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Belyi's Theorem and Dessins d'Enfants

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"Les grandes personnes ne comprennent
jamais rien toutes seules, et c'est fatigant,
pour les enfants, de toujours et toujours
leur donner des explications..."

Antoine de Saint-Exupéry

Contents

Introduction

The theory of *dessins d'enfants* was launched by Alexander Grothendieck in the 1980s, in his writing *Esquisse d'un Programme.* This was meant to be a proposal for a long-term research submitted to the French *Centre National de la Recherche Scientifique.* The proposal was only partially successful, as it was accompanied by a list of obligations that Grothendieck would have refused to fulfill. He managed to obtain a special position where, while keeping his affiliation at the University of Montpellier, he was paid by the CNRS and released of his teaching obligations. However, the manuscript remained unpublished until 1997, as the author "could not be found, much less his permission requested".

One of the primary aspects that caught Grothendieck's attention, inspiring some of the ideas contained in *Esquisse d'un Programme*, was the result proved by Belyi in 1979. It was indeed a consequence of some deep works by Weil, that if a complex algebraic curve can be expressed as a covering of the Riemann sphere with ramification occurring at most over three points, then the curve is defined over the algebraic closure of the rational numbers. Belyi, with a simple and clever argument, came up with a proof of the opposite implication, thereby lending his name to the entire theorem.

Grothendieck noticed a correspondence between this type of coverings and objects he called *dessins d'enfants*, which means *children's drawings.* These objects are just some graphs embedded in complex curves, that hence resemble drawings scrawled on a bit of paper. In this way, via Belyi's Theorem, one can see the absolute Galois group as a transforming agent acting on *dessins*, whose nature is merely topological and combinatorial. Grothendieck's hope was to use such action to extract insights about the absolute Galois group, whose structure was, and remains to this day, rather obscure.

Beyond these considerations, *Esquisse d'un Programme* is a work filled with many other interesting mathematical ideas, which nowadays continue to inspire research in the fields of algebraic geometry and Galois theory. In this thesis, we will delve into the topics just mentioned.

The first chapter is intended to serve as a quick introduction to the theoretical tools that will be employed in the following chapters. Essentially, we will cover classic topics in algebraic topology and Riemann surfaces, providing the necessary definitions and stating the most important results, such as the Galois correspondence for regular coverings, Riemann-Hurwitz formula and Riemann-Roch Theorem.

The second chapter presents a complete proof of Belyi's Theorem. To prove one half of the result, we will essentially rely on the ingenious algorithm proposed by Belyi in 1979. For the other half, the one traditionally based on Weil's results, we will introduce the notion of *moduli field* of a finite morphism. This will elegantly allow us to divide the result into two distinct and manageable parts, simplifying the exposition.

The third chapter constitutes the core of the thesis: here we will introduce the *dessins d'enfants*. Drawing directly from Grothendieck's ideas, we will describe the correspondence that associates them to the ramified coverings of the Riemann sphere addressed in Belyi's Theorem. We will focus on describing these objects in mutually equivalent ways. In particular, we will utilise both the language of function field extensions and that of permutations. Additionally, we will define a property of *regularity* for dessins, which finds its counterpart in the notion of regularity for coverings, that will allow us to state a Galois correspondence for *dessins*.

Finally, in the fourth chapter, we will define the absolute Galois group and its action on the set of *dessins*. We will study the faithfulness of this action, even when restricted to smaller sets. The richness of the languages with which we can describe *dessins*, presented in the previous chapter, will allow us to define a morphism from the absolute Galois group to the group of outer automorphisms of the profinite completion of the free group generated by two elements. The injectivity of this morphism, which follows from the faithfulness of the Galois action on *dessins*, will provide us with an embedding into a group whose description seems to be completely disconnected from the absolute Galois group.

Chapter 1

Riemann Surfaces

The purpose of this first chapter is to provide a brief and informal introduction to the tools necessary for the discussion in the next chapters. Essentially, it is a collection of classic topics of topology and Riemann surfaces, primarily drawn from [Mir95], [Ful95] and [Mas77].

We will start with the basics of algebraic topology, defining the *fundamental group* and the *coverings* of a topological space. After explaining the notion of *regularity* for coverings, we will state a Galois correspondence between the intermediate coverings of a regular one, and the subgroups of its automorphism group.

Then, we will introduce Riemann surfaces and, in particular, holomorphic maps between them. We will notice how every holomorphic map between compact Riemann surfaces is a ramified covering, and we will reverse this process, constructing a holomorphic map from certain coverings of Riemann surfaces. This correlation between coverings and holomorphic maps suggests also a connection with the topological nature of the surfaces considered. For this reason, after defining the Euler-Poincaré characteristic, which is a topological invariant for surfaces obtained through the use of triangulations, we will introduce Riemann-Hurwitz formula. This expression encodes precisely this relationship, linking the Euler-Poincaré characteristics of the involved surfaces to the branching properties of the considered map.

The discussion continues defining meromorphic functions, differential 1-forms, and divisors on compact Riemann surfaces, which are the necessary elements to present the important Riemann-Roch Theorem, which provides information about the dimension of spaces of certain meromorphic functions on a compact Riemann surface.

1.1 Fundamental Group and Coverings

Let X be a topological space, a **loop** with base point $x_0 \in X$ is a continuous function $f : [0;1] \to X$ such that $f(0) = f(1) = x_0$. We say that two loops f_0, f_1 with base points $x_0 \in X$ are **homotopic** if there exists a continuous function $H : [0;1] \times [0;1] \to X$ such that $H(s,0) = H(s,1) = x_0$ for any $s \in [0;1]$ and $H(0,t) = f_0(t)$, $H(1,t) = f_1(t)$ for any $t \in [0;1]$. Homotopy defines an equivalence relation on the set of loops with fixed base point, thus, we can define the **fundamental group** of X with base point $x_0 \in X$, denoted by $\pi_1(X, x_0)$, to be the group of equivalence classes of loops on X with base point x_o.

From now on, we will assume $\widetilde{X}$ and X to be arcwise and locally arcwise connected topological spaces. An **unramified covering** of X (sometimes called *unbranched covering*, or, very often, just *covering*) is a continuous surjective map $p : \widetilde{X} \to X$ such that for any $x \in X$ there exists an open neighborhood U of p satisfying that:

- $p^{-1}(U) = \bigcup_i U_i$ is a disjoint union of open subsets U_i of $\widetilde{X}$;
- $p|_{U_i} : U_i \to U$ is a homeomorphism for every i.

For any covering $p : \widetilde{X} \to X$, the cardinality of the preimage of a point is constant. We call it the **degree** of the covering.

Let $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ be two coverings of X. A **homomorphism of coverings** is a continuous map $\varphi : \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$. Moreover, if there exists a homomorphism of coverings $\psi : \widetilde{X}' \to \widetilde{X}$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps, then the two coverings p and p' are said to be **isomorphic**. An **automorphism of coverings** is an isomorphism of a covering onto itself. We denote with $\mathrm{Aut}_X(\widetilde{X})$ the group of automorphisms of the cover $p : \widetilde{X} \to X$.

Notice that if $p : \widetilde{X} \to X$ is a covering, and $\widetilde{x}_0 \in \widetilde{X}$ is such that $p(\widetilde{x}_0) = x_0$, then we have an embedding $\pi_1(\widetilde{X}, \widetilde{x}_0) \hookrightarrow \pi_1(X, x_0)$. We say that $p : \widetilde{X} \to X$ is a **universal covering** for X if the fundamental group of $\widetilde{X}$ is trivial. The universal covering is unique up to isomorphism.

A covering is said to be **regular** when its automorphism group acts transitively on the fibres. We have a Galois correspondence concerning the intermediate coverings of a regular covering and the subgroups of its automorphism group. Recall that a covering $q : Z \to X$ is an **intermediate covering** of $p : \widetilde{X} \to X$ if there is a continuous map $r : \widetilde{X} \to Z$ such that $q \circ r = p$. Notice that, in this situation, the map $r : \widetilde{X} \to Z$ is a covering, and if $p : \widetilde{X} \to X$ is regular, then also $r : \widetilde{X} \to Z$ is a regular.

Theorem 1.1. *Let $p : \widetilde{X} \to X$ be a regular covering. Then we have a bijection between the intermediate coverings of $p : \widetilde{X} \to X$ and the subgroups of its automorphism group*

$$\begin{aligned} \{\textit{Intermediate coverings of } \widetilde{X} \xrightarrow{p} X\} &\leftrightarrow \{\textit{Subgroups of } \mathrm{Aut}_X(\widetilde{X})\} \\ \widetilde{X} \xrightarrow{r} Z \xrightarrow{q} X &\longmapsto \mathrm{Aut}_Z(\widetilde{X}) \\ \widetilde{X}/H \to X &\longleftarrow H \end{aligned}.$$

The degree of the covering corresponds to the index of the subgroup, and isomorphic coverings correspond to conjugated subgroups. Furthermore, an intermediate covering is regular if and only if the corresponding subgroup of $\mathrm{Aut}_X \widetilde{X}$ *is normal.*

Since if $p : \widetilde{X} \to X$ is the universal covering of X, then $\mathrm{Aut}_X(\widetilde{X}) \cong \pi_1(X, x)$, this result provides us a bijection between the conjugation classes of subgroups of $\pi_1(X, x)$ and the isomorphism classes of coverings of X.

We say that a continuous surjective function $p : \widetilde{X} \to X$ is a **ramified covering** if there is a discrete subset $S \subseteq \widetilde{X}$, whose image is discrete in X, such that $p|_{\widetilde{X} \setminus S} : \widetilde{X} \setminus S \to X \setminus p(S)$ is an unramified covering. The points of $\widetilde{X}$ where p is not an unramified covering are called **ramification points**, and their set is called **ramification locus**. The image of the ramification locus is called **branched locus**, and its points are called **branched points**.

1.2 Riemann Surfaces and Holomorphic Maps

Let X be a topological space. A **complex chart** on X is a homeomorphism $\phi : U \to V$, where U is an open subset of X and V is an open subset of the complex plane $\mathbb{C}$. We say that the chart is **centered at** $p \in U$ if $\phi(p) = 0$. Two complex charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are said to be **compatible** if either U_1 and U_2 are disjoint, or the transition map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is a biholomorphism.

A **complex atlas** $\mathcal{A}$ on a topological space X is a collection $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ of pairwise compatible complex charts such that $X = \bigcup_\alpha U_\alpha$. Two complex atlas are said to be **equivalent** if their union is also a complex atlas. A **complex structure** on X is an equivalence class of complex atlases on

X. Then, a **Riemann surface** is a second countable, connected, Hausdorff topological space with a complex structure.

We can use the canonical morphism $\mathbb{C} \cong \mathbb{R}^2$ to see Riemann surfaces as 2-dimensional real manifolds. Thus, applying Cauchy-Riemann equation to the transition maps, we get that every Riemann surface is orientable.

Given X and Y two Riemann surfaces and a map $F : X \to Y$, we say that F is **holomorphic at** $p \in X$ if there exist complex charts $\phi : U \to V$ of X with $p \in U$, and $\psi : W \to Z$ of Y with $F(p) \in W$ such that the composition

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(F(U) \cap W)$$

is holomorphic. We say that F is holomorphic on $X' \subseteq X$ if it is holomorphic at every point of X'.

An **isomorphism**, or **biholomorphism**, between two Riemann surfaces X and Y is a bijective holomorphic map $F : X \to Y$ whose inverse $F^{-1} : Y \to X$ is holomorphic.

Proposition 1.2. *Let X and Y be two Riemann surfaces, and $f : X \to Y$ a non-constant holomorphic mapping. Then, for any $x \in X$ there exist a complex chart $\phi : U \to V$ centered at x, a complex chart $\psi : U' \to V'$ centered at $f(x)$ and an integer $k \geq 1$ such that $\phi(U) \subseteq U'$ and the map $F := \psi \circ f \circ \phi^{-1}$ is given by*

$$F(z) = z^k \quad \textit{for all } z \in V.$$

Proof. Clearly, we can find charts $\phi : U \to V$ centered at x and $\psi : U' \to V'$ centered at $f(x)$ such that $\phi(U) \subseteq U'$. Now, $f_1 = \psi \circ f \circ \phi^{-1}$ is a non-constant holomorphism, such that $f_1(0) = 0$. Hence $f_1(z) = z^k g(z)$ for some integer $k > 1$ and a holomorphic function g such that $g(0) \neq 0$. So, there exists a holomorphic function h defined on a neighborhood of 0 such that $h^k = g$. Consider $\alpha(z) := zh(z)$, it is a holomorphic mapping defined on an open neighborhood V_1 of 0. Replacing V with V_1 and ϕ with $\alpha \circ \phi$, we get the result. □

Notice that the map $F(z) = (\psi \circ f \circ \phi^{-1})(z) = z^k$ maps 0 to 0, and outside the origin is a covering of degree k. This suggests us to give the following definitions. Given a holomorphic mapping $f : X \to Y$ between Riemann surfaces, the **ramification index** of f at $p \in X$, denoted $e_f(p)$, is the unique integer k such that there exist complex charts as described in the previous proposition realizing f as $z \mapsto z^k$. A point $p \in X$ such that $e_f(p) > 1$ is called a **ramification point** for f. The choice of this terminology appears judicious in light of the following proposition.

Proposition 1.3. *Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Let $R \subseteq X$ be the set of ramification points of f, and $S = f(R) \subseteq Y$. Then:*

(1) There is a finite number of ramification points for f.

(2) The map $f : X \smallsetminus f^{-1}(S) \to Y \smallsetminus S$ is a covering of degree n, for some integer n.

(3) For any point $Q \in Y$, we have that

$$\sum_{P \in f^{-1}(Q)} e_f(P) = n.$$

Proof. (1) follows from the fact that any point admits a neighborhood that contains no other ramification point. So, using the compactness of X, we find a finite number of such neighborhoods. Similarly, for any $Q \in Y$, $f^{-1}(Q)$ is finite, otherwise it would admit an accumulation point, and this would imply f to be constant.

To prove (2), let $Q \notin S$, and let $f^{-1}(Q) = \{P_1, \cdots, P_n\}$. There are neighborhoods U_i of P_i and V_i of Q such that U_i and U_j are disjoint for $i \neq j$, and f maps homeomorphically U_i onto V_i. We may assume that each V_i contains no point of S. For any connected neighborhood V of Q contained in $V_1 \cap \cdots \cap V_n$, let $U'_i = U_i \cap f^{-1}(V)$. We are going to prove that for V small enough, $f^{-1}(V)$ is the disjoint union of the sets U'_i, from which it follows that f is a covering in a neighborhood of Q. By contradiction, assume that there is a sequence $(N_i)_i$ of neighborhoods of Q whose intersection is $\{Q\}$, such that there is a point P'_i in $f^{-1}(N_i)$ with P'_i not in $U_1 \cup \cdots \cup U_n$. By compactness of X, a subsequence of these P'_i must converge to a point $P' \in X$. By continuity, $f(P') = Q$, so $P' = P_j$ for some j, contradicting that the points P'_i are not in U_j for any i and j.

For (3), let $Q \in Y$ and $f^{-1}(Q) = \{P_1, \cdots, P_m\}$, and find neighborhoods U_i of P_i and V_i of Q such that f maps U_i onto V_i. Thus, locally, it is the map $z \mapsto z^{e_f(P_i)}$, which is a map $e_f(P_i) : 1$ except at the point P_i. Again, if V is a neighborhood of Q contained in the intersection of the V_i, but not containing any other point of S, then there are $\sum e_f(P_i)$ points over a point Q' in V, except for the point Q. By (2), this must be the number n of sheets of the covering. □

So, a holomorphic map $f : X \to Y$ between compact Riemann surfaces is a ramified covering, and the branched points are the critical values for f.

We would like to reverse this process, or, in other words, given a Riemann surface Y, a finite subset S of Y, and a covering $p : X^\circ \to Y \smallsetminus S$, we want to embed X° into a Riemann surface X and to construct a map $f : X \to Y$, extending p, that is a holomorphism of Riemann surfaces.

Theorem 1.4. *Let Y be a Riemann surface, S a finite subset of Y and $p : X^\circ \to Y \setminus S$ a covering of finite degree, with X° connected. Then there is an embedding of X° as an open subset of a Riemann surface X that is the union of X° and a finite set, so that p extends to a proper holomorphic map $X \to Y$. Moreover, such X is unique up to biholomorphic fibre-preserving maps.*

Proof. Firstly, notice that if Y is a Riemann surface, also $Y \setminus S$ is a Riemann surface, and if $p : X^\circ \to Y \setminus S$ is a covering, then there is a unique structure of a Riemann surface on X° so that p is a holomorphic mapping. In fact, one can choose for any complex chart $\phi : U \to Z$ of $Y \setminus S$, where U is evenly covered by p, meaning that $p^{-1}(U) = \bigcup_i V_i$ with $V_i \xrightarrow{\sim} U$, the composition

$$V_i \xrightarrow{p} U \xrightarrow{\phi} Z.$$

This gives an atlas of complex charts for X°.

Now consider $Q \in S$, and a complex chart of Y, centered at Q, $\varphi : U \to \mathbb{D}$, where $\mathbb{D}$ is the unit open disk in the complex plane and U does not contain any other point of S. Let $U^\circ = U \setminus \{Q\}$, then p restricts to a covering of U°, where $p^{-1}(U^\circ)$ is a disjoint union of connected open sets $V_1^\circ, \cdots, V_m^\circ$, where each $V_i^\circ \to U^\circ$ is a connected covering of degree e_i. So, $V_i^\circ \to U^\circ \to \mathbb{D}^\circ$ is a covering of $\mathbb{D}^\circ$ of degree e_i. It is, thus, isomorphic to the covering $q_{e_i} : \mathbb{D}^\circ \to \mathbb{D}^\circ$ sending $z \mapsto z^{e_i}$. In other words, there exists an homeomorphism $\psi_i : V_i^\circ \to \mathbb{D}^\circ$ such that the following diagram

$$\begin{tikzcd}
\mathbb{D}^\circ \arrow[d, "q_{e_i}"'] & V_i^\circ \arrow[l, "\psi_i"'] \arrow[d, "p"] \\
\mathbb{D}^\circ & U^\circ \arrow[l, "\varphi"]
\end{tikzcd}$$

commutes. We can therefore add one point to V_i°, getting V_i, so that ψ_i extends to a homeomorphism from V_i to $\mathbb{D}$. Doing this for any i and any other point of S, and taking these extension as charts, we find a Riemann surface X that is the union of X° with a finite number of points. Moreover, the map $V_i^\circ \to U^\circ$ extends to a holomorphic map $V_i \to U$ with ramification index e_i at the added point, so p extends to a mapping $f : X \to Y$, which is clearly proper and holomorphic.

To show uniqueness up to fibre-preserving biholomorphisms, assume that $g : X' \to Y$ is another proper holomorphic map with branched locus S such that there is a fibre-preserving biholomorphism s from X° to $X' \smallsetminus g^{-1}(S)$. We will show that s can be extended to a fibre-preserving biholomorphism s' from X to X'. Let $Q \in S$, and consider an open neighborhood U of Q such that both f and g are unbranched over $U^\circ = U \smallsetminus \{Q\}$ and there is a complex chart $U \to \mathbb{D}$ centered in Q. Let $V_1, \cdots, V_n$ and $W_1, \cdots, W_m$ the connected components of $f^{-1}(U)$ and, respectively, $g^{-1}(U)$. Since the restriction of s to $f^{-1}(U^\circ)$ is a biholomorphism between $f^{-1}(U^\circ)$ and $g^{-1}(U^\circ)$, we get that $n = m$, and hence we can relabel so that $s(V_i^\circ) = W_i^\circ$. Since $V_i^\circ \to U \to \mathbb{D}^\circ$ is a covering of $\mathbb{D}^\circ$ of finite degree, it is isomorphic to $\mathbb{D}^\circ \to \mathbb{D}^\circ$ sending $z \mapsto z^k$ for some integer k, and this yields that $V_i \cap f^{-1}(Q)$ consists of just a point P_i. Similarly, $W_i \cap g^{-1}(Q)$ consists of just a point P_i'. So, $s|_{f^{-1}(U^\circ)} : f^{-1}(U^\circ) \to g^{-1}(U^\circ)$ can be extended sending P_i to P_i' for any i. Since $f|_{V_i}$ and $g|_{W_i}$ are both proper maps, the continuation is a homeomorphism, and by Riemann's removable singularity theorem it is biholomorphic as well. Applying the construction to every point of S, we get the desired extension $s' : X \to X'$. □

1.3 Riemann-Hurwitz Formula

As we saw, non-constant holomorphic maps between compact Riemann surfaces are ramified coverings. This suggests that whenever there is such a map between two compact Riemann surfaces, there should be a deep relation concerning their topology. This is indeed the case, and the desired relation is encoded in the *Riemann-Hurwitz formula*. In order to state it, we need to give some definitions.

