ALGANT MASTER THESIS

Fano Manifolds as Mori Dream Spaces

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Abstract

The aim of this thesis is to study Fano manifolds as Mori Dream Spaces and, in partiular, to find bounds on the Picard number. We will mainly focus on Fano manifolds admitting a divisorial contraction sending a divisor into a curve.

Introduction

One of the main goals of algebraic geometry is to classify projective varieties over the complex numbers up to birational equivalence.

One approach in this direction is given by the Mori Theory, also called the Minimal Model Program (or shortly MMP). The MMP started developing in the '80s as an attempt to generalise the work of the Italian school.

The first advance in this direction was given by Mori in [Mor82], who introduced a new approach to the study of projective 3-folds. He studied projective manifolds of dimension 3 with non-nef canonical divisor, i.e. X a 3-fold with canonical divisor K_X such that there exists a curve with negative intersection with K_X.

We recall the Cone Theorem (Theorem (1.3.1)) and the Contraction Theorem (Theorem (1.3.8)) from the results raised from Mori's work. The Cone Theorem describes the Mori cone, i.e. the closure of the cone spanned by numerical classes of effective one-cycles. In particular, it states that the negative part of the cone with respect to the canonical divisor is locally polyhedral, generated by at most countably many rays and that every ray is given by the class of a rational curve. The Contraction Theorem allows us to associate with each $(-K_X)$-negative ray, or $(-K_X)$-negative face of the Mori cone, a morphism onto a normal projective variety with connected fibers, called extremal contraction.

The variety obtained after an extremal contraction of a variety of dimension ≥ 3 may present some singularities. Mori's work was then generalized to varieties of higher dimensions, allowing some mild of singularities. This was a joint work of several authors, among them, we recall Kawamata [Kaw84], Shokurov [Sho86], and Reid [Rei83].

This allowed us to find a good representative inside the birational class of a variety called the Minimal Model, i.e. a variety with a numerically effective canonical divisor.

Some problems arise; the exceptional locus of the ray contracted may be of codimension at least 2, i.e. a small ray. In this case, the variety obtained by contracting this ray is not even $\mathbb{Q}$-factorial. To solve this problem one can use the notion of "flip" (Definition (1.3.17)). This gives a new variety on which it is possible to continue the MMP. The existence and the termination of flips (i.e. there are no infinite sequences of flips) are the key points for carrying on the MMP.

The problem of the existence of flips has been settled in any dimension in [BCHM10], but termination is still an open problem.

In [HK00], to overcome the problem of termination of flips, a new category of varieties has been introduced: the Mori Dreams Spaces, or shortly MDS (Definition (2.1)). MDS are varieties with nice properties with respect to the Mori Theory. In particular, a Mori program exists for every divisor [HK00].

In [BCHM10] it has been proven that a Fano manifold of any dimension is a MDS. This allowed us to apply results that hold for MDS to Fano manifolds; in particular, given a Fano manifold and a divisor on it, there is always a MMP for that divisor.

This result enables us to study Fano varieties from a new perspective. An example of this new approach can be found in the work of Casagrande [Cas09] and [Cas12b], where the author focuses on Mori programs for prime divisors. This allows us to obtain information on the geometry of the starting Fano n-folds and bounds on the Picard number. Some of these results can be found in [Cas09] and in [Cas12b].

Here we go into more details about this essay. In the first section we will briefly review some background results that will be used frequently throughout the thesis. We will therefore start with some preliminaries related to intersection theory, singularities, and some first results in Mori Theory, such as the Cone and the Contraction Theorem. We will then recall some results on the theory of extremal rays and associated contractions. Moreover, we will go through some results by Wiśniewski [Wiś91] for projective manifolds such as Ionescu-Wiśniewski inequality (Theorem (1.3.20)). Finally, we will conclude this introductory section by exposing some results on Fano manifolds.

In the second section, we will introduce the notion of Mori Dream Space, and we will state some main features with respect to the Mori Theory. We will start by defining a Mori program for a divisor and by recalling that for a MDS X it always exists a MMP for every divisor in X.

We will later look at a Mori program for $-D \subset X$, where D is a prime divisor and X is a MDS. This type of Mori program was first introduced in [Cas09] and studied also in [Cas12b]. This approach is somewhat opposite to the classical one, since at every step we will consider a ray with positive intersection with the divisor. We will observe some properties, such as the fact that every Mori program for $-D$ in which $D \subset X$ is a prime divisor in X ends with a contraction of fiber type.

In the remaining part of the section, we will focus on Fano manifolds by viewing them as MDS. As a result, for each divisor $D \subset X$, we will consider a Mori program. Moreover, we will prove that for a Fano manifold, there is a suitable choice of extremal rays involved in the MMP whose contractions have positive anticanonical degree. We will call this Mori program Special Mori program. By studying the contractions involved in a Special Mori program, we will see that if the ray contracted is not contained in the linear subspace of $\mathrm{N}_1(X)$ spanned by classes of curves of the divisor D, then the contraction is the blow-up of a smooth subvariety of codimension 2; we will call it of type $(n-1, n-2)^{sm}$.

We will conclude by proving that, for a Fano manifold not birationally equivalent to a projective variety with a contraction of type $(n-1, n-2)^{sm}$, the Lefschetz defect δ_X, i.e. $\delta_X = \max\{\operatorname{codim} \mathrm{N}_1(D, X) | D \subset X \text{ prime divisor of } X\}$, is at most one.

In the third section we will focus on Fano n-folds X of dimension $n \geq 3$ with a prime divisor $D \subset X$ such that $\dim \mathrm{N}_1(D, X) \leq 2$. We will prove a bound on the Picard number in some cases. To be more specific, Tsukioka proved in [Tsu06] that a Fano n-fold of dimension at least 3 admitting a divisor D with $\rho_D = 1$ has Picard number $\rho_X \leq 3$. Casagrande generalized this result in [Cas08] to Fano n-folds with a prime divisor D and $\dim \mathrm{N}_1(D, X) = 1$.

In the remaining part of this section, we will focus on Fano manifolds containing a prime divisor D with $\dim \mathrm{N}_1(D, X) = 2$. This will allow us to bound the Picard number in some cases. We will look at two different situations: first, what happens when a ray R is positive on D, and then what happens when a ray R has some fibers contained in D and some that are disjoint from D. For the first case, we will see that either the Picard number is smaller than 5, or R is either small or a blow-up of a codimension 2 smooth subvariety and $R \not\subset \mathrm{N}_1(D, X)$. In the latter case, either the Picard number $\rho_X \leq 4$, or there exists an extremal ray R of type $(n-1, n-2)^{sm}$.

In the fourth section, we will focus on Fano n-folds with a divisorial extremal ray sending a divisor into a curve. The purpose of this section is to show that $\rho_X \leq 5$. This will be established in Theorem (4.9) except for one remaining case that will be treated separately in Proposition (4.10).

Furthermore, we will give an application to the Fano 4-folds case. In Corollary (4.11) we will show that for a Fano 4-fold X either ρ_X is at most 6, or X is a product and $\rho_X \leq 11$, or every elementary contraction of X is either divisorial sending a divisor onto a surface, or is small.

In the fifth section, we will give some application to Fano 5-folds with two divisorial contractions with the same exceptional locus sending a divisor into a surface. We will denote with E_0 the exceptional divisor of this contraction. First, we will consider a Fano 5-fold as before with pseudoindex $i_X > 1$ (Definition 1.4.3). Then $\rho_X = 3$, and $E_0 = \mathbb{P}^2 \times \mathbb{P}^2$. We will conclude by considering a Fano 5-fold X with two divisorial extremal rays $R_0, R_1 \subset \mathrm{NE}(X)$ sending a divisor onto a surface such that $R_0 \cdot E_1 < 0$, where E_1 is the exceptional divisor associated to R_1. By considering an extremal ray R_2 positive on E_0 we will obtain a bound on the Picard number of X in some cases.

1 Preliminars

1.1 Divisors/ one cycles / intersection/ cone of curves

Let X be a normal projective variety over $\mathbb{C}$ of dimension $\dim X = n$. We denote:
$\mathrm{Z}_1(X) \doteq$ abelian group of one-cycles.
$\mathrm{Z}^1(X) \doteq$ abelian group of Weil divisors.
$\mathrm{Pic}(X) \doteq$ abelian group of invertible sheaves.
$\mathrm{Div}(X) \doteq$ abelian group of Cartier divisors.
For a more detalied introduction, see [Har13, Chapter II, Section 6] or [Deb01, Chapter 1].

Remark 1.1.1. [Har13, Proposition II.6.13] Since X is a projective variety then $\mathrm{Pic}(X)$ is isomorphic to $\frac{\mathrm{Div}(X)}{(\sim)}$ where with $(\sim)$ we denote the linear equivalence.

A Weil divisor D is said to be a $\mathbb{Q}$-Cartier Divisor if mD is an element of $\mathrm{Div}(X)$ for some $m \in \mathbb{N}$.

Let $\rho : C \to X$ be a curve in X. Let $D \in \mathrm{Div}(X)$ be a Cartier divisor. We define an *intersection number* of C and D as follow:

$$D \cdot C \doteq \deg(\rho^*(\mathcal{O}_X(D))),$$

where with $\mathcal{O}_X(D)$ we denote the line bundle associated to D. This can be extended to a bilinear form

$$(\cdot) : \mathrm{Z}_1(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}.$$

$$(C, D) \to D \cdot C$$

This bilinear pairing allow us to define a notion of equivalence, both on $\mathrm{Z}_1(X)$ and on $\mathrm{Div}(X)$. We say that $D, D' \in \mathrm{Div}(X)$ are *numerically equivalent* if

$$D \cdot C = D' \cdot C$$

for every C curve in X. Analogously, we say that $C, C' \in \mathrm{Z}_1(X)$ are *numerically equivalent* if

$$D \cdot C = D \cdot C'$$

for every $D \in \mathrm{Div}(X)$. We denote the numerical equivalent relation by $\equiv$. We define the following $\mathbb{R}$-vector spaces

$$\mathrm{N}_1(X) \doteq \frac{\mathrm{Z}_1(X)}{\equiv} \otimes \mathbb{R}$$

$$\mathrm{N}^1(X) \doteq \frac{\mathrm{Div}(X)}{\equiv} \otimes \mathbb{R}.$$

Hence the first denotes the $\mathbb{R}$-vector space of one-cycles up to numerical equivalence and the second one the $\mathbb{R}$-vector space of Cartier divisors up to numerical equivalence. The map $(\cdot)$ induces the following non-degenerate pairing

$$(\cdot) : \mathrm{N}^1(X) \times \mathrm{N}_1(X) \longrightarrow \mathbb{R}.$$

and $\mathrm{N}_1(X)$ and $\mathrm{N}^1(X)$ are dual via $(\cdot)$. By the *Néron-Severi theorem* [Laz17, Prop. 1.1.16], they are finite dimensional. We define the *Picard number* of X as $\rho_X = \dim \mathrm{N}_1(X) < +\infty$.

The inclusion map $i : D \hookrightarrow X$ defines a push-foward of one-cycles

$$i_* : \mathrm{N}_1(D) \to \mathrm{N}_1(X).$$

We will denote by $\mathrm{N}_1(D, X)$ the image of $\mathrm{N}_1(D)$ under this linear map. It is the vector subspace generated by the numerical classes of curves of X contained in D.

Remark 1.1.2. By definition of $\mathrm{N}_1(D, X)$ we get that $\dim \mathrm{N}_1(D, X) \leq \rho_D$ and $\dim \mathrm{N}_1(D, X) \leq \rho_X$.

Note that this holds also if we consider a closed subset Z of X, instead of a divisor $D \subset X$ of X.

Inside $\mathrm{N}_1(X)$ we denote by $\mathrm{NE}(X) \subset \mathrm{N}_1(X)$ the convex cone of effective one-cycles

$$\mathrm{NE}(X) \doteq \{C \in \mathrm{N}_1(X) | C = \sum r_i C_i, r_i \in \mathbb{R} \text{ and } r_i \geq 0\}$$

where C_i are irreducible curves.

Let $\overline{\mathrm{NE}}(X)$ be the closure of $\mathrm{NE}(X)$ inside $\mathrm{N}_1(X)$; it is called *Mori cone of X*.

Let $D \in \mathrm{Div}(X)$, set

$$D_{\geq 0} \doteq \{x \in \mathrm{N}_1(X) | D \cdot x \geq 0\},$$

$$D^{\perp} \doteq \{x \in \mathrm{N}_1(X) | D \cdot x = 0\}$$

and analogously for $\leq, >$ and $<$. We will use the following notation $\overline{\mathrm{NE}}(X)_{D\geq 0} \doteq \overline{\mathrm{NE}}(X) \cap D_{\geq 0,}$ and similarly for $\leq, <, >$, and $=$ [KM98, Def. 1.17].

Definition 1.1.3. Let D be a divisor of X. We say that it is *semiample* if a multiple of D it is *base point free (b.p.f.)*, i.e. there exists $m \in \mathbb{N}$ such that mD induces a morphism

$$\varphi : X \to \mathbb{P}^n$$

for some $n \in \mathbb{N}$. Equivalently D is b.p.f. if $|D|$ has no base point, i.e. $Bs(D) = \bigcap_{D' \in |D|} D' = \emptyset$ where with $|D|$ we denote the linear system associated with D.

Definition 1.1.4. Let D be a divisor of X. We say that it is *ample* if a multiple of D is very ample, i.e. there exists $m \in \mathbb{N}$ such that mD induces a closed embedding

$$\varphi : X \hookrightarrow \mathbb{P}^n$$

for some $n \in \mathbb{N}$.

For a description of the construction of the morphism induced by a divisor see for example [Har13, Chapter II, Section 7].

Ampleness is a numerical property. Indeed, there exists the following numerical characterization of ampleness due to Kleimann [Kle66].

Theorem 1.1.5 (Kleimann's Ampleness Criterion)**.** *[KM98, Theorem 1.18] Let X be a projective variety and let D be a Cartier divisor of X. Then D is ample if and only if*

$$\overline{NE}(X) \subset D_{>0}$$

Being ample is not a stable property under pull-back.

Example 1.1.6. *Let $X \doteq \mathbb{P}^1 \times \mathbb{P}^1$ and consider the contraction given by the first projection on $Y \doteq \mathbb{P}^1$, i.e. $\varphi : X \to \mathbb{P}^1$. Consider a hyperplane H in Y. Then H is ample but, by the projection formula [Deb01, Section 1.9], $\varphi^* H$ is not ample.*

Therefore, we may want to relax the notion of ampleness.

Definition 1.1.7. A divisor D of X is said to be *numerically effective (nef)* if and only if

$$D \cdot C \geq 0$$

for every curve $C \subset X$.

Equivalently, nefness can be described with respect to the Mori cone. A divisor D is nef if and only if D is non-negative on $\overline{\mathrm{NE}}(X)$.

Note that, by the projection formula, the pull-back of a nef divisor is nef. Hence, being nef is stable under pull-back.

Definition 1.1.8. A Cartier divisor D on X is said to be *big* if $D^n > 0$ where $n \doteq \dim X$.

Definition 1.1.9. A Cartier divisor D of X is said to be *effective* if $D = \sum a_i D_i$ where D_i are prime divisors and $a_i \in \mathbb{Z}$ are all non-negative.

Lemma 1.1.10. *[KM98, Lemma 3.39] Let $f : X \to Y$ be a proper birational morphisms between normal varieties. Let $-B$ be an f-nef $\mathbb{Q}$-Cartier, $\mathbb{Q}$-divisor on X. Then*

B is effective if and only if $f_* B$ is effective.

Definition 1.1.11. An effective Cartier divisor D of X is said to be *movable* if its stable base locus

$$B(D) \doteq \bigcap_{m \in \mathbb{Z}_{>0}} Bs(mD)$$

has codimension at least 2.

The previous definitions allow us to define some convex cones inside $\mathrm{N}^1(X)$. The *Effective cone* $\mathrm{Eff}(X)$ is the convex cone in $\mathrm{N}^1(X)$ spanned by effective divisors. In general, it is not closed [Deb01, 1.35] but we will see that for Mori Dream Spaces (Definition (2.1)) it is rational polyhedral, in particular it is

closed.
The *Ample cone* $\mathrm{Amp}(X)$ is the open convex cone in $\mathrm{N}^1(X)$ of ample divisors.
The *Nef cone* $\mathrm{Nef}(X)$ is the closed cone of classes of nef divisors.
The *Movable cone* $\mathrm{Mov}(X)$ is the cone generated by classes of movable divisors.

The following inclusions hold:

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}(X)} \subseteq \overline{\mathrm{Eff}(X)}$$

$$\mathrm{Amp}(X) = \mathring{\mathrm{Nef}}(X)$$

$$\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$$

1.2 Singularities

In this part, we will collect some definitions and results regarding singularities. For a more detailed description, see [KM98, 2.3.].

We denote by K_X the canonical divisor of X.

Definition 1.2.1. Let X be a normal variety. X is $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier. X is $\mathbb{Q}$-*Gorenstein* if exists an integer $m \in \mathbb{N}$ such that mK_X is a Cartier divisor, i.e. $mK_X \in \mathrm{Div}(X)$.

Let X be a normal $\mathbb{Q}$-Gorenstein variety. We say that X has *terminal singularities* if there exists a resolution of singularities $f : Y \to X$ such that

$$mK_Y = f^*(mK_X) + \sum_{E\ f-\text{exceptional}} m \cdot a(E;X)E$$

with $a(E;X) \in \mathbb{Q}$ and $a(E;X) > 0$ for m big enough.

We call $a(E;X)$ the discrepancy of E with respect to X. We say that X is *terminal* if $a(E;X) > 0$ for every f-exceptional divisor with f a resolution.

Remark 1.2.2. Discrepancies do not depend on f. Moreover, if $K_X \in \mathrm{Div}(X)$, then $a(E;X) \in \mathbb{Z}$.

1.3 Cone Theorem and contractions

The Cone theorem is the first main step of the Mori program. It allows to describe the negative part of the Mori cone with respect to the canonical divisor of X. Mori in [Mor82] provides a proof in the non-singular case. The extension to the singular case is due to several authors; we recall Kawamata [Kaw84], Reid [Rei83] and Shokurov [Sho86].

In the following section, we will mainly follow [KM98, Chapter 3], where a comprehensive proof of the Cone Theorem and related results can be found.

Theorem 1.3.1 (Cone theorem)**.** *[KM98, Theorem 3.7] Let X be a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities. Then there are countably many R_l such that*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum R_l$$

where $R_l = \mathbb{R}_{>0}\Gamma_l$ with Γ_l rational curve on X such that $0 < -K_X \cdot \Gamma_l \leq \dim(X) + 1$.

Moreover for any ample divisor H and $\varepsilon > 0$,

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\varepsilon H)\geq 0} + \sum_{finite} R_l.$$

Definition 1.3.2. A K_X-negative face of the cone $\overline{\mathrm{NE}}(X)$ is called *extremal face* of the Mori Cone. The half-lines R_l of the previous theorem are called *Mori extremal rays* of the Mori cone and the rational curves Γ_l *extremal curves*.

In the next part of this section, we collect some fundamental results that allow us to associate to an extremal face of the Mori cone a contraction and to obtain the *Contraction Theorem*. The Contraction theorem is one main tool of the MMP.

Theorem 1.3.3 (Rationality theorem)**.** *[KM98, Theorem 3.5] Let X be a projective $\mathbb{Q}$-factorial variety with at most terminal singularities, and L an ample Cartier divisor of X. If K_X is not nef, then*

$$r \doteq \max\{t \in \mathbb{R} | L + tK_X \textit{ is nef}\}$$

is a rational number.

The next Theorem allows us to associate to each extremal face F of the Mori cone a supporting divisor, i.e. a nef Cartier divisor L of X such that $F = \overline{\mathrm{NE}}(X) \cap L^{\perp}$. A proof of this result can be found in [KM98, Proof of Th. 3.15 Step 6.].

Theorem 1.3.4. *[KM98] Let X be a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities and let F be an extremal face of $\overline{NE}(X)$. Then it exists a nef divisor $L \in \mathrm{Div}(X)$ such that*

$$F = \overline{NE}(X) \cap L^{\perp}$$

and $aL - K_X$ is nef and big for a big enough. L is called a supporting divisor of F.

Remark 1.3.5. Let R be an extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$, and let L be a supporting divisor. Let S be an extremal ray of $\overline{\mathrm{NE}}(X)$ such that $R \neq S$. By construction of a supporting divisor L, $S \cdot L > 0$.

Theorem 1.3.6. *[KM98, Theorem 3.3] Let X be a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities. Let $D \in \mathrm{Div}(X)$ be a nef Cartier divisor such that $aD - K_X$ is nef and big for $a \in \mathbb{N}$ big enough. Then $|aD|$ is base point free.*

Remark 1.3.7. Let F be an extremal face of the Mori cone $\overline{\mathrm{NE}}(X)$ and let L be a supporting divisor for F. By using Theorem (1.3.6), we see that the linear system associated to the divisor mL for m big enough, is base point free. Hence it defines a morphism

$$\varphi_{|mL|} : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(mL))).$$

Consider the Stein factorization of $\varphi_{|mL|}$ [Mat13, Proposition 2.16]:

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{|mL|}} & \mathbb{P}(H^0(X, \mathcal{O}_X(mL))) \; . \\ & {\scriptstyle \phi} \searrow \quad \nearrow {\scriptstyle \psi} & \\ & Y & \end{array}$$

Then ϕ is a morphism with connected fibers onto a normal projective variety Y and ψ is finite.

The previous results allow us to obtain the *Contraction Theorem*:

Theorem 1.3.8 (Contraction Theorem)**.** *[KM98, Theorem 3.7.(3)] Let X be a projective variety with at most terminal singularities, let F be an extremal face of the Mori cone $\overline{NE}(X)$ and let L be a supporting divisor of F. Then there is a unique morphism $\varphi : X \to Y$ such that:*

1. *Y is a normal projective variety;*
2. *φ has connected fibers;*
3. *an irreducible curve $C \subset X$ is mapped into a point if and only if $[C] \in F$, or equivalently an irreducible curve $C \subset X$ is mapped into a point if and only if $L \cdot C = 0$.*

Definition 1.3.9. Let X be a normal projective variety. A *contraction* of X is a surjective morphism $\varphi : X \to Y$ with connected fibers onto a normal variety Y. A contraction is said to be *elementary* if $\rho_X - \rho_Y = 1$.

Definition 1.3.10. Let φ be a contraction. The map $\varphi : X \to Y$ is:

1. of *fiber type* if $\dim X > \dim Y$;
2. *birational* if $\dim X = \dim Y$.

The *exceptional locus* of φ, $\mathrm{Exc}(\varphi)$, is the smallest subset such that φ is an isomorphism on $X \setminus \mathrm{Exc}(\varphi)$.

Definition 1.3.11. Let φ be a birational elementary contraction. The contraction φ is said to be *divisorial* (*small*) if its exceptional locus has codimension 1 (≥ 2).

Definition 1.3.12. Let φ be a contraction of X. We say that φ is of *type* (a, b), if $\dim \mathrm{Exc}(\varphi) = a$ and $\dim \varphi(\mathrm{Exc}(\varphi)) = b$.

It is possible to ask furthermore conditions on the anticanonical degree of curves in the fibers. Consider a contraction $\varphi : X \to Y$, we say that φ is a *Mori contraction* if $-K_X$ is φ-ample, i.e. if F is a non-trivial fiber of φ and $C \subset F$ is a curve then $-K_X \cdot C > 0$.

Let $\varphi : X \to Y$ be a contraction. Then we define the *relative cone* of φ as the convex subcone $\mathrm{NE}(\varphi)$ of $\mathrm{NE}(X)$ generated by all the one-cycles contracted by φ. If we consider $\varphi_* : \mathrm{N}_1(X) \to \mathrm{N}_1(Y)$ the push forward of one-cycles induced by φ. Then we can get an equivalence description of $\mathrm{NE}(\varphi)$: $\mathrm{NE}(\varphi) = \mathrm{NE}(X) \cap \ker(\varphi_*)$ [Deb01, Section 1.12].

