of the Little group

$e$ Space-like direction of a string

$H_m^+$ Forward mass hyperboloid

$M^{\mu\nu}$ Generators of the Lorentz Group

$P^\mu$ Generators of the translations

$p_{\text{ref}}$ Reference momentum for the orbits

$W^\mu$ Pauli-Lubanski vector

# Contents

**To my parents,**
**with love and gratitude**

# 1 Introduction to the problem

In high energy physics stable elementary particles are naturally associated with particular representations, the irreducible representations of universal cover of Poincaré group $\mathcal{P}$ according to the groundbreaking paper of Wigner published in 1939( [1]). All particles inside the Standard Model fall inside some of these representations, which are labelled by the value of a continuous parameter called mass $m$ and by the value of the spin $s$ in the massive case or by the value of the helicity $h$ in the massless case. But the most general type of massless particles allowed included the so-called infinite-spin representations which are characterized by a dimensionful scale $\kappa$ and reduce to familiar helicity particles only in the limit $\kappa \mapsto 0$. These particles are also called continuous spin particles (CSP). Two classes of objections were raised in the literature against the existence of these particles, one more of physical concern regarding the infinite number of degrees of freedom (and related to a continuous parameter $\kappa$) and the other related to the difficulty of building covariant quantum fields.

The first problem, at least in the axiomatic quantum field theory framework, was put in a mathematical form by Yngvason in 1970 where it was clarified that the existence of a pointlike quantum (free) field for the infinite spin representation is in contradiction with the local commutativity and the covariance transformation law axioms( [2]). At that time there were also many attempts to canonically quantize fields that transform covariantly under Poincaré transformations for the infinite-spin representation, using mostly (singular) CSP wavefunctions derived from Wigner's equations. But all of them failed to obtain a consistent CSP theory, since they found obstructions to build properly causal fields and/or local Hamiltonian( [3] [4] [5]). After the development of modular localization concepts and the natural introduction of string-localized fields, it became clear that fields localized in semi-infinite strings extending to space–like infinity can provide CSP quantum fields that also satisfy all axioms in the axiomatic framework. In particular the main result is the reconciliation of the infinite-spin representations with the principle of causality in a QFT framework. The main work, with the explicit construction of string-localized fields for CSPs, is summarized in the paper of Mund-Schroer-Yngvason published in 2004( [6]). An interesting point is that string-localized fields can be used to describe both bosonic and fermionic massive particles( [7] [8]) other than massless particles, and also they can be built over pointlike fields via integration. This class of fields can also provide a reformulation of some perturbative gauge theories, like QED, solving the old clash between Hilbert space positivity and pointlike localization by substituting a massless gauge potential with a string-localized potential.

More recently Schuster and Toro worked on these infinite-spin representations from a different point of view, finding new equations (different from Wigner's ones) for covariant CSPs wavefunctions and trying to build up a consistent gauge theory for CSPs. Working in one-particle quantum mechanical setting, they managed to identify new "smooth" solutions in addition to the "singular" ones and they wrote a gauge theory action for bosonic (free) CSPs using pointlike fields in a spacetime enlarged by an extra coordinate -a sort of "spin" space- $\eta$ ( [9] [10]). The action likely has simple generalizations to describe fermionic and/or supersymmetric CSPs, also in other spacetimes like (A)dS, and can also be extended to admit couplings with suitable conserved background currents( [11] [12] [13]). The action is local, Lorentz invariant, localized on the hyperbolic surface $\eta^2 = -1$ and the fields show standard commutation relations, but the canonical fields have non-trivial gauge variation and therefore they are not directly observable as expected from the previous considerations( [11]). It is also possible to find a local "gauge-fixed" Hamiltonian, but at the price of imposing a spatially non-local gauge fixing condition( [10]).

For the second problem, the continuous spin particles contain indeed an infinite tower of helicity-eigenstate with integer-spaced eigenvalues in the general case $\kappa \neq 0$. Therefore if all CSPs states thermalized democratically and rapidly enough, it would lead to rapid supercooling of all thermal systems due to infinite heat capacity per unit volume. But luckily along with particles of spin $s \leq 2$, CSPs are the only massless states possessing covariant soft factors, which opens the possibility that CSPs may mediate long-range forces. Indeed for a candidate theory of interacting massless particles of higher helicities in a flat spacetime (higher-spin gravity) the two exotic properties of CSPs, namely the presence of infinite degrees of freedom per spacetime point and the presence of a continuous mass scale $\kappa > 0$, could be two positive features. A standard feature of higher-spin theories in dimension four (and higher) is an infinite spectrum of helicities, and the spectrum of helicities in the CSPs case coincides with the one in higher-spin gravity([35]). Secondly higher-spin vertices are typically higher-derivative, and a dimensionful parameter for weighting them is a necessary feature of any interacting theory. Therefore continuous spin gauge fields might be able to circumvent the (Weinberg's) no-go theorems preventing the existence of interacting particles of spin greater than two in flat spacetime and might provide a "subtle flat spacetime analogue" of higher-spin gravity. Thanks to the hierarchical coupling structure in soft factors, also the second problem could be solved in real systems that are only approximately in thermodynamic equilibrium and only for a finite time. If additional CSP states are sufficiently weakly coupled, then the timescale for quasi-thermal systems to dissipate energy into those states is long enough to be physically irrelevant. The problem can then be traced to find a theory which produces such CSP covariant soft factors, as Schuster and Toro explained in their papers( [9]). We still don't know yet if these type of particles exist, but the interest on the topic has increased a lot over recent years due to new striking discoveries we mentioned.

The scope of the thesis is to study the general form of quantum (free) fields for CSP, particularly focusing on the structure of the intertwiners (the coefficients of the annihilation and creation operators in the mode expansion of the fields). I will first establish, parametrizing directly the orbits of the little group for massless particles $E(2)$, a new way to find the "smooth" and "singular" solutions of Schuster and Toro papers. Then I will make a deep connection between the general structure of Mund-Schroer-Yngvason intertwiners with Schuster-Toro smooth wavefunctions via Gaussian integration. Moreover I will emphasize the role of localization for infinite-spin intertwiners, considering both Mund-Schroer-Yngvason and Schuster and Toro's works from different perspectives. The Mund-Schroer-Yngvason bound ( [6]) regards the admissible class of infinite-spin intertwiners. We will give an estimate about a special type of intertwiner, showing that it will fulfill the Mund-Schroer-Yngvason bound if it is multiplied by a suitable factor. Finally I will discuss the properties of the 2-point function using the general structure of infinite-spin intertwiners.

# 2 Poincaré representations in Quantum Field Theory

## 2.1 The importance of the Poincaré symmetry

Spacetime plays a fundamental role in physics: it is not only the stage in which particle interactions take place, but it also exhibits its own dynamics. The underlying symmetries of spacetime are divided into two classes (the nomenclature is dictated by group theory), namely discrete symmetries (charge conjugation C, parity inversion P and time reversal T) and continuous symmetries (Poincaré symmetry, that comprised Lorentz symmetry and translational symmetry).

Poincaré symmetry is a cornerstone of both of our current best theories of physics: General Relativity (GR) and Standard Model of particle physics (SM). In fact laws of physics in the absence of gravity are Poincaré covariant, that is they are invariant in form both under spacetime translations and under the action of the Lorentz group (Lorentz invariance is a global symmetry of SM in a flat spacetime) [14]. Moreover in GR usually the formulation of Einstein Equivalence Principle (SEP) includes local position invariance and local Lorentz invariance for all test experiments (including gravity). Regarding CPT symmetry that relates a particle to its antiparticle, the SM is CPT-invariant by construction [15]. Considering the broad field of applicability of these symmetries, searches for Poincaré symmetry breaking or CPT-invariance breaking would provide a powerful test of fundamental physics (it could be a signature of some quantum gravity effects).

In Quantum Field Theory (QFT), on which the whole SM rely, the well known CPT theorem ensures CPT symmetry if the theory satisfies some mild physical requirements: the spectral condition (energy positivity), locality and relativistic covariance (Lorentz symmetry) ( [16], [17] [18]). It can be proved that also CPT breaking implies Lorentz violation under quite similar assumptions ("anti-CPT theorem" [19]), but the converse of this statement is not true. Therefore Lorentz symmetry is intimately related to CPT symmetry. Now let us suppose that translational symmetry is broken: then the energy-momentum tensor $T_{\mu\nu}$, which is the generator of translations, is tipically no longer conserved. Then in general the angular momentum tensor $M_{\mu\nu}$, defined as $M_{\mu\nu} = \int \mathrm{d}^3x \, (T_{0\mu}x_\nu - T_{0\nu}x_\mu)$, will show a nontrivial dependence on time: with the exception of special cases, translation-symmetry violation leads to Lorentz breakdown( [20]). This is the reason why a dedicated effective field theory has been developed in order to systematically consider all hypothetical violations of the Lorentz invariance (LV) -and also general CPT violation-, that is called Standard Model Extension (SME). Actually this is not a model, it is just a test framework. The lagrangian is ( [21] [22] [23])

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{SM} + \delta\mathcal{L}_{LV} \tag{1.1}$$

It contains the SM of particle physics ($\mathcal{L}_{SM}$), GR (Einstein-Hilbert lagrangian $\mathcal{L}_{EH}$) and all possible Lorentz-violating terms ($\delta\mathcal{L}_{LV}$) that can be constructed at the level of the Lagrangian, introducing a large number of new coefficients (multiplied by a Lorentz-violating operator in a coordinate-independent product) that can be constrained experimentally. Any experimental signal for Lorentz violation can be expressed in terms of one or more of these coefficients, and it is important to identify and analyze suitable experiments that can provide ultra-sensitive tests.

The most recent and complete review on the subject is "Data Tables for Lorentz and CPT Violation" of Kostelecky and Russell( [24]). From all data they extracted four summary tables covering the sectors for matter (electrons, protons, neutrons, and their antiparticles), photons, neutrinos, and gravity with the best estimates for the maximal attained sensitivities to the relevant SME. The conclusion is that at the moment there is *no confirmed experimental evidence supporting Lorentz violation*.

## 2.2 Symmetries in quantum field theory

In relativistic quantum mechanics, we are interested in the concept of Poincaré transformations acting on quantum mechanical states of the theory as symmetries, since the laws of physics should be inertial frame invariant. A physical experiment should come up with the same results regardless of where, when, or what orientation the experiment is done in. The results of an experiment should also be invariant whether the experiment is done at different uniform and constant velocities. When we combine several systems together, the overall symmetry of the system should be related to the individual symmetries of its components ( [25]). The most elementary systems are identified with the concept of "elementary particles", although the definition of elementary particle is not entirely rigorous as discussed later on. In the following we will study the relationship between elementary particles and irreducible representations of the (double cover of) restricted Poincaré group, building them step by step.

Let's define a separable complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with an inner product $\langle\cdot,\cdot\rangle$ that induces a norm $\|\phi\| := \langle\phi,\phi\rangle^{\frac{1}{2}}$ and hence a topology. In quantum mechanics pure states (physical states) are represented by rays $\rho_{|\phi\rangle} = \{|\psi\rangle \in \mathcal{H} | \exists\lambda \in \mathbb{C}\backslash\{0\} \text{ s.t. } |\psi\rangle = \lambda\,|\phi\rangle\}$. If we define the equivalence relation $|\phi\rangle \sim |\psi\rangle \Leftrightarrow |\phi\rangle\,, |\psi\rangle \in \rho_{|\psi\rangle}$ in $\mathcal{H}$, then the proper "physical" Hilbert space would be given by the quotient space $\mathbb{P}(\mathcal{H}) := (\mathcal{H} \setminus \{0\})/\sim$ (actually a projective space of one dimensional linear subspaces of $\mathcal{H}$).

Let's recall that a unitary operator $U$ on $\mathcal{H}$ is a $\mathbb{C}$-linear bijective map $U : \mathcal{H} \to \mathcal{H}$ leaving the inner product invariant: $|\psi\rangle\,, |\phi\rangle \in \mathcal{H} \Rightarrow \langle\psi,\phi\rangle = \langle U\psi, U\phi\rangle$. The composition $U \circ V$ of two unitary operators $U$, $V$ is always unitary and the inverse $U^{-1} : \mathcal{H} \to \mathcal{H}$ of a unitary operator $U : \mathcal{H} \to \mathcal{H}$ is unitary as well. The composition of operators defines the structure of a group on the set of all unitary operators on $\mathcal{H}$, called $U(\mathcal{H})$ (the unitary group of $\mathcal{H}$). Let $\pi : \mathcal{H} \setminus \{0\} \to \mathbb{P}(\mathcal{H})$ be the canonical map with respect to the equivalence relation $\sim$ defined previously, that is $\rho_{|\psi\rangle} := \pi(|\psi\rangle)$. We define the "transition probability" $\mathrm{Pr} : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ as

$$\mathrm{Pr}(\rho_{|\psi\rangle}, \rho_{|\phi\rangle}) := \frac{|\langle\phi|\,\psi\rangle|^2}{\|\phi\|^2\|\psi\|^2} \tag{1.2}$$

This map defines a topology on $\mathbb{P}(\mathcal{H})$ generated by the open subsets $\{\rho_{|\psi\rangle} \in \mathbb{P}(\mathcal{H}) : \Pr(\rho_{|\psi\rangle}, \rho_{|\phi\rangle}) < r\}$, $r \in \mathbb{R}$, $\rho_{|\phi\rangle} \in \mathbb{P}(\mathcal{H})$. This is the quotient topology on $\mathbb{P}(\mathcal{H})$ with respect to the quotient map $\pi$: a subset $W \subseteq \mathbb{P}(\mathcal{H})$ is open iff $\pi^{-1}(W) \subseteq \mathcal{H}$ is open in the Hilbert space topology.

A Wigner symmetry (projective transformation) $\Sigma$ is defined[1] as an automorphism (bijective map) on $\mathbb{P}(\mathcal{H})$ that preserves the transition probabilities[2]

$$\Pr(\rho_{|\psi\rangle}, \rho_{|\phi\rangle}) = \Pr(\Sigma(\rho_{|\psi\rangle}), \Sigma(\rho_{|\phi\rangle}))\, \forall\ |\psi\rangle\,, |\phi\rangle \in \mathcal{H} \tag{1.3}$$

We define the group $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ of projective transformations to be the set of all projective transformations where the group structure is again given by composition. For every $U \in U(\mathcal{H})$ we define a map $\hat{\pi}(U) : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ by

$$\hat{\pi}(U)(\rho_{|\psi\rangle}) := \pi(U(|\psi\rangle)) \quad \forall\, \rho_{|\psi\rangle} = \pi(|\psi\rangle) \in \mathbb{P}(\mathcal{H}) \tag{1.4}$$

Therefore $\hat{\pi}(U) \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, and the same can be done for every other antiunitary operator $V : \mathcal{H} \to \mathcal{H}$. It can be easily shown that $\hat{\pi} : U(\mathcal{H}) \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ is a group homomorphism.

Wigner's theorem states that

**Theorem 2.1** (Wigner). *The symmetry $\Sigma \in Aut(\mathbb{P}(\mathcal{H}))$ is induced by an unitary or antiunitary operator $U : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ determined up to a phase factor, which satifies*

$$\Sigma = \hat{\pi}(U) \Leftrightarrow U\rho = \Sigma(\rho) \quad \forall\, \rho \in \mathbb{P}(\mathcal{H}) \tag{1.5}$$

We can define the group of unitary projective transformations $U(\mathbb{P}(\mathcal{H})) := \hat{\pi}(U(\mathcal{H})) \subseteq \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, so that the sequence

$$1 \to U(1) \xrightarrow{\iota} U(\mathcal{H}) \xrightarrow{\hat{\pi}} U(\mathbb{P}(\mathcal{H})) \to 1 \tag{1.6}$$

with $\iota(\lambda) = \lambda\, id_{\mathcal{H}}$, $\lambda \in U(1)$ defines an exact sequence of homomorphism and hence a central extension of $U(\mathbb{P}(\mathcal{H}))$ by $U(1)$.

From now on we will follow mainly the nice (and logical) presentation of the topic written by Schottenloher in his conformal field theory's book( [27]), with some remarks taken from the quantum field theory's book of Weinberg ( [28]) and the book on quantum mechanics written by Moretti ( [26]).

Let us consider a topological group $(G, \cdot, e)$ as a group of transformations of states acting on a physical system described by an Hilbert space $\mathcal{H}$, in such a way that every $g \in G$ is associated to a Wigner symmetry $\Sigma_g$. Then the notion of symmetry can be extended to a group of symmetry operations, and we can suppose the map $G \ni g \mapsto \Sigma_g$ to be a group homomorphism from $G$ to $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ (projective representation of $G$):

$$\Sigma_g \circ \Sigma_{g'} = \Sigma_{gg'} \qquad \Sigma_e = id \qquad \Sigma_{g^{-1}} = (\Sigma_g)^{-1} \tag{1.7}$$

[1] we are considering implicitly the case of a single sector of Hilbert space $\mathbb{P}(\mathcal{H})$, for the general case in which $\mathbb{P}(\mathcal{H})$ splits coherently see [26]

[2] more general kind of symmetries are allowed in the quantum mechanics framework, such as Kadison symmetries and Jordan-Segal automorphisms (see Moretti [26])

where $id$ is the identity automorphism.

This is the natural setting for the quantization of a classical phase space $Y$: we can physically associate to each classical symmetry $\tau : G \to \text{Aut}(Y)$, where $\text{Aut}(Y)$ is a group of transformations that leave invariant the physics of the classical system, a transformation of the quantum phase space $T_\tau : G \to \text{Aut}(\mathbb{P}(\mathcal{H}))$. $T_\tau$ has to respect the natural structures of $G$ and $\text{Aut}(\mathbb{P}(\mathcal{H}))$: since $\tau$ is required to be a group homomorphism, then $T_\tau$ should be also a group homomorphism which respects the physics of the quantum system and so $T_\tau(g) : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ has to preserve transition probabilities. If $T_\tau$ is injective, $T_\tau$ is called (faithful) projective representation on $\mathbb{P}(\mathcal{H})$. If $\tau$ is continuous between the natural topologies of G and Aut(Y), then we would require $T_\tau$ to be continuous also between the topologies of G and $\text{Aut}(\mathbb{P}(\mathcal{H}))$. The topology on $\text{Aut}(\mathbb{P}(\mathcal{H}))$ is intended to be the strong operator topology (which in this case corresponds to the topology of pointwise convergence), namely $W(\Sigma_0, r) := \{\Sigma \in \text{Aut}(\mathbb{P}(\mathcal{H})) : \text{Pr}(\Sigma_0(\rho_\psi), \Sigma(\rho_\psi)) < r\}$ are the open neighborhoods of $\Sigma_0 \in \text{Aut}(\mathbb{P}(\mathcal{H}))$. In a similar manner we can construct the strong topology for $U(\mathcal{H})$ and $U(\mathbb{P}(\mathcal{H}))$, that are also connected and metrizable as topological spaces.

The important point is that in general the maps $\Sigma_g : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ are not linear, but with the Wigner theorem we can associate to each $\Sigma_g$ a unitary (linear) or an antiunitary (antilinear) operator $U_g$, defined up to a phase, acting on the Hilbert space $\mathcal{H}$. It is then interesting to ask ourselves if the map $G \ni g \mapsto U_g$ could be a unitary (linear) or an antiunitary (antilinear) representation of the group G on the Hilbert space $\mathcal{H}$. Reformulating the question, it is equivalent to ask if (for the unitary case) given $T' : G \to U(\mathbb{P}(\mathcal{H}))$, there could be a lift $S' : G \to U(\mathcal{H})$ such that $T' = \hat{\pi} \circ S$. The general answer is no. The reason is due to the fact that the property of composition $U_g \circ U_{g'} = U_{gg'} \,\forall\, g, g' \in G$ can't be satisfied, since it holds only

$$U_g \circ U_{g'} = \omega(g, g') U_{gg'} \,\forall\, g, g' \in G \tag{1.8}$$

with $\omega(g, g') \in \mathbb{C}$, $|\omega(g, g')| = 1$ ( $\omega(g, g') \in U(1)$ ).

But it is possible to prove explicitly that every projective unitary representation of a group $G$ is the restriction of a unitary representation of a suitable central extension $\widehat{G_\omega}$ (see Appendix A). Hence, by knowing central extensions of G whose multipliers are not equivalent and their unitary representations, we actually know the equivalence classes of projective unitary representations of $G$, and so all projective unitary representations of $G$.

Let's define the second cohomology group of the group $G$ with coefficients in $U(1)$ as

$$H_2(G, U(1)) := \{\omega : G \times G \to U(1) \mid \omega \text{ is a 2-cocycle}\}/\sim \tag{1.9}$$

where the equivalence relation $\omega \sim \omega'$ holds iff there is a $\chi : G \to U(1)$ with $\chi(g \cdot g') = \omega(g, g')\, \omega'(g, g')^{-1}\, \chi(g)\, \chi(g')$. We can notice that $H_2(G, U(1))$ is an abelian group with the multiplication induced by the pointwise multiplication of the maps $\omega$, and that $H_2(G, U(1))$ is in one-to-one correspondence with the equivalence classes of central extensions of $G$ by $U(1)$.

As a final comment to this question, let's consider the physical meaning of $\widehat{G_\omega}$ when there are no unitary representations of G, but only projective unitary representations. If we have a Wigner symmetry $G \ni g \mapsto \Sigma_g$ for our physical system with an Hilbert space (of states) $\mathcal{H}$, hence a projective representation on $\mathbb{P}(\mathcal{H})$ that is not describable by means of a unitary representation. We can take the central extension from $G$ to $\widehat{G_\omega}$ using the multipliers found, and then choose $\widehat{G_\omega}$ as the true symmetry group of our physical system, since the group action on the states of the system $\widehat{G_\omega} \ni (\chi, g) \mapsto \chi U_g$ is unitary. A specific central extension $\widehat{G_\omega}$ can be selected only by giving a physical meaning to the unitary representation of $\widehat{G_\omega}$: this can be done if $G$ turns into a Lie group.

Let's assume that $G$ is a Lie group from now on. If $G$ is also connected, for any projective representation $G \ni g \mapsto \Sigma_g$ the images $\Sigma_g$ can be associated to unitary operators only, according to Wigner's theorem. We assume then also that the projective unitary representation $G \ni g \mapsto U_g$ is continuous, and in general we can choose the phases (multipliers) so that the representation $G \ni g \mapsto U_g$ is strongly continuous[3], but only around the identity $e \in G$. Anyway, there exist in our hypothesis a central extension $\widehat{G_\omega}$ and a strongly continuous unitary representation $\widehat{G_\omega} \ni (\chi, g) \mapsto V_{(\chi,g)}$ with $\omega(e,e) = 1$, $V_{(\chi,e)} = \chi\, 1\ \forall\, \chi \in U(1)$.

Moreover $\widehat{G_\omega}$ is a connected Lie group; the canonical inclusion $\iota : U(1) \to \hat{G_\omega}$ and the canonical projection $\mathrm{pr}_2 : \widehat{G_\omega} \to G$ are Lie group homomorphisms. Also $\widehat{G_\omega}$ as a differentiable manifold is, around the identity, the local product of $U(1)$ and $G$ and the map $(1, g) \mapsto V_{(1,g)}$ is a strongly continuous projective unitary representation that induces $G \ni g \mapsto \Sigma_g$: $\Sigma_g(\rho) = V_{(1,g)} \rho V^{-1}_{(1,g)}\ \forall\, g \in G, \rho \in \mathbb{P}(\mathcal{H})$.

At this point, we need only another result before Bargmann's theorem. If $G$ is not simply connected, we can equivalently consider the continuous unitary projective representations of the universal covering group $\tilde{G}$ of $G$ ($\pi : \tilde{G} \to G$) in place of those of $G$. Infact continuous projective representation $\Sigma : G \ni g \mapsto \Sigma_g$ of G on the Hilbert space $\mathcal{H}$ arises from the continuous projective representation $\tilde{\Sigma} : \tilde{G} \ni g \mapsto \Sigma_g$ on $\mathcal{H}$ such that $\ker(\pi) \subset \ker(\Sigma \circ \pi)$, induced by $G \simeq \tilde{G}/\ker(\pi)$. This is due to the fact that since $\pi : \tilde{G} \to G$ is a continuous homomorphism of topological groups and $\Sigma : G \ni g \mapsto \Sigma_g$ is a continuous projective $G$-representation, then $\Sigma \circ \pi : \tilde{G} \ni h \mapsto (\Sigma \circ \pi)(h)$ is a continuous projective $\tilde{G}$-representation.

We have finally the following important

**Theorem 2.2** (Bargmann)**.** *Let $G$ be a connected and simply connected finite dimensional Lie group with $H_2(\mathfrak{g}, \mathbb{R}) = 0$, where $\mathfrak{g}$ is the Lie algebra of $G$. Then every continuous projective $G$-representation on the Hilbert space $\mathcal{H}$ is induced by a strongly continuous unitary representation on $\mathcal{H}$: every continuous projective representation $T : G \to U(\mathbb{P}(\mathcal{H}))$ has a lift as a strongly unitary representation $S : G \to U(\mathcal{H})$ where $T = \hat{\pi} \circ S$.*

Bargmann's theorem holds for simply connected Lie groups $G$ whose Lie algebra is simple or semisimple. Physically important cases are $SL(2, \mathbb{C})$ (the universal covering of the Lorentz group) and the universal covering of the Poincaré group, since the Lie algebras of those groups are semisimple. Therefore, dealing with relativistic quantum theories[4] one can always take advantage of Bargmann's theorem dealing with spacetime symmetries.

[3] the map $g \mapsto U_g$ is strongly continuous if $U_g \rho_{|\psi\rangle} \to U_{g_0} \rho_{|\psi\rangle}$ as $g \to g_0$ for each $\rho_{|\psi\rangle} \in \mathbb{P}(\mathcal{H})$

[4] Conversely, for example, the treatment of spacetime symmetries in Galilean quantum mechanics is much more complicated

## 2.3 Poincaré group and relativistic wave equations

The Poincaré group arises in different contexts, as a manifestation of spacetime symmetries. The spacetime is usually conceived as a 4-dimensional differential manifold $\mathcal{M}$[5], with events labelled by coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ (Minkowskian coordinates dependent on the reference system of the observer, see [29]). The spacetime interval between two events $x, y \in \mathcal{M}$ is defined as $(x-y)^2 := (x-y)^\mu \eta_{\mu\nu}(x-y)^\nu$: it is said to be time-like, light-like or spacelike if respectively $(x-y)^2 > 0$, $(x-y)^2 = 0$[6] or $(x-y)^2 < 0$.

Poincaré group arises as full symmetry group of Minkowski flat space (pseudo-Riemannian manifold) $(\mathcal{M}, \eta) := (\mathbb{R}^4, \eta)$, with $\eta =$diag$(1, -1, -1, -1)$ non degenerate bilinear symmetric form on $\mathbb{R}^4$. The metric has signature $(1, 3)$, so $(\mathcal{M}, \eta)$ is a Lorentzian manifold. Each tangent vector space of a 4-dimensional Lorentzian manifold is isomorphic to Minkowski space-time, hence the automorphism group (group of diffeomorphic isometries of $(\mathcal{M}, \eta)$) of such a tangent vector space $T_x\mathcal{M}$ is the semi-direct product $\mathcal{P} := \mathbb{R}^{1,3} \rtimes O(1, 3)$, the largest possible local symmetry group of the spacetime.