Let $\Delta \subseteq \mathbb{R}^2$ be the triangle of vertices $(0,0)$, $(0,1)$ and $(1,0)$. A **triangle** in a compact Riemann surface X is a continuous injective map $\tau : \Delta \to X$. We call **edges** and **vertices** the images under τ of the edges and the vertices of Δ. A **triangulation** of a compact Riemann surface X is a collection of triangles on X such that every point of X belong either to the interior of exactly one triangle, or on the edge of exactly two triangles or on a vertex of finitely many triangles. It can be shown that every compact Riemann surface admits a finite triangulation.

Let X be a compact Riemann surface, and consider a triangulation on X. Let T denote the number of triangles, E the number of edges and V the number of vertices. Then we can define the **Euler-Poincaré characteristic** of X to be

$$\chi(X) = V - E + T.$$

It can be proved that this number is independent on the chosen triangulation, and it is a topological invariant of the surface.

We can define the **genus** of a compact Riemann surface X to be the integer g_X such that

$$\chi(X) = 2 - 2g_X.$$

Compact Riemann surfaces can be topologically classified by their genus. In particular, a compact Riemann surface of genus 0 is homeomorphic to a sphere, one of genus 1 is homeomorphic to a torus, and one of genus $g \geq 2$ is homeomorphic to a connected sum of g tori.

Finally, we can give Riemann-Hurwitz formula involving holomorphic maps between compact Riemann surfaces.

Theorem 1.5 (Riemann-Hurwitz). *Let $f : X \to Y$ be a holomorphic map of degree n between compact Riemann surfaces. Then the genus g_X of X and the genus g_Y of Y are related by the formula*

$$2g_X - 2 = (2g_Y - 2)n + \sum_{P \in X} (e_f(P) - 1).$$

The proof of this theorem and a comprehensive discussion of the Euler-Poincaré characteristic can be found in [Ful95].

1.4 Riemann-Roch Theorem

In this section, we are going to present Riemann-Roch Theorem, which allows us to compute the dimensions of certain vector spaces of meromorphic functions on a compact Riemann surface. Before stating it, we need to introduce some objects.

Let X be a Riemann surface. Given $f : X \to \mathbb{C} \cup \{\infty\}$, we say that f is **meromorphic at** $p \in X$ if there is a complex chart $\phi : U \to V$, with $p \in U$, for which $f \circ \phi^{-1}$ is meromorphic. We say that f is meromorphic on an open subset W of X if it is meromorphic on every point of W. We will denote with $\mathscr{M}(X)$ the set of all meromorphic function on X, which is a field.

Notice that there is a correspondence between the field of meromorphic functions on X and the set of holomorphic maps from X to the Riemann sphere $\widehat{\mathbb{C}}$ which are not identically ∞. Moreover, since the Riemann sphere $\widehat{\mathbb{C}}$ and the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ are biholomorphic as Riemann surfaces, the same correspondence holds between meromorphic maps on X and holomorphic maps $X \to \mathbb{P}^1_{\mathbb{C}}$. In the next chapters we will refer both to $\widehat{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$ with the name of *Riemann sphere.*

Let f be a meromorphic function on a Riemann surface X. Then, if z is a complex chart centered at $p \in X$, $f(z)$ can be expressed with a Laurent series as $\sum_n a_n z^n$. The **order** of f at p, denoted by $\operatorname{ord}_p f$ is defined to be

$$\operatorname{ord}_p f = \min\{n \mid a_n \neq 0\}.$$

A **differential 1-form** ω on a Riemann surface X is a family $\{(U_i, \omega_i)_i\}$, where $(U_i)_i$ forms an open cover of X with complex charts $\phi_i : U_i \to V_i$, and $\omega_i = f_i(z)dz$ are differential 1-forms on the charts such that, for any i and j,

$$f_j = (f_i \circ \psi_{ij})\psi'_{ij}$$

for any transition function $\psi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$. The differential 1-form ω is said to be **holomorphic**, or **meromorphic**, if every f_i is holomorphic, or meromorphic.

We can define a notion of order also for meromorphic 1-forms ω on a point $P \in X$ in the following way:

$$\operatorname{ord}_P \omega = \operatorname{ord}_P f,$$

if $\omega = f dz$ is the local form of ω in an open $U \ni P$.

For a compact Riemann surface X, we define its **group of divisors** $\operatorname{Div}(X)$ to be the free abelian group generated by the points of X. So, a divisor $\mathcal{D} \in \operatorname{Div}(X)$ can be written as

$$\mathcal{D} = \sum_{P \in X} D_P P,$$

where $D_P \in \mathbb{Z}$ and $D_P = 0$ for every $P \in X$ but a finite number. We say that the **order** of a divisor $\mathcal{D} = \sum_{P \in X} D_P P$ in P is $\operatorname{ord}_P \mathcal{D} = D_P$. We can assign an order relation in the group of divisors: for $\mathcal{D}, \mathcal{D}' \in \operatorname{Div}(X)$, we write that $\mathcal{D} \leq \mathcal{D}'$ if $\operatorname{ord}_P \mathcal{D} \leq \operatorname{ord}_P \mathcal{D}'$ for every $P \in X$. There is a degree function $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ defined by

$$\deg \mathcal{D} = \sum_{P \in X} \operatorname{ord}_P \mathcal{D}\,.$$

Furthermore, to every meromorphic function $f \in \mathscr{M}(X)^*$ we can assign a divisor by setting $\operatorname{div} f = \sum_{P \in X} \operatorname{ord}_P(f)P$. Similarly, we can assign a divisor to any non-zero meromorphic differential 1-form ω on X by setting $\operatorname{div} \omega = \sum_{P \in X} \operatorname{ord}_P(\omega)P$. A divisor $\mathcal{D} \in \operatorname{Div}(X)$ is called a **principal divisor** if $\mathcal{D} = \operatorname{div} f$ for some $f \in \mathscr{M}(X)$, and its called **canonical divisor** if $\mathcal{D} = \operatorname{div} \omega$ for some meromorphic differential 1-form ω on X.

For a compact Riemann surface X and a divisor $\mathcal{D} \in \operatorname{Div}(X)$, we can define the $\mathbb{C}$-vector space of the meromorphic functions controlled by $\mathcal{D}$ as

$$\mathscr{L}(\mathcal{D}) \coloneqq \{f \in \mathscr{M}(X) \mid \operatorname{div} f + \mathcal{D} \geq 0\}.$$

This is the space of meromorphic functions that can have poles of order $\geq -\operatorname{ord}_P \mathcal{D}$ if $\operatorname{ord}_P \mathcal{D} > 0$, and that must be holomorphic elsewhere, with zeros of order $\geq -\operatorname{ord}_P \mathcal{D}$ if $\operatorname{ord}_P \mathcal{D} < 0$.

We are finally able to state Riemann-Roch Theorem.

Theorem 1.6 (Riemann-Roch). *Let X be a compact Riemann surface, and $\mathcal{D} \in \operatorname{Div}(X)$, then we have that*

$$\dim_{\mathbb{C}} \mathscr{L}(\mathcal{D}) = \deg \mathcal{D} + 1 - g_X + \dim_{\mathbb{C}} \mathscr{L}(K - D),$$

where K is a canonical divisor and g_X is the genus of X.

For a comprehensive discussion of Riemann-Roch theorem and its proof, we suggest consulting [Mir95].

Chapter 2

Belyi's Theorem

In this chapter, we are going to give a complete proof of Belyi's Theorem, stating that a complex algebraic curve X is defined over $\overline{\mathbb{Q}}$ if and only if X can be realized as a ramified covering of $\mathbb{P}^1_{\mathbb{C}}$, with branched locus consisting of at most three points.

The *if* direction of this statement is known in the literature as the *obvious part* of the theorem, since its proof, invoking a general result of Weil on the field of definition of a variety, was already known before Belyi, in 1979, came up with a proof for the *only if* part, for whose simplicity, the entire theorem was given his name.

Nevertheless, reconstructing the proof of the *obvious part* is much more complicated than proving the *only if* direction using Belyi's idea. In order to do this, we will follow the approach used by Köck in [Kö04], that introduces the notion of the *moduli field* of both a complex curve and a morphism $t : X \to \mathbb{P}^1_{\mathbb{C}}$ to split the proof in two parts: the first one consists in showing that X is defined over a finite extension of the moduli field of t, and the other one in proving that this moduli field is a finite extension of $\mathbb{Q}$.

This theorem is the starting point of the theory of *dessins d'enfants*, started by Grothendieck, who, deeply inspired by this result, writes in [Gro97]:

> $[\cdots]$ jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes! Sous la forme où l'énonce Bielyi, son résultat dit essentiellement que toute courbe algébrique définie sur un corps de nombres peut s'obtenir comme revêtement de la droite projective ramifié seulement en les points 0, 1, ∞. Ce résultat semble être passé plus ou moins inaperçu. Pourtant, il m'apparâıt d'une portée considérable. Pour moi, son message essentiel a été qu'il y a une identité profonde entre la combinatoire des cartes finies d'une part, et la géométrie des courbes algébriques définies sur des corps de nombres, de l'autre.

2.1 Some Scheme Theory

Let K be a field. We define **a curve over** K to be a smooth, projective, geometrically connected variety of dimension 1 over K, where a **variety over** K is an integral, separated scheme X, with a structure morphism $X \to \operatorname{Spec} K$ of finite type. We say that a variety over K is defined over a subfield k of K if there exists a k-scheme $\widetilde{X}$, meaning a scheme $\widetilde{X}$ together with a morphism $\widetilde{X} \to \operatorname{Spec} k$, such that

$$X \cong \widetilde{X} \times_{\operatorname{Spec} k} \operatorname{Spec} K,$$

or, in other words, if X can be covered by affine varieties given by polynomials with coefficients in k. If X and Y are varieties over K and $\varphi : X \to Y$ is a morphism between them, we say that φ is defined over k if both X and Y are defined over k and there exist a morphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ such that the morphism

$$\widetilde{\varphi} \times \mathrm{id}_{\operatorname{Spec} K} : \widetilde{X} \times_{\operatorname{Spec} k} \operatorname{Spec} K \longrightarrow \widetilde{Y} \times_{\operatorname{Spec} k} \operatorname{Spec} K,$$

coming from the universal property of the fibre product, as represented in the following diagram, is φ

$$\begin{tikzcd}
\widetilde{X} \times_{\operatorname{Spec} k} \operatorname{Spec} K \arrow[rr] \arrow[d] \arrow[drr, dashed] & & \operatorname{Spec} K \arrow[dr, "\mathrm{id}_{\operatorname{Spec} K}"] & \\
\widetilde{X} \arrow[drr, "\widetilde{\varphi}"'] & & \widetilde{Y} \times_{\operatorname{Spec} k} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\
& & \widetilde{Y} \arrow[r] & \operatorname{Spec} k
\end{tikzcd}.$$

Given a variety X over K with structure morphism $s : X \to \operatorname{Spec} K$, and an automorphism $\sigma \in \operatorname{Aut}(K)$, we denote by ${}^{\sigma}X$ the K-scheme consisting again in X as scheme, but in

$$X \xrightarrow{s} \operatorname{Spec} K \xrightarrow{\operatorname{Spec} \sigma^{-1}} \operatorname{Spec} K$$

as structure morphism.

Notice that this action corresponds to conjugating by σ the coefficients of the polynomials defining the variety X. For instance, if $f_1, \cdots, f_k \in \mathbb{C}[x_1, \cdots, x_n]$, and $X = \operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_k))$ (with structure morphism s induced by $\mathbb{C} \to \mathbb{C}[x_1, \cdots, x_n]$), then ${}^{\sigma}X$ is isomorphic to

$X = \operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_n]/(\sigma(f_1), \cdots, \sigma(f_k)))$ (with structure morphism s' induced by $\mathbb{C} \to \mathbb{C}[x_1, \cdots x_n]$). Indeed, denoting with τ the isomorphism

$$\mathbb{C}[x_1, \cdots, x_n]/(\sigma(f_1), \cdots, \sigma(f_k)) \xrightarrow{\tau} \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_k)$$

given extending σ^{-1} to the quotients of $\mathbb{C}[x_1, \cdots, x_n]$, we get that $\operatorname{Spec}\tau$ is an isomorphism making to commute the diagram

$$\begin{tikzcd}
\operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_k)) \arrow[r, "s"] \arrow[d, "\operatorname{Spec}\tau"'] & \operatorname{Spec}\mathbb{C} \arrow[d, "\operatorname{Spec}\sigma^{-1}"] \\
\operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_n]/(\sigma(f_1), \cdots, \sigma(f_k))) \arrow[r, "s'"'] & \operatorname{Spec}\mathbb{C}
\end{tikzcd} .$$

Thus, ${}^\sigma X$ and $\operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_n]/(\sigma(f_1), \cdots, \sigma(f_k)))$ are isomorphic as varieties over $\mathbb{C}$.

In other words, the $\mathbb{C}$-variety ${}^\sigma X$ is isomorphic to the $\mathbb{C}$-variety

$$X \times_\sigma \operatorname{Spec}\mathbb{C},$$

where the fibre product is taken between the morphisms $s : X \to \operatorname{Spec}\mathbb{C}$ and $\operatorname{Spec}\sigma : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. Indeed, the morphisms $\operatorname{Spec}\sigma^{-1} \circ s$ and id_X give rise to a morphism $\varphi : {}^\sigma X \to X \times_\sigma \operatorname{Spec}\mathbb{C}$, by the universal property of the fibre product, as represented in the following diagram:

$$\begin{tikzcd}
{}^\sigma X \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\operatorname{Spec}\sigma^{-1} \circ s"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_\sigma \operatorname{Spec}\mathbb{C} \arrow[r] \arrow[d, "\pi_X"'] & \operatorname{Spec}\mathbb{C} \arrow[d, "\operatorname{Spec}\sigma"] \\
& X \arrow[r, "s"'] & \operatorname{Spec}\mathbb{C}
\end{tikzcd} .$$

From this commutative diagram, it is clear that φ is a morphism of $\mathbb{C}$-varieties between ${}^\sigma X$ and $X \times_\sigma \operatorname{Spec}\mathbb{C}$ such that $\pi_X \circ \varphi = \mathrm{id}_X$. The fact that $\varphi \circ \pi_X = \mathrm{id}_{X \times_\sigma \operatorname{Spec}\mathbb{C}}$ comes applying the universal property of fibre product. Thus, φ is an isomorphism of $\mathbb{C}$-varieties.

Example. *In the affine case, consider* $X = \operatorname{Spec}(\mathbb{C}[x, y]/(y - ix))$ *with structure morphism induced by* $\mathbb{C} \to \mathbb{C}[x, y]/(y - ix)$*, and the conjugation* σ *as automorphism of* $\mathbb{C}$*. Then,* ${}^\sigma X$ *is again* $\operatorname{Spec}(\mathbb{C}[x, y]/(y - ix))$*, but the structure morphism is induced by the ring morphism* $\mathbb{C} \to \mathbb{C}[x, y]$ *such that* $z \mapsto \bar{z}$*.*

${}^{\sigma}X$ is thus isomorphic, as variety over $\mathbb{C}$, to $\operatorname{Spec}(\mathbb{C}[x,y]/(y+ix))$*, with structure morphism induced by the map $\mathbb{C} \to \mathbb{C}[x,y]/(y+ix)$. The explicit isomorphism of $\mathbb{C}$-varieties is indeed given by the morphism of spectra induced by the extension of conjugation to $\mathbb{C}[x,y]/(y+ix) \to \mathbb{C}[x,y]/(y-ix)$, i.e. the morphism such that $z \mapsto \overline{z}$ for $z \in \mathbb{C}$, $x \mapsto x$ and $y \mapsto y$.*

2.2 The Moduli Field

The **moduli field** $M(X)$ of a variety X over $\mathbb{C}$ is $\mathbb{C}^{U(X)}$, namely, the field fixed by the subgroup of $\operatorname{Aut}(\mathbb{C})$

$$U(X) = \{\sigma \in \operatorname{Aut}(\mathbb{C}) \text{ such that } {}^{\sigma}X \cong X \text{ as varieties over } \mathbb{C}\}.$$

For instance, if a variety X over $\mathbb{C}$ is defined over K, then clearly the subgroup $\operatorname{Aut}(\mathbb{C}\,/K)$ of $\operatorname{Aut}(\mathbb{C})$ is contained in $U(X)$, and hence, by the lemma we are about to state, $M(X)$ is contained in K.

Lemma 2.1. *Let K be a subfield of $\mathbb{C}$. Then, any automorphism of K can be extended to an automorphism of $\mathbb{C}$. Furthermore, $\mathbb{C}^{\operatorname{Aut}(\mathbb{C}\,/K)} = K$.*

Proof. To show the first assertion, consider $\varphi \in \operatorname{Aut}(K)$, and apply Zorn's Lemma to the set of all automorphisms of some subfield of $\mathbb{C}$ extending φ. We get a maximal element $\Phi \in \operatorname{Aut}(L)$, with $K \leq L \leq \mathbb{C}$. To prove that L is indeed $\mathbb{C}$, take $a \in \mathbb{C} \smallsetminus L$: if a is algebraic, then we can extend Φ to the normal closure of $K(a)$ by well-known results of Galois Theory, otherwise a is transcendental over K, so we can extend Φ to $K(a)$ just setting $\Phi(a) = a$. In any case, this contradicts the maximality of Φ.

We proceed with the second part. Clearly, $K \subseteq \mathbb{C}^{\operatorname{Aut}(\mathbb{C}\,/K)}$. To show the reverse inclusion, for any $x \in \mathbb{C} \smallsetminus K$ we construct a K-automorphism of $\mathbb{C}$ that does not fix x. If x is algebraic, take a K-conjugate $y \in \mathbb{C}$ distinct from x. Again by Galois theory, there exists an K-automorphism of the normal closure of $K(x)$, mapping x to y, that can be extended to an automorphism of $\mathbb{C}$ by the previous part. If x is transcendent over K, we can consider the automorphism of $K(x)$ sending x to $-x$, and again extend it to an automorphism of $\mathbb{C}$. $\square$

We will say that a subgroup U of $\operatorname{Aut}(\mathbb{C})$ is **closed** if there is a subfield K of $\mathbb{C}$ such that $U = \operatorname{Aut}(\mathbb{C}\,/K)$. The previous lemma yields a Galois correspondence between the set of subfields of $\mathbb{C}$ and the set of closed subgroups of $\operatorname{Aut}(\mathbb{C})$. In particular, $U = \operatorname{Aut}(\mathbb{C}\,/\,\mathbb{C}^{U})$ for any closed subgroup U of $\operatorname{Aut}(\mathbb{C})$.

Lemma 2.2. *Let U be a subgroup of* $\mathrm{Aut}(\mathbb{C})$ *such that there exists a finite field extension $K/\mathbb{C}^U$ with* $\mathrm{Aut}(\mathbb{C}/K) \subseteq U$*, then U is closed.*

Proof. We may assume that $K/\mathbb{C}^U$ is a Galois extension. Then, $\mathbb{C}^U$ is the field fixed by the image B of $U/\mathrm{Aut}(\mathbb{C}/K)$ under the canonical isomorphism

$$\mathrm{Aut}(\mathbb{C}/\mathbb{C}^U)/\mathrm{Aut}(\mathbb{C}/K) \longrightarrow \mathrm{Aut}(K/\mathbb{C}^U).$$

Thus, $B = \mathrm{Aut}(K/\mathbb{C}^U)$, and so $U = \mathrm{Aut}(\mathbb{C}/\mathbb{C}^U)$ is closed. □

Lemma 2.3. *Let U be a subgroup of* $\mathrm{Aut}(\mathbb{C})$ *and let V be a subgroup of U of finite index. Then the field extension $\mathbb{C}^V/\mathbb{C}^U$ is finite. Moreover, if V is a normal subgroup of U or if U is closed, then $\left|\mathbb{C}^V : \mathbb{C}^U\right| \leq |U : V|$. If V is closed, then equality holds.*

Proof. A normal subgroup W of U contained in V and of finite index always exists, and it is the normal core of U in V, which is defined as

$$W \coloneqq \bigcap_{g \in U} gVg^{-1}.$$

It is clearly contained in V normal in U. To show it has finite index, consider the action of U given by right multiplication on the set of right cosets of V. We get then a morphism from U to the symmetric group on $|U : V|$ elements, whose kernel is exactly W.