We recall some properties of small and divisorial contractions.

Proposition 1.3.13. *[KM98, Prop. 2.5] Let X be a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities and let $\varphi : X \to Y$ be the contraction of a divisorial extremal ray $R \subset \overline{NE}(X)$. Then $E \doteq \mathrm{Exc}(\varphi)$ is a prime divisor and it is the unique divisor with negative intersection with R.*

Theorem 1.3.14. *[Kaw89, Theorem 1.1] Let X be a 4-fold, and let $\varphi : X \to Y$ be a small elementary contraction. Then the exceptional locus E of φ is a disjoint union of its irreducible components E_i for $i = 1, \dots, s$. Furthermore, $E_i \cong \mathbb{P}^2$ for every $i = 1, \dots, s$.*

Proposition 1.3.15. *[Mat13, Proposition 8.2.1] Let X be a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities and let $\varphi : X \to Y$ be a Mori contraction of divisorial type. Then Y has just terminal singularities.*

Proof. Consider $K_X = \varphi^*(K_y) + aE$ with $a \in \mathbb{Q}$. By intersecting with a curve $C \subset X$ with $[C] \in \mathrm{NE}(\varphi)$, and using projection formula, we get

$$a(E \cdot C) = (\varphi^*(K_y) + aE) \cdot C = K_X \cdot C < 0.$$

Hence $a > 0$. Consider a resolution $f : Z \to X$ such that $\varphi \circ f : Z \to Y$ is a resolution of Y. Then

$$K_Z = f^*(K_X) + \sum a(E_i; X)E_i = f^*\varphi^* K_Y + af^*E + \sum a(E_i; Y)E_i,$$

so by using that X has terminal singularities also Y has terminal singularities. □

Definition 1.3.16. [KM98, Notation 0.4.11.] Let $\varphi : X \dashrightarrow Y$ be a rational map between varieties. Let Z be a subvariety of X such that φ is defined on a open dense subset $Z^0 \subset Z$. The closure of $\varphi(Z^0)$ in Y is called the *strict transform* of Z under φ.

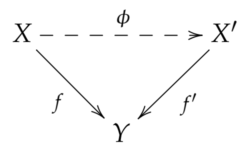
Definition 1.3.17. [HK00, Definition 1.9.] Let $f : X \to Y$ be a small elementary contraction, and let $D \subset X$ be a $\mathbb{Q}$-Cartier divisor such that $\mathrm{NE}(f)$ is negative on D, i.e. $D \cdot \mathrm{NE}(f) < 0$. A *$D$-flip of f* is a small birational morphism $f' : X' \to Y$ such that the strict transform of D in X' is $\mathbb{Q}$-Cartier and f'-ample. Flips are usually described by the following diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X'. \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y & \end{array}$$

We will also call ϕ D-flip of f.

The next Lemma shows that discrepancies do not decrease after flips of a small ray R with positive anticanonical degree. Hence if a variety X has terminal singularities, then after a flip of a ray with $(-K_X)$-positive degree, the variety obtained has still terminal singularities.

Lemma 1.3.18. *[KM98, Lemma 3.38] Consider a commutative diagram*



where X, X' and Y are normal varieties and f and f' proper birational morphism. Assume that:

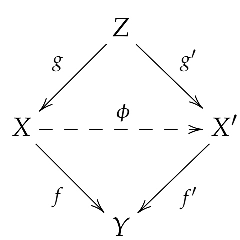
1. *$-K_X$ is $\mathbb{Q}$-Cartier and f-nef;*
2. *$K_{X'}$ is $\mathbb{Q}$-Cartier and f'-nef.*

Then for an f-exceptional divisor E over Y, we have

$$a(E;X) \leq a(E;X')$$

where $a(E;X)$ and $a(E;X')$ are the discrepancies of E with respect to X and X' respectively.

Proof. Consider a common resolution of X and X'



Set $h \doteq f \circ g = f' \circ g'$ and let m be an integer large enough such that mK_X and $mK_{X'}$ are Cartier divisors. Then

$$-m\,(K_Z) = -mg^*(K_X) - \sum a(E_i;X)E,$$
$$m\,(K_Z) = mg'^*(K'_X) + \sum a(E_i;X')E_i.$$

Consider $H \doteq \sum(a(E_i;X) - a(E_i;X'))E_i$, note that $H = mg'^*(K'_X) - mg^*(K_X)$ hence it is h-nef. All the coefficients are non-positive. Note that $h_*(-H) = 0$ hence $-H$ is effective (Lemma (1.1.10)). Thus $a(E;X) \leq a(E,X')$. $\square$

Definition 1.3.19. [HK00, Definition 1.8.] A *small $\mathbb{Q}$-factorial modification (SQM)* of X a normal projective $\mathbb{Q}$-factorial variety is a birational map $g : X \dashrightarrow Y$, where Y is normal, projective and $\mathbb{Q}$-factorial and g is an isomorphism in codimension 1.

One important class of examples of SQMs are flips.

Let R be a ray (we do not require to have negative anticanonical degree) of $\overline{\mathrm{NE}}(X)$. We define the $\mathrm{Locus}(R)$ to be the locus of curves whose classes lie in R [Wiś91, 1]. Note that it coincides with the exceptional locus of the contraction associated with R, and if R is of fiber type, then $\mathrm{Locus}(R) = X$.

The *length* of R is defined as

$$l(R) \doteq \min\{-K_X \cdot C | C \text{ rational curve and } [C] \in R\}$$

Theorem 1.3.20 (Ionescu-Wiśniewski)**.** *[Wiś91, Theorem 1.1][Ion86, Theorem 0.4] Let X be a projective manifold and let $R \subset \overline{NE}(X)$ be an extremal ray. Let F be an irreducible component of a non-trivial fiber of the contraction of R. Then*

$$\dim(F) + \dim(\mathrm{Locus}(R)) \geq \dim(X) + l(R) - 1.$$

Corollary 1.3.21. *[Wiś91] Let X be a projective manifold and let $R \subset \overline{NE}(X)$ be a ray. Let $\varphi : X \to Y$ be the associated contraction to R. Suppose that:*

1. *$-K_X \cdot R > 0$;*
2. *R is small.*

Then φ cannot have one-dimensional fibers.

Proof. Set $n \doteq \dim X$ and let F be an irreducible component of a non-trivial fiber of φ. First note that $l(R) \geq 1$, so

$$\dim(F) + \dim(\mathrm{Locus}(R)) \geq n.$$

R is small, so $\dim(\mathrm{Locus}\, R) \leq n - 2$. Hence

$$\begin{aligned}\dim(F) &\geq n - \dim(\mathrm{Locus}(R))\\ &\geq n - n + 2 = 2.\end{aligned}$$

Therefore F cannot be one-dimensional. □

Corollary 1.3.22. *Let X be a projective manifold of dimension n and let $R \subset \overline{NE}(X)$ be a small extremal ray with fibers of dimension at most 2. Then the associated contraction $\varphi : X \to Y$ is of type $(n-2, n-4)$.*

Proof. Foremost note that φ is equidimensional on $E \doteq \mathrm{Exc}(\varphi)$, and every non-trivial fiber of φ is two-dimensional. Indeed, by Corollary (1.3.21) it has no non-trivial fiber of dimension 1. Let F be a non-trivial fiber or φ. Consider the Ionescu-Wiśniewski inequality (1.3.20),

$$\dim(F) + \dim(\mathrm{Locus}(R)) \geq n + l(R) - 1.$$

R has positive anticanonical degree, so $l(R) \geq 1$. Hence

$$\dim(F) + \dim(\mathrm{Locus}(R)) \geq n + l(R) - 1 \geq n.$$

$\dim F = 2$, so $\dim \mathrm{Locus}(R) \geq n - 2$. Since φ is small $\dim \mathrm{Locus}(R) = n - 2$. Hence φ is small of type $(n-2, n-4)$. □

The next Lemma characterises the contractions of extremal divisorial rays of length $l(R)$ with fibers of dimension $l(R)$.

Lemma 1.3.23. *[AO02, Theorem 5.2] Let X be a projective manifold. The following are equivalent:*

1. *there exists an extremal ray R such that the contraction associated to R is divisorial and the fibers have dimension $= l(R)$;*
2. *There exists a morphism $\varphi : X \to Y$ into a smooth projective variety Y which is the blow-up of Y along a smooth subvariety of codimension $l(R) + 1$.*

Moreover the contraction of R and φ coincid.

The next Theorem is due to Wiśniewski and will be of frequent use throughout this thesis, and it describes the Mori contractions with at most one-dimensional fibers.

Theorem 1.3.24. *[Wiś91, Theorem 1.2.] Let X be a projective manifold, $\varphi : X \to Y$ a Mori contraction such that every fiber of φ has dimension at most one. Then one of the following holds:*

1. *φ is of fiber type;*
2. *if φ is birational then is of type $(n-1, n-2)^{sm}$, i.e. it is a blow-up of a smooth codimension 2 subvariety of Y.*

If φ is of fiber type, then we will call it a conic bundle.

Lemma 1.3.25. *[AW97, Lemma 2.12 and Theorem 4.1] Let X be a projective manifold, $\varphi : X \to Y$ be a Mori contraction, and F be a fiber with an irreducible component F_0 of dimension 1. Then Y is smooth in $\varphi(F_0)$. Either φ is of fiber type and F has two irreducible components (both isomorphic to $\mathbb{P}^1$) or φ is birational and $F = F_0 \cong \mathbb{P}^1$.*

Remark 1.3.26. Let X be a projective manifold, and let $\varphi : X \to Y$ be an elementary extremal contraction. If every non-trivial fiber of a Mori contraction $\varphi : X \to Y$ is one-dimensional, then Y is smooth.

For the singular case, we recall the following lemma:

Lemma 1.3.27. *[Ish91, Lemma 1.1.] Let X be a projective variety with at most terminal singularities, and let $\varphi : X \to Y$ be a birational Mori contraction with fibers of dimension at most 1. Let F be an irreducible component of a non-trivial fiber, and suppose that F contains a Gorenstein point of X. Then $F \cong \mathbb{P}^1$ and $-K_X \cdot F \leq 1$.*

Remark 1.3.28. Let X be a projective variety with at most terminal singularities and let $D \subset X$ be a prime divisor in X. Suppose that exists a ray $R \subset \overline{\mathrm{NE}}(X)$ such that $R \not\subset \mathrm{N}_1(D, X)$. Then $R \cap \mathrm{N}_1(D, X) = \{0\}$.

The following lemma will be of frequent use in our proofs:

Lemma 1.3.29. *Let X be a projective variety and let $D \subset X$ be a prime divisor of X. Suppose there exists a ray $R \subseteq \overline{NE}(X)$ associated to a contraction $\varphi : X \to Y$ such that:*

1. $R \cdot D > 0$*;*

2. $R \not\subset N_1(D, X)$*.*

Then every non-trivial fiber of the associated contraction φ *is a curve.*

Proof. Let F be an irreducible component of a non-trivial fiber of φ. Since $D \cdot R > 0$, then $F \cap D \neq \emptyset$. Note that since $R \not\subset N_1(D, X)$, then φ is finite on D and $F \not\subset D$. Then

$$\dim F - 1 \leq \dim(D \cap F) = \dim \varphi(D \cap F) = 0. \qquad \square$$

Corollary 1.3.30. *Let* X *be a projective variety with at most terminal singularities of dimension* n*. Let* $D \subset X$ *be a prime divisor. Suppose there exists a ray* R *such that:*

1. $D \cdot R > 0$*;*

2. the contraction associated to R *has fibers of dimension* > 1*.*

Then $R \subset N_1(D, X)$*.*

Proof. Suppose that $R \not\subset N_1(D, X)$. Then every non-trivial fiber is one-dimensional by Lemma (1.3.29). Hence $R \subset N_1(D, X)$. $\square$

Remark 1.3.31. Let X be a manifold of dimension n, and let D be a prime divisor. Suppose there exists an elementary contraction $\varphi : X \to Y$ of an extremal ray $R \subset NE(X)$ such that R is birational, $D \cdot R > 0$ and $R \not\subset N_1(D, X)$. Then every non-trivial fiber is one-dimensional by Lemma (1.3.29), so it is of type $(n-1, n-2)^{sm}$ by Theorem (1.3.24) . Then $E \doteq \mathrm{Exc}(\varphi)$ has a $\mathbb{P}^1$-bundle structure given by the restriction of the contraction φ to E, i.e. $\varphi_{|_E} : E \to W$.

Moreover for every fiber f of $\varphi_{|_E}$, the following hold:

1. $D \cdot f > 0$, because $D \cdot R > 0$;

2. $E \cdot f = -1$, φ is of type $(n-1, n-2)^{sm}$;

3. $f \not\subset D$, because $R \not\subset N_1(D, X)$.

We end this section by recalling a technical lemma that we will use in the proof of the Proposition (4.10), and a theorem due to Lazarsfeld.

Lemma 1.3.32. *[Cas09, Lemma 4.9] Let* E *be a projective manifold and* $\pi : E \to W$ *be a smooth morphism with fibers* $\mathbb{P}^r$*. Suppose that* E *has a Mori contraction* $\phi : E \to \mathbb{P}^r$ *which is finite on fibers of* π*. Then* $E \cong W \times \mathbb{P}^r$*.*

Theorem 1.3.33. *[Laz84, Theorem 4.1] Let* X *be a projective manifold of dimension* $n \geq 1$*, and let*

$$f : \mathbb{P}^n \to X$$

be a surjective morphism. Then $X \cong \mathbb{P}^n$*.*

1.4 Fano manifolds

Definition 1.4.1. A projective manifold is said to be *Fano* if the Cartier divisor $-K_X$ is ample.

Fano manifolds in dimension two are called *Del Pezzo surfaces*. Examples of del Pezzo surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Bl}_{\{P,Q\}} \mathbb{P}^2$ where P and Q are two points in $\mathbb{P}^2$.

Remark 1.4.2. By the Cone Theorem (1.3.1), if X is Fano then $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ and the Mori cone is polyhedral.

Definition 1.4.3. Let X be a Fano manifold. We define the *pseudoindex* of a Fano manifold X as

$$i_X \doteq \min\{-K_X \cdot C | C \text{ is a rational curve in } X\}.$$

Remark 1.4.4. Let X ba a Fano manifold of dimension n. Suppose that X has a structure of blow-up of a smooth subvariety of codimension 2, i.e. X has one extremal ray $R \subset \mathrm{NE}(X)$ of type $(n-1, n-2)^{sm}$. By the Ionescu-Wiśniewski inequality (1.3.20), $l(R) = 1$. Hence $i_X = 1$.

Example 1.4.5. *Consider the Fano manifold X obtained as the blow-up of $\mathbb{P}^5$ along a $\mathbb{P}^3$, i.e. $X = \mathrm{Bl}_{\mathbb{P}^3} \mathbb{P}^5$. Then $\rho_X = 2$ and $i_X = 1$.*

Example 1.4.6. *Consider $X \doteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Bl}_P \mathbb{P}^3$ where P is a point in $\mathbb{P}^3$. X is a Fano manifold of dimension 5 with $\rho_X = 4$ and pseudoindex $i_X = 2$. Note that it has a ray giving to X a structure of blow-up of a smooth subvariety, and three rays of fiber type.*

Example 1.4.7. *Consider $X \doteq \mathbb{P}^1 \times \mathrm{Bl}_P \mathbb{P}^4$ where P is a point in $\mathbb{P}^4$. X is a Fano manifold of dimension 5 with $\rho_X = 3$ and pseudoindex $i_X = 2$. Note that it has one divisorial ray which correspond to the blow-up of a point, and two rays of fiber type.*

Definition 1.4.8. Let X be a Fano manifold. We define the *Lefschetz defect* as

$$\delta_X = \max\{\mathrm{codim}\, \mathrm{N}_1(D, X) | D \subset X \text{ prime divisor of } X\}.$$

In general, $\dim \mathrm{N}_1(D, X)$ can be smaller than ρ_X. For example, take X to be the blow-up of $\mathbb{P}^2$ in one point and let E be the exceptional divisor of the blow-up. Note that, $\rho_X = 2$ and $\dim \mathrm{N}_1(E, X) = 1$.

It can happen that $\dim \mathrm{N}_1(D, X) = \rho_X$. For example, consider X Fano manifold of dimension n and D a principal ample divisor of X. Then, by Lefschetz Theorem on the Picard group [Laz17, Example 3.1.25], $\mathrm{N}_1(D, X) = \mathrm{N}_1(X)$. So the dimensions also coincide.

Lemma 1.4.9. *Let X be a Fano manifold of dimension n. Let $R \subset NE(X)$ be an extremal ray whose contraction $\varphi : X \to Y$ is the blow-up of a smooth subvariety $Z \subset Y$ of codimension at least 2. Let $E \doteq \mathrm{Exc}(\varphi)$ be the exceptional divisor of φ. Suppose that for every extremal ray $S \subset \mathrm{N}_1(E, X)$ such that $S \neq R$, S is non-negative on E.*

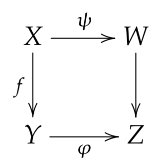
Then Y is Fano.

Proof. By contradiction suppose that Y is not Fano. Then there exists a ray $R \subset \mathrm{NE}(Y)$ such that $-K_Y \cdot R \leq 0$. Let $R_X \subset \mathrm{NE}(X)$ be a ray, not contracted by φ such that $R = \varphi_*(R_X)$. Then $-\varphi^*(K_Y) \cdot R_X \leq 0$. φ correspond to the blow-up of a smooth subvariety of Y of codimension ≥ 2, therefore

$$-K_X = -\varphi^*(K_Y) - (\operatorname{codim} Z - 1)E$$

with $\operatorname{codim} Z - 1 \geq 1$. Intersecting with R_X, $E \cdot R_X < 0$ thus $R_X \subset \mathrm{N}_1(E, X)$. Since $R_X \neq R$ and every ray $S \neq R$ contained in $\mathrm{N}_1(E, X)$ is non-negative on E, we get a contradiction. □

In the next Remark we will see that, if we consider an elementary contraction $f : X \to Y$ of a Fano manifold X and an elementary contraction φ of Y, it always exists a *lift* for φ, i.e. the elementary contraction $\psi : X \to W$ of X such that $\mathrm{NE}(\varphi \circ f) = \mathrm{NE}(f) + \mathrm{NE}(\psi)$.



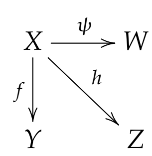
Remark 1.4.10. [Cas08, Section 2.5] Let X be a Fano variety and let $f : X \to Y$ be a contraction. Let $\alpha \subseteq \mathrm{NE}(Y)$ be a face of the Mori cone of Y, and let $\widehat{\alpha} \subseteq \mathrm{NE}(X)$ be the unique face of $\mathrm{NE}(X)$ containing $\mathrm{NE}(f)$ and such that $f_*(\widehat{\alpha}) = \alpha$. Then

$$\dim(\widehat{\alpha}) - \dim \alpha = \dim \mathrm{NE}(f).$$

Since $\mathrm{NE}(f)$ is a face of $\mathrm{NE}(X)$ contained in $\widehat{\alpha}$, then $\mathrm{NE}(f)$ is a face of $\widehat{\alpha}$. Then we can find another face of $\widehat{\alpha}$, $\widetilde{\alpha}$ such that the followings hold:

1. $\dim(\alpha) = \dim(\widetilde{\alpha})$;
2. $\widetilde{\alpha} \cap \mathrm{NE}(f) = \{0\}$.

Suppose that $\dim \alpha = 1$, i.e. the contraction φ associated to α is an elementary contraction. Then $\widetilde{\alpha}$ is an extremal ray of $\mathrm{NE}(X)$, because $\widetilde{\alpha}$ is a face of $\mathrm{NE}(X)$ of dimension $\dim(\widetilde{\alpha}) = \dim(\alpha) = 1$, $f_*(\widetilde{\alpha}) = \alpha$ and the choice of $\widetilde{\alpha}$ is unique. Since X is Fano, by the Contraction Theorem (1.3.8), there exists two contractions $h : X \to Z$ and $\psi : X \to W$ such that $\mathrm{NE}(h) = \widehat{\alpha}$ and $\mathrm{NE}(\psi) = \widetilde{\alpha}$.



Then there exists two contractions $\widetilde{\varphi} : Y \to Z$ and $g : W \to Z$ such that they make the following diagramm commutes:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & W \\ {\scriptstyle f}\downarrow & \searrow^{h} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\widetilde{\varphi}]{} & Z. \end{array}$$

Since $f_*(\widetilde{\alpha}) = \alpha$, then $\mathrm{NE}(\widetilde{\varphi}) = \alpha$. Thus $\varphi = \widetilde{\varphi}$ and $\mathrm{NE}(\varphi \circ f) = \mathrm{NE}(f) + \mathrm{NE}(\psi)$. We call ψ a *lift* of φ.

Lemma 1.4.11. *Let X a Fano manifold . Let $D \subset X$ be an effective divisor in X. Then it is always possible to find an extremal ray of the Mori cone $R \subset NE(X)$ such that $D \cdot R > 0$.*

Proof. By contradiction, let $D \subset X$ be an effective divisor such that $D \cdot R \leq 0$ for every extremal ray $R \subset \mathrm{NE}(X)$. Since D is effective, then it exists an irreducible curve $C \subset X$ positive on D, i.e. $D \cdot C > 0$. Since $[C] \in \mathrm{NE}(X)$ then $[C] = a_1[C_1] + \cdots + a_s[C_s]$ with $a_i \in \mathbb{R}_{\geq 0}$, not all 0, and $[C_i]$ class of a curve in an extremal ray R_i for every $i \in \{1, \cdots, s\}$. Therefore $0 < D \cdot C = D \cdot (a_1[C_1] + \cdots + a_s[C_s]) \leq 0$, a contradiction. □

Lemma 1.4.12. *[Cas09, Remark 4.6.] Let X be a Fano manifold. Suppose there exists a divisorial contraction associated to a ray S_1 with exceptional divisor G_1 such that $G_1 \cdot S \geq 0$ for every extremal ray $S \neq S_1$. Let S_2 be a birational extremal ray of $NE(X)$ with $G_1 \cdot S_2 = 0$. Then $S_1 + S_2$ is a face of $NE(X)$.*

Proof. By contradiction. Suppose that $S_1 + S_2$ is not a face and let C_i be a curve of X such that $[C_i] \in S_i$ for $i = 1, 2$. Let $\lambda_i \in \mathbb{Q}_{>0}$ for $i = 1, 2$. Then

$$\lambda_1 C_1 + \lambda_2 C_2 \equiv \sum_{k=3}^{m} \lambda_k C_k$$

with $\lambda_k \in \mathbb{Q}_{>0}$ for every $k \in \{1, \cdots, m\}$; moreover $C_k \in S_k$ where S_k is an extremal ray with non-negative intersection with G_1 for every $k \in \{3, \cdots, m\}$. Then, intersecting with G_1, we obtain

$$0 > (\lambda_1 C_1 + \lambda_2 C_2) \cdot G_1 = \left(\sum_{k=3}^{m} \lambda_k C_k \right) \cdot G_1 > 0,$$

which is a contradiction. □

We conclude this subsection with two results proved by C. Casagrande in [Cas12b], that allow us to obtain a bound on the Picard number of Fano 4-folds admitting some contractions of fiber type.

Corollary 1.4.13. *(Elementary contraction onto a surface) Let X be a Fano 4-fold. If X has an elementary contraction onto a surface S and $\rho_X \geq 4$, then $X \cong \mathbb{P}^2 \times S$ with S del Pezzo. Hence $\rho_X \leq 10$.*

Corollary 1.4.14. *(Elementary contraction onto a threefold) Let X be a Fano 4-fold. If X has an elementary contraction onto a threefold Y and $\rho_X \geq 7$, then either $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S$ or $X \cong \mathbb{F}_1 \times S$ with S del Pezzo. Hence $\rho_X \leq 11$*

2 Mori program for a MDS

The notion of Mori dream space (or shortly MDS) was introduced by Hu and Keel in [HK00] where it is shown that they have many important features with respect to the Mori theory. For example, we will recall that for a MDS X it always exists a MMP for every divisor in X.