The Poincaré group arises also from another point of view, namely the space of solutions of a relativistic equation (see [30]). We want to find what field equations are compatible with the given transformation law for the Poincaré group. In general, a phase space can be thought of as the space of initial conditions for an equation of motion. In a non relativistic field theory, the equation of motion is the first order in time Schrödinger equation, and the phase space is the space of fields (wavefunctions) at a specified initial time, say at $t = 0$. This space carries a representation of the time-translation group $\mathbb{R}$ and the Euclidean group $E(3) := \mathbb{R}^3 \rtimes SO(3)$. To construct a relativistic quantum field theory, we want to find an analog of this space of wavefunctions, that will be some sort of linear space of functions satisfying a specific equation of motion (and that we can then quantize later by applying "harmonic oscillator" methods). The space of solutions of this equation of motion provides a representation of the group of spacetime symmetries of the theory, that is Poincaré group. To construct representations of $\mathcal{P}$, we begin to define an action of $\mathcal{P}$ on $n$-component wavefunctions $\psi(x)$ by

$$\psi(x) \mapsto U(\Lambda, a)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}(x-a)) \tag{1.10}$$

where $S$ is a suitable matrix. This is the action one gets by identifying $n$-component wavefunctions with functions on $\mathcal{M} \otimes \mathbb{C}^n$ and using the induced action on functions for the first factor in the tensor product, on the second factor taking $S(\Lambda)$ to be an $n$-dimensional representation of the Lorentz group. One then chooses a differential operator $D$ on $n$-component wavefunctions that commutes with the group action, so that

$$U(\Lambda, a)DU(\Lambda, a)^{-1} = D \tag{1.11}$$

[5] we will not consider quantum gravity's theories in the following, according to which the spacetime can be composed of discrete pieces at sufficiently short scales

[6] if two events are light-like separated, they are connected with a light signal of speed $c$. This is due to the postulate of costancy of the speed of light in special relativity.

Thefore the $U(\Lambda, a)$ give a representation of $\mathcal{P}$ on the space of solutions to the wave equation $D\psi = c\psi$ with constant $c$. If the space of solutions is not irreducible an additional set of "subsidiary conditions" can be used to pick out a subspace of solutions on which the representation is irreducible.

In the same manner, if the wavefunction $|\psi\rangle$ refer to a free particle and satisfy some relativistic wave equations, there exist a correspondence between the wavefunctions describing the same state in different Lorentz frames. We choose to work in the Heisenberg representation. Let $|\psi\rangle$ and $|\psi'\rangle$ be the wave functions of the same state in two Lorentz frames L and L′. Then $|\psi'\rangle = U(\Lambda, a)\,|\psi\rangle$ where $U(\Lambda, a)$ is a linear unitary operator. Thus a vector space contains, together with $|\psi\rangle$, all transforms $U(\Lambda, a)\,|\psi\rangle$, where $(\Lambda, a)$ is any element of the Poincaré group. A classification of all unitary representations of Poincaré group amounts to a classification of all possible relativistic wave equations. Two representations $U$ and $V\,U\,V^{-1}$ are equivalent if $V$ is a fixed unitary operator. If the system is described by wavefunctions $|\psi\rangle$, the description by

$$\left|\psi'\right\rangle = V\,|\psi\rangle \tag{1.12}$$

is isomorphic with respect to linear superposition, with respect to forming the inner product of two wave functions, and also with respect to the transition from one Lorentz frame to another. They belong to the same class of equivalent wave equations. The present discussion is not based on any hypothesis about the structure of the wave equations provided that they are covariant. In particular, it is not necessary to assume differential equations in configuration space (as we did previously). But it is a result of the group-theoretical analysis that every irreducible wave equation is equivalent, in the sense of the equation, to a system of differential equations for fields on Minkowski spacetime( [31]).

It turns out that the Poincaré group, to perform transformations between any inertial frames of reference, implements the relativity principle[7]. In quantum mechanics the inertial frame of reference is connected with the totality of macroscopic devices (laboratory), necessary for the complete description of elementary particles system under study. The devices participate in the quantum mechanics description twice: at first, by means of some device a quantum object state is prepared and then measurements of objects are performed with another device.

To the Poincaré group of transformations may be assigned a double meaning. On the one hand, it can describe a change of spacetime location of a physical system during two measurements in the same inertial frame of reference, that is: difference in location of two identical physical systems with respect to the given inertial frame of reference (active viewpoint). On the other hand, this transformation can characterize the difference between two inertial reference frames, in which one and the same system is being studied (passive viewpoint). We will use the active interpretation of the Poincare group transformation( [32]).

The Poincaré transformations $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$ with an element of the Poincaré group $g = (a, \Lambda)$ leave the spacetime interval $(y-x)^2$ between two events $x, y \in \mathcal{M}$ invariant. The

[7] which can be stated in the form "All systems of reference are equivalent with respect to the formulation of the fundamental laws of physics"

multiplication law in the Poincaré group is $(a_2, \Lambda_2) \cdot (a_1, \Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2\Lambda_1)$. Under the active interpretation of the Poincaré transformation, every initial physical state $\Psi^{(in)}$ is compared with a transformed state $\Psi_g^{(in)}$, which differs from the initial state by arrangement of preparing devices. The state $\Psi_g$ is related to a facility with preparing devices which has been shifted, turned or is evenly moving with respect to the previous position. In the same way, every state $\Psi^{(f)}$, registered by measuring devices, can be compared with a transformed state $\Psi_g^{(f)}$. To find the probability of a transition $\Psi_g^{(in)} \to \Psi_g^{(f)}$, which we denote by $R_{i_g} \to R_{f_g}$, we must repeat the experiment $\Psi^{(in)} \to \Psi^{(out)}$ (with an identical physical system) in which all the preparing and and measuring devices must be transformed with respect to the initial state according to the geometrical sense of g. The existence of the Poincaré spacetime symmetry exhibits in the equality

$$R_{i_g} \to R_{f_g} = R_i \to R_f \tag{1.13}$$

This is the reason why we can define the Wigner symmetries for the Poincaré group, and the whole apparatus we built in the last chapter can then be used.

The main properties of the Poincaré group are listed in the appendix B, to which the interested reader are referred to. The Lie algebra associated to Poincaré group is the Poincaré algebra $\mathfrak{iso}(3,1)$, presented by the generators $\{P^\mu, M^{\nu\rho}\}$ and by the commutation relations

$$-i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\rho\nu} - \eta_{\mu\sigma}M_{\nu\rho} \tag{1.14}$$

$$-i[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \tag{1.15}$$

$$-i[P_\mu, P_\nu] = 0 \tag{1.16}$$

There are two notable subalgebras of the Poincaré algebra, namely the abelian Lie algebra of the translation group $\mathbb{R}^4$ presented by the generators $\{P^\mu\}$ and by the commutation relations (1.16) and the (non abelian) Lorentz algebra $\mathfrak{so}(3,1)$ presented by the generators $\{M^{\nu\rho}\}$ and by the commutations relations (1.14). It is easy to see that (1.15) implies that $\mathbb{R}^4$ is an ideal of the Poincaré algebra, so that the Poincaré algebra is the semidirect sum $\mathfrak{iso}(3,1) = \mathbb{R}^4 \niplus \mathfrak{so}(3,1)$.

The Casimir elements of a Lie algebra $\mathfrak{g}$ are homogeneous polynomials in the generators of $\mathfrak{g}$, and for a finite dimensional Lie algebra they form a distinguished basis of the center of the universal enveloping algebra $\mathbb{Z}(U(\mathfrak{g}))$. Regarding the Lorentz algebra, there are only one quadratic Casimir invariant $\mathcal{C}_2(\mathfrak{so}(3,1)) = -\frac{1}{2}M^{\mu\nu}M_{\mu\nu}$ by Racah's theorem since the algebra is not only semisimple but also simple (and thus it has rank[8] 1). Instead Poincaré algebra has rank 2, and therefore it has 2 Casimir invariants:

$$\mathcal{C}_2(\mathfrak{iso}(3,1)) = P^\mu P_\mu \tag{1.17}$$

$$\mathcal{C}_4(\mathfrak{iso}(3,1)) = W^\mu W_\mu \tag{1.18}$$

where $W^\mu := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma$ is called Pauli-Lubanski vector.

[8] defined as the dimension of its maximal Cartan subalgebra

Let's recap some basic notions about helicity and spin. Given the 3-vector of rotation generators for the Lorentz group $J^i = \frac{1}{2}\epsilon^{ijk}M_{jk}$ and the massless particle's 3-momentum $p^i$, the (non-covariant) definition of the helicity $h$ is

$$\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}\,|\psi_h\rangle = h\,|\psi_h\rangle \tag{1.19}$$

where $|\psi_h\rangle$ is an helicity eigenstate. On the other hand in the case of massive particles, the definition of the spin value $s$ relies on the relationship

$$J^2\,|\psi_s\rangle = -\frac{W^2}{m^2}\,|\psi_s\rangle = s(s+1)\,|\psi_s\rangle \tag{1.20}$$

where $|\psi_s\rangle$ is a spin eigenstate.

## 2.4 Classification of irreducible representation of Poincaré group

The Poincaré group is not semisimple and moreover is not compact nor connected. For finite-dimensional representations, the non-semisimplicity means that we cannot use Weyl's theorem on complete reducibility which ensures the fact that all representations are built from the irreducible ones (unlike the Lorentz group case, for example). The lack of connectedness means that time reversal and space inversion has to dealt with separatly for representations of the full group, so in the following we will consider just the restricted Poincaré group. So the representations of the Poincaré group do not follow from a general framework of representation theory.

The representation theory of semi-direct products of the type $N \rtimes K$, where $N$ and $K$ are topological groups, will in general be rather complicated. However when $N$ is commutative we can define the set of character of $N$, called $\hat{N}$, that is the set of functions $\alpha : N \to \mathbb{C}$ that satisfy the homomorphism property $\alpha(n_1 n_2) = \alpha(n_1)\alpha(n_2)$ (therefore $\hat{N}$ is a group). Since $N$ is a Lie group, we can focus on the character that are differentiable functions on $N$, and in the particular case $N = \mathbb{R}^4$ the differentiable irreducible representations are one dimensional and given by $\alpha_{\mathbf{p}}(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}}$ with $\mathbf{x} \in N$. The character group is thus $\hat{N} = \mathbb{R}^4$, with elements labelled by the vector $\mathbf{p}$. From the structure of the semi-direct product we have an automorphism $\Phi_k$ of $N$ for each $k \in K$, such that $\Phi_k : n \in N \mapsto \Phi_k(n) \in N$ and the semi-direct product $N \rtimes K$ is defined as the set of pairs $(n, k) \in N \times K$ with group law

$$(n_1, k_1)\cdot(n_2, k_2) = (n_1\Phi_{k_1}(n_2), k_1 k_2) \tag{1.21}$$

Therefore from this action we get an induced action on $\hat{N}$ given by $\hat{\Phi}_k : \hat{N} \ni \alpha \to \hat{\Phi}_k(\alpha) \in \hat{N}$, with $\hat{\Phi}_k(\alpha)(n) = \alpha(\Phi_k^{-1}(n))$. Let's consider representations $(\pi, V)$ on a space $V$ of $N \rtimes K$: one can focus on the $N$ action, decomposing $V$ into subspaces $V_\alpha$ where $N$ acts according to the index $\alpha$ ($v \in V_\alpha \Leftrightarrow \pi(n,1)v = \alpha(n)v$). Acting by $K$ on the element $v \in V_\alpha$ will take it to $\pi(0,k)v \in V_{\hat{\Phi}_k(\alpha)}$, as could be easily proved. Therefore for each $\alpha \in \hat{N}$ one can look at its orbit under the action of $K$ by $\hat{\Phi}_k$, which will give a subset $O_\alpha \subseteq \hat{N}$. The little group (or stabilizer group) is the subgroup $K_\alpha \subseteq K$ of elements $k \in K$ such that $\hat{\Phi}_k(\alpha) = \alpha$ for any given $\alpha \in \hat{N}$. Now the action of $\pi$ on $V$ is completely characterized in terms of the $K$-orbits $O_\alpha$ for $\alpha \in \hat{N}$ and an irreducible representation of the little group $K_\alpha$.

Regarding the physical Poincaré group $\mathcal{P}_+^\uparrow$ it is clear that $\hat{N} = \mathbb{R}^4$ is the space of characters of the translation group of Minkowski space and each element $\alpha$ is labelled by $p^\mu$, that are the eigenvalues of the 4-momentum operator $P^\mu$[9]. It follows that the space-time dependence of the wavefunctions is of the form $e^{ip_\mu x^\mu}$. The irreducible representations are then characterized by the values of the scalar $P^2$, and the by the values of the other scalar $W^2$ or by the representation of the little group $K_p$ on the eigenspace of the momentum operators with eigenvalue $p$. We know that the action of the Lorentz group on the momentum space $\mathbb{R}^4$ is $p \mapsto \Lambda p$ and therefore we can classify representations by orbits, restricting ourselves (for simplicity) to the $(p_0, p_3)$ plane. For each orbit, we will choose a reference momentum $p_{\text{ref}}$. It can be easily seen that there are 6 types of orbits: ( [30], [33])

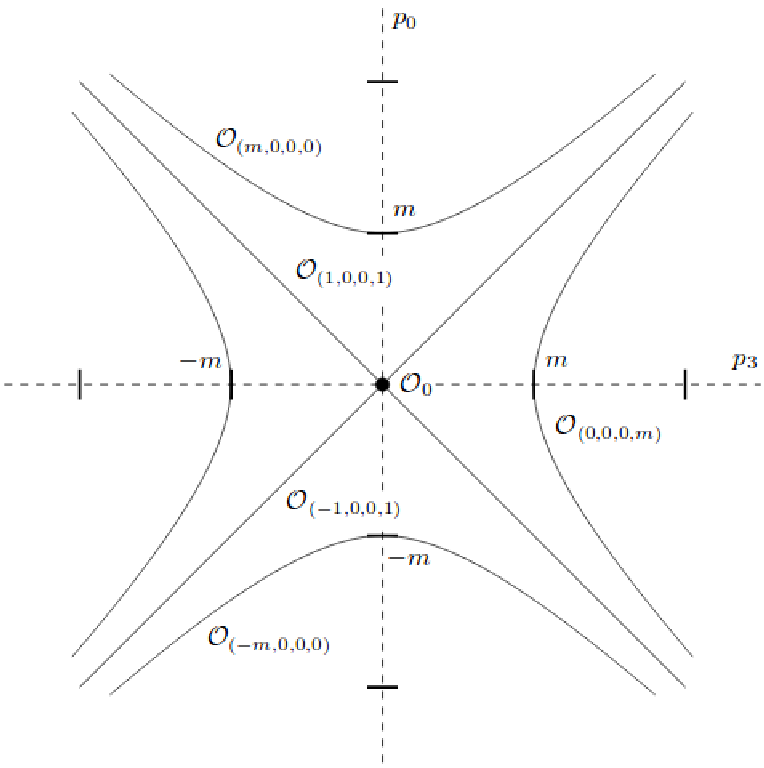


Figure 1.1: The classification of orbits of the Lorentz group action in the momentum space (taken from [30])

- Positive energy time-like orbits ($P^2 = m^2 > 0$, $p_{\text{ref}} = (m, 0, 0, 0)$): this is the upper positive energy sheet $\mathcal{O}_{(m,0,0,0)}$ of the massive hyperboloid $p^2 = m^2$, where the little group of $K_{(m,0,0,0)}$ is a subgroup of $SO^\uparrow(1,3)$ isomorphic to $SO(3)$. Irreducible representations of this group are classified by the spin $s$ as we can see also from the values of the scalar $W^2 = -m^2 J^2 = -m^2 s(s+1)$, where $J^2$ is the Casimir operator of

[9]from the equation (1.16) we can see that all the translation generators commute with each other

the subgroup of spatial rotations. Only integer values of the spin are allowed in this case, whereas for the the representations of the double cover of the Poincaré group also half-integer values are allowed. Functions on the hyperboloid both correspond to the space of all (positive energy) solutions to an appropriate wave equation (Klein-Gordon equation for spin $0$, massive Dirac equation for spin $\frac{1}{2}$, ...) and also carry an irreducible representation of the Poincaré group.

- Negative energy time-like orbits ($P^2 = m^2 > 0$, $p_{\text{ref}} = (-m, 0, 0, 0)$): this is the lower negative energy sheet $\mathcal{O}_{(-m,0,0,0)}$ of the massive hyperboloid $p^2 = m^2$, where the little group of $K_{(-m,0,0,0)}$ is again a subgroup of $SO^{\uparrow}(1,3)$ isomorphic to $SO(3)$. Irreducible representations of this group are classified again by the spin $s$ ($W^2 = -m^2 J^2 = -m^2 s(s+1)$), and functions on the hyperboloid both correspond to the space of all (negative energy) solutions to an appropriate wave equation and also carry an irreducible representation of the Poincaré group like the previous case.

- Space-like orbits ($P^2 = -m^2 < 0$, $p_{\text{ref}} = (0, 0, 0, m)$): this is a one sheet hyperboloid $\mathcal{O}_{(0,0,0,m)}$ where $p^2 = -m^2$, where the little group of $K_{(0,0,0,m)}$ is isomorphic to $SO^{\uparrow}(2,1) \simeq SL(2,\mathbb{R})$. There is no finite dimensional unitary representations of the Poincaré group because $SL(2,\mathbb{R})$ is simple and not compact.

- Positive energy null orbits ($P^2 = 0$, $p_{\text{ref}} = (|\vec{p}|, 0, 0, |\vec{p}|)$): this is the upper half $\mathcal{O}_{(|\vec{p}|,0,0,|\vec{p}|)}$ of the full null cone $p^2 = 0$, where the little group of $K_{(|\vec{p}|,0,0,|\vec{p}|)}$ is isomorphic to $E(2)$, the Euclidean group of the plane. We can distinguish two main classes of irreducible representations: the ones in which the translation subgroup of $E(2)$ acts trivially and the ones in which it doesn't. The first class is related to the one dimensional irreducible representations of $SO(2)$ group, labelled by an integer $h$ called helicity (that will become an half-integer when we will consider the double cover of the Poincaré group), whereas the second one are in correspondence with the infinite dimensional irreducible representation on a space of functions on a circle of radius $\rho$. In the first case $W^2 = 0$ and the eigenvalue of $J_3$ on the space of energy-momentum vectors $p_{\text{ref}}$ is $h$ and different values of $h$ bring to different wave equations (Klein-Gordon equation for massless scalars with $h = 0$, Weyl equation for Weyl spinors with $h = \pm\frac{1}{2}$, Maxwell equations for Photons with $h = \pm 1$,...). In the second case $W^2 = -\rho^2$, where $\rho$ has the dimension of a mass, and the representations are called infinite spin (or continuous spin) representations.

- Negative energy null orbits ($P^2 = 0$, $p_{\text{ref}} = (-|\vec{p}|, 0, 0, |\vec{p}|)$): this is the bottom half $\mathcal{O}_{(-|\vec{p}|,0,0,|\vec{p}|)}$ of the full null cone $p^2 = 0$, where the little group of $K_{(-|\vec{p}|,0,0,|\vec{p}|)}$ is isomorphic again to $E(2)$. The classification is completely analogous to the previous case.

- Zero orbit ($P^2 = 0$, $p_{\text{ref}} = (0, 0, 0, 0)$): this is a single point orbit $\mathcal{O}_{(0,0,0,0)}$ and the stabilizer group $K_{(0,0,0,0)}$ is the whole Lorentz group $SO^{\uparrow}(1,3)$. Of course $W^2 = 0$, and moreover for each finite dimensional representation of the Lorentz group one gets a corresponding representation of the Poincaré group, but not a unitary one.

Let's recap some definitions about the four-vectors $v^\mu$ on a Lorentzian manifold $\mathcal{M}$. The vector $v^\mu$ is said to be timelike, spacelike or lightlike if $v \cdot v > 0$, $v \cdot v < 0$ or $v \cdot v = 0$. Since the tangent space $T_x\mathcal{M}$ at an arbitrary point $x$ of a Lorentzian manifold is isomorphic to a Minkowski space $\mathcal{M}$, then the (dual) cotangent space $T_x^*\mathcal{M} \simeq \mathbb{R}^{3,1}$ can be regarded as the Minkowski energy-momentum space. Let us introduce an indefinite metric on $T_x^*\mathcal{M}$ given by $\langle p, p\rangle := p_0^2 - |\vec{p}|^2$. We can thus define some important structures:

- Forward mass hyperboloid $H_m^+ := \{p \in \mathcal{M} : \langle p, p\rangle = m^2, p_0 > 0\}$
- Backward mass hyperboloid $H_m^- := \{p \in \mathcal{M} : \langle p, p\rangle = m^2, p_0 < 0\}$
- Forward light cone $\partial V^+ := \{p \in \mathcal{M} : \langle p, p\rangle = 0, p_0 > 0\}$
- Backward light cone $\partial V^- := \{p \in \mathcal{M} : \langle p, p\rangle = 0, p_0 < 0\}$

Accordingly, in the following table we summarize the extension of our previous result for the unitary irreducible representations of $\mathcal{P}_+$, grouping together positive and negative energy orbits ( [31]):

| Spectrum of p | Orbit | Stability subgroup | UIR[10] |
|---|---|---|---|
| $\langle p, p\rangle = m^2 > 0$ | Mass (timelike) hyperboloid $H_m^+ \cup H_m^-$ | $SO(3)$ | Massive |
| $\langle p, p\rangle = -m^2 < 0$ | Spacelike hyperboloid | $SO^\uparrow(2,1)$ | Tachyonic |
| $\langle p, p\rangle = 0$ | Light cone $\partial V^+ \cup \partial V^-$ | $E(2)$ | Massless |
| $p^\mu = 0$ | Origin | $SO^\uparrow(3,1)$ | Zero-momentum |

where we can notice that the hypersurfaces of constant momentum square $\langle p, p\rangle$ are quadric of curvature radius $m > 0$.

This concludes a generic discussion about the (unitary) irreducible representations of the restricted Poincaré group $\mathcal{P}_+^\uparrow$, that as we already mentioned permit to classify the unitary action $(\psi(x) \to U(\Lambda, a)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}(x - a)))$ on n-component wavefunctions (solutions of some wave equation). However in relativistic particle physics we actually need the unitary irreducible representations of the double cover of the physical Poincaré group $\widetilde{\mathcal{P}_+^\uparrow} := \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ in order to classify relativistic one-particle states.

## 2.5 The method of induced representations

Let's consider a relativistic state vector $|\Psi\rangle \in \mathcal{H}_1$ (where $\mathcal{H}_1$ is the one-particle physical Hilbert space) on which the Poincaré group acts like

$$|\Psi\rangle \mapsto U(\Lambda, a)\,|\Psi\rangle \tag{1.22}$$

We notice from the equation (1.16) that the components of the energy-momentum vector all commute with each other. Therefore it is natural to label physical states $|\Psi\rangle$ in terms of eigenvectors of the translation generators $P^\mu$, introducing a label $\sigma$ to denote all other degrees of freedom. We take as part of the definition of one-particle state that the label $\sigma$ is purely discrete [11], so that

$$P^\mu |\Psi(p,\sigma)\rangle = p^\mu |\Psi(p,\sigma)\rangle \tag{1.23}$$

Since the unitary operator for infinitesimal transformations of the Poincaré group

$$U(\epsilon, 1+\omega) = 1 + \frac{1}{2} i\omega_{\rho\sigma} M^{\rho\sigma} - \epsilon_\alpha P^\alpha \tag{1.24}$$

where $\omega_{\rho\sigma}$ and $\epsilon_\alpha$ are infinitesimal parameters, then for finite translations $U(a,1)|\Psi(p,\sigma)\rangle = e^{-ip_\mu a^\mu} |\Psi(p,\sigma)\rangle$. To see the effect of an homogeneous Lorentz transformation $\Lambda^\mu{}_\nu$ on a state $|\Psi(p,\sigma)\rangle$, we apply the energy-momentum operator $P^\mu$ to the expression $U(0,\Lambda)|\Psi(p,\sigma)\rangle$:

$$\begin{aligned} P^\mu U(0,\Lambda)|\Psi(p,\sigma)\rangle =& U(0,\Lambda)U^{-1}(0,\Lambda)P^\mu U(0,\Lambda)|\Psi(p,\sigma)\rangle \\ =& U(0,\Lambda)(\Lambda^{-1}{}_\rho{}^\mu P^\rho)|\Psi(p,\sigma)\rangle = \Lambda^\mu{}_\rho p^\rho U(0,\Lambda)|\Psi(p,\sigma)\rangle \end{aligned} \tag{1.25}$$

where we used the equation (1.15) and the fact that $\Lambda^{-1}{}_\rho{}^\mu = \Lambda^\mu{}_\rho$. Therefore $U(0,\Lambda)|\Psi(p,\sigma)\rangle$ is an eigenvector of the energy-momentum operator $P^\mu$ with the eigenvalue $\Lambda p$ and it must be a linear combination of the form

$$U(0,\Lambda)|\Psi(p,\sigma)\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)\,|\Psi(\Lambda p,\sigma')\rangle \tag{1.26}$$

where he transformation matrix $C_{\sigma'\sigma}(\Lambda,p)$ should be unitary with respect to the norm

$$\langle \Psi(p,\sigma) | \Psi(p',\sigma') \rangle = 2p_0 \delta^3(\vec{p} - \vec{p'})\delta_{\sigma\sigma'} \tag{1.27}$$

In general it is possible to choose the $\sigma$ labels in such a way that each $|\Psi(p,\sigma)\rangle$ within the $\sigma$-block furnish an unitary representation of the Poincaré group, that is each $C_{\sigma'\sigma}(\Lambda,p)$ is block-diagonal. It is quite natural to identify the components of an irreducible unitary representation of the Poincaré group with the states of a specific particle type[12], since it cannot be further decomposed in this way. Thus all the states $|\Psi(p,\sigma)\rangle$ in an unitary irreducible representation of the Poincaré group have momenta $p$ belonging to the orbit of a single reference momentum $p_{\text{ref}}$. The classification of all orbits for different $p_{\text{ref}}$ under the action of the Lorentz group follows directly from the table (2.4) (and also from the detailed list before, if we consider separatly the values of $p^2_{\text{ref}}$ and $p^0_{\text{ref}}$). Moreover one can define the $|\Psi(p,\sigma)\rangle$ of momentum $p^\mu$ by

$$|\Psi(p,\sigma)\rangle = N(p)U(B_p)|\Psi(p_{\text{ref}},\sigma)\rangle \tag{1.28}$$

[11] in the general case for many unbound particles the label $\sigma$ has to include also continuous labels
[12] with some exceptions

where $B_p$ is a standard boost such that $p = B_p p_{\text{ref}}$ for each momentum of the orbit of reference momentum $p_{\text{ref}}$ and $N(p)$ is just a normalization factor. Considering an arbitrary Lorentz transformation one finds

$$U(0,\Lambda)\,|\Psi(p,\sigma)\rangle = N(p)U(\Lambda B_p)\,|\Psi(p_{\text{ref}},\sigma)\rangle = \tag{1.29}$$

$$= N(p)U(B_{\Lambda p})U(B^{-1}_{\Lambda p}\Lambda B_p)\,|\Psi(p_{\text{ref}},\sigma)\rangle \tag{1.30}$$

We see that $W(\Lambda,p) := B^{-1}_{\Lambda p}\Lambda B_p$ takes $p_{\text{ref}}$ to $B_p p_{\text{ref}} = p$, then to $\Lambda p$ and finally back to $p_{\text{ref}}$, so it belongs to the stabilizer (little group) $Stab_p \subseteq \mathcal{L}_+^\uparrow$ corresponding to the reference momentum $p_{\text{ref}}$. For any $W, \overline{W} \in Stab_p$ one has

$$U(W)\,|\Psi(p_{\text{ref}},\sigma)\rangle = \sum_{\sigma'} d_{\sigma'\sigma}(W)\,\big|\Psi(p_{\text{ref}},\sigma')\big\rangle \tag{1.31}$$

and $d_{\sigma'\sigma} = \sum_{\sigma''} d_{\sigma'\sigma''}(\overline{W})d_{\sigma''\sigma}(W)$: the coefficients $d(W)$ furnish a representation of the little group. Therefore

$$U(0,\Lambda)\,|\Psi(p,\sigma)\rangle = N(p)\sum_{\sigma'} d_{\sigma'\sigma}(W(\Lambda,p))U(B_{\Lambda p})\,\big|\Psi(p_{\text{ref}},\sigma')\big\rangle = \tag{1.32}$$

$$= \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} d_{\sigma'\sigma}(W(\Lambda,p))\,\big|\Psi(\Lambda p,\sigma')\big\rangle$$

Finally the problem of determining the coefficients $C_{\sigma'\sigma}$ has been reduced to the problem of finding the coefficients $d_{\sigma'\sigma}$. This is the physical picture that lies behind Mackey's machinery[13] and that allows us to consider just the stabilizer (Little group) $Stab_p$ and the possible orbits for our reference vectors in order to classify all unitary irreducible representations of the Poincaré group.