Then, $\mathbb{C}^U$ is the fixed field by the image B of the canonical morphism $U/W \to \mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^U)$. Thus, $\mathbb{C}^W/\mathbb{C}^U$ is a finite Galois extension, and $B = \mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^U)$. Hence, we have:

$$\left|\mathbb{C}^W : \mathbb{C}^U\right| = \left|\mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^U)\right| \leq |U/W| = |U : W| < +\infty.$$

This implies that $\left|\mathbb{C}^V : \mathbb{C}^U\right| \leq \left|\mathbb{C}^W : \mathbb{C}^U\right|$ is finite. Furthermore, if V is already normal in U, then replacing W by V in the above relation we get $\left|\mathbb{C}^V : \mathbb{C}^U\right| \leq |U : V|$. Moreover, we have

$$\begin{aligned}\left|\mathbb{C}^V : \mathbb{C}^U\right| &= \frac{\left|\mathbb{C}^W : \mathbb{C}^U\right|}{\left|\mathbb{C}^W : \mathbb{C}^V\right|} = \frac{\left|\mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^U)\right|}{\left|\mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^V)\right|} = \left|\frac{\mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^U)}{\mathrm{Aut}(\mathbb{C}^W/\mathbb{C}^V)}\right| = \\ &= \left|\frac{\frac{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^U)}{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^W)}}{\frac{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^V)}{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^W)}}\right| = \left|\frac{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^U)}{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^V)}\right|,\end{aligned}$$

which implies the inequality $\left|\mathbb{C}^V : \mathbb{C}^U\right| \leq |U : V|$ whenever $U = \mathrm{Aut}(\mathbb{C}/\mathbb{C}^U)$ is closed. Finally, if V is closed, then also U is closed by the previous lemma, and then $\left|\frac{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^U)}{\mathrm{Aut}(\mathbb{C}/\mathbb{C}^V)}\right| = |U/V|$, getting the desired equality. □

Recall that if X and Y are schemes, a morphism $f : X \to Y$ is said to be **finite** if for any affine open $V = \operatorname{Spec} R$ of Y, its inverse image $f^{-1}(V) = \operatorname{Spec} A$ is affine in X, and the associated morphism of ring $R \to A$ is finite. If $f : X \to Y$ is a finite morphism of curves over $\mathbb{C}$, then all points P of Y, but a finite number, are such that $|f^{-1}(P)| = \deg f$ for a natural number $\deg f$. We say that the critical values of f are the points of Y having less than $\deg f$ preimages, and the ramification locus of f is the preimage under f of the set of critical values.

Consider now a curve X over $\mathbb{C}$ and a finite morphism $t : X \to \mathbb{P}^1_{\mathbb{C}}$. The **moduli field** $M(X,t)$ of t is $\mathbb{C}^{U(X,t)}$, namely, the field fixed by the subgroup $U(X,t)$ of $U(t)$, consisting of all $\sigma \in \operatorname{Aut}(\mathbb{C})$ such that there exists an isomorphism ${}_{\sigma}f : {}^{\sigma}X \to X$ of varieties over $\mathbb{C}$ making the following diagram to commute:

$$\begin{tikzcd} {}^{\sigma}X \arrow[r, "{}_{\sigma}f"] \arrow[d, "{}^{\sigma}t"'] & X \arrow[d, "t"] \\ {}^{\sigma}\mathbb{P}^1_{\mathbb{C}} \arrow[r, "\overline{\sigma}"'] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd} .$$

In this setting, both X and $\mathbb{P}^1_{\mathbb{C}}$ are thought as $\mathbb{C}$-schemes, with their structure morphisms $s : X \to \operatorname{Spec}\mathbb{C}$ and $s' : \mathbb{P}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$, and ${}^{\sigma}t$ is the morphism of $\mathbb{C}$-schemes t, but where the structure morphisms are taken with the composition with $\operatorname{Spec}\sigma^{-1}$:

$$\begin{tikzcd} {}^{\sigma}X \arrow[rr, "{}^{\sigma}t"] \arrow[dr, "s"'] & & {}^{\sigma}\mathbb{P}^1_{\mathbb{C}} \arrow[dl, "s'"] \\ & \operatorname{Spec}\mathbb{C} \arrow[d, "\operatorname{Spec}\sigma^{-1}"] & \\ & \operatorname{Spec}\mathbb{C} & \end{tikzcd} .$$

The morphism $\overline{\sigma}$ is the morphism

$$\mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} \times \operatorname{Spec}\sigma^{-1} : \mathbb{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C} \longrightarrow \mathbb{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$$

induced by the universal property of the fibre product, where the first $\mathbb{P}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$ has structure morphism the composition of s' with $\operatorname{Spec}\sigma^{-1}$, and the second one has structure morphism just s'.

Given a point P in X, as a morphism $P : \operatorname{Spec}\mathbb{C} \to X$, we can consider the corresponding point ${}^{\sigma}P$ in ${}^{\sigma}X$ looking at the following diagram:

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C} \arrow[r, "P"] \arrow[dr, "\mathrm{id}"'] & X \arrow[d, "s"] \\
 & \operatorname{Spec}\mathbb{C} \arrow[d, "\operatorname{Spec}\sigma^{-}1"] \\
 & \operatorname{Spec}\mathbb{C} \arrow[uul, bend left, "\operatorname{Spec}\sigma"]
\end{tikzcd} .$$

Then, the point ${}^{\sigma}P$ corresponds to the morphism $P \circ \operatorname{Spec}\sigma$, and asking for the condition ${}_{\sigma}f({}^{\sigma}P) = P$ means to require that ${}_{\sigma}f \circ P \circ \operatorname{Spec}\sigma = P$, or, in other words, that the following diagram commutes

$$\begin{tikzcd}
{}^{\sigma}X \arrow[r, "{}_{\sigma}f"] & X \\
\operatorname{Spec}\mathbb{C} \arrow[u, "P"] \arrow[r, "\operatorname{Spec}\sigma^{-1}"'] & \operatorname{Spec}\mathbb{C} \arrow[u, "P"']
\end{tikzcd} .$$

Example. *Consider $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x]$, with structure morphism induced by the canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[x]$, and take as $\sigma \in \operatorname{Aut}(\mathbb{C})$ the conjugation. Define the point P to be the morphism $P : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}[x]$ corresponding to the ring morphism $\mathbb{C}[x] \to \mathbb{C}$ such that $x \mapsto i$. Then ${}^{\sigma}P$ is the morphism $P \circ \operatorname{Spec}\sigma$, that correspond to the ring morphism $\mathbb{C}[x] \to \mathbb{C}$ such that $x \mapsto -i$ and acting on $\mathbb{C} \subseteq \mathbb{C}[x]$ as the conjugation.*

In both cases, the point on the topological space of the spectrum that the morphism is representing is $(x - i)$, which is the kernel of both the ring morphisms associated to P and ${}^{\sigma}P$. Nevertheless, P and ${}^{\sigma}P$ are different if seen through the scheme-theoretic language, that interprets points as morphisms, since they have different ring morphisms associated.

Proposition 2.4. *Let X be a curve over $\mathbb{C}$ and let $t : X \to \mathbb{P}^1_{\mathbb{C}}$ be a finite morphism. Then both X and t are defined over a finite extension of $M(X, t)$.*

Proof. Choose a $\mathbb{Q}$-rational point $Q \in \mathbb{P}^1_{\mathbb{C}}$ that is not a critical value of t, and consider $P \in t^{-1}(Q)$. Applying Riemann-Roch Theorem to the divisor $\mathcal{D} := (g + 1)P$, where g is the genus of X, we get

$$\dim_{\mathbb{C}} \mathscr{L}(\mathcal{D}) - \dim_{\mathbb{C}} \mathscr{L}(K - \mathcal{D}) = \deg \mathcal{D} - g + 1 = 2,$$

for any canonical divisor K. From this, it follows that $\dim_{\mathbb{C}} \mathscr{L}(\mathcal{D}) \geq 2$. In other words, there exists a meromorphic function $z \in \mathscr{M}(X) \smallsetminus \mathbb{C}$ with just one pole at P.

Now we use the anti-equivalence of categories holding between projective, smooth algebraic curves over an algebraically closed field K and finitely generated field extension of K of transcendence degree 1. So, consider the morphisms of curves induced by the field extensions $\mathscr{M}(X)/\mathbb{C}(t,z)$, $\mathscr{M}(X)/\mathbb{C}(t)$ and $\mathscr{M}(X)/\mathbb{C}(z)$. Then, denoting with $X_{\mathbb{C}(t,z)}$ the curve corresponding to the function field $\mathbb{C}(t,z)$, and recalling that the curve associated to both $\mathbb{C}(t)$ and $\mathbb{C}(z)$ is $\mathbb{P}^1_{\mathbb{C}}$, we get the diagram

$$\begin{tikzcd} & X \arrow[d, "\gamma"] \arrow[ddl, bend right, "t"'] \arrow[ddr, bend left, "z"] & \\ & X_{\mathbb{C}(t,z)} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^1_{\mathbb{C}} & & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd},$$

where, by our assumptions, $t = \alpha \circ \gamma$ is unramified at P and $z = \beta \circ \gamma$ is totally ramified at P. It follows that γ is both unramified and totally ramified, and so $\mathscr{M}(X) = \mathbb{C}(t,z)$.

Assume now we chose z such that the order of its pole $m := -\mathrm{ord}_P(z) \in \mathbb{N}$ is minimal. Define

$$V := \mathscr{L}(mP) = \{x \in \mathscr{M}(X) : \ \mathrm{ord}_P(x) \geq -m, \ \mathrm{ord}_Q(x) \geq 0 \ \ \forall Q \neq P\}.$$

Clearly, $\mathbb{C} \oplus \mathbb{C}\, z \subseteq V$. We show that equality holds. Let $r \in V$ with $\mathrm{ord}_P(r) = -m$, then there exists a constant $\alpha \in \mathbb{C}$ such that $-\mathrm{ord}_P(r - \alpha z) < m$. By minimality on m, this implies that $\mathrm{ord}_P(r - \alpha z) \geq 0$, but then, since the order is non-negative for any other $Q \neq P$, we get that $r - \alpha z$ is a holomorphic function. Thus, $r - \alpha z \in \mathbb{C}$, and hence $r \in \mathbb{C} \oplus \mathbb{C}\, z$.

Since Q is not a critical value of t, the meromorphic function $t - Q$ yields a local chart of X centred in P, it is a local parameter on X in P. Let z' be the unique function in V such that in its Laurent expansion with respect to $t - Q$ the coefficient of $(t-Q)^m$ is 1, and the coefficient of $(t-Q)^0$ is 0. Without loss of generality, we may assume that z is already this z'.

Now, the extension $\mathbb{C}(t,z)/\mathbb{C}(t)$ is finite, because the transcendence degree of $\mathbb{C}(t,z)$ is 1, so we can consider the minimal polynomial of z over $\mathbb{C}(t)$. The claim we are going to prove is that such polynomial has coefficients in $k(t)$, where k is a finite extension of $M(X,t)$. Meaning that the inclusion $\mathscr{M}(X)/\mathbb{C}(t)$ is defined over k, this would imply the statement.

To prove this claim, denote by $U(X,t,P)$ the subgroup of $U(X,t)$ consisting of all $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that there exists an isomorphism ${}_\sigma f : {}^\sigma X \to X$

of curves over $\mathbb{C}$, making the diagram

$$\begin{tikzcd} {}^{\sigma}X \arrow[r, "{}_{\sigma}f"] \arrow[d, "{}_{\sigma}t"'] & X \arrow[d, "t"] \\ {}^{\sigma}\,\mathbb{P}^1_{\mathbb{C}} \arrow[r, "\overline{\sigma}"'] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

commute, and such that ${}_{\sigma}f({}^{\sigma}P) = P$, with all the notations as explained above. Since Q is not a critical value of t, $\mathrm{Aut}(t)$ acts freely on $t^{-1}(Q)$. This means that ${}_{\sigma}f$ is unique: if ${}_{\sigma}\tilde{f}$ had it same properties, we would have ${}_{\sigma}f \circ {}_{\sigma}\tilde{f}^{-1}(P) = P$, meaning ${}_{\sigma}f \circ {}_{\sigma}\tilde{f}^{-1} = \mathrm{id}$, by freeness. So, mapping σ to ${}_{\sigma}f^{\#} \in \mathrm{Aut}(\mathscr{M}(X))$ yields an action of $U(X,t,P)$ on $\mathscr{M}(X)$ which fixes t. The meromorphic function $z \in \mathscr{M}(X)$, and hence its minimal polynomial over $\mathbb{C}(t)$, is invariant under the action of $\sigma \in U(X,t,P)$, since its image has the same properties of z defined above. Moreover, $U(X,t,P)$ has finite index in $U(X,t)$, since it is the stabilizer of P under the action of $U(X,t)$ on $t^{-1}(Q)$ defined by $(\sigma, P) \mapsto {}_{\sigma}f(P)$. So, we can conclude the proof applying the Lemma 2.3, with $M(X,t) = \mathbb{C}^{U(X,t)}$ and the extension where z and its minimal polynomial are defined as $\mathbb{C}^{U(X,t,P)}$. $\square$

2.3 Belyi's Theorem

To proceed with our investigation and get closer to Belyi's result, we need to focus on the pairs (X,t), where X is a curve over $\mathbb{C}$, and $t : X \to \mathbb{P}^1_{\mathbb{C}}$ is a finite morphism of variety over $\mathbb{C}$. We will say that two such pairs (X_1, t_1) and (X_2, t_2) are isomorphic if there exists an isomorphism $f : X_1 \to X_2$ of varieties over $\mathbb{C}$ such that $t_2 \circ f = t_1$.

Proposition 2.5. *Let S be a finite set of point of $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$, and let $d \geq 1$ be a natural number. Then there are at most finitely many isomorphism classes of pairs (X,t), where X is a curve over $\mathbb{C}$ and $t : X \to \mathbb{P}^1_{\mathbb{C}}$ is a finite morphism of varieties over $\mathbb{C}$ of degree d and whose critical values lie in S.*

Proof. We can consider a map from the set of isomorphism classes of Belyi pairs to the set $\mathcal{M}$ of homeomorphism classes of unramified topological coverings of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus S$ of degree d. This is defined as follows: considering a finite morphism $t : X \to \mathbb{P}^1_{\mathbb{C}}$ we pass to the continuous map $t(\mathbb{C}) : X(\mathbb{C}) \to \mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$, and we restrict it to the preimage of $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \smallsetminus S$. This map is injective. Indeed, if (X_1, t_1) and (X_2, t_2) are two Belyi pairs, and $g : X_1(\mathbb{C}) \smallsetminus {t_1}^{-1}(S) \to X_2(\mathbb{C}) \smallsetminus {t_2}^{-1}(S)$ is a homeomorphism such that $t_2(\mathbb{C}) \circ g = t_1(\mathbb{C})$, then g is also biholomorphic. Moreover, as we saw in the proof of Theorem 1.4, g can be extended to a biholomorphism $h : X_1(\mathbb{C}) \to X_2(\mathbb{C})$, with $t_2(\mathbb{C}) \circ h = t_1(\mathbb{C})$. Since any biholomorphic map between compact complex curves is algebraic, we get that (X_1, t_1) and (X_2, t_2) are isomorphic as Belyi pairs.

Thus, it suffices to prove that the set $\mathcal{M}$ is finite. This comes from the fact that any unramified topological covering of degree d of $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \smallsetminus S$ is an intermediate covering of degree d of the universal one, and these coverings are in bijection with the subgroups of $\pi_1(\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \smallsetminus S)$ of index d. Since $\pi_1(\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \smallsetminus S)$ is the free group generated by $|S| - 1$ elements, it is finitely generated, and hence it has only finitely many subgroups of a given finite index. □

Corollary 2.6. *Let X be a curve over $\mathbb{C}$, let $t : X \to \mathbb{P}^1_{\mathbb{C}}$ be a finite morphism, and let K be a subfield of $\mathbb{C}$ such that the critical value of t are K-rational. Then the moduli field of t is contained in a finite extension of K.*

Proof. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ the critical values of the map $t(\sigma)$ defined by ${}^{\sigma}X \xrightarrow{{}^{\sigma}t} {}^{\sigma}\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\overline{\sigma}} \mathbb{P}^1_{\mathbb{C}}$ are the same as the ones of t, and the degree of the map is also the same. Thus, by Proposition 2.5, the orbits of the isomorphism classes of pairs (X, t), under the action of $\mathrm{Aut}(\mathbb{C}, K)$ defined by $(X, t) \overset{\sigma}{\mapsto} ({}^{\sigma}X, t(\sigma))$, is finite. So, the stabilizer of (X, t) has finite index in $\mathrm{Aut}(\mathbb{C}/K)$, and it is obviously contained in $U(X, t)$. Thus, by Lemma 2.1 and Lemma 2.3, the moduli field $M(X, t) = \mathbb{C}^{U(X,t)}$ is contained in a finite extension of $\mathbb{C}^{\mathrm{Aut}(\mathbb{C}/K)} = K$. □

We are finally ready to prove Belyi's Theorem. The *if* direction descends as a consequence of the theory we developed, while the *only if* one is just Belyi's smart algorithm to produce a finite morphism with critical values in $\{0, 1, \infty\}$.

Theorem 2.7 (Belyi). *A complex curve X is defined over $\overline{\mathbb{Q}}$ if and only if there exists a finite morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ of varieties over $\mathbb{C}$ with critical values lying in the set $\{0, 1, \infty\}$.*

Proof. ($\Longleftarrow$) : Assume we have a morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ as above. By Theorem 2.4 X is defined over a finite extension of $M(X, \beta)$, which is, by Corollary 2.6, a finite extension of $\mathbb{Q}$. Thus, X is defined over $\overline{\mathbb{Q}}$.

($\Longrightarrow$) : Let X be a complex curve defined over $\overline{\mathbb{Q}}$, and let $\widetilde{X}$ be the corresponding curve over $\overline{\mathbb{Q}}$. Since $\widetilde{X}$ is projective, there exists a finite morphism $\widetilde{\beta} : \widetilde{X} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$, and hence there exists morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\beta"'] & \widetilde{X} \arrow[d, "\widetilde{\beta}"] \\ \mathbb{P}^1_{\mathbb{C}} \arrow[r, "\psi"'] & \mathbb{P}^1_{\overline{\mathbb{Q}}} \end{tikzcd}$$

commute. But then, denoting the ramification locus of a morphism f as $\mathcal{R}(f)$ and its critical values as $\mathrm{Crit}(f)$, we get that

$$\mathrm{Crit}(\beta) = \beta(\mathcal{R}(\beta)) \subseteq \beta(\varphi^{-1}(\mathcal{R}(\widetilde{\beta}))) \subseteq \psi^{-1}(\widetilde{\beta}(\mathcal{R}(\widetilde{\beta}))) \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}),$$

where the first inclusion comes from the fact that since ψ maps every point of $\mathbb{P}^1_{\mathbb{C}}$ that is not $\overline{\mathbb{Q}}$-rational into the generic point, because $\overline{\mathbb{Q}}$ is algebraically closed, and the generic point cannot be a critical value, then the critical points of β have to be $\overline{\mathbb{Q}}$-rational. Hence, we got the existence of a finite morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ with critical values in $\overline{\mathbb{Q}} \cup \{\infty\}$.

Let $S \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ be a finite set containing the critical values of β. We claim that there exists a non-constant polynomial $p \in \mathbb{Q}[x]$, such that $p(S)$ and the critical values of $p : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ are in $\mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$. Enlarge S so that it becomes $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$-invariant. We proceed by induction on $n = |S|$. If $n = 1$, then the only element of S is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, and hence it belongs to $\mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$, thus we can just take $p(x) = x$. If $n > 1$, define p_1 to be the product of the minimal polynomials of all the elements in S. By construction, $p_1 \in \mathbb{Q}[x]$, and $p_1(S) = 0$. Then, setting $S_1 = p_1(\{z \in \overline{\mathbb{Q}} : p_1'(z) = 0\})$, the set of the critical values of p_1 is $S_1 \cup \{\infty\}$. Since S_1 has at most $n-1$ elements, by the inductive hypothesis there exists a polynomial $p_2 \in \mathbb{Q}[x]$ such that $p_2(S_1)$ and the critical values of p_2 lie in $\mathbb{Q} \cup \{\infty\}$. The composition $p_2 \circ p_1$ is such that

$$\mathrm{Crit}(p_2 \circ p_1) = \mathrm{Crit}(p_2) \cup p_2(\mathrm{Crit}(p_1)) = \mathrm{Crit}(p_2) \cup p_2(S_1 \cup \{\infty\}) \subseteq \mathbb{Q} \cup \{\infty\},$$

and $(p_2 \circ p_1)(S) = p_2(\{0\}) \subseteq \mathbb{Q}$, as wished.