In this section, we will collect some of these features.

Definition 2.1. [HK00, Def. 1.10] Let X be a normal $\mathbb{Q}$-factorial projective variety. X is said to be a *Mori dream space (MDS)* if it satisfies the following properties:

1. $\mathrm{Pic}(X)$ is finitely generated;
2. $\mathrm{Nef}(X)$ is generated by classes of finitely many semiample divisors;
3. there is a finite collection of SQM $g_i : X \dashrightarrow X_i$ for $i = 1, \cdots, r$ such that every X_i satisfies (1) and (2), and

$$\mathrm{Mov}(X) = \bigcup_{i=1}^{r} g_i^*(\mathrm{Nef}(X_i))$$

One of the main characteristics of a MDS is that the Effective cone is rational polyhedral, hence closed [Cas12a, Corollary 4.8.]. This allows us to prove the following Lemma:

Lemma 2.2. *Let X be a MDS and let $D \subset X$ be an effective divisor in X such that $[D] \not\equiv 0$. Then $-D$ is not nef.*

Proof. Since D is a non-zero effective divisor, $-D$ is not effective. By [Cas12a, Corollary 4.8.], the Effective cone is rational polyhedral. Hence it is closed, so $\overline{\mathrm{Eff}(X)} = \mathrm{Eff}(X)$. Thus

$$\mathrm{Nef}(X) \subseteq \mathrm{Eff}(X)$$

and $-D$ cannot be nef. □

Definition 2.3. Let X be a normal $\mathbb{Q}$-factorial projective variety and let $D \subset X$ be a divisor in X. A *Mori program for $D \doteq D_0$* is a finite sequence

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{k-2}}{\dashrightarrow} X_{k-1} \overset{\sigma_k-1}{\dashrightarrow} X_k$$

such that:

1. for all $i \in \{0, \ldots, k\}$ X_i is a normal and $\mathbb{Q}$-factorial projective variety;
2. for all $i \in \{0, \ldots, k-1\}$ there is a ray R_i of X_i such that $D_i \cdot Ri < 0$, where D_i is the transform of D_{i-1} (Def. (1.3.16)) if σ_{i-1} is a flip or $D_i = (\sigma_{i-1})_*(D_{i-1})$ if σ_{i-1} is a contraction of divisorial type. Moreover $\mathrm{Locus}(R_i) \subset X_i$, and σ_i is either the contraction of the ray R_i, in case R_i is of divisorial type, or the D_i-flip of R_i, in case R_i is small;

3. if $i = k$, then D_k is either nef or there exists a D_k negative contraction of a ray $R_k \subset NE(X_k)$ of fiber type $\varphi : X_k \to Y$.

It has been proved that if the starting variety X is a MDS and $D \subset X$ is a divisor in X, it always exists a Mori program for D. For a proof of the existence, see [HK00, Prop 1.11 (1)] or [Cas12a, Section 4].

Proposition 2.4. *[HK00, Prop 1.11 (1)] Let X be a MDS and $D \subset X$ a divisor in X. Then X admits a Mori program for D. Moreover, with the notation of Definition (2.3), the choice of the ray R_i is arbitrary among the D_i-negative ones.*

Remark 2.5. Consider X a MDS, $D \subset X$ a divisor in X and a Mori program for D. With the notation of Definition (2.3), every X_i is a MDS. [Cas12a, Proof of Theorem 4.2.]

In the remaining part of this section, we will consider a Mori program for $-D \subset X$ where D is a prime divisor and X is a MDS. This type of Mori program was first introduced [Cas09] and studied in detail in [Cas12b]. It is somewhat opposite to the classical approach since at every step we will consider a ray with positive intersection with the divisor.

As a corollary of the existence of a Mori Program on a MDS for every divisor, we obtain the following result (which is a generalization of Lemma (1.4.11))

Corollary 2.6. *Let X be a MDS and $D \subset X$ a prime divisor in X. Then D is positive on at least one ray of the Mori cone $NE(X)$.*

Next Lemma will be of frequent use in our proof. Indeed, let X be a MDS. By Remark (2.5), given a Mori program for $-D$ where D is a prime divisor $D \subset X$, at every step of the Mori program we can apply Corollary (2.6).

Lemma 2.7. *[Cas09, Remark 2.5] Let X be a MDS and let $D \subset X$ be a prime divisor in X, then it exists an elementary contraction $\varphi : X \to Y$ such that $D \cdot NE(\varphi) > 0$ and D intersects every non-trivial fiber of φ.*
Moreover one of the following occours:

1. *if φ is of fiber type, then $\rho_X \leq \dim \mathrm{N}_1(D, X) + 1$;*
2. *if φ is birational, then $\mathrm{Exc}(\varphi) \neq D$, $\varphi(D)$ is a divisor in Y and one of the following occours:*
   - *(a) $NE(\varphi) \subset \mathrm{N}_1(D, X)$ and $\dim \mathrm{N}_1(D, X) = \dim(\varphi(D), Y) + 1$;*
   - *(b) $NE(\varphi) \not\subset (D, X)$ and $\dim \mathrm{N}_1(D, X) = \dim(\varphi(D), Y)$.*

Proof. By Corollary (2.6) D is positive on at least one ray of $\mathrm{NE}(X)$. Consider a ray R positive on D and let $\varphi : X \to Y$ be the associated elementary contraction. By the positivity of D in $\mathrm{NE}(\varphi)$, D intersects every not trivial fiber of φ. A contraction can be either of fiber type or birational.
Suppose that φ is of fiber type. Since D intersects every non-trivial fiber of φ then

$$\varphi(X) = \varphi(D) = Y.$$

Consequently $(\varphi)_*(\mathrm{N}_1(D,X)) = \mathrm{N}_1(Y)$, hence $\rho_Y \leq \dim \mathrm{N}_1(D,X)$ and $\rho_X \leq \dim \mathrm{N}_1(D,X) + 1$.

Suppose now that φ is birational. Since $D \cdot R > 0$ then $\mathrm{Exc}(\varphi)$ intersects D. We will now prove that $\mathrm{Exc}(\varphi) \neq D$. If φ is small, then it is clear. Suppose that φ is of divisorial type and $D = \mathrm{Exc}(\varphi)$. By Proposition (1.3.13) $D \cdot \mathrm{NE}(\varphi) = \mathrm{Exc}(\varphi) \cdot \mathrm{NE}(\varphi) < 0$. Hence $D \neq \mathrm{Exc}(\varphi)$, thus $\varphi(D)$ is a divisor of Y. The ray contracted can either be contained in $\mathrm{N}_1(D,X)$ or not. Suppose that $\mathrm{NE}(\varphi) \subset \mathrm{N}_1(D,X)$; then $\dim \mathrm{N}_1(D,X) = \dim(\varphi(D),Y) + 1$. Otherwise if $\mathrm{NE}(\varphi) \not\subset \mathrm{N}_1(D,X)$, then $\dim \mathrm{N}_1(D,X) = \dim(\varphi(D),Y)$. Note that in this last case $\varphi|_D$ is finite, hence by Lemma (1.3.29) every fiber of $\varphi|_D$ has dimension at most 1. □

In the next Lemma we will observe that every Mori program for $-D$ where $D \subset X$ is a prime divisor in X ends with a contraction of fiber type.

Lemma 2.8. *[Cas12b, Lemma. 2.6] Let X be a MDS and $D \subset X$ be a prime divisor in X. Consider a Mori program for $-D$ as in (2.3):*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{k-2}}{\dashrightarrow} X_{k-1} \overset{\sigma_{k-1}}{\dashrightarrow} X_k$$

Then the following hold:

1. *at every step D_i is a prime divisor of X_i;*
2. *the ray R_k is of fiber type, hence the program ends with an elementary contraction of fiber type $\varphi : X_k \to Y$ such that $NE(\varphi) = R_k$ and $\varphi(D_k) = Y$;*

Proof. Let $i \in \{1, \cdots, k\}$ be such that σ_i is a flip. Then by the flip construction (Def. (1.3.17)), D_{i+1} is a prime divisor. To prove that at each step D_i is a prime divisor of X_i, it is enough to see it when σ_i is of divisorial type. Let $i \in \{1, \cdots, k\}$ be such that σ_i is a divisorial contraction. Since $D_i \cdot R_i > 0$, then $D_i \cap \mathrm{Exc}(\sigma_i) \neq \emptyset$ but $D_i \neq \mathrm{Exc}(\sigma_i)$ otherwise R_i it would have been negative on D_i by Proposition (1.3.13). So $D_{i+1} = \sigma_i(D_i)$ is a prime divisor in X_{i+1}. Thus at every step D_i is a prime divisor of X_i.

For $i = k$ we have that D_k is a prime divisor of X_k, then $-D_k$ cannot be nef by Lemma (2.2) and by Remark (2.5). Hence the Mori program ends with a contraction of fiber type. Let $\varphi : X_k \to Y$ be the contraction of fiber type associated to R_k. Since $D_k \cdot R_k > 0$, D_k intersects every non-trivial fiber of φ. Then $\varphi(D_k) = Y$. □

Lemma 2.9. *[Cas12b, Lemma. 2.6] Let X be a MDS and $D \subset X$ be a prime divisor. Consider a Mori program for $-D$ as in (2.8):*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{k-2}}{\dashrightarrow} X_{k-1} \overset{\sigma_{k-1}}{\dashrightarrow} X_k$$

Set $c_i \doteq \mathrm{codim}\, \mathrm{N}_1(D_i, X_i)$ for every $i \in \{0, \cdots, k\}$. For every $i \in \{0, \cdots, k-1\}$ we have

$$c_{i+1} = \begin{cases} c_i & \text{if } R_i \subset \mathrm{N}_1(D_i, X_i) \\ c_i - 1 & \text{if } R_i \not\subset \mathrm{N}_1(D_i, X_i) \end{cases}$$

and

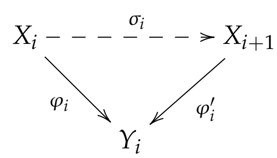
$$c_k = \begin{cases} 0 & \textit{if } R_k \subset \mathrm{N}_1(D_k, X_k) \\ 1 & \textit{if } R_k \not\subset \mathrm{N}_1(D_k, X_k). \end{cases}$$

Furthermore $\#\{i \in \{0, \cdots, k-1\} | R_i \not\subset \mathrm{N}_1(D_i, X_i)\} = \operatorname{codim} \mathrm{N}_1(D, X)$.

Proof. By definition of Mori program for $-D$, we have $D_i \cdot R_i > 0$ for every $i \in \{1, \cdots, k\}$.

Let $i \in \{1, \cdots, k\}$ be such that σ_i is a contraction of divisorial type and consider the push-foward of 1-cycles $(\sigma_i)_* : \mathrm{N}_1(X_i) \to \mathrm{N}_1(X_{i+1})$. Since D_i is positive on R_i and σ_i is the elementary contraction associated to the ray R_i, $\ker((\sigma_i)_*) = \mathbb{R}R_i$ and $\mathrm{N}_1(D_{i+1}, X_{i+1}) = (\sigma_i)_*(\mathrm{N}_1(D_i, X_i))$. Since we have $\rho_{X_{i+1}} = \rho_{X_i} - 1$, then $c_{i+1} = c_i$ if $R_i \subset \mathrm{N}_1(D_i, X_i)$, otherwise $c_{i+1} = c_i - 1$ if $R_i \not\subset \mathrm{N}_1(D_i, X_i)$.

Let $i \in \{0, \cdots, k-1\}$ be such that σ_i is a $-D_i$-flip and consider the flip diagram:



where φ_i is the contraction of R_i and φ'_i is its flip. Since $\varphi_i(D_i) = \varphi'_i(D_{i+1})$, then

$$(\varphi_i)_*(\mathrm{N}_1(D_i, X_i)) = \mathrm{N}_1(\varphi_i(D_i), Y_i) = \mathrm{N}_1(\varphi'_i(D_{i+1}), Y_i) = (\varphi'_i)_*(\mathrm{N}_1(D_{i+1}, X_{i+1})).$$

Note that $\mathrm{NE}(\varphi'_i) \subset \mathrm{N}_1(D_{i+1}, X_{i+1})$, since $D_{i+1} \cdot \mathrm{NE}(\varphi'_i) < 0$ by Definition of flip (1.3.17). Then $\ker(\varphi'_i)_* \subseteq \mathrm{N}_1(D_{i+1}, X_{i+1})$, so we have

$$c_{i+1} = \operatorname{codim} \mathrm{N}_1(\varphi_i(D_i), Y_i).$$

Therefore, $c_{i+1} = c_i$ if $R_i \subset \mathrm{N}_1(D_i, X_i)$, otherwise $c_{i+1} = c_i - 1$ if $R_i \not\subset \mathrm{N}_1(D_i, X_i)$. By Lemma (2.8), $\varphi_k(D_k) = Y$, so

$$(\varphi_k)_*(\mathrm{N}_1(D_k, X_k)) = \mathrm{N}_1(Y).$$

Hence either $c_k = 0$ if $R_k \subset \mathrm{N}_1(D_k, X_k)$ or $c_k = 1$ if $R_k \not\subset \mathrm{N}_1(D_k, X_k)$.

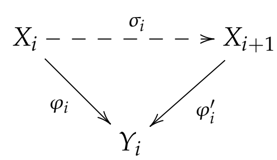
The last part of the statement follows immediately. □

Lemma 2.10. *[Cas12b, Lemma 2.6] Let X be a smooth MDS and* $D \subset X$ *be a prime divisor. Consider a Mori program for* $-D$*. Define* $A_l \subset X_l$ *for* $l \in \{1, \cdots, k\}$ *as follows: let* $A_1 \subset X_1$ *be the indeterminacy locus of* σ_0^{-1}*, and for* $i \in \{2, \ldots, k\}$ *let* $A_i \subset X_i$ *be the union of* $\sigma_{i-1}(A_{i-1})$ *and the indeterminacy locus of* σ_{i-1}^{-1}*, if* σ_{i-1} *is of divisorial type, or let* $A_i \subset X_i$ *be the union of the transform of* A_{i-1} *and the indeterminacy locus of* σ_{i-1}^{-1}*, if* σ_{i-1} *is a flip.*

Then for every $i \in \{1, \cdots, k\}$*,* $\mathrm{Sing}(X_i) \subseteq A_i \subset D_i$ *and* $X_i \setminus A_i$ *is isomorphic to an open subset of X, so it is smooth.*

Proof. Let $i \in \{0, \cdots, k-1\}$ be such that σ_i is a contraction of divisorial type. Since D_i is positive on R_i, it intersects every non-trivial fiber of σ_i. So $D_i \cap \operatorname{Exc}(\sigma_i) \neq \emptyset$ and $\sigma_i(\operatorname{Exc}(\sigma_i)) \subseteq D_{i+1}$. So D_{i+1} contains the indeterminacy locus of σ_i^{-1}.

Let $i \in \{1, \cdots, k\}$ be such that σ_i is a $-D_i$-flip and consider the flip diagram:



where φ_i is the contraction of R_i and φ_i' is its $-D_i$-flip. By construction D_{i+1} is negative on $\operatorname{NE}(\varphi_i')$, then $\operatorname{Exc}(\varphi_i') \subset D_{i+1}$. Hence D_{i+1} contains the indeterminacy locus of σ_i^{-1},

For every i D_i contains the indeterminacy locus of σ_{i-1}^{-1}, so we see that $A_i \subset D_i$. To conclude, note that A_i contains the indeterminacy locus of $(\sigma_{i-1} \circ \cdots \sigma_0)^{-1}$ and that $X_i \setminus A_i$ is smooth because X is smooth. □

Remark 2.11. Note that if σ_{i-1} is small, then $\dim A_i > 0$. Hence, $\dim A_i = 0$ occurs only if R_{i-1} is divisorial.

2.1 Fano as MDS

In [HK00, Corollary 2.16] it has been proved that Fano 3-folds are MDS and it has been conjectured that the same holds for arbitrary dimensions. In [BCHM10] it has been proved that any Fano manifold of any dimension is a MDS. This enables us to consider a Mori program for every divisor $D \subset X$. In the following section, we will also show that there is a suitable choice of extremal rays involved in the MMP whose contractions have positive anticanonical degree. We will call this Mori program *special Mori program*.

From now, X is fixed to be a Fano manifold of dimension at least 3.

Theorem 2.1.1. *[BCHM10, Corollary 2.16] Let X be a Fano manifold. Then X is a Mori Dream Space.*

Corollary 2.1.2. *Let X be a Fano manifold and let $D \subset X$ be a divisor in X. Then it exists a Mori program as in (2.3) for X and D.*

Lemma 2.1.3. *[Cas09, Lemma 3.8.] Let X be a Fano manifold and let $D \subset X$ be a prime divisor in X. Consider a Mori program for $-D$ as in Lemma (2.8), let $i \in \{0, \cdots k\}$ and suppose that for every $j \in \{0, \cdots, i-1\}$ the R_j is $-K_{X_j}$-positive. Then for every $s \in \{0, \ldots, i\}$ X_s has terminal singularities.*

Let $A_i \subset X_i$ as in Lemma (2.10). If $C \subset X_i$ is an irreducible curve not contained in A_i and $C_0 \subset X$ is the proper transform of C in X, the following holds:

$$-K_X \cdot C_0 \leq -K_{X_i} \cdot C.$$

Moreover if $C \cap A_i \neq \emptyset$ *then*

$$-K_X \cdot C_0 < -K_{X_i} \cdot C.$$

Proof. Fix $i \in \{1, \cdots, k\}$. Suppose that the statement holds for $i-1$ and take $\sigma_{i-1} : X_{i-1} \dashrightarrow X_i$. We will distinguish two cases, when σ_{i-1} is a $-D_{i-1}$-flip and when σ_{i-1} is divisorial. Suppose that σ_{i-1} is a $-D_{i-1}$-flip. Now consider a common resolution of X_i and X_{i-1} and the standard flip diagram:

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X_{i-1} & \overset{\sigma_{i-1}}{\dashrightarrow} & X_i. \\ {}_{\varphi_{i-1}}\searrow & & \swarrow_{\varphi'_{i-1}} \\ & Y & \end{array}$$

Foremost we want to see that, if X_{i-1} has terminal singularities, so does X_i. Let $E_1, \cdots, E_r \subset Z$ be the exceptional divisors of the resolutions, then

$$K_Z = f^*(K_{X_{i-1}}) + \sum_{k=1}^{r} b_k E_k = g^*(K_{X_i}) + \sum_{k=1}^{r} a_k E_k$$

where a_k's and b_k's are the discrepancies of E_k in X_i and X_{i-1} respectively, and $a_k, b_k \in \mathbb{Q}$. X_{i-1} has terminal singularities, hence $b_k > 0$. Since the contraction is a Mori contraction, discrepancies do not decrease after flips, also X_i has terminal singularities and $a_k \geq b_k > 0$ for every $k = 1, \cdots, r$ (Lemma 1.3.18).

Consider a curve $C \subset X_i$ as in the statement and let $C_Z \subset Z$ and $C_{i-1} \subset X_{i-1}$ be its strict transform in Z and X_{i-1} respectively. First, note that C_Z cannot be contained in the exceptional divisor E_k for every k, hence $E_k \cdot C_Z \geq 0$ for every k. Notice that the following holds:

$$g^*(K_{X_i}) = f^*(K_{X_{i-1}}) + \sum_{k=1}^{r} (b_k - a_k) E_k,$$

hence by the projection formula we get

$$-K_{X_i} \cdot C = -K_{X_{i-1}} \cdot C_{i-1} - \sum_{k=1}^{r} (b_k - a_k)(E_k \cdot C_Z).$$

Since $a_i \geq b_i > 0$, $E_i \cdot C_Z \geq 0$ and $-K_{X_{i-1}} \cdot C_{i-1} \geq -K_X \cdot C_0$, then

$$-K_{X_i} \cdot C \geq -K_{X_{i-1}} \cdot C_{i-1} \geq -K_X \cdot C_0.$$

It is left to prove the last part of the statement, i.e. if $C \cap A_i \neq \emptyset$ then $-K_X \cdot C_0 < -K_{X_i} \cdot C$.

Suppose that $C \cap A_i \neq \emptyset$, we can consider two different cases:

1. $C_{i-1} \cap A_{i-1} \neq \emptyset$. By hypothesis $-K_{X_{i-1}} \cdot C_{i-1} > -K_X \cdot C_0$, hence $-K_{X_i} \cdot C > -K_X \cdot C_0$.

2. $C_{i-1} \cap A_{i-1} = \emptyset$ but $C \cap A_i \neq \emptyset$. By the construction of the A_i's, C_{i-1} has to intersect $\text{Locus}(R_{i-1})$, hence C_Z must have positive intersection with E_j and $f(E_j) \subset \text{Locus}(R_{i-1})$ for some j.

   Now, since $-K_{X_{i-1}}$ is f-ample and f is not an isomorphism over the center of E_j in Y, then $a_j > b_j$. Hence, $-K_X \cdot C_0 \leq -K_{X_{i-1}} < -K_{X_i} \cdot C$.

Suppose now that $\sigma_{i-1} : X_{i-1} \to X_i$ is divisorial. Then X_i is terminal by Lemma (1.3.15), so it is left to prove the second part of the statement.

Consider an irreducible curve $C \subset X_i$ not contained in A_i and let $C_0 \subset X$ be the proper transform of C in X. We have that $-K_{X_{i-1}} = \sigma_{i-1}^*(-K_i) - aE$ where E is the exceptional divisor of σ_{i-1} and $a > 0$. Let C_{i-1} be the proper transform of C in X_{i-1}. Observe that $C_{i-1} \not\subset E$ because $C \not\subset A_i$, so $E \cdot C_{i-1} \geq 0$. Recall that $-K_{X_{i-1}} \cdot C_{i-1} \geq -K_X \cdot C_0$, so

$$-K_{X_i} \cdot C = (-K_{X_{i-1}} + aE) \cdot C_{i-1} \geq -K_{X_{i-1}} \cdot C_{i-1} \geq -K_X \cdot C_0.$$

By a similar argument as above, we can conclude. Indeed, suppose that $C \cap A_i \neq \emptyset$; we can distinguish two different cases:

1. $C_{i-1} \cap A_{i-1} \neq \emptyset$. Since $-K_{X_{i-1}} \cdot C_{i-1} > -K_X \cdot C_0$, then $-K_{X_i} \cdot C > -K_X \cdot C_0$.

2. $C_{i-1} \cap A_{i-1} = \emptyset$ but $C \cap A_i \neq \emptyset$. Then $E \cap C_{i-1} \neq \emptyset$ but the curve is not contained in the exceptional divisor, hence $E \cdot C_{i-1} > 0$ and we can conclude. □

Remark 2.1.4. Consider X, Y be $\mathbb{Q}$-factorial projective varieties such that $f : X \to Y$ is the blow-up of $A \subset Y_{reg}$ and let X be Fano. Then let C be an irreducible curve of Y not contained in A such that $A \cap C \neq \emptyset$. Then $-K_Y \cdot C \geq 2$.

Corollary 2.1.5. *Let X be a Fano manifold and let $D \subset X$ be a prime divisor in X. Let $\varphi : X \to Y$ be a divisorial contraction with associated ray R such that Y is not Fano and $D \cdot R > 0$. Then it exists a ray $R' \subset NE(Y)$ such that the following hold:*

1. *R' has non-positive anticanonical degree;*

2. *the contraction associated to R' is small.*

Proof. We will prove that every ray $R' \subset \text{NE}(Y)$ with non-positive anticanonical degree is a small ray. Set $A \doteq \varphi(\text{Exc}(\varphi))$ and let $D' \doteq \varphi(D)$ a prime divisor in Y. Since Y is not Fano, it exists a ray $R' \subset \text{NE}(Y)$ with non-positive anticanonical degree. Consider the contraction associated with this ray, say $\psi : Y \to Z$. Let $C \subset \text{Locus}(R')$ be a curve. Then $C \cdot (-K_Y) \leq 0$. By Lemma (2.1.3), C is contained in $A = \varphi(\text{Exc}(\varphi))$, otherwise it would have positive anticanonical degree. Hence $\text{Locus}(\psi) \subseteq A \subset D'$. Since φ is divisorial, $\dim A \leq \dim X - 2$. Hence ψ is small. □

Remark 2.1.6. In the previous Corollary (2.1.5), $\mathrm{Locus}(\psi) \subseteq A \subset D$. Thus, if F is a fiber of ψ, $\dim \varphi^{-1}(F) > \dim F$.