The mathematical formulation of the method of the induced representations is called Mackey's machinery. The line of reasoning is quite similar to the one we took previously, and the strategy will turn out to be particularly useful in order to characterize completely relativistic one-particle states. Since $\widetilde{\mathcal{P}_+^\uparrow}$ is a locally compact group, we denote by $Ind_{H\uparrow\widetilde{\mathcal{P}_+^\uparrow}} U_H$ the unitary representation of $\widetilde{\mathcal{P}_+^\uparrow}$ induced by $U_H$, where $H$ is a closed (normal) subgroup of $\widetilde{\mathcal{P}_+^\uparrow}$ and $U_H$ is a unitary representation of $H$. The translation group $\mathbb{R}^4$ is a normal subgroup of $\widetilde{\mathcal{P}_+^\uparrow}$, and so it is useful to consider the stabilizer $\overline{Stab_p}$ of the character $p$ (dual group of the translations) for the (adjoint) action of $\widetilde{\mathcal{P}_+^\uparrow}$. It is clear that this is related to the stabilizer (Little group) $Stab_p$ for the action of $\mathcal{L}_+^\uparrow := SO^\uparrow(1,3)$ acting naturally on $\mathbb{R}^4$ via $p \mapsto \Lambda p$ via $\overline{Stab_p} = Stab_p \rtimes \mathbb{R}^4$, since the translation group acts trivially on itself.

Then the Mackey's therem states that given an irreducible unitary representation $U$ of $\widetilde{\mathcal{P}_+^\uparrow}$, this is induced $U = Ind_{\overline{Stab_p}\uparrow\widetilde{\mathcal{P}_+^\uparrow}} U_{\overline{Stab_p}}$ from an irreducible unitary representation $U_{\overline{Stab_p}}$ of $\overline{Stab_p}$ ( [34]). Therefore we have reduced the problem to considering only the unitary representations of the form $U_{\overline{Stab_p}}$, with $\overline{Stab_p} = Stab_p \rtimes \mathbb{R}^4$. The following relation holds

$$U_{\overline{Stab_p}}(g,x) = d(g)p(x) \quad g \in Stab_p, x \in \mathbb{R}^4 \tag{1.33}$$

[13]see the following paragraph

where $d$ is an irreducible representation of $Stab_p$ and $p$ is the character of $\mathbb{R}^4$.

We will focus on positive energy, massless unitary representations of $\widetilde{\mathcal{P}_+^\uparrow}$ from now on and thus we choose the reference vector for the boundary of the forward light cone $\partial V^+$ as $p^\mu_{\text{ref}} = (1,0,0,1)$. From the table (2.4) we recognize that the stabilizer (Little group) for the action of the Lorentz group $\mathcal{L}_+^\uparrow$ is isomorphic to the two-dimensional Euclidean group $E(2)$, but since in this case we deal with the the double cover of the Lorentz group $\widetilde{\mathcal{L}_+^\uparrow}$ then $Stab_p = \widetilde{E(2)} = \widetilde{\mathbb{R}^2 \rtimes O}(2) = \mathbb{R}^2 \rtimes \mathbb{T}$. $\widetilde{E(2)}$ is a 3-dimensional simply connected and non-compact Lie group, with 2 degrees of freedom related to translational symmetries and 1 degree of freedom to rotational symmetry. We can parametrize this Little group $\widetilde{E(2)}$ as

$$Stab_p = \widetilde{E(2)} = \left\{ \begin{bmatrix} e^{i\theta} & a_1 + ia_2 \\ 0 & e^{-i\theta} \end{bmatrix} | a_1, a_2 \in \mathbb{R}, \theta \in [0, 2\pi[ \right\} \tag{1.34}$$

Since $\widetilde{E(2)} = \mathbb{R}^2 \rtimes \mathbb{T}$, the unitary irreducible representation $d$ of $\widetilde{E(2)}$ fits into one of the following two classes:

- the restriction of $d$ to $\mathbb{R}^2$ is trivial: considering the dual of $\mathbb{T}$, in this case UIR are labelled by integers $h \in \mathbb{Z}$ or half-integers $h \in \frac{\mathbb{Z}}{2}$;
- the restriction of $d$ to $\mathbb{R}^2$ is non-trivial: in this case UIR are labelled by the Pauli-Lubanski parameter $\kappa > 0$, that is related to the square of the Pauli-Lubanski vector $W^\mu$ since $W^2 = -\kappa^2$ (instead in the massive case $W^2 = -m^2 s(s+1)$, with $P^2 = m^2$).

To the first class belongs all massless particles of finite helicity like photons, whereas the second class corresponds to infinite (continuous) spin particles. In particular in four dimensions there exist only two infinite spin representations: the single-valued (bosonic) and the double-valued (fermionic) ones, which contains a countably infinite tower of all integer or half-integer helicity degrees of freedom respectively. It is worth mentioning that here we are considering helicity as the eigenvalue of the helicity operator defined in the equation (1.19). Indeed given an energy $E$, the ratio $\frac{\kappa}{E}$ controls the mixing of adiacent helicity states under Lorentz boosts (in analogy for massive particles the ratio $\frac{\sqrt{-W^2}}{E}$ controls the mixing of adiacent spin states, since $W^2 = -m^2 s(s+1)$), and only in the limit $\kappa \to 0$ helicity becomes a boost-invariant quantum number( [9]). The continuous -and unconstrained- parameter $\kappa > 0$ is dimensionful (a kind of euclidean mass) and so the conformal invariance holds only for the first class of UIR, where $\kappa = 0$. This is the origin of the unfortunate terminology "continuous spin" particles (CSPs), even if the "spin" of these particles are by no means continuous (in contrast with anyons in 2+1 dimensions). On the other hand, the (discrete) helicities in the spectrum of these CSPs -for both bosonic and fermionic cases- are unbounded. Since one definition of the “spin” is as the bound on the (absolute value of the) helicity eigenvalues, this fact gives rise to the terminology "infinite spin" historically[14]. All these properties will be discussed with more details in the next chapter, after the development of all the mathematical formalism required for the explicit description of such type of representations.

[14]this will become clearer when we will consider the so-called infinite-spin limit

In order to understand better the structure of $\widetilde{E(2)}$ and the implications for the infinite spin representation, let's define the so-called light-cone coordinates. The reference lightlike momentum points along the third spatial direction, so it is useful to define

$$x^{\pm} = x^0 \pm x^3 \quad x'^1 = x^1 \quad x'^2 = x^2 \tag{1.35}$$

where the Minkowski metric reads $\eta_{++} = \eta_{--} = 0$, $\eta_{+-} = \eta_{-+} = 2$ and $\eta_{ij} = -\delta_{ij}$ for $i, j \in \{1, 2\}$. In this reference frame the reference vector becomes $p_{\text{ref}} = (p^+, 0, 0, 0) = (2|\vec{p}|, 0, 0, 0)$ and the (transversal part of) Poincaré algebra adapted to these coordinates, given by the generators $M_{ij}$ and $\pi_i := p^{\mu}_{\text{ref}} M_{\mu i} = p^{+}_{\text{ref}} M_{+i}$, can be written as

$$-i[M_{ij}, M_{rs}] = \eta_{jr} M_{is} + \eta_{js} M_{ir} - \eta_{ir} M_{rj} - \eta_{is} M_{jr} \tag{1.36}$$

$$-i[\pi_i, M_{rs}] = \delta_{ir}\pi_s - \delta_{is}\pi_r \tag{1.37}$$

$$-i[\pi_i, \pi_j] = 0 \tag{1.38}$$

It is worth noticing that the generators $\{M_{ij}, \pi_i\}$ span the Lie algebra of the two-dimensional Euclidean group and that the quadratic Casimir of the Euclidean algebra $\mathfrak{iso}(2)$ is the square of the translation generators (joint spectrum)

$$\mathcal{C}_2(\mathfrak{iso}(2)) = \pi^i \pi_i \tag{1.39}$$

The method of induced representation can be applied also to the classification of the UIR of the (double cover of) two-dimensional Euclidean group $E(2)$ ( [31]). Considering the one particle states $|\Psi(p, \sigma)\rangle$ labelled by the (fixed) reference momentum $p^{\mu}$ and the other physical degrees of freedom $\sigma$, we can express $|\Psi(p, \sigma)\rangle$ in terms of the eigenvectors $\xi^i$ of the translation generators $\pi^i$ since the equation (1.38) holds. Introducing a new label $\zeta$ to denote all remaining physical components, one thus considers states $|\Psi(p, \xi, \zeta)\rangle$ such that

$$\pi^i \, |\Psi(p, \xi, \zeta)\rangle = \xi^i \, |\Psi(p, \xi, \zeta)\rangle \tag{1.40}$$

Repeating the previous arguments, all UIR of the massless little group $\widetilde{E(2)}$ has been reduced to the problem of finding all UIR of the stability subgroup of the two-dimensional vector $\xi$, called short little group and isomorphic to $SO(2)$. According to the value of $\xi^2$, the non-trivial representation of the (short) little group are divided into two categories: the helicity representation and the continuous spin representation. The first one corresponds to $\xi^i = 0$ (the action of the translation operators $\pi^i$ is trivial) whereas the second one correspons to $\xi^2 = \kappa^2 \Leftrightarrow \xi^i \in \mathbb{S}^1 \subseteq \mathbb{R}^2$ (the action of the translation operators $\pi^i$ is non-trivial). In the last case the non trivial orbits $O_{\kappa} := \{\xi \in \mathbb{R}^2 | \xi^2 = \kappa^2\}$ are infact circles in $\mathbb{R}^2$ of radius $\kappa$. Since we are dealing with first-quantized elementary particle, the number of physical components is given by the Hilbert space carrying the UIR of the little group: in the infinite spin case the little group is non-compact, so we have an infinite number of components. In fact the most exotic property of CSPs is the presence of an infinite number of degrees of freedom per spacetime point.

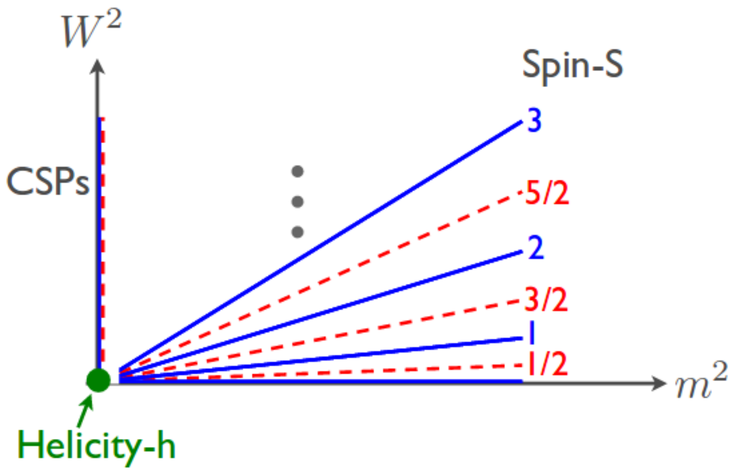


Figure 1.2: UIR of the Poincaré group can be (almost) completely characterized by the Casimir invariants $W^2$ and $P^2$. The massless (with spin fixed) limit, the helicity limit and the Pauli-Lubanski limit can be easily understood from this picture (taken from [9])

Finally, let us consider three interesting limits of the UIR of the (double cover of) Poincaré group we have considered so far(see [35]):

- The massless limit $m^2 \to 0$ (with spin fixed) of a spin $s$ massive representation gives the direct sum of helicity representations of helicity $|h| = s, s-1, ..., 0$;
- The helicity limit $\kappa \to 0$ of a single (bosonic or fermionic) continuous spin representation gives the direct sum of an infinite tower of helicity representations (either all integer spins or all half-integer spins);
- The zero-mass/infinite-spin (or Pauli-Lubanski) limit $m \to 0$, $s \to +\infty$ with the product $\kappa = ms$ fixed transforms the spin $s$ massive representation into the continuous spin representation (either bosonic or fermionic).

The last limit provides an interpretation of a continuous spin particle as the high-energy ($E \gg m$) large-spin ($s \gg 1$) limit of a massive particle in the regime $E \sim \kappa = ms$. In particular the dimensionful energy scale $\kappa$ can be seen as the remnant of the mass $m$ in a suitable massless limit (Pauli-Lubanski limit). And this limit can explain clearly the exotic properties of CSPs starting from the properties of the well-known massive representations. In the figure 1.2, the massless (respectively helicity) limit corresponds to going towards the origin along a spin $s$ massive line (respectively along a vertical line) whereas the Pauli-Lubanski limit corresponds to increasing the slope of the massive line till it becomes a vertical line.

## 2.6 The concept of elementary particles

In high energy physics, the phrase elementary particle is unfortunately used for truly elementary particles (these are stable particles, which do not decay into others) but also for a lot of unstable particles. Unstable particles are elementary particles that can decay by themselves, without interacting with another particle, into other elementary particles.

Let me start with the stable elementary particles. The very fact that they are called elementary means that all mathematical structures needed to describe their properties should be such that these cannot be decomposed into smaller entities. Otherwise, it would be extremely likely that the particle described with these mathematical structures could also be decomposed into "smaller" particles, it would not be stable, or it could at least be fragmented by external forces. Thus, if properties of particles are related to representations of (Lie) groups, stable elementary particles should be related to the smallest possible representations, i.e. the basic building block representations (which happen to be irreducible). All the other particles, which happen to be unstable, will ultimately decay into stable elementary particles. Thus, they should be related to representations which can be decomposed into the smallest ones. A particle can only decay into other particles, if the tensor product of all the representations related to the fragments contains the representation of the original particle. High energy experiments produce quite a lot of states which physicists like to associate with particles or excited states of particles. Since the compact Lie groups, which typically appear in physics, have the property that all of their finite dimensional representations can be decomposed into direct sums of irreducible representations, it became customary to associate the word particle in the sense of building block of physical entities to irreducible representations. But as explained above, only stable elementary particles are naturally associated with particular representations, the irreducible representations of universal cover of Poincaré group. For the others, it is merely a convenient way to get some order into the zoo of high energy physics. Therefore working with the irreducible representations is a convenient choice of basis. Regarding the question of stability, in that case there should be -also from an intuitive point of view- a time asymmetry (there is one favourite direction for the flowing of time) and indeed some authors considerer representations of a semigroup containing the Lorentz group and space-time translations into the future (see [36] [37])

To specify the free elementary particles capable of existing in a universe, i.e., the free elementary "particle ontology", one therefore specifies the projective, unitary, irreducible representations of a "local" space–time symmetry group. However, because because free particles are physical idealisations, one works backwards from the space–time symmetry group of interacting systems, and one bestows this symmetry group on free particles to ensure that when the idealisation is removed, and interactions are included, the space–time symmetry group is correct. Thus the symmetry group for free particles can be a subgroup determined by the space–time symmetry of interacting particles, and not necessarily the largest local space–time symmetry group permitted by the structure of space–time. In turn, the space–time symmetry group of interacting particles is determined by which gauge fields exist, and the way in which matter fields couple to those gauge fields. Infact in our case the parity and time-reversal symmetries are violated in nature, and the space–time symmetry group of interacting particles turns out to be only the physical Poincaré group $\mathcal{P}_+^\uparrow$ and not the full group of isometries of Minkowski space $\mathcal{P} = \mathbb{R}^4 \rtimes O(1,3)$. However, even with the local symmetry group fixed, the projective, unitary, irreducible representations of this group only determine the set of possible free particles. In our universe, only a finite number of elementary free particle types, of specific mass and spin, have been selected from the infinite number of possible free elementary particle types. Thus, in terms of describing the particle world in our universe at least, there is a type of 'surplus structure' in the SM. The discrepancy between actual and possible particle types in the standard model is either (i) an indication of the incomplete nature of the SM, or (ii) an indication that the masses and spins of the actual particle types in a universe is a matter of contingency ( [38]).

In terms of the Wigner representation, first quantization is the process of obtaining a Hilbert space of cross-sections of a vector bundle over mass hyperboloids $H_m^\pm$ and light cones $\partial V^\pm$. The second quantization then permits to construct a Fock space using such Hilbert space, treating it as a "one-particle" state space. In particular free particles of mass $m$ and spin $s$ correspond to vector bundles $E_{m,s}$ over mass hyperboloids $H_m^\pm$ and light cones $\partial V^\pm$ in the Minkowski energy momentum space $T_x^*\mathcal{M} \simeq \mathbb{R}^{1,3}$. The physically relevant irreducible (strongly continuous) unitary representation of the universal cover of the Poincaré group $\widetilde{\mathcal{P}_+^\uparrow}$ are provided by the Hilbert spaces $\mathcal{H}_{m,s}^\pm$ of square integrable (with respect to the base space) sections $\Gamma_{L^2}(E_{m,s}^\pm)$ of these vector bundles $E_{m,s}$. Moreover we notice that the irreducible unitary representation of $\widetilde{\mathcal{P}_+^\uparrow}$ on the space $\Gamma_{L^2}(E_{m,s}^+)$ of the square-integrable cross-sections of $E_{m,s}^+$ (over the forward mass hyperboloid or light cone) for a particle of mass $m$ and spin $s$ is unique up to unitary equivalence. The antiparticle os mass $m$ and spin $s$ is represented by the conjugate representation on the space $\Gamma_{L^2}(E_{m,s}^-)$ of the square-integrable cross sections of the vector bundle $E_{m,s}^-$ (over the backward mass hyperboloid or light cone). Therefore the particle is represented by the Hilbert space $\mathcal{H}$, then the antiparticle is represented by the conjugate Hilbert space $\overline{\mathcal{H}}$ and the two representations are related by an antiunitary transformation.

There is also another notion of elementary particles, according to which each different type of elementary particle is specified by the values of invariant properties, such as mass, spin and charge. This is compatible with the previous notion, since as we saw in the previous paragraphs the positive, unitary and irreducible representations of the physical Poincaré group are labelled by the value of the mass $m$ and of the spin $s$. Regarding charges, in relativistic quantum field theory there are superselection sectors that are eigenspaces of the charge operators. Therefore the most general one-particle Hilbert space can be always decomposed as the direct sum

$$\mathcal{H} = \bigoplus_{q,l,b\in\mathbb{Z}} \mathcal{H}(q,l,b) \tag{1.41}$$

where the indexes $q$, $l$ and $b$ are related to the electric, leptonic and baryonic charge respectively. The entire algebra of observables is represented on the full $\mathcal{H}$, and within this algebra there are also self-adjoint operators which represent electric charge $\mathcal{Q}$, leptonic charge $\mathcal{L}$ and baryonic charge $\mathcal{B}$ and which commute will all other operators representing physical observables. $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{B}$ possess integer-valued spectra: each superselection sector corresponds to a different combination $(q, l, b) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and possesses an UIR of the (double cover of) Poincaré group.

# 3 The infinite spin representation and string localized fields

## 3.1 The Hilbert one particle space in the infinite spin representation

As we saw in the previous chapter, an irreducible positive energy representation $U_1$ of $\mathcal{P}_+^\uparrow$ is characterized by the mass value $m^2 := \langle p, p\rangle$ and by an unitary irreducible representation $d$ of the little group $Stab_p$ acting in a Hilbert space $\mathfrak{h}$, where $p_{\text{ref}} \in H_m^+$ (forward mass hyperboloid) for $m > 0$ or $p_{\text{ref}} \in \partial V^+$ (forward light cone) for $m = 0$.

For the massive case, $Stab_p \simeq SU(2)$ for each $p_{\text{ref}} \in H_m^+$ and the irreducible representations of $SU(2)$ are parametrized by $s \in \frac{1}{2}\mathbb{Z}^+$. One possible realization of these $SU(2)$ representations can be the space $V_s$ of homogeneous polynomials of degree $s$ on $\mathbb{C}^2$. Therefore (by the method of induced representation) for a particle of mass $m$ and arbitrary spin $s$ we obtain a vector bundle $E_{m,s}^+$ over $H_m^+$ with typical fibre $V_s$ and an irreducible unitary representation of $\mathcal{P}_+^\uparrow$ upon the space $\Gamma_{L^2}(E_{m,s}^+)$ of square-integrable sections of the vector bundle $E_{m,s}^+$. For the massless case, $Stab_p \simeq \widetilde{E(2)}$ and the orbits of $\widetilde{E(2)}$ acting on $\mathbb{R}^2$ consist of concentric circles and the point at the origin. Considering the irreducible representations related to the circle (infinite-spin case), the method of induced representation provides a vector bundle over each circle and a representation of $\widetilde{E(2)}$ on the space of sections of each such vector bundle. Therefore we obtain a vector bundle $E_\kappa^+$ over $\partial V^+$ with an infinite-dimensional fibre. In the other case (helicity case), the isotropy group of the $\widetilde{E(2)}$-action is $\widetilde{SO(2)}$ and its irreducible representations are 1-dimensional and parametrized by $h \in \frac{1}{2}\mathbb{Z}^+$. For each such representation one has a vector bundle over a single point, with typical fibre isomorphic to $\mathbb{C}^1$ and a representation of $\widetilde{E(2)}$ upon the sections of such vector bundle. This can be extended to a representation of $\mathbb{R}^{(1,3)} \rtimes \widetilde{E(2)}$, and for each such representation parametrized by the helicity $h \in \frac{1}{2}\mathbb{Z}^+$ there is a vector bundle $E_{0,h}^+$ over $\partial V^+$, with typical fibre isomorphic to $\mathbb{C}^1$.

Therefore the representation $U_1$ acts on

$$\mathcal{H}_1 = L^2(H_m^+, \mathrm{d}\mu) \otimes \mathfrak{h} \qquad \mathrm{d}\mu(p) = \Theta(p_0)\delta(p^2 - m^2)\,\mathrm{d}^4p \tag{2.1}$$

for the massive case or on

$$\mathcal{H}_1 = L^2(\partial V^+, \mathrm{d}\mu) \otimes \mathfrak{h} \qquad \mathrm{d}\mu(p) = \Theta(p_0)\delta(p^2)\,\mathrm{d}^4p \tag{2.2}$$

for the massless case, where $\mathrm{d}\mu$ is the Lorentz invariant measure $\mathrm{d}\mu(p)$ on $H_m^+$ or $\partial V^+$ respectively. Since the forward light cone $\partial V^+$ is parametrized as $p_0 > 0$ and $|p_\perp|^2 = |p_1 + ip_2|^2 = (p_0 - p_3)(p_0 + p_3) = p_- p_+$ where $p_\perp = p_1 + ip_2$ and $p_\pm = p_0 \pm p_3$, then the $SO^\uparrow(1,3)$-invariant measure $\mathrm{d}\mu(p)$ on $\partial V^+$ has the form $\mathrm{d}\mu(p) = \Theta(p_0)\delta(p^2)d^4p = \Theta(p_0)\frac{\mathrm{d}p_+}{p_+}\,\mathrm{d}p_\perp\,\mathrm{d}p_\perp^*$.

Considering one particle states for the infinite-spin case, the Pauli-Lubanski parameter $\kappa$ labels nonequivalent representations of $\widetilde{E(2)}$ and therefore the representation space of $Stab_p$ is the Hilbert space $\mathcal{H}_\kappa$ of functions of $k \in \mathbb{R}^2$ square integrable with respect to the measure $\mathrm{d}\nu_\kappa = \delta(|k|^2 - \kappa^2)\,\mathrm{d}^2k$. Let $\psi(p)$ be an $\mathcal{H}_\kappa$-valued function of $p \in \mathbb{R}^4$, square integrable with respect to the Lorentz invariant measure $\mathrm{d}\mu$ on $\partial V^+$. Therefore the unitary representation $U_1$ of $\widetilde{P_+^\uparrow}$ on one particle Hilbert state

$$\mathcal{H}_1 = L^2(\partial V^+, \mathrm{d}\mu) \otimes \mathcal{H}_\kappa \qquad \mathcal{H}_\kappa = L^2(k, \mathrm{d}\nu_\kappa) \tag{2.3}$$

acts as ($U_1(a,\Lambda) \in \mathcal{B}(\mathcal{H}_1)$, where $\mathcal{B}(\mathcal{H}_1)$ is the space of all bounded linear operator in $\mathcal{H}_1$)

$$(U_1(a,\Lambda)\psi)(p) = e^{ip\cdot a} d_\kappa(W(\Lambda,p))\psi(\Lambda^{-1}p) \tag{2.4}$$

where $W(\Lambda,p) = B_p^{-1}\Lambda B_{\Lambda^{-1}p} \in \widetilde{E(2)}$ denotes the Wigner rotation and $B_p$ is a boost that is properly chosen for the reference vector $p_{\text{ref}} = (1,0,0,1)$. The mapping $\partial V^+ \to SL(2,\mathbb{C})$ given by $p_{\text{ref}} \mapsto B_p$ constitutes indeed a family of (Wigner) boosts such that $B_p p_{\text{ref}} = p \quad \forall\, p \in \partial V^+$. This representation extends to the full Poincaré group by adjoining representers for the space reflection $P$ (or parity transformation) and the time reflection $T = -P$ ( [38]).

In order to describe many-particle systems consisting of a finite number of one-particle systems, we will follow the second quantization procedure. In this paragraph we will avoid the bra-ket notation. Let us denote by $\mathcal{H}_S^{\otimes N}$ the symmetrized (bosonic), by $\mathcal{H}_A^{\otimes N}$ the anti-symmetrized (fermionic) $N$-fold tensor product $\mathcal{H}^{\otimes N} = \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_1$ of the Hilbert space $\mathcal{H}_1$ ( [39]):

$$\mathcal{H}_S^{\otimes N} = E_+^N(\mathcal{H}^{\otimes N}) = \frac{1}{N!}\sum_P S_P(\mathcal{H}^{\otimes N}) \tag{2.5}$$

$$\mathcal{H}_A^{\otimes N} = E_-^N(\mathcal{H}^{\otimes N}) = \frac{1}{N!}\sum_P sgn(P) S_P(\mathcal{H}^{\otimes N}) \tag{2.6}$$

where $E_\pm^N$ are projection operators[15], $S_P$ denotes the operator representating the permutation of particles $S_P(\psi_1 \otimes \psi_2 \otimes .... \otimes \psi_N) = (\psi_{P(1)} \otimes \psi_{P(2)} \otimes .... \otimes \psi_{P(N)})$ (representation of the permutation group) and $sgn(P) = \pm 1$ for even/odd permutations respectively. The scalar product of two basic vectors $\Psi = (\psi_1 \otimes \psi_2 \otimes .... \otimes \psi_N) \in \mathcal{H}^{\otimes N}$ and $\Phi = (\phi_1 \otimes \phi_2 \otimes .... \otimes \phi_N) \in \mathcal{H}^{\otimes N}$ is defined as $\langle \Psi, \Phi \rangle_N = \langle \psi_1, \phi_1 \rangle_1 \cdot \langle \psi_2, \phi_2 \rangle_1 \cdot ... \cdot \langle \psi_N, \phi_N \rangle_1$, where $\langle \cdot, \cdot \rangle_1$ is the scalar product in the single particle Hilbert space $\mathcal{H}_1$. This definition can be easily extended to the general state $\Psi \in \mathcal{H}_{S/A}^{\otimes N}$, where $\Psi$ can be a linear (symmetrized or antisymmetrized) combination of tensor product of single particle states belonging to $\mathcal{H}_1$. We have thus the additional feature that the particles cannot be distinguished for a many particle system of the same kind of a particles (bosonic/fermionic statistics), and we consider only observables (and so operators) which commute with all $S_P$ and their irreducible representations.