So, the composition $\beta' = p_2 \circ p_1 \circ \beta$ is a finite morphism with critical values in $\mathbb{Q} \cup \{\infty\}$. Now our goal is to show that it is possible, composing by a morphism of $\mathbb{P}^1_{\mathbb{C}}$, to make β' to have its critical values lying in a subset of $\{0, 1, \infty\}$. Let $S' \subseteq \mathbb{Q} \cup \{\infty\}$ be the finite set of critical values of β'. If $|S'| \leq 3$, this is clear by taking an opportune Möbius transformation. If $|S'| > 3$, choose three ordered points of S': again by applying a Möbius transformation, we can send them to 0, 1 and $\frac{m}{m+n}$ for some positive integers n and m. Then, the transformation

$$z \mapsto \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

maps both 0 and 1 to 0, and $\frac{m}{m+n}$ to 1, and has critical values contained in $\{0, 1, \infty\}$. Thus, composing β' with such a morphism, we get a finite morphism with set of critical values of cardinality less or equal than $|S'| - 1$. Repeating this procedure a finite number of times, we produce a finite morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ having set of critical values contained in $\{0, 1, \infty\}$. $\square$

Notice that we could have stated the theorem replacing the set $\{0, 1, \infty\}$ with any other set of three points, or have just put the condition that β have at most three critical values, since we can move those points with any Möbius transformation. Moreover, recall that by GAGA principle, since we are dealing with algebraic projective curves, finite morphisms correspond to holomorphic maps. Thus, the statement can be read as the equivalence, for an algebraic projective curve X over $\mathbb{C}$, of being defined over $\overline{\mathbb{Q}}$ and of the existence of an holomorphic map $X \to \mathbb{P}^1_{\mathbb{C}}$ with critical values lying in the set $\{0, 1, \infty\}$. Furthermore, we remark that, by Theorem 1.3, a non-constant holomorphic map between compact connected Riemann surfaces is a ramified covering. Thus, Belyi's Theorem provides a strong relation between a purely algebraic property, that is, being defined over $\overline{\mathbb{Q}}$, and a purely topological property, concerning ramified coverings. In the next chapter we are going to explore in detail this last aspect.

We conclude giving some examples of such maps, whose critical values lie in the set $\{0, 1, \infty\}$, that go under the name of *Belyi morphism*.

Example. *Take* $X = \mathbb{P}^1_{\mathbb{C}}$, *and consider the map* $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ *defined by* $z \mapsto z^n$ *for an integer* $n \geq 2$.

Then, β *is ramified only over* 0 *and* ∞, *thus it is a Belyi morphism.*

Example. *Let* $X = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} \mid x^n + y^n = z^n\}$ *be the Fermat curve of degree* n, *and consider* $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ *the morphism given by*

$$[x : y : z] \mapsto [x^n : y^n].$$

On the affine part $z = 1$ *of* X *we have* $\beta : (x, y) \mapsto x^n$, *so* $\deg \beta = n^2$.

Let $\zeta_n = e^{2\pi i/n}$, *we have that* $|\beta^{-1}(P)| < n^2$ *if:*

- P *is an affine point such that* $\beta(P) = 0$, *so* $P = (0, \zeta_n^k)$ *for some* k;
- P *is an affine point such that* $\beta(P) = 1$, *so* $P = (\zeta_n^k, 0)$ *for some* k;
- P *is an infinity point, so that* $P = [1 : y : 0]$, *with* $y^n = -1$. *In this case,* $\beta(P) = \infty$.

In each of the three cases, there are n *critical points, and the ramification index is* n *at each of them. Since the values of* β *at the critical points are just* $0, 1$ *and* ∞, β *is a Belyi morphism.*

Chapter 3

Dessins d'Enfants

In this chapter, we are going to introduce our main objects of study: *dessins d'enfants*. This name was introduced by Grothendieck, since he noticed that their pictures resembled some children's drawings scrawled on a bit of paper. Indeed, they are just some type of graphs, whose vertices satisfy a particular condition, embedded into Riemann surfaces.

We will follow the approach suggested by Grothendieck in [Gro97] to prove a particular bijection between some subsets of dessins d'enfants and Belyi morphisms. In particular, the introduction of a *marking* on a dessin, that is, a particular choice of points in the graph, will allow us to define a set of triangles, called *flags*, attached to a dessin. Then, defining a group acting on the set of flags, we will be able to state such bijection.

Next, we will focus on other possible descriptions for dessins, seeing that any dessin with N edges determines, and can be reconstructed from, a pair of permutations in S_N generating a transitive subgroup. This pair will allow us to define an action of the free group on two elements F_2 on the set of edges of a dessin. Interpreting a finite index subgroup of F_2 as the stabilizer of an edge of a dessin for such action, we will establish another bijection involving dessins that will generalize Grothendieck's one to all dessins and all Belyi morphisms. These observations will enable us to see dessins as various different objects.

Finally, we will study the automorphism group of a dessin, defining *regular* dessins to be the ones for which their automorphism group acts transitively on the set of the edges. We will see that this notion, which closely resembles the notion of regularity for coverings, can indeed be easily transferred to the other elements for which we have a bijection with dessins.

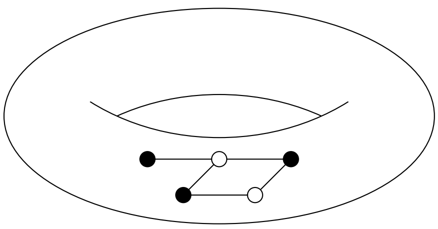
3.1 Definition of a Dessin

A **dessin d'enfants** $\mathcal{D}$ is a triple (X_0, X_1, X_2), with $X_0 \subseteq X_1 \subseteq X_2$, where X_2 is the topological space underlying a compact connected Riemann surface, X_0 is a finite set of points, called vertices, and X_1, whose elements are called edges, is a set such that $X_1 \smallsetminus X_0$ is a finite disjoint union of segments homeomorphic to the open interval $]0;1[$ and $X_2 \smallsetminus X_1$ is a finite disjoint union of open cells, each of which is homeomorphic to the topological open disk, called faces. Moreover, a dessin d'enfants must admit a bipartite structure on the set of vertices X_0, namely, the vertices can be marked with two different marks so that each edge connects two vertices with different mark. We will represent such marks using black and white colors.

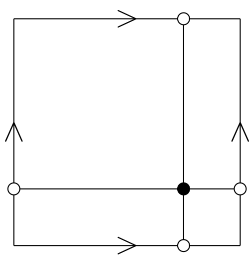
Two dessins d'enfants $\mathcal{D} = (X_0, X_1, X_2)$ and $\mathcal{D}' = (X_0', X_1', X_2')$ are said to be isomorphic if there exists an orientation-preserving homeomorphism from X_2 onto X_2' mapping X_1 to X_1' and X_0 to X_0', and respecting the bipartite structure, namely, sending vertices with same mark to vertices with same mark. We will frequently refer to a dessin d'enfants just using the shorter version *dessin*.

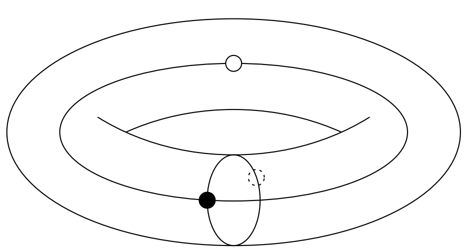
We can think of dessins as special types of bipartite graphs, that is, graph admitting a bipartite structure, on connected and compact surfaces. Notice that the condition imposed on the faces and the fact that X_2 is assumed to be connected imply that every path connecting two points on X_1 is homotopic to a path entirely contained in X_1. This means that the graph has to be connected.

Example. *Consider the following bipartite graph in the torus:*



This is not a dessin, since one of the two open cells is not isomorphic to an open disk. Instead, the following graph is an example of a dessin in the torus:



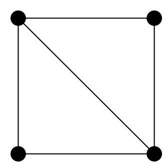
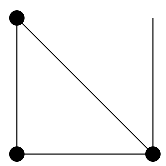


Given a connected, projective, smooth complex curve X over $\mathbb{C}$ that is defined over $\overline{\mathbb{Q}}$, by Belyi's Theorem, there exists a morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ whose critical values lie in the set $\{0, 1, \infty\}$. We will call such β a **Belyi morphism**, and we will call the couple (X, β) a **Belyi pair**. Two Belyi pairs (X, β) and (Y, α) are said to be isomorphic if there exists an isomorphism of curves $\phi : X \to Y$ such that $\beta = \alpha \circ \phi$.

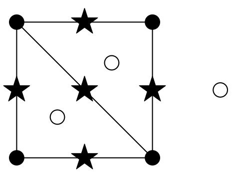
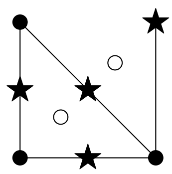
In order to study the correspondence between Belyi pairs and dessins as Grothendieck did in [Gro97], we need to simplify the definition we just gave. Dropping the request of a bipartite structure, we say that a **pre-clean dessin** is a triple (X_0, X_1, X_2), with $X_0 \subseteq X_1 \subseteq X_2$, where X_2 is the topological space underlying a compact connected Riemann surface, X_0 is a finite set of points, called vertices, and X_1, whose elements are called edges, is such that $X_1 \smallsetminus X_0$ is a finite disjoint union of segments homeomorphic either to the interval $]0;1[$ or to $]0;1]$, and $X_2 \smallsetminus X_1$ is a finite disjoint union of open cells, called faces, each of which is homeomorphic to the topological open disk. Notice that in this definition we also allow the edges to be homeomorphic to $]0;1]$, which means that we allow the presence of tails, meaning edges with an end without vertex. A **clean dessin d'enfants** is a pre-clean dessin having no tails, namely, such that each edge has a vertex at both its ends.

A **marking** on a pre-clean dessin is the fixed choice of one point in each component of $X_1 \smallsetminus X_0$, that will be denoted by $\star$, and a point in each open cell of $X_2 \smallsetminus X_1$, that will be denoted by $\circ$. The vertices in X_0 will be denoted by $\bullet$. Clearly, it is always possible to associate a marking to a pre-clean dessin d'enfants, and the choice of a marking on a pre-clean dessin induces a choice of a marking on every dessin in its isomorphism class.

Example. *The followings are both pre-clean dessins d'enfants on a sphere:*

A marking on each of them can be done as follows

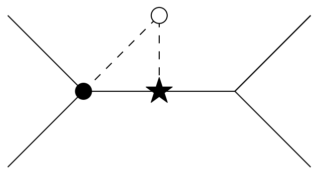



As we can notice, the left pre-clean dessin is also clean, while the other one is not, since it has a tail, which is the right vertical edge.

3.2 The Cartographical Group

In this section and in section 3.3 we are dealing only with pre-clean dessins.

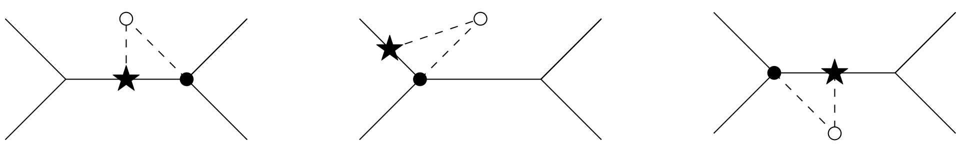
Given a pre-clean dessin $\mathcal{D}$ with marking, the **flag set** of $\mathcal{D}$ is the set $\mathscr{F}(\mathcal{D})$ of the triangles whose three vertices are marked with $\bullet$, $\circ$ and $\star$ in such a way that $\bullet$ is in the closure of the edge containing $\star$, and this edge is in the closure of the open cell containing $\circ$. The **oriented set flag** $\mathscr{F}^+(\mathcal{D})$ is the subset of $\mathscr{F}(\mathcal{D})$ containing the flags whose order of the vertices is $\circ - \bullet - \star$ when read counterclockwise. For instance, the following is the picture of an oriented flag F:



The **cartographical group** $\mathcal{C}_2$ is defined to be the group

$$\mathcal{C}_2 \coloneqq \left\langle \sigma_0, \sigma_1, \sigma_2 \ \mid \ \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = 1, \ (\sigma_0\, \sigma_2)^2 = 1 \right\rangle,$$

where σ_0, σ_1 and σ_2 act on $\mathscr{F}(\mathcal{D})$ in the following way: if $F \in \mathscr{F}(\mathcal{D})$ is the flag represented above, then $\sigma_0(F)$, $\sigma_1(F)$ and $\sigma_2(F)$ are respectively



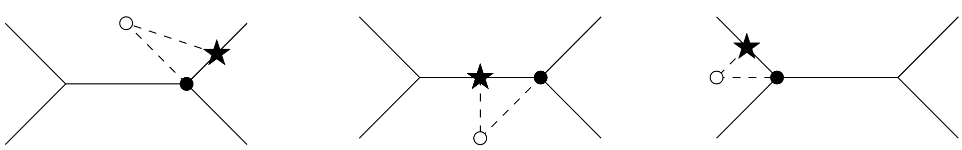
which is to say that σ_0 sends F into the flag with same $\circ$ and $\star$ vertices as F, but with the $\bullet$ vertex taken to be the other vertex of the edge containing $\star$. Similar definitions hold for σ_1 and σ_2.

We can also define the **oriented cartographical group** $\mathcal{C}_2^+$ as the subgroup of index 2 of $\mathcal{C}_2$ given by all its even words, whose presentation is

$$\mathcal{C}_2^+ = \left\langle \rho_0, \rho_1, \rho_2 \mid \rho_1^2 = 1, \ \rho_0\rho_1\rho_2 = 1 \right\rangle,$$

where $\rho_0 = \sigma_1\, \sigma_0$, $\rho_1 = \sigma_0\, \sigma_2$ and $\rho_2 = \sigma_2\, \sigma_1$.

The oriented cartographical group acts on the set $\mathscr{F}^+(\mathcal{D})$ as follows: if F is again the oriented flag represented above, then $\rho_0(F)$, $\rho_1(F)$ and $\rho_2(F)$ are respectively



We remark we are considering the element of $\mathcal{C}_2$ as acting on the left, meaning that in $\rho_0 = \sigma_1 \sigma_0$ the first to be applied on F is σ_0.

Notice that both the actions of $\mathcal{C}_2$ on $\mathscr{F}(\mathcal{D})$ and of $\mathcal{C}_2^+$ on $\mathscr{F}^+(\mathcal{D})$ do not depend on the choice of a marking on the dessin.

Lemma 3.1. *Let $\mathcal{D}$ be a pre-clean dessin with a marking and let $F \in \mathscr{F}^+(\mathcal{D})$ be a fixed flag. Then the stabilizer of F in $\mathcal{C}_2^+$, $\mathrm{Stab}_{\mathcal{C}_2^+}(F)$, has finite index in $\mathcal{C}_2^+$, and for any other flag $F' \in \mathscr{F}^+(\mathcal{D})$, its stabilizer $\mathrm{Stab}_{\mathcal{C}_2^+}(F')$ is conjugated to $\mathrm{Stab}_{\mathcal{C}_2^+}(F)$ in $\mathcal{C}_2^+$.*

Proof. Since $\mathscr{F}^+(\mathcal{D})$ is finite, the orbit of F under $\mathcal{C}_2^+$ is finite, and hence $\mathrm{Stab}_{\mathcal{C}_2^+}(F)$ has finite index in $\mathcal{C}_2^+$. Moreover, notice that the action of $\mathcal{C}_2^+$ on $\mathscr{F}^+(\mathcal{D})$ is transitive. Indeed, as shown in the above pictures, ρ_1 sends an oriented flag $F \in \mathscr{F}^+(\mathcal{D})$ to the oriented flag constructed on the same edge of the dessin, but with $\circ$ vertex in the other face, while the orbit of F under the action of ρ_0 is the set of all the oriented flags with $\circ$ vertex in the same face as F. Since the dessin is connected, with the actions of these two elements we are able to send F to any other arbitrary oriented flag. Thus, the action of $\mathcal{C}_2^+$ on $\mathscr{F}^+(\mathcal{D})$ is transitive, and so there exists an element $\sigma \in \mathcal{C}_2^+$ such that $\sigma(F') = F$, so that $\mathrm{Stab}_{\mathcal{C}_2^+}(F') = \sigma^{-1}\,\mathrm{Stab}_{\mathcal{C}_2^+}(F)\,\sigma$. □

3.3 The Grothendieck Correspondence

The action of the cartographical group allows us to establish a relation between the isomorphism classes of pre-clean dessins d'enfants and the conjugacy classes of subgroups of $\mathcal{C}_2^+$ of finite index.

Theorem 3.2. *There is a bijection between the set of isomorphism classes of pre-clean dessins d'enfants and the set of conjugacy classes of subgroups of $\mathcal{C}_2^+$ of finite index. Furthermore, the dessin corresponding to the conjugacy class of a subgroup B of $\mathcal{C}_2^+$ is clean if and only if $\rho_1 \notin B'$ for any B' in the conjugacy class of B.*

Proof. Consider an isomorphism class of a pre-clean dessin with a marking, then by Lemma 3.1 we can associate to it a conjugacy class of a subgroup of finite index of $\mathcal{C}_2^+$, just considering the conjugacy class of $\mathrm{Stab}_{\mathcal{C}_2^+}(F)$ for any choice of $F \in \mathscr{F}^+(\mathcal{D})$.

Conversely, let B be a subgroup of $\mathcal{C}_2^+$ of finite index, and consider the coset space $H = \mathcal{C}_2/B$. We will construct a pre-clean dessin $\mathcal{D}$ such that $\mathscr{F}(\mathcal{D})$ will be bijective to H, so that $\mathscr{F}^+(\mathcal{D})$ will be bijective to $\mathcal{C}_2^+/B$, and such that the action of $\mathcal{C}_2$ on $\mathscr{F}(\mathcal{D})$ is given by the action of $\mathcal{C}_2$ in H by left multiplication. The flag corresponding to the coset B will be fixed by the action of B.

To construct such a $\mathcal{D}$, notice that σ_0, σ_1 and σ_2 act on H, dividing its elements into orbits. In particular, two cosets, so two flags, will be in the same σ_0-orbit if their $\circ - \bigstar$ segment is the same, they will be in the same σ_1-orbit if their $\circ - \bullet$ segment is the same, and in the same σ_2-orbit if their $\bullet - \bigstar$ segment is the same. In particular, every σ_i-orbit contains at most two elements. So, we can begin to reconstruct $\mathcal{D}$ by noting the numbers of each three types of edge as the orders of the quotient spaces $\langle \sigma_i \rangle \backslash H$.

Consider then the action of σ_1 and σ_2 on the quotient space $\langle \sigma_0 \rangle \backslash H$. Identifying each element of $\langle \sigma_0 \rangle \backslash H$ with a $\circ - \bigstar$ edge, the σ_0-orbits that are in the same orbit under the action of σ_1 are those having the same $\circ$ point, and those identified under σ_2 have the same $\bigstar$ point. Analogously, when σ_0 and σ_2 act on $\langle \sigma_1 \rangle \backslash H$, considered as the set of $\circ - \bullet$ edges, they should identify edges having the same $\circ$ and $\bullet$ points respectively, and in the very same way, when σ_0 and σ_1 act on $\langle \sigma_2 \rangle \backslash H$ considered as the set of $\bigstar - \bullet$ edges, they identify edges having the same $\bigstar$ and $\bullet$ points respectively.

We can use this information to give the orders of the sets of vertices, edges and open cells of $\mathcal{D}$ respectively as the orders of $\langle \sigma_1, \sigma_2 \rangle \backslash H$, $\langle \sigma_0, \sigma_2 \rangle \backslash H$ and $\langle \sigma_0, \sigma_1 \rangle \backslash H$. Moreover, along the same ideas, we can define how these components glue together in the following way. Take a vertex x in $\langle \sigma_1, \sigma_2 \rangle \backslash H$ and an edge y in $\langle \sigma_0, \sigma_2 \rangle \backslash H$: they can be glued together, meaning that there is a flag in $\mathscr{F}(\mathcal{D})$ with vertices x and y, if they both occur in the same σ_2-orbit. Similarly, a vertex and an open cell, as elements of the quotient spaces, can be glued if the occur in the same σ_1-orbit, and an edge and an open cell can be glued if they occur in the same σ_0-orbit. So, the decomposition of H into orbits under the action of σ_0, σ_1 and σ_2 allows us to construct a pre-clean dessin $\mathcal{D}$. The situation is visualized in the following two diagrams, where we are denoting as $\mathscr{F}_{\circ\bigstar}$, $\mathscr{F}_{\circ\bullet}$ and $\mathscr{F}_{\bullet\bigstar}$ the sets of $\circ - \bigstar$, $\circ - \bullet$ and $\bullet - \bigstar$ edges respectively, while $\mathscr{F}_{\bullet}$, $\mathscr{F}_{\bigstar}$ and $\mathscr{F}_{\circ}$ denote the sets of vertices, edges and open cells respectively:

$$\begin{tikzcd}
 & \mathscr{F}(\mathcal{D}) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathscr{F}_{\circ\bigstar} \arrow[d] \arrow[dr] & \mathscr{F}_{\circ\bullet} \arrow[dl] \arrow[dr] & \mathscr{F}_{\bullet\bigstar} \arrow[dl] \arrow[d] \\
\mathscr{F}_{\circ} \arrow[dr] & \mathscr{F}_{\bigstar} \arrow[d] & \mathscr{F}_{\bullet} \arrow[dl] \\
 & \cdot &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle \sigma_0 \rangle \backslash H \arrow[d] \arrow[dr] & \langle \sigma_1 \rangle \backslash H \arrow[dl] \arrow[dr] & \langle \sigma_2 \rangle \backslash H \arrow[dl] \arrow[d] \\
\langle \sigma_0, \sigma_1 \rangle \backslash H \arrow[dr] & \langle \sigma_0, \sigma_2 \rangle \backslash H \arrow[d] & \langle \sigma_1, \sigma_2 \rangle \backslash H \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

Moreover, we notice that changing B to $\sigma^{-1}\,B\,\sigma$ for any $\sigma \in \mathcal{C}_2^+$ does not change any of the above objects considered as sets together with the action of the elements of $\mathcal{C}_2$, because the action of $\mathcal{C}_2$ on $\mathcal{C}_2\,/\,\sigma^{-1}\,B\,\sigma$ is the same as on $\mathcal{C}_2\,/B$. So, the pre-clean dessin $\mathcal{D}$ obtained is independent on the conjugacy class of B. Moreover, if we construct $\mathcal{D}$ as above from a subgroup B of $\mathcal{C}_2^+$, and then consider the subgroup fixing some given flag of the dessin, we find exactly a subgroup conjugated to B, since the set $H = \mathcal{C}_2\,/B$ is the flag set of the dessin, and B clearly fixes one flag, that is the one corresponding to the coset B, and by Lemma 3.1 all subgroups fixing the different flags of a dessin are conjugate.