Lemma 2.1.7. *[Cas09, Lemma 3.9.] Let X be a Fano manifold of dimension n and let $D \subset X$ be a prime divisor in X. Consider a Mori program for $-D$ as in Proposition (2.8):*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{k-2}}{\dashrightarrow} X_{k-1} \overset{\sigma_{k-1}}{\dashrightarrow} X_k$$

Suppose there exists $i \in \{0, \cdots, k\}$ such that $-K_{X_j} \cdot R_j > 0$ for every $j \in \{0, \cdots, i-1\}$. Let $\varphi : X_i \to Y$ be an elementary birational contraction such that $NE(\varphi) \not\subset \mathrm{N}_1(D_i, X_i)$, and $NE(\varphi) \cdot D_i > 0$. Let $A_i \subset X_i$ be as in Lemma (2.10).

Then φ is a Mori contraction, $\mathrm{Exc}(\varphi)$ is disjoint from A_i and $\varphi|_{X_i \setminus A_i}$ is a Mori contraction of type $(n-1, n-2)^{sm}$.

Proof. Set $R_i \doteq \mathrm{NE}(\varphi)$. By hypothesis $D_i \cdot R_i > 0$, so D_i intersects every not trivial fiber of σ_i. $R_i \not\subset \mathrm{N}_1(D_i, X_i)$, hence σ_i is finite on D_i. Let F' be an irreducible component of a non-trivial fiber F of σ_i; then F' is a curve by Lemma (1.3.29). F intersects D_i in finitely many points, so F cannot be contained in A_i and $\dim(\mathrm{Sing}(X_i \cap F)) = 0$. By Lemma (2.1.3), since X is Fano we have that $-K_{X_i} \cdot F \geq 1$. Then σ_i is a Mori contraction. Hence, by Lemma (1.3.27), $-K_{X_i} \cdot F' \leq 1$. By applying again Lemma (2.1.3), we obtain that $A_i \cap F = \emptyset$, so $\mathrm{Exc}(\sigma_i) \subseteq X_i \setminus \mathrm{Sing}(X_i)$. By Theorem (1.3.25), we can conclude that $\sigma_i|_{X_i \setminus A_i}$ is of type $(n-1, n-2)^{sm}$. ☐

Lemma 2.1.8. *[Cas09, Lemma 3.10] Let X be a Fano manifold and let $D \subset X$ be a prime divisor in X. Consider a Mori program for $-D$, let $i \in \{0, \cdots k\}$ and suppose that for every $j \in \{0, \cdots, i-1\}$ the ray R_j is $-K_{X_j}$-positive. Let $A_i \subset X_i$ be as in Lemma (2.10).*

If $\dim \mathrm{N}_1(D_i, X_i) = 1$ and $\dim A_i > 0$, then $i = k$ $\rho_{X_k} \leq 2$, and every D_k-positive elementary contraction $\psi : X_k \to Y$ is of fiber type.

Proof. Let ψ be an elementary contraction such that $D_i \cdot \mathrm{NE}(\psi) > 0$. By contradiction, suppose $\psi : X_i \to Y$ to be birational. $\mathrm{NE}(\psi)$ can be contained in $\mathrm{N}_1(D_i, X_i)$ or not. Assume $\mathrm{NE}(\psi) \subset \mathrm{N}_1(D_i, X_i)$. $\dim \mathrm{N}_1(D_i, X_i) = 1$, so the image of D_i under ψ will be a point and $D_i \cdot \mathrm{NE}(\psi) < 0$, which is not possibile because $\mathrm{NE}(\psi)$ is positive on D_i. Hence $\mathrm{NE}(\psi) \not\subset \mathrm{N}_1(D_i, X_i)$.

Thus ψ is a birational contraction with $\mathrm{NE}(\psi) \not\subset \mathrm{N}_1(D_i, X_i)$. Therefore, A_i is disjoint from $\mathrm{Exc}(\psi)$ and $\psi|_{X_i \setminus A_i}$ is a Mori contraction of type $(n-1, n-2)^{sm}$ (by Lemma (2.1.7)). Since $D_i \cdot \mathrm{Exc}(\psi) > 0$ and $D_i \cap \mathrm{Exc}(\psi) \neq \emptyset$, we can find an irreducible curve $C \subset \mathrm{Exc}(\psi) \cap D_i$ such that $\mathrm{Exc}(\psi) \cdot C > 0$. Since $\dim \mathrm{N}_1(D_i, X_i) = 1$, then every curve in D_i will be numerically proportional to C. Recall that $A_i \subset D_i$ is a positive dimensional subset, hence also every curve inside A_i will have positive intersection with $\mathrm{Exc}(\psi)$. This leads us to a contradiction because A_i and $\mathrm{Exc}(\psi)$ are disjoint. Hence ψ has to be of fiber type, so $i = k$. By Lemma (2.7) $\rho_{X_k} \leq \dim(D_i, X_i) + 1 = 2$. ☐

Lemma 2.1.9. *Let X be a Fano manifold. Let $D \subset X$ be a prime divisor in X and consider a Mori program for $-D$ as in Definition (2.8). Suppose that there exists $i = 0, \cdots, k$ such that* $\dim \mathrm{N}_1(D_i, X_i) = 1$.

Let $\varphi : X_i \to Y$ be an elementary birational contraction such that $D_i \cdot NE(\varphi) > 0$. Then the following hold:

1. *φ is finite on D_i;*

2. *every non-trivial fiber of φ is a curve.*

Proof. Suppose by contradiction that $\mathrm{NE}(\varphi) \subset \mathrm{N}_1(D_i, X_i)$. Then φ maps D_i into a point and $\mathrm{Exc}(\varphi) = D_i$. Therefore $D_i \cdot \mathrm{NE}(\varphi) = \mathrm{Exc}(\varphi) \cdot \mathrm{NE}(\varphi) < 0$, which contradicts $D_i \cdot \mathrm{NE}(\varphi) > 0$. Hence $\mathrm{NE}(\varphi) \not\subset \mathrm{N}_1(D_i, X_i)$. By Lemma (1.3.29) every non-trivial fiber of φ is one-dimensional. □

Definition 2.1.10. A *Special Mori program* for a divisor D in X is a Mori Program as in Definition (2.3) where every contraction involved is a Mori contraction.

The next proposition allows us to obtain that for a Fano manifold there always exists a suitable choice of rays such that the considered Mori program is a Special Mori program.

Proposition 2.1.11. *[Cas12b, Proposition 2.4] Let X be a Fano manifold and let $D \subset X$ be a divisor in X. There exists a Mori program for D as in (2.3) where the rays R_i are chosen among the K_{X_i}-negative ones for every $i \in \{0, \cdots, k\}$.*

Proof. By (2.1.2) a Mori program for D always exists, and the choice of R_i is arbitrary among the D_i's negative ones. Therefore, we have to prove that we can choose at each step of the program a ray R_i such that $K_i \cdot R_i < 0$ and $D_i \cdot R_i < 0$ for every $i \in \{0, \cdots, k\}$.

If D is nef, then there is nothing to prove because we have that $k = 0$ and X is assumed to be Fano. Hence, we can assume D not to be nef.

Define

$$\lambda_0 \doteq \sup\{\lambda \in \mathbb{R} | \; \lambda D + (1-\lambda)(-K_X) \text{ is nef}\}.$$

Since D is not nef and X is Fano and ampleness is an open property, then $0 < \lambda_0 < 1$. Furthermore by $\mathbb{Q}$-factoriality of X, $\lambda_0 \in \mathbb{Q}$. Set $H_0 \doteq \lambda_0 D + (1-\lambda_0)(-K_X)$; since ampleness is an open property and by the definition of λ_0, H_0 is nef but not ample. So, by construction of λ_0, it exists an extremal ray of the Mori cone of X, $R_0 \subset \mathrm{NE}(X)$ say R_0, such that $H_0 \cdot R_0 = 0$ and $D \cdot R_0 < 0$. Furthermore, $K_X \cdot R_0 < 0$ since $H_0 \cdot R_0 = 0$.

If R_0 is of fiber type, then we are done. Otherwise, $\sigma_0 : X_0 \dashrightarrow X_1$ is the contraction of R_0 if R_0 is divisorial, or σ_0 is the flip of R_0 if R_0 is small. Note that the divisor $\lambda_0 D_1 + (1-\lambda_0)(-K_{X_1}) \subset X_1$ is nef. As before, if D_1 is nef we are done, otherwise set

$$\lambda_1 \doteq \sup\{\lambda \in \mathbb{R} | \; \lambda D_1 + (1-\lambda)(-K_{X_1}) \text{ is nef}\}.$$

Using similar arguments as before we have that $\lambda_0 \leq \lambda_1 < 1$, $\lambda_1 \in \mathbb{Q}$ and $H_1 \doteq \lambda_1 D_1 + (1-\lambda_1)(-K_{X_1}) \subset X_1$ is nef but not ample. There is a ray R_1 of

$\mathrm{NE}(X_1)$ st $H_1 \cdot R_1 = 0$, $D_1 \cdot R_1 < 0$, thus $K_{X_1} \cdot R_1 < 0$. Now we can iterate the procedure. □

As a corollary of Lemma (2.1.7), applied to a special Mori program, we obtain the following result:

Corollary 2.1.12. *[Cas12b, Lemma 2.7] Let X be a Fano manifold of dimension n and $D \subset X$ a prime divisor in X. Consider a special Mori program for $-D$:*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{k-1}}{\dashrightarrow} X_{k-1} \overset{\sigma_k}{\dashrightarrow} X_k$$

Let $i \in \{0, \cdots, k-1\}$ be such that $R_i \not\subset \mathrm{N}_1(D_i, X_i)$ and R_i is birational. Then R_i is of type $(n-1, n-2)^{sm}$, i.e. σ_i is the blow-up of a smooth subvariety of codimension 2. Furthermore, $\mathrm{Exc}(\sigma_i)$ does not intersect the exceptional loci of the maps σ_j for $j < i$.

Proof. Fix $i \in \{0, \cdots, k-1\}$ such that $R_i \not\subset \mathrm{N}_1(D_i, X_i)$; by Lemma (2.1.7), $\sigma_i|_{X_i \setminus A_i}$ is divisorial of type $(n-1, n-2)^{sm}$ and $\mathrm{Exc}(\sigma_i) \cap A_i = \emptyset$. We can therefore conclude because σ_i is an isomorphism on A_i. □

Lemma 2.1.13. *[Cas12b, Lemma 2.7] Let X be a Fano manifold and let $D \subset X$ be a prime divisor in X. Consider a special Mori program for $-D$. Then the following hold:*

1. *Set $s \doteq \#\{i \in \{0, \cdots, k-1\} | R_i \not\subset \mathrm{N}_1(D_i, X_i)\}$. We have that either $s = \operatorname{codim} \mathrm{N}_1(D, X)$ and $\mathrm{N}_1(D_k, X_k) = \mathrm{N}_1(X_k)$ or $s = \operatorname{codim} \mathrm{N}_1(D, X) - 1$, $R_k \not\subset \mathrm{N}_1(D_k, X_k)$ and $\operatorname{codim} \mathrm{N}_1(D_k, X_k) = 1$.*
2. *Set $\{i_1, \cdots, i_s\} = \{i \in \{0, \cdots, k-1\} | R_i \not\subset \mathrm{N}_1(D_i, X_i)\}$, and let $E_j \subset X$ be the transform of $\mathrm{Exc}(\sigma_{i_j}) \subset X_{i_j}$ for every $j = 1, \cdots, s$.*
   Then E_j is a smooth $\mathbb{P}^1$-bundle with fiber f_j. Furthermore $E_j \cdot f_j = -1$, $D_j \cdot f_j > 0$, and $[f_j] \notin \mathrm{N}_1(D, X)$. Moreover $E_j \cap D \neq \emptyset$ and $E_j \neq D$;
3. *$E_1, \cdots E_s$ are pairwise disjoint.*

Proof. By Lemma (2.9) we have

$$s = \begin{cases} \operatorname{codim} \mathrm{N}_1(D, X) & \text{if } R_k \subset \mathrm{N}_1(D_k, X_k) \\ \operatorname{codim} \mathrm{N}_1(D, X) - 1 & \text{if } R_k \not\subset \mathrm{N}_1(D_k, X_k). \end{cases}$$

Because

$$c_{i+1} = \begin{cases} c_i & \text{if } R_i \subset \mathrm{N}_1(D_i, X_i) \\ c_i - 1 & \text{if } R_i \not\subset \mathrm{N}_1(D_i, X_i) \end{cases}$$

for $i \in \{0, \cdots, k-1\}$ and

$$c_k = \begin{cases} 0 & \text{if } R_k \subset \mathrm{N}_1(D_k, X_k) \\ 1 & \text{if } R_k \not\subset \mathrm{N}_1(D_k, X_k). \end{cases}$$

So the first part of the statement holds.

Let $j \in \{i_1, \cdots, i_s\}$. By Corollary (2.1.12) R_j is of type $(n-1, n-2)^{sm}$. Then by Remark (1.3.31), $\mathrm{Exc}(\sigma_j)$ is a $\mathbb{P}^1$-bundle with the $\mathbb{P}^1$-bundle structure given by the contraction of R_j. Since $E_j \cong \mathrm{Exc}(\sigma_j)$, also E_j has a $\mathbb{P}^1$-bundle structure. Let $\pi : E_j \to Y$ be the morphism giving the $\mathbb{P}^1$-bundle structure on E_j and let $f_j \subset E_j$ be the fiber of π. By Theorem (1.3.20), we have that $-K_X \cdot f_j = -1$. Furthermore $E_j \cdot f_j > 0$, since $E_{i_j} \cdot R_{i_j} > 0$. So $E_j \cap D \neq \emptyset$ and $E_j \neq D$. Since $R_{i_j} \not\subset \mathrm{N}_1(D_{i_j}, X_{i_j})$, $[f_j] \notin \mathrm{N}_1(D, X)$.

The E_j are pairwise disjoint because $\mathrm{Exc}(\sigma_{i_j})$ does not intersect the transform of the exceptional loci of the maps σ_l for $l < i$. □

Definition 2.1.14. The $E_1, \cdots, E_s$ determined in the Lemma (2.1.13) are called the $\mathbb{P}^1$-*bundles determined by the special Mori program for* $-D$.

As a straightforward consequence, the following lemma holds:

Proposition 2.1.15. *Let X be a Fano manifold, $D \subset X$ be a prime divisor in X such that $\mathrm{codim}(D, X) > 0$. Then there exist pairwise disjoint prime divisors $E_1, \cdots, E_s$ with $s = \mathrm{codim}(D, X)$ or $s = \mathrm{codim}(D, X) - 1$ such that $E_j \cdot f_j = -1$, $D_j \cdot f_j > 0$, and $[f_j] \notin \mathrm{N}_1(D, X)$. So $E_j \cap D \neq \emptyset$, $E_j \neq D$ and E_j are pairwise disjoint.*

2.2 Further results on a Mori program for $-D$

Let X be a Fano manifold. In this section we will consider a Mori program on X for $-D$ as in Lemma (2.3), but we will not always consider all the steps untill the contraction of fiber type: we will stop the program when either the contracted ray R_m is of fiber type or when it is birational and such that $R_m \not\subset \mathrm{N}_1(D_m, X_m)$. Note that for every $i \in \{0, \cdots, m-1\}$ R_i is a birational ray such that $R_i \subseteq \mathrm{N}_1(D_i, X_i)$.

If we ask furthermore that every contraction is a Mori contraction, then in the second case the program ends with a contraction of type $(n-1, n-2)^{sm}$.

We will consider the following Set up:

Set Up 2.2.1. *Let X be a Fano manifold and let $D \subset X$ be a prime divisor in X. Consider a Mori program for $-D$ as in Lemma 2.3. Let $m \in \{0, \cdots, k\}$ be the first index such that for every $i \in \{0, \cdots, m-1\}$ R_i is birational, $R_i \subset \mathrm{N}_1(D_i, X_i)$, and either R_m is birational with $R_m \not\subset \mathrm{N}_1(D_m, X_m)$, or R_m is of fiber type.*

Remark 2.2.2. The sequence

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m$$

in Set Up (2.2.1) satisfies the following:

1. for every $i \in \{0, \cdots, m-1\}$ R_i is birational and $R_i \subset \mathrm{N}_1(D_i, X_i)$;
2. R_m is either of fiber type, or R_m is birational and $R_m \not\subset \mathrm{N}_1(D_m, X_m)$.

Considering a Mori program as in Set up (2.2.1). With the following lemma we can follow what happen to $\dim \mathrm{N}_1(D_i, X_i)$ at every step.

Lemma 2.2.3. *Let X be a Fano manifold and let $D \subset X$ be a divisor in$X$. Consider a sequence as in Set Up (2.2.1)*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m.$$

Then for every $i \in \{0, \cdots, m-1\}$ we have the following:

$$\dim \mathrm{N}_1(D_{i+1}, X_{i+1}) = \begin{cases} \dim \mathrm{N}_1(D_i, X_i) - 1 & \textit{if } R_i \textit{ is divisorial} \\ \dim \mathrm{N}_1(D_i, X_i) & \textit{if } R_i \textit{ is small.} \end{cases}$$

Proof. By construction of Set Up (2.2.1), m is the smallest integer such that $R_m \not\subset \mathrm{N}_1(D_m, X_m)$, so $R_i \subset \mathrm{N}_1(D_i, X_i)$. Therefore, if R_i is divisorial

$$\dim \mathrm{N}_1(D_{i+1}, X_{i+1}) = \dim \mathrm{N}_1(D_i, X_i) - 1.$$

If R_i is associated with a small contraction, then we have to consider its $-D_i$-flip:

$$\begin{array}{ccc} X_i & \overset{\sigma_i}{\dashrightarrow} & X_{i+1} \\ & {\scriptstyle \varphi_i} \searrow \quad \swarrow {\scriptstyle \varphi_i'} & \\ & Y_i & \end{array}$$

where $\varphi_i' : X_{i+1} \to Y_i$ is the flip of the contraction associated to R_i. Let R_i' be the ray corresponding to φ_i', then $D_{i+1} \cdot R_i' < 0$ by the definition of $-D_i$-flip (Definition 1.3.17). Therefore $R_i' \subset \mathrm{N}_1(D_{i+1}, X_{i+1})$. Since $\varphi_i(D_i) = \varphi_i'(D_{i+1})$, the following equalities hold:

$$\begin{aligned} \dim \mathrm{N}_1(D_i, X_i) &= \dim \mathrm{N}_1(\varphi_i(D_i), Y_i) + 1 \\ &= \dim \mathrm{N}_1(\varphi_i'(D_{i+1}), Y_i) + 1 = \dim \mathrm{N}_1(D_{i+1}, X_{i+1}). \end{aligned}$$

Hence $\dim \mathrm{N}_1(D_i, X_i) = \dim \mathrm{N}_1(D_{i+1}, X_{i+1})$. $\square$

Remark 2.2.4. Let X be a Fano manifold, and let $D \subset X$ be a prime divisor in X. Consider a sequence as in Set Up (2.2.1):

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m.$$

As a consequence of Lemma (2.2.3), we obtain that

$$\rho_{X_i} - \dim(\mathrm{N}_1(D_i, X_i))$$

is constant for every i. Hence, with the notation of Lemma (2.9), c_i is constant at every step. Indeed recall that if the ray R_i is :

1. small, then $\rho_{X_{i+1}} = \rho_{X_i}$;
2. divisorial, then $\rho_{X_{i+1}} = \rho_{X_i} - 1$.

Hence $\rho_{X_i} - \dim(\mathrm{N}_1(D_i, X_i))$ is constant by Lemma (2.2.3).

Proposition 2.2.5. *[Cas09, Corollary 3.7] Let X be a Fano manifold and let $D \subset X$ be a divisor in X. Consider a sequence as in Set Up (2.2.1)*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m.$$

Suppose that the ray R_m is of fiber type. Then the following bound holds:

$$\rho_X \leq 1 + \dim \mathrm{N}_1(D, X).$$

Proof. If R_m is of fiber type, then by Lemma (2.7) the hollowing holds:

$$\rho_{X_m} \leq 1 + \dim \mathrm{N}_1(D_m, X_m).$$

Note that, by Remark (2.2.4), $\rho_{X_i} - \dim \mathrm{N}_1(D_i, X_i)$ is constant at every step of the Mori program. Hence

$$\rho_X - \dim \mathrm{N}_1(D, X) = \rho_{X_m} - \dim \mathrm{N}_1(D_m, X_m) \leq 1. \qquad \square$$

Remark 2.2.6. In the next section we will consider a Fano manifold X of dimension n with an extremal ray of type $(n-1, 1)$. Let E be the exceptional divisor of the contraction. Observe that $\dim \mathrm{N}_1(E, X) = 2$, since it is the contraction of the divisor onto a curve.

We can consider a Mori program for $-E$. If it ends with a contraction of fiber type, then $\rho_X \leq 2 + 1 = 3$.

Proposition 2.2.7. *Let X be a Fano manifold and let $D \subset X$ be a divisor in X. Consider a sequence as in Set Up (2.2.1)*

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m.$$

Suppose that the ray R_m is of fiber type. Then $\operatorname{codim} \mathrm{N}_1(D, X) \leq 1$.

Proof. Let $\varphi : X_m \to Y$ be the contraction of fiber type associated with R_m. By Lemma (2.9), when the Mori program ends with a contraction of fiber type, the following holds:

$$\operatorname{codim} \mathrm{N}_1(D_m, X_m) = \begin{cases} 0 & \text{if } R_m \subset \mathrm{N}_1(D_m, X_m) \\ 1 & \text{if } R_m \not\subset \mathrm{N}_1(D_m, X_m). \end{cases}$$

Then by Remark (2.2.4), $\operatorname{codim} \mathrm{N}_1(D_i, X_i) \leq 1$ for every $i \in \{0, \ldots, m\}$. Hence $\operatorname{codim}(D, X) \leq 1$. $\square$

In the next Proposition we would like to find a bound for the Lefschetz defect of a Fano manifold X not birationally equivalent to a variety with an extremal ray of type $(n-1, n-2)^{sm}$. Recall that the Lefschetz defect δ_X of a Fano manifold X is defined as following:

$$\delta_X = \max\{\operatorname{codim} \mathrm{N}_1(D, X) | D \subset X \text{ prime divisor of } X\}.$$

Since X is a Fano manifold, then by Lemma (2.1.11) we can furthermore ask that every ray contracted is K_X-negative. Then we obtain the following bound on the Lefschetz defect.

Proposition 2.2.8. *Let X be a Fano manifold of dimension n such that it is not birationally equivalent to a variety with an extremal ray of type $(n-1, n-2)^{sm}$. Then $\delta_X \leq 1$.*

Proof. Let D be a prime divisor in X. Then consider a special Mori program for $-D$, and consider m as in Set Up (2.2.1). The program has to end with a contraction of fiber type. By contradiction, suppose that R_m is birational and such that $R_m \not\subset \mathrm{N}_1(X_m, D_m)$. By Corollary (2.1.12) R_m is of type $(n-1, n-2)^{sm}$, a contradiction.

Hence $\operatorname{codim} \mathrm{N}_1(D, X) \leq 1$ for every prime divisor $D \subset X$. Consequently $\delta_X \leq 1$. □

3 Divisors $D \subset X$ with $\dim \mathrm{N}_1(D, X) \leq 2$

Let X be a Fano manifold of dimension $n \geq 3$. Tsukioka in [Tsu06] proved the following result:

Theorem 3.1. *[Tsu06, Proposition 5] Let X be a Fano manifold of dimension $n \geq 3$ and let $D \subset X$ be a prime divisor in X with $\rho_D = 1$. Then $\rho_X \leq 3$.*

Casagrande in [Cas08] generalized this result in order to obtain a bound on the Picard number when X contains a prime divisor D with $\dim \mathrm{N}_1(D, X) = 1$.