Having the description of multi-particle systems with a fixed number of particles $N$, we now want to describe a system with an undertermined number of particles: the particle

[15]The projection operators $E_\pm^N$ satisfy the properties

$$(E_\pm^N)^2 = E_\pm^N \tag{2.7}$$

$$(E_\pm^N)^\dagger = (E_\pm^N) \tag{2.8}$$

where $^\dagger$ denotes the hermitian conjugate.

number $N$ becomes a dynamical variable ( [40]) and we define the boson/fermion Fock spaces as:

$$\mathcal{F}_{S/A}(\mathcal{H}_1) = \bigoplus_{N=0}^{+\infty} \mathcal{H}_{S/A}^{\otimes N} \tag{2.9}$$

where $\mathcal{H}_0 = \mathbb{C}$ is a one-dimensional space representing the vacuum state in which no particle is present. We choose one vector $\Omega = (1, 0, 0, ....) \in \mathcal{H}_{S/A}^{\otimes N}$ and we call it the Fock vacuum. An element of $\mathcal{F}_{S/A}(\mathcal{H}_1)$ is an infinite sequence of states $\Phi = (C, \Phi_1, \Phi_2, ....)$ with $C \in \mathbb{C}$ and $\Phi_N \in \mathcal{H}_{S/A}^{\otimes N}$. The Fock space is also an Hilbert space with a scalar product defined as $\langle \Phi, \Psi \rangle = \sum_{N=0}^{+\infty} \langle \Phi_N, \Psi_N \rangle$ and all vectors $\Phi \in \mathcal{F}_{S/A}(\mathcal{H}_1)$ ( [41]). Second quantization is a functor between Hilbert spaces that associates to the original one-particle space the suitable (anti)symmetric Fock space, to self adjoint operators $V$ their second quantization $\mathrm{d}\mathcal{F}_{S/A}(V)$ and to unitary operators $e^{-itV}$ the second quantization $\mathcal{F}_{S/A}(e^{-itV}) = e^{-it\,\mathrm{d}\mathcal{F}_{S/A}(V)}$. Indeed the second quantization $U$ of the one-particle unitary operator $U_1$ acts on $\mathcal{F}_{S/A}(\mathcal{H}_1)$ by $U\Lambda = \Lambda$ and $(U\Phi)_N = (\bigoplus_{j=0}^{N} U_1)\Phi_N$ and so it preserves the number of particles. To introduce annihilation and creation operators one has to use "standard" quantization procedure: for a vector $\phi \in \mathcal{H}_1$, they are denoted as $a(\phi)$ and $a^\dagger(\phi)$. These operators satisfy

$$[a(\phi), a(\psi)]_\pm = [a^\dagger(\phi), a^\dagger(\psi)]_\pm = 0 \qquad [a(\phi), a^\dagger(\psi)]_\pm = \langle \phi, \psi \rangle \mathbf{1} \qquad \forall\, \phi, \psi \in \mathcal{H}_1 \tag{2.10}$$

$$a(\phi)\Omega = 0 \qquad \forall\, \phi \in \mathcal{H}_1 \tag{2.11}$$

where the $+$ sign stands for commutation (bosons) and the $-$ sign stands for anticommutation (fermions). These properties pick out just one (the Fock one) of the infinitely many unitarily inequivalent irreducible representations of the canonical commutation relations ($C^*$-algebra) built on such one-particle space $\mathcal{H}_1$.

## 3.2 Free quantum fields for the finite spin representation

Experiments in high energy physics are described in terms of collision processes of particles. The S matrix, central object of the description of scattering experiments involving the interactions between the given particles, is related to the scattering amplitudes of the physical processes and motivates the introduction of local quantum fields. Indeed the construction of relativistic interactions $V(t)$ using local fields can be motivated by invoking the spatial clustering principle for relativistic scattering amplitudes[16] ( [28]). Despite that there exist also other ways to construct (consistent) relativistic quantum mechanical theories, like direct particle interactions theories (DPI)[17], which involves only particles and satisfy Poincaré covariance, unitarity and macro-causality of the resulting S-matrix (which includes spatial cluster factorization, see [42] [43]).

[16] Since the S matrix has to be Lorentz invariant, $V(t)$ can be written as $\int \mathrm{d}^3x\, H_{\text{int}}(\vec{x}, t)$, with $H_{\text{int}}(x)$ scalar built up with creation and annihilation operators and satisfying $U_1(a, \Lambda) H_{\text{int}}(x) U_1(a, \Lambda)^{-1} = H_{\text{int}}(\Lambda x + a)$ and $[H_{\text{int}}(x), H_{\text{int}}(x')] = 0$ whenever $x$ and $x'$ are spacelike separated.

[17] this is mainly a phenomenological framework

QFT is usually formulated in terms of pointlike (covariant and local) quantum fields $\Psi_k(x)$ , $k = 1, .., N$, which are operator-valued tempered distributions in $\mathcal{H}$, that are maps $\Psi_k : \mathcal{S}(\mathcal{M}) \mapsto \mathcal{O}(\mathcal{H})$ such that there exists a common dense subspace $B \subseteq \mathcal{H}$ satisfying

- for each $f \in \mathcal{S}(\mathcal{M})$ and each $k = 1, ..., N$ the domain of definition of the quantum fields $D_{\Psi_k}$ contains $B$
- the induced map $\mathcal{S}(\mathcal{M}) \to \mathrm{End}(B), f \mapsto \Phi_k(f)|_B$ is linear
- for each $v \in B$ and $w \in \mathcal{H}$ the assignment $f \mapsto \langle w, \Phi_k(f)(v)\rangle$ is a tempered distribution

where $\mathcal{S}(\mathcal{M})$ denotes the Schwartz space of functions over the Minkowski space and $\mathcal{O}(\mathcal{H})$ the set of all densely defined operators in the Hilbert space $\mathcal{H}$ ( [27])[18]. The concept of a quantum field as an operator-valued distribution corresponds better to the actual physical situation than the more familiar notion of a field as a quantity defined at each point of spacetime. Indeed, in experiments the field strength is always measured not at a point $x$ of spacetime but rather in some region of space and in a finite time interval. Therefore, such a measurement is naturally described by the expectation value of the field as a distribution applied to a test function with support in the given spacetime region.

The relativistic invariance of the theory is formalized by the following relativistic properties of these fields $\Psi$:

- (Covariance) If one represents the operator-valued distribution $\Psi_k$ symbolically by a function $\Psi_k(x) \in \mathcal{O}(\mathcal{H})$, then the fields transform covariantly under the Poincare transformations $U(a, \Lambda) = U(a, 1)U(0, \Lambda)$:

$$U(a, \Lambda(A))\Psi_{k;l}(x)U(a, \Lambda(A))^{-1} = \sum_m D_k(A^{-1})_{lm}\Psi_{k;m}(\Lambda x + a) \qquad \forall\, (a, A) \in \widetilde{P_+^\uparrow} \tag{2.12}$$

  where $U(\Lambda(A))$ ($A \in SL(2, \mathbb{C})$) are (strongly continuous) unitary operators and $D(A)$ is a finite dimensional representation of $SL(2, \mathbb{C})$. We notice that there are many representations, including the scalar, the vector and a host of tensor and spinor representations. These particular representations are irreducible, but we do not require at this point that $D(A)$ be irreducible; in general it is a block-diagonal matrix with an arbitrary array of irreducible representations in the blocks. Therefore the index $l$ includes a label that runs over the types of particle described and the irreducible representations in the different blocks, as well as another that runs over the components of the individual irreducible representations. This is a purely formal way of writing the equivariance between the actions on $\mathcal{S}$ and $\mathcal{H}$ where $\widetilde{P_+^\uparrow}$ acts on $\mathrm{End}(D)$ by conjugation, i.e.

$$U(a, \Lambda(A))\Psi_{k;l}(f)U(a, \Lambda(A))^{-1} = \sum_m D_k(A^{-1})_{lm}\Psi_{k;m}((a, \Lambda)f) \quad \forall\, f \in \mathcal{S}(\mathcal{M}), (a, A) \in \widetilde{P_+^\uparrow} \tag{2.13}$$

[18] Usually it is assumed also that the vacuum state vector $\Omega$, with the property of being the unique translationally invariant state in $\mathcal{H}$, is a cyclic vector for the fields, i.e. that by applying polynomials of the (smeared) fields to the vacuum one obtains a dense set $D_0 \subseteq \mathcal{H}$

Moreover the common domain $B$ is Poincaré invariant and also invariant under the action of the fields:

$$U(a,\Lambda(A))B \subseteq B \;\; \forall\,(a,A) \in \widetilde{P_+^\uparrow} \qquad \Psi_k(f)B \subseteq B \quad \forall\, f \in \mathcal{S}(\mathcal{M}), k = 1, ..., N \tag{2.14}$$

- (Microcausality or locality) The fields either commute or anticommute at spacelike separated points

$$[\Psi_k(x), \Psi_p(y)]_\mp = 0 \;\; \forall\, k,p = 1, ..., N \qquad \text{for} \quad (x-y)^2 < 0 \tag{2.15}$$

In this construction we mention also the fact that the joint spectrum of the generators $P^\mu$ is contained in the forward closed cone $\overline{V^+} = \{p^\mu : p^2 \geq 0, p_0 \geq 0\}$.

For the explicit construction of the pointlike free fields of arbitrary mass and finite spin we will follow Weinberg ( [28]). In the following we consider the $k$ index of the set of fields to be fixed (and so we drop it). The labels $\sigma$ and $n$ we will use stand for the particle's spin $z$-components and for the species of the particle. As we discussed before, it is natural[19] to build the interaction density operator $H_{\text{int}}$ out of annihilation $\psi_l^+(x)$ and creation fields $\psi_l^-(x)$ defined as

$$\psi_l^+(x) = \sum_{\sigma n} \int \mathrm{d}^3 p \, u_l(x; \vec{p}, \sigma, n) a(\vec{p}, \sigma, n) \tag{2.16}$$

$$\psi_l^-(x) = \sum_{\sigma n} \int \mathrm{d}^3 p \, v_l(x; \vec{p}, \sigma, n) a^\dagger(\vec{p}, \sigma, n) \tag{2.17}$$

with coefficients -called intertwiners- $u_l(x; \vec{p}, \sigma, n)$ and $v_l(x; \vec{p}, \sigma, n)$ chosen so that under Lorentz transformation the fields transform as in equation (2.12). Since the transformation rules for the annihilation $a(\vec{p}, \sigma, n)$ and creation operators $a^\dagger(\vec{p}, \sigma, n)$ can be read directly from the trasformation properties of single-particle states[20] we can write explicitly for the intertwiners

$$\sum_{\overline{\sigma}} u_{\bar{l}}(\Lambda x + a; \overrightarrow{\Lambda p}, \overline{\sigma}) d^{(s_n)}_{\overline{\sigma}\sigma}(W(\Lambda, \vec{p})) = \sqrt{\frac{p_0}{(\Lambda p)_0}} \sum_l D_{\bar{l}l}(\Lambda) e^{i(\Lambda p)\cdot a} u_l(x; \vec{p}, \sigma, n) \tag{2.18}$$

$$\sum_{\overline{\sigma}} v_{\bar{l}}(\Lambda x + a; \overrightarrow{\Lambda p}, \overline{\sigma}) d^{(s_n)*}_{\overline{\sigma}\sigma}(W(\Lambda, \vec{p})) = \sqrt{\frac{p_0}{(\Lambda p)_0}} \sum_l D_{\bar{l}l}(\Lambda) e^{-i(\Lambda p)\cdot a} v_l(x; \vec{p}, \sigma, n) \tag{2.19}$$

where $s_n$ is the spin of particles of species $n$ and we are considering the group action of the Poincaré group with elements $(a, \Lambda)$. Considering translations, it is possible to write

$$u_l(x; \vec{p}, \sigma, n) = (2\pi)^{-\frac{3}{2}} e^{ip\cdot x} u_l(\vec{p}, \sigma, n) \qquad v_l(x; \vec{p}, \sigma, n) = (2\pi)^{-\frac{3}{2}} e^{-ip\cdot x} v_l(\vec{p}, \sigma, n) \tag{2.20}$$

[19] due to cluster decomposition and Lorentz invariance

[20] under Poincaré group action, it is easy to show that $a(\vec{p}, \sigma, n)$ transforms as

$$U(\Lambda, b) a(\vec{p}, \sigma, n) U^{-1}(\Lambda, b) = e^{i(\Lambda p)\cdot b} \sqrt{\frac{p_0}{(\Lambda p)_0}} \sum_{\overline{\sigma}} d^{(s_n)}_{\sigma\overline{\sigma}}(W(\Lambda, \vec{p})^{-1}) a(\overrightarrow{\Lambda p}, \overline{\sigma}, n)$$

In order to respect the locality property, we have to combine combine annihilation and creation fields in linear combinations to construct fields

$$\psi_i(x) = \kappa_i \psi_i^+(x) + \lambda_i \psi_i^-(x) \tag{2.21}$$

with the constants $\kappa_i$ and $\lambda_i$ and any other arbitrary constants in the fields adjusted to satisfy the following equations (see (2.15)):

$$[\psi_l(x), \psi_m(y)]_\mp = [\psi_l(x), \psi_m^\dagger(y)]_\mp = 0 \quad \forall\, l, m \qquad \text{for} \quad (x-y)^2 < 0 \tag{2.22}$$

Moreover, in order that interaction density operator $H_{\text{int}}$ should commute with the charge operator $Q$ (or some other symmetry generator) it is necessary that it is formed out of fields that have simple commutation relations with $Q$:

$$[Q, \psi_j(x)] = -q_j \psi_j(x) \tag{2.23}$$

where $q_j$ is the charge $q(n)$ of the field. Therefore we have to construct $H_{\text{int}}$ as a sum of products of fields $\psi_{j_1}\psi_{j_2}...$ and their adjoints $\psi_{i_1}^\dagger \psi_{i_2}^\dagger...$ such that $q_{j_1}+q_{j_2}+...-q_{i_1}-q_{i_2}-... = 0$. Using equation (2.23), it is easy to show that to conserve quantum numbers (like electric charge), there must be a doubling of particle species carrying non-zero values of $q(n)$: if a if a particular component of the annihilation field destroys a particle of species $n$, then the same component of the creation field must create particles of a species $n$, known as the antiparticles of the particles of species $n$, which have opposite values of all conserved quantum numbers. This is the reason of the existence of antiparticles( [28]).

For simplicity we will restrict our attention, from now on, to fields that destroy only a single type of particle (so dropping the label $n$) and create the corresponding antiparticle and also that transform irreducibly under the Lorentz group. In the general case, one also need to include the charge-conjugate annihilation $a^c(\vec{p},\sigma)$ and charge-conjugate creation operator $a^{c,\dagger}(\vec{p},\sigma)$ in order to describe particles with distinct antiparticles. We recall that the general (finite-dimensional) irreducible representation of the Lorentz algebra $\mathfrak{so}(3,1)$ is non unitary and is classified by an ordered pair of half-integers $A$ and $\dot{B}$, that is with the couple $(A,\dot{B})$[21]. This is due to the fact that in order to classify all (real linear) representations of the semisimple Lie algebra $\mathfrak{so}(3,1)$, we need only the irreducible complex linear representations[22] of the complexified Lie algebra $\mathfrak{so}(3,1)_{\mathbb{C}}$. From the generators $M^{\mu\nu}$ of the Lorentz algebra $\mathfrak{so}(3,1)$, namely the canonical generators of boost $K^i = M^{0i}$ and of rotations $J^i = \frac{1}{2}\epsilon^{ijk}M_{jk}$, we can define the complexified generators $A^i = \frac{J^i+iK^i}{2}$ and $B^i = \frac{J^i-iK^i}{2}$ which satisfy the $\mathfrak{su}(2)$ algebra:

$$[A_i, A_j] = i\epsilon_{ijk}A_k \qquad [B_i, B_j] = i\epsilon_{ijk}B_k \qquad [A_i, B_j] = 0 \ \forall\, i,j,k \in \{1,2,3\} \tag{2.24}$$

One has the following isomorphisms, at the level of representations( [44])

$$\mathfrak{so}(3,1) \hookrightarrow \mathfrak{so}(3,1)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus i\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \tag{2.25}$$

[21] this is the so called dotted/undotted spinorial notation
[22] due to the semisimplicity property

and so the complex linear irreducible representations of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ are indexed by a couple of half-integers $A$ and $\dot{B}$, and has dimensionality $(2A+1)(2\dot{B}+1)$. Then we can associate to a general $(m,s)$ Wigner representation a covariant quantum field

$$\psi_{(A,\dot{B})}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\sigma} \int \mathrm{d}^3p \, \left[\kappa a(\vec{p},\sigma) u_{(A,\dot{B})}(\vec{p},\sigma) e^{ip\cdot x} + \lambda a^{c,\dagger}(\vec{p},\sigma) v_{(A,\dot{B})}(\vec{p},\sigma) e^{-ip\cdot x}\right] \tag{2.26}$$

where dotted spinorial indices are related to the physical spin $s$ through the following inequalities (therefore for any $s$ there are infinitely many spinorial representation indices)

$$|A-\dot{B}| \le s \le |A+\dot{B}| \ \text{ for } m>0 \qquad |A-\dot{B}| = |h| \ \text{ for } m=0 \tag{2.27}$$

It is possible to show (see Weinberg [28]) that the formula for the generic $(A,\dot{B})$ field of a given particle, that is unique up to overall scale, is

$$\psi_{(A,\dot{B})}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\sigma} \int \mathrm{d}^3p \, \left[a(\vec{p},\sigma) u_{(A,\dot{B})}(\vec{p},\sigma) e^{ip\cdot x} + (-1)^{2\dot{B}} a^{c,\dagger}(\vec{p},\sigma) v_{(A,\dot{B})}(\vec{p},\sigma) e^{-ip\cdot x}\right] \tag{2.28}$$

We notice that whereas in the massive case the relation of the physical spin $s$ with the formal spin in the spinorial pointlike fields follows the angular momentum composition rules which leads to the spinorial restrictions (2.27) (but still there are infinitely many possibilities to represent the same massive particle), the zero mass finite helicity family has a significantly reduced number of spinorial pointlike descriptions.

One notices that in the zero mass case the vector representation $(A,\dot{B}) = \left(\frac{1}{2},\frac{1}{2}\right)$ for $|h|=1$ and the $(A,\dot{B}) = (1,1)$ for $|h|=2$ are missing, i.e. precisely those fields which correspond to the classic electromagnetic vector potential and to the metric tensor. For helicity $|h|=1$ the best one can do is to work with covariant field strength $F^{\mu\nu}$ which in the spinorial formalism correspond to $(1,0)\oplus(0,1)$ instead of working with classical vector potential $A^\mu$ ( $(A,\dot{B}) = \left(\frac{1}{2},\frac{1}{2}\right)$ ) which simply does not occur in local QFT with second quantization on the Hilbert space. I want to recap briefly here the argument for the simplest case. Let's try to construct a four vector $A^\mu$ ($(\frac{1}{2},\frac{1}{2})$) for a massless particle of helicity $h=\pm1$, using the standard creation and annihilation operators:

$$A^\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\sigma=\pm1} \int \mathrm{d}^3p \, \left[a(\vec{p},\sigma) u^\mu(\vec{p},\sigma) e^{ip\cdot x} + a^\dagger(\vec{p},\sigma) u^{\mu,*}(\vec{p},\sigma) e^{-ip\cdot x}\right] \tag{2.29}$$

where we used all equations 2.18 to put $v^\mu(\vec{p},\sigma) = u^{\mu,*}(\vec{p},\sigma)$ and we normalize the coefficients $\lambda$ and $\kappa$. In the four-vector representation $D(\Lambda)^\mu{}_\nu = \Lambda^\mu{}_\nu$ and we decide to normalize intertwiners as $u^\mu(\vec{p},\sigma) = (2p^0)^{-\frac{1}{2}} e^\mu(\vec{p},\sigma)$ with the so-called polarization vectors $e^\mu(\vec{p},\sigma)$. The transformation law for the polarization vectors should then be

$$e^{i\sigma\theta} e^\mu(\vec{p},\sigma) = W(R(\theta),\vec{t})^\mu{}_\nu e^\nu(\vec{p},\sigma) \qquad \forall\, \sigma=\pm1 \tag{2.30}$$

where we decomposed the little group $\widetilde{E(2)}$ into a rotation $R(\theta)$ of an angle $\theta$ and two translation generators $\vec{t} = (t_1,t_2)$. It is easy to see that the two vectors (amplitudes for circular polarized plane waves with wave vector $p^0$)

$$e^\mu_{\text{pol}}(\pm) = \frac{1}{\sqrt{2}}(0,1,\pm i,0) \tag{2.31}$$

satisfies (2.30) for elements of the little group $W(R(\theta))$ depending only on the rotation part. For the general transformation of the little group $W(R(\theta),\vec{t})$, it can be shown that the law holds only up to a vector proportional to $p^0$ ( [45] [28]). If we define, for any momentum $\vec{p}$, $e^\mu(\vec{p},\pm) = B_p e^\mu_{\text{pol}}(\pm)$ then the inhomogeneous transformation law for $u^\mu(\vec{p},\pm)$ is

$$W(R(\theta),\vec{t})^\mu{}_\nu u^\nu(\vec{p},\pm) = e^{i\sigma\theta(W,p))}\left(u^\nu(\overrightarrow{\Lambda p},\pm) + \gamma_\pm(\Lambda p)^\nu\right) \tag{2.32}$$

with some coefficients $\gamma_\pm \in \mathbb{C}$. This means that the fields $A^\mu_\pm(x)$ defined as

$$A^\mu_\pm(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3p\ \left[a(\vec{p},\pm)u^\mu(\vec{p},\pm)e^{ip\cdot x} + a^\dagger(\vec{p},\pm)u^{\mu,*}(\vec{p},\pm)e^{-ip\cdot x}\right] \tag{2.33}$$

transform as gauge fields with gauge parameters $\Gamma_\pm$:

$$U(a,\Lambda)A^\mu_\pm(x)U(a,\Lambda)^{-1} = (\Lambda^{-1})^\mu{}_\nu\left(A^\nu_\pm(\Lambda x + a) + \partial^\nu\Gamma_\pm\right) \tag{2.34}$$

We have thus come to the conclusion that no four-vector field $A^\mu(x) = A^\mu_+(x) + A^\mu_-(x)$ can be constructed from the annihilation and creation operators for a massless particle with helicity $|h| = 1$.

These gaps in the massless case have important physical consequences.( [46]) The explanation of this dilemma, which also leads to its cure, is that the loss of pointlike quantum potentials is the result of a clash between the Hilbert space structure (positivity)[23] and pointlike localization (or modular localization[24]). There are two ways out: gauge theory (in covariant gauge) which sacrifies the Hilbert space and keeps the pointlike formalism and the use of stringlike potentials which allows to preserve the Hilbert space. In the last case the missing spinorial fields reappear after relaxing the localization from pointlike to stringlike.

The first is the standard path, namely to keep pointlike fields. The second way is unusual, and linked somehow to the infinite spin representation case we haven't discussed so far. The following No-Go theorem states clearly that the (axiomatic) Wightman framework of QFT is incompatible with this class of Poincaré representations, and we need to relax some of the assumptions of this theorem in order to go further with the construction of a quantum (free) field.

Independently of this choice, in the pointlike case the well-studied perturbative formalism usually comes with the introduction of additional unphysical degrees of freedom (ghosts) due to the indefinite metric setting. There are many formalisms (Gupta-Bleuler, BRST for pure Yang Mills theories and Batalin–Vilkovisky for general Lagrangian gauge theories) already developed in order to deal with the ghost structure of the QFT. Since in typical perturbation calculations one does not use the Hilbert space norm for control of convergence (as Schwartz inequality) there is no problem at this stage, but at the end one has to reconvert the calculated correlation functions into a Hilbert space setting (gauge

[23]notice that instead in the classical theory Hilbert space positivity doesn't concern us
[24]we will explain briefly later the meaning of this term

invariant objects). The use of string-localized fields deflect the formal problems of extracting quantum data from an unphysical indefinite metric setting to the ambitious problem of extending perturbation theory to the realm of string-localized fields. For example one recent idea regards an extension Epstein-Glaser construction of time-ordered products to the string-localized fields setting.

## 3.3 The No-Go theorem by Yngvason and the modular localization

Within the (generalized) axiomatic framework of Wightman theory, a result indicating the incompatibility of the infinite spin representations with local commutativity (that is, the microcausality condition we mentioned in equation (2.15)) and covariant transformation law has been proved by J.Yngvason( [2]). In addition to the usual requirement of the existence of an Hilbert space which carries a unitary representation of the Poincaré group, the energy-momentum spectrum condition and the uniqueness of the vacuum he considered a local quantum theory composed of fields that are operator-valued distributions and which satisfy a (generalized) covariance transformation law[25] and microcausality, as well as the the property that repeatedly applying the field operators to the vacuum creates a dense subspace of the Hilbert space (cyclicity of the vacuum). Under these assumptions it can be shown that the one-particle states which the Wightman fields create from the vacuum are orthogonal to any irreducible representation subspace of the Hilbert space for zero mass and infinite spin, that is there are no nonvanishing matrix elements between the vacuum and the infinite spin one-particle state. The only way to circumvent this no-go theorem is to relax one of the assumptions, giving up pointlike localization of the Wightman fields.

An idea on how to solve this problem (as well as the previous one on the missing spinorial fields) comes from the concept of causal localization in QFT. It is well-known that no measurements taking place in regions of Minkowski space which are spacelike separated, i.e. which cannot be connected by a light beam, should have an influence on each other.