Finally, notice that $\rho_1 \in \sigma^{-1}\,B\,\sigma$ for some $\sigma \in \mathcal{C}_2^+$ if and only the flag corresponding to the coset $\sigma^{-1}\,B$ is fixed by all the elements of $\sigma^{-1}\,B\,\sigma$, and so, in particular, it is fixed by $\rho_1 = \sigma_0\,\sigma_2$. This is possible if and only if the flag corresponding to $\sigma^{-1}\,B$ is a tail, and hence the arising pre-clean dessin is not clean. $\square$

Example. *Consider the subgroup B of $\mathcal{C}_2^+$ generated by ρ_0^2, ρ_2^2, $\rho_2\rho_0\rho_2^{-1}\rho_0^{-1}$ and ρ_1. Recall that, since $\rho_1 = (\rho_2\rho_0)^{-1}$, we can see $\mathcal{C}_2^+$ as $\langle\rho_0, \rho_2\rangle$, and notice that the subgroups of the abelian group $\mathcal{C}_2^+\,/\,\langle\rho_2\rho_0\rho_2^{-1}\rho_0^{-1}\rangle$ are in bijection with the normal subgroups of $\mathcal{C}_2^+$ containing $\langle\rho_2\rho_0\rho_2^{-1}\rho_0^{-1}\rangle$. So, since B contains the commutator of ρ_0 and ρ_2, we get that B is normal in $\mathcal{C}_2^+$.*

Then the quotient group

$$H = \mathcal{C}_2/B = \left\langle \sigma_0, \sigma_1, \sigma_2 \;\middle|\; \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_0\,\sigma_2 = (\sigma_0\,\sigma_1)^2 = 1 \right\rangle$$

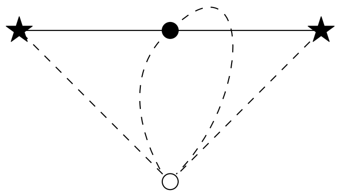
is the group formed just by the elements $1, \sigma_0, \sigma_1$ and $\sigma_0\,\sigma_1$. This set is in correspondence with the flag set $\mathscr{F}(\mathcal{D})$, so the dessin has four flags.

We study the orbits of those elements under the action of the elements σ_0, σ_1 and σ_2. In particular, we have:

- *Two σ_0-orbits: $\{1, \sigma_0\}$ and $\{\sigma_1, \sigma_0\,\sigma_1\}$;*
- *Two σ_1-orbits: $\{1, \sigma_1\}$ and $\{\sigma_0, \sigma_0\,\sigma_1\}$;*
- *Two σ_2-orbits: $\{1, \sigma_0\}$ and $\{\sigma_1, \sigma_0\,\sigma_1\}$;*
- *One $\langle\sigma_0, \sigma_1\rangle$-orbit: $\{1, \sigma_0, \sigma_1, \sigma_0\,\sigma_1\}$;*
- *One $\langle\sigma_1, \sigma_2\rangle$-orbit: $\{1, \sigma_0, \sigma_1, \sigma_0\,\sigma_1\}$;*
- *Two $\langle\sigma_0, \sigma_2\rangle$-orbits: $\{1, \sigma_0\}$ and $\{\sigma_1, \sigma_0\,\sigma_1\}$.*

We deduce, by the correspondence illustrated in the proof the theorem, the presence of one vertex, two edges, one open cell, two $\circ - \bullet$ edges, two $\bullet - \star$

edges and two $\star - \circ$ edges. A picture of the flag set is the following, that has to be thought as a triangulation of a sphere:



As we can see, the pre-clean dessin arising, which is

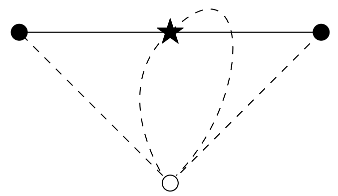
is not clean, as ρ_1 was an element of the subgroup B of $\mathcal{C}_2^+$ we considered.

Example. *Consider the subgroup B of $\mathcal{C}_2^+$ generated by $\rho_2\rho_0\rho_2^{-1}\rho_0^{-1}$, ρ_0^2 and ρ_2. Since we can see $\mathcal{C}_2^+$ as $\langle\rho_0, \rho_2\rangle$, and B contains the commutator of ρ_0 and ρ_2, we get that B is normal in $\mathcal{C}_2^+$.*

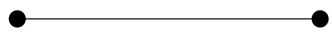
Then the quotient group

$$H = \mathcal{C}_2/B = \left\langle \sigma_0, \sigma_1, \sigma_2 \;\middle|\; \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_1\,\sigma_2 = (\sigma_0\,\sigma_1)^2 = 1 \right\rangle$$

is the group formed just by the elements $1, \sigma_0, \sigma_1$ and $\sigma_0\,\sigma_1$. Notice that this group is essentially the same as the one in the previous example, where the roles of σ_0 and σ_1 are exchanged. Then, it is not hard to see that, following the same procedure, one gets a dessin with two vertices, one edge, one open cell, two $\circ - \bullet$ edges, two $\bullet - \star$ edges and two $\star - \circ$ edges. So, in this case, the flag set, again thought as a triangulation of a sphere, is the following:



and the pre-clean dessin arising is simply



which is a clean dessin. Indeed, notice that ρ_1 does not belong to B, and hence it does not belong to any of the subgroups in its conjugacy class since B is normal.

In order to formulate properly the statement about the Grothendieck correspondence between dessins d'enfants and Belyi pairs, we need to restrict our attention to a special type of Belyi morphisms. We say that a finite covering $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ ramified only over $\{0, 1, \infty\}$ is a **clean Belyi morphism** if all the ramification indices over 1 are exactly equal to 2, and, in this case, we say that (X, β) is a **clean Belyi pair**. In fact, this definition is not so restrictive, as the following corollary to Belyi's Theorem suggests.

Corollary 3.3. *An algebraic curve defined over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if there exists a clean Belyi morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$.*

Proof. If $\alpha : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Belyi morphism, then $\beta = 4\alpha(1-\alpha)$ is a clean one. Indeed, if $\deg \alpha = d$, then $|\beta^{-1}(k)| = 2d$ for any $k \in \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$. Moreover, $|\beta^{-1}(1)| = |\alpha^{-1}(\frac{1}{2})| = d$, and the ramification degrees of the preimages of 1 under β are given by the ramification degrees of the preimages of $\frac{1}{2}$ under $t \mapsto 4t(1-t)$, which all are 2. $\square$

We can finally state the main result of this section.

Theorem 3.4. *There is a bijection between the set of the isomorphism classes of pre-clean dessins d'enfants and the set of pre-clean Belyi pairs. Moreover, this bijection restricts to a correspondence between the set of the isomorphism classes of clean dessins d'enfants and the set of clean Belyi pairs.*

Proof. The proof relies on the claim that there is a bijection between the conjugacy classes of subgroups of finite index of the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$, that is

$$\pi_1 = \langle l_0, l_1, l_\infty \mid l_0 l_1 l_\infty = 1 \rangle ,$$

and the isomorphism classes of finite coverings of X of $\mathbb{P}^1_{\mathbb{C}}$ ramified only over 0, 1 and ∞. This implies a bijection between the conjugacy classes of subgroups of finite index of $\pi_1' = \pi_1 / l_1^2$ and the isomorphism classes of pre-clean Belyi pairs. Since π_1' and $\mathcal{C}_2^+$ are canonically isomorphic via $\rho_0 \mapsto l_0$, $\rho_1 \mapsto l_1$ and $\rho_2 \mapsto l_\infty$, the thesis follows by Theorem 3.2. Notice that restricting to the conjugacy classes of a subgroups B of π_1' such that $l_1 \notin B'$ for any B' in the conjugacy class of B, we are considering exactly isomorphism classes of clean Belyi pairs, so the thesis in the clean case follows again by Theorem 3.2.

The proof of the claim relies on arguments of algebraic topology. Indeed, by Theorem 1.1, if $\widetilde{X}$ is the universal covering of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$, then there is a Galois bijection between the isomorphism classes of finite unbranched coverings $X' \to \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ of degree d and the conjugacy classes of subgroups of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\})$ of degree d. Furthermore, given an unbranched covering $X' \to \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$, it can be extended, by Theorem 1.4, to a ramified covering of $\mathbb{P}^1_{\mathbb{C}}$ in a unique way, up to biholomorphic fibre-preserving maps from X' into itself. This justifies the claim. $\square$

We can describe Grothendieck correspondence in a more concrete way, as follows. Given a pre-clean Belyi pair (X, β), it gives rise to the pre-clean dessin having X_2 as the topological space underlying X, X_1 as $\beta^{-1}([0;1])$, where $[0;1]$ is the segment of the real line on $\mathbb{P}^1_{\mathbb{C}}$, and the set of vertices X_0, marked with $\bullet$, as $\beta^{-1}(0)$. Requiring the Belyi function to be pre-clean is equivalent to asking that there be one edge to every preimage of 1, while if the Belyi function is, in particular, clean, then there is a vertex at the end of every edge. In any case, this suggests that the points $\beta^{-1}(1)$ can be chosen as the points marked with $\star$. To complete the marking, we wish to say that the $\circ$ points can be given by $\beta^{-1}(\infty)$, and this is indeed guaranteed by the following technical lemma.

Lemma 3.5. *Let (X, β) be a pre-clean Belyi pair, and let $\mathcal{D}$ be the pre-clean dessin given by the procedure above. Then every open cell of $\mathcal{D}$ is homeomorphic to an open disk, and it contains exactly one preimage of ∞ under β.*

Proof. Let $\beta^{-1}(\infty) = \{x_1, \cdots x_k\}$. Then, since β is a ramified covering, there exists a neighbourhood V of ∞, homeomorphic to the unit open disk $\mathbb{D}$, such that

$$\beta^{-1}(V) = \bigsqcup_{i=1}^{k} U_i,$$

where the union is disjoint and each U_i contains x_i. Consider, now, $V^\circ = V \smallsetminus \{\infty\}$ and $U_i^\circ = U_i \smallsetminus \{x_i\}$. Then $f|_{U_i^\circ} : U_i^\circ \to V^\circ$ is a covering map, and, it is thus isomorphic to $p : \mathbb{D}^\circ \to \mathbb{D}^\circ$ such that $z \mapsto z^{m_i}$, where $\mathbb{D}^\circ$ is the open unit disk with the origin removed. In other words, we have the following diagram

$$\begin{array}{ccc} U_i^\circ & \xrightarrow{f|_{U_i^\circ}} & V^\circ \\ \downarrow & & \downarrow \\ \mathbb{D}^\circ & \xrightarrow[p]{} & \mathbb{D}^\circ \end{array},$$

where the vertical arrows are analytic isomorphisms. By Riemann's removable singularity theorem, we can extend them to isomorphisms sending x_i to 0 and ∞ to 0 respectively. In particular, all U_i are isomorphic to an open disk and $\beta^{-1}(\mathbb{P}^1_{\mathbb{C}} \smallsetminus [0;1])$, where $[0;1]$ is the segment of real numbers, is a disjoint union of open disks, each one containing exactly one preimage of ∞. $\square$

Conversely, given a pre-clean dessin $\mathcal{D}$, to reconstruct the pre-clean Belyi morphism associated, we can consider the set of flags $\mathscr{F}(\mathcal{D})$, where the flags are considered as triangles. This set paves the topological surface X_2 with quadrilaterals formed by two flags: one positively and one negatively oriented, with the $\circ - \bullet$ side in common. Identifying the two $\star$ and their edges, we get something isomorphic to the sphere, that we can identify with $\mathbb{P}^1_{\mathbb{C}}$, making to correspond $\bullet$ with 0, $\star$ with 1 and $\circ$ with ∞. Applying this procedure for any of the described quadrilaterals of $\mathscr{F}(\mathcal{D})$, we get a morphism $\beta : X_2 \to \mathbb{P}^1_{\mathbb{C}}$, ramified only over the points 0, 1 and ∞. We can put on X_2 a Riemann surface structure requiring β to be a rational function.

3.4 Other Descriptions of Dessins

We refocus our attention on the general definition of dessins d'enfants, therefore considering, from now on, those equipped with a bipartite structure. Since we are representing the mark on the vertices using the colors black and white, we will speak about *white vertices* and *black vertices*.

We can give a description of dessins d'enfants using permutations. Consider a dessin $\mathcal{D}$ with N edges, and label them with numbers from 1 to N. Let $X = \{e_1, \cdots, e_N\}$ be the set of labeled edges of $\mathcal{D}$. Then we can define two permutations $\sigma_0, \sigma_1 \in S_N$ as follows. Draw a small circle around each of the white vertices, and set $\sigma_0(i) = j$ if the edges e_i and e_j have a white vertex in common and the edge e_j follows e_i under a counterclockwise rotation. The permutation σ_1 is obtained similarly, considering black vertices. Then, (σ_0, σ_1) is the **permutation pair** representing the dessin $\mathcal{D}$, and the subgroup $G = \langle \sigma_0, \sigma_1 \rangle$ of S_N is called **monodromy group**. Remark that the permutation pair representing a dessin is defined up to (simultaneous) conjugation in S_N, since a conjugation of both the elements corresponds to a relabelling of the edges. Notice, moreover, that the connectedness of $\mathcal{D}$ is equivalent to G being a transitive subgroup of S_N

We define a right action of the monodromy group G on the set X of the labeled edges of $\mathcal{D}$, setting $(e_i)^\alpha = e_{\alpha(i)}$ for $\alpha \in S_N$. In order to do so, we are adopting the convention for which, if $\alpha, \gamma \in G$, the product $\alpha\gamma$ is defined to be the permutation $i \mapsto \gamma(\alpha(i))$, so that the action on the right makes sense, meaning that $(e_i^\alpha)^\gamma = e_i^{\alpha\gamma} = e_{\gamma(\alpha(i))}$.

Now, the cycles of σ_0 are in bijection with the white vertices of $\mathcal{D}$, and the length of each cycle is the degree of the corresponding vertex, where the degree of a vertex is the number of edges having an end on it. The same holds for σ_1 and the black vertices. Moreover, the cycles of $\sigma_1\,\sigma_0$ (or, equivalently, those of $\sigma_0\,\sigma_1$) are in bijection with the faces of $\mathcal{D}$, since the orbit of e_i under

$\sigma_1\,\sigma_0$ enumerates in clockwise direction half of the edges of a face containing e_i, while its orbit under $\sigma_0\,\sigma_1$ enumerates in counterclockwise direction half of the edges of the other face of $\mathcal{D}$ containing e_i. Notice, indeed, that since $\mathcal{D}$ has a bipartite structure, its faces are bounded by an even number of edges. Remark that in these type of considerations we are including also 1-cycles. For instance, if $\sigma_0 = (12)(456) \in S_6$, we will have three white vertices, respectively of degree 1, 2 and 3.

Notice that we can recover the genus of the surface X_2 where $\mathcal{D}$ lies using Euler-Poincaré characteristic:

$$2 - 2g = |\{\text{cycles of } \sigma_0\}| + |\{\text{cycles of } \sigma_1\}| - N + |\{\text{cycles of } \sigma_1\,\sigma_0\}|.$$

As the following result explains, we can do also the converse construction, getting a correspondence.

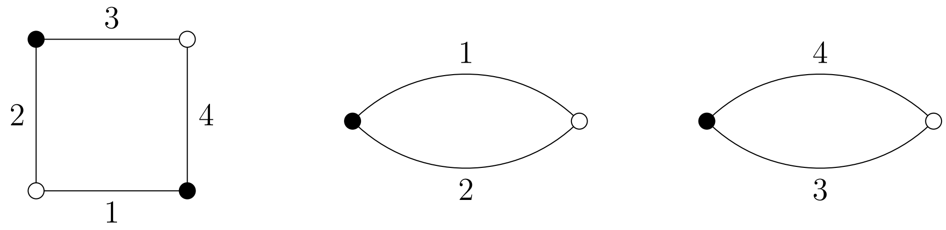
Theorem 3.6. *There is a bijection between the isomorphism classes of dessins d'enfants with N edges and the conjugacy classes of permutation pairs (σ_0, σ_1) in S_N such that $\langle\sigma_0, \sigma_1\rangle$ is a transitive subgroup.*

Proof. We have already explained how to associate a conjugacy class of a permutation pair (σ_0, σ_1) to the isomorphism class of a dessin.

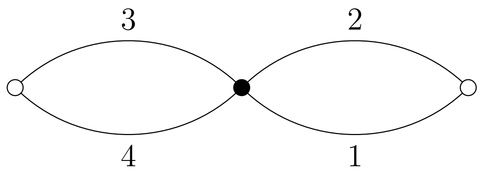
Now, given a permutation pair (σ_0, σ_1) in S_N such that $\langle\sigma_0, \sigma_1\rangle$ is a transitive subgroup, write $\sigma_1\,\sigma_0 = \tau_1 \cdots \tau_k$ for the decomposition in disjoint cycles of $\sigma_1\,\sigma_0$, where τ_j has order n_j, and clearly $n_1 + \cdots + n_j = N$. Construct k disjoint polygons with $2n_1, \cdots, 2n_k$ edges, and assign black and white colours to their vertices. Use the cycles of $\sigma_1\,\sigma_0$ to label half of the edges of each polygon, and use σ_0 to label the remaining edges. Now glue together the edges with same label. The transitivity of $\langle\sigma_0, \sigma_1\rangle$ implies that we get a connected object. We got a compact topological surface with no boundary, and the natural dessin drawn on it is given by the edges of the polygons (which are N, since we had $2N$ edges identified two by two) and their vertices, with the bipartite structure as assigned at the beginning. Notice that the permutation pair corresponding to this dessin is exactly (σ_0, σ_1).

Since the procedure described produces two isomorphic dessins if and only if we start from a pair conjugated to (σ_0, σ_1), this defines a bijection between the set of isomorphism classes of dessins d'enfants with N edges and the conjugacy classes of pairs (σ_0, σ_1) of permutations of S_N generating a transitive subgroup. □

Example. *Consider the permutation pair given by $\sigma_0 = (12)(34)$ and $\sigma_1 = (1234)$ in S_4. Then $\sigma_1\,\sigma_0 = (13)$, so we can compute the resulting genus to be $g = 0$, namely, we will get a dessin in a sphere. Following the procedure described in the theorem, we construct a polygon with four edges and two polygons with two edges, and we label all the edges using $\sigma_1\,\sigma_0$ and σ_0.*



Identifying edges with same labels, we find the following dessin, thought as embedded in the sphere:



Notice that the obtained dessin has permutation pair exactly (σ_0, σ_1).

Consider now the free group on two elements F_2. Given a dessin $\mathcal{D}$ with N edges, with permutation pair (σ_0, σ_1), by the universal property of free groups, mapping one generator of F_2 to σ_0 and the other one to σ_1 gives rise to a unique surjective morphism $F_2 \to \langle \sigma_0, \sigma_1 \rangle$, which allows us to define an action of F_2, again on the right, with same convention as before, on the set of edges of a dessin $\mathcal{D}$. Fixing an edge of $\mathcal{D}$, its stabilizer K has index N in F_2, and a different choice of the edge leads us to a subgroup of F_2 conjugated to K. Notice, furthermore, that starting from a permutation pair conjugated to (σ_0, σ_1) leads also to a conjugated subgroup of K. This allows us to associate the conjugacy class of a finite index subgroup of F_2 to a dessin, and the index in F_2 of such a subgroup corresponds to the number of edges of the dessin.