Proposition 3.2. *[Cas08, Proposition 3.16] Let X be a Fano manifold of dimension $n \geq 3$ and let $D \subset X$ be a prime divisor in X with* $\dim \mathrm{N}_1(D, X) = 1$*. Then $\rho_X \leq 3$.*

Remark 3.3. Let X be a Fano manifold of dimension $n \geq 3$ and suppose it exists an elementary contraction φ of type $(n-1, 0)$. Since φ is elementary, then $\dim \mathrm{N}_1(\mathrm{Exc}(\varphi), X) = 1$. Indeed, if $\dim(\mathrm{N}_1(\mathrm{Exc}(\varphi), X)) \geq 2$, then we would have $\dim(\varphi(D)) > 0$. Hence, by Proposition (3.2), $\rho_X \leq 3$.

In this section, we will focus on Fano manifolds containing a prime divisor D with $\dim \mathrm{N}_1(D, X) = 2$. This will allow us to bound the Picard number in some cases by considering an extremal ray positive on the prime divisor D. First of all, we will start by considering some intermediate results. Foremost observe that if a Fano manifold X contains a divisor as before, then $\rho_X \geq 2$.

In the next Lemma we will consider a Mori program for $-D \subset X$ with m as in Set up (2.2.1):

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m$$

where R_m is of fiber type, $m \geq 1$, and σ_0 is divisorial not of type $(n-1, 0)$. We will prove that either the Picard number of X is 3, or σ_0 is of type $(n-1, n-2)^{sm}$.

Lemma 3.4. *Let X be a Fano manifold of dimension n with $\rho_X \geq 3$. Let $D \subset X$ be a prime divisor such that* $\dim \mathrm{N}_1(D, X) = 2$*. Consider a Mori program for $-D$ with m as in Set up (2.2.1). Assume σ_0 divisorial contraction not of type $(n-1, 0)$, $m \geq 1$ and R_m of fiber type.*

Then the following hold:

1. $\rho_X = 3$;
2. σ_0 *is of type* $(n-1, n-2)^{sm}$.

Proof. Since $m \geq 1$, then $\mathrm{NE}(\sigma_0) \subset \mathrm{N}_1(D, X)$ by construction of Set Up (2.2.1). By Proposition (2.2.5) , we obtain that $\rho_X = 3$. We will prove that σ_0 is of type $(n-1, n-2)^{sm}$. Since $R_0 \subset \mathrm{N}_1(D, X)$,

$$\dim \mathrm{N}_1(D_1, X_1) = \dim \mathrm{N}_1(D, X) - 1 = 2 - 1 = 1.$$

Hence, we are considering only two steps by Lemma (2.1.8):

$$X \xrightarrow{\sigma_0} X_i \xrightarrow{\psi} Y$$

where the first step is the divisorial contraction σ_0 and the second step is an elementary contraction of fiber type ψ such that $\mathrm{NE}(\psi) \cdot D_1 > 0$. Then $\dim Y \geq 1$ since $\rho_{X_1} = 2$, and $\rho_Y = 1$. Note that $\dim(\mathrm{N}_1(D_1, X_1)) = 1$, i.e. all the curve in D_1 are numerically proportional. Hence ψ is finite on D_1. By Lemma (1.3.29), every non-trivial fiber is 1-dimensional and $\dim Y = \dim X - 1 = n - 1$.

We will prove that ψ is injective on A_1.

Note that $A_1 = \sigma_0(\mathrm{Exc}(\sigma_0)) \subset D_1$, hence ψ is finite on A_1 because $\mathrm{NE}(\psi) \cdot D_1 > 0$.

Let $x_1 \in A_1$ and consider the fiber on $\psi(x_1)$, $\psi^{-1}(\psi(x_1))$. Note that it is a fiber of dimension 1. But, since ψ is finite on A_1, $\psi^{-1}(\psi(x_1))$ intersects A_1 but it cannot be contained in it, $\sigma_0^{-1}(\psi^{-1}(\psi(x_1)))$ is a one-dimensional fiber and $\psi \circ \sigma_0$ is of fiber type. By applying Lemma (1.3.25), $\sigma_0^{-1}(\psi^{-1}(\psi(x_1)))$ has two irreducible components both isomorphic to $\mathbb{P}^1$. Note that $\psi^{-1}(\psi(x_1)) \cap A_1 = \{x_1\}$ because if the fiber intersects A_1 in more than 1 point, then $\sigma_0^{-1}(\psi^{-1}(x_1))$ should have at least three irreducible components. Thus ψ is injective on A_1. The two irreducibile components, both isomorphic to $\mathbb{P}^1$, are the strict transform of $\psi^{-1}(\psi(x_1))$ and the fiber of σ_0 on x_1, $\sigma_0^{-1}(x_1)$. Then every non-trivial fiber of σ_0 has dimension 1. By applying again Lemma (1.3.25), X_1 is smooth and σ_0 is of type $(n-1, n-2)^{sm}$. □

In the next Lemma we will consider again a Mori program for $-D \subset X$ with m as in Set up (2.2.1):

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m$$

where R_m is birational such that $R_m \not\subset \mathrm{N}_1(D_m, X_m)$, $m \geq 1$, and σ_0 is not of type $(n-1, 0)$. We will prove that R_0 is small and every contraction involved in the Mori program is a Mori contraction.

Lemma 3.5. *Let X be a Fano manifold of dimension n with $\rho_X \geq 3$. Let $D \subset X$ be a prime divisor such that $\dim \mathrm{N}_1(D, X) = 2$. Consider a Mori program for $-D \subset X$ with m as in Set up (2.2.1). Assume σ_0 birational contraction not of type $(n-1, 0)$, $m \geq 1$ and R_m birational with $R_m \not\subset \mathrm{N}_1(D, X)$. Then, for every i, R_i is a ray with positive anticanonical degree.*

Furthermore R_0 is a small extremal ray and $R_0 \subset \mathrm{N}_1(D, X)$; moreover it exists a smooth prime divisor $D' \subset X$ with a $\mathbb{P}^1$-bundle structure which is disjoint from $\mathrm{Exc}(\sigma_0)$. Furthermore if $\pi : D' \to Y$ is the map giving the bundle structure on D', then for every fiber f of π the following hold:

1. *$D \cdot f > 0$;*
2. *$D' \cdot f = -1$;*
3. *$f \not\subset D$.*

Proof. Foremost we will prove that every contraction of the Mori program is a Mori contraction, i.e. $-K_{X_i} \cdot R_i > 0$ for every $i \in \{1, \cdots, m\}$. Note that it is true for $i = 0$. Fix $i \in \{1, \ldots, m\}$. Suppose that $-K_{X_j} \cdot R_j > 0$ for $j = 1, \cdots i-1$. To prove that also R_i has positive anticanonical degree, we need to show the following:

$$\dim A_i > 0$$

where the A_i's are subsets of Y_i's for every $i \in \{1, \cdots, m\}$, as defined in Lemma (2.10). By construction of the A_i's, this holds if $i = 1$ or if $i > 1$ and R_{i-1} is small. Suppose that $i > 1$ and R_{i-1} is divisorial. There cannot be another divisorial ray among R_j for $j \in \{1, \cdots, m-1\}$, by Lemma (2.2.3). Consequently, R_{i-2} is small. Hence the indeterminacy locus L of σ_{i-2}^{-1} is positive dimensional and, $\sigma_{i-1}(L) \subseteq A_i \subset D_i$.

Since $\sigma_{i-1}(L)$ is the locus of the contraction of a small ray of $\mathrm{NE}(X_i)$, then σ_i is finite on L. Hence $\sigma_{i-1}(L)$ and A_i are positive dimensional. Since R_m is birational, then by Lemma (2.1.8), $\dim \mathrm{N}_1(D_i, X_i) = 2$. So R_{i-1} is small.

Let $R'_{i-1} \subset \mathrm{NE}(X_i)$ be the ray whose associated contraction is the flip of R_{i-1}. By the flip construction (1.3.17), $-K_{X_i} \cdot R'_{i-1} < 0$ and $D_i \cdot R'_{i-1} < 0$. By the negativity of R'_{i-1} on D_i, $R'_{i-1} \subset \mathrm{N}_1(D_i, X_i)$. Since $\dim \mathrm{N}_1(D_i, X_i) = 2$, then

$$\mathrm{N}_1(D_i, X_i) \cap \mathrm{NE}(X_i) = R_i + R'_{i-1}.$$

Since $A_i \subset D_i$ by Lemma (2.10), consider a curve $C \subset D_i$ not contained in A_i. By Lemma (2.1.3), since X is Fano, then C has positive anticanonical degree. Hence some curves of D_i have to have positive anticanonical degree. Since $-K_{X_i} \cdot R'_{i-1} < 0$, then $-K_{X_i} \cdot R_i > 0$.

We showed that $\dim \mathrm{N}_1(D_i, X_i) = 2$ for every $i \in \{1, \cdots, m\}$. By Lemma (2.2.3), every $i \in \{0, \cdots, m-1\}$ R_i is small, so also σ_0 is small.

Every contraction is a Mori contraction, hence by Lemma (2.1.7), R_m is of type $(n-1, n-2)^{sm}$ and $\mathrm{Locus}(R_m) \cap A_m = \emptyset$. By Remark (1.3.31), $\mathrm{Locus}(R_m)$ is a divisor with a $\mathbb{P}^1$-bundle structure given by σ_m.

Now we define the divisor D' as the strict transform in X of $\mathrm{Locus}(R_m)$, we obtain a divisor with a $\mathbb{P}^1$-bundle structure which is disjoint from $\mathrm{Exc}(\sigma_0)$. By Remark (1.3.31) if $\pi : D' \to Y$ is the map giving the bundle structure, then for every fiber f of π the following hold:

1. $D \cdot f > 0$: since R_m is positive on D_m;
2. $D' \cdot f = -1$: R_m is divisorial of type $(n-1, n-2)^{sm}$. Then by Theorem (1.3.20) and the ramification formula we obtain that $f \cdot \mathrm{Locus}(R_m) = -1$;
3. $f \not\subset D$: since $R_m \not\subset \mathrm{N}_1(D_m, X_m)$. □

Theorem 3.6. *[Cas09, Theorem 3.2.] Let X be a Fano manifold of dimension n and $D \subset X$ be a prime divisor such that $\dim \mathrm{N}_1(D, X) = 2$. Let $\varphi : X \to Y$ be an elementary contraction of X such that $D \cdot NE(\varphi) > 0$. Then one of the following occurs:*

1. $\rho_X = 2$;
2. $\rho_X = 3$ *and* φ *is either:*
   (a) *a conic bundle, or*
   (b) *of type* $(n-1, 0)$, *or*
   (c) *of type* $(n-1, n-2)^{sm}$ *and* $NE(\varphi) \subset \mathrm{N}_1(D, X)$, *or*
   (d) *small and* $NE(\varphi) \subset \mathrm{N}_1(D, X)$.
3. $\rho_X \geq 3$ *and either:*
   (a) φ *is of type* $(n-1, n-2)^{sm}$ *and* $NE(\varphi) \not\subset \mathrm{N}_1(D, X)$, *or*
   (b) φ *is small and* $NE(\varphi) \subset \mathrm{N}_1(D, X)$; *moreover it exists a smooth prime divisor* $D' \subset X$ *with a* $\mathbb{P}^1$*-bundle structure which is disjoint from* $\mathrm{Exc}(\varphi)$. *Furthermore if* $\pi : D' \to Y$ *is the map giving the bundle structure on* D', *then for every fiber* f *of* π *the following hold:*
      i. $D \cdot f > 0$;
      ii. $D' \cdot f = -1$;
      iii. $f \not\subset D$.

Proof. Suppose that $\rho_X \geq 3$. We have to distinguish between two cases: φ of fiber type and φ birational.

Suppose that φ is of fiber type. Then by Lemma (2.7) and Lemma (2.9) we see that $\varphi(D) = Y$, φ is finite on D and $\rho_X = 3$. Every non-trivial fiber of φ is one dimensional by Lemma (1.3.29). Hence, φ is a conic bundle, and we have 2.(a) of the statement.

Suppose that φ is birational. If φ is of type $(n-1, 0)$, then by Proposition (3.2) we have that $\rho_X \leq 3$ so we have 2.(b). If $\mathrm{NE}(\varphi) \not\subset \mathrm{N}_1(D, X)$, then by Lemma (2.1.7) we obtain 3.(a).

We can therefore assume that φ is birational, not of type $(n-1, 0)$, and $\mathrm{NE}(\varphi) \subset \mathrm{N}_1(D, X)$. Since $\mathrm{NE}(\varphi) \cdot D > 0$ and the choice of the ray is arbitrary, we can consider a Mori program for $-D$ with φ as first step, so $R_0 \doteq \mathrm{NE}(\varphi)$. Therefore in Set up (2.2.1) we have

$$X \doteq X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} \cdots \overset{\sigma_{m-2}}{\dashrightarrow} X_{m-1} \overset{\sigma_{m-1}}{\dashrightarrow} X_m,$$

where $m \geq 1$ because the ray R_0 is birational and $\mathrm{NE}(\varphi) \subset \mathrm{N}_1(D, X)$. Consider A_i as in Lemma (2.10). Since the contraction of R_0 is birational map and not of type $(n-1, 0)$, then $A_1 = \varphi(\mathrm{Exc}(\varphi))$ and $\dim A_1 > 0$.

Recall that in Set up (2.2.1), we have two possibilities:

- R_m is of fiber type, or
- R_m is birational with $R_m \not\subset \mathrm{N}_1(D_m, X_m)$.

First case: R_m **of fiber type.** By Proposition (2.2.5), $\rho_X = 3$. If φ is divisorial, then we obtain 2.(c) by Lemma (3.4); otherwise φ is small and we obtain 2.(d).

Second case: R_m **birational.** By Lemma (3.5), then φ is small and $\mathrm{NE}(\varphi) \subset \mathrm{N}_1(D, X)$; moreover it exists a smooth prime divisor $D' \subset X$ with a $\mathbb{P}^1$-bundle structure which is disjoint from $\mathrm{Exc}(\varphi)$. Furthermore if $\pi : D' \to Y$ is the map giving the bundle structure on D', then for every fiber f of π the following hold:

1. $D \cdot f > 0$;
2. $D' \cdot f = -1$;
3. $f \not\subset D$.

Hence we obtain 3.(b). □

In the last part of this section we will study the case when an extremal ray R of X, with associated map φ, is 0 on D but $D \cap \mathrm{Exc}(\varphi) \neq \emptyset$.

Before starting the study of this case, let's fix some notation. If we denote with $\varphi_i : X \to Y_i$ the contraction of an extremal ray R_i of $\mathrm{NE}(X)$, then we set $E_i \doteq \mathrm{Exc}(\varphi_i)$. Unless otherwise stated.

Lemma 3.7. *[Cas09, Lemma 3.11] Let X be a Fano manifold of dimension n. Let $\varphi_1 : X \to Y_1$ be an elementary divisorial contraction and $\psi : Y_1 \to Z$ be a birational contraction with one-dimensional non-trivial fibers. Let $\varphi_2 : X \to Y_2$ be the elementary contraction such that $NE(\psi \circ \varphi_1) = NE(\varphi_1) + NE(\varphi_2)$:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi_2} & Y_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow \\ Y_1 & \xrightarrow[\psi]{} & Z \end{array}$$

Then φ_2 is a contraction of type $(n-1, n-2)^{sm}$, Y_2 is smooth, $\mathrm{Exc}(\psi) = \varphi_1(E_2)$. Furthermore, one of the following holds:

1. *ψ is a divisorial Mori contraction, $E_1 \cdot NE(\varphi_2) = 0$, $\mathrm{Exc}(\psi) \cap \varphi_1(E_1)$ is a union of fibers of ψ and $E_2 \neq E_1$.*
2. *ψ is small, $\mathrm{Exc}(\psi) = \varphi_1(E_1)$, $E_1 \cdot NE(\varphi_2) < 0$ and $E_1 = E_2$.*

Proof. In the first part of the proof, we will show that φ_2 is birational with fibers of dimension at most one, hence it is a contraction of type $(n-1, n-2)^{sm}$ and Y_2 is smooth, by Lemma (1.3.25).

Let F_2 be a non-trivial fiber of φ_2. Recall that $\mathrm{NE}(\psi \circ \varphi_1) = \mathrm{NE}(\varphi_1) + \mathrm{NE}(\varphi_2)$ and ψ is a birational contraction. Hence $\mathrm{NE}(\varphi_1) \neq \mathrm{NE}(\varphi_2)$ and φ_1 is finite on every non-trivial fiber of φ_2. Therefore φ_1 is finite on F_2 and $\varphi_1(F_2) \subset \mathrm{Exc}(\psi)$. Hence $\dim \varphi_1(F_2) > 0$ and it is contained in a non-trivial fiber of ψ. Since every non-trivial fiber of ψ is one-dimensional, then $\dim \varphi_1(F_2) = 1$.

Therefore F_2 is one-dimensional, so φ_2 has fibers with dimension at most 1. Now, φ_2 cannot be of fiber type, since φ_1 and ψ are birational. Since X is Fano and φ_2 has one-dimensional fibers, then by Corollary (1.3.21) φ_2 cannot be small. Hence φ_2 is divisorial with at most one-dimensional fibers. Hence φ_2 is a contraction of type $(n-1, n-2)^{sm}$ and Y_2 is smooth, by Lemma (1.3.25).

Consider F a non-trivial fiber for ψ, then F is one-dimensional and $\varphi_1^{-1}(F)$ is a non-trivial fiber for $\psi \circ \varphi_1$. We are going to prove that, if $\varphi_1^{-1}(F)$ has an irreducible component of dimension 1, then $\mathrm{Exc}(\psi) \cap \varphi_1(E_1)$ is a union of fibers of ψ.

Suppose that $\varphi_1^{-1}(F)$ has an irreducible component of dimension 1. Observe that $\psi \circ \varphi_1$ is a Mori contraction. By Lemma (1.3.25), since $\psi \circ \varphi_1$ is birational, $\varphi_1^{-1}(F) \cong \mathbb{P}^1$. Then either $F \cap \varphi_1(E_1) = \emptyset$ or $F \subset \varphi_1(E_1)$. In fact, suppose that $F \cap \varphi_1(E_1) \neq \emptyset$ but $F \not\subset \varphi_1(E_1)$; then $\varphi_1^{-1}(F)$ would be reducibile. Therefore $\mathrm{Exc}(\psi) \cap \varphi_1(E_1)$ is a union of fibers of ψ.

We will proceed as follow:

1. if $E_1 \neq E_2$, we will show that ψ is a divisorial Mori contraction and $E_1 \cdot \mathrm{NE}(\varphi_2) = 0$;
2. if $E_1 = E_2$, we will obtain that ψ is small, $\mathrm{Exc}(\psi) = \varphi_1(E_1)$ and $E_1 \cdot \mathrm{NE}(\varphi_2) < 0$.

Suppose that $E_1 = E_2$. Then $E_1 \cdot \mathrm{NE}(\varphi_2) = E_2 \cdot \mathrm{NE}(\varphi_2) < 0$. Every curve in $\mathrm{NE}(\psi \circ \varphi_1) = \mathrm{NE}(\varphi_1) + \mathrm{NE}(\varphi_2)$ has negative intersection with E_1. Then $\mathrm{Exc}(\psi \circ \varphi_1) \subset E_1$. On the other hand $E_1 \subset \mathrm{Exc}(\psi \circ \varphi_1)$, so they coincide. Since $\dim \varphi_1(E_1) \leq n-2$, $\varphi_1(E_1) = \mathrm{Exc}(\psi)$ and ψ is small.

Assume that $E_1 \neq E_2$. Since $\mathrm{NE}(\psi \circ \varphi_1) = \mathrm{NE}(\varphi_1) + \mathrm{NE}(\varphi_2)$ then $\varphi_1(E_2)$ is a prime divisor contained in $\mathrm{Exc}(\psi)$. Hence ψ is divisorial and $\varphi_1(E_2) = \mathrm{Exc}(\psi)$. Since φ_1 is divisorial, then Y_1 is $\mathbb{Q}$-factorial and

$$K_X = \varphi_1^*(K_{Y_1}) + aE_1$$

with $a \in \mathbb{Q} \setminus \{0\}$ discrepancy of E_1. Since E_1 and E_2 are exceptional divisors such that $E_1 \neq E_2$, then there are non-trivial fibers of ψ disjoint form $\varphi_1(E_1)$.

Consider C an irreducible curve of Y_1 contained in a non-trivial fiber of ψ disjoint from $\varphi_1(E_1)$. Let $\widetilde{C} \subset X$ be its strict transform in X under φ_1. Then $E_1 \cdot \widetilde{C} = 0$. Therefore ψ is a Mori contraction:

$$0 < -K_X \cdot \widetilde{C} = -\varphi_1^*(K_{Y_1}) \cdot \widetilde{C} = -K_{Y_1} \cdot C.$$

There are non-trivial fibers of ψ disjoint from $\varphi_1(E_1)$ and $\mathrm{Exc}(\psi) = \varphi_1(E_2)$, so there are fibers of φ_2 disjoint from E_1. Thus $E_1 \cdot \mathrm{NE}(\varphi_2) = 0$. □

Remark 3.8. Let $\varphi : X \to Y$ be a divisorial contraction and let $D \subset X$ be a prime divisor with $\dim \mathrm{N}_1(D, X) = 2$. Suppose that one of the following situations occurs:

1. $D \cdot \mathrm{NE}(\varphi) = 0$ and $D \cap \mathrm{Exc}(\varphi) \neq \emptyset$;

2. $D \cdot \mathrm{NE}(\varphi) > 0$.

Set $A \doteq \varphi(\mathrm{Exc}(\varphi))$. Consider the first case, i.e. $D \cdot \mathrm{NE}(\varphi) = 0$ and $D \cap \mathrm{Exc}(\varphi) \neq \emptyset$. Then some non-trivial fibers of φ are contained in D and some are not. Therefore $A \not\subset \varphi(D)$. Now suppose that $D \cdot \mathrm{NE}(\varphi) > 0$, then every non-trivial fiber of φ intersects D. Hence $A \subset \varphi(D)$.

In Theorem (3.6), we analized the second situation. In the next Lemma, we will analize the first case instead. The differences highlighted in Remark (3.8), are the main differences between Theorem (3.6) and the following lemma.

Lemma 3.9. *[Cas09, Lemma 3.3.] Let X be a Fano manifold with dimension $n \geq 3$ and consider a prime divisor $D \subset X$ such that* $\dim \mathrm{N}_1(D, X) = 2$.

Suppose X has an elementary divisorial contraction $\varphi : X \to Y$ such that $D \cdot NE(\varphi) = 0$ and $D \cap \mathrm{Exc}(\varphi) \neq \emptyset$. Then $\rho_X \geq 2$ and at least one of the following occurs:

1. *$\rho_X \leq 4$;*

2. *it exists an extremal ray R such that:*

   (a) *$R \neq NE(\varphi)$;*
   (b) *R is of type $(n-1, n-2)^{sm}$;*
   (c) *$R \cdot \mathrm{Exc}(\varphi) < 0$;*
   (d) *$R + NE(\varphi)$ is a face of $NE(X)$.*

Proof. Set $D_Y \doteq \varphi(D)$ and $A \doteq \varphi(\mathrm{Exc}(\varphi))$. Observe that we are in the first case of Remark (3.8), hence $A \not\subset \varphi(D)$. There are some fibers disjoint from D and others contained in D, so $\dim \mathrm{N}_1(D_Y, Y) = 1$.

D_Y is a prime divisor in Y, hence it exists a ray, whose associated elementary contraction $\psi : Y \to Z$, is positive on D_Y, i.e. $D \cdot \mathrm{NE}(\psi) > 0$. The contraction ψ can be either birational or of fiber type.