In non-relativistic quantum mechanics (QM), Born's principle of localization is as follows: for a single particle, if a wave function $\psi_K(x) \coloneqq \langle x| P_K |\psi\rangle$ vanishes outside a bounded spatial region $K$, it is said to be localized in $K$. Here $P_K \coloneqq \int \mathrm{d}^3x\ |x\rangle \langle x|$ denotes the projection operator which projects state vectors $\phi$ in the Hilbert space $\mathcal{H}$ to state vectors with support in $K$, and it is clear that if a spatial region $K$ is disjoint from a spatial region $K'$, a wave function $\psi_K(x)$ localized in $K$ is orthogonal to $\psi_{K'}(x)$ (due to the properties of the projection operator). Here there is also a sort of "natural duality" between localization of states and localization of observables: for any observable $A$, due to the equality $\langle\psi_K| A |\chi_K\rangle = \int \mathrm{d}^3x\ \mathrm{d}^3y\ \langle\psi| P_K A P_K |\chi\rangle$ valid $\forall\ |\psi\rangle, |\chi\rangle$ we can equally restrict states (to $|\psi_K\rangle, |\chi_K\rangle$) or observables (to $P_K A P_K$). This notion is limited to non-relativistic physics, in which $\vec{x}$ trasforms under the Galilei group. ( [47])

The generalization of such a principle of localization for the relativistic case is called Newton-Wigner (BNW) localization. One may characterize the BNW localization in a

[25] in order to include infinite spin representations one has to consider infinite dimensional representation of $SL(2, \mathbb{C})$

bounded spatial region $K$ at a given time in terms of a projector $P_K$ which appears in the spectral decomposition of the selfadjoint position operator $\vec{x}$, adapting the previous construction to the case of an invariant scalar product of relativistic wave functions. Such (family of) spectral projectors are supposed to measure the probability of detecting a (single) particle in different space-time regions. To the region $K$ corresponds the subspace $L^2(K) \subseteq L^2(\mathbb{R}^3)$ of wavefunctions with probability amplitude vanishing (almost everywhere) outside of $K$ and -via an unitary transformation- a subspace of the Hilbert space $\mathcal{H}$. The lack of covariance of BNW localization in finite time propagation leads to frame-dependence and superluminal effects; but still the basis for time-dependent scattering theory in QM and QFT (and also various settings of measurament theory) rely on this concept of localization. This is due to the fact that in the asymptotic limit of large timelike separation as required in scattering theory, the covariance, frame-independence and causal relations are recovered. Anyway, such a principle of localization isn't compatible with relativistic covariance and causality in QFT (or interacting point particles), except in an approximate sense for distances of the order of the Compton wavelength or smaller: in that case we loose the duality between localization of states and localization of observables, and we have to look for a more intrinsic type of localization focusing only on observables.( [48], [42])

The only concept of causal localization compatible with relativistic covariance and causality is modular localization, that is intrinsically defined within the representation theory of the Poincaré group but draws its motivation from local quantum field theory. This concept does not refer anymore to individual operators (like the position ones) but rather to local measurements of observables (that is, an ensemble of observables which share the same localization region). Having defined the algebra $\mathcal{A}$ of observables in QFT, to each spacetime region $\mathcal{O}$ we can assign a subalgebra $\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}$ generated by the smeared field operators $\Phi(f)$[26] with test functions $f$ supported in $\mathcal{O}$. The net structure of the observables allows a local comparison of states: two states are locally equal in a region $\mathcal{O}$ iff the expectation values of all operators in $\mathcal{A}(\mathcal{O})$ are the same in both states. Local deviation from any state, and in particular from the vacuum state, can be measured by local comparison: this is particularly useful in order to define "strictly localized states" for a region $\mathcal{O}$, that is states which give the same expectation values as the vacuum for all measurements in the causal complement of $\mathcal{O}$. Thus regarding causality, this is mathematically expressed by local commutativity (i.e. mutual commutativity of the algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O}'))$ and physically connected to the possibility of preparing states that exhibit no mutual correlations for a given pair of causally disjoint regions $\mathcal{O}$ and $\mathcal{O}'$. The space $\tilde{H}(\mathcal{O})$ obtained by applying operators in $\mathcal{A}(\mathcal{O})$ to the vacuum is, for any open region $\mathcal{O}$, dense in the Hilbert space and thus far from being orthogonal to $\tilde{H}(\mathcal{O}')$ (differently from the non-relativistic case we mentioned before). This is due to unavoidable correlations in the vacuum state between measurements at an arbitrary distance, that is a direct consequence of Reeh-Schlieder property (by which any state can be approximated to arbitrary precision by acting on the vacuum

[26]or their neutral currents in case the fields are charged

with an operator selected from the local algebra). At the end, using Tomita-Takesaki theory it is also possible to recover localized states from local algebras.

Coming now back to the original problem, there is a natural localization structure on the representation space for any positive energy representation of the restricted Poincaré group $\widetilde{\mathcal{P}_+^\uparrow}$ which upon second quantization gives rise to a (local) net of operator algebras on the Fock space over the representation Hilbert space $\mathcal{H}$. The role of quantum fields is simply to "coordinatize" localized algebras by playing the role of singular generators of all localized algebras. Skipping many technical details that the interested reader can find in the beautiful paper ( [49]), the outcome is that the generating fields are pointlike operator-valued distributions for the massive case and for the helicity representation whereas they are semiinfinite spacelike string-localized operator-valued distributions for the continuous spin case. In the last case, to be more precise, the generating fields are distributions on $\mathbb{R}^4 \times \{e \in \mathbb{R}^4 : e \cdot e = -1\}$; moreover they are localized -in the sense of commutation relations- along semiinfinite strings of the form $x + \mathbb{R}^+ e$ (with $e \cdot e = -1$, that means spacelike strings). Both kind of fields are singular limits of operators localized in causally closed regions; pointlike fields in case of double cone localized and semi-infinite string-localized fields in case of spacelike cone localized operators. The physical reason of this difference is that in the continuous spin case the faithful representation of the little group $\widetilde{E(2)}$ contains infinitely many degrees of freedom and doesn't allow a compact localization.

### 3.4 String-localized fields

The "string" is a ray which extends from a point $x \in \mathbb{R}^4$ to infinity in a space–like direction. Let's denote by $H^3 \coloneqq \{e \in \mathbb{R}^4 : e \cdot e = -1\}$ the hyperboloid of space-like directions and by $\alpha$ a collection of dotted or undotted spinor indices, that labels the representations of the 3-dimensional de Sitter group $dS_3$ (3-dimensional de Sitter space can be thought as a one-sheet hyperboloid embedded in a 3+1 Minkowski spacetime). Let $U$ be a unitary irreducible (positive energy) ray representation of the restricted Poincaré group acting on a Hilbert space $\mathcal{H}$ with a unique invariant vector $\Omega$, which contains an irreducible representation $U_1$ acting on $\mathcal{H}_1$. A string-localized covariant quantum field is an operator-valued (tempered) distribution $\Phi^\alpha(x, e)$, where $(x, e) \in \mathbb{R}^4 \times H^3$ and satisfies ( [7], [50])

- (String locality) if $x_1 + \mathbb{R}^+ e_1'$ and $x_2 + \mathbb{R}^+ e_2$ are spacelike separated for all $e_1'$ in an open neighborhood of $e_1$ then $[\Phi^\alpha(x_1, e_1), \Phi^{\alpha'}(x_2, e_2)]_\mp = 0$ (spacelike bosonic/fermionic commutation rule)

- (Covariance) The transformation law is consistent with localization properties

$$U(a, \Lambda)\Phi^\alpha(x, e)U(a, \Lambda)^{-1} = \sum_{\alpha'} D(\Lambda^{-1})^{\alpha\alpha'} \Phi^{\alpha'}(\Lambda x + a, \Lambda e) \tag{2.35}$$

- (Reeh-Schlieder property) After smearing $\Phi^\alpha(x, e)$ with test functions $f$ and $g$ in $x$ and $e$, the field operators generate a dense set in Fock space when applied to the vacuum vector $\Omega$ ($\Omega$ is cyclic for the polynomial algebra of fields $\Phi^\alpha(x, e)$)

The field is called free if it creates only single particle states from the vacuum vector $\Omega$, that is $\Phi^\alpha(f,g)\Omega \in \mathcal{H}_1$.

The most direct (and general) method to construct string-localized fields is to smear a point-like field over a semi-infinite space-like line

$$\Phi_{\text{smeared}}(x,e) = \int_0^{+\infty} \mathrm{d}t\, f(t) \sum_r \phi_r(x+te) z(e)_r \tag{2.36}$$

where $f(t)$ is supported in the interval $[0,+\infty[$ and $z(e)$ is a tensor formed from $e$ which is Lorentz-invariant in the sense that

$$D(\Lambda) z(\Lambda^{-1}e) = z(e) \qquad \forall\, \Lambda \in \mathcal{L}_+^\uparrow \tag{2.37}$$

Using a non-trivial theorem stated in the MSY paper ( [6]) that we will explain later, one can show that such $\Phi_{\text{smeared}}(x,e)$ satisfies the "String-locality" and the "Covariance" property we mentioned before.

Restricting our attention to the single particle vector $\Psi^\alpha(x,e) \coloneqq \Phi^\alpha(x,e)$ for free fields, we notice that it enjoys certain specific properties reflecting the covariance and locality of the field. These properties are intrinsic to the representation $U_1$ and can be formulated without reference to the field, using the concept of a modular localization we mentioned in the previous paragraph. We will call a $\mathcal{H}_1$-valued distribution satisfying the ensuing properties a string–localized covariant wave function for $U_1$. Our strategy, that we will explain in some details in the following paragraph, is to construct such a $\mathcal{H}_1$-valued distribution $\Psi^\alpha(x,e)$ for given $U_1$ and then to obtain the field via second quantization. Since $U_1$ is induced by a representation $d$ of a subgroup (little group) $G$ of the Lorentz group, then $U_1$ is contained in $U_0 \otimes V$, where $U_0$ is the scalar representation and $V$ is one of the extension of $d$ to the full Lorentz group. Thus the problem can be separated, and the $U_0$ part can be solved via Fourier transformation whereas for the $V$ representation it is possible to (implicitely) construct a localized covariant wave function living on $H^3$. Consider then the tensor product of a wave function localized at $x$ for $U_0$ and a wave function localized at $e \in H^3$ for $V$. The main result is that the projection onto $U_1$ of this vector turns out to be a vector which is localized for $U_1$ in the string with initial point $x$ and direction $e$.

We recall that the representation $U_1$ is induced as by $d$ as follows. For the massive case representation space $\mathcal{H}_1 \coloneqq L^2(H_{\mathrm{m}}^+, \mathrm{d}\mu) \otimes \mathfrak{h}$, where $\mathrm{d}\mu$ is a Lorentz invariant measure on $H_{\mathrm{m}}^+$ and $\mathfrak{h}$ is the representation space of $d$. On the $\mathcal{H}_1$ Hilbert space $U_1$ acts according to

$$(U_1(a,\Lambda)\psi)(p) = e^{ia\cdot p} d(W(\Lambda,p))\psi(\Lambda^{-1}p) \tag{2.38}$$

where $W(\Lambda,p)$ is the Wigner rotation defined by $W(\Lambda,p) = B_p^{-1}\Lambda B_{\Lambda^{-1}p}$ and for almost all $p \in H_{\mathrm{m}}^+$, $B_p$ is a Lorentz transformation which maps $p_{\text{ref}}$ to $p$. Exactly the same argument holds for the massless case, with the substitution $H_0^+ \leftrightarrow \partial V^+$. The equation (2.38) is not suitable for the construction of covariant and local fields by second quantization. The first reason is that the transformation matrix $d(W(\Lambda,p))$ depends generically on $p$ (except in the scalar case), and so in the $x$ space the transformation law is nonlocal after Fourier transformation. The second one is that Wigner rotation factors have singularities which cause problems with the property of local commutativity( [6]).

These two problems in the pointlike free field case are solved by using the so-called intertwiner functions, rectangular $(2A+1)(2\dot{B}+1)\otimes(2s+1)$ functions which intertwine between unitary $2s+1$-component Wigner representation (the representer of the Wigner rotation factor $W(\Lambda,p)$) and covariant $(2A+1)(2\dot{B}+1)$-dimensional spinorial representation labeled by the semi-integers $A$ and $\dot{B}$ (with a representer of $\Lambda$), that is (recall equations (2.18)).

$$\sum_{\bar{\sigma}} u_{\bar{l}}(\overrightarrow{\Lambda p},\bar{\sigma},n)d^{(s_n)}_{\bar{\sigma}\sigma}(W(\Lambda,p)) = \sqrt{\frac{p_0}{(\Lambda p)_0}}\sum_l D_{\bar{l}l}(\Lambda)u_l(\vec{p},\sigma,n) \tag{2.39}$$

$$\sum_{\bar{\sigma}} v_{\bar{l}}(\overrightarrow{\Lambda p},\bar{\sigma},n)d^{(s_n)*}_{\bar{\sigma}\sigma}(W(\Lambda,p)) = \sqrt{\frac{p_0}{(\Lambda p)_0}}\sum_l D_{\bar{l}l}(\Lambda)v_l(\vec{p},\sigma,n) \tag{2.40}$$

Indeed these intertwiners connect the $(m,s)$ irreducible one-particle Wigner representation with wave functions (and their associated quantum fields) transforming under certain finite-dimensional (non-unitary) representations $D$ of the Lorentz group.

In this case of pointlike localization the intertwiners above are determined by the covariant transformation law for the field $\psi_{(A,\dot{B})}(x)$, but we would be lead to the same family of distribution valued intertwiners if modular localization was required instead. In other words, covariance in the sense of the (classical) tensor/spinor calculus is in this case equivalent to the quantum requirement of modular localization. The continuous spin case instead required stringlike localization, as we will see soon.

## 3.5 Construction of Mund-Schroer-Yngvason intertwiners for the infinite spin representation

The new solution, which in contrast to the mentioned above works also for the massless infinite spin representations which remained outside the covariant spinorial formalism, is to look at intertwiner functions $u(e,\cdot)$ which depends as a distribution on the points $e$ in the set $H^3$ of space-like directions, and absorb the Wigner rotation factor $W(\Lambda,p)$ by trading it with a transformation $e\mapsto\Lambda e$. We decide here to follow closely the paper of Mund, Schroer and Yngvason ( [6]).

Let's define the complexification $H^{3,c}$ of $H^3$ as $H^{3,c}=\{e\in\mathbb{C}^4 : e\cdot e=-1\}$ where the dot denotes bilinear extension of the Minkowski metric to $\mathbb{C}^4$, that is $e\cdot e := e'\cdot e' - e''\cdot e'' + 2ie'\cdot e''$ with $e=e'+ie''$. Let further $\mathcal{T}_+$ be the tuboid consisting of all $e=e'+ie''\in H^{3,c}$ such that $e''$ is in the interior of the forward light cone $\partial V^+$. The set of $p$ for which the canonical boost $B_p$ is defined will be denoted by $\dot{H}^+_m$. We will consider (compact) subsets $\Omega$ of $\mathcal{T}_+$ of the form $\Theta = H^{3,c}\cap(\Omega_1 + i\mathbb{R}^+\Omega_2)$ where $\Omega_1$ and $\Omega_2$ are compact subsets of $\mathbb{R}^4$ and $\Omega_2$ is contained in the forward light cone.

Considering the irreducible representation $d$ of the little group, in the context of string-localized fields we call a function $u:\mathcal{T}_+\times\dot{H}^+_m\to\mathfrak{h}$ an intertwiner if it satisfies the intertwiner property

$$d(W(\Lambda,p))u(\Lambda^{-1}e,\Lambda^{-1}p)=u(e,p) \tag{2.41}$$

for $(e,p)\in\mathcal{T}_+\times\dot{H}_m^+, \Lambda\in\mathcal{L}_+^\uparrow$ and for almost all $p$ the function $e\mapsto u(e,p)$ is analytic in the tuboid $\mathcal{T}_+$. Moreover it is usually required that there is a constant $N\in\mathbb{N}_0$ and a function $M$ on $\dot{H}_m^+$ which is locally in $L^2(\dot{H}_m^+,\mathrm{d}\mu)$ and polynomially bounded, and for every $\Omega\subset\mathcal{T}_+$ (specified as above, in particular compact) there is a constant $c_\Omega$ such that for all $e\in\Omega$ holds

$$\|u(e,p)\|\le c_\Omega M(p)|e''|^{-N} \tag{2.42}$$

(here $|\cdot|$ denotes any norm in $\mathbb{R}^4$). This bound is chosen so that for fixed $p$ the function $e\mapsto u(e,p)$ is of moderate growth near the real boundary $H^3$ and therefore admits a distributional boundary value in $\mathcal{D}'(H^3)$:

Let's now focus on the bosonic case for simplicity, restricting thus our attention to single valued representations of the restricted Poincaré group. It is possible to prove a fundamental theorem about existence and uniqueness of string-localized fields:

**Theorem 3.1** (String-localized fields (MSY [6], Theorem 3.3)). *Let $U^1$ be any irreducible positive energy representation of the Poincaré group with faithful or trivial representation of the little group. It follows:*
*i) Let $u$ be an intertwiner function for $d$, and let $u_c$ be the conjugate intertwiner*[27]*. Then the field $\phi(x,e)$ defined by*

$$\phi(x,e):=\int_{H_m^+}\sum_\sigma \mathrm{d}\mu(p)\,\{e^{ip\cdot x}u(e,p,\sigma)a^*(p,\sigma)+e^{-ip\cdot x}\overline{u_c(e,p,\sigma)}a(p,\sigma)\} \tag{2.43}$$

*satisfies "String-locality" and "Covariance" property. It further satisfies Reeh-Schlieder property and moreover if the growth order $N$ of $e\mapsto u(e,p)$ is zero, then the field is a function in $e$, and the commutativity property already holds if $x_1+\mathbb{R}_0^+e_1$ is space-like separated from $x_2+\mathbb{R}_0^+e_2$*
*ii) A non-trivial intertwiner function $u$ with these properties exists for all mentioned representations. It is unique up to multiplication with a function of $e\cdot p$, which is meromorphic in the upper half plane (that is to say, if $\hat{u}$ is another intertwiner function, then for almost all $e\in\mathcal{T}_+$ and for almost all $p$, $u(e,p)=F(e,p)\hat{u}(e,p)$ where $F$ is a numerical function, meromorphic on the complex upper half-plane)*
*iii) Conversely, let $\phi$ be a free string-localized field that satisfies "String-locality" and "Covariance" property and which satisfies in addition the Bisognano-Wichmann property*[28]*. Then it is of the form* (2.43)*, up to unitary equivalence, with the intertwiner $u$ and its conjugate intertwiner $u_c$*

[27] for the explicit definition, see the paper [6]

[28] We don't have the intention to explain completely the exact mathematical meaning of this property in the context of algebraic QFT, but it is a sort of compatibility requirement between algebraic properties and unitary group representation properties (for a wedge)

If for a string-like field $\phi(x,e)$ the space-like commutation property depend only on $x$ and not on $e$, then the field turns out to be point-like localized. According to the following theorem, this is related to analiticity property of $u$:

**Theorem 3.2** (Point-localized fields (MSY [6], Proposition 3.4))**.** *A string-localized field in the sense of covariance transformation property, is point-like localized if, and only if, the function $e \mapsto u(e,p)$ is analytic on the entire complexified $H^{3,c}$ and the bound on $u$ in equation* (2.42)*, with growth order $N = 0$, holds for all compact subsets of $H^{3,c}$.*

The reader interested in the proof of these theorems is referred directly to the MSY paper.

It is useful to notice that the string-like field $\Phi_{\text{smeared}}(x,e)$ that we obtained by smearing a point-like field, in agreement with the theorem, can be written as in the equation (2.43), with the intertwiner given by

$$u(p,e) = \tilde{f}(e \cdot p) u_{\text{point}}(p,e) = \tilde{f}(e \cdot p) \sum_r v(p)_{k,r} z(e)_r \tag{2.44}$$

where $k = -s, ..., s$ and $\tilde{f}$ is the Fourier transform of $f$, being analytic in the upper half-plane (but not in the whole plane, in which case one falls back to point-like localization).

In order to build up the intertwiners for the infinite spin representation, we need to exploit the correspondence betweeen all irreducible unitary representations $d$ of the little group $G$ and the representation acting naturally on functions on suitable $G$-orbits. Let $\Gamma$ be the $G$-orbit defined by $\Gamma := \{q \in H_m^+ : q \cdot p_{\text{ref}} = 1\}$, where we have identified $H_0^+$ with $\partial V^+$. Then it is easy to show that $\Gamma$ is isometric to the sphere $S^2$ for $m > 0$, and to $\mathbb{R}^2$ for $m = 0$. Since every isomorphism of $\Gamma$ extends, by linearity, to a Lorentz transformation which leaves $p_{\text{ref}}$ invariant, it follows that $G$ is precisely the isometry group of $\Gamma$. Thus, the isometry $\Gamma \simeq S^2$ or $\mathbb{R}^2$ establishes the isomorphism $G \simeq SO(3)$ or $E(2)$, respectively for $m > 0$ or $m = 0$. Let now $\mathrm{d}\nu$ denote the $G$-invariant measure on $\Gamma$, and let $\tilde{d}$ be the unitary representation of $G$ acting on $L^2(\Gamma, \mathrm{d}\nu)$ as

$$(\tilde{d}(R)v)(q) := v(R^{-1}q) \tag{2.45}$$

with $R \in G$. It is known that $\tilde{d}$ decomposes into the direct sum of all irreducible representations of $G \simeq SO(3)$ for $m > 0$ and into a direct integral of all faithful irreducible representation of $G \simeq E(2)$ for $m = 0$. Hence for any faithful representation $d$ of $G$ there exists a partial isometry $V$ from $L^2(\Gamma, \mathrm{d}\nu)$ into $\mathfrak{h}$ which intertwines the representations $\tilde{d}$ and $d$: $d(R)V = V\tilde{d}(R)$ with $R \in G$.

We can now solve the intertwiner equation (2.41) by projecting a corresponding $L^2(\Gamma, \mathrm{d}\nu)$-valued solution $\tilde{u}(e,p)$ onto $\mathfrak{h}$. Let $F$ be an arbitrary numerical function of the argument $q \cdot B_p^{-1} e$ and define

$$\tilde{u}(e,p)(q) := F(q \cdot B_p^{-1} e) \tag{2.46}$$

$$u(e,p) := V\tilde{u}(e,p) \tag{2.47}$$

With these definitions, then $\tilde{u}$ solves

$$\tilde{d}(W(\Lambda,p))\tilde{u}(\Lambda^{-1}e,\Lambda^{-1}p)=\tilde{u}(e,p) \tag{2.48}$$

since $u$ has to solve equation (2.41). Since the imaginary part of $q\cdot e$ is strictly positive if $q\in H_0^+$ and $e\in\mathcal{T}_+$, then the analyticity property can be satisfied if $F$ has an analytic extension into the upper complex half plane.

A good choice for $F$ is a generic power $w^\alpha$ for suitable $\alpha\in\mathbb{C}$. In case $\alpha\notin\mathbb{Z}$, the power $w^\alpha$ is understood via the branch of the logarithm on $\mathbb{C}\setminus\mathbb{R}_0^-$ with $\ln 1=0$, and by continuous extension from the upper half plane if $w\in\mathbb{R}^-$, i.e. $\lim_{\epsilon\to 0+}(w+i\epsilon)^\alpha$. In general, the function $q\mapsto(q\cdot e)^\alpha$ will be in $L^2(\Gamma,\mathrm{d}\nu)$ only after smearing with a test function $h\in\mathcal{D}(H^3)$, $\int\mathrm{d}\sigma(e)\,h(e)(q\cdot e)^\alpha$. The representation $\tilde{d}$ extends naturally to the Lorentz group on the (dense) set of functions of this form via push-forward.

We want to briefly mention here that in the massive case $m>0$ with spin $s$ we can write the explicit expression for the intertwiners and for the two-point function in the case of string-like localization. It is possible to show that the general structure of intertwiners is $u(e,p)=F(e\cdot p)u^{|s|}(e,p)$, where $F$ is an analytic function on the upper half plane which is polynomially bounded at infinity and has moderate growth near the reals and $u^{|s|}(e,p)$ can be computed explicitly. In particular since we are working in $4$ dimensions $u^{|s|}(e,p)=u_0^s(B_p^{-1}e)$ and each component $u_0^s(e)_k$ $(k=-s,...,s)$ is given by

$$u_0^s(e)_k=i^s\sqrt{\frac{(s+k)!}{(s-k)!}}\{(e_1+ie_2)\partial_{e_3}-(\partial_{e_1}+i\partial_{e_2})e_3\}^{s-k}(e_1-ie_2)^s \tag{2.49}$$

and for real $e\in H^3$

$$u_0^s(e)_k=(-i)^s(1+e_0^2)^{\frac{s}{2}}Y^*_{s,k}(n(e)) \tag{2.50}$$

where $Y_{s,k}$ are the spherical harmonics and $n(e):-(1+e_0^2)^{-\frac{1}{2}}(e_1,e_2,e_3)\in S^3$. The analiticity and covariance transformation properties are now evident from the equations (2.49) and (2.50) respectively. Moreover, in the massive case it can be shown that every string-localized free field can be written as an integral, along the string, of a point-localized tensor field $\Phi_{\text{smeared}}(x,e)$.

In the massless infinite spin case and for $F=w^\alpha$, we wish to obtain a family of intertwiners $u^\alpha(e,p)$ labeled by $\alpha\in\mathbb{C}$ with $\Re\alpha<0$[29]. I want to recall that for $m=0$ the little group $G$ is isomorphic to the euclidean group $E(2)$ and the faithful irreducible unitary representations of $E(2)$ are labeled by $\kappa\in\mathbb{R}^+$. The representation $d=d_\kappa$ acts on $\mathfrak{h}=L^2(\mathbb{R}^2,\mathrm{d}\nu_\kappa(k))$ as

$$(d(t,R)u)(k):=e^{it\cdot k}u(R^{-1}k) \tag{2.51}$$

where $(t,R)\in E(2)$ and $d\nu_\kappa(k):=\delta(|k|^2-\kappa^2)\,\mathrm{d}^2k$. For $m=0$ and base-point $p_{\text{ref}}=(1,0,0,1)$ in $H_0^+$, the set $\Gamma$ of all $q\in H_0^+$ with $q\cdot p_{\text{ref}}=1$ is isometric to the euclidean space

[29] This restriction is due to the fact that we want a unitary $\tilde{d}$ representation in $L^2(\Gamma,\mathrm{d}\nu)$, since in that case $\Re\alpha$ has to be $-\frac{d-2}{2}$

$\mathbb{R}^2$ via the parametrization of $\Gamma$ given by (in 4-dimension)

$$\xi(z) = \left(\frac{1}{2}\left(z^2+1\right), z_1, z_2, \frac{1}{2}\left(z^2-1\right)\right) \tag{2.52}$$

where $z \in \mathbb{R}^2$ and $z^2 := z_1^2 + z_2^2$. This isomorphism $\xi$ from $\mathbb{R}^2$ to $\Gamma$ identifies the action of $G$ in $\Gamma$ with the action of $E(2)$ in $\mathbb{R}^2$. One gets generalized intertwiners $V = V_\kappa$ from the representation $\tilde{d}$, to the irreducible representation $d = d_\kappa$ as, where the restrictions of the Fourier transforms to a fixed value $\kappa$ make sense,

$$(V_\kappa v)(k) := \tilde{v}(k) = \int_{\mathbb{R}^2} \mathrm{d}^2 z\, e^{ik\cdot z} v(\xi(z)) \tag{2.53}$$

with $|k|^2 = \kappa^2$. Using equations (2.46), we can write explicitly the following intertwiners defined for $e \in \mathcal{T}_+$ and $p \in \dot{H}_0^+$

$$u_F(e,p) = \int \mathrm{d}^2 z e^{ik\cdot z} F(B_p \xi(z)\cdot e) \tag{2.54}$$

and in the particular case of $F = w^\alpha$ we have

$$u^\alpha(e,p) = \int_{\mathbb{R}^2} \mathrm{d}^2 z\, e^{ik\cdot z} (B_p\xi(z)\cdot e)^\alpha \tag{2.55}$$

with $|k| = \kappa$. Let's study the structure of the intertwiner in details. The function $z \mapsto B_p\xi(z)\cdot e$ is a quadratic polynomial in $z$. We write the scalar product in Minkowski space with light-cone coordinates as $x\cdot p = \frac{1}{2}(x_+p_- + x_-p_+) - x_1p_1 - x_2p_2$, where $x_\pm = x_0 \pm x_3$, so that $\xi(z)_+ = z^2$, $\xi(z)_- = 1$, $\xi(z)_1 = z_1$ and $\xi(z)_2 = z_2$. Since the standard boost $B_p$ for the choice of $p_{\text{ref}}$ can be written as (in the matrix $2x2$ notation)

$$B_p = \frac{1}{\sqrt{2(p_0+p_3)}} \begin{bmatrix} p_0+p_3 & p_1 - ip_2 \\ 0 & 1 \end{bmatrix} \tag{2.56}$$

therefore we have

$$B_p\xi(z)\cdot e = az^2 + b\cdot z + c \tag{2.57}$$

with $a = \frac{1}{2}(p\cdot e)$, $b = -((B_p^{-1}e)_1, (B_p^{-1}e)_2) \in \mathbb{C}^2$ and $c = \frac{1}{2}(B_p^{-1}e)_+$. Taking account that $4ac - b^2 = e^2 = -1$ and of $2a = p\cdot e > 0$, we have

$$B_p\xi(z)\cdot e = a\left(z + \frac{b}{2a}\right)^2 - \frac{1}{4a} \tag{2.58}$$

It is worth noticing that $z \mapsto B_p\xi(z)\cdot e$ is a polynomial in $z$ without any real zeroes if $e \in \mathcal{T}_+$, so that the integral in equation (2.54) exists and defines a continuous function $u^\alpha(e,p)$ of $k$. It is a proper intertwiner in the sense of the definition of intertwiner we gave before, and it is also bounded in $p$ for $\Re\alpha \in [-2, 0.5)$ after smearing with a test function $h \in \mathcal{D}(H)$.