We can do also the converse operation. Given finite index subgroup K of $F_2 = \langle a, b \rangle$, consider the set of cosets $K \backslash F_2 = \{Ke_1, \cdots, Ke_N\}$. We want to interpret it as the set of edges of some dessin. So, making a and b act by right multiplication, we can construct $\sigma_0, \sigma_1 \in S_N$ in the following way: for $i = 1, \cdots N$, σ_0 is the permutation sending i to j, where j is such that $Ke_i a = Ke_j$, and σ_1 is the one sending i to j, where j is such that $Ke_i b = Ke_j$. We constructed a permutation pair (σ_0, σ_1), which clearly generates a transitive subgroup of S_N. Since starting with a subgroup conjugated to K leads us to a permutation pair conjugated to (σ_0, σ_1), we explained how to associate a dessin to a conjugacy class of a finite index subgroup of F_2. All this proves the following result.

Theorem 3.7. *There is a bijection between the conjugacy classes of finite index subgroups of the free group on two elements and the isomorphism classes* *of dessins d'enfants.*

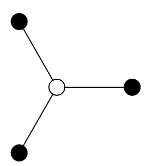
This result generalizes Grothendieck correspondence stated in the previous section. Indeed, the free group on two generators can be seen as the fundamental group of a sphere minus three points. As we did in the proof of Theorem 3.4, we can apply Galois correspondence for coverings, and the extension result provided by Theorem 1.4 to get a bijection between the isomorphism classes of dessins d'enfants and the isomorphism classes of Belyi pairs. The previous section suggests, moreover, how to make clear the correspondence.

Let $\mathcal{D}$ be a dessin d'enfants, we can consider a triangle decomposition of it, which is an analogous of the flag set for pre-clean dessins, choosing a point in every open cell, that will be denoted with $\times$, and joining it with non-intersecting segments to all the vertices of the corresponding cell. With a technique similar to the one explained in the previous section, we can construct a Belyi morphism $\beta : X_2 \to \mathbb{P}^1_{\mathbb{C}}$. Indeed, for any triangle $\bullet - \circ - \times$, we can identify the point $\times$ and the two edges $\bullet - \times$ and $\circ - \times$ with the ones of the triangle with $\bullet - \circ$ common edge. We get a topological sphere, so that, as before, we can send $\bullet$ to 0, $\circ$ to 1 and $\times$ to ∞. Applying this procedure for all pairs of triangles with the $\bullet - \circ$ edge in common, we get the desired morphism $\beta : X_2 \to \mathbb{P}^1_{\mathbb{C}}$.

Conversely, consider a Belyi pair (X, β), and take X_2 as the topological space underlying X, X_1 as $\beta^{-1}([0;1])$, and X_0 as $\beta^{-1}(0) \cup \beta^{-1}(1)$, where the vertices that are sent to 0 are marked as $\bullet$, and the ones sent to 1 are marked as $\circ$. This gives rise to a dessin. If we mark the preimages of ∞ as $\times$, and consider the procedure above concerning the triangle decomposition, we get a Belyi pair isomorphic to (X, β).

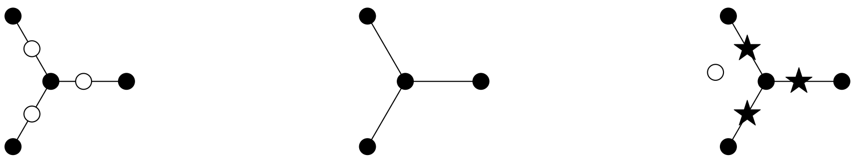
The following example should clarify the relationship between dessins d'enfants (the ones with the bipartite structure, that we can call here *bipartite dessins*, to avoid confusion) and clean dessins discussed in the previous sections.

Example. *Consider the morphism $\alpha : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ such that $x \mapsto 1 - x^3$. This is a Belyi morphism since its critical values are 1 and ∞. Using the general bijection, holding between bipartite dessins d'enfants and Belyi pairs, we can draw the corresponding dessin, where we are marking with $\bullet$ the preimages of 0 and with $\circ$ the preimages of 1. Then, we obtain the following graph, thought as embedded in the Riemann sphere:*



However, α is not a clean Belyi morphism, so, following the procedure of Lemma 3.3, we can obtain out of it a clean one, that is $\beta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defined by $x \mapsto 4(1-x^3)x^3$. Then we have that $\beta^{-1}(0) = \{0, 1, \varepsilon, \varepsilon^2\}$, where ε is a primitive third root of the unity, $\beta^{-1}(1) = \left\{ \frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\varepsilon, \frac{1}{\sqrt[3]{2}}\varepsilon^2 \right\}$ and, finally, $\beta^{-1}(\infty) = \{\infty\}$.

We are going to draw below three pictures. If we apply the general correspondence for bipartite dessins d'enfants to β, we must take as vertices marked with $\bullet$ the preimages of 0 and as vertices marked with $\circ$ the preimages of 1: this is the picture on the left. If we apply the Grothendieck correspondence for clean dessins, we have to consider as set of vertices just the preimages of 0: this is the central picture. We can assign a marking on this clean dessin, marking the preimages of 1 with $\star$ and the preimages of ∞ with $\circ$: this is the picture in the right. Recall that points marked with $\star$ are not considered as vertices, but just as points in the edges. Notice that the following graphs should be thought as embedded in the Riemann sphere, and the $\circ$ point should be thought to be at the infinity point.



It is now clear what is the effect of considering $\beta = 4\alpha(1-\alpha)$ instead of the Belyi morphism α: since $t \mapsto 4t(1-t)$ maps 0 to 0, 1 to 0, ∞ to ∞ and $\frac{1}{2}$ to 1, the bipartite dessin corresponding to β in the general correspondence is obtained starting from the one corresponding to α, marking with $\bullet$ all its vertices, including the $\circ$ ones, and adding vertices marked $\circ$ in the middle of every edge. Furthermore, this explains also how Grothendieck correspondence is related to the general one: a clean dessin corresponds to a bipartite dessin, where we consider as set of vertices only the ones marked with $\bullet$, and we delete the ones marked with $\circ$, coming from the preimages of 1. This operation is indeed allowed by the condition of being clean.

Finally, we can look at Belyi morphisms also with the language of function fields extensions. In particular, if X is defined over $\overline{\mathbb{Q}}$, and $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Belyi morphism, we can look at the corresponding morphism over $\overline{\mathbb{Q}}$, that is, $\widetilde{\beta} : \widetilde{X} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$. Consider, then, the associated extension of function fields $\mathscr{M}(\widetilde{X})/\overline{\mathbb{Q}}(x)$, where $\mathscr{M}(\widetilde{X})$ is the function field of $\widetilde{X}$. We need to understand how to translate the condition about ramification.

Let k be a field, we say that a Galois extension $L/k(x)$ is **not ramified** at 0 if it embeds into the extension $k((x))/k(x)$. For a general $s \in k$, we define $L_s = L \otimes_{k(x)} k(x)$, where $k(x)$ is seen as a $k(x)$-algebra via the map $k(x) \to k(x)$ sending x to $x+s$. Then we say that $L/k(x)$ is not ramified at s if L_s is not ramified at 0. We can give the same definition for ∞, constructing L_∞ with the morphism sending x to x^{-1}. For any extension $L/k(x)$, we say that it does not ramify at s if its Galois closure $\widetilde{L}/k(x)$ does not.

The point is that if $p : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is a ramified covering, with X connected, then $s \in \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is a ramification point if and only if the extension $\mathscr{M}(X)/\,\overline{\mathbb{Q}}(x)$ ramifies at s, as explained above.

Putting all these consideration together, we gained that there exist bijections between the isomorphism classes of dessins d'enfants, the isomorphism classes of Belyi pairs, the conjugacy classes of finite index subgroups of F_2 and the finite extensions $L/\,\overline{\mathbb{Q}}(x)$, up to $\overline{\mathbb{Q}}(x)$-isomorphism, that do not ramify outside of the set $\{0, 1, \infty\}$.

3.5 Regular Dessins

In this section, we are going to present a special type of dessins. Before doing so, we need to define what is an automorphism of a dessin. Firstly, a **bipartite graph automorphism** is a permutation σ of the vertices preserving the bipartite structure, namely, sending black vertices to black vertices, such that two vertices are connected by an edge if and only if their images under σ are connected by an edge. Then, the **automorphism group** $\operatorname{Aut}\mathcal{D}$ of a dessin d'enfants $\mathcal{D}$ is the group of bipartite graph automorphisms that are induced by an orientation-preserving homeomorphism of X_2.

Let $\mathcal{D}$ be a dessin with N edges, and let $X = \{e_1, \cdots, e_N\}$ be the set of the labeled edges of $\mathcal{D}$. Notice that an element ψ of $\operatorname{Aut}\mathcal{D}$ induces a permutation of the edges of the graph $h \in S_N$. So, we can define a left action of $\operatorname{Aut}\mathcal{D}$ on X as ${}^{\psi}e_i = e_{h^{-1}(i)}$. The inverse is taken accordingly to the composition law we defined in the previous section, that sees $\alpha\gamma$ to be the permutation $i \mapsto \gamma(\alpha(i))$. In this way, the action makes sense, since if $\varphi, \psi \in \operatorname{Aut}\mathcal{D}$ induce respectively $g, h \in S_N$, then

$$^{\varphi}(^{\psi}e_i) = {}^{\varphi}e_{h^{-1}(i)} = e_{g^{-1}(h^{-1}(i))} = e_{(h^{-1}g^{-1})(i)} = e_{(gh)^{-1}(i)} = {}^{\varphi\psi}e_i.$$

Lemma 3.8. *The automorphism group of a dessin $\mathcal{D}$ with N edges, with permutation pair (σ_0, σ_1), is isomorphic to the centralizer of $\langle\sigma_0, \sigma_1\rangle$ in S_N.*

Proof. Let $\mathcal{D}$ be a dessin with labeled edges $\{e_1, \cdots, e_N\}$, with permutation pair (σ_0, σ_1), and let $\psi \in \operatorname{Aut}\mathcal{D}$ inducing a permutation $h \in S_N$ of the labeled edges of $\mathcal{D}$. As $e_i^{\sigma_0}$ is the edge next to e_i within a given open cell of $\mathcal{D}$, and ψ comes from an orientation-preserving homeomorphism of X_2, ${}^{\psi}(e_i^{\sigma_0})$ must be the edge next to ${}^{\psi}e_i$, that is to say that the right action of G and the left action of $\operatorname{Aut}\mathcal{D}$ commute. This implies that $e_{h^{-1}(\sigma_0(i))} = {}^{\psi}(e_i^{\sigma_0}) = ({}^{\psi}e_i)^{\sigma_0} = e_{\sigma_0(h^{-1}(i))}$. The same, clearly, holds for σ_1. So, $h \in C_{S_N}(\langle \sigma_0, \sigma_1 \rangle)$.

Conversely, given a permutation $h \in S_N$ commuting with σ_0 and σ_1, consider a triangular decomposition of $\mathcal{D}$, namely, choose a point in every open cell, mark it with $\times$, and connect it with non-intersecting segments to the vertices of the corresponding open cell. In this way, we get a set of triangles paving the surface X_2. We will call T_i^+ the triangle with $\bullet - \circ$ edge labelled with i in which we read $\bullet - \circ - \times$ clockwise, and T_i^- the triangle with $\bullet - \circ$ edge labelled with i in which we read $\circ - \bullet - \times$ clockwise. We claim that we can choose homeomorphisms

$$H_i^{\pm} : T_i^{\pm} \to T_{h(i)}^{\pm}$$

such that they can be glued to form a well-defined orientation-preserving homeomorphism of X_2 whose restriction to $T_i^{\pm}$ coincide with $H_i^{\pm}$.

For instance, gluing $H_i^-, H_{\sigma_0(i)}^-$ and $H_{\sigma_1^{-1}(i)}^-$ requires that

$$\begin{aligned} H_i^+ &= H_i^- && \text{on} \quad T_i^+ \cap T_i^- \\ H_i^+ &= H_{\sigma_0(i)}^- && \text{on} \quad T_i^+ \cap T_{\sigma_0(i)}^- \\ H_i^+ &= H_{\sigma_1^{-1}(i)}^- && \text{on} \quad T_i^+ \cap T_{\sigma_1^{-1}(i)}^- \end{aligned}$$

The first one can be achieved with no special assumption on h, and the remaining ones are obtained since h commutes with σ_0 and σ_1.

Thus from such a $h \in S_N$ we constructed an automorphism of $\mathcal{D}$ inducing h as permutation of the labeled edges of $\mathcal{D}$. $\square$

Lemma 3.9. *Let $\mathcal{D}$ be a dessin with permutation pair (σ_0, σ_1), let X be the set of labeled edges of $\mathcal{D}$. Let $e \in X$, and let $G = \langle \sigma_0, \sigma_1 \rangle$, and $H = \operatorname{Stab}_G(e)$. Then $\operatorname{Aut}(\mathcal{D}) \cong N(H)/H$, where $N(H)$ is the normalizer of H in G.*

Proof. We can identify X with $H \backslash G$, and define, for each $g \in N(H)$, a map $X \to X$ sending $Hx \mapsto Hgx$. This map is well-defined because if $x' = hx$ for some $h \in H$, then

$$Hgx' = Hghx = Hghg^{-1}gx = Hgx$$

since $g \in N(H)$. This map is bijective since its inverse can be constructed using g^{-1}. Moreover, these maps clearly commute with the action of G, so, by Lemma 3.8, $Hx \mapsto Hgx$ gives rise to an automorphism of $\mathcal{D}$.

Conversely, any automorphism ψ is determined by ${}^{\psi}H := Hg$, since ${}^{\psi}(Hx) = {}^{\psi}(H^x) = ({}^{\psi}H)^x = (Hg)^x = Hgx$. The fact that h is well-defined implies that $g \in N(H)$. So, we got a surjective map $N(H) \to \operatorname{Aut}(\mathcal{D})$, whose kernel is clearly H. $\square$

A dessin d'enfants is said to be **regular** if its automorphism group acts transitively on the set of the edges. We can translate this condition into equivalent conditions for the other settings described in the previous section. So, regular dessins $\mathcal{D}$ come from Belyi morphisms that, restricted to the preimage of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ are regular topological coverings, or from finite index normal subgroups of F_2, or from Galois extensions L of $\overline{\mathbb{Q}}(x)$. In the last case, we have that $\mathrm{Gal}(L/\overline{\mathbb{Q}}(x))$ is indeed $\mathrm{Aut}\,\mathcal{D}$.

There are several ways to think of regular dessins, as the following proposition suggests.

Proposition 3.10. *Let $\mathcal{D}$ be a dessin with N edges and permutation pair (σ_0, σ_1), and denote with set of the edges X. Let $x \in X$, and let $G = \langle \sigma_0, \sigma_1 \rangle$, and $H = \mathrm{Stab}_G(x)$. The following are equivalent.*

(1) $\mathcal{D}$ is regular;

(2) G acts freely on X;

(3) H is normal in G;

(4) H is trivial

(5) $G \cong \mathrm{Aut}\,\mathcal{D}$

(6) G and $\mathrm{Aut}\,\mathcal{D}$ have both order N

Proof. (1) $\implies$ (2): Let $x \in X$ and assume $x = x^g$ for some $g \in G$. Then $({}^h x)^g = {}^h(x^g) = {}^h x$ for any $h \in \mathrm{Aut}\,\mathcal{D}$. Since $\mathcal{D}$ is regular, $\mathrm{Aut}\,\mathcal{D}$ acts transitively on X, hence this implies that $g = 1$.
(2) $\implies$ (4) $\implies$ (3) are immediate.
(3) $\implies$ (1): The normality of H in G implies that $N(H)/H = G/H$, and the description of the action of $N(H)/H$ on X of the previous lemma makes clear the transitivity of $\mathrm{Aut}\,\mathcal{D}$ on X.
(4) $\implies$ (5) $\implies$ (6): $H = \{1\}$ implies that $\mathrm{Aut}\,\mathcal{D} \cong N(H)/H \cong G$.
(6) $\implies$ (4): $|\mathrm{Aut}\,\mathcal{D}| = |G| = |X| = N$ implies $|H| = 1$, since $X = H \backslash G$. $\square$

In order to state a Galois correspondence for regular dessins, we need to give the following definition. Given two dessins $\mathcal{D}$ and $\mathcal{D}'$, whose corresponding Belyi morphisms are $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ and $\beta' : X' \to \mathbb{P}^1_{\mathbb{C}}$ respectively, we say that $\mathcal{D}'$ is an **intermediate dessin** of $\mathcal{D}$ if there is a factorization of β as

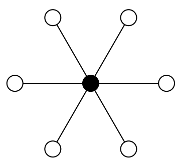
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\beta"'] & X' \arrow[r, "\beta'"] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd},$$

for a continuous mapping $f : X \to X'$.

As finite extensions of $\overline{\mathbb{Q}}(x)$, an intermediate dessin is provided by a tower $\overline{\mathbb{Q}}(x) \subseteq L' \subseteq L$ of finite field extensions. So, the fundamental theorem of Galois theory applied for the finite extensions of $\overline{\mathbb{Q}}(x)$ provides us the following result.

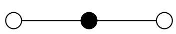
Proposition 3.11. *Let $\mathcal{D}$ be a regular dessin. There is a bijection between the set of isomorphism classes of intermediate dessins of $\mathcal{D}$ and the conjugacy classes of subgroups of of* $\mathrm{Aut}(\mathcal{D})$. *Normal subgroups correspond to regular intermediate dessins.*

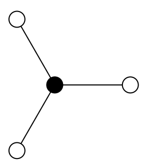
Example. *Consider the permutation pair $\sigma_0 = \mathrm{id}$ and $\sigma_1 = (123456)$ in S_6. Then $G = \langle \sigma_0, \sigma_1 \rangle = \langle (123456) \rangle$ has order six and it is a transitive subgroup of S_6. It can be easily seen that it corresponds to the following dessin $\mathcal{D}$ on the sphere*



This is a regular dessin, and this can be checked in any of the ways suggested by Proposition 3.10. For instance, fixing an edge, its stabilizer under the action of G is clearly trivial, or we can notice that the automorphism group of $\mathcal{D}$ is of order six, so the same as G, since it is generated by the rotation of angle $\frac{\pi}{3}$ around the $\bullet$ vertex.

We can describe the Galois correspondence on this dessin. It is easy to see that it corresponds to the Belyi morphism $\beta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto z^6$. Its automorphism group, as we saw, is cyclic of order 6, so it has two non-trivial subgroups of order 2 and 3, both cyclic. They correspond respectively to the intermediate coverings $z \mapsto z^2$ and $z \mapsto z^3$. The dessins they give rise are respectively the followings





Since G is abelian, every its subgroup is normal, so we expect those dessins to be regular. This is indeed the case, since the very same arguments we used to prove the regularity of $\mathcal{D}$ hold also in those cases.

Chapter 4

The Galois Action

In the previous chapters, we saw that we can associate a dessin d'enfants to any Belyi morphism, and that the existence of a Belyi morphism characterizes the complex algebraic curves that are defined over $\overline{\mathbb{Q}}$. It is a natural idea, then, to let the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}} \,/\, \mathbb{Q})$ act on the set of dessins d'enfants.

This would allow us to see the group $\mathrm{Gal}(\overline{\mathbb{Q}} \,/\, \mathbb{Q})$, which nowadays is still quite mysterious, as a group acting on really simple objects, whose nature is merely topological-combinatorial. This was, indeed, the intuition expressed by Grothendieck in [Gro97].

Therefore, in this chapter, we are going to describe the action of the absolute Galois group on the set of dessins d'enfants. We will prove that this action is faithful, and, more generally, it remains faithful even when restricted to the subfamilies of dessins in Riemann surfaces of genus g, for any fixed $g \geq 0$.

The relevance of the faithfulness of the Galois action on the set of dessins is given by the fact that it will be the key that will allow us to prove an embedding of the Galois group into $\mathrm{Out}(\widehat{F_2})$, the group of outer automorphisms of the profinite completion of the free group on two generators.

The final part is devoted to the description of this embedding. We will construct a family of finite index normal subgroups of F_2. They will allow us to describe $\mathrm{Out}(\widehat{F_2})$ as the inverse limit of the outer automorphism groups of the quotient of F_2 by them. Then, using the different description of dessins we presented in the previous chapter, we will define a morphism from $\mathrm{Gal}(\overline{\mathbb{Q}} \,/\, \mathbb{Q})$ to this group, whose injectivity will be a mere consequence of the faithfulness of the Galois action on the set of dessins d'enfants.

4.1 The Galois Group and Dessins

The absolute Galois group, defined as

$$\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q}) := \{\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}) \text{ such that } \sigma|_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}\},$$

acts on the set of the isomorphism classes complex curves defined over $\overline{\mathbb{Q}}$, just conjugating the coefficients of the polynomials defining the curves. In the same way, given a complex projective curve X defined over $\overline{\mathbb{Q}}$, if we consider a Belyi morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$, the absolute Galois group acts also on the morphism β, again conjugating the coefficients of the polynomials defining β.