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

First case: $\mathbf{NE}(\psi)$ of fiber type. By Lemma (2.7), the following bounds on the Picard numbers hold: $\rho_Z \leq 1$, $\rho_Y \leq 2$ and $\rho_X \leq 3$. Hence $\rho_X = 2$ or $\rho_X = 3$.

Second case: $\mathbf{NE}(\psi)$ birational. If ψ is birational, than it can be either small or divisorial. Observe that by Lemma (2.1.9) the following holds:

1. ψ is finite on D_Y because $\mathrm{NE}(\psi) \not\subset \mathrm{N}_1(D_Y, Y)$;

2. the dimension of every non-trivial fiber of ψ is 1.

Consider the elementary contraction $\varphi_2 : X \to Y_2$ such that $\mathrm{NE}(\psi \circ \varphi) = \mathrm{NE}(\varphi) + \mathrm{NE}(\varphi_2)$. We are in the following situation:

$$\begin{array}{ccc} X & \xrightarrow{\varphi_2} & Y_2 \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ Y & \xrightarrow[\psi]{} & Z \end{array}$$

By Lemma (3.7), φ_2 is of type $(n-1, n-2)^{sm}$ and one of the following holds:

1. ψ is a divisorial Mori contraction, $\mathrm{Exc}(\varphi) \cdot \mathrm{NE}(\varphi_2) = 0$, $\mathrm{Exc}(\psi) \cap A$ is a union of fibers of ψ and $\mathrm{Exc}(\varphi) \neq \mathrm{Exc}(\varphi_2)$;

2. ψ is small, $\mathrm{Exc}(\psi) = \varphi(\mathrm{Exc}(\varphi))$, $\mathrm{Exc}(\varphi) \cdot \mathrm{NE}(\varphi_2) < 0$ and $\mathrm{Exc}(\varphi) = \mathrm{Exc}(\varphi_2)$.

Suppose ψ is small. Then the ray $R \doteq \mathrm{NE}(\varphi_2)$ is an extremal ray as in the second part of the statement.

Suppose ψ is a divisorial Mori contraction. Since ψ is finite on D_Y, then $\mathrm{NE}(\psi) \not\subset \mathrm{N}_1(D_Y, Y)$. Hence $D_Z \doteq \psi(D_Y)$ is a prime divisor in Z and

$$\dim \mathrm{N}_1(D_Z, Z) = 1.$$

D_Y intersects every non-trivial fiber of ψ, so $\psi(\mathrm{Exc}(\psi)) \subset D_Z$.

We can consider an elementary contraction $\xi : Z \to W$ such that $D_Z \cdot \mathrm{NE}(\xi) > 0$. Therefore we are in the following situation:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi_2} & Y_2 & & \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \\ Y & \xrightarrow[\psi]{} & Z & \xrightarrow[\xi]{} & W, \end{array}$$

where ξ can either birational or of fiber type.

Suppose ξ is of fiber type. Since $\dim \mathrm{N}_1(D_Z, Z) = 1$, then $\rho_W \leq 1$ by Lemma (2.7). Since every contraction in the previous diagram is elementary, $\rho_X \leq 4$. Then $\rho_X = 3$ or $\rho_X = 4$.

If ξ is birational then, by Lemma (2.1.9), it is finite on D_Z and every fiber of ξ had dimension at most 1.

Let $\psi_2 : Y \to Z_2$ be the elementary contraction such that $\mathrm{NE}(\xi \circ \psi) = \mathrm{NE}(\psi) + \mathrm{NE}(\psi_2)$. Since $\mathrm{NE}(\xi \circ \psi) = \mathrm{NE}(\psi) + \mathrm{NE}(\psi_2)$, then $D_Y \cdot \mathrm{NE}(\psi_2) > 0$ because $D_Y \cdot \mathrm{NE}(\psi) > 0$ and $D_Z \cdot \mathrm{NE}(\xi) > 0$.

Note that ψ_2 cannot be of fiber type because ξ and ψ are birational. Thus ψ_2 is birational, finite on D_Y and with fibers at most of dimension 1 by Lemma (2.1.9)

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi_2} & Y_2 \\
{\scriptstyle \varphi}\downarrow & & \downarrow \\
Y & \xrightarrow[\psi]{} & Z \\
{\scriptstyle \psi_2}\downarrow & & \downarrow{\scriptstyle \xi} \\
Z_2 & & W
\end{array}
$$

As before, we can consider the elementary contraction $\varphi_3 : X \to Y_3$ such that $\mathrm{NE}(\psi_2 \circ \varphi) = \mathrm{NE}(\varphi) + \mathrm{NE}(\varphi_3)$:

$$
\begin{array}{ccccccc}
X_3 & \xleftarrow{\varphi_3} & X & \xrightarrow{\varphi_2} & Y_2 & & \\
\downarrow & & {\scriptstyle \varphi}\downarrow & & \downarrow & & \\
Z_2 & \xleftarrow{\psi_2} & Y & \xrightarrow[\psi]{} & Z & \xrightarrow[\xi]{} & W.
\end{array}
$$

By applying Lemma (3.7), φ_3 is divisorial of type $(n-1, n-2)^{sm}$. Furthermore one of the following holds:

1. ψ_2 is a divisorial Mori contraction, $\mathrm{Exc}(\varphi) \cdot \mathrm{NE}(\varphi_3) = 0$, $\mathrm{Exc}(\psi_2) \cap A$ is an union of fibers of ψ_2 and $\mathrm{Exc}(\varphi) \neq \mathrm{Exc}(\varphi_3)$;

2. ψ_2 is small, $\mathrm{Exc}(\psi_2) = \varphi(\mathrm{Exc}(\varphi))$, $E \cdot \mathrm{NE}(\varphi_3) < 0$ and $\mathrm{Exc}(\varphi) = \mathrm{Exc}(\varphi_3)$.

If ψ_2 is small, then $R \doteq \mathrm{NE}(\varphi_3)$ gives an extremal ray R as in the statement.

In the last part of the proof, we will show that ψ_2 cannot be divisorial. Suppose by contradiction that ψ_2 is divisorial. Recall that by construction $\mathrm{NE}(\xi \circ \psi) = \mathrm{NE}(\psi) + \mathrm{NE}(\psi_2)$. Since ψ and ψ_2 are Mori contraction, then $\eta \doteq \xi \circ \psi$ is a Mori contraction.

The following holds: $\mathrm{Exc}(\psi) \neq \mathrm{Exc}(\psi_2)$. Indeed if $\mathrm{Exc}(\psi) = \mathrm{Exc}(\psi_2)$, then $\mathrm{Exc}(\psi) \cdot \mathrm{NE}(\eta) < 0$ hence $\mathrm{Exc}(\eta) = \mathrm{Exc}(\psi)$ and $\mathrm{Exc}(\xi) = \psi(\mathrm{Exc}(\psi))$. Foremost note $\mathrm{Exc}(\psi) \subseteq \mathrm{Exc}(\eta)$ by construction. Since $\mathrm{Exc}(\psi) \cdot \mathrm{NE}(\psi_2) < 0$, then all curves with class in $\mathrm{NE}(\eta)$ are contained in $\mathrm{Exc}(\psi)$. Thus $\mathrm{Exc}(\psi) = \mathrm{Exc}(\eta)$. Then $\mathrm{NE}(\eta) \cdot \mathrm{Exc}(\psi) < 0$ and $\psi(\mathrm{Exc}(\psi)) = \mathrm{Exc}(\xi)$. This is impossible because ξ is finite on D_Z and $\psi(\mathrm{Exc}(\psi)) \subset D_Z$.

Then $\mathrm{Exc}(\psi) \neq \mathrm{Exc}(\psi_2)$ and $\psi(\mathrm{Exc}(\psi_2))$ is a divisor of Z contained in $\mathrm{Exc}(\xi)$. Thus $\mathrm{Exc}(\xi) = \psi(\mathrm{Exc}(\psi_2))$ and ξ is divisorial.

Recall that $D_Z \cdot \mathrm{NE}(\xi) > 0$, hence $\mathrm{Exc}(\xi) \cap D_Z \neq \emptyset$. By Lemma (2.1.9), ξ is finite on D_Z and we can find a curves $C \subset D_Z$ such that $C \cdot \mathrm{Exc}(\xi) > 0$. Since all the curve in D_Z are numerically equivalent, then $\mathrm{Exc}(\xi)$ has to intersect every curve in D_Z. Recall that every non-trivial fiber of ψ is one-dimensional. Hence $\dim \psi(\mathrm{Exc}(\psi)) = n - 2 \geq 1$. Then $\mathrm{Exc}(\xi) \cap \psi(\mathrm{Exc}(\psi)) \neq \emptyset$. Therefore $\dim(\psi(\mathrm{Exc}(\psi)) \cap \mathrm{Exc}(\xi)) \geq n - 3$.

Recall that ξ is finite on D_Z, hence

$$\dim \xi(\mathrm{Exc}(\xi) \cap \psi(\mathrm{Exc}(\psi))) \geq n-3.$$

We claim that

$$\dim \xi(\mathrm{Exc}(\xi) \cap \psi(\mathrm{Sing}(Y)) \leq n-4.$$

Consider A and note that $\dim A \leq n-2$ because φ is divisorial.

Suppose that $\dim A = n-2$; A is normal and $\sin(Y) \subseteq A$ then $\dim \mathrm{Sing}(Y) \leq n-4$. Hence the claim holds.

Suppose that $\dim A < n-2$. By Lemma (2.10), then $\mathrm{Sing}(Y) \subseteq A$, thus if $\dim \xi(\mathrm{Exc}(\xi) \cap \psi(A) \leq n-4$, the assertion follows. If $\psi(A) \not\subset \mathrm{Exc}(\xi)$ the claim holds. If $\psi(A) \subset \mathrm{Exc}(\xi)$, then

$$A = \psi^{-1}(\psi(A)) \subseteq \psi^{-1}(\mathrm{Exc}(\xi)) \subset \mathrm{Exc}(\psi) \cup \mathrm{Exc}(\psi_2)$$

(note that this last inclusion holds because $\mathrm{Exc}(\xi) = \psi(\mathrm{Exc}(\psi_2)))$. Since A is irreducibile, then A is contained either in $\mathrm{Exc}(\psi)$ or in $\mathrm{Exc}(\psi_2)$. Since $\mathrm{NE}(\eta) = \mathrm{NE}(\psi) + \mathrm{NE}(\psi_2)$, then $\dim(\eta(A)) \leq n-4$.

Recall that η and ξ are Mori contractions with at most one-dimensional fibers. Hence $\mathrm{Sing}(W) \subseteq \eta(\mathrm{Sing}(Y))$.

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi_2} & Z_2 \\ & & {\scriptstyle\psi}\downarrow & \searrow^{\eta} & \\ & & Z & \xrightarrow[\xi]{} & W \end{array}$$

Since $\dim \xi(\mathrm{Exc}(\xi) \cap \psi(\mathrm{Sing}(Y)) \leq n-4$, then it exists $w_0 \in W \setminus \mathrm{Sing}(W)$ such that $\xi^{-1}(w_0)$ has dimension 1 and intersects $\psi(\mathrm{Exc}(\psi))$. Consider the contraction η, and its restriction $\widetilde{\eta}$ to $Y \setminus \eta^{-1}(\mathrm{Sing}(W))$. Hence we have the following contraction

$$\widetilde{\eta} : Y \setminus \eta^{-1}(\mathrm{Sing}(W)) \to W \setminus \mathrm{Sing}(W).$$

Set $F \doteq \eta^{-1}(w_0)$, then $\psi^{-1}(F)$ is a fiber of $(\xi \circ \psi)$. Since ψ is a contraction with one dimensional fiber , then $\psi^{-1}(F)$ has an irreducibile component of dimension 1. Hence $\psi^{-1}(F) \cong \mathbb{P}^1$ by Lemma (1.3.25). Therefore either $F \subset \psi(\mathrm{Exc}(\psi))$, or $F \cap \psi(\mathrm{Exc}(\psi)) = \emptyset$. By the choice of w_0, $F \cap \psi(\mathrm{Exc}(\psi)) \neq \emptyset$, so $F \subset \psi(\mathrm{Exc}(\psi))$. This is not possibile since $\eta^{-1}(w_0) \cong \mathbb{P}^1$. □

4 Elementary contractions of type (n-1,1)

Let X be a Fano manifold of dimension n with a divisorial extremal ray R_1 whose associated contraction $\varphi_1 : X \to Y_1$ sends a divisor to a curve. We would like to use the results of the previous sections to find a bound for ρ_X. We will show that $\rho_X \leq 5$.

Since X admits an extremal contraction of type $(n-1,1)$, then $\rho_X \geq 2$. Set $E_1 \doteq \mathrm{Exc}(\varphi_1)$. Then $\dim \mathrm{N}_1(\varphi_1(E_1), Y_1) = 1$, since $\varphi_1(E_1)$ is a curve. Hence $\dim \mathrm{N}_1(E_1, X) = 2$. We can therefore apply Theorem (3.6) and Lemma(3.9) to E_1.

By Lemma (1.4.11), we may choose an extremal ray R_2 positive on E_1 with associated contraction φ_2. By Theorem (3.6) we have that one of the following holds:

1. $\rho_X \leq 3$;
2. R_2 is of type $(n-1, n-2)^{sm}$ and $\mathrm{NE}(\varphi_2) \not\subset \mathrm{N}_1(E_1, X)$.
3. φ_2 is small and it exists a smooth prime divisor $D' \subset X$ with a $\mathbb{P}^1$-bundle structure which is disjoint from E_2. Furthermore if $\pi : D' \to Y$ is the map giving the bundle structure, then for every fiber f of π the following hold:
   (a) $E_1 \cdot f > 0$;
   (b) $D' \cdot f = -1$;
   (c) $f \not\subset E_1$.

Since we want to prove that $\rho_X \leq 5$, we are left only to consider the case (2) and the case (3). Before starting with the proof of the main theorem, let us fix some notations and prove some preliminary results.

Notation: if we denote a ray with R_i, than its exceptional locus will be denoted with E_i, the associated contraction with $\varphi_i : X \to Y_i$, and a general fiber with F_i, unless otherwise stated.

Lemma 4.1. *Let X be a Fano manifold of dimension $n \geq 4$. Suppose there exist a divisorial extremal ray $R_1 \subset NE(X)$ of type $(n-1,1)$. Let R_2 be an extremal ray such that:*

1. *R_2 is positive on E_1, i.e. $E_1 \cdot R_2 > 0$;*
2. *R_2 is divisorial;*
3. *$R_1 \cdot E_2 > 0$;*
4. *$R_2 \not\subset \mathrm{N}_1(E_1, X)$;*
5. *Y_2 is Fano.*

Then $\rho_X \leq 4$.

Proof. Set $D \doteq \varphi_2(E_1)$ and $A \doteq \varphi_1(E_2)$; note that $\dim \mathrm{N}_1(D, X) = 2$. We remark that by Corollary (2.6) we may consider a ray R with positive intersection with D. Let φ be the associated contraction. Apply Theorem (3.6) to D and R. If $\rho_{Y_2} \leq 3$, then $\rho_X = \rho_{Y_2} + 1 \leq 4$. Otherwise one of the following holds:

1. φ is of type $(n-1, n-2)^{sm}$ and $\mathrm{NE}(\varphi) \not\subset \mathrm{N}_1(D, X)$;

2. φ is small and it exists a smooth prime divisor $D' \subset X$ with a $\mathbb{P}^1$-bundle structure which is disjoint from $\mathrm{Exc}(\varphi)$. Furthermore if $\pi : D' \to Y$ is the map giving the $\mathbb{P}^1$-bundle structure, then for every fiber f of π the following hold:

   (a) $D \cdot f > 0$;

   (b) $D' \cdot f = -1$;

   (c) $f \not\subset D$.

Note that in both cases it exists a smooth prime divisor $D' \subset Y_2$ with a $\mathbb{P}^1$-bundle structure, such that for every fiber f of the $\mathbb{P}^1$-bundle, $D \cdot f > 0$, $D' \cdot f = -1$ and $f \not\subset D$. If φ is small then this is clear. If φ is divisorial, then set D' to be the exceptional divisor of φ by Remark (1.3.31). Since $D' \cdot f = -1$, then $f \cap A = \emptyset$ by Lemma (2.1.3). Thus $A \cap D' = \emptyset$. Hence $D'' \doteq \varphi_2^{-1}(D')$ is a prime divisor of X, with a $\mathbb{P}^1$-bundle structure, that intersects E_1 but is disjoint from E_2. Let us show that it is impossible. We have two possibilities:

1. if $D'' \cdot R_1 = 0$ then D'' contains some non trivial fibers of R_1 because $D'' \cap E_1 \neq \emptyset$. This would imply that $D'' \cap E_2 \neq \emptyset$, a contradiction.

2. if $D'' \cdot R_1 > 0$, then D'' intersects every non trivial fiber of R_1. Moreover, by Lemma (1.3.25), $R_1 \subset \mathrm{N}_1(E_2, X)$ since φ_1 has at least one fiber of dimension more than 1 and it is positive on E_2. Hence we can find some irreducible curve C of E_2 with numerical equivalence class in R_1 intersecting D''. This is not possibile because $D'' \cap E_2 = \emptyset$. □

Remark 4.2. From the proof of Lemma (4.1) we see that this Lemma still holds also with weaker assumptions. Indeed, one could just ask that R_1 is a divisorial contraction with at least one fiber of dimension ≥ 2, $\dim \mathrm{N}_1(E_1, X) = 2$, and $1., \cdots, 5.$ still holds.

The next lemma is a technical lemma that we will use in the proof of Lemma (4.8). We will just give a sketch of the proof since it involves results from the analysis of families of rational curves on varieties. One can find this results, for example, in [Kol13] or in [Deb01].

Lemma 4.3. *Let X be a Fano manifold of dimension n. Suppose that*

1. *$\exists\, R_1 \subset NE(X)$ a ray of type $(n-1, 1)$;*

2. *$\exists\, R_2 \subset NE(X)$ a ray of type $(n-1, n-2)^{sm}$ such that $E_1 \cdot R_2 > 0$, $E_2 \cdot R_1 > 0$, and $R_2 \not\subset \mathrm{N}_1(E_1, X)$;*

3. $\exists\ R_3 \subset NE(X)$ *a ray of type* $(n-1, n-2)^{sm}$ *such that* $E_2 \cdot R_3 < 0$ *and* $R_2 + R_3$ *is a face of* $NE(X)$.

Then the following hold:

$$E_1 \cdot R_3 = 0, \text{ and for every curve } C \subset E_2 \text{ we have } [C] \in R_1 + R_2 + R_3.$$

Proof. Foremost we prove that $N_1(E_2, X) = \mathbb{R}(R_1 + R_2 + R_3)$. By Lemma (1.3.29) φ_2 has one-dimesional non-trivial fibers, so by Theorem (1.3.24) R_2 is birational of type $(n-1, n-2)^{sm}$. Observe that E_2 is smooth, since it is the exceptional locus of a birational contraction of type $(n-1, n-2)^{sm}$. Both R_2 and R_3 are divisorial and negative on E_2, so $\varphi_2|_{E_2}$ and $\varphi_3|_{E_2}$ gives a $\mathbb{P}^1$-bundle on E_2 by Remark (1.3.31).

We claim that $N_1(E_2, X) = \mathbb{R}(R_1 + R_2 + R_3)$.

Since $R_2 \cdot E_2 < 0$ and $R_3 \cdot E_2 < 0$, then $R_2, R_3 \subset N_1(E_2, X)$. By assumption $E_2 \cdot R_1 > 0$ and R_1 is birational of type $(n-1, 1)$. Thus $R_1 \subset N_1(E_2, X)$ by Corollary (1.3.30). Hence $R_i \subset N_1(E_2, X)$ for $i \in \{1, 2, 3\}$. Observe that E_1 meets every non-trivial fiber of $\varphi_2|_{E_2}$, so $\varphi_2(E_1 \cap E_2) = \varphi_2(E_2)$ and

$$\mathbb{R}(R_1 + R_2 + R_3) \subseteq N_1(E_2, X) \subseteq \mathbb{R}R_2 + N_1(E_1, X).$$

Since all three subspaces are three dimensional, then $N_1(E_2, X) = \mathbb{R}(R_1 + R_2 + R_3)$.

No we consider the normalization $\nu : T \to Y_1$ of $\varphi_1(E_2)$ in Y_1 nad the contraction $\xi : E_2 \to T$ induced by the inclusion $i : E_2 \hookrightarrow X$ and by the restiction $\varphi_{1|_{E_2}}$:

$$\begin{array}{ccc} E_2 & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ T & \underset{\nu}{\longrightarrow} & Y_1. \end{array}$$

Then the following holds:

1. $i_*(\ker(\xi_*)) = \ker(\varphi_1|_{E_1}) = \ker(\varphi_{1*}) = \mathbb{R}R_1$
2. ξ is birational;
3. $\mathrm{Exc}(\xi) = E_1 \cap E_2$;
4. $\xi(\mathrm{Exc}(\xi)) \subset T$ is a curve.

By construction ρ_T is the codimension of $\ker(\xi_*)$ in $N_1(E_2)$, hence $\rho_T \geq 2$ and $\rho_T = 2$ if and only if $\ker(i_*) \subset \ker(\xi_*)$. Then we construct a proper, covering family of irreducible rational curves in T (see [Deb01, Chapter 5]). Let $A \doteq \varphi_2(E_2)$ and consider the following diagram

$$\begin{array}{ccc} E_2 & \overset{\xi}{\longrightarrow} & T\ . \\ {\scriptstyle \varphi_2|_{E_2}}\downarrow & & \\ A & & \end{array}$$

This family induces an E_2-equivalence relation as [Deb01, Section 5.4]. This allows us to construct a contraction $\alpha : T \to C$ onto a smooth curve C where every fiber of $\alpha : T \to C$ is an equivalence class for the E_2-equivalence relation.

As a consequence one can see that $\rho_T = 2$, $\ker(i_*) \subset \ker(\xi_*)$, and $\mathrm{NE}(\varphi_3|_{E_2}) \not\subset \ker(\alpha \circ \xi)_*$.

Recall that $R_3 \cdot E_2 < 0$, so $E_1 \cdot R_3 \geq 0$ but one can see that $R_3 \cdot E_1 = 0$. Hence $R_3 \subset \mathrm{N}_1(E_1, X)$, $\mathrm{N}_1(E_1, X) = \mathbb{R}(R_1 + R_3)$, and

$$\mathrm{N}_1(E_1, X) \cap \mathrm{NE}(X) = R_1 + R_3.$$

To conlcude the proof of this lemma one can use what we proved so far to obtain that id C is a curve in E_2 then

$$[C] \in R_1 + R_2 + R_3. \qquad \square$$

Lemma 4.4. *Let X be a Fano manifold of dimension n. Suppose that there are three divisorial rays R_1, R_2 and R_4 such that:*

1. *$R_2 + R_4$ is a face of $NE(X)$;*
2. *R_1 is of type $(n-1, 1)$;*
3. *$E_1 \cdot R_4 > 0$;*
4. *$E_2 \cdot R_1 > 0$;*
5. *$R_2 \not\subset \mathrm{N}_1(E_1, X)$;*
6. *$R_4 \not\subset \mathrm{N}_1(E_1, X)$.*

Then $(\varphi_2)_*(R_4)$ is of fiber type and $\rho_X \leq 4$.