### 3.6 Construction of CSP one particle states

We decide here to follow closely the recent paper of Schuster and Toro ( [9]), adapting the notation to our conventions. A one-particle quantum mechanical approach to continuous-spin particle (CSP) is look for covariant wavefunctions $\psi$ whose defining property relates the action of a generic Lorentz transformation $\Lambda$ to the unitary (but momentum dependent) little group action it induces, as the intertwiners condition (2.39). Thus such wavefunctions behave like CSP single particle states. In particular, denoting the Lorentz labels with $l$, $\bar{l}$ and the Little group labels with $a$, $a'$ we require

$$\sum_{a'} \psi(\overrightarrow{\Lambda p}, a', \bar{l}) d_{aa'}(W(\Lambda, p)) = \sum_{l} D_{\bar{l}l}(\Lambda)\psi(\vec{p}, a, l) \tag{2.59}$$

We will focus now on the explicit construction of CSP one particle states. In the massless case CSP one particle states are characterized by $W^2 = -\kappa^2 \neq 0$ (instead $W^2 = 0$ for the usual helicity states) and going to a light-cone frame with momentum $k_+ \neq 0$, $k_- = k_1 = k_2 = 0$ the Pauli-Lubanski vector has components $W^+ = 0$, $W^- = -\frac{1}{2}k_+\epsilon^{ij}J_{ij}$ and $W^i = -\frac{1}{2}k_+\epsilon^{ij}J_{j-}$, so that $W^2 = -W^iW^i$ and the helicity operator is $h = -W^-/k_+$. A basis with vectors $|\kappa, h\rangle$ which are simultaneously eigenvectors of $W^2$ and $h$, with eigenvalues $-\kappa^2$ and $h$, respectively, must satisfy

$$W^2 |\kappa, h\rangle = -\kappa^2 |\kappa, h\rangle \qquad \kappa^2 > 0 \tag{2.60}$$

$$h |\kappa, h\rangle = h |\kappa, h\rangle \qquad h = 0, \pm 1, \pm 2, ... \tag{2.61}$$

$$W_\pm |\kappa, h\rangle = \pm i\kappa |\kappa, h \pm 1\rangle \tag{2.62}$$

where $W_\pm = W^1 \pm iW^2$ increases/decreases the helicity by one unit so that the irreducible representation comprises all basis vectors $\{|\kappa, h\rangle, h = 0, \pm 1, \pm 2, ...\}$ and hence it is infinite dimensional ( [13]).

To classify Lorentz transformations that leave a momentum $p^\mu_{\text{ref}}$ invariant, we can consider the independent components of $w^\mu := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}p_{\text{ref},\nu}M_{\rho\sigma}$. The condition $w \cdot p_{\text{ref}} = 0$ restricts the independent components to the three generators of the little group $G$, and this motivates a decomposition of $w^\mu$ into a "rotation" component proportional to $p^\mu_{\text{ref}}$ and two "translation" components along polarization directions $\epsilon^\mu_{1,2}$ with $\epsilon_{1,2} \cdot k = 0$

$$w^\mu = -p^\mu_{\text{ref}}R + \epsilon^\mu_1 t_1 + \epsilon^\mu_2 t_2 \tag{2.63}$$

Since $p_{\text{ref}} = (1, 0, 0, 1)$, then we can expand $w^\mu$ in components as

$$w^\mu = -p^\mu_{\text{ref}}M_{12} + \hat{e}^\mu_x(M_{32} + M_{02}) + \hat{e}^\mu_y(-(M_{31} + M_{01})) \tag{2.64}$$

where $M_{12}$, $(M_{32} + M_{02})$ and $(M_{31} + M_{01})$ are the generators for Lorentz trasformations that leave $p_{\text{ref}}$ invariant. If $q$ is the unique vector satisying

$$q^2 = 0 \qquad p_{\text{ref}} \cdot q = 1 \qquad q \cdot \epsilon_{1,2} = 0 \tag{2.65}$$

then the components $R$ and $t_{1,2}$ can be extracted as

$$R = q \cdot w \qquad t_{1,2} = \epsilon_{1,2} \cdot w \tag{2.66}$$

Using the group structure of the Little group, inferred from the Pauli-Lubanski pseudo-vector's commutation relations (we just have to recall the Lorentz algebra)

$$[W^\mu, W^\nu] = -i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma$$

we can obtain commutators

$$[R, t_{1,2}] = \pm i t_{2,1} \qquad [t_1, t_2] = 0 \tag{2.67}$$

so that the little group $G$ has the structure of $E(2)$, the isometry group of a Euclidean plane. Defining the conjugate raising and lowering generators $t_\pm = t_1 \pm i t_2$, we can decompose any $G$- element in a canonical way as

$$W(\theta, \beta) = e^{\frac{i}{\sqrt{2}}(\beta t_- + \beta^* t_+)} e^{-i\theta R} \tag{2.68}$$

where $\theta \in [0, 2\pi[$, $\beta$ is complex and has dimensions of length (since translation generators have units of mass). It is also useful to define a two-vector of little group translations $\vec{t} = (t_1, t_2)$.For (one-dimensional) helicity representations, labelled by $h$, the non-compact translation generators $\vec{t}$ act trivially and the rotation $R$ acts as a phase on single particle states

$$U(W(\theta, \beta)) \left| p_{\text{ref}}, h \right\rangle = e^{ih\theta} \left| p_{\text{ref}}, h \right\rangle \tag{2.69}$$

where the requirement of periodicity under rotations by $4\pi$ restricts $h$ to be integer or half-integer. Since we found previously that also the set of equations (2.60) holds, this means that continuous spin one particle states are a superposition of all helicity eigenstates, either all integer or all half-integer (bosonic and fermionic case respectively). Remembering that the invariant $W^2 = w^2 = -t_- t_+$ can be used to classify representations, for all helicity representations $W^2 = -\kappa^2 = 0$: these are the only finite-dimensional representations of the little group $G$.

In the continuous spin case $W^2 = -\kappa^2 \neq 0$ and there is a countable tower of states on which all generators of the little group $G$ act non-trivially. We can describe the action in two different bases: eigenstates of R (the "spin basis", labelled by an integer or half-integer) or simultaneous eigenstates of $t_1$ and $t_2$ (labelled by an angle in $[0, 2\pi[$).

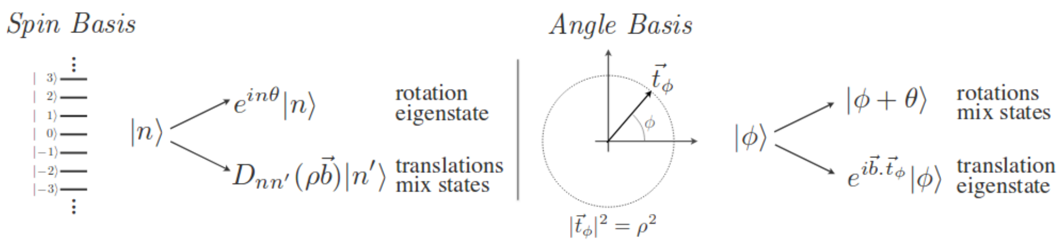


Figure 2.3: Comparison between the action of little group elements in the spin basis and in the angle basis (taken from [9], in our conventions $\rho \mapsto \kappa$ and $D \mapsto d$)

Let's start from the angle basis. Eigenstates of $t_1$ and $t_2$, written as $t_{1,2}\left|p_{\text{ref}},\vec{t}\right\rangle$, are plane-wave states in $\mathbb{R}^2$ on which $E(2)$ acts. A useful parametrization of these kind of states is given by $\vec{t}_\phi = (\kappa\cos(\phi), \kappa\sin(\phi))$ for fixed $\kappa$ with $W^2 = -\kappa^2$. Therefore states are labelled by the polar angle $\phi$, with the periodic identification $|0\rangle = |2\pi\rangle$[30]. Under rotations of the little group,

$$U(W(\theta,\beta))\left|p_{\text{ref}},\phi\right\rangle = e^{\frac{i}{\sqrt{2}}[(\beta+\beta^*)t_1 - it_2(\beta-\beta^*)]}\left|p_{\text{ref}},\phi+\theta\right\rangle = \tag{2.70}$$

$$= e^{i\vec{b}\cdot\vec{t}_{\phi+\theta}}\left|p_{\text{ref}},\phi+\theta\right\rangle = e^{i\kappa\Re(\sqrt{2}\beta e^{-i(\phi+\theta)})}\left|p_{\text{ref}},\phi+\theta\right\rangle = \tag{2.71}$$

$$= \int \frac{\mathrm{d}\phi'}{2\pi}\, d_{\phi\phi'}(\theta,\beta)\left|p_{\text{ref}},\phi'\right\rangle \tag{2.72}$$

with $d_{\phi\phi'}(\theta,\beta) = (2\phi)\delta(\phi'-\phi-\theta)e^{i\kappa\Re(\sqrt{2}\beta e^{-i(\phi+\theta)})}$ and $\vec{b} = \sqrt{2}(\Re(\beta),\Im(\beta))$[31]. Conjugate states $\langle k,\phi|$ transform according to

$$\left\langle p_{\text{ref}},\phi\right| U^\dagger(W(\theta,\beta)) = e^{-i\vec{b}\cdot\vec{t}_{\phi+\theta}}\left\langle p_{\text{ref}},\phi\right| = \tag{2.73}$$

$$= \int \frac{\mathrm{d}\phi'}{2\pi}\, (d_{\phi\phi'}(\theta,\beta))^*\left\langle p_{\text{ref}},\phi'\right| \tag{2.74}$$

The above transformations are unitary with respect to the inner product

$$\left\langle p_{\text{ref}},\phi \middle| p'_{\text{ref}},\phi'\right\rangle = 2p^0_{\text{ref}}\delta^3(p_{\text{ref}}-p'_{\text{ref}})2\pi\delta(\phi-\phi') \tag{2.75}$$

With this formalism, we can write a generic little-group state as $|\psi_f\rangle = \int \frac{\mathrm{d}\phi}{2\pi} f(\phi)\,|\phi\rangle$.

The spin basis, obtained by Fourier transforming in $\phi$, diagolizes $R$ and makes contact with the helicity representation. Infact by defining, for any integer $n$,

$$\left|p_{\text{ref}},n\right\rangle := \int \frac{\mathrm{d}\phi}{2\pi}\, e^{in\phi}\left|p_{\text{ref}},\phi\right\rangle \tag{2.76}$$

we obtain the transformation rule

$$U(W(\theta,\beta))\left|p_{\text{ref}},n\right\rangle = \sum_{n'} d_{nn'}(\theta,\beta)\left|p_{\text{ref}},n'\right\rangle \tag{2.77}$$

with

$$d_{nn'}(\theta,\beta) = \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi'}{(2\pi)^2}\, d_{\phi\phi'}(\theta,\beta)e^{in\phi}e^{-in'\phi'} = e^{-in\theta}(ie^{i\alpha})^{n-n'}J_{n-n'}(\kappa\sqrt{2}|\beta|) \tag{2.78}$$

where $\beta = |\beta|e^{i\alpha}$. Conjugate states transform as

$$\left\langle p_{\text{ref}},n\right| U^\dagger(W(\theta,\beta)) = \sum_{n'}\left\langle p_{\text{ref}},n'\right|(d_{nn'}(\theta,\beta))^* \tag{2.79}$$

and these transformations are unitary with respect to the inner product

$$\left\langle p_{\text{ref}},n \middle| p'_{\text{ref}},n'\right\rangle = 2p^0_{\text{ref}}\delta^3(p_{\text{ref}}-p'_{\text{ref}})\delta_{n'n} \tag{2.80}$$

[30]this is single-valued (bosonic) type of CSP; in the double-valued (fermionic( type we make the antiperiodic identification $|0\rangle = -|2\pi\rangle$ and the spin basis states happen to be labelled by half-integer $n$

[31]This was introduced just for exhibit the $E(2)$ structure

It is worth noticing that the Bessel functions are the representation functions for the group $E(2)$ due to their additivity property.

Considering the full unitary action of Lorentz transformations on single-particle CSP states,

$$U(\Lambda)\,|p_{\text{ref}},\phi\rangle = U(W(\theta_{\Lambda,p},\beta_{\Lambda,p}))\,|\Lambda p_{\text{ref}},\phi\rangle \tag{2.81}$$

$$= \int \frac{\mathrm{d}\phi'}{2\pi}\, d_{\phi\phi'}(\theta_{\Lambda,p},\beta_{\Lambda,p})\,\left|\Lambda p_{\text{ref}},\phi'\right\rangle \tag{2.82}$$

where $W(\theta_{\Lambda,p},\beta_{\Lambda,p}) = B^{-1}_{\Lambda p}\Lambda B_p$ is used to define the little group rotation $\theta_{\Lambda,p}$ and translation $\beta_{\Lambda,p}$ induced by the Lorentz transformation $\Lambda$. We want to underline that the standard boost $B_p$ is uniquely chosen, for a given choice of frames for all $p_{\text{ref}}$, as in equation (2.56). Moreover we notice that in the limit $\kappa \mapsto 0$, $J_{n-n'}(\kappa\sqrt{2}|\beta|)$ approaches zero for $n \neq n'$ and $1$ for $n = n'$, so that the tranformation rule reduces to

$$d_{nn'}(\theta,\beta) \mapsto e^{-in\theta}\delta_{nn'} \tag{2.83}$$

and we recover as a limit the direct sum of all integer-helicity states by considering all continuous spin states $|p_{\text{ref}},n\rangle$.

Let's recap what is our first goal. We want to find the family of wavefunctions which behave like continuous spin single particle states. Such wavefunctions have to transform in infinite-dimensional Lorentz representations and since it is complicated to study infinite-dimensional irreducible representation of $SL(2,\mathbb{C})$, we decide to work with reducible ones. In particular we consider wavefunctions $\psi(\eta)$ that are functions of an auxiliary vector $\eta^\mu$, on which Lorentz transformation act as

$$D(\Lambda)\psi(\eta,x) = \psi(\Lambda\eta,\Lambda x) \tag{2.84}$$

The reducibility property comes from the fact that we are restricting the domain of $\psi(\eta,x)$ to the orbits of $\eta$, that is starting from one value of $\eta$ we can reach only other points on the same orbit $\Lambda\eta$. In the angle basis, the covariance equation (2.59) becomes

$$\int \frac{\mathrm{d}\phi'}{2\pi}\, d_{\phi\phi'}(W(\Lambda,k))\psi(\{\overrightarrow{\Lambda p_{\text{ref}}},\phi\},\eta) = \psi(\{\overrightarrow{p_{\text{ref}}},\phi\},\Lambda^{-1}\eta) \tag{2.85}$$

In this case it is worth noticing that the generator of Lorentz trasformations should be written, for the transformation in the space of Lorentz four-vectors, as

$$(M_{\mu\nu})^{\alpha}{}_{\beta} = i\left(\delta^{\alpha}{}_{\mu}g_{\nu\beta} - \delta^{\alpha}{}_{\nu}g_{\mu\beta}\right) \tag{2.86}$$

whereas in the space of coordinates $(x,\eta)$

$$M^{\mu\nu} = i\left(\eta^{[\mu}\partial_{\eta}^{\nu]} + p^{[\mu}\partial_{p}^{\nu]}\right) \tag{2.87}$$

To go further with calculations it is more convenient to consider an appropriate basis in which one of the vector basis is the reference momentum vector $p_{\text{ref}}$, and the other three vectors are related to the polarization basis $\epsilon_1$ and $\epsilon_2$. In general there is an ambiguity in the choice of the basis, due to the fact that we can choose different polarization vectors $\epsilon_1$ and $\epsilon_2$ for the same reference vector $p_{\text{ref}}$, and we can get different generators $t_\pm$ and $R$ for the same little group. This is the reason why it is important to fix $\epsilon_1(p_{\text{ref}})$ and $\epsilon_2(p_{\text{ref}})$ for a specific value of $p_{\text{ref}}$. Having defined

$$\epsilon_\pm(p_{\text{ref}}) = \frac{\epsilon_1(p_{\text{ref}}) \pm i\epsilon_2(p_{\text{ref}})}{\sqrt{2}} \tag{2.88}$$

and remember the definition of the four vector $q^\mu$ according to equations (2.65), we can choose as the "standard" basis

$$p^\mu_{\text{ref}} = (p, 0, 0, p) \qquad q^\mu = (q, 0, 0, -q) \qquad \epsilon_\pm(k) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \tag{2.89}$$

It is clear that the standard boost $B_p$ will take $p^\mu_{\text{ref}}$ to a generic $p^\mu$ and $\epsilon_\pm(p_{\text{ref}})$ to new complex null $\epsilon_\pm(p)$ at the same time. The following relations

$$(\epsilon_+(p))^* = (\epsilon_-(p)) \qquad \epsilon_+(p) \cdot \epsilon_-(p) = -1 \qquad \epsilon(p) \cdot p = 0 \tag{2.90}$$

are preserved so that the choice of the reference frame at $p$ is consistent. Since the frame form a basis for all Lorentz vectors, we can decompose the invariants $\epsilon^{\mu\nu\rho\sigma}$ and $g^{\mu\nu}$ according to this standard basis. So using the definition (2.66), it is possible to extract $R$ and $t_\pm$ ( [9])

$$R = i\epsilon^\rho_+ \epsilon^\sigma_- M_{\rho\sigma} \tag{2.91}$$

$$t_\pm = \pm i\sqrt{2} p^\rho_{\text{ref}} \epsilon^\sigma_\pm M_{\rho\sigma} \tag{2.92}$$

To solve the covariance equation for $\phi(x, \eta)$, we need to find the action of the little group generators $R$, $t_\pm$ in the space of coordinates $(x, \eta)$:

$$R = -\epsilon^\rho_+ \epsilon^\sigma_- (\eta_{[\rho} \partial_{\eta,\sigma]}) \tag{2.93}$$

$$t_\pm = \pm\sqrt{2} p^\rho_{\text{ref}} \epsilon^\sigma_\pm (\eta_{[\rho} \partial_{\eta,\sigma]}) \tag{2.94}$$

where we used equation (2.87).

## 3.7 Differential wave equations for continuous-spin particles and relation with Wigner's original wave equations

Let's start from the covariance equation (2.85). Let's study the special case $\Lambda = B_{\Lambda p} B_p^{-1}$ in which $W(\Lambda, p_{\text{ref}}) = 1$ and the equation becomes

$$\psi(\{\overrightarrow{\Lambda p_{\text{ref}}}, \phi\}, \Lambda\eta) = \psi(\{\overrightarrow{p_{\text{ref}}}, \phi\}, \eta) \tag{2.95}$$

This is satisfied only iff $\psi$ is a scalar-valued function of $p_{\text{ref}}$, $\eta$ and $\epsilon_\pm(p_{\text{ref}})$. Since any Lorentz transformation can be decomposed in a product of boosts $B_p$ and little group elements, any "scalar" $\psi$ that solves (2.85) for Lorentz transformations $W \in E(2)_{p_{\text{ref}}}$ will also solve (2.85) for general $\Lambda$.

Remembering the action of little group elements on angle-basis states (2.70), we can expand $W$ in equation (2.68) for infinitesimal values of the parameters $\beta$, $\beta^*$ and $\theta$:

$$\psi - i\theta R\psi = \psi + \theta\partial_\phi\psi \tag{2.96}$$

$$\psi + \frac{i}{\sqrt{2}}\beta(t_1 - it_2)\psi = \phi + \frac{i}{\sqrt{2}}\kappa e^{-i\phi}\psi \tag{2.97}$$

$$\psi + \frac{i}{\sqrt{2}}\beta^*(t_1 + it_2)\psi = \phi + \frac{i}{\sqrt{2}}\kappa e^{i\phi}\psi \tag{2.98}$$

where we used the fact that

$$e^{i\vec{b}\cdot\vec{t}_\phi} = e^{i\sqrt{2}\Re(\beta)t_1 + i\sqrt{2}\Im(\beta)t_2} \tag{2.99}$$

$$e^{i\frac{\kappa}{\sqrt{2}}\left[\beta e^{-i\phi}+\beta^* e^{i\phi}\right]} = e^{i\left[\kappa\cos(\phi)\sqrt{2}\left(\frac{\beta+\beta^*}{2}\right)+\kappa\sin(\phi)\sqrt{2}\left(\frac{\beta-\beta^*}{2i}\right)\right]} \tag{2.100}$$

from which (using (2.70))

$$e^{i\sqrt{2}\Re(\beta)t_1}\psi = e^{i\left[\kappa\cos(\phi)\sqrt{2}\left(\frac{\beta+\beta^*}{2}\right)\right]}\psi \tag{2.101}$$

$$e^{i\sqrt{2}\Im(\beta)t_2}\psi = e^{i\left[\kappa\sin(\phi)\sqrt{2}\left(\frac{\beta-\beta^*}{2i}\right)\right]}\psi \tag{2.102}$$

$$\tag{2.103}$$

Rewriting the system of equations (2.96) explicitly

$$-i\left[(\eta\cdot\epsilon_-)(\epsilon_+\cdot\partial_\eta)\right]\psi = \partial_\phi\psi \tag{2.104}$$

$$-\left[(\eta\cdot\epsilon_-)(p\cdot\partial_\eta) - (\eta\cdot p)(\epsilon_-\cdot\partial_\eta)\right]\psi = \frac{i}{\sqrt{2}}\kappa e^{-i\phi}\psi \tag{2.105}$$

$$\left[(\eta\cdot\epsilon_+)(p\cdot\partial_\eta) - (\eta\cdot p)(\epsilon_+\cdot\partial_\eta)\right]\psi = \frac{i}{\sqrt{2}}\kappa e^{i\phi}\psi \tag{2.106}$$

We can notice that the set of partial differential equations (2.104) are homogeneous in $\eta$ and Fourier-conjugate to themselves[32]. Any family of solutions $\psi$ to this system of equations forms a basis of solutions to a particular covariant wave equation. It can be easily shown (using equation (2.87)) that these differential equations imply

$$\left[2(p\cdot\eta)(p\cdot\partial_\eta)(\eta\cdot\partial_\eta) - (p\cdot\eta)^2\partial_\eta^2 - \eta^2(p\cdot\partial_\eta)^2 + \kappa^2\right]\psi = (W^2+\kappa^2)\psi = 0 \tag{2.107}$$

Following the paper of Schuster-Toro ( [9]), the solutions can be divided into two different classes:

- solutions smooth in $\eta$ near $\eta\cdot p = 0$
- singular solutions supported on $\delta(\eta\cdot p)$

It is interesting also to find a connection with Wigner's covariant wave equations ( [51]), the solution space of which carries a continuous-spin unitary irreducible representation of the Poincaré group. While the singular solutions are related to a basis of Wigner equations, the smooth solutions solve a new class of wave equations.

[32]Later in the next chapter we will analyze deeply the consequences of this fact

For later purposes, we want to define the quantity

$$\epsilon(p,\phi) = \frac{i}{\sqrt{2}}\left(\epsilon_+(p_{\text{ref}})e^{-i\phi} - \epsilon_-(p_{\text{ref}})e^{i\phi}\right) \tag{2.108}$$

in order to associate the angle $\phi$ with a direction in the space-time, since the origin $\phi = 0$ has no invariant significance. Let's start from the singular solutions, that are parametrized by a generic function $f(r)$ which specifies an arbitrary profile of $\psi$ under rescalings of $\eta$:

$$\psi(\{p,\phi,f\},\eta) = \int \mathrm{d}r f(r) \int \mathrm{d}\tau \delta^4(\eta - r\epsilon(p,\phi) - r\tau p)e^{i-\tau p} \tag{2.109}$$

These solutions satisfies the Wigner's equations for continuous spin particles

$$\eta^2 + 1 = 0 \tag{2.110}$$
$$p^2\psi = 0 \tag{2.111}$$
$$p\cdot\eta\psi = 0 \tag{2.112}$$
$$(W^2+\kappa^2)\psi = (-\eta^2(p\cdot\partial_\eta)^2+\kappa^2)\psi = 0 \tag{2.113}$$

where we used equation (2.107). To recover a basis of solutions to the Wigner's equations we may choose $f(r) = \delta(r-1)$. But this is not mandatory, and one can choose an ordinary differential equation in the $\eta$-space to single out a specific profile $f(r)$: the resulting $\psi(p,\phi,f,\eta)$ will then satisfy such equation and the set of Wigner's PDE (2.110) and will provide a basis of covariant wavefunctions. A convenient choice can be to impose the (zeroth order) homogeneity condition in $\eta$:

$$\eta\cdot\partial_\eta\psi = 0 \tag{2.114}$$

Regarding smooth solutions, they are parametrized by an arbitrary function $f(\eta\cdot p,\eta^2)$ depending on two parameters $\eta\cdot p$ and $\eta^2$

$$\psi(\{p,\phi,f\},\eta) = f(\eta\cdot p,\eta^2)e^{i\kappa\frac{\eta\cdot\epsilon(p,\phi)}{\eta\cdot p}} \tag{2.115}$$

These solutions satisfy

$$p^2\psi = 0 \tag{2.116}$$
$$(W^2+\kappa^2)\psi = 0 \tag{2.117}$$

and this time we need two more wave equations to fix completely the functional form of $f(\eta{\cdot}p,\eta^2)$. In order to get contact with the transverse-traceless gauge-fixed form of Fronsdal equations for massless particles with integer $n$-helicity particles ( [52]), we want to choose the following two equations

$$\eta\cdot\partial_\eta\psi = n\psi \tag{2.118}$$
$$p\cdot\partial_\eta\psi = 0 \tag{2.119}$$

where we fix the degree of homogeneity in the variable $\eta$ to be $n$. It is worth noticing (for future purposes) that the choice $n = 0$ implies $f = 1$.

# 4 The structure of intertwiners in the infinite spin representation

## 4.1 General solution of the intertwiner condition for the infinite spin representation

Let's consider the intertwiner property

$$d(W(\Lambda,p))u(\Lambda^{-1}e,\Lambda^{-1}p)=u(e,p)\Leftrightarrow d(W(\Lambda,p))u(e,p)=u(\Lambda e,\Lambda p) \tag{3.1}$$

where $(e,p)\in\mathcal{T}_+\times\dot{H}_m^+$, $p=B_p p_{\text{ref}}$, $\Lambda\in\mathcal{L}_+^\uparrow$ and for almost all $p$ the function $e\mapsto u(e,p)$ is analytic in the tuboid $\mathcal{T}_+$. For the infinite spin representation, the Pauli-Lubanski parameter $\kappa>0$ labels the faithful representations of the little group $\widetilde{E(2)}$ and $d_\kappa$ acts on $L^2(\mathbb{R}^2,\mathrm{d}\nu_\kappa)$ according to ( [53])

$$(d_\kappa(\vec{a},R)f)(\vec{k})=e^{i\vec{a}\cdot\vec{k}}f(R^{-1}k) \tag{3.2}$$

with $(\vec{a},R)\in\widetilde{E(2)}$.

Using the surjective homomorphism $SL(2,\mathbb{C})\to SO^\uparrow(1,3)$, we can identify the space of Hermitian matrices with Minkowski spacetime $\mathcal{M}$ in such a way that the determinant of a Hermitian matrix is the squared length of the corresponding vector in Minkowski spacetime, and the Lorentz transformation $x\mapsto\Lambda x$ corresponds to the action (by conjugation, with $S\in SL(2,\mathbb{C})$) $X\mapsto S(\Lambda)XS^\dagger(\Lambda)$ of $SL(2,\mathbb{C})$ on $X$ [33].