We can give a more explicit and satisfying description of this action using the scheme theoretic language. Let $\widetilde{X}$ be a curve over $\mathbb{C}$ defined over $\overline{\mathbb{Q}}$, and let X be the corresponding curve over $\overline{\mathbb{Q}}$ with structure morphism $s : X \to \mathrm{Spec}(\overline{\mathbb{Q}})$. Then, let $\beta : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ be a morphism of varieties over $\overline{\mathbb{Q}}$, and consider $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$. Then σ acts on X by changing its structure morphism, as we saw in the Chapter 2, so that ${}^{\sigma}X$ will be X with structure morphism $\mathrm{Spec}\,\sigma^{-1} \circ s$. Denoting with s' the structure morphism of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, and with $\overline{\sigma} : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ the morphism $\mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} \times \mathrm{Spec}\,\sigma^{-1}$ coming from the universal property of fibre morphism, we have that ${}^{\sigma}\beta : {}^{\sigma}X \to \mathbb{P}^1_{\mathbb{C}}$ is given by $\overline{\sigma} \circ \beta$, as shown in the following diagram:

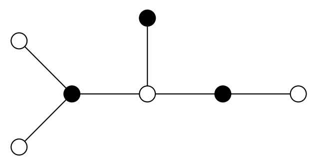
$$\begin{tikzcd}
X \arrow[dr, "s"] \arrow[drr, "\mathrm{Spec}\,\sigma^{-1}\circ s"] \arrow[dd, "\beta"] \arrow[ddr, bend right=60, "{}^{\sigma}\beta"'] & & \\
& \mathrm{Spec}\,\overline{\mathbb{Q}} \arrow[r, "\mathrm{Spec}\,\sigma^{-1}"'] & \mathrm{Spec}\,\overline{\mathbb{Q}} \\
\mathbb{P}^1_{\overline{\mathbb{Q}}} \arrow[ur, "s'"] \arrow[r, "\overline{\sigma}"] & \mathbb{P}^1_{\overline{\mathbb{Q}}} \arrow[ur, "s'"'] &
\end{tikzcd},$$

where the structure morphisms are taken as follows:

$$\begin{tikzcd}
X \arrow[rr, "{}^{\sigma}\beta"] \arrow[dr, "\mathrm{Spec}\,\sigma^{-1}\circ s"'] & & \mathbb{P}^1_{\overline{\mathbb{Q}}} \arrow[dl, "s'"] \\
& \mathrm{Spec}\,\overline{\mathbb{Q}} &
\end{tikzcd}.$$

The absolute Galois group acts also on the set of dessins d'enfants: if $\mathcal{D}_\beta$ is the dessin corresponding to the Belyi morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ then $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ acts on $\mathcal{D}_\beta$ as ${}^{\sigma}\mathcal{D}_\beta = \mathcal{D}_{\sigma\beta}$.

Example. *Consider the following dessin $\mathcal{D}$ in the sphere:*



We want to reconstruct the associated Belyi morphism $F : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. Being a graph with just one open cell and six edges, F must be a polynomial of degree 6. Let us assume that the left black vertex, with degree 3, is placed in 0, the right black one, with degree 2, is placed in 1 and the other black one is placed in some value $a \in \overline{\mathbb{Q}}$ to be determined. Then, $F : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is defined as

$$F(z) = Cz^3(z-1)^2(z-a)$$

for some constant $C \in \overline{\mathbb{Q}}$.

We study the critical points of F by computing its first derivative:

$$F'(z) = z^2(z-1)(6z^2 + (-5a-4)z + 3a).$$

Since $\mathcal{D}$ has a white vertex of degree 3, F must have a ramification point α, distinct from 0 and 1, of ramification index 3, that has to occur as a double root of F'. So, imposing the discriminant of $P(z) = 6z^2 + (-5a-4)z + 3a$ to be zero, we find two values for a: $a_+ = \frac{4}{25}(4+3i)$ and $a_- = \frac{4}{25}(4-3i)$. Each of these two values for a gives rise to a root of P: they are $\alpha_+ = \frac{5a_+ + 4}{12} = \frac{3+i}{5}$ and $\alpha_- = \frac{5a_- + 4}{12} = \frac{3-i}{5}$. Notice that a_+ and a_- are defining two different functions $F_+, F_- : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

We can determinate the constants C_+ and C_- for F_+ and F_- imposing the conditions $F_+(\alpha_+) = F_-(\alpha_-) = 1$. Hence we get constants $C_+ = \frac{-29+278i}{4}$ and $C_- = \frac{-29-278i}{4}$.

So, we constructed two functions $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$

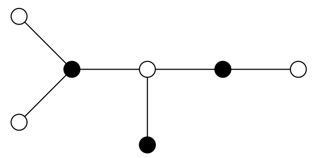
$$F_+(z) = \frac{-29+278i}{4}\left(z - \frac{16+12i}{25}\right) z^3(z-1)^2,$$

$$F_-(z) = \frac{-29-278i}{4}\left(z - \frac{16-12i}{25}\right) z^3(z-1)^2.$$

Notice that, by the conditions we imposed, they have ramification locus respectively $\{0,1,\alpha_+,\infty\}$ and $\{0,1,\alpha_-,\infty\}$, with critical values both $\{0,1,\infty\}$.

So, they are both Belyi morphisms. Moreover, notice that considering the element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ *as the complex conjugation, we get that* ${}^{\sigma}F_{+} = F_{-}$.

The morphism F_{+} *is associated to the dessin presented at the beginning of this example, while* F_{-} *is associated to the dessin*



These two dessins are not equivalent, since there is no orientation-preserving homeomorphism of the sphere sending one to the other. This, in particular, provides us an example of a non-trivial action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ *on a dessin.*

4.2 Faithfulness of the Galois Action

Grothendieck's idea was to study the absolute Galois group, whose nature is still quite mysterious, through its action on the set of dessins d'enfants. To do so, it is useful to make sure that this action is faithful. The easiest proof of this fact concerns dessins in genus 1, which means dessins whose topological surface X_2 has genus 1. Using the j-invariant, it is possible to prove that the Galois action is faithful even in this subfamily of dessins.

Theorem 4.1. *The action of the absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is faithful on the set of dessins in genus* 1.

Proof. Recall that genus 1 Riemann surfaces are classified by the j-invariant, which is a $\mathbb{Q}$-rational expression in the coefficients of the polynomial defining the curve. This means that, given such a curve E and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, we have that $j({}^{\sigma}E) = \sigma(j(E))$.

So, given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, with $\sigma \neq \mathrm{id}$, and $j \in \overline{\mathbb{Q}}$ such that $\sigma(j) \neq j$, to construct a dessin that is not fixed by the action of σ, we consider a genus 1 curve E having j-invariant equal to j. Clearly, E is defined over $\overline{\mathbb{Q}}$ since j is so, and hence it admits a Belyi morphism $\beta : E \to \mathbb{P}^1_{\mathbb{C}}$. This corresponds, up to isomorphism, to a dessin D_β.

The curve ${}^{\sigma}E$ has j-invariant $\sigma(j) \neq j$, so σ cannot act trivially on β nor on its corresponding dessin, since ${}^{\sigma}E \ncong E$. $\square$

More can be said: a proof by Lenestra and reported by Schneps in [Sch94] shows that the Galois action is faithful also in the set of trees, that is, dessins being connected and acyclic graphs. To achieve this result, we fist need two technical lemmas.

Lemma 4.2. *Let F be a polynomial of degree n, and let $d \mid n$. If there exist a monic polynomial H of degree d such that $H(0) = 0$, and a polynomial G such that $F = G \circ H$, then such a polynomial H is unique.*

Proof. Let $deg(G) = m$, so that $n = md$, and write $G = \lambda_m z^m + \cdots + \lambda_0$ and $H = T^d + h_{d-1}T^{d-1} \cdots + h_1 T$. Thus, we have

$$F = \lambda_m H^m + \lambda_{m-1} H^{m-1} + \cdots + \lambda_0.$$

Looking at the right-hand side of this relation, we notice that the terms of degree $n, \cdots, n-d+1$ are contributed entirely from the leading term $\lambda_m H^m$. So, from these terms, one can uniquely get the d highest coefficients of H: the leading term is 1 by assumption, and for $n-d+1 \leq i \leq n-1$ the coefficient of the degree i term in H^m is a polynomial in $h_{i-n+d}, h_{i-n+d+1}, \cdots, h_{d-1}$ which is linear in h_{i-n+d}. Thus, the coefficients $h_{d-1}, \cdots, h_1$ of H are determined, and since by assumption the constant term $h_0 = 0$, H is completely and uniquely determined. $\square$

Lemma 4.3. *Let G, H, $\tilde{G}$ and $\tilde{H}$ be polynomials such that $G \circ H = \tilde{G} \circ \tilde{H}$ and such that $deg(H) = deg(\tilde{H})$. Then there exist two constants c and d such that $\tilde{H} = cH + d$.*

Proof. Let μ and $\tilde{\mu}$ be the leading coefficients of H and $\tilde{H}$ respectively, and let ν and $\tilde{\nu}$ be the constant terms of H/μ and $\tilde{H}/\tilde{\mu}$ respectively. Then

$$G_1 \circ (H/(\mu - \nu)) = G \circ H = \tilde{G} \circ \tilde{H} = G_2 \circ (\tilde{H}/(\tilde{\mu} - \tilde{\nu})),$$

where G_1 is just $G \circ P$, with $P(x) = (x + \nu)\mu$, and G_2 is analogous.

$H/(\mu - \nu)$ and $\tilde{H}/(\tilde{\mu} - \tilde{\nu})$ are both monic with constant coefficient 0, and they have the same degree. Therefore, by the previous lemma, they are equal. To conclude, setting $c = \tilde{\mu}/\mu$ and $d = \tilde{\mu}(\tilde{\nu} - \nu)$, we have $\tilde{H} = cH + d$. $\square$

We can finally state the theorem concerning the faithfulness of the Galois action on the set of trees. Notice that this result is strongly linked to dessins in genus 0. Indeed, if the dessin is a tree, then $X_2 \setminus X_1$ consists of just one open cell, which is homeomorphic, by our definition of a dessin, to an open disk. This is the same as requiring X_2 minus a point to be simply connected, which is possible only if X_2 has genus 0, that is, if X_2 is isomorphic to the Riemann sphere. So, if a dessin is a tree, it must be a dessin in genus 0, and hence the following theorem implies the faithfulness of the Galois action on the set of dessin in genus 0.

Theorem 4.4. *The action of* $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ *on the set of trees is faithful.*

Proof. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, such that $\sigma \neq \mathrm{id}$, and let $\alpha \in \overline{\mathbb{Q}}$ be such that $\gamma := \sigma(\alpha) \neq \alpha$. We are going to construct a tree on which σ acts non-trivially. Notice that if a dessin is a tree, then it is a dessin on the Riemann sphere having only one open cell, that is, having only one preimage of ∞. Thus, we can consider a tree D_β and its corresponding Belyi function $\beta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. As β has just one pole, we may assume, up to isomorphism, that $\beta(\infty) = \infty$. In other words, β is a polynomial.

Notice that D_β and ${}^\sigma D_\beta$ are isomorphic if and only if there exists a Möbius transformation ${}_\sigma f$ such that ${}^\sigma\beta = \beta \circ {}_\sigma f$. Since both β and ${}^\sigma\beta$ have only one pole, at ∞, we deduce that ${}_\sigma f(\infty) = \infty$, that is, ${}_\sigma f(z) = az + b$, with $a \neq 0$, is an affine transformation. Thus, our goal is to show that there really exists a polynomial β such that ${}^\sigma\beta(z) \neq \beta(az + b)$ for any $a, b \in \mathbb{C}$.

Consider a polynomial $p_\alpha \in \mathbb{Q}(\alpha)[z]$ such that $p'_\alpha(z) = z^3(z-1)^2(z-\alpha)$. As shown in the proof of Belyi's Theorem, there exists a polynomial $p \in \mathbb{Q}[z]$ such that $\beta := p \circ p_\alpha$ is a Belyi morphism. Moreover, we have that ${}^\sigma\beta = p \circ p_\gamma$. If ${}^\sigma\beta(z) = \beta(az+b)$, then $p \circ p_\alpha(az+b) = p \circ p_\gamma(z)$, and by Lemma 4.3 there exist some constants c, d such that

$$p_\alpha(az + b) = cp_\gamma(z) + d.$$

Differentiating both sides, we get

$$a(az + b)^3(az + b - 1)^2(az + b - \alpha) = cz^3(z - 1)^2(z - \gamma).$$

Since the roots of these polynomial are the same, with same multiplicity, we get the following conditions on a, b, α and γ:

$$\begin{cases} b = 0 \\ a + b = 1 \\ a\gamma + b = \alpha \end{cases} \implies \begin{cases} b = 0 \\ a = 1 \\ \alpha = \gamma \end{cases}$$

The equality $\alpha = \gamma$ contradicts our assumptions, so it follows that the action of σ is not trivial on the dessin D_β. □

Corollary 4.5. $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ *acts faithfully on the set of dessins in genus* 0.

The faithfulness of the Galois action has been generalised to dessins in genus g, for a fixed $g \geq 2$, by E. Girondo and G. Gonzáles-Diez in [GGD07], where they proved that this result holds even in the subfamily of hyperelliptic curves, as the following theorem states.

Theorem 4.6. *For any fixed $g \geq 2$, the action of the Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is faithful on the set of dessins on hyperelliptic curves of genus g.*

Proof. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, with $\sigma \neq \mathrm{id}$, and $a \in \overline{\mathbb{Q}}$ such that $\sigma(a) \neq a$. Consider then the hyperelliptic curve

$$C_n \coloneqq \{y^2 = (x-1)(x-2)\cdots(x-(2g+1))(x-(a+n))\}.$$

Suppose by contradiction that ${}^{\sigma}C_n \cong C_n$ for all $n \in \mathbb{N}$, then there exists a Möbius transformation $M_n \in \mathbb{P}SL_2(\mathbb{C})$, for every n, such that

$$M_n(\{1, 2, \cdots, 2g+1, a+n\}) = \{1, 2, \cdots, 2g+1, \sigma(a+n)\}.$$

Recalling that given three distinct points $z_1, z_2, z_3 \in \mathbb{P}^1_{\mathbb{C}}$ and other three distinct points $w_1, w_2, w_3 \in \mathbb{P}^1_{\mathbb{C}}$, there exists a unique Möbius transformation sending $z_i \mapsto w_i$, we have the following facts:

(i) Since it maps three rational points to three rational points, $M_n \in \mathbb{P}SL_2(\mathbb{Q})$.

(ii) Since $a+n \notin \mathbb{Q}$, $M_n(\{1, 2, \cdots, 2g+1\}) = \{1, 2, \cdots, 2g+1\}$ by (i).

(iii) $M_n(a+n) = \sigma(a+n) = \sigma(a) + n$, by (ii).

(iv) The set $\{M_n \mid n \in \mathbb{N}\}$ is finite, since by (ii) its cardinality is upper bounded by the number of permutations of $2g+1$ elements.

In particular, there exist three different numbers $p, q, r \in \mathbb{N}$ such that $M_p = M_q = M_r$. Therefore, we have that

$$\begin{cases} M_p(a+p) = \sigma(a) + p \\ M_p(a+q) = M_q(a+q) = \sigma(a) + q \\ M_p(a+r) = M_r(a+r) = \sigma(a) + r \end{cases} \Longrightarrow \begin{cases} M_p(a+p) - \sigma(a) = p \\ M_p(a+q) - \sigma(a) = q \\ M_p(a+r) - \sigma(a) = r \end{cases}$$

Consider, now, the Möbius transformation $\widetilde{M}(z) \coloneqq M_p(a+z) - \sigma(a)$. Since $\widetilde{M}(p) = M_p(a+p) - \sigma(a) = \sigma(a) + p - \sigma(a) = p$, and, similarly, $\widetilde{M}(q) = q$ and $\widetilde{M}(r) = r$, it follows that $\widetilde{M} = \mathrm{id}$, since it fixes three points, and so $M_p(a+z) = z + \sigma(a)$. Computing it in $z - a$, we deduce that $M_p(z) = z + \sigma(a) - a$.

Using this relation, we get that $M_p(N) = N + \sigma(a) - a$, for any N in the set $\{1, 2, \cdots, 2g+1\}$. This yields

$$\sigma(a) - a = M_p(1) - 1 = M_p(2) - 2 = \cdots = M_p(2g+1) - (2g+1).$$

But, since $M_p(N) \in \{1, 2, \cdots, 2g+1\}$, we deduce both $M_p(1) - 1 \geq 0$ and $M_p(2g+1) - (2g+1) \leq 0$, meaning $M_p(N) - N = 0$ for any $N \in \{1, 2, \cdots, 2g+1\}$, and hence $\sigma(a) = a$, which contradicts our assumptions. $\square$

4.3 An Embedding of the Galois Group

In this section, we are going to define an injective group morphism from $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ to the group of outer automorphisms of the profinite completion of the free group on two generators. In order to do so, we need to express such a profinite group as an inverse limit of suitable finite groups, so that we will be able to handle it better. The construction that follows suits our needs.

Let $F_2 := \langle \sigma, \alpha \rangle$ be the free group on two generators. For any group G, denote as $G^{(n)}$ the intersection of all normal subgroups of G whose index is less or equal to n. Define, finally, the group

$$H_n := F_2/F_2^{(n)}.$$

Lemma 4.7. *The group H_n is finite, and $H_n^{(n)} = \{1\}$.*

Proof. Firstly, notice that there exist only a finite number of normal subgroups of F_2 with index $\leq n$, since each of them is the kernel of a morphism $F_2 \to G$, with G a group of cardinality $|G| \leq n$, and there are only finitely many groups of cardinality n and finitely many morphisms from F_2 to G. So, it suffices to show that if N and M are normal subgroup of F_2 of index $\leq n$, then $|F_2 : N \cap M| < \infty$. This is because

$$\begin{aligned} |F_2 : N \cap M| =& |F_2 : N||N : N \cap M| \leq |F_2 : N||NM : M| \leq \\ \leq& |F_2 : N||F_2 : M| < \infty. \end{aligned}$$

So, $F_2^{(n)}$ has finite index in F_2 and hence H_n is finite.

The fact that $H_n^{(n)} = \{1\}$ follows by the correspondence theorem for groups. Indeed, N is a normal subgroup of F_2 such that $|F_2 : N| \leq n$ if and only if $N/F_2^{(n)}$ is normal in H_n, and $|H_n : N/F_2^{(n)}| \leq n$. Hence, the intersection of all normal subgroups of H_n whose index id $\leq n$ is trivial. □

Proposition 4.8. *With the above notation, we have the following facts:*

(i) *For any group G of order $|G| \leq n$ and for any $g_1, g_2 \in G$, there exists a homomorphism $H_n \to G$ sending σ to g_1 and α to g_2.*

(ii) *If g_1 and g_2 are generators of a group G such that $G^{(n)} = \{1\}$, then there exists a surjective map $H_n \to G$ sending σ to g_1 and α to g_2.*

(iii) *If h_1 and h_2 are generators of H_n, then there exists an automorphism of H_n sending $\overline{\sigma}$ to h_1 and $\overline{\alpha}$ to h_2, where $\overline{\sigma}$ and $\overline{\alpha}$ are the images of σ and α, generators of F_2, in H_n.*

Proof. (i) The mapping $\sigma \mapsto g_1$ and $\alpha \mapsto g_2$ gives rise, by the universal property of free groups, to a group morphism $\varphi : F_2 \to G$.

Moreover, we have that

$$|F_2 : \ker \varphi| = |\operatorname{im} \varphi| \leq |G| \leq n,$$

implying, since $\ker \varphi$ is normal in F_2, that

$$F_2^{(n)} = \bigcap_{\substack{|F_2:N| \leq n \\ N \trianglelefteq F_2}} N \subseteq \ker \varphi.$$

Thus, φ descends to a homomorphism $H_n \to G$.

(ii) The mapping $\sigma \mapsto g_1$ and $\alpha \mapsto g_2$ gives rise, by the universal property of free groups, to a homomorphism $\varphi : F_2 \to G$, which is surjective since G is generated by g_1 and g_2.

Consider now a normal subgroup N of G such that $|G : N| \leq n$, then we have that $\varphi^{-1}(N)$ is normal in F_2, and also $|F_2 : \varphi^{-1}(N)| \leq n$. Thus, we have

$$\ker \varphi = \varphi^{-1}(1) = \varphi^{1}(G^{(n)}) = \bigcap_{\substack{|G:N| \leq n \\ N \trianglelefteq G}} \varphi^{-1}(N) \supseteq \bigcap_{\substack{|F_2:M| \leq n \\ M \trianglelefteq F_2}} M = F_2^{(n)}.$$

Hence, we get the desired surjective morphism $H_n \to G$ by the universal property of the quotient.