Proof. Set $D \doteq \varphi_2(E_1)$. Since $R_2 + R_4$ is a face, then $(\varphi_2)_*(R_4)$ is a ray of $\mathrm{NE}(Y_2)$. Since $E_1 \cdot R_4 > 0$ and φ_2 is finite on E_1, then $D \cdot (\varphi_2)_*(R_4) > 0$. Since the locus of $(\varphi_2)_*(R_4)$ contain $\varphi_2(E_4)$, then the contraction is either divisorial or of fiber type. By Lemma(1.3.29), and by Theorem (1.3.24), R_4 is of type $(n-1, n-2)^{sm}$. Consider a non-trivial fiber of φ_4, $F \subset X$. Since $E_2 \cdot R_1 > 0$, then $\varphi_2(F) \cdot \varphi_2(E_4) \geq 0$. Therefore φ_2 cannot be divisorial. Thus $(\varphi_2)_*(R_4)$ is of fiber type. By using Lemma (2.7), $\rho_{Y_2} \leq 3$ hence $\rho_X = \rho_{Y_2} + 1 \leq 4$. $\square$

Remark 4.5. Observe that the Lemma (4.4), holds also with weaker hypotesis. Indeed one could ask that R_1 is divisorial and $\dim \mathrm{N}_1(E_1, X) = 2$.

Lemma 4.6. *Let X ba a Fano manifold of dimension n. Suppose there are R_i divisorial rays for $i = 1, 2, 3, 4$ such that :*

1. *$R_2 + R_4$ is not a face of $NE(X)$;*
2. *$\mathrm{N}_1(E_1, X) = \mathbb{R}(R_1 + R_3)$;*
3. *$E_1 \cap E_4 = \emptyset$;*

4. $E_1 \cdot R_3 = 0$.

Let S be a ray of $NE(Y_4)$ positive on $\varphi_4(E_1)$. Let R_5 be the extremal ray of $NE(X)$ such that $R_4 + R_5$ is a face and $\varphi_{4*}R_5 = S$.

Then one of the following occours:

1. $\rho_X \leq 4$;

2. *R is of type $(n-1, n, 2)^{sm}$ and $R \not\subset \mathrm{N}_1(E_1, X)$.*

Proof. By contruction $R_2 \neq R_5$. Since $E_4 \cap E_1 = \emptyset$, then $\varphi_4^{-1}(\varphi_4(E_1)) = E_1$. Hence $E_1 \cdot R_5 > 0$. Thus $R_3 \neq R_5$ and $R_1 \neq R_5$.

Since $\mathrm{N}_1(E_1, X) \cap \mathrm{NE}(X) = R_1 + R_3$, then $R_5 \not\subset \mathrm{N}_1(E_1, X)$. Therefore the contraction of R_5 has every non-trivial fiber of dimension 1. Since X is Fano, R_5 is not small by Corollary (1.3.21). By Theorem (3.6), either $\rho_X \leq 4$ or R_5 is of type $(n-1, n-2)^{sm}$ with $R_5 \not\subset \mathrm{N}_1(E_1, X)$. □

Lemma 4.7. *[Cas09, Remark 4.7] Let X be a Fano manifold of dimension $n \geq 4$ with a divisorial ray R_1 of type $(n-1, 1)$. Suppose that exists a birational extremal ray R_2 such that $E_1 \cdot R_2 > 0$, $E_2 \cdot R_1 = 0$ and $R_2 \not\subset \mathrm{N}_1(E_1, X)$. Then the following hold:*

1. $\mathrm{N}_1(E_2, X) = \mathbb{R}R_2 + \mathbb{R}R_1$;

2. *R_2 is of type $(n-1, n-2)^{sm}$.*

Furthermore Y_2 is Fano, it has an elementary contraction σ of type $(n-1, 1)$ and φ_2 is the blow-up of a smooth fiber of σ.

Proof. Every fiber of φ_2 is at most one-dimensional by Lemma (1.3.29). Then by Theorem (1.3.24) φ_2 is of type $(n-1, n-2)^{sm}$. Observe that $E_1 \cap E_2 \neq \emptyset$. Since $E_2 \cdot R_1 = 0$, then E_2 has to contain some fibers F of φ_1 of dimension $n-2 = \dim \varphi_2(E_2)$. Note that $R_2 \not\subset \mathrm{N}_1(F, X)$ since $R_1 \neq R_2$. Thus φ_2 is finite on F, so $\varphi_2(F) = \varphi_2(E_2)$. Hence

$$\mathrm{N}_1(E_2, X) = \mathbb{R}R_2 + \mathrm{N}_1(F, X) = \mathbb{R}(R_1 + R_2).$$

This allow us to conclude that every extremal ray $S \subset \mathrm{NE}(X)$ different from R_2 must have non-negative intersection with E_2, i.e. $E_2 \cdot S \geq 0$. Indeed, $\mathrm{N}_1(E_2, X) = \mathbb{R}R_2 + \mathrm{N}_1(F, X) = \mathbb{R}R_2 + \mathbb{R}R_1$ and $R_1 \cdot E_1 = 0$. Hence Y_2 is Fano by Lemma (1.4.9).

By Lemma (1.4.12), $R_1 + R_2$ is a face whose contraction is birational because both contractions are birational. Note that E_1 cannot be sent to a point by the contraction associated with the face $R_1 + R_2$, otherwise we would have $\mathrm{N}_1(E_1, X) = \mathbb{R}(R_1 + R_2)$. However, this is not possible because $R_2 \not\subset \mathrm{N}_1(E_1, X)$. Then $(\varphi_2)_*(R_1)$ is a ray of type $(n-1, 1)$ of $\mathrm{NE}(Y_2)$ that send $\varphi_2(E_1)$ to a curve. The associated contraction $\sigma : Y_2 \to Z$ has exceptional divisor $\varphi_2(E_1)$ and φ_2 is the blow-up of a fiber of such contraction. □

Lemma 4.8. *Let X be a Fano manifold of dimension $n \geq 4$. Suppose there exists a divisorial extremal ray $R_1 \subset NE(X)$ of type $(n-1, 1)$. Moreover suppose that $\dim \mathrm{N}_1(E_1, X) = 2$. Let R_2 be an extremal ray such that:*

1. *R_2 is positive on E_1, i.e. $E_1 \cdot R_2 > 0$;*

2. *R_2 is divisorial;*

3. *$R_1 \cdot E_2 > 0$;*

4. *$R_2 \not\subset \mathrm{N}_1(E_1, X)$;*

5. *Y_2 is not Fano.*

Then $\rho_X \leq 4$.

Proof. Since Y_2 is not Fano, by Corollary (2.1.5), there is a ray $\widetilde{R}$ with non-positive anticanonical degree such that the associated contraction $\widetilde{\psi} : Y_2 \to \widetilde{Z}$ is small.

Let us show that every non-trivial fiber is at most one dimensional. By contradiction, suppose that $\widetilde{\psi}$ admits a fiber F with dimension at least 2. By Remark (2.1.6) $\dim \varphi_2(F) \geq 3$, hence $\dim(\varphi_2^{-1}(F) \cap E_1) \geq 2$. Consider a non-trivial fiber of φ_1, F_1. F_1 has dimension $n-2$. Hence $\varphi_2^{-1}(F) \cap E_1$ and F_1 have to intersect in a subset of dimension at least 1, so $\varphi_2^{-1}(F) \cap E_1$ contains a curve with numerical class in R_1. Thus $R_1 \subset \mathrm{N}_1(\varphi_2^{-1}(F), X)$ and $\mathrm{NE}(\widetilde{\psi}) = (\varphi_2)_*(R_1)$, so $D \subseteq \mathrm{Exc}(\widetilde{\psi})$. This is not possible because $\mathrm{Exc}(\widetilde{\psi})$ is strictly smaller than D.

Then $\widetilde{\psi}$ is a small contraction with fibers of dimension at most one. Consider the elementary contraction φ_3 such that

$$\mathrm{NE}(\widetilde{\psi} \circ \varphi_2) = \mathrm{NE}(\varphi_2) + \mathrm{NE}(\varphi_3) = R_2 + R_3.$$

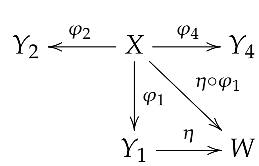
Note that φ_3 exists because if X is a Fano variety, then it always exists a lift of $\widetilde{\psi}$ by Remark (1.4.10).

We can apply Lemma (3.7). Then R_3 is of type $(n-1, n-2)^{sm}$, $R_3 \cdot E_2 < 0$ and $R_2 + R_3$ is a face of $\mathrm{NE}(X)$. X has at least 3 different rays, so $\rho_X \geq 3$.

By Lemma (4.3) we know that

$$E_1 \cdot R_3 = 0, \text{ and for every curve } C \subset E_2 \text{ we have } [C] \in R_1 + R_2 + R_3.$$

Furthermore note that $\mathrm{N}_1(E_1, X) = \mathbb{R}(R_1 + R_3)$. Consider $\varphi_1 : X \to Y_1$. Recall that it is divisorial of type $(n-1, 1)$ and consider the prime divisor $\varphi_1(E_2) \subset Y_1$. By Corollary (2.6), we may take a ray $\mathrm{NE}(\eta) \subset \mathrm{NE}(Y_1)$ positive on it, i.e. $\varphi_1(E_2) \cdot \mathrm{NE}(\eta) > 0$. Denote $\eta : Y_1 \to W$. Let R_4 be the extremal ray of $\mathrm{NE}(X)$ such that $R_1 + R_4$ is a face and $(\varphi_1)_*(R_4) = \mathrm{NE}(\eta)$.



By Lemma (4.3), $N_1(E_2, X) = \mathbb{R}(R_1 + R_2 + R_3)$. Hence $\dim(N_1(\varphi_1(E_2), Y_1)) = 2$, because $N_1(\varphi_1(E_2), Y_1) = \mathbb{R}(\varphi_{1*}R_2 + \varphi_{1*}R_3)$. By Lemma (2.7), if η is of fiber type, then $\rho_W \leq 2$ and $\rho_X \leq 4$. Suppose now that η is birational. Let us prove that every non-trivial fiber is one-dimensional. First we will show that η is finite on $\varphi_1(E_2)$. Indeed, since $NE(\eta \circ \varphi_1) = NE(\varphi_4) + NE(\varphi_1)$, then if η is not finite on $\varphi_1(E_2)$, it exists a curve $[C] \in R_4$ such that $C \subset E_1 \cup E_2$. Then $[C] \in R_4$ and $[C] \in R_1 + R_2 + R_3$. Moreover, since $N_1(E_1, X) \cap N_1(X) = R_1 + R_3$, then $R_4 = R_2$ or $R_4 = R_3$. Both cases cannot happen, otherwise we would get $\mathrm{Exc}(\eta) = \varphi_1(E_2)$ and $\varphi_1(E_2) \cdot NE(\eta) < 0$, a contradiction.

By Lemma (1.3.29), then every non-trivial fiber of η has dimension 1. Then by Lemma (3.7) the following hold:

1. R_4 is of type $(n-1, n-2)^{sm}$;
2. $\mathrm{Exc}(\eta) = \varphi_1(E_4)$.

Note that $R_1 \not\subset (E_4, X)$. Otherwise $R_4 = R_3$, which we already know that it is not possible. Hence $\varphi_1(E_4)$ is a divisor. Again by Lemma (3.7), the following hold:

1. $E_1 \neq E_4$;
2. η is a divisorial Mori contraction and $\mathrm{Exc}(\eta) \cap \varphi(E_1)$ is a union of fibers of η;
3. $R_4 \cdot E_1 = 0$.

We know that $\mathrm{Exc}(\eta) \cap \varphi_1(E_1)$ is a union of fibers of η but η is finite on $\varphi_1(E_1)$ because $\varphi_1(E_1) \subset \varphi_1(E_2)$. Hence $\mathrm{Exc}(\eta) \cap \varphi_1(E_1) = \emptyset$ and $E_1 \cap E_4 = \emptyset$. Note that $\mathrm{Exc}(\eta)$ must intersect $\varphi_1(E_2)$, so $E_4 \cap E_2 \neq \emptyset$. E_2 cannot contain curves with numerical classi in R_4 because $N_1(E_2, X) = R_1 + R_2 + R_3$ and $R_4 \neq R_i$ for $i = 1, 2, 3$. Then $E_2 \cdot R_4 > 0$.

Now consider the subspace $R_2 + R_4$ which can be a face or not.

Suppose that $R_2 + R_4$ is a face. By Lemma (4.4), then $\rho_X \leq 4$.

Suppose that $R_2 + R_4$ is not a face of $NE(X)$. By Corollary (2.6), there exists a ray S of $NE(Y_4)$ positive on the divisor $\varphi_4(E_1)$, i.e. $\varphi_4(E_1) \cdot S > 0$. Consider the extremal ray $R_5 \subset NE(X)$ such that $R_4 + R_5$ is a face and $(\varphi_4)_*(R_5) = S$. By Lemma (4.6), either $\rho_X \leq 4$ or R_5 is of type $(n-1, n-2)^{sm}$ with $R_5 \not\subset N_1(E_1, X)$. We are going to see that the latter cannot happen.

We apply similar arguments to R_5 to the one applied for the case R_2 divisorial. First note that $R_5 \cdot E_1 > 0$ and $R_5 \not\subset N_1(E_1, X)$. By Lemma (1.3.29), every non-trivial fiber is one-dimensional. Since X is Fano, R_5 is either of fiber type or divisorial by Corollary (1.3.21). If R_5 is of fiber type, $\rho_X \leq 3$ by Lemma (2.7). If R_5 is of divisorial type, then it is of type $(n-1, n-2)^{sm}$ by Theorem (1.3.24). Note that $R_1 \cdot E_5 \geq 0$ and $E_1 \cap E_5 \neq \emptyset$. We will distinguish the two cases $E_5 \cdot R_1 > 0$, and $E_5 \cdot R_1 = 0$.

Suppose $E_5 \cdot R_1 = 0$. By Lemma (3.9) applied to E_5 and φ_1 either $\rho_X \leq 4$, or there exists an extremal ray $R_0 \neq R_1$ of $NE(X)$ of type $(n-1, n-2)^{sm}$

such that $E_1 \cdot R_0 < 0$. We will see that the last one cannot occour. Indeed $R_0 \subset \mathrm{N}_1(E_1, X)$, so $R_0 = R_1$ but they are of different rays (for instance they have different intersection with E_1).

Suppose $E_5 \cdot R_1 > 0$ and let $\varphi_5 : X \to Y_5$ be associated contraction to R_5. If Y_5 is Fano, then $\rho_X \leq 4$ by Lemma (4.1). Suppose Y_5 not Fano. By a similar argument to the one applied for R_2, it exists an extremal ray $R_6 \neq R_5$ of type $(n-1, n-2)^{sm}$ such that $E_5 \cdot R_6 < 0$, $R_6 + R_5$ is a face, and $E_1 \cdot R_6 = 0$.

Summarizing one of the following holds:

- $\rho_X \leq 4$;
- it exists a ray $R_6 \neq R_5$ of type $(n-1, n-2)^{sm}$ such that $E_5 \cdot R_6 < 0$, $R_6 + R_5$ is a face, and $E_1 \cdot R_6 = 0$.

If $\rho_X \leq 4$ we are done. Suppose we are in the last case, so $R_6 \subset \mathrm{N}_1(E_1, X) = \mathbb{R}R_1 + \mathbb{R}R_3$. Note that $R_6 \neq R_1$ because they are of different types. Hence $R_3 = R_6$. Then $E_6 = E_3 = E_2$. By $E_5 \cdot R_6 < 0$, $E_5 = E_6$. Hence $E_5 = E_2$. Consider a curve $C \subset E_5$ with numerical class in $[C] \in R_5$, then $C \subset E_2$. Therefore $[C] = R_1 + R_2 + R_3$. This is impossible because $R_5 \neq R_i$ for $i = 1, 2, 3$. □

The next Theorem allows us to obtain a bound on the Picard number of a Fano manifold admitting a contraction sending a divisor onto a curve. We will obtain that the Picard number is at most 5. A bound for the case 6. of the next Theorem will be shown separately in Proposition (4.10).

Theorem 4.9. *[Cas09, Theorem 4.2.] Let X be a Fano manifold of dimension $n \geq 4$ and let R_1 be an extremal ray of type $(n-1, 1)$. Let R_2 be an E_1-positive ray. Then one of the following holds:*

1. *φ_2 is either of type $(n, n-2)$, or $(n, n-1)$, or $(n-1, n-3)$, and $\rho_X = 2$;*
2. *φ_2 is a conic bundle and $\rho_X = 3$;*
3. *φ_2 is of type $(n-2, n-4)$ and $\rho_X \leq 3$;*
4. *$n = 4$, φ_2 is of type $(2, 0)$ and $\rho_X = 4$;*
5. *$\rho_X \leq 4$ and either φ_2 is of type $(n-1, n-2)^{sm}$, or φ_2 is of type $(n-1, n-2)$;*
6. *φ_2 is of type $(n-1, n-2)^{sm}$, $E_2 \cdot R_1 = 0$, and there exists an extremal ray $R_0 \neq R_1$ such that $E_1 \cdot R_0 < 0$.*

Proof. Since $R_2 \cdot E_1 > 0$, we infer that E_1 intersects every non-trivial fiber of φ_2. Let F be an irreducible component of a non-trivial fiber of φ_2; then φ_1 is finite on $E_1 \cap F$ because φ_1 and φ_2 correspond to contractions of different extremal rays and F is a fiber of φ_2. Thus

$$\dim F - 1 \leq \dim(F \cap E_1) = \dim \varphi_1(E_1 \cap F) \leq 1,$$

so every non-trivial fiber of φ_2 is at most two-dimensional:

Since $\dim \mathrm{N}_1(E_1, X) = 2$ and $R_2 \cdot E_1 > 0$, we can apply Theorem (3.6). Recall that, for the Corollary (1.3.22), if R_2 is small is of type $(n-2, n-4)$ so the cases (i) and (ii) of Theorem (3.6) gives the cases 1., 2., 3., and 5. with $\rho_X \leq 3$ of the statement.

We are left to consider the following cases:

1. φ_2 is of type $(n-1, n-2)^{sm}$ and $R_2 \not\subset \mathrm{N}_1(E_1, X)$;

2. φ_2 is small and it exists a smooth prime divisor $D' \subset X$ with a $\mathbb{P}^1$-bundle structure which is disjoint from E_2. Furthermore if $\pi : D' \to Y$ is the map giving the bundle structure, then for every fiber f of π the following hold:

    (a) $E_1 \cdot f > 0$;

    (b) $D' \cdot f = -1$;

    (c) $f \not\subset E_1$.

The two cases will be treated separately.

Case 1: φ_2 is of type $(n-1, n-2)^{sm}$. Since R_1 and R_2 are two different divisorial rays and $R_2 \not\subset \mathrm{N}_1(E_1, X)$, then $E_2 \cdot R_1 \geq 0$. Hence, either $E_2 \cdot R_1 = 0$ or $E_2 \cdot R_1 > 0$.

Suppose $E_2 \cdot R_1 = 0$. Then by Lemma (4.7)

$$\mathrm{N}_1(E_2, X) = \mathbb{R}R_2 + \mathbb{R}R_1 = \mathbb{R}(R_1 + R_2).$$

By applying Lemma (3.9) to E_2 and φ_1, we obtain either $\rho_X \leq 4$ ond we gat case 5. of the statement, or φ_2 is divisorial of type $(n-1, n-2)^{sm}$, $E_2 \cdot R_1 = 0$, and there exists an extremal ray $R_0 \neq R_1$ of $\mathrm{NE}(X)$ such that $E_1 \cdot R_0 < 0$. Hence we heve case 6. of the statement.

Suppose $E_2 \cdot R_1 > 0$. In this case $D \doteq \varphi_2(E_1) \subset Y_2$ is a prime divisor, E_2 intersect every non-trivial fiber of φ_1 and $A \doteq \varphi_2(E_2)$ is a smooth subvariety of dimension $n-2$ (recall that R_2 is of type $(n-1, n-2)^{sm}$). Let C be an irreducible curve of Y_2 not contained in A. Lemma (2.1.3) yields to $-K_{Y_2} \cdot C \geq 1$ and the inequality is strict whenever C intersects A. If Y_2 is Fano, by Lemma (4.1), $\rho_X \leq 4$. If Y_2 is not Fano, by Lemma (4.8), $\rho_X \leq 4$. In both cases we obtain case 5. of the statement.

Case 2: φ_2 small. Recall that every fiber of φ_2 is at most two-dimensional and that exists a divisor $D' \subset X$ with a $\mathbb{P}^1$-bundle structure which is disjoint from E_2. Furthermore, if $\pi : D' \to Y$ is the map giving the bundle structure, then for every fiber f of π the following hold:

1. $E_1 \cdot f > 0$;

2. $D' \cdot f = -1$;

3. $f \not\subset E_1$.

X is Fano and every fiber is at most two-dimensional. We can apply Corollary (1.3.22) to φ_2, so φ_2 is of type $(n-2, n-4)$. Note that $E_1 \cap E_2 \neq \emptyset$, so $\dim(E_1 \cap E_2) \geq n-3$. Hence $\dim(E_1 \cap E_2) = n-3$. Let F_2 be a non-trivial fiber of φ_2. Since $E_1 \cap F_2 \neq \emptyset$, then $\dim(E_1 \cap F_2) = 1$ so $\varphi_2(E_1 \cap F_2) = \varphi_1(E_1)$. Hence F_2 intersects horizzontaly φ_1. Thus E_2 intersects every non-trivial fiber of φ_1. Then every non-trivial fiber of φ_1 cannot be contained in D' because $D' \cap E_2 = \emptyset$. Therefore $D' \cdot R_1 > 0$ and D' intersects every non-trivial fiber of φ_1. Since D' and E_2 are disjoint, then φ_1 is finite on E_2. Hence

$$n-3 = \dim(E_1 \cap E_2) = \dim \varphi_1(E_1 \cap E_2) \leq 1,$$

hence $n = 4$ and φ_2 is of type $(2, 0)$. Since R_2 is small and $E_1 \cdot R_2 > 0$, then $R_2 \subset \mathrm{N}_1(E_1, X)$ by Corollary (1.3.30). Thus $\mathrm{N}_1(E_1, X) \cap \mathrm{NE}(X) = R_1 + R_2$. Note that $[f] \notin \mathrm{N}_1(E_1, X)$. Otherwise $[f] = aR_1 + bR_2$ with $a, b \geq 0$. This would imply:

$$0 > [f] \cdot D' = aR_1 \cdot D' + b \cdot D' = a(R_1 \cdot D'),$$

and $a < 0$, a contradiction. Moreover $\pi(D' \cap E_1) = Y$, since $D' \cdot f > 0$. Thus

$$\mathrm{N}_1(D', X) = \mathbb{R}[f] \oplus \mathbb{R}R_1 \oplus \mathbb{R}R_2$$

and $\dim \mathrm{N}_1(D', X) = 3$.

Since $D' \cdot f < 0$, there is an extremal ray $\widetilde{R}_2 \subset \mathrm{NE}(X)$ with negative intersection with f, i.e. $\widetilde{R}_2 \cdot D' < 0$. Foremost, note that $\widetilde{R}_2$ is not of fiber type, so $\widetilde{R}_2$ is birational. Let F be and irreducibile component of the exceptional locus $\widetilde{E_2}$ of $\widetilde{R}_2$. If $\widetilde{R}_2$ were small, by Theorem (1.3.14) $F \cong \mathbb{P}^2$. Hence $\pi(F) = Y$, and $\dim \mathrm{N}_1(D', X) = 2$, which would give a contradiction. Therefore $\widetilde{R}_2$ is divisorial. Since $\widetilde{R}_2 \cdot D' < 0$ and D' is a prime divisor, D' is the exceptional divisor of $\widetilde{R}_2$ by Proposition (1.3.13). Since $\mathrm{N}_1(E_1, X) \cap \mathrm{NE}(X) = R_1 + R_2$, $D' \cdot R_1 > 0$, and $D' \cdot R_2 = 0$, $\widetilde{R}_2 \not\subset \mathrm{N}_1(E_1, X)$. Hence $\widetilde{R}_2$ is of type $(3, 2)^{sm}$, and $E_1 \cdot \widetilde{R}_2 > 0$.

We apply similar arguments to $\widetilde{R}_2$ to the one applied when R_2 is of divisorial type. Let $\widetilde{\varphi_2} : X \to W$ be the contraction associated with $\widetilde{R}_2$ and $\widetilde{E_2}$ be the exceptional divisor. Note that $\widetilde{E_2} \cdot R_1 \geq 0$. We will distinguish the two cases $\widetilde{E_2} \cdot R_1 > 0$ and $\widetilde{E_2} \cdot R_1 = 0$.