Since we can parametrize the little group $\widetilde{E(2)}\subseteq SL(2,\mathbb{C})$ with the two-dimensional translation generators and the one-dimensional rotation generator

$$A(\vec{a})=\begin{bmatrix}1 & a_1-ia_2\\ 0 & 1\end{bmatrix}\qquad A(\phi)=\begin{bmatrix}e^{-i\phi} & 0\\ 0 & e^{i\phi}\end{bmatrix} \tag{3.4}$$

every generic element $P\in\widetilde{E(2)}$ can be expressed as a product of these generators

$$P=\begin{bmatrix}e^{-i\phi} & a_1-ia_2\\ 0 & e^{i\phi}\end{bmatrix}=A\left(\frac{\phi}{2}\right)A\left(\vec{a}\right)A\left(\frac{\phi}{2}\right) \tag{3.5}$$

It is also worth noticing that the Lorentz transformation corresponding to $A(\phi)$ acts like the counter-clockwise rotation $R(2\phi)$ in the $1-2$ plane because

$$A(\phi)XA^{-1}(\phi)=\begin{bmatrix}x_0+x_3 & e^{-2i\phi}(x_1-ix_2)\\ e^{2i\phi}(x_1+ix_2) & x_0+x_3\end{bmatrix} \tag{3.6}$$

[33] remember that, if $\{\sigma_\mu\}$ is the set of Pauli matrices,

$$x_\mu=(x_0,x_1,x_2,x_3)\leftrightarrow X=x^\mu\sigma_\mu=\begin{bmatrix}x_0+x_3 & x_1-ix_2\\ x_1+ix_2 & x_0-x_3\end{bmatrix} \tag{3.3}$$

We want now to consider special cases of the intertwiner condition (3.1). For $\Lambda = B_p^{-1}$, this becomes

$$u(p_{\text{ref}}, B_p^{-1}e) = u(e, p) \Leftrightarrow u(p_{\text{ref}}, e) = u(p, B_p e) \tag{3.7}$$

since $\Lambda p = p_{\text{ref}}$ and so $B_{\Lambda p} = id$. We can therefore restrict our attention to the function $u(p_{\text{ref}}, e)$ for all $e$. Moreover by choosing $\Lambda_{\text{ref}} \in \widetilde{E(2)}$, one has also

$$u(p_{\text{ref}}, \Lambda_{\text{ref}} e) = d_\kappa(\Lambda_{\text{ref}}) u(p_{\text{ref}}, e) \tag{3.8}$$

and so it suffices to know $u(p_{\text{ref}}, e)$ for one point $e$ on each orbit of $\widetilde{E(2)}$.

Recalling the definition of the $G$-orbit $\Gamma = \{q \in H_0^+ : q \cdot p_{\text{ref}} = 1\}$, where $p_{\text{ref}} = (1, 0, 0, 1) \in H_0^+$, it is useful to consider the invariants of the orbit. From the parametrization of the orbit $\xi(z)$, let's define $\xi_0 := \xi(0) = \frac{1}{2}(1, 0, 0, -1)$ so that $\xi_0 \cdot p_{\text{ref}}$. Since $\widetilde{E(2)}$ fixes precisely $p_{\text{ref}}$, its orbits are labelled by the only (Lorentz) invariants we can build, namely $X = e^2$ and $Y = e \cdot p_{\text{ref}}$. Using the fact that $\widetilde{E(2)}$ has three parameters and choosing a reference point $e = \alpha p_{\text{ref}} + \beta \xi_0$, we notice that the $G$-action on $e$ is such that

$$T(\vec{a}) R(\phi) e = T(\vec{a}) R(\phi) e = T(\vec{a}) \left(\alpha p_{\text{ref}} + \beta \xi_0\right) \tag{3.9}$$

and so the orbits are really two-dimensional surfaces. This is the reason why we chose a two-parametric family to exhaust all points of the orbits. With our choice of the reference point $e$, we have (calculations are pretty straightforward, since $p_{\text{ref}} \cdot p_{\text{ref}} = \xi_0 \cdot \xi_0 = 0$) $Y = \beta$ and $X = 2\alpha\beta$.

We already know that the Lorentz transformation corresponding to $A(\phi)$ acts like $R(2\phi)$ in the 1-2 plane and $p_{\text{ref}}$ and $\xi_0$ are invariant under $R(2\phi)$. It follows from equation (3.8) that $u(p_{\text{ref}}, \alpha p_{\text{ref}} + \beta \xi_0)$ is invariant under $d_\kappa(\Lambda_{\text{ref}})$, hence $u(p_{\text{ref}}, \alpha p_{\text{ref}} + \beta \xi_0)$ as a function of $\vec{k} \in \kappa S^1$ is a multiple $g(\alpha, \beta) \cdot 1$ of the constant function 1 with an arbitrary function $g(\alpha, \beta)$. Moreover the Lorentz transformation corresponding to $A(\vec{a})$, that is $T(\vec{a})$, is such that

$$e = T(\vec{a}) \left(\alpha p_{\text{ref}} + \beta \xi_0\right) = \alpha p_{\text{ref}} + \beta \xi(\vec{a}) \tag{3.10}$$

where

$$\xi(\vec{a}) = T(\vec{a}) \xi_0 = \left(\frac{1}{2}(|\vec{a}|^2 + 1), \vec{a}, \frac{1}{2}(|\vec{a}|^2 - 1)\right) \tag{3.11}$$

Therefore it follows that

$$u(p, e) = u(p_{\text{ref}}, T(\vec{a} \left(\alpha p_{\text{ref}} + \beta \xi_0\right))(\vec{k}) = \tag{3.12}$$

$$= g(\alpha, \beta)(d_\kappa(A(\vec{a}))1)(\vec{k})) = g(\alpha, \beta) \cdot e^{i\vec{k} \cdot \vec{a}} \tag{3.13}$$

In order to express $u(p, e)$ as a function of $e$, we introduce the four-vector $E(\vec{k}) = (0, \vec{k}, 0)$ such that $\vec{k} \cdot \vec{a} = -E(\vec{k}) \cdot \xi(\vec{a}) = -E(\vec{k}) \cdot e/\beta$. Thus

$$u(p_{\text{ref}}, e)(\vec{k}) = g\left(\frac{e^2}{2e \cdot p_{\text{ref}}}, e \cdot p_{\text{ref}}\right) \cdot e^{-i\frac{e \cdot E(\vec{k})}{e \cdot p_{\text{ref}}}} \tag{3.14}$$

Using also the condition (3.7), we can determine the intertwiner function everywhere

$$u_f(p,e)(\vec{k}) = g\left(\frac{e^2}{2e\cdot p}, e\cdot p\right)\cdot e^{-i\frac{e\cdot E_p(\vec{k})}{e\cdot p}} = f(e^2, e\cdot p)\cdot e^{-i\frac{e\cdot E_p(\vec{k})}{e\cdot p}} \tag{3.15}$$

where $E_p(\vec{k}) := B_p E(\vec{k})$. These solutions are parametrized by an arbitrary function $f(e^2, e\cdot p) := g\left(\frac{e^2}{2e\cdot p}, e\cdot p\right)$ and they are smooth Schuster and Toro's wavefunctions, up to an irrelevant sign change of $\vec{k}$.

The parametrization we used so far misses out the singular orbits with $e\cdot p_{\text{ref}} = 0$ but $e^2 \neq 0$, that would require $\alpha = \infty$. In that case we can parametrize the orbit via

$$e = cp_{\text{ref}} + E(\vec{e}) \tag{3.16}$$

such that $e^2 = E(\vec{e})^2 = -\vec{e}^2$. Acting by the translation operators $T(\vec{a})$ we get

$$A(\vec{a})E(\vec{e})A^\dagger(\vec{a}) = \begin{bmatrix} 1 & a_1 - ia_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & e_1 - ie_2 \\ e_1 + ie_2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a_1 + ia_2 & 1 \end{bmatrix} = \tag{3.17}$$

$$= \begin{bmatrix} 1 & a_1 - ia_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (e_1 - ie_2)a_1 + a_2(e_2 + ie_1) & e_1 - ie_2 \\ e_1 + ie_2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2(a_1 e_1 + a_2 e_2) & e_1 - ie_2 \\ e_1 + ie_2 & 0 \end{bmatrix}$$

and coming back to the vector notation $T(\vec{a})E(\vec{e}) = E(\vec{a}) + \vec{a}\cdot\vec{e}p_{\text{ref}}$. Therefore in this case $e$ is invariant under the translations orthogonal to $\vec{e}$ and $u(p_{\text{ref}}, e)$ has support only at $\vec{k}$ parallel to $\vec{e}$, i.e. it is an $L^2$-valued distribution containing a $\delta$-function $\delta(E_p(\vec{k})^2 e^2 - (E_p(\vec{k})\cdot e)^2)$, like the singular Schuster and Toro's wavefunctions.

## 4.2 Comparison of Schuster-Toro wavefunctions with Mund-Schroer-Yngvason intertwiners

The Mund-Schroer-Yngvason intertwiner has the general form

$$u_F(p,e)(\vec{k}) = \int \mathrm{d}^2 z e^{i\vec{k}\cdot\vec{z}} F(e\cdot B_p\xi(\vec{z})) \tag{3.18}$$

with certain bounds and analyticity properties of $F$ in the upper half plane, explicitly stated in the theorem 2.1. Following the proof of the theorem in the MSY paper ( [6]), it can be easily shown that as long as $u_F(p,e)$ is analytic and bounded in the tube $e = e' + ie'' \in H_1^{\mathbb{C}}$, $e'' \in V_+$ the function $F(e\cdot B_p\xi(\vec{z}))$ may be generalized to depend also on $(e\cdot p)$. We can notice that multiplying a Mund-Schroer-Yngvason intertwiner with any positive power of $(p\cdot e)$ does not change the localization properties, because it is just a derivative in the real $x-$space. Moreover multiplying it by an inverse power with the proper $i\epsilon$ also respects the localization along the string because this is just an integration along the string ( [54]); but it may introduce an IR problem at $p \mapsto 0$.

In order to find the relation between $f(e^2, e\cdot p)$ and $F(e\cdot p, e\cdot B_p\xi(\vec{z}))$ such that $u_f = u_F$, we can write $F(e\cdot p, e\cdot B_p\xi(\vec{z}))$ as

$$F(e\cdot p, e\cdot B_p\xi(\vec{z})) = \int_0^{+\infty} \mathrm{d}s e^{i\kappa s(e\cdot B_p\xi(\vec{z}))}\hat{F}(e\cdot p, s) \tag{3.19}$$

Indeed looking at the dependence of $F$ and $\hat{F}$ on the second argument, $\hat{F}$ should be supported on $\mathbb{R}_+$ since $F$ has to be analytic in $\mathbb{C}_+ \coloneqq \{z \in \mathbb{C} : \Im z > 0\}$.

It is very useful to evaluate the gaussian integral in the $z$ variable, in order to compare explicitly the structure of the MSY intertwiner with the Schuster-Toro approach. Let's consider the following standard integrals

$$\int \mathrm{d}z_i \exp\left[is\left(A_i z_i^2 + B_i z_i + C_i\right)\right] = \tag{3.20}$$
$$= e^{isC_i} \int \mathrm{d}z_i \exp\left[isA_i\left(z_i + \frac{B_i}{2A_i}\right)^2\right] \exp\left[-is\frac{B_i^2}{4A_i}\right] = e^{isC_i} e^{-is\frac{B_i^2}{4A_i}} \sqrt{\frac{i\pi}{sA_i}} \forall\, i = 1,2$$

where we require $\Re(-iA_i) > 0 \Leftrightarrow \Im A > 0$ due to the convergence condition. Therefore

$$\int \mathrm{d}^2 z \exp\left[is\left(\sum_i A_i z_i^2 + B_i z_i + C_i\right)\right] = \tag{3.21}$$
$$e^{is(C_1+C_2)} \frac{i\pi}{s\sqrt{A_1 A_2}} e^{-is\left[\frac{B_1^2}{4A_1} + \frac{B_2^2}{4A_2}\right]} \tag{3.22}$$

In our case, identifying the letters $A_i$,$B_i$ and $C_i$ with the following expressions

$$A_1 = A_2 = \frac{p \cdot e}{2} \tag{3.23}$$
$$B_1 = -\kappa(B_p^{-1}e)_1 + \frac{k_1}{s} \tag{3.24}$$
$$B_2 = -\kappa(B_p^{-1}e)_2 + \frac{k_2}{s} \tag{3.25}$$
$$C_1 = \frac{1}{2}(B_p^{-1}e)_+ \tag{3.26}$$
$$C_2 = 0 \tag{3.27}$$

where $(B_p^{-1}e)$ is the 4-vector in the light-cone coordinate frame. Using equation (3.21), we obtain directly

$$\int \mathrm{d}^2 z \exp\left[is\left(\sum_i A_i z_i^2 + B_i z_i + C_i\right)\right] = \tag{3.28}$$
$$= e^{is\kappa\frac{1}{2}(B_p^{-1}e)_+} \frac{2i\pi}{s(p\cdot e)} e^{-is\left(\frac{k_1^2+k_2^2}{2s^2\kappa(p\cdot e)}\right)} e^{-is\left(\frac{\kappa((B_p^{-1}e)_1^2+(B_p^{-1}e)_2^2)}{2(p\cdot e)}\right)} e^{-i\left(\frac{-k_1(B_p^{-1}e)_1-k_2(B_p^{-1}e)_2}{(p\cdot e)}\right)} =$$
$$= e^{is\kappa\frac{1}{2}(B_p^{-1}e)_+} \frac{2i\pi}{s(p\cdot e)} e^{-i\left(\frac{\kappa}{2s(p\cdot e)}\right)} e^{-is\left(\frac{\kappa(-(B_p^{-1}e)^2+(B_p^{-1}e)_0^2-(B_p^{-1}e)_3^2}{2(p\cdot e)}\right)} e^{i\left(\frac{E(\vec{k})_1(B_p^{-1}e)_1+E(\vec{k})_2(B_p^{-1}e)_2}{(p\cdot e)}\right)} =$$
$$= e^{is\kappa\frac{1}{2}(B_p^{-1}e)_+} e^{is\frac{\kappa}{2(p\cdot e)}\left(-(B_p^{-1}e)_0^2+(B_p^{-1}e)_3^2\right)} \frac{2i\pi}{s(p\cdot e)} e^{-i\left(\frac{\kappa}{2s(p\cdot e)}\right)} e^{-is\left(\frac{-\kappa(B_p^{-1}e)^2}{2(p\cdot e)}\right)} e^{-i\left(\frac{E(\vec{k})\cdot(B_p^{-1}e)}{(p\cdot e)}\right)}$$

In the light-cone coordinate frame, $x \cdot p = \frac{1}{2}\left(x_+ p_- + x_- p_+\right) - x_1 p_1 - x_2 p_2$ and so $x \cdot x = x_+ x_- - x_1^2 - x_2^2$. Therefore, since $(B_p^{-1}e)_- = p_{\text{ref}} \cdot (B_p^{-1}e) = p \cdot e$, we can write

$$e^{is\kappa\frac{1}{2}(B_p^{-1}e)_+} = e^{is\kappa\frac{(B_p^{-1}e)_+(B_p^{-1}e)_-}{2(p\cdot e)}} = \tag{3.29}$$
$$e^{is\kappa\frac{(B_p^{-1}e)^2+(B_p^{-1}e)_1^2+(B_p^{-1}e)_2^2}{2(p\cdot e)}}$$

and therefore

$$\int \mathrm{d}^2 z \exp\left[is\left(\sum_i A_i z_i^2 + B_i z_i + C_i\right)\right] = \tag{3.30}$$

$$= e^{is\kappa\frac{(B_p^{-1}e)^2+(B_p^{-1}e)_1^2+(B_p^{-1}e)_2^2}{2(p\cdot e)}} e^{is\frac{\kappa}{2(p\cdot e)}\left(-(B_p^{-1}e)_0^2+(B_p^{-1}e)_3^2\right)} \frac{2i\pi}{s(p\cdot e)} e^{-i\left(\frac{\kappa}{2s(p\cdot e)}\right)} e^{-is\left(\frac{-\kappa(B_p^{-1}e)^2}{2(p\cdot e)}\right)} e^{-i\left(\frac{E(\vec{k})\cdot(B_p^{-1}e)}{(p\cdot e)}\right)} =$$

$$= \frac{2i\pi}{s(p\cdot e)} e^{-i\left(\frac{\kappa}{2s(p\cdot e)}\right)} e^{-is\left(\frac{-\kappa(B_p^{-1}e)^2}{2(p\cdot e)}\right)} e^{-i\left(\frac{E(\vec{k})\cdot(B_p^{-1}e)}{(p\cdot e)}\right)}$$

Rewriting $u_F(p,e)(\vec{k})$ in the following Lorentz invariant expression

$$u_F(p,e)(\vec{k}) = e^{-i\left(\frac{E(\vec{k})\cdot(B_p^{-1}e)}{(p\cdot e)}\right)} \frac{2i\pi}{s(p\cdot e)} \int_0^{+\infty} \frac{\mathrm{d}s}{s} \hat{F}((e\cdot p), s) e^{i\frac{\kappa}{2(p\cdot e)}(se^2 - 1/s)} \tag{3.31}$$

we can recognize the similarity with the Schuster-Toro form. Indeed, for $e^2 = -1$, we have

$$f(-1, e\cdot p) = \frac{2i\pi}{(p\cdot e)} \int_0^{+\infty} \frac{\mathrm{d}s}{s} \hat{F}((e\cdot p), s) e^{i\frac{\kappa}{2(p\cdot e)}(se^2 - 1/s)} \tag{3.32}$$

and trading $s$ for $t \coloneqq \frac{1}{2}\log(s)$ we obtain (remember the hyperbolic identity $\cosh(2x) = 1 + 2\sinh^2(x)$)

$$\begin{aligned} f(-1, e\cdot p) &= \frac{4i\pi}{(p\cdot e)} \int_{\mathbb{R}} \mathrm{d}t \hat{F}((e\cdot p), e^{2t}) e^{-i\frac{\kappa}{2(p\cdot e)}(e^{2t}+e^{-2t})} = \\ &= \frac{4i\pi}{(p\cdot e)} e^{-i\frac{\kappa}{(p\cdot e)}} \int_{\mathbb{R}} \mathrm{d}t \hat{F}((e\cdot p), e^{2t}) e^{-i\frac{\kappa}{(p\cdot e)}\sinh^2(t)} \end{aligned} \tag{3.33}$$

## 4.3 Properties of the Fourier transform of a Wigner intertwiner for the infinite spin representation

Let's consider again the intertwiner condition in equation (3.1). Although the intertwiner gives a string-localized field only for spacelike $e$, it is defined for all $e \in \mathbb{R}^4$ and we may take the Fourier transform with respect to $e$. Under a Fourier transformation $u_F(p,e)(\vec{k})$ becomes

$$\hat{u}(p,\eta)(\vec{k}) = \int \mathrm{d}^4 e e^{-i(\eta\cdot e)} u_F(p,e)(\vec{k}) \tag{3.34}$$

and it is easy to see that the intertwiner condition is invariant under such a transformation, due to the fact that sending $e \mapsto \Lambda^{-1}e$

$$\int \mathrm{d}^4 e e^{-i(\eta\cdot e)} u_F(p, \Lambda e)(\vec{k}) \mapsto \int \mathrm{d}^4 e e^{-i(\Lambda\eta\cdot e)} u_F(p,e)(\vec{k}) \tag{3.35}$$

Our purpose is to show that $\hat{u}(p,\eta)(\vec{k})$ is supported at $\eta^2 = 0$. Instead of following a direct (but complicated) route to prove this statement (e.g. looking to the Fourier trasformation), we prefer to take an easier route but that requires new ideas.

We want to focus on the differential equations that $\hat{u}(p,\eta)(\vec{k})$ solves. We can apply the d'Alembert operator (Laplace operator of the Minkowski space) $\Box_e$ in the variable $e$ to $\hat{u}(p,\eta)(\vec{k})$:

$$\Box_e u_F(p,e)(\vec{k}) = \Box_e \left[\int \mathrm{d}^2 z \int \mathrm{d}s e^{i\vec{k}\cdot\vec{z}} e^{i\kappa s(e\cdot B_p\xi(\vec{z}))} \hat{F}(e\cdot p, s)\right] = \tag{3.36}$$

$$= \int \mathrm{d}^2 z \int \mathrm{d}s e^{i\vec{k}\cdot\vec{z}} \Big[\Box_e \left(e^{i\kappa s(e\cdot B_p\xi(\vec{z}))}\right) \hat{F}(e\cdot p, e\cdot B_p\xi(\vec{z})) + e^{i\kappa s(e\cdot B_p\xi(\vec{z}))}\Box_e \left(\hat{F}(e\cdot p, s)\right) +$$
$$2\frac{\partial}{\partial e^\mu}\left(e^{i\kappa s(e\cdot B_p\xi(\vec{z}))}\right)\frac{\partial}{\partial e_\mu}\left(\hat{F}(e\cdot p, s)\right)\Big] \tag{3.37}$$

Since (remember that $\xi(\vec{z})$ and $p$ have to be null vectors)

$$\Box_e \left(e^{i\kappa s(e\cdot B_p\xi(\vec{z}))}\right) = -\kappa^2 s^2 \left(B_p\xi(\vec{z})\cdot B_p\xi(\vec{z})\right) e^{i\kappa s(e\cdot B_p\xi(\vec{z}))} = 0 \tag{3.38}$$

and

$$\Box_e \left(\hat{F}(e\cdot p, s)\right) = \frac{\partial}{\partial e_\mu}\frac{\partial}{\partial e^\mu}\left(\hat{F}(e\cdot p, s)\right) = \tag{3.39}$$
$$= p^2 \frac{\partial}{\partial(e\cdot p)}\frac{\partial}{\partial(e\cdot p)}\left(\hat{F}(e\cdot p, s)\right) = 0$$

then

$$\Box_e u_F(p,e)(\vec{k}) = \int \mathrm{d}^2 z \int \mathrm{d}s e^{i\vec{k}\cdot\vec{z}} 2i\kappa s(p\cdot B_p\xi(\vec{z})) e^{i\kappa s(e\cdot B_p\xi(\vec{z}))} \frac{\partial \hat{F}(e\cdot p, s)}{\partial(e\cdot p)} = \tag{3.40}$$
$$= \int \mathrm{d}^2 z \int \mathrm{d}s\, 2i\kappa s\, e^{i\vec{k}\cdot\vec{z}} e^{i\kappa s(e\cdot B_p\xi(\vec{z}))} \frac{\partial \hat{F}(e\cdot p, s)}{\partial(e\cdot p)}$$

So the last expression (3.40) vanishes iff $\hat{F}(e\cdot p, s)$ does not depend on $(p\cdot e)$, as in the original formulation of Mund-Schroer-Yngvason. In this case, and only in this case, Fourier transforming the expression (3.40) gives an algebraic equation in the space of distributions $-\eta^2 \hat{u}_F(p,\eta)(\vec{k}) = (|\vec{\eta}|^2 - \eta_0^2)\hat{u}_F(p,\eta)(\vec{k}) = 0$. It is then obvious that the space of solutions is supported on $\eta^2 = 0$. To be more precise, there are two classes of solutions to this equation according to the fact that $|\vec{\eta}| = 0$ or $|\vec{\eta}| \neq 0$, but both solutions are supported on $\eta^2 = 0$ ( [55]).

Schuster and Toro's work regards a one-particle quantum mechanical setting, in which localization doesn't play an important role and we have to talk more properly about supports of the wavefunctions. Since they found (smooth or singular) solutions to the intertwiner equation and such equation is invariant under Fourier transformation, it is clear that such wavefunctions $\psi(p,\eta)$ are null-supported for an appropriate choice of the auxiliary variable $\eta$. There is no dynamics in $\eta$-space - it is just a useful book-keeping device for compactly manipulating many tensors simultaneously - and moreover $\eta$ has not direct geometrical meaning related to localization properties.

We want to underline that there cannot be any meaningful notion of "string-localization of wavefunctions", since such notion makes sense only in terms of commutators on the QFT setting. Indeed regarding $\eta$ as a label that carries the Lorentz representation $D$ the wavefunctions $\psi(x,\eta)$ should be considered as localized at $x$ only; conversely regarding $\eta$ as a variable then the wavefunctions $\psi(x,\eta)$ should be considered as localized at $(x,\eta)$. This is due to the fact that we are working in a one-particle quantum mechanical setting.

That extra-variable can be "meaningful" only after promoting wavefunctions to gauge fields $\Psi(p,\eta)$, for which the role of $\eta^\mu$ is to extend Minkowski spacetime to a cotangent bundle over Minkowski spacetime where the gauge field $\Psi(p,\eta)$ lives ([13]).

## 4.4 Bounds on the Schuster-Toro smooth solutions

Let us control the bounds on $u_f$ in equation (3.15) in the tube $e = e' + ie'' \in H_1^{\mathbb{C}}$, where $e'' \in V_+$. In particular we have to consider the modulus of the exponential factor

$$e^{-i\left(\frac{E_p(\vec{k})\cdot e}{(p\cdot e)}\right)} \tag{3.41}$$

and to provide appropriate bounds on it. If we define $f := B_p^{-1}e = f' + if''$, with $f', f'' \in \mathbb{R}$ and $f' \cdot f'' = 0$[34], then $f'' \in V_+$ and $f'^2 - f''^2 = -1$ since $e'' \in V_+$ and $e^2 = -1$. It is then possible to parametrize $f'$ and $f''$ as

$$f' = \alpha' p_0 + \beta' \xi_0 + E(\vec{f}') \tag{3.42}$$

$$f'' = \alpha'' p_0 + \beta'' \xi_0 + E(\vec{f}'') \tag{3.43}$$

From the previous conditions

$$\begin{gathered} f' \cdot f'' = 0 \Rightarrow \alpha'\beta'' + \alpha''\beta' - E(\vec{f}') \cdot E(\vec{f}'') = 0 \Rightarrow \vec{f}' \cdot \vec{f}'' = \alpha'\beta'' + \alpha''\beta' \\ f'^2 - f''^2 = -1 \Rightarrow 2\alpha'\beta' - \vec{f}'^2 - 2\alpha''\beta'' + \vec{f}''^2 = -1 \\ f'' \in V_+ \Leftrightarrow f''^2 > 0 \quad \text{and} \quad f_0 > 0 \Rightarrow 2\alpha''\beta'' > \vec{f}''^2 \quad \text{and} \quad \alpha'' + \frac{1}{2}\beta'' > 0 \end{gathered} \tag{3.44}$$

and then also the following holds

$$\begin{gathered} 2\alpha'\beta' - \vec{f}'^2 - 2\alpha''\beta'' + \vec{f}''^2 = -1 \qquad \text{and} \qquad 2\alpha''\beta'' > \vec{f}''^2 \Rightarrow 2\alpha'\beta' + 1 > \vec{f}'^2 \\ 2\alpha''\beta'' > \vec{f}''^2 \quad \text{and} \quad \alpha'' + \frac{1}{2}\beta'' > 0 \Rightarrow \alpha'' > 0 \quad \text{and} \quad \beta'' > 0 \end{gathered} \tag{3.45}$$

We can now proceed with the calculation. Using the following identities

$$e \cdot p = e \cdot B_p p_{\text{ref}} = f \cdot p_{\text{ref}} \tag{3.46}$$

$$e \cdot E_p(\vec{k}) = f \cdot E(\vec{k}) = -\vec{f} \cdot \vec{k} \tag{3.47}$$

we can rewrite the exponential factor as

$$e^{i\left(\frac{(\vec{f}\cdot\vec{k})(f\cdot p_{\text{ref}})^*}{(f\cdot p_{\text{ref}})^2}\right)} \tag{3.48}$$

[34] $2f' \cdot f'' = \Im(f^2) = \Im(e^2)$, but the complex tube is contained in the set of complex vectors with $e^2 = -1 \in \mathbb{R}$

so that the following inequalities hold

$$
\begin{aligned}
&e^{i\left(\frac{[(\vec{f}'+i\vec{f}'')\cdot\vec{k}][(\vec{f}'+i\vec{f}'')\cdot p_{\text{ref}}]}{|f\cdot p_{\text{ref}}|^2}\right)} \leq \\
&\leq e^{\Re\left[i\left(\frac{[(\vec{f}'+i\vec{f}'')\cdot\vec{k}][(\vec{f}'+i\vec{f}'')\cdot p_{\text{ref}}]}{|f\cdot p_{\text{ref}}|^2}\right)\right]} \leq \\
&\leq e^{\left[\frac{(\vec{f}'\cdot\vec{k})(\vec{f}''\cdot p_{\text{ref}})-(\vec{f}''\cdot\vec{k})(\vec{f}'\cdot p_{\text{ref}})}{|f\cdot p_{\text{ref}}|^2}\right]}
\end{aligned}
\tag{3.49}
$$

But $\vec{f}''\cdot p_{\text{ref}} = \beta''$ and $\vec{f}'\cdot p_{\text{ref}} = \beta'$, so using Schwartz inequality

$$
\begin{aligned}
&e^{\left[\frac{(\vec{f}'\beta''-\vec{f}''\beta')\cdot\vec{k}}{|f\cdot p_{\text{ref}}|^2}\right]} \leq \\
&\leq e^{\left[\frac{|\vec{f}'\beta''-\vec{f}''\beta'|\kappa}{|f\cdot p_{\text{ref}}|^2}\right]}
\end{aligned}
\tag{3.50}
$$

Now, using also the inequalities we found in equation (3.44) and (3.45)

$$
\begin{aligned}
|\vec{f}'\beta''-\vec{f}''\beta'| = (\beta'')^2|\vec{f}'|^2+(\beta')^2|\vec{f}''|^2-2\beta'\beta''\vec{f}'\cdot\vec{f}'' \leq \\
\leq (\beta'')^2(2\alpha'\beta'+1)+(\beta')^2(2\alpha''\beta'')-2\beta'\beta''(\alpha'\beta''+\alpha''\beta') \leq (\beta'')^2
\end{aligned}
\tag{3.51}
$$

and therefore finally

$$
\begin{aligned}
e^{-i\left(\frac{E_p(\vec{k})\cdot e}{(p\cdot e)}\right)} \leq e^{\frac{\kappa\beta''}{|f\cdot p_{\text{ref}}|^2}} = e^{\frac{\kappa\Re(-if\cdot p_{\text{ref}})}{|e\cdot p|^2}} = e^{\Re\left[\frac{-i\kappa}{(e\cdot p)^*}\right]} = \\
= |e^{\left(\frac{-i\kappa}{(e\cdot p)^*}\right)}| = |e^{\left(\frac{i\kappa}{(e\cdot p)}\right)}|
\end{aligned}
\tag{3.52}
$$

As a result, one may multiply with $e^{-i\frac{\kappa}{(e\cdot p)}}$ to get something bounded in the entire tube. This property is certainly stronger than the bounds provided in the Mund-Schroer-Yngvason paper ( [6]) in every compact $\Omega$ that we already mentioned in the chapter 2.