(iii) Since $H_n^{(n)} = \{1\}$, we can apply (ii) to get the existence of a surjective morphism $H_n \to H_n$ sending $\overline{\sigma}$ to h_1 and $\overline{\alpha}$ to h_2, which is bijective since H_n is finite.

□

Notice that, since $F_2^{(n+1)} \leq F_2^{(n)}$ for any $n \geq 1$, we have surjective maps $\varphi_{n+1} : H_{n+1} \to H_n$ obtained from the canonical projection $F_2 \to H_n$ using the universal property of the quotient. These maps maps allow us to see $(H_n)_{n \geq 1}$ as an inverse system, whose inverse limit will be some profinite group.

Recall that a **profinite group** is a compact and totally disconnected topological group, or, equivalently, it is a topological group isomorphic to the inverse limit of an inverse system of discrete finite groups. Given a group G, its **profinite completion** $\widehat{G}$ is the inverse limit $\lim G/N$, where N runs over all the normal subgroups of finite index of G. The natural morphism $j : G \to \widehat{G}$ is continuous and it has dense image in $\widehat{G}$. Moreover, it is injective if and only if G is residually finite, meaning that the intersection of all normal subgroup of finite index of G is trivial. Notice that this is the case for F_2, since free groups are always residually finite.

Lemma 4.9. *The inverse limit* $\lim_n H_n$ *is isomorphic to* $\widehat{F}_2$*, the profinite completion of* F_2.

Proof. By definition, the profinite completion of F_2 is

$$\widehat{F}_2 = \lim F_2/N,$$

where the inverse limit runs over all normal subgroups N of F_2 of finite index. Since each N contains some $F_2^{(n)}$ for n large enough, the family $(F_2^{(n)})_n$ is final in the inverse limit, implying the result. $\square$

Notice that the kernel of the map $\varphi_{n+1} : H_{n+1} \to H_n$ is $H_{n+1}^{(n)}$, which is a characteristic subgroup of H_{n+1}, meaning that $\alpha(H_{n+1}^{(n)}) = H_{n+1}^{(n)}$ for any automorphism $\alpha \in \mathrm{Aut}(H_{n+1})$. This implies that the map $\Lambda_{n+1} : \mathrm{Aut}(H_{n+1}) \to \mathrm{Aut}(H_n)$, defined by $\psi \mapsto \varphi_{n+1} \circ \psi \circ \varphi_{n+1}^{-1}$, is well-defined, since, for any $h \in H_n$ and for a $y \in \varphi_{n+1}^{-1}(h)$ we have

$$h \xmapsto{\varphi_{n+1}^{-1}} y \ker \varphi_{n+1} \overset{\psi}{\mapsto} \psi(y) \ker \varphi_{n+1} \xmapsto{\varphi_{n+1}} \varphi_{n+1}(\psi(y)).$$

Notice, indeed, that the choice of a different $y' \in \varphi_{n+1}^{-1}(h)$ does not change the image under $\varphi_{n+1} \circ \psi \circ \varphi_{n+1}^{-1}$. Furthermore, if ψ is the conjugation by $g \in H_{n+1}$, then $\Lambda_{n+1}(\psi) = \varphi_{n+1} \circ \psi \circ \varphi_{n+1}^{-1}$ is the conjugation by $\varphi_{n+1}(g)$, so the map $\mathrm{Aut}(H_{n+1}) \to \mathrm{Aut}(H_n)$ gives rise to a map $\mathrm{Out}(H_{n+1}) \to \mathrm{Out}(H_n)$.

We are now going to study the group

$$\mathrm{Out}(\widehat{F}_2) \coloneqq \mathrm{Aut}_c(\widehat{F}_2)/Inn(\widehat{F}_2),$$

where $\mathrm{Aut}_c(\widehat{F}_2)$ is the group of continuous automorphisms of $\widehat{F}_2$.

Proposition 4.10. *There is an isomorphism*

$$\mathrm{Out}(\widehat{F}_2) \cong \lim_n \mathrm{Out}(H_n).$$

Proof. We use the identification of Lemma 4.9 to construct the map

$$\lim_n \mathrm{Aut}(H_n) \to \mathrm{Aut}_c(\lim_n H_n) = \mathrm{Aut}_c(\widehat{F}_2),$$

which associates the automorphism $(h_n)_n \mapsto (\psi_n(h_n))_n$ to any element $(\psi_n)_n$ of $\lim_n \mathrm{Aut}(H_n)$. This morphism is well-defined since

$$\varphi_n(\psi_n(h_n)) = \psi_{n-1}(\varphi_n(h_n)) = \psi_{n-1}(h_{n-1}).$$

Conversely, we construct the map going the other way,

$$\mathrm{Aut}_c(\widehat{F}_2) \to \lim_n \mathrm{Aut}(H_n).$$

The image of $\phi \in \mathrm{Aut}_c(\widehat{F}_2)$ under this map is defined to be $(\psi_n)_n$, where $\psi_n \in \mathrm{Aut}(H_n)$ is defined as $\psi_n(h + F_2^{(n)}) = j^{-1}(\phi(j(h))) + F_2^{(n)}$ for any $h + F_2^{(n)} \in H_n$, with $j : F_2 \to \widehat{F}_2$ the natural morphism of the profinite completion. This map is well-defined, since the closure of $F_2^{(n)}$ in $\widehat{F}_2$, which is the kernel of $\widehat{F}_2 \to H_n$ coming from the canonical projection $F_2 \to H_n$ by the universal property of the profinite completion, is preserved by all continuous automorphisms of $\widehat{F}_2$.
These maps are inverses to one another.

Next, we study the corresponding map

$$\pi : \lim_n \mathrm{Aut}(H_n) \to \lim_n \mathrm{Out}(H_n).$$

We show it is surjective. Consider an element $(\gamma_n)_n$ of $\lim_n \mathrm{Out}(H_n)$, and choose a representative $\widetilde{\gamma}_n \in \mathrm{Aut}(H_n)$ of $\gamma_n \in \mathrm{Out}(H_n)$ for any n. It may not be the case that $\widetilde{\gamma}_{n+1}$ maps to $\widetilde{\gamma}_n$ under $\Lambda_{n+1} : \mathrm{Aut}(H_{n+1}) \to \mathrm{Aut}(H_n)$, but the two differ by an inner automorphism of H_n, that is, $\Lambda_{n+1}(\widetilde{\gamma}_{n+1}) = \widetilde{\gamma}_n \circ \iota_g$, with ι_g conjugation by $g \in H_n$. But since $\varphi_{n+1} : H_{n+1} \to H_n$ is surjective, there exists an element $h \in H_{n+1}$ such that $\varphi_{n+1}(h) = g$, so that $\Lambda_{n+1}(\widetilde{\gamma}_{n+1} \circ \iota_h) = \widetilde{\gamma}_n$, where ι_h is the conjugation by $h \in H_{n+1}$. So we can replace $\widetilde{\gamma}_{n+1}$ with $\widetilde{\gamma}_{n+1} \circ \iota_h$. Doing so for every n, we get an element $(\widetilde{\gamma}_n)_n \in \lim_n \mathrm{Aut}(H_n)$ that is mapped in $(\gamma_n)_n$, proving surjectivity.

We show that $\ker \pi = \mathrm{Inn}(\widehat{F}_2)$. We rely on a result by Jarden (see [Jar80]), stating that any automorphism of $\widehat{F}_2$ fixing all open normal subgroups is inner. We will also use the fact that normal subgroups of finite index in F_2 are in bijection with open, normal subgroups of $\widehat{F}_2$, which are automatically closed and of finite index, under the closure operation $N \mapsto \bar{N}$: in fact, the quotient map $F_2 \to F_2/N$ extends to a map $\widehat{F}_2 \to F_2/N$ whose kernel is $\bar{N}$.

Let us therefore consider an element $\beta \in \ker \pi$: it must satisfy Jarden's assumptions, since each open, normal subgroup of $\widehat{F}_2$ is the closure $\bar{N}$ of a normal subgroup N of finite index in F_2, and each such subgroup contains some $F_2^{(n)}$ for n large enough. Hence, if β induces an inner automorphism of H_n, it must fix $\bar{N}$. We get that $\beta \in \mathrm{Inn}(\widehat{F}_2)$, and since clearly $\mathrm{Inn}(\widehat{F}_2)$ is contained in $\ker \pi$, we get that $\ker \pi = \mathrm{Inn}(\widehat{F}_2)$, and the thesis follows. □

We are going to use the described isomorphism to construct a morphism from $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ to $\mathrm{Out}(\widehat{F}_2)$. Make use of the axiom of choice to select an algebraic closure Ω of $\overline{\mathbb{Q}}(x)$.

The finite group H_n, being a quotient of F_2 by a finite index, normal subgroup, gives rise to a regular dessin, which can be interpreted, as we saw in the previous chapter, as a finite Galois extension $L_n/\,\overline{\mathbb{Q}}(x)$, which can be chosen with $L_n \subseteq \Omega$, whose Galois group is $\mathrm{Gal}(L_n/\,\overline{\mathbb{Q}}(x)) = H_n$. Notice that

L_n is the unique subfield of Ω with such proprieties. Indeed, if $L'_n \subseteq \Omega$ were another one, we would have an isomorphism of field extension $L_n \to L'_n$, but by basic Galois theory any map $L_n \to \Omega$ has its values in L_n, hence $L_n = L'_n$. Similarly, notice that if $L/\overline{\mathbb{Q}}(x)$ is any extension isomorphic to $L_n/\overline{\mathbb{Q}}(x)$, then any two isomorphisms $L_n \to L$ differ by an element of $\mathrm{Gal}(L_n/\overline{\mathbb{Q}}(x))$.

Now, let $\lambda \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and extend it to a map $\overline{\mathbb{Q}}(x) \to \overline{\mathbb{Q}}(x)$ fixing x. We will denote also this map with λ. Then consider

$$^{\lambda}L = L \otimes_{\lambda} \overline{\mathbb{Q}}(x),$$

which is the tensor product $L \otimes_{\overline{\mathbb{Q}}(x)} \overline{\mathbb{Q}}(x)$, where $\overline{\mathbb{Q}}(x)$ is seen as a $\overline{\mathbb{Q}}(x)$-module via the morphism λ. We turn $^{\lambda}L$ into a $\overline{\mathbb{Q}}(x)$-algebra via the morphism $t \mapsto 1 \otimes t$. This describes the action of $\lambda \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the finite field extension L of $\overline{\mathbb{Q}}(x)$. Remark that there is the following commutative diagram

$$\begin{tikzcd}
\overline{\mathbb{Q}}(x) \arrow[r, "\lambda"] \arrow[d] & \overline{\mathbb{Q}}(x) \arrow[d] \\
L \arrow[r, "\Lambda"'] & {}^{\lambda}L
\end{tikzcd},$$

where $\Lambda : L = L \otimes_{\mathrm{id}} \overline{\mathbb{Q}}(x) \to L \otimes_{\lambda} \overline{\mathbb{Q}}(x) = {}^{\lambda}L$ is defined by $y \otimes s \mapsto y \otimes \lambda(s)$. It follows that there is an isomorphism

$$\lambda^* : \mathrm{Gal}(L/\overline{\mathbb{Q}}(x)) \xrightarrow{\sim} \mathrm{Gal}({}^{\lambda}L/\overline{\mathbb{Q}}(x)),$$

given by $\sigma \mapsto \Lambda \circ \sigma \circ \Lambda^{-1}$.

Applying this to our case, we get that $^{\lambda}L_n$ is a regular dessin, since L_n is so, and it corresponds to the choice of two new generators for the group H_n. Nevertheless, by Proposition 4.8, there is an automorphism of H_n sending these new generators to the previous ones, meaning that L_n and $^{\lambda}L_n$ are isomorphic. Thus, there exists an isomorphism $\iota : L_n \to {}^{\lambda}L_n$ of extensions of $\overline{\mathbb{Q}}(x)$, defined up to precomposition by an element of $\mathrm{Gal}(L_n/\overline{\mathbb{Q}}(x)) = H_n$.

Let now $h \in H_n = \mathrm{Gal}(L_n/\overline{\mathbb{Q}}(x))$, and consider the following diagram, which does not commute

$$\begin{tikzcd}
L_n \arrow[r, "\iota"] \arrow[d, "h"'] & {}^{\lambda}L_n \arrow[d, "\lambda^*(h)"] \\
L_n \arrow[r, "\iota"'] & {}^{\lambda}L_n
\end{tikzcd}.$$

The map $\iota^{-1} \circ \lambda^*(h) \circ \iota$ depends on the choice of ι, which was defined up to precomposition by an element of H_n. Thus the mapping $h \mapsto \iota^{-1} \circ \lambda^*(h) \circ \iota$ induces a well-defined element in $\mathrm{Out}(H_n)$, depending only on $\lambda \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$.

Using this association we can define a morphism $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \to \mathrm{Out}(\widehat{F}_2)$, whose injectivity is essentially a consequence of the faithfulness of the Galois action on the set of dessins d'enfants.

Theorem 4.11. *There exists an injective homomorphism of groups*

$$\Gamma : \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \to \lim_{n} \mathrm{Out}(H_n) \cong \mathrm{Out}(\widehat{F}_2).$$

Proof. We have just explained how to associate to $\lambda \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ an element in $\mathrm{Out}(H_n)$. We need to check that it gives indeed a homomorphism

$$\Gamma_n : \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \to \mathrm{Out}(H_n),$$

for each fixed n. Assume that $\Gamma_n(\lambda_i)$ is represented by $h \mapsto \iota_i^{-1} \circ \lambda_i^*(h) \circ \iota_i$ for $i = 1, 2$. Then $\Gamma_n(\lambda_1) \circ \Gamma_n(\lambda_2)$ is represented by their composition, which is

$$\begin{aligned} h &\mapsto \iota_1^{-1} \circ \Lambda_1 \circ \iota_2^{-1} \circ \Lambda_2 \circ h \circ \Lambda_2^{-1} \circ \iota_2 \circ \Lambda_1^{-1} \circ \iota_1 \\ &= \iota_1^{-1} \circ \Lambda_1 \circ \iota_2^{-1} \circ \widetilde{\Lambda}^{-1} \circ \widetilde{\Lambda} \circ \Lambda_2 \circ h \circ \Lambda_2^{-1} \circ \widetilde{\Lambda}^{-1} \circ \widetilde{\Lambda} \circ \iota_2 \circ \Lambda_1^{-1} \circ \iota_1 \\ &= \iota_1^{-1} \circ {}^{\lambda_1}\iota_2{}^{-1} \circ (\lambda_1\lambda_2)^*(h) \circ {}^{\lambda_1}\iota_2 \circ \iota_1, \end{aligned}$$

where $\widetilde{\Lambda}$ is the morphism

$${}^{\lambda_2}L = L \otimes_{\lambda_2} \overline{\mathbb{Q}}(x) \to L \otimes_{\lambda_1\lambda_2} \overline{\mathbb{Q}}(x) = {}^{\lambda_1\lambda_2}L$$

given by $y \otimes s \mapsto y \otimes \lambda_1(s)$, and ${}^{\lambda_1}\iota_2 = \widetilde{\Lambda} \circ \iota_2 \circ \Lambda_1^{-1}$. Now, since ${}^{\lambda_1}\iota_2 \circ \iota_1$ is an isomorphism $L_n \xrightarrow{\sim} {}^{\lambda_1\lambda_2}L_n$, we see that this automorphism represents $\Gamma_n(\lambda_1\lambda_2)$. Hence $\Gamma_n(\lambda_1\lambda_2) = \Gamma_n(\lambda_1) \circ \Gamma_n(\lambda_2)$, as wished.

We need to check the compatibility of the morphisms Γ_n with the maps $\mathrm{Out}(H_{n+1}) \to \mathrm{Out}(H_n)$. We have that $L_n \subseteq L_{n+1}$, and, in the Galois correspondence, L_n corresponds to $H_{n+1}^{(n)}$, which is a characteristic subgroup of H_{n+1}. Thus, any isomorphism $L_{n+1} \to {}^{\lambda}L_{n+1}$ must send L_n onto ${}^{\lambda}L_n$. Moreover, $\lambda^* : \mathrm{Gal}(L_{n+1}/\overline{\mathbb{Q}}(x)) \to \mathrm{Gal}({}^{\lambda}L_{n+1}/\overline{\mathbb{Q}}(x))$ sends $\mathrm{Gal}(L_n/\overline{\mathbb{Q}}(x))$ to $\mathrm{Gal}({}^{\lambda}L_{n+1}/\overline{\mathbb{Q}}(x))$, so we get the desired compatibility.

Finally, we prove that Γ is injective. Since, as we saw, the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ on the set of dessins is faithful, it suffices to show that whenever $\Gamma(\lambda) = 1$, the action of λ on dessins is trivial. So, pick a dessin as a finite extension $L/\overline{\mathbb{Q}}(x)$. It is contained in L_n for some n, and it corresponds, via

Galois correspondence, to the subgroup $K = \mathrm{Gal}(L_n/L)$ of H_n. Furthermore, since $\lambda^*(K) = \mathrm{Gal}({}^\lambda L_n/{}^\lambda L)$, the subfield ${}^\lambda L$ of ${}^\lambda L_n$ corresponds to the subgroup $\lambda^*(K)$. The condition $\Gamma(\lambda) = 1$ means that $h = \iota^{-1} \circ \lambda^*(h) \circ \iota$, so if we identify ${}^\lambda L_n$ with L_n via the isomorphism ι, the map λ^* becomes the conjugation by a certain element of H_n. Hence, the subfield ${}^\lambda L$ corresponds to $\lambda^*(K)$, which is a conjugate of K, and thus, again by Galois correspondence, it is isomorphic to L. $\square$

One may ask whether this morphism is also surjective, giving, so, an isomorphism concerning the Galois group. This is not true. In fact, it is known that the image of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ in $\mathrm{Out}(\widehat{F_2})$ lies in a proper subgroup $\widehat{\mathcal{GT}}$, called the Grothendieck-Teichmüller group (see, for instance, [Iha94]). It is still an open question whether $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ is isomorphic to $\widehat{\mathcal{GT}}$ or not.

Bibliography

[Mas77] William S. Massey. *Algebraic Topology: An Introduction*. Springer, 1977. ISBN: 978-0-387-90271-5.

[Bel80] Gennadiĭ Vladimirovich Belyĭ. "On Galois extensions of a maximal cyclotomic field". In: *Mathematics of the USSR-Izvestiya* 14 (2 1980), pp. 247–256.

[Jar80] Moshe Jarden. "Normal automorphisms of free profinite groups". In: *Journal of Algebra* 62 (1 1980), pp. 118–123.

[For81] Otto Forster. *Lectures on Riemann Surfaces*. Springer, 1981. ISBN: 978-0-387-90617-1.

[Iha94] Yasutaka Ihara. "On the embedding of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ into $\widehat{\mathcal{GT}}$". In: *The Grothendieck Theory of Dessins d'Enfants*. Ed. by L. Schneps. Vol. London Math. Soc. Lecture Notes 200. Cambridge Univ. Press, 1994, pp. 289–306.

[Sch94] Leila Schneps. "Dessin d'Enfants on the Riemann Sphere". In: *The Grothendieck Theory of Dessins d'Enfants*. Ed. by L. Schneps. Vol. London Math. Soc. Lecture Notes 200. Cambridge Univ. Press, 1994, pp. 47–78.

[Ful95] William Fulton. *Algebraic Topology. A First Course*. Springer, 1995. ISBN: 978-0-387-94327-5.

[Mir95] Rick Miranda. *Algebraic Curves and Riemann Surfaces*. American Mathematical Society, 1995. ISBN: 978-0-8218-0268-7.

[Gro97] Alexander Grothendieck. "Esquisse d'un Programme". In: *Geometric Galois Actions*. Ed. by L. Schneps and P. Lochak. Vol. London Math. Soc. Lecture Notes 242. Cambridge Univ. Press, 1997, pp. 5–48.

[Kö04] Bernhard Köck. "Belyi's Theorem Revisited". In: *Beiträge zur Algebra und Geometrie* 45 (1 2004), pp. 253–265.

[GGD07] Ernesto Girondo and Gabino González-Diez. "A Note on the Action of the Absolute Galois Group on Dessins". In: *Bulletin of the London Mathematical Society* 39 (5 2007), pp. 721–723.

[GGD12] Ernesto Girondo and Gabino Gonzáles-Diez. *Introduction to Compact Riemann Surfaces and Dessins d'Enfants*. Cambridge Univ. Press, 2012. ISBN: 978-0-521-74022-7.

[Gui14] Pierre Guillot. "An elementary approach to dessins d'enfants and the Grothendieck-Teichmüller group". In: *L'Enseignement Mathématique* 60 (3 2014), pp. 293–375.

[GW16] A. Jones Gareth and Jürgen Wolfart. *Dessins d'Enfants on Riemann Surfaces*. Springer, 2016. ISBN: 978-3-319-24709-0.