Suppose $\widetilde{E_2} \cdot R_1 = 0$. By Lemma (3.9) applied to E_5 and φ_1 either $\rho_X \leq 4$, or there exists an extremal ray $\widetilde{R}_0 \neq R_1$ of $\mathrm{NE}(X)$ of type $(n-1, n-2)^{sm}$ such that $E_1 \cdot \widetilde{R}_0 < 0$. Suppose that $\rho_X \geq 5$. Then $\widetilde{R}_0$ si contained in $\mathrm{N}_1(E_1, X)$ but $\widetilde{R}_0$ is different both from R_1 and form R_3. Hence we obtain a contradiction. If $\widetilde{E_2} \cdot R_1 > 0$. Then we will distinguish the two cases: W Fano or W, not Fano.

Suppose W Fano, then $\rho_X \leq 4$ by Lemma (4.1).

Suppose W not Fano. It exists an extremal ray R_7 of divisorial type such that $R_7 \neq \widetilde{R}_2$, $\widetilde{E_2} \cdot R_7 < 0$ and $E_1 \cdot R_7 = 0$. This last case leads to a contradiction because R_7 is different form R_1 and R_2. □

The next Proposition allows us to conclude that a Fano manifold with a divisorial contraction of type $(n-1,1)$ has Picard number at most 5.

Proposition 4.10. *[Cas09, Proposition 4.8.] Let X a Fano manifold of dimension $n \geq 4$ with a divisorial contraction of type $(n-1,1)$. Suppose that there exists a ray $R_0 \neq R_1$ with $E_1 \cdot R_0 < 0$. Then $\rho_X \leq 5$ and $R_0 + R_1$ is a face of $NE(X)$. Furthermore, $E_1 \cong W \times \mathbb{P}^1$ where W is a Fano manifold, and φ_0 is the blow-up of a smooth subvariety isomorphic to W.*

If $\rho_X = 5$, then there exists a Fano manifold Z with $\rho_Z = 3$ and $\dim Z = n$, having an elementary contraction of type $(n-1,1)$, such that X is the blow-up of Z along two fibers of such contraction.

Proof. Let F_0 be a non-trivial fiber of φ_0. Every non-trivial fibers of φ_1 have dimension $n-2$ and $R_0 \neq R_1$, so F_0 is one-dimensional. Since φ_0 is a Mori contraction, φ_0 is not small by Corollary (1.3.21). Furthermore, φ_0 is not of fiber type because every non-trivial fiber is contained in E_1. Hence φ_0 is birational with fibers of dimension at most one. Hence φ_0 is of type $(n-1,n-2)^{sm}$, Y_0 is smooth, and E_1 is smooth by Theorem (1.3.24). Let $W \subset Y_0$ be the smooth codimension 2 subvariety blowed-up by φ_0, then E_1 has a $\mathbb{P}^1$-bundle structure over W given by φ_0 as in Remark (1.3.31).

Note that $\mathrm{N}_1(E_1, X) = \mathbb{R}(R_0 + R_1)$ and $\mathrm{N}_1(E_1, X) \cap \mathrm{NE}(X) = R_0 + R_1$. Hence there are no other extremal ray with negative intersection with E_1.

In the next step we will prove that $R_0 + R_1$ is a face of the Mori cone $\mathrm{NE}(X)$. For $i = 0, 1$ let C_i be a curve with class in R_i and consider a supporting nef divisor H_i for R_i (1.3.4). i.e. H_i is a nef divisor such that it intersect the Mori exactly on R_i. Hence if $S \subset \mathrm{NE}(X)$ an extremal ray, then $H_i \cdot S \geq 0$, and $H_i \cdot S = 0$ if and only if $S = R_i$. Consider the divisor

$$H \doteq (H_0 \cdot C_1)H_1 + (H_1 \cdot C_0)(-E_1 \cdot C_1)H_0 + (H_0 \cdot C_1)(H_1 \cdot C_0)E_1.$$

Let R be a ray of $\mathrm{NE}(X)$ such that $R \neq R_i$ for $i = 0, 1$. By the construction of supporting nef divisor for a ray (Theorem 1.3.4), $R \cdot H_i > 0$ for $i = 0.1$. Furthermore $H_0 \cdot C_1 > 0$, $H_1 \cdot C_0 > 0$, and $-E_1 \cdot C_1 > 0$. Hence $H \cdot R > 0$. Note that $H \cdot R_0 = 0$ and $H \cdot R_1 = 0$. Hence H is nef. To summarize, $R_0 + R_1$ is a face of the Mori cone with supporting nef divisor H.

Now we will prove that E_1 is Fano. Consider $\gamma \in \overline{\mathrm{NE}}(E_1) \setminus \{0\}$, then by projection and adjunction formula

$$-K_{E_1} \cdot \gamma = -(K_X + E_1)|_{E_1} \cdot \gamma = -(K_X + E_1) \cdot i_*(\gamma)$$

where $i : E_1 \hookrightarrow X$ is the inclusion. Consider A an ample divisor on X, then by projection formula

$$A \cdot i_*(\gamma) = A|_{E_1} \cdot \gamma > 0$$

because $A|_{E_1}$ is still ample. Hence $i_*(\gamma)$ is not zero. Since $\gamma \in \overline{\mathrm{NE}}(E_1) \setminus \{0\}$, we get $i_*(\gamma) \in R_0 + R_1$. Then $E_1 \cdot i_*(\gamma) < 0$. Thus E_1 is Fano.

The map $\varphi_1|_{E_1} : E_1 \to \varphi_1(E_1)$ is a surjective morphism with connected fibers sending E_1 to a curve. Recall that E_1 is also the exceptional divisor of φ_0, whose fibers are isomorphic to $\mathbb{P}^1$. Hence E_1 is covered by fibers of $\varphi_0|_{E_1}$ and $\varphi_1(E_1)$ is rational. By Theorem (1.3.33) $\varphi_0|_{E_1}$ induces a Mori contraction

$$\phi : E_1 \to \mathbb{P}^1$$

that does not contract the fibers of $\varphi_0|_{E_1}$. Then by Lemma (1.3.32), $E_1 = W \times \mathbb{P}^1$.

The next part of the proof will be focused on finding that $\rho_X \leq 5$.

By Lemma (1.4.11), we may consider R_2 a positive ray on E_1. Then $R_2 \neq R_0$ and $R_2 \neq R_1$, so $R_2 \not\subset \mathrm{N}_1(E_1, X)$. Thus $\rho(X) \geq 3$ and φ_2 is finite on E_1. By Theorem (1.3.25), either φ_2 is a conic bundle or φ_2 is a divisorial contraction of type $(n-1, n-2)^{sm}$. Suppose first that φ_2 is a conic bundle. By Lemma (2.7) $\rho_{Y_2} \leq 2$ and $\rho_X \leq 3$. Hence $\rho_{Y_2} = 2$ and $\rho_X = 3$.

Suppose that φ_2 is a divisorial contraction of type $(n-1, n-2)^{sm}$. Then Y_2 is smooth, $A \doteq \varphi_2(E_2)$ is a smooth subvariety and it is contained in the prime divisor $D \doteq \varphi_2(E_1)$ since E_1 intersects every non-trivial fiber of φ_2, i.e. $A \subset D \subset Y_2$.

Since E_1 intersects every non-trivial fiber of φ_2, then $\varphi_2(E_1 \cap E_2) = \varphi_2(E_2)$. Since $E_1 \neq E_2$ and $\mathrm{NE}(X) \cap \mathrm{N}_1(E_1, X) = R_0 + R_1$ then for every curve $C \subset E_1$ $E_2 \cdot C \geq 0$. Since φ_2 is of type $(n-1, n-2)^{sm}$, then

$$-K_X = \varphi_2^*(-K_{Y_2}) - E_2.$$

Consider a curve $C_2 \subset Y_2$. Let $C \subset X$ be the strict transform of C_2. If C_2 is not contained in $\varphi_2(E_2)$ then $C \cdot E_2 \geq 0$, hence

$$-K_{Y_2} \cdot C_2 = (\varphi_2^*(-K_{Y_2})) \cdot C = (-K_X + E_2) \cdot C > 0$$

If C_2 is contained in $\varphi_2(E_2)$, since $\varphi_2(E_1 \cap E_2) = \varphi_2(E_2)$ then C can be considered inside E_1. For every curve $C \subset E_1$, $E_2 \cdot C \geq 0$, so by using the projection formula as before, $-K_{Y_2} \cdot C_2 > 0$. Hence Y_2 is Fano.

Consider an elementary contraction $\psi : Y_2 \to Z$ such that $D \cdot \mathrm{NE}(\psi) > 0$. If ψ is of fiber type, then by Lemma (2.7), $\rho_Z \leq 2$ and $\rho_X \leq 4$. Suppose that ψ is birational. Then ψ is finite on D, because

$$\mathrm{N}_1(D, Y_2) \cap \mathrm{NE}(Y_2) = (\varphi_2)_*(\mathrm{N}_1(E_2, X) \cap \mathrm{NE}(X)) = (\varphi_2)_*(R_0) + (\varphi_2)_*(R_1).$$

Suppose was not finite on D, then $\mathrm{NE}(\psi) = (\varphi_2)_*(R_0)$ or $\mathrm{NE}(\psi) = (\varphi_2)_*(R_1)$. In both cases $\mathrm{Exc}(\psi) = D$ and this is not possible because $\mathrm{NE}(\psi)$ is positive on D. Since ψ is finite on D and $\mathrm{NE}(\psi) \cdot D > 0$, then by Lemma (1.3.29) every non-trivial fiber is one-dimensional. Thus ψ cannot be small by Corollary (2.1.5), so ψ is divisorial. Therefore, ψ is of type $(n-1, n-2)^{sm}$ and Z is smooth by Lemma (1.3.25). Since the hypothesis of Lemma (3.7) are satisfied and ψ is birational, than $\mathrm{Exc}(\psi) \cap A$ is a union of fibers of ψ. Since ψ is finite on A, then $\mathrm{Exc}(\psi) \cap A = \emptyset$. The composition $\psi \circ \varphi_2$ is the blow-up of two disjoint, smooth subvarieties of codimension 2.

$$\begin{array}{ccc} X & & \\ {\scriptstyle \varphi_2}\downarrow & \searrow^{\psi\circ\varphi_2} & \\ Y_2 & \xrightarrow[\psi]{} & Z \end{array}$$

Set $\widetilde{E_2} \doteq \varphi_2^{-1}(\mathrm{Exc}(\psi))$; $\mathrm{Exc}(\psi \circ \varphi_2) = E_2 \cup \widetilde{E_2}$.

Note that $E_2 \cap E_1$ has pure dimension at least 2. Since $\varphi_1(E_1)$ is a curve, then the map $\varphi_1|_{E_1 \cap E_2} : E_1 \cap E_2 \to \varphi_1(E_1)$ has fibers of positive dimension. Let F be a such fiber, and let $C \subset F$ be a curve. C is in a fiber, so $C \in R_1$ and $C \subset E_1 \cap E_2 \subset E_2$. Hence C is disjoint from $\widetilde{E_2}$ and $\widetilde{E_2} \cdot R_1 = 0$. In the same way $E_2 \cdot R_1 = 0$. Then $E_1 \cap E_2$ and $E_1 \cap \widetilde{E_2}$ are union of finitely many fibers of φ_1. By Lemma (1.4.12), $R_1 + R_2$ is a face on $\mathrm{NE}(X)$ and $S_1 \doteq (\varphi_2)_*(R_1)$ is a ray of type $(n-1, 1)$ of $\mathrm{NE}(Y_2)$ with exceptional divisor D. Note that the other possible ray contained in $\mathrm{N}_1(D, X)$ is $(\varphi_2)_*(R_0)$.

Observe that $E_2 \cdot R_0 > 0$. If $E_2 \cdot R_0 < 0$ then $E_0 = E_2$ but $E_0 = E_1$, hence $E_2 = E_1$. This is not possible since $R_2 \not\subset \mathrm{N}_1(E_1, X)$. If $E_2 \cdot R_0 = 0$, then some curve contracted by φ_0 are contained in E_2 and some are not. Hence $R_0, R_1, R_2 \subset \mathrm{N}_1(E_2, X)$. Hence E_2 contains a fiber F_i of R_i for $i = 1, 2, 0$. This is not possible because F_1 is of dimension $n-2$, and F_i of dimension 1 for $i = 0, 2$.

The ray S_1 is the only ray with negative intersection with D. Indeed, the other ray contained in $\mathrm{N}_1(D, X_2)$ is $S_0 \doteq (\varphi_2)_*(R_0)$, but $D \cdot \varphi_{2*}(R_0) \geq 0$.

Theorem (4.9) applied to Y_2, D and S_1 yelds to $\rho_{Y_2} \leq 4$ and $\rho_X \leq 5$.

Recall that $\mathrm{NE}(\psi) \subset \mathrm{NE}(Y_2)$ is a ray of type $(n-1, n-2)^{sm}$. Moreover $\widetilde{E_2} \cdot R_1 = 0$ and $\widetilde{E_2} \doteq \varphi_2^{-1}(\mathrm{Exc}(\psi))$ yields $\mathrm{Exc}(\psi) \cdot S_1 = 0$. Hence by Lemma (1.4.12), Z is Fano, $\psi_*(S_1)$ is of type $(n-1, 1)$ with exceptional divisor $\psi(D)$ and X is the consecutive blow-up of Z along two fibers of the associated contractions. Since φ_2 is finite on E_1 and ψ is finite on D, then $\psi \circ \varphi_2$ is finite on E_1. Hence the normalization of $\psi(D)$ is $W \times \mathbb{P}^1$. □

We will now give an application to the case of a Fano manifold of dimension 4.

Corollary 4.11. *[Cas09, Corollary 1.3] Let X be a Fano 4-fold. Then one of the following holds:*

1. *$\rho_X \leq 6$;*
2. *X is a product and $\rho_X \leq 11$;*
3. *every contraction of X is of type $(3, 2)$ or $(2, 0)$.*

Proof. Let X be a Fano 4-fold with $\rho_X \geq 7$. Then X cannot have elementary contractions of type:

1. $(3, 0)$, since otherwise $\rho_X \leq 3$;

2. $(3,1)$, since otherwise $\rho_X \leq 5$;
3. $(2,1)$ and $(1,0)$ because small K_X-negative contraction cannot have fibers of dimension 1 by Corollary (1.3.21);
4. $(4,0)$, since otherwise $\rho_X = 1$;
5. $(4,1)$, since otherwise $\rho_X = 2$.

Therefore the only possible elementary contractions are of type $(4,3)$, $(4,2)$, $(3,2)$ or $(2,0)$. If X has a contraction of type $(4,2)$ then by Corollary (1.4.13) $X \cong \mathbb{P}^2 \times S$ with S del Pezzo surface thus $\rho_X \leq 10$. By using again Corollary (1.4.14), if X has a contraction of type $(4,3)$, either $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S$ or $X \cong \mathbb{F}_1 \times S$ hence $\rho_X \leq 11$. If X is not a product, then it can only have contractions of type $(3,2)$ and $(2,0)$. □

5 Some applications to Fano 5-folds

We will now give some applications for the 5-fold case.

Notation: if we denote a ray with R_i, then its exceptional locus will be denoted with E_i, the associated contraction with $\varphi_i : X \to Y_i$, and a general fiber with F_i, unless otherwise stated.

Lemma 5.1. *Let X be a Fano 5-fold with $i_X > 1$. Suppose there exists an extremal ray R_0 of type $(4,2)$. Suppose moreover that there exists an extremal ray $R_1 \subset NE(X)$ such that $R_1 \neq R_0$, and $E_0 \cdot R_1 < 0$.*

Then $\rho_X = 3$, R_1 is of type $(4,2)^{sm}$, and $E_0 = \mathbb{P}^2 \times \mathbb{P}^2$.

Proof. First of all, observe that φ_1 is not of fiber type, so φ_1 is birational. Let F_1 be a non-trivial fiber of φ_1. Since every non-trivail fiber of φ_0 has dimension at least 2, then $\dim F_1 \leq 2$.

Suppose that $\dim F_1 = 1$, then φ_1 is of divisorial type by Corollary (1.3.21). Hence $E_0 = E_1$ and $l(R_1) = 1$ by Ionescu-Wiśniewski inequality (1.3.20). Thus this case cannot occurs, since $i_X > 1$.

Suppose $\dim F_1 = 2$. Then φ_1 can be either divisorial or small. Suppose that φ_2 is small. By Ionescu-Wiśniewski inequality (1.3.20) $l(R_1) = 1$, which contradicts $i_X > 1$. Hence φ_1 is a divisorial contraction. Thus $E_0 = E_1$ and every non-trivial fiber of φ_0 and of φ_1 is two-dimensional.

Note that R_0 and R_1 are rays of length $l(R_1) = l(R_2) = 2$ with fibers of dimension 2 in X, then by Lemma (1.3.23) R_0 and R_1 are of type $(4,2)^{sm}$.

Furthermore R_0 and R_1 are the only rays of $\mathrm{NE}(X)$ in E_0, so $\mathrm{N}_1(E_0, X) = \mathbb{R}(R_1 + R_0)$.

By Lemma (1.4.11), we can consider an extremal ray $R \subset \mathrm{NE}(X)$ positive on E_0 and let $\varphi : X \to Y$ be the associated contraction. Note that $\rho_X \geq 3$, and $R \not\subset \mathrm{N}_1(E_0, X)$. Then by Theorem (1.3.24), eihter φ is a conic bundle or φ is of type $(4,3)^{sm}$. If φ is of type $(4,3)^{sm}$, then $l(R) = 1$ a contradiction. Thus φ is a conic bundle and $\rho_X = 3$ by Lemma (2.7).

Consider the restrictions of φ_0 and φ_1 on E_0, $\varphi_{0|_{E_0}}$ and $\varphi_{1|_{E_0}}$. Then, by [Sat85, Theorem A (1)], $E_2 \cong \mathbb{P}^2 \times \mathbb{P}^2$. □

The next example is an example of a Fano 5-fold that satisfies the hypothesis of the previous Lemma.

Example 5.2. [CO06, Example j.1] Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)$. Consider $X = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{F})$. Let $\pi : X \to \mathbb{P}^2 \times \mathbb{P}^2$ be the projection map, ξ be the tautological line bundle on X, and E be the section that corresponds to the surjection

$$\mathcal{F} \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \to 0.$$

Hence

$$-K_X = 2\xi - \pi^*(K_{\mathbb{P}^2 \times \mathbb{P}^2} + \det \mathcal{F}) = 2(\xi + \pi^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2})).$$

Note that $\pi^*(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2})$ is nef, and by projection formula it vanishes only on the fibers f of π. Furthermore $\xi \cdot f = 1$. Therefore$X$ is a Fano variety and $i_X = 2$.

Observe that X has three extremal rays. One is of fiber type and corresponds to the projection π. The other two rays are of type $(4,2)$, have exceptional divisor E, and each one corresponds to the contraction of one of the "rulings" of E.

Lemma 5.3. *Let X be a Fano 5-fold. Suppose there are two extremal rays $R_0, R_1 \subset NE(X)$ of type $(4,2)$ such that $R_0 \cdot E_1 < 0$. Let $R_2 \subset NE(X)$ be an extremal ray positive on E_1. Then one of the following occour:*

1. *$\rho_X = 3$ and φ_2 is a conic bundle;*
2. *$\rho_X \in \{3,4\}$ and φ_2 is of type $(4,3)^{sm}$;*
3. *φ_2 is of type $(4,3)^{sm}$ and $R_0 \cdot E_2 = R_1 \cdot E_2 = 0$.*

Proof. First note that $E_1 = E_0$ and $\mathrm{N}_1(E_1, X) = \mathbb{R}(R_0 + R_1)$. Moreover the non-trivial fibers of φ_0 and φ_1 are equidimensional, thus every non-trivial fiber has dimension 2. By Lemma (1.4.11), we may consider an extremal ray $R_2 \subset \mathrm{NE}(X)$ positive on E_1. Since R_2 is different from R_1 and from R_0, then $\rho_X \geq 3$. Note that $R_2 \not\subset \mathrm{N}_1(E_1, X)$, so by Lemma (1.3.29) every non-trivial fiber is one-dimensional. Hence by Theorem (1.3.24), one of the following occours:

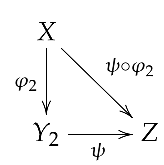
1. φ_2 is a conic bundle;
2. φ_2 is of type $(4,3)^{sm}$.

If φ_2 is a conic bundle then by Lemma (2.7), $\rho_X \leq 3$. Hence we obtain 1. of the statement. Now suppose that φ_2 is divisorial of type $(4,3)^{sm}$, so Y_2 is smooth and φ_2 is the blow-up of a smooth subvariety $A \doteq \varphi_2(E_2)$. Set $D \doteq \varphi_2(E_1)$, then $A \subset D$. Note that $\varphi_2(E_1 \cap E_2) = \varphi_2(E_2)$, and $C \cdot E_2 \geq 0$ for every $C \subset E_1$. Since

$$\varphi_2^*(-K_{Y_2}) = -K_X + E_2,$$

by projection formula Y_2 is Fano. Consider an elementary contraction $\psi : Y_2 \to Z$ such that $D \cdot \mathrm{NE}(\psi) > 0$. If ψ is of fiber type, then $\rho_Z \leq 2$, and $\rho_X \leq 4$. Hence $\rho_X \in \{3,4\}$ and we obtain 2. of the statement.

Suppose ψ birational; ψ is finite on D. Suppose by contradiction ψ not finite on D. Recall that $\mathrm{NE}(E_1, X) = \mathbb{R}(R_0 + R_1)$, then either $\mathrm{NE}(\psi) = (\psi)_*(R_0)$, or $\mathrm{N}_1(\psi) = (\psi)_*(R_1)$. In both cases $\mathrm{Exc}(\psi) = \varphi_2(E_1)$, but this is not possibile because $\mathrm{NE}(\psi) \cdot D > 0$. Thus ψ is finite on D, so by Theorem (1.3.24) ψ is of type $(4,3)^{sm}$ and Z is smooth.



By Lemma (3.7), $\mathrm{Exc}(\psi) \cap A$ is a union of fibers of ψ, but ψ is finite on A so $\mathrm{Exc}(\psi) \cap A = \emptyset$. Hence $\psi \circ \varphi_2$ is the blow-up of two disjoint subverieties of Z. Set $\widetilde{E}_2 \doteq \varphi_2^{-1}(\mathrm{Exc}(\psi))$, so $\mathrm{Exc}(\psi \circ \varphi_2) = E_2 \cup \widetilde{E}_2$ and $\widetilde{E}_2 \cap E_2 = \emptyset$.

Consider $E_1 \cap \widetilde{E}_2$; since $\mathrm{NE}(\psi) \cdot D > 0$, then $E_1 \cap \widetilde{E}_2 \neq \emptyset$. Therefore $E_1 \cap \widetilde{E}_2$ is of pure dimension, $\dim(E_1 \cap \widetilde{E}_2) \geq n - 2 = 3$. So $E_1 \cap \widetilde{E}_2$ has dimension 3, furthermore recall that $\dim \varphi_1(E_1) = 1$. Therefore

$$\varphi_1|_{E_1 \cap \widetilde{E}_2} : E_1 \cap \widetilde{E}_2 \to \varphi_1(E_1)$$

has fibers of dimension 1. Consider a curve $\widetilde{C}$ in one of this fiber, then $\widetilde{C} \subset \widetilde{E}_2$ and $[\widetilde{C}] \in R_1$. Since $E_2 \cap \widetilde{E}_2 = \emptyset$, then $E_2 \cap C = \emptyset$. Therefore $E_2 \cdot R_1 = 0$.

In the same way, by considering $\varphi_0|_{E_1 \cap \widetilde{E}_2} : E_1 \cap \widetilde{E}_2 \to \varphi_1(E_1)$, we obtain $E_2 \cdot R_0 = 0$. □

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