Therefore if we write

$$
f(-1, e\cdot p) = e^{-i\frac{\kappa}{(e\cdot p)}} g((e\cdot p)) \tag{3.53}
$$

where

$$
g((e\cdot p)) = \frac{4i\pi}{(p\cdot e)}\int_{\mathbb{R}} \mathrm{d}t\hat{F}((e\cdot p), e^{2t})e^{-i\frac{\kappa}{(p\cdot e)}\sinh^2(t)} \tag{3.54}
$$

then $u_f(p,e)(\vec{k})$ is bounded in the tube iff $g((e\cdot p))$ is bounded in $\mathbb{C}_+$.

## 4.5 Two-point function for the infinite spin representation

Let's consider the structure of a Mund-Schroer-Yngvason string-localized field associated with the infinite spin representation $U_\kappa$, for a generic function $F$:

$$
\phi_F(x,e) \coloneqq \int_{H_m^+} \mathrm{d}\mu(p)\int_{S^1}\mathrm{d}\mu(\vec{k})\,\{e^{ip\cdot x}u_F(e,p,\vec{k})a^*(p,\vec{k}) + e^{-ip\cdot x}\overline{u_F}(e,p,\vec{k})a(p,\vec{k})\} \tag{3.55}
$$

The computation of the two-point function $M(x,e;x',e')$ for our problem is then pretty straightforward:

$$M(x,e;x',e') = (\phi_F(x,e)\Omega, \phi_F(x',e')\Omega) = \int_{H_m^+} \mathrm{d}\mu(p) e^{-ip(x-x')} \int \frac{\mathrm{d}\phi}{2\pi} \overline{u_F}(e,p,\vec{k}_\phi) u_F(e',p,\vec{k}_\phi) \tag{3.56}$$

where we used the angle basis $\phi$ for the variable $\vec{k}$ (we will choose latex $\phi$ as the angle between $\vec{k}$ and a specific vector $\vec{f}$). The kernel of the two-point function is defined as

$$M(x,e;x',e') = \int_{H_m^+} \mathrm{d}\mu(p) e^{-ip(x-x')} M(p,e,e') \tag{3.57}$$

and using the previous equation (3.15) regarding the structure of smooth intertwiners we can single out the $\vec{k}-$dependence

$$M(p,e,e') = \int \frac{\mathrm{d}\phi}{2\pi} e^{i\frac{e'\cdot E_p(\vec{k})}{e'\cdot p}} e^{-i\frac{e\cdot E_p(\vec{k})}{e\cdot p}} \overline{f}(e^2, e\cdot p) f(e^2, e\cdot p) \tag{3.58}$$

The exponent of the exponential factor is linear in $\vec{k}$ and can be written as

$$\begin{aligned} i\left(\frac{e'\cdot E_p(\vec{k})}{e'\cdot p} - \frac{e\cdot E_p(\vec{k})}{e\cdot p}\right) &= \\ = iB_p^{-1}\left(\frac{e'}{e'\cdot p} - \frac{e}{e\cdot p}\right)\cdot E(\vec{k}) &=: -i\vec{f}\cdot\vec{k} \end{aligned} \tag{3.59}$$

where $\vec{f}$ is the transverse $1-2$ part of the four-vector

$$f = B_p^{-1}\left(\frac{e'}{e'\cdot p} - \frac{e}{e\cdot p}\right) \tag{3.60}$$

Since

$$(f\cdot p_{\text{ref}})(f\cdot\xi_0) = \frac{1}{2}(f_0 - f_3)(f_0 + f_3) = \frac{1}{2}(f_0^2 - f_3^2) \tag{3.61}$$

then due to the fact that $f\cdot p_{\text{ref}}$ vanishes identically we have

$$2(f\cdot p_{\text{ref}})(f\cdot\xi_0) - |\vec{f}|^2 = f^2 == \left(\frac{e'}{e'\cdot p} - \frac{e}{e\cdot p}\right)^2 = -|\vec{f}|^2 \tag{3.62}$$

Coming back to the integral, we can denote the angle between $\vec{k}$ and $\vec{f}$ as $\phi$ so that

$$\begin{aligned} M(p,e,e') = \overline{f}(e^2,e\cdot p)f(e^2,e\cdot p)\int\frac{\mathrm{d}\phi}{2\pi}e^{-i\vec{k}\cdot\vec{f}} &= \overline{f}(e^2,e\cdot p)f(e^2,e\cdot p)\int\frac{\mathrm{d}\phi}{2\pi}e^{-i\kappa|\vec{f}|\cos(\phi)} = \\ &= \overline{f}(e^2,e\cdot p)f(e^2,e\cdot p)J_0(\kappa|\vec{f}|) \end{aligned} \tag{3.63}$$

where we used the integral representation of the first type (zeroth order) Bessel function ( [56])

$$J_n(z) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi} e^{i(n\phi - z\sin(\phi))} \tag{3.64}$$

From the final result in equation (3.63) we can investigate the behaviour of the two-point function in various directions of string configuration space. We want to check first the singular values of the argument of the Bessel function, that is the directions where $e \cdot p = 0$ (or $e' \cdot p = 0$). It is easy to see from the expression (3.62) for $|\vec{f}|$ that the argument of the Bessel function behaves like $\kappa/|e \cdot p|$ (or $\kappa/|e' \cdot p|$ respectively) in such directions. Since at large arguments the Bessel function behaves like

$$J_0(z) \sim \sqrt{\frac{2}{\pi|z|}} \cos\Big(|u| - \frac{\pi}{4}\Big) \tag{3.65}$$

then for $p \cdot e \mapsto 0_+$

$$\begin{aligned} M(p,e,e') &\overset{p\cdot e \mapsto 0_+}{\sim} |e \cdot p|^{\frac{1}{2}} e^{i\kappa\left(\frac{1}{e'\cdot p} - \frac{1}{e\cdot p}\right)} \overline{g}(e \cdot p) g(e \cdot p) (e^{i\left(\frac{\kappa}{|p\cdot e|} - \frac{\pi}{4}\right)} + e^{-i\left(\frac{\kappa}{|p\cdot e|} - \frac{\pi}{4}\right)}) \\ &\overset{p\cdot e \mapsto 0_+}{\sim} |e \cdot p|^{\frac{1}{2}} \overline{g}(0) g(0) (e^{-i\left(\frac{\pi}{4}\right)} + e^{-i\left(2\frac{\kappa}{p\cdot e} - \frac{\pi}{4}\right)}) \end{aligned} \tag{3.66}$$

whereas for $p \cdot e \mapsto 0_-$

$$M(p,e,e') \overset{p\cdot e \mapsto 0_-}{\sim} |e \cdot p|^{\frac{1}{2}} \overline{g}(0) g(0) (e^{+i\left(\frac{\pi}{4}\right)} + e^{-i\left(2\frac{\kappa}{p\cdot e} + \frac{\pi}{4}\right)}) \tag{3.67}$$

For $p \cdot e' \mapsto 0_\pm$ the asymptotic limits of $M(p,e,e')$ are the same, with $e'$ in place of $e$. Both asymptotic behaviours indicate that $M(p,e,e')$ admits a distributional boundary value from $\Im(e) \in V^+$.

Away from the singular directions, the two-point kernel $M(p,e,e')$ is regular in the $UV$ for $p \mapsto +\infty$ ($\Rightarrow |\vec{f}| \mapsto 0$), since at small arguments of the Bessel function

$$J_0(z) \sim 1 + \mathcal{O}(z^2) \tag{3.68}$$

and then

$$M(p,e,e') \overset{p \mapsto +\infty}{\sim} \overline{g}(+\infty) g(+\infty) \left(1 + \mathcal{O}(\kappa^2 |\vec{f}|^2)\right) \tag{3.69}$$

At coinciding directions $e = e' \Rightarrow |\vec{f}| = 0$ and so $M(p,e,e')$ is exactly zero.

Regarding the IR behaviour away from singular directions, the Bessel function is again dominated by rapid oscillations of a cosine function that decays proportional to $|p \cdot e|^{\frac{1}{2}}$.

# 5 Conclusion

In the first chapter we built explicitly in a rigorous way the unitary irreducible (positive energy) representations (UIR) of the Poincaré group, using the method of induced representations and classifying all representations by the value of the two Casimir invariants, $P^\mu P_\mu$ and $W^\mu W_\mu$. In four spacetime dimensions the infinite spin representation, characterized by $P^\mu P_\mu = 0$ and $W^\mu W_\mu = -\kappa^2$ with a continuous (Pauli-Lubanski) parameter $\kappa > 0$, according to Wigner interpretation of elementary particles corresponds to a new type of exotic particles called continuous spin particles (CSPs). The exotic properties of CSPs are the presence of infinite degrees of freedom per spacetime point and the presence of the dimensionful scale $\kappa$ althought the fact that these particles are massless. In particular it is possible to interpret physically the infinite spin representations as a limit (Pauli-Lubanski limit) of the well-known massive spin $s$ representations for $m \to 0$, $s \to +\infty$ while keeping the product $\kappa := ms$ fixed, that is CSPs are considered as high energy ($E \gg m$), large $s$ ($s \gg 1$) limit of massive particles in the regime $E \sim ms$.

In the second chapter we started to discuss how to construct quantum free fields and one particle states for CSPs, starting from the finite spin case and showing which problems arise for the extension to the infinite spin case. In particular, according to Yngvason's theorem, the pointlike localization for quantum free fields is incompatible with the Wightman's framework of axiomatic QFT in the infinite spin case. In fact the algebraic concept of modular localization applied to UIR of the Poincaré group, that deals with the (sub)algebra of operators $\mathcal{A}(\mathcal{O})$ in a given spacetime region $\mathcal{O}$, and the missing spinorial fiels in the Wigner construction of covariant $(m, s)$ quantum free fields motivate the introduction of string-localized fields to describe CSPs. Mathematically, it is thus necessary to introduce a new variable $e$ that belongs to the hyperboloid of space-like directions $H^3$, and the quantum free fields (or the associated intertwiners) will depend distributionally on this additional variable and will satisfy a natural extension of the Wightman's axioms (covariance, locality, etc.). For bosonic CSPs, we wrote explicitly the structure of a general type of infinite spin intertwiner ($B_p$ is the standard boost here)

$$u_F(e,p) = \int \mathrm{d}^2 z e^{ik\cdot z} F(B_p \xi(z) \cdot e) \tag{3.1}$$

with a dependence on a generic function $F$ that has an analytic extension to the upper complex half plane following the paper of Mund, Schroer and Yngvason([6]). Moreover we reviewed the structure of CSPs states, both in the angle and in the spin basis, starting from the construction of one particle Hilbert space in the infinite spin case $\mathcal{H}_1 = L^2(\partial V^+, \mathrm{d}\mu) \otimes L^2(k, \mathrm{d}\nu_\kappa)$ with $\mathrm{d}\nu_\kappa = \delta(|k|^2 - \kappa^2)$. Following Schuster and Toro ([9]), we considered general solution in form of covariant wavefunctions $\psi(x, \eta)$ of the intertwiner (little group) condition $\tilde{d}(W(\Lambda, p))u(\Lambda^{-1}e, \Lambda^{-1}p) = u(e, p)$. There are both smooth and singular solutions and that solutions satisfy different sets of partial differential equations.

In the third chapter, which consists entirely of original work, we developed the general solution of the intertwiner condition using a suitable parametrization of the orbits of the little group for massless particles, and we established that the possible solutions are in a bijective correspondence with Schuster and Toro smooth and singular wavefunctions. In particular the smooth solutions have the structure (here $(E_p(\vec{k}) = B_p E(\vec{k})$ and $E(\vec{k}) = (0, \vec{k}, 0))$

$$u_f(p,e)(\vec{k}) = f(e^2, e \cdot p) \cdot e^{-i\frac{e \cdot E_p(\vec{k})}{e \cdot p}} \tag{3.2}$$

and are parametrized by an arbitrary function $f(e^2, e \cdot p) := g\left(\frac{e^2}{2e \cdot p}, e \cdot p\right)$, whereas the singular solution are proportional to $\delta(E_p(\vec{k})^2 e^2 - (E_p(\vec{k}) \cdot e)^2)$. Moreover, using the explicit form of $u_F(e,p)$ found in Mund, Schroer and Yngvason's paper, we found a direct correspondence with $u_f(p,e)$ by performing a Gaussian integration in the variable $z$. This is the main result, which allows to establish a new relationship between $F$ and $f$. In addition to this, we found that the Fourier transform of $u_F(p,e)$ is supported on $\eta^2 = 0$ by showing that $\Box_e \hat{u}(p,e)_F = 0$, where $\eta$ is the Fourier conjugate variable of $e$. This helped us to make some deep considerations on the role of locality regarding the infinite spin representation, comparing the works of Schuster, Toro and Mund, Schroer and Yngvason. Then we found an estimate about a particular type of intertwiner $u_f(p,e)(\vec{k})$ with a function $f$ of the form $f(-1, e \cdot p) = e^{-i\frac{\kappa}{(e \cdot p)}} g((e \cdot p))$: $u_f(p,e)(\vec{k})$ is bounded in the tube iff $g((e \cdot p))$ is bounded in $\mathbb{C}_+$. At the end we considered the structure of the two-point function $M(x,e;x',e') = (\phi_F(x,e)\Omega, \phi_F(x',e')\Omega)$, where $\phi_F(x,e)$ is the infinite spin bosonic field with intertwiner $u_F(x,e)$. By computing thw two point kernel $M(p,e,e')$ explicitly, we found the behaviour in various directions of string configuration space: it behaves like a rapid oscillating function modulated by the factor $|p \cdot e|^{\frac{1}{2}}$ in singular directions $p \cdot e \to 0^+$ or $p \cdot e' \to 0^+$, it is zero at coinciding directions $e = e'$, it is -away from singular directions- finite in the UV and again oscillating in the IR.

A future research objective is to study the Pauli-Lubanski limit in more detail (starting from well-known objects in massive representations of the Poincaré group) and and also to consider de Sitter infinite-spin fields of the form $\phi_F(x,e) = \phi_F(Re, e)$ from the point of view of local commutativity (which give rise to some paradoxes). It would be also interesting to study the dynamics of infinite-spin fields, classifying the possible consistent vertices for CSPs, either self-interacting or interacting with lower spin matter.

# Appendices

# Appendix A

# Relation between projective unitary representation of a group $G$ and a unitary representation of a suitable central extension

The functions $\omega(g, g')$ are not totally arbitrary, because the law of associativity holds

$$(U_g\, U_{g'})\, U_{g''} = U_g\, (U_{g'}\, U_{g''}) \Rightarrow \omega(g, g')\, \omega(g \cdot g', g'') = \omega(g, g' \cdot g'')\, \omega(g', g'') \tag{A.1}$$

and of course $\omega(e, e) = 1$. These properties define the so called 2-cocycle on $G$ with values in $U(1)$. Given two unitary projective representations $G \ni g \mapsto U_g$ and $G \ni g \mapsto U'_g$, they are said to be unitarily equivalent if there exist a unitary operator $V : \mathcal{H} \to \mathcal{H}$ and a map $\chi : G \to U(1)$ satisfying $\chi(g) V U_g V^{-1} = U'_g\ \forall\, g \in G$. In order to ask whether a projective unitary representation $G \ni g \mapsto \Sigma_g$ of a group $G$ can be described by a unitary representation of $G$, we fix an arbitrary representative inside the equivalence class and we consider its multipliers.

If there is a map $\chi : G \ni g \to \chi(g) \in \mathbb{C}$ such that

$$|\chi(g)| = 1 \qquad \text{and} \qquad \chi(g \cdot g') = \omega(g, g')\, \chi(g)\chi(g')\ \forall\, g, g' \in G \tag{A.2}$$

then the projective unitary representation is equivalent to a unitary representation according to our previous definition, and conversely if the multipliers of $G \ni g \mapsto U_g$ are trivial, the $\chi$ satisfies the equation (A.2).

The product $\hat{G_\omega} := U(1) \times_\omega G$ endowed with the multiplication defined by $(\chi, g) \cdot (\eta, g') = (\omega(g, g')\chi\eta, g \cdot g')\ \forall\, (\chi, g), (\eta, g') \in U(1) \times_\omega G$ is a group owe to equation (A.1), and so we can associate to each 2-cocycle $\omega$ a central extension of $G$ by $U(1)$ (exact sequence):

$$1 \to U(1) \xrightarrow{\iota} \hat{G_\omega} \xrightarrow{pr_2} G \to 1 \tag{A.3}$$

with $\iota : \chi \mapsto (\chi, e)$ as the canonical injection (injective homomorphism) and $\mathrm{pr}_2 : (\chi, g) \mapsto g$ as the canonical projection (surjetive homomorphism). Since $G$ is a topological group (as

$U(1)$) then for a cocycle $\omega : G \times G \to U(1)$ which is continuous the extension $\hat{G_\omega}$ is a topological group and the inclusion and projection in the exact sequence are continuous homomorphisms. The kernel of the canonical projection $\mathrm{pr}_2$ is a normal subgroup $N$ (that is the range of the canonical injection, and so isomorphic to $U(1)$) of pairs $(\chi, e)$ with $\chi \in U(1)$. Since by definition $N$ is contained in the centre of $\hat{G_\omega}$, in practice the group $G$ is naturally identified with the quotient $\hat{G_\omega}/N$.

Therefore we can outline the procedure for obtaining all projective unitary representations $G \ni g \mapsto U_g$ of G:

- If $G \ni g \mapsto U_g$ has a multiplier function $\omega$, the map $\hat{G_\omega} \ni g \mapsto V_{(\chi,g)} := \chi U_g$ is a unitary $\hat{G_\omega}$ representation on $\mathcal{H}$. Infact $V_{(\chi,g)} : \mathcal{H} \to \mathcal{H}$ is unitary, so $V(\omega(e,e)^{-1}, e) = 1$ and $V_{(\chi,g)} V_{(\chi',g')} = \chi\, U_g\, \chi'\, U_{g'} = \chi\, \chi'\, \omega(g,g')\, U_{gg'} = V_{(\chi,g)\circ(\chi',g')}$
- The representation of $G$ $g \mapsto U_g := V_{(1,g)}$ obtained from any unitary representation of $\hat{G_\omega} \ni (\chi, g) \mapsto V(\chi, g)$ by restricting the domain of $V$ to elements $(1, g)$, $g \in G$, is a projective unitary representation iff the following condition holds

$$V_{(\chi,e)} = \chi\, \omega(e,e)\, 1 \;\forall\, \chi \in U(1) \tag{A.4}$$

  Infact $V_{(\chi,g)} = \chi U_g$ implies $V(\chi, e) = \chi\, U_e = \chi\, \omega(e,e)\, 1$. Conversely, from $(\chi, g) = (\chi\, \omega(e,e)^{-1}, e) \circ (1, g)$, if the previous condition (A.4) holds then we could write $V(\chi, g) = V(\chi\, \omega(e,e)^{-1}, e) V(1, g) = \chi V(1, g) = \chi U_g$.

In conclusion every projective unitary representation of a group $G$ is the restriction of a unitary representation of a suitable central extension $\hat{G_\omega}$ whose multiplier function satisfies (A.4).

However, we don't need to know all central extension in order to classify projective unitary representations, but only the ones which have non equivalent multipliers. Two multiplier functions of the same group $\omega_a(g, g') \in U(1)$ and $\omega_b(g, g') \in U(1)$ are called equivalent if there is a map $\chi : G \to U(1)$ such that $\omega_a(g, g') = \frac{\chi(g \cdot g')}{\chi(g)\, \chi(g')} \omega_b(g, g') \;\forall\, g, g' \in G$. If two projective unitary representations $U_a$, $U_b$ of $G$ are equivalent, they are restrictions of unitary representations of central extensions $\widehat{G_{\omega_a}}$, $\widehat{G_{\omega_b}}$ with equivalent multiplier functions $\omega_a$, $\omega_b$.

# Appendix B

# Properties of the Poincaré group

The Poincaré group is defined as $\mathcal{P} := \mathbb{R}^{1,3} \rtimes O(1,3)$. The semi-orthogonal group $\mathcal{L} := O(1,3)$, called also Lorentz group, is the group of linear transformations which preserve a signature $(1,3)$ symmetric bilinear form[1] on a real vector space. Clearly $O(1,3) = \{A \mid A^T \eta A = \eta\}$, that is the group of linear isometries of $(\mathcal{M}, \eta)$. $\mathbb{R}^{1,3}$ is a $(3+1)$-dimensional Euclidean space equipped with a symmetric bilinear form of signature $(1,3)$ and treated as a group of translations in this context. In our universe particle interactions involving the weak force violate space-reflection symmetry ("parity") and time reversal symmetry, hence the Lorentzian manifold appears to be equipped with a local time orientation and a local space orientation. Infact time reversal operations, parity transformations and combinations thereof, are not to be local spacetime symmetries. Hence the local spacetime symmetry group of our universe appears to be a subgroup of the Poincaré group, called the restricted (proper orthochronous) Poincaré group $\mathcal{P}_+^\uparrow := \mathbb{R}^{1,3} \rtimes \mathcal{L}_+^\uparrow$ with the restricted (proper orthochronous) Lorentz group $\mathcal{L}_+^\uparrow = SO^\uparrow(1,3)$ ( [38]).

There are four generators $P^\mu$ of $\mathbb{R}^{(1,3)}$, corresponding to the space-time translations. The group $O(1,3)$ is semisimple, non compact and has four connected components. A Lorentz transformation may or may not preserve the direction of time, and it may or may not preserve the orientation of space; these choices correspond to the four connected components. The two components that preserve time form the subgroup $O^\uparrow(1,3)$, whereas the two components that preserve the orientation of space form the subgroup $SO(1,3)$, which are the elements with determinant 1.

The identity component, which preserves both time and space orientations, is the subgroup $SO^\uparrow(1,3)$, that is the group of proper Lorentz transformations. There are six types of transformations that generate $SO^\uparrow(1,3)$: three of the generators $J^i$ are simple angular momenta, and the other three $K^i$ are time-space operators known as boosts. Accordingly, the Poincaré group can be divided into four connected components (classes) differing according to signs of the quantity $\det \Lambda$ and ${\Lambda^0}_0$ :

- $\mathcal{P}_+^\uparrow$ $(\det \Lambda = 1,\ {\Lambda^0}_0 > 0)$
- $\mathcal{P}_+^\downarrow = T\mathcal{P}_+^\uparrow$ $(\det \Lambda = 1,\ {\Lambda^0}_0 < 0)$

[1] this symmetric bilinear form is defined, if $x, y \in \mathcal{M}$, as $\langle x, y\rangle := x^T \eta y = x^\mu \eta_{\mu\nu} y^\nu$

- $\mathcal{P}_-^{\uparrow} = P\mathcal{P}_+^{\uparrow}$ ($\det \Lambda = -1$, ${\Lambda^0}_0 > 0$)
- $\mathcal{P}_-^{\downarrow} = PT\mathcal{P}_+^{\uparrow}$ ($\det \Lambda = -1$, ${\Lambda^0}_0 < 0$)

where we introduce the parity inversion matrix $P = diag(1, -1, -1, -1)$ and the temporal inversion matrix $T = diag(-1, 1, 1, 1)$. We can write the full Poincaré group as the union of his connected components $\mathcal{P} = \mathcal{P}_+^{\uparrow} \cup T\mathcal{P}_+^{\uparrow} \cup P\mathcal{P}_+^{\uparrow} \cup PT\mathcal{P}_+^{\uparrow}$.

The abelian group of translations is a normal subgroup, while the Lorentz group is also a subgroup, the stabilizer of the origin. The Poincaré group itself is also the minimal subgroup of the affine group which includes all translations and Lorentz transformations. The proper Lorentz group $SO^{\uparrow}(1,3)$ has a (two-fold) universal cover Spin$(3,1) \cong SL(2,\mathbb{C})^2$, which is the group of 2 by 2 complex matrices with unit determinant. We identify $\mathbb{R}^4$ with the space of 2 by 2 complex self-adjoint matrices by

$$(x_0, x_1, x_2, x_3) \leftrightarrow \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} \tag{B.1}$$

with

$$\det\left(\begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}\right) = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{B.2}$$

The linear transformation

$$\begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} \to A\left(\begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}\right)A^{\dagger} \tag{B.3}$$

for $A \in SL(2,\mathbb{C})$ preserves the determinant and thus the inner product and it also takes self-adjoint matrices to self-adjoints, and thus $\mathbb{R}^4$ to $\mathbb{R}^4$. Both $A$ and $-A$ give the same linear transformation when they act by conjugation, and all elements of $SO^{\uparrow}(1,3)$ arise as such conjugation maps( [30]). The covering homomorphism $\Lambda : SL(2,\mathbb{C}) \to SO^{\uparrow}(1,3)$ can be extended to a covering homomorphism $\tilde{\Lambda} : \mathbb{R}^4 \rtimes SL(2,\mathbb{C}) \to \mathbb{R}^4 \rtimes SO^{\uparrow}(1,3)$ of the restricted Poincaré group.

[2] just as the groups $SO(3)$ have double covers Spin$(n)$

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